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F. Smarandache第 80问题与除数问题

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摘要 研究了 $\sum_{n\leqslant x} d(a_k(n))$ 的渐进公式, 这里 $a_k(n)$ 是 F. Smarandache教授提出的 "only problems, not solutions" 中第 80问题的数列。

关键词 算术函数 均值 渐进公式

中图法分类号 O156; 文献标识码 A

1993年, F. Smarandache教授在其所著的 "Only problems, not solutions"[1]中提出了 100个问题, 这引起了学者们的极大兴趣。其中, 第 80个问题是:

幂根: 0, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 8, 8 ……。研究这个数列的性质。

我们把第 80个问题中的数列定义为 $a(n)$, 不难发现 $a(n)=[\sqrt{n}]$, 其中 $[x]$ 是不超过 x 的最大整数。更一般地, 设 $k\geqslant 2$ 是任意固定整数, 定义 $a_k(n)=[n^{1/k}]$。何晓林和郭金保在 "On the 80th Problem of F. Smarandache(Ⅱ)"[2]中研究了除数问题在序列 $a_k(n)$ 中的推广。他们证明了

$$\sum_{n\leqslant x} d(a_2(n)) = \sum_{n\leqslant x} d([\sqrt{n}]) = \frac{1}{2}x\log x + \left(2\gamma - \frac{1}{2}\right)x + O(x^{3/4}),$$

$$\sum_{n\leqslant x} d(a_k(n)) = \sum_{n\leqslant x} d([n^{1/k}]) = \frac{1}{k}x\log x + O(x)。$$

本文将改进上述结果, 并证明其他相关的结论。

1 $\sum_{n\leqslant x} d(a(n))$ 的渐进公式

基本定理 令

$$\Delta(x):=\sum_{n\leqslant x} d(n) - x\log x - (2\gamma-1)x;$$

$$E_k(x):=\sum_{n\leqslant x} d([n^{1/k}]) - \frac{1}{k}x\log x - \left(2\gamma - \frac{1}{k}\right)x。$$

这里 γ 是 Euler常数, 则

$$E_k(x) = kx^{1-(1/k)}\Delta(x^{1/k}) + O(x^{1-(1/k)+\varepsilon}) \quad (1)$$

证明 对任意正数 x 一定存在整数 N, 使得 $N^k\leqslant x<(N+1)^k$, 所以

$$\sum_{n\leqslant x} d([n^{1/k}]) = \sum_{j\leqslant N}\sum_{j^k\leqslant i<(j+1)^k} d([i^{1/k}]) - \sum_{x<i<(N+1)^k} d([i^{1/k}]) = \sum_{j\leqslant N}((j+1)^k - j^k)d(j) + O(N^{k-1}d(N)) = k\sum_{j\leqslant N} j^{k-1}d(j) + O(\sum_{j\leqslant N} j^{k-2}d(j)) + O(N^{k-1}d(N)) = k\sum_{j\leqslant x^{1/k}} j^{k-1}d(j) + O(x^{1-(1/k)+\varepsilon}) \quad (2)$$

这里用到了 $\sum_{n\leqslant x} d(n) = x\log x + (2\gamma-1)x + O(\sqrt{x})$ 及 $d(N)\ll N^{\varepsilon}$。

令 $D(t)=\sum_{j\leqslant t} d(j)$, 则

$$\sum_{j\leqslant x^{1/k}} j^{k-1}d(j) = \int_1^{x^{1/k}} t^{k-1}\,\mathrm{d}D(t) = \int_1^{x^{1/k}} t^{k-1}(\log x + 2\gamma)\,\mathrm{d}t + \int_1^{x^{1/k}} t^{k-1}\,\mathrm{d}\Delta(t) = \mathrm{I} + \mathrm{II} \quad (3)$$

易得

$$I = \frac{1}{k^2}x\log x + \frac{1}{k}2\gamma x - \frac{1}{k^2}x + O(1) \quad (4)$$

设 $G(t)=\int_1^t \Delta(x)\,\mathrm{d}x$, Voronoi[3]证明了

$$\int_1^T \Delta(x)\,\mathrm{d}x = \frac{T}{4} + O(T^{3/4}) \quad (5)$$

所以

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$$\int_1^{x^{1/k}} t^{k-2}\Delta(t)\,dt = \int_1^{x^{1/k}} t^{k-2}\,dG(t) = x^{1-(2/k)}G(x^{1/k}) - (k-2)\int_1^{x^{1/k}} t^{k-3}G(t)\,dt \ll x^{1-(1/k)}。$$

从而由分部积分得

$$\mathrm{II} = x^{1-(1/k)}\Delta(x^{1/k}) + O(1) - (k-1)\int_1^{x^{(1/k)}} t^{k-2}\Delta(t)\,dt =$$

$$x^{1-(1/k)}\Delta(x^{1/k}) + O(x^{1-(1/k)}) \quad (6)$$

由(2)式,(3)式,(4)式和(6)式,得

$$\sum_{n\leqslant x} d([n^{1/k}]) = \frac{1}{k}x\log x + \left(2\gamma - \frac{1}{k}\right)x + kx^{1-(1/k)}\times \Delta(x^{1/k}) + O(x^{1-(1/k)+\varepsilon}) \quad (7)$$

定理 1 若 $\Delta(x)\leqslant x^{\theta}$,则

$$E_k(x) \leqslant x^{1-(1/k)+(\theta/k)} \quad (8)$$

特别地,

$$E_k(x) \ll x^{1-285/416k}(\log x)^{26957/8320} \quad (9)$$

证明 (8)式由基本定理直接可得。Huxley[4]证明了 $\Delta(x)\ll x^{131/416}(\log x)^{26957/8320}$,代入(8)式知(9)式成立。

$$E_k(x) \ll x^{1-285/416k}(\log x)^{26957/8320}。$$

定理 2

$$E_k(x) = \Omega_+(x^{1-3/4k}(\log x)^{1/4}\times (\log\log x)^{(3+\log 4)/4}\exp(-c\sqrt{\log\log\log x}) \quad (10)$$

$$E_k(x) = \Omega_-(x^{1-3/4k}\exp(c'(\log\log x)^{1/4}\times (\log\log\log x)^{-3/4})) \quad (11)$$

其中 c, c'是绝对正常数。

证明 Hafner[5]证明了

$$\Delta(x) = \Omega_+(x^{1/4}(\log x)^{1/4}\times(\log\log x)^{(3+\log 4)/4}\exp(-c\sqrt{\log\log\log x})),\ c>0 \quad (12)$$

K. Corradi和 I K'atai[6]证明了

$$\Delta(x) = \Omega_-(x^{1/4}\exp(c'(\log\log x)^{1/4}\times(\log\log\log x)^{-3/4})),\ c'>0 \quad (13)$$

由基本定理,(12)式和(13)式可得定理 2。

定理 3

$$\int_1^T E_k^2(x)\,dx = \frac{c_1k^3}{2k-1}T^{3-3/2k} + O(T^{3-7/4k+\varepsilon}) \quad (14)$$

其中 $c_1 = \frac{(\zeta(3/2))^4}{6\pi^2\zeta(3)}$。

证明

$$\int_1^T E_k^2(x)\,dx = k^2\int_1^T x^{2-(2/k)}\Delta^2(x^{1/k})\,dx + 2k\int_1^T x^{1-(1/k)}\times\Delta(x^{1/k})O(x^{1-(1/k)+\varepsilon})\,dx + \int_1^T O(x^{2-(2/k)+2\varepsilon})\,dx = \mathrm{I} + \mathrm{II} + \mathrm{III} \quad (15)$$

令 $F(t) = \int_1^t \Delta^2(x)\,dx$,由文献[7]有

$$F(T) = c_1T^{3/2} + G(T),\ G(T) = O(T\log^5 T) \quad (16)$$

所以

$$\mathrm{I} = k^3\int_1^{T^{1/k}} t^{3k-3}\,dF(t) = \frac{3}{2}c_1k^3\int_1^{T^{1/k}} t^{3k-3+1/2}\,dt + k^3\int_1^{T^{1/k}} t^{3k-3}\,dG(t) = \frac{c_1k^3}{2k-1}T^{3-(3/2k)} + k^3T^{3-(3/k)}G(T^{1/k}) + O(1) - k^3(3k-3)\int_1^{T^{1/k}} t^{3k-4}G\,dt = \frac{c_1k^3}{2k-1}T^{3-(3/2k)} + O(T^{3-2/k}\log^5 T) \quad (17)$$

由 Cauchy不等式及(16)式得

$$\mathrm{II} = 2k^2\int_1^{T^{1/k}} t^{2k-2}\Delta(t)O(t^{k-1+\varepsilon})\,dt \ll 2k^2\times \left(\int_1^{T^{1/k}} t^{4k-4}O(t^{2k-2+2\varepsilon})\,dt\right)^{1/2}\left(\int_1^{T^{1/k}}\Delta^2(t)\,dt\right)^{1/2} \ll T^{3-7/4k+\varepsilon} \quad (18)$$

又

$$\mathrm{III} \ll T^{3-2/k+\varepsilon} \quad (19)$$

由(15)式,(17)式,(18)式和(19)式得(14)式。

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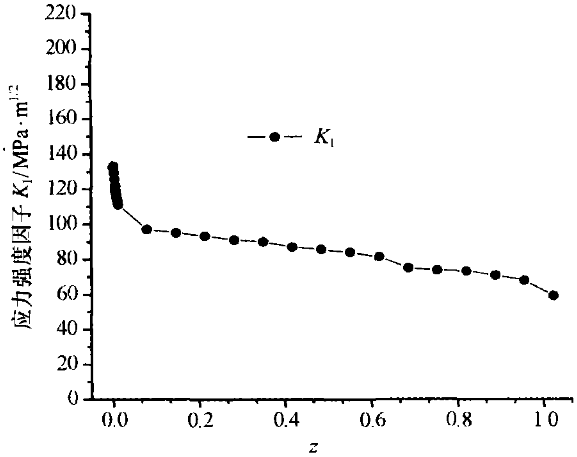

图5 β=75°，K_I 沿 z 方向的转变

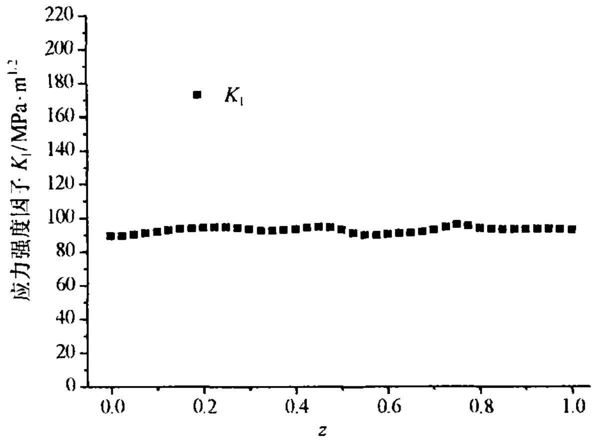

图6 β=90°，即裂纹长度相等时 K_I 沿 z 方向的转变

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Calculation of Stress Intensity Factors for Oblique through Cracks

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[Abstract] Stress intensity factors(SIFs) of cracks having different length which angle β is equal to 45°, 60° and 75° respectively, are calculated by using finite element method(FEM). Then distribution of value of SIFs are obtained along with crack tips and angle, which fetches up handbooks of stress intensity factors in oblique through cracks.

[Key words] stress intensity factors(SIFs) oblique through cracks finite element method(FEM)

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F. Smarandache 80th Problem

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[Abstract] The main purpose is to study the mean value of $d(a_k(n))$, where $d(n)$ is divisor function and $a_k(n)$ is the sequence in Problem 80 of Professor F. Smarandache's book "only problems, not solutions".

[Key words] divisor function mean-value asymptotic formula