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Combination of beliefs on hybrid DSm models

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Abstract: *This chapter presents a general method for combining uncertain and paradoxical (i.e. highly conflicting) sources of evidence for a wide class of fusion problems. From the foundations of the DSmT we show how the DSm rule of combination can be extended to take into account all possible integrity constraints (if any) of the problem under consideration due to the true nature of elements/concepts involved into it. We show how Shafer's model can be considered as a specific hybrid DSm model and can be easily handled by the DSmT and one presents here a new efficient alternative to Dempster's rule of combination, following steps of previous researchers towards this quest. Several simple didactic examples are also provided to show the efficiency and the generality of the approach proposed in this work.*

4.1 Introduction

According to each model occurring in real-world fusion problems, we present a general hybrid DSm rule which combines two or more masses of independent sources of information and takes care of constraints, i.e. of sets which might become empty at time t_l or new sets/elements that might arise in the frame at time t_{l+1}. The hybrid DSm rule is applied in a real time when the hyper-power set D^Θ changes (i.e. the set of all propositions built from elements of frame Θ with $\cup$ and $\cap$ operators - see [3] for details), either increasing or decreasing its focal elements, or when even Θ decreases or increases influencing the D^Θ as well, thus the dynamicity of our DSmT.

This chapter introduces the reader to the independence of sources of evidences, which needs to be studied deeper in the future, then one defines the models and the hybrid DSm rule, which is different from other rules of combination such as Dempster's, Yager's, Smets', Dubois-Prade's and gives seven numerical examples of applying the hybrid DSm rule in various models and several examples of dynamicity of DSmT, then the Bayesian hybrid DSm models mixture.

4.2 On the independence of the sources of evidences

The notion of independence of the sources of evidence plays a major role in the development of efficient information fusion algorithms but is very difficult to formally establish when manipulating uncertain and paradoxical (i.e. highly conflicting) sources of information. Some attempts to define the independence of uncertain sources of evidences have been proposed by P. Smets and al. in Dempster-Shafer Theory (DST) and Transferable Belief Model in [12, 13, 14] and by other authors in possibility theory [1, 2, 5, 8, 10]. In the following, we consider that n sources of evidences are independent if the internal mechanism by which each source provides its own basic belief assignment doesn't depend on the mechanisms of other sources (i.e. there is no internal relationship between all mechanisms) or if the sources don't share (even partially) same knowledge/experience to establish their own basic belief assignment. This definition doesn't exclude the possibility for independent sources to provide the same (numerical) basic belief assignments. The fusion of dependent uncertain and paradoxical sources is much more complicated because, one has first to identify precisely the piece of redundant information between sources in order to remove it before applying the fusion rules. The problem of combination of dependent sources is under investigation.

4.3 DSm rule of combination for free-DSm models

4.3.1 Definition of the free-DSm model $\mathcal{M}^f(\Theta)$

Let's consider a finite frame $\Theta = \{\theta_1, \ldots \theta_n\}$ of the fusion problem under consideration. We abandon Shafer's model by assuming here that the fuzzy/vague/relative nature of elements θ_i $i = 1, \ldots, n$ of Θ can be non-exclusive. We assume also that no refinement of Θ into a new finer *exclusive frame of discernment* Θ^{ref} is possible. This is the free-DSm model $\mathcal{M}^f(\Theta)$ which can be viewed as the opposite (if we don't introduce non-existential constraints - see next section) of Shafer's model, denoted $\mathcal{M}^0(\Theta)$ where all θ_i are forced to be exclusive and therefore fully discernable.

4.3.2 Example of a free-DSm model

Let's consider the frame of the problem $\Theta = \{\theta_1, \theta_2, \theta_3\}$. The free Dedekind lattice $D^\Theta = \{\alpha_0, \dots, \alpha_{18}\}$ over Θ owns the following 19 elements (see chapter 2)

Elements of D^Θ for $\mathcal{M}^f(\Theta)$

$\alpha_0 \triangleq \emptyset$	
$\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \neq \emptyset$	$\alpha_{10} \triangleq \theta_2 \neq \emptyset$
$\alpha_2 \triangleq \theta_1 \cap \theta_2 \neq \emptyset$	$\alpha_{11} \triangleq \theta_3 \neq \emptyset$
$\alpha_3 \triangleq \theta_1 \cap \theta_3 \neq \emptyset$	$\alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \neq \emptyset$
$\alpha_4 \triangleq \theta_2 \cap \theta_3 \neq \emptyset$	$\alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \neq \emptyset$
$\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 \neq \emptyset$	$\alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \neq \emptyset$
$\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \neq \emptyset$	$\alpha_{15} \triangleq \theta_1 \cup \theta_2 \neq \emptyset$
$\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \neq \emptyset$	$\alpha_{16} \triangleq \theta_1 \cup \theta_3 \neq \emptyset$
$\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \neq \emptyset$	$\alpha_{17} \triangleq \theta_2 \cup \theta_3 \neq \emptyset$
$\alpha_9 \triangleq \theta_1 \neq \emptyset$	$\alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3 \neq \emptyset$

The free-DSm model $\mathcal{M}^f(\Theta)$ assumes that all elements α_i, $i > 0$, are non-empty. This corresponds to the following Venn diagram where in Smarandache's codification "i" denotes the part of the diagram which belongs to θ_i only, "ij" denotes the part of the diagram which belongs to θ_i and θ_j only, "ijk" denotes the part of the diagram which belongs to θ_i and θ_j and θ_k only, etc [3]. On such Venn diagram representation of the model, we emphasize the fact that all boundaries of intersections must be seen/interpreted as only vague boundaries just because the nature of elements θ_i can be, in general, only vague, relative and even imprecise (see chapter 6).

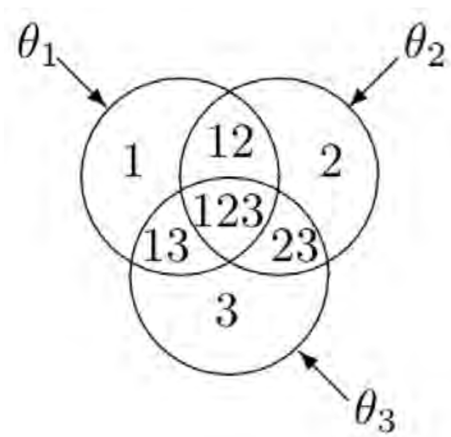

Figure 4.1: Venn Diagram for $\mathcal{M}^f(\Theta)$

For the chapter to be self-contained, we recall here the *classical* DSm rule of combination based on $\mathcal{M}^f(\Theta)$ over the free Dedekind's lattice built from elements of Θ with $\cap$ and $\cup$ operators, i.e. D^Θ.

4.3.3 Classical DSm rule for 2 sources for free-DSm models

For two *independent* uncertain and paradoxical (i.e. highly conflicting) sources of information (experts/-bodies of evidence) providing generalized basic belief assignment $m_1(.)$ and $m_2(.)$ over D^Θ (or over any subset of D^Θ), the classical DSm conjunctive rule of combination $m_{\mathcal{M}_f(\Theta)}(.) \triangleq [m_1 \oplus m_2](.)$ is given by

$$\forall A \neq \emptyset \in D^\Theta, \quad m_{\mathcal{M}^f(\Theta)}(A) \triangleq [m_1 \oplus m_2](A) = \sum_{\substack{X_1, X_2 \in D^\Theta \\ (X_1 \cap X_2) = A}} m_1(X_1) m_2(X_2) \tag{4.1}$$

$m_{\mathcal{M}^f(\Theta)}(\emptyset) = 0$ by definition, unless otherwise specified in special cases when some source assigns a non-zero value to it (like in the Smets TBM approach [9]). This DSm rule of combination is commutative and associative. This rule, dealing with both uncertain and paradoxical/conflicting information, requires no normalization process and can always been applied.

4.3.4 Classical DSm rule for $k \geq 2$ sources for free-DSm models

The above formula can be easily generalized for the free-DSm model $\mathcal{M}^f(\Theta)$ with $k \geq 2$ independent sources in the following way:

$$\forall A \neq \emptyset \in D^\Theta, \quad m_{\mathcal{M}^f(\Theta)}(A) \triangleq [m_1 \oplus \ldots m_k](A) = \sum_{\substack{X_1, \ldots, X_k \in D^\Theta \\ (X_1 \cap \ldots \cap X_k) = A}} \prod_{i=1}^{k} m_i(X_i) \tag{4.2}$$

$m_{\mathcal{M}^f(\Theta)}(\emptyset) = 0$ by definition, unless otherwise specified in special cases when some source assigns a non-zero value to it. This DSm rule of combination is still commutative and associative.

4.4 Presentation of hybrid DSm models

4.4.1 Definition

Let Θ be the general frame of the fusion problem under consideration with n elements θ_1, θ_2, $\ldots$, θ_n. A hybrid DSm model $\mathcal{M}(\Theta)$ is defined from the free-DSm model $\mathcal{M}^f(\Theta)$ by introducing some *integrity constraints* on some elements A of D^Θ if one knows with certainty the exact nature of the model corresponding to the problem under consideration. An integrity constraint on A consists in forcing A to be empty (vacuous element), and we will denote such constraint as $A \overset{\mathcal{M}}{\equiv} \emptyset$ which means that A has been forced to $\emptyset$ through the model $\mathcal{M}(\Theta)$. This can be justified by the knowledge of the true nature of each element θ_i of Θ. Indeed, in some fusion problems, some elements θ_i and θ_j of Θ can be fully discernable because they are truly exclusive while other elements cannot be refined into finer exclusive elements. Moreover, it is also possible that for some reason with some new knowledge on the problem, an element or several elements θ_i have to be forced to the empty set (especially if dynamical fusion problems are considered, i.e when Θ varies with space and time). For example, if we consider a list of three potential suspects into a police investigation, it can occur that, during the investigation, one of the suspects can be withdrawn of the initial frame of the problem if his innocence is proven with an ascertainable alibi. The initial basic belief masses provided by sources of information one had on the three suspects, must then be modified by taking into account this new knowledge on the model of the problem.

There exists several possible kinds of integrity constraints which can be introduced in any free-DSm model $\mathcal{M}^f(\Theta)$ actually. The first kind of integrity constraint concerns *exclusivity constraints* by taking into account that some conjunctions of elements $\theta_i, \ldots, \theta_k$ are truly impossible (i.e. $\theta_i \cap \ldots \cap \theta_k \overset{\mathcal{M}}{\equiv} \emptyset$). The second kind of integrity constraint concerns the *non-existential constraints* by taking into account that some disjunctions of elements $\theta_i, \ldots, \theta_k$ are also truly impossible (i.e. $\theta_i \cup \ldots \cup \theta_k \overset{\mathcal{M}}{\equiv} \emptyset$). We exclude from our presentation the completely degenerate case corresponding to the constraint $\theta_1 \cup \ldots \cup \theta_n \overset{\mathcal{M}}{\equiv} \emptyset$ (total ignorance) because there is no way and no interest to treat such a vacuous problem. In such a degenerate case, we can just set $m(\emptyset) \triangleq 1$ which is useless because the problem remains vacuous and D^Θ reduces to $\emptyset$. The last kind of possible integrity constraint is a mixture of the two previous ones, like for example $(\theta_i \cap \theta_j) \cup \theta_k$ or any other hybrid proposition/element of D^Θ involving both $\cap$ and $\cup$ operators such that at least one element θ_k is a subset of the constrained proposition. From any $\mathcal{M}^f(\Theta)$, we can thus build several hybrid DSm models depending on the number of integrity constraints one needs to fully characterize the nature of the problem. The introduction of a given integrity constraint $A \overset{\mathcal{M}}{\equiv} \emptyset \in D^\Theta$ implies necessarily the set of inner constraints $B \overset{\mathcal{M}}{\equiv} \emptyset$ for all $B \subset A$. Moreover the introduction of two integrity constraints, say on A and B in D^Θ implies also necessarily the constraint on the emptiness of the disjunction $A \cup B$ which belongs also to D^Θ (because D^Θ is closed under $\cap$ and $\cup$ operators). This implies the emptiness of all $C \in D^\Theta$ such that $C \subset (A \cup B)$. The same remark has to be extended for the case of the introduction of n integrity constraints as well. Shafer's model is the unique and most constrained hybrid DSm model including all possible exclusivity constraints *without non-existential constraint* since all $\theta_i \neq \emptyset \in \Theta$ are forced to be mutually exclusive. Shafer's model is denoted $\mathcal{M}^0(\Theta)$ in the sequel. We denote by $\boldsymbol{\emptyset}_{\mathcal{M}}$ the set of elements of D^Θ which have been forced to be empty in the hybrid DSm model $\mathcal{M}$.

4.4.2 Example 1 : hybrid DSm model with an exclusivity constraint

Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$ be the general frame of the problem under consideration and let's consider the following hybrid DSm model $\mathcal{M}_1(\Theta)$ built by introducing the following exclusivity constraint $\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_1}{\equiv} \emptyset$. This exclusivity constraint implies however no other constraint because α_1 doesn't contain other elements of D^Θ but itself. Therefore, one has now the following set of elements for D^Θ

Elements of D^Θ for $\mathcal{M}_1(\Theta)$

$\alpha_0 \triangleq \emptyset$	
$\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_1}{\equiv} \emptyset$	$\alpha_{10} \triangleq \theta_2 \neq \emptyset$
$\alpha_2 \triangleq \theta_1 \cap \theta_2 \neq \emptyset$	$\alpha_{11} \triangleq \theta_3 \neq \emptyset$
$\alpha_3 \triangleq \theta_1 \cap \theta_3 \neq \emptyset$	$\alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \neq \emptyset$
$\alpha_4 \triangleq \theta_2 \cap \theta_3 \neq \emptyset$	$\alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \neq \emptyset$
$\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 \neq \emptyset$	$\alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \neq \emptyset$
$\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \neq \emptyset$	$\alpha_{15} \triangleq \theta_1 \cup \theta_2 \neq \emptyset$
$\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \neq \emptyset$	$\alpha_{16} \triangleq \theta_1 \cup \theta_3 \neq \emptyset$
$\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \neq \emptyset$	$\alpha_{17} \triangleq \theta_2 \cup \theta_3 \neq \emptyset$
$\alpha_9 \triangleq \theta_1 \neq \emptyset$	$\alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3 \neq \emptyset$

Hence the initial basic belief mass over D^Θ has to be transferred over the new constrained hyper-power set $D^\Theta(\mathcal{M}_1(\Theta))$ with the 18 elements defined just above (including actually 17 non-empty elements). The mechanism for the transfer of basic belief masses from D^Θ onto $D^\Theta(\mathcal{M}_1(\Theta))$ will be obtained by the hybrid DSm rule of combination presented in the sequel.

4.4.3 Example 2 : hybrid DSm model with another exclusivity constraint

As the second example for a hybrid DSm model $\mathcal{M}_2(\Theta)$, let's consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and the following exclusivity constraint $\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$. This constraint implies also $\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$ since $\alpha_1 \subset \alpha_2$. Therefore, one has now the following set of elements for $D^\Theta(\mathcal{M}_2(\Theta))$

Elements of D^Θ for $\mathcal{M}_2(\Theta)$

$\alpha_0 \triangleq \emptyset$	
$\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$	$\alpha_{10} \triangleq \theta_2 \neq \emptyset$
$\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$	$\alpha_{11} \triangleq \theta_3 \neq \emptyset$
$\alpha_3 \triangleq \theta_1 \cap \theta_3 \neq \emptyset$	$\alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_2}{\equiv} \alpha_{11} \neq \emptyset$
$\alpha_4 \triangleq \theta_2 \cap \theta_3 \neq \emptyset$	$\alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \neq \emptyset$
$\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 \neq \emptyset$	$\alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \neq \emptyset$
$\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \alpha_4 \neq \emptyset$	$\alpha_{15} \triangleq \theta_1 \cup \theta_2 \neq \emptyset$
$\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_2}{\equiv} \alpha_3 \neq \emptyset$	$\alpha_{16} \triangleq \theta_1 \cup \theta_3 \neq \emptyset$
$\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_2}{\equiv} \alpha_5 \neq \emptyset$	$\alpha_{17} \triangleq \theta_2 \cup \theta_3 \neq \emptyset$
$\alpha_9 \triangleq \theta_1 \neq \emptyset$	$\alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3 \neq \emptyset$

Note that in this case several non-empty elements of $D^\Theta(\mathcal{M}_2(\Theta))$ coincide because of the constraint ($\alpha_6 \overset{\mathcal{M}_2}{\equiv} \alpha_4$, $\alpha_7 \overset{\mathcal{M}_2}{\equiv} \alpha_3$, $\alpha_8 \overset{\mathcal{M}_2}{\equiv} \alpha_5$, $\alpha_{12} \overset{\mathcal{M}_2}{\equiv} \alpha_{11}$). $D^\Theta(\mathcal{M}_2(\Theta))$ has now only 13 different elements. Note that the introduction of both constraints $\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$ and $\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$ doesn't change the construction of $D^\Theta(\mathcal{M}_2(\Theta))$ because $\alpha_1 \subset \alpha_2$.

4.4.4 Example 3 : hybrid DSm model with another exclusivity constraint

As the third example for a hybrid DSm model $\mathcal{M}_3(\Theta)$, let's consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and the following exclusivity constraint $\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_3}{\equiv} \emptyset$. This constraint implies now $\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$ since $\alpha_1 \subset \alpha_6$, but also $\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_3}{\equiv} \emptyset$ because $\alpha_2 \subset \alpha_6$ and $\alpha_4 \triangleq \theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$ because $\alpha_4 \subset \alpha_6$. Therefore, one has now the following set of elements for $D^\Theta(\mathcal{M}_3(\Theta))$

Elements of D^Θ for $\mathcal{M}_3(\Theta)$

$\alpha_0 \triangleq \emptyset$	
$\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$	$\alpha_{10} \triangleq \theta_2 \neq \emptyset$
$\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_3}{\equiv} \emptyset$	$\alpha_{11} \triangleq \theta_3 \neq \emptyset$
$\alpha_3 \triangleq \theta_1 \cap \theta_3 \neq \emptyset$	$\alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_3}{\equiv} \alpha_{11} \neq \emptyset$
$\alpha_4 \triangleq \theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$	$\alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \neq \emptyset$
$\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \alpha_3 \neq \emptyset$	$\alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \overset{\mathcal{M}_3}{\equiv} \alpha_9 \neq \emptyset$
$\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_3}{\equiv} \emptyset$	$\alpha_{15} \triangleq \theta_1 \cup \theta_2 \neq \emptyset$
$\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_3}{\equiv} \alpha_3 \neq \emptyset$	$\alpha_{16} \triangleq \theta_1 \cup \theta_3 \neq \emptyset$
$\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_3}{\equiv} \alpha_5 \neq \emptyset$	$\alpha_{17} \triangleq \theta_2 \cup \theta_3 \neq \emptyset$
$\alpha_9 \triangleq \theta_1 \neq \emptyset$	$\alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3 \neq \emptyset$

$D^\Theta(\mathcal{M}_3(\Theta))$ has now only 10 different elements.

4.4.5 Example 4 : Shafer's model

As the fourth particular example for a hybrid DSm model $\mathcal{M}_4(\Theta)$, let's consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and the following exclusivity constraint $\alpha_8 \triangleq \{(\theta_1 \cap \theta_2) \cup \theta_3\} \cap (\theta_1 \cup \theta_2) \overset{\mathcal{M}_4}{\equiv} \emptyset$. Therefore, one has now the following set of elements for $D^\Theta(\mathcal{M}_4(\Theta))$

Elements of D^Θ for $\mathcal{M}_4(\Theta)$ (Shafer's model)

$\alpha_0 \triangleq \emptyset$	
$\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \emptyset$	$\alpha_{10} \triangleq \theta_2 \neq \emptyset$
$\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_4}{\equiv} \emptyset$	$\alpha_{11} \triangleq \theta_3 \neq \emptyset$
$\alpha_3 \triangleq \theta_1 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \emptyset$	$\alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_4}{\equiv} \alpha_{11} \neq \emptyset$
$\alpha_4 \triangleq \theta_2 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \emptyset$	$\alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \overset{\mathcal{M}_4}{\equiv} \alpha_{10} \neq \emptyset$
$\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \emptyset$	$\alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \overset{\mathcal{M}_4}{\equiv} \alpha_9 \neq \emptyset$
$\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_4}{\equiv} \emptyset$	$\alpha_{15} \triangleq \theta_1 \cup \theta_2 \neq \emptyset$
$\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_4}{\equiv} \emptyset$	$\alpha_{16} \triangleq \theta_1 \cup \theta_3 \neq \emptyset$
$\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_4}{\equiv} \emptyset$	$\alpha_{17} \triangleq \theta_2 \cup \theta_3 \neq \emptyset$
$\alpha_9 \triangleq \theta_1 \neq \emptyset$	$\alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3 \neq \emptyset$

This model corresponds actually to Shafer's model $\mathcal{M}^0(\Theta)$ because this constraint includes all possible exclusivity constraints between elements θ_i, $i = 1, 2, 3$ since $\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \subset \alpha_8$, $\alpha_2 \triangleq \theta_1 \cap \theta_2 \subset \alpha_8$, $\alpha_3 \triangleq \theta_1 \cap \theta_3 \subset \alpha_8$ and $\alpha_4 \triangleq \theta_2 \cap \theta_3 \subset \alpha_8$. $D^\Theta(\mathcal{M}_4(\Theta))$ has now $2^{|\Theta|} = 8$ different elements and coincides obviously with the classical power set 2^Θ. This corresponds to Shafer's model and serves as the foundation for Dempster-Shafer Theory.

4.4.6 Example 5 : hybrid DSm model with a non-existential constraint

As the fifth example for a hybrid DSm model $\mathcal{M}_5(\Theta)$, let's consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and the following *non-existential* constraint $\alpha_9 \triangleq \theta_1 \overset{\mathcal{M}_5}{\equiv} \emptyset$. In other words, we remove θ_1 from the initial frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$. This non-existential constraint implies $\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_5}{\equiv} \emptyset$, $\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_5}{\equiv} \emptyset$, $\alpha_3 \triangleq \theta_1 \cap \theta_3 \overset{\mathcal{M}_5}{\equiv} \emptyset$ and $\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_5}{\equiv} \emptyset$. Therefore, one has now the following set of elements for $D^\Theta(\mathcal{M}_5(\Theta))$

Elements of D^Θ for $\mathcal{M}_5(\Theta)$

$$
\begin{array}{ll}
\alpha_0 \triangleq \emptyset & \\
\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_5}{\equiv} \emptyset & \alpha_{10} \triangleq \theta_2 \neq \emptyset \\
\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_5}{\equiv} \emptyset & \alpha_{11} \triangleq \theta_3 \neq \emptyset \\
\alpha_3 \triangleq \theta_1 \cap \theta_3 \overset{\mathcal{M}_5}{\equiv} \emptyset & \alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_5}{\equiv} \alpha_{11} \neq \emptyset \\
\alpha_4 \triangleq \theta_2 \cap \theta_3 \neq \emptyset & \alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \overset{\mathcal{M}_5}{\equiv} \alpha_{10} \neq \emptyset \\
\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_5}{\equiv} \alpha_4 \neq \emptyset & \alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \overset{\mathcal{M}_5}{\equiv} \alpha_4 \neq \emptyset \\
\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_5}{\equiv} \alpha_4 \neq \emptyset & \alpha_{15} \triangleq \theta_1 \cup \theta_2 \overset{\mathcal{M}_5}{\equiv} \alpha_{10} \neq \emptyset \\
\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_5}{\equiv} \emptyset & \alpha_{16} \triangleq \theta_1 \cup \theta_3 \overset{\mathcal{M}_5}{\equiv} \alpha_{11} \neq \emptyset \\
\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_5}{\equiv} \alpha_4 \neq \emptyset & \alpha_{17} \triangleq \theta_2 \cup \theta_3 \neq \emptyset \\
\alpha_9 \triangleq \theta_1 \overset{\mathcal{M}_5}{\equiv} \emptyset & \alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3 \overset{\mathcal{M}_5}{\equiv} \alpha_{17} \neq \emptyset
\end{array}
$$

$D^\Theta(\mathcal{M}_5(\Theta))$ has now 5 different elements and coincides obviously with the hyper-power set $D^{\Theta \setminus \theta_1}$.

4.4.7 Example 6 : hybrid DSm model with two non-existential constraints

As the sixth example for a hybrid DSm model $\mathcal{M}_6(\Theta)$, let's consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and the following two *non-existential* constraints $\alpha_9 \triangleq \theta_1 \overset{\mathcal{M}_6}{\equiv} \emptyset$ and $\alpha_{10} \triangleq \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$. Actually, these two constraints are equivalent to choose only the following constraint $\alpha_{15} \triangleq \theta_1 \cup \theta_2 \overset{\mathcal{M}_5}{\equiv} \emptyset$. In other words, we remove now both θ_1 and θ_2 from the initial frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$. These non-existential constraints implies now $\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset$, $\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$, $\alpha_3 \triangleq \theta_1 \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset$, $\alpha_4 \triangleq \theta_2 \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset$, $\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset$, $\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$, $\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_6}{\equiv} \emptyset$, $\alpha_8 \triangleq \{(\theta_1 \cap \theta_2) \cup \theta_3\} \cap (\theta_1 \cup \theta_2) \overset{\mathcal{M}_6}{\equiv} \emptyset$, $\alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$, $\alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \overset{\mathcal{M}_6}{\equiv} \emptyset$. Therefore, one has now the following set of elements for $D^\Theta(\mathcal{M}_6(\Theta))$:

Elements of D^Θ for $\mathcal{M}_6(\Theta)$

$$\begin{array}{ll}
\alpha_0 \triangleq \emptyset & \\
\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset & \alpha_{10} \triangleq \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset \\
\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset & \alpha_{11} \triangleq \theta_3 \neq \emptyset \\
\alpha_3 \triangleq \theta_1 \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset & \alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \alpha_{11} \neq \emptyset \\
\alpha_4 \triangleq \theta_2 \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset & \alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset \\
\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset & \alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \overset{\mathcal{M}_6}{\equiv} \emptyset \\
\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset & \alpha_{15} \triangleq \theta_1 \cup \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset \\
\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_6}{\equiv} \emptyset & \alpha_{16} \triangleq \theta_1 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \alpha_{11} \neq \emptyset \\
\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_6}{\equiv} \emptyset & \alpha_{17} \triangleq \theta_2 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \alpha_{11} \neq \emptyset \\
\alpha_9 \triangleq \theta_1 \overset{\mathcal{M}_6}{\equiv} \emptyset & \alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \alpha_{11} \neq \emptyset
\end{array}$$

$D^\Theta(\mathcal{M}_6(\Theta))$ reduces now to only two different elements $\emptyset$ and θ_3. $D^\Theta(\mathcal{M}_6(\Theta))$ coincides obviously with the hyper-power set $D^{\Theta\setminus\{\theta_1,\theta_2\}}$. Because there exists only one possible non empty element in $D^\Theta(\mathcal{M}_6(\Theta))$, such kind of a problem is called a *trivial problem*. If one now introduces all non-existential constraints in the free-DSm model, then the initial problem reduces to a *vacuous problem* also called the *impossible problem* corresponding to $m(\emptyset) \equiv 1$ (such kind of a "problem" is not related to reality). Such kinds of trivial or vacuous *problems* are not considered anymore in the sequel since they present no real interest for engineering information fusion problems.

4.4.8 Example 7 : hybrid DSm model with a mixed constraint

As the seventh example for a hybrid DSm model $\mathcal{M}_7(\Theta)$, let's consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and the following *mixed exclusivity and non-existential* constraint $\alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$. This mixed constraint implies $\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\alpha_3 \triangleq \theta_1 \cap \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\alpha_4 \triangleq \theta_2 \cap \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\alpha_8 \triangleq \{(\theta_1 \cap \theta_2) \cup \theta_3\} \cap (\theta_1 \cup \theta_2) \overset{\mathcal{M}_7}{\equiv} \emptyset$ and $\alpha_{11} \triangleq \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$. Therefore, one has now the following set of elements for $D^\Theta(\mathcal{M}_7(\Theta))$

Elements of D^Θ for $\mathcal{M}_7(\Theta)$

$$
\begin{array}{ll}
\alpha_0 \triangleq \emptyset & \\
\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset & \alpha_{10} \triangleq \theta_2 \neq \emptyset \\
\alpha_2 \triangleq \theta_1 \cap \theta_2 \overset{\mathcal{M}_7}{\equiv} \emptyset & \alpha_{11} \triangleq \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset \\
\alpha_3 \triangleq \theta_1 \cap \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset & \alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset \\
\alpha_4 \triangleq \theta_2 \cap \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset & \alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \overset{\mathcal{M}_7}{\equiv} \alpha_{10} \neq \emptyset \\
\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset & \alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \overset{\mathcal{M}_7}{\equiv} \alpha_9 \neq \emptyset \\
\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_7}{\equiv} \emptyset & \alpha_{15} \triangleq \theta_1 \cup \theta_2 \neq \emptyset \\
\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_7}{\equiv} \emptyset & \alpha_{16} \triangleq \theta_1 \cup \theta_3 \overset{\mathcal{M}_7}{\equiv} \alpha_9 \neq \emptyset \\
\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_7}{\equiv} \emptyset & \alpha_{17} \triangleq \theta_2 \cup \theta_3 \overset{\mathcal{M}_7}{\equiv} \alpha_{10} \neq \emptyset \\
\alpha_9 \triangleq \theta_1 \neq \emptyset & \alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3 \overset{\mathcal{M}_7}{\equiv} \alpha_{15} \neq \emptyset
\end{array}
$$

$D^\Theta(\mathcal{M}_7(\Theta))$ reduces now to only four different elements $\emptyset$, θ_1, θ_2, and $\theta_1 \cup \theta_2$.

4.5 DSm rule of combination for hybrid DSm models

In this section, we present a general DSm-hybrid rule of combination able to deal with any hybrid DSm models (including Shafer's model). We will show how this new general rule of combination works with all hybrid DSm models presented in the previous section and we list interesting properties of this new useful and powerful rule of combination.

4.5.1 Notations

Let $\Theta = \{\theta_1, \dots \theta_n\}$ be a frame of *partial discernment* (i.e. a frame Θ for which at least one conjunctive element of $D^\Theta \setminus \{\emptyset\}$ is known to be truly empty) of the constrained fusion problem, and D^Θ the free distributive lattice (hyper-power set) generated by Θ and the empty set $\emptyset$ under $\cap$ and $\cup$ operators. We need to distinguish between the empty set $\emptyset$, which belongs to D^Θ, and by $\emptyset$ we understand a set which is empty all the time (we call it *absolute emptiness* or *absolutely empty*) independent of time, space and model, and all other sets from D^Θ. For example $\theta_1 \cap \theta_2$ or $\theta_1 \cup \theta_2$ or only θ_i itself, $1 \leq i \leq n$, etc, which could be or become empty at a certain time (if we consider a fusion dynamicity) or in a particular model $\mathcal{M}$ (but could not be empty in other model and/or time) (we call a such element *relative emptiness* or *relatively empty*). We'll denote by $\boldsymbol{\emptyset}_{\mathcal{M}}$ the set of relatively empty such elements of D^Θ (i.e. which become empty in a particular model $\mathcal{M}$ or at a specific time). $\boldsymbol{\emptyset}_{\mathcal{M}}$ is the set of integrity constraints which depends on the DSm model $\mathcal{M}$ under consideration, and the model $\mathcal{M}$ depends on the structure of its corresponding *fuzzy Venn Diagram* (number of elements in Θ, number of non-empty intersections, and time in case of dynamic fusion). Through our convention $\emptyset \notin \boldsymbol{\emptyset}_{\mathcal{M}}$. Let's note by $\boldsymbol{\emptyset} \triangleq \{\emptyset, \boldsymbol{\emptyset}_{\mathcal{M}}\}$ the set of all relatively and absolutely empty elements.

For any $A \in D^\Theta$, let $\phi(A)$ be the *characteristic non emptiness function* of the set A, i.e. $\phi(A) = 1$ if $A \notin \boldsymbol{\emptyset}$ and $\phi(A) = 0$ otherwise. This function assigns the value zero to all relatively or absolutely empty elements of D^Θ through the choice of hybrid DSm model $\mathcal{M}$. Let's define the total ignorance on $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ as $I_t \triangleq \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ and the set of relative ignorances as $I_r \triangleq \{\theta_{i_1} \cup \ldots \cup \theta_{i_k},$ where $i_1, ..., i_k \in \{1, 2, ..., n\}$ and $2 \leq k \leq n-1\}$, then the set of all kind of ignorances as $I = I_t \cup I_r$. For any element A in D^Θ, one considers $u(A)$ as the union of all singletons θ_i that compose A. For example, if A is a singleton then $u(A) = A$; if $A = \theta_1 \cap \theta_2$ or $A = \theta_1 \cup \theta_2$ then $u(A) = \theta_1 \cup \theta_2$; if $A = (\theta_1 \cap \theta_2) \cup \theta_3$ then $u(A) = \theta_1 \cup \theta_2 \cup \theta_3$. ; by convention $u(\emptyset) \triangleq \emptyset$. The second summation of the hybrid DSm rule (see eq. (4.3) and (4.5) and denoted S_2 in the sequel) transfers the mass of $\emptyset$ [if any; sometimes, in rare cases, $m(\emptyset) > 0$ (for example in Smets' work); we want to catch this particular case as well] to the total ignorance $I_t = \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$. The other part of the mass of relatively empty elements, θ_i and θ_j together for example, $i \neq j$, goes to the partial ignorance/uncertainty $m(\theta_i \cup \theta_j)$. S_2 multiplies, naturally following the DSm classic network architecture, only the elements of columns of absolutely and relatively empty sets, and then S_2 transfers the mass $m_1(X_1)m_2(X_2) \ldots m_k(X_k)$ either to the element $A \in D^\theta$ in the case when $A = u(X_1) \cup u(X_2) \cup \ldots \cup u(X_k)$ is not empty, or if $u(X_1) \cup u(X_2) \cup \ldots \cup u(X_k)$ is empty then the mass $m_1(X_1)m_2(X_2) \ldots m_k(X_k)$ is transferred to the total ignorance. We include all degenerate problems/models in this new DSmT hybrid framework, but the degenerate/vacuous DSm-hybrid model $\mathcal{M}_\emptyset$ defined by the constraint $I_t = \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n \overset{\mathcal{M}_\emptyset}{\equiv} \emptyset$ which is meaningless and useless.

4.5.2 Programming of the $u(X)$ function

We provide here the issue for programming the calculation of $u(X)$ from the binary representation of any proposition $X \in D^\Theta$ expressed in the Dezert-Smarandache order (see chapters 2 and 3). Let's consider the Smarandache codification of elements $\theta_1, \ldots, \theta_n$. One defines the anti-absorbing relationship as follows: element i anti-absorbs element ij (with $i < j$), and let's use the notation $i << ij$, and also $j << ij$; similarly $ij << ijk$ (with $i < j < k$), also $jk << ijk$ and $ik << ijk$. This relationship is transitive, therefore $i << ij$ and $ij << ijk$ involve $i << ijk$; one can also write $i << ij << ijk$ as a chain; similarly one gets $j << ijk$ and $k << ijk$. The anti-absorbing relationship can be generalized for parts with any number of digits, i.e. when one uses the Smarandache codification for the corresponding Venn diagram on $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, with $n \geq 1$. Between elements ij and ik, or between ij and jk there is no anti-absorbing relationship, therefore the anti-absorbing relationship makes a partial order on the parts of the Venn diagram for the free DSm model. If a proposition X is formed by a part only, say $i_1 i_2 \ldots i_r$, in the Smarandache codification, then $u(X) = \theta_{i_1} \cup \theta_{i_2} \cup \ldots \cup \theta_{i_r}$. If X is formed by two or more parts, the first step is to eliminate all anti-absorbed parts, ie. if $A << B$ then $u(A, B) = u(A)$; generally speaking, a part B is anti-absorbed by part A if all digits of A belong to B; for an anti-absorbing chain $A_1 << A_2 << ... << A_s$ one takes A_1 only and the others are eliminated; afterwards, when X is anti-absorbingly irreducible, $u(X)$ will be the unions of all singletons whose indices occur in the remaining parts of X - if one digit occurs many times it is taken only once. For convenience, one provides below the MatLab[1] source code for computing $u(X), X \in D^\Theta$. The input variable u_n of this routine corresponds to the DSm base encoding and can be obtained by the method proposed in chapter 2.

```
%**************************************************
function [UX]=GetUX(u_n,X);
%**************************************************
% GetUX computes the function u(X) involved
% in the DSm hybrid rule of combination.
% Inputs : u_n => Dezert-Smarandache base encoding
%          X   => Element of D^Theta in base u_n
% Example for n=3: if Theta={theta1,theta2,theta3}
%                  then u_3=[1 2 12 3 13 23 123]
% Output : Ux => u(X) expressed in base u_n
% Copyrights (c) 2003 - J. Dezert & F. Smarandache
%**************************************************
UX=zeros(1,size(u_n,2));XP=u_n(find(X==1))';
AF=zeros(size(XP,1),1); XC=[];
for jj=1:size(XP,1)
if(AF(jj)==0),ujj=num2str(XP(jj));
for kk=1:size(XP,1)
if(AF(kk)==0)
ukk=num2str(XP(kk));w=intersect(ujj,ukk);
if(isempty(w)==0),
if((isequal(w,ujj)+isequal(w,ukk))>0)
XC=[XC;str2num(w)];
if(size(ujj,2)<size(ukk,2)),AF(kk)=1;end
if(size(ukk,2)<size(ujj,2)),AF(jj)=1;end
end; end; end; end; end; end
XC=unique(XC);XCS=unique(num2str(XC'));
for ii=1:size(XCS,2),if(XCS(ii)~=' ')
for jj=1:size(u_n,2)
if(isempty(intersect(XCS(ii),num2str(u_n(jj))))==0)
UX(jj)=1;end;end;end;end
```

Matlab source code for computing $u(X), X \in D^\Theta$

Here are some examples for the case $n = 3$: $12 << 123$, i.e. 12 anti-absorbs 123. Between 12 and 23 there is no anti-absorbing relationship.

- If $X = 123$ then $u(X) = \theta_1 \cup \theta_2 \cup \theta_3$.
- If $X = \{23, 123\}$, then $23 << 123$, thus $u(\{23, 123\}) = u(23)$, because 123 has been eliminated, hence $u(X) = u(23) = \theta_2 \cup \theta_3$.
- If $X = \{13, 123\}$, then $13 << 123$, thus $u(\{13, 123\}) = u(13) = \theta_1 \cup \theta_3$.

[1] Matlab is a trademark of The MathWorks, Inc.

- If $X = \{13, 23, 123\}$, then $13 << 123$, thus $u(\{13, 23, 123\}) = u(\{13, 23\}) = \theta_1 \cup \theta_2 \cup \theta_3$ (one takes as theta indices each digit in the $\{13, 23\}$) - if one digit is repeated it is taken only once; between 13 and 23 there is no relation of anti-absorbing.

- If $X = \{3, 13, 23, 123\}$, then $u(X) = u(\{3, 13, 23\})$ because $23 << 123$, then $u(\{3, 13, 23\}) = u(\{3, 13\})$ because $3 << 23$, then $u(\{3, 13\}) = u(3) = \theta_3$ because $3 << 13$.

- If $X = \{1, 12, 13, 23, 123\}$, then one has the anti-absorbing chain: $1 << 12 << 123$, thus $u(X) = u(\{1, 13, 23\}) = u(\{1, 23\})$ because $1 << 13$, and finally $u(X) = \theta_1 \cup \theta_2 \cup \theta_3$.

- If $X = \{1, 2, 12, 13, 23, 123\}$, then $1 << 12 << 123$ and $2 << 23$ thus $u(X) = u(\{1, 2, 13\}) = u(\{1, 2\})$ because $1 << 13$, and finally $u(X) = \theta_1 \cup \theta_2$.

- If $X = \{2, 12, 3, 13, 23, 123\}$, then $2 << 23 << 123$ and $3 << 13$ thus $u(X) = u(\{2, 12, 3\})$, but $2 << 12$ hence $u(X) = u(\{2, 3\}) = \theta_2 \cup \theta_3$.

4.5.3 The hybrid DSm rule of combination for 2 sources

To eliminate the *degenerate vacuous fusion problem* from the presentation, we assume from now on that the given hybrid DSm model $\mathcal{M}$ under consideration is always different from the vacuous model $\mathcal{M}_\emptyset$ (i.e. $I_t \neq \emptyset$). The hybrid DSm rule of combination, associated to a given hybrid DSm model $\mathcal{M} \neq \mathcal{M}_\emptyset$, for two sources is defined for all $A \in D^\Theta$ as:

$$
\begin{aligned}
m_{\mathcal{M}(\Theta)}(A) \triangleq \phi(A)\Big[&\sum_{\substack{X_1,X_2\in D^\Theta\\(X_1\cap X_2)=A}} m_1(X_1)m_2(X_2)\\
&+ \sum_{\substack{X_1,X_2\in \boldsymbol{\emptyset}\\ [(u(X_1)\cup u(X_2))=A]\vee[(u(X_1)\cup u(X_2)\in\boldsymbol{\emptyset})\wedge(A=I_t)]}} m_1(X_1)m_2(X_2)\\
&+ \sum_{\substack{X_1,X_2\in D^\Theta\\(X_1\cup X_2)=A\\ X_1\cap X_2\in\boldsymbol{\emptyset}}} m_1(X_1)m_2(X_2)\Big]
\end{aligned}
\tag{4.3}
$$

The first sum entering in the previous formula corresponds to mass $m_{\mathcal{M}^f(\Theta)}(A)$ obtained by the classic DSm rule of combination (4.1) based on the free-DSm model $\mathcal{M}^f$ (i.e. on the free lattice D^Θ), i.e.

$$
m_{\mathcal{M}^f(\Theta)}(A) \triangleq \sum_{\substack{X_1,X_2\in D^\Theta\\(X_1\cap X_2)=A}} m_1(X_1)m_2(X_2) \tag{4.4}
$$

The second sum entering in the formula of the DSm-hybrid rule of combination (4.3) represents the mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances. The third sum entering in the formula of the DSm-hybrid rule of combination (4.3) transfers the sum of relatively empty sets to the non-empty sets in a similar way as it was calculated following the DSm classic rule.

4.5.4 The hybrid DSm rule of combination for $k \geq 2$ sources

The previous formula of hybrid DSm rule of combination can be generalized in the following way for all $A \in D^\Theta$:

$$
\begin{aligned}
m_{\mathcal{M}(\Theta)}(A) \triangleq \phi(A)\Big[& \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1\cap X_2\cap\ldots\cap X_k)=A}} \prod_{i=1}^{k} m_i(X_i) \\
& + \sum_{\substack{X_1,X_2,\ldots,X_k \in \boldsymbol{\emptyset} \\ [(u(X_1)\cup u(X_2)\cup\ldots\cup u(X_k))=A]\vee[(u(X_1)\cup u(X_2)\cup\ldots\cup u(X_k)\in\boldsymbol{\emptyset})\wedge(A=I_t)]}} \prod_{i=1}^{k} m_i(X_i) \\
& + \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1\cup X_2\cup\ldots\cup X_k)=A \\ X_1\cap X_2\cap\ldots\cap X_k \in \boldsymbol{\emptyset}}} \prod_{i=1}^{k} m_i(X_i)\Big]
\end{aligned}
\tag{4.5}
$$

The first sum entering in the previous formula corresponds to mass $m_{\mathcal{M}^f(\Theta)}(A)$ obtained by the classic DSm rule of combination (4.2) for k sources of information based on the free-DSm model $\mathcal{M}^f$ (i.e. on the free lattice D^Θ), i.e.

$$
m_{\mathcal{M}^f(\Theta)}(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1\cap X_2\cap\ldots\cap X_k)=A}} \prod_{i=1}^{k} m_i(X_i) \tag{4.6}
$$

4.5.5 On the associativity of the hybrid DSm rule

From (4.5) and (4.6), the previous general formula can be rewritten as

$$
m_{\mathcal{M}(\Theta)}(A) \triangleq \phi(A)\Big[S_1(A) + S_2(A) + S_3(A)\Big] \tag{4.7}
$$

where

$$
S_1(A) \equiv m_{\mathcal{M}^f(\Theta)}(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1\cap X_2\cap\ldots\cap X_k)=A}} \prod_{i=1}^{k} m_i(X_i) \tag{4.8}
$$

$$
S_2(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in \boldsymbol{\emptyset} \\ [(u(X_1)\cup u(X_2)\cup\ldots\cup u(X_k))=A]\vee[(u(X_1)\cup u(X_2)\cup\ldots\cup u(X_k)\in\boldsymbol{\emptyset})\wedge(A=I_t)]}} \prod_{i=1}^{k} m_i(X_i) \tag{4.9}
$$

$$
S_3(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1\cup X_2\cup\ldots\cup X_k)=A \\ X_1\cap X_2\cap\ldots\cap X_k \in \boldsymbol{\emptyset}}} \prod_{i=1}^{k} m_i(X_i) \tag{4.10}
$$

This rule of combination can be viewed actually as a two-step procedure as follows:

- **Step 1:** Evaluate the combination of the sources over the free lattice D^Θ by the classical DSm rule of combination to get for all $A \in D^\Theta$, $S_1(A) = m_{\mathcal{M}^f(\Theta)}(A)$ using (4.6). This step preserves the commutativity and associativity properties of the combination. When there is no constraint (when using the free DSm model), the hybrid DSm rule reduces to the classic DSm rule because $\boldsymbol{\emptyset} = \{\emptyset\}$ and $m_i(\emptyset) = 0$, $i = 1, \ldots k$ and therefore $\Phi(A) = 1$ and $S_2(A) = S_3(A) = 0$ $\forall A \neq \emptyset \in D^\Theta$. For $A = \emptyset$, $\Phi(A) = 0$ and thus $m_{\mathcal{M}^f}(\emptyset) = 0$.

- **Step 2:** Transfer the masses of the integrity constraints of the hybrid DSm model $\mathcal{M}$ according to formula (4.7). Note that this step is necessary only if one has reliable information about the real integrity constraints involved in the fusion problem under consideration. More precisely, when some constraints are introduced to deal with a given hybrid DSm model $\mathcal{M}(\Theta)$, there exists some propositions $A \overset{\mathcal{M}}{\equiv} \emptyset$ for which $\Phi(A) = 0$. For these propositions, it is actually not necessary to compute $S_1(A)$, $S_2(A)$ and $S_3(A)$ since the product $\Phi(A)[S_1(A) + S_2(A) + S_3(A)]$ equals zero because $\Phi(A) = 0$. This reduces the cost of computations. For propositions $A \overset{\mathcal{M}}{\not\equiv} \emptyset$ characterized by $\Phi(A) = 1$, the derivation of $S_1(A)$, $S_2(A)$ and $S_3(A)$ is necessary to get $m_{\mathcal{M}(\Theta)}(A)$. The last part of the hybrid DSm combination mechanism (called compression step) consists in gathering (summing) all masses corresponding to same proposition because of the constraints of the model. As example, if one considers the 3D frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$ with the constraint $\theta_2 \cap \theta_3 \overset{\mathcal{M}}{\equiv} \emptyset$, then the mass resulting from the hybrid DSm fusion rule (4.7) $m_{\mathcal{M}(\Theta)}(\theta_1 \cup (\theta_2 \cap \theta_3))$ will have to be added to $m_{\mathcal{M}(\Theta)}(\theta_1)$ because $\theta_1 \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}}{\equiv} \theta_1$ due to the constraint $\theta_2 \cap \theta_3 \overset{\mathcal{M}}{\equiv} \emptyset$.

The second step does not preserve the full associativity of the rule (same remark applies also with Yager's or Dubois & Prade's rules), but this is not a fundamental requirement because this problem can be easily circumvented by keeping in parallel the two previous steps 1 and 2. The fusion has to start always on the free-DSm model. The second step is applied only when some integrity constraints are introduced and before the decision-making. In other words, if one has only 2 independent sources of information giving $m_1(.)$ and $m_2(.)$ and some integrity constraints on the frame Θ, one applies step 1 to get[2] $m^{1,2}_{\mathcal{M}^f(\Theta)}(.) = [m_1 \oplus m_2](.)$ defined on the free-DSm model and then one applies step 2 to get the final result $m^{1,2}_{\mathcal{M}(\Theta)}(.)$ on the hybrid-model. If a third source of information is introduced, say $m_3(.)$, one combines it with the two previous ones by step 1 again to get $m^{1,2,3}_{\mathcal{M}^f(\Theta)}(.) = [m_3 \oplus m^{1,2}_{\mathcal{M}^f(\Theta)}](.)$ and then one applies step 2 to get the final result $m^{1,2,3}_{\mathcal{M}(\Theta)}(.)$ on the hybrid-model $\mathcal{M}(\Theta)$.

There is no technical difficulty to process the fusion in this way and that's why the full associativity of the fusion rule is not so fundamental despite of all criticisms against the alternatives to Dempster's rules emerging in litterature over the years. The full/direct associativity property is realized only through Demspter's rule of combination when working on Shafer's model. This is one of reasons for which Dempster's rule is usually preferred to the other fusion rules, but in turn this associativity property (through the normalization factor $1 - m(\emptyset)$) is also one of the main sources of the criticisms for more than twenty years because one knows that Dempster's rule fails to provide coherent results when conflicts become high (see chapters 5 and 12 for examples) and something else must be carried out anyway to prevent problems. This matter of fact is quite paradoxical.

[2]We introduce here the notation $m^{1,2}(.)$ to explicitly express that the resulting mass is related to the combination of sources 1 and 2 only.

To avoid the loss of information in the fusion, one has first to combine all sources using DSm rule on free-DSm model and then to adapt the belief masses according to the integrity constraints of the model $\mathcal{M}$. If one first adapts the local masses $m_1(.), ...m_k(.)$ to the hybrid-model $\mathcal{M}$ and afterwards one applies the combination rule, the fusion becomes only suboptimal because some information is lost forever during the transfer of masses of integrity constraints. The same remark holds if the transfer of masses of integrity constraints is done at some intermediate steps after the fusion of m sources with $m < k$.

Let's note also that this formula of transfer is more general (because we include the possibilities to introduce both exclusivity constraints and non-existential constraints as well) and more precise (because we explicitly consider all different relative emptiness of elements into the general transfer formula (4.7)) than the generic transfer formulas used in the DST framework proposed as alternative rules to Dempster's rule of combination [6] and discussed in section 4.5.10.

4.5.6 Property of the hybrid DSm Rule

The following equality holds:

$$\sum_{A \in D^\Theta} m_{\mathcal{M}(\Theta)}(A) = \sum_{A \in D^\Theta} \phi(A)\Big[S_1(A) + S_2(A) + S_3(A)\Big] = 1 \tag{4.11}$$

Proof: Let's first prove that $\sum_{A \in D^\Theta} m(A) = 1$ where all masses $m(A)$ are obtained by the DSm classic rule. Let's consider each mass $m_i(.)$ provided by the ith source of information, for $1 \leq i \leq k$, as a vector of $d = \mid D^\Theta \mid$ dimension, whose sum of components is equal to one, i.e. $m_i(D^\Theta) = [m_{i1}, m_{i2}, \dots, m_{id}]$, and $\sum_{j=1,d} m_{ij} = 1$. Thus, for $k \geq 2$ sources of information, the mass matrix becomes

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1d} \\ \dots & \dots & \dots & \dots \\ m_{k1} & m_{k2} & \dots & m_{kd} \end{bmatrix}$$

If one denotes the sets in D^Θ by A_1, A_2, ..., A_d (it doesn't matter in what order one lists them) then the column (j) in the matrix represents the masses assigned to A_j by each source of information s_1, s_2, $\dots$, s_k; for example $s_i(A_j) = m_{ij}$, where $1 \leq i \leq k$. According to the DSm network architecture [3], all the products in this network will have the form $m_{1j_1} m_{2j_2} \dots m_{kj_k}$, i.e. one element only from each matrix row, and no restriction about the number of elements from each matrix column, $1 \leq j_1, j_2, \dots, j_k \leq d$. Each such product will enter in the fusion mass of one set only from D^Θ. Hence the sum of all components of the fusion mass is equal to the sum of all these products, which is equal to

$$\prod_{i=1}^{k} \sum_{j=1}^{d} m_{ij} = \prod_{i=1}^{k} 1 = 1 \tag{4.12}$$

The hybrid DSm rule has three sums S_1, S_2, and S_3. Let's separate the mass matrix $\mathbf{M}$ into two disjoint sub-matrices $\mathbf{M}_\emptyset$ formed by the columns of all absolutely and relatively empty sets, and $\mathbf{M}_N$ formed by the columns of all non-empty sets. According to the DSm network architecture (for $k \geq 2$ rows):

- S_1 is the sum of all products resulted from the multiplications of the columns of $\mathbf{M}_N$ following the DSm network architecture such that the intersection of their corresponding sets is non-empty, i.e. the sum of masses of all non-empty sets before any mass of absolutely or relatively empty sets could be transferred to them;

- S_2 is the sum of all products resulted from the multiplications of $\mathbf{M}_\emptyset$ following the DSm network architecture, i.e. a partial sum of masses of absolutely and relatively empty sets transferred to the ignorances in $I \triangleq I_t \cup I_r$ or to singletons of Θ.

- S_3 is the sum of all the products resulted from the multiplications of the columns of $\mathbf{M}_N$ and $\mathbf{M}_\emptyset$ together, following the DSm network architecture, but such that at least a column is from each of them, and also the sum of all products of columns of $\mathbf{M}_N$ such that the intersection of their corresponding sets is empty (what did not enter into the previous sum S_1), i.e. the remaining sum of masses of absolutely or relatively empty sets transferred to the non-empty sets of the hybrid DSm model $\mathcal{M}$.

If one now considers all the terms (each such term is a product of the form $m_{1j_1}m_{2j_2}\ldots m_{kj_k}$) of these three sums, we get exactly the same terms as in the DSm network architecture for the DSm classic rule, thus the sum of all terms occurring in S_1, S_2, and S_3 is 1 (see formula (4.12)) which completes the proof. The hybrid DSm rule naturally derives from the DSm classic rule. Entire masses of relatively and absolutely empty sets in a given hybrid DSm model $\mathcal{M}$ are transferred to non-empty sets according to the formula (4.7) and thus

$$\forall A \in \boldsymbol{\emptyset} \subset D^\Theta, \quad m_{\mathcal{M}(\Theta)}(A) = 0 \tag{4.13}$$

The entire mass of a relatively empty set (from D^Θ) which has in its expression θ_{j_1}, θ_{j_2}, ..., θ_{j_r}, with $1 \leq r \leq n$ will generally be distributed among the θ_{j_1}, θ_{j_2}, ..., θ_{j_r} or their unions or intersections, and the distribution follows the way of multiplication from the DSm classic rule, explained by the DSm network architecture [3]. Thus, because nothing is lost, nothing is gained, the sum of all $m_{\mathcal{M}(\Theta)}(A)$ is equal to 1 as just proven previously, and fortunately no normalization constant is needed which could bring a loss of information in the fusion rule. The three summations $S_1(.)$, $S_3(.)$ and $S_3(.)$ are disjoint because:

- $S_1(.)$ multiplies the columns corresponding to non-empty sets only - but such that the intersections of the sets corresponding to these columns are non-empty [from the definition of DSm classic rule];

- $S_2(.)$ multiplies the columns corresponding to absolutely and relatively empty sets only;

- $S_3(.)$ multiplies:

a) either the columns corresponding to absolutely or relatively empty sets with the columns corresponding to non-empty sets such that at least a column corresponds to an absolutely or relatively emptyset and at least a column corresponds to a non-emptyset,

b) or the columns corresponding to non-empty sets - but such that the intersections of the sets corresponding to these columns are empty.

The multiplications are following the DSm network architecture, i.e. any product has the above general form: $m_{1j_1}m_{2j_2}\ldots m_{kj_k}$, i.e. any product contains as factor one element only from each row of the mass matrix $\mathbf{M}$ and the total number of factors in a product is equal to k. The function $\phi(A)$ automatically assigns the value zero to the mass of any empty set, and allows the calculation of masses of all non-empty sets.

4.5.7 On the programming of the hybrid DSm rule

We briefly give here an issue for a fast programming of the DSm rule of combination. Let's consider $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, the sources $\mathcal{B}_1$, $\mathcal{B}_2$,..., $\mathcal{B}_k$, and $p = \min\{n, k\}$. One needs to check only the focal sets, i.e. sets (i.e. propositions) whose masses assigned to them by these sources are not all zero. Thus, if $\mathbf{M}$ is the mass matrix, and we consider a set A_j in D^Θ, then the column (j) corresponding to A_j, i.e. $(m_{1j}\ m_{2j}\ \ldots\ m_{kj})$ transposed has not to be identical to the null-vector of k-dimension $(0\ 0\ \ldots\ 0)$ transposed. Let $D^\Theta(\text{step}_1)$ be formed by all focal sets at the beginning (after sources $\mathcal{B}_1$, $\mathcal{B}_2$,..., $\mathcal{B}_k$ have assigned masses to the sets in D^Θ). Applying the DSm classic rule, besides the sets in $D^\Theta(\text{step}_1)$ one adds r-intersections of sets in $D^\Theta(\text{step}_1)$, thus:

$$D^\Theta(\text{step}_2) = D^\Theta(\text{step}_1) \vee \{A_{i_1} \wedge A_{i_2} \wedge \ldots \wedge A_{i_r}\}$$

where A_{i_1}, A_{i_2}, ..., A_{i_r} belong to $D^\Theta(\text{step}_1)$ and $2 \leq r \leq p$.

Applying the hybrid DSm rule, due to its S_2 and S_3 summations, besides the sets in $D^\Theta(\text{step}_2)$ one adds r-unions of sets and the total ignorance in $D^\Theta(\text{step}_2)$, thus:

$$D^\Theta(\text{step}_3) = D^\Theta(\text{step}_2) \vee I_t \vee \{A_{i_1} \vee A_{i_2} \vee \ldots \vee A_{i_r}\}$$

where A_{i_1}, A_{i_2}, ..., A_{i_r} belong to $D^\Theta(\text{step}_2)$ and $2 \leq r \leq p$.

This means that instead of computing the masses of all sets in D^Θ, one needs to first compute the masses of all focal sets (step 1), second the masses of their r-intersections (step 2), and third the masses of r-unions of all previous sets and the mass of total ignorance (step 3).

4.5.8 Application of the hybrid DSm rule on previous examples

We present in this section some numerical results of the hybrid DSm rule of combination for 2 independent sources of information. We examine the seven previous examples in order to help the reader to check by himself (or herself) the validity of our new general formula. We will not go in details in the derivations, but we just present the main intermediary results $S_1(A)$, $S_2(A)$ and $S_3(A)$ (defined in (4.8), (4.9), (4.10)) involved into the general formula (4.3) with setting the number of sources to combine to $k = 2$. Now let's consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and two independent bodies of evidence with the generalized basic belief assignments[3] $m_1(.)$ and $m_2(.)$ given in the following table[4].

Element A of D^Θ	$m_1(A)$	$m_2(A)$	$m_{\mathcal{M}^f(\Theta)}(A)$
$\emptyset$	0	0	0
$\theta_1 \cap \theta_2 \cap \theta_3$	0	0	0.16
$\theta_2 \cap \theta_3$	0	0.20	0.19
$\theta_1 \cap \theta_3$	0.10	0	0.12
$(\theta_1 \cup \theta_2) \cap \theta_3$	0	0	0.01
θ_3	0.30	0.10	0.10
$\theta_1 \cap \theta_2$	0.10	0.20	0.22
$(\theta_1 \cup \theta_3) \cap \theta_2$	0	0	0.05
$(\theta_2 \cup \theta_3) \cap \theta_1$	0	0	0
$(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$	0	0	0
$(\theta_1 \cap \theta_2) \cup \theta_3$	0	0	0
θ_2	0.20	0.10	0.03
$(\theta_1 \cap \theta_3) \cup \theta_2$	0	0	0
$\theta_2 \cup \theta_3$	0	0	0
θ_1	0.10	0.20	0.08
$(\theta_2 \cap \theta_3) \cup \theta_1$	0	0	0.02
$\theta_1 \cup \theta_3$	0.10	0.20	0.02
$\theta_1 \cup \theta_2$	0.10	0	0
$\theta_1 \cup \theta_2 \cup \theta_3$	0	0	0

The right column of the table gives the result obtained by the DSm rule of combination based on the free-DSm model. The following sections give the results obtained by the hybrid DSm rule on the seven previous examples of section 4.3. The tables show the values of $\phi(A)$, $S_1(A)$, $S_2(A)$ and $S_3(A)$ to help the reader to check the validity of these results. It is important to note that the values of $S_1(A)$, $S_2(A)$ and $S_3(A)$ when $\phi(A) = 0$ do not need to be computed in practice but are provided here only for verification.

[3] A general example with $m_1(A) > 0$ and $m_2(A) > 0$ for all $A \neq \emptyset \in D^\Theta$ will be briefly presented in next section.

[4] The order of elements of D^Θ is the order obtained from the generation of isotone Boolean functions - see chapter 2.

4.5.8.1 Application of the hybrid DSm rule on example 1

Here is the numerical result corresponding to example 1 with the hybrid-model $\mathcal{M}_1$ (i.e with the exclusivity constraint $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_1}{\equiv} \emptyset$). The right column of the table provides the result obtained using the hybrid DSm rule, ie. $\forall A \in D^\Theta$, $m_{\mathcal{M}_1(\Theta)}(A) = \phi(A)\big[S_1(A) + S_2(A) + S_3(A)\big]$

Element A of D^Θ	$\phi(A)$	$S_1(A)$	$S_2(A)$	$S_3(A)$	$m_{\mathcal{M}_1(\Theta)}(A)$
$\emptyset$	0	0	0	0	0
$\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_1}{\equiv} \emptyset$	0	0.16	0	0	0
$\theta_2 \cap \theta_3$	1	0.19	0	0	0.19
$\theta_1 \cap \theta_3$	1	0.12	0	0	0.12
$(\theta_1 \cup \theta_2) \cap \theta_3$	1	0.01	0	0.02	0.03
θ_3	1	0.10	0	0	0.10
$\theta_1 \cap \theta_2$	1	0.22	0	0	0.22
$(\theta_1 \cup \theta_3) \cap \theta_2$	1	0.05	0	0.02	0.07
$(\theta_2 \cup \theta_3) \cap \theta_1$	1	0	0	0.02	0.02
$\{(\theta_1 \cap \theta_2) \cup \theta_3\} \cap (\theta_1 \cup \theta_2)$	1	0	0	0	0
$(\theta_1 \cap \theta_2) \cup \theta_3$	1	0	0	0.07	0.07
θ_2	1	0.03	0	0	0.03
$(\theta_1 \cap \theta_3) \cup \theta_2$	1	0	0	0.01	0.01
$\theta_2 \cup \theta_3$	1	0	0	0	0
θ_1	1	0.08	0	0	0.08
$(\theta_2 \cap \theta_3) \cup \theta_1$	1	0.02	0	0.02	0.04
$\theta_1 \cup \theta_3$	1	0.02	0	0	0.02
$\theta_1 \cup \theta_2$	1	0	0	0	0
$\theta_1 \cup \theta_2 \cup \theta_3$	1	0	0	0	0

$$\mathbf{D}_{\mathcal{M}_1} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 1 & 1 & 1\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 1\\
0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 1\\
0 & 1 & 1 & 0 & 0 & 1\\
0 & 1 & 1 & 0 & 1 & 1\\
0 & 1 & 1 & 1 & 1 & 1\\
1 & 0 & 1 & 0 & 1 & 0\\
1 & 0 & 1 & 0 & 1 & 1\\
1 & 0 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 0 & 1 & 1\\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

From the previous table of this first numerical example, we see in column corresponding to $S_3(A)$ how the initial combined mass $m_{\mathcal{M}^f(\Theta)}(\theta_1 \cap \theta_2 \cap \theta_3) \equiv S_1(\theta_1 \cap \theta_2 \cap \theta_3) = 0.16$ is transferred (due to the constraint of $\mathcal{M}_1$) only onto the elements $(\theta_1 \cup \theta_2) \cap \theta_3$, $(\theta_1 \cup \theta_3) \cap \theta_2$, $(\theta_2 \cup \theta_3) \cap \theta_1$, $(\theta_1 \cap \theta_2) \cup \theta_3$, $(\theta_1 \cap \theta_3) \cup \theta_2$, and $(\theta_2 \cap \theta_3) \cup \theta_1$ of D^Θ. We can easily check that the sum of the elements of the column for $S_3(A)$ is equal to $m_{\mathcal{M}^f(\Theta)}(\theta_1 \cap \theta_2 \cap \theta_3) = 0.16$ (i.e. to the sum of $S_1(A)$ for which $\phi(A) = 0$) and that the sum of $S_2(A)$ for which $\phi(A) = 1$ is equal to the sum of $S_3(A)$ for which $\phi(A) = 0$ (in this example the sum is zero). Thus after introducing the constraint, the initial hyper-power set D^Θ reduces to 18 elements as follows

$$\begin{aligned}
D^\Theta_{\mathcal{M}_1} = \{&\emptyset, \theta_2 \cap \theta_3, \theta_1 \cap \theta_3, (\theta_1 \cup \theta_2) \cap \theta_3, \theta_3, \theta_1 \cap \theta_2, (\theta_1 \cup \theta_3) \cap \theta_2, (\theta_2 \cup \theta_3) \cap \theta_1, \{(\theta_1 \cap \theta_2) \cup \theta_3\} \cap (\theta_1 \cup \theta_2),\\
&(\theta_1 \cap \theta_2) \cup \theta_3, \theta_2, (\theta_1 \cap \theta_3) \cup \theta_2, \theta_2 \cup \theta_3, \theta_1, (\theta_2 \cap \theta_3) \cup \theta_1, \theta_1 \cup \theta_3, \theta_1 \cup \theta_2, \theta_1 \cup \theta_2 \cup \theta_3\}
\end{aligned}$$

As detailed in chapter 2, the elements of $D^{\Theta}_{\mathcal{M}_1}$ can be described and encoded by the matrix product $\mathbf{D}_{\mathcal{M}_1} \cdot \mathbf{u}_{\mathcal{M}_1}$ with $\mathbf{D}_{\mathcal{M}_1}$ given above and the basis vector $\mathbf{u}_{\mathcal{M}_1}$ defined[5] as $\mathbf{u}_{\mathcal{M}_1} = [<1><2><12><3><13><23>]'$. Actually $\mathbf{u}_{\mathcal{M}_1}$ is directly obtained from $\mathbf{u}_{\mathcal{M}^f}$ by removing its component $<123>$ corresponding to the constraint introduced by the model $\mathcal{M}_1$. In general, the encoding matrix $\mathbf{D}_{\mathcal{M}}$ for a given hybrid DSm model $\mathcal{M}$ is obtained from $\mathbf{D}_{\mathcal{M}^f}$ by removing all its columns corresponding to the constraints of the chosen model $\mathcal{M}$ and all the rows corresponding to redundant/equivalent propositions. In this particular example with model $\mathcal{M}_1$, we will just have to remove the last column of $\mathbf{D}_{\mathcal{M}^f}$ to get $\mathbf{D}_{\mathcal{M}_1}$ and no row is removed from $\mathbf{D}_{\mathcal{M}^f}$ because there is no redundant/equivalent proposition involved in this example. This suppression of some rows of $\mathbf{D}_{\mathcal{M}^f}$ will however occur in the next examples.

4.5.8.2 Application of the hybrid DSm rule on example 2

Here is the numerical result corresponding to example 2 with the hybrid-model $\mathcal{M}_2$ (i.e with the exclusivity constraint $\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset \Rightarrow \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$). One gets now

Element A of D^{Θ}	$\phi(A)$	$S_1(A)$	$S_2(A)$	$S_3(A)$	$m_{\mathcal{M}_2(\Theta)}(A)$
$\emptyset$	0	0	0	0	0
$\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$	0	0.16	0	0	0
$\theta_2 \cap \theta_3$	1	0.19	0	0	0.19
$\theta_1 \cap \theta_3$	1	0.12	0	0	0.12
$(\theta_1 \cup \theta_2) \cap \theta_3$	1	0.01	0	0.02	0.03
θ_3	1	0.10	0	0	0.10
$\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$	0	0.22	0	0.02	0
$(\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \theta_2 \cap \theta_3$	1	0.05	0	0.02	0.07
$(\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_2}{\equiv} \theta_1 \cap \theta_3$	1	0	0	0.02	0.02
$(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_2}{\equiv} (\theta_1 \cup \theta_2) \cap \theta_3$	1	0	0	0	0
$(\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_2}{\equiv} \theta_3$	1	0	0	0.07	0.07
θ_2	1	0.03	0	0.05	0.08
$(\theta_1 \cap \theta_3) \cup \theta_2$	1	0	0	0.01	0.01
$\theta_2 \cup \theta_3$	1	0	0	0	0
θ_1	1	0.08	0	0.04	0.12
$(\theta_2 \cap \theta_3) \cup \theta_1$	1	0.02	0	0.02	0.04
$\theta_1 \cup \theta_3$	1	0.02	0	0.04	0.06
$\theta_1 \cup \theta_2$	1	0	0.02	0.07	0.09
$\theta_1 \cup \theta_2 \cup \theta_3$	1	0	0	0	0

[5] $\mathbf{D}_{\mathcal{M}^f}$ was denoted $\mathbf{D}_n$ and $\mathbf{u}_{\mathcal{M}^f}$ as $\mathbf{u}_n$ in chapter 2.

From the previous table of this numerical example, we see in the column corresponding to $S_3(A)$ how the initial combined masses $m_{\mathcal{M}^f(\Theta)}(\theta_1 \cap \theta_2 \cap \theta_3) \equiv S_1(\theta_1 \cap \theta_2 \cap \theta_3) = 0.16$ and $m_{\mathcal{M}^f(\Theta)}(\theta_1 \cap \theta_2) \equiv S_1(\theta_1 \cap \theta_2) = 0.22$ are transferred (due to the constraint of $\mathcal{M}_2$) onto some elements of D^Θ. We can easily check that the sum of the elements of the column for $S_3(A)$ is equal to $0.16 + 0.22 = 0.38$ (i.e. to the sum of $S_1(A)$ for which $\phi(A) = 0$) and that the sum of $S_2(A)$ for which $\phi(A) = 1$ is equal to the sum of $S_3(A)$ for which $\phi(A) = 0$ (this sum is 0.02). Because some elements of D^Θ are now equivalent due to the constraints of $\mathcal{M}_2$, we have to sum all the masses corresponding to same equivalent propositions/elements (by example $\{(\theta_1 \cap \theta_2) \cup \theta_3\} \cap (\theta_1 \cup \theta_2) \overset{\mathcal{M}_2}{\equiv} (\theta_1 \cup \theta_2) \cap \theta_3$). This can be viewed as the final compression step. One then gets the reduced hyper-power set $D^\Theta_{\mathcal{M}_2}$ having now 13 different elements with the combined belief masses presented in the following table.

The basis vector $\mathbf{u}_{\mathcal{M}_2}$ and the encoding matrix $\mathbf{D}_{\mathcal{M}_2}$ for the elements of $D^\Theta_{\mathcal{M}_2}$ are given by $\mathbf{u}_{\mathcal{M}_2} = [<1><2><3><13><23>]'$ and below. Actually $\mathbf{u}_{\mathcal{M}_2}$ is directly obtained from $\mathbf{u}_{\mathcal{M}^f}$ by removing its components $<12>$ and $<123>$ corresponding to the constraints introduced by the model $\mathcal{M}_2$.

Element A of $D^\Theta_{\mathcal{M}_2}$	$m_{\mathcal{M}_2(\Theta)}(A)$
$\emptyset$	0
$\theta_2 \cap \theta_3$	$0.19 + 0.07 = 0.26$
$\theta_1 \cap \theta_3$	$0.12 + 0.02 = 0.14$
$(\theta_1 \cup \theta_2) \cap \theta_3$	$0.03 + 0 = 0.03$
θ_3	$0.10 + 0.07 = 0.17$
θ_2	0.08
$(\theta_1 \cap \theta_3) \cup \theta_2$	0.01
$\theta_2 \cup \theta_3$	0
θ_1	0.12
$(\theta_2 \cap \theta_3) \cup \theta_1$	0.04
$\theta_1 \cup \theta_3$	0.06
$\theta_1 \cup \theta_2$	0.09
$\theta_1 \cup \theta_2 \cup \theta_3$	0

and

$$\mathbf{D}_{\mathcal{M}_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

4.5.8.3 Application of the hybrid DSm rule on example 3

Here is the numerical result corresponding to example 3 with the hybrid-model $\mathcal{M}_3$ (i.e with the exclusivity constraint $(\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_3}{\equiv} \emptyset$). This constraint implies directly $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$, $\theta_1 \cap \theta_2 \overset{\mathcal{M}_3}{\equiv} \emptyset$ and $\theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$. One gets now

Element A of D^Θ	$\phi(A)$	$S_1(A)$	$S_2(A)$	$S_3(A)$	$m_{\mathcal{M}_3(\Theta)}(A)$
$\emptyset$	0	0	0	0	0
$\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$	0	0.16	0	0	0
$\theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$	0	0.19	0	0	0
$\theta_1 \cap \theta_3$	1	0.12	0	0	0.12
$(\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \theta_1 \cap \theta_3$	1	0.01	0	0.02	0.03
θ_3	1	0.10	0	0.06	0.16
$\theta_1 \cap \theta_2 \overset{\mathcal{M}_3}{\equiv} \emptyset$	0	0.22	0	0.02	0
$(\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_3}{\equiv} \emptyset$	0	0.05	0	0.02	0
$(\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_3}{\equiv} \theta_1 \cap \theta_3$	1	0	0	0.02	0.02
$(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_3}{\equiv} \theta_1 \cap \theta_3$	1	0	0	0	0
$(\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_3}{\equiv} \theta_3$	1	0	0	0.07	0.07
θ_2	1	0.03	0	0.09	0.12
$(\theta_1 \cap \theta_3) \cup \theta_2$	1	0	0	0.01	0.01
$\theta_2 \cup \theta_3$	1	0	0	0.05	0.05
θ_1	1	0.08	0	0.04	0.12
$(\theta_2 \cap \theta_3) \cup \theta_1 \overset{\mathcal{M}_3}{\equiv} \theta_1$	1	0.02	0	0.02	0.04
$\theta_1 \cup \theta_3$	1	0.02	0	0.06	0.08
$\theta_1 \cup \theta_2$	1	0	0.02	0.09	0.11
$\theta_1 \cup \theta_2 \cup \theta_3$	1	0	0.02	0.05	0.07

We see in the column corresponding to $S_3(A)$ how the initial combined masses $m_{\mathcal{M}^f(\Theta)}((\theta_1 \cup \theta_3) \cap \theta_2) \equiv S_1((\theta_1 \cup \theta_3) \cap \theta_2) = 0.05$, $m_{\mathcal{M}^f(\Theta)}(\theta_1 \cap \theta_2 \cap \theta_3) \equiv S_1(\theta_1 \cap \theta_2 \cap \theta_3) = 0.16$, $m_{\mathcal{M}^f(\Theta)}(\theta_2 \cap \theta_3) \equiv S_1(\theta_2 \cap \theta_3) = 0.19$ and $m_{\mathcal{M}^f(\Theta)}(\theta_1 \cap \theta_2) \equiv S_1(\theta_1 \cap \theta_2) = 0.22$ are transferred (due to the constraint of $\mathcal{M}_3$) onto some elements of D^Θ. We can easily check that the sum of the elements of the column for $S_3(A)$ is equal to $0.05 + 0.16 + 0.19 + 0.22 = 0.62$ (i.e. to the sum of $S_1(A)$ for which $\phi(A) = 0$) and that the sum of $S_2(A)$ for which $\phi(A) = 1$ is equal to $0.02 + 0.02 = 0.04$ (i.e. to the sum of $S_3(A)$ for which $\phi(A) = 0$). Due to the model $\mathcal{M}_3$, one has to sum all the masses corresponding to same equivalent propositions. Thus after the final compression step, one gets the reduced hyper-power set $D^\Theta_{\mathcal{M}_3}$ having only 10 different elements with the following combined belief masses. The basis vector $\mathbf{u}_{\mathcal{M}_3}$ is given by $\mathbf{u}_{\mathcal{M}_3} = [<1><2><3><13>]'$ and the encoding matrix $\mathbf{D}_{\mathcal{M}_3}$ is shown just right after.

Element A of $D^{\Theta}_{\mathcal{M}_3}$	$m_{\mathcal{M}_3(\Theta)}(A)$
$\emptyset$	0
$\theta_1 \cap \theta_3$	$0.12 + 0.03 + 0.02 + 0 = 0.17$
θ_3	$0.16 + 0.07 = 0.23$
θ_2	0.12
$(\theta_1 \cap \theta_3) \cup \theta_2$	0.01
$\theta_2 \cup \theta_3$	0.05
θ_1	$0.12 + 0.04 = 0.16$
$\theta_1 \cup \theta_3$	0.08
$\theta_1 \cup \theta_2$	0.11
$\theta_1 \cup \theta_2 \cup \theta_3$	0.07

and

$$\mathbf{D}_{\mathcal{M}_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

4.5.8.4 Application of the hybrid DSm rule on example 4 (Shafer's model)

Here is the result obtained with the hybrid-model $\mathcal{M}_4$, i.e. Shafer's model.

Element A of D^{Θ}	$\phi(A)$	$S_1(A)$	$S_2(A)$	$S_3(A)$	$m_{\mathcal{M}_4(\Theta)}(A)$
$\emptyset$	0	0	0	0	0
$\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \emptyset$	0	0.16	0	0	0
$\theta_2 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \emptyset$	0	0.19	0	0	0
$\theta_1 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \emptyset$	0	0.12	0	0	0
$(\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \emptyset$	0	0.01	0	0.02	0
θ_3	1	0.10	0	0.07	0.17
$\theta_1 \cap \theta_2 \overset{\mathcal{M}_4}{\equiv} \emptyset$	0	0.22	0	0.02	0
$(\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_4}{\equiv} \emptyset$	0	0.05	0	0.02	0
$(\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_4}{\equiv} \emptyset$	0	0	0	0.02	0
$(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_4}{\equiv} \emptyset$	0	0	0	0	0
$(\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_4}{\equiv} \theta_3$	1	0	0	0.07	0.07
θ_2	1	0.03	0	0.09	0.12
$(\theta_1 \cap \theta_3) \cup \theta_2 \overset{\mathcal{M}_4}{\equiv} \theta_2$	1	0	0	0.01	0.01
$\theta_2 \cup \theta_3$	1	0	0	0.05	0.05
θ_1	1	0.08	0	0.06	0.14
$(\theta_2 \cap \theta_3) \cup \theta_1 \overset{\mathcal{M}_4}{\equiv} \theta_1$	1	0.02	0	0.02	0.04
$\theta_1 \cup \theta_3$	1	0.02	0	0.15	0.17
$\theta_1 \cup \theta_2$	1	0	0.02	0.09	0.11
$\theta_1 \cup \theta_2 \cup \theta_3$	1	0	0.06	0.06	0.12

From the previous table of this numerical example, we see in column corresponding to $S_3(A)$ how the initial combined masses of the eight elements forced to the empty set by the constraints of the model $\mathcal{M}_4$ are transferred onto some elements of D^Θ. We can easily check that the sum of the elements of the column for $S_3(A)$ is equal to $0.16 + 0.19 + 0.12 + 0.01 + 0.22 + 0.05 = 0.75$ (i.e. to the sum of $S_1(A)$ for which $\phi(A) = 0$) and that the sum of $S_2(A)$ for which $\phi(A) = 1$ is equal to the sum of $S_3(A)$ for which $\phi(A) = 0$ (this sum is $0.02 + 0.06 = 0.08 = 0.02 + 0.02 + 0.02 + 0.02$).

After the final compression step (i.e. the clustering of all equivalent propositions), one gets the reduced hyper-power set $D^\Theta_{\mathcal{M}_4}$ having only $2^3 = 8$ (corresponding to the classical power set 2^Θ) with the following combined belief masses:

Element A of $D^\Theta_{\mathcal{M}_4}$	$m_{\mathcal{M}_4(\Theta)}(A)$
$\emptyset$	0
θ_3	$0.17 + 0.07 = 0.24$
θ_2	$0.12 + 0.01 = 0.13$
$\theta_2 \cup \theta_3$	0.05
θ_1	$0.14 + 0.04 = 0.18$
$\theta_1 \cup \theta_3$	0.17
$\theta_1 \cup \theta_2$	0.11
$\theta_1 \cup \theta_2 \cup \theta_3$	0.12

and

$$\mathbf{D}_{\mathcal{M}_4} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The basis vector $\mathbf{u}_{\mathcal{M}_4}$ is given by $\mathbf{u}_{\mathcal{M}_4} = [< 1 >< 2 >< 3 >]'$ and the encoding matrix $\mathbf{D}_{\mathcal{M}_4}$ is shown just above.

4.5.8.5 Application of the hybrid DSm rule on example 5

The following table presents the numerical result corresponding to example 5 with the hybrid-model $\mathcal{M}_5$ including the non-existential constraint $\theta_1 \overset{\mathcal{M}_5}{\equiv} \emptyset$. This non-existential constraint implies $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_5}{\equiv} \emptyset$, $\theta_1 \cap \theta_2 \overset{\mathcal{M}_5}{\equiv} \emptyset$, $\theta_1 \cap \theta_3 \overset{\mathcal{M}_5}{\equiv} \emptyset$ and $(\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_5}{\equiv} \emptyset$.

From the table, we see in the column corresponding to $S_3(A)$ how the initial combined masses of the 5 elements forced to the empty set by the constraints of the model $\mathcal{M}_5$ are transferred onto some elements of D^Θ. We can easily check that the sum of the elements of the column for $S_3(A)$ is equal to $0 + 0.16 + 0.12 + 0.22 + 0 + 0.08 = 0.58$ (i.e. to the sum of $S_1(A)$ for which $\phi(A) = 0$) and that the sum of $S_2(A)$ for which $\phi(A) = 1$ is equal to the sum of $S_3(A)$ for which $\phi(A) = 0$ (this sum is $0.02 + 0.06 + 0.04 = 0.12 = 0.02 + 0.02 + 0.08$).

Element A of D^Θ	$\phi(A)$	$S_1(A)$	$S_2(A)$	$S_3(A)$	$m_{\mathcal{M}_5(\Theta)}(A)$
$\emptyset$	0	0	0	0	0
$\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_5}{\equiv} \emptyset$	0	0.16	0	0	0
$\theta_2 \cap \theta_3$	1	0.19	0	0	0.19
$\theta_1 \cap \theta_3 \overset{\mathcal{M}_5}{\equiv} \emptyset$	0	0.12	0	0	0
$(\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_5}{\equiv} \theta_2 \cap \theta_3$	1	0.01	0	0.02	0.03
θ_3	1	0.10	0	0.01	0.11
$\theta_1 \cap \theta_2 \overset{\mathcal{M}_5}{\equiv} \emptyset$	0	0.22	0	0.02	0
$(\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_5}{\equiv} \theta_2 \cap \theta_3$	1	0.05	0	0.02	0.07
$(\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_5}{\equiv} \emptyset$	0	0	0	0.02	0
$(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_5}{\equiv} \theta_2 \cap \theta_3$	1	0	0	0	0
$(\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_5}{\equiv} \theta_3$	1	0	0	0.07	0.07
θ_2	1	0.03	0	0.05	0.08
$(\theta_1 \cap \theta_3) \cup \theta_2 \overset{\mathcal{M}_5}{\equiv} \theta_2$	1	0	0	0.01	0.01
$\theta_2 \cup \theta_3$	1	0	0	0	0
$\theta_1 \overset{\mathcal{M}_5}{\equiv} \emptyset$	0	0.08	0.02	0.08	0
$(\theta_2 \cap \theta_3) \cup \theta_1 \overset{\mathcal{M}_5}{\equiv} \theta_2 \cap \theta_3$	1	0.02	0	0.02	0.04
$\theta_1 \cup \theta_3 \overset{\mathcal{M}_5}{\equiv} \theta_3$	1	0.02	0.02	0.17	0.21
$\theta_1 \cup \theta_2 \overset{\mathcal{M}_5}{\equiv} \theta_2$	1	0	0.06	0.09	0.15
$\theta_1 \cup \theta_2 \cup \theta_3 \overset{\mathcal{M}_5}{\equiv} \theta_2 \cup \theta_3$	1	0	0.04	0	0.04

After the final compression step (i.e. the clustering of all equivalent propositions), one gets the reduced hyper-power set $D^\Theta_{\mathcal{M}_5}$ having only 5 different elements according to:

Element A of $D^\Theta_{\mathcal{M}_5}$	$m_{\mathcal{M}_5(\Theta)}(A)$
$\emptyset$	0
$\theta_2 \cap \theta_3$	$0.19 + 0.03 + 0.07 + 0 + 0.04 = 0.33$
θ_3	$0.11 + 0.07 + 0.21 = 0.39$
θ_2	$0.08 + 0.01 + 0.15 = 0.24$
$\theta_2 \cup \theta_3$	$0 + 0.04 = 0.04$

and

$$\mathbf{D}_{\mathcal{M}_5} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The basis vector $\mathbf{u}_{\mathcal{M}_5}$ is given by $\mathbf{u}_{\mathcal{M}_5} = [< 2 >< 3 >< 23 >]'$. and the encoding matrix $\mathbf{D}_{\mathcal{M}_5}$ is shown just above.

4.5.8.6 Application of the hybrid DSm rule on example 6

Here is the numerical result corresponding to example 6 with the hybrid-model $\mathcal{M}_6$ including the two non-existential constraint $\theta_1 \overset{\mathcal{M}_6}{\equiv} \emptyset$ and $\theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$. This is a degenerate example actually, since no uncertainty arises in such trivial model. We just want to show here that the hybrid DSm rule still works in this example and provide a legitimate result. By applying the hybrid DSm rule of combination, one now gets:

Element A of D^Θ	$\phi(A)$	$S_1(A)$	$S_2(A)$	$S_3(A)$	$m_{\mathcal{M}_6(\Theta)}(A)$
$\emptyset$	0	0	0	0	0
$\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0.16	0	0	0
$\theta_2 \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0.19	0	0	0
$\theta_1 \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0.12	0	0	0
$(\theta_1 \cup \theta_2) \cap \theta_3 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0.01	0	0.02	0
θ_3	1	0.10	0	0.07	0.17
$\theta_1 \cap \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0.22	0	0.02	0
$(\theta_1 \cup \theta_3) \cap \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0.05	0	0.02	0
$(\theta_2 \cup \theta_3) \cap \theta_1 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0	0	0.02	0
$(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0	0	0	0
$(\theta_1 \cap \theta_2) \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \theta_3$	1	0	0	0.07	0.07
$\theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0.03	0.02	0.11	0
$(\theta_1 \cap \theta_3) \cup \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0	0	0.01	0
$\theta_2 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \theta_3$	1	0	0.04	0.05	0.09
$\theta_1 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0.08	0	0.08	0
$(\theta_2 \cap \theta_3) \cup \theta_1 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0.02	0	0.02	0
$\theta_1 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \theta_3$	1	0.02	0.02	0.19	0.23
$\theta_1 \cup \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$	0	0	0.21	0.12	0
$\theta_1 \cup \theta_2 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \theta_3$	1	0	0.36	0.08	0.44

We can still verify that the sum of $S_3(A)$ (i.e. 0.88) equals the sum of $S_1(A)$ for which $\phi(A) = 0$ and that the sum of $S_2(A)$ for which $\phi(A) = 1$ (i.e. 0.42) equals the sum of $S_3(A)$ for which $\phi(A) = 0$. After the clustering of all equivalent propositions, one gets the reduced hyper-power set $D^\Theta_{\mathcal{M}_6}$ having only 2 different elements according to:

Element A of $D^\Theta_{\mathcal{M}_6}$	$m_{\mathcal{M}_6(\Theta)}(A)$
$\emptyset$	0
θ_3	$0.17 + 0.07 + 0.09 + 0.23 + 0.44 = 1$

The encoding matrix $\mathbf{D}_{\mathcal{M}_6}$ and the basis vector $\mathbf{u}_{\mathcal{M}_6}$ for the elements of $D^\Theta_{\mathcal{M}_6}$ reduce to $\mathbf{D}_{\mathcal{M}_6} = [01]'$ and $\mathbf{u}_{\mathcal{M}_6} = [< 3 >]$.

4.5.8.7 Application of the hybrid DSm rule on example 7

Here is the numerical result corresponding to example 7 with the hybrid-model $\mathcal{M}_7$ including the *mixed exclusivity and non-existential* constraint $(\theta_1\cap\theta_2)\cup\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$. This mixed constraint implies $\theta_1\cap\theta_2\cap\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\theta_1\cap\theta_2 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\theta_1\cap\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\theta_2\cap\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $(\theta_1\cup\theta_2)\cap\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $(\theta_1\cup\theta_3)\cap\theta_2 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $(\theta_2\cup\theta_3)\cap\theta_1 \overset{\mathcal{M}_7}{\equiv} \emptyset$, $\{(\theta_1\cap\theta_2)\cup\theta_3\}\cap(\theta_1\cup\theta_2) \overset{\mathcal{M}_7}{\equiv} \emptyset$ and $\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$. By applying the hybrid DSm rule of combination, one gets:

Element A of D^Θ	$\phi(A)$	$S_1(A)$	$S_2(A)$	$S_3(A)$	$m_{\mathcal{M}_7(\Theta)}(A)$
$\emptyset$	0	0	0	0	0
$\theta_1\cap\theta_2\cap\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$	0	0.16	0	0	0
$\theta_2\cap\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$	0	0.19	0	0	0
$\theta_1\cap\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$	0	0.12	0	0	0
$(\theta_1\cup\theta_2)\cap\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$	0	0.01	0	0.02	0
$\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$	0	0.10	0.03	0.10	0
$\theta_1\cap\theta_2 \overset{\mathcal{M}_7}{\equiv} \emptyset$	0	0.22	0	0.02	0
$(\theta_1\cup\theta_3)\cap\theta_2 \overset{\mathcal{M}_7}{\equiv} \emptyset$	0	0.05	0	0.02	0
$(\theta_2\cup\theta_3)\cap\theta_1 \overset{\mathcal{M}_7}{\equiv} \emptyset$	0	0	0	0.02	0
$(\theta_1\cap\theta_2)\cup(\theta_1\cap\theta_3)\cup(\theta_2\cap\theta_3) \overset{\mathcal{M}_7}{\equiv} \emptyset$	0	0	0	0	0
$(\theta_1\cap\theta_2)\cup\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$	0	0	0	0.07	0
θ_2	1	0.03	0	0.09	0.12
$(\theta_1\cap\theta_3)\cup\theta_2 \overset{\mathcal{M}_7}{\equiv} \theta_2$	1	0	0	0.01	0.01
$\theta_2\cup\theta_3 \overset{\mathcal{M}_7}{\equiv} \theta_2$	1	0	0.06	0.05	0.11
θ_1	1	0.08	0	0.06	0.14
$(\theta_2\cap\theta_3)\cup\theta_1 \overset{\mathcal{M}_7}{\equiv} \theta_1$	1	0.02	0	0.02	0.04
$\theta_1\cup\theta_3 \overset{\mathcal{M}_7}{\equiv} \theta_1$	1	0.02	0.01	0.22	0.25
$\theta_1\cup\theta_2$	1	0	0.02	0.09	0.11
$\theta_1\cup\theta_2\cup\theta_3 \overset{\mathcal{M}_7}{\equiv} \theta_1\cup\theta_2$	1	0	0.16	0.06	0.22

After the clustering of all equivalent propositions, one gets the reduced hyper-power set $D^\Theta_{\mathcal{M}_7}$ having only 4 different elements according to:

Element A of $D^\Theta_{\mathcal{M}_7}$	$m_{\mathcal{M}_7(\Theta)}(A)$
$\emptyset$	0
θ_2	$0.12+0.01+0.11=0.24$
θ_1	$0.14+0.04+0.25=0.43$
$\theta_1\cup\theta_2$	$0.11+0.22=0.33$

The basis vector $\mathbf{u}_{\mathcal{M}_7}$ and the encoding matrix $\mathbf{D}_{\mathcal{M}_7}$ for the elements of $D^{\Theta}_{\mathcal{M}_7}$ are given by

$$\mathbf{u}_{\mathcal{M}_7} = [<1><2>]' \qquad \text{and} \qquad \mathbf{D}_{\mathcal{M}_7} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

We can still verify that the sum of $S_3(A)$ (i.e. 0.85) equals the sum of $S_1(A)$ for which $\phi(A) = 0$ and that the sum of $S_2(A)$ for which $\phi(A) = 1$ (i.e. 0.25) equals the sum of $S_3(A)$ for which $\phi(A) = 0$.

4.5.9 Example with more general basic belief assignments $m_1(.)$ and $m_2(.)$

We present in this section the numerical results of the hybrid DSm rule of combination applied upon the seven previous models $\mathcal{M}_i$, $i = 1, ..., 7$ with two general basic belief assignments $m_1(.)$ and $m_2(.)$ such that $m_1(A) > 0$ and $m_2(A) > 0$ for all $A \neq \emptyset \in D^{\Theta=\{\theta_1,\theta_2,\theta_3\}}$. We just provide here the results. The verification is left to the reader. The following table presents the numerical values chosen for $m_1(.)$ and $m_2(.)$ and the result of the fusion obtained by the classical DSm rule of combination

Element A of D^{Θ}	$m_1(A)$	$m_2(A)$	$m_{\mathcal{M}^f}(A)$
$\emptyset$	0	0	0
$\theta_1 \cap \theta_2 \cap \theta_3$	0.01	0.40	0.4389
$\theta_2 \cap \theta_3$	0.04	0.03	0.0410
$\theta_1 \cap \theta_3$	0.03	0.04	0.0497
$(\theta_1 \cup \theta_2) \cap \theta_3$	0.01	0.02	0.0257
θ_3	0.03	0.04	0.0311
$\theta_1 \cap \theta_2$	0.02	0.20	0.1846
$(\theta_1 \cup \theta_3) \cap \theta_2$	0.02	0.01	0.0156
$(\theta_2 \cup \theta_3) \cap \theta_1$	0.03	0.04	0.0459
$(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$	0.04	0.03	0.0384
$(\theta_1 \cap \theta_2) \cup \theta_3$	0.04	0.03	0.0296
θ_2	0.02	0.01	0.0084
$(\theta_1 \cap \theta_3) \cup \theta_2$	0.01	0.02	0.0221
$\theta_2 \cup \theta_3$	0.20	0.02	0.0140
θ_1	0.01	0.02	0.0109
$(\theta_2 \cap \theta_3) \cup \theta_1$	0.02	0.01	0.0090
$\theta_1 \cup \theta_3$	0.04	0.03	0.0136
$\theta_1 \cup \theta_2$	0.03	0.04	0.0175
$\theta_1 \cup \theta_2 \cup \theta_3$	0.40	0.01	0.0040

The following table shows the results obtained by the hybrid DSm rule before the final compression step of all redundant propositions for the hybrid DSm models presented in the previous examples.

Element A of D^Θ	$m_{\mathcal{M}_1}(A)$	$m_{\mathcal{M}_2}(A)$	$m_{\mathcal{M}_3}(A)$	$m_{\mathcal{M}_4}(A)$	$m_{\mathcal{M}_5}(A)$	$m_{\mathcal{M}_6}(A)$	$m_{\mathcal{M}_7}(A)$
$\emptyset$	0	0	0	0	0	0	0
$\theta_1 \cap \theta_2 \cap \theta_3$	0	0	0	0	0	0	0
$\theta_2 \cap \theta_3$	0.0573	0.0573	0	0	0.0573	0	0
$\theta_1 \cap \theta_3$	0.0621	0.0621	0.0621	0	0	0	0
$(\theta_1 \cup \theta_2) \cap \theta_3$	0.0324	0.0324	0.0335	0	0.0334	0	0
θ_3	0.0435	0.0435	0.0460	0.0494	0.0459	0.0494	0
$\theta_1 \cap \theta_2$	0.1946	0	0	0	0	0	0
$(\theta_1 \cup \theta_3) \cap \theta_2$	0.0323	0.0365	0	0	0.0365	0	0
$(\theta_2 \cup \theta_3) \cap \theta_1$	0.0651	0.0719	0.0719	0	0	0	0
$(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$	0.0607	0.0704	0.0743	0	0.0764	0	0
$(\theta_1 \cap \theta_2) \cup \theta_3$	0.0527	0.0613	0.0658	0.0792	0.0687	0.0792	0
θ_2	0.0165	0.0207	0.0221	0.0221	0.0207	0	0.0221
$(\theta_1 \cap \theta_3) \cup \theta_2$	0.0274	0.0309	0.0340	0.0375	0.0329	0	0.0375
$\theta_2 \cup \theta_3$	0.0942	0.1346	0.1471	0.1774	0.1518	0.1850	0.1953
θ_1	0.0151	0.0175	0.0175	0.0195	0	0	0.0195
$(\theta_2 \cap \theta_3) \cup \theta_1$	0.0182	0.0229	0.0243	0.0295	0.0271	0	0.0295
$\theta_1 \cup \theta_3$	0.0299	0.0385	0.0419	0.0558	0.0489	0.0589	0.0631
$\theta_1 \cup \theta_2$	0.0299	0.0412	0.0452	0.0544	0.0498	0	0.0544
$\theta_1 \cup \theta_2 \cup \theta_3$	0.1681	0.2583	0.3143	0.4752	0.3506	0.6275	0.5786

The next tables present the final results of the hybrid DSm rule of combination after the compression step (the merging of all equivalent redundant propositions) presented in previous examples.

Element A of $D^\Theta_{\mathcal{M}_7}$	$m_{\mathcal{M}_7(\Theta)}(A)$
$\emptyset$	0
θ_2	0.2549
θ_1	0.1121
$\theta_1 \cup \theta_2$	0.6330

On example no. 7

Element A of $D^\Theta_{\mathcal{M}_6}$	$m_{\mathcal{M}_6(\Theta)}(A)$
$\emptyset$	0
θ_3	1

On example no. 6

Element A of $D^\Theta_{\mathcal{M}_5}$	$m_{\mathcal{M}_5(\Theta)}(A)$
$\emptyset$	0
$\theta_2 \cap \theta_3$	0.2307
θ_3	0.1635
θ_2	0.1034
$\theta_2 \cup \theta_3$	0.5024

On example no. 5

Element A of $D^{\Theta}_{\mathcal{M}_4}$	$m_{\mathcal{M}_4(\Theta)}(A)$
$\emptyset$	0
θ_3	0.1286
θ_2	0.0596
$\theta_2 \cup \theta_3$	0.1774
θ_1	0.0490
$\theta_1 \cup \theta_3$	0.0558
$\theta_1 \cup \theta_2$	0.0544
$\theta_1 \cup \theta_2 \cup \theta_3$	0.4752

On example no 4

Element A of $D^{\Theta}_{\mathcal{M}_3}$	$m_{\mathcal{M}_3(\Theta)}(A)$
$\emptyset$	0
$\theta_1 \cap \theta_3$	0.2418
θ_3	0.1118
θ_2	0.0221
$(\theta_1 \cap \theta_3) \cup \theta_2$	0.0340
$\theta_2 \cup \theta_3$	0.1471
θ_1	0.0418
$\theta_1 \cup \theta_3$	0.0419
$\theta_1 \cup \theta_2$	0.0452
$\theta_1 \cup \theta_2 \cup \theta_3$	0.3143

On example no 3

Element A of $D^{\Theta}_{\mathcal{M}_2}$	$m_{\mathcal{M}_2(\Theta)}(A)$
$\emptyset$	0
$\theta_2 \cap \theta_3$	0.0938
$\theta_1 \cap \theta_3$	0.1340
$(\theta_1 \cup \theta_2) \cap \theta_3$	0.1028
θ_3	0.1048
θ_2	0.0207
$(\theta_1 \cap \theta_3) \cup \theta_2$	0.0309
$\theta_2 \cup \theta_3$	0.1346
θ_1	0.0175
$(\theta_2 \cap \theta_3) \cup \theta_1$	0.0229
$\theta_1 \cup \theta_3$	0.0385
$\theta_1 \cup \theta_2$	0.0412
$\theta_1 \cup \theta_2 \cup \theta_3$	0.2583

On example no 2

Element A of $D^{\Theta}_{\mathcal{M}_1}$	$m_{\mathcal{M}_1(\Theta)}(A)$
$\emptyset$	0
$\theta_2 \cap \theta_3$	0.0573
$\theta_1 \cap \theta_3$	0.0621
$(\theta_1 \cup \theta_2) \cap \theta_3$	0.0324
θ_3	0.0435
$\theta_1 \cap \theta_2$	0.1946
$(\theta_1 \cup \theta_3) \cap \theta_2$	0.0323
$(\theta_2 \cup \theta_3) \cap \theta_1$	0.0651
$(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$	0.0607
$(\theta_1 \cap \theta_2) \cup \theta_3$	0.0527
θ_2	0.0165
$(\theta_1 \cap \theta_3) \cup \theta_2$	0.0274
$\theta_2 \cup \theta_3$	0.0942
θ_1	0.0151
$(\theta_2 \cap \theta_3) \cup \theta_1$	0.0182
$\theta_1 \cup \theta_3$	0.0299
$\theta_1 \cup \theta_2$	0.0299
$\theta_1 \cup \theta_2 \cup \theta_3$	0.1681

On example no 1

4.5.10 The hybrid DSm rule versus Dempster's rule of combination

In its essence, the hybrid DSm rule of combination is close to Dubois and Prade's rule of combination (see chapter 1 and [4]) but more general and precise because it works on $D^\Theta \supset 2^\Theta$ and allows us to include all possible exclusivity and non-existential constraints for the model one has to work with. The advantage of using the hybrid DSm rule is that it does not require the calculation of weighting factors, nor a normalization. The hybrid DSm rule of combination is definitely not equivalent to Dempster's rule of combination as one can easily prove in the following very simple example:

Let's consider $\Theta = \{\theta_1, \theta_2\}$ and the two sources in full contradiction providing the following basic belief assignments

$$m_1(\theta_1) = 1 \qquad m_1(\theta_2) = 0$$

$$m_2(\theta_1) = 0 \qquad m_2(\theta_2) = 1$$

Using the classic DSm rule of combination working with the free DSm model $\mathcal{M}^f$, one gets

$$m_{\mathcal{M}^f}(\theta_1) = 0 \qquad m_{\mathcal{M}^f}(\theta_2) = 0 \qquad m_{\mathcal{M}^f}(\theta_1 \cap \theta_2) = 1 \qquad m_{\mathcal{M}^f}(\theta_1 \cup \theta_2) = 0$$

If one forces θ_1 and θ_2 to be exclusive to work with Shafer's model $\mathcal{M}^0$, then the Dempster's rule of combination can not be applied in this limit case because of the full contradiction of the two sources of information. One gets the undefined operation $0/0$. But the hybrid DSm rule can be applied in such limit case because it transfers the mass of this empty set ($\theta_1 \cap \theta_2 \equiv \emptyset$ because of the choice of the model $\mathcal{M}^0$) to non-empty set(s), and one gets:

$$m_{\mathcal{M}^0}(\theta_1) = 0 \qquad m_{\mathcal{M}^0}(\theta_2) = 0 \qquad m_{\mathcal{M}^0}(\theta_1 \cap \theta_2) = 0 \qquad m_{\mathcal{M}^0}(\theta_1 \cup \theta_2) = 1$$

This result is coherent in this very simple case with Yager's and Dubois-Prade's rule of combination [11, 4].

Now let examine the behavior of the numerical result when introducing a small variation $\epsilon > 0$ on initial basic belief assignments $m_1(.)$ and $m_2(.)$ as follows:

$$m_1(\theta_1) = 1 - \epsilon \qquad m_1(\theta_2) = \epsilon \qquad \text{and} \qquad m_2(\theta_1) = \epsilon \qquad m_2(\theta_2) = 1 - \epsilon$$

As shown in figure 4.2, $\lim_{\epsilon \to 0} m_{DS}(.)$, where $m_{DS}(.)$ is the result obtained from the Dempster's rule of combination, is given by

$$m_{DS}(\theta_1) = 0.5 \qquad m_{DS}(\theta_2) = 0.5 \qquad m_{DS}(\theta_1 \cap \theta_2) = 0 \qquad m_{DS}(\theta_1 \cup \theta_2) = 0$$

This result is very questionable because it assigns same belief on θ_1 and θ_2 which is more informational than to assign all the belief to the total ignorance. The assignment of the belief to the total ignorance appears to be more justified from our point of view because it properly reflects the almost total contradiction between the two sources and in such cases, it seems legitimate that the information can be drawn from the fusion. When we apply the hybrid DSm rule of combination (using Shafer's model $\mathcal{M}^0$), one gets the expected belief assignment on the total ignorance, i.e. $m_{\mathcal{M}^0}(\theta_1 \cup \theta_2) = 1$. The figure below shows the evolution of belief assignments on θ_1, θ_2 and $\theta_1 \cup \theta_2$ with ϵ obtained with the classical Dempster rule and the hybrid DSm rule based on Shafer's model $\mathcal{M}^0$ (i.e. $\theta_1 \cap \theta_2 \overset{\mathcal{M}_0}{\equiv} \emptyset$).

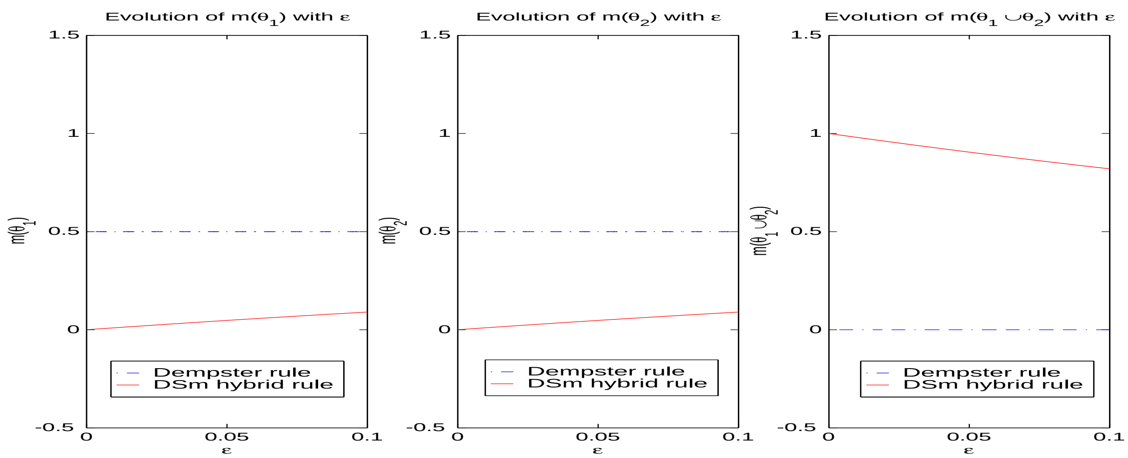

Figure 4.2: Comparison of Dempster's rule with the hybrid DSm rule on $\Theta = \{\theta_1, \theta_2\}$

4.6 Dynamic fusion

The hybrid DSm rule of combination presented in this paper has been developed for *static* problems, but is also directly applicable for easily handling *dynamic fusion* problems in real time as well, since at each temporal change of the models, one can still apply such a hybrid rule. If D^Θ changes, due to the dynamicity of the frame Θ, from time t_l to time t_{l+1}, i.e. some of its elements which at time t_l were not empty become (or are proven) empty at time t_{l+1}, or vice versa: if new elements, empty at time t_l, arise non-empty at time t_{l+1}, this hybrid DSm rule can be applied again at each change. If Θ stays the same but its set non-empty elements of D^Θ increases, then again apply the hybrid DSm rule.

4.6.1 Example 1

Let's consider the testimony fusion problem[6] with the frame

$$\Theta(t_l) \triangleq \{\theta_1 \equiv \text{young}, \theta_2 \equiv \text{old}, \theta_3 \equiv \text{white hairs}\}$$

with the following two basic belief assignments

$$m_1(\theta_1) = 0.5 \qquad m_1(\theta_3) = 0.5 \qquad \text{and} \qquad m_2(\theta_2) = 0.5 \qquad m_2(\theta_3) = 0.5$$

[6]This problem has been proposed to the authors in a private communication by L. Cholvy in 2002.

By applying the classical DSm fusion rule, one then gets

$$m_{\mathcal{M}^f(\Theta(t_l))}(\theta_1 \cap \theta_2) = 0.25 \qquad m_{\mathcal{M}^f(\Theta(t_l))}(\theta_1 \cap \theta_3) = 0.25$$

$$m_{\mathcal{M}^f(\Theta(t_l))}(\theta_2 \cap \theta_3) = 0.25 \qquad m_{\mathcal{M}^f(\Theta(t_l))}(\theta_3) = 0.25$$

Suppose now that at time t_{l+1}, one knows that young people don't have white hairs (i.e $\theta_1 \cap \theta_3 \equiv \emptyset$). How can we update the previous fusion result with this new information on the model of the problem? We solve it with the hybrid DSm rule, which transfers the mass of the empty sets (imposed by the constraints on the new model $\mathcal{M}$ available at time t_{l+1}) to the non-empty sets of D^Θ, going on the track of the DSm classic rule. Using the hybrid DSm rule with the constraint $\theta_1 \cap \theta_3 \equiv \emptyset$, one then gets:

$$m_{\mathcal{M}}(\theta_1 \cap \theta_2) = 0.25 \qquad m_{\mathcal{M}}(\theta_2 \cap \theta_3) = 0.25 \qquad m_{\mathcal{M}}(\theta_3) = 0.25$$

and the mass $m_{\mathcal{M}}(\theta_1 \cap \theta_3) = 0$, because $\theta_1 \cap \theta_3 = \{\text{young}\} \cap \{\text{white hairs}\} \overset{\mathcal{M}}{\equiv} \emptyset$ and its previous mass $m_{\mathcal{M}^f(\Theta(t_l))}(\theta_1 \cap \theta_3) = 0.25$ is transferred to $m_{\mathcal{M}}(\theta_1 \cup \theta_3) = 0.25$ by the hybrid DSm rule.

4.6.2 Example 2

Let $\Theta(t_l) = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be a list of suspects and let's consider two observers who eyewitness the scene of plunder at a museum in Bagdad and who testify to the radio and TV the identities of thieves using the basic beliefs assignments $m_1(.)$ and $m_2(.)$ defined on $D^{\Theta(t_l)}$, where t_l represents the time of the observation. Afterwards, at time t_{l+1}, one finds out that one suspect, among this list $\Theta(t_l)$, say θ_i, could not be a suspect because he was on duty in another place, evidence which was certainly confirmed. Therefore he has to be taken off the suspect list $\Theta(t_l)$, and a new frame of discernment results in $\Theta(t_{l+1})$. If this one changes again, one applies again the hybrid DSm of combining of evidences, and so on. This is a typically dynamical example where models change with time and where one needs to adapt fusion results with the current model over time. In the meantime, one can also take into account new observations/testimonies in the hybrid DSm fusion rule as soon as they become available to the fusion system.

If Θ (and therefore D^Θ) diminish (i.e. some of their elements are proven to be empty sets) from time t_l to time t_{l+1}, then one applies the hybrid DSm rule in order to transfer the masses of empty sets to the non-empty sets (in the way of the DSm classic rule) getting an updated basic belief assignment $m_{t_{l+1}|t_l}(.)$. Contrarily, if Θ and D^Θ increase (i.e. new elements arise in Θ, and/or new elements in D^Θ are proven different from the empty set and as a consequence a basic belief assignment for them is required), then new masses (from the same or from the other sources of information) are needed to describe these new elements, and again one combines them using the hybrid DSm rule.

4.6.3 Example 3

Let's consider a fusion problem at time t_l characterized by the frame $\Theta(t_l) \triangleq \{\theta_1, \theta_2\}$ and two independent sources of information providing the basic belief assignments $m_1(.)$ and $m_2(.)$ over $D^{\Theta(t_l)}$ and assume that at time t_{l+1} a new hypothesis θ_3 is introduced into the previous frame $\Theta(t_l)$ and a third source of evidence available at time t_{l+1} provides its own basic belief assignment $m_3(.)$ over $D^{\Theta(t_{l+1})}$ where

$$\Theta(t_{l+1}) \triangleq \{\Theta(t_l), \theta_3\} \equiv \{\theta_1, \theta_2, \theta_3\}$$

To solve such kind of dynamical fusion problems, we just use the classical DSm fusion rule as follows:

- combine $m_1(.)$ and $m_2(.)$ at time t_l using classical DSm fusion rule to get $m_{12}(.) = [m_1 \oplus m_2](.)$ over $D^{\Theta(t_l)}$
- because $D^{\Theta(t_l)} \subset D^{\Theta(t_{l+1})}$, $m_{12}(.)$ assigns the combined basic belief on a subset of $D^{\Theta(t_{l+1})}$, it is still directly possible to combine $m_{12}(.)$ with $m_3(.)$ at time t_{l+1} by the classical DSm fusion rule to get the final result $m_{123}(.)$ over $D^{\Theta(t_{l+1})}$ given by

$$m_{t_{l+1}}(.) \triangleq m_{123}(.) = [m_{12} \oplus m_3](.) = [(m_1 \oplus m_2) \oplus m_3](.) \equiv [m_1 \oplus m_2 \oplus m_3](.)$$

- eventually apply hybrid DSm rule if some integrity constraints have to be taken into account in the model $\mathcal{M}$ of the problem

This method can be directly generalized to any number of sources of evidences and, in theory, to any structure/dimension of the frames $\Theta(t_l)$, $\Theta(t_{l+1})$, ... In practice however, due to the huge number of elements of hyper-power sets, the dimension of the frames $\Theta(t_l)$, $\Theta(t_{l+1})$, $\ldots$ must be not too large. This practical limitation depends on the computer resources available for the real-time processing. Specific suboptimal implementations of DSm rule will have to be developed to deal with fusion problems of large dimension.

It is also important to point out here that DSmT can easily deal, not only with dynamical fusion problems but with decentralized fusion problems as well working on non exhaustive frames. For example, let's consider a set of two independent sources of information providing the basic belief assignments $m_1(.)$ and $m_2(.)$ over $D^{\Theta_{12}(t_l)=\{\theta_1,\theta_2\}}$ and another group of three independent sources of information providing the basic belief assignments $m_3(.)$, $m_4(.)$ and $m_5(.)$ over $D^{\Theta_{345}(t_l)=\{\theta_3,\theta_4,\theta_5,\theta_6\}}$, then it is still possible to combine all information in a decentralized manner as follows:

- combine $m_1(.)$ and $m_2(.)$ at time t_l using classical DSm fusion rule to get $m_{12}(.) = [m_1 \oplus m_2](.)$ over $D^{\Theta_{12}(t_l)}$.
- combine $m_3(.)$, $m_4(.)$ and $m_5(.)$ at time t_l using classical DSm fusion rule to get $m_{345}(.) = [m_3 \oplus m_4 \oplus m_5](.)$ over $D^{\Theta_{345}(t_l)}$.

- consider now the global frame $\Theta(t_l) \triangleq \{\Theta_{12}(t_l), \Theta_{345}(t_l)\}$.
- eventually apply hybrid DSm rule if some integrity constraints have to be taken into account in the model $\mathcal{M}$ of the problem.

Note that this *static* decentralized fusion can also be extended to decentralized dynamical fusion also by mixing the two previous approaches.

One can even combine all five masses together by extending the vectors $m_i(.)$, $1 \leq i \leq 5$, with null components for the new elements arisen from enlarging Θ to $\{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ and correspondingly enlarging D^Θ, and using the hybrid DSm rule for $k = 5$. And more general combine the masses of any $k \geq 2$ sources.

We give now several simple numerical examples for such dynamical fusion problems involving non exclusive frames.

4.6.3.1 Example 3.1

Let's consider $\Theta(t_l) \triangleq \{\theta_1, \theta_2\}$ and the two following basic belief assignments available at time t_l:

$$m_1(\theta_1) = 0.1 \qquad m_1(\theta_2) = 0.2 \qquad m_1(\theta_1 \cup \theta_2) = 0.3 \qquad m_1(\theta_1 \cap \theta_2) = 0.4$$

$$m_2(\theta_1) = 0.5 \qquad m_2(\theta_2) = 0.3 \qquad m_2(\theta_1 \cup \theta_2) = 0.1 \qquad m_2(\theta_1 \cap \theta_2) = 0.1$$

The classical DSm rule of combination gives

$$m_{12}(\theta_1) = 0.21 \qquad m_{12}(\theta_2) = 0.17 \qquad m_{12}(\theta_1 \cup \theta_2) = 0.03 \qquad m_{12}(\theta_1 \cap \theta_2) = 0.59$$

Now let's consider at time t_{l+1} the frame $\Theta(t_{l+1}) \triangleq \{\theta_1, \theta_2, \theta_3\}$ and a third source of evidence with the following basic belief assignment

$$m_3(\theta_3) = 0.4 \qquad m_3(\theta_1 \cap \theta_3) = 0.3 \qquad m_3(\theta_2 \cup \theta_3) = 0.3$$

Then the final result of the fusion is obtained by combining $m_3(.)$ with $m_{12}(.)$ by the classical DSm rule of combination. One thus obtains:

$$m_{123}(\theta_1 \cap \theta_2 \cap \theta_3) = 0.464 \quad m_{123}(\theta_2 \cap \theta_3) = 0.068 \quad m_{123}(\theta_1 \cap \theta_3) = 0.156 \quad m_{123}((\theta_1 \cup \theta_2) \cap \theta_3) = 0.012$$

$$m_{123}(\theta_1 \cap \theta_2) = 0.177 \quad m_{123}(\theta_1 \cap (\theta_2 \cup \theta_3)) = 0.063 \quad m_{123}(\theta_2) = 0.051 \quad m_{123}((\theta_1 \cap \theta_3) \cup \theta_2) = 0.009$$

4.6.3.2 Example 3.2

Let's consider $\Theta(t_l) \triangleq \{\theta_1, \theta_2\}$ and the two previous following basic belief assignments $m_1(.)$ and $m_2(.)$ available at time t_l. The classical DSm fusion rule gives gives as before

$$m_{12}(\theta_1) = 0.21 \qquad m_{12}(\theta_2) = 0.17 \qquad m_{12}(\theta_1 \cup \theta_2) = 0.03 \qquad m_{12}(\theta_1 \cap \theta_2) = 0.59$$

Now let's consider at time t_{l+1} the frame $\Theta(t_{l+1}) \triangleq \{\theta_1, \theta_2, \theta_3\}$ and the third source of evidence as in previous example with the basic belief assignment

$$m_3(\theta_3) = 0.4 \qquad m_3(\theta_1 \cap \theta_3) = 0.3 \qquad m_3(\theta_2 \cup \theta_3) = 0.3$$

The final result of the fusion obtained by the classical DSm rule of combination corresponds to the result of the previous example, but suppose now one finds out that the integrity constraint $\theta_3 = \emptyset$ holds, which implies also constraints $\theta_1 \cap \theta_2 \cap \theta_3 = \emptyset$, $\theta_1 \cap \theta_3 = \emptyset$, $\theta_2 \cap \theta_3 = \emptyset$ and $(\theta_1 \cup \theta_2) \cap \theta_3 = \emptyset$. This is the hybrid DSm model $\mathcal{M}$ under consideration here. We then have to readjust the mass $m_{123}(.)$ of the previous example by the hybrid DSm rule and one finally gets

$$m_{\mathcal{M}}(\theta_1) = 0.147 \qquad m_{\mathcal{M}}(\theta_2) = 0.060 + 0.119 = 0.179$$

$$m_{\mathcal{M}}(\theta_1 \cup \theta_2) = 0 + 0 + 0.021 = 0.021 \qquad m_{\mathcal{M}}(\theta_1 \cap \theta_2) = 0.240 + 0.413 = 0.653$$

Therefore, when we restrain back $\theta_3 = \emptyset$ and apply the hybrid DSm rule, we don't get back the same result (i.e. $m_{\mathcal{M}}(.) \neq m_{12}(.)$) because still remains some information from $m_3(.)$ on θ_1, θ_2, $\theta_1 \cup \theta_2$, or $\theta_1 \cap \theta_2$, i.e. $m_3(\theta_2) = 0.3 > 0$.

4.6.3.3 Example 3.3

Let's consider $\Theta(t_l) \triangleq \{\theta_1, \theta_2\}$ and two previous following basic belief assignments $m_1(.)$ and $m_2(.)$ available at time t_l. The classical DSm fusion rule gives as before

$$m_{12}(\theta_1) = 0.21 \qquad m_{12}(\theta_2) = 0.17 \qquad m_{12}(\theta_1 \cup \theta_2) = 0.03 \qquad m_{12}(\theta_1 \cap \theta_2) = 0.59$$

Now let's consider at time t_{l+1} the frame $\Theta(t_{l+1}) \triangleq \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and another third source of evidence with the following basic belief assignment

$$m_3(\theta_3) = 0.5 \qquad m_3(\theta_4) = 0.3 \qquad m_3(\theta_3 \cap \theta_4) = 0.1 \qquad m_3(\theta_3 \cup \theta_4) = 0.1$$

Then, the DSm rule applied at time t_{l+1} provides the following combined belief assignment

$$m_{123}(\theta_1 \cap \theta_3) = 0.105 \quad m_{123}(\theta_1 \cap \theta_4) = 0.063 \quad m_{123}(\theta_1 \cap (\theta_3 \cup \theta_4)) = 0.021 \quad m_{123}(\theta_1 \cap \theta_3 \cap \theta_4) = 0.021$$

$$m_{123}(\theta_2 \cap \theta_3) = 0.085 \quad m_{123}(\theta_2 \cap \theta_4) = 0.051 \quad m_{123}(\theta_2 \cap (\theta_3 \cup \theta_4)) = 0.017 \qquad m_{123}(\theta_2 \cap \theta_3 \cap \theta_4) = 0.017$$

$$m_{123}(\theta_3 \cap (\theta_1 \cup \theta_2)) = 0.015 \quad m_{123}(\theta_4 \cap (\theta_1 \cup \theta_2)) = 0.009 \quad m_{123}((\theta_1 \cup \theta_2) \cap (\theta_3 \cup \theta_4)) = 0.003$$

$$m_{123}((\theta_1 \cup \theta_2) \cap (\theta_3 \cap \theta_4)) = 0.003 \quad m_{123}(\theta_1 \cap \theta_2 \cap \theta_3) = 0.295 \quad m_{123}(\theta_1 \cap \theta_2 \cap \theta_4) = 0.177$$

$$m_{123}((\theta_1 \cap \theta_2) \cap (\theta_3 \cup \theta_4)) = 0.059 \quad m_{123}(\theta_1 \cap \theta_2 \cap \theta_3 \cap \theta_4) = 0.059$$

Now suppose at time t_{l+2} one finds out that $\theta_3 = \theta_4 = \emptyset$, then one applies the hybrid DSm rule after re-adjusting the combined belief mass $m_{123}(.)$ by cumulating the masses of all empty sets. Using the hybrid DSm rule, one finally gets:

$$m_{t_{l+2}}(\theta_1) = m_{123}(\theta_1) + \{m_{12}(\theta_1)m_3(\theta_3) + m_{12}(\theta_1)m_3(\theta_4) + m_{12}(\theta_1)m_3(\theta_3 \cup \theta_4) + m_{12}(\theta_1)m_3(\theta_3 \cap \theta_4)\}$$
$$= 0 + \{(0.21 \times 0.5) + (0.21 \times 0.3) + (0.21 \times 0.1) + (0.21 \times 0.1)\} = 0.21$$

$$m_{t_{l+2}}(\theta_2) = m_{123}(\theta_2) + \{m_{12}(\theta_2)m_3(\theta_3) + m_{12}(\theta_2)m_3(\theta_4) + m_{12}(\theta_2)m_3(\theta_3 \cup \theta_4) + m_{12}(\theta_2)m_3(\theta_3 \cap \theta_4)\}$$
$$= 0 + \{(0.17 \times 0.5) + (0.17 \times 0.3) + (0.17 \times 0.1) + (0.17 \times 0.1)\} = 0.17$$

$$\begin{aligned} m_{t_{l+2}}(\theta_1 \cup \theta_2) = m_{123}(\theta_1 \cup \theta_2) &+ \{m_{12}(\theta_1 \cup \theta_2)m_3(\theta_3) + m_{12}(\theta_1 \cup \theta_2)m_3(\theta_4) \\ &+ m_{12}(\theta_1 \cup \theta_2)m_3(\theta_3 \cup \theta_4) + m_{12}(\theta_1 \cup \theta_2)m_3(\theta_3 \cap \theta_4)\} \\ &+ \sum_{X_1, X_2 \in \{\theta_3, \theta_4, \theta_3 \cup \theta_4, \theta_3 \cap \theta_4\}} m_{12}(X_1)m_3(X_2) \\ &= 0 + \{(0.03 \times 0.5) + (0.03 \times 0.3) + (0.03 \times 0.1) + (0.03 \times 0.1)\} + \{0\} = 0.03 \end{aligned}$$

$$\begin{aligned} m_{t_{l+2}}(\theta_1 \cap \theta_2) = m_{123}(\theta_1 \cap \theta_2) &+ \{m_{12}(\theta_1 \cap \theta_2)m_3(\theta_3) + m_{12}(\theta_1 \cap \theta_2)m_3(\theta_4) \\ &+ m_{12}(\theta_1 \cap \theta_2)m_3(\theta_3 \cup \theta_4) + m_{12}(\theta_1 \cap \theta_2)m_3(\theta_3 \cap \theta_4)\} \\ &= 0 + \{(0.59 \times 0.5) + (0.59 \times 0.3) + (0.59 \times 0.1) + (0.59 \times 0.1)\} = 0.59 \end{aligned}$$

Thus we get the same result as for $m_{12}(.)$ at time t_l, which is normal.

Remark: note that if the third source of information doesn't assign non-null masses to θ_1, or θ_2 (or to their combinations using $\cup$ or $\cap$ operators), then one obtains the same result at time t_{l+2} as at time t_l as in this example 3.3, i.e. $m_{l+2}(.) = m_l(.)$, when imposing back $\theta_3 = \theta_4 = \emptyset$. But, if the third source of information assigns non-null masses to either θ_1, or θ_2, or to some of their combinations $\theta_1 \cup \theta_2$ or $\theta_1 \cap \theta_2$, then when one returns from 4 singletons to 2 singletons for Θ, replacing $\theta_3 = \theta_4 = \emptyset$ and using the hybrid DSm rule, the fusion results at time t_{l+2} is different from that at time t_l, and this is normal because some information/mass is left from the third source and is now fusioned with that of the previous sources (as in example 3.2 or in the next example 3.4).

In general, let's suppose that the fusion of $k \geq 2$ masses provided by the sources $\mathcal{B}_1$, $\mathcal{B}_2$, ..., $\mathcal{B}_k$ has been done at time t_l on $\Theta(t_l) = \{\theta_1, \theta_2, ..., \theta_n\}$. At time t_{l+1} new non-empty elements θ_{n+1}, θ_{n+2}, $\ldots$, θ_{n+m} appear, $m \geq 1$, thus $\Theta(t_{l+1}) = \{\theta_1, \theta_2, ..., \theta_n, \theta_{n+1}, \theta_{n+2}, \ldots, \theta_{n+m}\}$ and of course one or more sources (i.e. bodies of evidences) $\mathcal{B}_{k+1}$, $\ldots$, $\mathcal{B}_{k+l}$, where $l \geq 1$, appear to assign masses to these new elements.

a) If all these new sources $\mathcal{B}_{k+1}, \ldots, \mathcal{B}_{k+l}$ assign null masses to all elements from $D^{\Theta(t_{l+1})}$ which contain in their structure/composition at least one of the singletons $\theta_1, \theta_2, \ldots, \theta_n$, then at time t_{l+2} if one sets back the constraints that $\theta_{n+1} = \theta_{n+2} = \ldots = \theta_{n+m} = \emptyset$, then using the hybrid DSm rule, one obtains the same result as at time t_l, i.e. $m_{l+2}(.) = m_l(.)$.

b) Otherwise, the fusion at time t_{l+2} will be different from the fusion at time t_l because there still remains some information/mass from sources $\mathcal{B}_{k+1}, \ldots, \mathcal{B}_{k+l}$ on singletons $\theta_1, \theta_2, \ldots, \theta_n$ or on some elements from $D^{\Theta(t_l)}$ which contain at least one such singleton, information/mass which fusions with the previous sources.

4.6.3.4 Example 3.4

Let's consider $\Theta(t_l) \triangleq \{\theta_1, \theta_2\}$ and the two following basic belief assignments available at time t_l:

$$m_1(\theta_1) = 0.6 \qquad m_1(\theta_2) = 0.4 \qquad \text{and} \qquad m_2(\theta_1) = 0.7 \qquad m_2(\theta_2) = 0.3$$

The classical DSm rule of combination gives $m_{12}(\theta_1) = 0.42$, $m_{12}(\theta_2) = 0.12$ and $m_{12}(\theta_1 \cap \theta_2) = 0.46$. Now let's consider at time t_{l+1} the frame $\Theta(t_{l+1}) \triangleq \{\theta_1, \theta_2, \theta_3\}$ and a third source of evidence with the following basic belief assignment $m_3(\theta_1) = 0.5$, $m_3(\theta_2) = 0.2$ and $m_3(\theta_3) = 0.3$. Then the final result obtained from the classical DSm rule of combination is still as before

$$m_{123}(\theta_1) = 0.210 \quad m_{123}(\theta_2) = 0.024 \quad m_{123}(\theta_1 \cap \theta_2) = 0.466 \quad m_{123}(\theta_1 \cap \theta_3) = 0.126$$
$$m_{123}(\theta_2 \cap \theta_3) = 0.036 \quad m_{123}(\theta_1 \cap \theta_2 \cap \theta_3) = 0.138$$

Suppose now one finds out that the integrity constraint $\theta_1 \cap \theta_3 = \emptyset$ which also implies $\theta_1 \cap \theta_2 \cap \theta_3 = \emptyset$. This is the hybrid DSm model $\mathcal{M}$ under consideration. By applying the hybrid DSm fusion rule, one forces $m_{\mathcal{M}}(\theta_1 \cap \theta_3) = 0$ and $m_{\mathcal{M}}(\theta_1 \cap \theta_2 \cap \theta_3) = 0$ and we transfer $m_{123}(\theta_1 \cap \theta_2 \cap \theta_3) = 0.138$ towards $m_{\mathcal{M}}((\theta_1 \cap \theta_2) \cup \theta_3)$ and the mass $m_{123}(\theta_1 \cap \theta_3) = 0.126$ has to be transferred towards $m_{\mathcal{M}}(\theta_1 \cup \theta_3)$. One then gets finally

$$m_{\mathcal{M}}(\theta_1) = 0.210 \quad m_{\mathcal{M}}(\theta_2) = 0.024 \quad m_{\mathcal{M}}(\theta_1 \cap \theta_2) = 0.466 \quad m_{\mathcal{M}}(\theta_2 \cap \theta_3) = 0.036$$
$$m_{\mathcal{M}}((\theta_1 \cap \theta_2) \cup \theta_3) = 0.138 \quad m_{\mathcal{M}}(\theta_1 \cup \theta_3) = 0.126$$

4.6.3.5 Example 3.5

Let's consider $\Theta(t_l) \triangleq \{\theta_1, \theta_2\}$ and the two previous basic belief assignments available at time t_l as in previous example, i.e.

$$m_1(\theta_1) = 0.6 \qquad m_1(\theta_2) = 0.4 \qquad \text{and} \qquad m_2(\theta_1) = 0.7 \qquad m_2(\theta_2) = 0.3$$

The classical DSm rule of combination gives

$$m_{12}(\theta_1) = 0.42 \qquad m_{12}(\theta_2) = 0.12 \quad m_{12}(\theta_1 \cap \theta_2) = 0.46$$

Now let's consider at time t_{l+1} the frame $\Theta(t_{l+1}) \triangleq \{\theta_1, \theta_2, \theta_3\}$ and a third source of evidence with the following basic belief assignment

$$m_3(\theta_1) = 0.5 \qquad m_3(\theta_2) = 0.2 \qquad m_3(\theta_3) = 0.3$$

Then the final result of the fusion is obtained by combining $m_3(.)$ with $m_{12}(.)$ by the classical DSm rule of combination. One thus obtains now

$$m_{123}(\theta_1) = 0.210 \quad m_{123}(\theta_2) = 0.024 \quad m_{123}(\theta_1 \cap \theta_2) = 0.466 \quad m_{123}(\theta_1 \cap \theta_3) = 0.126$$
$$m_{123}(\theta_2 \cap \theta_3) = 0.036 \quad m_{123}(\theta_1 \cap \theta_2 \cap \theta_3) = 0.138$$

But suppose one finds out that the integrity constraint is now $\theta_3 = \emptyset$ which implies necessarily also $\theta_1 \cap \theta_3 = \theta_2 \cap \theta_3 = \theta_1 \cap \theta_2 \cap \theta_3 \equiv \emptyset$ and $(\theta_1 \cup \theta_2) \cap \theta_3 = \emptyset$ (this is our new hybrid DSm model $\mathcal{M}$ under consideration in this example). By applying the hybrid DSm fusion rule, one gets finally the non-null masses

$$m_{\mathcal{M}}(\theta_1) = 0.336 \qquad m_{\mathcal{M}}(\theta_2) = 0.060 \qquad m_{\mathcal{M}}(\theta_1 \cap \theta_2) = 0.604$$

4.6.3.6 Example 3.6

Let's consider $\Theta(t_l) \triangleq \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and the following basic belief assignments available at time t_l :

$$\begin{cases} m_1(\theta_1) = 0.5 \qquad m_1(\theta_2) = 0.4 \qquad m_1(\theta_1 \cap \theta_2) = 0.1 \\ m_2(\theta_1) = 0.3 \qquad m_2(\theta_2) = 0.2 \qquad m_2(\theta_1 \cap \theta_3) = 0.1 \qquad m_2(\theta_4) = 0.4 \end{cases}$$

The classical DSm rule of combination gives

$$m_{12}(\theta_1) = 0.15 \qquad m_{12}(\theta_2) = 0.08 \qquad m_{12}(\theta_1 \cap \theta_2) = 0.27 \qquad m_{12}(\theta_1 \cap \theta_3) = 0.05 \qquad m_{12}(\theta_1 \cap \theta_4) = 0.20$$
$$m_{12}(\theta_2 \cap \theta_4) = 0.16 \qquad m_{12}(\theta_1 \cap \theta_2 \cap \theta_3) = 0.05 \qquad m_{12}(\theta_1 \cap \theta_2 \cap \theta_4) = 0.04$$

Now assume that at time t_{l+1} one finds out that $\theta_1 \cap \theta_2 \overset{\mathcal{M}}{\equiv} \theta_1 \cap \theta_3 \overset{\mathcal{M}}{\equiv} \emptyset$. Using the hybrid DSm rule, one gets:

$$\begin{cases} m_{\mathcal{M}}(\theta_1 \cap \theta_2) = m_{\mathcal{M}}(\theta_1 \cap \theta_3) = m_{\mathcal{M}}(\theta_1 \cap \theta_2 \cap \theta_3) = m_{\mathcal{M}}(\theta_1 \cap \theta_2 \cap \theta_4) = 0 \\ m_{\mathcal{M}}(\theta_1) = m_{12}(\theta_1) + m_2(\theta_1)m_1(\theta_1 \cap \theta_2) + m_1(\theta_1)m_2(\theta_1 \cap \theta_3) = 0.15 + 0.03 + 0.05 = 0.23 \\ m_{\mathcal{M}}(\theta_2) = m_{12}(\theta_2) + m_2(\theta_2)m_1(\theta_1 \cap \theta_2) + m_1(\theta_2)m_2(\theta_1 \cap \theta_3) = 0.08 + 0.02 + 0.04 = 0.14 \\ m_{\mathcal{M}}(\theta_4) = m_{12}(\theta_4) + m_1(\theta_1 \cap \theta_2)m_2(\theta_4) = 0 + 0.04 = 0.04 \\ m_{\mathcal{M}}(\theta_1 \cap \theta_4) = m_{12}(\theta_1 \cap \theta_4) = 0.20 \\ m_{\mathcal{M}}(\theta_2 \cap \theta_4) = m_{12}(\theta_2 \cap \theta_4) = 0.16 \\ m_{\mathcal{M}}(\theta_1 \cup \theta_2) = m_{12}(\theta_1 \cup \theta_2) + m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2) + m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cap \theta_2) = 0.22 \\ m_{\mathcal{M}}(\theta_1 \cup \theta_2 \cup \theta_3) = m_{12}(\theta_1 \cup \theta_2 \cup \theta_3) + m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cap \theta_3) + m_2(\theta_1 \cap \theta_2)m_1(\theta_1 \cap \theta_3) \\ \qquad\qquad +m_1(\theta_1 \cap \theta_2 \cap \theta_3)m_2(\theta_1 \cap \theta_2 \cap \theta_3) = 0.01 \end{cases}$$

4.6.3.7 Example 3.7

Let's consider $\Theta(t_l) \triangleq \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and the following basic belief assignments available at time t_l :

$$\begin{cases} m_1(\theta_1) = 0.2 \quad m_1(\theta_2) = 0.4 \quad m_1(\theta_1 \cap \theta_2) = 0.1 \quad m_1(\theta_1 \cap \theta_3) = 0.2 \quad m_1(\theta_4) = 0.1 \\ m_2(\theta_1) = 0.1 \quad m_2(\theta_2) = 0.3 \quad m_2(\theta_1 \cap \theta_2) = 0.2 \quad m_2(\theta_1 \cap \theta_3) = 0.1 \quad m_2(\theta_4) = 0.3 \end{cases}$$

The classical DSm rule of combination gives

$$m_{12}(\theta_1) = 0.02 \quad m_{12}(\theta_2) = 0.12 \quad m_{12}(\theta_1 \cap \theta_2) = 0.28 \quad m_{12}(\theta_1 \cap \theta_3) = 0.06 \quad m_{12}(\theta_4) = 0.03$$

$$m_{12}(\theta_1 \cap \theta_4) = 0.07 \quad m_{12}(\theta_2 \cap \theta_4) = 0.15 \quad m_{12}(\theta_1 \cap \theta_2 \cap \theta_3) = 0.15$$

$$m_{12}(\theta_1 \cap \theta_2 \cap \theta_4) = 0.05 \quad m_{12}(\theta_1 \cap \theta_3 \cap \theta_4) = 0.07$$

Now assume that at time t_{l+1} one finds out that $\theta_1 \cap \theta_2 \overset{\mathcal{M}}{\equiv} \theta_1 \cap \theta_3 \overset{\mathcal{M}}{\equiv} \emptyset$. Using the hybrid DSm rule, one gets:

$$\begin{cases} m_{\mathcal{M}}(\theta_1 \cap \theta_2) = m_{\mathcal{M}}(\theta_1 \cap \theta_3) = m_{\mathcal{M}}(\theta_1 \cap \theta_2 \cap \theta_3) = m_{\mathcal{M}}(\theta_1 \cap \theta_2 \cap \theta_4) = 0 \\ m_{\mathcal{M}}(\theta_1) = m_{12}(\theta_1) + m_1(\theta_1)m_2(\theta_1 \cap \theta_2) + m_2(\theta_1)m_1(\theta_1 \cap \theta_2) + m_1(\theta_1)m_2(\theta_1 \cap \theta_3) \\ \qquad\qquad + m_2(\theta_1)m_1(\theta_1 \cap \theta_3) = 0.11 \\ m_{\mathcal{M}}(\theta_2) = m_{12}(\theta_2) + m_1(\theta_2)m_2(\theta_1 \cap \theta_2) + m_2(\theta_2)m_1(\theta_1 \cap \theta_2) + m_1(\theta_2)m_2(\theta_1 \cap \theta_3) \\ \qquad\qquad + m_2(\theta_2)m_1(\theta_1 \cap \theta_3) = 0.33 \\ m_{\mathcal{M}}(\theta_4) = m_{12}(\theta_4) + m_1(\theta_4)m_2(\theta_1 \cap \theta_2) + m_2(\theta_4)m_1(\theta_1 \cap \theta_2) + m_1(\theta_4)m_2(\theta_1 \cap \theta_3) \\ \qquad\qquad + m_2(\theta_4)m_1(\theta_1 \cap \theta_3) = 0.15 \\ m_{\mathcal{M}}(\theta_1 \cap \theta_4) = m_{12}(\theta_1 \cap \theta_4) = 0.07 \\ m_{\mathcal{M}}(\theta_2 \cap \theta_4) = m_{12}(\theta_2 \cap \theta_4) = 0.15 \\ m_{\mathcal{M}}(\theta_1 \cup \theta_2) = m_{12}(\theta_1 \cup \theta_2) + m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cap \theta_2) + m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2) = 0.12 \\ m_{\mathcal{M}}(\theta_1 \cup \theta_3) = m_{12}(\theta_1 \cup \theta_3) + m_1(\theta_1 \cap \theta_3)m_2(\theta_1 \cap \theta_3) = 0.02 \\ m_{\mathcal{M}}(\theta_1 \cup \theta_2 \cup \theta_3) = m_{12}(\theta_1 \cup \theta_2 \cup \theta_3) + m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cap \theta_3) + m_2(\theta_1 \cap \theta_2)m_1(\theta_1 \cap \theta_3) = 0.05 \end{cases}$$

4.7 Bayesian mixture of hybrid DSm models

In the preceding, one has first shown how to combine generalized basic belief assignments provided by $k \geq 2$ independent and equally reliable sources of information with the general hybrid DSm rule of combination for dealing with all possible kinds of integrity constraints involved in a model. This approach implicitly assumes that one knows/trusts with certainty that the model $\mathcal{M}$ (usually a hybrid DSm model) of the problem is valid and corresponds to the true model. In some complex fusion problems however (static or dynamic ones), one may have some doubts about the validity of the model $\mathcal{M}$ on which is based the fusion because of the nature and evolution of elements of the frame Θ. In such situations, we propose to consider a set of exclusive and exhaustive models $\{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_K\}$ with some probabilities $\{P\{\mathcal{M}_1\}, P\{\mathcal{M}_2\}, \ldots, P\{\mathcal{M}_K\}\}$. We don't go here deeper on the justification/acquisition of such probabilities because this is highly dependent on the nature of the fusion problem under consideration. We just assume here that such probabilities are available at any given time t_l when the fusion has to be done. We propose then to use the Bayesian mixture of combined masses $m_{\mathcal{M}_i(\Theta)}(.)$ $i = 1, \ldots, K$ to obtain the final result :

$$\forall A \in D^{\Theta}, \qquad m_{\mathcal{M}_1,\ldots,\mathcal{M}_K}(A) = \sum_{i=1}^{K} P\{\mathcal{M}_i\} m_{\mathcal{M}_i(\Theta)}(A) \tag{4.14}$$

4.8 Conclusion

In this chapter we have extended the DSmT and the classical DSm rule of combination to the case of any kind of hybrid model for the frame Θ involved in many complex fusion problems. The free-DSm model (which assumes that none of the elements of the frame is refinable) can be interpreted as the opposite of Shafer's model (which assumes that all elements of the frame are truly exclusive) on which is based the mathematical theory of evidence (Dempster-Shafer Theory - DST). Between these two extreme models, there exists actually many possible hybrid models for the frames Θ depending on the real intrinsic nature of elements of the fusion problem under consideration. For real problems, some elements of Θ can appear to be truly exclusive whereas some others cannot be considered as fully discernable or refinable. This present research work proposes a new hybrid DSm rule of combination for hybrid models based on the DSmT. The hybrid DSm rule works in any model and is involved in calculation of mass fusion of any number of sources of information, no matter how big is the conflict/paradoxism of sources, and on any frame (exhaustive or non-exhaustive, with elements which may be exclusive or non-exclusive or both). This is an important rule since does not require the calculation of weighting factors, neither normalization as other rules do, and the transfer of masses of empty-sets to the masses of non-empty sets is naturally done following the DSm network architecture which is derived from the DSm classic rule. DSmT together with hybrid DSm rule is a new solid alternative to classical approaches and to existing combination rules. This new result is appealing for the development of future complex (uncertain/incomplete/paradoxical/dynamical) information fusion systems.

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