# Bidirectional projection method for multiple attribute group decision making with neutrosophic numbers

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**Abstract**: Neutrosophic numbers are very suitable for expressing indeterminate evaluation information in complex decision making problems, and then projection measure is a useful method for handling the decision making problems. However, due to the lack of engineering applications of neutrosophic numbers and some shortcoming implied in general projection measures in some cases. Therefore, the paper proposes a bidirectional projection measure of interval numbers to overcome the shortcoming and extend it to the bidirectional projection measure of neutrosophic numbers, and then develops a bidirectional projection-based multiple attribute group decision making method with neutrosophic numbers. Through the bidirectional projection measure between each alternative decision matrix and the ideal alternative matrix, all the alternatives can be ranked to select the best one. Finally, an illustrative example demonstrates the application of the proposed method. The effectiveness and advantages of the proposed method are shown by the comparative analysis with existing relative methods.

**Keywords:** Bidirectional projection measure; Group decision making; Neutrosophic number; Interval number

## 1 Introduction

Multiple attribute group decision making is an important branch of decision theory, which has been widely applied in many fields. Then, projection measure is a very suitable method for dealing with decision making problems because it can consider not only the distance but also the included angle between objects evaluated [1, 2]. Therefore, projection methods have been applied successfully to decision making. First, the projection methods were used for uncertain multiple attribute decision making with preference information [1, 2]. Then, the projection model-based approaches were applied to intuitionistic fuzzy and interval valued intuitionistic fuzzy multiple attribute decision making problems [3]. Further, grey relational projection methods with crisp values were presented and applied to multiple attribute decision making problems [4, 5]]. Projection model-based approaches were applied to intuitionistic fuzzy multiple attribute decision making problems [6]. A group decision making approach based on interval multiplicative and fuzzy preference relations was developed by using projection [7]. Projection methods were used for multiple attribute group decision making with intuitionistic fuzzy information [8-10]. A projection method was introduced for group decision making problems with incomplete weight information in linguistic setting [11]. A projection measure was introduced to deal with a group decision making method with hybrid intuitionistic fuzzy information [12]. However, the general projection measures imply some shortcoming (see examples in Subsection 2.2 and Section 3) in some cases and need to be improved to

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overcome the shortcoming implied in projection measures.

In multiple attribute group decision making, because of the indeterminacy of human thinking and the complexity of objective things, the attribute values expressed by the crisp numbers have difficulty in conveying people's thinking about objective things. Hence, Smarandache [13-15] firstly proposed the concept of a neutrosophic number, which consists of two parts: a determinate part and an indeterminate part. The neutrosophic number can be expressed by $N = d + uI$ for $d, u \in R$ ($R$ is all real numbers), where $d$ is its determinate part, $uI$ is its indeterminate part, and $I$ means indeterminacy. In the worst scenario, $N$ can be expressed as the unknown part $N = uI$ ($d = 0$). In the best scenario, $N$ can be expressed as the determinate part $N = d$ ($uI = 0$). Obviously, it is very suitable for the expression of indeterminate evaluation information in complex decision making problems. Therefore, Ye [16] firstly proposed a group decision making method with neutrosophic numbers based on a de-neutrosophication method and a possibility degree ranking method for neutrosophic numbers. Then, Kong et al. [17] developed a cosine similarity measure of neutrosophic numbers and applied it to the misfire fault diagnosis of gasoline engines with neutrosophic number information. Till this day, the study on the neutrosophic numbers used for handling indeterminate problems has made a little progress in scientific and engineering fields. Therefore, it is necessary to propose a new method based on the neutrosophic numbers to handle group decision making problems. In order to do so, the main purposes of this paper are: (1) to develop a bidirectional projection measure of interval numbers to overcome the shortcoming of the general projection measure, (2) to introduce a bidirectional projection measure of neutrosophic numbers based on the bidirectional projection measure of interval numbers, and (3) to develop a bidirectional projection-based multiple attribute group decision making method with neutrosophic numbers.

The rest of the paper is organized as follows. Section 2 briefly describes some basic concepts of neutrosophic numbers and the general projection measure of interval numbers. Section 3 proposes a bidirectional projection measure between interval numbers to overcome some shortcoming implied in the general projection measure. In Section 4, we develop a bidirectional projection-based multiple attribute group decision making method with neutrosophic numbers. In Section 5, an illustrative example is presented to demonstrate the application of the proposed method, and then the effectiveness and advantages of the proposed method are demonstrated by the comparative analysis with existing relative methods. Finally, Section 6 contains conclusions and future work.

## 2. Preliminaries

### 2.1 Some concepts of neutrosophic numbers

A neutrosophic number, proposed by Smarandache [13-15], consists of the determinate part and the indeterminate part, which is denoted by $N = d + uI$, where $d$ and $u$ are real numbers, and $I$ is indeterminacy, such that $I^n = I$ for n > 0, $0\times I = 0$, and $uI/kI$ = undefined for any real number $k$.

For example, assume that there is a neutrosophic number $N = 7 + 3I$. If $I \in [0, 0.2]$, it is equivalent to $N \in [7, 7.6]$ for sure $N \geq 7$, this means that its determinate part is 7 and its indeterminate part is $3I$ for the indeterminacy $I \in [0, 0.2]$ and the possibility for the number "$N$" is within the interval [7, 7.6].

Let $N_1 = d_1 + u_1I$ and $N_2 = d_2 + u_2I$ be two neutrosophic numbers for $d_1, u_1, d_2, u_2 \in R$. their operational relations are as follows [16-18]:

(1) $N_1 + N_2 = d_1 + d_2 + (u_1 + u_2)I$;

(2) $N_1 - N_2 = d_1 - d_2 + (u_1 - u_2)I$;;

(3) $N_1 \times N_2 = d_1d_2 + (d_1u_2 + u_1d_2 + u_1u_2)I$;

(4) $N_1^2 = (d_1 + u_1I)^2 = d_1^2 + (2d_1u_1 + u_1^2)I$;

(5) $\dfrac{N_1}{N_2} = \dfrac{d_1 + u_1 I}{d_2 + u_2 I} = \dfrac{d_1}{d_2} + \dfrac{d_2 u_1 - d_1 u_2}{d_2(d_2 + u_2)} \cdot I$ for $d_2 \neq 0$ and $d_2 \neq -u_2$;

(6) $$\sqrt{N_1} = \sqrt{d_1 + u_1 I} = \begin{cases} \sqrt{d_1} - (\sqrt{d_1} + \sqrt{d_1 + u_1})I \\ \sqrt{d_1} - (\sqrt{d_1} - \sqrt{d_1 + u_1})I \\ -\sqrt{d_1} + (\sqrt{d_1} + \sqrt{d_1 + u_1})I \\ -\sqrt{d_1} + (\sqrt{d_1} - \sqrt{d_1 + u_1})I \end{cases}.$$

**Definition 1.** Let $N = d + uI$ be a neutrosophic number. If $d$, $u \geq 0$, then $N$ is called positive neutrosophic numbers.

In the following, all neutrosophic numbers are considered to be positive and are called neutrosophic numbers for short, unless they are stated.

### 2.2 Projection measure of interval numbers

Kaufmann and Gupta [18] introduced interval numbers and defined as follows.

**Definition 2** [18]. If $a = [a^l, a^u] = \{x \mid a^l \leq x \leq a^u, a^l, a^u \in R\}$, then $a$ is called an interval number. If $a = [a^l, a^u] = \{x \mid 0 < a^l \leq x \leq a^u\}$, then $a$ is called a positive interval number. If $a^l = a^u$, then $a$ is reduced to a real number (crisp value).

In the following, all interval numbers are considered to be positive and are called interval numbers for short, unless they are stated.

**Definition 3** [2]. Let $a = ([a_1^l, a_1^u], [a_2^l, a_2^u], ..., [a_n^l, a_n^u])$ and $b = ([b_1^l, b_1^u], [b_2^l, b_2^u], ..., [b_n^l, b_n^u])$ be two interval vectors, then the modules of $a$ and $b$ are defined as $\|a\| = \sqrt{\sum_{j=1}^n (a_j^l)^2 + (a_j^u)^2}$ and $\|b\| = \sqrt{\sum_{j=1}^n (b_j^l)^2 + (b_j^u)^2}$ respectively, the inner product between $a$ and $b$ is defined as $a \cdot b = \sum_{j=1}^n (a_j^l b_j^l + a_j^u b_j^u)$, and then

$$\cos(a, b) = \frac{a \cdot b}{\|a\| \|b\|} \tag{1}$$

is called the cosine of the included angle between $a$ and $b$.

**Definition 4** [2]. Let $a = ([a_1^l, a_1^u], [a_2^l, a_2^u], ..., [a_n^l, a_n^u])$ and $b = ([b_1^l, b_1^u], [b_2^l, b_2^u], ..., [b_n^l, b_n^u])$ be two interval vectors, then

$$\mathrm{Pr}oj_b(a) = \|a\| \cos(a, b) = \frac{a \cdot b}{\|b\|} = \frac{\sum_{j=1}^n (a_j^l b_j^l + a_j^u b_j^u)}{\sqrt{\sum_{j=1}^n \left[(b_j^l)^2 + (b_j^u)^2\right]}} \tag{2}$$

is called the projection of the vector $a$ on the vector $b$.

The projection measure $\mathrm{Proj}_b(a)$ can include both the distance and the included angle between $a$ and $b$. In general, the larger the value of $\mathrm{Proj}_b(a)$ is, the closer $a$ is to $b$ [2]. However, this case is not always reasonable in some case. For example, let $a = b = ([a_1^l, a_1^u],[a_2^l, a_2^u],...,[a_n^l, a_n^u])$ and $c = ([2a_1^l, 2a_1^u],[2a_2^l, 2a_2^u],...,[2a_n^l, 2a_n^u])$, then $\mathrm{Proj}_b(a) = \|b\|$ and $\mathrm{Proj}_b(c) = 2\|b\|$. Clearly, $\mathrm{Proj}_b(c)$ is larger than $\mathrm{Proj}_b(a)$. In fact, $a$ is closer to $b$ than $c$. Hence, the projection cannot accurately depict the degree of $a$ close to $b$. However, the "closeness" between two vectors for the general projection measure introduced by Xu [2] is not also always reasonable in some case. Then, author notices that when $a$ is equal to $b$, $\mathrm{Proj}_b(a)$ should be equal to 1, conversely $\mathrm{Proj}_a(b)$ should be also equal to 1. Hence, author proposes a bidirectional projection measure for interval numbers and neutrosophic numbers below to overcome the shortcoming.

## 3 Bidirectional projection measures of interval numbers and neutrosophic numbers

This section firstly proposes a bidirectional projection measure between interval numbers to overcome the shortcoming of the general projection measure of interval numbers, and then extend it to the bidirectional projection measure between neutrosophic numbers.

To overcome the shortcoming implied in the existing projection, author proposes a bidirectional projection measure between interval numbers below.

**Definition 5.** Let $a = ([a_1^l, a_1^u],[a_2^l, a_2^u],...,[a_n^l, a_n^u])$ and $b = ([b_1^l, b_1^u],[b_2^l, b_2^u],...,[b_n^l, b_n^u])$ be two interval vectors, then

$$B\mathrm{Pr}oj(a,b) = \frac{1}{1 + \left| \frac{a \cdot b}{\|a\|} - \frac{a \cdot b}{\|b\|} \right|} = \frac{\|a\|\|b\|}{\|a\|\|b\| + \big|\|a\| - \|b\|\big| a \cdot b} \tag{3}$$

is called the bidirectional projection between $a$ and $b$, where $\|a\| = \sqrt{\sum_{j=1}^{n} (a_j^l)^2 + (a_j^u)^2}$ and $\|b\| = \sqrt{\sum_{j=1}^{n} (b_j^l)^2 + (b_j^u)^2}$ are the modules of $a$ and $b$ respectively, and $a \cdot b = \sum_{j=1}^{n} (a_j^l b_j^l + a_j^u b_j^u)$ is the inner product between $a$ and $b$.

The bidirectional projection measure can include not only both the distance and the included angle between $a$ and $b$ but also the bidirectional projection magnitude between two vectors $a$ and $b$.

Obviously, the closer the value of BProj($a$, $b$) is to 1, the closer $a$ is to $b$, and then there are BProj($a$, $b$) = 1 if and only if $a = b$ and $0 \le$ BProj($a$, $b$) $\le 1$ for any two interval vectors $a$ and $b$, which is a normalized measure.

To demonstrate the rationality and effectiveness of the bidirectional projection measure, we give the following example.

**Example 1**. Let $a$ = ([0, 4], [0, 6]) and $b$ = ([0, 3], [0, 4]) be two interval vectors. Since a·b = 36, ||a|| = $\sqrt{52}$ and ||b|| = 5, firstly according to Eq. (2), we have that $\mathrm{Proj}_b(a)$ = 36/5 = 7.2 and $\mathrm{Proj}_b(b)$ = 25/5 = 5. In this case, we have that $\mathrm{Proj}_b(b) < \mathrm{Proj}_b(a)$. Since $a \ne b$, $b$ should be much closer to itself rather than to $a$. So, the projection measure is not reasonable. Then according to Eq. (3), we have that BProj($a$, $b$) = 1/(1+36/$\sqrt{52}$ – 36/5) = 0.3118 and BProj($b$, $b$) =1/(1+ 25/5 – 25/5) = 1. Hence BProj($b$, $b$) > BProj($a$, $b$).

Obviously, the bidirectional projection measure is reasonable and effective.

Then, the bidirectional projection measure of interval numbers can be extended to the bidirectional projection measure of neutrosophic numbers. Assume that there is a set of neutrosophic numbers $A = \{N_1, N_2, \ldots, N_n\}$ for $d_j$, $u_j \geq 0$ and $j = 1, 2, \ldots, n$. A neutrosophic number $N_j = d_j + u_jI$ for $j = 1, 2, \ldots, n$ can be transformed into an interval number based on the maximum and minimum ranges for $I$, where the lower limit of indeterminacy $I$ is denoted by inf $I$ and the upper limit of indeterminacy $I$ is denoted by sup $I$. Thus, the neutrosophic number $N_j = d_j + u_jI$ is equivalent to $N_j = [d_j + u_j \inf I, d_j + u_j \sup I]$ for $j = 1, 2, \ldots, n$. If the two sets of neutrosophic numbers $A = \{N_{A1}, N_{A2}, \ldots, N_{An}\}$ and $B = \{N_{B1}, N_{B2}, \ldots, N_{Bn}\}$ are considered as two neutrosophic number vectors, based on the bidirectional projection measure between interval vectors, we can give the bidirectional projection measure between two neutrosophic numbers.

**Definition 6**. Let $A = (N_{A1}, N_{A2}, \ldots, N_{An})$ and $B = (N_{B1}, N_{B2}, \ldots, N_{Bn})$ be two neutrosophic number vectors, where $N_{Aj} = d_{Aj} + u_{Aj}I$ and $N_{Bj} = d_{Bj} + u_{Bj}I$ ($j = 1, 2, \ldots, n$) for $d_{Aj}$, $u_{Aj}$, $d_{Bj}$, $u_{Bj} \geq 0$ and $I \in$ [inf $I$, sup $I$]. Then,

$$\|A\| = \sqrt{\sum_{j=1}^{n}\left[(d_{Aj} + u_{Aj}\inf I)^2 + (d_{Aj} + u_{Aj}\sup I)^2\right]},$$

$$\|B\| = \sqrt{\sum_{j=1}^{n}\left[(d_{Bj} + u_{Bj}\inf I)^2 + (d_{Bj} + u_{Bj}\sup I)^2\right]}$$

are called the modules of $A$ and $B$ respectively,

$$A \cdot B = \sum_{j=1}^{n}\left[(d_{Aj} + u_{Aj}\inf I)(d_{Bj} + u_{Bj}\inf I) + (d_{Aj} + u_{Aj}\sup I)(d_{Bj} + u_{Bj}\sup I)\right]$$

is called the inner product of $A$ and $B$, and then

$$\cos(A, B) = \frac{A \cdot B}{\|A\|\|B\|} \tag{4}$$

is called the cosine of the included angle between $A$ and $B$.

**Definition 7**. Let $A = (N_{A1}, N_{A2}, \ldots, N_{An})$ and $B = (N_{B1}, N_{B2}, \ldots, N_{Bn})$ be two neutrosophic number vectors, where $N_{Aj} = d_{Aj} + u_{Aj}I$ and $N_{Bj} = d_{Bj} + u_{Bj}I$ ($j = 1, 2, \ldots, n$) for $d_{Aj}$, $u_{Aj}$, $d_{Bj}$, $u_{Bj} \geq 0$ and $I \in$ [inf $I$, sup $I$]. Then

$$B\Pr oj(A, B) = \frac{1}{1 + \left|\frac{A \cdot B}{\|B\|} - \frac{A \cdot B}{\|A\|}\right|} = \frac{\|A\|\|B\|}{\|A\|\|B\| + \left|\|A\| - \|B\|\right|A \cdot B} \tag{5}$$

is called the bidirectional projection measure between $A$ and $B$, where $\|A\| = \sqrt{\sum_{j=1}^{n}\left[(d_{Aj} + u_{Aj}\inf I)^2 + (d_{Aj} + u_{Aj}\sup I)^2\right]}$, $\|B\| = \sqrt{\sum_{j=1}^{n}\left[(d_{Bj} + u_{Bj}\inf I)^2 + (d_{Bj} + u_{Bj}\sup I)^2\right]}$, and $A \cdot B = \sum_{j=1}^{n}\left[(d_{Aj} + u_{Aj}\inf I)(d_{Bj} + u_{Bj}\inf I) + (d_{Aj} + u_{Aj}\sup I)(d_{Bj} + u_{Bj}\sup I)\right]$.

Obviously, the closer the value of BProj($A$, $B$) is to 1, the closer $A$ is to $B$, and then there are BProj($A$, $B$) = 1 if and only if $A = B$ and $0 \leq$ BProj($A$, $B$) $\leq 1$ for any two neutrosophic number vectors $A$ and $B$, which is a normalized measure.

In the following, we introduce the projection between neutrosophic number matrices whose elements are all neutrosophic numbers.

**Definition 8**. Let $X = (x_{kj})_{t \times n}$ and $Y = (y_{kj})_{t \times n}$ be two neutrosophic number matrices, where $x_{kj} = d_{xkj} + u_{xkj}I$ and $y_{kj} = d_{ykj} + u_{ykj}I$ for $d_{xkj}$, $u_{xkj}$, $d_{ykj}$, $u_{ykj} \geq 0$ and $I \in$ [inf $I$, sup $I$] ($k = 1, 2, \ldots, t$; $j = 1, 2, \ldots, n$). Then the bidirectional projection measure between $X$ and $Y$ are defined as

$$B\Pr oj(X,Y) = \frac{\|X\|\|Y\|}{\|X\|\|Y\| + \big\|\|X\| - \|Y\|\big\| X \cdot Y}, \tag{6}$$

where $\|X\| = \sqrt{\sum_{k=1}^{t} \sum_{j=1}^{n} \left[ (d_{xkj} + u_{xkj} \inf\ I)^2 + (d_{xkj} + u_{xkj} \sup I)^2 \right]}$, $\|Y\| = \sqrt{\sum_{k}^{t} \sum_{j=1}^{n} \left[ (d_{ykj} + u_{ykj} \inf\ I)^2 + (d_{ykj} + u_{ykj} \sup I)^2 \right]}$,

and $X \cdot Y = \sum_{k=1}^{t} \sum_{j=1}^{n} \left[ (d_{xkj} + u_{xkj} \inf\ I)(d_{ykj} + u_{ykj} \inf\ I) + (d_{xkj} + u_{xkj} \sup I)(d_{ykj} + u_{ykj} \sup I) \right]$. Especially when $d_{xkj}$ and $d_{ykj}$ are some real numbers (crisp values) for $I$ = 0, which are the special cases of the neutrosophic numbers, the interval numbers of the equality of the upper and lower limits are also represented by $x_{kj} = [d_{xkj}, d_{xkj}]$ and $y_{kj} = [d_{ykj}, d_{ykj}]$ to apply Eq. (6) conveniently.

**Example 2**. Suppose that there are the two neutrosophic number matrices:

$$X = \begin{bmatrix} 5 & 3+I] \\ 2+2I & 4 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 6 & 7 \\ 5+I & 4+2I \end{bmatrix} \text{ for } I \in [0, 2].$$

Then, the two matrices can be transformed into the following forms:

$$X = \begin{bmatrix} [5,5] & [3,5] \\ [2,6] & [4,4] \end{bmatrix} \text{ and } Y = \begin{bmatrix} [6,6] & [7,7] \\ [5,7] & [4,8] \end{bmatrix}.$$

Hence, we firstly give the following calculations:

$$\|X\| = \sqrt{(5^2 + 5^2 + 3^2 + 5^2) + (2^2 + 6^2 + 4^2 + 4^2)} = \sqrt{156},$$

$$\|Y\| = \sqrt{(6^2 + 6^2 + 7^2 + 7^2) + (5^2 + 7^2 + 4^2 + 8^2)} = \sqrt{324},$$

$$\|X\|\|Y\| = (5 \times 6 + 5 \times 6 + 3 \times 7 + 5 \times 7) + (2 \times 5 + 6 \times 7 + 4 \times 4 + 4 \times 8) = 216.$$

Then, by using Eq. (6), we have the following bidirectional projection measure result:

$$B\Pr oj(X,Y) = \frac{\|X\|\|Y\|}{\|X\|\|Y\| + \big\|\|X\| - \|Y\|\big\| X \cdot Y} = \frac{\sqrt{156} \times \sqrt{324}}{\sqrt{156} \times \sqrt{324} + (\sqrt{324} - \sqrt{156}) \times 216} = 0.1589.$$

## 4 Group decision making method based the bidirectional projection measure

In this section, we present a handling method for multiple attribute group decision making problems with neutrosophic numbers by using the bidirectional projection measure of neutrosophic numbers.

In a multiple attribute group decision making problem with neutrosophic numbers, let $S$ = {$S_1$, $S_2$, . . . , $S_m$} be a set of alternatives, $A$ = {$A_1$, $A_2$, . . . , $A_n$} be a set of attributes, and $E$ = {$E_1$, $E_2$, . . . , $E_t$} be a set of decision makers or experts. If the decision maker $E_k$ ($k$ = 1, 2,…, $t$) provide an evaluation value of the attribute $A_j$ ($j$ = 1, 2,…, $n$) for the alternative $S_i$ ($i$ = 1, 2, . . . , $m$) by using a scale from 1 (less fit) to 10 (more fit) with indeterminacy $I$, which is represented by the form of a neutrosophic number $x_{kj}^i = d_{kj}^i + u_{kj}^i I$ for $d_{kj}^i, u_{kj}^i \geq 0$ and $d_{kj}^i, u_{kj}^i \in R$ ($k$ = 1, 2,…, $t$; $j$ = 1, 2,…, $n$; $i$ = 1, 2,…, $m$) and $I \in$ [inf $I$, sup $I$]. Thus, we can establish the alternative decision matrix of neutrosophic numbers $X^i$ ($i$ = 1, 2, . . . , $m$):

$$X^i = \begin{bmatrix} x_{11}^i & x_{12}^i & \cdots & x_{1n}^i \\ x_{21}^i & x_{22}^i & \cdots & x_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ x_{t1}^i & x_{t2}^i & \cdots & x_{tn}^i \end{bmatrix}.$$

In the following statements, $X^i$ is called the alternative decision matrix for short.

If the weights of attributes and decision makers are considered as the different importance of each attribute $A_j$ ($j$ = 1, 2, . . . , $n$) and each decision maker $E_k$ ($k$ = 1, 2,…, $t$), the weight vector of attributes is $W = (w_1, w_2,\ldots, w_n)^T$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$ and the weight vector of decision makers is $V = (v_1, v_2,\ldots, v_t)^T$ with $v_k \geq 0$ and $\sum_{k=1}^{t} v_j = 1$.

Then, the procedure of the group decision making problem is described as follows:

**Step 1:** For de-neutrosophication in the decision making problem, each alternative decision matrix of neutrosophic numbers $X^i$ can be transformed into an alternative decision matrix of interval numbers when a neutrosophic number $x_{kj}^i = d_{kj}^i + u_{kj}^i I$ is transformed into $x_{kj}^i = [d_{kj}^i + u_{kj}^i \inf I, d_{kj}^i + u_{kj}^i \sup I] = [x_{kj}^{li}, x_{kj}^{ui}]$ with respect to the specified indeterminacy $I \in [\inf I, \sup I]$ according to decision makers' and real requirements.

**Step 2:** By calculating $y_{kj}^i = [y_{kj}^{li}, y_{kj}^{ui}] = [w_j x_{kj}^{li}, w_j x_{kj}^{lu}]$ ($k$ = 1, 2,…, $t$; $j$ = 1, 2,…, $n$; $i$ = 1, 2,…, $m$) for $X^i$ ($i$ = 1, 2, …, $m$), a weighted alternative decision matrix is obtained as follows:

$$Y^i = \begin{bmatrix} y_{11}^i & y_{12}^i & \cdots & y_{1n}^i \\ y_{21}^i & y_{22}^i & \cdots & y_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ y_{t1}^i & y_{t2}^i & \cdots & y_{tn}^i \end{bmatrix}.$$

**Step 3:** According to $y_{kj}^* = [y_{kj}^{l*}, y_{kj}^{u*}] = [\max_i(y_{kj}^{li}), \max_i(y_{kj}^{ui})]$ ($k$ = 1, 2,…, $t$; $j$ = 1, 2,…, $n$; $i$ = 1, 2,…, $m$), the ideal alternative matrix is determined as follows:

$$Y^* = \begin{bmatrix} y_{11}^* & y_{12}^* & \cdots & y_{1n}^* \\ y_{21}^* & y_{22}^* & \cdots & y_{2n}^* \\ \vdots & \vdots & \ddots & \vdots \\ y_{t1}^* & y_{t2}^* & \cdots & y_{tn}^* \end{bmatrix}.$$

**Step 4:** According to Eq. (6), the bidirectional projection measure between each weighted alternative decision matrix $Y^i$ ($i$ = 1, 2,…, $m$) and the ideal alternative matrix $Y^*$ can be calculated by

$$B\Pr oj(Y^i, Y^*) = \frac{\|Y^i\|\|Y^*\|}{\|Y^i\|\|Y^*\| + \|Y^i\| - \|Y^*\| Y^i \cdot Y^*}, \tag{7}$$

where $\left\| Y^i \right\| = \sqrt{\sum_{k}^{t} v_j \sum_{j=1}^{n} \left[ (y_{kj}^{li})^2 + (y_{kj}^{ui})^2 \right]}$ , $\left\| Y^* \right\| = \sqrt{\sum_{k=1}^{t} v_k \sum_{j=1}^{n} \left[ (y_{kj}^{*})^2 + (y_{kj}^{*})^2 \right]}$ , and $Y^i \cdot Y^* = \sum_{k=1}^{t} v_j \sum_{j=1}^{n} \left[ y_{kj}^{li} y_{kj}^{l*} + y_{kj}^{ui} y_{kj}^{u*} \right]$.

**Step 5:** The alternatives are ranked in a descending order according to the values of BProj($Y^i$, $Y^*$) for $i$ = 1, 2, …, $m$. The greater value of BProj($Y^i$, $Y^*$) means the better alternative $S_i$.

**Step 6:** End.

## 5. Example analysis

In this section, an illustrative example above a multiple attribute group decision making problem with neutrosophic numbers is given to show the applications and effectiveness of the proposed group decision making method in realistic scenarios.

### 5.1 Illustrative example

An illustrative example about investment alternatives for a multiple attribute group decision-making problem adopted from [16] is given to demonstrate the applications of the proposed group decision-making method with neutrosophic numbers. Assume that an investment company wants to invest a sum of money for the best option. To invest the money, there are four possible alternatives: (1) $S_1$ is a car company; (2) $S_2$ is a food company; (3) $S_3$ is a computer company; (4) $S_4$ is an arms company. The investment company must take a decision according to the three attributes: (1) $A_1$ is the risk factor; (2) $A_2$ is the growth factor; (3) $A_3$ is the environmental factor. Assume that the weighting vector of the attributes is $W$ = (0.35, 0.25, 0.4)$^T$. If three experts are required in the evaluation process and their weighting vector is $V$ = (0.37, 0.33, 0.3)$^T$, the expert $E_k$ ($k$ = 1, 2, 3) evaluates the four possible alternatives of $S_i$ ($i$ = 1, 2, 3, 4) with respect to the three attributes of $A_j$ ($j$ = 1, 2, 3) by the form of neutrosophic numbers $x_{kj}^i = d_{kj}^i + u_{kj}^i I$ for $d_{kj}^i, u_{kj}^i \geq 0$ and $d_{kj}^i, b_{kj}^i \in R$ ($k$ = 1, 2, 3; $j$ = 1, 2, 3; $i$ = 1, 2, 3, 4).

For example, the first expert $E_1$ gives the neutrosophic number of an attribute $A_1$ for an alternative $S_1$ as $x_{11}^1$ = 4 + $I$ by using a scale from 1 (less fit) to 10 (more fit) with indeterminacy $I$, which indicates that the grade of the attribute $A_1$ with respect to the alternative $S_1$ is the determinate degree 4 with an indeterminacy $I$. Thus, when the four possible alternatives with respect to the three attributes are evaluated by the three experts, we can establish the following four alternative decision matrices, respectively:

$$X^1 = \begin{bmatrix} 4+I & 5 & 3+I \\ 5 & 4 & 4 \\ 4 & 5+I & 4 \end{bmatrix},$$

$$X^2 = \begin{bmatrix} 6 & 6 & 5 \\ [5+I & 6 & 6 \\ 6 & 7 & 5+I \end{bmatrix},$$

$$X^3 = \begin{bmatrix} 3 & 5+I & 6 \\ 4 & 5 & 5+I \\ 4+I & 5 & 6 \end{bmatrix},$$

$$X^4 = \begin{bmatrix} 7 & 6 & 4+I \\ 6+I & 6 & 5 \\ 8 & 6 & 4+I \end{bmatrix}.$$

.

Then, the developed approach is applied to the decision making problem and described by the following steps:

**Step 1:** Assume that the specified indeterminacy is $I \in [0, 0.5]$ according to the decision makers' and real requirements. Then, the four alternative decision matrices of $X^i$ ($i$ = 1, 2, …, $m$) can be transformed into the following forms, respectively:

$$X^1 = \begin{bmatrix} [4,4.5] & [5,5] & [3,3.5] \\ [5,5] & [4,4] & [4,4] \\ [4,4] & [5,5.5] & [4,4] \end{bmatrix},$$

$$X^2 = \begin{bmatrix} [6,6] & [6,6] & [5,5] \\ [5,5.5] & [6,6] & [6,6] \\ [6,6] & [7,7] & [5,5.5] \end{bmatrix},$$

$$X^3 = \begin{bmatrix} [3,3] & [5,5.5] & [6,6] \\ [4,4] & [5,5] & [5,5.5] \\ [4,4.5] & [5,5] & [6,6] \end{bmatrix},$$

$$X^4 = \begin{bmatrix} [7,7] & [6,6] & [4,4.5] \\ [6,6.5] & [6,6] & [5,5] \\ [8,8] & [6,6] & [4,4.5] \end{bmatrix}.$$

**Step 2:** By calculating $y_{kj}^{i} = [y_{kj}^{li}, y_{kj}^{ui}] = [w_j x_{kj}^{li}, w_j x_{kj}^{lu}]$ ($k$ = 1, 2,…, $t$; $j$ = 1, 2,…, $n$; $i$ = 1, 2,…, $m$) for $X^i$ ($i$ = 1, 2, …, $m$), the four weighted alternative decision matrices are obtained, respectively, as follows:

$$Y^1 = \begin{bmatrix} [1.4,1.575] & [1.25,1.25] & [1.2,1.4] \\ [1.75,1.75] & [1,1] & [1.6,1.6] \\ [1.4,1.4] & [1.25,1.375] & [1.6,1.6] \end{bmatrix},$$

$$Y^2 = \begin{bmatrix} [2.1,2.1] & [1.5,1.5] & [2,2] \\ [1.75,1.925] & [1.5,1.5] & [2.4,2.4] \\ [2.1,2.1] & [1.75,1.75] & [2,2.2] \end{bmatrix},$$

$$Y^3 = \begin{bmatrix} [1.05,1.05] & [1.25,1.375] & [2.4,2.4] \\ [1.4,1.4] & [1.25,1.25] & [2,2.2] \\ [1.4,1.575] & [1.25,1.25] & [2.4,2.4] \end{bmatrix},$$

$$Y^4 = \begin{bmatrix} [2.45,2.45] & [1.5,1.5] & [1.6,1.8] \\ [2.1,2.275] & [1.5,1.5] & [2,2] \\ [2.8,2.8] & [1.5,1.5] & [1.6,1.8] \end{bmatrix}.$$

**Step 3:** According to $y_{kj}^{*} = [y_{kj}^{l*}, y_{kj}^{u*}] = [\max_i (y_{kj}^{li}), \max_i (y_{kj}^{ui})]$ ($k$ = 1, 2,…, $t$; $j$ = 1, 2,…, $n$; $i$ = 1, 2,…, $m$), the ideal alternative matrix is determined as follows:

$$Y^* = \begin{bmatrix} [2.45,2.45] & [1.5,1.5] & [2.4,2.4] \\ [2.1,2.275] & [1.5,1.5] & [2.4,2.4] \\ [2.8,2.8] & [1.75,1.75] & [2.4,2.4] \end{bmatrix}.$$

**Step 4:** According to Eq. (7), the bidirectional projection measure values between each weighted alternative decision matrix $Y^i$ ($i$ = 1, 2, 3, 4) and the ideal alternative matrix $Y^*$ can be obtained as follows:

$BProj(Y^1, Y^*) = 0.3505$, $BProj(Y^2, Y^*) = 0.6185$, $BProj(Y^3, Y^*) = 0.4632$, and $BProj(Y^4, Y^*) = 0.6814$.

**Step 5:** Since the values of the bidirectional projection measure are $BProj(Y^4, Y^*) > BProj(Y^2, Y^*) > BProj(Y^3, Y^*) > BProj(Y^1, Y^*)$, the ranking order of the four alternatives is $S_4 > S_2 > S_3 > S_1$. Hence, the alternative $S_4$ is the best choice among all the alternatives.

## 5.2 Comparative analysis

In this subsection, we give the comparative analysis with the general projection measure and the decision making method proposed by Ye [16] to illustrate the effectiveness and advantages of the developed method.

For the above-mentioned example, if Eq.(7) are replaced by the general projection measure:

$$\mathrm{Pr}\,oj_{Y^*}(Y^i) = \frac{Y^i \cdot Y^*}{\left\| Y^* \right\|}. \tag{8}$$

Then, the projections of $Y^i$ ($i$ = 1, 2, 3, 4) on the ideal decision $R^*$ can be calculated by Step 3 and using Eq. (8). The results are shown as follows:

$Proj_{Y^*}(Y^1) = 3.4422$, $Proj_{Y^*}(Y^2) = 4.7239$, $Proj_{Y^*}(Y^3) = 4.0006$, and $Proj_{Y^*}(Y^4) = 4.7818$.

Since the values of the projection measure are $Proj_{Y^*}(Y^4) > Proj_{Y^*}(Y^2) > Proj_{Y^*}(Y^3) > Proj_{Y^*}(Y^1)$, the ranking order of the four alternatives is also $S_4 > S_2 > S_3 > S_1$. Hence, the alternative $S_4$ is also the best choice among all the alternatives.

Obviously, the two ranking orders obtained by using the general projection method and the bidirectional projection method are identical. Compared with the general projection measure decision making method, the proposed bidirectional projection measure decision making method is superior to the general projection measure decision making method because the bidirectional projection measure can consider not only the distance and included angle between objects evaluated but also the bidirectional projection magnitudes, while the general projection measure only consider the single directional projection magnitude between objects evaluated rather than the bidirectional projection magnitudes and implies some unreasonable phenomena in some case. As mentioned above, furthermore, the bidirectional projection measure can overcome the shortcoming of the general projection measure and is superior to the general projection measure.

Compared with the decision making method proposed by Ye [16], the bidirectional projection decision making method demonstrates the same ranking order as Ye's method [16]. For convenient comparison, if we consider different ranges of the indeterminate degree for *I*, by Steps 1-4, all the results are shown in Table 1.

Table 1. Ranking alternatives in different indeterminate ranges for *I*

| $I$ | Ranking order of the bidirectional projection method | Ranking order of Ye's method [16] |
|---|---|---|
| $I = 0$ | $S_4 > S_2 > S_3 > S_1$ | $S_2 > S_4 > S_3 > S_1$ |
| $I \in [0, 0.2]$ | $S_4 > S_2 > S_3 > S_1$ | $S_2 > S_4 > S_3 > S_1$ |
| $I \in [0, 0.4]$ | $S_4 > S_2 > S_3 > S_1$ | $S_4 > S_2 > S_3 > S_1$ |
| $I \in [0, 0.6]$ | $S_4 > S_2 > S_3 > S_1$ | $S_4 > S_2 > S_3 > S_1$ |
| $I \in [0, 0.8]$ | $S_4 > S_2 > S_3 > S_1$ | $S_4 > S_2 > S_3 > S_1$ |
| $I \in [0, 1]$ | $S_4 > S_2 > S_3 > S_1$ | $S_4 > S_2 > S_3 > S_1$ |

From Table 1, it is obvious that the ranking orders indicate their difference in the indeterminate degree for $I \in [0, 0.2]$ based on different methods, while the ranking orders are identical in the indeterminate degree from $I \in [0, 0.4]$ to $I \in [0, 1]$. The method proposed by Ye [16] is based on the de-neutrosophication process and possibility degree ranking order of neutrosophic numbers, then it cannot evaluate the proximity between an ideal solution (an ideal alternative) and alternatives; while the new method proposed in this paper is based on the bidirectional projection between each alternative decision matrix and the ideal alternative matrix, and then it shows that the closer the alternative is to the ideal alternative, the better the alternative is. Therefore, the new method is more reasonable than the existing method [16].

However, the main advantages of the proposed method are outlined as follows:

(1) The bidirectional projection method is more reasonable than the general projection method because the former can overcome the shortcoming of the latter, then the bidirectional projection measure value is bounded within [0, 1], which is a normalized measure.

(2) The bidirectional projection method is more comprehensive than the general projection method because the bidirectional projection can consider not only the distance and the included angle between objects evaluated but also the bidirectional projection magnitudes.

(3) The bidirectional projection-based multiple attribute group decision making method with neutrosophic numbers is reasonable and effective and provide a new decision making method under a neutrosophic number environment

## 6 Conclusion

This paper firstly proposed a bidirectional projection measure of interval numbers to overcome the shortcomings of the general projection measure, and then extended it to the bidirectional projection measure between neutrosophic numbers. Further, a bidirectional projection-based multiple attribute group decision making method was developed under a neutrosophic number environment. Through the bidirectional projection measure between each alternative decision matrix and the ideal alternative matrix, the ranking order of all alternatives can be determined to select the best alternative. Finally, an illustrative example demonstrated the application of the developed method, and then the effectiveness and rationality of the developed method are demonstrated by the comparative analysis with existing relative methods.

In the future work, we shall extend the bidirectional projection method to other decision data, such as intuitionistic fuzzy sets and neutrosophic sets, and develop the applications such as pattern recognition and medical diagnosis.

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| $I \in [0, 0.2]$ | $S_4 > S_2 > S_3 > S_1$ | $S_2 > S_4 > S_3 > S_1$ |
| $I \in [0, 0.4]$ | $S_4 > S_2 > S_3 > S_1$ | $S_4 > S_2 > S_3 > S_1$ |
| $I \in [0, 0.6]$ | $S_4 > S_2 > S_3 > S_1$ | $S_4 > S_2 > S_3 > S_1$ |
| $I \in [0, 0.8]$ | $S_4 > S_2 > S_3 > S_1$ | $S_4 > S_2 > S_3 > S_1$ |
| $I \in [0, 1]$ | $S_4 > S_2 > S_3 > S_1$ | $S_4 > S_2 > S_3 > S_1$ |