The Explicit Formula for the Smarandache Function and Solutions of Related Equations

LIAO Qunying, LUO Wenli

(*Institute of Mathematics and Software Science, Sichuan Normal University, Chengdu* 610066, *Sichuan*)

Abstract: Let $\varphi(n)$ and $S(n)$ be the Euler function and Smarandache function for a positive integer n, respectively. By using elementary methods and techniques, the explicit formula for $S(p^\alpha)$ is obtained, where p is a prime and α is a positive integer. As a corollary, some properties for positive integer solutions of the equations $\varphi(n)=S(n^k)$ or $\sigma(2^\alpha q)/S(2^\alpha q)$ are given, where q is an odd prime and $\sigma(n)$ is the sum of different positive factors for n.

Keywords: Smarandache function; Euler function; Gauss function; perfect number

2010 MSC: 12E20; 12E30; 11T99

doi: 10.3969/j.issn.1001-8395.2017.01.001

1 Introduction and Main Results

In 1918, Kempner[1] studied the formula of the value $\min\{m: m\in\mathbf{N}, n\mid m!\}$ for a fixed positive integer n. In 1993, Smarandache raised some interesting number theory problems, and put forword the definition of the Smarandache function $S(n)=\min\{m: m\in\mathbf{N}, n\mid m!\}$ for a positive integer n. From the definition, $S(1)=1, S(2)=2, S(3)=3$, and so on. So far, there are some good related results[1-9]. For example, in [2], the distribution of $S(n)$ was discussed, and the asymptotic formula of $S(n)$ was given as follows

$$\sum_{n\leqslant x}(S(n)-P(n))^2=\frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x}+o(\frac{x^{\frac{3}{2}}}{\ln^2 x}),$$

where $P(n)$ is the maximum prime factor of n, and $\zeta(s)$ is the Riemann-zeta function. In [3], Farris studied the bound of $S(n)$ and got the following upper and lower bounds

$$(p-1)\alpha+1\leqslant S(p^\alpha)\leqslant(p-1)[1+\alpha+\log_p\alpha],$$

where p is a prime. For a positive integer $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, where $p_1,\cdots,p_r$ are different primes and $\alpha_1,\cdots,\alpha_r$ are positive integers. From the definition, it is easy to show that $S(n)=\max\{S(p_i^{\alpha_i})\mid 1\leqslant i\leqslant r\}$. So it is enough to compute $S(p^\alpha)$, where p is a prime and α is a positive integer, which has not been solved completely yet.

On the other hand, a lot of number theory equations related to $S(n)$ have been studied in recent years. Especially, for a given positive integer k, many properties for positive integer solutions of the equation $\varphi(n)=S(n^k)$ were studied, where φ is the Euler function. Easy to see that this is equivalent to solve the equation

$$\varphi(p^\alpha m)=S(p^{\alpha k}),\qquad(*)$$

where p is a prime, $\gcd(p,m)=1$ and $S(p^{\alpha k})\geqslant S(m^k)$.

By using elementary methods and techniques, the present paper gives the explicit formula for $S(p^\alpha)$, where p is a prime and α is a positive integer, and then some properties for positive integer solutions of the equations $\varphi(n)=S(n^k)$ or $\frac{\sigma(2^\alpha q)}{S(2^\alpha q)}$ are given, where q is an odd prime and $\sigma(n)$ is the sum of the different positive factors for n. In fact, we prove the following main results.

Theorem 1.1 Let p be a prime and α be a positive integer.

Received date: 2016-01-03

Foundation Items: This work is supported by National Natural Science Foundation of China (No. 11401408) and Project of Science and Technology Department of Sichuan Province (No. 2016JY0134)

1) For any positive integer r and $\alpha = p^r$, we have
$$S(p^\alpha) = p^{r+1} - p^r + p.$$

2) For any positive integer r, $t \in [1, r]$ and $\alpha = p^r - t$, we have
$$S(p^\alpha) = p^{r+1} - p^r.$$

3) For any positive integer r, $t \in [r+1, p^r - p^{r-1}]$ and $\alpha = p^r - t$.

(I) If
$$\alpha = p^r - r - \sum_{i=1}^{n-1} (-1)^{i-1}(p^{k_i} - k_i) + (-1)^n p^{k_n}$$
with
$$k_i < p^{k_i - 1}(p-1) - 1, \quad 1 \leqslant i \leqslant n-1,$$
then we have
$$S(p^\alpha) = (p-1)\Big(p^r + \sum_{i=1}^{n} (-1)^i p^{k_i}\Big) + (-1)^n p. \qquad (1)$$

(II) If
$$\alpha = p^r - r - \sum_{i=1}^{n-1} (-1)^{i-1}(p^{k_i} - k_i) + (-1)^n (p^{k_n} - t)$$
with $t \in [1, k_n]$ and
$$k_i < p^{k_i - 1}(p-1) - 1, \quad 1 \leqslant i \leqslant n-1,$$
then
$$S(p^\alpha) = (p-1)\Big(p^r + \sum_{i=1}^{n} (-1)^i p^{k_i}\Big). \qquad (2)$$

Corollary 1.2 Let α be a positive integer. If
$$\alpha = \sum_{i=1}^{n} 2^{k_i} - n, \quad 1 \leqslant k_1 < k_2 < \cdots < k_n,$$
then we have $S(2^\alpha) = \alpha + n$.

For $k = 2, 3, 4$, the solutions of the equation (*) have been discussed in [7]. In the present paper, we complement their results and obtain some necessary conditions for solutions of the equation (*).

Theorem 1.3 1) For any positive integer k, there are no any prime p and positive integer m coprime with p, such that $\varphi(pm) = S(p^k)$ and $S(p^k) \geqslant S(m^k)$.

2) For any positive integer k, if there are some prime p and positive integer m coprime with p, such that $\varphi(p^2 m) = S(p^{2k})$ and $S(p^{2k}) \geqslant S(m^k)$. Then $p = 2k+1$ or $2 \leqslant p \leqslant k$. Furthermore,

(I) if $2k+1 = p$, then
$$(p, m) = (2k+1, 1), (2k+1, 2), (2, 3);$$

(II) otherwise, i.e., $2 \leqslant p \leqslant k$, then $k \geqslant 3$ and
$$\begin{cases} 2 \leqslant \varphi(m) \leqslant \dfrac{2k^2 + k - 1}{3}, & k \equiv 2 \pmod 3, \\ 2 \leqslant \varphi(m) \leqslant \dfrac{2k^2 + k}{3}, & \text{otherwise}. \end{cases}$$

3) For any positive integer k, if there are some prime p and positive integer m coprime with p, such that $\varphi(p^\alpha m) = S(p^{\alpha k})$ and $S(p^{\alpha k}) \geqslant S(m^k)$. Then $\alpha k + 1 > p^{\alpha - 3}(p^2 - 1)$ and $1 \leqslant \varphi(m) \leqslant q$, where
$$\alpha k + 1 = q p^{\alpha - 3}(p^2 - 1) + r,$$
$$0 \leqslant r < p^{\alpha - 3}(p^2 - 1).$$

4) For any positive integer k, there exist some prime p and positive integer m coprime with p, such that $\varphi(p^3 m) = S(p^{3k})$ and $S(p^{3k}) \geqslant S(m^k)$, namely, $m = 1, 2$.

Theorem 1.4 1) For any prime p, there is no any positive integer α such that $\dfrac{\sigma(p^\alpha)}{S(p^\alpha)}$ is a positive integer.

2) Let p be an odd prime, $\alpha \geqslant 1$ and $n = 2^\alpha p$.

(I) If $\sum_{i=1}^{\infty} \left[\dfrac{p}{2^i}\right] \geqslant \alpha$ and $\dfrac{\sigma(n)}{S(n)}$ is a positive integer, then $2^{\alpha+1} \equiv 1 \pmod p$.

(II) If $\sum_{i=1}^{\infty} \left[\dfrac{p}{2^i}\right] < \alpha$ and $\dfrac{\sigma(n)}{S(n)}$ is a positive integer, then $\dfrac{\sigma(n)}{S(n)} = m \dfrac{2^{\alpha+1} - 1}{d}$ and $p = m \dfrac{S(2^\alpha)}{d} - 1$, where $d = \gcd(2^{\alpha+1} - 1, S(2^\alpha))$ and $0 < m \leqslant d$.

Corollary 1.5 1) Let r be a positive integer and $2^r + 1$ be a prime. If $n = 2^{2^r}(2^r + 1)$, then $\dfrac{\sigma(n)}{S(n)} = 2^{2^r + 1} - 1$.

2) If $n = 2^{p-1}(2^p - 1)$ is an even perfect number, i.e., $\sigma(n) = 2n$, then $\dfrac{\sigma(n)}{S(n)} = 2^p$.

3) If $2^r - 1$ is a prime and $n = 2^{2^r - 1}(2^r - 1)$, then
$$\frac{\sigma(n)}{S(n)} = 2^{2^r} - 1.$$

Remark For convenience, throughout the paper we denote $[\cdot]$ to be the Gauss function.

2 The Proofs for Our Main Results

Before proving our main results, the following Lemmas are necessary.

Lemma 2.1[4] 1) Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is a positive integer, where $p_1, \cdots, p_r$ are different primes and $\alpha_1, \cdots, \alpha_r$ are positive integers. Then

$$S(n) = \max\{S(p_i^{\alpha_i}) \mid 1 \leqslant i \leqslant r\}.$$

2) For any prime p and positive integer k with $k \leqslant p$, we have

$$S(p^k) = kp.$$

Lemma 2.2 For any positive integer α and prime p, we have $S(p^\alpha) \leqslant (\alpha - k_\alpha) p$, where $k_\alpha (p+1) \leqslant \alpha < (k_\alpha + 1)(p+1)$.

Proof For $0 < \alpha < p+1$, by 2) of Lemma 2.1, we have $S(p^\alpha) = \alpha p = (\alpha - k_\alpha) p$, i.e., $k_\alpha = 0$. Namely, in this case Lemma 2.2 is true.

Now for $\alpha = m \geqslant p+1$, if $S(p^m) = (m - k_m) p$ with

$$k_m (p+1) \leqslant m < (k_m + 1)(p+1),$$

then

$$k_m (p+1) + 1 \leqslant m + 1 < (k_m + 1)(p+1) + 1.$$

Thus for $\alpha = m+1$, we know that

$$k_{m+1}(p+1) \leqslant m+1 < (k_{m+1}+1)(p+1).$$

Hence we have two cases as following.

(I) If

$$k_m (p+1) + 1 \leqslant m+1 < (k_m+1)(p+1),$$

then $k_{m+1} = k_m$. By the definition of $S(n)$, we have $S(p^{m+1}) \leqslant S(p^m) + p$, and so

$$S(p^{m+1}) \leqslant S(p^m) + p \leqslant (m - k_m) p + p = (m+1-k_m) p = (m+1-k_{m+1}) p,$$

therefore in this case Lemma 2.2 is true.

(II) Otherwise, we have $m+1 = (k_m+1)(p+1)$, and then $k_{m+1} = k_m + 1$ and $m - k_m = (k_m+1) p$, where

$$S(p^m) \leqslant (m - k_m) p = (k_m+1) p^2.$$

Note that

$$\sum_{i=1}^{\infty} \left[\frac{(k_m+1) p^2}{p^i}\right] = (k_m+1) p + k_m + 1 + \sum_{i=3}^{\infty} \left[\frac{(k_m+1) p^2}{p^i}\right] \geqslant (m+1),$$

therefore

$$S(p^{m+1}) \leqslant (m - k_m) p = ((m+1) - (k_m+1)) p = ((m+1) - k_{m+1}) p.$$

This means that Lemma 2.2 is true.

By the definition of $S(n)$, we immediately have the following.

Lemma 2.3 Let p be a prime and m be a positive integer. Then

$$S(p^{m+1}) = \begin{cases} S(p^m) + p, & m = \sum_{i=1}^{\infty} \left[\frac{S(p^m)}{p^i}\right], \\ S(p^m), & m < \sum_{i=1}^{\infty} \left[\frac{S(p^m)}{p^i}\right]. \end{cases}$$

The Proof for Theorem 1.1 1) Since p is a prime, and so

$$\sum_{i=1}^{\infty} \left[\frac{p^{r+1} - p^r + p}{p^i}\right] = \sum_{i=1}^{r} \left[\frac{p^{r+1} - p^r + p}{p^i}\right] = \sum_{i=1}^{r} \left[p^{r-i}(p-1) + \frac{p}{p^i}\right] = (p-1)(p^{r-1} + \cdots + p^0) + 1 = (p-1)\frac{p^r - 1}{p-1} + 1 = p^r.$$

Thus, by the definition of $S(n)$, we have $S(p^{p^r}) = p^{r+1} - p^r + p$, and then (1) of Theorem 1.1 is proved.

2) Since

$$\sum_{i=1}^{\infty} \left[\frac{p^{r+1} - p^r}{p^i}\right] = \sum_{i=1}^{r} \left[\frac{p^{r+1} - p^r}{p^i}\right] = \sum_{i=1}^{r} \left[p^{r-i}(p-1)\right] = (p-1)(p^{r-1} + \cdots + p^0) = (p-1)\frac{p^r - 1}{p-1} = p^r - 1,$$

and $p^r \parallel (p^{r+1} - p^r)$, and so for any positive integer r and $\alpha = p^r - t$ with $t \in [1, r]$, we have $S(p^\alpha) = p^{r+1} - p^r$, thus (2) of Theorem 1.1 is true.

3) For $\alpha = p^r - t$ with $r+1 < p^r - p^{r-1}$ and $t \in [r+1, p^r - p^{r-1}]$. Set $m = t - r$, then $\alpha = p^r - r - m$ $(1 \leqslant m \leqslant p^r - p^{r-1} - r)$, i.e., $r + m \in [r+1, p^r - p^{r-1}]$. We can conclude that

$$S(p^\alpha) = p^{r+1} - p^r - S(p^{t-r}) = p^{r+1} - p^r - S(p^m). \tag{3}$$

In fact, for $m = 1$, i.e., $\alpha = p^r - r - 1$, we have

$$\sum_{i=1}^{\infty} \left[\frac{p^{r+1} - p^r - p}{p^i}\right] = p^r - r - 1,$$

$$\sum_{i=1}^{\infty} \left[\frac{p^{r+1} - p^r - 2p}{p^i}\right] = p^r - r - 2.$$

And then by the definition of $S(n)$, we can obtain

$$S(p^{p^r-r-1}) = p^{r+1} - p^r - p = p^{r+1} - p^r - S(p^1),$$

which means that (3) is true for $m=1$. Now suppose that (3) is true for any $m=k(\geqslant 1)$, i.e.,

$$S(p^{p^r-r-k}) = p^{r+1} - p^r - S(p^k).$$

Then for $m=k+1$, by Lemma 2.3, we have

$$S(p^{p^r-r-k-1}) = p^{r+1} - p^r - S(p^k) - p, \quad (A)$$

or

$$S(p^{p^r-r-k-1}) = p^{r+1} - p^r - S(p^k). \quad (B)$$

For the case (A), by Lemma 2.3, we have

$$p^r - r - k - 1 = \sum_{i=1}^{\infty}\left[\frac{p^{r+1}-p^r-S(p^k)-p}{p^i}\right] = \sum_{i=1}^{r}\left[\frac{p^{r+1}-p^r-S(p^k)-p}{p^i}\right] = \sum_{i=1}^{r}\left[\frac{p^{r+1}-p^r-S(p^k)}{p^i}\right] - r \leqslant \sum_{i=1}^{r}\left[\frac{p^{r+1}-p^r}{p^i}\right] - \sum_{i=1}^{r}\left[\frac{S(p^k)}{p^i}\right] - r = p^r - r - 1 - \sum_{i=1}^{r}\left[\frac{S(p^k)}{p^i}\right],$$

and then

$$k \geqslant \sum_{i=1}^{r}\left[\frac{S(p^k)}{p^i}\right].$$

Thus by the definition of $S(p^k)$, we have $k \leqslant \sum_{i=1}^{\infty}\left[\frac{S(p^k)}{p^i}\right]$, hence $k = \sum_{i=1}^{\infty}\left[\frac{S(p^k)}{p^i}\right]$. Then by Lemma 2.3, $S(p^{k+1}) = S(p^k) + p$, and so

$$S(p^{p^r-r-k-1}) = p^{r+1} - p^r - S(p^k) - p = p^{r+1} - p^r - S(p^{k+1}),$$

which means that (3) is true.

Now for the case (B), we have $k < \sum_{i=1}^{r}\left[\frac{S(p^k)}{p^i}\right]$. Otherwise, by $k = \sum_{i=1}^{r}\left[\frac{S(p^k)}{p^i}\right]$ we have the case (A), which is a contradiction. Hence $k < \sum_{i=1}^{r}\left[\frac{S(p^k)}{p^i}\right]$, thus by Lemma 2.3, we have $S(p^{k+1}) = S(p^k)$, and so

$$S(p^{p^r-r-k-1}) = p^{r+1} - p^r - S(p^k) = p^{r+1} - p^r - S(p^{k+1}),$$

which means that the identity (3) is satisfied.

From the above, the identity (3) is true.

Now we prove (3) of Theorem 1.1.

1) Suppose that for any positive integer k_1 and $m = p^{k_1}$ such that $\alpha = p^r - r - p^{k_1}$. From $r+m \in [r+1, p^r - p^{r-1}]$, we have $r + p^{k_1} \in [r+1, p^r - p^{r-1}]$, thus by the identity (3) and (1) of Theorem 1.1, we can obtain

$$S(p^\alpha) = S(p^{p^r-r-p^{k_1}}) = p^{r+1} - p^r - S(p^{p^{k_1}}) = p^{r+1} - p^r - (p^{k_1+1} - p^{k_1} + p) = (p-1)(p^r - p^{k_1}) - p.$$

2) Suppose that for any positive integer k_1, $s \in [1, k_1]$ and $m = p^{k_1} - s$, such that $\alpha = p^r - r - (p^{k_1} - s)$. From $r+m \in [r+1, p^r - p^{r-1}]$, i.e., $r + p^{k_1} - s \in [r+1, p^r - p^{r-1}]$, (3) and (2) of Theorem 1.1, we have

$$S(p^\alpha) = S(p^{p^r-r-(p^{k_1}-s)}) = p^{r+1} - p^r - S(p^{p^{k_1}-s}) = p^{r+1} - p^r - (p^{k_1+1} - p^{k_1}) = (p-1)(p^r - p^{k_1}).$$

3) Suppose that there is some positive integer k_1 and $e \in [k_1+1, p^{k_1} - p^{k_1-1}]$, such that $m = p^{k_1} - e$, namely, $\alpha = p^r - r - (p^{k_1} - e)$. From $r+m \in [r+1, p^r - p^{r-1}]$ we have $r + p^{k_1} - e \in [r+1, p^r - p^{r-1}]$. Now set

$$m_1 = p^{k_1} - k_1 - m,$$
$$1 \leqslant m_1 \leqslant p^{k_1-1}(p-1) - k_1,$$

then

$$r + p^{k_1} - k_1 - m_1 \in [r+1, p^r - p^{r-1}].$$

Similar to the previous discussions, we have the following three cases.

1′) If there is some positive integer k_2 such that $m_1 = p^{k_2}$, i.e.,

$$\alpha = p^r - r - (p^{k_1} - k_1) + p^{k_2},$$

and

$$r + p^{k_1} - k_1 - p^{k_2} \in [r+1, p^r - p^{r-1}].$$

Thus by (3) and (1) of Theorem 1.1, we have

$$S(p^\alpha) = p^{r+1} - p^r - S(p^{p^{k_1}-k_1-m_1}) = p^{r+1} - p^r - (p^{k_1+1} - p^{k_1} - S(p^{m_1})) = p^{r+1} - p^r - (p^{k_1+1} - p^{k_1} - S(p^{p^{k_2}})) = p^{r+1} - p^r - (p^{k_1+1} - p^{k_1} - (p^{k_2+1} - p^{k_2} + p)) = (p-1)(p^r - p^{k_1} + p^{k_2}) + p = (p-1)(p^r + (-1)^1 p^{k_1} + (-1)^2 p^{k_2}) + (-1)^2 p,$$

which satisfies (1) of Theorem 1.1.

2′) Suppose that there is some positive integer k_2 and $t_1 \in [1,k_2]$, such that $m_1 = p^{k_2} - t_1$, i.e.,

$$\alpha = p^r - r - (p^{k_1} - k_1) + (p^{k_2} - t_1),$$
$$t_1 \in [1,k_2],$$

and so

$$r + p^{k_1} - k_1 - (p^{k_2} - t_1) \in [r+1, p^r - p^{r-1}].$$

Thus by (3) and (2) of Theorem 1.1, we have

$$\begin{aligned}S(p^{\alpha}) &= p^{r+1} - p^r - S(p^{p^{k_1} - k_1 - m_1}) = \\ &p^{r+1} - p^r - (p^{k_1+1} - p^{k_1} - S(p^{m_1})) = \\ &p^{r+1} - p^r - (p^{k_1+1} - p^{k_1} - S(p^{p^{k_2} - t})) = \\ &p^{r+1} - p^r - (p^{k_1+1} - p^{k_1} - (p^{k_2+1} - p^{k_2})) = \\ &(p-1)(p^r - p^{k_1} + p^{k_2}) = \\ &(p-1)(p^r + (-1)^1 p^{k_1} + (-1)^2 p^{k_2}),\end{aligned}$$

which satisfies (2) of Theorem 1.1.

3′) Suppose that there is some positive integer k_2 and $t_1 \in [k_2+1, p^{k_2} - p^{k_2-1}]$, such that $m_1 = p^{k_2} - t_1$, i.e.,

$$\alpha = p^r - r - (p^{k_1} - k_1) + (p^{k_2} - t_1).$$

Now set

$$m_2 = p^{k_2} - k_2 - m_1,$$
$$1 \leqslant m_2 \leqslant p^{k_2-1}(p-1) - k_2,$$

then

$$\alpha = p^r - r - (p^{k_1} - k_1) + (p^{k_2} - k_2) - m_2,$$

and so

$$r + (p^{k_1} - k_1) - (p^{k_2} - k_2) + m_2 \in [r+1, p^r - p^{r-1}].$$

Similar to the previous discussions, we know that $\alpha \in [p^{r-1}, p^r]$ is a positive integer. Thus, one can repeat the above discussions 1)－3).

From the above discussions, Theorem 1.1 is proved.

The Proof for Corollary 1.2 For any positive integers $k_i (1 \leqslant i \leqslant n)$ with $1 \leqslant k_1 < k_2 < \cdots < k_n$,

$$\sum_{j=1}^{\infty}\left[\frac{\sum_{i=1}^{n} 2^{k_i}}{2^j}\right] = \sum_{j=1}^{k_n}\left[\frac{\sum_{i=1}^{n} 2^{k_i}}{2^j}\right] = \sum_{j=1}^{k_1}\left[\frac{\sum_{i=1}^{n} 2^{k_i}}{2^j}\right] + \sum_{j=k_1+1}^{k_2}\left[\frac{\sum_{i=1}^{n} 2^{k_i}}{2^j}\right] + \cdots + \sum_{j=k_{n-1}+1}^{k_n}\left[\frac{\sum_{i=1}^{n} 2^{k_i}}{2^j}\right]. \qquad (**)$$

Note that for any $k_m (1 \leqslant m \leqslant n-1)$, we have

$$\begin{aligned}&\sum_{j=k_m+1}^{k_{m+1}}\left[\frac{\sum_{i=1}^{n} 2^{k_i}}{2^j}\right] = \\ &\left[\frac{\sum_{i=1}^{n} 2^{k_i}}{2^{k_m+1}}\right] + \left[\frac{\sum_{i=1}^{n} 2^{k_i}}{2^{k_m+2}}\right] + \cdots + \left[\frac{\sum_{i=1}^{n} 2^{k_i}}{2^{k_{m+1}}}\right] = \\ &\sum_{j=m+1}^{n}\left[2^{k_j-(k_m+1)}\right] + \sum_{j=m+1}^{n}\left[2^{k_j-(k_m+2)}\right] + \cdots + \sum_{j=m+1}^{n}\left[2^{k_j-(k_{m+1})}\right] = \\ &\left(\left[2^{k_{m+1}-(k_m+1)}\right] + \left[2^{k_{m+2}-(k_m+1)}\right] + \cdots + \left[2^{k_n-(k_m+1)}\right]\right) + \\ &\left(\left[2^{k_{m+1}-(k_m+2)}\right] + \left[2^{k_{m+1}-(k_m+2)}\right] + \cdots + \left[2^{k_n-(k_m+2)}\right]\right) + \\ &\cdots + \left(\left[2^{(k_{m+1})-(k_{m+1})}\right] + \left[2^{k_{m+2}-(k_{m+1})}\right] + \cdots + \left[2^{k_n-(k_{m+1})}\right]\right) = \\ &\sum_{j=k_m+1}^{k_{m+1}}\left[2^{k_n-j}\right] + \left(\left[2^{(k_m+1)-(k_m+1)}\right] + \left[2^{(k_m+2)-(k_m+1)}\right] + \cdots + \left[2^{(k_{n-1})-(k_m+1)}\right]\right) + \\ &\left(\left[2^{(k_m+2)-(k_m+2)}\right] + \left[2^{(k_m+3)-(k_m+2)}\right] + \cdots + \left[2^{(k_{n-1})-(k_m+2)}\right]\right) + \\ &\cdots + \left(\left[2^{(k_{m+1})-(k_{m+1})}\right] + \left[2^{(k_{m+2})-(k_{m+1})}\right] + \cdots + \left[2^{(k_{n-1})-(k_{m+1})}\right]\right) = \\ &\sum_{j=k_m+1}^{k_{m+1}}\left[2^{k_{m+1}-j}\right] + \sum_{j=k_m+1}^{k_{m+1}}\left[2^{k_{m+2}-j}\right] + \cdots + \sum_{j=k_m+1}^{k_{m+1}}\left[2^{k_n-j}\right].\end{aligned}$$

Thus from (**) we can get

$$\begin{aligned}&\sum_{j=1}^{\infty}\left[\frac{\sum_{i=1}^{n} 2^{k_i}}{2^j}\right] = \\ &\left(\sum_{j=1}^{k_1}\left[2^{k_1-j}\right] + \sum_{j=1}^{k_1}\left[2^{k_2-j}\right] + \cdots + \sum_{j=1}^{k_1}\left[2^{k_n-j}\right]\right) + \\ &\left(\sum_{j=k_1+1}^{k_2}\left[2^{k_2-j}\right] + \sum_{j=k_1+1}^{k_2}\left[2^{k_3-j}\right] + \cdots + \sum_{j=k_1+1}^{k_2}\left[2^{k_n-j}\right]\right) + \\ &\cdots + \left(\sum_{j=k_{n-1}+1}^{k_n}\left[2^{k_n-j}\right]\right) = \\ &\sum_{j=1}^{k_1}\left[2^{k_1-j}\right] + \sum_{j=1}^{k_1}\left[2^{k_2-j}\right] + \cdots + \sum_{j=1}^{k_1}\left[2^{k_n-j}\right] + \\ &\sum_{j=k_1+1}^{k_2}\left[2^{k_2-j}\right] + \sum_{j=k_1+1}^{k_2}\left[2^{k_3-j}\right] + \cdots + \\ &\sum_{j=k_1+1}^{k_2}\left[2^{k_n-j}\right] + \cdots + \sum_{j=k_{n-1}+1}^{k_n}\left[2^{k_n-j}\right] =\end{aligned}$$

$$\sum_{j=1}^{k_1}[2^{k_1-j}]+\Big(\sum_{j=1}^{k_1}[2^{k_2-j}]+\sum_{j=k_1+1}^{k_2}[2^{k_2-j}]\Big)+\cdots+\Big(\sum_{j=1}^{k_1}[2^{k_n-j}]+\sum_{j=k_1+1}^{k_2}[2^{k_n-j}]+\cdots+\sum_{j=k_{n-1}+1}^{k_n}[2^{k_n-j}]\Big).$$

Hence

$$\sum_{j=1}^{\infty}\Big[\frac{\sum_{i=1}^{n}2^{k_i}}{2^j}\Big]=$$

$$\sum_{j=1}^{k_1}[2^{k_1-j}]+\sum_{j=1}^{k_2}[2^{k_2-j}]+\cdots+\sum_{j=1}^{k_n}[2^{k_n-j}]=$$

$$(2^{k_1}-1)+(2^{k_2}-1)+(\cdots+2^{k_n}-1)=$$

$$\sum_{i=1}^{n}2^{k_i}-n.$$

Thus by the definition of $S(n)$, we know that $S(2^\alpha)=\sum_{i=1}^{n}2^{k_i}$ for

$$\alpha=\sum_{i=1}^{n}2^{k_i}-n,\quad 1\leqslant k_1<k_2<\cdots<k_n.$$

Thus Corollary 1.2 is proved.

The Proof for Theorem 1.3 1) If there are some prime p and positive integer m coprime with p, such that $S(p^k)=\varphi(pm)$ and $S(p^k)\geqslant S(m^k)$. Then for $p=2$, we have

$$\varphi(2m)=\varphi(m)=S(2^k)\equiv 0(\bmod 2),$$
$$S(2^k)\geqslant S(m^k).$$

By $\varphi(2m)\equiv 0(\bmod 2)$ we have $m\geqslant 3$. While by $S(2^k)\geqslant S(m^k)$, we have $m=1$, this is a contradiction. And so $p\geqslant 3$, thus from the definition of $S(n)$ and the assumption that p is coprime with m, we have

$$(p-1)\varphi(m)=\varphi(pm)=$$
$$S(p^k)\equiv 0(\bmod p),$$

hence $\varphi(m)\equiv 0(\bmod p)$. In particular, if $m=\prod_{i=1}^{r}p_i^{\alpha_i}$ is the prime factors decomposition, then

$$\varphi(m)=\prod_{i=1}^{r}p_i^{\alpha_i-1}(p_i-1)\equiv 0(\bmod p).$$

Note that $\gcd(p,m)=1$, therefore there exists some $p_i(1\leqslant i\leqslant r)$ such that $p\mid p_i-1$, and so $p_i>p$, namely, $S(p_i^k)>S(p^k)$. On the other hand, $S(m^k)=\max\{S(p_j^k),1\leqslant j\leqslant r\}$, this is a contradiction to the assumption $S(p^k)\geqslant S(m^k)$. Thus we prove (1) of Theorem 1.3.

2) Suppose that there exist some positive integer α, prime p and positive integer m coprime with p, such that $\varphi(p^2m)=S(p^{2k})$ and $S(p^{2k})\geqslant S(m^k)$.

(Ⅰ) For the case $2k\leqslant p$, by (2) of Lemma 2.1, we have

$$2kp=S(p^{2k})=\varphi(p^2m)=p(p-1)\varphi(m),$$

i.e., $2k=(p-1)\varphi(m)$. Note that p is a prime, if $p=2$, then by $2k\leqslant p=2$ we have $k=1$, and so $\varphi(m)=2$, thus $m=3,4,6$. Hence from $\gcd(p,m)=1$ and $p=2$, we can get $m=3$. In this case,

$$S(p^2)=S(4)=4,\quad \varphi(p^2m)=\varphi(12)=4,$$

which means that $(p,m)=(2,3)$ is a solution.

Now for $p\geqslant 3$, by $2k\leqslant p$ we have

$$\varphi(m)=\frac{2k}{p-1}\leqslant\frac{p}{p-1}<2,$$

and so $\varphi(m)=1$, i.e., $m=1$ or 2 and $p=2k+1$, hence

$$(p,m)=(2k+1,1),(2k+1,2).$$

(Ⅱ) For the case $2k>p$, suppose that t_1 and t_2 are both nonnegative integers such that

$$S(p^{2k})=(2k-t_1)p,\tag{4}$$

and

$$t_2(p+1)\leqslant 2k<(t_2+1)(p+1).$$

Then by $S(p^{2k})=\varphi(p^2m)$ and Lemma 2.2, we have

$$t_1\geqslant t_2>\frac{2k}{p+1}-1,\tag{5}$$

and

$$(2k-t_1)p=\varphi(p^2m)=p(p-1)\varphi(m).$$

Now from (5), we know that

$$2k-(p-1)\varphi(m)=t_1>\frac{2k}{p+1}-1,$$

which means that

$$2kp>(p^2-1)\varphi(m)-(p+1).\tag{6}$$

Note that $2k>p$, i.e., $2kp>p^2$, thus we have three cases as following.

1) For the case

$$(p^2-1)\varphi(m)-(p+1)=p^2,$$

which means that

$$\varphi(m)=\frac{p^2+p+1}{p^2-1}=1+\frac{p+2}{p^2-1},$$

and so $p^2-1\mid p+2$, i.e., $p+2\geqslant p^2-1$. While

$$p+2\geqslant p^2-1\Leftrightarrow p^2-p-3\leqslant 0\Leftrightarrow$$
$$p(p-1)\leqslant 3\Leftrightarrow p=2.$$

Hence $p=2$, and then we have $\frac{p+2}{p^2-1}=\frac{4}{3}\notin \mathbf{Z}^+$, which is a contradiction.

2) For the case

$$(p^2-1)\varphi(m)-(p+1)<p^2,$$

i. e.,

$$\varphi(m)<\frac{p^2+p+1}{p^2-1}=1+\frac{p+2}{p^2-1}.$$

Thus if $p=2$, then $\varphi(m)\leqslant 2$, and so $m=1,2,3,4$ or 6. Note that $\gcd(p,m)=1$, hence $m=1$ or 3. From $m=1$, we can get $S(2^{2k})=\varphi(2^2 m)=2$, this is a contradiction since $S(2^{2k})\geqslant S(4)=4$. From $m=3$, we have $S(2^{2k})=\varphi(2^2\cdot 3)=4$, and so $k=1$, this is a contradiction to the assumption $2=2k>p=2$. Therefore $p\geqslant 3$, thus $0<\frac{p+2}{p^2-1}<1$, and so

$$\varphi(m)=1,\quad S(p^{2k})=\varphi(p^2)=p(p-1).$$

Now from $\sum_{i=1}^{\infty}\left[\frac{p(p-1)}{p^i}\right]=p-1$ and the definition of $S(p^{2k})$, we have $2k\leqslant p-1$, this is a contradiction to the assumption $2k>p$.

3) Therefore we must have $(p^2-1)\varphi(m)-(p+1)>p^2$, namely,

$$\varphi(m)>\frac{p^2+p+1}{p^2-1}=1+\frac{p+2}{p^2-1}.$$

By (6), we have

$$\frac{2kp+p+1}{p^2-1}>\varphi(m)>\frac{p^2+p+1}{p^2-1}=1+\frac{p+2}{p^2-1},$$

i. e.,

$$2\leqslant\varphi(m)\leqslant\frac{(2k+1)p+1}{p^2-1},\tag{7}$$

thus $2p^2-(2k+1)p-3\leqslant 0$, and so

$$p(2p-(2k+1))\leqslant 3.\tag{8}$$

Note that p is a prime, and so $2p-(2k+1)\leqslant 1$. If $2p-(2k+1)=1$, then by (8), we know that $(p,k)=(2,1),(3,2)$. From $(p,k)=(2,1)$, we have $2k=p=2$, this is a contradiction to $2k>p$. So $k=2,p=3$ or $2p<2k+1$. By $k=2,p=3$ and (7), we have $\varphi(m)=2$, and then $m=3,4,6$. Note that $\gcd(p,m)=1$ and then for $p=3$, we have $m=4$, thus

$$12=\varphi(3^2\cdot 4)\neq S(3^4\cdot 4^2)=S(3^4)=9,$$

which is a contradiction. Therefore $2p<2k+1$, i. e., $p<k+\frac{1}{2}$. From (8) and p is a prime, we have $2\leqslant p\leqslant k$. Now by (7),

$$2\leqslant\varphi(m)\leqslant\frac{2kp+p+1}{p^2-1}\leqslant\frac{2kp+p+1}{3}\leqslant\frac{2k^2+k+1}{3},$$

namely,

$$2\leqslant\varphi(m)\leqslant\left[\frac{2k^2+k+1}{3}\right].\tag{9}$$

Note that

$$\left[\frac{2k^2+k+1}{3}\right]=\begin{cases}\frac{2k^2+k-1}{3}, & k\equiv -1 \pmod 3,\\ \frac{2k^2+k}{3}, & \text{otherwise},\end{cases}$$

thus $\min\left\{\frac{2k^2+k-1}{3},\frac{2k^2+k}{3}\right\}\geqslant 2$, i. e., $k\geqslant 2$. Hence by $k=2$ and $2\leqslant p\leqslant k$, we have $p=k=2$. Now by (9) we have $\varphi(m)=2$, and so $m=3,4,6$. Since $p=2$ and $\gcd(p,m)=1$, and so $m=3$, thus

$$\varphi(p^2m)=\varphi(12)=4\neq 8=S(2^4)=S(p^{2k}),$$

which is a contradiction. Hence $k\geqslant 3$, thus we prove (2) of Theorem 1.3.

3) For $\alpha\geqslant 3$. If $\alpha k\leqslant p$, then by (2) of Lemma 2.1, we have

$$\alpha kp=S(p^{\alpha k})=\varphi(p^{\alpha}m)=p^{\alpha-1}(p-1)\varphi(m),$$

thus

$$p\geqslant\alpha k=p^{\alpha-2}(p-1)\varphi(m)\geqslant p(p-1),$$

hence $\alpha k=p=2$, which is a contradiction to the assumption $\alpha\geqslant 3$. And so $\alpha k>p$. Now suppose that t_1 and t_2 are both nonnegative integers such that

$$S(p^{\alpha k})=(\alpha k-t_1)p,\tag{10}$$

and

$$t_2(p+1)\leqslant\alpha k<(t_2+1)(p+1).\tag{11}$$

Then by Lemma 2.2, we can obtain $t_1\geqslant t_2>\frac{\alpha k}{p+1}-1$. Now by $p^{\alpha-1}(p-1)\varphi(m)=S(p^{\alpha k})$ and (10), we know that

$$(\alpha k-t_1)p=p^{\alpha-1}(p-1)\varphi(m),$$

namely,

$$\alpha k - p^{\alpha-2}(p-1)\varphi(m) = t_1 > \frac{\alpha k}{p+1} - 1,$$

thus

$$\alpha k > p^{\alpha-3}(p^2-1)\varphi(m) - 1 - \frac{1}{p},$$

and so

$$\alpha k + 1 \geqslant p^{\alpha-3}(p^2-1)\varphi(m),$$

i. e.,

$$\varphi(m) \leqslant \frac{\alpha k+1}{p^{\alpha-3}(p^2-1)}. \quad (12)$$

Note that for any positive integer m, we have $\varphi(m) \geqslant 1$, therefore we must have $\alpha k + 1 \geqslant p^{\alpha-3}(p^2-1)$.

If

$$\alpha k + 1 = p^{\alpha-3}(p^2-1) = p^{\alpha-1} - p^{\alpha-3},$$

i. e., $\varphi(m) = 1$. In this case, for $\alpha = 3$ we have $3k + 1 = p^2 - 1$, i. e., $p^2 = 3k + 2$, which is impossible. So $\alpha > 3$, and then

$$\alpha k = p^{\alpha-1} - p^{\alpha-3} - 1 =$$
$$[p^{\alpha-1} - (\alpha-1)] - [p^{\alpha-3} - (\alpha-3)] + 1.$$

We can conclude that

$$\alpha - 3 < p^{\alpha-4}(p-1) - 1. \quad (13)$$

Otherwise, from $\alpha - 3 \geqslant p^{\alpha-4}(p-1) - 1$, we have

$$\alpha \geqslant p^{\alpha-4}(p-1) + 2. \quad (14)$$

It is easy to see that for $\alpha \geqslant 4$ there is no any prime $p > 5$ satisfying (14). Hence $p = 2$ or 3. By $p = 3$ and (14) we have $\alpha \geqslant 2(3^{\alpha-4} + 1)$. While $2(3^{\alpha-4} + 1) > \alpha$ for $\alpha \geqslant 5$. Therefore from (14) we have $\alpha = 4$, and then $4k + 1 = 3^{\alpha-1} - 3^{\alpha-3} = 24$, which is a contradiction. Thus we must have $p = 2$.

Now from $p = 2$ and (14), we have $\alpha > 2^{\alpha-4} + 2$, and so $\alpha = 4, 5, 6$. Thus by $\alpha k + 1 = p^{\alpha-1} - p^{\alpha-3}$ and $\alpha = 4$, we have $\alpha k + 1 = 4k + 1 = 2^3 - 2 = 6$, which is a contradiction. For $\alpha = 5$, we have $5k + 1 = 12$, which is also a contradiction. For $\alpha = 6$, $6k + 1 = 24$, it is also a contradiction. Hence (14) is not true, and so $\alpha - 3 < p^{\alpha-4}(p-1) - 1$. Thus by (I) of (3) for Theorem 1.1, we have

$$S(p^{\alpha k}) = p^{\alpha} - p^{\alpha-1} - p^{\alpha-2} + p^{\alpha-3} + p.$$

Now by $\varphi(m) = 1$, $\gcd(p, m) = 1$ and $\varphi(p^{\alpha} m) = S(p^{\alpha k})$, we have

$$p^{\alpha-1}(p-1) = p^{\alpha} - p^{\alpha-1} - p^{\alpha-2} + p^{\alpha-3} + p,$$

this means $p^{\alpha-3} + p = p^{\alpha-2}$. Note that p is a prime, thus we have $p = 2$ and $\alpha = 4$. And so $4k + 1 = 2^3 - 2 = 6$, which is a contradiction.

From the above we must have $\alpha k + 1 > p^{\alpha-3}(p^2-1) \geqslant p + 1$. Without loss of the generality, set

$$\alpha k + 1 = q p^{\alpha-3}(p^2-1) + r,$$
$$0 \leqslant r < p^{\alpha-3}(p^2-1).$$

Now by (12), we have

$$\varphi(m) \leqslant \frac{\alpha k+1}{p^{\alpha-3}(p^2-1)} =$$
$$q + \frac{r}{p^{\alpha-3}(p^2-1)}, \quad (15)$$

and so $1 \leqslant \varphi(m) \leqslant q$. Thus we prove (3) of Theorem 1.3.

4) For $\alpha = 3$, $q = 1$ and $r = 1$. By (15) we have $3k + 1 = p^2$, and so $p = \sqrt{3k+1}$. Since $q = 1$ and $\varphi(m) = 1$, hence $m = 1, 2$. Thus $\varphi(p^3 m) = p^2(p-1)\varphi(m) = p^2(p-1)$. Now by $S(p^{3k} m^k) = S(p^{3k}) = S(p^{p^2-1})$ and (2) of Theorem 1.1, we can get

$$S(p^{p^2-1}) = p^3 - p^2 = p^2(p-1) = \varphi(p^3 m).$$

Thus we prove (4) of Theorem 1.3.

From the above Theorem 1.3 is proved.

The Proof for Theorem 1.4 1) If there is some positive integer α such that $\frac{\sigma(p^{\alpha})}{S(p^{\alpha})} = k$ is a positive integer, i. e., $\sigma(p^{\alpha}) = kS(p^{\alpha})$. Note that $\sigma(p^{\alpha}) = \sum_{i=0}^{\alpha} p^i \equiv 1 \pmod p$ and $S(p^{\alpha}) \equiv 0 \pmod p$, then $1 \equiv 0 \pmod p$, which is a contradiction. Thus we prove (1) of Theorem 1.4.

2) Note that p is an odd prime and $n = 2^{\alpha} p = \max\{S(2^{\alpha}), S(p)\}$. For the case $\sum_{i=1}^{\infty} \left[\frac{p}{2^i}\right] \geqslant \alpha$, we have $S(2^{\alpha}) \leqslant S(p)$, and then $S(n) = S(p) = p$. Now from

$$\sigma(n) = \sigma(2^{\alpha} p) = (2^{\alpha+1} - 1)(1 + p),$$

we have

$$\frac{\sigma(n)}{S(n)} = \frac{(2^{\alpha+1}-1)(1+p)}{p} =$$
$$(2^{\alpha+1} - 1) + \frac{2^{\alpha+1}-1}{p} \in \mathbf{Z}^+,$$

i. e.,

$$2^{\alpha+1} \equiv 1 \pmod p.$$

For the case $\sum_{i=1}^{\infty}\left[\frac{p}{2^i}\right] < \alpha$, we have $S(n) = S(2^\alpha)$. Now set $\frac{\sigma(n)}{S(n)} = k \in \mathbf{Z}^+$, i.e.,

$$\sigma(n) = kS(n). \quad (16)$$

Thus from

$$\sigma(2^\alpha p) = (2^{\alpha+1}-1)(1+p) \equiv 0(\bmod 2^{\alpha+1}-1),$$

we have

$$kS(2^\alpha) \equiv 0(\bmod 2^{\alpha+1}-1). \quad (17)$$

Set $\gcd(S(2^\alpha), 2^{\alpha+1}-1) = d$, $S(2^\alpha) = sd$ and $2^{\alpha+1}-1 = td$, where $\gcd(s,t) = 1$. Then $d \equiv 1(\bmod 2)$, and by (16)-(17) we have $t \mid k$ and $\frac{t(1+p)}{s} = k = mt$, which means that $p = ms-1$. Now from

$$sd = S(2^\alpha) = S(n) = \max\{S(2^\alpha), S(p)\} > p = ms-1,$$

we can obtain $1 \leqslant m \leqslant d$.

Thus we prove Theorem 1.4.

The Proof for Corollary 1.5 1) If $p = 2^r+1$ is a prime and $\alpha = 2^r$, $n = 2^{2^r}(2^r+1)$. Then

$$\sum_{i=1}^{\infty}\left[\frac{p}{2^i}\right] = \sum_{i=1}^{\infty}\left[\frac{2^r+1}{2^i}\right] = 2^{r-1} < 2^r = \alpha.$$

Thus by (2) of Theorem 1.4 and (1) of Theorem 1.1, we have $p = m \cdot \frac{S(2^\alpha)}{d} - 1 = m \cdot \frac{2^r+2}{d} - 1$, by $p = 2^r+1$, we have $m = d$, and by (II) of Theorem 1.4(2), we have $\frac{\sigma(n)}{S(n)} = 2^{2^r+1}-1$.

On the other hand, by the definition of $\sigma(n)$ and (1) of Theorem 1.1, we also have

$$\frac{\sigma(n)}{S(n)} = \frac{(2^r+2)(2^{2^r+1}-1)}{2^r+2} = 2^{2^r+1}-1.$$

Thus from $n = 2^{2^r}(2^r+1)$ and 2^r+1 is a prime, we know that $\frac{\sigma(n)}{S(n)} = 2^{2^r}-1$ is a positive integer.

2) Since $n = 2^{p-1}(2^p-1)$ is a perfect number, so $\sigma(n) = 2^p(2^p-1)$. Thus from (1) of Lemma 2.1 and 2^p-1 is a prime number, we have

$$S(n) = \max\{S(2^{p-1}), S(2^p-1)\} = \max\{S(2^{p-1}), 2^p-1\}.$$

Note that

$$S(2^{p-1}) \leqslant 1+p-1+\log_2(p-1) = p+\log_2(p-1) < 2^p-1,$$

and so

$$S(n) = \max\{S(2^{p-1}), 2^p-1\} = 2^p-1,$$

thus $\frac{\sigma(n)}{S(n)} = 2^p$.

By the similar way, we can prove part (3).

Thus we prove Corollary 1.5.

3 Some Examples

In this section, some examples for both Theorem 1.1 and Corollary 1.2 are given.

Example 3.1 Let $p = 3$, $\alpha = 3^5 = 243$, then by (1) of Theorem 1.1 we have

$$S(3^{243}) = 3^6 - 3^5 + 3 = 489.$$

On the other hand, from

$$\sum_{i=1}^{\infty}\left[\frac{489}{3^i}\right] = \sum_{i=1}^{6}\left[\frac{489}{3^i}\right] = 163+54+18+6+2+0 = 243,$$

and the definition of $S(n)$, we also have $S(3^{243}) = 489$.

Example 3.2 Let $p = 3$, $\alpha = 3^6 - 4 = 725$. Namely, be taking $r = 6$, $t = 4$ in (2) of Theorem 1.1, we know that

$$S(3^{725}) = 3^7 - 3^6 = 2 \times 3^6 = 1\,458.$$

On the other hand, from

$$\sum_{i=1}^{\infty}\left[\frac{1\,458}{3^i}\right] = \sum_{i=1}^{7}\left[\frac{1\,458}{3^i}\right] = 486+162+54+18+6+2+0 = 728,$$

$$\sum_{i=1}^{\infty}\left[\frac{1\,457}{3^i}\right] = \sum_{i=1}^{7}\left[\frac{1\,457}{3^i}\right] = 485+161+53+17+5+1+0 = 722,$$

and the definition of $S(n)$, we also have $S(3^{725}) = 1\,458$.

Example 3.3 Let $p = 3$, $\alpha = 5\,017$, i.e.,

$$\alpha = (3^8-8)-(3^7-7)+(3^6-6)-(3^4-3),$$

thus from (2) of Theorem 1.1, we have

$$S(3^{5\,017}) = 2 \times (3^8 - 3^7 + 3^6 - 3^4) = 2 \times (2 \times 3^7 + 8 \times 3^4) = 4 \times 2\,187 + 16 \times 81 = 10\,044.$$

On the other hand, from

$$\sum_{i=1}^{\infty}\left[\frac{10\,044}{3^i}\right] = \sum_{i=1}^{8}\left[\frac{10\,044}{3^i}\right] =$$

$$3\,348 + 1\,116 + 372 + 124 + 41 + 13 + 4 + 1 = 5\,019,$$

and

$$\sum_{i=1}^{\infty}\left[\frac{10\,043}{3^i}\right] = \sum_{i=1}^{8}\left[\frac{10\,043}{3^i}\right] = 3\,347 + 1\,115 + 371 + 123 + 41 + 13 + 4 + 1 = 5\,015,$$

we also have $S(3^{5\,017}) = 10\,044$.

4 Conclusion

In the present paper, by using elementary methods and techniques, we obtain the explicit formula for $S(p^\alpha)$, where p is a prime and α is a positive integer. As a corollary, some properties for positive integer solutions of the equations $\varphi(n) = S(n^k)$ or $\frac{\sigma(2^\alpha q)}{S(2^\alpha q)}$ are given, where q is an odd prime, $\varphi(n)$ and $\sigma(n)$ are the Euler function and the sum of the different positive factors of n, respectively. In [1], Kempner studied the formula of $S(n)$ in induction way, namely, by fixing the value of $S(n)$ to solve the corresponding positive integer n. While, the present paper obtains the formula of $S(n)$ in a direct way. Our method is much more universal.

References

[1] KEMPNER A J. Miscellanea [J]. American Mathematical Monthly, 1918, 25(5): 201-210.
[2] XU Z F. The value distribution of Smarandache function [J]. Acta Mathematica Sinica, 2006, 49(5): 1009-1012.
[3] FARRIS M, MITSHELL P. Bounding the Smarandache function [J]. Smarandache Notions J, 2002, 13: 37-42.
[4] SMARANDACHE F. Only Problems, Not Solution [M]. Chicago: Xiquan Publishing House, 1993.
[5] GORSKI D. The pseudo-Smarandache function [J]. Smarandache Notions J, 2002, 13(1/2/3): 140-149.
[6] LE M H. A lower bound for $S(2^{p-1}(2^p-1))$ [J]. Smarandache Notions J, 2001, 12(1): 217-218.
[7] LIU Y M. On the solutions of an equation invloving the Smarandache function [J]. Scientia Magna, 2006, 2(1): 76-79.
[8] 温田丁. Smarandache 函数的一个下界估计 [J]. 纯粹数学与应用数学, 2010, 26(3): 413-416.
[9] YI Y. An equation in volving the Euler function and Smarandache function [J]. Scientia Magna, 2005, 1(2): 172-175.

Smarandache 函数的准确计算公式以及相关数论方程的求解

廖群英， 罗文力

(四川师范大学 数学与软件科学学院，四川 成都 610066)

摘要: 设 $\varphi(n)$ 和 $S(n)$ 分别为正整数 n 的欧拉函数和 Smarandache 函数. 熟知, $S(n)$ 的准确计算公式是一个尚未解决的公开问题. 利用初等的方法与技巧, 给出了 $S(p^\alpha)$ 的准确计算公式, 其中 p 为质数, α 为正整数, 从而完全解决了上述公开问题. 由此得到方程 $\varphi(n) = S(n^k)$ 的正整数解 (n,k) 的性质, 以及 $\frac{\sigma(2^\alpha q)}{S(2^\alpha q)}$ 为正整数的几个必要条件, 其中 q 为奇质数, $\sigma(n)$ 表示 n 的全部不同正因数的和.

关键词: Smarandache 函数; 欧拉函数; 高斯函数; 完全数

中图分类号: O156.4 **文献标志码:** A **文章编号:** 1001-8395(2017)01-0001-10

(编辑 周 俊)

收稿日期: 2016-01-03

基金项目: 国家自然科学基金(11401408)和四川省科技厅研究项目(2016JY0134)

第一作者简介: 廖群英(1974—), 女, 教授, 主要从事编码和密码学理论的研究, E-mail: qunyingliao@sicnu.edu.cn