pISSN : 2093-9310
eISSN : 2287-6235

Annals of Fuzzy Mathematics and Informatics

AFMI

VOL.14 NO.1 July 2017

KYUNG MOON SA

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 1, (July 2017), pp. 1–120
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Contents
[The special issue for "Neutrosophic Sets"]

Special issue on "Neutrosophic Sets and their Applications"

The authors and co-authors, listed in the order of their published neutrosophic papers: Muhammad Akram, Muzzamal Sitara, A. A. A. Agboola, B. Davvaz, F. Smarandache, Ali Hassan, Muhammad Aslam Malik, Said Broumi, Assia Bakali, Mohamed Talea, K. Hur, P. K. Lim, J. G. Lee, J. Kim, Young Bae Jun, Maryam Nasir, and A. Borumand Saeid, would like to thank Prof. Kul Hur, the Editor-in-Chief of the international journal Annals of Fuzzy Mathematics and Informatics (AFMI), for dedicating the whole Vol. 14, No.1, published on 25 July 2017, to the neutrosophic theories and applications. The papers included in this volume are especially referring to neutrosophic (single-valued and interval-valued) graphs and bipolar graphs, and their applications in multi-criteria decision making (MCDM), and to neutrosophic algebraic structures, such as: category of neutrosophic crisp sets, neutrosophic quadruple algebraic hyperstructures, and neutrosophic subalgebras of BCK/BCI-algebras. We would also like to bring our gratitude to many reviewers of the neutrosophic community, from around the world, community that has grew to over eight hundred peoples (students, faculty, and researchers).

Prof. Florentin Smarandache, Ph. D.
University of New Mexico
Gallup Campus, USA

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 1, (July 2017), pp. 1–27
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Application of intuitionistic neutrosophic graph structures in decision-making

Muhammad Akram, Muzzamal Sitara

Received 9 January 2017; Revised 8 February 2017; Accepted 16 February 2017

Abstract. In this research study, we present concept of intuitionistic neutrosophic graph structures. We introduce the certain operations on intuitionistic neutrosophic graph structures and elaborate them with suitable examples. Further, we investigate some remarkable properties of these operators. Moreover, we discuss a highly worthwhile real-life application of intuitionistic neutrosophic graph structures in decision-making. Lastly, we elaborate general procedure of our application by designing an algorithm.

2010 AMS Classification: 03E72, 05C72, 05C78, 05C99

Keywords: Intuitionistic neutrosophic graph structures, Decision making

Corresponding Author: Muhammad Akram (m.akram@pucit.edu.pk)

1. Introduction

Graphical models are extensively useful tools for solving combinatorial problems of different fields including optimization, algebra, computer science, topology and operations research etc. Fuzzy graphical models are comparatively more close to nature, because in nature vagueness and ambiguity occurs. There are many complex phenomena and processes in science and technology having incomplete information. To deal such cases we needed a theory different from classical mathematics. Graph structures as generalized simple graphs are widely used for study of edge colored and edge signed graphs, also helpful and copiously used for studying large domains of computer science. Initially in 1965, Zadeh [29] proposed the notion of fuzzy sets to handle uncertainty in a lot of real applications. Fuzzy set theory is finding large number of applications in real time systems, where information inherent in systems has various levels of precision. Afterwards, Turksen [26] proposed the idea of interval-valued fuzzy set. But in various systems, there are membership and non-membership values, membership value is in favor of an event and non-membership value is against of that event. Atanassov [8] proposed the notion of intuitionistic

fuzzy set in 1986. The intuitionistic fuzzy sets are more practical and applicable in real-life situations. Intuitionistic fuzzy set deal with incomplete information, that is, degree of membership function, non-membership function but not indeterminate and inconsistent information that exists definitely in many systems, including belief system, decision-support systems etc. In 1998, Smarandache [24] proposed another notion of imprecise data named as neutrosophic sets. "Neutrosophic set is a part of neutrosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra". Neutrosophic set is recently proposed powerful formal framework. For convenient usage of neutrosophic sets in real-life situations, Wang et al. [27] proposed single-valued neutrosophic set as a generalization of intuitionistic fuzzy set[8]. A neutrosophic set has three independent components having values in unit interval [0, 1]. On the other hand, Bhowmik and Pal [10, 11] introduced the notions of intuitionistic neutrosophic sets and relations. Kauffman [16] defined fuzzy graph on the basis of Zadeh's fuzzy relations [30]. Rosenfeld [21] investigated fuzzy analogue of various graph-theoretic ideas in 1975. Later on, Bhattacharya gave some remarks on fuzzy graph in 1987. Bhutani and Rosenfeld discussed M-strong fuzzy graphs with their properties in [12]. In 2011, Dinesh and Ramakrishnan [15] put forward fuzzy graph structures and investigated its properties. In 2016, Akram and Akmal [1] proposed the notion of bipolar fuzzy graph structures. Broumi et al. [13] portrayed single-valued neutrosophic graphs. Akram and Shahzadi [2] introduced the notion of neutrosophic soft graphs with applications. Akram and Shahzadi [4] highlighted some flaws in the definitions of Broumi et al. [13] and Shah-Hussain [22]. Akram et al. [5] also introduced the single-valued neutrosophic hypergraphs. Representation of graphs using intuitionistic neutrosophic soft sets was discussed in [3]. In this paper, we present concept of intuitionistic neutrosophic graph structures. We introduce the certain operations on intuitionistic neutrosophic graph structures and elaborate them with suitable examples. Further, we investigate some remarkable properties of these operators. Moreover, we discuss a highly worthwhile real-life application of intuitionistic neutrosophic graph structures in decision-making. Lastly, we elaborate general procedure of our application by designing an algorithm.
We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [3,6, 7, 9, 13, 14, 17, 18, 20, 22, 23, 25, 28, 30].

2. Intuitionistic Neutrosophic Graph Structures

Definition 2.1. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \ldots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, Cartesian product of $\check{G}_1$ and $\check{G}_1$ is defined as:

$$\check{G}_1 \times \check{G}_2 = (P \times P', P_1 \times P'_1, P_2 \times P'_2, \ldots, P_r \times P'_r),$$

where $P_h \times P'_h = \{(k_1 l)(k_2 l) \mid l \in P',\ k_1 k_2 \in P_h\} \cup \{(kl_1)(kl_2) \mid k \in p, l_1 l_2 \in P'_h\}$, $h = (1, 2, \ldots, r)$.

Definition 2.2. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \ldots, P_n)$ and $\check{G}_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, cross product of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 * \check{G}_2 = (P * P', P_1 * P'_1, P_2 * P'_2, \ldots, P_r * P'_r),$$

where $P_h * P'_h = \{(k_1l_1)(k_2l_2) \mid k_1k_2 \in P_h,\, l_1l_2 \in P'_h\}$, $h = (1, 2, \dots, r)$.

Definition 2.3. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \dots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \dots, P'_r)$ be two GSs, lexicographic product of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 \bullet \check{G}_2 = (P \bullet P', P_1 \bullet P'_1, P_2 \bullet P'_2, \dots, P_r \bullet P'_r),$$

where $P_h \bullet P'_h = \{(kl_1)(kl_2) \mid k \in P,\, l_1l_2 \in P'_h\} \cup \{(k_1l_1)(k_2l_2) \mid k_1k_2 \in P_h, l_1l_2 \in P'_h\}$, $h = (1, 2, \dots, r)$.

Definition 2.4. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \dots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \dots, P'_r)$ be two GSs, strong product of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 \boxtimes \check{G}_2 = (P \boxtimes P', P_1 \boxtimes P'_1, P_2 \boxtimes P'_2, \dots, P_r \boxtimes P'_r),$$

where $P_h \boxtimes P'_h = \{(k_1l)(k_2l) \mid l \in P',\, k_1k_2 \in P_h\} \cup \{(kl_1)(kl_2) \mid k \in P, l_1l_2 \in P'_h\} \cup \{(k_1l_1)(k_2l_2) \mid k_1k_2 \in P_h,\, l_1l_2 \in P'_h\}$, $h = (1, 2, \dots, r)$.

Definition 2.5. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \dots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \dots, P'_n)$ be two GSs, composition of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 \circ \check{G}_2 = (P \circ P', P_1 \circ P'_1, P_2 \circ P'_2, \dots, P_r \circ P'_r),$$

where $P_h \circ P'_h = \{(k_1l)(k_2l) \mid l \in P',\, k_1k_2 \in P_h\} \cup \{(kl_1)(kl_2) \mid k \in P, l_1l_2 \in P'_h\} \cup \{(k_1l_1)(k_2l_2) \mid k_1k_2 \in P_h,\, l_1, l_2 \in P'$ such that $l_1 \neq l_2\}$, $h = (1, 2, \dots, r)$.

Definition 2.6. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \dots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \dots, P'_r)$ be two GSs, union of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 \cup \check{G}_2 = (P \cup P', P_1 \cup P'_1, P_2 \cup P'_2, \dots, P_r \cup P'_r).$$

Definition 2.7. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \dots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \dots, P'_r)$ be two GSs, join of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 + \check{G}_2 = (P + P', P_1 + P'_1, P_2 + P'_2, \dots, P_r + P'_r),$$

where $P + P' = P \cup P'$, $P_h + P'_h = P_h \cup P'_h \cup P''_h$ for $h = (1, 2, \dots, r)$. P''_h consists of all those edges which join the vertices of P and P'.

Definition 2.8. ([19]). Let V be a fixed set. A generalized intuitionistic fuzzy set I of V is an object having the form I=$\{(v, \mu_I(v), \nu_I(v)) | v \in V\}$, where the functions $\mu_I : V \to [0, 1]$ and $\nu_I : V \to [0, 1]$ define the degree of membership and degree of nonmembership of an element $v \in V$, respectively, such that

$$\min\{\mu_I(v), \nu_I(v)\} \leq 0.5, \text{ for all } v \in V.$$

This condition is called the generalized intuitionistic condition.

Definition 2.9. ([10, 11]). A set $I = \{T_I(v), I_I(v), F_I(v) : v \in V\}$ is said to be an intuitionistic neutrosophic (IN)set, if

(i) $\{T_I(v) \wedge I_I(v)\} \leq 0.5$, $\{I_I(v) \wedge F_I(v)\} \leq 0.5$, $\{F_I(v) \wedge T_I(v)\} \leq 0.5$,
(ii) $0 \leq T_I(v) + I_I(v) + F_I(v) \leq 2$.

Definition 2.10. An intuitionistic neutrosophic graph is a pair $G = (A, B)$ with underlying set V, where T_A, F_A, $I_A : V \to [0, 1]$ denote the truth, falsity and indeterminacy membership values of the vertices in V and T_B, F_B, $I_B : E \subseteq V \times V \to [0, 1]$ denote the truth, falsity and indeterminacy membership values of the edges $kl \in E$ such that

(i) $T_B(kl) \leq T_A(k) \wedge T_A(l), \quad F_B(kl) \leq F_A(k) \vee F_A(l), \quad I_B(kl) \leq I_A(k) \wedge I_A(l),$
(ii) $T_B(kl) \wedge I_B(kl) \leq 0.5, \quad T_B(kl) \wedge F_B(kl) \leq 0.5, \quad I_B(kl) \wedge F_B(lk) \leq 0.5,$
(iii) $0 \leq T_B(kl) + F_B(kl) + I_B(kl) \leq 2, \forall\, k, l \in V.$

Definition 2.11. $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ is said to be an intuitionistic neutrosophic graph structure(INGS) of graph structure $\check{G} = (P, P_1, P_2, \ldots, P_r)$, if $O = < k, T(k), I(k), F(k) >$ and $O_h = < kl, T_h(kl), I_h(kl), F_h(kl) >$ are the intuitionistic neutrosophic(IN) sets on the sets P and P_h, respectively such that
(i) $T_h(kl) \leq T(k) \wedge T(l), \quad I_h(kl) \leq I(k) \wedge I(l), \quad F_h(kl) \leq F(k) \vee F(l),$
(ii) $T_h(kl) \wedge I_h(kl) \leq 0.5, \quad T_h(kl) \wedge F_h(kl) \leq 0.5, \quad I_h(kl) \wedge F_h(kl) \leq 0.5,$
(iii) $0 \leq T_h(kl) + I_h(kl) + F_h(kl) \leq 2, \quad for\ all\ kl \in O_h,\ h \in \{1, 2, \ldots, r\},$
where, O and O_h are underlying vertex and h-edge sets of INGS $\check{G}_i$, $h \in \{1, 2, \ldots, r\}$.

Example 2.12. An intuitionistic neutrosophic graph structure is represented in Fig. 1.

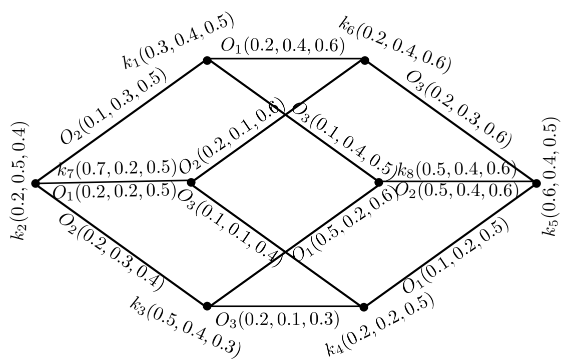


Figure 1. An intuitionistic neutrosophic graph structure

Now we define the operations on INGSs.

Definition 2.13. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, \ldots, Q_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively.
Cartesian product of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \times \check{G}_{i2} = (O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}, \ldots, O_{1r} \times O_{2r}),$$

is defined as:

$$\text{(i)} \begin{cases} T_{(O_1 \times O_2)}(kl) = (T_{O_1} \times T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1 \times O_2)}(kl) = (I_{O_1} \times I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1 \times O_2)}(kl) = (F_{O_1} \times F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$$

for all $kl \in P_1 \times P_2$,

$$\text{(ii)} \begin{cases} T_{(O_{1h} \times O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \times T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1 l_2) \\ I_{(O_{1h} \times O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \times I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1 l_2) \\ F_{(O_{1h} \times O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \times F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1 l_2) \end{cases}$$

for all $k \in P_1$, $l_1 l_2 \in P_{2h}$,

(iii) $\begin{cases} T_{(O_{1h}\times O_{2h})}(k_1l)(k_2l) = (T_{O_{1h}} \times T_{O_{2h}})(k_1l)(k_2l) = T_{O_2}(l) \wedge T_{O_{1h}}(k_1k_2) \\ I_{(O_{1h}\times O_{2h})}(k_1l)(k_2l) = (I_{O_{1h}} \times I_{O_{2h}})(k_1l)(k_2l) = I_{O_2}(l) \wedge I_{O_{2h}}(k_1k_2) \\ F_{(O_{1h}\times O_{2h})}(k_1l)(q_2l) = (F_{O_{1h}} \times F_{O_{2h}})(k_1l)(k_2l) = F_{O_2}(l) \vee F_{O_{2h}}(k_1k_2) \end{cases}$

for all $l \in P_2$, $k_1k_2 \in P_{1h}$.

Example 2.14. Consider $\check{G}_{i1} = (O_1, O_{11}, O_{12})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22})$ are two INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12})$ and $\check{G}_2 = (P_2, P_{21}, P_{22})$ respectively, as represented in Fig. 2, where $P_{11} = \{k_1k_2\}$, $P_{12} = \{k_3k_4\}$, $P_{21} = \{l_1l_2\}$, $P_{22} = \{l_2l_3\}$.

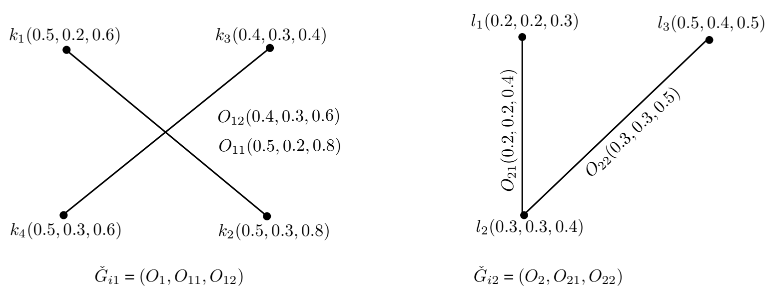
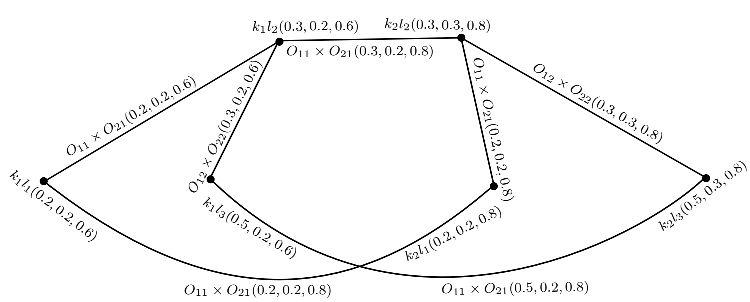


FIGURE 2. Two INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$

Cartesian product of $\check{G}_{i1}$ and $\check{G}_{i2}$ defined as $\check{G}_{i1} \times \check{G}_{i2} = \{O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}\}$ is represented in Fig. 3.



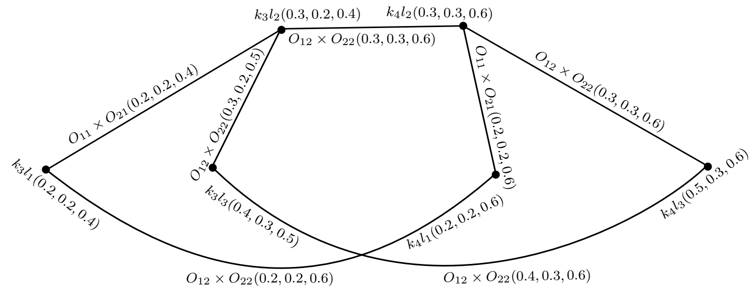


FIGURE 3. $\check{G}_{i1} \times \check{G}_{i2}$

Theorem 2.15. *Cartesian product* $\check{G}_{i1} \times \check{G}_{i2} = (O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}, \ldots, O_{1r} \times O_{2r})$ *of two INGSs of GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1 \times \check{G}_2$.

Proof. We consider two cases:

Case 1: For $k \in P_1$, $l_1l_2 \in P_{2h}$

$$\begin{aligned} T_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ &\leq T_{O_1}(k) \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\ &= [T_{O_1}(k) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k) \wedge T_{O_2}(l_2)] \\ &= T_{(O_1 \times O_2)}(kl_1) \wedge T_{(O_1 \times O_2)}(kl_2), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) &= I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ &\leq I_{O_1}(k) \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\ &= [I_{O_1}(k) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k) \wedge I_{O_2}(l_2)] \\ &= I_{(O_1 \times O_2)}(kl_1) \wedge I_{(O_1 \times O_2)}(kl_2), \end{aligned}$$

$$\begin{aligned} F_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) &= F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \\ &\leq F_{O_1}(k) \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)] \\ &= [F_{O_1}(k) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k) \vee F_{O_2}(l_2)] \\ &= F_{(O_1 \times O_2)}(kl_1) \vee F_{(O_1 \times O_2)}(kl_2), \end{aligned}$$

for $kl_1, kl_2 \in P_1 \times P_2$.

Case 2: For $k \in P_2$, $l_1l_2 \in P_{1h}$

$$\begin{aligned} T_{(O_{1h} \times O_{2h})}((l_1k)(l_2k)) &= T_{O_2}(k) \wedge T_{O_{1h}}(l_1l_2) \\ &\leq T_{O_2}(k) \wedge [T_{O_1}(l_1) \wedge T_{O_1}(l_2)] \\ &= [T_{O_2}(k) \wedge T_{O_1}(l_1)] \wedge [T_{O_2}(k) \wedge T_{O_1}(l_2)] \\ &= T_{(O_1 \times O_2)}(l_1k) \wedge T_{(O_1 \times O_2)}(l_2k), \end{aligned}$$

$$
\begin{aligned}
I_{(O_{1h}\times O_{2h})}((l_1k)(l_2k)) &= I_{O_2}(k)\wedge I_{O_{1h}}(l_1l_2)\\
&\le I_{O_2}(k)\wedge[I_{O_1}(l_1)\wedge I_{O_1}(l_2)]\\
&= [I_{O_2}(k)\wedge I_{O_1}(l_1)]\wedge[I_{O_2}(k)\wedge I_{O_1}(l_2)]\\
&= I_{(O_1\times O_2)}(l_1k)\wedge I_{(O_1\times O_2)}(l_2k),
\end{aligned}
$$

$$
\begin{aligned}
F_{(O_{1h}\times O_{2h})}((l_1k)(l_2k)) &= F_{O_2}(k)\vee F_{O_{1h}}(l_1l_2)\\
&\le F_{O_2}(k)\vee[F_{O_1}(l_1)\vee F_{O_1}(l_2)]\\
&= [F_{O_2}(k)\vee F_{O_1}(l_1)]\vee[F_{O_2}(k)\vee F_{O_1}(l_2)]\\
&= F_{(O_1\times O_2)}(l_1k)\vee F_{(O_1\times O_2)}(l_2k),
\end{aligned}
$$

for $l_1k, l_2k \in P_1\times P_2$.

Both cases exists $\forall h\in\{1,2,\ldots,r\}$. This completes the proof. □

Definition 2.16. Let $\check{G}_{i1}=(O_1,O_{11},O_{12},\ldots,Q_{1r})$ and $\check{G}_{i2}=(O_2,O_{21},O_{22},\ldots,Q_{2r})$ be INGSs of GSs $\check{G}_1=(P_1,P_{11},P_{12},\ldots,P_{1r})$ and $\check{G}_2=(P_2,P_{21},P_{22},\ldots,P_{2r})$, respectively. Cross product of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1}*\check{G}_{i2}=(O_1*O_2,O_{11}*O_{21},O_{12}*O_{22},\ldots,O_{1r}*O_{2r}),$$

is defined as:

(i) $$\begin{cases} T_{(O_1*O_2)}(kl)=(T_{O_1}*T_{O_2})(kl)=T_{O_1}(k)\wedge T_{O_2}(l)\\ I_{(O_1*O_2)}(kl)=(I_{O_1}*I_{O_2})(kl)=I_{O_1}(k)\wedge I_{O_2}(l)\\ F_{(O_1*O_2)}(kl)=(F_{O_1}*F_{O_2})(kl)=F_{O_1}(k)\vee F_{O_2}(l)\end{cases}$$

for all $kl\in P_1\times P_2$,

(ii) $$\begin{cases} T_{(O_{1h}*O_{2h})}(k_1l_1)(k_2l_2)=(T_{O_{1h}}*T_{O_{2h}})(k_1l_1)(k_2l_2)=T_{O_{1h}}(k_1k_2)\wedge T_{O_{2h}}(l_1l_2)\\ I_{(O_{1h}*O_{2h})}(k_1l_1)(k_2l_2)=(I_{O_{1h}}*I_{O_{2h}})(k_1l_1)(k_2l_2)=I_{O_{1h}}(k_1k_2)\wedge I_{O_{2h}}(l_1l_2)\\ F_{(O_{1h}*O_{2h})}(k_1l_1)(k_2l_2)=(F_{O_{1h}}*F_{O_{2h}})(k_1l_1)(k_2l_2)=F_{O_{1h}}(k_1k_2)\vee F_{O_{2h}}(l_1l_2)\end{cases}$$

for all $k_1k_2\in P_{1h}$, $l_1l_2\in P_{2h}$.

Example 2.17. Cross product of INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as $\check{G}_{i1}*\check{G}_{i2}=\{O_1*O_2,O_{11}*O_{21},O_{12}*O_{22}\}$ and is represented in Fig. 4.

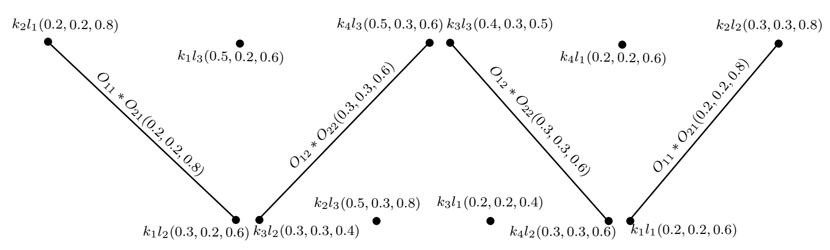


FIGURE 4. $\check{G}_{i1}*\check{G}_{i2}$

Theorem 2.18. *Cross product* $\check{G}_{i1}*\check{G}_{i2}=(O_1*O_2,O_{11}*O_{21},O_{12}*O_{22},\ldots,O_{1r}*O_{2r})$ *of two INGSs of GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1*\check{G}_2$.

Proof. For all $k_1l_1, k_2l_2 \in P_1 * P_2$

$$\begin{aligned} T_{(O_{1h}*O_{2h})}((k_1l_1)(k_2l_2)) &= T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2) \\ &\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\ &= [T_{O_1}(k_1) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k_2) \wedge T_{O_2}(l_2)] \\ &= T_{(O_1*O_2)}(k_1l_1) \wedge T_{(O_1*O_2)}(k_2l_2), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h}*O_{2h})}((k_1l_1)(k_2l_2)) &= I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}l_1l_2) \\ &\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\ &= [I_{O_1}(k_1) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k_2) \wedge I_{O_2}(l_2)] \\ &= I_{(O_1*O_2)}(k_1l_1) \wedge I_{(O_1*O_2)}(k_2l_2), \end{aligned}$$

$$\begin{aligned} F_{(O_{1h}*O_{2h})}((k_1l_1)(k_2l_2)) &= F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2) \\ &\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)] \\ &= [F_{O_1}(k_1) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k_2) \vee F_{O_2}(l_2)] \\ &= F_{(O_1*O_2)}(k_1l_1) \vee F_{(O_1*O_2)}(k_2l_2), \end{aligned}$$

for $h \in \{1, 2, \ldots, r\}$. This completes the proof. □

Definition 2.19. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. Lexicographic product of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \bullet \check{G}_{i2} = (O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}, \ldots, O_{1r} \bullet O_{2r}),$$

is defined as:

(i) $$\begin{cases} T_{(O_1 \bullet O_2)}(kl) = (T_{O_1} \bullet T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1 \bullet O_2)}(kl) = (I_{O_1} \bullet I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1 \bullet O_2)}(kl) = (F_{O_1} \bullet F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$$
for all $kl \in P_1 \times P_2$

(ii) $$\begin{cases} T_{(O_{1h} \bullet O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \bullet T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h} \bullet O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \bullet I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h} \bullet O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \bullet F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \end{cases}$$
for all $k \in P_1$, $l_1l_2 \in P_{2h}$,

(iii) $$\begin{cases} T_{(O_{1h} \bullet O_{2h})}(k_1l_1)(k_2l_2) = (T_{O_{1h}} \bullet T_{O_{2h}})(k_1l_1)(k_2l_2) = T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h} \bullet O_{2h})}(k_1l_1)(k_2l_2) = (I_{O_{1h}} \bullet I_{O_{2h}})(k_1l_1)(k_2l_2) = I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h} \bullet O_{2h})}(k_1l_1)(k_2l_2) = (F_{O_{1h}} \bullet F_{O_{2h}})(k_1l_1)(k_2l_2) = F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2) \end{cases}$$
for all $k_1k_2 \in P_{1h}$, $l_1l_2 \in P_{2h}$.

Example 2.20. Lexicographic product of INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as $\check{G}_{i1} \bullet \check{G}_{i2} = \{O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}\}$ and is represented in Fig. 5.

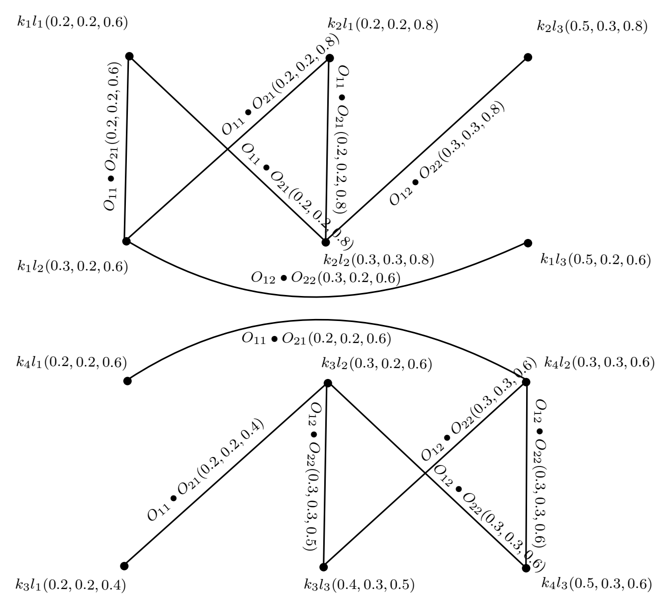


FIGURE 5. $\check{G}_{i1} \bullet \check{G}_{i2}$

Theorem 2.21. *Lexicographic product* $\check{G}_{i1} \bullet \check{G}_{i2} = (O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}, \ldots, O_{1r} \bullet O_{2r})$ *of two INGSs of the GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1 \bullet \check{G}_2$.

Proof. We consider two cases:

Case 1: For $k \in P_1$, $l_1l_2 \in P_{2h}$

$$\begin{aligned} T_{(O_{1h} \bullet O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ &\leq T_{O_1}(k) \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\ &= [T_{O_1}(k) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k) \wedge T_{O_2}(l_2)] \\ &= T_{(O_1 \bullet O_2)}(kl_1) \wedge T_{(O_1 \bullet O_2)}(kl_2), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h} \bullet O_{2i})}((kl_1)(kl_2)) &= I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ &\leq I_{O_1}(k) \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\ &= [I_{O_1}(k) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k) \wedge I_{O_2}(l_2)] \\ &= I_{(O_1 \bullet O_2)}(kl_1) \wedge I_{(O_1 \bullet O_2)}(kl_2), \end{aligned}$$

$$\begin{aligned} F_{(O_{1h} \bullet O_{2i})}((kl_1)(kl_2)) &= F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \\ &\leq F_{O_1}(k) \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)] \\ &= [F_{O_1}(k) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k) \vee F_{O_2}(l_2)] \\ &= F_{(O_1 \bullet O_2)}(kl_1) \vee F_{(O_1 \bullet O_2)}(kl_2), \end{aligned}$$

for $kl_1, kl_2 \in P_1 \bullet P_2$.

Case 2: For $k_1k_2 \in P_{1h}, l_1l_2 \in P_{2h}$

$$\begin{aligned} T_{(O_{1h}\bullet O_{2h})}((k_1l_1)(k_2l_2)) &= T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2) \\ &\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\ &= [T_{O_1}(k_1) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k_2) \wedge T_{O_2}(l_2)] \\ &= T_{(O_1\bullet O_2)}(k_1l_1) \wedge T_{(O_1\bullet O_2)}(k_2l_2), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h}\bullet O_{2h})}((k_1l_1)(k_2l_2)) &= I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}(l_1l_2) \\ &\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\ &= [I_{O_1}(k_1) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k_2) \wedge I_{O_2}(l_2)] \\ &= I_{(O_1\bullet O_2)}(k_1l_1) \wedge I_{(O_1\bullet O_2)}(k_2l_2), \end{aligned}$$

$$\begin{aligned} F_{(O_{1h}\bullet O_{2h})}((k_1l_1)(k_2l_2)) &= F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2) \\ &\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)] \\ &= [F_{O_1}(k_1) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k_2) \vee F_{O_2}(l_2)] \\ &= F_{(O_1\bullet O_2)}(k_1l_1) \vee F_{(O_1\bullet O_2)}(k_2l_2), \end{aligned}$$

for $k_1l_1, k_2l_2 \in P_1 \bullet P_2$.
Both cases hold for $h \in \{1, 2, \ldots, r\}$. This completes the proof.

□

Definition 2.22. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, \ldots, O_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. Strong product of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \boxtimes \check{G}_{i2} = (O_1 \boxtimes O_2, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}, \ldots, O_{1r} \boxtimes O_{2r}),$$

is defined as:

(i) $$\begin{cases} T_{(O_1\boxtimes O_2)}(kl) = (T_{O_1} \boxtimes T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1\boxtimes O_2)}(kl) = (I_{O_1} \boxtimes I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1\boxtimes O_2)}(kl) = (F_{O_1} \boxtimes F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$$

for all $kl \in P_1 \times P_2$,

(ii) $$\begin{cases} T_{(O_{1h}\boxtimes O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \boxtimes T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h}\boxtimes O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \boxtimes I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h}\boxtimes O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \boxtimes F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \end{cases}$$

for all $k \in P_1$, $l_1l_2 \in P_{2h}$,

(iii) $$\begin{cases} T_{(O_{1h}\boxtimes O_{2h})}(k_1l)(k_2l) = (T_{O_{1h}} \boxtimes T_{O_{2h}})(k_1l)(k_2l) = T_{O_2}(l) \wedge T_{O_{1h}}(k_1k_2) \\ I_{(O_{1h}\boxtimes O_{2h})}(k_1l)(k_2l) = (I_{O_{1h}} \boxtimes I_{O_{2h}})(k_1l)(k_2l) = I_{O_2}(l) \wedge I_{O_{2h}}(k_1k_2) \\ F_{(O_{1h}\boxtimes O_{2h})}(k_1l)(k_2l) = (F_{O_{1h}} \boxtimes F_{O_{2h}})(k_1l)(k_2l) = F_{O_2}(l) \vee F_{O_{2h}}(k_1k_2) \end{cases}$$

for all $l \in P_2$, $k_1k_2 \in P_{1h}$,

(iv) $$\begin{cases} T_{(O_{1h}\boxtimes O_{2h})}(k_1l_1)(k_2l_2) = (T_{O_{1h}} \boxtimes T_{O_{2h}})(k_1l_1)(k_2l_2) = T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h}\boxtimes O_{2h})}(k_1l_1)(k_2l_2) = (I_{O_{1h}} \boxtimes I_{O_{2h}})(k_1l_1)(k_2l_2) = I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h}\boxtimes O_{2h})}(k_1l_1)(k_2l_2) = (F_{O_{1h}} \boxtimes F_{O_{2h}})(k_1l_1)(k_2l_2) = F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2) \end{cases}$$

for all $k_1k_2 \in P_{1h}$, $l_1l_2 \in P_{2h}$.

Example 2.23. Strong product of INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as $\check{G}_{i1} \boxtimes \check{G}_{i2} = \{O_1 \boxtimes O_2, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}\}$ and is represented in Fig. 6.

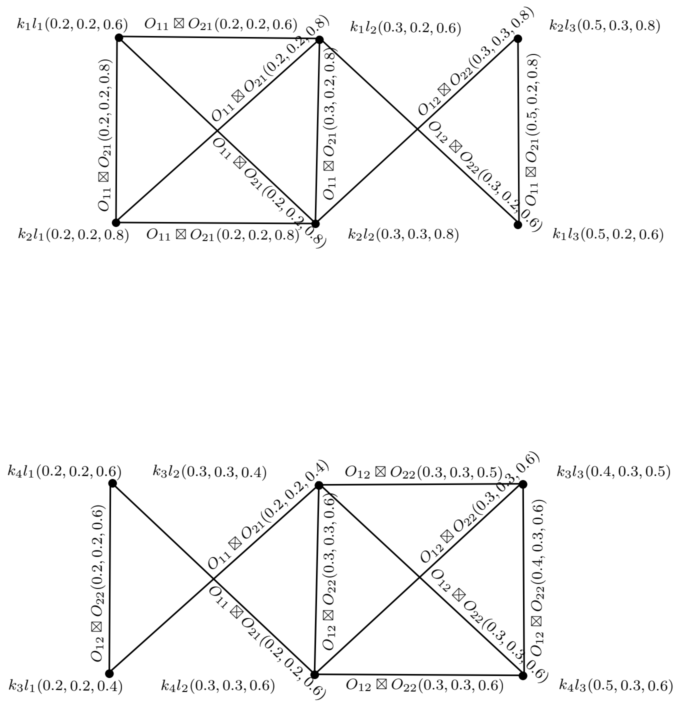


FIGURE 6. $\check{G}_{i1} \boxtimes \check{G}_{i2}$

Theorem 2.24. *Strong product* $\check{G}_{i1} \boxtimes \check{G}_{i2} = (O_1 \boxtimes O_2, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}, \dots, O_{1r} \boxtimes O_{2r})$ *of two INGSs of the GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1 \boxtimes \check{G}_2$.

Proof. There are three cases:

Case 1: For $k \in P_1$, $l_1l_2 \in P_{2h}$

$$\begin{aligned}
T_{(O_{1h} \boxtimes O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\
&\leq T_{O_1}(k) \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\
&= [T_{O_1}(k) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k) \wedge T_{O_2}(l_2)] \\
&= T_{(O_1 \boxtimes O_2)}(kl_1) \wedge T_{(O_1 \boxtimes O_2)}(kl_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h} \boxtimes O_{2h})}((kl_1)(kl_2)) &= I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\
&\leq I_{O_1}(k) \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\
&= [I_{O_1}(k) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k) \wedge I_{O_2}(l_2)] \\
&= I_{(O_1 \boxtimes O_2)}(kl_1) \wedge I_{(O_1 \boxtimes O_2)}(kl_2),
\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\boxtimes O_{2h})}((kl_1)(kl_2)) &= F_{O_1}(k)\vee F_{O_{2h}}(l_1l_2)\\&\leq F_{O_1}(k)\vee[F_{O_2}(l_1)\vee F_{O_2}(l_2)]\\&=[F_{O_1}(k)\vee F_{O_2}(l_1)]\vee[F_{O_1}(k)\vee F_{O_2}(l_2)]\\&=F_{(O_1\boxtimes O_2)}(kl_1)\vee F_{(O_1\boxtimes O_2)}(kl_2),\end{aligned}$$

for $kl_1, kl_2 \in P_1 \boxtimes P_2$.

Case 2: For $k \in P_2$, $l_1l_2 \in P_{1h}$

$$\begin{aligned}T_{(O_{1h}\boxtimes O_{2h})}((l_1k)(l_2k)) &= T_{O_2}(k)\wedge T_{O_{1h}}(l_1l_2)\\&\leq T_{O_2}(k)\wedge[T_{O_1}(l_1)\wedge T_{O_1}(l_2)]\\&=[T_{O_2}(k)\wedge T_{O_1}(l_1)]\wedge[T_{O_2}(k)\wedge T_{O_1}(l_2)]\\&=T_{(O_1\boxtimes O_2)}(l_1k)\wedge T_{(O_1\boxtimes O_2)}(l_2k),\end{aligned}$$

$$\begin{aligned}I_{(O_{1h}\boxtimes O_{2h})}((l_1k)(l_2k)) &= I_{O_2}(k)\wedge I_{O_{1h}}(l_1l_2)\\&\leq I_{O_2}(k)\wedge[I_{O_1}(l_1)\wedge I_{O_1}(l_2)]\\&=[I_{O_2}(k)\wedge I_{O_1}(l_1)]\wedge[I_{O_2}(k)\wedge I_{O_1}(l_2)]\\&=I_{(O_1\boxtimes O_2)}(l_1k)\wedge I_{(O_1\boxtimes O_2)}(l_2k),\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\boxtimes O_{2h})}((l_1k)(l_2k)) &= F_{O_2}(k)\vee F_{O_{1h}}(l_1l_2)\\&\leq F_{O_2}(k)\vee[F_{O_1}(l_1)\vee F_{O_1}(l_2)]\\&=[F_{O_2}(k)\vee F_{O_1}(l_1)]\vee[F_{O_2}(k)\vee F_{O_1}(l_2)]\\&=F_{(O_1\boxtimes O_2)}(l_1k)\vee F_{(O_1\boxtimes O_2)}(l_2k),\end{aligned}$$

for $l_1k, l_2k \in P_1 \boxtimes P_2$.

Case 3: For every $k_1k_2 \in P_{1h}, l_1l_2 \in P_{2h}$

$$\begin{aligned}T_{(O_{1h}\boxtimes O_{2h})}((k_1l_1)(k_2l_2)) &= T_{O_{1h}}(k_1k_2)\wedge T_{O_{2h}}(l_1l_2)\\&\leq[T_{O_1}(k_1)\wedge T_{O_1}(k_2)]\wedge[T_{O_2}(l_1)\wedge T_{O_2}(l_2)]\\&=[T_{O_1}(k_1)\wedge T_{O_2}(l_1)]\wedge[T_{O_1}(k_2)\wedge T_{O_2}(l_2)]\\&=T_{(O_1\boxtimes O_2)}(k_1l_1)\wedge T_{(O_1\boxtimes O_2)}(k_2l_2),\end{aligned}$$

$$\begin{aligned}I_{(O_{1h}\boxtimes O_{2h})}((k_1l_1)(k_2l_2)) &= I_{O_{1h}}(k_1k_2)\wedge I_{O_{2h}}(l_1l_2)\\&\leq[I_{O_1}(k_1)\wedge I_{O_1}(k_2)]\wedge[I_{O_2}(l_1)\wedge I_{O_2}(l_2)]\\&=[I_{O_1}(k_1)\wedge I_{O_2}(l_1)]\wedge[I_{O_1}(k_2)\wedge I_{O_2}(l_2)]\\&=I_{(O_1\boxtimes O_2)}(k_1l_1)\wedge I_{(O_1\boxtimes O_2)}(k_2l_2),\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\boxtimes O_{2h})}((k_1l_1)(k_2l_2)) &= F_{O_{1h}}(k_1k_2)\vee F_{O_{2h}}(l_1l_2)\\&\leq[F_{O_1}(k_1)\vee F_{O_1}(k_2)]\vee[F_{O_2}(l_1)\vee F_{O_2}(l_2)]\\&=[F_{O_1}(k_1)\vee F_{O_2}(l_1)]\vee[F_{O_1}(k_2)\vee F_{O_2}(l_2)]\\&=F_{(O_1\boxtimes O_2)}(k_1l_1)\vee F_{(O_1\boxtimes O_2)}(k_2l_2),\end{aligned}$$

for $k_1l_1, k_2l_2 \in P_1 \boxtimes P_2$, and $h = 1, 2, \ldots, r$.

This completes the proof. □

Definition 2.25. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. The composition of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \circ \check{G}_{i2} = (O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}, \ldots, O_{1r} \circ O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1 \circ O_2)}(kl) = (T_{O_1} \circ T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1 \circ O_2)}(kl) = (I_{O_1} \circ I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1 \circ O_2)}(kl) = (F_{O_1} \circ F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$

for all $kl \in P_1 \times P_2$,

(ii) $\begin{cases} T_{(O_{1h} \circ O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \circ T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h} \circ O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \circ I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h} \circ O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \circ F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \end{cases}$

for all $k \in P_1$, $l_1l_2 \in P_{2h}$,

(iii) $\begin{cases} T_{(O_{1h} \circ O_{2h})}(k_1l)(k_2l) = (T_{O_{1h}} \circ T_{O_{2h}})(k_1l)(k_2l) = T_{O_2}(l) \wedge T_{O_{1h}}(k_1k_2) \\ I_{(O_{1h} \circ O_{2h})}(k_1l)(k_2l) = (I_{O_{1h}} \circ I_{O_{2h}})(k_1l)(k_2l) = I_{O_2}(l) \wedge I_{O_{2h}}(k_1k_2) \\ F_{(O_{1h} \circ O_{2h})}(k_1l)(k_2l) = (F_{O_{1h}} \circ F_{O_{2h}})(k_1l)(k_2l) = F_{O_2}(l) \vee F_{O_{2h}}(k_1k_2) \end{cases}$

for all $l \in P_2$, $k_1k_2 \in P_{1h}$,

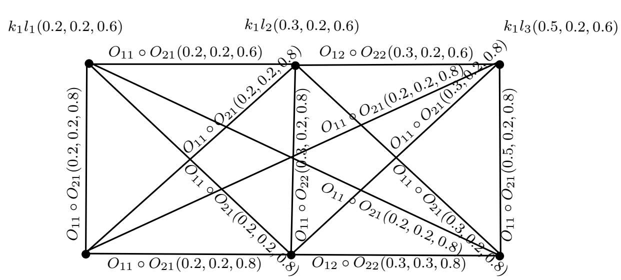
(iv) $\begin{cases} T_{(O_{1h} \circ O_{2h})}(k_1l_1)(k_2l_2) = (T_{O_{1h}} \circ T_{O_{2h}})(k_1l_1)(k_2l_2) = T_{O_{1h}}(k_1k_2) \wedge T_{O_2}(l_1) \wedge T_{O_2}(l_2) \\ I_{(O_{1h} \circ O_{2h})}(k_1l_1)(k_2l_2) = (I_{O_{1h}} \circ I_{O_{2h}})(k_1l_1)(k_2l_2) = I_{O_{1h}}(k_1k_2) \wedge I_{O_2}(l_1) \wedge I_{O_2}(l_2) \\ F_{(O_{1h} \circ O_{2h})}(k_1l_1)(k_2l_2) = (F_{O_{1h}} \circ F_{O_{2h}})(k_1l_1)(k_2l_2) = F_{O_{1h}}(k_1k_2) \vee F_{O_2}(l_1) \vee F_{O_2}(l_2) \end{cases}$

for all $k_1k_2 \in P_{1h}$, $l_1l_2 \in P_{2h}$ such that $l_1 \neq l_2$.

Example 2.26. The composition of INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as:

$$\check{G}_{i1} \circ \check{G}_{i2} = \{O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}\}$$

and is represented in Fig. 7.



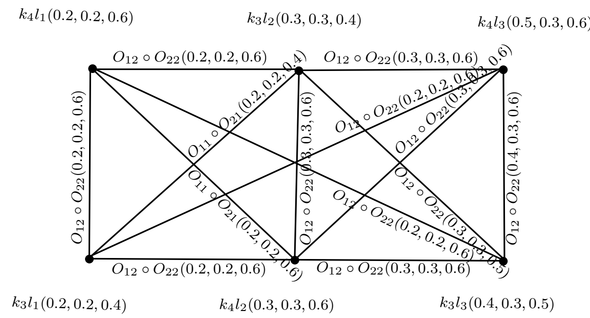


FIGURE 7. $\check{G}_{i1} \circ \check{G}_{i2}$

Theorem 2.27. *The composition* $\check{G}_{i1} \circ \check{G}_{i2} = (O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}, \ldots, O_{1r} \circ O_{2r})$ *of two INGSs of GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1 \circ \check{G}_2$*.*

Proof. We consider three cases:

Case 1: For $k \in P_1$, $l_1l_2 \in P_{2h}$

$$\begin{aligned}
T_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\
&\leq T_{O_1}(k) \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\
&= [T_{O_1}(k) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k) \wedge T_{O_2}(l_2)] \\
&= T_{(O_1 \circ O_2)}(kl_1) \wedge T_{(O_1 \circ O_2)}(kl_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) &= I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\
&\leq I_{O_1}(k) \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\
&= [I_{O_1}(k) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k) \wedge I_{O_2}(l_2)] \\
&= I_{(O_1 \circ O_2)}(kl_1) \wedge I_{(O_1 \circ O_2)}(kl_2),
\end{aligned}$$

$$\begin{aligned}
F_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) &= F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \\
&\leq F_{O_1}(k) \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)] \\
&= [F_{O_1}(k) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k) \vee F_{O_2}(l_2)] \\
&= F_{(O_1 \circ O_2)}(kl_1) \vee F_{(O_1 \circ O_2)}(kl_2),
\end{aligned}$$

for $kl_1, kl_2 \in P_1 \circ P_2$.

Case 2: For $k \in P_2$, $l_1l_2 \in P_{1h}$

$$\begin{aligned}
T_{(O_{1h} \circ O_{2h})}((l_1k)(l_2k)) &= T_{O_2}(k) \wedge T_{O_{1h}}(l_1l_2) \\
&\leq T_{O_2}(k) \wedge [T_{O_1}(l_1) \wedge T_{O_1}(l_2)] \\
&= [T_{O_2}(k) \wedge T_{O_1}(l_1)] \wedge [T_{O_2}(k) \wedge T_{O_1}(l_2)] \\
&= T_{(O_1 \circ O_2)}(l_1k) \wedge T_{(O_1 \circ O_2)}(l_2k),
\end{aligned}$$

$$\begin{aligned}I_{(O_{1h}\circ O_{2h})}((l_1k)(l_2k)) &= I_{O_2}(k)\wedge I_{O_{1h}}(l_1l_2)\\ &\leq I_{O_2}(k)\wedge[I_{O_1}(l_1)\wedge I_{O_1}(l_2)]\\ &=[I_{O_2}(k)\wedge I_{O_1}(l_1)]\wedge[I_{O_2}(k)\wedge I_{O_1}(l_2)]\\ &= I_{(O_1\circ O_2)}(l_1k)\wedge I_{(O_1\circ O_2)}(l_2k),\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\circ O_{2h})}((l_1k)(l_2k)) &= F_{O_2}(k)\vee F_{O_{1h}}(l_1l_2)\\ &\leq F_{O_2}(k)\vee[F_{O_1}(l_1)\vee F_{O_1}(l_2)]\\ &=[F_{O_2}(k)\vee F_{O_1}(l_1)]\vee[F_{O_2}(k)\vee F_{O_1}(l_2)]\\ &= F_{(O_1\circ O_2)}(l_1k)\vee F_{(O_1\circ O_2)}(l_2k),\end{aligned}$$

for $l_1k, l_2k \in P_1\circ P_2$.

Case 3: For $k_1k_2\in P_{1h}, l_1, l_2\in P_2$ such that $l_1\neq l_2$

$$\begin{aligned}T_{(O_{1h}\circ O_{2h})}((k_1l_1)(k_2l_2)) &= T_{O_{1h}}(k_1k_2)\wedge T_{O_2}(l_1)\wedge T_{O_2}(l_2)\\ &\leq [T_{O_1}(k_1)\wedge T_{O_1}(k_2)]\wedge T_{O_2}(l_1)\wedge T_{O_2}(l_2)\\ &=[T_{O_1}(k_1)\wedge T_{O_2}(l_1)]\wedge[T_{O_1}(k_2)\wedge T_{O_2}(l_2)]\\ &= T_{(O_1\circ O_2)}(k_1l_1)\wedge T_{(O_1\circ O_2)}(k_2l_2),\end{aligned}$$

$$\begin{aligned}I_{(O_{1h}\circ O_{2h})}((k_1l_1)(k_2l_2)) &= I_{O_{1h}}(k_1k_2)\wedge I_{O_2}(l_1)\wedge I_{O_2}(l_2)\\ &\leq [I_{O_1}(k_1)\wedge I_{O_1}(k_2)]\wedge[I_{O_2}(l_1)\wedge I_{O_2}(l_2)]\\ &=[I_{O_1}(k_1)\wedge I_{O_2}(l_1)]\wedge[I_{O_1}(k_2)\wedge I_{O_2}(l_2)]\\ &= I_{(O_1\circ O_2)}(k_1l_1)\wedge I_{(O_1\circ O_2)}(k_2l_2),\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\circ O_{2h})}((k_1l_1)(k_2l_2)) &= F_{O_{1h}}(k_1k_2)\vee F_{O_2}(l_1)\vee F_{O_2}(l_2)\\ &\leq [F_{O_1}(k_1)\vee F_{O_1}(k_2)]\vee[F_{O_2}(l_1)\vee F_{O_2}(l_2)]\\ &=[F_{O_1}(k_1)\vee F_{O_2}(l_1)]\vee[F_{O_1}(k_2)\vee F_{O_2}(l_2)]\\ &= F_{(O_1\circ O_2)}(k_1l_1)\vee F_{(O_1\circ O_2)}(k_2l_2),\end{aligned}$$

for $k_1l_1, k_2l_2\in P_1\circ P_2$.

All cases holds for $h = 1, 2, \ldots, r$. This completes the proof. □

Definition 2.28. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. The union of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1}\cup\check{G}_{i2} = (O_1\cup O_2, O_{11}\cup O_{21}, O_{12}\cup O_{22}, \ldots, O_{1r}\cup O_{2r}),$$

is defined as:

(i) $$\begin{cases} T_{(O_1\cup O_2)}(k) = (T_{O_1}\cup T_{O_2})(k) = T_{O_1}(k)\vee T_{O_2}(k)\\ I_{(O_1\cup O_2)}(k) = (I_{O_1}\cup I_{O_2})(k) = I_{O_1}(k)\vee I_{O_2}(k)\\ F_{(O_1\cup O_2)}(k) = (F_{O_1}\cup F_{O_2})(k) = F_{O_1}(k)\wedge F_{O_2}(k)\end{cases}$$

for all $k\in P_1\cup P_2$,

(ii) $$\begin{cases} T_{(O_{1h}\cup O_{2h})}(kl) = (T_{O_{1h}}\cup T_{O_{2h}})(kl) = T_{O_{1h}}(kl)\vee T_{O_{2h}}(kl)\\ I_{(O_{1h}\cup O_{2h})}(kl) = (I_{O_{1h}}\cup I_{O_{2h}})(kl) = I_{O_{1h}}(kl)\vee I_{O_{2h}}(kl)\\ F_{(O_{1h}\cup O_{2h})}(kl) = (F_{O_{1h}}\cup F_{O_{2h}})(kl) = F_{O_{1h}}(kl)\wedge F_{O_{2h}}(kl)\end{cases}$$

for all $kl\in P_{1h}\cup P_{2h}$.

Example 2.29. The union of two INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as

$$\check{G}_{i1} \cup \check{G}_{i2} = \{O_1 \cup O_2, O_{11} \cup O_{21}, O_{12} \cup O_{22}\}$$

and is represented in Fig. 8.

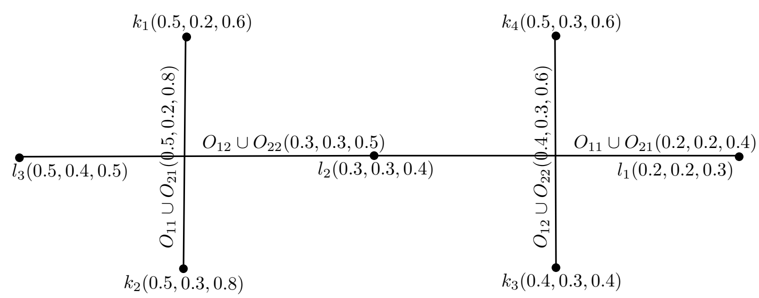


Figure 8. $\check{G}_{i1} \cup \check{G}_{i2}$

Theorem 2.30. *The union* $\check{G}_{i1} \cup \check{G}_{i2} = (O_1 \cup O_2, O_{11} \cup O_{21}, O_{12} \cup O_{22}, \ldots, O_{1r} \cup O_{2r})$ *of two INGSs of the GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1 \cup \check{G}_2$.

Proof. Let $k_1k_2 \in P_{1h} \cup P_{2h}$. There are two cases:

Case 1: For $k_1, k_2 \in P_1$, by definition 2.28, $T_{O_2}(k_1) = T_{O_2}(k_2) = T_{O_{2h}}(k_1k_2) = 0$, $I_{O_2}(k_1) = I_{O_2}(k_2) = I_{O_{2h}}(k_1k_2) = 0$, $F_{O_2}(k_1) = F_{O_2}(k_2) = F_{O_{2h}}(k_1k_2) = 1$. Thus,

$$\begin{aligned}
T_{(O_{1h} \cup O_{2h})}(k_1k_2) &= T_{O_{1h}}(k_1k_2) \vee T_{O_{2h}}(k_1k_2) \\
&= T_{O_{1h}}(k_1k_2) \vee 0 \\
&\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \vee 0 \\
&= [T_{O_1}(k_1) \vee 0] \wedge [T_{O_1}(k_2) \vee 0] \\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)] \\
&= T_{(O_1 \cup O_2)}(k_1) \wedge T_{(O_1 \cup O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h} \cup O_{2h})}(k_1k_2) &= I_{O_{1h}}(k_1k_2) \vee I_{Q_{2h}}(k_1k_2) \\
&= I_{O_{1h}}(k_1k_2) \vee 0 \\
&\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \vee 0 \\
&= [I_{O_1}(k_1) \vee 0] \wedge [I_{O_1}(k_2) \vee 0] \\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)] \\
&= I_{(O_1 \cup O_2)}(k_1) \wedge I_{(O_1 \cup O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned} F_{(O_{1h}\cup O_{2h})}(k_1k_2) &= F_{O_{1h}}(k_1k_2) \wedge F_{O_{2h}}(k_1k_2) \\ &= F_{O_{1i}}(k_1k_2) \wedge 1 \\ &\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \wedge 1 \\ &= [F_{O_1}(k_1) \wedge 1] \vee [F_{O_1}(k_2) \wedge 1] \\ &= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_1}(k_2) \wedge F_{O_2}(k_2)] \\ &= F_{(O_1\cup O_2)}(k_1) \vee F_{(O_1\cup O_2)}(k_2), \end{aligned}$$

for $k_1, k_2 \in P_1 \cup P_2$.

Case 2: For $k_1, k_2 \in P_2$, by definition 2.28, $T_{O_1}(k_1) = T_{O_1}(k_2) = T_{O_{1h}}(k_1k_2) = 0$, $I_{O_1}(k_1) = I_{O_1}(k_2) = I_{O_{1h}}(k_1k_2) = 0$, $F_{O_1}(k_1) = F_{O_1}(q_2) = F_{O_{1h}}(k_1k_2) = 1$, so

$$\begin{aligned} T_{(O_{1h}\cup O_{2h})}(k_1k_2) &= T_{O_{1h}}(k_1k_2) \vee T_{O_{2i}}(k_1k_2) \\ &= T_{O_{2i}}(k_1k_2) \vee 0 \\ &\leq [T_{O_2}(k_1) \wedge T_{O_2}(k_2)] \vee 0 \\ &= [T_{O_2}(k_1) \vee 0] \wedge [T_{O_2}(k_2) \vee 0] \\ &= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)] \\ &= T_{(O_1\cup O_2)}(k_1) \wedge T_{(O_1\cup O_2)}(k_2), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h}\cup O_{2h})}(q_1k_2) &= I_{O_{1h}}(k_1k_2) \vee I_{O_{2h}}(k_1k_2) \\ &= I_{O_{2h}}(k_1k_2) \vee 0 \\ &\leq [I_{O_2}(k_1) \wedge I_{O_2}(k_2)] \vee 0 \\ &= [I_{O_2}(k_1) \vee 0] \wedge [I_{O_2}(k_2) \vee 0] \\ &= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)] \\ &= I_{(O_1\cup O_2)}(k_1) \wedge I_{(O_1\cup O_2)}(k_2), \end{aligned}$$

$$\begin{aligned} F_{(O_{1h}\cup O_{2h})}(k_1k_2) &= F_{O_{1h}}(k_1k_2) \wedge F_{O_{2h}}(k_1k_2) \\ &= F_{O_{2h}}(k_1k_2) \wedge 1 \\ &\leq [F_{O_2}(k_1) \vee F_{O_2}(k_2)] \wedge 1 \\ &= [F_{O_2}(k_1) \wedge 1] \vee [F_{O_2}(k_2) \wedge 1] \\ &= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_1}(k_2) \wedge F_{O_2}(k_2)] \\ &= F_{(O_1\cup O_2)}(k_1) \vee F_{(O_1\cup O_2)}(k_2), \end{aligned}$$

for $k_1, k_2 \in P_1 \cup P_2$.

Both cases hold $\forall h \in \{1, 2, \ldots, r\}$. This completes the proof. □

Theorem 2.31. *Let $\check{G} = (P_1 \cup P_2, P_{11} \cup P_{21}, P_{12} \cup P_{22}, \ldots, P_{1r} \cup P_{2r})$ be the union of two GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, ..., P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, ..., P_{2r})$. Then every INGS $\check{G}_i = (O, O_1, O_2, ..., O_r)$ of $\check{G}$ is union of the two INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ of GSs $\check{G}_1$ and $\check{G}_2$, respectively.*

Proof. Firstly, we define O_1, O_2, O_{1h} and O_{2h} for $h \in \{1, 2, \ldots, r\}$ as:
$T_{O_1}(k) = T_O(k), I_{O_1}(k) = I_O(k), F_{O_1}(k) = F_O(k)$, if $k \in P_1$,
$T_{O_2}(k) = T_O(k), I_{O_2}(k) = I_O(k), F_{O_2}(k) = F_O(k)$, if $k \in P_2$,
$T_{O_{1h}}(k_1k_2) = T_{O_h}(k_1k_2), I_{O_{1h}}(k_1k_2) = I_{O_h}(k_1k_2), F_{O_{1h}}(k_1k_2) = F_{O_h}(k_1k_2)$, if $k_1k_2 \in P_{1h}$,
$T_{O_{2h}}(k_1k_2) = T_{O_h}(k_1k_2), I_{O_{2h}}(k_1k_2) = I_{O_h}(k_1k_2), F_{O_{2h}}(k_1k_2) = F_{O_h}(k_1k_2)$, if $k_1k_2 \in P_{2h}$.
Then $O = O_1 \cup O_2$ and $O_h = O_{1h} \cup O_{2h}$, $h \in \{1, 2, \ldots, r\}$.
Now for $k_1k_2 \in P_{lh}$, $l = 1, 2$, $h = 1, 2, \ldots, r$,
$T_{O_{lh}}(k_1k_2) = T_{O_h}(k_1k_2) \leq T_O(k_1) \wedge T_O(k_2) = T_{O_l}(k_1) \wedge T_{O_l}(k_2)$,
$I_{O_{lh}}(k_1k_2) = I_{O_h}(k_1k_2) \leq I_O(k_1) \wedge I_O(k_2) = I_{O_l}(k_1) \wedge I_{O_l}(k_2)$,
$F_{O_{lh}}(k_1k_2) = F_{O_h}(k_1k_2) \leq F_O(k_1) \vee F_O(k_2) = F_{O_l}(k_1) \vee F_{O_l}(k_2)$, i.e.,
$\check{G}_{il} = (O_l, O_{l1}, O_{l2}, ..., O_{lr})$ is an INGS of $\check{G}_l$, l = 1,2.
Thus $\check{G}_i = (O, O_1, O_2, ..., O_r)$, an INGS of $\check{G} = \check{G}_1 \cup \check{G}_2$, is the union of the two INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$. □

Definition 2.32. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively and let $P_1 \cap P_2 = \varnothing$. *Join* of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} + \check{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1+O_2)}(k) = T_{(O_1 \cup O_2)}(k) \\ I_{(O_1+O_2)}(k) = I_{(O_1 \cup O_2)}(k) \\ F_{(O_1+O_2)}(k) = F_{(O_1 \cup O_2)}(k) \end{cases}$
for all $k \in P_1 \cup P_2$,

(ii) $\begin{cases} T_{(O_{1h}+O_{2h})}(kl) = T_{(O_{1h} \cup O_{2h})}(kl) \\ I_{(O_{1h}+O_{2h})}(kl) = I_{(O_{1h} \cup O_{2h})}(kl) \\ F_{(O_{1h}+O_{2h})}(kl) = F_{(O_{1h} \cup O_{2h})}(kl) \end{cases}$
for all $kl \in P_{1h} \cup P_{2h}$,

(iii) $\begin{cases} T_{(O_{1h}+O_{2h})}(kl) = (T_{O_{1h}} + T_{O_{2h}})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_{1h}+O_{2h})}(kl) = (I_{O_{1h}} + I_{O_{2h}})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_{1h}+O_{2h})}(kl) = (F_{O_{1h}} + F_{O_{2h}})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$
for all $k \in P_1$, $l \in P_2$.

Example 2.33. The join of two INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as $\check{G}_{i1} + \check{G}_{i2} = \{O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}\}$ and is represented in the Fig. 9.

Theorem 2.34. *The join $\check{G}_{i1} + \check{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r})$ of two INGSs of GSs $\check{G}_1$ and $\check{G}_2$ is INGS of $\check{G}_1 + \check{G}_2$.*

Proof. Let $k_1k_2 \in P_{1h} + P_{2h}$. There are three cases:

Case 1: For $k_1, k_2 \in P_1$, by definition 2.32, $T_{O_2}(k_1) = T_{O_2}(k_2) = T_{O_{2h}}(k_1k_2) = 0$, $I_{O_2}(k_1) = I_{O_2}(k_2) = I_{O_{2h}}(k_1k_2) = 0$, $F_{O_2}(k_1) = F_{O_2}(k_2) = F_{O_{2h}}(k_1k_2) =$

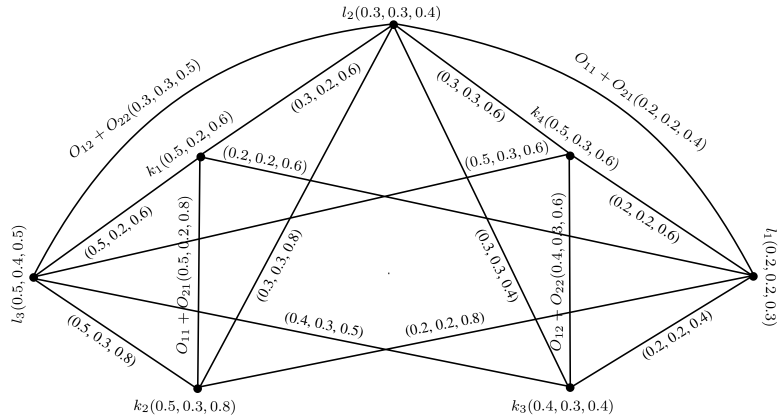


FIGURE 9. $\check{G}_{i1} + \check{G}_{i2}$

1, so,

$$\begin{aligned}
T_{(O_{1h}+O_{2h})}(k_1k_2) &= T_{O_{1h}}(k_1k_2) \vee T_{O_{2h}}(k_1k_2)\\
&= T_{O_{1h}}(k_1k_2) \vee 0\\
&\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \vee 0\\
&= [T_{O_1}(k_1) \vee 0] \wedge [T_{O_1}(q_2) \vee 0]\\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)]\\
&= T_{(O_1+O_2)}(k_1) \wedge T_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h}+O_{2h})}(k_1k_2) &= I_{O_{1h}}(k_1k_2) \vee I_{O_{2h}}(k_1k_2)\\
&= I_{O_{1h}}(k_1k_2) \vee 0\\
&\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \vee 0\\
&= [I_{O_1}(k_1) \vee 0] \wedge [I_{O_1}(k_2) \vee 0]\\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)]\\
&= I_{(O_1+O_2)}(k_1) \wedge I_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
F_{(O_{1h}+O_{2h})}(k_1k_2) &= F_{O_{1h}}(k_1k_2) \wedge F_{O_{2h}}(k_1k_2)\\
&= F_{O_{1h}}(k_1k_2) \wedge 1\\
&\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \wedge 1\\
&= [F_{O_1}(k_1) \wedge 1] \vee [F_{O_1}(k_2) \wedge 1]\\
&= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_1}(k_2) \wedge F_{O_2}(k_2)]\\
&= F_{(O_1+O_2)}(k_1) \vee F_{(O_1+O_2)}(k_2),
\end{aligned}$$

for $k_1, k_2 \in P_1 + P_2$.

Case 2: For $k_1, k_2 \in P_2$, by definition 2.32, $T_{O_1}(k_1) = T_{O_1}(k_2) = T_{O_{1h}}(k_1k_2) = 0$, $I_{O_1}(k_1) = I_{O_1}(k_2) = I_{O_{1h}}(k_1k_2) = 0$, $F_{O_1}(k_1) = F_{O_1}(k_2) = F_{O_{1h}}(k_1k_2) = 1$, so

$$\begin{aligned}
T_{(O_{1h}+O_{2h})}(k_1k_2) &= T_{O_{1i}}(k_1k_2) \vee T_{O_{2i}}(k_1k_2)\\
&= T_{O_{2i}}(k_1k_2) \vee 0\\
&\leq [T_{O_2}(k_1) \wedge T_{O_2}(k_2)] \vee 0\\
&= [T_{O_2}(k_1) \vee 0] \wedge [T_{O_2}(k_2) \vee 0]\\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)]\\
&= T_{(O_1+O_2)}(k_1) \wedge T_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h}+O_{2h})}(k_1k_2) &= I_{O_{1h}}(k_1k_2) \vee I_{O_{2h}}(k_1k_2)\\
&= I_{O_{2h}}(k_1k_2) \vee 0\\
&\leq [I_{O_2}(k_1) \wedge I_{O_2}(k_2)] \vee 0\\
&= [I_{O_2}(k_1) \vee 0] \wedge [I_{O_2}(k_2) \vee 0]\\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)]\\
&= I_{(O_1+O_2)}(k_1) \wedge I_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
F_{(O_{1h}+O_{2h})}(k_1k_2) &= F_{O_{1h}}(k_1k_2) \wedge F_{O_{2h}}(k_1k_2)\\
&= F_{O_{2h}}(k_1k_2) \wedge 1\\
&\leq [F_{O_2}(k_1) \vee F_{O_2}(k_2)] \wedge 1\\
&= [F_{O_2}(k_1) \wedge 1] \vee [F_{O_2}(k_2) \wedge 1]\\
&= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_1}(k_2) \wedge F_{O_2}(k_2)]\\
&= F_{(O_1+O_2)}(k_1) \vee F_{(O_1+O_2)}(k_2),
\end{aligned}$$

for $q_1, q_2 \in P_1 + P_2$.

Case 3: For $k_1 \in P_1$, $k_2 \in P_2$, by definition 2.32, $T_{O_1}(k_2) = T_{O_2}(k_1) = 0$, $I_{O_1}(k_2) = I_{O_2}(k_1) = 0$, $F_{O_1}(k_2) = F_{O_2}(k_1) = 1$, so

$$\begin{aligned}
T_{(O_{1h}+O_{2h})}(k_1k_2) &= T_{O_1}(q_1) \wedge T_{O_2}(k_2)\\
&= [T_{O_1}(k_1) \vee 0] \wedge [T_{O_2}(k_2) \vee 0]\\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_2}(k_2) \vee T_{O_1}(k_2)]\\
&= T_{(O_1+O_2)}(k_1) \wedge T_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h}+O_{2h})}(k_1k_2) &= I_{O_1}(k_1) \wedge I_{O_2}(k_2)\\
&= [I_{O_1}(k_1) \vee 0] \wedge [I_{O_2}(k_2) \vee 0]\\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_2}(k_2) \vee I_{O_1}(k_2)]\\
&= I_{(O_1+O_2)}(k_1) \wedge I_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned} F_{(O_{1h}+O_{2h})}(k_1k_2) &= F_{O_1}(k_1) \vee F_{O_2}(k_2) \\ &= [F_{O_1}(k_1) \wedge 1] \vee [F_{O_2}(k_2) \wedge 1] \\ &= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_2}(k_2) \wedge F_{O_1}(k_2)] \\ &= F_{(O_1+O_2)}(k_1) \vee F_{(O_1+O_2)}(k_2), \end{aligned}$$

for $k_1, k_2 \in P_1 + P_2$.

All these cases hold $\forall h \in \{1, 2, \ldots, r\}$. This completes the proof. □

Theorem 2.35. *If $\check{G} = (P_1 + P_2, P_{11} + P_{21}, P_{12} + P_{22}, \ldots, P_{1r} + P_{2r})$ is the join of the two GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, ..., P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, ..., P_{2r})$. Then each strong INGS $\check{G}_i = (O, O_1, O_2, ..., O_r)$ of $\check{G}$, is join of the two strong INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ of GSs $\check{G}_1$ and $\check{G}_2$, respectively.*

Proof. We define O_l and O_{lh} for $l = 1, 2$ and $h = 1, 2, \ldots, r$ as:
$T_{O_l}(k) = T_O(k), I_{O_k}(k) = I_O(k), F_{O_l}(k) = F_O(k)$, if $k \in P_l$,
$T_{O_{lh}}(k_1k_2) = T_{O_h}(k_1k_2), I_{O_{lh}}(k_1k_2) = I_{O_h}(k_1k_2), F_{O_{lh}}(k_1k_2) = F_{O_h}(k_1k_2)$, if $k_1k_2 \in P_{lh}$.
Now for $k_1k_2 \in P_{lh}$, $l = 1, 2$, $h = 1, 2, \ldots, r$,
$T_{O_{lh}}(k_1k_2) = T_{O_h}(k_1k_2) = T_O(k_1) \wedge T_O(k_2) = T_{O_l}(k_1) \wedge T_{O_l}(k_2)$,
$I_{O_{lh}}(k_1k_2) = I_{O_h}(k_1k_2) = I_O(k_1) \wedge I_O(k_2) = I_{O_l}(k_1) \wedge I_{O_l}(k_2)$,
$F_{O_{lh}}(k_1k_2) = F_{O_h}(k_1k_2) = F_O(k_1) \vee F_O(k_2) = F_{O_l}(k_1) \vee F_{O_l}(k_2)$, i.e.,
$\check{G}_{il} = (O_l, O_{l1}, O_{l2}, ..., O_{lr})$ is strong INGS of $\check{G}_l$, $l = 1, 2$.
Moreover, $\check{G}_i$ is the join of $\check{G}_{i1}$ and $\check{G}_{i2}$ as shown:
According to the definitions 2.28 and 2.32, $O = O_1 \cup O_2 = O_1 + O_2$ and $O_h = O_{1h} \cup O_{2h} = O_{1h} + O_{2h}$, $\forall k_1k_2 \in P_{1h} \cup P_{2h}$.
When $k_1k_2 \in P_{1h} + P_{2h}$ $(P_{1h} \cup P_{2h})$, i.e., $k_1 \in P_1$ and $k_2 \in P_2$,
$T_{O_h}(k_1k_2) = T_O(k_1) \wedge T_O(k_2) = T_{O_l}(k_1) \wedge T_{O_l}(k_2) = T_{(O_{1h}+O_{2h})}(k_1k_2)$,
$I_{O_h}(k_1k_2) = I_O(k_1) \wedge I_O(k_2) = I_{O_l}(k_1) \wedge I_{O_l}(k_2) = I_{(O_{1h}+O_{2h})}(k_1k_2)$, $F_{O_h}(k_1k_2) = F_O(k_1) \vee F_O(k_2) = F_{O_l}(k_1) \vee F_{O_l}(k_2) = F_{(O_{1h}+O_{2h})}(k_1k_2)$,
when $k_1 \in P_2$, $k_2 \in P_1$, we get similar calculations. It's true for $h = 1, 2, \ldots, r$. This completes the proof. □

3. **Application**

According to IMF data, 1.75 billion people are living in poverty, their living is estimated to be less than two dollars a day. Poverty changes by region, for example in Europe it is 3%, and in the Sub-Saharan Africa it is up to 65%. We rank the countries of the World as poor or rich, using their GDP per capita as scale. Poor countries are trying to catch up with rich or developed countries. But this ratio is very small, that's why trade of poor countries among themselves is very important. There are different types of trade among poor countries, for example: agricultural or food items, raw minerals, medicines, textile materials, industrials goods etc. Using INGS, we can estimate between any two poor countries which trade is comparatively stronger than others. Moreover, we can decide(judge) which country has large number of resources for particular type of goods and better circumstances for its trade. We can figure out, for which trade, an external investor can invest his money in these poor countries. Further, it will be easy to judge that in which field these poor countries are trying to

Table 1. IN set O of nine poor countries on globe

Poor Country	T	I	F
Congo	0.5	0.3	0.2
Liberia	0.4	0.4	0.3
Burundi	0.4	0.4	0.4
Tanzania	0.5	0.5	0.4
Uganda	0.4	0.4	0.5
Sierra Leone	0.5	0.4	0.4
Zimbabwe	0.3	0.4	0.4
Kenya	0.5	0.3	0.3
Zambia	0.4	0.4	0.4

be better, and can be helped. It will also help in deciding that in which trade they are weak, and should be facilitated, so that they can be independent and improve their living standards.
We consider a set of nine poor countries in the World:

$$P = \{\text{Congo}, \text{Liberia}, \text{Burundi}, \text{Tanzania}, \text{Ugenda}, \text{SierriaLeone}, \text{Zimbabwe}, \text{Kenya}, \text{Zambia}\}.$$

Let O be the IN set on P, as defined in Table 1. In Table 1, symbol T demonstrates the positive aspects of that poor country, symbol I indicates its negative aspects, whereas F denotes the percentage of ambiguity of its problems for the World. Let we use following alphabets for country names:
CO = Congo, L = Liberia, B = Burundi, T = Tanzania, U = Uganda, SL = Sierra Leone, ZI = Zimbabwe, K = Kenya, ZA = Zambia. For every pair of poor countries in set P, different trades with their T, I and F values are demonstrated in Tables $2-8$, where T, F and I indicates the percentage of occurrence, non-occurrence and uncertainty, respectively of a particular trade between those two poor countries

Table 2. IN set of different types of trade between Congo and other poor countries in P

Type of trade	(CO, L)	(CO, B)	(CO, T)	(CO, U)	(CO, K)
Food items	(0.1, 0.2, 0.3)	(0.4, 0.2, 0.1)	(0.2, 0.1, 0.4)	(0.4, 0.3, 0.5)	(0.2, 0.1, 0.3)
Chemicals	(0.2, 0.4, 0.3)	(0.1, 0.2, 0.1)	(0.1, 0.2, 0.4)	(0.3, 0.2, 0.4)	(0.5, 0.1, 0.1)
Oil	(0.4, 0.2, 0.1)	(0.4, 0.3, 0.2)	(0.5, 0.1, 0.2)	(0.4, 0.2, 0.2)	(0.5, 0.3, 0.1)
Raw minerals	(0.3, 0.1, 0.1)	(0.4, 0.3, 0.3)	(0.4, 0.2, 0.2)	(0.4, 0.1, 0.2)	(0.5, 0.1, 0.1)
Textile products	(0.2, 0.3, 0.3)	(0.1, 0.3, 0.4)	(0.1, 0.2, 0.4)	(0.1, 0.3, 0.2)	(0.2, 0.1, 0.3)
Gold and diamonds	(0.4, 0.1, 0.1)	(0.4, 0.2, 0.2)	(0.2, 0.2, 0.4)	(0.2, 0.2, 0.4)	(0.1, 0.3, 0.3)

Table 3. IN set of different types of trade between Liberia and other poor countries in P

Type of trade	(L, B)	(L, T)	(L, U)	(L, SL)	(L, ZI)
Food items	(0.4, 0.2, 0.2)	(0.4, 0.3, 0.2)	(0.3, 0.3, 0.4)	(0.3, 0.3, 0.3)	(0.2, 0.3, 0.3)
Chemicals	(0.2, 0.2, 0.4)	(0.1, 0.4, 0.3)	(0.3, 0.3, 0.3)	(0.2, 0.2, 0.4)	(0.1, 0.3, 0.3)
Oil	(0.1, 0.1, 0.4)	(0.2, 0.3, 0.3)	(0.1, 0.1, 0.4)	(0.2, 0.4, 0.3)	(0.2, 0.2, 0.3)
Raw minerals	(0.3, 0.1, 0.3)	(0.2, 0.2, 0.3)	(0.2, 0.1, 0.4)	(0.3, 0.2, 0.3)	(0.2, 0.1, 0.3)
Textile products	(0.1, 0.3, 0.4)	(0.1, 0.3, 0.3)	(0.2, 0.1, 0.3)	(0.1, 0.2, 0.3)	(0.2, 0.2, 0.3)
Gold and diamonds	(0.2, 0.1, 0.4)	(0.2, 0.1, 0.3)	(0.3, 0.1, 0.3)	(0.4, 0.1, 0.1)	(0.3, 0.1, 0.1)

Table 4. IN set of different types of trade between Burundi and other poor countries in P

Type of trade	(B, T)	(B, U)	(B, SL)	(B, ZI)	(B, K)
Food items	(0.3, 0.2, 0.2)	(0.4, 0.1, 0.2)	(0.3, 0.3, 0.1)	(0.3, 0.3, 0.2)	(0.3, 0.2, 0.2)
Chemicals	(0.1, 0.2, 0.3)	(0.2, 0.1, 0.3)	(0.2, 0.4, 0.3)	(0.3, 0.4, 0.3)	(0.3, 0.3, 0.1)
Oil	(0.1, 0.1, 0.4)	(0.2, 0.3, 0.4)	(0.2, 0.4, 0.3)	(0.2, 0.2, 0.5)	(0.1, 0.3, 0.4)
Raw minerals	(0.2, 0.1, 0.3)	(0.4, 0.2, 0.3)	(0.4, 0.2, 0.4)	(0.3, 0.2, 0.2)	(0.4, 0.2, 0.2)
Textile products	(0.3, 0.1, 0.1)	(0.2, 0.4, 0.3)	(0.3, 0.2, 0.2)	(0.3, 0.2, 0.1)	(0.4, 0.1, 0.2)
Gold and diamonds	(0.3, 0.2, 0.3)	(0.3, 0.4, 0.3)	(0.1, 0.4, 0.2)	(0.2, 0.4, 0.2)	(0.2, 0.3, 0.4)

Table 5. IN set of different types of trade between Tanzania and other poor countries in P

Type of trade	(T, U)	(T, SL)	(T, ZI)	(T, K)	(T,ZA)
Food items	(0.4, 0.2, 0.1)	(0.5, 0.1, 0.1)	(0.3, 0.1, 0.2)	(0.4, 0.3, 0.2)	(0.3, 0.2, 0.2)
Chemicals	(0.2, 0.3, 0.3)	(0.2, 0.3, 0.4)	(0.2, 0.3, 0.3)	(0.4, 0.1, 0.4)	(0.3, 0.4, 0.4)
Oil	(0.1, 0.3, 0.3)	(0.4, 0.1, 0.3)	(0.3, 0.4, 0.2)	(0.2, 0.3, 0.3)	(0.1, 0.3, 0.3)
Raw minerals	(0.3, 0.3, 0.4)	(0.4, 0.3, 0.3)	(0.3, 0.2, 0.1)	(0.4, 0.2, 0.3)	(0.3, 0.2, 0.3)
Textile products	(0.2, 0.4, 0.3)	(0.2, 0.4, 0.4)	(0.1, 0.3, 0.4)	(0.2, 0.3, 0.2)	(0.4, 0.1, 0.2)
Gold and diamonds	(0.3, 0.4, 0.3)	(0.4, 0.3, 0.4)	(0.3, 0.1, 0.1)	(0.2, 0.2, 0.2)	(0.4, 0.3, 0.3)

Table 6. IN set of different types of trade between Sierra Leone and other poor countries in P

Type of trade	(SL, ZI)	(SL, K)	(SL, ZA)	(SL, CO)	(L, K)
Food items	(0.3, 0.3, 0.2)	(0.4, 0.2, 0.1)	(0.2, 0.4, 0.3)	(0.5, 0.1, 0.1)	(0.4, 0.1, 0.2)
Chemicals	(0.2, 0.3, 0.4)	(0.3, 0.2, 0.2)	(0.2, 0.4, 0.4)	(0.2, 0.2, 0.3)	(0.2, 0.3, 0.3)
Oil	(0.1, 0.3, 0.4)	(0.2, 0.2, 0.3)	(0.3, 0.4, 0.2)	(0.5, 0.2, 0.1)	(0.3, 0.3, 0.3)
Raw minerals	(0.3, 0.2, 0.2)	(0.5, 0.2, 0.1)	(0.3, 0.1, 0.1)	(0.3, 0.3, 0.3)	(0.4, 0.1, 0.2)
Textile products	(0.2, 0.4, 0.2)	(0.3, 0.2, 0.3)	(0.2, 0.2, 0.4)	(0.2, 0.2, 0.3)	(0.3, 0.3, 0.2)
Gold and diamonds	(0.3, 0.1, 0.1)	(0.1, 0.2, 0.4)	(0.2, 0.3, 0.3)	(0.4, 0.1, 0.2)	(0.3, 0.2, 0.3)

Table 7. IN set of different types of trade between Zimbabwe and other poor countries in P

Type of trade	(ZI, K)	(ZI, ZA)	(ZI, U)	(ZI, CO)
Food items	(0.3, 0.2, 0.2)	(0.3, 0.1, 0.1)	(0.3, 0.1, 0.1)	(0.2, 0.1, 0.1)
Chemicals	(0.3, 0.3, 0.2)	(0.2, 0.4, 0.3)	(0.3, 0.2, 0.2)	(0.2, 0.1, 0.2)
Oil	(0.1, 0.3, 0.3)	(0.1, 0.4, 0.4)	(0.3, 0.2, 0.1)	(0.3, 0.1, 0.1)
Raw minerals	(0.3, 0.1, 0.2)	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.3)	(0.2, 0.3, 0.1)
Textile products	(0.2, 0.2, 0.2)	(0.2, 0.4, 0.3)	(0.2, 0.3, 0.3)	(0.2, 0.3, 0.1)
Gold and diamonds	(0.3, 0.3, 0.1)	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.2)	(0.3, 0.2, 0.1)

Table 8. IN set of different types of trade between Zambia and other poor countries in P

Type of trade	(ZA, CO)	(ZA, L)	(ZA, B)	(ZA, K)
Food items	(0.3, 0.1, 0.2)	(0.3, 0.1, 0.2)	(0.4, 0.2, 0.1)	(0.3, 0.1, 0.3)
Chemicals	(0.2, 0.2, 0.2)	(0.2, 0.2, 0.1)	(0.3, 0.2, 0.2)	(0.3, 0.1, 0.1)
Oil	(0.4, 0.1, 0.1)	(0.2, 0.1, 0.1)	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.2)
Raw minerals	(0.3, 0.1, 0.1)	(0.4, 0.1, 0.1)	(0.4, 0.2, 0.2)	(0.4, 0.1, 0.1)
Textile products	(0.2, 0.2, 0.2)	(0.2, 0.2, 0.3)	(0.2, 0.3, 0.2)	(0.3, 0.1, 0.2)
Gold and diamonds	(0.1, 0.2, 0.4)	(0.4, 0.3, 0.2)	(0.2, 0.3, 0.2)	(0.3, 0.2, 0.1)

Many relations can be defined on the set P, we define following relations on set P as:
P_1 = Food items, P_2 = Chemicals, P_3 = Oil, P_4 = Raw minerals, P_5 = Textile products, P_6 = Gold and diamonds, such that $(P, P_1, P_2, P_3, P_4, P_5, P_6)$ is a GS. Any element of a relation demonstrates a particular trade between those two poor countries. As $(P, P_1, P_2, P_3, P_4, P_5, P_6)$ is GS, that's why any element can appear in only one relation. Therefore, any element will be considered in that relation, whose value of T is high, and values of I, F are comparatively low, using data of above tables.
Write down T, I and F values of the elements in relations according to above data, such that O_1, O_2, O_3, O_4, O_5, O_6 are IN sets on relations P_1, P_2, P_3, P_4, P_5, P_6, respectively.
Let $P_1 = \{(\text{Burundi}, \text{Congo}), (\text{SierraLeone}, \text{Congo}), (\text{Burundi}, \text{Zambia})\}$, $P_2 = \{(Kenya, Congo)\}$,
$P_3 = \{(\text{Congo}, \text{Zambia}), (\text{Congo}, \text{Tanzania}), (\text{Zimbabwe}, \text{Congo})\}$,
$P_4 = \{(\text{Congo}, \text{Uganda}), (\text{SierraLeone}, \text{Kenya}), (\text{Zambia}, \text{Kenya})\}$,
$P_5 = \{(\text{Burundi}, \text{Zimbabwe}), (\text{Tanzania}, \text{Burundi})\}$,
$P_6 = \{(\text{SierraLeone}, \text{Liberia}), (\text{Uganda}, \text{SierraLeone}), (\text{Zimbabwe}, \text{SierraLeone})\}$.
Let $O_1 = \{((B, CO), 0.4, 0.2, 0.1), ((SL, CO), 0.5, 0.1, 0.1), ((B, ZA), 0.4, 0.2, 0.1)\}$,
$O_2 = \{((K, CO), 0.5, 0.1, 0.1)\}$, $O_3 = \{((CO, ZA), 0.4, 0.1, 0.1), ((CO, T), 0.5, 0.1, 0.2),$
$((ZI, CO), 0.3, 0.1, 0.1)\}$, $O_4 = \{((CO, U), 0.4, 0.1, 0.2), ((SL, K), 0.5, 0.2, 0.1), ((ZA, K), 0.4, 0.1, 0.1)\}$,
$O_5 = \{((B, ZI), 0.3, 0.2, 0.1), ((T, B), 0.3, 0.1, 0.1)\}$, $O_6 = \{((SL, L), 0.4, 0.1, 0.1), ((U, SL), 0.4, 0.2, 0.1),$
$((ZI, SL), 0.3, 0.1, 0.1)\}$. Obviously, $(O, O_1, O_2, O_3, O_4, O_5, O_6)$ is an INGS as shown in Fig. 10.

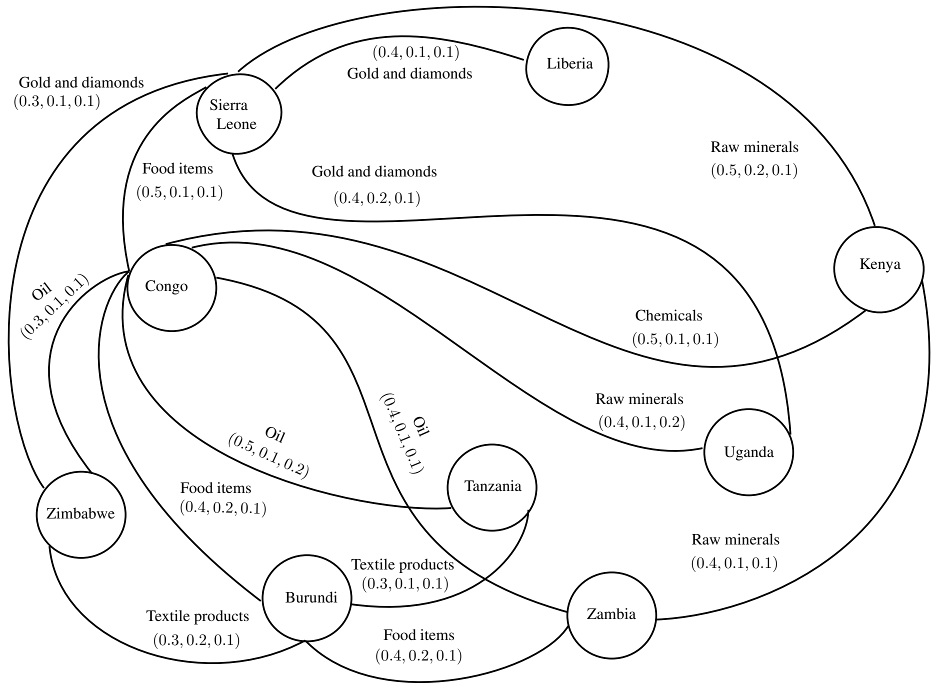


FIGURE 10. INGS indicating eminent trade between any two poor countries

Every edge of this INGS demonstrates the prominent trade between two poor countries, for example prominent trade between Congo and Zambia is Oil, its T, F and I values are 0.4, 0.1 and 0.1, respectively. According to these values, despite of poverty, circumstances of Congo and Zambia are 40% favorable for oil trade, 10% are unfavorable, and 10% are uncertain, that is, sometimes they may be favorable and sometimes unfavorable. We can observe that Congo is vertex with highest vertex degree for relation oil and Sierra Leone is vertex with highest vertex degree for relation gold and diamonds. That is, among these nine poor countries, Congo is most favorable for oil trade, and Sierra Leone is most favorable for trade of gold and diamonds. This INGS will be useful for those investors, who are interested to invest in these nine poor countries. For example an investor can invest in oil in Congo. And if someone wants to invest in gold and diamonds, this INGS will help him that Sierra Leone is most favorable.

A big advantage of this INGS is that United Nations, IMF, World Bank, and rich countries can be aware of the fact that in which fields of trade, these poor countries are trying to be better and can be helped to make their economic conditions better. Moreover, INGS of poor countries can be very beneficial for them, it may increase trade as well as foreign aid and economic help from the World, and can present their better aspects before the World.

We now explain general procedure of this application by following algorithm.

Algorithm:

1. Input a vertex set $P = \{C_1, C_2, \ldots, C_n\}$ and a IN set O defined on set P.
2. Input IN set of trade of any vertex with all other vertices and calculate T, F, and I of each pair of vertices using, $T(C_iC_j) \leq \min(T(C_i), T(C_j))$, $F(C_iC_j) \leq \max(F(C_i), F(C_j))$, $I(C_iC_j) \leq \min(I(C_i), I(C_j))$.
3. Repeat Step 2 for each vertex in set P.
4. Define relations $P_1, P_2, \ldots, P_n$ on the set P such that $(P, P_1, P_2, \ldots, P_n)$ is a GS.
5. Consider an element of that relation, for which its value of T is comparatively high, and its values of F and I are low than other relations.
6. Write down all elements in relations with T, F and I values, corresponding relations $O_1, O_2, \ldots, O_n$ are IN sets on $P_1, P_2, P_3, \ldots, P_n$, respectively and $(O, O_1, O_2, \ldots, O_n)$ is an INGS.

4. Conclusions

Fuzzy graphical models are highly utilized in applications of computer science. Especially in database theory, cluster analysis, image capturing, data mining, control theory, neural networks, expert systems and artificial intelligence. In this research paper, we have introduced certain operations on intuitionistic neutrosophic graph structures. We have discussed a novel and worthwhile real-life application of intuitionistic neutrosophic graph structure in decision-making. We have intensions to generalize our concepts to (1) Applications of IN soft GSs in decision-making (2) Applications of IN rough fuzzy GSs in decision-making, (3) Applications of IN fuzzy soft GSs in decision-making, and (4) Applications of IN rough fuzzy soft GSs in decision-making.

Acknowledgment: The authors are thankful to Editor-in-Chief and the referees for their valuable comments and suggestions.

References

[1] M. Akram and R. Akmal, Application of bipolar fuzzy sets in graph structures, Applied Computational Intelligence and Soft Computing (2016) 13 pages.

[2] M. Akram and S. Shahzadi, Neutrosophic soft graphs with application, Journal of Intelligent & Fuzzy Systems 32 (2017) 841–858.

[3] M. Akram and S. Shahzadi, Representation of graphs using intuitionistic neutrosophic soft sets, Journal of Mathematical Analysis 7 (6) (2016) 31–53.

[4] M. Akram and G. Shahzadi, Operations on single-valued neutrosophic graphs, Journal of Uncertain System 11 (2017) 1–26.

[5] M. Akram, S. Shahzadi, and A. Borumand saeid, Single-valued neutrosophic hypergraphs, TWMS Journal of Applied and Engineering Mathematics, 2016 (In press).

[6] M. Akram and M. Sitara, Novel applications of single-valued neutrosophic graph structures in decision making, Journal of Applied Mathematics and Computing, 2016, DOI 10.1007/s12190-017-1084-5 1–32.

[7] M. Akram and A. Adeel, Representation of labeling tree based on m- polar fuzzy sets, Ann. Fuzzy Math. Inform. 13 (2) (2017) 1–9.

[8] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[9] P. Bhattacharya, Some remarks on fuzzy graphs, Pattern Recognition Letters 6 (5) (1987) 297–302.
[10] M. Bhowmik and M. Pal, Intuitionistic neutrosophic set, Journal of Information and Computing Science 4 (2) (2009) 142–152.
[11] M. Bhowmik and M. Pal, Intuitionistic neutrosophic set relations and some of its properties, Journal of Information and Computing Science 5 (3) 183–192.
[12] K. R. Bhutani and A. Rosenfeld, Strong arcs in fuzzy graphs, Inform. Sci. 152 (2003) 319–326.
[13] S. Broumi, M. Talea, A. Bakali, and F. Smarandache, Single-valued neutrosophic graphs, Journal of New Theory 10 (2016) 86–101.
[14] T. Dinesh, A study on graph structures, incidence algebras and their fuzzy analogues [Ph.D.thesis], Kannur University, Kannur India 2011.
[15] T. Dinesh and T. V. Ramakrishnan, On generalised fuzzy graph structures, Applied Mathematical Sciences 5 (4) (2011) 173–180.
[16] A. Kauffman, Introduction a la Theorie des Sous-emsembles Flous, Masson et Cie Vol.1 1973.
[17] M. G. Karunambigai and R. Buvaneswari, Degrees in intuitionistic fuzzy graphs, Ann. Fuzzy Math. Inform. 13 (2017) 1–13.
[18] J. N. Mordeson and P. Chang-Shyh, Operations on fuzzy graphs, Inform. Sci. 79 (1994) 159–170.
[19] T.K. Mondal and S.K. Samanta, Generalized intuitionistic fuzzy sets, J. Fuzzy Math. 10 (4) (2002) 839–862.
[20] J. J. Peng, J. Q. Wang, H. Y. Zhang and X. H. Chen, An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets, Applied Soft Computing 25 (2014) 336–346.
[21] A. Rosenfeld, Fuzzy graphs, Fuzzy Sets and their Applications(L.A.Zadeh, K.S.Fu, M.Shimura, Eds.), Academic Press, New York (1975) 77–95.
[22] N. Shah and A. Hussain, Neutrosophic soft graphs, Neutrosophic Set and Systems 11 (2016) 31–44.
[23] E. Sampathkumar, Generalized graph structures, Bulletin of Kerala Mathematics Association 3 (2) (2006) 65–123.
[24] F. Smarandache, Neutrosophy Neutrosophic Probability, Set, and Logic, Amer Res Press Rehoboth USA 1998.
[25] F. Smarandache, A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth 1999.
[26] I. Turksen, Interval-valued fuzzy sets based on normal form, Fuzzy Sets and Systems 20 (1986) 191–210.
[27] H. Wang, F. Smarandache, Y. Q. Zhang and R. Sunderraman, Single valued neutrosophic sets, Multispace and Multistructure 4 (2010) 410-413.
[28] J. Ye, Single-valued neutrosophic minimum spanning tree and its clustering method, Journal of Intelligent Systems 23 (3) (2014) 311–324.
[29] L. A. Zadeh, Fuzzy sets, Information and control 8 (1965) 338–353.
[30] L. A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.

Muhammad Akram (m.akram@pucit.edu.pk, makrammath@yahoo.com)
Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan

Muzzamal Sitara (muzzamalsitara@gmail.com)
Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 1, (July 2017), pp. 29–42
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Neutrosophic quadruple algebraic hyperstructures

A. A. A. Agboola, B. Davvaz, F. Smarandache

Received 13 February 2017; Revised 23 February 2017; Accepted 12 March 2017

Abstract. The objective of this paper is to develop neutrosophic quadruple algebraic hyperstructures. Specifically, we develop neutrosophic quadruple semihypergroups, neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings and we present elementary properties which characterize them.

2010 AMS Classification: 03B60, 20A05, 20N20, 97H40

Keywords: Neutrosophy, Neutrosophic quadruple number, Neutrosophic quadruple semihypergroup, Neutrosophic quadruple canonical hypergroup, Neutrosophic quadruple hyperrring.

Corresponding Author: A. A. A. Agboola (aaaola2003@yahoo.com)

1. Introduction

The concept of neutrosophic quadruple numbers was introduced by Florentin Smarandache [18]. It was shown in [18] how arithmetic operations of addition, subtraction, multiplication and scalar multiplication could be performed on the set of neutrosophic quadruple numbers. In [1], Akinleye et.al. introduced the notion of neutrosophic quadruple algebraic structures. Neutrosophic quadruple rings were studied and their basic properties were presented. In the present paper, two hyperoperations $\hat{+}$ and $\hat{\times}$ are defined on the neutrosophic set NQ of quadruple numbers to develop new algebraic hyperstructures which we call neutrosophic quadruple algebraic hyperstructures. Specifically, it is shown that $(NQ, \hat{\times})$ is a neutrosophic quadruple semihypergroup, $(NQ, \hat{+})$ is a neutrosophic quadruple canonical hypergroup and $(NQ, \hat{+}, \hat{\times})$ is a neutrosophic quadruple hyperrring and their basic properties are presented.

Definition 1.1 ([18])**.** A neutrosophic quadruple number is a number of the form (a, bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and $a, b, c, d \in \mathbb{R}$ or $\mathbb{C}$. The set NQ defined by

$$NQ = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C}\} \tag{1.1}$$

is called a neutrosophic set of quadruple numbers. For a neutrosophic quadruple number (a, bT, cI, dF) representing any entity which may be a number, an idea, an object, etc, a is called the known part and (bT, cI, dF) is called the unknown part.

Definition 1.2. Let $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQ$. We define the following:

$$a + b = (a_1 + b_1, (a_2 + b_2)T, (a_3 + b_3)I, (a_4 + b_4)F), \tag{1.2}$$

$$a - b = (a_1 - b_1, (a_2 - b_2)T, (a_3 - b_3)I, (a_4 - b_4)F). \tag{1.3}$$

Definition 1.3. Let $a = (a_1, a_2T, a_3I, a_4F) \in NQ$ and let α be any scalar which may be real or complex, the scalar product $\alpha.a$ is defined by

$$\alpha.a = \alpha.(a_1, a_2T, a_3I, a_4F) = (\alpha a_1, \alpha a_2T, \alpha a_3I, \alpha a_4F). \tag{1.4}$$

If $\alpha = 0$, then we have $0.a = (0, 0, 0, 0)$ and for any non-zero scalars m and n and $b = (b_1, b_2T, b_3I, b_4F)$, we have:

$$\begin{aligned}(m + n)a &= ma + na,\\ m(a + b) &= ma + mb,\\ mn(a) &= m(na),\\ -a &= (-a_1, -a_2T, -a_3I, -a_4F).\end{aligned}$$

Definition 1.4 ([18])**.** [Absorbance Law] Let X be a set endowed with a total order $x < y$, named " x prevailed by y" or "x less stronger than y" or "x less preferred than y". $x \leq y$ is considered as "x prevailed by or equal to y" or "x less stronger than or equal to y" or "x less preferred than or equal to y".

For any elements $x, y \in X$, with $x \leq y$, absorbance law is defined as

$$x.y = y.x = \text{absorb}(x, y) = \max\{x, y\} = y \tag{1.5}$$

which means that the bigger element absorbs the smaller element (the big fish eats the small fish). It is clear from (1.5) that

$$x.x = x^2 = \text{absorb}(x, x) = \max\{x, x\} = x \quad \text{and} \tag{1.6}$$

$$x_1.x_2 \cdots x_n = \max\{x_1, x_2, \cdots, x_n\}. \tag{1.7}$$

Analogously, if $x > y$, we say that "x prevails to y" or "x is stronger than y" or "x is preferred to y". Also, if $x \geq y$, we say that "x prevails or is equal to y" or "x is stronger than or equal to y" or "x is preferred or equal to y".

Definition 1.5. Consider the set $\{T, I, F\}$. Suppose in an optimistic way we consider the prevalence order $T > I > F$. Then we have:

$$TI = IT = \max\{T, I\} = T, \tag{1.8}$$

$$TF = FT = \max\{T, F\} = T, \tag{1.9}$$

$$IF = FI = \max\{I, F\} = I, \tag{1.10}$$

$$TT = T^2 = T, \tag{1.11}$$

$$II = I^2 = I, \tag{1.12}$$

$$FF = F^2 = F. \tag{1.13}$$

Analogously, suppose in a pessimistic way we consider the prevalence order $T < I < F$. Then we have:

$$TI = IT = \max\{T, I\} = I, \tag{1.14}$$
$$TF = FT = \max\{T, F\} = F, \tag{1.15}$$
$$IF = FI = \max\{I, F\} = F, \tag{1.16}$$
$$TT = T^2 = T, \tag{1.17}$$
$$II = I^2 = I, \tag{1.18}$$
$$FF = F^2 = F. \tag{1.19}$$

Except otherwise stated, we will consider only the prevalence order $T < I < F$ in this paper.

Definition 1.6. Let $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQ$. Then

$$\begin{aligned} a.b &= (a_1, a_2T, a_3I, a_4F).(b_1, b_2T, b_3I, b_4F) \\ &= (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)T, (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I, \\ &\quad (a_1b_4 + a_2b_4, a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)F). \end{aligned} \tag{1.20}$$

Theorem 1.7 ([1])**.** *$(NQ, +)$ is an abelian group.*

Theorem 1.8 ([1])**.** *$(NQ, .)$ is a commutative monoid.*

Theorem 1.9 ([1])**.** *$(NQ, .)$ is not a group.*

Theorem 1.10 ([1])**.** *$(NQ, +, .)$ is a commutative ring.*

Definition 1.11. Let NQR be a neutrosophic quadruple ring and let NQS be a nonempty subset of NQR. Then NQS is called a neutrosophic quadruple subring of NQR, if $(NQS, +, .)$ is itself a neutrosophic quadruple ring. For example, $NQR(n\mathbb{Z})$ is a neutrosophic quadruple subring of $NQR(\mathbb{Z})$ for $n = 1, 2, 3, \cdots$.

Definition 1.12. Let NQJ be a nonempty subset of a neutrosophic quadruple ring NQR. NQJ is called a neutrosophic quadruple ideal of NQR, if for all $x, y \in NQJ, r \in NQR$, the following conditions hold:

(i) $x - y \in NQJ$,
(ii) $xr \in NQJ$ and $rx \in NQJ$.

Definition 1.13 ([1])**.** Let NQR and NQS be two neutrosophic quadruple rings and let $\phi : NQR \to NQS$ be a mapping defined for all $x, y \in NQR$ as follows:

(i) $\phi(x + y) = \phi(x) + \phi(y)$,
(ii) $\phi(xy) = \phi(x)\phi(y)$,
(iii) $\phi(T) = T$, $\phi(I) = I$ and $\phi(F) = F$,
(iv) $\phi(1, 0, 0, 0) = (1, 0, 0, 0)$.

Then ϕ is called a neutrosophic quadruple homomorphism. Neutrosophic quadruple monomorphism, endomorphism, isomorphism, and other morphisms can be defined in the usual way.

Definition 1.14. Let $\phi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism.

(i) The image of ϕ denoted by $Im\phi$ is defined by the set

$$Im\phi = \{y \in NQS : y = \phi(x), \text{for some } x \in NQR\}.$$

(ii) The kernel of ϕ denoted by $Ker\phi$ is defined by the set

$$Ker\phi = \{x \in NQR : \phi(x) = (0, 0, 0, 0)\}.$$

Theorem 1.15 ([1]). *Let $\phi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism. Then:*

(1) *$Im\phi$ is a neutrosophic quadruple subring of NQS,*

(2) *$Ker\phi$ is not a neutrosophic quadruple ideal of NQR.*

Theorem 1.16 ([1]). *Let $\phi : NQR(\mathbb{Z}) \to NQR(\mathbb{Z})/NQR(n\mathbb{Z})$ be a mapping defined by $\phi(x) = x + NQR(n\mathbb{Z})$ for all $x \in NQR(\mathbb{Z})$ and $n = 1, 2, 3, \ldots$. Then ϕ is not a neutrosophic quadruple ring homomorphism.*

Definition 1.17. Let H be a non-empty set and let $+$ be a hyperoperation on H. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

(i) $x + y = y + x$, for all $x, y \in H$,

(ii) $x + (y + z) = (x + y) + z$, for all $x, y, z \in H$,

(iii) there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, for all $x \in H$,

(iv) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,

(v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in H$.

A nonempty subset A of H is called a subcanonical hypergroup, if A is a canonical hypergroup under the same hyperaddition as that of H that is, for every $a, b \in A$, $a - b \in A$. If in addition $a + A - a \subseteq A$ for all $a \in H$, A is said to be normal.

Definition 1.18. A hyperring is a tripple $(R, +, .)$ satisfying the following axioms:

(i) $(R, +)$ is a canonical hypergroup,

(ii) $(R, .)$ is a semihypergroup such that $x.0 = 0.x = 0$ for all $x \in R$, that is, 0 is a bilaterally absorbing element,

(iii) for all $x, y, z \in R$,

$$x.(y + z) = x.y + x.z \text{ and } (x + y).z = x.z + y.z.$$

That is, the hyperoperation . is distributive over the hyperoperation $+$.

Definition 1.19. Let $(R, +, .)$ be a hyperring and let A be a nonempty subset of R. A is said to be a subhyperring of R if $(A, +, .)$ is itself a hyperring.

Definition 1.20. Let A be a subhyperring of a hyperring R. Then

(i) A is called a left hyperideal of R if $r.a \subseteq A$ for all $r \in R, a \in A$,

(ii) A is called a right hyperideal of R if $a.r \subseteq A$ for all $r \in R, a \in A$,

(iii) A is called a hyperideal of R if A is both left and right hyperideal of R.

Definition 1.21. Let A be a hyperideal of a hyperring R. A is said to be normal in R, if $r + A - r \subseteq A$, for all $r \in R$.

For full details about hypergroups, canonical hypergroups, hyperrings, neutrosophic canonical hypergroups and neutrosophic hyperrings, the reader should see [3, 14]

2. Development of neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings

In this section, we develop two neutrosophic hyperquadruple algebraic hyperstructures namely neutrosophic quadruple canonical hypergroup and neutrosophic quadruple hyperring . In what follows, all neutrosophic quadruple numbers will be real neutrosophic quadruple numbers i.e $a, b, c, d \in \mathbb{R}$ for any neutrosophic quadruple number $(a, bT, cI, dF) \in NQ$.

Definition 2.1. Let $+$ and . be hyperoperations on $\mathbb{R}$ that is $x+y \subseteq \mathbb{R}, x.y \subseteq \mathbb{R}$ for all $x, y \in \mathbb{R}$. Let $\hat{+}$ and $\hat{\times}$ be hyperoperations on NQ. For $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F) \in NQ$ with $x_i, y_i \in \mathbb{R}, i = 1, 2, 3, 4$, define:

$$\begin{aligned} x\hat{+}y &= \{(a, bT, cI, dF) : a \in x_1 + y_1, b \in x_2 + y_2, \\ &\quad c \in x_3 + y_3, d \in x_4 + y_4\}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} x\hat{\times}y &= \{(a, bT, cI, dF) : a \in x_1.y_1, b \in (x_1.y_2) \cup (x_2.y_1) \cup (x_2.y_2), c \in (x_1.y_3) \\ &\quad \cup (x_2.y_3) \cup (x_3.y_1) \cup (x_3.y_2) \cup (x_3.y_3), d \in (x_1.y_4) \cup (x_2.y_4) \\ &\quad \cup (x_3.y_4) \cup (x_4.y_1) \cup (x_4.y_2) \cup (x_4.y_3) \cup (x_4.y_4)\}. \end{aligned} \tag{2.2}$$

Theorem 2.2. $(NQ, \hat{+})$ *is a canonical hypergroup.*

Proof. Let $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F), z = (z_1, z_2T, z_3I, z_4F) \in NQ$ be arbitrary with $x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4$.

(i) To show that $x\hat{+}y = y\hat{+}x$, let

$x\hat{+}y = \{a = (a_1, a_2T, a_3I, a_4F) : a_1 \in x_1 + y_1, a_2 \in x_2 + y_2, a_3 \in x_3 + y_3,$
$a_4 \in x_4 + y_4\}$,

$y\hat{+}x = \{b = (b_1, b_2T, b_3I, b_4F) : b_1 \in y_1 + x_1, b_2 \in y_2 + x_2, b_3 \in y_3 + b_3,$
$b_4 \in y_4 + x_4\}$.

Since $a_i, b_i \in \mathbb{R}, i = 1, 2, 3, 4$, it follows that $x\hat{+}y = y\hat{+}x$.

(ii) To show that that $x\hat{+}(y\hat{+}z) = (x\hat{+}y)\hat{+}z$, let

$y\hat{+}z = \{w = (w_1, w_2T, w_3I, w_4F) : w_1 \in y_1 + z_1, w_2 \in y_2 + z_2,$
$w_3 \in y_3 + z_3, w_4 \in y_4 + z_4\}$. Now,

$$\begin{aligned} x\hat{+}(y\hat{+}z) &= x\hat{+}w \\ &= \{p = (p_1, p_2T, p_3I, p_4F) : p_1 \in x_1 + w_1, p_2 \in x_2 + w_2, p_3 \in x_3 + w_3, \\ &\quad p_4 \in x_4 + w_4\} \\ &= \{p = (p_1, p_2T, p_3I, p_4F) : p_1 \in x_1 + (y_1 + z_1), p_2 \in x_2 + (y_2 + z_2), \\ &\quad p_3 \in x_3 + (y_3 + z_3), p_4 \in x_4 + (y_4 + z_4)\}. \end{aligned}$$

Also, let $x\hat{+}y = \{u = (u_1, u_2T, u_3I, u_4F) : u_1 \in x_1 + y_1, u_2 \in x_2 + y_2, u_3 \in x_3 + y_3, u_4 \in x_4 + y_4\}$ so that

$$\begin{aligned} (x\hat{+}y)\hat{+}z &= u\hat{+}z \\ &= \{q = (q_1, q_2T, q_3I, q_4F) : q_1 \in u_1 + z_1, q_2 \in u_2 + z_2, q_3 \in u_3 + z_3, \\ &\quad q_4 \in u_4 + z_4\} \\ &= \{q = (q_1, q_2T, q_3I, q_4F) : q_1 \in (x_1 + y_1) + z_1, q_2 \in (x_2 + y_2) + z_2, \\ &\quad q_3 \in (x_3 + y_3) + z_3, q_4 \in (x_4 + y_4) + z_4\}. \end{aligned}$$

Since $u_i, p_i, q_i, w_i, x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4$, it follows that $x\hat{+}(y\hat{+}z) = (x\hat{+}y)\hat{+}z$.

(iii) To show that $0 = (0, 0, 0, 0) \in NQ$ is a neutral element, consider

$$\begin{aligned} x\hat{+}(0,0,0,0) &= \{a = (a_1, a_2T, a_3I, a_4F) : a_1 \in x_1 + 0, a_2 \in x_2 + 0, a_3 \in x_3 + 0, \\ &\quad a_4 \in x_4 + 0\} \\ &= \{a = (a_1, a_2T, a_3I, a_4F) : a_1 \in \{x_1\}, a_2 \in \{x_2\}, a_3 \in \{x_3\}, \\ &\quad a_4 \in \{x_4\}\} \\ &= \{x\}. \end{aligned}$$

Similarly, it can be shown that $(0, 0, 0, 0)\hat{+}x = \{x\}$. Hence $0 = (0, 0, 0, 0) \in NQ$ is a neutral element.

(iv) To show that that for every $x \in NQ$, there exists a unique element $\hat{-}x \in NQ$ such that $0 \in x\hat{+}(\hat{-}x) \cap (\hat{-}x)\hat{+}x$, consider

$$\begin{aligned} x\hat{+}(\hat{-}x) \cap (\hat{-}x)\hat{+}x &= \{a = (a_1, a_2T, a_3I, a_4F) : a_1 \in x_1 - x_1, a_2 \in x_2 - x_2, \\ &\quad a_3 \in x_3 - x_3, a_4 \in x_4 - x_4\} \cap \{b = (b_1, b_2T, b_3I, b_4F) : \\ &\quad b_1 \in -x_1 + x_1, b_2 \in -x_2 + x_2, b_3 \in -x_3 + x_3, b_4 \in -x_4 + x_4\} \\ &= \{(0, 0, 0, 0)\}. \end{aligned}$$

This shows that for every $x \in NQ$, there exists a unique element $\hat{-}x \in NQ$ such that $0 \in x\hat{+}(\hat{-}x) \cap (\hat{-}x)\hat{+}x$.

(v) Since for all $x, y, z \in NQ$ with $x_i, y_1, z_i \in \mathbb{R}, i = 1, 2, 3, 4$, it follows that $z \in x\hat{+}y$ implies $y \in \hat{-}x\hat{+}z$ and $x \in z\hat{+}(\hat{-}y)$. Hence, $(NQ, \hat{+})$ is a canonical hypergroup. □

Lemma 2.3. *Let $(NQ, \hat{+})$ be a neutrosophic quadruple canonical hypergroup. Then*

(1) $\hat{-}(\hat{-}x) = x$ *for all* $x \in NQ$,
(2) $0 = (0, 0, 0, 0)$ *is the unique element such that for every* $x \in NQ$, *there is an element* $\hat{-}x \in NQ$ *such that* $0 \in x\hat{+}(\hat{-}x)$,
(3) $\hat{-}0 = 0$,
(4) $\hat{-}(x\hat{+}y) = \hat{-}x\hat{-}y$ *for all* $x, y \in NQ$.

Example 2.4. Let $NQ = \{0, x, y\}$ be a neutrosophic quadruple set and let $\hat{+}$ be a hyperoperation on NQ defined in the table below.

$\hat{+}$	0	x	y
0	0	x	y
x	x	$\{0, x, y\}$	y
y	y	y	$\{0, y\}$

Then $(NQ, \hat{+})$ is a neutrosophic quadruple canonical hypergroup.

Theorem 2.5. $(NQ, \hat{\times})$ *is a semihypergroup.*

Proof. Let $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F), z = (z_1, z_2T, z_3I, z_4F) \in NQ$ be arbitrary with $x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4$.

(i)

$$\begin{aligned} x\hat{\times}y &= \{a=(a_1,a_2T,a_3I,a_4F): a_1\in x_1y_1, a_2\in x_1y_2\cup x_2y_1\cup x_2y_2, a_3\in x_1y_3 \\ &\quad \cup x_2y_3\cup x_3y_1\cup x_3y_2\cup x_3y_3, a_4\in x_1y_4\cup x_2y_4 \\ &\quad \cup x_3y_4\cup x_4y_1\cup x_4y_2\cup x_4y_3\cup x_4y_4\} \\ &\subseteq NQ. \end{aligned}$$

(ii) To show that $x\hat{\times}(y\hat{\times}z)=(x\hat{\times}y)\hat{\times}z$, let

$$\begin{aligned} y\hat{\times}z &= \{w=(w_1,w_2T,w_3I,w_4F): w_1\in y_1z_1, w_2\in y_1z_2\cup y_2z_1\cup y_2z_2, \\ &\quad w_3\in y_1z_3\cup y_2z_3\cup y_3z_1\cup y_3z_2\cup y_3z_3, w_4\in y_1z_4)\cup y_2z_4 \\ &\quad \cup y_3z_4\cup y_4z_1\cup y_4z_2\cup y_4z_3\cup y_4z_4\} \end{aligned} \tag{2.3}$$

so that

$$\begin{aligned} x\hat{\times}(y\hat{\times}z) &= x\hat{\times}w \\ &= \{p=(p_1,p_2T,p_3I,p_4F): p_1\in x_1w_1, p_2\in x_1w_2\cup x_2w_1\cup x_2w_2, \\ &\quad p_3\in x_1w_3\cup x_2w_3\cup x_3w_1\cup x_3w_2\cup x_3y_3, p_4\in x_1w_4\cup x_2w_4 \\ &\quad \cup x_3w_4\cup x_4w_1\cup x_4w_2\cup x_4w_3\cup x_4w_4\}. \end{aligned} \tag{2.4}$$

Also, let

$$\begin{aligned} x\hat{\times}y &= \{u=(u_1,u_2T,u_3I,u_4F): u_1\in x_1y_1, u_2\in x_1y_2\cup x_2y_1\cup x_2y_2, u_3\in x_1y_3 \\ &\quad \cup x_2y_3\cup x_3y_1\cup x_3y_2\cup x_3y_3, u_4\in x_1y_4\cup x_2y_4 \\ &\quad \cup x_3y_4\cup x_4y_1\cup x_4y_2\cup x_4y_3\cup x_4y_4\} \end{aligned} \tag{2.5}$$

so that

$$\begin{aligned} (x\hat{\times}y)\hat{\times}z &= u\hat{\times}z \\ &= \{q=(q_1,q_2T,q_3I,q_4F): q_1\in u_1z_1, q_2\in u_1z_2\cup u_2z_1\cup u_2z_2, \\ &\quad q_3\in u_1z_3\cup u_2z_3\cup u_3z_1\cup u_3z_2\cup u_3z_3, q_4\in u_1z_4\cup u_2z_4 \\ &\quad \cup u_3z_4\cup u_4z_1\cup u_4z_2\cup u_4z_3\cup u_4z_4\}. \end{aligned} \tag{2.6}$$

Substituting w_i of (2.3) in (2.4) and also substituting u_i of (2.5) in (2.6), where $i=1,2,3,4$ and since $p_i,q_i,u_i,w_i,x_i,z_i\in\mathbb{R}$, it follows that $x\hat{\times}(y\hat{\times}z)=(x\hat{\times}y)\hat{\times}z$. Consequently, $(NQ,\hat{\times})$ is a semihypergroup which we call neutrosophic quadruple semihypergroup. □

Remark 2.6. $(NQ,\hat{\times})$ is not a hypergroup.

Definition 2.7. Let $(NQ,\hat{+})$ be a neutrosophic quadruple canonical hypergroup. For any subset NH of NQ, we define

$$\hat{-}NH=\{\hat{-}x : x\in NH\}.$$

A nonempty subset NH of NQ is called a neutrosophic quadruple subcanonical hypergroup, if the following conditions hold:

(i) $0=(0,0,0,0)\in NH$,
(ii) $x\hat{-}y\subseteq NH$ for all $x,y\in NH$.

A neutrosophic quadruple subcanonical hypergroup NH of a netrosophic quadruple canonical hypergroup NQ is said to be normal, if $x\hat{+}NH\hat{-}x\subseteq NH$ for all $x\in NQ$.

Definition 2.8. Let $(NQ, \hat{+})$ be a neutrosophic quadruple canonical hypergroup. For $x_i \in NQ$ with $i = 1, 2, 3 \ldots, n \in \mathbb{N}$, the heart of NQ denoted by NQ_ω is defined by

$$NQ_\omega = \bigcup \sum_{i=1}^{n} \left(x_i \hat{-} x_i \right).$$

In Example 2.4, $NQ_\omega = NQ$.

Definition 2.9. Let $(NQ_1, \hat{+})$ and $(NQ_2, \hat{+}')$ be two neutrosophic quadruple canonical hypergroups. A mapping $\phi : NQ_1 \to NQ_2$ is called a neutrosophic quadruple strong homomorphism, if the following conditions hold:

(i) $\phi(x \hat{+} y) = \phi(x) \hat{+}' \phi(y)$ for all $x, y \in NQ_1$,
(ii) $\phi(T) = T$,
(iii) $\phi(I) = I$,
(iv) $\phi(F) = F$,
(v) $\phi(0) = 0$.

If in addition ϕ is a bijection, then ϕ is called a neutrosophic quadruple strong isomorphism and we write $NQ_1 \cong NQ_2$.

Definition 2.10. Let $\phi : NQ_1 \to NQ_2$ be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups. Then the set $\{x \in NQ_1 : \phi(x) = 0\}$ is called the kernel of ϕ and it is denoted by $Ker\phi$. Also, the set $\{\phi(x) : x \in NQ_1\}$ is called the image of ϕ and it is denoted by $Im\phi$.

Theorem 2.11. $(NQ, \hat{+}, \hat{\times})$ *is a hyperring.*

Proof. That $(NQ, \hat{+})$ is a canonical hypergroup follows from Theorem 2.2. Also, that $(NQ, \hat{\times})$ is a semihypergroup follows from Theorem 2.4.

Next, let $x = (x_1, x_2T, x_3I, x_4F) \in NQ$ be arbitrary with $x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4$. Then

$$\begin{aligned} x \hat{\times} 0 &= \{u = (u_1, u_2T, u_3I, u_4F) : u_1 \in x_1.0, u_2 \in x_1.0 \cup x_2.0 \cup x_2.0, u_3 \in x_1.0 \\ &\quad \cup x_2.0 \cup x_3.0 \cup x_3.0 \cup x_3.0, u_4 \in x_1.0 \cup x_2.0 \cup x_3.0 \cup x_4.0 \cup x_4.0 \\ &\quad \cup x_4.0 \cup x_4.0\} \\ &= \{u = (u_1, u_2T, u_3I, u_4F) : u_1 \in \{0\}, u_2 \in \{0\}, u_3 \in \{0\}, u_4 \in \{0\}\} \\ &= \{0\}. \end{aligned}$$

Similarly, it can be shown that $0 \hat{\times} x = \{0\}$. Since x is arbitrary, it follows that $x \hat{\times} 0 = 0 \hat{\times} x = \{0\}$, for all $x \in NQ$. Hence, $0 = (0, 0, 0, 0)$ is a bilaterally absorbing element.

To complete the proof, we have to show that $x \hat{\times} (y \hat{+} z) = (x \hat{\times} y) \hat{+} (x \hat{\times} z)$, for all $x, y, z \in NQ$. To this end, let $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F), z = (z_1, z_2T, z_3I, z_4F) \in NQ$ be arbitrary with $x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4$. Let

$$\begin{aligned} y \hat{+} z &= \{w = (w_1, w_2T, w_3I, w_4F) : w_1 \in y_1 + z_1, w_2 \in y_2 + z_2, w_3 \in y_3 + z_3, \\ &\quad w_4 \in y_4 + z_4\} \end{aligned} \tag{2.7}$$

so that

$$\begin{aligned} x\hat{\times}(y\hat{+}z) &= x\hat{\times}w \\ &= \{p=(p_1,p_2T,p_3I,p_4F): p_1\in x_1w_1, p_2\in x_1w_2\cup x_2w_1\cup x_2w_2, \\ &\quad p_3\in x_1w_3\cup x_2w_3\cup x_3w_1\cup x_3w_2\cup x_3y_3, p_4\in x_1w_4\cup x_2w_4 \\ &\quad \cup x_3w_4\cup x_4w_1\cup x_4w_2\cup x_4w_3\cup x_4w_4\}. \end{aligned} \tag{2.8}$$

Substituting $w_i, i=1,2,3,4$ of (2.7) in (2.8), we obtain the following:

$$p_1\in x_1(y_1+z_1), \tag{2.9}$$

$$p_2\in x_1(y_2+z_2)\cup x_2(y_1+z_1)\cup x_2(y_2+z_2), \tag{2.10}$$

$$p_3\in x_1(y_3+z_3)\cup x_2(y_3+z_3)\cup x_3(y_1+z_1)\cup x_3(y_2+z_2)\cup x_3(y_3+z_3), \tag{2.11}$$

$$\begin{aligned} &p_4\in x_1(y_4+z_4)\cup x_2(y_4+z_4)\cup x_3(y_4+z_4)\cup x_4(y_1+z_1)\cup x_4(y_2+z_2), \\ &\cup x_4(y_3+z_3)\cup x_4(y_4+z_4). \end{aligned} \tag{2.12}$$

Also, let

$$\begin{aligned} x\hat{\times}y &= \{u=(u_1,u_2T,u_3I,u_4F): u_1\in x_1y_1, u_2\in x_1y_2\cup x_2y_1\cup x_2y_2, \\ &\quad u_3\in x_1y_3\cup x_2y_3\cup x_3y_1\cup x_3y_2\cup x_3y_3, u_4\in x_1y_4\cup x_2y_4 \\ &\quad \cup x_3y_4\cup x_4y_1\cup x_4y_2\cup x_4y_3\cup x_4y_4\} \end{aligned} \tag{2.13}$$

$$\begin{aligned} x\hat{\times}z &= \{v=(v_1,v_2T,v_3I,v_4F): v_1\in x_1z_1, v_2\in x_1z_2\cup x_2z_1\cup x_2z_2, \\ &\quad v_3\in x_1z_3\cup x_2z_3\cup x_3z_1\cup x_3z_2\cup x_3z_3, v_4\in x_1z_4\cup x_2z_4 \\ &\quad \cup x_3z_4\cup x_4z_1\cup x_4z_2\cup x_4z_3\cup x_4z_4\} \end{aligned} \tag{2.14}$$

so that

$$\begin{aligned} (x\hat{\times}y)\hat{+}(x\hat{\times}z) &= u\hat{+}v \\ &= \{q=(q_1,q_2T,q_3I,q_4F): q_1\in u_1+v_1, q_2\in u_2+v_2, \\ &\quad q_3\in u_3+v_3, q_4\in u_4+v_4\}. \end{aligned} \tag{2.15}$$

Substituting u_i of (2.13) and v_i of (2.14) in (2.15), we obtain the following:

$$q_1\in u_1+v_1\subseteq x_1y_1+x_1z_1\subseteq x_1(y_1+z_1), \tag{2.16}$$

$$\begin{aligned} &q_2\in u_2+v_2\subseteq (x_1y_2\cup x_2y_1\cup x_2y_2) \\ &+(x_1z_2\cup x_2z_1\cup x_2(z_2) \\ &\subseteq x_1(y_2+z_2)\cup x_2(y_1+z_1)\cup x_2(y_2+z_2), \end{aligned} \tag{2.17}$$

$$\begin{aligned} &q_3\in u_3+v_3\subseteq (x_1y_3\cup x_2y_3\cup x_3y_1)\cup x_3y_2\cup x_3y_3) \\ &+(x_1z_3\cup x_2z_3\cup x_3z_1)\cup x_3z_2\cup x_3z_3) \\ &\subseteq x_1(y_3+z_3)\cup x_2(y_3+z_3)\cup x_3(y_1+z_1)\cup x_3(y_2+z_2)\cup x_3(y_3+z_3). \end{aligned} \tag{2.18}$$

$$\begin{aligned} &q_4\in u_4+v_4\subseteq (x_1y_4\cup x_2y_4\cup x_3y_4)\cup x_4y_1\cup x_4y_2)\cup x_4y_3\cup x_4y_4) \\ &+(x_1z_4\cup x_2z_4\cup x_3z_4)\cup x_4z_1\cup x_4z_2)\cup x_4z_3\cup x_4z_4) \\ &\subseteq x_1(y_4+z_4)\cup x_2(y_4+z_4)\cup x_3(y_4+z_4)\cup x_4(y_1+z_1)\cup x_4(y_2+z_2) \\ &\cup x_4(y_3+z_3)\cup x_4(y_4+z_4). \end{aligned} \tag{2.19}$$

Comparing (2.9), (2.10), (2.11) and (2.12) respectively with (2.16), (2.17), (2.18) and (2.19), we obtain $p_i=q_i, i=1,2,3,4$. Hence, $x\hat{\times}(y\hat{+}z)=(x\hat{\times}y)\hat{+}(x\hat{\times}z)$, for all

$x, y, z \in NQ$. Thus, $(NQ, \hat{+}, \hat{\times})$ is a hyperring which we call neutrosophic quadruple hyperring. □

Theorem 2.12. *$(NQ, \hat{+}, \circ)$ is a Krasner hyperring where $\circ$ is an ordinary multiplicative binary operation on NQ.*

Definition 2.13. Let $(NQ, \hat{+}, \hat{\times})$ be a neutrosophic quadruple hyperring. A nonempty subset NJ of NQ is called a neutrosophic quadruple subhyperring of NQ, if $(NJ, \hat{+}, \hat{\times})$ is itself a neutrosophic quadruple hyperring.

NJ is called a neutrosophic quadruple hyperideal if the following conditions hold:

(i) $(NJ, \hat{+})$ is a neutrosophic quadruple subcanonical hypergroup.

(ii) For all $x \in NJ$ and $r \in NQ$, $x\hat{\times}r, r\hat{\times}x \subseteq NJ$.

A neutrosophic quadruple hyperideal NJ of NQ is said to be normal in NQ, if $x\hat{+}NJ\hat{-}x \subseteq NJ$, for all $x \in NQ$.

Definition 2.14. Let $(NQ_1, \hat{+}, \hat{\times})$ and $(NQ_2, \hat{+}', \hat{\times}')$ be two neutrosophic quadruple hyperrings. A mapping $\phi : NQ_1 \rightarrow NQ_2$ is called a neutrosophic quadruple strong homomorphism, if the following conditions hold:

(i) $\phi(x\hat{+}y) = \phi(x)\hat{+}'\phi(y)$, for all $x, y \in NQ_1$,

(ii) $\phi(x\hat{\times}y) = \phi(x)\hat{\times}'\phi(y)$, for all $x, y \in NQ_1$,

(iii) $\phi(T) = T$,

(iv) $\phi(I) = I$,

(v) $\phi(F) = F$,

(vi) $\phi(0) = 0$.

If in addition ϕ is a bijection, then ϕ is called a neutrosophic quadruple strong isomorphism and we write $NQ_1 \cong NQ_2$.

Definition 2.15. Let $\phi : NQ_1 \rightarrow NQ_2$ be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple hyperrings. Then the set $\{x \in NQ_1 : \phi(x) = 0\}$ is called the kernel of ϕ and it is denoted by $Ker\phi$. Also, the set $\{\phi(x) : x \in NQ_1\}$ is called the image of ϕ and it is denoted by $Im\phi$.

Example 2.16. Let $(NQ, \hat{+}, \hat{\times})$ be a neutrosophic quadruple hyperring and let NX be the set of all strong endomorphisms of NQ. If $\oplus$ and $\odot$ are hyperoperations defined for all $\phi, \psi \in NX$ and for all $x \in NQ$ as

$$\begin{aligned} \phi \oplus \psi &= \{\nu(x) : \nu(x) \in \phi(x)\hat{+}\psi(x)\}, \\ \phi \odot \psi &= \{\nu(x) : \nu(x) \in \phi(x)\hat{\times}\psi(x)\}, \end{aligned}$$

then $(NX, \oplus, \odot)$ is a neutrosophic quadruple hyperring.

3. Characterization of neutrosophic quadruple canonical hypergroups and neutrosophic hyperrings

In this section, we present elementary properties which characterize neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings.

Theorem 3.1. *Let NG and NH be neutrosophic quadruple subcanonical hypergroups of a neutrosophic quadruple canonical hypergroup $(NQ, \hat{+})$. Then*

(1) *$NG \cap NH$ is a neutrosophic quadruple subcanonical hypergroup of NQ,*

(2) *$NG \times NH$ is a neutrosophic quadruple subcanonical hypergroup of NQ.*

Theorem 3.2. *Let NH be a neutrosophic quadruple subcanonical hypergroup of a neutrosophic quadruple canonical hypergroup $(NQ, \hat{+})$. Then*

(1) $NH\hat{+}NH = NH$,
(2) $x\hat{+}NH = NH$, *for all* $x \in NH$.

Theorem 3.3. *Let $(NQ, \hat{+})$ be a neutrosophic quadruple canonical hypergroup. NQ_ω, the heart of NQ is a normal neutrosophic quadruple subcanonical hypergroup of NQ.*

Theorem 3.4. *Let NG and NH be neutrosophic quadruple subcanonical hypergroups of a neutrosophic quadruple canonical hypergroup $(NQ, \hat{+})$.*

(1) *If $NG \subseteq NH$ and NG is normal, then NG is normal.*
(2) *If NG is normal, then $NG\hat{+}NH$ is normal.*

Definition 3.5. Let NG and NH be neutrosophic quadruple subcanonical hypergroups of a neutrosophic quadruple canonical hypergroup $(NQ, \hat{+})$. The set $NG\hat{+}NH$ is defined by

$$NG\hat{+}NH = \{x\hat{+}y : x \in NG, y \in NH\}. \tag{3.1}$$

It is obvious that $NG\hat{+}NH$ is a neutrosophic quadruple subcanonical hypergroup of $(NQ, \hat{+})$.

If $x \in NH$, the set $x\hat{+}NH$ is defined by

$$x\hat{+}NH = \{x\hat{+}y : y \in NH\}. \tag{3.2}$$

If x and y are any two elements of NH and τ is a relation on NH defined by $x\tau y$ if $x \in y\hat{+}NH$, it can be shown that τ is an equivalence relation on NH and the equivalence class of any element $x \in NH$ determined by τ is denoted by $[x]$.

Lemma 3.6. *For any $x \in NH$, we have*

(1) $[x] = x\hat{+}NH$,
(2) $[\hat{-}x] = \hat{-}[x]$.

Proof. (1)

$$\begin{aligned} [x] &= \{y \in NH : x\tau y\} \\ &= \{y \in NH : y \in x\hat{+}NH\} \\ &= x\hat{+}NH. \end{aligned}$$

(2) Obvious. □

Definition 3.7. Let NQ/NH be the collection of all equivalence classes of $x \in NH$ determined by τ. For $[x], [y] \in NQ/NH$, we define the set $[x]\hat{\oplus}[y]$ as

$$[x]\hat{\oplus}[y] = \{[z] : z \in x\hat{+}y\}. \tag{3.3}$$

Theorem 3.8. *$(NQ/NH, \hat{\oplus})$ is a neutrosophic quadruple canonical hypergroup.*

Proof. Same as the classical case and so omitted. □

Theorem 3.9. *Let $(NQ, \hat{+})$ be a neutrosophic quadruple canonical hypergroup and let NH be a normal neutrosophic quadruple subcanonical hypergroup of NQ. Then, for any $x, y \in NH$, the following are equivalent:*

(1) $x \in y\hat{+}NH$,
(2) $y\hat{-}x \subseteq NH$,
(3) $(y\hat{-}x) \cap NH \neq \varnothing$

Proof. Same as the classical case and so omitted. □

Theorem 3.10. *Let $\phi : NQ_1 \to NQ_2$ be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups. Then*

(1) *$Ker\phi$ is not a neutrosophic quadruple subcanonical hypergroup of NQ_1,*
(2) *$Im\phi$ is a neutrosophic quadruple subcanonical hypergroup of NQ_2.*

Proof. (1) Since it is not possible to have $\phi((0,T,0,0)) = \phi((0,0,0,0))$, $\phi((0,0,I,0)) = \phi((0,0,0,0))$ and $\phi((0,0,0,F)) = \phi((0,0,0,0))$, it follows that $(0,T,0,0), (0,0,I,0)$ and $(0,0,0,F)$ cannot be in the kernel of ϕ. Consequently, $Ker\phi$ cannot be a neutrosophic quadruple subcanonical hypergroup of NQ_1.

(2) Obvious. □

Remark 3.11. If $\phi : NQ_1 \to NQ_2$ is a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups, then $Ker\phi$ is a subcanonical hypergroup of NQ_1.

Theorem 3.12. *Let $\phi : NQ_1 \to NQ_2$ be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups. Then*

(1) *$NQ_1/Ker\phi$ is not a neutrosophic quadruple canonical hypergroup,*
(2) *$NQ_1/Ker\phi$ is a canonical hypergroup.*

Theorem 3.13. *Let NH be a neutrosophic quadruple subcanonical hypergroup of the neutrosophic quadruple canonical hypergroup $(NQ, \hat{+})$. Then the mapping $\phi : NQ \to NQ/NH$ defined by $\phi(x) = x\hat{+}NH$ is not a neutrosophic quadruple strong homomorphism.*

Remark 3.14. Isomorphism theorems do not hold in the class of neutrosophic quadruple canonical hypergroups.

Lemma 3.15. *Let NJ be a neutrosophic quadruple hyperideal of a neutrosophic quadruple hyperring $(NQ, \hat{+}, \hat{\times})$. Then*

(1) $\hat{-}NJ = NJ$,
(2) $x\hat{+}NJ = NJ$, *for all* $x \in NJ$,
(3) $x\hat{\times}NJ = NJ$, *for all* $x \in NJ$.

Theorem 3.16. *Let NJ and NK be neutrosophic quadruple hyperideals of a neutrosophic quadruple hyperring $(NQ, \hat{+}, \hat{\times})$. Then*

(1) *$NJ \cap NK$ is a neutrosophic quadruple hyperideal of NQ,*
(2) *$NJ \times NK$ is a neutrosophic quadruple hyperideal of NQ,*
(3) *$NJ\hat{+}NK$ is a neutrosophic quadruple hyperideal of NQ.*

Theorem 3.17. *Let NJ be a normal neutrosophic quadruple hyperideal of a neutrosophic quadruple hyperring $(NQ, \hat{+}, \hat{\times})$. Then*

(1) $(x\hat{+}NJ)\hat{+}(y\hat{+}NJ) = (x\hat{+}y)\hat{+}NJ$, *for all* $x, y \in NJ$,
(2) $(x\hat{+}NJ)\hat{\times}(y\hat{+}NJ) = (x\hat{\times}y)\hat{+}NJ$, *for all* $x, y \in NJ$,
(3) $x\hat{+}NJ = y\hat{+}NJ$, *for all* $y \in x\hat{+}NJ$.

Theorem 3.18. *Let* NJ *and* NK *be neutrosophic quadruple hyperideals of a neutrosophic quadruple hyperring* $(NQ, \hat{+}, \hat{\times})$ *such that* NJ *is normal in* NQ. *Then*
(1) $NJ \cap NK$ *is normal in* NJ,
(2) $NJ\hat{+}NK$ *is normal in* NQ,
(3) NJ *is normal in* $NJ\hat{+}NK$.

Let NJ be a neutrosophic quadruple hyperideal of a neutrosophic quadruple hyperring $(NQ, \hat{+}, \hat{\times})$. For all $x \in NQ$, the set NQ/NJ is defined as

$$NQ/NJ = \{x\hat{+}NJ : x \in NQ\}. \tag{3.4}$$

For $[x], [y] \in NQ/NJ$, we define the hyperoperations $\hat{\oplus}$ and $\hat{\otimes}$ on NQ/NJ as follows:

$$[x]\hat{\oplus}[y] = \{[z] : z \in x\hat{+}y\}, \tag{3.5}$$

$$[x]\hat{\otimes}[y] = \{[z] : z \in x\hat{\times}y\}. \tag{3.6}$$

It can easily be shown that $(NQ/NH, \hat{\oplus}, \hat{\otimes})$ is a neutrosophic quadruple hyperring.

Theorem 3.19. *Let* $\phi : NQ \to NR$ *be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple hyperrings and let* NJ *be a neutrosophic quadruple hyperideal of* NQ. *Then*
(1) $Ker\phi$ *is not a neutrosophic quadruple hyperideal of* NQ,
(2) $Im\phi$ *is a neutrosophic quadruple hyperideal of* NR,
(3) $NQ/Ker\phi$ *is not a neutrosophic quadruple hyperring*,
(4) $NQ/Im\phi$ *is a neutrosophic quadruple hyperring*,
(5) *The mapping* $\psi : NQ \to NQ/NJ$ *defined by* $\psi(x) = x\hat{+}NJ$, *for all* $x \in NQ$ *is not a neutrosophic quadruple strong homomorphism.*

Remark 3.20. The classical isomorphism theorems of hyperrings do not hold in neutrosophic quadruple hyperrings.

4. Conclusion

We have developed neutrosophic quadruple algebraic hyperstrutures in this paper. In particular, we have developed new neutrosophic algebraic hyperstructures namely neutrosophic quadruple semihypergroups, neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings. We have presented elementary properties which characterize the new neutrosophic algebraic hyperstructures.

Acknowledgements. The authors thank all the anonymous reviewers for useful observations and critical comments which have improved the quality of the paper.

References

[1] S. A. Akinleye, F. Smarandache and A. A. A. Agboola, On Neutrosophic Quadruple Algebraic Structures, Neutrosophic Sets and Systems 12 (2016) 122–126.
[2] S. A. Akinleye, E. O. Adeleke and A. A. A. Agboola, Introduction to Neutrosophic Nearrings, Ann. Fuzzy Math. Inform. 12 (3) (2016) 7–19.

[3] A. A. A. Agboola, On Refined Neutrosophic Algebraic Structures I, Neutrosophic Sets and Systems 10 (2015) 99–101.
[4] A. A. A. Agboola and B. Davvaz, On Neutrosophic Canonical Hypergroups and Neutrosophic Hyperrings, Neutrosophic Sets and Systems 2 (2014) 34–41.
[5] A. Asokkumar and M. Velrajan, Characterization of regular hyperrings , Italian Journal of Pure and Applied Mathematics 22 (2007) 115–124.
[6] A. R. Bargi, A class of hyperrings, J. Disc. Math. Sc. and Cryp. (6) (2003) 227–233.
[7] P. Corsini, Prolegomena of Hypergroup Theory, Second edition, Aviain editore 1993.
[8] P. Corsini and V. Leoreanu, Applications of Hyperstructure Theory, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht 2003.
[9] B. Davvaz, Polygroup Theory and Related Systems, World Sci. Publ. 2013.
[10] B. Davvaz, Isomorphism theorems of hyperrings, Indian J. Pure Appl. Math. 35 (3) (2004) 321–333.
[11] B. Davvaz, Approximations in hyperrings, J. Mult.-Valued Logic Soft Comput. 15 (5-6) (2009) 471–488.
[12] B. Davvaz and A. Salasi, A realization of hyperrings, Comm. Algebra 34 (2006) 4389–4400.
[13] B. Davvaz and T. Vougiouklis, Commutative rings obtained from hyperrings (H_v-rings) with α^*-relations, Comm. Algebra 35 (2007) 3307–3320.
[14] B. Davvaz and V. Leoreanu-Fotea, Hyperring Theory and Applications, International Academic Press, USA 2007.
[15] B. Davvaz , Isomorphism theorems of hyperrings, Indian Journal of Pure and Applied Mathematics 23 (3) (2004) 321–331.
[16] M. De Salvo, Hyperrings and hyperfields, Annales Scientifiques de l'Universite de Clermont-Ferrand II, 22 (1984) 89–107.
[17] F. Smarandache, Neutrosophy/Neutrosophic Probability, Set, and Logic, American Research Press, Rehoboth, USA 1998.
http://fs.gallup.unm.edu/eBook-otherformats.htm
[18] F. Smarandache, Neutrosophic Quadruple Numbers, Refined Neutrosophic Quadruple Numbers, Absorbance Law, and the Multiplication of Neutrosophic Quadruple Numbers, Neutrosophic Sets and Systems 10 (2015) 96–98.
[19] F. Smarandache, (t,i,f) - Neutrosophic Structures and I-Neutrosophic Structures, Neutrosophic Sets and Systems 8 (2015) 3–10.
[20] F. Smarandache, n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Progress in Physics 4 (2013) 143–146.
[21] M. Velrajan and A. Asokkumar, Note on Isomorphism Theorems of Hyperrings, Int. J. Math. & Math. Sc. (2010) ID 376985 1–12.

A. A. A. Agboola (agboolaaaa@funaab.edu.ng)

Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria

B. Davvaz (davvaz@yazd.ac.ir)

Department of Mathematics, Yazd University, Yazd, Iran

F. Smarandache (smarand@unm.edu)

Department of Mathematics and Science, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 1, (July 2017), pp. 43–54
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

http://www.kyungmoon.com

The category of neutrosophic crisp sets

K. Hur, P. K. Lim, J. G. Lee, J. Kim

Received 18 Februry 2017; Revised 1 March 2017; Accepted 19 March 2017

Abstract. We introduce the category **NCSet** consisting of neutrosophic crisp sets and morphisms between them. And we study **NCSet** in the sense of a topological universe and prove that it is Cartesian closed over **Set**, where **Set** denotes the category consisting of ordinary sets and ordinary mappings between them.

2010 AMS Classification: 03E72, 18B05

Keywords: Neutrosophic crisp set, Cartesian closed category, Topological universe, Neutrosophic crisp set.

Corresponding Author: J. G. Lee (jukolee@wku.ac.kr)

1. Introduction

In 1965, Zadeh [20] had introduced a concept of a fuzzy set as the generalization of a crisp set. In 1986, Atanassove [1] proposed the notion of intuitionistic fuzzy set as the generalization of fuzzy sets considering the degree of membership and non-membership. In 1998 Smarandache [19] introduced the concept of a neutrosophic set considering the degree of membership, the degree of indeterminacy and the degree of non-membership. Moreover, Salama et al. [15, 16, 18] applied the concept of neutrosophic crisp sets to topology and relation.

After that time, many researchers [2, 3, 4, 5, 7, 8, 10, 12, 13, 14] have investigated fuzzy sets in the sense of category theory, for instance, **Set**(**H**), $\mathbf{Set_f}$(**H**), $\mathbf{Set_g}$(**H**), **Fuz**(**H**). Among them, the category **Set**(**H**) is the most useful one as the "standard" category, because **Set**(**H**) is very suitable for describing fuzzy sets and mappings between them. In particular, Carrega [2], Dubuc [3], Eytan [4], Goguen [5], Pittes [12], Ponasse [13, 14] had studied **Set**(**H**) in topos view-point. However Hur et al. investigated **Set**(**H**) in topological view-point. Moreover, Hur et al. [8] introduced the category **ISet**(**H**) consisting of intuitionistic H-fuzzy sets and morphisms between them, and studied **ISet**(**H**) in the sense of topological universe. Recently, Lim et al [10] introduced the new category **VSet**(**H**) and investigated it in the sense of topological universe.

The concept of a topological universe was introduced by Nel [11], which implies a Cartesian closed category and a concrete quasitopos. Furthermore the concept has already been up to effective use for several areas of mathematics.

In this paper, first, we obtain some properties of neutrosophic crisp sets proposed by Salama and Smarandache [17] in 2015. Second, we introduce the category **NCSet** consisting of neutrosophic crisp sets and morphisms between them. And we prove that the category **NCSet** is topological and cotopological over **Set** (See Theorem 4.6 and Corollary 4.8), where **Set** denotes the category consisting of ordinary sets and ordinary mappings between them. Furthermore, we prove that final episinks in **NCSet** are preserved by pullbacks(See Theorem 4.10) and **NCSet** is Cartesian closed over **Set** (See Theorem 4.15).

2. Preliminaries

In this section, we list some basic definitions and well-known results from [6, 9, 11] which are needed in the next sections.

Definition 2.1 ([9])**.** Let **A** be a concrete category and $((Y_j, \xi_j))_J$ a family of objects in **A** indexed by a class J. For any set X, let $(f_j : X \to Y_j)_J$ be a source of mappings indexed by J. Then an **A**-structure ξ on X is said to be initial with respect to (in short, w.r.t.) $(X, (f_j), (Y_j, \xi_j))_J$, if it satisfies the following conditions:

(i) for each $j \in J$, $f_j : (X, \xi) \to (Y_j, \xi_j)$ is an **A**-morphism,

(ii) if (Z, ρ) is an **A**-object and $g : Z \to X$ is a mapping such that for each $j \in J$, the mapping $f_j \circ g : (Z, \rho) \to (Y_j, \xi_j)$ is an **A**-morphism, then $g : (Z, \rho) \to (X, \xi)$ is an **A**-morphism.

In this case, $(f_j : (X, \xi) \to (Y_j, \xi_j))_J$ is called an initial source in **A**.

Dual notion: cotopological category.

Result 2.2 ([9], Theorem 1.5)**.** *A concrete category* **A** *is topological if and only if it is cotopological.*

Result 2.3 ([9], Theorem 1.6)**.** *Let* **A** *be a topological category over Set, then it is complete and cocomplete.*

Definition 2.4 ([9])**.** Let **A** be a concrete category.

(i) The **A**-fibre of a set X is the class of all **A**-structures on X.

(ii) **A** is said to be properly fibred over **Set**, it satisfies the followings:

(a) (Fibre-smallness) for each set X, the **A**-fibre of X is a set,

(b) (Terminal separator property) for each singleton set X, the **A**-fibre of X has precisely one element,

(c) if ξ and η are **A**-structures on a set X such that $id : (X, \xi) \to (X, \eta)$ and $id : (X, \eta) \to (X, \xi)$ are **A**-morphisms, then $\xi = \eta$.

Definition 2.5 ([6])**.** A category **A** is said to be Cartesian closed, if it satisfies the following conditions:

(i) for each **A**-object A and B, there exists a product $A \times B$ in **A**,

(ii) exponential objects exist in A, i.e., for each **A**-object A, the functor $A \times - : A \to A$ has a right adjoint, i.e., for any **A**-object B, there exist an **A**-object B^A and a **A**-morphism $e_{A,B} : A \times B^A \to B$ (called the evaluation) such that for any

A-object C and any **A**-morphism $f : A \times C \to B$, there exists a unique **A**-morphism $\bar{f} : C \to B^A$ such that the diagram commutes:

Definition 2.6 ([6])**.** A category **A** is called a topological universe over **Set**, if it satisfies the following conditions:

(i) **A** is well-structured, i.e. (a) **A** is concrete category; (b) **A** satisfies the fibre-smallness condition; (c) **A** has the terminal separator property,

(ii) **A** is cotopological over **Set**,

(iii) final episinks in **A** are preserved by pullbacks, i.e., for any episink $(g_j : X_j \to Y)_J$ and any **A**-morphism $f : W \to Y$, the family $(e_j : U_j \to W)_J$, obtained by taking the pullback f and g_j, for each $j \in J$, is again a final episink.

3. Neutrosophic crisp sets

In [17], Salama and Smarandache introduced the concept of a neutrosophic crisp set in a set X and defined the inclusion between two neutrosophic crisp sets, the intersection [union] of two neutrosophic crisp sets, the complement of a neutrosophic crisp set, neutrosophic crisp empty [resp., whole] set as more than two types. And they studied some properties related to neutrosophic crisp set operations. However, by selecting only one type, we define the inclusion, the intersection [union], and neutrosophic crisp empty [resp., whole] set again and find some properties.

Definition 3.1. Let X be a non-empty set. Then A is called a neutrosophic crisp set (in short, NCS) in X if A has the form $A = (A_1, A_2, A_3)$,
where A_1, A_2, and A_3 are subsets of X,

The neutrosophic crisp empty [resp., whole] set, denoted by ϕ_N [resp., X_N] is an NCS in X defined by $\phi_N = (\phi, \phi, X)$ [resp., $X_N = (X, X, \phi)$]. We will denote the set of all NCSs in X as $NCS(X)$.

In particular, Salama and Smarandache [17] classified a neutrosophic crisp set as the followings.

A neutrosophic crisp set $A = (A_1, A_2, A_3)$ in X is called a:

(i) neutrosophic crisp set of Type 1 (in short, NCS-Type 1), if it satisfies

$$A_1 \cap A_2 = A_2 \cap A_3 = A_3 \cap A_1 = \phi,$$

(ii) neutrosophic crisp set of Type 2 (in short, NCS-Type 2), if it satisfies
$$A_1 \cap A_2 = A_2 \cap A_3 = A_3 \cap A_1 = \phi \text{ and } A_1 \cup A_2 \cup A_3 = X,$$

(iii) neutrosophic crisp set of Type 3 (in short, NCS-Type 3), if it satisfies
$$A_1 \cap A_2 \cap A_3 = \phi \text{ and } A_1 \cup A_2 \cup A_3 = X.$$

We will denote the set of all NCSs-Type 1 [resp., Type 2 and Type 3] as $NCS_1(X)$ [resp., $NCS_2(X)$ and $NCS_3(X)$].

Definition 3.2. Let $A = (A_1, A_2, A_3), B = (B_1, B_2, B_3) \in NCS(X)$. Then

(i) A is said to be contained in B, denoted by $A \subset B$, if
$$A_1 \subset B_1,\ A_2 \subset B_2 \text{ and } A_3 \supset B_3,$$

(ii) A is said to equal to B, denoted by $A = B$, if
$$A \subset B \text{ and } B \subset A,$$

(iii) the complement of A, denoted by A^c, is an NCS in X defined as:

$$A^c = (A_3, A_2^c, A_1),$$

(iv) the intersection of A and B, denoted by $A \cap B$, is an NCS in X defined as:

$$A \cap B = (A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3),$$

(v) the union of A and B, denoted by $A \cup B$, is an NCS in X defined as:

$$A \cup B = (A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3).$$

Let $(A_j)_{j \in J} \subset NCS(X)$, where $A_j = (A_{j,1}, A_{j,2}, A_{j,3})$. Then

(vi) the intersection of $(A_j)_{j \in J}$, denoted by $\bigcap_{j \in J} A_j$ (simply, $\bigcap A_j$), is an NCS in X defined as:

$$\bigcap A_j = (\bigcap A_{j,1}, \bigcap A_{j,2}, \bigcup A_{j,3}),$$

(vii) the the union of $(A_j)_{j \in J}$, denoted by $\bigcup_{j \in J} A_j$ (simply, $\bigcup A_j$), is an NCS in X defined as:

$$\bigcup A_j = (\bigcup A_{j,1}, \bigcup A_{j,2}, \bigcap A_{j,3}).$$

The followings are the immediate results of Definition 3.2.

Proposition 3.3. *Let $A, B, C \in NCS(X)$. Then*
(1) $\phi_N \subset A \subset X_N$,
(2) *if* $A \subset B$ *and* $B \subset C$, *then* $A \subset C$,
(3) $A \cap B \subset A$ *and* $A \cap B \subset B$,
(4) $A \subset A \cup B$ *and* $B \subset A \cup B$,
(5) $A \subset B$ *if and only if* $A \cap B = A$,
(6) $A \subset B$ *if and only if* $A \cup B = B$.

Also the followings are the immediate results of Definition 3.2.

Proposition 3.4. *Let $A, B, C \in NCS(X)$. Then*
(1) (Idempotent laws): $A \cup A = A$, $A \cap A = A$,
(2) (Commutative laws): $A \cup B = B \cup A$, $A \cap B = B \cap A$,
(3) (Associative laws): $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$,
(4) (Distributive laws): $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws): $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$,
(6) (DeMorgan's laws): $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$,
(7) $(A^c)^c = A$,
(8) (8a) $A \cup \phi_N = A$, $A \cap \phi_N = \phi_N$,
(8b) $A \cup X_N = X_N$, $A \cap X_N = A$,
(8c) $X_N^c = \phi_N$, $\phi_N^c = X_N$,
(8d) *in general*, $A \cup A^c \neq X_N$, $A \cap A^c \neq \phi_N$.

Proposition 3.5. *Let $A \in NCS(X)$ and let $(A_j)_{j \in J} \subset NCS(X)$. Then*
(1) $(\bigcap A_j)^c = \bigcup A_j^c$, $(\bigcup A_j)^c = \bigcap A_j^c$,
(2) $A \cap (\bigcup A_j) = \bigcup (A \cap A_j)$, $A \cup (\bigcap A_j) = \bigcap (A \cup A_j)$.

Proof. (1) $A_j = (A_{j,1}, A_{j,2}, A_{j,3})$. Then $\bigcap A_j = (\bigcap A_{j,1}, \bigcap A_{j,2}, \bigcup A_{j,3})$. Thus

$$(\bigcap A_j)^c = (\bigcup A_{j,3}, (\bigcap A_{j,2})^c, \bigcap A_{j,1}) = (\bigcup A_{j,3}, \bigcup A_{j,2}^c, \bigcap A_{j,1}) = \bigcup A_j^c.$$

Similarly, the second part is proved.

(2) Let $A = (A_1, A_2, A_3)$. Then

$$A \cup (\bigcap A_j) = (A_1 \cup (\bigcap A_{j,1}), A_2 \cup (\bigcap A_{j,2}), A_3 \cap (\bigcup A_{j,3}))$$

$= (\bigcap(A_1 \cup A_{j,1}), \bigcap(A_2 \cup A_{j,2}), \bigcup(A_3 \cap A_{j,3})$
$= \bigcap(A \cup A_j).$

Similarly, the first part is proved. □

Definition 3.6. Let $f : X \to Y$ be a mapping, and let $A = (A_1, A_2, A_3) \in NCS(X)$ and $B = (B_1, B_2, B_3) \in NCS(Y)$. Then

(i) the image of A under f, denoted by $f(A)$, is an NCS in Y defined as:

$$f(A) = (f(A_1), f(A_2), f(A_3)),$$

(ii) the preimage of B, denoted by $f^{-1}(B)$, is an NCS in X defined as:

$$f^{-1}(B) = (f^{-1}(B_1), f^{-1}(B_2), f^{-1}(B_3)).$$

Proposition 3.7. *Let $f : X \to Y$ be a mapping and let $A, B, C \in NCS(X)$, $(A_j)_{j\in J} \subset NCS(X)$ and $D, E, F \in NCS(Y)$, $(D_k)_{k\in K} \subset NCS(Y)$. Then the followings hold:*

(1) *if $B \subset C$, then $f(B) \subset f(C)$ and if $E \subset F$, then $f^{-1}(E) \subset f^{-1}(F)$.*
(2) *$A \subset f^{-1}f(A))$ and if f is injective, then $A = f^{-1}f(A))$,*
(3) *$f(f^{-1}(D)) \subset D$ and if f is surjective, then $f(f^{-1}(D)) = D$,*
(4) *$f^{-1}(\bigcup D_k) = \bigcup f^{-1}(D_k)$, $f^{-1}(\bigcap D_k) = \bigcap f^{-1}(D_k)$,*
(5) *$f(\bigcup A_j) = \bigcup f(A_j)$, $f(\bigcap A_j) \subset \bigcap f(A_j)$,*
(6) *$f(A) = \phi_N$ if and only if $A = \phi_N$ and hence $f(\phi_N) = \phi_N$, in particular if f is surjective, then $f(X_N) = Y_N$,*
(7) *$f^{-1}(Y_N) = Y_N$, $f^{-1}(\phi_N) = \phi$.*

Definition 3.8 ([17])**.** Let $A = (A_1, A_2, A_3) \in NCS(X)$, where X is a set having at least distinct three points. Then A is called a neutrosophic crisp point (in short, NCP) in X, if A_1, A_2 and A_3 are distinct singleton sets in X.

Let $A_1 = \{p_1\}$, $A_2 = \{p_2\}$ and $A_3 = \{p_3\}$, where $p_1 \neq p_2 \neq p_3 \in X$. Then $A = (A_1, A_2, A_3)$ is an NCP in X. In this case, we will denote A as $p = (p_1, p_2, p_3)$. Furthermore, we will denote the set of all NCPs in X as $NCP(X)$.

Definition 3.9. Let $A = (A_1, A_2, A_3) \in NCS(X)$ and let $p = (p_1, p_2, p_3) \in NCP(X)$. Then p is said to belong to A, denoted by $p \in A$, if $\{p_1\} \subset A_1$, $\{p_2\} \subset A_2$ and $\{p_3\}^c \supset A_3$, i.e., $p_1 \in A_1$, $p_2 \in A_2$ and $p_3 \in A_3^c$.

Proposition 3.10. *Let $A = (A_1, A_2, A_3) \in NCS(X)$. Then*

$$A = \bigcup\{p \in NCP(X) : p \in A\}.$$

Proof. Let $p = (p_1, p_2, p_3) \in NCP(X)$. Then

$\bigcup\{p \in NCP(X) : p \in A\}$
$= (\bigcup\{p_1 \in X : p_1 \in A_1\}, \bigcup\{p_2 \in X : p_2 \in A_2\}, \bigcap\{p_3 \in X : p_3 \in A_3^c\}$
$= A.$ □

Proposition 3.11. *Let $A = (A_1, A_2, A_3), B = (B_1, B_2, B_3) \in NCS(X)$. Then $A \subset B$ if and only if $p \in B$, for each $p \in A$.*

Proof. Suppose $A \subset B$ and let $p = (p_1, p_2, p_3) \in A$. Then

$$A_1 \subset B_1, A_2 \subset B_2, A_3 \supset B_3$$

and

$$p_1 \in A_1, p_2 \in A_2, p_3 \in A_3^c.$$

Thus $p_1 \in B_1, p_2 \in B_2, p_3 \in B_3^c$. So $p \in B$. □

Proposition 3.12. *Let $(A_j)_{j \in J} \subset NCS(X)$ and let $p \in NCP(X)$.*
(1) *$p \in \bigcap A_j$ if and only if $p \in A_j$ for each $j \in J$.*
(2) *$p \in \bigcup A_j$ if and only if there exists $j \in J$ such that $p \in A_j$.*

Proof. Let $A_j = (A_{j,1}, A_{j,2}, A_{j,3})$ for each $j \in J$ and let $p = (p_1, p_2, p_3)$.
(1) Suppose $p \in \bigcap A_j$. Then $p_1 \in \bigcap A_{j,1}, p_2 \in \bigcap A_{j,2}, p_3 \in \bigcup A_{j,3}^c$. Thus
$p_1 \in A_{j,1}, p_2 \in A_{j,2}, p_3 \in A_{j,3}^c$, for each $j \in J$. So $p \in A_j$ for each $j \in J$.
We can easily see that the sufficient condition holds.
(2) suppose the necessary condition holds. Then there exists $j \in J$ such that

$$p_1 \in A_{j,1}, p_2 \in A_{j,2}, p_3 \in A_{j,3}^c.$$

Thus $p_1 \in \bigcup A_{j,1}, p_2 \in \bigcup A_{j,2}, p_3 \in (\bigcap A_{j,3})^c$. So $p \in \bigcup A_j$.
We can easily prove that the necessary condition holds. □

Definition 3.13. Let $f : X \to Y$ be an injective mapping, where X, Y are sets having at least distinct three points. Let $p = (p_1, p_2, p_3) \in NCP(X)$. Then the image of p under f, denoted by $f(p)$, is an NCP in Y defined as:

$$f(p) = (f(p_1), f(p_2), f(p_3)).$$

Remark 3.14. In Definition 3.13, if either X or Y has two points, or f is not injective, then $f(p)$ is not an NCP in Y.

Definition 3.15 ([17]). Let $A = (A_1, A_2, A_3) \in NCS(X)$ and $B = (B_1, B_2, B_3) \in NCS(Y)$. Then the Cartesian product of A and B, denoted by $A \times B$, is an NCS in $X \times Y$ defined as: $A \times B = (A_1 \times B_1, A_2 \times B_2, A_3 \times B_3)$.

4. Properties of **NCSet**

Definition 4.1. A pair (X, A) is called a neutrosophic crisp space (in short, NCSp), if $A \in NCS(X)$.

Definition 4.2. A pair (X, A) is called a neutrosophic crisp space-Type j (in short, NCSp-Type j), if $A \in NCS_j(X)$, $j = 1, 2, 3$.

Definition 4.3. Let (X, A_X), (Y, A_Y) be two NCSps or NCSps-Type j, $j = 1, 2, 3$ and let $f : X \to Y$ be a mapping. Then $f : (X, A_X) \to (Y, A_Y)$ is called a morphism, if $A_X \subset f^{-1}(A_Y)$, equivalently,
$A_{X,1} \subset f^{-1}(A_{Y,1})$, $A_{X,2} \subset f^{-1}(A_{Y,2})$ and $A_{X,3} \supset f^{-1}(A_{Y,3})$,
where $A_X = (A_{X,1}, A_{X,2}, A_{X,3})$ and $A_Y = (A_{Y,1}, A_{Y,2}, A_{Y,3})$.
In particular, $f : (X, A_X) \to (Y, A_Y)$ is called an epimorphism [resp., a monomorphism and an isomorphism], if it is surjective [resp., injective and bijective].

From Definitions 3.9, 4.3 and Proposition 3.11, it is obvious that
$f : (X, A_X) \to (Y, A_Y)$ is a morphism
if and only if
$p = (p_1, p_2, p_3) \in f^{-1}(A_Y)$, for each $p = (p_1, p_2, p_3) \in A_X$, i.e.,
$f(p_1) \in A_{Y,1}$, $f(p_2) \in A_{Y,2}$, $f(p_3) \notin A_{Y,3}$, i.e.,

$$f(p) = (f(p_1), f(p_2), f(p_3)) \in A_Y.$$

The following is an immediate result of Definitions 4.3.

Proposition 4.4. *For each NCSp or each NCSps-Type j* (X, A_X), $j = 1, 2, 3$, *the identity mapping* $id : (X, A_X) \to (X, A_X)$ *is a morphism.*

Proposition 4.5. *Let* (X, A_X), (Y, A_Y), (Z, A_Z) *be NCSps or NCSps-Type j*, $j = 1, 2, 3$ *and let* $f : X \to Y$, $g : Y \to Z$ *be mappings. If* $f : (X, A_X) \to (Y, A_Y)$ *and* $f : (Y, A_Y) \to (Z, A_Z)$ *are morphisms, then* $g \circ f : (X, A_X) \to (Z, A_Z)$ *is a morphism.*

Proof. Let $A_X = (A_{X,1}, A_{X,2}, A_{X,3})$, $A_Y = (A_{Y,1}, A_{Y,2}, A_{Y,3})$ and $A_Z = (A_{Z,1}, A_{Z,2}, A_{Z,3})$. Then by the hypotheses, $A_X \subset f^{-1}(A_Y)$ and $A_Y \subset g^{-1}(A_Z)$. Thus by Definition 4.3,

$$A_{X,1} \subset f^{-1}(A_{Y,1}),\ A_{X,2} \subset f^{-1}(A_{Y,2}),\ A_{X,3} \supset f^{-1}(A_{Y,3})$$

and

$$A_{Y,1} \subset g^{-1}(A_{Z,1}),\ A_{Y,2} \subset g^{-1}(A_{Z,2}),\ A_{Y,3} \supset g^{-1}(A_{Z,3}).$$

So $A_{X,1} \subset f^{-1}(g^{-1}(A_{Z,1}))$, $A_{X,2} \subset f^{-1}(g^{-1}(A_{Z,2}))$, $A_{X,3} \supset f^{-1}(g^{-1}(A_{Z,3}))$.
Hence $A_{X,1} \subset (g \circ f)^{-1}(A_{Z,1})$, $A_{X,2} \subset (g \circ f)^{-1}(A_{Z,2})$, $A_{X,3} \supset (g \circ f)^{-1}(A_{Z,2})$.
Therefore $g \circ f$ is a morphism. □

From Propositions 4.4 and 4.5, we can form the concrete category **NCSet** [resp., $\mathbf{NCSet_j}$] consisting of NCSs [resp., -Type j, $j = 1, 2, 3$] and morphisms between them. Every **NCSet** [resp., $\mathbf{NCSet_j}$, $j = 1, 2, 3$]-morphism will be called a **NCSet** [resp., $\mathbf{NCSet_j}$, $j = 1, 2, 3$]-mapping.

Theorem 4.6. *The category* **NCSet** *is topological over* **Set**.

Proof. Let X be any set and let $((X_j, A_j))_{j \in J}$ be any families of NCSps indexed by a class J. Suppose $(f_j : X \to (X_j, A_j))_J$ is a source of ordinary mappings. We define the NCS A_X in X by $A_X = \bigcap f_j^{-1}(A_j)$ and $A_X = (A_{X,1}, A_{X,2}, A_{X,3})$.
Then clearly, $A_{X,1} = \bigcap f_j^{-1}(A_{j,1})$, $A_{X,2} = \bigcap f_j^{-1}(A_{j,2})$, $A_{X,3} = \bigcup f_j^{-1}(A_{j,3})$.
Thus (X, A_X) is an NCSp and $A_{X,1} \subset f_j^{-1}(A_{j,1})$, $A_{X,2} \subset f_j^{-1}(A_{j,2})$ and $A_{X,3} \supset f_j^{-1}(A_{j,3})$. So each $f_j : (X, A_X) \to (X_j, A_j)$ is an **NCSet**-mapping.

Now let (Y, A_Y) be any NCSp and suppose $g : Y \to X$ is an ordinary mapping for which $f_j \circ g : (Y, A_Y) \to (X_j, A_j)$ is a **NCSet**-mapping for each $j \in J$. Then for each $j \in J$, $A_Y \subset (f_j \circ g)^{-1}(A_j) = g^{-1}(f_j^{-1}(A_j))$. Thus

$$A_Y \subset (f_j \circ g)^{-1}(A_j) = g^{-1}(\bigcap f_j^{-1}(A_j)) = g^{-1}(A_X).$$

So $g : (Y, A_Y) \to (X, A_X)$ is an **NCSet**-mapping. Hence $(f_j : (X, A_X) \to (X_j, A_j)_J$ is an initial source in **NCSet**. This completes the proof. □

Example 4.7. (1) Let X be a set, let (Y, A_Y) be an NCSp and let $f : X \to Y$ be an ordinary mapping. Then clearly, there exists a unique NCS A_X in X for which $f : (X, A_X) \to (Y, A_Y)$ is an **NCSet**-mapping. In fact, $A_X = f^{-1}(A_Y)$.

In this case, A_X is called the inverse image under f of the NCS structure A_Y.

(2) Let $((X_j, A_j))_{j\in J}$ be any family of NCSps and let $X = \Pi_{j\in J}X_j$. For each $j \in J$, let $pr_j : X \to X_j$ be the ordinary projection. Then there exists a unique NCS A_X in X for which $pr_j : (X, A_X \to (X_j, A_j)$ is an **NCSet**-mapping for each $j \in J$.

In this case, A_X is called the product of $(A_j)_{j\in J}$, denoted by

$$A_X = \Pi A_j = (\Pi A_{j,1}, \Pi A_{j,2}, \Pi A_{j,3})$$

and $(\Pi X_j, \Pi A_j)$ is called the product NCSp of $((X_j, A_j))_{j\in J}$.

In fact, $A_X = \bigcap_{j\in J} pr_j^{-1}(A_j)$.

In particular, if $J = \{1, 2\}$, then $A_1 \times A_2 = (A_{1,1} \times A_{2,1}, A_{1,2} \times A_{2,2}, A_{1,3} \times A_{2,3})$, where $A_1 = (A_{1,1}, A_{1,2}, A_{1,3}) \in NCS(X_1)$ and $A_2 = (A_{2,1}, A_{2,2}, A_{2,3}) \in NCS(X_2)$.

The following is obvious from Result 2.2. But we show directly it.

Corollary 4.8. *The category* **NCSet** *is cotopological over* **Set***.*

Proof. Let X be any set and let $((X_j, A_j))_J$ be any family of NCSps indexed by a class J. Suppose $(f_j : X_j \to X)_J$ is a sink of ordinary mappings. We define A_X as $A_X = \bigcup f_j(A_j)$, where $A_X = (A_{X,1}, A_{X,2}, A_{X,3})$ and $A_j = (A_{j,1}, A_{j,2}, A_{j,3})$. Then clearly, $A_X \in NCS(X)$ and each $f_j : (X_j, A_j) \to (X, A_X)$ is an **NCSet**-mapping.

Now for each NCSp (Y, A_Y), let $g : X \to Y$ be an ordinary mapping for which each $g \circ f_j : (X_j, A_j) \to (Y, A_Y)$ is an **NCSet**-mapping. Then clearly for each $j \in J$,

$$A_j \subset (g \circ f_j)^{-1}(A_Y), \text{ i.e., } A_j \subset f_j^{-1}(g^{-1}(A_Y)).$$

Thus $\bigcup A_j \subset \bigcup f_j^{-1}(g^{-1}(A_Y))$. So $f_j(\bigcup A_j) \subset f_j(\bigcup f_j^{-1}(g^{-1}(A_Y)))$. By Proposition 3.7 and the definition of A_X,

$$f_j(\bigcup A_j) = \bigcup f_j(A_j) = A_X$$

and

$$f_j(\bigcup f_j^{-1}(g^{-1}(A_Y))) = \bigcup (f_j \circ f_j^{-1})(g^{-1}(A_Y)) = g^{-1}(A_Y).$$

Hence $A_X \subset g^{-1}(A_Y)$. Therefore $g : (X, A_X) \to (Y, A_Y)$ is an **NCSet**-mapping. This completes the proof. □

The following is proved similarly as the proof of Theorem 4.6.

Corollary 4.9. *The category* $\mathbf{NCSet_j}$ *is topological over* **Set** *for* $j = 1, 2, 3$.

The following is proved similarly as the proof of Corollary 4.8.

Corollary 4.10. *The category* $\mathbf{NCSet_j}$ *is cotopological over* **Set** *for* $j = 1, 2, 3$.

Theorem 4.11. *Final episinks in* **NCSet** *are prserved by pullbacks.*

Proof. Let $(g_j : (X_j, A_j) \to (Y, A_Y))_J$ be any final episink in **NCSet** and let $f : (W, A_W) \to (Y, A_Y)$ be any **NCSet**-mapping. For each $j \in J$, let

$$U_j = \{(w, x_j) \in W \times X_j : f(w) = g_j(x_j)\}.$$

For each $j \in J$, we define the NCS $A_{U_j} = (A_{U_{j,1}}, A_{U_{j,2}}, A_{U_{j,3}})$ in U_j by:

$$A_{U_{j,1}} = A_{W,1} \times A_{j,1}, A_{U_{j,2}} = A_{W,2} \times A_{j,2}, A_{U_{j,3}} = A_{W,3} \times A_{j,3}.$$

For each $j \in J$, let $e_j : U_j \to W$ and $p_j : U_j \to X_j$ be ordinary projections of U_j. Then clearly,

$$A_{U_{j,1}} \subset e_j^{-1}(A_{W,1}), A_{U_{j,2}} \subset e_j^{-1}(A_{W,2}), A_{U_{j,3}} \supset e_j^{-1}(A_{W,3})$$

and

$$A_{U_{j,1}} \subset p_j^{-1}(A_{j,1}), A_{U_{j,2}} \subset p_j^{-1}(A_{j,2}), A_{U_{j,3}} \supset p_j^{-1}(A_{j,3}).$$

Thus $A_{U_j} \subset e_j^{-1}(A_W)$ and $A_{U_j} \subset p_j^{-1}(A_j)$. So $e_j : (U_j, A_{U_j}) \to (W, A_W)$ and $p_j : (U_j, A_{U_j}) \to (X_j, A_j)$ are **NCSet**-mappings. Moreover, $g_h \circ p_h = f \circ e_j$ for each $j \in J$, i.e., the diagram is a pullback square in **NCSet**:

$$\begin{array}{ccc} (U_j, A_{U_j}) & \xrightarrow{\quad p_j \quad} & (X_j, A_j) \\ e_j \downarrow & & \downarrow g_j \\ (W, A_W) & \xrightarrow[\quad f \quad]{} & (Y, A_Y). \end{array}$$

Now in order to prove that $(e_j)_J$ is an episink in **NCSet**, i.e., each e_j is surjective, let $w \in W$. Since $(g_j)_J$ is an episink, there exists $j \in J$ such that $g_j(x_j) = f(w)$ for some $x_j \in X_j$. Thus $(w, x_j) \in U_j$ and $w = e_j(w, x_j)$. So $(e_j)_J$ is an episink in **NCSet**.

Finally, let us show that $(e_j)_J$ is final in **NCSet**. Let A_W^* be the final structure in W w.r.t. $(e_j)_J$ and let $w = (w_1, w_2, w_3) \in A_W$. Since $f : (W, A_W) \to (Y, A_Y)$ is an **NCSet**-mapping, by Definition 3.9,

$w_1 \in A_{W,1} \cap f^{-1}(A_{Y,1})$, $w_2 \in A_{W,2} \cap f^{-1}(A_{Y,2})$ and $w_3 \in A_{W,3}^c \cap (f^{-1}(A_{Y,3}))^c$.

Thus

$w_1 \in A_{W,1}, f(w_1) \in A_{Y,1}$, $w_2 \in A_{W,2}, f(w_2) \in A_{Y,2}$ and $w_3 \in A_{W,3}^c, f(w_3) \in A_{Y,3}^c$.

Since $(g_j)_J$ is final,

$$w_1 \in A_{W,1}, x_{j,1} \in \bigcup_J \bigcup_{x_{j,1} \in g_j^{-1}(f(w))} A_{j,1},$$

$$w_2 \in A_{W,2}, x_{j,2} \in \bigcup_J \bigcup_{x_{j,2} \in g_j^{-1}(f(w))} A_{j,2}$$

and

$$w_3 \in A_{W,3}^c, x_{j,3} \in (\bigcap_J \bigcap_{x_{j,3} \in g_j^{-1}(f(w))} A_{j,3})^c.$$

So $(w_1, x_{j,1}) \in A_{U_{j,1}}$, $(w_2, x_{j,2}) \in A_{U_{j,2}}$ and $(w_3, x_{j,3}) \in A_{U_{j,1}}^c$. Since A_W^* is the final structure in W w.r.t. $(e_j)_J$, $w \in A_W^*$, i.e., $A_W \subset A_W^*$. On the other hand, since $(e_j : (U_j, A_{U_j}) \to (W, A_W))_J$ is final, $1_W : (W, A_W^*) \to (W, A_W)$ is an **NCSet**-mapping and thus $A_W^* \subset A_W$. Hence $A_W^* = A_W$. Therefore $(e_j)_J$ is final. This completes the proof. □

The following is proved similarly as the proof of Theorem 4.9.

Corollary 4.12. *Final episinks in* $\mathbf{NCSet_j}$ *are prserved by pullbacks, for* $J = 1, 2, 3$.

For any singleton set $\{a\}$, NCS $A_{\{a\}}$ [resp., NCS-Type j $A_{\{a\},j}$, for $j = 1, 2, 3$] on $\{a\}$ is not unique, the category **NCSet** [resp., $\mathbf{NCSet_j}$, for $j = 1, 2, 3$] is not properly fibred over **Set**. Then by Definition 2.6, Corollary 4.8 and Theorem 4.11 [resp., Corollaries 4.10 and 4.12], we have the following result.

Theorem 4.13. *The category* **NCSet** *[resp.,* $\mathbf{NCSet_j}$*, for* $j = 1, 2, 3$*] satisfies all the conditions of a topological universe over* **Set** *except the terminal separator property.*

The following is an immediate result of Definitions 3.9 and 3.15.

Proposition 4.14. *Let* $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in NCP(X)$ *and let* $A = (A_1, A_2, A_3), B = (B_1, B_2, B_3) \in NCS(X)$*. Then* $(p, q) \in A \times B$ *if and only if* $(p_1, q_1) \in A_1 \times B_1$*,* $(p_2, q_2) \in A_2 \times B_2$ *and* $(p_3, q_3) \in (A_2 \times B_2)^c$*, i.e.,* $p_3 \in A_3^c$ *or* $q_3 \in B_3^c$*.*

Theorem 4.15. *The category* **NCSet** *is Cartesian closed over* **Set***.*

Proof. It is clear that **NCSet** has products by Theorem 4.6. Then it is sufficient to see that **NCSet** has exponential objects.

For any NCSps $\mathbf{X} = (X, A_X)$ and $\mathbf{Y} = (Y, A_Y)$, let Y^X be the set of all ordinary mappings from X to Y. We define the NCS $A_{Y^X} = (A_{Y^X,1}, A_{Y^X,2}, A_{Y^X,3})$ in Y^X by: for each $f = (f_1, f_2, f_3) \in Y^X$, $f \in A_{Y^X}$ if and only if $f(x) \in A_Y$, for each $x = (x_1, x_2, x_3) \in NCP(X)$, i.e.,

$$f_1 \in A_{Y^X,1}, f_2 \in A_{Y^X,2}, f_3 \notin A_{Y^X,3}$$

if and only if

$$f_1(x_1) \in A_{Y,1}, f_2(x_2) \in A_{Y,2}, f_3(x_3) \notin A_{Y,3}.$$

In fact,

$$A_{Y^X,1} = \{f_1 \in Y^X : f_1(x_1) \in A_{Y,1} \text{ for each } x_1 \in X\},$$

$$A_{Y^X,2} = \{f_2 \in Y^X : f_2(x_2) \in A_{Y,2} \text{ for each } x_2 \in X\},$$

$$A_{Y^X,3} = \{f_3 \in Y^X : f_3(x_3) \notin A_{Y,3} \text{ for some } x_3 \in X\}.$$

Then clearly, (Y^X, A_{Y^X}) is an NCSp.

Let $\mathbf{Y^X} = (Y^X, A_{Y^X})$. Then by the definition of A_{Y^X},

$$A_{Y^X,1} \subset f^{-1}(A_{Y,1}),\ A_{Y^X,2} \subset f^{-1}(A_{Y,2}) \text{ and } A_{Y^X,3} \supset f^{-1}(A_{Y,3}).$$

We define $e_{X,Y} : X \times Y^X \to Y$ by $e_{X,Y}(x, f) = f(x)$, for each $(x, f) \in X \times Y^X$. Let $(x, f) \in A_X \times A_{Y^X}$, where $x = (x_1, x_2, x_3)$, $f = (f_1, f_2, f_3)$. Then by Proposition 4.14 and the definition of $e_{X,Y}$,

$$(x_1, f_1) \in A_{X,1} \times A_{Y^X,1},\ (x_2, f_2) \in A_{X,2} \times A_{Y^X,2},\ (x_3, f_3) \in (A_{X,3} \times A_{Y^X,3})^c$$

and

$$e_{X,Y}(x_1, f_1) = f_1(x_1),\ e_{X,Y}(x_2, f_2) = f_2(x_2),\ e_{X,Y}(x_3, f_3) = f_3(x_3).$$

Thus by the definition of A_{Y^X},

$$(x_1, f_1) \in f^{-1}(A_{Y,1}) \times f^{-1}(A_{Y,1}),$$

$$(x_2, f_2) \in f^{-1}(A_{X,2}) \times f^{-1}(A_{X,2}),$$

$$(x_3, f_3) \in (f^{-1}(A_{X,3}) \times (f^{-1}(A_{X,3}))^c.$$

So $(x_1, f_1) \in e^{-1}_{X,Y}(A_{Y,1})$, $(x_2, f_2) \in e^{-1}_{X,Y}(A_{Y,2})$ and $(x_3, f_3) \in (e^{-1}_{X,Y}(A_{Y,3}))^c$. Hence $A_X \times A_{Y^X} \subset e^{-1}_{X,Y}(A_Y)$. Therefore $e_{X,Y} : \mathbf{X} \times \mathbf{Y^X} \to \mathbf{Y}$ is an **NCSet**-mapping.

For any $\mathbf{Z} = (Z, A_Z) \in \mathbf{NCSet}$, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be an **NCSet**-mapping. We define $\bar{h} : Z \to Y^X$ by $[\bar{h}(z)](x) = h(x, z)$, for each $z \in Z$ and each $x \in X$. Let $(x, z) \in A_X \times A_Z$, where $x = (x_1, x_2, x_3)$ and $z = (z_1, z_2, z_3)$. Since $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ is an **NCSet**-mapping,

$A_{X,1} \times A_{Z,1} \subset h^{-1}(A_{Y,1})$, $A_{X,2} \times A_{Z,2} \subset h^{-1}(A_{Y,2})$, $A_{X,3} \times A_{Z,3} \supset h^{-1}(A_{Y,1})$.
Then by Proposition 4.14,
$(x_1, z_1) \in h^{-1}(A_{Y,1})$, $(x_2, z_2) \in h^{-1}(A_{Y,2})$, $(x_3, z_3) \in (h^{-1}(A_{Y,3}))^c$.
Thus $h((x_1, z_1)) \in A_{Y,1}$, $h((x_2, z_2)) \in A_{Y,2}$, $h((x_3, z_3)) \in (A_{Y,3})^c$.
By the definition of $\bar{h}$,
$[\bar{h}(z_1)](x_1) \in A_{Y,1}$, $[\bar{h}(z_2)](x_2) \in A_{Y,2}$, $[\bar{h}(z_3)](x_3) \in (A_{Y,3})^c$.
By the definition of A_{Y^X},
$[\bar{h}(z_1)](A_{Z,1}) \subset A_{Y^X,1}$, $[\bar{h}(z_2)](A_{Z,2}) \subset A_{Y^X,2}$, $[\bar{h}(z_3)](A_{Z,3}) \supset A_{Y^X,3}$.
So $A_Z \subset \bar{h}^{-1}(A_{Y^X})$. Hence $\bar{h} : \mathbf{Z} \to \mathbf{Y}^{\mathbf{X}}$ is an **NCSet**-mapping. Furthermore, $\bar{h}$ is the unique **NCSet**-mapping such that $e_{X,Y} \circ (1_X \times \bar{h}) = h$. This completes the proof. □

The following is proved similarly as the proof of Theorem 4.15.

Corollary 4.16. *The category* $\mathbf{NCSet_j}$ *is Cartesian closed over* **Set** *for* $j = 1, 2, 3$.

5. Conclusions

For a non-empty set X, by defining a neutrosophic crisp set $A = (A_1, A_2, A_3)$ and an intuitionistic crisp set $A = (A_1, A_2)$ in X, respectively as follows:
(i) $A_1 \subset X, A_2 \subset X, A_3 \subset X$,
(ii) $A_1 \subset A_3^c, A_3 \subset A_2^c$,
and
(i) $A_1 \subset X, A_2 \subset X$,
(ii) $A_1 \subset A_2^c$,
we can form another categories $\mathbf{NCSet}_*$ and **ICSet**. Furthermore, we will study them in view points of a topological universe and obtain some relationship between them.

Acknowledgements. We are very grateful for the positive advices Prof. Smarandache and Prof. Agboola have given us.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[2] J. C. Carrega, The category $Set(H)$ and $Fzz(H)$, Fuzzy sets and systems 9 (1983) 327–332.
[3] E. J. Dubuc, Concrete quasitopoi Applications of Sheaves, Proc. Dunham1977, Lect. Notes in Math. 753 (1979) 239–254.
[4] M. Eytan, Fuzzy sets:a topological point of view, Fuzzy sets and systems 5 (1981) 47–67.
[5] J. A. Goguen, Categories of V-sets, Bull. Amer. Math. Soc. 75 (1969) 622–624.
[6] H. Herrlich, Catesian closed topological categories, Math. Coll. Univ. Cape Town 9 (1974) 1–16.
[7] K. Hur, A Note on the category $Set(H)$, Honam Math. J. 10 (1988) 89–94.
[8] K. Hur, H. W. Kang and J. H. Ryou, Intutionistic H-fuzzy sets, J. Korea Soc. Math. Edu. Ser. B:Pure Appl. Math. 12 (1) (2005) 33–45.
[9] C. Y. Kim, S. S. Hong, Y. H. Hong and P. H. Park, Algebras in Cartesian closed topological categories, Lecture Note Series Vol. 1 1985.
[10] P. K. Lim, S. R. Kim and K. Hur, The category $VSet(H)$, International Journal of Fuzzy Logic and Intelligent Systems 10 (1) (2010) 73–81.
[11] L. D. Nel, Topological universes and smooth Gelfand Naimark duality, mathematical applications of category theory, Prc. A. M. S. Spec. Sessopn Denver,1983, Contemporary Mathematics 30 (1984) 224–276.

[12] A. M. Pittes, Fuzzy sets do not form a topos, Fuzzy sets and Systems 8 (1982) 338–358.
[13] D. Ponasse, Some remarks on the category $Fuz(H)$ of M. Eytan, Fuzzy sets and Systems 9 (1983) 199–204.
[14] D. Ponasse, Categorical studies of fuzzy sets, Fuzzy sets and Systems 28 (1988) 235–244.
[15] A. A. Salama, Said Broumi and Florentin Smarandache, Neutrosophic Crisp Open Set and Neutrosophic Crisp Continuity via Neutrosophic Crisp Ideals, in Neutrosophic Theory and Its Applications. Collected Papers, Vol. I, EuropaNova asbl, pp. 199-205, Brussels, EU 2014. See http://fs.gallup.unm.edu/NeutrosophicTheoryApplications.pdf
[16] A. A. Salama, Said Broumi and Florentin Smarandache, Some Types of Neutrosophic Crisp Sets and Neutrosophic Crisp Relations, in Neutrosophic Theory and Its Applications. Collected Papers, Vol. I, EuropaNova asbl, pp. 378-385, Brussels, EU 2014.
[17] A. A. Salama and F. Smarandache, Neutrosophic Crisp Set Theory, The Educational Publisher Columbus, Ohio 2015.
[18] A. A. Salama, Florentin Smarandache and Valeri Kroumov, Neutrosophic Crisp Sets and Neutrosophic Crisp Topological Spaces, in Neutrosophic Theory and Its Applications. Collected Papers, Vol. I, EuropaNova asbl, pp. 206-212, Brussels, EU 2014.
[19] F. Smarandache, Neutrosophy Neutrisophic Property, Sets, and Logic, Amer Res Press, Rehoboth, USA 1998.
[20] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

K. Hur (kulhur@wku.ac.kr)
Division of Mathematics and Informational Statistics, Institute of Basic Natural Science, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

P. K. Lim (pklim@wku.ac.kr)
Division of Mathematics and Informational Statistics, Institute of Basic Natural Science, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

J. G. Lee (jukolee@wku.ac.kr)
Division of Mathematics and Informational Statistics, Institute of Basic Natural Science, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

J. Kim (junhikim@wku.ac.kr)
Department of Mathematics Education, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 1, (July 2017), pp. 55–73
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Special types of bipolar single valued neutrosophic graphs

Ali Hassan, Muhammad Aslam Malik, Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache

Received 16 February 2017; Revised 28 February 2017; Accepted 20 March 2017

Abstract. Neutrosophic theory has many applications in graph theory, bipolar single valued neutrosophic graphs (BSVNGs) is the generalization of fuzzy graphs and intuitionistic fuzzy graphs, SVNGs. In this paper we introduce some types of BSVNGs, such as subdivision BSVNGs, middle BSVNGs, total BSVNGs and bipolar single valued neutrosophic line graphs (BSVNLGs), also investigate the isomorphism, co weak isomorphism and weak isomorphism properties of subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs.

2010 AMS Classification: 05C99

Keywords: Bipolar single valued neutrosophic line graph, Subdivision BSVNG, middle BSVNG, total BSVNG.

Corresponding Author: Broumi Said (broumisaid78@gmail.com)

1. Introduction

Neutrosophic set theory (NS) is a part of neutrosophy which was introduced by Smarandache [43] from philosophical point of view by incorporating the degree of indeterminacy or neutrality as independent component for dealing problems with indeterminate and inconsistent information. The concept of neutrosophic set theory is a generalization of the theory of fuzzy set [50], intuitionistic fuzzy sets [5], interval-valued fuzzy sets [47] interval-valued intuitionistic fuzzy sets [6]. The concept of neutrosophic set is characterized by a truth-membership degree (T), an indeterminacy-membership degree (I) and a falsity-membership degree (f) independently, which are within the real standard or nonstandard unit interval $]^-0, 1^+[$. Therefore, if their range is restrained within the real standard unit interval $[0, 1]$: Nevertheless, NSs are hard to be apply in practical problems since the values of the functions of truth, indeterminacy and falsity lie in $]^-0, 1^+[$. The single valued neutrosophic set was introduced for the first time by Smarandache [43]. The concept of single valued neutrosophic sets is a subclass of neutrosophic sets in which the value of truth-membership, indeterminacy membership and falsity-membership degrees are intervals of numbers instead of the real numbers. Later on, Wang et al. [49] studied some properties related to single valued neutrosophic sets. The concept of neutrosophic sets and its extensions such as single valued neutrosophic sets, interval neutrosophic sets, bipolar neutrosophic sets and so on have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine and economic and can be found in [9, 15, 16, 30, 31, 32, 33, 34, 35, 36, 37, 51]. Graphs are the most powerful tool used in representing information involving relationship between objects and concepts. In a crisp graphs two vertices are either related or not related to each other, mathematically, the degree of relationship is either 0 or 1. While in fuzzy graphs, the degree of relationship takes values from $[0, 1]$. Atanassov [42] defined the concept of intuitionistic fuzzy graphs (IFGs) using five types of Cartesian products. Theconcept fuzzy graphs, intuitionistic fuzzy graphs and their extensions such interval valued fuzzy graphs, bipolar fuzzy graph, bipolar intuitionitsic fuzzy graphs, interval valued intuitionitic fuzzy graphs, hesitancy fuzzy graphs, vague graphs and so on, have been studied deeply by several researchers in the literature. When description of the object or their relations or both is indeterminate and inconsistent, it cannot be handled by fuzzy intuitionistic fuzzy, bipolar fuzzy, vague and interval valued fuzzy graphs. So, for this purpose, Smaranadache [45] proposed the concept of neutrosophic graphs based on literal indeterminacy (I) to deal with such situations. Later on, Smarandache [44] gave another definition for neutrosphic graph theory using the neutrosophic truth-values (T, I, F) without and constructed three structures of neutrosophic graphs: neutrosophic edge graphs, neutrosophic vertex graphs and neutrosophic vertex-edge graphs. Recently, Smarandache [46] proposed new version of neutrosophic graphs such as neutrosophic offgraph, neutrosophic bipolar/tripola/multipolar graph. Recently several researchers have studied deeply the concept of neutrosophic vertex-edge graphs and presented several extensions neutrosophic graphs. In [1, 2, 3]. Akram et al. introduced the concept of single valued neutrosophic hypergraphs, single valued neutrosophic planar graphs, neutrosophic soft graphs and intuitionstic neutrosophic soft graphs. Then, followed the work of Broumi et al. [7, 8, 9, 10, 11, 12, 13, 14, 15], Malik and Hassan [38] defined the concept of single valued neutrosophic trees and studied some of their properties. Later on, Hassan et Malik [17] introduced some classes of bipolar single valued neutrosophic graphs and studied some of their properties, also the authors generalized the concept of single valued neutrosophic hypergraphs and bipolar single valued neutrosophic hypergraphs in [19, 20]. In [23, 24] Hassan et Malik gave the important types of single (interval) valued neutrosophic graphs, another important classes of single valued neutrosophic graphs have been presented in [22] and in [25] Hassan et Malik introduced the concept of m-Polar single valued neutrosophic graphs and its classes. Hassan et al. [18, 21] studied the concept on regularity and total regularity of single valued neutrosophic hypergraphs and bipolar single valued neutrosophic hypergraphs. Hassan et al. [26, 27, 28] discussed the isomorphism properties on SVNHGs, BSVNHGs and IVNHGs. Nasir et al. [40] introduced a new type of graph called neutrosophic soft graphs and established a link between graphs and neutrosophic soft sets. The authors also studeied some basic operations of neutrosophic soft graphs such as union, intersection and complement. Nasir and Broumi [41] studied the concept of irregular neutrosophic graphs and investigated some of their related properties. Ashraf et al. [4], proposed some novels concepts of edge regular, partially edge regular and full edge regular single valued neutrosophic graphs and investigated some of their properties. Also the authors, introduced the notion of single valued neutrosophic digraphs (SVNDGs) and presented an application of SVNDG in multi-attribute decision making. Mehra and Singh [39] introduced a new concept of neutrosophic graph named single valued neutrosophic Signed graphs (SVNSGs) and examined the properties of this concept with suitable illustration. Ulucay et al. [48] proposed a new extension of neutrosophic graphs called neutrosophic soft expert graphs (NSEGs) and have established a link between graphs and neutrosophic soft expert sets and studies some basic operations of neutrosophic soft experts graphs such as union, intersection and complement. The neutrosophic graphs have many applications in path problems, networks and computer science. Strong BSVNG and complete BSVNG are the types of BSVNG. In this paper, we introduce others types of BSVNGs such as subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs and these are all the strong BSVNGs, also we discuss their relations based on isomorphism, co weak isomorphism and weak isomorphism.

2. Preliminaries

In this section we recall some basic concepts on BSVNG. Let G denotes BSVNG and $G^* = (V, E)$ denotes its underlying crisp graph.

Definition 2.1 ([10])**.** Let X be a crisp set, the single valued neutrosophic set (SVNS) Z is characterized by three membership functions $T_Z(x), I_Z(x)$ and $F_Z(x)$ which are truth, indeterminacy and falsity membership functions, $\forall x \in X$

$$T_Z(x), I_Z(x), F_Z(x) \in [0, 1].$$

Definition 2.2 ([10])**.** Let X be a crisp set, the bipolar single valued neutrosophic set (BSVNS) Z is characterized by membership functions $T^+_Z(x)$, $I^+_Z(x)$, $F^+_Z(x)$, $T^-_Z(x)$, $I^-_Z(x)$, and $F^-_Z(x)$. That is $\forall x \in X$

$$T^+_Z(x), I^+_Z(x), F^+_Z(x) \in [0, 1],$$
$$T^-_Z(x), I^-_Z(x), F^-_Z(x) \in [-1, 0].$$

Definition 2.3 ([10])**.** A bipolar single valued neutrosophic graph (BSVNG) is a pair $G = (Y, Z)$ of G^*, where Y is BSVNS on V and Z is BSVNS on E such that

$$T^+_Z(\beta\gamma) \leq \min(T^+_Y(\beta), T^+_Y(\gamma)),\ I^+_Z(\beta\gamma) \geq \max(I^+_Y(\beta), I^+_Y(\gamma)),$$
$$I^-_Z(\beta\gamma) \leq \min(I^-_Y(\beta), I^-_Y(\gamma)),\ F^-_Z(\beta\gamma) \leq \min(F^-_Y(\beta), F^-_Y(\gamma)),$$
$$F^+_Z(\beta\gamma) \geq \max(F^+_Y(\beta), F^+_Y(\gamma)),\ T^-_Z(\beta\gamma) \geq \max(T^-_Y(\beta), T^-_Y(\gamma)),$$

where

$$0 \leq T^+_Z(\beta\gamma) + I^+_Z(\beta\gamma) + F^+_Z(\beta\gamma) \leq 3$$
$$-3 \leq T^-_Z(\beta\gamma) + I^-_Z(\beta\gamma) + F^-_Z(\beta\gamma) \leq 0$$

$\forall\ \beta, \gamma \in V$.

In this case, D is bipolar single valued neutrosophic relation (BSVNR) on C. The BSVNG $G = (Y, Z)$ is complete (strong) BSVNG, if

$$T_Z^+(\beta\gamma) = \min(T_Y^+(\beta), T_Y^+(\gamma)),\ I_Z^+(\beta\gamma) = \max(I_Y^+(\beta), I_Y^+(\gamma)),$$

$$I_Z^-(\beta\gamma) = \min(I_Y^-(\beta), I_Y^-(\gamma)),\ F_Z^-(\beta\gamma) = \min(F_Y^-(\beta), F_Y^-(\gamma)),$$

$$F_Z^+(\beta\gamma) = \max(F_Y^+(\beta), F_Y^+(\gamma)),\ T_Z^-(\beta\gamma) = \max(T_Y^-(\beta), T_Y^-(\gamma)),$$

$\forall\ \beta, \gamma \in V (\forall\ \beta\gamma \in E)$. The order of BSVNG $G = (A, B)$ of G^*, denoted by $O(G)$, is defined by

$$O(G) = (O_T^+(G), O_I^+(G), O_F^+(G), O_T^-(G), O_I^-(G), O_F^-(G)),$$

where

$$O_T^+(G) = \sum_{\alpha\in V} T_A^+(\alpha),\ O_I^+(G) = \sum_{\alpha\in V} I_A^+(\alpha),\ O_F^+(G) = \sum_{\alpha\in V} F_A^+(\alpha),$$

$$O_T^-(G) = \sum_{\alpha\in V} T_A^-(\alpha),\ O_I^-(G) = \sum_{\alpha\in V} I_A^-(\alpha),\ O_F^-(G) = \sum_{\alpha\in V} F_A^-(\alpha).$$

The size of BSVNG $G = (A, B)$ of G^*, denoted by $S(G)$, is defined by

$$S(G) = (S_T^+(G), S_I^+(G), S_F^+(G), S_T^-(G), S_I^-(G), S_F^-(G)),$$

where

$$S_T^+(G) = \sum_{\beta\gamma\in E} T_B^+(\beta\gamma),\ S_T^-(G) = \sum_{\beta\gamma\in E} T_B^-(\beta\gamma),$$

$$S_I^+(G) = \sum_{\beta\gamma\in E} I_B^+(\beta\gamma),\ S_I^-(G) = \sum_{\beta\gamma\in E} I_B^-(\beta\gamma),$$

$$S_F^+(G) = \sum_{\beta\gamma\in E} F_B^+(\beta\gamma),\ S_F^-(G) = \sum_{\beta\gamma\in E} F_B^-(\beta\gamma).$$

The degree of a vertex β in BSVNG $G = (A, B)$ of G^*,, denoted by $d_G(\beta)$, is defined by

$$d_G(\beta) = (d_T^+(\beta), d_I^+(\beta), d_F^+(\beta), d_T^-(\beta), d_I^-(\beta), d_F^-(\beta)),$$

where

$$d_T^+(\beta) = \sum_{\beta\gamma\in E} T_B^+(\beta\gamma),\ d_T^-(\beta) = \sum_{\beta\gamma\in E} T_B^-(\beta\gamma),$$

$$d_I^+(\beta) = \sum_{\beta\gamma\in E} I_B^+(\beta\gamma),\ d_I^-(\beta) = \sum_{\beta\gamma\in E} I_B^-(\beta\gamma),$$

$$d_F^+(\beta) = \sum_{\beta\gamma\in E} F_B^+(\beta\gamma),\ d_F^-(\beta) = \sum_{\beta\gamma\in E} F_B^-(\beta\gamma).$$

3. Types of BSVNGs

In this section we introduce the special types of BSVNGs such as subdivision, middle and total and intersection BSVNGs, for this first we give the basic definitions of homomorphism, isomorphism, weak isomorphism and co weak isomorphism of BSVNGs which are very useful to understand the relations among the types of BSVNGs.

Definition 3.1. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two BSVNGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then the homomorphism $\chi : G_1 \to G_2$ is a mapping $\chi : V_1 \to V_2$ which satisfies the following conditions:

$$T^+_{C_1}(p) \le T^+_{C_2}(\chi(p)),\ I^+_{C_1}(p) \ge I^+_{C_2}(\chi(p)),\ F^+_{C_1}(p) \ge F^+_{C_2}(\chi(p)),$$

$$T^-_{C_1}(p) \ge T^-_{C_2}(\chi(p)),\ I^-_{C_1}(p) \le I^-_{C_2}(\chi(p)),\ F^-_{C_1}(p) \le F^-_{C_2}(\chi(p)),$$

$\forall\ p \in V_1$,

$$T^+_{D_1}(pq) \le T^+_{D_2}(\chi(p)\chi(q)),\ T^-_{D_1}(pq) \ge T^-_{D_2}(\chi(p)\chi(q)),$$

$$I^+_{D_1}(pq) \ge I^+_{D_2}(\chi(p)\chi(q)),\ I^-_{D_1}(pq) \le I^-_{D_2}(\chi(p)\chi(q)),$$

$$F^+_{D_1}(pq) \ge F^+_{D_2}(\chi(p)\chi(q)),\ F^-_{D_1}(pq) \le F^-_{D_2}(\chi(p)\chi(q)),$$

$\forall\ pq \in E_1$.

Definition 3.2. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two BSVNGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then the weak isomorphism $\upsilon : G_1 \to G_2$ is a bijective mapping $\upsilon : V_1 \to V_2$ which satisfies following conditions:

υ is a homomorphism such that

$$T^+_{C_1}(p) = T^+_{C_2}(\upsilon(p)),\ I^+_{C_1}(p) = I^+_{C_2}(\upsilon(p)),\ F^+_{C_1}(p) = F^+_{C_2}(\upsilon(p)),$$

$$T^-_{C_1}(p) = T^-_{C_2}(\upsilon(p)),\ I^-_{C_1}(p) = I^-_{C_2}(\upsilon(p)),\ F^-_{C_1}(p) = F^-_{C_2}(\upsilon(p)),$$

$\forall\ p \in V_1$.

Remark 3.3. The weak isomorphism between two BSVNGs preserves the orders.

Remark 3.4. The weak isomorphism between BSVNGs is a partial order relation.

Definition 3.5. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two BSVNGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then the co-weak isomorphism $\kappa : G_1 \to G_2$ is a bijective mapping $\kappa : V_1 \to V_2$ which satisfies following conditions:

κ is a homomorphism such that

$$T^+_{D_1}(pq) = T^+_{D_2}(\kappa(p)\kappa(q)),\ T^-_{D_1}(pq) = T^-_{D_2}(\kappa(p)\kappa(q)),$$

$$I^+_{D_1}(pq) = I^+_{D_2}(\kappa(p)\kappa(q)),\ I^-_{D_1}(pq) = I^-_{D_2}(\kappa(p)\kappa(q)),$$

$$F^+_{D_1}(pq) = F^+_{D_2}(\kappa(p)\kappa(q)),\ F^-_{D_1}(pq) = F^-_{D_2}(\kappa(p)\kappa(q)),$$

$\forall\ pq \in E_1$.

Remark 3.6. The co-weak isomorphism between two BSVNGs preserves the sizes.

Remark 3.7. The co-weak isomorphism between BSVNGs is a partial order relation.

Table 1. BSVNSs of BSVNG.

A	T_A^+	I_A^+	F_A^+	T_A^-	I_A^-	F_A^-
a	0.2	0.1	0.4	-0.3	-0.1	-0.4
b	0.3	0.2	0.5	-0.5	-0.4	-0.6
c	0.4	0.7	0.6	-0.2	-0.6	-0.2
B	T_B^+	I_B^+	F_B^+	T_B^-	I_B^-	F_B^-
p	0.2	0.4	0.5	-0.2	-0.5	-0.6
q	0.3	0.8	0.6	-0.1	-0.7	-0.8
r	0.1	0.7	0.9	-0.1	-0.8	-0.5

Definition 3.8. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two BSVNGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then the isomorphism $\psi : G_1 \to G_2$ is a bijective mapping $\psi : V_1 \to V_2$ which satisfies the following conditions:

$$T_{C_1}^+(p) = T_{C_2}^+(\psi(p)),\ I_{C_1}^+(p) = I_{C_2}^+(\psi(p)),\ F_{C_1}^+(p) = F_{C_2}^+(\psi(p)),$$

$$T_{C_1}^-(p) = T_{C_2}^-(\psi(p)),\ I_{C_1}^-(p) = I_{C_2}^-(\psi(p)),\ F_{C_1}^-(p) = F_{C_2}^-(\psi(p)),$$

$\forall\ p \in V_1$,

$$T_{D_1}^+(pq) = T_{D_2}^+(\psi(p)\psi(q)),\ T_{D_1}^-(pq) = T_{D_2}^-(\psi(p)\psi(q)),$$

$$I_{D_1}^+(pq) = I_{D_2}^+(\psi(p)\psi(q)),\ I_{D_1}^-(pq) = I_{D_2}^-(\psi(p)\psi(q)),$$

$$F_{D_1}^+(pq) = F_{D_2}^+(\psi(p)\psi(q)),\ F_{D_1}^-(pq) = F_{D_2}^-(\psi(p)\psi(q)),$$

$\forall\ pq \in E_1$.

Remark 3.9. The isomorphism between two BSVNGs is an equivalence relation.

Remark 3.10. The isomorphism between two BSVNGs preserves the orders and sizes.

Remark 3.11. The isomorphism between two BSVNGs preserves the degrees of their vertices.

Definition 3.12. The subdivision SVNG be $sd(G) = (C, D)$ of $G = (A, B)$, where C is a BSVNS on $V \cup E$ and D is a BSVNR on C such that

(i) $C = A$ on V and $C = B$ on E,

(ii) if $v \in V$ lie on edge $e \in E$, then

$$T_D^+(ve) = \min(T_A^+(v), T_B^+(e)),\ I_D^+(ve) = \max(I_A^+(v), I_B^+(e))$$

$$I_D^-(ve) = \min(I_A^-(v), I_B^-(e)),\ F_D^-(ve) = \min(F_A^-(v), F_B^-(e))$$

$$F_D^+(ve) = \max(F_A^+(v), F_B^+(e)),\ T_D^-(ve) = \max(T_A^-(v), T_B^-(e))$$

else

$$D(ve) = O = (0, 0, 0, 0, 0, 0).$$

Example 3.13. Consider the BSVNG $G = (A, B)$ of a $G^* = (V, E)$, where $V = \{a, b, c\}$ and $E = \{p = ab, q = bc, r = ac\}$, the crisp graph of G is shown in Fig. 1. The BSVNSs A and B are defined on V and E respectively which are defined in Table 1. The SDBSVNG $sd(G) = (C, D)$ of a BSVNG G, the underlying crisp graph of $sd(G)$ is given in Fig. 2. The BSVNSs C and D are defined in Table 2.

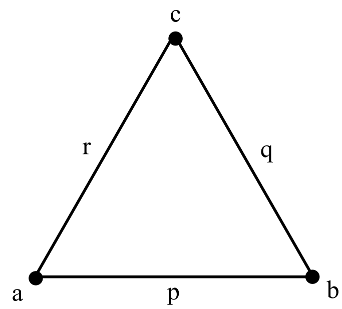


Figure 1. Crisp Graph of BSVNG.

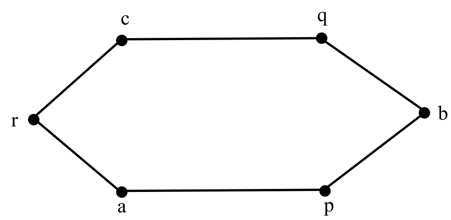


Figure 2. Crisp Graph of SDBSVNG.

Table 2. BSVNSs of SDBSVNG.

C	T_C^+	I_C^+	F_C^+	T_C^-	I_C^-	F_C^-
a	0.2	0.1	0.4	-0.3	-0.1	-0.4
p	0.2	0.4	0.5	-0.2	-0.5	-0.6
b	0.3	0.2	0.5	-0.5	-0.4	-0.6
q	0.3	0.8	0.6	-0.1	-0.7	-0.8
c	0.4	0.7	0.6	-0.2	-0.6	-0.2
r	0.1	0.7	0.9	-0.1	-0.8	-0.5
D	T_D^+	I_D^+	F_D^+	T_D^-	I_D^-	F_D^-
ap	0.2	0.4	0.5	-0.2	-0.5	-0.6
pb	0.2	0.4	0.5	-0.2	-0.5	-0.6
bq	0.3	0.8	0.6	-0.1	-0.7	-0.8
qc	0.3	0.8	0.6	-0.1	-0.7	-0.8
cr	0.1	0.7	0.9	-0.1	-0.8	-0.5
ra	0.1	0.7	0.9	-0.1	-0.8	-0.5

Proposition 3.14. *Let G be a BSVNG and $sd(G)$ be the SDBSVNG of a BSVNG G, then $O(sd(G)) = O(G) + S(G)$ and $S(sd(G)) = 2S(G)$.*

Remark 3.15. Let G be a complete BSVNG, then $sd(G)$ need not to be complete BSVNG.

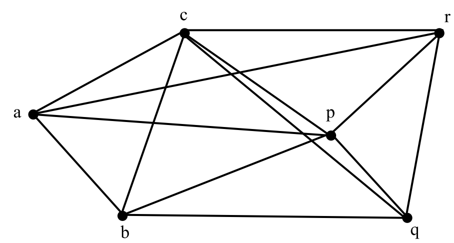


Figure 3. Crisp Graph of TSVNG.

Definition 3.16. The total bipolar single valued neutrosophic graph (TBSVNG) is $T(G) = (C, D)$ of $G = (A, B)$, where C is a BSVNS on $V \cup E$ and D is a BSVNR on C such that

(i) $C = A$ on V and $C = B$ on E,

(ii) if $v \in V$ lie on edge $e \in E$, then

$$T_D^+(ve) = \min(T_A^+(v), T_B^+(e)),\ I_D^+(ve) = \max(I_A^+(v), I_B^+(e))$$

$$I_D^-(ve) = \min(I_A^-(v), I_B^-(e)),\ F_D^-(ve) = \min(F_A^-(v), F_B^-(e))$$

$$F_D^+(ve) = \max(F_A^+(v), F_B^+(e)),\ T_D^-(ve) = \max(T_A^-(v), T_B^-(e))$$

else

$$D(ve) = O = (0, 0, 0, 0, 0, 0),$$

(iii) if $\alpha\beta \in E$, then

$$T_D^+(\alpha\beta) = T_B^+(\alpha\beta),\ I_D^+(\alpha\beta) = I_B^+(\alpha\beta),\ F_D^+(\alpha\beta) = F_B^+(\alpha\beta)$$

$$T_D^-(\alpha\beta) = T_B^-(\alpha\beta),\ I_D^-(\alpha\beta) = I_B^-(\alpha\beta),\ F_D^-(\alpha\beta) = F_B^-(\alpha\beta),$$

(iv) if $e, f \in E$ have a common vertex, then

$$T_D^+(ef) = \min(T_B^+(e), T_B^+(f)),\ I_D^+(ef) = \max(I_B^+(e), I_B^+(f))$$

$$I_D^-(ef) = \min(I_B^-(e), I_B^-(f)),\ F_D^-(ef) = \min(F_B^-(e), F_B^-(f))$$

$$F_D^+(ef) = \max(F_B^+(e), F_B^+(f)),\ T_D^-(ef) = \max(T_B^-(e), T_B^-(f))$$

else

$$D(ef) = O = (0, 0, 0, 0, 0, 0).$$

Example 3.17. Consider the Example 3.13 the TBSVNG $T(G) = (C, D)$ of underlying crisp graph as shown in Fig. 3. The BSVNS C is given in Example 3.13. The BSVNS D is given in Table 3.

Proposition 3.18. *Let G be a BSVNG and $T(G)$ be the TBSVNG of a BSVNG G, then $O(T(G)) = O(G) + S(G) = O(sd(G))$ and $S(sd(G)) = 2S(G)$.*

Proposition 3.19. *Let G be a BSVNG, then $sd(G)$ is weak isomorphic to $T(G)$.*

TABLE 3. BSVNS of TBSVNG.

D	T_D^+	I_D^+	F_D^+	T_D^-	I_D^-	F_D^-
ab	0.2	0.4	0.5	-0.2	-0.5	-0.6
bc	0.3	0.8	0.6	-0.1	-0.7	-0.8
ca	0.1	0.7	0.9	-0.1	-0.8	-0.5
pq	0.2	0.8	0.6	-0.1	-0.7	-0.8
qr	0.1	0.8	0.9	-0.1	-0.8	-0.8
rp	0.1	0.7	0.9	-0.1	-0.8	-0.6
ap	0.2	0.4	0.5	-0.2	-0.5	-0.6
pb	0.2	0.4	0.5	-0.2	-0.5	-0.6
bq	0.3	0.8	0.6	-0.1	-0.7	-0.8
qc	0.3	0.8	0.6	-0.1	-0.7	-0.8
cr	0.1	0.7	0.9	-0.1	-0.8	-0.5
ra	0.1	0.7	0.9	-0.1	-0.8	-0.5

Definition 3.20. The middle bipolar single valued neutrosophic graph (MBSVNG) $M(G) = (C, D)$ of G, where C is a BSVNS on $V \cup E$ and D is a BSVNR on C such that

(i) $C = A$ on V and $C = B$ on E, else $C = O = (0, 0, 0, 0, 0, 0)$,

(ii) if $v \in V$ lie on edge $e \in E$, then

$$T_D^+(ve) = T_B^+(e),\ I_D^+(ve) = I_B^+(e),\ F_D^+(ve) = F_B^+(e)$$

$$T_D^-(ve) = T_B^-(e),\ I_D^-(ve) = I_B^-(e),\ F_D^-(ve) = F_B^-(e)$$

else

$$D(ve) = O = (0, 0, 0, 0, 0, 0),$$

(iii) if $u, v \in V$, then

$$D(uv) = O = (0, 0, 0, 0, 0, 0),$$

(iv) if $e, f \in E$ and e and f are adjacent in G, then

$$T_D^+(ef) = T_B^+(uv),\ I_D^+(ef) = I_B^+(uv),\ F_D^+(ef) = F_B^+(uv)$$

$$T_D^-(ef) = T_B^-(uv),\ I_D^-(ef) = I_B^-(uv),\ F_D^-(ef) = F_B^-(uv).$$

Example 3.21. Consider the BSVNG $G = (A, B)$ of a G^*, where $V = \{a, b, c\}$ and $E = \{p = ab, q = bc\}$ the underlaying crisp graph is shown in Fig. 4. The BSVNSs A and B are defined in Table 4. The crisp graph of MBSVNG $M(G) = (C, D)$ is shown in Fig. 5. The BSVNSs C and D are given in Table 5.

Remark 3.22. Let G be a BSVNG and $M(G)$ be the MBSVNG of a BSVNG G, then $O(M(G)) = O(G) + S(G)$.

Remark 3.23. Let G be a BSVNG, then $M(G)$ is a strong BSVNG.

Remark 3.24. Let G be complete BSVNG, then $M(G)$ need not to be complete BSVNG.

Proposition 3.25. *Let G be a BSVNG, then $sd(G)$ is weak isomorphic with $M(G)$.*

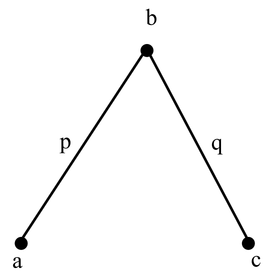


FIGURE 4. Crisp Graph of BSVNG.

TABLE 4. BSVNSs of BSVNG.

A	T_A^+	I_A^+	F_A^+	T_A^-	I_A^-	F_A^-
a	0.3	0.4	0.5	-0.2	-0.1	-0.3
b	0.7	0.6	0.3	-0.3	-0.3	-0.2
c	0.9	0.7	0.2	-0.5	-0.4	-0.6
B	T_B^+	I_B^+	F_B^+	T_B^-	I_B^-	F_B^-
p	0.2	0.6	0.6	-0.1	-0.4	-0.3
q	0.4	0.8	0.7	-0.3	-0.5	-0.6

TABLE 5. BSVNSs of MBSVNG.

C	T_C^+	I_C^+	F_C^+	T_C^-	I_C^-	F_C^-
a	0.3	0.4	0.5	-0.2	-0.1	-0.3
b	0.7	0.6	0.3	-0.3	-0.3	-0.2
c	0.9	0.7	0.2	-0.5	-0.4	-0.6
e_1	0.2	0.6	0.6	-0.1	-0.4	-0.3
e_2	0.4	0.8	0.7	-0.3	-0.5	-0.6
D	T_D^+	I_D^+	F_D^+	T_D^-	I_D^-	F_D^-
pq	0.2	0.8	0.7	-0.1	-0.5	-0.6
ap	0.2	0.6	0.6	-0.1	-0.4	-0.3
bp	0.2	0.6	0.6	-0.1	-0.4	-0.3
bq	0.2	0.6	0.6	-0.3	-0.5	-0.6
cq	0.4	0.8	0.7	-0.3	-0.5	-0.6

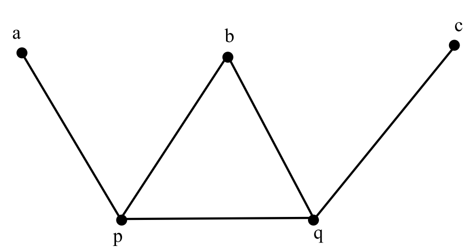


FIGURE 5. Crisp Graph of MBSVNG.

Proposition 3.26. *Let G be a BSVNG, then $M(G)$ is weak isomorphic with $T(G)$.*

Proposition 3.27. *Let G be a BSVNG, then $T(G)$ is isomorphic with $G \cup M(G)$.*

Definition 3.28. Let $P(X) = (X, Y)$ be the intersection graph of a G^*, let C_1 and D_1 be BSVNSs on V and E, respectively and C_2 and D_2 be BSVNSs on X and Y respectively. Then bipolar single valued neutrosophic intersection graph (BSVNIG) of a BSVNG $G = (C_1, D_1)$ is a BSVNG $P(G) = (C_2, D_2)$ such that,

$$T^+_{C_2}(X_i) = T^+_{C_1}(v_i),\ I^+_{C_2}(X_i) = I^+_{C_1}(v_i),\ F^+_{C_2}(X_i) = F^+_{C_1}(v_i)$$
$$T^-_{C_2}(X_i) = T^-_{C_1}(v_i),\ I^-_{C_2}(X_i) = I^-_{C_1}(v_i),\ F^-_{C_2}(X_i) = F^-_{C_1}(v_i)$$
$$T^+_{D_2}(X_iX_j) = T^+_{D_1}(v_iv_j),\ T^-_{D_2}(X_iX_j) = T^-_{D_1}(v_iv_j),$$
$$I^+_{D_2}(X_iX_j) = I^+_{D_1}(v_iv_j),\ I^-_{D_2}(X_iX_j) = I^-_{D_1}(v_iv_j),$$
$$F^+_{D_2}(X_iX_j) = F^+_{D_1}(v_iv_j),\ F^-_{D_2}(X_iX_j) = F^-_{D_1}(v_iv_j)$$

$\forall\ X_i, X_j \in X$ and $X_iX_j \in Y$.

Proposition 3.29. *Let $G = (A_1, B_1)$ be a BSVNG of $G^* = (V, E)$, and let $P(G) = (A_2, B_2)$ be a BSVNIG of $P(S)$. Then BSVNIG is a also BSVNG and BSVNG is always isomorphic to BSVNIG.*

Proof. By the definition of BSVNIG, we have

$$\begin{aligned}
T^+_{B_2}(S_iS_j) &= T^+_{B_1}(v_iv_j) \le \min(T^+_{A_1}(v_i), T^+_{A_1}(v_j)) = \min(T^+_{A_2}(S_i), T^+_{A_2}(S_j)),\\
I^+_{B_2}(S_iS_j) &= I^+_{B_1}(v_iv_j) \ge \max(I^+_{A_1}(v_i), I^+_{A_1}(v_j)) = \max(I^+_{A_2}(S_i), I^+_{A_2}(S_j)),\\
F^+_{B_2}(S_iS_j) &= F^+_{B_1}(v_iv_j) \ge \max(F^+_{A_1}(v_i), F^+_{A_1}(v_j)) = \max(F^+_{A_2}(S_i), F^+_{A_2}(S_j)),\\
T^-_{B_2}(S_iS_j) &= T^-_{B_1}(v_iv_j) \ge \max(T^-_{A_1}(v_i), T^-_{A_1}(v_j)) = \max(T^-_{A_2}(S_i), T^-_{A_2}(S_j)),\\
I^-_{B_2}(S_iS_j) &= I^-_{B_1}(v_iv_j) \le \min(I^-_{A_1}(v_i), I^-_{A_1}(v_j)) = \min(I^-_{A_2}(S_i), I^-_{A_2}(S_j)),\\
F^-_{B_2}(S_iS_j) &= F^-_{B_1}(v_iv_j) \le \min(F^-_{A_1}(v_i), F^-_{A_1}(v_j)) = \min(F^-_{A_2}(S_i), F^-_{A_2}(S_j)).
\end{aligned}$$

This shows that BSVNIG is a BSVNG.

Next define $f : V \to S$ by $f(v_i) = S_i$ for $i = 1, 2, 3, \ldots, n$ clearly f is bijective. Now $v_iv_j \in E$ if and only if $S_iS_j \in T$ and $T = \{f(v_i)f(v_j) : v_iv_j \in E\}$. Also

$$T^+_{A_2}(f(v_i)) = T^+_{A_2}(S_i) = T^+_{A_1}(v_i),\ I^+_{A_2}(f(v_i)) = I^+_{A_2}(S_i) = I^+_{A_1}(v_i),$$
$$F^+_{A_2}(f(v_i)) = F^+_{A_2}(S_i) = F^+_{A_1}(v_i),\ T^-_{A_2}(f(v_i)) = T^-_{A_2}(S_i) = T^-_{A_1}(v_i),$$
$$I^-_{A_2}(f(v_i)) = I^-_{A_2}(S_i) = I^-_{A_1}(v_i),\ F^-_{A_2}(f(v_i)) = F^-_{A_2}(S_i) = F^-_{A_1}(v_i),$$

$\forall\ v_i \in V$,

$$T^+_{B_2}(f(v_i)f(v_j)) = T^+_{B_2}(S_iS_j) = T^+_{B_1}(v_iv_j),$$
$$I^+_{B_2}(f(v_i)f(v_j)) = I^+_{B_2}(S_iS_j) = I^+_{B_1}(v_iv_j),$$
$$F^+_{B_2}(f(v_i)f(v_j)) = F^+_{B_2}(S_iS_j) = F^+_{B_1}(v_iv_j),$$
$$T^-_{B_2}(f(v_i)f(v_j)) = T^-_{B_2}(S_iS_j) = T^-_{B_1}(v_iv_j),$$
$$I^-_{B_2}(f(v_i)f(v_j)) = I^-_{B_2}(S_iS_j) = I^-_{B_1}(v_iv_j),$$
$$F^-_{B_2}(f(v_i)f(v_j)) = F^-_{B_2}(S_iS_j) = F^-_{B_1}(v_iv_j),$$

$\forall\ v_iv_j \in E$. □

Table 6. BSVNSs of BSVNG.

A_1	$T^+_{A_1}$	$I^+_{A_1}$	$F^+_{A_1}$	$T^-_{A_1}$	$I^-_{A_1}$	$F^-_{A_1}$
α_1	0.2	0.5	0.5	-0.1	-0.4	-0.5
α_2	0.4	0.3	0.3	-0.2	-0.3	-0.2
α_3	0.4	0.5	0.5	-0.3	-0.2	-0.6
α_4	0.3	0.2	0.2	-0.4	-0.1	-0.3
B_1	$T^+_{B_1}$	$I^+_{B_1}$	$F^+_{B_1}$	$T^-_{B_1}$	$I^-_{B_1}$	$F^-_{B_1}$
x_1	0.1	0.6	0.7	-0.1	-0.4	-0.5
x_2	0.3	0.6	0.7	-0.2	-0.3	-0.6
x_3	0.2	0.7	0.8	-0.3	-0.2	-0.6
x_4	0.1	0.7	0.8	-0.1	-0.4	-0.5

Definition 3.30. Let $G^* = (V, E)$ and $L(G^*) = (X, Y)$ be its line graph, where A_1 and B_1 be BSVNSs on V and E, respectively. Let A_2 and B_2 be BSVNSs on X and Y, respectively. The bipolar single valued neutrosophic line graph (BSVNLG) of BSVNG $G = (A_1, B_1)$ is BSVNG $L(G) = (A_2, B_2)$ such that,

$$T^+_{A_2}(S_x) = T^+_{B_1}(x) = T^+_{B_1}(u_x v_x),\ I^+_{A_2}(S_x) = I^+_{B_1}(x) = I^+_{B_1}(u_x v_x),$$
$$I^-_{A_2}(S_x) = I^-_{B_1}(x) = I^-_{B_1}(u_x v_x),\ F^-_{A_2}(S_x) = F^-_{B_1}(x) = F^-_{B_1}(u_x v_x),$$
$$F^+_{A_2}(S_x) = F^+_{B_1}(x) = F^+_{B_1}(u_x v_x),\ T^-_{A_2}(S_x) = T^-_{B_1}(x) = T^-_{B_1}(u_x v_x),$$

$\forall\ S_x, S_y \in X$ and

$$T^+_{B_2}(S_x S_y) = \min(T^+_{B_1}(x), T^+_{B_1}(y)),\ I^+_{B_2}(S_x S_y) = \max(I^+_{B_1}(x), I^+_{B_1}(y)),$$
$$I^-_{B_2}(S_x S_y) = \min(I^-_{B_1}(x), I^-_{B_1}(y)),\ F^-_{B_2}(S_x S_y) = \min(F^-_{B_1}(x), F^-_{B_1}(y)),$$
$$F^+_{B_2}(S_x S_y) = \max(F^+_{B_1}(x), F^+_{B_1}(y)),\ T^-_{B_2}(S_x S_y) = \max(T^-_{B_1}(x), T^-_{B_1}(y)),$$

$\forall\ S_x S_y \in Y$.

Remark 3.31. Every BSVNLG is a strong BSVNG.

Remark 3.32. The $L(G) = (A_2, B_2)$ is a BSVNLG corresponding to BSVNG $G = (A_1, B_1)$.

Example 3.33. Consider the $G^* = (V, E)$ where $V = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $E = \{x_1 = \alpha_1\alpha_2, x_2 = \alpha_2\alpha_3, x_3 = \alpha_3\alpha_4, x_4 = \alpha_4\alpha_1\}$ and $G = (A_1, B_1)$ is BSVNG of $G^* = (V, E)$ which is defined in Table 6. Consider the $L(G^*) = (X, Y)$ such that $X = \{\Gamma_{x_1}, \Gamma_{x_2}, \Gamma_{x_3}, \Gamma_{x_4}\}$ and $Y = \{\Gamma_{x_1}\Gamma_{x_2}, \Gamma_{x_2}\Gamma_{x_3}, \Gamma_{x_3}\Gamma_{x_4}, \Gamma_{x_4}\Gamma_{x_1}\}$. Let A_2 and B_2 be BSVNSs of X and Y respectively, then BSVNLG $L(G)$ is given in Table 7.

Proposition 3.34. *The $L(G) = (A_2, B_2)$ is a BSVNLG of some BSVNG $G = (A_1, B_1)$ if and only if*

$$T^+_{B_2}(S_x S_y) = \min(T^+_{A_2}(S_x), T^+_{A_2}(S_y)),$$
$$T^-_{B_2}(S_x S_y) = \max(T^-_{A_2}(S_x), T^-_{A_2}(S_y)),$$
$$I^+_{B_2}(S_x S_y) = \max(I^+_{A_2}(S_x), I^+_{A_2}(S_y)),$$
$$F^-_{B_2}(S_x S_y) = \min(F^-_{A_2}(S_x), F^-_{A_2}(S_y)),$$

Table 7. BSVNSs of BSVNLG.

A_1	$T^+_{A_1}$	$I^+_{A_1}$	$F^+_{A_1}$	$T^-_{A_1}$	$I^-_{A_1}$	$F^-_{A_1}$
Γ_{x_1}	0.1	0.6	0.7	-0.1	-0.4	-0.5
Γ_{x_2}	0.3	0.6	0.7	-0.2	-0.3	-0.6
Γ_{x_3}	0.2	0.7	0.8	-0.3	-0.2	-0.6
Γ_{x_4}	0.1	0.7	0.8	-0.1	-0.4	-0.5
B_1	$T^+_{B_1}$	$I^+_{B_1}$	$F^+_{B_1}$	$T^-_{B_1}$	$I^-_{B_1}$	$F^-_{B_1}$
$\Gamma_{x_1}\Gamma_{x_2}$	0.1	0.6	0.7	-0.1	-0.4	-0.6
$\Gamma_{x_2}\Gamma_{x_3}$	0.2	0.7	0.8	-0.2	-0.3	-0.6
$\Gamma_{x_3}\Gamma_{x_4}$	0.1	0.7	0.8	-0.1	-0.4	-0.6
$\Gamma_{x_4}\Gamma_{x_1}$	0.1	0.7	0.8	-0.1	-0.4	-0.5

$$I^-_{B_2}(S_xS_y) = \min(I^-_{A_2}(S_x), I^-_{A_2}(S_y)),$$
$$F^+_{B_2}(S_xS_y) = \max(F^+_{A_2}(S_x), F^+_{A_2}(S_y)),$$

$\forall\ S_xS_y \in Y$.

Proof. Assume that,

$$T^+_{B_2}(S_xS_y) = \min(T^+_{A_2}(S_x), T^+_{A_2}(S_y)),$$
$$T^-_{B_2}(S_xS_y) = \max(T^-_{A_2}(S_x), T^-_{A_2}(S_y)),$$
$$I^+_{B_2}(S_xS_y) = \max(I^+_{A_2}(S_x), I^+_{A_2}(S_y)),$$
$$F^-_{B_2}(S_xS_y) = \min(F^-_{A_2}(S_x), F^-_{A_2}(S_y)),$$
$$I^-_{B_2}(S_xS_y) = \min(I^-_{A_2}(S_x), I^-_{A_2}(S_y)),$$
$$F^+_{B_2}(S_xS_y) = \max(F^+_{A_2}(S_x), F^+_{A_2}(S_y)),$$

$\forall\ S_xS_y \in Y$. Define

$$T^+_{A_1}(x) = T^+_{A_2}(S_x),\ I^+_{A_1}(x) = I^+_{A_2}(S_x),\ F^+_{A_1}(x) = F^+_{A_2}(S_x),$$
$$T^-_{A_1}(x) = T^-_{A_2}(S_x),\ I^-_{A_1}(x) = I^-_{A_2}(S_x),\ F^-_{A_1}(x) = F^-_{A_2}(S_x)$$

$\forall\ x \in E$. Then

$$I^+_{B_2}(S_xS_y) = \max(I^+_{A_2}(S_x), I^+_{A_2}(S_y)) = \max(I^+_{A_2}(x), I^+_{A_2}(y)),$$
$$I^-_{B_2}(S_xS_y) = \min(I^-_{A_2}(S_x), I^-_{A_2}(S_y)) = \min(I^-_{A_2}(x), I^-_{A_2}(y)),$$
$$T^+_{B_2}(S_xS_y) = \min(T^+_{A_2}(S_x), T^+_{A_2}(S_y)) = \min(T^+_{A_2}(x), T^+_{A_2}(y)),$$
$$T^-_{B_2}(S_xS_y) = \max(T^-_{A_2}(S_x), T^-_{A_2}(S_y)) = \max(T^-_{A_2}(x), T^-_{A_2}(y)),$$
$$F^-_{B_2}(S_xS_y) = \min(F^-_{A_2}(S_x), F^-_{A_2}(S_y)) = \min(F^-_{A_2}(x), F^-_{A_2}(y)),$$
$$F^+_{B_2}(S_xS_y) = \max(F^+_{A_2}(S_x), F^+_{A_2}(S_y)) = \max(F^+_{A_2}(x), F^+_{A_2}(y)).$$

A BSVNS A_1 that yields the property

$$T^+_{B_1}(xy) \leq \min(T^+_{A_1}(x), T^+_{A_1}(y)),\ I^+_{B_1}(xy) \geq \max(I^+_{A_1}(x), I^+_{A_1}(y)),$$
$$I^-_{B_1}(xy) \leq \min(I^-_{A_1}(x), I^-_{A_1}(y)),\ F^-_{B_1}(xy) \leq \min(F^-_{A_1}(x), F^-_{A_1}(y)),$$
$$F^+_{B_1}(xy) \geq \max(F^+_{A_1}(x), F^+_{A_1}(y)),\ T^-_{B_1}(xy) \geq \max(T^-_{A_1}(x), T^-_{A_1}(y))$$

will suffice. Converse is straight forward. □

Proposition 3.35. *If $L(G)$ be a BSVNLG of BSVNG G, then $L(G^*) = (X, Y)$ is the crisp line graph of G^*.*

Proof. Since $L(G)$ is a BSVNLG,

$$T^+_{A_2}(S_x) = T^+_{B_1}(x),\ I^+_{A_2}(S_x) = I^+_{B_1}(x),\ F^+_{A_2}(S_x) = F^+_{B_1}(x),$$

$$T^-_{A_2}(S_x) = T^-_{B_1}(x),\ I^-_{A_2}(S_x) = I^-_{B_1}(x),\ F^-_{A_2}(S_x) = F^-_{B_1}(x)$$

$\forall\ x \in E$, $S_x \in X$ if and only if $x \in E$, also

$$T^+_{B_2}(S_xS_y) = \min(T^+_{B_1}(x), T^+_{B_1}(y)),\ I^+_{B_2}(S_xS_y) = \max(I^+_{B_1}(x), I^+_{B_1}(y)),$$

$$I^-_{B_2}(S_xS_y) = \min(I^-_{B_1}(x), I^-_{B_1}(y)),\ F^-_{B_2}(S_xS_y) = \min(F^-_{B_1}(x), F^-_{B_1}(y)),$$

$$F^+_{B_2}(S_xS_y) = \max(F^+_{B_1}(x), F^+_{B_1}(y)),\ T^-_{B_2}(S_xS_y) = \max(T^-_{B_1}(x), T^-_{B_1}(y)),$$

$\forall\ S_xS_y \in Y$. Then $Y = \{S_xS_y : S_x \cap S_y \neq \phi, x, y \in E, x \neq y\}$. □

Proposition 3.36. *The $L(G) = (A_2, B_2)$ be a BSVNLG of BSVNG G if and only if $L(G^*) = (X, Y)$ is the line graph and*

$$T^+_{B_2}(xy) = \min(T^+_{A_2}(x), T^+_{A_2}(y)),\ I^+_{B_2}(xy) = \max(I^+_{A_2}(x), I^+_{A_2}(y)),$$

$$I^-_{B_2}(xy) = \min(I^-_{A_2}(x), I^-_{A_2}(y)),\ F^-_{B_2}(xy) = \min(F^-_{A_2}(x), F^-_{A_2}(y)),$$

$$F^+_{B_2}(xy) = \max(F^+_{A_2}(x), F^+_{A_2}(y)),\ T^-_{B_2}(xy) = \max(T^-_{A_2}(x), T^-_{A_2}(y)),$$

$\forall\ xy \in Y$.

Proof. It follows from propositions 3.34 and 3.35. □

Proposition 3.37. *Let G be a BSVNG, then $M(G)$ is isomorphic with $sd(G) \cup L(G)$.*

Theorem 3.38. *Let $L(G) = (A_2, B_2)$ be BSVNLG corresponding to BSVNG $G = (A_1, B_1)$.*

(1) *If G is weak isomorphic onto $L(G)$ if and only if $\forall\ v \in V,\ x \in E$ and G^* to be a cycle, such that*

$$T^+_{A_1}(v) = T^+_{B_1}(x),\ I^+_{A_1}(v) = T^+_{B_1}(x),\ F^+_{A_1}(v) = T^+_{B_1}(x),$$

$$T^-_{A_1}(v) = T^-_{B_1}(x),\ I^-_{A_1}(v) = T^-_{B_1}(x),\ F^-_{A_1}(v) = T^-_{B_1}(x).$$

(2) *If G is weak isomorphic onto $L(G)$, then G and $L(G)$ are isomorphic.*

Proof. By hypothesis, G^* is a cycle. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E = \{x_1 = v_1v_2, x_2 = v_2v_3, \ldots, x_n = v_nv_1\}$, where $P : v_1v_2v_3 \ldots v_n$ is a cycle, characterize a BSVNS A_1 by $A_1(v_i) = (p_i, q_i, r_i, p_i^{'}, q_i^{'}, r_i^{'})$ and B_1 by $B_1(x_i) = (a_i, b_i, c_i, a_i^{'}, b_i^{'}, c_i^{'})$ for $i = 1, 2, 3, \ldots, n$ and $v_{n+1} = v_1$. Then for $p_{n+1} = p_1$, $q_{n+1} = q_1$, $r_{n+1} = r_1$,

$$a_i \leq \min(p_i, p_{i+1}),\ b_i \geq \max(q_i, q_{i+1}),\ c_i \geq \max(r_i, r_{i+1}),$$

$$a_i^{'} \geq \max(p_i^{'}, p_{i+1}^{'}),\ b_i^{'} \leq \min(q_i^{'}, q_{i+1}^{'}),\ c_i^{'} \leq \min(r_i^{'}, r_{i+1}^{'}),$$

for $i = 1, 2, 3, \ldots, n$.

Now let $X = \{\Gamma_{x_1}, \Gamma_{x_2}, \ldots, \Gamma_{x_n}\}$ and $Y = \{\Gamma_{x_1}\Gamma_{x_2}, \Gamma_{x_2}\Gamma_{x_3}, \ldots, \Gamma_{x_n}\Gamma_{x_1}\}$. Then for $a_{n+1} = a_1$, we obtain

$$A_2(\Gamma_{x_i}) = B_1(x_i) = (a_i, b_i, c_i, a_i^{'}, b_i^{'}, c_i^{'})$$

and $B_2(\Gamma_{x_i}\Gamma_{x_{i+1}}) = (\min(a_i, a_{i+1}), \max(b_i, b_{i+1}), \max(c_i, c_{i+1}), \max(a_i^{'}, a_{i+1}^{'}), \min(b_i^{'}, b_{i+1}^{'}), \min(c_i^{'}, c_{i+1}^{'}))$ for $i = 1, 2, 3, \ldots, n$ and $v_{n+1} = v_1$. Since f preserves adjacency, it induce permutation π of $\{1, 2, 3, \ldots, n\}$,

$$f(v_i) = \Gamma_{v_{\pi(i)}v_{\pi(i)+1}}$$

and

$$v_iv_{i+1} \to f(v_i)f(v_{i+1}) = \Gamma_{v_{\pi(i)}v_{\pi(i)+1}}\Gamma_{v_{\pi(i+1)}v_{\pi(i+1)+1}},$$

for $i = 1, 2, 3, \ldots, n-1$. Thus

$$p_i = T^+_{A_1}(v_i) \leq T^+_{A_2}(f(v_i)) = T^+_{A_2}(\Gamma_{v_{\pi(i)}v_{\pi(i)+1}}) = T^+_{B_1}(v_{\pi(i)}v_{\pi(i)+1}) = a_{\pi(i)}.$$

Similarly, $p_i^{'} \geq a_{\pi(i)}^{'}$, $q_i \geq b_{\pi(i)}$, $r_i \geq c_{\pi(i)}$, $q_i^{'} \leq b_{\pi(i)}^{'}$, $r_i^{'} \leq c_{\pi(i)}^{'}$ and

$$\begin{aligned} a_i &= T^+_{B_1}(v_iv_{i+1}) \leq T^+_{B_2}(f(v_i)f(v_{i+1})) \\ &= T^+_{B_2}(\Gamma_{v_{\pi(i)}v_{\pi(i)+1}}\Gamma_{v_{\pi(i+1)}v_{\pi(i+1)+1}}) \\ &= \min(T^+_{B_1}(v_{\pi(i)}v_{\pi(i)+1}), T^+_{B_1}(v_{\pi(i+1)}v_{\pi(i+1)+1})) \\ &= \min(a_{\pi(i)}, a_{\pi(i)+1}). \end{aligned}$$

Similarly, $b_i \geq \max(b_{\pi(i)}, b_{\pi(i)+1})$, $c_i \geq \max(c_{\pi(i)}, c_{\pi(i)+1})$, $a_i^{'} \geq \max(a_{\pi(i)}^{'}, a_{\pi(i)+1}^{'})$, $b_i^{'} \leq \min(b_{\pi(i)}^{'}, b_{\pi(i)+1}^{'})$ and $c_i^{'} \leq \min(c_{\pi(i)}^{'}, c_{\pi(i)+1}^{'})$ for $i = 1, 2, 3, \ldots, n$. Therefore

$$p_i \leq a_{\pi(i)},\ q_i \geq b_{\pi(i)},\ r_i \geq c_{\pi(i)},\ p_i^{'} \geq a_{\pi(i)}^{'},\ q_i^{'} \leq b_{\pi(i)}^{'},\ r_i^{'} \leq c_{\pi(i)}^{'}$$

and

$$\begin{gathered} a_i \leq \min(a_{\pi(i)}, a_{\pi(i)+1}),\ a_i^{'} \geq \max(a_{\pi(i)}^{'}, a_{\pi(i)+1}^{'}), \\ b_i \geq \max(b_{\pi(i)}, b_{\pi(i)+1}),\ b_i^{'} \leq \min(b_{\pi(i)}^{'}, b_{\pi(i)+1}^{'}), \\ c_i \geq \max(c_{\pi(i)}, c_{\pi(i)+1}),\ c_i \leq \min(c_{\pi(i)}^{'}, c_{\pi(i)+1}^{'}) \end{gathered}$$

thus

$$a_i \leq a_{\pi(i)},\ b_i \geq b_{\pi(i)},\ c_i \geq c_{\pi(i)},\ a_i^{'} \geq a_{\pi(i)}^{'},\ b_i^{'} \leq b_{\pi(i)}^{'},\ c_i^{'} \leq c_{\pi(i)}^{'}$$

and so

$$\begin{gathered} a_{\pi(i)} \leq a_{\pi(\pi(i))},\ b_{\pi(i)} \geq b_{\pi(\pi(i))},\ c_{\pi(i)} \geq c_{\pi(\pi(i))} \\ a_{\pi(i)}^{'} \geq a_{\pi(\pi(i))}^{'},\ b_{\pi(i)}^{'} \leq b_{\pi(\pi(i))}^{'},\ c_{\pi(i)}^{'} \leq c_{\pi(\pi(i))}^{'} \end{gathered}$$

$\forall\ i = 1, 2, 3, \ldots, n$. Next to extend,

$$\begin{gathered} a_i \leq a_{\pi(i)} \leq \ldots \leq a_{\pi^j(i)} \leq a_i,\ a_i^{'} \geq a_{\pi(i)}^{'} \geq \ldots \geq a_{\pi^j(i)}^{'} \geq a_i^{'} \\ b_i \geq b_{\pi(i)} \geq \ldots \geq b_{\pi^j(i)} \geq b_i,\ b_i^{'} \leq b_{\pi(i)}^{'} \leq \ldots \leq b_{\pi^j(i)}^{'} \leq b_i^{'} \\ c_i \geq c_{\pi(i)} \geq \ldots \geq c_{\pi^j(i)} \geq c_i,\ c_i^{'} \leq c_{\pi(i)}^{'} \leq \ldots \leq c_{\pi^j(i)}^{'} \leq c_i^{'} \end{gathered}$$

where π^{j+1} identity. Hence

$$a_i = a_{\pi(i)},\ b_i = b_{\pi(i)},\ c_i = c_{\pi(i)},\ a_i^{'} = a_{\pi(i)}^{'},\ b_i^{'} = b_{\pi(i)}^{'},\ c_i^{'} = c_{\pi(i)}^{'}$$

$\forall\ i = 1, 2, 3, \ldots, n$. Thus we conclude that

$$\begin{gathered} a_i \leq a_{\pi(i+1)} = a_{i+1},\ b_i \geq b_{\pi(i+1)} = b_{i+1},\ c_i \geq c_{\pi(i+1)} = c_{i+1} \\ a_i^{'} \geq a_{\pi(i+1)}^{'} = a_{i+1}^{'},\ b_i^{'} \leq b_{\pi(i+1)}^{'} = b_{i+1}^{'},\ c_i^{'} \leq c_{\pi(i+1)}^{'} = c_{i+1}^{'} \end{gathered}$$

which together with

$$a_{n+1} = a_1,\ b_{n+1} = b_1,\ c_{n+1} = c_1,\ a'_{n+1} = a'_1,\ b'_{n+1} = b'_1,\ c'_{n+1} = c'_1$$

which implies that

$$a_i = a_1,\ b_i = b_1,\ c_i = c_1,\ a'_i = a'_1,\ b'_i = b'_1,\ c'_i = c'_1$$

$\forall\ i = 1, 2, 3, \ldots, n$. Thus we have

$$\begin{aligned} a_1 = a_2 = \ldots = a_n = p_1 = p_2 = \ldots = p_n \\ a'_1 = a'_2 = \ldots = a'_n = p'_1 = p'_2 = \ldots = p'_n \\ b_1 = b_2 = \ldots = b_n = q_1 = q_2 = \ldots = q_n \\ b'_1 = b'_2 = \ldots = b'_n = q'_1 = q'_2 = \ldots = q'_n \\ c_1 = c_2 = \ldots = c_n = r_1 = r_2 = \ldots = r_n \\ c'_1 = c'_2 = \ldots = c'_n = r'_1 = r'_2 = \ldots = r'_n \end{aligned}$$

Therefore (a) and (b) holds, since converse of result (a) is straight forward. □

4. Conclusion

The neutrosophic graphs have many applications in path problems, networks and computer science. Strong BSVNG and complete BSVNG are the types of BSVNG. In this paper, we discussed the special types of BSVNGs, subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs of the given BSVNGs. We investigated isomorphism properties of subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs.

References

[1] M. Akram, Single-Valued Neutrosophic Planar Graphs, International Journal of Algebra and Statistics 2 (2016) 157–167.

[2] M. Akram and S. Shahzadi, Neutrosophic soft graphs with application, Journal of Intelligent and Fuzzy Systems DOI:10.3233/JIFS-16090 (2016) 1–18.

[3] M. Akram and S. Shahzadi, Representation of graphs using intuitionistic neutrosophic soft sets, Journal of Mathematical Analysis 7 (2016) 1–23.

[4] S. Ashraf, S. Naz, H. Rashmanlou and M. A. Malik, Regularity of graphs in single valued neutrosophic environment (2016) (In press).

[5] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[6] K. Atanassov and G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (1989) 343–349.

[7] S. Broumi, M. Talea, A. Bakali and F. Smarandache, Single Valued Neutrosophic Graphs, Journal of New Theory 10 (2016) 86–101.

[8] S. Broumi, M. Talea, F. Smarandache and A. Bakali, Single Valued Neutrosophic Graphs: Degree, Order and Size. IEEE World Congress on Computational Intelligence (2016) 2444–2451.

[9] S. Broumi, M. Talea, A. Bakali and F. Smarandache, On Bipolar Single Valued Neutrosophic Graphs, Journal of New Theory 11 (2016) 84–102.

[10] S. Broumi, F. Smarandache, M. Talea and A. Bakali, An Introduction to Bipolar Single Valued Neutrosophic Graph Theory. Applied Mechanics and Materials 841 (2016) 184–191.

[11] S. Broumi, A. Bakali, M. Talea, F. Smarandache and L. Vladareanu, Applying Dijkstra algorithm for solving neutrosophic shortest path problem, Proceedings of the 2016 International Conference on Advanced Mechatronic Systems, Melbourne, Australia (2016) 412–416.

[12] S. Broumi, A. Bakali, M. Talea and F.Smarandache, An Isolated Interval Valued Neutrosophic Graphs, Critical Review XIII (2016) 67–80.
[13] S. Broumi, F. Smarandache, M. Talea and A. Bakali, Decision-Making Method Based On the Interval Valued Neutrosophic Graph, IEEE Confernce Future Technologie (2016) 44–50.
[14] S. Broumi, A. Bakali, M. Talea, F. Smarandache and M. Ali, Shortest Path Problem under Bipolar Neutrosphic Setting, Applied Mechanics and Materials 859 (2016) 59–66.
[15] S. Broumi, A. Bakali, M, Talea and F, Smarandache, Isolated Single Valued Neutrosophic Graphs. Neutrosophic Sets and Systems 11 (2016) 74–78.
[16] S. Fathi, H. Elchawalby and A. A. Salama, A neutrosophic graph similarity measures, Chapter in Book entitled by: New Trends in Neutrosophic Theory and Applications- Florentin Smarandache and Surpati Pramanik (Editors) (2016) 223–230.
[17] A. Hassan and M. A. Malik, The classes of bipolar single valued neutrosophic graphs, TWMS Journal of Applied and Engineering Mathematics (2016) (In press).
[18] A. Hassan, M. A. Malik, S. Broumi and F. Smarandache, Regular single valued neutrosophic hypergraphs, Neutrosophic Sets and Systems 13 (2016) 84–89.
[19] A. Hassan and M. A. Malik, Generalized bipolar single valued neutrosophic hypergraphs, TWMS Journal of Applied and Engineering Mathematics (2016) (In press).
[20] A. Hassan and M. A. Malik, Generalized neutrosophic hypergraphs, TWMS Journal of Applied and Engineering Mathematics (2016) (In press).
[21] A. Hassan, M. A. Malik, S. Broumi and F. Smarandache, Regular bipolar single valued neutrosophic hypergraphs, Neutrosophic Sets and Systems (2016) (In press).
[22] A. Hassan and M. A. Malik, Studies on neutrosophic graphs, TWMS Journal of Applied and Engineering Mathematics (2016) (In press).
[23] A. Hassan and M. A. Malik, Special types of single valued neutrosophic graphs, TWMS Journal of Applied and Engineering Mathematics (2016) (In press).
[24] A. Hassan and M. A. Malik, Special types of interval valued neutrosophic graphs, Punjab University Journal of Mathematics (2017) (submitted).
[25] A. Hassan and M. A. Malik, The m-Polar single valued neutrosophic graphs, TWMS Journal of Applied and Engineering Mathematics (2016) (In press).
[26] A. Hassan, M. A. Malik, S. Broumi, M. Talea, A. Bakali and F. Smarandache, Isomorphism on single valued neutrosophic Hypergraphs, Critical Review XIII (2016) 19–40.
[27] A. Hassan, M. A. Malik, S. Broumi, M. Talea, A. Bakali and F. Smarandache, Isomorphism on interval valued neutrosophic Hypergraphs, Critical Review XIII (2016) 41–65.
[28] A. Hassan, M. A. Malik, S. Broumi, M. Talea, A. Bakali and F. Smarandache, Isomorphism on bipolar single valued neutrosophic Hypergraphs, Critical Review XIII (2016) 79–102.
[29] K. Hur, P. K. Lim, J. G. Lee and J. Kim, The category of neutrosophic crisp sets, Ann. Fuzzy Math. Inform. (2017) 12 pages.
[30] P. D. Liu and Y. M. Wang, Multiple Attribute Decision-Making Method Based on Single Valued Neutrosophic Normalized Weighted Bonferroni Mean, Neural Computing and Applications (2014) 2001–2010.
[31] P. D. Liu and Y. C. Chu, Y. W. Li and Y. B. Chen, Some generalized neutrosophic number Hamacher aggregation operators and their application to Group Decision Making, International Journal of Fuzzy Systems 16 (2014) 242–255.
[32] P. D. Liu and Y. M. Wang, Interval neutrosophic prioritized OWA operator and its application to multiple attribute decision making, Journal of Systems Science and Complexity 29 (2016) 681–697.
[33] P. D. Liu and L. L. Shi, The Generalized Hybrid Weighted Average Operator Based on Interval Neutrosophic Hesitant Set and Its Application to Multiple Attribute Decision Making, Neural Computing and Applications 26 (2015) 457–471.
[34] P. D. Liu and G. L. Tang, Some power generalized aggregation operators based on the interval neutrosophic numbers and their application to decision making, Journal of Intelligent and Fuzzy Systems 30 (2016) 2517–2528.
[35] P. D. Liu and H. G. Li, Multiple attribute decision making method based on some normal neutrosophic Bonferroni mean operators, Neural Computing and Applications 28 (2017) 179–194.

[36] P. D. Liu and F. Teng, Multiple attribute decision making method based on normal neutrosophic generalized weighted power averaging operator, international journal of machine learning and cybernetics 10.1007/s13042-015-0385-y.
[37] P. D. Liu and L. L. Shi, Some Neutrosophic Uncertain Linguistic Number Heronian Mean Operators and Their Application to Multi-attribute Group Decision making, Neural Computing and Applications doi:10.1007/s00521-015-2122-6.
[38] M. A. Malik and A. Hassan, Single valued neutrosophic trees, TWMS Journal of Applied and Engineering Mathematics (2016) (In press).
[39] S. Mehra and M. Singh, Single valued neutrosophic signedgarphs,International Journal of computer Applications 9 (2017) 31–37.
[40] N. Shah and A. Hussain, Neutrosophic Soft Graphs, Neutrosophic Sets and Systems 11 (2016) 31–44.
[41] N. Shah and S. Broumi, Irregular Neutrosophic Graphs, Neutrosophic Sets and Systems 13 (2016) 47–55.
[42] A. Shannon and K. Atanassov, A First Step to a Theory of the Intuitionistic Fuzzy Graphs, Proc. of the First Workshop on Fuzzy Based Expert Systems (D. akov, Ed.), Sofia (1994) 59–61.
[43] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set and Logic, ProQuest Information and Learning, Ann Arbor, Michigan, USA, http://fs.gallup.unm.edu/eBookneutrosophics6.pdf (last edition online).
[44] F. Smarandache, Refined literal indeterminacy and the multiplication law of sub indeterminacies, Neutrosophic Sets and Systems 9 (2015) 58–63.
[45] F. Smarandache, Symbolic Neutrosophic Theory (Europanova asbl, Brussels, Belgium (2015) 195–200.
[46] F. Smarandache, Neutrosophic overset, neutrosophic underset, Neutrosophic offset, Similarly for Neutrosophic Over-/Under-/OffLogic, Probability, and Statistic, Pons Editions, Brussels (2016) 170–170.
[47] I. Turksen, Interval valued fuzzy sets based on normal forms, Fuzzy Sets and Systems 20 (1986) 191–210.
[48] V. Ulucay, M. Sahin, S Broumi, A. Bakali, M. Talea and F. Smarandache, Decision-Making Method based on Neutrosophic Soft Expert Graphs, (submited).
[49] H. Wang, F. Smarandache, Y. Zhang and R. Sunderraman, Single valued Neutrosophic Sets, Multisspace and Multistructure, 4 (2010) 410–413.
[50] L. Zadeh, Fuzzy sets, Inform and Control 8 (1965) 338–353.
[51] More information on http://fs.gallup.unm.edu/NSS/.

Ali Hassan (alihassan.iiui.math@gmail.com)
Department of Mathematics, University of Punjab (New campus), Lahore (Pakistan)

Muhammad Aslam Malik (aslam@math.pu.edu.pk)
Department of Mathematics, University of Punjab (New campus), Lahore (Pakistan)

Said Broumi (broumisaid78@gmail.com)
Laboratory of Information processing, Faculty of Science Ben MÆSik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco

Assia Bakali (assiabakali@yahoo.fr)
Ecole Royale Navale, Boulevard Sour Jdid, B.P 16303 Casablanca, Morocco

Mohamed Talea (taleamohamed@yahoo.fr)
Laboratory of Information processing, Faculty of Science Ben MÆSik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco

Florentin Smarandache (fsmarandache@gmail.com)

Department of Mathematics, University of New Mexico,705 Gurley Avenue, Gallup, NM 87301, USA

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 1, (July 2017), pp. 75–86
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Neutrosophic subalgebras of several types in BCK/BCI-algebras

Young Bae Jun

Received 24 March 2017; Revised 12 April 2017; Accepted 10 May 2017

Abstract. Given $\Phi, \Psi \in \{\in, q, \in \vee q\}$, the notion of $(\Phi,\ \Psi)$-neutrosophic subalgebras of a BCK/BCI-algebra are introduced, and related properties are investigated. Characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in \vee q)$-neutrosophic subalgebra are provided. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets, conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets to be subalgebras are discussed. Conditions for a neutrosophic set to be a $(q, \in \vee q)$-neutrosophic subalgebra are considered.

2010 AMS Classification: 06F35, 03B60, 03B52.

Keywords: Neutrosophic set, neutrosophic $\in$-subset, neutrosophic q-subset, neutrosophic $\in \vee q$-subset, $(\in, \in)$-neutrosophic subalgebra, $(\in, q)$-neutrosophic subalgebra, $(q, \in)$-neutrosophic subalgebra, (q, q)-neutrosophic subalgebra, $(\in, \in \vee q)$-neutrosophic subalgebra, $(q, \in \vee q)$-neutrosophic subalgebra,

Corresponding Author: Y. B. Jun (skywine@gmail.com)

1. Introduction

The concept of neutrosophic set (NS) developed by Smarandache [5, 6, 7] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part. For further particulars I refer readers to the site http://fs.gallup.unm.edu/neutrosophy.htm. Agboola et al. [1] studied neutrosophic ideals of neutrosophic BCI-algebras. Agboola et al. [2] also introduced the concept of neutrosophic BCI/BCK-algebras, and presented elementary properties of neutrosophic BCI/BCK-algebras.

In this paper, we introduce the notion of $(\Phi,\ \Psi)$-neutrosophic subalgebra of a BCK/BCI-algebra X for $\Phi, \Psi \in \{\in, q, \in \vee q\}$, and investigate related properties.

We provide characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in\vee q)$-neutrosophic subalgebra. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets, we provide conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets to be subalgebras. We consider conditions for a neutrosophic set to be a $(q, \in\vee q)$-neutrosophic subalgebra.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \Rightarrow x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [3] and [4] for further information regarding BCK/BCI-algebras.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [6]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

3. Neutrosophic subalgebras of several types

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$T_\in(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha\},$$
$$I_\in(A; \beta) := \{x \in X \mid A_I(x) \geq \beta\},$$
$$F_\in(A; \gamma) := \{x \in X \mid A_F(x) \leq \gamma\},$$
$$T_q(A; \alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},$$
$$I_q(A; \beta) := \{x \in X \mid A_I(x) + \beta > 1\},$$
$$F_q(A; \gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},$$
$$T_{\in\vee q}(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},$$
$$I_{\in\vee q}(A; \beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},$$
$$F_{\in\vee q}(A; \gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.$$

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are *neutrosophic $\in$-subsets*; $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are *neutrosophic q-subsets*; and $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are *neutrosophic $\in\vee\, q$-subsets*. For $\Phi \in \{\in, q, \in\vee\, q\}$, the element of $T_\Phi(A;\alpha)$ (resp., $I_\Phi(A;\beta)$ and $F_\Phi(A;\gamma)$) is called a *neutrosophic T_Φ-point* (resp., *neutrosophic I_Φ-point* and *neutrosophic F_Φ-point*) with value α (resp., β and γ). It is clear that

$$T_{\in\vee\, q}(A;\alpha) = T_\in(A;\alpha) \cup T_q(A;\alpha), \tag{3.1}$$

$$I_{\in\vee\, q}(A;\beta) = I_\in(A;\beta) \cup I_q(A;\beta), \tag{3.2}$$

$$F_{\in\vee\, q}(A;\gamma) = F_\in(A;\gamma) \cup F_q(A;\gamma). \tag{3.3}$$

Proposition 3.1. *For any neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we have*

$$\alpha \in [0, 0.5] \;\Rightarrow\; T_{\in\vee\, q}(A;\alpha) = T_\in(A;\alpha), \tag{3.4}$$

$$\beta \in [0, 0.5] \;\Rightarrow\; I_{\in\vee\, q}(A;\beta) = I_\in(A;\beta), \tag{3.5}$$

$$\gamma \in [0.5, 1] \;\Rightarrow\; F_{\in\vee\, q}(A;\gamma) = F_\in(A;\gamma), \tag{3.6}$$

$$\alpha \in (0.5, 1] \;\Rightarrow\; T_{\in\vee\, q}(A;\alpha) = T_q(A;\alpha), \tag{3.7}$$

$$\beta \in (0.5, 1] \;\Rightarrow\; I_{\in\vee\, q}(A;\beta) = I_q(A;\beta), \tag{3.8}$$

$$\gamma \in [0, 0.5) \;\Rightarrow\; F_{\in\vee\, q}(A;\gamma) = F_q(A;\gamma). \tag{3.9}$$

Proof. If $\alpha \in [0, 0.5]$, then $1 - \alpha \in [0.5, 1]$ and $\alpha \leq 1 - \alpha$. It is clear that $T_\in(A;\alpha) \subseteq T_{\in\vee\, q}(A;\alpha)$ by (3.1). If $x \notin T_\in(A;\alpha)$, then $A_T(x) < \alpha \leq 1 - \alpha$, i.e., $x \notin T_q(A;\alpha)$. Hence $x \notin T_{\in\vee\, q}(A;\alpha)$, and so $T_{\in\vee\, q}(A;\alpha) \subseteq T_\in(A;\alpha)$. Thus (3.4) is valid. Similarly, we have the result (3.5). If $\gamma \in [0.5, 1]$, then $1 - \gamma \in [0, 0.5]$ and $\gamma \geq 1 - \gamma$. It is clear that $F_\in(A;\gamma) \subseteq F_{\in\vee\, q}(A;\gamma)$ by (3.3). Let $z \in F_{\in\vee\, q}(A;\gamma)$. Then $z \in F_\in(A;\gamma)$ or $z \in F_q(A;\gamma)$. If $z \notin F_\in(A;\gamma)$, then $A_F(z) > \gamma \geq 1 - \gamma$, i.e., $A_F(z) + \gamma > 1$. Thus $z \notin F_q(A;\gamma)$, and so $z \notin F_{\in\vee\, q}(A;\gamma)$. This is a contradiction. Hence $z \in F_\in(A;\gamma)$, and therefore $F_{\in\vee\, q}(A;\gamma) \subseteq F_\in(A;\gamma)$. Let $\beta \in (0.5, 1]$. Then $\beta > 1 - \beta$. Note that $I_q(A;\beta) \subseteq I_{\in\vee\, q}(A;\beta)$ by (3.2). Let $y \in I_{\in\vee\, q}(A;\beta)$. Then $y \in I_\in(A;\beta)$ or $y \in I_q(A;\beta)$. If $y \notin I_q(A;\beta)$, then $A_I(y) + \beta \leq 1$ and so $A_I(y) \leq 1 - \beta < \beta$, i.e., $y \notin I_\in(A;\beta)$. Thus $y \notin I_{\in\vee\, q}(A;\beta)$, a contradiction. Hence $y \in I_q(A;\beta)$. Therefore $I_{\in\vee\, q}(A;\beta) \subseteq I_q(A;\beta)$. This shows that (3.8) is true. The result (3.7) is proved by the similar way. Let $\gamma \in [0, 0.5)$ and $z \in F_{\in\vee\, q}(A;\gamma)$. Then $1 - \gamma > \gamma$ and $z \in F_\in(A;\gamma)$ or $z \in F_q(A;\gamma)$. If $z \notin F_q(A;\gamma)$, then $A_F(z) + \gamma \geq 1$ and so $A_F(z) \geq 1 - \gamma > \gamma$, i.e., $z \notin F_\in(A;\gamma)$. Thus $z \notin F_{\in\vee\, q}(A;\gamma)$, which is a contradiction. Hence $F_{\in\vee\, q}(A;\gamma) \subseteq F_q(A;\gamma)$. The reverse inclusion is by (3.3). □

Definition 3.2. Given $\Phi, \Psi \in \{\in, q, \in\vee\, q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called a $(\Phi,\, \Psi)$*-neutrosophic subalgebra* of X if the following assertions are valid.

$$\begin{aligned} &x \in T_\Phi(A;\alpha_x),\ y \in T_\Phi(A;\alpha_y) \;\Rightarrow\; x * y \in T_\Psi(A;\alpha_x \wedge \alpha_y),\\ &x \in I_\Phi(A;\beta_x),\ y \in I_\Phi(A;\beta_y) \;\Rightarrow\; x * y \in I_\Psi(A;\beta_x \wedge \beta_y),\\ &x \in F_\Phi(A;\gamma_x),\ y \in F_\Phi(A;\gamma_y) \;\Rightarrow\; x * y \in F_\Psi(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.10}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y, \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Theorem 3.3. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \geq A_T(x) \wedge A_T(y) \\ A_I(x * y) \geq A_I(x) \wedge A_I(y) \\ A_F(x * y) \leq A_F(x) \vee A_F(y) \end{array} \right). \tag{3.11}$$

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X. If there exist $x, y \in X$ such that $A_T(x * y) < A_T(x) \wedge A_T(y)$, then

$$A_T(x * y) < \alpha_t \leq A_T(x) \wedge A_T(y)$$

for some $\alpha_t \in (0, 1]$. It follows that $x, y \in T_\in(A; \alpha_t)$ but $x * y \notin T_\in(A; \alpha_t)$. Hence $A_T(x * y) \geq A_T(x) \wedge A_T(y)$ for all $x, y \in X$. Similarly, we show that

$$A_I(x * y) \geq A_I(x) \wedge A_I(y)$$

for all $x, y \in X$. Suppose that there exist $a, b \in X$ and $\gamma_f \in [0, 1]$ be such that $A_F(a * b) > \gamma_f \geq A_F(a) \vee A_F(b)$. Then $a, b \in F_\in(A; \gamma_f)$ and $a * b \notin F_\in(A; \gamma_f)$, which is a contradiction. Therefore $A_F(x * y) \leq A_F(x) \vee A_F(y)$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which satisfies the condition (3.11). Let $x, y \in X$ be such that $x \in T_\in(A; \alpha_x)$ and $y \in T_\in(A; \alpha_y)$. Then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$, which imply that $A_T(x*y) \geq A_T(x) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y$, that is, $x * y \in T_\in(A; \alpha_x \wedge \alpha_y)$. Similarly, if $x \in I_\in(A; \beta_x)$ and $y \in I_\in(A; \beta_y)$ then $x * y \in I_\in(A; \beta_x \wedge \beta_y)$. Now, let $x \in F_\in(A; \gamma_x)$ and $y \in F_\in(A; \gamma_y)$ for $x, y \in X$. Then $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$, and so $A_F(x * y) \leq A_F(x) \vee A_F(y) \leq \gamma_x \vee \gamma_y$. Hence $x * y \in F_\in(A; \gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X. □

Theorem 3.4. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then neutrosophic q-subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ whenever they are nonempty.*

Proof. Let $x, y \in T_q(A; \alpha)$. Then $A_T(x) + \alpha > 1$ and $A_T(y) + \alpha > 1$. It follows that

$$\begin{aligned} A_T(x * y) + \alpha &\geq (A_T(x) \wedge A_T(y)) + \alpha \\ &= (A_T(x) + \alpha) \wedge (A_T(y) + \alpha) > 1 \end{aligned}$$

and so that $x * y \in T_q(A; \alpha)$. Hence $T_q(A; \alpha)$ is a subalgebra of X. Similarly, we can prove that $I_q(A; \beta)$ is a subalgebra of X. Now let $x, y \in F_q(A; \gamma)$. Then $A_F(x) + \gamma < 1$ and $A_F(y) + \gamma < 1$, which imply that

$$\begin{aligned} A_F(x * y) + \gamma &\leq (A_F(x) \vee A_F(y)) + \gamma \\ &= (A_F(x) + \alpha) \vee (A_F(y) + \alpha) < 1. \end{aligned}$$

Hence $x * y \in F_q(A; \gamma)$ and $F_q(A; \gamma)$ is a subalgebra of X. □

Theorem 3.5. *If $A = (A_T, A_I, A_F)$ is a $(q, \in \vee\, q)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then neutrosophic q-subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0, 5)$ whenever they are nonempty.*

Proof. Let $x, y \in T_q(A; \alpha)$. Then $x * y \in T_{\in\vee q}(A; \alpha)$, and so $x * y \in T_\in(A; \alpha)$ or $x*y \in T_q(A; \alpha)$. If $x*y \in T_\in(A; \alpha)$, then $A_T(x*y) \geq \alpha > 1-\alpha$ since $\alpha > 0.5$. Hence $x * y \in T_q(A; \alpha)$. Therefore $T_q(A; \alpha)$ is a subalgebra of X. Similarly, we prove that $I_q(A; \beta)$ is a subalgebra of X. Let $x, y \in F_q(A; \gamma)$. Then $x * y \in F_{\in\vee q}(A; \gamma)$, and so $x * y \in F_\in(A; \gamma)$ or $x * y \in F_q(A; \gamma)$. If $x * y \in F_\in(A; \gamma)$, then $A_F(x * y) \leq \gamma < 1 - \gamma$ since $\gamma \in [0, 0, 5)$. Hence $x * y \in F_q(A; \gamma)$, and therefore $F_q(A; \gamma)$ is a subalgebra of X. □

We provide characterizations of an $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 3.6. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in\vee q)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(3.12) \qquad (\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} \\ A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y).0.5\} \\ A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \end{array} \right).$$

Proof. Suppose that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X and let $x, y \in X$. If $A_T(x) \wedge A_T(y) < 0.5$, then $A_T(x * y) \geq A_T(x) \wedge A_T(y)$. For, assume that $A_T(x * y) < A_T(x) \wedge A_T(y)$ and choose α_t such that

$$A_T(x * y) < \alpha_t < A_T(x) \wedge A_T(y).$$

Then $x \in T_\in(A; \alpha_t)$ and $y \in T_\in(A; \alpha_t)$ but $x * y \notin T_\in(A; \alpha_t)$. Also $A_T(x * y) + \alpha_t < 1$, i.e., $x * y \notin T_q(A; \alpha_t)$. Thus $x * y \notin T_{\in\vee q}(A; \alpha_t)$, a contradiction. Therefore $A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$ whenever $A_T(x) \wedge A_T(y) < 0.5$. Now suppose that $A_T(x) \wedge A_T(y) \geq 0.5$. Then $x \in T_\in(A; 0.5)$ and $y \in T_\in(A; 0.5)$, which imply that $x*y \in T_{\in\vee q}(A; 0.5)$. Hence $A_T(x*y) \geq 0.5$. Otherwise, $A_T(x*y)+0.5 < 0.5+0.5 = 1$, a contradiction. Consequently, $A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$ for all $x, y \in X$. Similarly, we know that $A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}$ for all $x, y \in X$. Suppose that $A_F(x) \vee A_F(y) > 0.5$. If $A_F(x*y) > A_F(x) \vee A_F(y) := \gamma_f$, then $x, y \in F_\in(A; \gamma_f)$, $x * y \notin F_\in(A; \gamma_f)$ and $A_F(x * y) + \gamma_f > 2\gamma_f > 1$, i.e., $x * y \notin F_q(A; \gamma_f)$. This is a contradiction. Hence $A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ whenever $A_F(x) \vee A_F(y) > 0.5$. Now, assume that $A_F(x) \vee A_F(y) \leq 0.5$. Then $x, y \in F_\in(A; 0.5)$ and so $x*y \in F_{\in\vee q}(A; 0.5)$. Thus $A_F(x*y) \leq 0.5$ or $A_F(x*y)+0.5 < 1$. If $A_F(x*y) > 0.5$, then $A_F(x * y) + 0.5 > 0.5 + 0.5 = 1$, a contradiction. Thus $A_F(x * y) \leq 0.5$, and so $A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ whenever $A_F(x) \vee A_F(y) \leq 0.5$. Therefore $A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which satisfies the condition (3.12). Let $x, y \in X$ and α_x, α_y, β_x, β_y, $\gamma_x, \gamma_y \in [0, 1]$. If $x \in T_\in(A; \alpha_x)$ and $y \in T_\in(A; \alpha_y)$, then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$. If $A_T(x * y) < \alpha_x \wedge \alpha_y$, then $A_T(x) \wedge A_T(y) \geq 0.5$. Otherwise, we have

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} = A_T(x) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y,$$

a contradiction. It follows that

$$A_T(x * y) + \alpha_x \wedge \alpha_y > 2A_T(x * y) \geq 2 \bigwedge\{A_T(x), A_T(y), 0.5\} = 1$$

and so that $x * y \in T_q(A; \alpha_x \wedge \alpha_y) \subseteq T_{\in\vee q}(A; \alpha_x \wedge \alpha_y)$. Similarly, if $x \in I_\in(A; \beta_x)$ and $y \in I_\in(A; \beta_y)$, then $x * y \in I_{\in\vee q}(A; \beta_x \wedge \beta_y)$. Now, let $x \in F_\in(A; \gamma_x)$ and

$y \in F_{\in}(A; \gamma_y)$. Then $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$. If $A_F(x * y) > \gamma_x \vee \gamma_y$, then $A_F(x) \vee A_F(y) \leq 0.5$ because if not, then

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq A_F(x) \vee A_F(y) \leq \gamma_x \vee \gamma_y,$$

which is a contradiction. Hence

$$A_F(x * y) + \gamma_x \vee \gamma_y < 2A_F(x * y) \leq 2\bigvee\{A_F(x), A_F(y), 0.5\} = 1,$$

and so $x * y \in F_q(A; \gamma_x \vee \gamma_y) \subseteq F_{\in\vee\, q}(A; \gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of X. □

Theorem 3.7. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then neutrosophic q-subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ whenever they are nonempty.*

Proof. Assume that $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are nonempty for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. Let $x, y \in T_q(A; \alpha)$. Then $A_T(x)+\alpha > 1$ and $A_T(y)+\alpha > 1$. It follows from Theorem 3.6 that

$$\begin{aligned} A_T(x * y) + \alpha &\geq \bigwedge\{A_T(x), A_T(y), 0.5\} + \alpha \\ &= \bigwedge\{A_T(x) + \alpha, A_T(y) + \alpha, 0.5 + \alpha\} \\ &> 1, \end{aligned}$$

that is, $x * y \in T_q(A; \alpha)$. Hence $T_q(A; \alpha)$ is a subalgebra of X. By the similar way, we can induce that $I_q(A; \beta)$ is a subalgebra of X. Now, let $x, y \in F_q(A; \gamma)$. Then $A_F(x) + \gamma < 1$ and $A_F(y) + \gamma < 1$. Using Theorem 3.6, we have

$$\begin{aligned} A_F(x * y) + \gamma &\leq \bigvee\{A_F(x), A_F(y), 0.5\} + \gamma \\ &= \bigvee\{A_F(x) + \gamma, A_F(y) + \gamma, 0.5 + \gamma\} \\ &< 1, \end{aligned}$$

and so $x * y \in F_q(A; \gamma)$. Therefore $F_q(A; \gamma)$ is a subalgebra of X. □

Theorem 3.8. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A; \alpha)$, $I_{\in\vee\, q}(A; \beta)$ and $F_{\in\vee\, q}(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of X.*

Proof. Let $T_{\in\vee\, q}(A; \alpha)$ be a subalgebra of X and assume that

$$A_T(x * y) < \bigwedge\{A_T(x), A_T(y), 0.5\}$$

for some $x, y \in X$. Then there exists $\alpha \in (0, 0.5]$ such that

$$A_T(x * y) < \alpha \leq \bigwedge\{A_T(x), A_T(y), 0.5\}.$$

It follows that $x, y \in T_{\in}(A; \alpha) \subseteq T_{\in\vee\, q}(A; \alpha)$, and so that $x * y \in T_{\in\vee\, q}(A; \alpha)$. Hence $A_T(x * y) \geq \alpha$ or $A_T(x * y) + \alpha > 1$. This is a contradiction, and so

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$$

for all $x, y \in X$. Similarly, we show that

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}$$

for all $x, y \in X$. Now let $F_{\in\vee\, q}(A;\gamma)$ be a subalgebra of X and assume that

$$A_F(x * y) > \bigvee\{A_F(x), A_F(y), 0.5\}$$

for some $x, y \in X$. Then

$$A_F(x * y) > \gamma \geq \bigvee\{A_F(x), A_F(y), 0.5\}, \tag{3.13}$$

for some $\gamma \in [0.5, 1)$, which implies that $x, y \in F_\in(A;\gamma) \subseteq F_{\in\vee\, q}(A;\gamma)$. Thus $x*y \in F_{\in\vee\, q}(A;\gamma)$. From (3.13), we have $x*y \notin F_\in(A;\gamma)$ and $A_F(x*y)+\gamma > 2\gamma \geq 1$, i.e., $x * y \notin F_q(A;\gamma)$. This is a contradiction, and hence

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$$

for all $x, y \in X$. Using Theorem 3.6, we know that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of X. □

Theorem 3.9. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Proof. Assume that $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are nonempty for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Let $x, y \in I_{\in\vee\, q}(A;\beta)$. Then

$$x \in I_\in(A;\beta) \text{ or } x \in I_q(A;\beta),$$

and

$$y \in I_\in(A;\beta) \text{ or } y \in I_q(A;\beta).$$

Hence we have the following four cases:

(i) $x \in I_\in(A;\beta)$ and $y \in I_\in(A;\beta)$,
(ii) $x \in I_\in(A;\beta)$ and $y \in I_q(A;\beta)$,
(iii) $x \in I_q(A;\beta)$ and $y \in I_\in(A;\beta)$,
(iv) $x \in I_q(A;\beta)$ and $y \in I_q(A;\beta)$.

The first case implies that $x * y \in I_{\in\vee\, q}(A;\beta)$. For the second case, $y \in I_q(A;\beta)$ induces $A_I(y) > 1-\beta \geq \beta$, that is, $y \in I_\in(A;\beta)$. Thus $x*y \in I_{\in\vee\, q}(A;\beta)$. Similarly, the third case implies $x*y \in I_{\in\vee\, q}(A;\beta)$. The last case induces $A_I(x) > 1-\beta \geq \beta$ and $A_I(y) > 1-\beta \geq \beta$, that is, $x \in I_\in(A;\beta)$ and $y \in I_\in(A;\beta)$. Hence $x*y \in I_{\in\vee\, q}(A;\beta)$. Therefore $I_{\in\vee\, q}(A;\beta)$ is a subalgebra of X for all $\beta \in (0, 0.5]$. By the similar way, we show that $T_{\in\vee\, q}(A;\alpha)$ is a subalgebra of X for all $\alpha \in (0, 0.5]$. Let $x, y \in F_{\in\vee\, q}(A;\gamma)$. Then

$$A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1,$$

and

$$A_F(y) \leq \gamma \text{ or } A_F(y) + \gamma < 1.$$

If $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$, then

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{\gamma, 0.5\} = \gamma$$

by Theorem 3.6, and so $x*y \in F_{\in}(A;\gamma) \subseteq F_{\in\vee\, q}(A;\gamma)$. If $A_F(x) \leq \gamma$ and $A_F(y)+\gamma < 1$, then

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{\gamma, 1-\gamma, 0.5\} = \gamma$$

by Theorem 3.6. Thus $x*y \in F_{\in}(A;\gamma) \subseteq F_{\in\vee\, q}(A;\gamma)$. Similarly, if $A_F(x)+\gamma < 1$ and $A_F(y) \leq \gamma$, then $x*y \in F_{\in\vee\, q}(A;\gamma)$. Finally, assume that $A_F(x)+\gamma < 1$ and $A_F(y)+\gamma < 1$. Then

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{1-\gamma, 0.5\} = 0.5 < \gamma$$

by Theorem 3.6. Hence $x*y \in F_{\in}(A;\gamma) \subseteq F_{\in\vee\, q}(A;\gamma)$. Consequently, $F_{\in\vee\, q}(A;\gamma)$ is a subalgebra of X for all $\gamma \in [0.5, 1)$. □

Theorem 3.10. *If $A = (A_T, A_I, A_F)$ is a $(q, \in\vee\, q)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$.*

Proof. Assume that $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are nonempty for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. Let $x, y \in T_{\in\vee\, q}(A;\alpha)$. Then

$$x \in T_{\in}(A;\alpha) \text{ or } x \in T_q(A;\alpha).$$

and

$$y \in T_{\in}(A;\alpha) \text{ or } y \in T_q(A;\alpha).$$

If $x \in T_q(A;\alpha)$ and $y \in T_q(A;\alpha)$, then obviously $x*y \in T_{\in\vee\, q}(A;\alpha)$. Suppose that $x \in T_{\in}(A;\alpha)$ and $y \in T_q(A;\alpha)$. Then $A_T(x)+\alpha \geq 2\alpha > 1$, i.e., $x \in T_q(A;\alpha)$. It follows that $x*y \in T_{\in\vee\, q}(A;\alpha)$. Similarly, if $x \in T_q(A;\alpha)$ and $y \in T_{\in}(A;\alpha)$, then $x*y \in T_{\in\vee\, q}(A;\alpha)$. Now, let $x, y \in F_{\in\vee\, q}(A;\gamma)$. Then

$$x \in F_{\in}(A;\gamma) \text{ or } x \in F_q(A;\gamma),$$

and

$$y \in F_{\in}(A;\gamma) \text{ or } y \in F_q(A;\gamma).$$

If $x \in F_q(A;\gamma)$ and $y \in F_q(A;\gamma)$, then clearly $x*y \in F_{\in\vee\, q}(A;\gamma)$. If $x \in F_{\in}(A;\gamma)$ and $y \in F_q(A;\gamma)$, then $A_F(x)+\gamma \leq 2\gamma < 1$, i.e., $x \in F_q(A;\gamma)$. It follows that $x*y \in F_{\in\vee\, q}(A;\gamma)$. Similarly, if $x \in F_q(A;\gamma)$ and $y \in F_{\in}(A;\gamma)$, then $x*y \in F_{\in\vee\, q}(A;\gamma)$. Finally, assume that $x \in F_{\in}(A;\gamma)$ and $y \in F_{\in}(A;\gamma)$. Then $A_F(x)+\gamma \leq 2\gamma < 1$ and $A_F(y)+\gamma \leq 2\gamma < 1$, that is, $x \in F_q(A;\gamma)$ and $y \in F_q(A;\gamma)$. Therefore $x*y \in F_{\in\vee\, q}(A;\gamma)$. Consequently, $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. □

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, we consider:

$$X_0^1 := \{x \in X \mid A_T(x) > 0,\ A_I(x) > 0,\ A_F(x) < 1\}.$$

Theorem 3.11. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. Suppose that $A_T(x*y) = 0$. Note that $x \in T_\in(A; A_T(x))$ and $y \in T_\in(A; A_T(y))$. But $x * y \notin T_\in(A; A_T(x) \wedge A_T(y))$ because $A_T(x * y) = 0 < A_T(x) \wedge A_T(y)$. This is a contradiction, and thus $A_T(x * y) > 0$. By the similar way, we show that $A_I(x * y) > 0$. Note that $x \in F_\in(A; A_F(x))$ and $y \in F_\in(A; A_F(y))$. If $A_F(x * y) = 1$, then $A_F(x * y) = 1 > A_F(x) \vee A_F(y)$, and so $x * y \notin F_\in(A; A_F(x) \vee A_F(y))$. This is impossible. Hence $x * y \in X_0^1$, and therefore X_0^1 is a subalgebra of X. □

Theorem 3.12. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, q)$-neutrosophic subalgebra of X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. If $A_T(x * y) = 0$, then

$$A_T(x * y) + A_T(x) \wedge A_T(y) = A_T(x) \wedge A_T(y) \leq 1.$$

Hence $x*y \notin T_q(A; A_T(x) \wedge A_T(y))$, which is a contradiction since $x \in T_\in(A; A_T(x))$ and $y \in T_\in(A; A_T(y))$. Thus $A_T(x*y) > 0$. Similarly, we get $A_I(x*y) > 0$. Assume that $A_F(x * y) = 1$. Then

$$A_F(x * y) + A_F(x) \vee A_F(y) = 1 + A_F(x) \vee A_F(y) \geq 1,$$

that is, $x * y \notin F_q(A; A_F(x) \vee A_F(y))$. This is a contradiction because of $x \in F_\in(A; A_F(x))$ and $y \in F_\in(A; A_F(y))$. Hence $A_F(x*y) < 1$. Consequently, $x*y \in X_0^1$ and X_0^1 is a subalgebra of X. □

Theorem 3.13. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is a $(q, \in)$-neutrosophic subalgebra of X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. It follows that $A_T(x)+1 > 1$, $A_T(y)+1 > 1$, $A_I(x)+1 > 1$, $A_I(y)+1 > 1$, $A_F(x)+0 < 1$ and $A_F(y)+0 < 1$. Hence $x, y \in T_q(A; 1) \cap I_q(A; 1) \cap F_q(A; 0)$. If $A_T(x*y) = 0$ or $A_I(x*y) = 0$, then $A_T(x*y) < 1 = 1 \wedge 1$ or $A_I(x*y) < 1 = 1 \wedge 1$. Thus $x*y \notin T_q(A; 1 \wedge 1)$ or $x*y \notin I_q(A; 1 \wedge 1)$, a contradiction. Hence $A_T(x*y) > 0$ and $A_I(x*y) > 0$. If $A_F(x*y) = 1$, then $x*y \notin F_q(A; 0 \vee 0)$ which is a contradiction. Thus $A_F(x * y) < 1$. Therefore $x * y \in X_0^1$ and the proof is complete. □

Theorem 3.14. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is a (q, q)-neutrosophic subalgebra of X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. Hence $A_T(x)+1 > 1$, $A_T(y)+1 > 1$, $A_I(x)+1 > 1$, $A_I(y)+1 > 1$, $A_F(x) + 0 < 1$ and $A_F(y) + 0 < 1$. Hence $x, y \in T_q(A; 1) \cap I_q(A; 1) \cap F_q(A; 0)$. If $A_T(x * y) = 0$ or $A_I(x * y) = 0$, then

$$A_T(x * y) + 1 \wedge 1 = 0 + 1 = 1$$

or

$$A_I(x * y) + 1 \wedge 1 = 0 + 1 = 1,$$

and so $x * y \notin T_q(A; 1 \wedge 1)$ or $x * y \notin I_q(A; 1 \wedge 1)$. This is impossible, and thus $A_T(x * y) > 0$ and $A_I(x * y) > 0$. If $A_F(x * y) = 1$, then $A_F(x * y) + 0 \vee 0 = 1$, that is, $x * y \notin F_q(A; 0 \vee 0)$. This is a contradiction, and so $A_F(x * y) < 1$. Therefore $x * y \in X_0^1$ and the proof is complete. □

Theorem 3.15. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is a (q, q)-neutrosophic subalgebra of X, then $A = (A_T, A_I, A_F)$ is neutrosophic constant on X_0^1, that is, A_T, A_I and A_F are constants on X_0^1.*

Proof. Assume that A_T is not constant on X_0^1. Then there exist $y \in X_0^1$ such that $\alpha_y = A_T(y) \neq A_T(0) = \alpha_0$. Then either $\alpha_y > \alpha_0$ or $\alpha_y < \alpha_0$. Suppose $\alpha_y < \alpha_0$ and choose $\alpha_1, \alpha_2 \in (0, 1]$ such that $1 - \alpha_0 < \alpha_1 \leq 1 - \alpha_y < \alpha_2$. Then $A_T(0) + \alpha_1 = \alpha_0 + \alpha_1 > 1$ and $A_T(y) + \alpha_2 = \alpha_y + \alpha_2 > 1$, which imply that $0 \in T_q(A; \alpha_1)$ and $y \in T_q(A; \alpha_2)$. Since

$$A_T(y * 0) + \alpha_1 \wedge \alpha_2 = A_T(y) + \alpha_1 = \alpha_y + \alpha_1 \leq 1,$$

we get $y * 0 \notin T_q(A; \alpha_1 \wedge \alpha_2)$, which is a contradiction. Next assume that $\alpha_y > \alpha_0$. Then $A_T(y) + (1 - \alpha_0) = \alpha_y + 1 - \alpha_0 > 1$ and so $y \in T_q(A; 1 - \alpha_0)$. Since

$$A_T(y * y) + (1 - \alpha_0) = A_T(0) + 1 - \alpha_0 = \alpha_0 + 1 - \alpha_0 = 1,$$

we have $y * y \notin T_q(A; (1 - \alpha_0) \wedge (1 - \alpha_0))$. This is impossible. Therefore A_T is constant on X_0^1. Similarly, A_I is constant on X_0^1. Finally, suppose that A_F is not constant on X_0^1. Then $\gamma_y = A_F(y) \neq A_F(0) = \gamma_0$ for some $y \in X_0^1$, and we have two cases:

$$\text{(i) } \gamma_y < \gamma_0 \text{ and (ii) } \gamma_y > \gamma_0.$$

The first case implies that $A_F(y) + 1 - \gamma_0 = \gamma_y + 1 - \gamma_0 < 1$, that is, $y \in F_q(A; 1 - \gamma_0)$. Hence $y * y \in F_q(A; (1 - \gamma_0) \vee (1 - \gamma_0))$, i.e., $0 \in F_q(A; 1 - \gamma_0)$, which is a contradiction since $A_F(0) + 1 - \gamma_0 = 1$. For the second case, there exist $\gamma_1, \gamma_2 \in (0, 1)$ such that

$$1 - \gamma_0 > \gamma_1 > 1 - \gamma_y > \gamma_2.$$

Then $A_F(y) + \gamma_2 = \gamma_y + \gamma_2 < 1$, i.e., $y \in F_q(A; \gamma_2)$, and $A_F(0) + \gamma_1 = \gamma_0 + \gamma_1 < 1$, i.e., $0 \in F_q(A; \gamma_1)$. It follows that $y * 0 \in F_q(A; \gamma_1 \vee \gamma_2)$. But

$$A_F(y * 0) + \gamma_1 \vee \gamma_2 = A_F(y) + \gamma_1 = \gamma_y + \gamma_1 > 1,$$

and so $y * 0 \notin F_q(A; \gamma_1 \vee \gamma_2)$. This is a contradiction. Therefore A_F is constant on X_0^1. This completes the proof. □

We provide conditions for a neutrosophic set to be a $(q, \in\vee\, q)$-neutrosophic subalgebra.

Theorem 3.16. *For a subalgebra S of a BCK/BCI-algebra X, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X such that*

$$(\forall x \in S)\,(A_T(x) \geq 0.5, A_I(x) \geq 0.5, A_F(x) \leq 0.5)\,, \tag{3.14}$$

$$(\forall x \in X \setminus S)\,(A_T(x) = 0, A_I(x) = 0, A_F(x) = 1)\,. \tag{3.15}$$

Then $A = (A_T, A_I, A_F)$ is a $(q, \in\vee\, q)$-neutrosophic subalgebra of X.

Proof. Assume that $x \in I_q(A; \beta_x)$ and $y \in I_q(A; \beta_y)$ for all $x, y \in X$ and $\beta_x, \beta_y \in [0, 1]$. Then $A_I(x) + \beta_x > 1$ and $A_I(y) + \beta_y > 1$. If $x * y \notin S$, then $x \in X \setminus S$ or $y \in X \setminus S$ since S is a subalgebra of X. Hence $A_I(x) = 0$ or $A_I(y) = 0$, which imply that $\beta_x > 1$ or $\beta_y > 1$. This is a contradiction, and so $x * y \in S$. If $\beta_x \wedge \beta_y > 0.5$, then $A_I(x*y)+\beta_x\wedge\beta_y>1$, i.e., $x*y\in I_q(A;\beta_x\wedge\beta_y)$. If $\beta_x\wedge\beta_y\leq 0.5$, then $A_I(x*y)\geq 0.5\geq\beta_x\wedge\beta_y$, i.e., $x*y\in I_\in(A;\beta_x\wedge\beta_y)$. Hence $x*y\in I_{\in\vee\, q}(A;\beta_x\wedge\beta_y)$. Similarly, if $x\in T_q(A;\alpha_x)$ and $y\in T_q(A;\alpha_y)$ for all $x,y\in X$ and $\alpha_x,\alpha_y\in[0,1]$, then $x*y\in T_{\in\vee\, q}(A;\alpha_x\wedge\alpha_y)$. Now let $x,y\in X$ and $\gamma_x,\gamma_y\in[0,1]$ be such that $x\in F_q(A;\gamma_x)$ and $y\in F_q(A;\gamma_y)$. Then $A_F(x)+\gamma_x<1$ and $A_F(y)+\gamma_y<1$. It follows that $x*y\in S$. In fact, if not then $x\in X\setminus S$ or $y\in X\setminus S$ since S is a subalgebra of X. Hence $A_F(x)=1$ or $A_F(y)=1$, which imply that $\gamma_x<0$ or $\gamma_y<0$. This is a contradiction, and so $x*y\in S$. If $\gamma_x\vee\gamma_y\geq 0.5$, then $A_F(x*y)\leq 0.5\leq\gamma_x\vee\gamma_y$, that is, $x*y\in F_\in(A;\gamma_x\vee\gamma_y)$. If $\gamma_x\vee\gamma_y<0.5$, then $A_F(x*y)+\gamma_x\vee\gamma_y<1$, that is, $x*y\in F_q(A;\gamma_x\vee\gamma_y)$. Hence $x*y\in F_{\in\vee\, q}(A;\gamma_x\vee\gamma_y)$, and consequently $A=(A_T,A_I,A_F)$ is a $(q,\in\vee\, q)$-neutrosophic subalgebra of X. □

Combining Theorems 3.5 and 3.16, we have the following corollary.

Corollary 3.17. *For a subalgebra S of X, if $A=(A_T,A_I,A_F)$ is a neutrosophic set in X satisfying conditions (3.14) and (3.15), then $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha,\beta\in(0.5,1]$ and $\gamma\in[0,0,5)$ whenever they are nonempty.*

Theorem 3.18. *Let $A=(A_T,A_I,A_F)$ be a $(q,\in\vee\, q)$-neutrosophic subalgebra of X in which A_T, A_I and A_F are not constant on X_0^1. Then there exist $x,y,z\in X$ such that $A_T(x)\geq 0.5$, $A_I(y)\geq 0.5$ and $A_F(z)\leq 0.5$. In particular, $A_T(x)\geq 0.5$, $A_I(y)\geq 0.5$ and $A_F(z)\leq 0.5$ for all $x,y,z\in X_0^1$.*

Proof. Assume that $A_T(x)<0.5$ for all $x\in X$. Since there exists $a\in X_0^1$ such that $\alpha_a=A_T(a)\neq A_T(0)=\alpha_0$, we have $\alpha_a>\alpha_0$ or $\alpha_a<\alpha_0$. If $\alpha_a>\alpha_0$, then we can choose $\delta>0.5$ such that $\alpha_0+\delta<1<\alpha_a+\delta$. It follows that $a\in T_q(A;\delta)$, $A_T(a*a)=A_T(0)=\alpha_0<\delta=\delta\wedge\delta$ and $A_T(a*a)+\delta\wedge\delta=A_T(0)+\delta=\alpha_0+\delta<1$ so that $a*a\notin T_{\in\vee\, q}(A;\delta\wedge\delta)$. This is a contradiction. Now if $\alpha_a<\alpha_0$, we can take $\delta>0.5$ such that $\alpha_a+\delta<1<\alpha_0+\delta$. Then $0\in T_q(A;\delta)$ and $a\in T_q(A;1)$, but $a*0\notin T_{\in\vee\, q}(A;1\wedge\delta)$ since $A_T(a)<0.5<\delta$ and $A_T(a)+\delta=\alpha_a+\delta<1$. This is also a contradiction. Thus $A_T(x)\geq 0.5$ for some $x\in X$. Similarly, we know that $A_I(y)\geq 0.5$ for some $y\in X$. Finally, suppose that $A_F(z)>0.5$ for all $z\in X$. Note that $\gamma_c=A_F(c)\neq A_F(0)=\gamma_0$ for some $c\in X_0^1$. It follows that $\gamma_c<\gamma_0$ or $\gamma_c>\gamma_0$. We first consider the case $\gamma_c<\gamma_0$. Then $\gamma_0+\varepsilon>1>\gamma_c+\varepsilon$ for some $\varepsilon\in[0,0.5)$, and so $c\in F_q(A;\varepsilon)$. Also $A_F(c*c)=A_F(0)=\gamma_0>\varepsilon$ and $A_F(c*c)+\varepsilon\vee\varepsilon=A_F(0)+\varepsilon=\gamma_0+\varepsilon>1$ which shows that $c*c\notin F_{\in\vee\, q}(A;\varepsilon\vee\varepsilon)$. This is impossible. Now, if $\gamma_c>\gamma_0$, then we can take $\varepsilon\in[0,0.5)$ and so that $\gamma_0+\varepsilon<1<\gamma_c+\varepsilon$. It follows that $0\in F_q(A;\varepsilon)$ and $c\in F_q(A;0)$. Since $A_F(c*0)=A_F(c)=\gamma_c>\varepsilon$ and $A_F(c*0)+\varepsilon=A_F(c)+\varepsilon=\gamma_c+\varepsilon>1$, we have $c*0\notin F_{\in\vee\, q}(A;\varepsilon)$. This is a contradiction, and therefore $A_F(z)<0.5$ for some $z\in X$. We now show that $A_T(0)\geq 0.5$, $A_I(0)\geq 0.5$ and $A_F(0)\leq 0.5$. Suppose that $A_T(0)=\alpha_0<0.5$. Since there exists $x\in X$ such that $A_T(x)=\alpha_x\geq 0.5$, it follows that $\alpha_0<\alpha_x$. Choose $\alpha_1\in[0,1]$ such that $\alpha_1>\alpha_0$ and $\alpha_0+\alpha_1<1<\alpha_x+\alpha_1$. Then $A_T(x)+\alpha_1=\alpha_x+\alpha_1>1$, and so $x\in T_q(A;\alpha_1)$. Now we have $A_T(x*x)+\alpha_1\wedge\alpha_1=A_T(0)+\alpha_1=\alpha_0+\alpha_1<1$ and $A_T(x*x)=A_T(0)=\alpha_0<\alpha_1=\alpha_1\wedge\alpha_1$. Thus $x*x\notin T_{\in\vee\, q}(A;\alpha_1\wedge\alpha_1)$, a contradiction. Hence $A_T(0)\geq 0.5$. Similarly, we have $A_I(0)\geq 0.5$. Assume that $A_F(0)=\gamma_0>0.5$. Note that $A_F(z)=\gamma_z\leq 0.5$ for some $z\in X$. Hence $\gamma_z<\gamma_0$, and so we can take $\gamma_1 \in [0,1]$ such that $\gamma_1 < \gamma_0$ and $\gamma_0 + \gamma_1 > 1 > \gamma_z + \gamma_1$. It follows that $A_F(z) + \gamma_1 = \gamma_z + \gamma_1 < 1$, that is, $z \in F_q(A; \gamma_1)$. Also $A_F(z*z) = A_F(0) = \gamma_0 > \gamma_1 = \gamma_1 \vee \gamma_1$, i.e., $z*z \notin F_\in(A; \gamma_1 \vee \gamma_1)$, and $A_F(z*z) + \gamma_1 \vee \gamma_1 = A_F(0) + \gamma_1 = \gamma_0 + \gamma_1 > 1$, i.e., $z * z \notin F_q(A; \gamma_1 \vee \gamma_1)$. Thus $z * z \notin F_{\in\vee\, q}(A; \gamma_1 \vee \gamma_1)$, a contradiction. Hence $A_F(0) \leq 0.5$. We finally show that $A_T(x) \geq 0.5$, $A_I(y) \geq 0.5$ and $A_F(z) \leq 0.5$ for all $x, y, z \in X_0^1$. We first assume that $A_I(y) = \beta_y < 0.5$ for some $y \in X_0^1$, and take $\beta > 0$ such that $\beta_y + \beta < 0.5$. Then $A_I(y) + 1 = \beta_y + 1 > 1$ and $A_I(0) + \beta + 0.5 > 1$, which imply that $y \in I_q(A; 1)$ and $0 \in I_q(A; \beta + 0.5)$. But $y * 0 \notin I_{\in\vee\, q}(A; \beta + 0.5)$ since $A_I(y * 0) = A_I(y) < \beta + 0.5 < 1 \wedge (\beta + 0.5)$ and $A_I(y*0) + 1 \wedge (\beta + 0.5) = A_I(y) + \beta + 0.5 = \beta_y + \beta + 0.5 < 1$. This is a contradiction. Hence $A_I(y) \geq 0.5$ for all $y \in X_0^1$. Similarly, we induces $A_T(x) \geq 0.5$ for all $x \in X_0^1$. Suppose $A_F(z) = \gamma_z > 0.5$ for some $z \in X_0^1$, and take $\gamma \in (0, 0.5)$ such that $\gamma_z > 0.5 + \gamma$. Then $z \in F_q(A; 0)$ and $A_F(0) + 0.5 - \gamma \leq 1 - \gamma < 1$, i.e., $0 \in F_q(A; 0.5 - \gamma)$. But $A_F(z*0) = A_F(z) > 0.5 > 0.5 - \gamma$ and $A_F(z*0) + 0.5 - \gamma = A_F(z) + 0.5 - \gamma = \gamma_z + 0.5 - \gamma > 1$, which imply that $z * 0 \notin F_{\in\vee\, q}(A; 0.5 - \gamma)$. This is a contradiction, and the proof is complete. □

Acknowledgements

The author wishes to thank the anonymous reviewers for their valuable suggestions.

References

[1] A.A.A. Agboola and B. Davvaz, On neutrosophic ideals of neutrosophic BCI-algebras, Critical Review. Volume X, (2015), 93–103.

[2] A.A.A. Agboola and B. Davvaz, Introduction to neutrosophic BCI/BCK-algebras, Inter. J. Math. Math. Sci. Volume 2015, Article ID 370267, 6 pages.

[3] Y. S. Huang, BCI-algebra, Science Press, Beijing, 2006.

[4] J. Meng and Y. B. Jun, BCK-algebra, Kyungmoon Sa Co. Seoul, 1994.

[5] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998. http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition online).

[6] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Reserch Press, Rehoboth, NM, 1999.

[7] F. Smarandache, Neutrosophic set-a generalization of the intuitionistic fuzzy set, Int. J. Pure Appl. Math. 24(3) (2005), 287–297.

Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 1, (July 2017), pp. 87–97
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Neutrosophic subalgebras of BCK/BCI-algebras based on neutrosophic points

A. Borumand Saeid, Young Bae Jun

Received 27 March 2017; Revised 28 April 2017; Accepted 20 May 2017

Abstract. Properties on neutrosophic $\in\vee q$-subsets and neutrosophic q-subsets are investigated. Relations between an $(\in, \in\vee q)$-neutrosophic subalgebra and a $(q, \in\vee q)$-neutrosophic subalgebra are considered. Characterization of an $(\in, \in\vee q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets are discussed. Conditions for an $(\in, \in\vee q)$-neutrosophic subalgebra to be a $(q, \in\vee q)$-neutrosophic subalgebra are provided.

2010 AMS Classification: 06F35, 03B60, 03B52.

Keywords: Neutrosophic set, neutrosophic $\in$-subset, neutrosophic q-subset, neutrosophic $\in\vee q$-subset, neutrosophic T_Φ-point, neutrosophic I_Φ-point, neutrosophic F_Φ-point.

Corresponding Author: Y. B. Jun (skywine@gmail.com)

1. Introduction

The concept of neutrosophic set (NS) developed by Smarandache [17, 18, 19] is a more general platform which extends the concepts of the classic set and fuzzy set (see [20], [21]), intuitionistic fuzzy set (see [1]) and interval valued intuitionistic fuzzy set (see [2]). Neutrosophic set theory is applied to various part (see [4], [5], [8], [9], [10], [11], [12], [13], [15], [16]). For further particulars, we refer readers to the site http://fs.gallup.unm.edu/neutrosophy.htm. Barbhuiya [3] introduced and studied the concept of $(\in, \in\vee q)$-intuitionistic fuzzy ideals of BCK/BCI-algebras. Jun [7] introduced the notion of neutrosophic subalgebras in BCK/BCI-algebras with several types. He provided characterizations of an $(\in,\in)$-neutrosophic subalgebra and an $(\in,\in\vee q)$-neutrosophic subalgebra. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets, he considered conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets to be subalgebras. He discussed conditions for a neutrosophic set to be a $(q, \in\vee q)$-neutrosophic subalgebra.

In this paper, we give relations between an $(\in, \in\vee q)$-neutrosophic subalgebra and a $(q, \in\vee q)$-neutrosophic subalgebra. We discuss characterization of an $(\in, \in\vee q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets. We provide conditions for an $(\in, \in\vee q)$-neutrosophic subalgebra to be a $(q, \in\vee q)$-neutrosophic subalgebra. We investigate properties on neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \;\Rightarrow\; x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [6] and [14] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [18]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

3. Neutrosophic subalgebras of several types

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$\begin{aligned}
&T_\in(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha\},\\
&I_\in(A;\beta) := \{x \in X \mid A_I(x) \geq \beta\},\\
&F_\in(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma\},\\
&T_q(A;\alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},\\
&I_q(A;\beta) := \{x \in X \mid A_I(x) + \beta > 1\},\\
&F_q(A;\gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},\\
&T_{\in\vee\, q}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},\\
&I_{\in\vee\, q}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},\\
&F_{\in\vee\, q}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.
\end{aligned}$$

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are *neutrosophic* $\in$*-subsets*; $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are *neutrosophic q-subsets*; and $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are *neutrosophic* $\in\vee\, q$*-subsets*. For $\Phi \in \{\in, q, \in\vee\, q\}$, the element of $T_\Phi(A;\alpha)$ (resp., $I_\Phi(A;\beta)$ and $F_\Phi(A;\gamma)$) is called a *neutrosophic* T_Φ*-point* (resp., *neutrosophic* I_Φ*-point* and *neutrosophic* F_Φ*-point*) with value α (resp., β and γ) (see [7]).

It is clear that

$$T_{\in\vee\, q}(A;\alpha) = T_\in(A;\alpha) \cup T_q(A;\alpha), \tag{3.1}$$

$$I_{\in\vee\, q}(A;\beta) = I_\in(A;\beta) \cup I_q(A;\beta), \tag{3.2}$$

$$F_{\in\vee\, q}(A;\gamma) = F_\in(A;\gamma) \cup F_q(A;\gamma). \tag{3.3}$$

Definition 3.1 ([7])**.** Given $\Phi, \Psi \in \{\in, q, \in\vee\, q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called a (Φ, Ψ)*-neutrosophic subalgebra* of X if the following assertions are valid.

$$\begin{aligned}
&x \in T_\Phi(A;\alpha_x),\ y \in T_\Phi(A;\alpha_y) \ \Rightarrow\ x * y \in T_\Psi(A;\alpha_x \wedge \alpha_y),\\
&x \in I_\Phi(A;\beta_x),\ y \in I_\Phi(A;\beta_y) \ \Rightarrow\ x * y \in I_\Psi(A;\beta_x \wedge \beta_y),\\
&x \in F_\Phi(A;\gamma_x),\ y \in F_\Phi(A;\gamma_y) \ \Rightarrow\ x * y \in F_\Psi(A;\gamma_x \vee \gamma_y)
\end{aligned} \tag{3.4}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Lemma 3.2 ([7])**.** *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X) \begin{pmatrix} A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} \\ A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), 0.5\} \\ A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \end{pmatrix}. \tag{3.5}$$

Theorem 3.3. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of X if and only if the neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. For any $x, y \in X$, let $\alpha \in (0, 0.5]$ be such that $x, y \in T_\in(A; \alpha)$. Then $A_T(x) \geq \alpha$ and $A_T(y) \geq \alpha$. It follows from (3.5) that

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} \geq \alpha \wedge 0.5 = \alpha$$

and so that $x * y \in T_\in(A; \alpha)$. Thus $T_\in(A; \alpha)$ is a subalgebra of X for all $\alpha \in (0, 0.5]$. Similarly, $I_\in(A; \beta)$ is a subalgebra of X for all $\beta \in (0, 0.5]$. Now, let $\gamma \in [0.5, 1)$ be such that $x, y \in F_\in(A; \gamma)$. Then $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. Hence

$$A_F(x * y) \leq \bigvee\{A_F x), A_F(y), 0.5\} \leq \gamma \vee 0.5 = \gamma$$

by (3.5), and so $x * y \in F_\in(A; \gamma)$. Thus $F_\in(A; \gamma)$ is a subalgebra of X for all $\gamma \in [0.5, 1)$.

Conversely, let $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ be such that $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are subalgebras of X. If there exist $a, b \in X$ such that

$$A_I(a * b) < \bigwedge\{A_I(a), A_I(b), 0.5\},$$

then we can take $\beta \in (0, 1)$ such that

$$A_I(a * b) < \beta < \bigwedge\{A_I(a), A_I(b), 0.5\}. \tag{3.6}$$

Thus $a, b \in I_\in(A; \beta)$ and $\beta < 0.5$, and so $a * b \in I_\in(A; \beta)$. But, the left inequality in (3.6) induces $a * b \notin I_\in(A; \beta)$, a contradiction. Hence

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}$$

for all $x, y \in X$. Similarly, we can show that

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$$

for all $x, y \in X$. Now suppose that

$$A_F(a * b) > \bigvee\{A_F(a), A_F(b), 0.5\}$$

for some $a, b \in X$. Then there exists $\gamma \in (0, 1)$ such that

$$A_F(a * b) > \gamma > \bigvee\{A_F(a), A_F(b), 0.5\}.$$

It follows that $\gamma \in (0.5, 1)$ and $a, b \in F_\in(A; \gamma)$. Since $F_\in(A; \gamma)$ is a subalgebra of X, we have $a * b \in F_\in(A; \gamma)$ and so $A_F(a * b) \leq \gamma$. This is a contradiction, and thus

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$$

for all $x, y \in X$. Using Lemma 3.2, $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

Using Theorem 3.3 and [7, Theorem 3.8], we have the following corollary.

Corollary 3.4. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in\vee q$-subsets $T_{\in\vee q}(A; \alpha)$, $I_{\in\vee q}(A; \beta)$ and $F_{\in\vee q}(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, then the neutrosophic $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Theorem 3.5. *Given neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ if and only if the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \vee 0.5 \geq A_T(x) \wedge A_T(y) \\ A_I(x * y) \vee 0.5 \geq A_I(x) \wedge A_I(y) \\ A_F(x * y) \wedge 0.5 \leq A_F(x) \vee A_F(y) \end{array} \right). \tag{3.7}$$

Proof. Assume that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. Suppose that there are $a, b \in X$ such that $A_T(a * b) \vee 0.5 < A_T(a) \wedge A_T(b) := \alpha$. Then $\alpha \in (0.5, 1]$ and $a, b \in T_\in(A;\alpha)$. Since $T_\in(A;\alpha)$ is a subalgebra of X, it follows that $a * b \in T_\in(A;\alpha)$, that is, $A_T(a * b) \geq \alpha$ which is a contradiction. Thus

$$A_T(x * y) \vee 0.5 \geq A_T(x) \wedge A_T(y)$$

for all $x, y \in X$. Similarly, we know that $A_I(x * y) \vee 0.5 \geq A_I(x) \wedge A_I(y)$ for all $x, y \in X$. Now, if $A_F(x * y) \wedge 0.5 > A_F(x) \vee A_F(y)$ for some $x, y \in X$, then $x, y \in F_\in(A;\gamma)$ and $\gamma \in [0, 0.5)$ where $\gamma = A_F(x) \vee A_F(y)$. But, $x * y \notin F_\in(A;\gamma)$ which is a contradiction. Hence $A_F(x * y) \wedge 0.5 \leq A_F(x) \vee A_F(y)$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying the condition (3.7). Let $x, y, a, b \in X$ and $\alpha, \beta \in (0.5, 1]$ be such that $x, y \in T_\in(A;\alpha)$ and $a, b \in I_\in(A;\beta)$. Then

$$A_T(x * y) \vee 0.5 \geq A_T(x) \wedge A_T(y) \geq \alpha > 0.5,$$
$$A_I(a * b) \vee 0.5 \geq A_I(a) \wedge A_I(b) \geq \beta > 0.5.$$

It follows that $A_T(x * y) \geq \alpha$ and $A_I(a * b) \geq \beta$, that is, $x * y \in T_\in(A;\alpha)$ and $a * b \in I_\in(A;\beta)$. Now, let $x, y \in X$ and $\gamma \in [0, 0.5)$ be such that $x, y \in F_\in(A;\gamma)$. Then $A_F(x * y) \wedge 0.5 \leq A_F(x) \vee A_F(y) \leq \gamma < 0.5$ and so $A_F(x * y) \leq \gamma$, i.e., $x * y \in F_\in(A;\gamma)$. This completes the proof. □

We consider relations between a $(q, \in\vee q)$-neutrosophic subalgebra and an $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 3.6. *In a BCK/BCI-algebra, every $(q, \in\vee q)$-neutrosophic subalgebra is an $(\in, \in\vee q)$-neutrosophic subalgebra.*

Proof. Let $A = (A_T, A_I, A_F)$ be a $(q, \in\vee q)$-neutrosophic subalgebra of a BCK/BCI-algebra X and let $x, y \in X$. Let $\alpha_x, \alpha_y \in (0, 1]$ be such that $x \in T_\in(A;\alpha_x)$ and $y \in T_\in(A;\alpha_y)$. Then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$. Suppose $x * y \notin T_{\in\vee q}(A;\alpha_x \wedge \alpha_y)$. Then

$$A_T(x * y) < \alpha_x \wedge \alpha_y, \tag{3.8}$$

$$A_T(x * y) + (\alpha_x \wedge \alpha_y) \leq 1. \tag{3.9}$$

It follows that

$$A_T(x * y) < 0.5. \tag{3.10}$$

Combining (3.8) and (3.10), we have

$$A_T(x*y) < \bigwedge\{\alpha_x, \alpha_y, 0.5\}$$

and so

$$\begin{aligned}1 - A_T(x*y) &> 1 - \bigwedge\{\alpha_x, \alpha_y, 0.5\}\\ &= \bigvee\{1-\alpha_x, 1-\alpha_y, 0.5\}\\ &\geq \bigvee\{1-A_T(x), 1-A_T(y), 0.5\}.\end{aligned}$$

Hence there exists $\alpha \in (0,1]$ such that

$$1 - A_T(x*y) \geq \alpha > \bigvee\{1-A_T(x), 1-A_T(y), 0.5\}. \tag{3.11}$$

The right inequality in (3.11) induces $A_T(x)+\alpha > 1$ and $A_T(y)+\alpha > 1$, that is, $x, y \in T_q(A;\alpha)$. Since $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic subalgebra of X, we have $x*y \in T_{\in\vee q}(A;\alpha)$. But, the left inequality in (3.11) implies that $A_T(x*y)+\alpha \leq 1$, i.e., $x*y \notin T_q(A;\alpha)$, and $A_T(x*y) \leq 1-\alpha < 1-0.5 = 0.5 < \alpha$, i.e., $x*y \notin T_\in(A;\alpha)$. Hence $x*y \notin T_{\in\vee q}(A;\alpha)$, a contradiction. Thus $x*y \in T_{\in\vee q}(A;\alpha_x \wedge \alpha_y)$. Similarly, we can show that if $x \in I_\in(A;\beta_x)$ and $y \in I_\in(A;\beta_y)$ for $\beta_x, \beta_y \in (0,1]$, then $x*y \in I_{\in\vee q}(A;\beta_x \wedge \beta_y)$. Now, let $\gamma_x, \gamma_y \in [0,1)$ be such that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$. If $x*y \notin F_{\in\vee q}(A;\gamma_x \vee \gamma_y)$, then

$$A_F(x*y) > \gamma_x \vee \gamma_y, \tag{3.12}$$

$$A_F(x*y) + (\gamma_x \vee \gamma_y) \geq 1. \tag{3.13}$$

It follows that

$$A_F(x*y) > \bigvee\{\gamma_x, \gamma_y, 0.5\}$$

and so that

$$\begin{aligned}1 - A_F(x*y) &< 1 - \bigvee\{\gamma_x, \gamma_y, 0.5\}\\ &= \bigwedge\{1-\gamma_x, 1-\gamma_y, 0.5\}\\ &\leq \bigwedge\{1-A_F(x), 1-A_F(y), 0.5\}.\end{aligned}$$

Thus there exists $\gamma \in [0,1)$ such that

$$1 - A_F(x*y) \leq \gamma < \bigwedge\{1-A_F(x), 1-A_F(y), 0.5\}. \tag{3.14}$$

It follows from the right inequality in (3.14) that $A_F(x)+\gamma < 1$ and $A_F(y)+\gamma < 1$, that is, $x, y \in F_q(A;\gamma)$, which implies that $x*y \in F_{\in\vee q}(A;\gamma)$. But, we have $x*y \notin F_{\in\vee q}(A;\gamma)$ by the left inequality in (3.14). This is a contradiction, and so $x*y \in F_{\in\vee q}(A;\gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

The following example shows that the converse of Theorem 3.6 is not true.

Table 1. Cayley table of the operation $*$

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.8	0.3
1	0.2	0.3	0.6
2	0.2	0.3	0.6
3	0.7	0.1	0.7
4	0.4	0.4	0.9

Example 3.7. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.
Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by
Then

$$T_{\in}(A;\alpha) = \begin{cases} \{0,3\} & \text{if } \alpha \in (0.4, 0.5], \\ \{0,3,4\} & \text{if } \alpha \in (0.2, 0.4], \\ X & \text{if } \alpha \in (0, 0.2], \end{cases}$$

$$I_{\in}(A;\beta) = \begin{cases} \{0\} & \text{if } \beta \in (0.4, 0.5], \\ \{0,4\} & \text{if } \beta \in (0.3, 0.4], \\ \{0,1,2,4\} & \text{if } \beta \in (0.1, 0.3], \\ X & \text{if } \beta \in (0, 0.1], \end{cases}$$

$$F_{\in}(A;\gamma) = \begin{cases} X & \text{if } \gamma \in (0.9, 1), \\ \{0,1,2,3\} & \text{if } \gamma \in [0.7, 0.9), \\ \{0,1,2\} & \text{if } \gamma \in [0.6, 0.7), \\ \{0\} & \text{if } \gamma \in [0.5, 0.6), \end{cases}$$

which are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Using Theorem 3.3, $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. But it is not a $(q, \in\vee q)$-neutrosophic subalgebra of X since $2 \in T_q(A; 0.83)$ and $3 \in T_q(A; 0.4)$, but $2 * 3 = 2 \notin T_{\in\vee q}(A; 0.4)$.

We provide conditions for an $(\in, \in\vee q)$-neutrosophic subalgebra to be a $(q, \in\vee q)$-neutrosophic subalgebra.

Theorem 3.8. *Assume that any neutrosophic T_Φ-point and neutrosophic I_Φ-point has the value α and β in $(0, 0.5]$, respectively, and any neutrosophic F_Φ-point has the value γ in $[0.5, 1)$ for $\Phi \in \{\in, q, \in\vee q\}$. Then every $(\in, \in\vee q)$-neutrosophic subalgebra is a $(q, \in\vee q)$-neutrosophic subalgebra.*

Proof. Let X be a BCK/BCI-algebra and let $A = (A_T, A_I, A_F)$ be an $(\in, \in\vee q)$-neutrosophic subalgebra of X. For $x, y, a, b \in X$, let $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 0.5]$ be such that $x \in T_q(A;\alpha_x)$, $y \in T_q(A;\alpha_y)$, $a \in I_q(A;\beta_a)$ and $b \in T_q(A;\beta_b)$. Then $A_T(x) + \alpha_x > 1$, $A_T(y) + \alpha_y > 1$, $A_I(a) + \beta_a > 1$ and $A_I(b) + \beta_b > 1$. Since $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 0.5]$, it follows that $A_T(x) > 1 - \alpha_x \geq \alpha_x$, $A_T(y) > 1 - \alpha_y \geq \alpha_y$, $A_I(a) > 1 - \beta_a \geq \beta_a$ and $A_I(b) > 1 - \beta_b \geq \beta_b$, that is, $x \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a \in I_\in(A;\beta_a)$ and $b \in I_\in(A;\beta_b)$. Also, let $x \in F_q(A;\gamma_x)$ and $y \in F_q(A;\gamma_y)$ for $x, y \in X$ and $\gamma_x, \gamma_y \in [0.5, 1)$. Then $A_F(x) + \gamma_x < 1$ and $A_F(y) + \gamma_y < 1$, and so $A_F(x) < 1 - \gamma_x \leq \gamma_x$ and $A_F(y) < 1 - \gamma_y \leq \gamma_y$ since $\gamma_x, \gamma_y \in [0.5, 1)$. This shows that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. It follows from (3.4) that $x * y \in T_{\in\vee\, q}(A;\alpha_x \wedge \alpha_y)$, $a * b \in I_{\in\vee\, q}(A;\beta_a \wedge \beta_b)$, and $x * y \in F_{\in\vee\, q}(A;\gamma_x \vee \gamma_y)$. Consequently, $A = (A_T, A_I, A_F)$ is a $(q, \in\vee\, q)$-neutrosophic subalgebra of X. □

Theorem 3.9. *Both $(\in, \in)$-neutrosophic subalgebra and $(\in\vee\, q, \in\vee\, q)$-neutrosophic subalgebra are an $(\in, \in\vee\, q)$-neutrosophic subalgebra.*

Proof. It is clear that $(\in, \in)$-neutrosophic subalgebra is an $(\in, \in\vee\, q)$-neutrosophic subalgebra. Let $A = (A_T, A_I, A_F)$ be an $(\in\vee\, q, \in\vee\, q)$-neutrosophic subalgebra of X. For any $x, y, a, b \in X$, let α_x, α_y, β_a, $\beta_b \in (0, 1]$ be such that $x \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a \in I_\in(A;\beta_a)$ and $b \in I_\in(A;\beta_b)$. Then $x \in T_{\in\vee\, q}(A;\alpha_x)$, $y \in T_{\in\vee\, q}(A;\alpha_y)$, $a \in I_{\in\vee\, q}(A;\beta_a)$ and $b \in I_{\in\vee\, q}(A;\beta_b)$ by (3.1) and (3.2). It follows that $x * y \in T_{\in\vee\, q}(A;\alpha_x \wedge \alpha_y)$ and $a * b \in I_{\in\vee\, q}(A;\beta_a \wedge \beta_b)$. Now, let $x, y \in X$ and $\gamma_x, \gamma_y \in [0, 1)$ be such that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. Then $x \in F_{\in\vee\, q}(A;\gamma_x)$ and $y \in F_{\in\vee\, q}(A;\gamma_y)$ by (3.3). Hence $x * y \in F_{\in\vee\, q}(A;\gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of X. □

The converse of Theorem 3.9 is not true in general. In fact, the $(\in, \in\vee\, q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ in Example 3.7 is neither an $(\in, \in)$-neutrosophic subalgebra nor an $(\in\vee\, q, \in\vee\, q)$-neutrosophic subalgebra.

Theorem 3.10. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in (0, 0.5)$, then*

$$\begin{aligned} & x \in T_\in(A;\alpha_x),\ y \in T_\in(A;\alpha_y) \ \Rightarrow\ x * y \in T_q(A;\alpha_x \vee \alpha_y), \\ & x \in I_\in(A;\beta_x),\ y \in I_\in(A;\beta_y) \ \Rightarrow\ x * y \in I_q(A;\beta_x \vee \beta_y), \\ & x \in F_\in(A;\gamma_x),\ y \in F_\in(A;\gamma_y) \ \Rightarrow\ x * y \in F_q(A;\gamma_x \wedge \gamma_y) \end{aligned} \tag{3.15}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0.5, 1]$ and $\gamma_x, \gamma_y \in (0, 0.5)$.

Proof. Let $x, y, a, b, u, v \in X$ and α_x, α_y, β_a, $\beta_b \in (0.5, 1]$ and $\gamma_u, \gamma_v \in (0, 0.5)$ be such that $x \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a \in I_\in(A;\beta_a)$, $b \in I_\in(A;\beta_b)$, $u \in F_\in(A;\gamma_u)$ and $v \in F_\in(A;\gamma_v)$. Then $A_T(x) \geq \alpha_x > 1 - \alpha_x$, $A_T(y) \geq \alpha_y > 1 - \alpha_y$, $A_I(a) \geq \beta_a > 1 - \beta_a$, $A_I(b) \geq \beta_b > 1 - \beta_b$, $A_F(u) \leq \gamma_u < 1 - \gamma_u$ and $A_F(v) \leq \gamma_v < 1 - \gamma_v$. It follows that $x, y \in T_q(A;\alpha_x \vee \alpha_y)$, $a, b \in I_q(A;\beta_a \vee \beta_b)$ and $u, v \in F_q(A;\gamma_u \wedge \gamma_v)$. Since $\alpha_x \vee \alpha_y, \beta_a \vee \beta_b \in (0.5, 1]$ and $\gamma_u \wedge \gamma_v \in (0, 0.5)$, we have $x * y \in T_q(A;\alpha_x \vee \alpha_y)$, $a * b \in I_q(A;\beta_a \vee \beta_b)$ and $u * v \in F_q(A;\gamma_u \wedge \gamma_v)$ by hypothesis. This completes the proof. □

The following corollary is by Theorem 3.10 and [7, Theorem 3.7].

Corollary 3.11. *Every $(\in, \in\vee\, q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X satisfies the condition (3.15).*

Corollary 3.12. *Every $(q, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X satisfies the condition (3.15).*

Proof. It is by Theorem 3.6 and Corollary 3.11. □

Theorem 3.13. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha,\beta\in(0,0.5]$ and $\gamma\in(0.5,1)$, then*

$$\begin{aligned} &x\in T_q(A;\alpha_x),\ y\in T_q(A;\alpha_y)\ \Rightarrow\ x*y\in T_\in(A;\alpha_x\vee\alpha_y),\\ &x\in I_q(A;\beta_x),\ y\in I_q(A;\beta_y)\ \Rightarrow\ x*y\in I_\in(A;\beta_x\vee\beta_y),\\ &x\in F_q(A;\gamma_x),\ y\in F_q(A;\gamma_y)\ \Rightarrow\ x*y\in F_\in(A;\gamma_x\wedge\gamma_y)\end{aligned}\tag{3.16}$$

for all $x,y\in X$, $\alpha_x,\alpha_y,\ \beta_x,\beta_y\in(0,0.5]$ and $\gamma_x,\gamma_y\in(0.5,1)$.

Proof. Let $x,y,a,b,u,v\in X$ and $\alpha_x,\ \alpha_y,\ \beta_a,\ \beta_b\in(0,0.5]$ and $\gamma_u,\gamma_v\in(0.5,1)$ be such that $x\in T_q(A;\alpha_x)$, $y\in T_q(A;\alpha_y)$, $a\in I_q(A;\beta_a)$, $b\in I_q(A;\beta_b)$, $u\in F_q(A;\gamma_u)$ and $v\in F_q(A;\gamma_v)$. Then $x,y\in T_q(A;\alpha_x\vee\alpha_y)$, $a,b\in I_q(A;\beta_a\vee\beta_b)$ and $u,v\in F_q(A;\gamma_u\wedge\gamma_v)$. Since $\alpha_x\vee\alpha_y,\beta_a\vee\beta_b\in(0,0.5]$ and $\gamma_u\wedge\gamma_v\in(0.5,1)$, it follows from the hypothesis that $x*y\in T_q(A;\alpha_x\vee\alpha_y)$, $a*b\in I_q(A;\beta_a\vee\beta_b)$ and $u*v\in F_q(A;\gamma_u\wedge\gamma_v)$. Hence

$$\begin{aligned}&A_T(x*y)>1-(\alpha_x\vee\alpha_y)\geq\alpha_x\vee\alpha_y,\text{ that is, }x*y\in T_\in(A;\alpha_x\vee\alpha_y),\\ &A_I(a*b)>1-(\beta_a\vee\beta_b)\geq\beta_a\vee\beta_b,\text{ that is, }a*b\in I_\in(A;\beta_a\vee\beta_b),\\ &A_F(u*v)<1-(\gamma_u\wedge\gamma_v)\leq\gamma_u\wedge\gamma_v,\text{ that is, }u*v\in F_\in(A;\gamma_u\wedge\gamma_v).\end{aligned}$$

Consequently, the condition (3.16) is valid for all $x,y\in X$, $\alpha_x,\alpha_y,\ \beta_x,\beta_y\in(0,0.5]$ and $\gamma_x,\gamma_y\in(0.5,1)$. □

Theorem 3.14. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in\vee q$-subsets $T_{\in\vee q}(A;\alpha)$, $I_{\in\vee q}(A;\beta)$ and $F_{\in\vee q}(A;\gamma)$ are subalgebras of X for all $\alpha,\beta\in(0,0.5]$ and $\gamma\in[0.5,1)$, then the following assertions are valid.*

$$\begin{aligned} &x\in T_q(A;\alpha_x),\ y\in T_q(A;\alpha_y)\ \Rightarrow\ x*y\in T_{\in\vee q}(A;\alpha_x\vee\alpha_y),\\ &x\in I_q(A;\beta_x),\ y\in I_q(A;\beta_y)\ \Rightarrow\ x*y\in I_{\in\vee q}(A;\beta_x\vee\beta_y),\\ &x\in F_q(A;\gamma_x),\ y\in F_q(A;\gamma_y)\ \Rightarrow\ x*y\in F_{\in\vee q}(A;\gamma_x\wedge\gamma_y)\end{aligned}\tag{3.17}$$

for all $x,y\in X$, $\alpha_x,\alpha_y,\ \beta_x,\beta_y\in(0,0.5]$ and $\gamma_x,\gamma_y\in[0.5,1)$.

Proof. Let $x,y,a,b,u,v\in X$ and $\alpha_x,\ \alpha_y,\ \beta_a,\ \beta_b\in(0,0.5]$ and $\gamma_u,\gamma_v\in[0.5,1)$ be such that $x\in T_q(A;\alpha_x)$, $y\in T_q(A;\alpha_y)$, $a\in I_q(A;\beta_a)$, $b\in I_q(A;\beta_b)$, $u\in F_q(A;\gamma_u)$ and $v\in F_q(A;\gamma_v)$. Then $x\in T_{\in\vee q}(A;\alpha_x)$, $y\in T_{\in\vee q}(A;\alpha_y)$, $a\in I_{\in\vee q}(A;\beta_a)$, $b\in I_{\in\vee q}(A;\beta_b)$, $u\in F_{\in\vee q}(A;\gamma_u)$ and $v\in F_{\in\vee q}(A;\gamma_v)$. It follows that $x,y\in T_{\in\vee q}(A;\alpha_x\vee\alpha_y)$, $a,b\in I_{\in\vee q}(A;\beta_a\vee\beta_b)$ and $u,v\in F_{\in\vee q}(A;\gamma_u\wedge\gamma_v)$ which imply from the hypothesis that $x*y\in T_{\in\vee q}(A;\alpha_x\vee\alpha_y)$, $a*b\in I_{\in\vee q}(A;\beta_a\vee\beta_b)$ and $u*v\in F_{\in\vee q}(A;\gamma_u\wedge\gamma_v)$. This completes the proof. □

Corollary 3.15. *Every $(\in, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the condition (3.17).*

Proof. It is by Theorem 3.14 and [7, Theorem 3.9]. □

Theorem 3.16. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in \vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$, then the following assertions are valid.*

$$\begin{aligned} & x \in T_q(A;\alpha_x),\ y \in T_q(A;\alpha_y) \ \Rightarrow\ x * y \in T_{\in\vee\, q}(A;\alpha_x \vee \alpha_y), \\ & x \in I_q(A;\beta_x),\ y \in I_q(A;\beta_y) \ \Rightarrow\ x * y \in I_{\in\vee\, q}(A;\beta_x \vee \beta_y), \\ & x \in F_q(A;\gamma_x),\ y \in F_q(A;\gamma_y) \ \Rightarrow\ x * y \in F_{\in\vee\, q}(A;\gamma_x \wedge \gamma_y) \end{aligned} \tag{3.18}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0.5, 1]$ and $\gamma_x, \gamma_y \in [0, 0.5)$.

Proof. It is similar to the proof Theorem 3.14. □

Corollary 3.17. *Every $(q, \in\vee\, q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the condition (3.18).*

Proof. It is by Theorem 3.16 and [7, Theorem 3.10]. □

Combining Theorems 3.14 and 3.16, we have the following corollary.

Corollary 3.18. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in \vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, then the following assertions are valid.*

$$\begin{aligned} & x \in T_q(A;\alpha_x),\ y \in T_q(A;\alpha_y) \ \Rightarrow\ x * y \in T_{\in\vee\, q}(A;\alpha_x \vee \alpha_y), \\ & x \in I_q(A;\beta_x),\ y \in I_q(A;\beta_y) \ \Rightarrow\ x * y \in I_{\in\vee\, q}(A;\beta_x \vee \beta_y), \\ & x \in F_q(A;\gamma_x),\ y \in F_q(A;\gamma_y) \ \Rightarrow\ x * y \in F_{\in\vee\, q}(A;\gamma_x \wedge \gamma_y) \end{aligned}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Conclusions

We have considered relations between an $(\in, \in\vee\, q)$-neutrosophic subalgebra and a $(q, \in\vee\, q)$-neutrosophic subalgebra. We have discussed characterization of an $(\in, \in\vee\, q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets, and have provided conditions for an $(\in, \in\vee\, q)$-neutrosophic subalgebra to be a $(q, \in\vee\, q)$-neutrosophic subalgebra. We have investigated properties on neutrosophic q-subsets and neutrosophic $\in\vee\, q$-subsets. Our future research will be focused on the study of generalization of this paper.

4. Acknowledgements

The authors wish to thanks the anonymous reviewers for their valuable suggestions.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[2] K. Atanassov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (1989) 343–349.
[3] S. R. Barbhuiya, $(\in, \in\vee q)$-intuitionistic fuzzy ideals of BCK/BCI-algebras, Notes on IFS 21 (1) (2015) 24–42.

[4] P. Biswas, S. Pramanik and B. C. Giri, TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment, Neural Computing and Applications 27 (2016) 727–737.

[5] P. Biswas, S. Pramanik and B. C. Giri, Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multi-attribute decision making, Neutrosophic Sets and Systems 12 (2016) 127–138.

[6] Y. S. Huang, BCI-algebra, Science Press, Beijing 2006.

[7] Y. B. Jun, Neutrosophic subalgebras of several types in BCK/BCI-algebras, Ann. Fuzzy Math. Inform. (submitted).

[8] P. D. Liu, Y. C. Chu, Y. W. Li and Y. B. Chen, Some generalized neutrosophic number Hamacher aggregation operators and their application to group decision making, Int. J. Fuzzy Syst. 16 (2) (2014) 242–255.

[9] P. D. Liu and Y. M. Wang, Multiple attribute decision-making method based on single valued neutrosophic normalized weighted Bonferroni mean, Neural Computing and Applications 25(7-8) (2014) 2001–2010.

[10] P. D. Liu and L. L. Shi, The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multiple attribute decision making, Neural Computing and Applications 26 (2) (2015) 457–471.

[11] [4] P. D. Liu and G. L. Tang, Some power generalized aggregation operators based on the interval neutrosophic numbers and their application to decision making, J. Intell. Fuzzy Systems 30 (2016) 2517–2528.

[12] P. D. Liu and Y. M. Wang, Interval neutrosophic prioritized OWA operator and its application to multiple attribute decision making, J. Syst. Sci . Complex. 29 (3) (2016) 681–697.

[13] P. D. Liu and H. G. Li, Multiple attribute decision making method based on some normal neutrosophic Bonferroni mean operators, Neural Computing and Applications 28 (1) (2017) 179–194.

[14] J. Meng and Y. B. Jun, BCK-algebra, Kyungmoon Sa Co. Seoul 1994.

[15] K. Mondal and S. Pramanik, Neutrosophic tangent similarity measure and its application to multiple attribute decision making, Neutrosophic Sets and Systems 9 (2015) 85–92.

[16] S. Pramanik, P. Biswas and B. C. Giri, Hybrid vector similarity measures and their applications to multi-attribute decision making under neutrosophic environment, Neural Computing and Applications 28 (2017) 1163–1176.

[17] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p. 1998. http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition online).

[18] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Reserch Press, Rehoboth, NM, 1999.

[19] F. Smarandache, Neutrosophic set-a generalization of the intuitionistic fuzzy set, Int. J. Pure Appl. Math. 24 (3) (2005) 287–297.

[20] L. A. Zadeh, Fuzzy sets, Inform and Control 8 (1965) 338–353.

[21] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Part 1, Inform. Sci. 8 (1975) 199–249.

Arsham Borumand Saeid (a_b_saeid@yahoo.com)
Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran

Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 1, (July 2017), pp. 99–120
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Interval-valued neutrosophic competition graphs

Muhammad Akram and Maryam Nasir

Received 19 April 2017; Revised 2 May 2017; Accepted 10 May 2017

Abstract. We first introduce the concept of interval-valued neutrosophic competition graphs. We then discuss certain types, including k-competition interval-valued neutrosophic graphs, p-competition interval-valued neutrosophic graphs and m-step interval-valued neutrosophic competition graphs. Moreover, we present the concept of m-step interval-valued neutrosophic neighbourhood graphs.

2010 AMS Classification: 03E72, 05C72, 05C78, 05C99

Keywords: Interval-valued neutrosophic digraphs, Interval-valued neutrosophic competition graphs.

Corresponding Author: Muhammad Akram (m.akram@pucit.edu.pk)

1. Introduction

In 1975, Zadeh [35] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [34] in which the values of the membership degrees are intervals of numbers instead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive part of fuzzy control is defuzzification [19]. Atanassov [12] proposed the extended form of fuzzy set theory by adding a new component, called, intuitionistic fuzzy sets. Smarandache [26, 27] introduced the concept of neutrosophic sets by combining the non-standard analysis. In neutrosophic set, the membership value is associated with three components: truth-membership (t), indeterminacy-membership (i) and falsity-membership (f), in which each membership value is a real standard or non-standard subset of the non-standard unit interval $]0^-, 1^+[$ and there is no restriction on their sum. Smarandache [28] and Wang et al. [29] presented the notion of single-valued neutrosophic sets to apply neutrosophic sets in real life problems more conveniently. In single-valued neutrosophic sets, three components are independent and their values are taken from the standard unit interval $[0, 1]$. Wang et al. [30] presented the concept of interval-valued neutrosophic sets, which is more precise and more flexible than the single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership (t, i, f) functions are independent, and their values belong to the unit interval $[0, 1]$.

Kauffman [18] gave the definition of a fuzzy graph. Fuzzy graphs were narrated by Rosenfeld [22]. After that, some remarks on fuzzy graphs were represented by Bhattacharya [13]. He showed that all the concepts on crisp graph theory do not have similarities in fuzzy graphs. Wu [32] discussed fuzzy digraphs. The concept of fuzzy k-competition graphs and p-competition fuzzy graphs was first developed by Samanta and Pal in [23], it was further studied in [11, 21, 25]. Samanta et al. [24] introduced the generalization of fuzzy competition graphs, called m-step fuzzy competition graphs. Samanta et al. [24] also introduced the concepts of fuzzy m-step neighbourhood graphs, fuzzy economic competition graphs, and m-step economic competition graphs. The concepts of bipolar fuzzy competition graphs and intuitionistic fuzzy competition graphs are discussed in [21, 25]. Hongmei and Lianhua [16], gave definition of interval-valued fuzzy graphs. Akram et al. [1, 2, 3, 4] have introduced several concepts on interval-valued fuzzy graphs and interval-valued neutrosophic graphs. Akram and Shahzadi [6] introduced the notion of neutrosophic soft graphs with applications. Akram [7] introduced the notion of single-valued neutrosophic planar graphs. Akram and Shahzadi [8] studied properties of single-valued neutrosophic graphs by level graphs. Recently, Akram and Nasir [5] have discussed some concepts of interval-valued neutrosophic graphs. In this paper, we first introduce the concept of interval-valued neutrosophic competition graphs. We then discuss certain types, including k-competition interval-valued neutrosophic graphs, p-competition interval-valued neutrosophic graphs and m-step interval-valued neutrosophic competition graphs. Moreover, we present the concept of m-step interval-valued neutrosophic neighbourhood graphs.

We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [6, 9, 10, 13, 14, 15, 17, 20, 26, 33, 36].

2. Interval-Valued Neutrosophic Competition Graphs

Definition 2.1 ([35])**.** The interval-valued fuzzy set A in X is defined by

$$A = \{(s, [t^l_A(s), t^u_A(s)]) : s \in X\},$$

where, $t^l_A(s)$ and $t^u_A(s)$ are fuzzy subsets of X such that $t^l_A(s) \leq t^u_A(s)$ for all $x \in X$. An interval-valued fuzzy relation on X is an interval-valued fuzzy set B in $X \times X$.

Definition 2.2 ([30, 31])**.** The interval-valued neutrosophic set (IVN-set) A in X is defined by

$$A = \{(s, [t^l_A(s), t^u_A(s)], [i^l_A(s), i^u_A(s)], [f^l_A(s), f^u_A(s)]) : s \in X\},$$

where, $t^l_A(s)$, $t^u_A(s)$, $i^l_A(s)$, $i^u_A(s)$, $f^l_A(s)$, and $f^u_A(s)$ are neutrosophic subsets of X such that $t^l_A(s) \leq t^u_A(s)$, $i^l_A(s) \leq i^u_A(s)$ and $f^l_A(s) \leq f^u_A(s)$ for all $s \in X$. An interval-valued neutrosophic relation (IVN-relation) on X is an interval-valued neutrosophic set B in $X \times X$.

Definition 2.3 ([5]). An interval-valued neutrosophic digraph (IVN-digraph) on a non-empty set X is a pair $G = (A, \overrightarrow{B})$, (in short, G), where $A = ([t^l_A, t^u_A], [i^l_A, i^u_A], [f^l_A, f^u_A])$ is an IVN-set on X and $B = ([t^l_B, t^u_B], [i^l_B, i^u_B], [f^l_B, f^u_B])$ is an IVN-relation on X, such that:

(i) $t^l_B\overrightarrow{(s,w)} \leq t^l_A(s) \wedge t^l_A(w)$, $\quad t^u_B\overrightarrow{(s,w)} \leq t^u_A(s) \wedge t^u_A(w)$,

(ii) $i^l_B\overrightarrow{(s,w)} \leq i^l_A(s) \wedge i^l_A(w)$, $\quad i^u_B\overrightarrow{(s,w)} \leq i^u_A(s) \wedge i^u_A(w)$,

(iii) $f^l_B\overrightarrow{(s,w)} \leq f^l_A(s) \wedge f^l_A(w)$, $\quad f^u_B\overrightarrow{(s,w)} \leq f^u_A(s) \wedge f^u_A(w)$, for all $s, w \in X$.

Example 2.4. We construct an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 1.

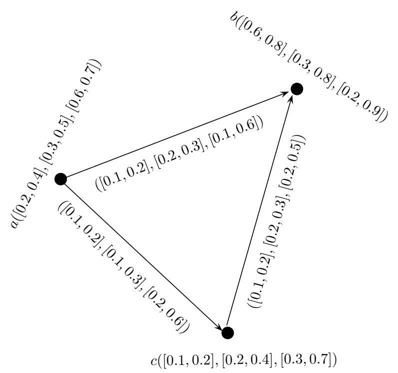


FIGURE 1. IVN-digraph

Definition 2.5. Let $\overrightarrow{G}$ be an IVN-digraph then interval-valued neutrosophic out-neighbourhoods (IVN-out-neighbourhoods) of a vertex s is an IVN-set

$$\mathbb{N}^+(s) = (X^+_s, [t^{(l)^+}_s, t^{(u)^+}_s], [i^{(l)^+}_s, i^{(u)^+}_s], [f^{(l)^+}_s, t^{(u)^+}_s]),$$

where

$$X^+_s = \{w | [t^l_B\overrightarrow{(s,w)} > 0, t^u_B\overrightarrow{(s,w)} > 0], [i^l_B\overrightarrow{(s,w)} > 0, i^u_B\overrightarrow{(s,w)} > 0], [f^l_B\overrightarrow{(s,w)} > 0, f^u_B\overrightarrow{(s,w)} > 0]\},$$

such that $t^{(l)^+}_s : X^+_s \to [0,1]$, defined by $t^{(l)^+}_s(w) = t^l_B\overrightarrow{(s,w)}$, $t^{(u)^+}_s : X^+_s \to [0,1]$, defined by $t^{(u)^+}_s(w) = t^u_B\overrightarrow{(s,w)}$, $i^{(l)^+}_s : X^+_s \to [0,1]$, defined by $i^{(l)^+}_s(w) = i^l_B\overrightarrow{(s,w)}$, $i^{(u)^+}_s : X^+_s \to [0,1]$, defined by $i^{(u)^+}_s(w) = i^u_B\overrightarrow{(s,w)}$, $f^{(l)^+}_s : X^+_s \to [0,1]$, defined by $f^{(l)^+}_s(w) = f^l_B\overrightarrow{(s,w)}$, $f^{(u)^+}_s : X^+_s \to [0,1]$, defined by $f^{(u)^+}_s(w) = f^u_B\overrightarrow{(s,w)}$.

Definition 2.6. Let $\overrightarrow{G}$ be an IVN-digraph then interval-valued neutrosophic in-neighbourhoods (IVN-in-neighbourhoods) of a vertex s is an IVN-set

$$\mathbb{N}^-(s) = (X^-_s, [t^{(l)^-}_s, t^{(u)^-}_s], [i^{(l)^-}_s, i^{(u)^-}_s], [f^{(l)^-}_s, t^{(u)^-}_s]),$$

where

$$X^-_s = \{w | [t^l_B\overrightarrow{(w,s)} > 0, t^u_B\overrightarrow{(w,s)} > 0], [i^l_B\overrightarrow{(w,s)} > 0, i^u_B\overrightarrow{(w,s)} > 0], [f^l_B\overrightarrow{(w,s)} > 0, f^u_B\overrightarrow{(w,s)} > 0]\},$$

such that $t_s^{(l)^-}: X_s^- \to [0,1]$, defined by $t_s^{(l)^-}(w) = t_B^l\overrightarrow{(w,s)}$, $t_s^{(u)^-}: X_s^- \to [0,1]$, defined by $t_s^{(u)^-}(w) = t_B^u\overrightarrow{(w,s)}$, $i_s^{(l)^-}: X_s^- \to [0,1]$, defined by $i_s^{(l)^-}(w) = i_B^l\overrightarrow{(w,s)}$, $i_s^{(u)^-}: X_s^- \to [0,1]$, defined by $i_s^{(u)^-}(w) = i_B^u\overrightarrow{(w,s)}$, $f_s^{(l)^-}: X_s^- \to [0,1]$, defined by $f_s^{(l)^-}(w) = f_B^l\overrightarrow{(w,s)}$, $f_s^{(u)^-}: X_s^- \to [0,1]$, defined by $f_s^{(u)^-}(w) = f_B^u\overrightarrow{(w,s)}$.

Example 2.7. Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 2.

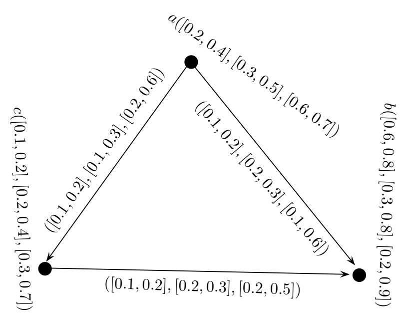


Figure 2. IVN-digraph

We have Table 1 and Table 2 representing interval-valued neutrosophic out and in-neighbourhoods, respectively.

Table 1. IVN-out-neighbourhoods

s	$\mathbb{N}^+(s)$
a	{(b, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.1,0.3],[0.2,0.6])}
b	∅
c	{(b, [0.1,0.2],[0.2,0.3],[0.2,0.5])}

Table 2. IVN-in-neighbourhoods

s	$\mathbb{N}^-(s)$
a	∅
b	{(a, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.2,0.3],[0.2,0.5])}
c	{(a, [0.1,0.2],[0.1,0.3],[0.2,0.6])}

Definition 2.8. The height of IVN-set $A = (s, [t_A^l, t_A^u], [i_A^l, i_A^u], [f_A^l, f_A^u])$ in universe of discourse X is defined as: for all $s \in X$,

$$h(A) = ([h_1^l(A), h_1^u(A)], [h_2^l(A), h_2^u(A)], [h_3^l(A), h_3^u(A)]),$$
$$= ([\sup_{s\in X} t_A^l(s), \sup_{s\in X} t_A^u(s)], [\sup_{s\in X} i_A^l(s), \sup_{s\in X} i_A^u(s)], [\inf_{s\in X} f_A^l(s), \inf_{s\in X} f_A^u(s)]).$$

Definition 2.9. An interval-valued neutrosophic competition graph (IVNC-graph) of an interval-valued neutrosophic graph (IVN-graph) $\overrightarrow{G} = (A, \overrightarrow{B})$ is an undirected IVN-graph $\mathbb{C}(\overrightarrow{G}) = (A, W)$ which has the same vertex set as in $\overrightarrow{G}$ and there is an edge between two vertices s and w if and only if $\mathbb{N}^+(s) \cap \mathbb{N}^+(w) \neq \varnothing$. The truth-membership, indeterminacy-membership and falsity-membership values of the edge (s, w) are defined as: for all $s, w \in X$,

(i) $t^l_W(s, w) = (t^l_A(s) \wedge t^l_A(w))h^l_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
$t^u_W(s, w) = (t^u_A(s) \wedge t^u_A(w))h^u_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
(ii) $i^l_W(s, w) = (i^l_A(s) \wedge i^l_A(w))h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
$i^u_W(s, w) = (i^u_A(s) \wedge i^u_A(w))h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
(iii) $f^l_W(s, w) = (f^l_A(s) \wedge f^l_A(w))h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
$f^u_W(s, w) = (f^u_A(s) \wedge f^u_A(w))h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$.

Example 2.10. Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 3.

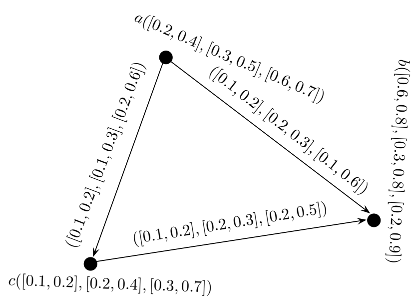


Figure 3. IVN-digraph

We have Table 3 and Table 4 representing interval-valued neutrosophic out and in-neighbourhoods, respectively.

Table 3. IVN-out-neighbourhoods

s	$\mathbb{N}^+(s)$
a	{(b, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.1,0.3],[0.2,0.6])}
b	∅
c	{(b, [0.1,0.2],[0.2,0.3],[0.2,0.5])}

Table 4. IVN-in-neighbourhoods

s	$\mathbb{N}^-(s)$
a	∅
b	{(a, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.2,0.3],[0.2,0.5])}
c	{(a, [0.1,0.2],[0.1,0.3],[0.2,0.6])}

Then IVNC-graph of Fig. 3 is shown in Fig. 4.

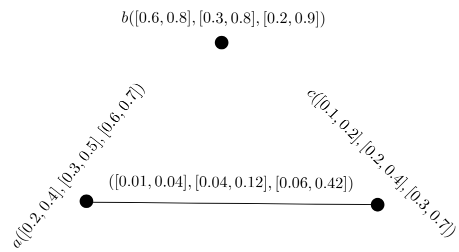


FIGURE 4. IVNC-graph

Definition 2.11. Consider an IVN-graph $G = (A, B)$, where $A = ([A_1^l, A_1^u], [A_2^l, A_2^u], [A_3^l, A_3^u)]$ and $B = ([B_1^l, B_1^u], [B_2^l, B_2^u], [B_3^l, B_3^u)]$. then, an edge (s, w), s, $w \in X$ is called independent strong, if

$$\begin{aligned}
\frac{1}{2}[A_1^l(s) \wedge A_1^l(w)] < B_1^l(s,w), \quad & \frac{1}{2}[A_1^u(s) \wedge A_1^u(w)] < B_1^u(s,w),\\
\frac{1}{2}[A_2^l(s) \wedge A_2^l(w)] < B_2^l(s,w), \quad & \frac{1}{2}[A_2^u(s) \wedge A_2^u(w)] < B_2^u(s,w),\\
\frac{1}{2}[A_3^l(s) \wedge A_3^l(w)] > B_3^l(s,w), \quad & \frac{1}{2}[A_3^u(s) \wedge A_3^u(w)] > B_3^u(s,w).
\end{aligned}$$

Otherwise, it is called weak.

We state the following theorems without their proofs.

Theorem 2.12. *Suppose $\overrightarrow{G}$ is an IVN-digraph. If $\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$ contains only one element of $\overrightarrow{G}$, then the edge (s, w) of $\mathbb{C}(\overrightarrow{G})$ is independent strong if and only if*

$$\begin{aligned}
|[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{t^l} > 0.5, \quad & |[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{t^u} > 0.5,\\
|[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{i^l} > 0.5, \quad & |[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{i^u} > 0.5,\\
|[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{f^l} < 0.5, \quad & |[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{f^u} < 0.5.
\end{aligned}$$

Theorem 2.13. *If all the edges of an IVN-digraph $\overrightarrow{G}$ are independent strong, then*

$$\begin{aligned}
\frac{B_1^l(s,w)}{(A_1^l(s) \wedge A_1^l(w))^2} > 0.5, \quad & \frac{B_1^u(s,w)}{(A_1^u(s) \wedge A_1^u(w))^2} > 0.5,\\
\frac{B_2^l(s,w)}{(A_2^l(s) \wedge A_2^l(w))^2} > 0.5, \quad & \frac{B_2^u(s,w)}{(A_2^u(s) \wedge A_2^u(w))^2} > 0.5,\\
\frac{B_3^l(s,w)}{(A_3^l(s) \wedge A_3^l(w))^2} < 0.5, \quad & \frac{B_3^u(s,w)}{(A_3^u(s) \wedge A_3^u(w))^2} < 0.5,
\end{aligned}$$

for all edges (s, w) in $\mathbb{C}(\overrightarrow{G})$.

Definition 2.14. The interval-valued neutrosophic open-neighbourhood (IVN-open-neighbourhood) of a vertex s of an IVN-graph $G = (A, B)$ is IVN-set $\mathbb{N}(s) = (X_s, [t_s^l, t_s^u], [i_s^l, i_s^u], [f_s^l, f_s^u])$, where

$$X_s = \{w|[B_1^l(s,w) > 0,\ B_1^u(s,w) > 0],\ [B_2^l(s,w) > 0,\ B_2^u(s,w) > 0],\ [B_3^l(s,w) > 0,\ B_3^u(s,w) > 0]\},$$

and $t_s^l : X_s \to [0,1]$ defined by $t_s^l(w) = B_1^l(s,\ w)$, $t_s^u : X_s \to [0,1]$ defined by $t_s^u(w) = B_1^u(s,\ w)$, $i_s^l : X_s \to [0,1]$ defined by $i_s^l(w) = B_2^l(s,\ w)$, $i_s^u : X_s \to [0,1]$ defined by $i_s^u(w) = B_2^u(s,\ w)$, $f_s^l : X_s \to [0,1]$ defined by $f_s^l(w) = B_3^l(s,\ w)$, $f_s^u : X_s \to [0,1]$ defined by $f_s^u(w) = B_3^u(s,\ w)$. For every vertex $s \in X$, the interval-valued neutrosophic singleton set, $\breve{A}_s = (s,\ [A_1^{l\prime},\ A_1^{u\prime}],\ [A_2^{l\prime},\ A_2^{u\prime}],\ [A_3^{l\prime},\ A_3^{u\prime})$ such that: $A_1^{l\prime} : \{s\} \to [0,1]$, $A_1^{u\prime} : \{s\} \to [0,1]$, $A_2^{l\prime} : \{s\} \to [0,1]$, $A_2^{u\prime} : \{s\} \to [0,1]$, $A_3^{l\prime} : \{s\} \to [0,1]$, $A_3^{u\prime} : \{s\} \to [0,1]$, defined by $A_1^{l\prime}(s) = A_1^l(s)$, $A_1^{u\prime}(s) = A_1^u(s)$, $A_2^{l\prime}(s) = A_2^l(s)$, $A_2^{u\prime}(s) = A_2^u(s)$, $A_3^{l\prime}(s) = A_3^l(s)$ and $A_3^{u\prime}(s) = A_3^u(s)$, respectively. The interval-valued neutrosophic closed-neighbourhood (IVN-closed-neighbourhood) of a vertex s is $\mathbb{N}[s] = \mathbb{N}(s) \cup A_s$.

Definition 2.15. Suppose $G = (A,\ B)$ is an IVN-graph. Interval-valued neutrosophic open-neighbourhood graph (IVN-open-neighbourhood-graph) of G is an IVN-graph $\mathbb{N}(G) = (A,\ B')$ which has the same IVN-set of vertices in G and has an interval-valued neutrosophic edge between two vertices $s,\ w \in X$ in $\mathbb{N}(G)$ if and only if $\mathbb{N}(s) \cap \mathbb{N}(w)$ is a non-empty IVN-set in G. The truth-membership, indeterminacy-membership, falsity-membership values of the edge $(s,\ w)$ are given by:

$$\begin{aligned}
B_1^{l\prime}(s,w) &= [A_1^l(s) \wedge A_1^l(w)]h_1^l(\mathbb{N}(s) \cap \mathbb{N}(w)),\\
B_2^{l\prime}(s,w) &= [A_2^l(s) \wedge A_2^l(w)]h_2^l(\mathbb{N}(s) \cap \mathbb{N}(w)),\\
B_3^{l\prime}(s,w) &= [A_3^l(s) \wedge A_3^l(w)]h_3^l(\mathbb{N}(s) \cap \mathbb{N}(w)),\\
B_1^{u\prime}(s,w) &= [A_1^u(s) \wedge A_1^u(w)]h_1^u(\mathbb{N}(s) \cap \mathbb{N}(w)),\\
B_2^{u\prime}(s,w) &= [A_2^u(s) \wedge A_2^u(w)]h_2^u(\mathbb{N}(s) \cap \mathbb{N}(w)),\\
B_3^{u\prime}(s,w) &= [A_3^u(s) \wedge A_3^u(w)]h_3^u(\mathbb{N}(s) \cap \mathbb{N}(w)), \text{ respectively.}
\end{aligned}$$

Definition 2.16. Suppose $G = (A,\ B)$ is an IVN-graph. Interval-valued neutrosophic closed-neighbourhood graph (IVN-closed-neighbourhood-graph) of G is an IVN-graph $\mathbb{N}(G) = (A,\ B')$ which has the same IVN-set of vertices in G and has an interval-valued neutrosophic edge between two vertices $s,\ w \in X$ in $\mathbb{N}[G]$ if and only if $\mathbb{N}[s] \cap \mathbb{N}[w]$ is a non-empty IVN-set in G. The truth-membership, indeterminacy-membership, falsity-membership values of the edge $(s,\ w)$ are given by:

$$\begin{aligned}
B_1^{l\prime}(s,w) &= [A_1^l(s) \wedge A_1^l(w)]h_1^l(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_2^{l\prime}(s,w) &= [A_2^l(s) \wedge A_2^l(w)]h_2^l(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_3^{l\prime}(s,w) &= [A_3^l(s) \wedge A_3^l(w)]h_3^l(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_1^{u\prime}(s,w) &= [A_1^u(s) \wedge A_1^u(w)]h_1^u(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_2^{u\prime}(s,w) &= [A_2^u(s) \wedge A_2^u(w)]h_2^u(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_3^{u\prime}(s,w) &= [A_3^u(s) \wedge A_3^u(w)]h_3^u(\mathbb{N}[s] \cap \mathbb{N}[w]), \text{ respectively.}
\end{aligned}$$

We now discuss the method of construction of interval-valued neutrospohic competition graph of the Cartesian product of IVN-digraph in following theorem which can be proof using similar method as used in [21], hence we omit its proof.

Theorem 2.17. *Let $\mathbb{C}(\overrightarrow{G_1}) = (A_1, B_1)$ and $\mathbb{C}(\overrightarrow{G_2}) = (A_2, B_2)$ be two IVNC-graphs of IVN-digraphs $\overrightarrow{G_1} = (A_1, \overrightarrow{L_1})$ and $\overrightarrow{G_2} = (A_2, \overrightarrow{L_2})$, respectively. Then $\mathbb{C}(\overrightarrow{G_1} \Box \overrightarrow{G_2}) = G_{\mathbb{C}(\overrightarrow{G_1})^* \Box \mathbb{C}(\overrightarrow{G_2})^*} \cup G^{\Box}$, where $G_{\mathbb{C}(\overrightarrow{G_1})^* \Box \mathbb{C}(\overrightarrow{G_2})^*}$ is an IVN-graph on the crisp graph $(X_1 \times X_2, E_{\mathbb{C}(\overrightarrow{G_1})^*} \Box E_{\mathbb{C}(\overrightarrow{G_2})^*})$, $\mathbb{C}(\overrightarrow{G_1})^*$ and $\mathbb{C}(\overrightarrow{G_2})^*$ are the crisp competition graphs of $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$, respectively. $G^{\Box}$ is an IVN-graph on $(X_1 \times X_2, E^{\Box})$ such that:*

(1) $E^{\Box} = \{(s_1, s_2)(w_1, w_2) : w_1 \in \mathbb{N}^-(s_1)^*, w_2 \in \mathbb{N}^+(s_2)^*\}$
$E_{\mathbb{C}(\overrightarrow{G_1})^*} \Box E_{\mathbb{C}(\overrightarrow{G_2})^*} = \{(s_1, s_2)(s_1, w_2) : s_1 \in X_1, s_2w_2 \in E_{\mathbb{C}(\overrightarrow{G_2})^*}\}$
$\cup\{(s_1, s_2)(w_1, s_2) : s_2 \in X_2, s_1w_1 \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\}$.

(2) $t^l_{A_1 \Box A_2} = t^l_{A_1}(s_1) \wedge t^l_{A_2}(s_2)$, $\quad i^l_{A_1 \Box A_2} = i^l_{A_1}(s_1) \wedge i^l_{A_2}(s_2)$, $\quad f^l_{A_1 \Box A_2} = f^l_{A_1}(s_1) \wedge f^l_{A_2}(s_2)$,
$t^u_{A_1 \Box A_2} = t^u_{A_1}(s_1) \wedge t^u_{A_2}(s_2)$, $\quad i^u_{A_1 \Box A_2} = i^u_{A_1}(s_1) \wedge i^u_{A_2}(s_2)$, $\quad f^u_{A_1 \Box A_2} = f^u_{A_1}(s_1) \wedge f^u_{A_2}(s_2)$.

(3) $t^l_B((s_1, s_2)(s_1, w_2)) = [t^l_{A_1}(s_1) \wedge t^l_{A_2}(s_2) \wedge t^l_{A_2}(w_2)] \times \vee_{a_2}\{t^l_{A_1}(s_1) \wedge t^l_{\overrightarrow{L_2}}(s_2a_2) \wedge t^l_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(4) $i^l_B((s_1, s_2)(s_1, w_2)) = [i^l_{A_1}(s_1) \wedge i^l_{A_2}(s_2) \wedge i^l_{A_2}(w_2)] \times \vee_{a_2}\{i^l_{A_1}(s_1) \wedge i^l_{\overrightarrow{L_2}}(s_2a_2) \wedge i^l_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(5) $f^l_B((s_1, s_2)(s_1, w_2)) = [f^l_{A_1}(s_1) \wedge f^l_{A_2}(s_2) \wedge f^l_{A_2}(w_2)] \times \vee_{a_2}\{f^l_{A_1}(s_1) \wedge f^l_{\overrightarrow{L_2}}(s_2a_2) \wedge f^l_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(6) $t^u_B((s_1, s_2)(s_1, w_2)) = [t^u_{A_1}(s_1) \wedge t^u_{A_2}(s_2) \wedge t^u_{A_2}(w_2)] \times \vee_{a_2}\{t^u_{A_1}(s_1) \wedge t^u_{\overrightarrow{L_2}}(s_2a_2) \wedge t^u_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(7) $i^u_B((s_1, s_2)(s_1, w_2)) = [i^u_{A_1}(s_1) \wedge i^u_{A_2}(s_2) \wedge i^u_{A_2}(w_2)] \times \vee_{a_2}\{i^u_{A_1}(s_1) \wedge i^u_{\overrightarrow{L_2}}(s_2a_2) \wedge i^u_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(8) $f^u_B((s_1, s_2)(s_1, w_2)) = [f^u_{A_1}(s_1) \wedge f^u_{A_2}(s_2) \wedge f^u_{A_2}(w_2)] \times \vee_{a_2}\{f^u_{A_1}(s_1) \wedge f^u_{\overrightarrow{L_2}}(s_2a_2) \wedge f^u_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(9) $t^l_B((s_1, s_2)(w_1, s_2)) = [t^l_{A_1}(s_1) \wedge t^l_{A_1}(w_1) \wedge t^l_{A_2}(s_2)] \times \vee_{a_1}\{t^l_{A_2}(s_2) \wedge t^l_{\overrightarrow{L_1}}(s_1a_1) \wedge t^l_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1, s_2)(w_1, s_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1 \in (\mathbb{N}^+(s_1) \cap \mathbb{N}^+(w_1))^*$.

(10) $i^l_B((s_1, s_2)(w_1, s_2)) = [i^l_{A_1}(s_1) \wedge i^l_{A_1}(w_1) \wedge i^l_{A_2}(s_2)] \times \vee_{a_1}\{i^l_{A_2}(s_2) \wedge i^l_{\overrightarrow{L_1}}(s_1a_1) \wedge i^l_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1, s_2)(w_1, s_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1 \in (\mathbb{N}^+(s_1) \cap \mathbb{N}^+(w_1))^*$.

(11) $f^l_B((s_1,s_2)(w_1,s_2)) = [f^l_{A_1}(s_1)\wedge f^l_{A_1}(w_1)\wedge f^l_{A_2}(s_2)]\times\vee_{a_1}\{t^l_{A_2}(s_2)\wedge f^l_{\overrightarrow{L_1}}(s_1a_1)\wedge f^l_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2)\in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1\in(\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.

(12) $t^u_B((s_1,s_2)(w_1,s_2)) = [t^u_{A_1}(s_1)\wedge t^u_{A_1}(w_1)\wedge t^u_{A_2}(s_2)]\times\vee_{a_1}\{t^u_{A_2}(s_2)\wedge t^u_{\overrightarrow{L_1}}(s_1a_1)\wedge t^u_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2)\in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1\in(\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.

(13) $i^u_B((s_1,s_2)(w_1,s_2)) = [i^u_{A_1}(s_1)\wedge i^u_{A_1}(w_1)\wedge i^u_{A_2}(s_2)]\times\vee_{a_1}\{i^u_{A_2}(s_2)\wedge i^u_{\overrightarrow{L_1}}(s_1a_1)\wedge i^u_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2)\in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1\in(\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.

(14) $f^u_B((s_1,s_2)(w_1,s_2)) = [f^u_{A_1}(s_1)\wedge f^u_{A_1}(w_1)\wedge f^u_{A_2}(s_2)]\times\vee_{a_1}\{t^u_{A_2}(s_2)\wedge f^u_{\overrightarrow{L_1}}(s_1a_1)\wedge f^u_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2)\in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1\in(\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.

(15) $t^l_B((s_1,s_2)(w_1,w_2)) = [t^l_{A_1}(s_1)\wedge t^l_{A_1}(w_1)\wedge t^l_{A_2}(s_2)\wedge t^l_{A_2}(w_2)]\times[t^l_{A_1}(s_1)\wedge t^l_{\overrightarrow{L_1}}(w_1s_1)\wedge t^l_{A_2}(w_2)\wedge t^l_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

(16) $i^l_B((s_1,s_2)(w_1,w_2)) = [i^l_{A_1}(s_1)\wedge i^l_{A_1}(w_1)\wedge i^l_{A_2}(s_2)\wedge i^l_{A_2}(w_2)]\times[i^l_{A_1}(s_1)\wedge i^l_{\overrightarrow{L_1}}(w_1s_1)\wedge i^l_{A_2}(w_2)\wedge i^l_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

(17) $f^l_B((s_1,s_2)(w_1,w_2)) = [f^l_{A_1}(s_1)\wedge f^l_{A_1}(w_1)\wedge f^l_{A_2}(s_2)\wedge f^l_{A_2}(w_2)]\times[f^l_{A_1}(s_1)\wedge f^l_{\overrightarrow{L_1}}(w_1s_1)\wedge f^l_{A_2}(w_2)\wedge f^l_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

(18) $t^u_B((s_1,s_2)(w_1,w_2)) = [t^u_{A_1}(s_1)\wedge t^u_{A_1}(w_1)\wedge t^u_{A_2}(s_2)\wedge t^u_{A_2}(w_2)]\times[t^u_{A_1}(s_1)\wedge t^u_{\overrightarrow{L_1}}(w_1s_1)\wedge t^u_{A_2}(w_2)\wedge t^u_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

(19) $i^u_B((s_1,s_2)(w_1,w_2)) = [i^u_{A_1}(s_1)\wedge i^u_{A_1}(w_1)\wedge i^u_{A_2}(s_2)\wedge i^u_{A_2}(w_2)]\times[i^u_{A_1}(s_1)\wedge i^u_{\overrightarrow{L_1}}(w_1s_1)\wedge i^u_{A_2}(w_2)\wedge i^u_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

(20) $f^u_B((s_1,s_2)(w_1,w_2)) = [f^u_{A_1}(s_1)\wedge f^u_{A_1}(w_1)\wedge f^u_{A_2}(s_2)\wedge f^u_{A_2}(w_2)]\times[f^u_{A_1}(s_1)\wedge f^u_{\overrightarrow{L_1}}(w_1s_1)\wedge f^u_{A_2}(w_2)\wedge f^u_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

A. k-competition interval-valued neutrosophic graphs

We now discuss an extension of IVNC-graphs, called k-competition IVN-graphs.

Definition 2.18. The cardinality of an IVN-set A is denoted by

$$|A| = \big([|A|_{t^l},\, |A|_{t^u}],\, [|A|_{i^l},\, |A|_{i^u}],\, [|A|_{f^l},\, |A|_{f^u}]\big).$$

Where $[|A|_{t^l},\, |A|_{t^u}]$, $[|A|_{i^l},\, |A|_{i^u}]$ and $[|A|_{f^l},\, |A|_{f^u}]$ represent the sum of truth-membership values, indeterminacy-membership values and falsity-membership values, respectively, of all the elements of A.

Example 2.19. The cardinality of an IVN-set $A = \{(a,\ [0.5,\ 0.7],\ [0.2,\ 0.8], [0.1,\ 0.3]), (b,\ [0.1,\ 0.2],\ [0.1,\ 0.5],\ [0.7,\ 0.9]), (c,\ [0.3,\ 0.5],\ [0.3,\ 0.8],\ [0.6,\ 0.9])\}$ in $X = \{a,\ b,\ c\}$ is

$$\begin{aligned}|A| &= \big([|A|_{t^l}, |A|_{t^u}], [|A|_{i^l}, |A|_{i^u}], [|A|_{f^l}, |A|_{f^u}]\big)\\ &= ([0.9, 1.4], [0.6, 2.1], [1.4, 2.1]).\end{aligned}$$

We now discuss k-competition IVN-graphs.

Definition 2.20. Let k be a non-negative number. Then k-competition IVN-graph $\mathbb{C}_k(\overrightarrow{G})$ of an IVN-digraph $\overrightarrow{G} = (A,\ \overrightarrow{B})$ is an undirected IVN-graph $G = (A,\ B)$ which has same IVN-set of vertices as in $\overrightarrow{G}$ and has an interval-valued neutrosophic edge between two vertices $s,\ w \in X$ in $\mathbb{C}_k(\overrightarrow{G})$ if and only if $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^l} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^u} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^l} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^u} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^l} > k$ and $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^u} > k$. The interval-valued truth-membership value of edge $(s,\ w)$ in $\mathbb{C}_k(\overrightarrow{G})$ is $t^l_B(s,\ w) = \frac{k^l_1 - k}{k^l_1}[t^l_A(s) \wedge t^l_A(w)]h^l_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^l_1 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^l}$ and $t^u_B(s,\ w) = \frac{k^u_1 - k}{k^u_1}[t^u_A(s) \wedge t^u_A(w)]h^u_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^u_1 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^u}$, the interval-valued indeterminacy-membership value of edge $(s,\ w)$ in $\mathbb{C}_k(\overrightarrow{G})$ is $i^l_B(s,\ w) = \frac{k^l_2 - k}{k^l_2}[i^l_A(s) \wedge i^l_A(w)]h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^l_2 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^l}$, and $i^u_B(s,\ w) = \frac{k^u_2 - k}{k^u_2}[i^u_A(s) \wedge i^u_A(w)]h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^u_2 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^u}$, the interval-valued falsity-membership value of edge $(s,\ w)$ in $\mathbb{C}_k(\overrightarrow{G})$ is $f^l_B(s,\ w) = \frac{k^l_3 - k}{k^l_3}[f^l_A(s) \wedge f^l_A(w)]h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^l_3 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^l}$, and $f^u_B(s,\ w) = \frac{k^u_3 - k}{k^u_3}[f^u_A(s) \wedge f^u_A(w)]h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^u_3 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^u}$.

Example 2.21. Consider an IVN-digraph $G = (A,\ \overrightarrow{B})$ on $X = \{s, w, a, b, c\}$, such that $A = \{(s,\ [0.4, 0.5],\ [0.5, 0.7],\ [0.8, 0.9]),\ (w,\ [0.6, 0.7],\ [0.4, 0.6],\ [0.2, 0.3]),\ (a,\ [0.2, 0.6],\ [0.3, 0.6],\ [0.2, 0.6]),\ (b,\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.6]),\ (c,\ [0.2, 0.7],\ [0.3, 0.5],\ [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)},\ [0.1, 0.4],\ [0.3, 0.6],\ [0.2, 0.6]),\ (\overrightarrow{(s, b)},\ [0.2, 0.4],\ [0.1, 0.5],\ [0.2, 0.6]),\ (\overrightarrow{(s, c)},\ [0.2, 0.5],\ [0.3, 0.5],\ [0.2, 0.6]),\ (\overrightarrow{(w, a)},\ [0.2, 0.5],\ [0.2, 0.5],\ [0.2, 0.3]),\ (\overrightarrow{(w, b)},\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.3]),\ (\overrightarrow{(w, c)},\ [0.2, 0.7],\ [0.3, 0.5],\ [0.2, 0.3])\}$, as shown in Fig. 5.

We calculate $\mathbb{N}^+(s) = \{(a,\ [0.1, 0.4],\ [0.3, 0.6],\ [0.2, 0.6]),\ (b,\ [0.2, 0.4],\ [0.1, 0.5],\ [0.2, 0.6]),\ (c,\ [0.2, 0.5],\ [0.3, 0.5],\ [0.2, 0.6])\}$ and $\mathbb{N}^+(w) = \{(a,\ [0.2, 0.5],\ [0.2, 0.5],\ [0.2, 0.3]),\ (b,\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.3]),\ (c,\ [0.2, 0.7],\ [0.3, 0.5],\ [0.2, 0.3])\}$. Therefore, $\mathbb{N}^+(s) \cap \mathbb{N}^+(w) = \{(a,\ [0.1, 0.4],\ [0.2, 0.5],\ [0.2, 0.3]),\ (b,\ [0.2, 0.4],\ [0.1, 0.5],\ [0.2, 0.3]),\ (c,\ [0.2, 0.5],\ [0.3, 0.5],\ [0.2, 0.3)\}$. So, $k^l_1 = 0.5$, $k^u_1 = 1.3$, $k^l_2 = 0.6$, $k^u_2 = 1.5$, $k^l_3 = 0.6$ and $k^u_3 = 0.9$. Let $k = 0.4$, then, $t^l_B(s,\ w) = 0.02$, $t^u_B(s,\ w) = 0.56$, $i^l_B(s,\ w) = 0.06$, $i^u_B(s,\ w) = 0.82$, $f^l_B(s,\ w) = 0.02$ and $f^u_B(s,\ w) = 0.11$. This graph is depicted in Fig. 6.

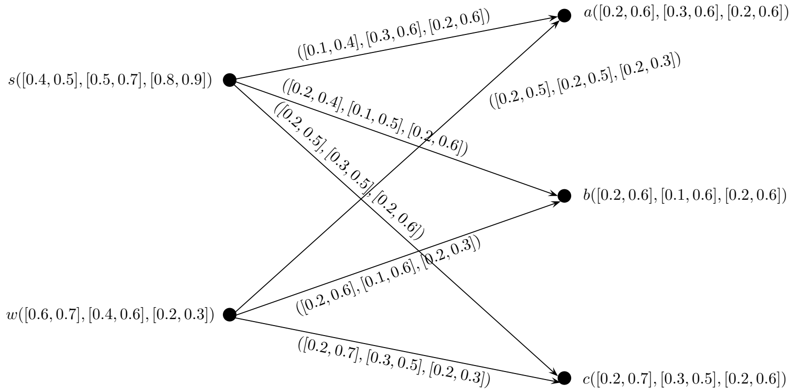


FIGURE 5. IVN-digraph

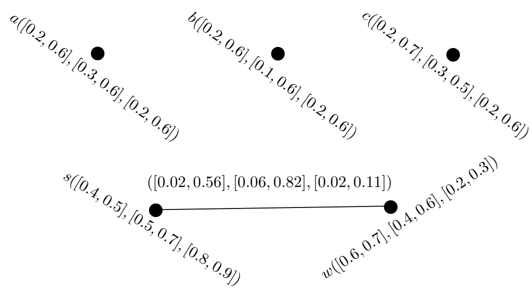


FIGURE 6. 0.4-Competition IVN-graph

Theorem 2.22. *Let $\overrightarrow{G} = (A,\ \overrightarrow{B})$ be an IVN-digraph. If*

$$h_1^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h_2^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h_3^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1,$$
$$h_1^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h_2^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h_3^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1,$$

and

$$|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^l} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^l} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^l} < 2k,$$
$$|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^u} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^u} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^u} < 2k,$$

Then the edge (s, w) is independent strong in $\mathbb{C}_k(\overrightarrow{G})$.

Proof. Let $\overrightarrow{G} = (A,\ \overrightarrow{B})$ be an IVN-digraph. Let $\mathbb{C}_k(\overrightarrow{G})$ be the corresponding k-competition IVN-graph.

If $h_1^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{t^l} > 2k$, then $k_1^l > 2k$. Thus,

$$t_B^l(s,w) = \frac{k_1^l - k}{k_1^l}[t_A^l(s)\wedge t_A^l(w)]h_1^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$$
$$\text{or,}\quad t_B^l(s,w) = \frac{k_1^l - k}{k_1^l}[t_A^l(s)\wedge t_A^l(w)]$$
$$\frac{t_B^l(s,w)}{[t_A^l(s)\wedge t_A^l(w)]} = \frac{k_1^l - k}{k_1^l} > 0.5.$$

If $h_1^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{t^u} > 2k$, then $k_1^u > 2k$. Thus,

$$t_B^u(s,w) = \frac{k_1^u - k}{k_1^u}[t_A^u(s)\wedge t_A^u(w)]h_1^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$$
$$\text{or,}\quad t_B^u(s,w) = \frac{k_1^u - k}{k_1^u}[t_A^u(s)\wedge t_A^u(w)]$$
$$\frac{t_B^u(s,w)}{[t_A^u(s)\wedge t_A^u(w)]} = \frac{k_1^u - k}{k_1^u} > 0.5.$$

If $h_2^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{i^l} > 2k$, then $k_2^l > 2k$. Thus,

$$i_B^l(s,w) = \frac{k_2^l - k}{k_2^l}[i_A^l(s)\wedge i_A^l(w)]h_2^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$$
$$\text{or,}\quad i_B^l(s,w) = \frac{k_2^l - k}{k_2^l}[i_A^l(s)\wedge i_A^l(w)]$$
$$\frac{i_B^l(s,w)}{[i_A^l(s)\wedge i_A^l(w)]} = \frac{k_2^l - k}{k_2^l} > 0.5.$$

If $h_2^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{i^u} > 2k$, then $k_2^u > 2k$. Thus,

$$i_B^u(s,w) = \frac{k_2^u - k}{k_2^u}[i_A^u(s)\wedge i_A^u(w)]h_2^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$$
$$\text{or,}\quad i_B^u(s,w) = \frac{k_2^u - k}{k_2^u}[i_A^u(s)\wedge i_A^u(w)]$$
$$\frac{i_B^u(s,w)}{[i_A^u(s)\wedge i_A^u(w)]} = \frac{k_2^u - k}{k_2^u} > 0.5.$$

If $h_3^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{f^l} < 2k$, then $k_3^l < 2k$. Thus,

$$f_B^l(s,w) = \frac{k_3^l - k}{k_3^l}[f_A^l(s)\wedge f_A^l(w)]h_3^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$$
$$\text{or,}\quad f_B^l(s,w) = \frac{k_3^l - k}{k_3^l}[f_A^l(s)\wedge f_A^l(w)]$$
$$\frac{f_B^l(s,w)}{[f_A^l(s)\wedge f_A^l(w)]} = \frac{k_3^l - k}{k_3^l} < 0.5.$$

If $h^u_3(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))=1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{f^u}<2k$, then $k^u_3<2k$. Thus,

$$f^u_B(s,w)=\frac{k^u_3-k}{k^u_3}[f^u_A(s)\wedge f^u_A(w)]h^u_3(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$$

$$\text{or,}\quad f^u_B(s,w)=\frac{k^u_3-k}{k^u_3}[f^u_A(s)\wedge f^u_A(w)]$$

$$\frac{f^u_B(s,w)}{[f^u_A(s)\wedge f^u_A(w)]}=\frac{k^u_3-k}{k^u_3}<0.5.$$

So, the edge (s,w) is independent strong in $\mathbb{C}_k(\overrightarrow{G})$. □

B. p-competition interval-valued neutrosophic graphs

We now define another extension of IVNC-graphs, called p-competition IVN-graphs.

Definition 2.23. The support of an IVN-set $A=(s,\,[t^l_A,\,t^u_A],\,[i^l_A,\,i^u_A],\,[f^l_A,\,f^u_A])$ in X is the subset of X defined by

$$supp(A)=\{s\in X:\,[t^l_A(s)\neq 0,\,t^u_A(s)\neq 0],\,[i^l_A(s)\neq 0,\,i^u_A(s)\neq 0],\,[f^l_A(s)\neq 1,\,f^u_A(s)\neq 1]\}$$

and $|supp(A)|$ is the number of elements in the set.

Example 2.24. The support of an IVN-set $A=\{(a,\,[0.5,\,0.7],\,[0.2,\,0.8],[0.1,\,0.3]),$ $(b,\,[0.1,\,0.2],\,[0.1,\,0.5],\,[0.7,\,0.9]),\,(c,\,[0.3,\,0.5],\,[0.3,\,0.8],\,[0.6,\,0.9]),\,(d,\,[0,\,0],\,[0,\,0],$ $[1,\,1])\}$ in $X=\{a,\,b,\,c,\,d\}$ is $supp(A)=\{a,b,c\}$ and $|supp(A)|=3$.

We now define p-competition IVN-graphs.

Definition 2.25. Let p be a positive integer. Then p-competition IVN-graph $\mathbb{C}^p(\overrightarrow{G})$ of the IVN-digraph $\overrightarrow{G}=(A,\,\overrightarrow{B})$ is an undirected IVN-graph $G=(A,\,B)$ which has same IVN-set of vertices as in $\overrightarrow{G}$ and has an interval-valued neutrosophic edge between two vertices $s,\,w\in X$ in $\mathbb{C}^p(\overrightarrow{G})$ if and only if $|supp(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|\geq p$. The interval-valued truth-membership value of edge $(s,\,w)$ in $\mathbb{C}^p(\overrightarrow{G})$ is $t^l_B(s,\,w)=\frac{(i-p)+1}{i}[t^l_A(s)\wedge t^l_A(w)]h^l_1(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, and $t^u_B(s,\,w)=\frac{(i-p)+1}{i}[t^u_A(s)\wedge t^u_A(w)]h^u_1(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, the interval-valued indeterminacy-membership value of edge $(s,\,w)$ in $\mathbb{C}^p(\overrightarrow{G})$ is $i^l_B(s,\,w)=\frac{(i-p)+1}{i}[i^l_A(s)\wedge i^l_A(w)]h^l_2(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, and $i^u_B(s,\,w)=\frac{(i-p)+1}{i}[i^u_A(s)\wedge i^u_A(w)]h^u_2(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, the interval-valued falsity-membership value of edge $(s,\,w)$ in $\mathbb{C}^p(\overrightarrow{G})$ is $f^l_B(s,\,w)=\frac{(i-p)+1}{i}[f^l_A(s)\wedge f^l_A(w)]h^l_3$ $(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, and $f^u_B(s,\,w)=\frac{(i-p)+1}{i}[f^u_A(s)\wedge f^u_A(w)]h^u_3(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, where $i=|supp(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|$.

Example 2.26. Consider an IVN-digraph $G=(A,\,\overrightarrow{B})$ on $X=\{s,w,a,b,c\}$, such that $A=\{(s,\,[0.4,0.5],\,[0.5,0.7],\,[0.8,0.9]),\,(w,\,[0.6,0.7],\,[0.4,0.6],\,[0.2,0.3]),\,(a,$ $[0.2,0.6],\,[0.3,0.6],\,[0.2,0.6]),\,(b,\,[0.2,0.6],\,[0.1,0.6],\,[0.2,0.6]),\,(c,\,[0.2,0.7],\,[0.3,0.5],$ $[0.2,0.6])\}$, and $B=\{(\overrightarrow{(s,a)},\,[0.1,0.4],\,[0.3,0.6],\,[0.2,0.6]),\,(\overrightarrow{(s,b)},\,[0.2,0.4],\,[0.1,0.5],$ $[0.2,0.6]),\,(\overrightarrow{(s,c)},\,[0.2,0.5],\,[0.3,0.5],\,[0.2,0.6]),\,(\overrightarrow{(w,a)},\,[0.2,0.5],\,[0.2,0.5],\,[0.2,0.3]),$ $(\overrightarrow{(w,b)},\,[0.2,0.6],\,[0.1,0.6],\,[0.2,0.3]),\,(\overrightarrow{(w,c)},\,[0.2,0.7],\,[0.3,0.5],\,[0.2,0.3])\}$, as shown in Fig. 7.

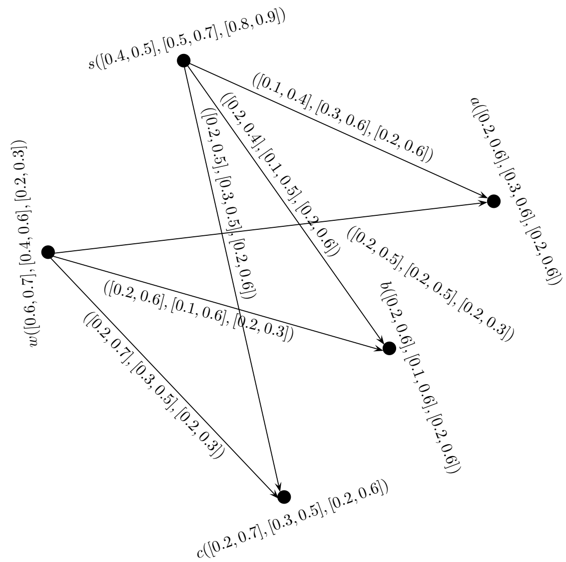


Figure 7. IVN-digraph

We calculate $\mathbb{N}^+(s) = \{(a,\ [0.1, 0.4],\ [0.3, 0.6],\ [0.2, 0.6]),\ (b,\ [0.2, 0.4],\ [0.1, 0.5],\ [0.2, 0.6]),\ (c,\ [0.2, 0.5],\ [0.3, 0.5],\ [0.2, 0.6])\}$ and $\mathbb{N}^+(w) = \{(a,\ [0.2, 0.5],\ [0.2, 0.5],\ [0.2, 0.3]),\ (b,\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.3]),\ (c,\ [0.2, 0.7],\ [0.3, 0.5],\ [0.2, 0.3])\}$. Therefore, $\mathbb{N}^+(s) \cap \mathbb{N}^+(w) = \{(a,\ [0.1, 0.4],\ [0.2, 0.5],\ [0.2, 0.3]),\ (b,\ [0.2, 0.4],\ [0.1, 0.5],\ [0.2, 0.3]),\ (c,\ [0.2, 0.5],\ [0.3, 0.5],\ [0.2, 0.3)\}$. Now, $i = |supp(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))| = 3$. For $p = 3$, we have, $t^l_B(s,\ w) = 0.02$, $t^u_B(s,\ w) = 0.08$, $i^l_B(s,\ w) = 0.04$, $i^u_B(s,\ w) = 0.1$, $f^l_B(s,\ w) = 0.01$ and $f^u_B(s,\ w) = 0.03$. This graph is depicted in Fig. 8.

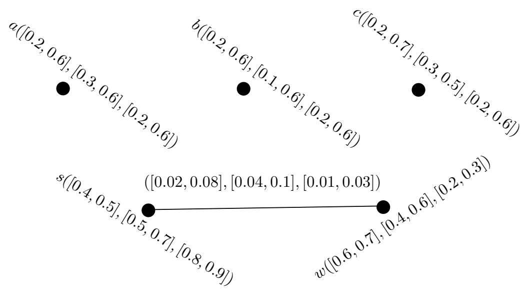


Figure 8. 3-Competition IVN-graph

We state the following theorem without its proof.

Theorem 2.27. *Let* $\overrightarrow{G} = (A,\ \overrightarrow{B})$ *be an IVN-digraph. If*

$$h^l_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 0,$$
$$h^u_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 0,$$

in $\mathbb{C}^{[\frac{i}{2}]}(\overrightarrow{G})$*, then the edge* (s,w) *is strong, where* $i = |supp(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|$*. (Note that for any real number* s*,* $[s]$*=greatest integer not esceeding* s*.)*

C. m-step interval-valued neutrosophic competition graphs

We now define another extension of IVNC-graph known as m-step IVNC-graph. We will use the following notations:

$P^m_{s,w}$: An interval-valued neutrosophic path of length m from s to w.
$\overrightarrow{P}^m_{s,w}$: A directed interval-valued neutrosophic path of length m from s to w.
$\mathbb{N}^+_m(s)$: m-step interval-valued neutrosophic out-neighbourhood of vertex s.
$\mathbb{N}^-_m(s)$: m-step interval-valued neutrosophic in-neighbourhood of vertex s.
$\mathbb{N}_m(s)$: m-step interval-valued neutrosophic neighbourhood of vertex s.
$\mathbb{N}_m(G)$: m-step interval-valued neutrosophic neighbourhood graph of the IVN-graph G.
$\mathbb{C}_m(\overrightarrow{G})$: m-step IVNC-graph of the IVN-digraph $\overrightarrow{G}$.

Definition 2.28. Suppose $\overrightarrow{G} = (A,\ \overrightarrow{B})$ is an IVN-digraph. The m-step IVN-digraph of $\overrightarrow{G}$ is denoted by $\overrightarrow{G}_m = (A,\ B)$, where IVN-set of vertices of $\overrightarrow{G}$ is same with IVN-set of vertices of $\overrightarrow{G}_m$ and has an edge between s and w in $\overrightarrow{G}_m$ if and only if there exists an interval-valued neutrosophic directed path $\overrightarrow{P}^m_{s,w}$ in $\overrightarrow{G}$.

Definition 2.29. The m-step interval-valued neutrosophic out-neighbourhood (IVN-out-neighbourhood) of vertex s of an IVN-digraph $\overrightarrow{G} = (A,\ \overrightarrow{B})$ is IVN-set

$$\mathbb{N}^+_m(s) = (X^+_s,\ [t^{(l)^+}_s,\ t^{(u)^+}_s],\ [i^{(l)^+}_s,\ i^{(u)^+}_s],\ [f^{(l)^+}_s,\ f^{(u)^+}_s]), \quad \text{where}$$

$X^+_s = \{w|$ there exists a directed interval-valued neutrosophic path of length m from s to w, $\overrightarrow{P}^m_{s,w}\}$, $t^{(l)^+}_s : X^+_s \to [0,\ 1]$, $t^{(u)^+}_s : X^+_s \to [0,\ 1]$, $i^{(l)^+}_s : X^+_s \to [0,\ 1]$, $i^{(u)^+}_s : X^+_s \to [0,\ 1]$, $f^{(l)^+}_s : X^+_s \to [0,\ 1]$ $f^{(u)^+}_s : X^+_s \to [0,\ 1]$ are defined by $t^{(l)^+}_s = \min\{t^l\overrightarrow{(s_1,\ s_2)},\ (s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $t^{(u)^+}_s = \min\{t^u\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $i^{(l)^+}_s = \min\{i^l\overrightarrow{(s_1,\ s_2)},\ (s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $i^{(u)^+}_s = \min\{i^u\overrightarrow{(s_1,\ s_2)},\ (s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $f^{(l)^+}_s = \min\{f^l\overrightarrow{(s_1,\ s_2)},\ (s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $f^{(u)^+}_s = \min\{f^u\overrightarrow{(s_1,\ s_2)},\ (s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, respectively.

Example 2.30. Consider an IVN-digraph $G = (A,\ \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s,\ [0.4, 0.5],\ [0.5, 0.7],\ [0.8, 0.9]),\ (w,\ [0.6, 0.7],\ [0.4, 0.6],\ [0.2, 0.3]),\ (a,\ [0.2, 0.6],\ [0.3, 0.6],\ [0.2, 0.6]),\ (b,\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.6]),\ (c,\ [0.2, 0.7],\ [0.3, 0.5],\ [0.2, 0.6]),\ d([0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)},\ [0.1, 0.4],\ [0.3, 0.6],\ [0.2, 0.6]),\ (\overrightarrow{(a, c)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.6]),\ (\overrightarrow{(a, d)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.4]),\ (\overrightarrow{(w, b)},\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.3]),\ (\overrightarrow{(b, c)},\ [0.2, 0.4],\ [0.1, 0.2],\ [0.1, 0.3]),\ (\overrightarrow{(b, d)},\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.4])\}$, as shown in Fig. 9.

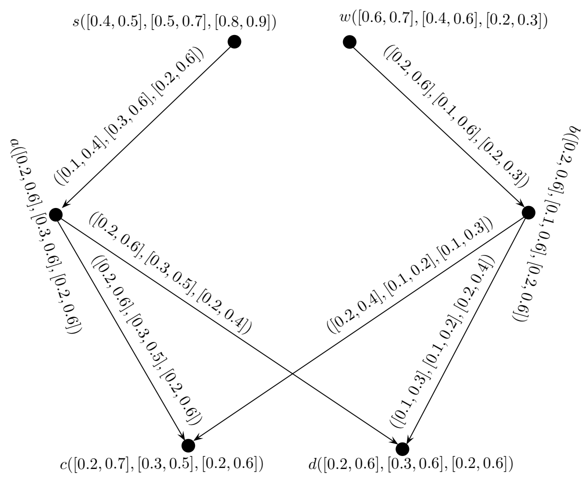


FIGURE 9. IVN-digraph

We calculate 2-step IVN-out-neighbourhoods as, $\mathbb{N}_2^+(s) = \{(c,\ [0.1, 0.4],\ [0.3, 0.5],\ [0.2, 0.6]),\ (d,\ [0.1, 0.4],\ [0.3, 0.5],\ [0.2, 0.4])\}$ and $\mathbb{N}_2^+(w) = \{(c,\ [0.2, 0.4],\ [0.1, 0.2],\ [0.1, 0.3]),\ (d,\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.3])\}$.

Definition 2.31. The m-step interval-valued neutrosophic in-neighbourhood (IVN-in-neighbourhood) of vertex s of an IVN-digraph $\overrightarrow{G} = (A,\ \overrightarrow{B})$ is IVN-set

$$\mathbb{N}_m^-(s) = (X_s^-,\ [t_s^{(l)^-},\ t_s^{(u)^-}],\ [i_s^{(l)^-},\ i_s^{(u)^-}],\ [f_s^{(l)^-},\ f_s^{(u)^-}]), \quad \text{where}$$

$X_s^- = \{w|$ there exists a directed interval-valued neutrosophic path of length m from w to s, $\overrightarrow{P}_{w,s}^m\}$, $t_s^{(l)^-} : X_s^- \to [0,\ 1]$, $t_s^{(u)^-} : X_s^- \to [0,\ 1]$, $i_s^{(l)^-} : X_s^- \to [0,\ 1]$, $i_s^{(u)^-} : X_s^- \to [0,\ 1]$, $f_s^{(l)^-} : X_s^- \to [0,\ 1]$ $f_s^{(u)^-} : X_s^- \to [0,\ 1]$ are defined by $t_s^{(l)^-} = \min\{t^l\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}_{w,s}^m\}$, $t_s^{(u)^-} = \min\{t^u\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}_{w,s}^m\}$, $i_s^{(l)^-} = \min\{i^l\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}_{w,s}^m\}$, $i_s^{(u)^-} = \min\{i^u\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}_{w,s}^m\}$, $f_s^{(l)^-} = \min\{f^l\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}_{w,s}^m\}$, $f_s^{(u)^-} = \min\{f^u\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}_{w,s}^m\}$, respectively.

Example 2.32. Consider an IVN-digraph $G = (A,\ \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s,\ [0.4, 0.5],\ [0.5, 0.7],\ [0.8, 0.9]),\ (w,\ [0.6, 0.7],\ [0.4, 0.6],\ [0.2, 0.3]),\ (a,\ [0.2, 0.6],\ [0.3, 0.6],\ [0.2, 0.6]),\ (b,\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.6]),\ (c,\ [0.2, 0.7],\ [0.3, 0.5],\ [0.2, 0.6]),\ d([0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)},\ [0.1, 0.4],\ [0.3, 0.6],\ [0.2, 0.6]),\ (\overrightarrow{(a, c)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.6]),\ (\overrightarrow{(a, d)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.4]),\ (\overrightarrow{(w, b)},\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.3]),\ (\overrightarrow{(b, c)},\ [0.2, 0.4],\ [0.1, 0.2],\ [0.1, 0.3]),\ (\overrightarrow{(b, d)},\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.4])\}$, as shown in Fig. 10.

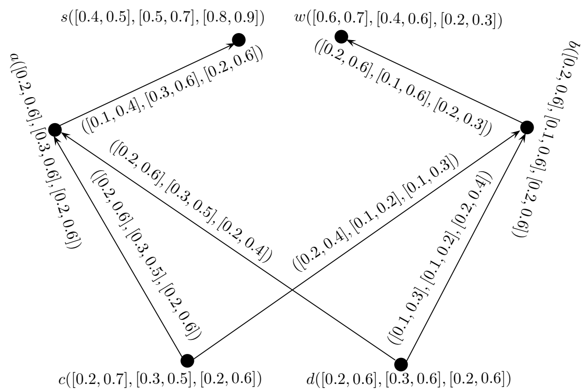


FIGURE 10. IVN-digraph

We calculate 2-step IVN-in-neighbourhoods as, $\mathbb{N}_2^-(s) = \{(c,$ [0.1,0.4], [0.3,0.5], [0.2,0.6]), (d, [0.1,0.4], [0.3,0.5], [0.2,0.4])$\}$ and $\mathbb{N}_2^-(w) = \{(c,$ [0.2,0.4], [0.1,0.2], [0.1,0.3]), (d, [0.1,0.3], [0.1,0.2], [0.2,0.3])$\}$.

Definition 2.33. Suppose $\overrightarrow{G} = (A,\ \overrightarrow{B})$ is an IVN-digraph. The m-step IVNC-graph of IVN-digraph $\overrightarrow{G}$ is denoted by $\mathbb{C}_m(\overrightarrow{G}) = (A,\ B)$ which has same IVN-set of vertices as in $\overrightarrow{G}$ and has an edge between two vertices $s,\ w \in X$ in $\mathbb{C}_m(\overrightarrow{G})$ if and only if $(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$ is a non-empty IVN-set in $\overrightarrow{G}$. The interval-valued truth-membership value of edge $(s,\ w)$ in $\mathbb{C}_m(\overrightarrow{G})$ is $t_B^l(s,\ w) = [t_A^l(s) \wedge t_A^l(w)]h_1^l(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$, and $t_B^u(s,\ w) = [t_A^u(s) \wedge t_A^u(w)]h_1^u(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$, the interval-valued indeterminacy-membership value of edge $(s,\ w)$ in $\mathbb{C}_m(\overrightarrow{G})$ is $i_B^l(s,\ w) = [i_A^l(s) \wedge i_A^l(w)]h_2^l(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$, and $i_B^u(s,\ w) = [i_A^u(s) \wedge i_A^u(w)]h_2^u(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$, the interval-valued falsity-membership value of edge $(s,\ w)$ in $\mathbb{C}_m(\overrightarrow{G})$ is $f_B^l(s,\ w) = [f_A^l(s) \wedge f_A^l(w)]h_3^l(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$, and $f_B^u(s,\ w) = [f_A^u(s) \wedge f_A^u(w)]h_3^u(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$.

The 2−step IVNC-graph is illustrated by the following example.

Example 2.34. Consider an IVN-digraph $G = (A,\ \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s,\ [0.4, 0.5],\ [0.5, 0.7],\ [0.8, 0.9]),\ (w,\ [0.6, 0.7],\ [0.4, 0.6],\ [0.2, 0.3]),\ (a,\ [0.2, 0.6],\ [0.3, 0.6],\ [0.2, 0.6]),\ (b,\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.6]),\ (c,\ [0.2, 0.7],\ [0.3, 0.5],\ [0.2, 0.6]),\ d([0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)},\ [0.1, 0.4],\ [0.3, 0.6],\ [0.2, 0.6]),\ (\overrightarrow{(a, c)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.6]),\ (\overrightarrow{(a, d)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.4]),\ (\overrightarrow{(w, b)},\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.3]),\ (\overrightarrow{(b, c)},\ [0.2, 0.4],\ [0.1, 0.2],\ [0.1, 0.3]),\ (\overrightarrow{(b, d)},\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.4])\}$, as shown in Fig. 11.

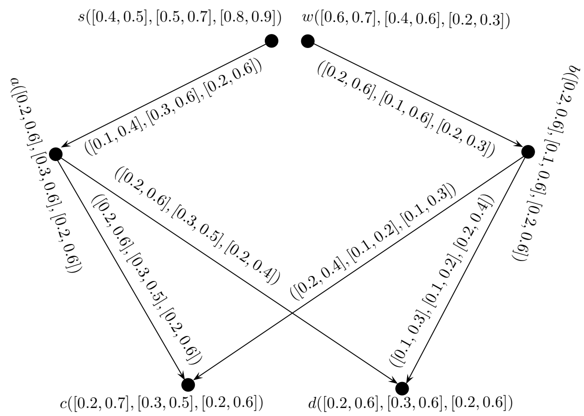


FIGURE 11. IVN-digraph

We calculate $\mathbb{N}_2^+(s) = \{(c,\ [0.1, 0.4],\ [0.3, 0.5],\ [0.2, 0.6]),\ (d,\ [0.1, 0.4],\ [0.3, 0.5],\ [0.2, 0.4])\}$ and $\mathbb{N}_2^+(w) = \{(c,\ [0.2, 0.4],\ [0.1, 0.2],\ [0.1, 0.3]),\ (d,\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.3])\}$. Therefore, $\mathbb{N}_2^+(s) \cap \mathbb{N}_2^+(w) = \{(c,\ [0.1, 0.4],\ [0.1, 0.2],\ [0.2, 0.6]),\ (d,\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.4])\}$. Thus, $t_B^l(s,\ w) = 0.04$, $t_B^u(s,\ w) = 0.20$, $i_B^l(s,\ w) = 0.04$, $i_B^u(s,\ w) = 0.12$, $f_B^l(s,\ w) = 0.04$ and $f_B^u(s,\ w) = 0.12$. This graph is depicted in Fig. 12.

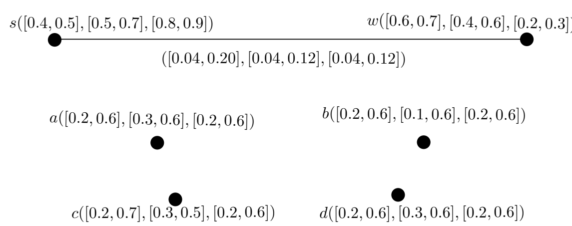


FIGURE 12. 2-Step IVNC-graph

If a predator s attacks one prey w, then the linkage is shown by an edge $\overrightarrow{(s, w)}$ in an IVN-digraph. But, if predator needs help of many other mediators s_1, s_2, $\ldots$, s_{m-1}, then linkage among them is shown by interval-valued neutrosophic directed path $\overrightarrow{P}_{s,w}^{m}$ in an IVN-digraph. So, m-step prey in an IVN-digraph is represented by a vertex which is the m-step out-neighbourhood of some vertices. Now, the strength of an IVNC-graphs is defined below.

Definition 2.35. Let $\overrightarrow{G} = (A,\ \overrightarrow{B})$ be an IVN-digraph. Let w be a common vertex of m-step out-neighbourhoods of vertices $s_1,\ s_2,\ \ldots,\ s_l$. Also, let $\overrightarrow{B_1^l}(u_1, v_1)$, $\overrightarrow{B_1^l}(u_2, v_2)$, $\ldots$, $\overrightarrow{B_1^l}(u_r, v_r)$ and $\overrightarrow{B_1^u}(u_1, v_1)$, $\overrightarrow{B_1^u}(u_2, v_2)$, $\ldots$, $\overrightarrow{B_1^u}(u_r, v_r)$ be the minimum interval-valued truth-membership values, $\overrightarrow{B_2^l}(u_1, v_1)$, $\overrightarrow{B_2^l}(u_2, v_2)$,$\ldots$, $\overrightarrow{B_2^l}(u_r, v_r)$ and $\overrightarrow{B_2^u}(u_1, v_1)$, $\overrightarrow{B_2^u}(u_2, v_2)$, $\ldots$, $\overrightarrow{B_2^u}(u_r, v_r)$ be the minimum indeterminacy-membership values, $\overrightarrow{B_3^l}(u_1, v_1)$, $\overrightarrow{B_3^l}(u_2, v_2)$, $\ldots$, $\overrightarrow{B_3^l}(u_r, v_r)$ and $\overrightarrow{B_3^u}(u_1, v_1)$, $\overrightarrow{B_3^u}(u_2, v_2)$, $\ldots$, $\overrightarrow{B_3^u}(u_r, v_r)$ be the maximum false-membership values, of edges of the paths $\overrightarrow{P}^m_{s_1,w}$, $\overrightarrow{P}^m_{s_2,w}$, $\ldots$, $\overrightarrow{P}^m_{s_r,w}$, respectively. The m-step prey $w \in X$ is strong prey if

$$\begin{aligned}&\overrightarrow{B_1^l}(u_i, v_i) > 0.5, \quad \overrightarrow{B_2^l}(u_i, v_i) > 0.5, \quad \overrightarrow{B_3^l}(u_i, v_i) < 0.5,\\ &\overrightarrow{B_1^u}(u_i, v_i) > 0.5, \quad \overrightarrow{B_2^u}(u_i, v_i) > 0.5, \quad \overrightarrow{B_3^u}(u_i, v_i) < 0.5, \text{ for all } i = 1, 2, \ldots, r.\end{aligned}$$

The strength of the prey w can be measured by the mapping $S : X \to [0, 1]$, such that:

$$\begin{aligned}S(w) =& \frac{1}{r}\Bigg\{\sum_{i=1}^{r}[\overrightarrow{B_1^l}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_1^u}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^l}(u_i, v_i)]\\ &+ \sum_{i=1}^{r}[\overrightarrow{B_2^u}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^l}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^u}(u_i, v_i)]\Bigg\}.\end{aligned}$$

Example 2.36. Consider an IVN-digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ as shown in Fig. 11, the strength of the prey c is equal to

$$\begin{aligned}\frac{(0.2 + 0.2) + (0.6 + 0.4) + (0.1 + 0.1) + (0.6 + 0.2) - (0.2 + 0.1) - (0.3 + 0.3)}{2} &= 1.5\\ &> 0.5.\end{aligned}$$

Hence, c is strong 2-step prey.

We state the following theorem without its proof.

Theorem 2.37. *If a prey w of $\overrightarrow{G} = (A, \overrightarrow{B})$ is strong, then the strength of w, $S(w) > 0.5$.*

Remark 2.38. The converse of the above theorem is not true, i.e. if $S(w) > 0.5$, then all preys may not be strong. This can be explained as:
Let $S(w) > 0.5$ for a prey w in $\overrightarrow{G}$. So,

$$\begin{aligned}S(w) =& \frac{1}{r}\Bigg\{\sum_{i=1}^{r}[\overrightarrow{B_1^l}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_1^u}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^l}(u_i, v_i)]\\ &+ \sum_{i=1}^{r}[\overrightarrow{B_2^u}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^l}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^u}(u_i, v_i)]\Bigg\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\Bigg\{\sum_{i=1}^{r}[\overrightarrow{B_1^l}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_1^u}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^l}(u_i, v_i)]\\ &+ \sum_{i=1}^{r}[\overrightarrow{B_2^u}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^l}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^u}(u_i, v_i)]\Bigg\} > \frac{r}{2}.\end{aligned}$$

This result does not necessarily imply that

$$\overrightarrow{B_1^l}(u_i, v_i) > 0.5, \quad \overrightarrow{B_2^l}(u_i, v_i) > 0.5, \quad \overrightarrow{B_3^l}(u_i, v_i) < 0.5,$$
$$\overrightarrow{B_1^u}(u_i, v_i) > 0.5, \quad \overrightarrow{B_2^u}(u_i, v_i) > 0.5, \quad \overrightarrow{B_3^u}(u_i, v_i) < 0.5, \text{ for all } i = 1, 2, \ldots, r.$$

Since, all edges of the directed paths $\overrightarrow{P}^m_{s_1,w}$, $\overrightarrow{P}^m_{s_2,w}$,$\ldots$,$\overrightarrow{P}^m_{s_r,w}$, are not strong. So, the converse of the above statement is not true i.e., if $S(w) > 0.5$, the prey w of $\overrightarrow{G}$ may not be strong. Now, m-step interval-valued neutrosophic neighbouhood graphs are defines below.

Definition 2.39. The m-step IVN-out-neighbourhood of vertex s of an IVN-digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ is IVN-set

$$\mathbb{N}_m(s) = (X_s, [t_s^l, t_s^u], [i_s^l, i_s^u], [f_s^l, f_s^u]), \quad \text{where}$$

$X_s = \{w|$ there exists a directed interval-valued neutrosophic path of length m from s to w, $\mathbb{P}^m_{s,w}\}$, $t_s^l : X_s \to [0, 1]$, $t_s^u : X_s \to [0, 1]$, $i_s^l : X_s \to [0, 1]$, $i_s^u : X_s \to [0, 1]$, $f_s^l : X_s \to [0, 1]$, $f_s^u : X_s \to [0, 1]$, are defined by $t_s^l = \min\{t^l(s_1, s_2)$, (s_1, s_2) is an edge of $\mathbb{P}^m_{s,w}\}$, $t_s^u = \min\{t^u(s_1, s_2)$, (s_1, s_2) is an edge of $\mathbb{P}^m_{s,w}\}$, $i_s^l = \min\{i^l(s_1, s_2)$, (s_1, s_2) is an edge of $\mathbb{P}^m_{s,w}\}$, $i_s^u = \min\{i^u(s_1, s_2)$, (s_1, s_2) is an edge of $\mathbb{P}^m_{s,w}\}$, $f_s^l = \min\{f^l(s_1, s_2)$, (s_1, s_2) is an edge of $\mathbb{P}^m_{s,w}\}$, $f_s^u = \min\{f^u(s_1, s_2)$, (s_1, s_2) is an edge of $\mathbb{P}^m_{s,w}\}$, respectively.

Definition 2.40. Suppose $G = (A, B)$ is an IVN-graph. Then m-step interval-valued neutrosophic neighbouhood graph $\mathbb{N}_m(G)$ is defined by $\mathbb{N}_m(G) = (A, \acute{B})$ where $A = ([A_1^l, A_1^u], [A_2^l, A_2^u], [A_3^l, A_3^u])$, $\acute{B} = ([\acute{B}_1^l, \acute{B}_1^u], [\acute{B}_2^l, \acute{B}_2^u], [\acute{B}_3^l, \acute{B}_3^u])$, $\acute{B}_1^l : X \times X \to [0, 1]$, $\acute{B}_1^u : X \times X \to [0, 1]$, $\acute{B}_2^l : X \times X \to [0,1]$, $\acute{B}_2^u : X \times X \to [0, 1]$, $\acute{B}_3^l : X \times X \to [0, 1]$, and $\acute{B}_3^u : X \times X \to [0, -1]$ are such that:

$$\acute{B}_1^l(s, w) = A_1^l(s) \wedge A_1^l(w) h_1^l(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}_2^l(s, w) = A_2^l(s) \wedge A_2^l(w) h_2^l(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}_3^l(s, w) = A_3^l(s) \wedge A_3^l(w) h_3^l(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}_1^u(s, w) = A_1^u(s) \wedge A_1^u(w) h_1^u(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}_2^u(s, w) = A_2^u(s) \wedge A_2^u(w) h_2^u(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}_3^u(s, w) = A_3^u(s) \wedge A_3^u(w) h_3^u(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)), \text{respectively.}$$

We state the following theorems without thier proofs.

Theorem 2.41. *If all preys of* $\overrightarrow{G} = (A, \overrightarrow{B})$ *are strong, then all edges of* $\mathbb{C}_m(\overrightarrow{G}) = (A, B)$ *are strong.*

A relation is established between m-step IVNC-graph of an IVN-digraph and IVNC-graph of m-step IVN-digraph.

Theorem 2.42. *If* $\overrightarrow{G}$ *is an IVN-digraph and* $\overrightarrow{G_m}$ *is the* m*-step IVN-digraph of* $\overrightarrow{G}$*, then* $\mathbb{C}(\overrightarrow{G}_m) = \mathbb{C}_m(\overrightarrow{G})$.

Theorem 2.43. *Let* $\overrightarrow{G} = (A, \overrightarrow{B})$ *be an IVN-digraph. If* $m > |X|$ *then* $\mathbb{C}_m(\overrightarrow{G}) = (A, B)$ *has no edge.*

Theorem 2.44. *If all the edges of IVN-digraph* $\overrightarrow{G} = (A, \overrightarrow{B})$ *are independent strong, then all the edges of* $\mathbb{C}_m(\overrightarrow{G})$ *are independent strong.*

3. Conclusions

Graph theory is an enjoyable playground for the research of proof techniques in discrete mathematics. There are many applications of graph theory in different fields. We have introduced IVNC-graphs and k-competition IVN-graphs, p-competition IVN-graphs and m-step IVNC-graphs as the generalized structures of IVNC-graphs. We have described interval-valued neutrosophic open and closed-neighbourhood. Also we have established some results related to them. We aim to extend our research work to (1) Interval-valued fuzzy rough graphs; (2) Interval-valued fuzzy rough hypergraphs, (3) Interval-valued fuzzy rough neutrosophic graphs, and (4) Decision support systems based on IVN-graphs.

Acknowledgment: The authors are thankful to Editor-in-Chief and the referees for their valuable comments and suggestions.

References

[1] M. Akram and W. A. Dudek, Interval-valued fuzzy graphs, Computers & Mathematics with Applications 61 (2011) 289–299.
[2] M. Akram, Interval-valued fuzzy line graphs, Neural Computing & Applications 21 (2012) 145–150.
[3] M. Akram, N. O. Al-Shehrie and W. A. Dudek, Certain types of interval-valued fuzzy graphs, Journal of Applied Mathematics Volume 2013 Article ID 857070 (2013) 11 pages.
[4] M. Akram, W. A. Dudek and M. Murtaza Yousaf, Self centered interval-valued fuzzy graphs, Afrika Matematika 26 (5-6) (2015) 887–898.
[5] M. Akram and M. Nasir, Concepts of interval-valued neutrosophic graphs, International Journal of Algebra and Statistics 6(1-2) (2017) DOI :10.20454/ijas.2017.1235 22–41.
[6] M. Akram and S. Shahzadi, Neutrosophic soft graphs with application, Journal of Intelligent & Fuzzy Systems 32 (1) (2017) 841–858.
[7] M. Akram, Single-valued neutrosophic planar graphs, International Journal of Algebra and Statistics 5 (2) (2016) 157–167.
[8] M. Akram and G. Shahzadi, Operations on single-valued neutrosophic graphs, Journal of Uncertain System 11 (3) (2017) 1–26.
[9] M. Akram and M. Sitara, Novel applications of single-valued neutrosophic graph structures in decision making, Journal of Applied Mathematics and Computing (2016) DOI 10.1007/s12190-017-1084-5 1–32.
[10] M. Akram and A. Adeel, Representation of labeling tree based on m- polar fuzzy sets, Ann. Fuzzy Math. Inform. 13 (2) (2017) 189–197.
[11] N. O. Al-Shehrie and M. Akram, Bipolar fuzzy competition graphs, Ars Combinatoria 121 (2015) 385–402.
[12] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and Systems 20 (1986) 87–96.
[13] P. Bhattacharya, Some remark on fuzzy graphs, Pattern Recognition Letters 6 (1987) 297–302.
[14] S. Broumi, M. Talea, A. Bakali and F. Smarandache, Single-valued neutrosophic graphs, Journal of New Theory 10 (2016) 86–101.
[15] T. Dinesh, A study on graph structures, incidence algebras and their fuzzy analogues [Ph.D.thesis], Kannur University Kannur India 2011.
[16] J. Hongmei and W. Lianhua, Interval-valued fuzzy subsemigroups and subgroups sssociated by interval-valued fuzzy graphs, 2009 WRI Global Congress on Intelligent Systems (2009) 484–487.

[17] M. G. Karunambigai and R. Buvaneswari, Degrees in intuitionistic fuzzy graphs, Ann. Fuzzy Math. Inform. 13 (3) (2017) 345–357.
[18] A. Kauffman, Introduction a la theorie des sousemsembles flous, Masson et cie Paris 1973.
[19] J. M. Mendel, Uncertain rule-based fuzzy logic systems: Introduction and new directions, Prentice-Hall, Upper Saddle River, New Jersey 2001.
[20] J. N. Mordeson and P. Chang-Shyh, Operations on fuzzy graphs, Inform. Sci. 79 (1994) 159–170.
[21] M. Nasir, S. Siddique and M. Akram, Novel properties of intuitionistic fuzzy competition graphs, Journal of Uncertain Systems 2 (1) (2017) 49–67.
[22] A. Rosenfeld, Fuzzy graphs, Fuzzy Sets and their Application, Academic press New York (1975) 77–95.
[23] S. Samanta and M. Pal, Fuzzy k-competition graphs and p-competition fuzzy graphs, Fuzzy Information and Engineering 5 (2013) 191–204.
[24] S. Samanta, M. Akram and M. Pal, m-step fuzzy competition graphs, Journal of Applied Mathematics and Computing 47 (2015) 461–472.
[25] M. Sarwar and M. Akram, Novel concepts of bipolar fuzzy competition graphs, Journal of Applied Mathematics and Computing (2016) DOI 10.1007/s12190-016-1021-z.
[26] F. Smarandache, A unifying field in logics. neutrosophy: neutrosophic probability, set and logic. Rehoboth:, American Research Press 1999.
[27] F. Smarandache, Neutrosophic set- a generalization of the intuitionistic fuzzy set, Granular Computing. 2006 IEEE International Conference (2006) DOI: 10.1109/GRC.2006.1635754 38–42.
[28] F. Smarandache, Neutrosophy. Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA 105 p. (1998).
[29] H. Wang, F. Smarandache, Y. Q. Zhang and R. Sunderraman, Single-valued neutrosophic sets, Multisspace and Multistruct 4 (2010) 410–413.
[30] H. Wang, Y. Zhang and R. Sunderraman, Truth-value based interval neutrosophic sets, Granular Computing, 2005 IEEE International Conference 1 (2005) DOI: 10.1109/GRC.2005.1547284 274-277.
[31] H. Wang, F. Smarandache, Y. Q. Zhang and R. Sunderram, An Interval neutrosophic sets and logic: theory and applications in computing, Hexis Arizona (2005).
[32] S. Y. Wu, The compositions of fuzzy digraphs, Journal of Research in Education Science 31 (1986) 603–628.
[33] J. Ye, Single-valued neutrosophic minimum spanning tree and its clustering method, Journal of Intelligent Systems 23 (3) (2014) 311–324.
[34] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[35] L. A. Zadeh, The concept of a linguistic and its application to approximate reasoning I, Inform. Sci. 8 (1975) 199–249.
[36] L. A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.

Muhammad Akram (makrammath@yahoo.com)
Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan

Maryam Nasir (maryamnasir912@gmail.com)
Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 1, (July 2017), pp. 1–120
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Contents
[The special issue for "Neutrosophic Sets"]