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A Limit Problem Involving the F.Smarandache Square Complementary

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Abstract: The main purpose of this paper is using the elementary method to study a limit problem involving the F.Smarandache square complementary number, and obtain its limit value.
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§1. Introduction and Results

For any positive integer n, we call $Ssc(n)$ as the square complementary of n if $Ssc(n)$ denotes the least positive integer such that $Ssc(n)n$ is a perfect square. For example, $Ssc(1) = 1$, $Ssc(2) = 2$, $Ssc(3) = 3$, $Ssc(4) = 1$, $Ssc(5) = 5$, $Ssc(6) = 6$, $Ssc(7) = 7$, $Ssc(8) = 2$, $Ssc(9) = 1$, $Ssc(10) = 10$, $\cdots$. In reference [1], Professor F.Smarandache asked us to study the properties of $Ssc(n)$. About this problem, many people had studied it, and obtained many interesting results, see [2] and [3]. For example, Felice Russo[3] studied the arithmetical properties of $Ssc(n)$, and proved many conclusions. At the same time, he also presented 21 unsolved problems. Maohua Le[4] solved the problems 5, 6 and 7. Maohua Le[6-7] solved problems 13 and 17 respectively. But the problem 16 still have not been solved at present. That is, whether there exists a limit for

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\frac{\ln(Ssc(k))}{\ln k}.$$

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In this paper, we use the elementary method to study this problem, and prove the following sharper conclusion:

Theorem Let $Ssc(n)$ denotes the square complementary of n, for any positive integer $n > 2$, we have the asymptotic formula

$$\sum_{k=2}^{n} \frac{\ln(Ssc(k))}{\ln k} = n + O\left(\frac{n \ln\ln n}{\ln n}\right).$$

From this Theorem we may immediately get the following:

Corollary Let $Ssc(n)$ denotes the square complementary of n, then we have the limit formula

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{n} \frac{\ln(Ssc(k))}{\ln k} = 1.$$

It is clear that this corollary solved problem 16 of [3].

§2. Proof of the Theorem

In this section, we will complete the proof of the theorem. For any positive integer n, it is clear that there exist unique positive integers l and m such that $n = m^2 l$, where l is a square-free number. So we have $Ssc(n) = Ssc(l) = l$ and

$$\begin{aligned}
&\sum_{k=2}^{n} \frac{\ln(Ssc(k))}{\ln k} \\
=& \sum_{m^2 l \le n} \frac{|\mu(l)| \ln l}{\ln(m^2 l)} \\
=& \sum_{l\le n} |\mu(l)| \ln l \sum_{m\le\sqrt{\frac{n}{l}}} \left(\frac{1}{\ln l} - \frac{2\ln m}{\ln l(2\ln m + \ln l)}\right) \\
=& \sum_{l\le n} |\mu(l)| \sum_{m\le\sqrt{\frac{n}{l}}} 1 - \sum_{l\le n} |\mu(l)| \sum_{m\le\sqrt{\frac{n}{l}}} \frac{2\ln m}{2\ln m + \ln l} \\
=& \sum_{m\le\sqrt{n}} \sum_{l\le\frac{n}{m^2}} |\mu(l)| - \sum_{l\le n} |\mu(l)| \sum_{m\le\sqrt{\frac{n}{l}}} \frac{2\ln m}{2\ln m + \ln l}.
\end{aligned} \tag{1.1}$$

Noting that

$$\sum_{n\le x} \frac{1}{n^2} = \zeta(2) + O\left(\frac{1}{x}\right);$$

$$\sum_{n\le x} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} + O\left(\frac{1}{x}\right);$$

$$\sum_{n\le x} \frac{1}{n^{\frac{1}{2}}} = 2\sqrt{x} + \zeta\left(\frac{1}{2}\right) + O\left(\frac{1}{\sqrt{x}}\right).$$

The first term on the right hand side of (1) is

$$
\begin{aligned}
&\sum_{m\le\sqrt{n}}\sum_{l\le\frac{n}{m^2}}|\mu(l)|\\
=&\sum_{m^2l\le n}|\mu(l)|\\
=&\sum_{m^2l\le n}\sum_{d^2|l}\mu(d)\\
=&\sum_{m^2d^2t\le n}\mu(d)\\
=&\sum_{d\le\sqrt{n}}\mu(d)\sum_{m^2t\le\frac{n}{d^2}}1\\
=&\sum_{d\le\sqrt{n}}\mu(d)\sum_{m\le\frac{\sqrt{n}}{d}}\sum_{t\le\frac{n}{m^2d^2}}1\\
=&\sum_{d\le\sqrt{n}}\mu(d)\sum_{m\le\frac{\sqrt{n}}{d}}\Big(\frac{n}{m^2d^2}+O(1)\Big)\\
=&n\sum_{d\le\sqrt{n}}\frac{\mu(d)}{d^2}\sum_{m\le\frac{\sqrt{n}}{d}}\frac{1}{m^2}+O\left(\sqrt{n}\sum_{d\le\sqrt{n}}\frac{|\mu(d)|}{d}\right)\\
=&n\sum_{d\le\sqrt{n}}\frac{\mu(d)}{d^2}\left(\zeta(2)+O\left(\frac{d}{\sqrt{n}}\right)\right)+O\left(\sqrt{n}\sum_{d\le\sqrt{n}}\frac{|\mu(d)|}{d}\right)\\
=&n\zeta(2)\cdot\frac{1}{\zeta(2)}+O\left(\sqrt{n}\sum_{d\le\sqrt{n}}\frac{|\mu(d)|}{d}\right)\\
=&n+O(\sqrt{n}\ln n). \qquad (1.2)
\end{aligned}
$$

The second term on the right hand side of (1) is

$$
\begin{aligned}
&\sum_{l\le n}|\mu(l)|\sum_{m\le\sqrt{\frac{n}{l}}}\frac{2\ln m}{2\ln m+\ln l}\\
=&\sum_{l\le\frac{n}{(\ln n)^2}}|\mu(l)|\sum_{m\le\sqrt{\frac{n}{l}}}\frac{2\ln m}{2\ln m+\ln l}+\sum_{\frac{n}{(\ln n)^2}<l\le n}|\mu(l)|\sum_{m\le\sqrt{\frac{n}{l}}}\frac{2\ln m}{2\ln m+\ln l}\\
=&O\left(\sum_{l\le\frac{n}{(\ln n)^2}}|\mu(l)|\sqrt{\frac{n}{l}}\right)+O\left(\sum_{\frac{n}{(\ln n)^2}<l\le n}|\mu(l)|\frac{\ln\ln n}{\ln n}\sqrt{\frac{n}{l}}\right)\\
=&O\left(\sqrt{n}\sum_{l\le\frac{n}{(\ln n)^2}}\frac{1}{\sqrt{l}}\right)+O\left(\frac{\sqrt{n}\ln\ln n}{\ln n}\sum_{\frac{n}{(\ln n)^2}<l\le n}\frac{1}{\sqrt{l}}\right)\\
=&O\left(\frac{n\ln\ln n}{\ln n}\right). \qquad (1.3)
\end{aligned}
$$

Combining (1), (2) and (3), we have

$$\sum_{k=2}^{n} \frac{\ln(Ssc(k))}{\ln k} = n + O\left(\frac{n \ln\ln n}{\ln n}\right).$$

This completes the proof of theorem.

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