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A method of ranking interval numbers based on degrees for multiple attribute decision making

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Abstract. In order to deal with the difficulty of ranking interval numbers in the multiple attribute decision making process, interval numbers are expressed in the Rectangular Coordinate System. On the basis of this, two-dimensional relations of interval numbers are analyzed. For interval numbers, their advantage degree functions of the symmetry axis and the length are deduced after an information mining process, and then the advantage degree function of interval numbers is defined. Procedures of ranking interval numbers based on degrees for multiple attribute decision making are given. Finally, the feasibility and the effectiveness of this method are verified through an example.

Keywords: Multiple attribute decision making, interval number, advantage degree function, ranking

1. Introduction

Multiple attribute decision making approach has been widely applied in areas such as economy, management and construction [1–3]. In the actual decision making process, the evaluation values are always not real numbers but intervals, *i.e.*, interval numbers, due to the fuzziness of human's mind and the uncertainty of the objective world [4, 5]. Using interval numbers to represent evaluation values for multiple attribute decision making is closer to the reality of uncertainty and more consistent with human's fuzzy mind than using real numbers [6, 22]. Especially, qualitative indexes are evaluated by linguistic information [26, 27, 29], interval numbers have good effect to represent them [28, 30].

Interval number was first proposed by Dwyer [7] in 1951, the formal system establishment and value evidence of it was provided by Moore [8] and Moore and Yang [9]. As its superiority, interval numbers input in multiple attribute decision making has been a very active field of research [1]. Ranking interval numbers is key in the multiple attribute decision making approach which uses interval numbers to indicate evaluation values, and has been studied by numerous scholars.

Regarding ranking methods as mentioned by Ishibuchi and Tanaka [10], Zhang [6], Nakahara [11], Xu [12], et al., the calculations are simple but the information of interval numbers is lost seriously and the accuracy of decision making is affected. For example, when comparing interval numbers with equal symmetry axis like [10, 20] and [14, 16], different interval lengths of them shows different levels of risks, simple ranking methods treat that they are equal but that is not in accordance with human's decision making habits. Regarding other ranking methods as mentioned by Sengupta and Pal [13], Ganesan [14], et al., the comparison results are effective, but calculation methods are complex, especially when comparing a group of interval numbers.

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Motivated by the aforementioned discussions, we focus on providing a simple method of ranking a group of interval numbers in multiple attribute decision making, which can also rank interval numbers with equal symmetry axis.

The main contributions of this work can be summarized below. (i) Relations of interval numbers are expressed in the Rectangular Coordinate System (RCS) firstly instead of on the Number Axis. The two-dimensional relations of interval numbers in RCS are analyzed. It may provide a new perspective of processing interval numbers. (ii) On the basis of this, Advantage Degree Function of Interval Numbers was proposed for ranking interval numbers simply and feasibly based on degrees, especially for a group of them. (iii) Interval numbers with equal symmetry axis can be ranked easily by using this method.

The remainder of this paper is organized as follows. In Section 2, a brief account of current works on comparing interval numbers was given. In Section 3, the basic knowledge of interval numbers was introduced, and the method of expressing them in RCS was proposed. In Section 4, the method of ranking interval numbers was presented, especially the Advantage Degree Function of Interval Numbers and its effectiveness. In Section 5, an example to verify the effectiveness and advantages of the developed approach is given. Conclusions are drawn in Section 6, with recommendations on future studies.

2. Related works

Moore [15] proposed a method of comparing two interval numbers in 1979, but this method cannot compare them when they have overlap range. Ishibuchi and Tanaka [10] defined weak preference order relation of two interval numbers in linear programming in 1990, which made a significant improvement. The shortage of it is that the relation does not discuss "how much higher" when one interval is known to be higher than another [13, 16]. Kundu [17] claimed that the selection of least (or most) preferred item in two interval numbers can be made by using Left(A, B) (or Right(A, B)) in 1997, which based on the calculation of the limits of interval numbers, but it rank interval numbers with equal symmetry axis. On the basis of Ishibuchi et al. [10] and Kundu [17], Sengupta and Pal [13] defined an acceptability index in 2000, to measure "how much higher or smaller" of one interval number than another including interval numbers with equal symmetry axis. Ruan et al. [16] formulated a preference-based index which could compare a mixture of crisp and interval numbers. Nakahara [11], Zhang [6] and Fan et al. [23] defined possibility degree functions to calculate the advantage possibility degree between two interval numbers respectively. The principles of these methods were similar to Kundu's. So, they had the same shortage with it. The typical related works were summarized in Table 1.

In addition, some scholars made other attempts to calculate interval numbers. For example, Kůrka [18] let an interval number system be given by an initial interval cover of the extended real line and by a finite system of nonnegative Möbius transformations; Xu [19] used normal distribution based method to assume the probability density function of interval numbers before measuring advantage possibility degree. Wei et al. [24] define operations on hesitant fuzzy linguistic term sets (HFLTSs) and give possibility degree formulas for comparing HFLTSs. Dong et al. [25] propose a consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets in the decision making problems with linguistic preference relations.

3. Preliminaries

3.1. The basic definitions of interval numbers

Definition 1. [13, 20] Let $\tilde{a} = [a^L,\ a^U] = \{a|a^L \leq a \leq a^U,\ a^L,\ a^U \in R\}$ be an interval number, where a^L and a^U are the upper and lower limits of $\tilde{a}$ on the real line R, respectively. Especially, if $a^L = a^U$, then $\tilde{a}$ degenerates into a real number (where $\tilde{a}$ is also called degenerate interval number).

Definition 2. [20] Let $\tilde{a} = [a^L,\ a^U]$ and $\tilde{b} = [b^L,\ b^U]$, then $\tilde{a} = \tilde{b}$, if $a^L = b^L$ and $a^U = b^U$.

Definition 3. [13, 14, 20] Let $\tilde{a} = [a^L,\ a^U]$ and $\tilde{b} = [b^L,\ b^U]$, let "⊕" and "⊗" be the arithmetic operations on the set of interval numbers, then $\tilde{a} \oplus \tilde{b} = [a^L + b^L,\ a^U + b^U]$; $\tilde{a} \otimes \tilde{b} = [a^L \cdot b^L,\ a^U \cdot b^U]$ where $a^L,\ b^L > 0$, $\tilde{a}$ and $\tilde{b}$ are positive interval numbers.

Definition 4. [12, 13] Let $\tilde{a} = [a^L,\ a^U]$, then $l^+(\tilde{a})$ and $l^-(\tilde{a})$ are defined as the symmetry axis and the length of the interval number $\tilde{a}$, respectively, *i.e.*, $l^+(\tilde{a}) = (a^L + a^U)/2$, $l^-(\tilde{a}) = (a^U + a^L)/2$.

Table 1
Related works on ranking interval numbers

Works on ranking interval numbers	Contributions	Whether to rank a group of interval numbers conveniently	Whether to rank interval numbers with equal symmetry axis	Whether to measure "how much higher"	Calculation burden
Moore [15]	Defined an simple order relation	No	No	No	–
Ishibuchi and Tanaka [10]	Defined weak preference order relation	No	No	No	–
Kundu [17]	Defined a leftness order relation to measure advantage possibility degree	Yes	No	Yes	Small
Sengupta and Pal [13]	Defined an acceptability index	No	Yes	Yes	Big
Nakahara [11], Fan et al. [23]	Defined an advantage possibility function	Yes	No	Yes	Small
Our work	Expressed interval numbers in RCS, improved the function to rank them with equal symmetry axis	Yes	Yes	Yes	Small

Definition 5. [6, 11] Let $\tilde{a} = [a^L, a^U]$ and $\tilde{b} = [b^L, b^U]$, define $P(\tilde{a} \succ \tilde{b})$ is the advantage degree of $\tilde{a}$ compared with $\tilde{b}$, $P(\tilde{a} \succ \tilde{b}) \in [0, 1]$, and $P(\tilde{a} \succ \tilde{b}) + P(\tilde{b} \succ \tilde{a}) = 1$, definitely. If $P(\tilde{a} \succ \tilde{b}) > 0.5$, then $\tilde{a} \succ \tilde{b}$; if $P(\tilde{a} \succ \tilde{b}) = 0.5$, then $\tilde{a} = \tilde{b}$; and if $P(\tilde{a} \succ \tilde{b}) < 0.5$, then $\tilde{b} \succ \tilde{a}$.

Definition 6. [6, 12] Let $\tilde{a} = [a^L, a^U]$ and $\tilde{b} = [b^L, b^U]$, if $a^L \geq b^U$, then $P(\tilde{a} \succ \tilde{b}) = 1$ and $P(\tilde{b} \succ \tilde{a}) = 0$.

Definition 7. [12] If $\tilde{a}$ and $\tilde{b}$ degenerate into real numbers a and b, the advantage degree of real numbers a compared with b is as follows:

$$P(a \succ b) = \begin{cases} 1 & a > b \\ 0.5 & a = b \\ 0 & a < b \end{cases}$$

3.2. The goal interval number (GIN)

Let $\{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\}$ be a group of interval numbers, and suppose $\tilde{a}_m$ is one of them. If $a_m^U = \max(a_1^U, a_2^U, \ldots a_n^U)$, then define $\tilde{a}_m = [a_m^L, a_m^U]$ as the Goal Interval Number (GIN) of the group of interval numbers. If $\max(a_1^U, a_2^U, \ldots, a_n^U) = a_c^U = a_d^U = \cdots = a_k^U$, $a_i^L = \max(a_c^L, a_d^L, \ldots, a_k^L)$ then the goal interval number is $\tilde{a}_i$. It means that if two or more interval numbers of the group have an equal upper limit, there is a need to compare their lower limits, and the one with the biggest lower limit is the goal interval number. The purpose of selecting the GIN is to determine a target before comparing a group of interval numbers. The method will reduce the time and burden of the comparison work, and make the comparison efficient.

3.3. Analyzing relations of interval numbers in RCS

In the decision making science, the upper and lower limits of interval numbers evaluation values are always positive real numbers. So, positive interval numbers and the situation of no degeneration are only focused on, *i.e.*, $\tilde{a} = [a^L, a^U] = \{a | a^L \leq a \leq a^U, a^L < a^U, a^L, a^U \in R^+\}$. Interval numbers are expressed in RCS as shown in Fig. 1. Additionally, descriptions of the figure are as follows:

(1) The upper and lower limits of the interval number are expressed by y-axis and x-axis of the RCS, respectively.

(2) Suppose $\tilde{a} = [a^L, a^U]$ being the GIN of a group of interval numbers, it is easy to obtain that the arbitrary interval number $\tilde{a}_*$ of the group corresponding to the Point (a_*^L, a_*^U) can be expressed in the triangle area which is bounded by lines $y = a^U$, $x = 0$ and $y = x$.

Use four other lines to divide the triangle area into five smaller areas, the regularity and the significance of each area and each line are summarized in Table 2.

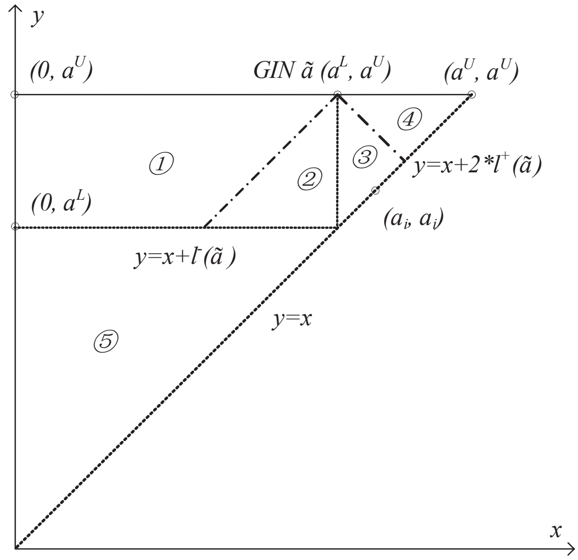


Fig. 1. Expressing interval numbers in RCS.

The propositions summarized from Fig. 1 and Table 2 are as follows:

Proposition 1. *The length of the arbitrary interval number in the triangle area* $\tilde{a}_* = [a_*^L,\ a_*^U]$ *is* $\sqrt{2}$ *times of the distance (which is named as* d_**) from the corresponded Point* $(a_*^L,\ a_*^U)$ *of the interval number to the Line* $y = x$*, i.e.,* $d_* = l^-(\tilde{a}_*)/\sqrt{2}$.

Proof: According to the distance formula of a point to a line, the distance from the Point $(a_*^L,\ a_*^U)$ to the Line $y = x$ is

$$d_* = \frac{\left|a_*^L - a_*^U + 0\right|}{\sqrt{1^2 + (-1)^2}} = \frac{l^-(\tilde{a}_*)}{\sqrt{2}}$$

Proposition 2. *If the arbitrary interval number (in the group)* $\tilde{a}_* = [a_*^L,\ a_*^U]$ *degenerates into a real number, i.e.,* $a_*^L = a_*^U$*, the length of interval number* $\tilde{a}_*(l^-(\tilde{a}_*))$ *is* 0*, and* $d_* = 0$*. So the real number is on the Line* $y = x$.

Proposition 3. *When the Point* $(a_*^L,\ a_*^U)$ *is on the Line* $y = x + l^-(\tilde{a})$*, the length of interval number* $\tilde{a}_* = [a_*^L,\ a_*^U]$ *is equal to that of the GIN, i.e.,* $l^+(\tilde{a}_*) = l^+(\tilde{a})$*. So the Line* $y = x + l^-(\tilde{a})$ *is named as the Interval Equal-length Function.*

Proof: When the Point $(a_*^L,\ a_*^U)$ is on the Line $y = x + l^-(\tilde{a})$, then

$$y - x = a_*^U - a_*^L = l^-(\tilde{a}_*) = l^-(\tilde{a}).$$

It therefore generates that, with the movement of Point $(a_*^L,\ a_*^U)$ to the upper left side from the Line $y = x$, d_* and the length of the interval number is increased; and when the point is on the Line $y = x + l^{-(\tilde{a})}$, then $d_* = d$. d is the distance of the corresponded point $(a^L,\ a^U)$ of the GIN to the Line $y = x$. If the Point $(a_*^L,\ a_*^U)$ keeps moving away from the Line $y = x + l^{-(\tilde{a})}$, then $d_* > d$ and $l^-(\tilde{a}_*) > l^{-(\tilde{a})}$. Therefore, when the Point $(a_*^L,\ a_*^U)$ is in the area (Area ①) which is above the line of the Interval Equal-length Function, the length of interval number $\tilde{a}_*$ is longer than that of the GIN $\tilde{a}$; when the Point $(a_*^L,\ a_*^U)$ is in the areas (Area ②, ③ and 174) which are below the line of the Interval Equal-length Function, the length of interval number $\tilde{a}_*$ is shorter than that of the GIN $\tilde{a}$.

Proposition 4. *When the Point* $(a_*^L,\ a_*^U)$ *in the Rectangular Plane Coordinate System corresponding to the arbitrary interval number* $\tilde{a}_*$ *of the group is on the Line* $y = -x + 2l^+(\tilde{a})$*, the symmetry axis of the interval number* $\tilde{a}_*$ *is equal to that of the GIN* $\tilde{a}$*, i.e.,* $l^+(\tilde{a}_*) = l^+(\tilde{a})$*. So the Line* $y = -x + 2l^+(\tilde{a})$ *is named as the Interval Equal-symmetry-axis Function.*

Proposition 5. *When the Point* $(a_*^L,\ a_*^U)$ *is in the area (Area ④) which is above the line of the Interval Equal-symmetry-axis Function, the symmetry axis of interval number* $\tilde{a}_*$ *is upper than that of the GIN* $\tilde{a}$*. When the Point* $(a_*^L,\ a_*^U)$ *is in the areas (Area ①, ② and ③) which are below the line of the Interval Equal-symmetry-axis Function, the symmetry axis of interval number* $\tilde{a}_*$ *is lower than that of the GIN* $\tilde{a}$.

Proof: When the Point $(a_*^L + a_*^U)$ is on the Line $y = -x + 2l^+(\tilde{a})$, then

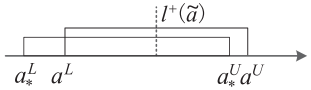
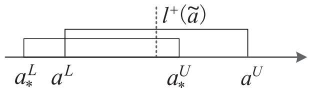
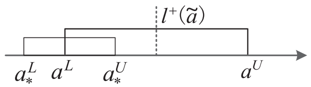
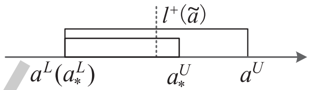
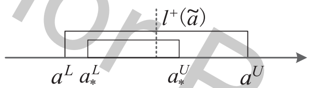
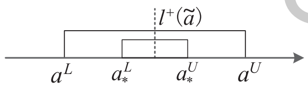
$$l^+(\tilde{a}) = (y + x)/2 = (a_*^L + a_*^U)/2 = l^+(\tilde{a}_*).$$

When the Point $(a_*^L,\ a_*^U)$ is in the area which is above the Line $y = -x + 2l^+(\tilde{a})$, then

$$a_*^L + a_*^U - 2l^+(\tilde{a}) > 0$$
$$\Rightarrow \left(a_*^L + a_*^U\right)/2 = l^+(\tilde{a}_*) > l^+(\tilde{a}).$$

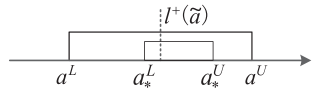
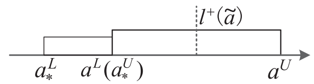
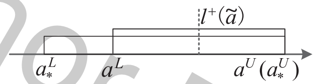
The symmetry axis relation of $\tilde{a}_*$ and $\tilde{a}$ when the Point $(a_*^L,\ a_*^U)$ is in the areas (Area ①, ② and ③) can be easily proved with the same approach.

Table 2
The regularity and significance of interval numbers in each area or on each line

Area or Line	Arbitrary interval number $\tilde{a}_* = \left[a_*^L,\ a_*^U\right]$ compared to GIN $\tilde{a} = \left[a^L,\ a^U\right]$ which is expressed in one area or on one line				
	Relations of constraints	Relations of lengths	Relations of symmetry axes	Relations of them when they are expressed on the Number Axis	Remarks
①	$a^L < a_*^U < a^U$ $a_*^L > 0$ $a_*^L - a_*^U + l^-(\tilde{a}) < 0$	$l^-(\tilde{a}_*) > l^-(\tilde{a})$	$l^+(\tilde{a}_*) < l^+(\tilde{a})$		Intersected, longer length, lower symmetry axis
$y = x + l^-(\tilde{a})$	$a^L < a_*^U < a^U$ $a_*^L - a_*^U + l^-(\tilde{a}) = 0$	$l^-(\tilde{a}_*) = l^-(\tilde{a})$	$l^+(\tilde{a}_*) < l^+(\tilde{a})$		Intersected, equal length, lower symmetry axis
②	$a_*^U > a^L$ $a_*^L < a^L$ $a_*^L - a_*^U + l^-(\tilde{a}) > 0$	$l^-(\tilde{a}_*) < l^-(\tilde{a})$	$l^+(\tilde{a}_*) < l^+(\tilde{a})$		Intersected, shorter length, lower symmetry axis
$x = a^L$	$a^L < a_*^U < a^U$ $a_*^L = a^L$	$l^-(\tilde{a}_*) < l^-(\tilde{a})$	$l^+(\tilde{a}_*) < l^+(\tilde{a})$		Contained, equal low limit, shorter length, lower symmetry axis
③	$a_*^L > a^L$ $a_*^L - a_*^U < 0$ $a_*^L + a_*^U - 2l^+(\tilde{a}) < 0$	$l^-(\tilde{a}_*) < l^-(\tilde{a})$	$l^+(\tilde{a}_*) < l^+(\tilde{a})$		Contained, shorter length, lower symmetry axis
$y = -x + 2l^+(\tilde{a})$	$a_*^L > a^L$ $a_*^L + a_*^U - l^+(\tilde{a}) = 0$	$l^-(\tilde{a}_*) < l^-(\tilde{a})$	$l^+(\tilde{a}_*) = l^+(\tilde{a})$		Contained, shorter length, equal symmetry axis

(Continued)

Table 2
(Continued)

Area or Line	Arbitrary interval number $\tilde{a}_* = \left[a_*^L,\ a_*^U\right]$ compared to GIN $\tilde{a} = \left[a^L,\ a^U\right]$ which is expressed in one area or on one line				
	Relations of constraints	Relations of lengths	Relations of symmetry axes	Relations of them when they are expressed on the Number Axis	Remarks
④	$a_*^U < a^U$ $a_*^L - a_*^U < 0$ $a_*^L + a_*^U - 2l^+(\tilde{a}) < 0$	$l^-(\tilde{a}_*) < l^-(\tilde{a})$	$l^+(\tilde{a}_*) > l^+(\tilde{a})$		Contained, shorter length, upper symmetry axis
⑤	$a_*^U < a^L$ $a_*^L > 0$ $a_*^L - a_*^U < 0$	—	$l^+(\tilde{a}_*) > l^+(\tilde{a})$		Deviated, lower symmetry axis
$y = a^L$	$a_*^U = a^L$ $0 < a_*^L < a^L$	—	$l^+(\tilde{a}_*) < l^+(\tilde{a})$		Deviated, lower symmetry axis
$y = a^U$ (the part above Area ①)	$a_*^U = a^U$ $0 < a_*^L < a^L$	$l^-(\tilde{a}_*) > l^-(\tilde{a})$	$l^+(\tilde{a}_*) < l^+(\tilde{a})$		Contained, equal upper limit, longer length, lower symmetry axis

Note: The part of the Line $y = a^U$ which is above Area ① is selected only as no interval number in the group has the equal upper limit and bigger lower limit compared to the GIN after the GIN is selected.

4. The method of ranking interval numbers

4.1. The advantage degree function of interval numbers

The Interval Numbers Advantage Degree Function which based on the principles of Limit and Piecewise function is summarized based on the following information:

- The symmetry axes and lengths of interval numbers.

- The variation rules of the distances between the points (which correspond to interval numbers) and the lines(which correspond to the Interval Equal-length Function and the Interval Equal-symmetry-axis Function) in RCS.

When the relation of two interval numbers is not deviated, the Advantage Degree Function of Interval Numbers Symmetry Axis S_1 and the Advantage Degree Function of Interval Numbers Length S_2 are as follows.

$$S_1(\tilde{a}_* \succ \tilde{a}) = \begin{cases} 0.5 - (l^+(\tilde{a}) - l^+(\tilde{a}_*))/a^U & l^+(\tilde{a}_*) < l^+(\tilde{a}) \\ 0.5 & l^+(\tilde{a}_*) = l^+(\tilde{a}) \quad a_*^U > a^L \\ (l^+(\tilde{a}_*) - l^+(\tilde{a}))/l^-(\tilde{a}) + 0.5 & l^+(\tilde{a}_*) > l^+(\tilde{a}) \end{cases} \tag{1}$$

$$S_2(\tilde{a}_* \succ \tilde{a}) = \begin{cases} (l^-(\tilde{a}) - l^-(\tilde{a}_*))/2l^-(\tilde{a}) + 0.5 & l^-(\tilde{a}_*) < l^-(\tilde{a}) \\ 0.5 & l^-(\tilde{a}_*) = l^-(\tilde{a}) \quad a_*^U > a^L \\ 0.5 - (l^-(\tilde{a}_*) - l^-(\tilde{a}))/2a^L & l^-(\tilde{a}_*) > l^-(\tilde{a}) \end{cases} \tag{2}$$

Function S_1 and S_2 should be continuous functions in the function range (0,1).

Proof: (1) Prove the continuity of the functions first.

It is easy to know that Function S_1 is a monotone and linear function for the independent variable $l^+(\tilde{a}_*)$ when $l^+(\tilde{a}_*) \neq l^+(\tilde{a})$, so S_1 is continuous when its independent variable locates in two piecewise ranges. To prove the continuity of S_1, the only thing needs to do is to prove S_1 is continuous when $l^+(\tilde{a}_*) = l^+(\tilde{a})$. The continuity of Function S_2 can be proved in the same way.

So there is a need to prove the left limit and right limit of the piecewise functions are both equal to the function value when $l^+(\tilde{a}_*) = l^+(\tilde{a})$. *i.e.*,

$$\lim_{l^+(\tilde{a}_*) \to l^+(\tilde{a})^-} (0.5 - (l^+(\tilde{a}) - l^+(\tilde{a}_*))/a^U)$$

$$= \lim_{l^+(\tilde{a}_*) \to l^+(\tilde{a})^+} ((l^+(\tilde{a}_*) - l^+(\tilde{a}_*))/l^-(\tilde{a}) + 0.5) = 0.5$$

$$\lim_{l^+(\tilde{a}_*) \to l^+(\tilde{a})^-} ((l^-(\tilde{a}) - l^-(\tilde{a}_*))/2l^-(\tilde{a}) + 0.5)$$

$$= \lim_{l^+(\tilde{a}_*) \to l^-(\tilde{a})^+} (0.5 - (l^-(\tilde{a}_*) - l^-(\tilde{a}))/2a^L) = 0.5$$

S_1 and S_2 are therefore both continuous functions.

(2) Then, prove the function ranges of S_1 and S_2 are both (0,1).

Referring to Fig. 1, the farthest points to both sides of the lines of the Interval Equal-length Function $y = x + l^-(\tilde{a})$ and the Interval Equal-symmetry-axis Function $y = -x + 2l^+(\tilde{a})$ of the GIN $\tilde{a}$ in Area ①, ②, ③ and ④ (plus the boundaries) are $(0,\ a^U)$, $(a_i,\ a_i)$, $(0,\ a^L)$ and $(a_U,\ a_U)$. Especially, $(a_i,\ a_i)$ is the arbitrary point on the Line $y = x (a^L < x < a^U)$. The symmetry axis $l^+(\tilde{a}_*)$ and the length $l^-(\tilde{a}_*)$ of the interval numbers which correspond to farthest points are extremums.

Because

$$\lim_{\substack{x \to 0^+ \\ y \to a^{L+}}} (0.5 - (l^+(\tilde{a}) - l^+(\tilde{a}_*))/a^U)$$

$$\Rightarrow \lim_{l^+(\tilde{a}_*) \to a^{L+}} (0.5 - (l^+(\tilde{a}) - (l^+(\tilde{a}_*))/a^U) = 0$$

$$\lim_{\substack{x \to a^{U-} \\ y \to a^{U-}}} ((l^+(\tilde{a}_*) - l^+(\tilde{a}))/l^-(\tilde{a}) + 0.5)$$

$$\Rightarrow \lim_{l^+(\tilde{a}_*) \to 2a^{U-}} ((l^+(\tilde{a}_*) - (l^+(\tilde{a}))/l^-(\tilde{a}) + 0.5) = 1$$

$$\lim_{\substack{x \to a_i^+ \\ y \to a_i^+}} ((l^+(\tilde{a}) - l^-(\tilde{a}_*))/2l^-(\tilde{a}) + 0.5)$$

$$\Rightarrow \lim_{l^+(\tilde{a}_*) \to 0^+} ((l^-(\tilde{a}) - (l^-(\tilde{a}_*))/2l^-(\tilde{a}) + 0.5) = 1$$

$$\lim_{\substack{x \to 0^+ \\ y \to a^{U+}}} (0.5 - (l^-(\tilde{a}_*) - l^-(\tilde{a}))/2a^L)$$

$$\Rightarrow \lim_{l^+(\tilde{a}_*) \to a^{U-}} (0.5 - (l^-(\tilde{a}_*) - (l^-(\tilde{a}_))/2a^L) = 0$$

Function S_1 and S_2 are both monotone continuous functions. So, their function ranges are both (0,1).

Then the Advantage Degree Function of Interval Numbers can be defined as follows.

$$P(\tilde{a}_* \succ \tilde{a}) = \begin{cases} S_1(\tilde{a}_* \succ \tilde{a}) & a_*^U > a^L, l^+(\tilde{a}_*) \neq l^+(\tilde{a}) \\ S_2(\tilde{a}_* \succ \tilde{a}) & a_*^U > a^L, l^+(\tilde{a}_*) = l^+(\tilde{a}) \\ 0 & a_*^U \leq a^L \end{cases} \tag{3}$$

Consider the interval number is a set of possible values, from the Advantage Degree Function of Interval Numbers, it can be found that when two interval numbers are compared, the one with an upper symmetry axis is superior to the other because it has a bigger average value. If two interval numbers have an equal symmetry axis, then the one with a shorter length is superior to the other because it has more concentrative value around the symmetry axis. So when two interval numbers are compared in the condition that one's lower limit is not bigger than the one's upper limit, their symmetry axes are compared first, then compare their lengths if they have an equal symmetry axis.

4.2. Procedures of the method

(i) Use the method which is introduced in Section 3.2, select the GIN $\tilde{a}$ from a group of interval numbers.

(ii) Analyze the relation of the arbitrary interval number (in the group) $\tilde{a}_*$ and $\tilde{a}$. Use the Advantage Degree Function of Interval Numbers which is given in Section 4.1. If $a_*^U \le a^L$, then $P(\tilde{a}_* \succ \tilde{a}) = 0$. If $a_*^U > a^L$, use Equation (1) to calculate the advantage degrees of symmetry axes of the interval numbers to the GIN, and rank the interval numbers according to the advantage degrees. If two or more interval numbers have equal symmetry axis, then use Equation (2) to calculate the advantage degrees of lengths of the interval numbers to the GIN, and rank them according to the results as a complementary to the initial ranking.

(iii) If two or more $P(\tilde{a}_* \succ \tilde{a}) = 0$, repeat Procedure (i) and (ii) for these interval numbers until all interval numbers of the group are ranked.

The detailed procedure of ranking interval numbers is described in Fig. 2.

Example. Let $\tilde{a}_1,\ \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6$, and $\tilde{a}_7$ be a group of interval numbers as such

$$\tilde{a}_1 = [10,\ 30],\ \tilde{a}_2 = [6,\ 18],\ \tilde{a}_3 = [24,\ 30],$$

$$\tilde{a}_4 = [4,\ 10],\ \tilde{a}_5 = [12,\ 20],\ \tilde{a}_6 = [15,\ 25] \text{ and } \tilde{a}_7 = [2,\ 8]$$

Try to rank all of the interval numbers.

Use the method of ranking interval numbers based on degrees which is introduced in Section 4.2.

(i) Compare the upper limits of the interval numbers, $a_1^U = \max(a_1^U,\ a_2^U, \ldots,\ a_7^U) = 30$, then the GIN is $\tilde{a}_1$.

(ii) Because $a_4^U \le a_1^L$ and $a_7^U \le a_1^L$, $P(\tilde{a}_4 \succ \tilde{a}_1) = 0 = P(\tilde{a}_7 \succ \tilde{a}_1)$.

Calculate the advantage degrees of symmetry axes of $\tilde{a}_2, \tilde{a}_3, \tilde{a}_5$ and $\tilde{a}_6$ to $\tilde{a}_1$(GIN) by using Equation (1), the results are as follows

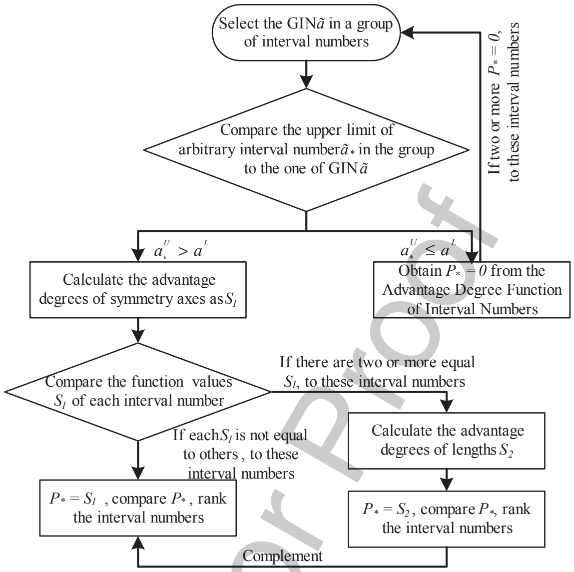


Fig. 2. Procedures of ranking interval numbers.

$$S_1(\tilde{a}_2 \succ \tilde{a}_1) = 7/30,\ S_1(\tilde{a}_3 \succ \tilde{a}_1) = 17/20,$$

$$S_1(\tilde{a}_5 \succ \tilde{a}_1) = 11/30,\ S_1(\tilde{a}_6 \succ \tilde{a}_1) = 1/2.$$

$P(\tilde{a}_3 \succ \tilde{a}_1) > 0.5 > P(\tilde{a}_5 \succ \tilde{a}_1) > P(\tilde{a}_2 \succ \tilde{a}_1)$, then $\tilde{a}_3 \succ \tilde{a}_1 \succ \tilde{a}_5 \succ \tilde{a}_2$.

For $S_1(\tilde{a}_6 \succ \tilde{a}_1) = S_1(\tilde{a}_1 \succ \tilde{a}_1) = 1/2$, calculate the advantage degrees of lengths of $\tilde{a}_6$ to $\tilde{a}_1$(GIN) by using Equation (2), and the result is $S_2(\tilde{a}_6 \succ \tilde{a}_1) = 5/8$.

$P(\tilde{a}_6 \succ \tilde{a}_1) > 0.5$, then $\tilde{a}_6 \succ \tilde{a}_1$. Add this rank to the one that is made in the last step, then the five interval numbers will be ranked as $\tilde{a}_3 \succ \tilde{a}_6 \succ \tilde{a}_1 \succ \tilde{a}_5 \succ \tilde{a}_2$.

(iii) For $P(\tilde{a}_4 \succ \tilde{a}_1) = P(\tilde{a}_7 \succ \tilde{a}_1) = 0$, they cannot be ranked directly, therefore Procedure (i) and (ii) are repeated for $\tilde{a}_4$ and $\tilde{a}_7$.

$\tilde{a}_4$ is the GIN of the group of interval numbers which is constituted by $\tilde{a}_4$ and $\tilde{a}_7$, then

$$S_1(\tilde{a}_7 \succ \tilde{a}_4) = 3/10,\ P(\tilde{a}_7 \succ \tilde{a}_4) < 0.5.$$

So $\tilde{a}_4 \succ a_7$, and the final rank of all interval numbers is $\tilde{a}_3 \succ \tilde{a}_6 \succ \tilde{a}_1 \succ \tilde{a}_5 \succ \tilde{a}_2 \succ \tilde{a}_4 \succ \tilde{a}_7$.

Stressed is that, $\tilde{a}_1$ and $\tilde{a}_6$ are equal-symmetry-axis interval numbers. It is obvious that this method is more advanced than Nakahara's method [6, 11]. The ranking method is in accordance with people's prospective and decision making habit, the interval number which has bigger length has more risk, showing effectiveness on some level.

5. An example of applying the method for resolving a multiple attribute decision making problem

Let Liu's [21] case be the application example.

The initial five mining methods are proposed according to the local experience and specific conditions of one mine. Eight evaluation indexes have been chosen to determine the final mining method. The weights and the interval-number-values of evaluation indexes are as follows (see Tables 3 and 4).

$$\begin{cases} a_{ij}^{L*} = a_{ij}^{L} \big/ max(a_{ij}^{U}), \quad a_{ij}^{U*} = a_{ij}^{U} \big/ max(a_{ij}^{U}) & \text{income index} \\ a_{ij}^{L*} = min(a_{ij}^{L}) \big/ a_{ij}^{U}, \quad a_{ij}^{U*} = min(a_{ij}^{L}) \big/ a_{ij}^{L} & \text{cost index} \end{cases}$$

$$\begin{cases} A_i^L = \sum_{j=1}^{n} a_{ij}^{L*} \cdot w_j \\ A_i^U = \sum_{j=1}^{n} a_{ij}^{U*} \cdot w_j \end{cases}$$

Through adopting the method to rank the comprehensive interval-number-values, the procedures are as follows:

(i) $A_3^U = \max(A_1^U, A_2^U, \ldots, A_5^U) = 0.9585$, so the GIN is $\tilde{A}_3$.

(ii) Because the upper limits A_2^U and A_4^U are smaller than GIN's lower limit, *i.e.*, $P(\tilde{A}_2 \succ \tilde{A}_3) = 0 = P(\tilde{A}_4 \succ \tilde{A}_3)$. Then the advantage degrees of symmetry axes of $\tilde{A}_1$ and $\tilde{A}_5$ to $\tilde{A}_3$ (GIN) can be obtained by using Equation (1), the results are as follows

$$S_1(\tilde{A}_1 \succ \tilde{A}_3) = 0.4046, \ S_1(\tilde{A}_5 \succ \tilde{A}_3) = 0.4445$$

According to Equation (3), the rank of advantage degrees of $\tilde{A}_1$ and $\tilde{A}_5$ to $\tilde{A}_3$ is $0.5 > P(\tilde{A}_5 \succ \tilde{A}_3) > P(\tilde{A}_1 \succ \tilde{A}_3)$. So the rank of the three interval numbers is $\tilde{A}_3 \succ \tilde{A}_5 \succ \tilde{A}_1$.

(iii) Repeat Procedure (i) and (ii) for $\tilde{A}_2$ and $\tilde{A}_4$. $A_4^U = \max(A_2^U, A_4^U) = 0.7962$, so the GIN of $\tilde{A}_2$ and $\tilde{A}_4$ is $\tilde{A}_4$. Because $A_2^U < A_4^L$, then $P(\tilde{A}_2 \succ \tilde{A}_4) = 0$, $\tilde{A}_4 \succ \tilde{A}_2$.

The final rank of all the interval numbers is $\tilde{A}_3 \succ \tilde{A}_5 \succ \tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2$, *i.e.*, the rank of the five methods is $X_3 \succ X_5 \succ X_1 \succ X_4 \succ X_2$, which is the same with the one of Liu's [21] but with less calculation burden. This proposed method, which is applicable, will create good profit for multiple attribute decision making approach which uses interval numbers to represent evaluation values.

6. Conclusion

In order to develop a method for resolving the multiple attribute decision making problem, interval numbers can be expressed in RCS. On the basis of this, interval numbers are expressed in different areas of the RCS according to their different properties after information mining. This approach seems to clarify relations of interval numbers.

Table 3
Interval-number-weights of evaluation indexes

Evaluation indexes	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
weights	0.145	0.076	0.078	0.188	0.146	0.141	0.133	0.094

Table 4
Interval-number-values of evaluation indexes

Evaluation indexes	Method X_1	Method X_2	Method X_3	Method X_4	Method X_5
Total taxation profit (G_1)/(RMB¥ · Mt^{-1})	[215, 232]	[205, 230]	[285, 310]	[270, 295]	[270, 290]
Coefficient of loss (G_2)/%	[8, 10]	[11, 13]	[8, 10]	[9, 11]	[8, 10]
Boulder frequency (G_3)/%	[5, 10]	[10, 20]	[5, 8]	[10, 15]	[6, 10]
Mining ratio (G_4)/(m · Mt^{-1})	[40, 44]	[30, 35]	[26, 30]	[31, 35]	[31, 35]
Mining safety (G_5)	[0.9, 1]	[0.4, 0.6]	[0.7, 0.8]	[0.6, 0.8]	[0.7, 0.9]
Ventilation condition (G_6)	[0.5, 0.7]	[0.1, 0.2]	[0.8, 1]	[0.9, 1]	[0.7, 0.9]
Technical Difficulty (G_7)	[0.9, 1]	[0.3, 0.5]	[0.7, 0.9]	[0.4, 0.6]	[0.7, 0.9]
Environment protection (G_8)	[0.9, 1]	[0.3, 0.4]	[0.9, 1]	[0.4, 0.6]	[0.9, 1]

G_1, G_2, G_3 and G_4 are quantitative indexes, G_1 is an income index, and the others are cost indexes. G_5, G_6, G_7 and G_8 are qualitative indexes, and all of them are income indexes. Suppose interval number $[a_{ij}^L, a_{ij}^U]$ is the value of Evaluation Index G_j of Method X_i, and $[a_{ij}^L, a_{ij}^U]$ is the dimensionalized value of it. The values of each evaluation index are made dimensionless (0, 1) through Equation (4). The dimensionalized interval-number-values of evaluation indexes show in Table 5. Suppose $[A_i^L, A_i^U]$ is the comprehensive values of Method X_i, and w_j is the weight of Evaluation Index G_j. Through Equation (5), calculate the comprehensive interval-number-values of each method, and the results show in Table 6.

Table 5
Dimensionalized interval-number-values of evaluation indexes

Evaluation indexes	Method X_1	Method X_2	Method X_3	Method X_4	Method X_5
G_1	[0.694,0.748]	[0.661,0.742]	[0.919,1]	[0.871,0.952]	[0.871,0.935]
G_2	[0.8,1]	[0.615,0.727]	[0.8,1]	[0.727,0.889]	[0.8,1]
G_3	[0.5,1]	[0.25,0.5]	[0.625,1]	[0.333,0.5]	[0.5,0.833]
G_4	[0.591,0.650]	[0.743,0.867]	[0.867,1]	[0.743,0.839]	[0.743,0.839]
G_5	[0.9,1]	[0.4,0.6]	[0.7,0.8]	[0.6,0.8]	[0.7,0.9]
G_6	[0.5,0.7]	[0.1,0.2]	[0.8,1]	[0.9,1]	[0.7,0.9]
G_7	[0.9,1]	[0.3,0.5]	[0.7,0.9]	[0.4,0.6]	[0.7,0.9]
G_8	[0.9,1]	[0.3,0.4]	[0.9,1]	[0.4,0.6]	[0.9,1]

Table 6
Comprehensive interval-number-values of each school

Method	X_1	X_2	X_3	X_4	X_5
Comprehensive interval-number-values	[0.7177, 0.8564]	[0.4424, 0.5847]	[0.7985, 0.9585]	[0.6525, 0.7962]	[0.7443, 0.9063]

The ranking procedures and results of the examples show that the method of ranking interval numbers based on degrees is feasible, simple and effective. In addition, as a practical method, the equal-symmetry-axis interval numbers can be ranked by using it.

There are still further works that we will perform. For example, in this work, the length of interval numbers is the second attribute of ranking work. Indexes can be given to make comprehensive consideration of symmetry and length. In addition, new two-dimensional relations of interval numbers in RCS could be mined, which can show different attitudes in the ranking work.

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