Chin. Quart. J. of Math.
2008, 23 (1): 115—119

On the Smarandache LCM Sequence

XUE Xi-feng

(*Department of Mathematics, Northwest University, Xi'an* 710069, *China*)

Abstract: For any positive integer n, let $L(n) = [1, 2, \cdots, n]$ be the least common multiple of the integers from 1 to n. Let k be any positive integer. The main purpose of this paper is to study the asymptotic property of $L(n^k)$, and give some interesting relevant results.
Key words: least common multiple; Smarandache LCM sequence; asymptotic formula; Limit
2000 MR Subject Classification: 11D
CLC number: O156.4 **Document code:** A
Article ID: 1002–0462 (2008) 01–0115–05

§1. Introduction and Results

For any positive integer n, let $L(n)$ denote the least common multiple of the natural numbers from 1 through n. That is

$$L(n) = [1, 2, \cdots, n].$$

The Smarandache LCM sequence is defined by(see [1]):

$$L(1),\ L(2),\ L(3),\ \cdots,\ L(n),\ \cdots$$

The first few numbers of Smarandache LCM sequence are: 1, 2, 6, 12, 60, 60, 420, 840, 2520, 2520, $\cdots$.

About some simple arithmetical properties of the least common multiply, there are many results in elementary number theory text books. For example, for any positive integers a, b and c, we have

$$[a, b] = \frac{ab}{(a, b)} \quad \text{and} \quad [a, b, c] = \frac{abc \cdot (a, b, c)}{(a, b)(b, c)(c, a)},$$

where $(a_1, a_2, \cdots, a_k)$ denotes the greatest common divisor of a_1, a_2, $\cdots$, a_{k-1} and a_k.

Received date: 2007-06-04
Foundation item: Supported by the NSF of China(10671155)
Biography: XUE Xi-feng(1961-), male, native of Weinan, Shannxi, an associate professor of Northwest University, engages in analytic number theory.

In [2], Xu Zhefeng defined a multiplicative function $g(n)$ by using the least common multiple of n and a fixed positive integer m as $g(n) = [m,n]/m$, then he proved

$$\sum_{n\leq x}\sigma_\alpha(g(n)) = x^{\alpha+1}\cdot\frac{\zeta(\alpha+1)}{\alpha+1}\prod_{p^\beta\|m}\left(1-\frac{1}{p}+\frac{1}{p^{(\beta+1)(\alpha+1)-\alpha}}\right)+O\left(x^{\alpha+\frac{1}{2}+\varepsilon}\right),$$

$$\sum_{n\leq x} d(g(n))$$

$$= x\cdot\prod_{p^\beta\|m}\left(1-\frac{1}{p}+\frac{1}{p^{\beta+1}}\right)\times\left(\log x+2C-1+\sum_{p^\beta\|m}\frac{\left(p^\beta-(\beta+1)p^{2(\beta+1)}\right)\log p}{p^{\beta+1}-p^\beta+1}\right)+O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

and

$$\sum_{n\leq x}\phi(g(n)) = \frac{x^2}{2\zeta(2)}\prod_{p^\beta\|m}\frac{1}{p^{2\beta}}\left(1+\frac{p^{2\beta+3}-p^3}{p^3+p^2-p-1}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, $d(n)$ is the Dirichlet divisor function, $\phi(n)$ the Euler function and C the Euler constant.

But about the deeply arithmetical properties of $L(n)$, it seems that none has been studied before. For any positive integer k, the main purpose of this paper is to study the asymptotic property of $L(n^k)$, and give an interesting limit theorem for it. That is, we shall prove the following:

Theorem For any positive integers n and k, we have the asymptotic formula

$$\left(\frac{L(n^k)}{\prod\limits_{p\leq n^k} p}\right)^{n^{-\frac{k}{2}}} = e+O\left(\exp\left(\frac{-c\left(\frac{k}{2}\ln n\right)^{\frac{3}{5}}}{(\ln k-\ln 2+\ln\ln n)^{\frac{1}{5}}}\right)\right),$$

where $\prod\limits_{p\leq n^k}$ denotes the production over all primes $p\leq n^k$.

From this theorem we may immediately deduce the following:

Corollary For any positive integer k, we have

$$\lim_{n\to\infty}\left(\frac{L(n^k)}{\prod\limits_{p\leq n^k} p}\right)^{n^{-\frac{k}{2}}} = e.$$

§2. Proof of the Theorem

In this section, we shall complete the proof of our theorem. First we need the following simple lemma.

Lemma For any $x > 0$, we have the asymptotic formula

$$\theta(x) = \sum_{p \le x} \ln p = x + O\left(x\exp\left(\frac{-c(\ln x)^{\frac{3}{5}}}{(\ln\ln x)^{\frac{1}{5}}} \right) \right),$$

where $c > 0$ is a constant, $\sum\limits_{p \le x}$ denotes the summation over all primes $p \le x$.

Proof See [3].

Now we complete the proof of the theorem.

Let

$$L(n^k) = [1,\ 2,\ \cdots,\ n^2] = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \tag{2.1}$$

be the factorization of $L(n^k)$ into prime powers. It is clear that $\alpha_i := \alpha(p_i)$ is the highest power of p_i in the factorization of $1, 2, 3, \cdots, n^k$.

First we write

$$\left(\frac{L(n^k)}{\prod\limits_{p \le n^k} p} \right)^{n^{-\frac{k}{2}}} = \exp\left(n^{-\frac{k}{2}} \ln \frac{L(n^k)}{\prod\limits_{p \le n^k} p} \right) = \exp\left(n^{-\frac{k}{2}} \left(\ln L(n^k) - \ln \prod_{p \le n^k} p \right) \right). \tag{2.2}$$

Now we calculate the inner term in (2). From (1) we can write

$$\begin{aligned}
\ln L(n^k) - \ln \prod_{p \le n^k} p &= \ln (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}) - \ln \prod_{p \le n^k} p \\
&= \sum_{p \le n^k} \alpha(p) \ln p - \sum_{p \le n^k} \ln p \\
&= \sum_{p \le n^k} (\alpha(p) - 1) \ln p \\
&= \sum_{p \le n^{\frac{k}{k+1}}} (\alpha(p) - 1) \ln p + \sum_{i=1}^{k} \sum_{n^{\frac{k}{i+1}} < p \le n^{\frac{k}{i}}} (\alpha(p) - 1) \ln p \\
&:= M_0 + \sum_{i=1}^{k} M_i.
\end{aligned} \tag{2.3}$$

Noting that $\alpha(p) = i$ if

$$n^{\frac{k}{i+1}} < p \le n^{\frac{k}{i}},$$

we have from the Lemma

$$
\begin{aligned}
M_i &= (i-1) \sum_{n^{\frac{k}{i+1}} < p \leq n^{\frac{k}{i}}} \ln p \\
&= (i-1) \left[\sum_{p \leq n^{\frac{k}{i}}} \ln p - \sum_{p \leq n^{\frac{k}{i+1}}} \ln p \right] \\
&= (i-1) \left[n^{\frac{k}{i}} - n^{\frac{k}{i+1}} + O\left(n^{\frac{k}{i}} \exp\left(\frac{-c\left(\frac{k}{i}\ln n\right)^{\frac{3}{5}}}{(\ln k - \ln i + \ln\ln n)^{\frac{1}{5}}} \right) \right) \right].
\end{aligned}
$$

Hence,

$$
\sum_{i=1}^{k} M_i = n^{\frac{k}{2}} + O\left(n^{\frac{k}{2}} \exp\left(\frac{-c\left(\frac{k}{2}\ln n\right)^{\frac{3}{5}}}{(\ln k - \ln 2 + \ln\ln n)^{\frac{1}{5}}} \right) \right). \tag{2.4}
$$

On the other hand, we know that

$$
M_0 = O\left(\frac{k}{k+1} \ln^2 n \sum_{p \leq n^{\frac{k}{k+1}}} 1 \right) = O\left(\frac{k}{k+1} \ln^2 n \frac{n^{\frac{k}{k+1}}}{\frac{k}{k+1}\ln n} \right) = O\left(n^{\frac{k}{k+1}} \ln n \right). \tag{2.5}
$$

Now combining (3), (4) and (5) we may get

$$
\ln L(n^k) - \ln \prod_{p \leq n^k} p = n^{\frac{k}{2}} + O\left(n^{\frac{k}{2}} \exp\left(\frac{-c\left(\frac{k}{2}\ln n\right)^{\frac{3}{5}}}{(\ln k - \ln 2 + \ln\ln n)^{\frac{1}{5}}} \right) \right). \tag{2.6}
$$

So from (2), (6) and noting that

$$
e^x = 1 + O(x) \ (x \to 0),
$$

we immediately obtain

$$
\begin{aligned}
&\left(\frac{L(n^k)}{\prod_{p \leq n^k} p} \right)^{n^{-\frac{k}{2}}} \\
&= \exp\left(n^{-\frac{k}{2}} \left(\ln L(n^k) - \ln \prod_{p \leq n^k} p \right) \right) \\
&= \exp\left[n^{-\frac{k}{2}} \left[n^{\frac{k}{2}} + O\left(n^{\frac{k}{2}} \exp\left(\frac{-c\left(\frac{k}{2}\ln n\right)^{\frac{3}{5}}}{(\ln k - \ln 2 + \ln\ln n)^{\frac{1}{5}}} \right) \right) \right] \right] \\
&= \exp\left[1 + O\left(\exp\left(\frac{-c\left(\frac{k}{2}\ln n\right)^{\frac{3}{5}}}{(\ln k - \ln 2 + \ln\ln n)^{\frac{1}{5}}} \right) \right) \right] \\
&= e \cdot \exp\left[O\left(\exp\left(\frac{-c\left(\frac{k}{2}\ln n\right)^{\frac{3}{5}}}{(\ln k - \ln 2 + \ln\ln n)^{\frac{1}{5}}} \right) \right) \right]
\end{aligned}
$$

$$= e\left[1 + O\left(\exp\left(\frac{-c\left(\frac{k}{2}\ln n\right)^{\frac{3}{5}}}{(\ln k - \ln 2 + \ln\ln n)^{\frac{1}{5}}}\right)\right)\right]$$

$$= e + O\left(\exp\left(\frac{-c\left(\frac{k}{2}\ln n\right)^{\frac{3}{5}}}{(\ln k - \ln 2 + \ln\ln n)^{\frac{1}{5}}}\right)\right).$$

This completes the proof of our Theorem.

[References]

[1] AMARNTH Murthy. Generalized Partitions and New Ideas On Number Theory and Smarandache Sequences[C]. Hexis, 2005, 20-22.

[2] XU Zhe-feng. Some new arithmetical functions and their mean value formulas[J]. Mathematics in Practice and Theory(in Chinese), 2006, 36(3): 300-303.

[3] PAN Cheng-dong, PAN Cheng-biao. The Foundation of Analytic Number Theory[M]. Beijing: Science Publication, 1999.

[4] TOM M. Apostol, Introduction to Analytic Number Theory[M]. New York: Springer-Verlag, 1976.