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We devote part of this issue of *Critical Review* to Lotfi Zadeh.
The Editors.

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Contents

Interval Valued Neutrosophic Graphs

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Abstract

The notion of interval valued neutrosophic sets is a generalization of fuzzy sets, intuitionistic fuzzy sets, interval valued fuzzy sets, interval valued intuitionstic fuzzy sets and single valued neutrosophic sets. We apply for the first time to graph theory the concept of interval valued neutrosophic sets, an instance of neutrosophic sets. We introduce certain types of interval valued neutrosophc graphs (IVNG) and investigate some of their properties with proofs and examples.

Keyword

Interval valued neutrosophic set, Interval valued neutrosophic graph, Strong interval valued neutrosophic graph, Constant interval valued neutrosophic graph, Complete interval valued neutrosophic graph, Degree of interval valued neutrosophic graph.

1 Introduction

Neutrosophic sets (NSs) proposed by Smarandache [13, 14] are powerful mathematical tools for dealing with incomplete, indeterminate and inconsistent information in real world. They are a generalization of fuzzy sets [31], intuitionistic fuzzy sets [28, 30], interval valued fuzzy set [23] and interval-valued intuitionistic fuzzy sets theories [29].

The neutrosophic sets are characterized by a truth-membership function (t), an indeterminacy-membership function (i) and a falsity-membership function (f) independently, which are within the real standard or nonstandard unit interval $]^-0, 1^+[$. In order to conveniently practice NS in real life applications, Smarandache [53] and Wang et al. [17] introduced the concept of single-valued neutrosophic set (SVNS), a subclass of the neutrosophic sets.

The same authors [16, 18] introduced as well the concept of interval valued neutrosophic set (IVNS), which is more precise and flexible than the single valued neutrosophic set. The IVNS is a generalization of the single valued neutrosophic set, in which three membership functions are independent, and their values included into the unit interval [0, 1].

More on single valued neutrosophic sets, interval valued neutrosophic sets and their applications may be found in [3, 4, 5,6, 19, 20, 21, 22, 24, 25, 26, 27, 39, 41, 42, 43, 44, 45, 49].

Graph theory has now become a major branch of applied mathematics and it is generally regarded as a branch of combinatorics. Graph is a widely used tool for solving a combinatorial problem in different areas, such as geometry, algebra, number theory, topology, optimization or computer science. Most important thing which is to be noted is that, when we have uncertainty regarding either the set of vertices or edges, or both, the model becomes a fuzzy graph.

The extension of fuzzy graph [7, 9, 38] theory have been developed by several researchers, including intuitionistic fuzzy graphs [8, 32, 40], considering the vertex sets and edge sets as intuitionistic fuzzy sets. In interval valued fuzzy graphs [33, 34], the vertex sets and edge sets are considered as interval valued fuzzy sets. In interval valued intuitionstic fuzzy graphs [2, 48], the vertex sets and edge sets are regarded as interval valued intuitionstic fuzzy sets. In bipolar fuzzy graphs [35, 36], the vertex sets and edge sets are considered as bipolar fuzzy sets. In m-polar fuzzy graphs [37], the vertex sets and edge sets are regarded as m-polar fuzzy sets.

But, when the relations between nodes (or vertices) in problems are indeterminate, the fuzzy graphs and their extensions fail. In order to overcome the failure, Smarandache [10, 11, 12, 51] defined four main categories of neutrosophic graphs: I-edge neutrosophic graph, I-vertex neutrosophic graph [1, 15, 50, 52], (t, i, f)-edge neutrosophic graph and (t, i, f)-vertex neutrosophic graph. Later on, Broumi et al. [47] introduced another neutrosophic graph model. This model allows the attachment of truth-membership (t), indeterminacy –membership (i) and falsity-membership (f) degrees both to vertices and edges. A neutrosophic graph model that generalizes the fuzzy graph and intuitionstic fuzzy graph is called single valued neutrosophic graph (SVNG). Broumi [46] introduced as well the neighborhood degree of a vertex and closed neighborhood degree of a vertex in single valued neutrosophic graph, as generalizations of neighborhood degree of a vertex and closed neighborhood degree of a vertex in fuzzy graph and intuitionistic fuzzy graph.

In this paper, we focus on the study of interval valued neutrosophic graphs.

2 Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic sets, fuzzy graph, intuitionistic fuzzy graph, single valued neutrosophic graphs, relevant to the present work. See especially [2, 7, 8, 13, 18, 47] for further details and background.

Definition 2.1 [13]

Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ X}, where the functions T, I, F: X→]$^-0,1^+$[define respectively the a truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element x ∈ X to the set A with the condition:

$$^-0 \leq T_A(x)+ I_A(x)+ F_A(x) \leq 3^+.$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of]$^-0,1^+$[.

Since it is difficult to apply NSs to practical problems, Wang et al. [16] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [17]

Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X $T_A(x)$, $I_A(x)$, $F_A(x)$ ∈ [0, 1]. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\}$$

Definition 2.3 [7]

A fuzzy graph is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ. i.e σ : V → [0,1] and μ: VxV→[0,1], such that μ(uv) ≤ σ(u) ⋀ σ(v) for all u, v ∈ V where *uv* denotes the edge between *u* and *v* and σ(u) ⋀ σ(v) denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of V and μ is called the fuzzy edge set of E.

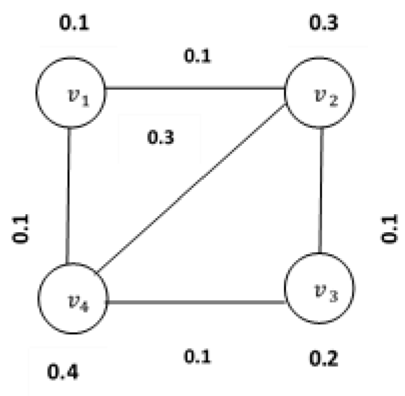

Figure 1: Fuzzy Graph

Definition 2.4 [7]

The fuzzy subgraph H = (τ, ρ) is called a fuzzy subgraph of G = (σ, μ), if τ(u) ≤ σ(u) for all u ∈ V and ρ(u, v) ≤ μ(u, v) for all u, v ∈ V.

Definition 2.5 [8]

An Intuitionistic fuzzy graph is of the form G =(V, E), where

i. V={v_1, v_2,…., v_n} such that μ_1: V→ [0,1] and γ_1: V → [0,1] denote the degree of membership and nonmembership of the element v_i ∈ V, respectively, and 0 ≤ $\mu_1(v_i)$ + $\gamma_1(v_i)$) ≤ 1 , for every v_i ∈ V, (i = 1, 2, ……. n);
ii. E ⊆ V x V where μ_2: VxV→[0,1] and γ_2: VxV→ [0,1] are such that $\mu_2(v_i, v_j) \leq \min [\mu_1(v_i), \mu_1(v_j)]$ and $\gamma_2(v_i, v_j) \geq \max [\gamma_1(v_i), \gamma_1(v_j)]$ and 0 ≤ $\mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \leq 1$ for every (v_i, v_j) ∈ E, (i, j = 1,2, ……. n)

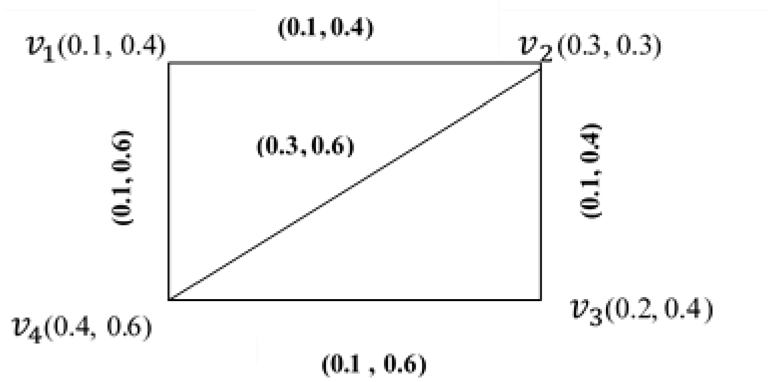

Figure 2: Intuitionistic Fuzzy Graph

Definition 2.6 [2]

An interval valued intuitionistic fuzzy graph with underlying set V is defined to be a pair G= (A, B), where

1) The functions M_A : V→ D [0, 1] and N_A : V→ D [0, 1] denote the degree of membership and non-membership of the element x ∈ V, respectively, such that 0≤M_A(x)+ N_A(x) ≤ 1 for all x ∈ V.

2) The functions M_B : E ⊆ $V \times V$ → D [0, 1] and N_B : : E ⊆ $V \times V$ → D [0, 1] are defined by:

$$M_{BL}(x,y))\leq \min\,(M_{AL}(x), M_{AL}(y)),$$

$$N_{BL}(x,y))\ \geq \max\,(N_{AL}(x), N_{AL}(y)),$$

$$M_{BU}(x,y))\leq \min\,(M_{AU}(x), M_{AU}(y)),$$

$$N_{BU}(x,y))\ \geq \max\,(N_{AU}(x), N_{AU}(y)),$$

such that

$$0\leq M_{BU}(x,y))+ N_{BU}(x,y)) \leq 1, \text{ for all } (x,y) \in \text{E}.$$

Definition 2.7 [47]

Let A = (T_A, I_A, F_A) and B = (T_B, I_B, F_B) be single valued neutrosophic sets on a set X. If A = (T_A, I_A, F_A) is a single valued neutrosophic relation on a set X, then A =(T_A, I_A, F_A) is called a single valued neutrosophic relation on B = (T_B, I_B, F_B), if

$$\mathrm{T_B}(\mathrm{x, y}) \leq \min(\mathrm{T_A}(\mathrm{x}), \mathrm{T_A}(\mathrm{y})),$$

$$\mathrm{I_B}(\mathrm{x, y}) \geq \max(\mathrm{I_A}(\mathrm{x}), \mathrm{I_A}(\mathrm{y})),$$

$$\mathrm{F_B}(\mathrm{x, y}) \geq \max(\mathrm{F_A}\mathrm{x}), \mathrm{F_A}(\mathrm{y})),$$

for all x, y ∈ X.
A single valued neutrosophic relation A on X *is* called symmetric if

$$T_A(x, y) = T_A(y, x),\ I_A(x, y) = I_A(y, x),\ F_A(x, y) = F_A(y, x)$$

$$T_B(x, y) = T_B(y, x),\ I_B(x, y) = I_B(y, x)$$

$$F_B(x, y) = F_B(y, x),$$

for all $x, y \in X$.

Definition 2.8 [47]

A single valued neutrosophic graph (SVN-graph) with underlying set V is defined to be a pair G = (A, B), where

1) The functions $\mathrm{T_A}$:V→[0, 1], $\mathrm{I_A}$:V→[0, 1] and $\mathrm{F_A}$:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element v_i ∈ V, respectively, and

$$0\leq \mathrm{T_A}(\mathrm{v_i}) + \mathrm{I_A}(\mathrm{v_i}) + \mathrm{F_A}(\mathrm{v_i}) \leq 3,$$

for all v_i ∈ V (i=1, 2, …, n).

2) The functions T_B: E ⊆ V x V →[0, 1], I_B:E ⊆ V x V →[0, 1] and F_B: E ⊆ V x V →[0, 1] are defined by

$$T_B(\{v_i, v_j\}) \leq \min [T_A(v_i), T_A(v_j)],$$

$$I_B(\{v_i, v_j\}) \geq \max [I_A(v_i), I_A(v_j)],$$

$$F_B(\{v_i, v_j\}) \geq \max [F_A(v_i), F_A(v_j)],$$

denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in$ E respectively, where

$$0 \leq T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \leq 3,$$

for all $\{v_i, v_j\} \in$ E (i, j = 1, 2, …, n).

We call A the single valued neutrosophic vertex set of V, and B the single valued neutrosophic edge set of E, respectively. Note that B is a symmetric single valued neutrosophic relation on A. We use the notation (v_i, v_j) for an element of E. Thus, G = (A, B) is a single valued neutrosophic graph of G*= (V, E) if

$$T_B(v_i, v_j) \leq \min [T_A(v_i), T_A(v_j)],$$

$$I_B(v_i, v_j) \geq \max [I_A(v_i), I_A(v_j)],$$

$$F_B(v_i, v_j) \geq \max [F_A(v_i), F_A(v_j)],$$

for all $(v_i, v_j) \in$ E.

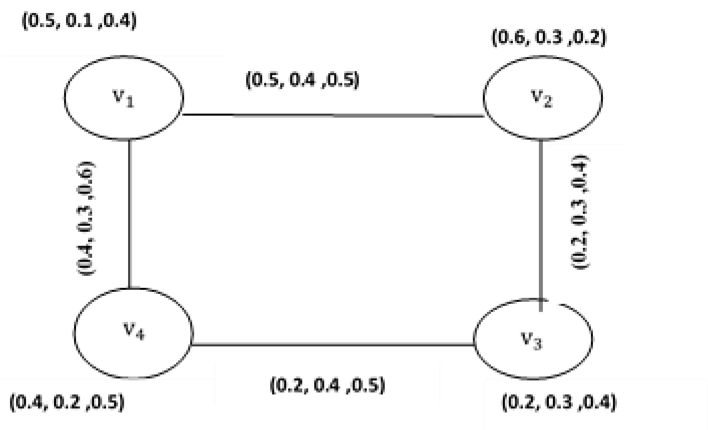

Figure 3: Single valued neutrosophic graph

Definition 2.9 [47]

A partial SVN-subgraph of SVN-graph G= (A, B) is a SVN-graph H = (V', E') such that

(i) $V' \subseteq V$, where $T'_A(v_i) \leq T_A(v_i)$, $I'_A(v_i) \geq I_A(v_i)$, $F'_A(v_i) \geq F_A(v_i)$, for all $v_i \in V$.

(ii) $E' \subseteq E$, where $T'_B(v_i, v_j) \leq T_B(v_i, v_j)$, $I'_{Bij} \geq I_B(v_i, v_j)$, $F'_B(v_i, v_j) \geq F_B(v_i, v_j)$, for all $(v_i\, v_j) \in E$.

Definition 2.10 [47]

A SVN-subgraph of SVN-graph G= (V, E) is a SVN-graph H = ($\boldsymbol{V'}$, $\boldsymbol{E'}$) such that

(i) $\boldsymbol{V'} = \boldsymbol{V}$, where $\boldsymbol{T'_A(v_i)} = \boldsymbol{T_A(v_i)}$, $\boldsymbol{I'_A(v_i)} = \boldsymbol{I_A(v_i)}$, $\boldsymbol{F'_A(v_i)} = \boldsymbol{F_A(v_i)}$for all $\boldsymbol{v_i}$ in the vertex set of $\boldsymbol{V'}$.

(ii) $\boldsymbol{E'} = \boldsymbol{E}$, where $\boldsymbol{T'_B(v_i, v_j)} = \boldsymbol{T_B(v_i, v_j)}$, $\boldsymbol{I'_B(v_i, v_j)} = \boldsymbol{I_B(v_i, v_j)}$, $\boldsymbol{F'_B(v_i, v_j)} = \boldsymbol{F_B(v_i, v_j)}$ for every $(\boldsymbol{v_i}\ \boldsymbol{v_j}) \in \boldsymbol{E}$ in the edge set of $\boldsymbol{E'}$.

Definition 2.11 [47]

Let G= (A, B) be a single valued neutrosophic graph. Then the degree of any vertex v is the sum of degree of truth-membership, sum of degree of indeterminacy-membership and sum of degree of falsity-membership of all those edges which are incident on vertex v denoted by d(v) = ($d_T(v)$, $d_I(v)$, $d_F(v)$), where

$d_T(v)$=$\sum_{u \neq v} T_B(u, v)$ denotes degree of truth-membership vertex,

$d_I(v)$=$\sum_{u \neq v} I_B(u, v)$ denotes degree of indeterminacy-membership vertex,

$d_F(v)$=$\sum_{u \neq v} F_B(u, v)$ denotes degree of falsity-membership vertex.

Definition 2.12 [47]

A single valued neutrosophic graph G=(A, B) of G^*= (V, E) is called strong single valued neutrosophic graph, if

$$T_B(v_i, v_j) = \min\,[T_A(v_i),\ T_A(v_j)],$$

$$I_B(v_i, v_j) = \max\,[I_A(v_i),\ I_A(v_j)],$$

$$F_B(v_i, v_j) = \max\,[F_A(v_i), F_A(v_j)],$$

for all $(v_i, v_j) \in$ E.

Definition 2.13 [47]

A single valued neutrosophic graph G = (A, B) is called complete if

$$T_B(v_i, v_j) = \min\,[T_A(v_i),\ T_A(v_j)],$$

$$I_B(v_i, v_j) = \max\,[I_A(v_i),\ I_A(v_j)],$$

$$F_B(v_i, v_j) = \max\,[F_A(v_i), F_A(v_j)],$$

for all $v_i, v_j \in$ V.

Definition 2.14 [47]

The complement of a single valued neutrosophic graph G (A, B) on G^* is a single valued neutrosophic graph $\bar{G}$ on G^*, where:

1. $\bar{A}$ =A
2. $\overline{T_A}(v_i)= T_A(v_i)$, $\overline{I_A}(v_i)= I_A(v_i)$, $\overline{F_A}(v_i) = F_A(v_i)$, for all $v_j \in$ V.
3. $\overline{T_B}(v_i, v_j)=$ min $[T_A(v_i), T_A(v_j)] - T_B(v_i, v_j)$

$\overline{I_B}(v_i, v_j)=$ max $[I_A(v_i), I_A(v_j)] - I_B(v_i, v_j)$, and

$\overline{F_B}(v_i, v_j)=$ max $[F_A(v_i), F_A(v_j)] - F_B(v_i, v_j)$,

for all $(v_i, v_j) \in$ E.

Definition 2.15 [18]

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS A) A in X is characterized by truth-membership function $\mathrm{T_A}(\mathrm{x})$, indeterminacy-membership function $\mathrm{I_A}(\mathrm{x})$ and falsity-membership function $\mathrm{F_A}(\mathrm{x})$. For each point x in X, we have that $\mathrm{T_A}(\mathrm{x})= [T_{AL}(\mathrm{x}), T_{AU}(\mathrm{x})]$, $\mathrm{I_A}(\mathrm{x}) = [I_{AL}(x), I_{AU}(x)]$, $\mathrm{F_A}(\mathrm{x}) = [F_{AL}(x), F_{AU}(x)] \subseteq [0, 1]$ and $0 \leq \mathrm{T_A}(\mathrm{x}) + \mathrm{I_A}(\mathrm{x}) + \mathrm{F_A}(\mathrm{x}) \leq 3$.

Definition 2.16 [18]

An IVNS A is contained in the IVNS B, A ⊆ B, if and only if $T_{AL}(\mathrm{x}) \leq T_{BL}(\mathrm{x})$, $T_{AU}(\mathrm{x}) \leq T_{BU}(\mathrm{x})$, $I_{AL}(\mathrm{x}) \geq I_{BL}(\mathrm{x})$, $I_{AU}(\mathrm{x}) \geq I_{BU}(\mathrm{x})$, $F_{AL}(\mathrm{x}) \geq F_{BL}(\mathrm{x})$ and $F_{AU}(\mathrm{x}) \geq F_{BU}(\mathrm{x})$ for any x in X.

Definition 2.17 [18]

The union of two interval valued neutrosophic sets A and B is an interval neutrosophic set C, written as C = A ∪ B, whose truth-membership, indeterminacy-membership, and false membership are related to A and B by

$$\mathrm{T_{CL}}(\mathrm{x}) = \max (\mathrm{T_{AL}}(\mathrm{x}),\ \mathrm{T_{BL}}(\mathrm{x}))$$

$$\mathrm{T_{CU}}(\mathrm{x}) = \max (\mathrm{T_{AU}}(\mathrm{x}),\ \mathrm{T_{BU}}(\mathrm{x}))$$

$$\mathrm{I_{CL}}(\mathrm{x}) = \min (\mathrm{I_{AL}}(\mathrm{x}),\ \mathrm{I_{BL}}(\mathrm{x}))$$

$$\mathrm{I_{CU}}(\mathrm{x}) = \min (\mathrm{I_{AU}}(\mathrm{x}),\ \mathrm{I_{BU}}(\mathrm{x}))$$

$$\mathrm{F_{CL}}(\mathrm{x}) = \min (\mathrm{F_{AL}}(\mathrm{x}),\ \mathrm{F_{BL}}(\mathrm{x}))$$

$$\mathrm{F_{CU}}(\mathrm{x}) = \min (\mathrm{F_{AU}}(\mathrm{x}),\ \mathrm{F_{BU}}(\mathrm{x}))$$

for all x in X.

Definition 2.18 [18]

Let X and Y be two non-empty crisp sets. An interval valued neutrosophic relation R(X, Y) is a subset of product space X × Y, and is characterized by the truth membership function $T_R(x, y)$, the indeterminacy membership function $I_R(x, y)$, and the falsity membership function $F_R(x, y)$, where $x \in X$ and $y \in Y$ and $T_R(x, y), I_R(x, y), F_R(x, y) \subseteq [0, 1]$.

3 Interval Valued Neutrosophic Graphs

Throughout this paper, we denote G^* = (V, E) a crisp graph, and G = (A, B) an interval valued neutrosophic graph.

Definition 3.1

By an interval-valued neutrosophic graph of a graph G^* = (V, E) we mean a pair G = (A, B), where A =< $[T_{AL},T_{AU}]$, $[I_{AL}, I_{AU}]$, $[F_{AL}, F_{AU}]$> is an interval-valued neutrosophic set on V; and B =< $[T_{BL}, T_{BU}]$, $[I_{BL}, I_{BU}]$, $[F_{BL}, F_{BU}]$> is an interval-valued neutrosophic relation on E satisfying the following condition:

1) V = $\{v_1, v_2, \ldots, v_n\}$, such that T_{AL}:V → [0, 1], T_{AU}:V → [0, 1], I_{AL}:V → [0, 1], I_{AU}:V→[0, 1] and F_{AL}:V→[0, 1], F_{AU}:V→[0, 1] denote the degree of truth-membership, the degree of indeterminacy-membership and falsity-membership of the element $y \in$ V, respectively, and

$$0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3,$$

for all $v_i \in$ V (i=1, 2, …,n)

2) The functions T_{BL}:V x V →[0, 1], T_{BU}:V x V →[0, 1], I_{BL}:V x V →[0, 1], I_{BU}:V x V →[0, 1] and F_{BL}:V x V →[0,1], F_{BU}:V x V →[0, 1] are such that

$$T_{BL}(\{v_i, v_j\}) \leq \min\,[T_{AL}(v_i), T_{AL}(v_j)],$$

$$T_{BU}(\{v_i, v_j\}) \leq \min\,[T_{AU}(v_i), T_{AU}(v_j)],$$

$$I_{BL}(\{v_i, v_j\}) \geq \max[I_{BL}(v_i), I_{BL}(v_j)],$$

$$I_{BU}(\{v_i, v_j\}) \geq \max[I_{BU}(v_i), I_{BU}(v_j)],$$

$$F_{BL}(\{v_i, v_j\}) \geq \max[F_{BL}(v_i), F_{BL}(v_j)],$$

$$F_{BU}(\{v_i, v_j\}) \geq \max[F_{BU}(v_i), F_{BU}(v_j)],$$

denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in$ E respectively, where

$$0 \leq T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \leq 3$$

for all $\{v_i, v_j\} \in$ E (i, j = 1, 2,…, n).

We call A the interval valued neutrosophic vertex set of V, and B the interval valued neutrosophic edge set of E, respectively. Note that B is a symmetric interval valued neutrosophic relation on A. We use the notation (v_i, v_j) for an element of E. Thus, G = (A, B) is an interval valued neutrosophic graph of G*= (V, E) if

$$T_{BL}(v_i, v_j) \leq \min[T_{AL}(v_i), T_{AL}(v_j)],$$

$$T_{BU}(v_i, v_j) \leq \min[T_{AU}(v_i), T_{AU}(v_j)],$$

$$I_{BL}(v_i, v_j) \geq \max[I_{BL}(v_i), I_{BL}(v_j)],$$

$$I_{BU}(v_i, v_j) \geq \max[I_{BU}(v_i), I_{BU}(v_j)],$$

$$F_{BL}(v_i, v_j) \geq \max[F_{BL}(v_i), F_{BL}(v_j)],$$

$$F_{BU}(v_i, v_j) \geq \max[F_{BU}(v_i), F_{BU}(v_j)] \text{ — for all } (v_i, v_j) \in \text{E}.$$

Example 3.2

Consider a graph G^*, such that V = $\{v_1, v_2, v_3, v_4\}$, E = $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let A be a interval valued neutrosophic subset of V and B a interval valued neutrosophic subset of E, denoted by

	v_1	v_2	v_3
T_{AL}	0.3	0.2	0.1
T_{AU}	0.5	0.3	0.3
I_{AL}	0.2	0.2	0.2
I_{AU}	0.3	0.3	0.4
F_{AL}	0.3	0.1	0.3
F_{AU}	0.4	0.4	0.5

	v_1v_2	v_2v_3	v_3v_1
T_{BL}	0.1	0.1	0.1
T_{BU}	0.2	0.3	0.2
I_{BL}	0.3	0.4	0.3
I_{BU}	0.4	0.5	0.5
F_{BL}	0.4	0.4	0.4
F_{BU}	0.5	0.5	0.6

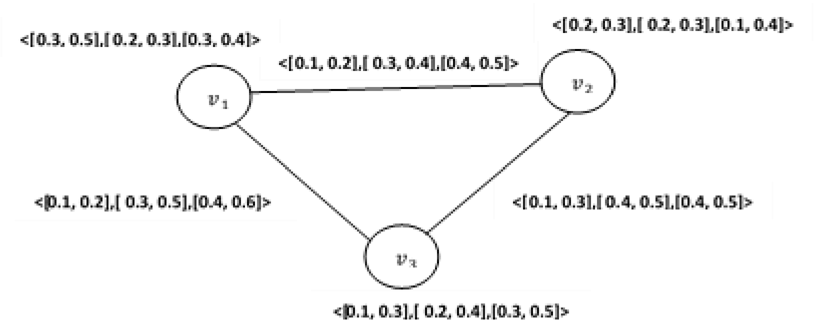

Figure 4: G: Interval valued neutrosophic graph

In *Figure 4*,

(i) (v_1, <[0.3, 0.5],[0.2, 0.3],[0.3, 0.4]>) is an interval valued neutrosophic vertex or IVN-vertex.

(ii) (v_1v_2, <[0.1, 0.2], [0.3, 0.4], [0.4, 0.5]>) is an interval valued neutrosophic edge or IVN-edge.

(iii) (v_1, <[0.3, 0.5], [0.2, 0.3], [0.3, 0.4]>) and (v_2, <[0.2, 0.3],[0.2, 0.3],[0.1, 0.4]>) are interval valued neutrosophic adjacent vertices.

(iv) (v_1v_2, <[0.1, 0.2], [0.3, 0.4], [0.4, 0.5]>) and (v_1v_3, <[0.1, 0.2],[0.3, 0.5], [0.4, 0.6]>) are an interval valued neutrosophic adjacent edge.

Remarks

(i) When $T_{BL}(v_i,v_j) = T_{BU}(v_i,v_j) = I_{BL}(v_i,v_j) = I_{BU}(v_i,v_j) = F_{BL}(v_i,v_j) = F_{BU}(v_i,v_j)$ for some *i* and *j*, then there is no edge between v_i and v_j . Otherwise there exists an edge between v_i and v_j .

(ii) If one of the inequalities is not satisfied in (1) and (2), then G is not an IVNG.

The interval valued neutrosophic graph G depicted in *Figure 3* is represented by the following adjacency matrix $\boldsymbol{M_G}$ —

$$\boldsymbol{M_G} = \begin{bmatrix} <[0.3,0.5],[0.2,0.3],[0.3,0.4]> & <[0.1,0.2],[0.3,0.4],[0.4,0.5]> & <[0.1,0.2],[0.3,0.5],[0.4,0.6]> \\ <[0.1,0.2],[0.3,0.4],[0.4,0.5]> & <[0.2,0.3],[0.2,0.3],[0.1,0.4]> & <[0.1,0.3],[0.4,0.5],[0.4,0.5]> \\ <[0.1,0.2],[0.3,0.5],[0.4,0.6]> & <[0.1,0.3],[0.4,0.5],[0.4,0.5]> & <[0.1,0.3],[0.2,0.4],[0.3,0.5]> \end{bmatrix}$$

Definition 3.3

A partial IVN-subgraph of IVN-graph G= (A, B) is an IVN-graph H = ($\boldsymbol{V'}$, $\boldsymbol{E'}$) such that —

(i) $\boldsymbol{V'} \subseteq \boldsymbol{V}$, where $\boldsymbol{T'_{AL}(v_i) \le T_{AL}(v_i)}$, $\boldsymbol{T'_{AU}(v_i) \le T_{AU}(v_i)}$, $\boldsymbol{I'_{AL}(v_i) \ge I_{AL}(v_i)}$, $\boldsymbol{I'_{AU}(v_i) \ge I_{AU}(v_i)}$, $\boldsymbol{F'_{AL}(v_i) \ge F_{AL}(v_i)}$, $\boldsymbol{F'_{AU}(v_i) \ge F_{AU}(v_i)}$, for all $\boldsymbol{v_i \in V}$.

(ii) $\boldsymbol{E' \subseteq E}$, where $\boldsymbol{T'_{BL}(v_i,v_j) \le T_{BL}(v_i,v_j)}$, $\boldsymbol{T'_{BU}(v_i,v_j) \le T_{BU}(v_i,v_j)}$, $\boldsymbol{I'_{BL}(v_i,v_j) \ge I_{BL}(v_i,v_j)}$, $\boldsymbol{I'_{BU}(v_i,v_j) \ge I_{BU}(v_i,v_j)}$, $\boldsymbol{F'_{BL}(v_i,v_j) \ge F_{BL}(v_i,v_j)}$, $\mathbf{F'_{BU}}\boldsymbol{(v_i,v_j) \ge F_{BU}(v_i,v_j)}$, for all $\boldsymbol{(v_i\, v_j) \in E}$.

Definition 3.4

An IVN-subgraph of IVN-graph G= (V, E) is an IVN-graph H = ($\boldsymbol{V'}$, $\boldsymbol{E'}$) such that

(i) $\boldsymbol{T'_{AL}(v_i) = T_{AL}(v_i)}$, $\boldsymbol{T'_{AU}(v_i) = T_{AU}(v_i)}$, $\boldsymbol{I'_{AL}(v_i) = I_{AL}(v_i)}$, $\boldsymbol{I'_{AU}(v_i) = I_{AU}(v_i)}$, $\boldsymbol{F'_{AL}(v_i) = F_{AL}(v_i)}$, $\boldsymbol{F'_{AU}(v_i) = F_{AU}(v_i)}$, for all $\boldsymbol{v_i}$ in the vertex set of $\boldsymbol{V'}$.

(ii) $\boldsymbol{E' = E}$, where $\boldsymbol{T'_{BL}(v_i,v_j) = T_{BL}(v_i,v_j)}$, $\boldsymbol{T'_{BU}(v_i,v_j) = T_{BU}(v_i,v_j)}$,

$\boldsymbol{I'_{BL}(v_i, v_j) = I_{BL}(v_i, v_j)}$, $\boldsymbol{I'_{BU}(v_i, v_j) = I_{BU}(v_i, v_j)}$, $\boldsymbol{F'_{BL}(v_i, v_j) = F_{BL}(v_i, v_j)}$, $\boldsymbol{F'_{BU}(v_i, v_j) = F_{BU}(v_i, v_j)}$, for every $\boldsymbol{(v_i\ v_j) \in E}$ in the edge set of $\boldsymbol{E'}$.

Example 3.5

$\mathbf{G_1}$ in *Figure 5* is an IVN-graph, $\mathbf{H_1}$ in *Figure 6* is a partial IVN-subgraph and $\mathbf{H_2}$ in *Figure 7* is a IVN-subgraph of $\mathbf{G_1}$.

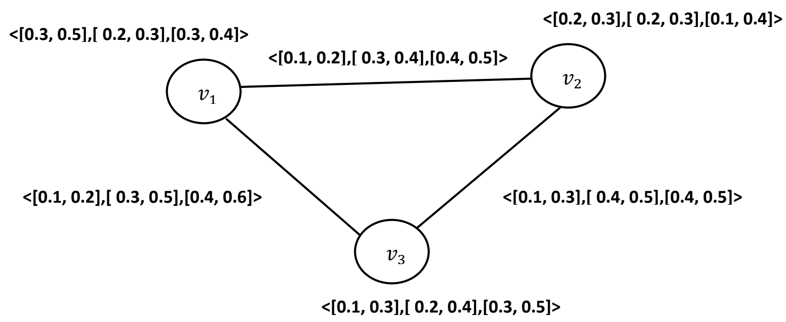

Figure 5: $\mathrm{G_1}$, an interval valued neutrosophic graph

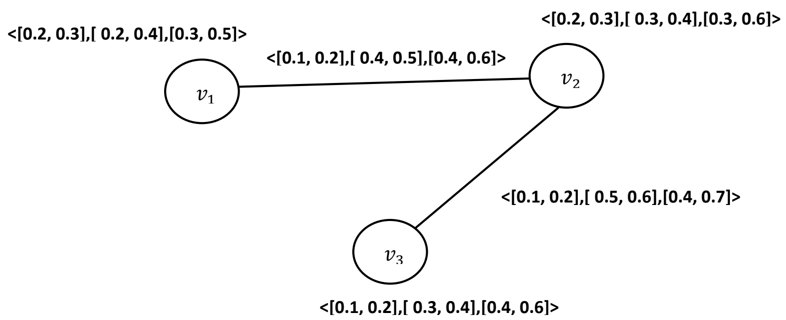

Figure 6: $\mathrm{H_1}$, a partial IVN-subgraph of $\mathrm{G_1}$

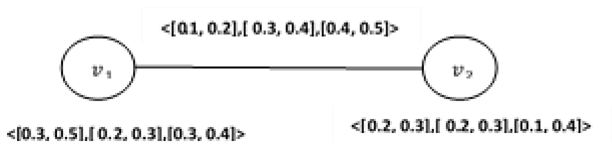

Figure 7: $\mathrm{H_2}$, an IVN-subgraph of $\mathrm{G_1}$

Definition 3.6

The two vertices are said to be adjacent in an interval valued neutrosophic graph G= (A, B) if —

$$T_{BL}(v_i, v_j) = \min[T_{AL}(v_i), T_{AL}(v_j)],$$
$$T_{BU}(v_i, v_j) = \min[T_{AU}(v_i), T_{AU}(v_j)],$$
$$I_{BL}(v_i, v_j) = \max[I_{AL}(v_i), I_{AL}(v_j)]$$
$$I_{BU}(v_i, v_j) = \max[I_{AU}(v_i), I_{AU}(v_j)]$$

$$F_{BL}(v_i, v_j) = \max[F_{AL}(v_i), F_{AL}(v_j)]$$
$$F_{BU}(v_i, v_j) = \max[F_{AU}(v_i), F_{AU}(v_j)]$$

In this case, v_i and v_j are said to be neighbours and (v_i, v_j) is incident at v_i and v_j also.

Definition 3.7

A path P in an interval valued neutrosophic graph G= (A, B) is a sequence of distinct vertices $v_0, v_1, v_3,\dots v_n$ such that $T_{BL}(v_{i-1}, v_i) > 0$, $T_{BU}(v_{i-1}, v_i) > 0$, $I_{BL}(v_{i-1}, v_i) > 0$, $I_{BU}(v_{i-1}, v_i) > 0$ and $F_{BL}(v_{i-1}, v_i) > 0$, $F_{BU}(v_{i-1}, v_i) > 0$ for $0 \leq i \leq 1$. Here $n \geq 1$ is called the length of the path P. A single node or vertex v_i may also be considered as a path. In this case, the path is of the length ([0, 0], [0, 0], [0, 0]). The consecutive pairs (v_{i-1}, v_i) are called edges of the path. We call P a cycle if $v_0 = v_n$ and $n \geq 3$.

Definition 3.8

An interval valued neutrosophic graph G= (A, B) is said to be connected if every pair of vertices has at least one interval valued neutrosophic path between them, otherwise it is disconnected.

Definition 3.9

A vertex $v_j \in V$ of interval valued neutrosophic graph G= (A, B) is said to be an isolated vertex if there is no effective edge incident at v_j.

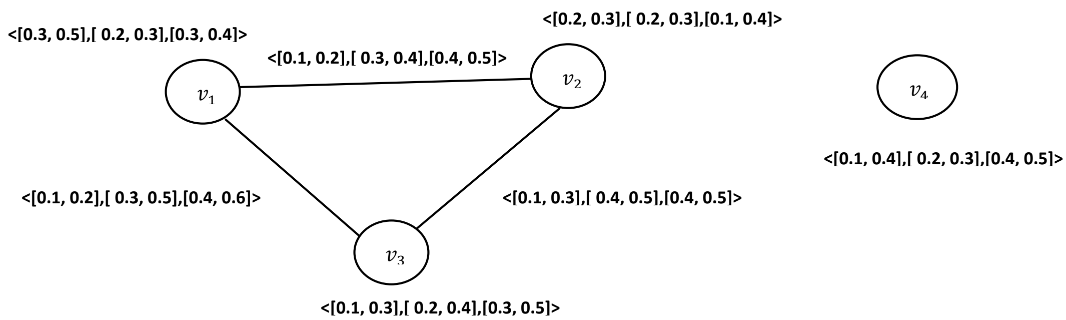

Figure 8. Example of interval valued neutrosophic graph

In *Figure 8*, the interval valued neutrosophic vertex v_4 is an isolated vertex.

Definition 3.10

A vertex in an interval valued neutrosophic G = (A, B) having exactly one neighbor is called *a pendent vertex*. Otherwise, it is called *non-pendent vertex*. An edge in an interval valued neutrosophic graph incident with a pendent vertex is called a *pendent edge*. Otherwise it is called *non-pendent edge*. A vertex in an interval valued neutrosophic graph adjacent to the pendent vertex is called a support of the pendent edge.

Definition 3.11

An interval valued neutrosophic graph G = (A, B) that has neither self-loops nor parallel edge is called simple interval valued neutrosophic graph.

Definition 3.12

When a vertex $\mathbf{v_i}$ is end vertex of some edges $(\mathbf{v_i}, \mathbf{v_j})$ of any IVN-graph G = (A, B). Then $\mathbf{v_i}$ and $(\mathbf{v_i}, \mathbf{v_j})$ are said to be incident to each other.

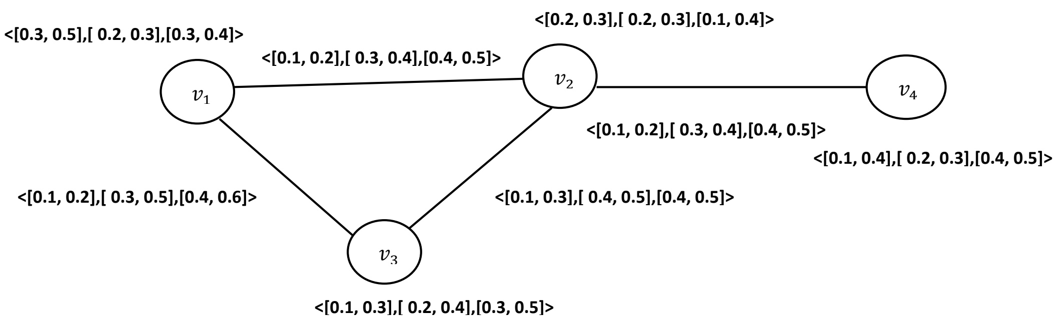

Figure 9. Incident IVN-graph.

In this graph v_2v_1, v_2v_3 and v_2v_4 are incident on v_2.

Definition 3.13

Let G = (A, B) be an interval valued neutrosophic graph. Then the degree of any vertex v is sum of degree of truth-membership, sum of degree of indeterminacy-membership and sum of degree of falsity-membership of all those edges which are incident on vertex v denoted by —

$$d(v) = ([d_{TL}(v), d_{TU}(v)], [d_{IL}(v), d_{IU}(v)], [d_{FL}(v), d_{FU}(v)]),$$

where:

$d_{TL}(v)=\sum_{u\neq v} T_{BL}(u,v)$ denotes the degree of lower truth-membership vertex;
$d_{TU}(v) = \sum_{u\neq v} T_{BU}(u,v)$ denotes the degree of upper truth-membership vertex;
$d_{IL}(v) = \sum_{u\neq v} I_{BL}(u,v)$ denotes the degree of lower indeterminacy-membership vertex;
$d_{IU}(v) = \sum_{u\neq v} I_{BU}(u,v)$ denotes the degree of upper indeterminacy-membership vertex;
$d_{FL}(v) = \sum_{u\neq v} F_{BL}(u,v)$ denotes the degree of lower falsity-membership vertex;
$d_{FU}(v) = \sum_{u\neq v} F_{BU}(u,v)$ denotes the degree of upper falsity-membership vertex.

Example 3.14

Let us consider an interval valued neutrosophic graph G = (A, B) of G^* = (V, E) where V = {v_1, v_2, v_3, v_4} and E = {v_1v_2, v_2v_3, v_3v_4 , v_4v_1}.

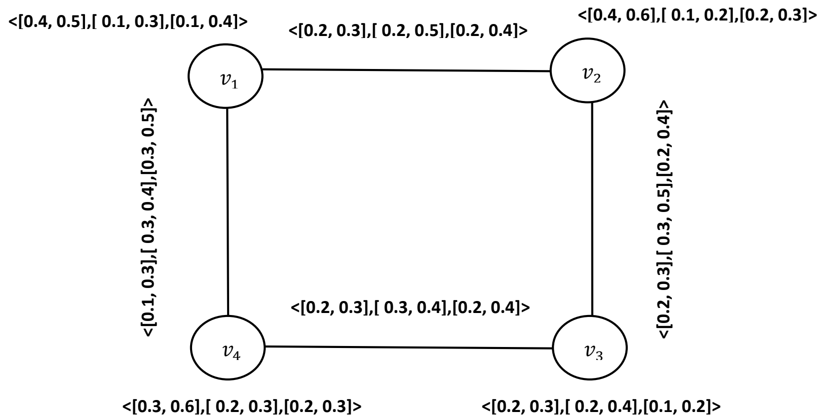

Figure 10: Degree of vertex of interval valued neutrosophic graph

We have the degree of each vertex as follows:

$d(v_1)$= ([0.3, 0.6], [0.5, 0.9], [0.5, 0.9]), $d(v_2)$= ([0.4, 0.6], [0.5, 1.0], [0.4, 0.8]), $d(v_3)$= ([0.4, 0.6], [0.6, 0.9], [0.4, 0.8]), $d(v_4)$= ([0.3, 0.6], [0.6, 0.8], [0.5, 0.9]).

Definition 3.15

An interval valued neutrosophic graph G= (A, B) is called constant if degree of each vertex is k =([k_{1L}, k_{1U}], [k_{2L}, k_{2U}], [k_{3L}, k_{3U}]). That is d(v) =([k_{1L}, k_{1U}], [k_{2L}, k_{2U}], [k_{3L}, k_{3U}]), for all $v \in$ V.

Example 3.16

Consider an interval valued neutrosophic graph G such that V = {v_1, v_2, v_3, v_4} and E = {v_1v_2, v_2v_3, v_3v_4 , v_4v_1}.

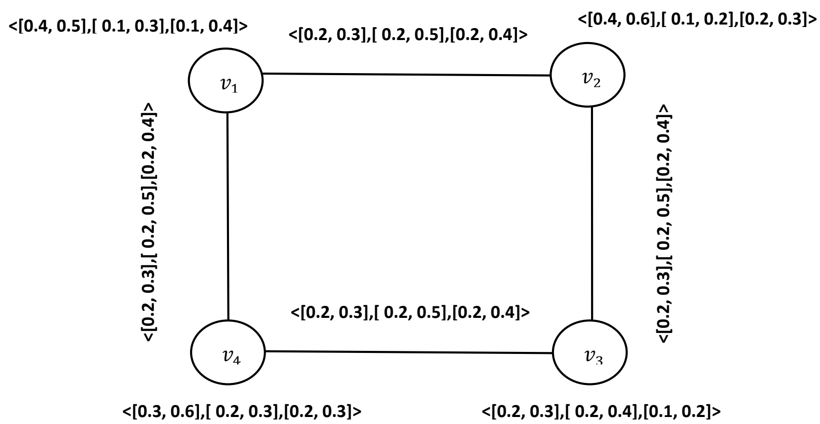

Figure 11. Constant IVN-graph.

Clearly, G is constant IVN-graph since the degree of $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\boldsymbol{v_3}$ and $\boldsymbol{v_4}$ is ([0.4, 0.6], [0.4, 1], [0.4, 0.8])

Definition 3.17

An interval valued neutrosophic graph G = (A, B) of G^*= (V, E) is called strong interval valued neutrosophic graph if

$T_{BL}(v_i, v_j) = \min[T_{AL}(v_i),\ T_{AL}(v_j)], T_{BU}(v_i, v_j) = \min[T_{AU}(v_i),$
$T_{AU}(v_j)]$

$I_{BL}(v_i, v_j) = \max[I_{AL}(v_i),\ I_{AL}(v_j)],\ I_{BU}(v_i, v_j) = \max[I_{AU}(v_i),$
$I_{AU}(v_j)]$

$F_{BL}(v_i, v_j) = \max[F_{AL}(v_i), F_{AL}(v_j)],\ F_{BU}(v_i, v_j) = \max[F_{AU}(v_i),$
$F_{AU}(v_j)]$, for all $(v_i, v_j) \in$ E.

Example 3.18

Consider a graph G^* such that V = $\{v_1, v_2, v_3, v_4\}$, E = $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let A be an interval valued neutrosophic subset of V and let B an interval valued neutrosophic subset of E denoted by:

	v_1	v_2	v_3
T_{AL}	0.3	0.2	0.1
T_{AU}	0.5	0.3	0.3
I_{AL}	0.2	0.2	0.2
I_{AU}	0.3	0.3	0.4
F_{AL}	0.3	0.1	0.3
F_{AU}	0.4	0.4	0.5

	v_1v_2	v_2v_3	v_3v_1
T_{BL}	0.2	0.1	0.1
T_{BU}	0.3	0.3	0.3
I_{BL}	0.2	0.2	0.2
I_{BU}	0.3	0.4	0.4
F_{BL}	0.3	0.3	0.3
F_{BU}	0.4	0.4	0.5

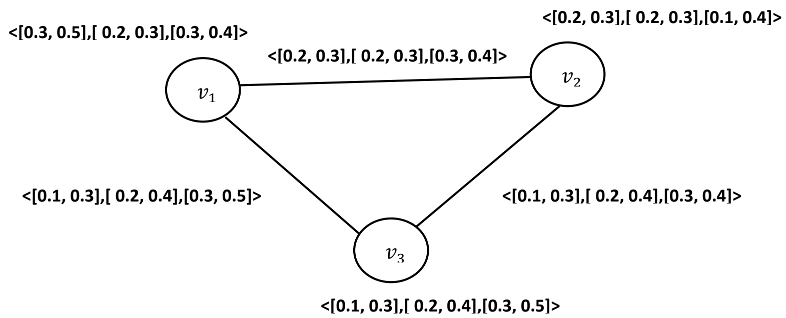

Figure 12. Strong IVN-graph.

By routing computations, it is easy to see that G is a strong interval valued neutrosophic of G^*.

Proposition 3.19

An interval valued neutrosophic graph is the generalization of interval valued fuzzy graph

Proof

Suppose G = (V, E) be an interval valued neutrosophic graph. Then by setting the indeterminacy-membership and falsity-membership values of vertex set and edge set equals to zero reduces the interval valued neutrosophic graph to interval valued fuzzy graph.

Proposition 3.20

An interval valued neutrosophic graph is the generalization of interval valued intuitionistic fuzzy graph

Proof

Suppose G = (V, E) is an interval valued neutrosophic graph. Then by setting the indeterminacy-membership values of vertex set and edge set equals to zero reduces the interval valued neutrosophic graph to interval valued intuitionistic fuzzy graph.

Proposition 3.21

An interval valued neutrosophic graph is the generalization of intuitionistic fuzzy graph.

Proof

Suppose G = (V, E) is an interval valued neutrosophic graph. Then by setting the indeterminacy-membership, upper truth-membership and upper falsity-membership values of vertex set and edge set equals to zero reduces the interval valued neutrosophic graph to intuitionistic fuzzy graph.

Proposition 3.22

An interval valued neutrosophic graph is the generalization of single valued neutrosophic graph.

Proof

Suppose G = (V, E) is an interval valued neutrosophic graph. Then by setting the upper truth-membership equals lower truth-membership, upper indeterminacy-membership equals lower indeterminacy-membership and upper falsity-membership equals lower falsity-membership values of vertex set and edge set reduces the interval valued neutrosophic graph to single valued neutrosophic graph.

Definition 3.23

The complement of an interval valued neutrosophic graph G (A, B) on G^* is an interval valued neutrosophic graph $\bar{G}$ on G^* where:

1. $\bar{A}$ =A

2. $\overline{T_{AL}}(v_i)= T_{AL}(v_i)$, $\overline{T_{AU}}(v_i)= T_{AU}(v_i)$, $\overline{I_{AL}}(v_i)= I_{AL}(v_i)$, $\overline{I_{AU}}(v_i)= I_{AU}(v_i)$, $\overline{F_{AL}}(v_i) = F_{AL}(v_i)$, $\overline{F_{AU}}(v_i) = F_{AU}(v_i)$,

for all $v_j \in$ V.

3. $\overline{T_{BL}}(v_i, v_j)=\min\left[T_{AL}(v_i), T_{AL}(v_j)\right] - T_{BL}(v_i, v_j)$,
$\overline{T_{BU}}(v_i, v_j)=\min\left[T_{AU}(v_i), T_{AU}(v_j)\right] - T_{BU}(v_i, v_j)$,

$\overline{I_{BL}}(v_i, v_j)=\max\left[I_{AL}(v_i), I_{AL}(v_j)\right] - I_{BL}(v_i, v_j)$, $\overline{I_{BU}}(v_i, v_j)=\max\left[I_{AU}(v_i), I_{AU}(v_j)\right] - I_{BU}(v_i, v_j)$,

and

$\overline{F_{BL}}(v_i, v_j)=\max\left[F_{AL}(v_i), F_{AL}(v_j)\right] - F_{BL}(v_i, v_j)$,
$\overline{F_{BU}}(v_i, v_j)=\max\left[F_{AU}(v_i), F_{AU}(v_j)\right] - F_{BU}(v_i, v_j)$,

for all $(v_i, v_j) \in$ E

Remark 3.24

If G = (V, E) is an interval valued neutrosophic graph on G^*. Then from above definition, it follow that $\bar{\bar{G}}$ is given by the interval valued neutrosophic graph $\bar{\bar{G}} = (\bar{\bar{V}}, \bar{\bar{E}})$ on G^* where $\bar{\bar{V}}$ =V and —

$$\overline{\overline{T_{BL}}}(v_i, v_j)=\min\left[T_{AL}(v_i), T_A(v_j)\right]-T_{BL}(v_i, v_j),$$

$$\overline{\overline{T_{BU}}}(v_i, v_j)=\min\left[T_{AU}(v_i), T_A(v_j)\right]-T_{BU}(v_i, v_j),$$

$$\overline{\overline{I_{BL}}}(v_i, v_j)=\max\left[I_{AL}(v_i), I_{AL}(v_j)\right]-I_{BL}(v_i, v_j),$$
$$\overline{\overline{I_{BU}}}(v_i, v_j)=\max\left[I_{AU}(v_i), I_{AU}(v_j)\right]-I_{BU}(v_i, v_j),$$

and

$$\overline{\overline{F_{BL}}}(v_i, v_j) = \max\left[F_{AL}(v_i), F_{AL}(v_j)\right]-F_{BL}(v_i, v_j), \overline{\overline{F_{BU}}}(v_i, v_j) = \max\left[F_{AU}(v_i), F_{AU}(v_j)\right]-F_{BU}(v_i, v_j), \text{ For all } (v_i, v_j) \in E.$$

Thus $\overline{\overline{T_{BL}}} = T_{BL}$, $\overline{\overline{T_{BU}}} = T_{BUL}$, $\overline{\overline{I_{BL}}} = I_{BL}$, $\overline{\overline{I_{BU}}} = I_{BU}$, and $\overline{\overline{F_{BL}}} = F_{BL}$, $\overline{\overline{F_{BU}}} = F_{BU}$ on V, where E =($[T_{BL}, T_{BU}]$, $[I_{BL}, I_{BU}]$, $[F_{BL}, F_{BU}]$) is the interval valued neutrosophic relation on V. For any interval valued neutrosophic graph G, $\bar{G}$ is strong interval valued neutrosophic graph and $G \subseteq \bar{G}$.

Proposition 3.25

G= $\overline{\overline{G}}$ if and only if G is a strong interval valued neutrosophic graph.

Proof

It is obvious.

Definition 3.26

A strong interval valued neutrosophic graph G is called self complementary if $G \cong \bar{G}$, where $\bar{G}$ is the complement of interval valued neutrosophic graph G.

Example 3.27

Consider a graph G^* = (V, E) such that V ={v_1, v_2, v_3, v_4}, E={v_1v_2, v_2v_3, v_3v_4, v_1v_4}. Consider an interval valued neutrosophic graph G.

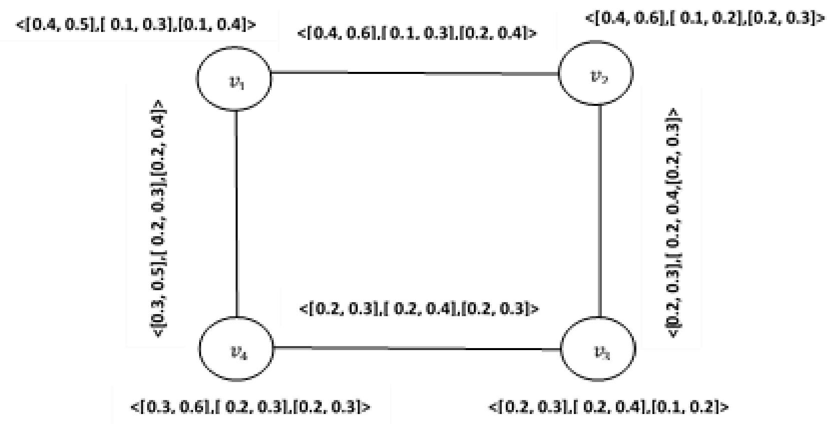

Figure 13. G: Strong IVN- graph

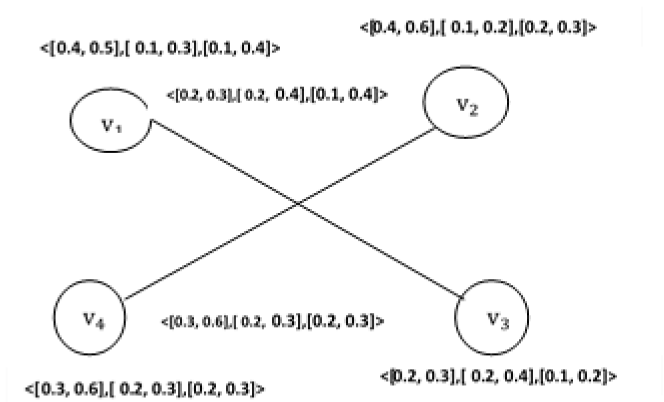

Figure 14. $\bar{G}$ Strong IVN- graph

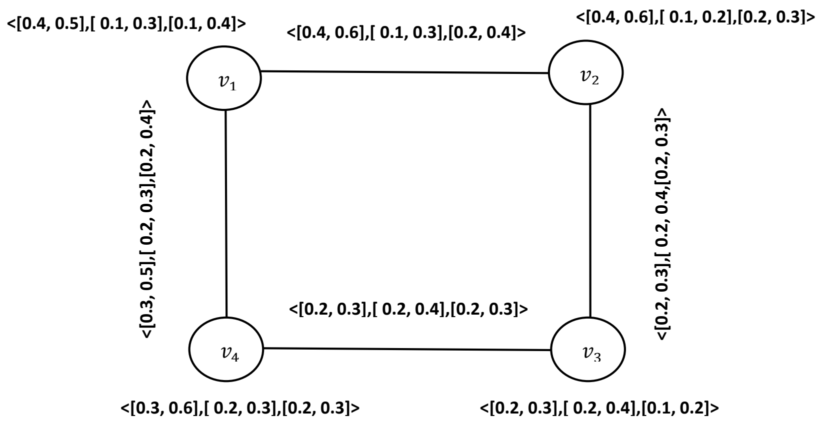

Figure 15. $\overline{\overline{G}}$ Strong IVN- graph

Clearly, G≅ $\overline{\overline{G}}$, hence G is self complementary.

Proposition 3.26

Let G=(A, B) be a *strong* interval valued neutrosophic graph. If –

$$T_{BL}(v_i, v_j) = \min\,[T_{AL}(v_i),\ T_{AL}(v_j)]$$

$$T_{BU}(v_i, v_j) = \min\,[T_{AU}(v_i),\ T_{AU}(v_j)]$$

$$I_{BL}(v_i, v_j) = \max\,[I_{AL}(v_i),\ I_{AL}(v_j)]$$

$$I_{BU}(v_i, v_j) = \max\,[I_{AU}(v_i),\ I_{AU}(v_j)]$$

$$F_{BL}(v_i, v_j) = \max\,[F_{AL}(v_i), F_{AL}(v_j)]$$

$$F_{BU}(v_i, v_j) = \max\,[F_{AU}(v_i), F_{AU}(v_j)]$$

for all v_i, v_j ∈ V, then G is self complementary.

Proof

Let G = (A, B) be a strong interval valued neutrosophic graph such that –

$$T_{BL}(v_i, v_j) = \min\,[T_{AL}(v_i),\ T_{AL}(v_j)];$$

$$T_{BU}(v_i, v_j) = \min\,[T_{AU}(v_i),\ T_{AU}(v_j)];$$

$$I_{BL}(v_i, v_j) = \max\,[I_{AL}(v_i),\ I_{AL}(v_j)];$$

$$I_{BU}(v_i, v_j) = \max\,[I_{AU}(v_i),\ I_{AU}(v_j)];$$

$$F_{BL}(v_i, v_j) = \max\,[F_{AL}(v_i), F_{AL}(v_j)];$$

$$F_{BU}(v_i, v_j) = \max\,[F_{AU}(v_i), F_{AU}(v_j)],$$

for all v_i, v_j ∈ V, then G≈ $\overline{\overline{G}}$ under the identity map *I*: V →V, hence G is self complementary.

Proposition 3.27

Let G be a self complementary interval valued neutrosophic graph. Then —

$$\sum_{v_i \neq v_j} T_{BL}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \min [T_{AL}(v_i), T_{AL}(v_j)]$$

$$\sum_{v_i \neq v_j} T_{BU}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \min [T_{AU}(v_i), T_{AU}(v_j)]$$

$$\sum_{v_i \neq v_j} I_{BL}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \max [I_{AL}(v_i), I_{AL}(v_j)]$$

$$\sum_{v_i \neq v_j} I_{BU}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \max [I_{AU}(v_i), I_{AU}(v_j)]$$

$$\sum_{v_i \neq v_j} F_{BL}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \max [F_{AL}(v_i), F_{AL}(v_j)]$$

$$\sum_{v_i \neq v_j} F_{BU}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \max [F_{AU}(v_i), F_{AU}(v_j)].$$

Proof

If G be a self complementary interval valued neutrosophic graph. Then there exist an isomorphism $f: V_1 \rightarrow V_1$ satisfying

$$\overline{T_{V_1}}(f(v_i)) = T_{V_1}(f(v_i)) = T_{V_1}(v_i)$$

$$\overline{I_{V_1}}(f(v_i)) = I_{V_1}(f(v_i)) = I_{V_1}(v_i)$$

$$\overline{\overline{F_{V_1}}}(f(v_i)) = F_{V_1}(f(v_i)) = F_{V_1}(v_i)$$

for all $v_i \in V_1$, and —

$$\overline{T_{E_1}}(f(v_i), f(v_j)) = T_{E_1}(f(v_i), f(v_j)) = T_{E_1}(v_i, v_j)$$

$$\overline{I_{E_1}}(f(v_i), f(v_j)) = I_{E_1}(f(v_i), f(v_j)) = I_{E_1}(v_i, v_j)$$

$$\overline{F_{E_1}}(f(v_i), f(v_j)) = F_{E_1}(f(v_i), f(v_j)) = F_{E_1}(v_i, v_j)$$

for all $(v_i, v_j) \in E_1$.

We have

$$\overline{T_{E_1}}(f(v_i), f(v_j)) = \min [\overline{T_{V_1}}(f(v_i)), \overline{T_{V_1}}(f(v_j))] - T_{E_1}(f(v_i), f(v_j))$$

i.e, $T_{E_1}(v_i, v_j) = \min [T_{V_1}(v_i), T_{V_1}(v_j)] - T_{E_1}(f(v_i), f(v_j))$

$$T_{E_1}(v_i, v_j) = \min [T_{V_1}(v_i), T_{V_1}(v_j)] - T_{E_1}(v_i, v_j).$$

That is —

$$\sum_{v_i \neq v_j} T_{E_1}(v_i, v_j) + \sum_{v_i \neq v_j} T_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \min [T_{V_1}(v_i), T_{V_1}(v_j)]$$

$$\sum_{v_i \neq v_j} I_{E_1}(v_i, v_j) + \sum_{v_i \neq v_j} I_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max [I_{V_1}(v_i), I_{V_1}(v_j)]$$

$$\sum_{v_i \neq v_j} F_{E_1}(v_i, v_j) + \sum_{v_i \neq v_j} F_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max [F_{V_1}(v_i), F_{V_1}(v_j)]$$

$$2\sum_{v_i \neq v_j} T_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \min\,[T_{V_1}(v_i), T_{V_1}(v_j)]$$
$$2\sum_{v_i \neq v_j} I_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max\,[I_{V_1}(v_i), I_{V_1}(v_j)]$$
$$2\sum_{v_i \neq v_j} F_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max\,[F_{V_1}(v_i), F_{V_1}(v_j)].$$

From these equations, *Proposition 3.27* holds.

Proposition 3.28

Let G_1 and G_2 be strong interval valued neutrosophic graph, $\overline{G_1} \approx \overline{G_2}$ (isomorphism).

Proof

Assume that G_1 and G_2 are isomorphic, there exists a bijective map $f\colon V_1 \rightarrow V_2$ satisfying

$$T_{V_1}(v_i) = T_{V_2}(f(v_i)),$$
$$I_{V_1}(v_i) = I_{V_2}(f(v_i)),$$
$$F_{V_1}(v_i) = F_{V_2}(f(v_i)),$$

for all $v_i \in V_1$, and

$$T_{E_1}(v_i, v_j) = T_{E_2}(f(v_i), f(v_j)),$$
$$I_{E_1}(v_i, v_j) = I_{E_2}(f(v_i), f(v_j)),$$
$$F_{E_1}(v_i, v_j) = F_{E_2}(f(v_i), f(v_j)),$$

for all $(v_i, v_j) \in E_1$.

By *Definition 3.21*, we have

$$\overline{T_{E_1}}(v_i, v_j) = \min\,[T_{V_1}(v_i), T_{V_1}(v_j)] - T_{E_1}(v_i, v_j)$$
$$= \min\,[T_{V_2}(f(v_i)),\ T_{V_2}(f(v_j))] - T_{E_2}(f(v_i), f(v_j)),$$
$$= \overline{T_{E_2}}(f(v_i), f(v_j)),$$
$$\overline{I_{E_1}}(v_i, v_j) = \max\,[I_{V_1}(v_i),\ I_{V_1}(v_j)] - I_{E_1}(v_i, v_j)$$
$$= \max[I_{V_2}(f(v_i)), I_{V_2}(f(v_j))] - I_{E_2}(f(v_i), f(v_j)),$$
$$= \overline{I_{E_2}}(f(v_i), f(v_j)),$$
$$\overline{F_{E_1}}(v_i, v_j) = \min\,[F_{V_1}(v_i), F_{V_1}(v_j)] - F_{E_1}(v_i, v_j)$$
$$= \min\,[F_{V_2}(f(v_i)), F_{V_2}(f(v_j))] - F_{E_2}(f(v_i), f(v_j)),$$
$$= \overline{F_{E_2}}(f(v_i), f(v_j)),$$

for all $(v_i, v_j) \in E_1$, hence $\overline{G_1} \approx \overline{G_2}$. The converse is straightforward.

4 Complete Interval Valued Neutrosophic Graphs

Definition 4.1

An interval valued neutrosophic graph G= (A, B) is called complete if

$T_{BL}(v_i, v_j) = \min(T_{AL}(v_i), T_{AL}(v_j))$, $T_{BU}(v_i, v_j) = \min(T_{AU}(v_i), T_{AU}(v_j))$,

$I_{BL}(v_i, v_j) = \max(I_A(v_i), I_A(v_j))$, $I_{BU}(v_i, v_j) = \max(I_{AU}(v_i), I_{AU}(v_j))$,

and

$F_{BL}(v_i, v_j) = \max(F_A(v_i), F_A(v_j))$, $F_{BU}(v_i, v_j) = \max(F_{AU}(v_i), F_{AU}(v_j))$,

for all $v_i, v_j \in$ V.

Example 4.2

Consider a graph G^* = (V, E) such that V = {v_1, v_2, v_3, v_4}, E = {v_1v_2, v_1v_3 , v_2v_3, v_1v_4, v_3v_4 , v_2v_4}, then G = (A, B) is a complete interval valued neutrosophic graph of G^*.

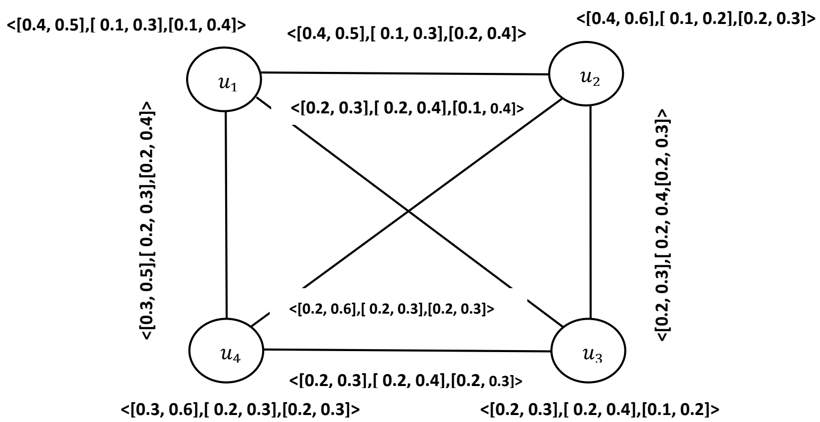

Figure17: Complete interval valued neutrosophic graph

Definition 4.3

The complement of a complete interval valued neutrosophic graph G = (A, B) of G^*= (V, E) is an interval valued neutrosophic complete graph $\bar{G}$= ($\bar{A}$, $\bar{B}$) on G^*= (V, $\bar{E}$), where

1. $\bar{V}$ =V

2. $\overline{T_{AL}}(v_i)$= $T_{AL}(v_i)$, $\overline{T_{AU}}(v_i)$= $T_{AU}(v_i)$, $\overline{I_{AL}}(v_i)$= $I_{AL}(v_i)$, $\overline{I_{AU}}(v_i)$= $I_{AU}(v_i)$, $\overline{F_{AL}}(v_i)$ = $F_{AL}(v_i)$, $\overline{F_{AU}}(v_i)$ = $F_{AU}(v_i)$, for all $v_j \in$ V.

3. $\overline{T_{BL}}(v_i, v_j)$= min $[T_{AL}(v_i), T_{AL}(v_j)]$ $- T_{BL}(v_i, v_j)$,

$\overline{T_{BU}}(v_i, v_j)$= min $[T_{AU}(v_i), T_{AU}(v_j)]$ $- T_{BU}(v_i, v_j)$,

$\overline{I_{BL}}(v_i, v_j)$= max $[I_{AL}(v_i), I_{AL}(v_j)]$ $- I_{BL}(v_i, v_j)$,

$\overline{I_{BU}}(v_i, v_j)$= max $[I_{AU}(v_i), I_{AU}(v_j)] - I_{BU}(v_i, v_j)$,

and

$\overline{F_{BL}}(v_i, v_j)$= max $[F_{AL}(v_i), F_{AL}(v_j)]$ $- F_{BL}(v_i, v_j)$,

$\overline{F_{BU}}(v_i, v_j)$= max $[F_{AU}(v_i), F_{AU}(v_j)]$ $- F_{BU}(v_i, v_j)$,

for all $(v_i, v_j) \in$ E.

Proposition 4.4

The complement of complete IVN-graph is a IVN-graph with no edge. Or if G is a complete, then in $\bar{G}$ the edge is empty.

Proof

Let G = (A, B) be a complete IVN-graph. So

$T_{BL}(v_i, v_j)$ = min$(T_{AL}(v_i), T_{AL}(v_j))$, $T_{BU}(v_i, v_j)$ = min$(T_{AU}(v_i), T_{AU}(v_j))$, $I_{BL}(v_i, v_j)$ = max $(I_{AL}(v_i), I_{AL}(v_j))$, $I_{BU}(v_i, v_j)$ = max $(I_{AU}(v_i), I_{AU}(v_j))$ and $F_{BL}(v_i, v_j)$ = max $(F_{AL}(v_i), F_{AL}(v_j))$, $F_{BU}(v_i, v_j)$ = max $(F_{AU}(v_i), F_{AU}(v_j))$, for all $v_i, v_j \in$ V

Hence in $\bar{G}$,

$\bar{T}_{BL}(v_i, v_j)$= min$(T_{AL}(v_i), T_{AL}(v_j)) - T_{BL}(v_i, v_j)$ for all i, j, ..., n

= min$(T_{AL}(v_i), T_{AL}(v_j))$ $-$ min$(T_{AL}(v_i), T_{AL}(v_j))$ for all i, j, …, n

= 0 for all i, j, ..., n

$\bar{T}_{BU}(v_i, v_j)$= min$(T_{AU}(v_i), T_{AU}(v_j)) - T_{BU}(v_i, v_j)$ for all i, j, ..., n

= min$(T_{AU}(v_i), T_{AU}(v_j))$ $-$ min$(T_{AU}(v_i), T_{AU}(v_j))$ for all i, j, ..., n

= 0 for all i, j, .., n.

and

$\bar{I}_{BL}(v_i, v_j)$= max$(I_{AL}(v_i), I_{AL}(v_j)) - I_{BL}(v_i, v_j)$ for all i, j, ..., n

= max$(I_{AL}(v_i), I_{AL}(v_j))$ $-$ max$(I_{AL}(v_i), I_{AL}(v_j))$ for all i, j, …, n

= 0 for all i, j, .., n

$\bar{I}_{BU}(v_i, v_j)$= $\max(I_{AU}(v_i), I_{AU}(v_j)) - I_{BU}(v_i, v_j)$ for all i, j, .., n

= $\max(I_{AU}(v_i), I_{AU}(v_j)) - \max(I_{AU}(v_i), I_{AU}(v_j))$ for all i, j, …, n

= 0 for all i, j, ..., n.

Also

$\bar{F}_{BL}(v_i, v_j)$= $\max(F_{AL}(v_i), F_{AL}(v_j)) - F_{BL}(v_i, v_j)$ for all i, j, ..., n.

= $\max(F_{AL}(v_i), I_{AL}(v_j)) - \max(F_{AL}(v_i), F_{AL}(v_j))$ for all i, j, …, n

= 0, for all i, j, ..., n.

$\bar{F}_{BU}(v_i, v_j)$= $\max(F_{AU}(v_i), F_{AU}(v_j)) - F_{BU}(v_i, v_j)$ for all i, j, ..., n

= $\max(F_{AU}(v_i), F_{AU}(v_j)) - \max(F_{AU}(v_i), F_{AU}(v_j))$ for all i, j, ...,n

= 0, for all i, j, ..., n.

Thus

([$\bar{T}_{BL}(v_i, v_j), \bar{T}_{BU}(v_i, v_j)$], [$\bar{I}_{BL}(v_i, v_j), \bar{I}_{BU}(v_i, v_j)$],
[$\bar{F}_{BL}(v_i, v_j), \bar{F}_{BU}(v_i, v_j)$]) = ([0, 0], [0, 0], [0, 0]).

Hence, the edge set of $\bar{G}$ is empty if G is a complete IVN-graph.

5 Conclusion

Interval valued neutrosophic sets is a generalization of the notion of fuzzy sets, intuitionistic fuzzy sets, interval valued fuzzy sets, interval valued intuitionstic fuzzy sets and single valued neutrosophic sets.

Interval valued neutrosophic model gives more precisions, flexibility and compatibility to the system as compared to the classical, fuzzy, intuitionistic fuzzy and single valued neutrosophic models.

In this paper, we have defined for the first time certain types of interval valued neutrosophic graphs, such as strong interval valued neutrosophic graph, constant interval valued neutrosophic graph and complete interval valued neutrosophic graphs.

In future study, we plan to extend our research to regular interval valued neutrosophic graphs and irregular interval valued neutrosophic.

References

[1] V. Devadoss, A. Rajkumar & N. J. P. Praveena. *A Study on Miracles through Holy Bible using Neutrosophic Cognitive Maps (NCMS)*, in "International Journal of Computer Applications", 69(3), 2013.

[2] Mohamed Ismayil and A. Mohamed Ali. *On Strong Interval-Valued Intuitionistic Fuzzy Graph*, in "International Journal of Fuzzy Mathematics and Systems", Volume 4, Number 2, 2014, pp. 161-168.

[3] Q. Ansari, R. Biswas & S. Aggarwal. *Neutrosophic classifier: An extension of fuzzy classifier*, in "Applied Soft Computing", 13, 2013, pp. 563-573, http://dx.doi.org/10.1016/j.asoc.2012.08.002.

[4] Aydoğdu, *On Similarity and Entropy of Single Valued Neutro-sophic Sets*, in "Gen. Math. Notes", Vol. 29, No. 1, 2015, pp. 67-74.

[5] Q. Ansari, R. Biswas & S. Aggarwal. *Neutrosophication of Fuzzy Models*, poster presentation at IEEE Workshop On Computational Intelligence: Theories, Applications and Future Directions (hosted by IIT Kanpur), 14th July 2013.

[6] Q. Ansari, R. Biswas & S. Aggarwal. *Extension to fuzzy logic representation: Moving towards neutrosophic logic - A new laboratory rat*, at the Fuzzy Systems IEEE International Conference, 2013, 1–8, doi:10.1109/FUZZ-IEEE.2013.6622412.

[7] Nagoor Gani and M. Basheer Ahamed. *Order and Size in Fuzzy Graphs*, in "Bulletin of Pure and Applied Sciences", Vol 22E, No. 1, 2003, pp. 145-148.

[8] Nagoor Gani, A. and S. Shajitha Begum. *Degree, Order and Size in Intuitionistic Fuzzy Graphs*, in "International Journal of Algorithms, Computing and Mathematics", (3)3, 2010.

[9] Nagoor Gani and S. R. Latha. *On Irregular Fuzzy Graphs*, in "Applied Mathematical Sciences", Vol. 6, no. 11, 2012, pp. 517-523.

[10] F. Smarandache. *Refined Literal Indeterminacy and the Multiplication Law of Sub-Indeterminacies*, in "Neutrosophic Sets and Systems", Vol. 9, 2015, pp. 58- 63.

[11] F. Smarandache. *Types of Neutrosophic Graphs and Neutrosophic Algebraic Structures together with their Applications in Technology*, seminar, Universitatea Transilvania din Brasov, Facultatea de Design de Produs si Mediu, Brasov, Romania, 6 June 2015.

[12] F. Smarandache, *Symbolic Neutrosophic Theory*, Europanova asbl, Brussels, 2015, 195p.

[13] F. Smarandache. *Neutrosophic set - a generalization of the intuitionistic fuzzy set,* Granular Computing, 2006 IEEE International Conference, (2006)38 – 42, DOI: 10.1109/GRC.2006.1635754.

[14] F. Smarandache. *A geometric interpretation of the neutrosophic set — A generalization of the intuitionistic fuzzy set*, Granular Computing (GrC), 2011 IEEE International Conference, 2011, pp. 602 – 606, DOI 10.1109/GRC.2011.6122665.

[15] Gaurav Garg, Kanika Bhutani, Megha Kumar and Swati Aggarwal. *Hybrid model for medical diagnosis using Neutrosophic Cognitive Maps with Genetic Algorithms*, FUZZ-IEEE 2015 (IEEE International conference on fuzzy systems).

[16] H. Wang,. Y. Zhang, R. Sunderraman. *Truth-value based interval neutrosophic sets*, Granular Computing, 2005 IEEE International Conference, Vol. 1, 2005, pp. 274 - 277
DOI: 10.1109/GRC.2005.1547284.

[17] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, *Single valued Neutrosophic Sets*, Multisspace and Multistructure 4, 2010, pp. 410-413.

[18] H. Wang, F. Smarandache, Y.Q. Zhang and R. Sunderram. *An Interval neutrosophic sets and logic: theory and applications in computing*. Hexis, Arizona, 2005.

[19] H. Y. Zhang , J. Q. Wang , X. H. Chen. *Interval neutrosophic sets and their application in multicriteria decision making problems*, The Scientific World Journal, (2014), DOI:10.1155/2014/ 645953.

[20] H, Zhang, J .Wang, X. Chen. *An outranking approach for multi-criteria decision-making problems with interval valued neutrosophic sets*, Neural Computing and Applications, 2015, pp. 1-13.

[21] H.Y. Zhang, P. Ji, J.Q. Wang & X. H. Chen. *An Improved Weighted Correlation Coefficient Based on Integrated Weight for Interval Neutrosophic Sets and its Application in Multi-criteria Decision-making Problems*, International Journal of Computational Intelligence Systems, V8, Issue 6 (2015), DOI:10.1080/18756891.2015.1099917.

[22] Deli, M. Ali, F. Smarandache. *Bipolar neutrosophic sets and their application based on multi-criteria decision making problems*, Advanced Mechatronic Systems (ICAMechS), International Conference, 2015, pp. 249 – 254.

[23] Turksen. *Interval valued fuzzy sets based on normal forms*, Fuzzy Sets and Systems, vol. 20, 1986, pp. 191-210 .

[24] J. Ye. *vector similarity measures of simplified neutrosophic sets and their application in multicriteria decision making*, International Journal of Fuzzy Systems, Vol. 16, No. 2, 2014, pp. 204-211.

[25] J. Ye. *Single-Valued Neutrosophic Minimum Spanning Tree and Its Clustering Method*, Journal of Intelligent Systems 23(3), 2014, pp. 311–324.

[26] J. Ye. *Similarity measures between interval neutrosophic sets and their applications in Multi-criteria decision-making*, Journal of Intelligent and Fuzzy Systems, 26, 2014, pp. 165-172.

[27] J. Ye. *Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute decision making*, Journal of Intelligent & Fuzzy Systems (27), 2014, pp. 2231-2241.

[28] K. Atanassov. *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, vol. 20, 1986, pp. 87-96.

[29] K. Atanassov and G. Gargov. *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, vol. 31, 1989, pp. 343-349.

[30] K. Atanassov. *Intuitionistic fuzzy sets: theory and applications*, Physica, New York, 1999.

[31] L. Zadeh. *Fuzzy sets*, Inform and Control, 8, 1965, pp. 338-353.

[32] M. Akram and B. Davvaz, *Strong intuitionistic fuzzy graphs*, Filomat, vol. 26, no. 1, 2012, pp. 177–196.

[33] M. Akram and W. A. Dudek, *Interval-valued fuzzy graphs*, Computers & Mathematics with Applications, vol. 61, no. 2, 2011, pp. 289–299.

[34] M. Akram, *Interval-valued fuzzy line graphs*, in "Neural Computing and Applications", vol. 21, pp. 145–150, 2012.

[35] M. Akram, *Bipolar fuzzy graphs*, Information Sciences, vol. 181, no. 24, 2011, pp. 5548–5564.

[36] M. Akram, *Bipolar fuzzy graphs with applications*, Knowledge Based Systems, vol. 39, 2013, pp. 1–8.

[37] M. Akram and A. Adeel, *m-polar fuzzy graphs and m-polar fuzzy line graphs*, Journal of Discrete Mathematical Sciences and Cryptography, 2015.

[38] P. Bhattacharya. *Some remarks on fuzzy graphs*, Pattern Recognition Letters 6, 1987, pp. 297-302.

[39] P. Liu and L. Shi. *The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multiple attribute decision making*, Neural Computing and Applications, 26 (2), 2015, pp. 457-471.

[40] R. Parvathi and M. G. Karunambigai. *Intuitionistic Fuzzy Graphs*, Computational Intelligence, Theory and applications, International Conference in Germany, Sept. 18-20, 2006.

[41] R. Rıdvan, A. Küçük,. *Subsethood measure for single valued neutrosophic sets*, Journal of Intelligent & Fuzzy Systems, vol. 29, no. 2, 2015, pp. 525-530, DOI: 10.3233/IFS-141304.

[42] R. Şahin. *Cross-entropy measure on interval neutrosophic sets and its applications in multicriteria decision making*, Neural Computing and Applications, 2015, pp. 1-11.

[43] S. Aggarwal, R. Biswas, A. Q. Ansari. *Neutrosophic modeling and control*, Computer and Communication Technology (ICCCT), International Conference, 2010, pp. 718 – 723, DOI: 10.1109/ ICCCT.2010.5640435

[44] S. Broumi, F. Smarandache. *New distance and similarity measures of interval neutrosophic sets*, Information Fusion (FUSION), 2014 IEEE 17th International Conference, 2014, pp. 1 – 7.

[45] S. Broumi, F. Smarandache, *Single valued neutrosophic trapezoid linguistic aggregation operators based multi-attribute decision making*, Bulletin of Pure & Applied Sciences - Mathematics and Statistics, 2014, pp. 135-155, DOI: 10.5958/2320-3226.2014.00006.X.

[46] S. Broumi, M. Talea, F. Smarandache, *Single Valued Neutrosophic Graphs: Degree, Order and Size*, 2016, submitted.

[47] S. Broumi, M. Talea, A. Bakali, F. Smarandache. *Single Valued Neutrosophic Graphs*, Journal of new theory, 2016, under process.

[48] S. N. Mishra and A. Pal, *Product of Interval Valued Intuitionistic fuzzy graph*, Annals of Pure and Applied Mathematics, Vol. 5, No. 1, 2013, pp. 37-46.

[49] Y. Hai-Long, G. She, Yanhonge, L. Xiuwu. *On single valued neutrosophic relations*, Journal of Intelligent & Fuzzy Systems, vol. Preprint, no. Preprint, 2015, pp. 1-12.

[50] W. B. Vasantha Kandasamy and F. Smarandache. *Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps*, 2013.

[51] W. B. Vasantha Kandasamy, K. Ilanthenral and Florentin Smarandache. *Neutrosophic Graphs: A New Dimension to Graph Theory*, Kindle Edition, 2015.

[52] W.B. Vasantha Kandasamy and F. Smarandache. *Analysis of social aspects of migrant laborers living with HIV/AIDS using Fuzzy Theory and Neutrosophic Cognitive Maps*, Xiquan, Phoenix, 2004.

[53] Florentin Smarandache, *Neutrosophy. Neutrosophic Probability, Set, and Logic*, Amer. Res. Press, Rehoboth, USA, 105 p., 1998; http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition); reviewed in Zentralblatt für Mathematik (Berlin, Germany), https://zbmath.org/?q=an:01273000

[54] F. Smarandache, *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics*, 168 p., Pons Editions, Bruxelles, Belgique, 2016, on Cornell University's website: https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf and in France at the international scientific database: https://hal.archives-ouvertes.fr/hal-01340830

Neutrosophic Crisp Probability Theory & Decision Making Process

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Abstract

Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. In neutrosophic set, indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are independent. For that purpose, it is natural to adopt the value from the selected set with highest degree of truth-membership, indeterminacy membership and least degree of falsity-membership on the decision set. These factors indicate that a decision making process takes place in neutrosophic environment. In this paper, we introduce and study the probability of neutrosophic crisp sets. After giving the fundamental definitions and operations, we obtain several properties and discuss the relationship between them. These notions can help researchers and make great use in the future in making algorithms to solve problems and manage between these notions to produce a new application or new algorithm of solving decision support problems. Possible applications to mathematical computer sciences are touched upon.

Keyword

Neutrosophic set, Neutrosophic probability, Neutrosophic crisp set, Intuitionistic neutrosophic set.

1 Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts [1, 2, 3, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 42] such as the neutrosophic set theory. The fundamental concepts of neutrosophic set, introduced by Smarandache in [37, 38, 39, 40], and Salama et al. in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21], provides a natural foundation for treating mathematically the neutrosophic phenomena which pervasively exist in our real world and for building new branches of neutrosophic mathematics.

In this paper, we introduce and study the probability of neutrosophic crisp sets. After giving the fundamental definitions and operations, we obtain several properties, and discuss the relationship between neutrosophic crisp sets and others.

2 Terminology

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [37, 38, 39, 40], and Salama et al. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Smarandache introduced the neutrosophic components *T, I, F* — which represent the membership, indeterminacy and non-membership values respectively, which are included into the nonstandard unit interval.

2.1 Example 2.1 [37, 39]

Let us consider a neutrosophic set, a collection of possible locations (positions) of particle *x* and let A and B two neutrosophic sets.

One can say, by language abuse, that any particle *x* neutrosophically belongs to any set, due to the percentages of truth/indeterminacy/falsity involved, which varies between $^{-}0$ and 1^{+}.

For example: *x* (0.5, 0.2, 0.3) belongs to A (which means a probability of 50% that the particle *x* is in A, a probability of 30% that *x* is not in A, and the rest is undecidable); or *y* (0, 0, 1) belongs to A (which normally means *y* is not for sure in A); or *z* (0, 1, 0) belongs to A (which means one does know absolutely nothing about *z* affiliation with A).

More general, *x*((0.2-0.3), (0.4—0.45) $\cup$ [0.50-0.51,{0.2,0.24,0.28}) belongs to the set, which means: with a probability in between 20-30%, the particle *x* is in a position of A (one cannot find an exact approximation because of various sources used); with a probability of 20% or 24% or 28%, *x* is not in A; the indeterminacy related to the appurtenance of *x* to A is in between 40-45% or between 50-51% (limits included).

The subsets representing the appurtenance, indeterminacy, and falsity may overlap, and, in this case, *n-sup* = 30% + 51% + 28% > 100.

Definition 2.1 [14, 15, 21]

A neutrosophic crisp set (NCS for short) $A = \langle A_1, A_2, A_3 \rangle$ can be identified to an ordered triple $\langle A_1, A_2, A_3 \rangle$ which are subsets on X, and every crisp set in X is obviously a NCS having the form $\langle A_1, A_2, A_3 \rangle$.

Definition 2.2 [21]

The object having the form $A = \langle A_1, A_2, A_3 \rangle$ is called

1) *Neutrosophic Crisp Set with Type I* if it satisfies $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$ (NCS-Type I for short).
2) *Neutrosophic Crisp Set with Type II* if it satisfies $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$ and $A_1 \cup A_2 \cup A_3 = X$ (NCS-Type II for short).
3) *Neutrosophic Crisp Set with Type III* if it satisfies $A_1 \cap A_2 \cap A_3 = \phi$ and $A_1 \cup A_2 \cup A_3 = X$ (NCS-Type III for short).

Definition 2.3

1. *Neutrosophic Set* [7]: Let X be a non-empty fixed set. A neutrosophic set (NS for short) A is an object having the form $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$, where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-membership (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where

$$0^- \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1^+$$

and

$$0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+.$$

2. *Neutrosophic Intuitionistic Set of Type 1* [8]: Let X be a non-empty fixed set. A neutrosophic intuitionistic set of type 1 (NIS1 for short) set A is an object having the form $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$, where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-membership (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where

$$0^- \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1^+$$

and the functions satisfy the condition

$$\mu_A(x) \wedge \sigma_A(x) \wedge \nu_A(x) \leq 0.5$$

and

$$0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+.$$

3. *Neutrosophic Intuitionistic Set of Type 2* [41]: Let X be a non-empty fixed set. A neutrosophic intuitionistic set of type 2 A (NIS2 for short) is an object having the form $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of

indeterminacy (namely $\sigma_A(x)$), and the degree of non-membership (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where

$$0.5 \le \mu_A(x), \sigma_A(x), \nu_A(x)$$

and the functions satisfy the condition

$$\mu_A(x) \wedge \sigma_A(x) \le 0.5,\ \mu_A(x) \wedge \nu_A(x) \le 0.5,\ \sigma_A(x) \wedge \nu_A(x) \le 0.5,$$

and

$$^{-}0 \le \mu_A(x) + \sigma_A(x) + \nu_A(x) \le 2^{+}.$$

A neutrosophic crisp with three types the object $A = \langle A_1, A_2, A_3 \rangle$ can be identified to an ordered triple $\langle A_1, A_2, A_3 \rangle$ which are subsets on X, and every crisp set in X is obviously a NCS having the form $\langle A_1, A_2, A_3 \rangle$. Every neutrosophic set $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ on X is obviously a NS having the form $\langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$.

Salama et al in [14, 15, 21] constructed the tools for developed neutrosophic crisp set and introduced the NCS ϕ_N, X_N in X.

Remark 2.1

The neutrosophic intuitionistic set is a neutrosophic set, but the neutrosophic set is not a neutrosophic intuitionistic set in general. Neutrosophic crisp sets with three types are neutrosophic crisp set.

3 The Probability of Neutrosophic Crisp Sets

If an experiment produces indeterminacy, that is called a neutrosophic experiment. Collecting all results, including the indeterminacy, we get the neutrosophic sample space (or the neutrosophic probability space) of the experiment. The neutrosophic power set of the neutrosophic sample space is formed by all different collections (that may or may not include the indeterminacy) of possible results. These collections are called neutrosophic events.

In classical experimental, the probability is

$$\left(\frac{\text{number of times event A occurs}}{\text{total number of trials}} \right).$$

Similarly, Smarandache in [16, 17, 18] introduced the Neutrosophic Experimental Probability, which is:

$$\left(\frac{\text{number of times event A occurs}}{\text{total number of trials}}, \frac{\text{number of times indeterminacy occurs}}{\text{total number of trials}}, \frac{\text{number of times event A does not occur}}{\text{total number of trials}}\right)$$

Probability of NCS is a generalization of the classical probability in which the chance that an event $A=\langle A_1, A_2, A_3\rangle$ to occur is:

$$P(A_1)\ true, P(A_2)\ \text{indeterminate}, P(A_3)\ false,$$

on a sample space X, or $NP(A)=\langle P(A_1), P(A_2), P(A_3)\rangle$.

A subspace of the universal set, endowed with a neutrosophic probability defined for each of its subsets, forms a probability neutrosophic crisp space.

Definition 3.1

Let X be a non- empty set and A be any type of neutrosophic crisp set on a space X, then the neutrosophic probability is a mapping $NP : X \to [0,1]^3$, $NP(A)=\langle P(A_1), P(A_2), P(A_3)\rangle$, that is the probability of a neutrosophic crisp set that has the property that —

$$NP(A)=\begin{cases}(p_1,p_2,p_3) & \text{where } p_{1,2,3} \in [0,1] \\ 0 & \text{if } p_1,p_2,p_3 < o\end{cases}.$$

Remark 3.1

1. In case if $A=\langle A_1, A_2, A_3\rangle$ is NCS, then

$$^-0 \le P(A_1)+P(A_2)+P(A_3) \le 3^+.$$

2. In case if $A=\langle A_1, A_2, A_3\rangle$ is NCS-Type I, then $0 \le P(A_1)+P(A_2)+P(A_3) \le 2$.

3. The Probability of NCS-Type II is a neutrosophic crisp set where

$$^-0 \le P(A_1)+P(A_2)+P(A_3) \le 2^+.$$

4. The Probability of NCS-Type III is a neutrosophic crisp set where

$$^-0 \le P(A_1)+P(A_2)+P(A_3) \le 3^+.$$

Probability Axioms of NCS Axioms

1. The Probability of neutrosophic crisp set and NCS-Type III A on X $NP(A)=\langle P(A_1), P(A_2), P(A_3)\rangle$ where $P(A_1)\ge 0, P(A_2)\ge 0, P(A_3)\ge 0$ or

$$NP(A)=\begin{cases}(p_1,p_2,p_3) & \text{where } p_{1,2,3} \in [0,1] \\ 0 & \text{if } p_1,p_2,p_3 < o\end{cases}.$$

2. The probability of neutrosophic crisp set and NCS-Type IIIs A on X $NP(A)=\langle P(A_1), P(A_2), P(A_3)\rangle$ where $^-0 \le p(A_1)+p(A_2)+p(A_3) \le 3^+$.

3. Bounding the probability of neutrosophic crisp set and NCS-Type III $NP(A) = \langle P(A_1), P(A_2), P(A_3) \rangle$ where $1 \geq P(A_1) \geq 0, P(A_2) \geq 0, P(A_3) \geq 0.$

4. Addition law for any two neutrosophic crisp sets or NCS-Type III

$$NP(A \cup B) =< (P(A_1) + P(B_1) - P(A_1 \cap B_1),$$
$$(P(A_2) + P(B_2) - P(A_2 \cap B_2),\ (P(A_3) + P(B_3) - P(A_3 \cap B_3) >$$

if

$A \cap B = \phi_N$, then $NP(A \cap B) = NP(\phi_N)$.

$$NP(A \cup B) =< NP(A_1) + NP(B_1) - NP(\phi_{N_1}), NP(A_2) + NP(B_2) - NP(\phi_{N_2}),$$
$$NP(A_3) + NP(B_3) - NP(\phi_{N_3}).$$

Since our main purpose is to construct the tools for developing probability of neutrosophic crisp sets, we must introduce the following —

1. Probability of neutrosophic crisp empty set with three types ($NP(\phi_N)$ for short) may be defined as four types:

Type 1: $NP(\phi_N) = \langle P(\phi), P(\phi), P(X) \rangle =< 0,0,1 >$;

Type 2: $NP(\phi_N) = \langle P(\phi), P(X), P(X) \rangle =< 0,1,1 >$;

Type 3: $NP(\phi_N) = \langle P(\phi), P(\phi), P(\phi) \rangle =< 0,0,0 >$;

Type 4: $NP(\phi_N) = \langle P(\phi), P(X), P(\phi) \rangle =< 0,1,0 >$.

2. Probability of neutrosophic crisp universal and NCS-Type III universal sets ($NP(X_N)$ for short) may be defined as four types —

Type 1: $NP(X_N) = \langle P(X), P(\phi), P(\phi) \rangle =< 1,0,0 >$;

Type 2: $NP(X_N) = \langle P(X), P(X), P(\phi) \rangle =< 1,1,0 >$;

Type 3: $NP(X_N) = \langle P(X), P(X), P(X) \rangle =< 1,1,1 >$;

Type 4: $NP(X_N) = \langle P(X), P(\phi), P(X) \rangle =< 1,0,1 >$.

Remark 3.2

$NP(X_N) = 1_N$, $NP(\phi_N) = O_N$, where $1_N, O_N$ are in Definition 2.1 [6], or equals any type for 1_N .

Definition 3.2 (Monotonicity)

Let X be a non-empty set, and NCSS A and B in the form $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$ with

$$NP(A) = \langle P(A_1), P(A_2), P(A_3) \rangle, NP(B) = \langle P(B_1), P(B_2), P(B_3) \rangle,$$

then we may consider two possible definitions for subsets ($A \subseteq B$) —

Type1:

$$NP(A) \leq NP(B) \Leftrightarrow P(A_1) \leq P(B_1), P(A_2) \leq P(B_2) \text{ and } \mathrm{P}(A_3) \geq P(B_3),$$

or Type2:

$$NP(A) \leq NP(B) \Leftrightarrow P(A_1) \leq P(B_1), P(A_2) \geq P(B_2) \text{ and } \mathrm{P}(A_3) \geq P(B_3).$$

Definition 3.3

Let X be a non-empty set, and NCSs A and B in the form $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$ be NCSs.

Then —

1. $NP(A \cap B)$ may be defined two types as —

Type1:

$$NP(A \cap B) = \langle P(A_1 \cap B_1), P(A_2 \cap B_2), P(A_3 \cup B_3) \rangle, \text{ or}$$

Type2:

$$NP(A \cap B) = \langle P(A_1 \cap B_1), P(A_2 \cup B_2), P(A_3 \cup B_3) \rangle.$$

2. $NP(A \cup B)$ may be defined two types as:

Type1:

$$NP(A \cup B) = \langle P(A_1 \cup B_1), P(A_2 \cap B_2), P(A_3 \cap B_3) \rangle,$$

or Type 2:

$$NP(A \cup B) = \langle P(A_1 \cup B_1), P(A_2 \cup B_2), P(A_3 \cap B_3) \rangle.$$

3. $NP(A^c)$ may be defined by three types:

Type1:

$$NP(A^c) = \left\langle P(A_1{}^c), P(A_2^c), P(A_3^c) \right\rangle = < (1 - A_1), (1 - A_2), (1 - A_3) >$$

or Type2:

$$NP(A^c) = \left\langle P(A_3), P(A_2^c), P(A_1) \right\rangle$$

or Type3:

$$NP(A^c) = \left\langle P(A_3), P(A_2), P(A_1) \right\rangle.$$

Proposition 3.1

Let A and B in the form $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle$ be NCSs on a non-empty set X.

Then —

$$NP(A)^c + NP(A) =< (1,1,1> \text{ or } NP(X_N) = 1_N \text{, or = any type of} 1_N .$$

$$NP(A-B) =< (P(A_1) - P(A_1 \cap B_1)), (P(A_2) - P(A_2 \cap B_2)),$$
$$(P(A_3) - P(A_3 \cap B_3)) >$$

$$NP(A/B) =< \frac{NP(A_1)}{NP(A_1 \cap B_1)}, \frac{NP(A_2)}{NP(A_2 \cap B_2)}, \frac{NP(A_3)}{NP(A_3 \cap B_3)} >.$$

Proposition 3.1

Let A and B in the form $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle$ are NCSs on a non-empty set X and p , p_N are NCSs.
Then

$$NP(p) = \left\langle \frac{1}{n(X)}, \frac{1}{n(X)}, \frac{1}{n(X)} \right\rangle;$$

$$NP(p_N) = \left\langle 0, \frac{1}{n(X)}, 1 - \frac{1}{n(X)} \right\rangle.$$

Example 3.1

1. Let $X = \{a,b,c,d\}$ and A, B are two neutrosophic crisp events on X defined by $A=\langle\{a\},\{b,c\},\{c,d\}\rangle$, $B=\langle\{a,b\},\{a,c\},\{c\}\rangle$, $p=\langle\{a\},\{c\},\{d\}\rangle$ then see that $NP(A)=\langle 0.25,0.5,0.5\rangle$, $NP(B)=\langle 0.5,0.5,0.25\rangle$, $NP(p)=\langle 0.25,0.25,0.25\rangle$, one can compute all probabilities from definitions.

2. If $A=\langle\{\phi\},\{b,c\},\{\phi\}\rangle$ and $B=\langle\{\phi\},\{d\},\{\phi\}\rangle$ are neutrosophic crisp sets on X. Then —

$$A \cap B = \langle\{\phi\},\{\phi\},\{\phi\}\rangle \text{ and } NP(A \cap B) = \langle 0,0,0\rangle = 0_N ,$$

$$A \cap B = \langle\{\phi\},\{b,c,d\},\{\phi\}\rangle \text{ and } NP(A \cap B) = \langle 0,0.75,0\rangle \neq 0_N .$$

Example 3.2

Let $X = \{a,b,c,d,e,f\}$,

$A=\langle\{a,b,c,d\},\{e\},\{f\}\rangle$, $D=\langle\{a,b\},\{e,c\},\{f,d\}\rangle$ be a NCS-Type 2,

$B = \langle \{a,b,c\},\{d\},\{e\} \rangle$ be a NCT-Type I but not NCS-Type II, III,

$C = \langle \{a,b\},\{c,d\},\{e,f,a\} \rangle$ be a NCS-Type III, but not NCS-Type I, II,

$E = \langle \{a,b,c,d,e\},\{c,d\},\{e,f,a\} \rangle$,

$F = \langle \{a,b,c,d,e\},\phi,\{e,f,a,d,c,b\} \rangle$.

We can compute the probabilities for NCSs by the following:

$$NP(A) = \left\langle \frac{4}{6},\frac{1}{6},\frac{1}{6} \right\rangle,$$

$$NP(D) = \left\langle \frac{2}{6},\frac{2}{6},\frac{2}{6} \right\rangle,$$

$$NP(B) = \left\langle \frac{3}{6},\frac{1}{6},\frac{1}{6} \right\rangle,$$

$$NP(C) = \left\langle \frac{2}{6},\frac{2}{6},\frac{3}{6} \right\rangle,$$

$$NP(E) = \left\langle \frac{4}{6},\frac{2}{6},\frac{3}{6} \right\rangle,$$

$$NP(F) = \left\langle \frac{5}{6},0,\frac{6}{6} \right\rangle.$$

Remark 3.3

The probabilities of a neutrosophic crisp set are neutrosophic sets.

Example 3.3

Let $X = \{a,b,c,d\}$, $A = \langle \{a,b\},\{c\},\{d\} \rangle$, $B = \langle \{a\},\{c\},\{d,b\} \rangle$ are NCS-Type I on X and $U_1 = \langle \{a,b\},\{c,d\},\{a,d\} \rangle$, $U_2 = \langle \{a,b,c\},\{c\},\{d\} \rangle$ are NCS-Type III on X; then we can find the following operations —

1. *Union, intersection, complement, difference and its probabilities.*

a) Type1: $A \cap B = \langle \{a\},\{c\},\{d,b\} \rangle$, $NP(A \cap B) = \langle 0.25,0.25,0.5\} \rangle$ and

Type 2,3: $A \cap B = \langle \{a\},\{c\},\{d,b\} \rangle$, $NP(A \cap B) = \langle 0.25,0.25,0.5\} \rangle$.

2. $NP(A-B)$ *may be equals.*

Type1: $NP(A-B) =< 0.25,0,0 >$, Type 2: $NP(A-B) =< 0.25,0,0 >$,

Type 3: $NP(A-B) =< 0.25,0,0 >$,

b) Type 2: $A \cup B = \langle \{a,b\},\{c\},\{d\} \rangle$, $NP(A \cup B) = \langle 0.5,0.25,0.25\} \rangle$ and

Type 2: $A \cup B = \langle \{a.b\},\{c\},\{d\} \rangle$ $NP(A \cup B) = \langle 0.5,0.25,0.25\} \rangle$.

c) Type1: $A^c = \langle \{c,d\} \ ,\{a,b,d\},\{a,b,c\}\rangle$ NCS-Type III set on X,

$NP(A^c) = \langle 0.5, 0.75, 0.75 \rangle$.

Type2: $A^c = \langle \{d\} \ ,\{a,b,d\},\{a,b\}\rangle$ NCS-Type III on X,

$NP(A^c) = \langle 0.25, 0.75, 0.5 \rangle$.

Type3: $A^c = \langle \{d\} \ ,\{c\},\{a,b\}\rangle$ NCS-Type III on X,

$NP(A^c) = \langle 0.75, 0.75, 0.5 \rangle$.

d) Type 1: $B^c = \langle \{b,c,d\},\{a,b,d\},\{a,c\}\rangle$ be NCS-Type III on X ,

$NP(B^c) = \langle 0.75, 0.75, 0.5 \rangle$

Type 2: $B^c = \langle \{b,d\},\{c\},\{a\}\rangle$ NCS-Type I on X, and $NP(B^c) =$ $\langle 0.5, 0.25, 0.25 \rangle$.

Type 3: $B^c = \langle \{b,d\},\{a,b,d\},\{a\}\rangle$ NCS-Type III on X and $NP(B^c) =$ $\langle 0.5, 0.75, 0.25 \rangle$.

e) Type 1: $U_1 \cup U_2 = \langle \{a,b,c\},\{c,d\},\{a,d\}\rangle$, NCS-Type III,

$NP(U_1 \cup U_2) = \langle \{0.75, 0.5, 0.5 \rangle$,

Type 2: $U_1 \cup U_2 = \langle \{a,b,c\},\{c\},\{a,d\}\rangle$, $NP(U_1 \cup U_2) = \langle \{0.75, 0.25, 0.5 \rangle$.

f) Type1: $U_1 \cap U_2 = \langle \{a,b\},\{c,d\},\{a,d\}\rangle$, NCS-Type III,

$NP(U_1 \cap U_2) = \langle 0.5, 0.5, 0.5 \rangle$,

Type2: $U_1 \cap U_2 = \langle \{a,b\},\{c\},\{a,d\}\rangle$, NCS-Type III, and

$NP(U_1 \cap U_2) = \langle 0.5, 0.25, 0.5 \rangle$,

g) Type 1: ${U_1}^c = \langle \{c,d\},\{a,b\},\{c,b\}\rangle$, NCS-Type III and

$NP({U_1}^c) = \langle 0.5, 0.5, 0.5 \rangle$

Type 2: ${U_1}^c = \langle \{a,d\},\{c,d\},\{a,b\}\rangle$, NCS-Type III and

$NP({U_1}^c) = \langle 0.5, 0.5, 0.5 \rangle$

Type3: ${U_1}^c = \langle \{a,d\},\{a,b\},\{a,b\}\rangle$, NCS-Type III and

$NP({U_1}^c) = \langle 0.5, 0.5, 0.5 \rangle$.

h) Type1: ${U_2}^c = \langle \{d\},\{a,b,d\},\{a,b,c\}\rangle$ NCS-Type III and

$NP({U_2}^c) = \langle 0.25, 0.75, 0.75 \rangle$, Type2: $U^c{}_2 = \langle \{d\},\{c\},\{a,b,c\}\rangle$ NCS-Type III and $NP({U_2}^c) = \langle 0.25, 0.25, 0.75 \rangle$, Type3: $U^c{}_2 = \langle \{d\},\{a,b,d\},\{a,b,c\}\rangle$ NCS-Type III. $NP({U_2}^c) = \langle 0.25, 0.75, 0.75 \rangle$.

3. *Probabilities for events*.

$NP(A)=\langle 0.5,0.25,0.25\rangle$, $NP(B)=\langle 0.25,0.25,0.5\rangle$, $NP(U_1)=\langle 0.5,0.5,0.5\rangle$,

$NP(U_2)=\langle 0.75,0.25,0.25\rangle$

$NP(U_1{}^c)=\langle 0.5,0.5,0.5\rangle$, $NP(U_2{}^c)=\langle 0.25,0.75,0.75\rangle$.

e) $(A\cap B)^c=\langle \{b,c,d\},\{a,b,d\},\{a,c\}\rangle$ be a NCS-Type III.

$NP(A\cap B)^c=\langle 0.75,0.75,0.25\rangle$ be a neutrosophic set.

f) $NP(A)^c\cap NP(B)^c=\langle 0.5,0.75,0.75\rangle$,

$NP(A)^c\cup NP(B)^c=\langle 0.75,0.75,0.5\rangle$

g) $NP(A\cup B)=NP(A)+NP(B)-NP(A\cap B)=\langle 0.5,0.25,0.25\}\rangle$

h) $NP(A)=\langle 0.5,0.25,0.25\rangle$, $NP(A)^c=\langle 0.5,0.75,0.75\rangle$,

$NP(B)=\langle 0.25,0.25,0.5\rangle$, $NP(B^c)=\langle 0.75,0.75,0.5\rangle$

4. *Probabilities for Products*. The product of two events given by —

$A\times B=\langle\{(a,a),(b,a)\},\{(c,c)\},\{(d,d),(d,b)\}\rangle$,

and $NP(A\times B)=\langle 2/16, 1/16, 2/16\rangle$

$B\times A=\langle\{(a,a),(a,b)\},\{(c,c)\},\{(d,d),(b,d)\}\rangle$

and $NP(B\times A)=\langle 2/16, 1/16, 2/16\rangle$

$A\times U_1=\langle\{(a,a),(b,a),(a,b),(b,b)\},\{(c,c),(c,d)\},\{(d,d),(d,a)\}\rangle$,

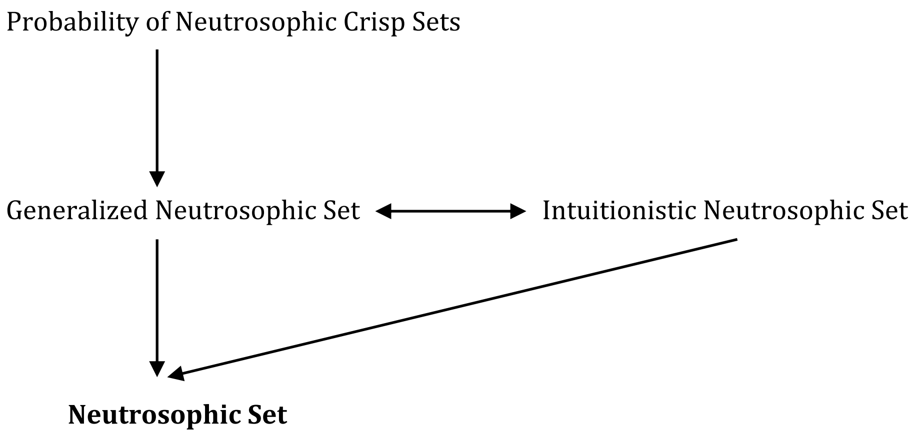
and $NP(A\times U_1)=\langle 4/16, 2/16, 2/16\rangle$

$U_1\times U_2=\langle\{(a,a),(b,a),(a,b),(b,b),(a,c),(b,c)\},\{(c,c),(d,c)\},\{(d,d),(a,d)\}\rangle$

and $NP(U_1\times U_2)=\langle 6/16, 2/16, 2/16\rangle$.

Remark 3.4

The following diagram represents the relation between neutrosophic crisp concepts and neutrosphic sets:



References:

[1] K. Atanassov. *Intuitionistic fuzzy sets*, in V. Sgurev, ed., ITKRS Session, Sofia, June 1983, Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984.

[2] K. Atanassov. *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 2087-96, 1986.

[3] K. Atanassov. *Review and new result on intuitionistic fuzzy sets*, preprint IM-MFAIS-1-88, Sofia, 1988.

[4] A. A. Salama. *Basic Structure of Some Classes of Neutrosophic Crisp Nearly Open Sets and Possible Application to GIS Topology*, Neutrosophic Sets and Systems, 2015, Vol. 7, pp18-22.

[5] A. A. Salama, Mohamed Eisa, S. A. Elhafeez and M. M. Lotfy. *Review of Recommender Systems Algorithms Utilized in Social Networks based e-Learning Systems & Neutrosophic System*, Neutrosophic Sets and Systems, 2015, Vol. 8, pp. 35-44.

[6] A. A. Salama and Said Broumi. *Roughness of Neutrosophic Sets*, Elixir Appl. Math. 74, 2014, pp. 33-37.

[7] A. A. Salama, Mohamed Abdelfattah and Mohamed Eisa. *Distances, Hesitancy Degree and Flexible Querying via Neutrosophic Sets*, International Journal of Computer Applications, Volume 101– No. 10, 2014, pp. 75–87.

[8] M. M. Lofty, A. A. Salama, H. A. El-Ghareeb and M. A. Eldosuky. *Subject Recommendation Using Ontology for Computer Science ACM Curricula*, International Journal of Information Science and Intelligent System, Vol. 3, 2014, pp. 199-205

[9] A.A. Salama, Haithem A. El-Ghareeb, Ayman M. Maine and Florentin Smarandache. *Introduction to Develop Some Software Programs for dealing with Neutrosophic Sets*, Neutrosophic Sets and Systems, 2014,Vol. 4, pp. 51-52.

[10] A. A. Salama, F. Smarandache, and M. Eisa. *Introduction to Image Processing via Neutrosophic Technique*, Neutrosophic Sets and Systems, 2014, Vol. 5, pp. 59-63.

[11] S. A. Alblowi, A.A. Salama and Mohmed Eisa. *New Concepts of Neutrosophic Sets*, International Journal of Mathematics and Computer Applications Research (IJMCAR), Vol. 4, Issue 1, 2014, pp. 59-66.

[12] A. A. Salama, Mohamed Eisa and M. M. Abdelmoghny. *Neutrosophic Relations Database*, International Journal of Information Science and Intelligent System, 3(1), 2014, pp. 33-46 .

[13] A. A. Salama, Haitham A. El-Ghareeb, Ayman M. Manie and M. M. Lotfy. *Utilizing Neutrosophic Set in Social Network Analysis e-Learning Systems*,

International Journal of Information Science and Intelligent System, 3(2), 2014, pp. 61-72.

[14] I. M. Hanafy, A. A. Salama and K. Mahfouz. *Correlation of Neutrosophic Data*, International Refereed Journal of Engineering and Science (IRJES) , Vol. 1, Issue 2, 2012, pp. 39-33.

[15] I. M. Hanafy, A. A. Salama and K. M. Mahfouz. *Neutrosophic Classical Events and Its Probability*, International Journal of Mathematics and Computer Applications Research (IJMCAR), Vol. 3, Issue 1, March 2013, pp. 171-178.

[16] A. A. Salama and S. A. Alblowi. *Generalized Neutrosophic Set and Generalized Neutrosophic Topological Spaces*, Journal Computer Sci. Engineering, Vol. 2, No. 7, 2012, pp. 129-132.

[17] A. A. Salama and S. A. Alblowi. *Neutrosophic Set and Neutrosophic Topological Spaces*, ISOR J. Mathematics, Vol. 3, Issue 3, 2012, pp. 31-35.

[18] A. A. Salama. *Neutrosophic Crisp Point & Neutrosophic Crisp Ideals*, Neutrosophic Sets and Systems, Vol.1, No. 1, 2013, pp. 50-54.

[19] A. A. Salama and F. Smarandache. *Filters via Neutrosophic Crisp Sets*, Neutrosophic Sets and Systems, Vol. 1, No. 1, 2013, pp. 34-38.

[20] A.A. Salama, and H. Elagamy. *Neutrosophic Filters*, International Journal of Computer Science Engineering and Information Technology Reseearch (IJCSEITR), Vol. 3, Issue 1, 2013, pp. 307-312.

[21] A. A. Salama, F. Smarandache and Valeri Kroumov. *Neutrosophic crisp Sets & Neutrosophic Crisp Topological Spaces*, Neutrosophic Sets and Systems, Vol. 2, pp. 25-30, 2014.

[22] A. A. Salama, Mohamed Eisa and M. M. Abdelmoghny. *Neutrosophic Relations Database*, International Journal of Information Science and Intelligent System, 3 (1), 2014.

[23] A. A. Salama, Florentin Smarandache and S. A. Alblowi. *New Neutrosophic Crisp Topological Concepts*, Neutrosophic Sets and Systems, (Accepted, 2014) .

[24] A. A. Salama, Said Broumi and Florentin Smarandache. *Neutrosophic Crisp Open Set and Neutrosophic Crisp Continuity via Neutrosophic Crisp Ideals*, I.J. Information Engineering and Electronic Business, 2014, Vol. 3, pp1-8.

[25] A. A. Salama, Said Broumi and Florentin Smarandache. *Some Types of Neutrosophic Crisp Sets and Neutrosophic Crisp Relations*, I.J. Information Engineering and Electronic Business, 2014,

[26] A. A. Salama, Haithem A. El-Ghareeb, Ayman. M. Maine and Florentin Smarandache. *Introduction to Develop Some Software Programes for dealing with Neutrosophic Sets*, Neutrosophic Sets and Systems, 2014,Vol. 3, pp. 51-52.

[27] A. A. Salama, Florentin Smarandache and S.A. Alblowi. *The Characteristic Function of a Neutrosophic Set*, Neutrosophic Sets and Systems, 2014, Vol. 3, pp. 14-18.

[28] A. A. Salama, Mohamed Abdelfattah and Mohamed Eisa. *Distances, Hesitancy Degree and Flexible Querying via Neutrosophic Sets*, International Journal of Computer Applications, Issue 4, Vol. 3, May 2014.

[29] A. A. Salama, F. Smarandache and Valeri Kroumov. *Neutrosophic Closed Set and Continuous Functions*, Neutrosophic Sets and Systems, 2014.

[30] A. A. Salama, Said Broumi and Florentin Smarandache. *Some Types of Neutrosophic Crisp Sets and Neutrosophic Crisp Relations*, I. J. Information Engineering and Electronic Business, 2014

[31] A. A. Salama, Florentin Smarandache, *Neutrosophic Ideal Theory Neutrosophic Local Function and Generated Neutrosophic Topology*, In Neutrosophic Theory and Its Applications. Collected Papers, Volume 1, EuropaNova, Bruxelles, 2014, pp. 213-218.

[32] M. E. Abd El-Monsef, A.A. Nasef, A. A. Salama. *Extensions of fuzzy ideals*, Bull. Calcutta Math. Soc. 92, No. 3, pp. 181-188, 2000.

[33] M.E. Abd El-Monsef, A.A. Nasef, A.A. Salama. *Some fuzzy topological operators via fuzzy ideals,* Chaos Solitons Fractals, Vol. 12, No. 13, pp. 2509-2515, 2001.

[34] M. E. Abd El-Monsef, A. A. Nasef, A. A. Salama. *Fuzzy L-open sets and fuzzy L-continuous functions*, Analele Universitatii de Vest din Timisoara, Seria Matematica-Informatica, Vol. 4, No. 2, pp. 3-13, 2002.

[35] I. M. Hanafy and A.A. Salama. *A unified framework including types of fuzzy compactness*, Conference Topology and Analysis in Applications Durban, 12-16 July, 2004. School of Mathematical Sciences, UKZN.

[36] A.A. Salama. *Fuzzy Hausdorff spaces and fuzzy irresolute functions via fuzzy ideals,* V Italian-Spanish Conference on General Topology and its Applications June 21-23, 2004, Almeria, Spain

[37] M.E. Abdel Monsef, A. Kozae, A. A. Salama and H. Elagamy. *Fuzzy Ideals and Bigranule Computing,* 20th Conference of Topology and its Applications 2007, Port Said Univ., Egypt.

[38] A.A. Salama. *Intuitionistic Fuzzy Ideals Theory and Intuitionistic Fuzzy Local Functions,* CTAC'08 the 14th Biennial Computational Techniques and Applications Conference13–16th July 2008. Australian National University, Canberra, ACT, Australia.

[39] A.A. Salama. *Fuzzy Bitopological Spaces Via Fuzzy Ideals*, Blast 2008, August 6-10, 2008, University of Denver, Denver, CO, USA.

[40] A.A. Salama. *A New Form of Fuzzy Compact spaces and Related Topics via Fuzzy Idealization,* Journal of fuzzy System and Mathematics Vol. 24, No. 2, 2010, pp 33-39.

[41] A. A. Salama and A. Hassan. *On Fuzzy Regression Model*, The Egyptian Journal for commercial Studies, Volume 34, No. 4. pp. 305-319, 2010.

[42] A.A. Salama and S.A. Alblowi. *Neutrosophic Set Theory and Neutrosophic Topological Ideal Spaces,* The First International Conference on Mathematics and Statistics, ICMS'10.

[43] A.A. Salama. *A New Form of Fuzzy Hausdorff Space and Related Topics via Fuzzy Idealization,* IOSR Journal of Mathematics (IOSR-JM), Volume 3, Issue 5 (Sep.-Oct. 2012), pp. 1-4.

[44] A. A. Salama and F. Smarandache. *Neutrosophic Crisp Set Theory*, Education Publishing, 2015.

[45] M. E. Abd El-Monsef, A. M. Kozae, A. A. Salama, H. Elagamy. *Fuzzy Biotopolgical Ideals Theory*, IOSR Journal of Computer Engineering (IOSRJCE), Vol. 6, Issue 4, 2012, pp. 1-5.

[46] I. M. Hanafy, A.A. Salama , M. Abdelfattah and Y. Wazery. *Security in Mant Based on Pki using Fuzzy Function,* IOSR Journal of Computer Engineering, Vol. 6, Issue 3, 2012, pp. 54-60.

[47] M. E. Abd El-Monsef, A. Kozae, A.A. Salama, and H. M. Elagamy. *Fuzzy Pairwise L-Open Sets and Fuzzy Pairwise L-Continuous Functions,* International Journal of Theoretical and Mathematical Physics, Vol. 3, No. 2, March 2013, pp. 69-72.

[48] Florentin Smarandache. *Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics,* University of New Mexico, Gallup, NM 87301, USA, 2002.

[49] Florentin Smarandache. *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic crisp Set, Neutrosophic Probability*. American Research Press, Rehoboth, NM, 1999.

[50] Florentin Smarandache. *Neutrosophic set, a generalization of the intuituionistics fuzzy sets*, Inter. J. Pure Appl. Math., 24, 2005, pp. 287 – 297.

[51] Florentin Smarandache. *Introduction To Neutrosophic Measure, Neutrosophic Integral and Neutrosophic Probability,* 2015 http://fs.gallup. unm.edu/ eBooks-otherformats.htm

[52] M. Bhowmik and M. Pal. *Intuitionistic Neutrosophic Set Relations and Some of its Properties*, Journal of Information and Computing Science, 5(3), pp. 183-192, 2010.

[53] L.A. Zadeh, *Fuzzy Sets*, Inform and Control, 8, pp. 338-353, 1965.

[54] F. Smarandache, *Operators on Single-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets*, Journal of Mathematics and Informatics, Vol. 5, 63-67, 2016; https://hal.archives-ouvertes.fr/hal-01340833.

On Strong Interval Valued Neutrosophic Graphs

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Abstract

In this paper, we discuss a subclass of interval valued neutrosophic graphs called strong interval valued neutrosophic graphs, which were introduced by Broumi et al. [41]. The operations of Cartesian product, composition, union and join of two strong interval valued neutrosophic graphs are defined. Some propositions involving strong interval valued neutrosophic graphs are stated and proved.

Keyword

Single valued neutrosophic graph, Interval valued neutrosophic graph, Strong interval valued neutrosophic graph, Cartesian product, Composition, Union, Join.

1 Introduction

Neutrosophic set proposed by Smarandache [13, 14] is a powerful tool to deal with incomplete, indeterminate and inconsistent information in real world. It is a generalization of the theory of fuzzy set [30], intuitionistic fuzzy sets [27, 29], interval-valued fuzzy sets [22] and interval-valued intuitionistic fuzzy sets [28]. The neutrosophic set is characterized by a truth-membership degree (t), an indeterminacy-membership degree (i) and a falsity-membership degree (f) independently, which are within the real standard or nonstandard unit interval $]^-0, 1^+[$. Therefore, if their range is restrained within the real standard unit interval [0, 1], the neutrosophic set is easily applied to engineering problems. For this purpose, Smarandache [48] and Wang et al. [17] introduced the concept of a single valued neutrosophic set (SVNS) as a subclass of the neutrosophic set. The same authors introduced the notion of interval valued neutrosophic sets [18] as subclass of neutrosophic sets in which the value of truth-membership, indeterminacy-membership and falsity-membership degrees are intervals of numbers instead of the real numbers. Recently, the concept of single valued neutrosophic set and interval valued neutrosophic sets have been applied in a wide variety of fields including computer science, enginnering, mathematics, medicine and economics [3, 4, 5, 6, 16, 19, 20, 21, 23, 24, 25, 26, 32, 34, 35, 36, 37, 38, 43].

Lots of works on fuzzy graphs and intuitionistic fuzzy graphs [7, 8, 9, 31, 33] have been carried out and all of them have considered the vertex sets and edge sets as fuzzy and /or intuitionistic fuzzy sets. But, when the relations between nodes (or vertices) in problems are indeterminate, the fuzzy graphs and intuitionistic fuzzy graphs fail.

For this purpose, Smarandache [10, 11] defined four main categories of neutrosophic graphs. Two are based on literal indeterminacy (I), called I-edge neutrosophic graph and I-vertex neutrosophic graph; these concepts are studied deeply and has gained popularity among the researchers due to their applications via real world problems [1, 12, 15, 44, 45, 46]. The two others graphs are based on (t, i, f) components and are called (t, i, f)-edge neutrosophic graph and (t, i, f)-vertex neutrosophic graph; these concepts are not developed at all. Later on, Broumi et al. [40] introduced a third neutrosophic graph model combining the (t, i, f)-edge and and the (t, i, f)-vertex neutrosophic graph, and investigated some of their properties. The third neutrosophic graph model is called single valued neutrosophic graph (SVNG for short).

The single valued neutrosophic graph is the generalization of fuzzy graph and intuitionistic fuzzy graph. The same authors [39] introduced neighborhood degree of a vertex and closed neighborhood degree of vertex in single valued neutrosophic graph as a generalization of neighborhood degree of a vertex and closed neighborhood degree of vertex in fuzzy graph and intuitionistic fuzzy graph. Broumi et al. [41] introduced the concept of interval valued neutrosophic graph, which is a generalization of fuzzy graph, intuitionistic fuzzy graph, interval valued fuzzy graph, interval valued intuitionistic fuzzy graph and single valued neutrosophic graph. Also, Broumi et al. [42] studied some operations on interval valued neutrosophic graphs.

In this paper, motivated by the operations on (crisp) graphs, such as Cartesian product, composition, union and join, we define the operations of Cartesian product, composition, union and join on strong interval valued neutrosophic graphs and investigate some of their properties.

2 Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic sets, fuzzy graph and intuitionistic fuzzy graph, interval valued intuitionstic fuzzy graph and interval valued neutrosophic graph, relevant to the present work.

See especially [2, 7, 8, 13, 17, 40, 41] for further details and background.

Definition 2.1 [13]

Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ X}, where the functions T, I, F: X→]−0,1+[define respectively the a truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element x ∈ X to the set A with the condition:

$$^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}. \quad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of]−0,1+[.

Since it is difficult to apply NSs to practical problems, Smarandache [48] and Wang et al. [16] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [17]

Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$.

For each point x in X $T_A(x)$, $I_A(x)$, $F_A(x)$ ∈ [0, 1].

A SVNS A can be written as —

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\}. \quad (2)$$

Definition 2.3 [7]

A fuzzy graph is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ. i.e σ : V → [0,1] and μ: VxV→[0,1] such that μ(uv) ≤ σ(u) ⋀ σ(v) for all u, v ∈ V where *uv* denotes the edge between *u* and *v* and σ(u) ⋀ σ(v) denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of V and μ is called the fuzzy edge set of E.

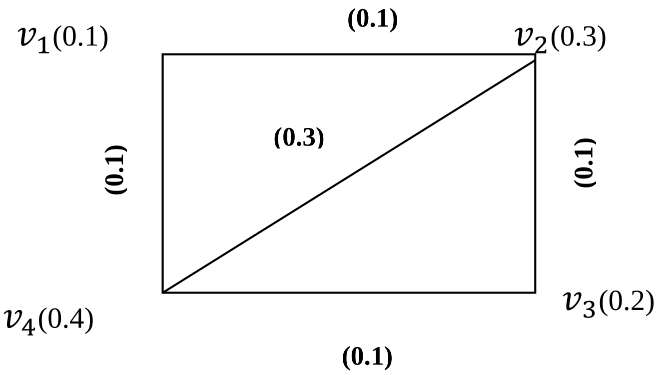

Figure 1. Fuzzy Graph

Definition 2.4 [7]

The fuzzy subgraph H = (τ, ρ) is called a fuzzy subgraph of G = (σ, μ), if $\tau(u) \leq \sigma(u)$ for all u ∈ V and $\rho(u, v) \leq \mu(u, v)$ for all u, v ∈ V.

Definition 2.5 [8]

An intuitionistic fuzzy graph is of the form G = (V, E), where

i. V={$v_1, v_2, \ldots, v_n$} such that μ_1: V→ [0,1] and γ_1: V → [0,1] denote the degree of membership and nonmembership of the element $v_i \in V$, respectively, and $0 \leq \mu_1(v_i) + \gamma_1(v_i)) \leq 1$ for every $v_i \in V$, (i = 1, 2, …, n),

ii. E ⊆ V x V where μ_2: VxV→[0,1] and γ_2: VxV→ [0,1] are such that $\mu_2(v_i, v_j) \leq \min [\mu_1(v_i), \mu_1(v_j)]$ and $\gamma_2(v_i, v_j) \geq \max [\gamma_1(v_i), \gamma_1(v_j)]$ and $0 \leq \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \leq 1$ for every $(v_i, v_j) \in E$, (i, j = 1,2, …, n)

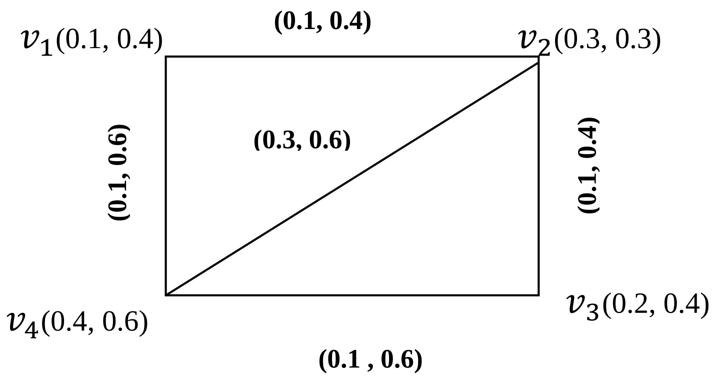

Figure 2. Intuitionistic Fuzzy Graph

Definition 2.6 [40]

Let A = (T_A, I_A, F_A) and B = (T_B, I_B, F_B) be single valued neutrosophic sets on a set X. If A = (T_A, I_A, F_A) is a single valued neutrosophic relation on a set X, then A =(T_A, I_A, F_A) is called a single valued neutrosophic relation on B = (T_B, I_B, F_B) if

$T_B(x, y) \leq \min(T_A(x), T_A(y))$

$I_B(x, y) \geq \max(I_A(x), I_A(y))$ and

$F_B(x, y) \geq \max(F_A x), F_A(y))$

for all x, y ∈ X.
A single valued neutrosophic relation A on X is called symmetric if $T_A(x, y) = T_A(y, x)$, $I_A(x, y) = I_A(y, x)$, $F_A(x, y) = F_A(y, x)$ and $T_B(x, y) = T_B(y, x)$, $I_B(x, y) = I_B(y, x)$ and $F_B(x, y) = F_B(y, x)$, for all x, y ∈ X.

Definition 2.7 [2]

An interval valued intuitionistic fuzzy graph with underlying set V is defined to be a pair G = (A, B), where

1) The functions M_A : V→ D [0, 1] and N_A : V→ D [0, 1] denote the degree of membership and non membership of the element x ∈ V, respectively, such that 0 such that $0 \leq M_A(x)+ N_A(x) \leq 1$ for all x ∈ V.

2) The functions $M_B : E \subseteq V \times V \rightarrow D$ [0, 1] and $N_B : E \subseteq V \times V \rightarrow D$ [0, 1] are defined by

$M_{BL}(x, y)) \leq \min\ (M_{AL}(x), M_{AL}(y))$ and $N_{BL}(x, y)) \geq \max\ (N_{AL}(x), N_{AL}(y))$,

$M_{BU}(x, y)) \leq \min\ (M_{AU}(x), M_{AU}(y))$ and $N_{BU}(x, y)) \geq \max\ (N_{AU}(x), N_{AU}(y))$,

such that

$0 \leq M_{BU}(x, y)) + N_{BU}(x, y)) \leq 1$,

for all $(x, y) \in E$.

Definition 2.8 [41]

By an interval-valued neutrosophic graph of a graph G^* = (V, E) we mean a pair G = (A, B), where A = <[T_{AL}, T_{AU}], [I_{AL}, I_{AU}], [F_{AL}, F_{AU}]> is an interval-valued neutrosophic set on V and B =< [T_{BL}, T_{BU}], [I_{BL}, I_{BU}], [F_{BL}, F_{BU}]> is an interval-valued neutrosophic relation on E satisfies the following conditions:

1. V= $\{v_1, v_2, \ldots, v_n\}$ such that T_{AL} :V → [0, 1], T_{AU} :V → [0, 1], I_{AL} :V → [0, 1], I_{AU}:V→[0, 1] and F_{AL}:V→[0, 1], F_{AU}:V→[0, 1] denote the degree of truth-membership, the degree of indeterminacy-membership and falsity-membership of the element y ∈ V, respectively, and

$0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$ for all $v_i \in$ V (I = 1, 2, …, n).

2. The functions T_{BL}:V x V →[0, 1], T_{BU}:V x V →[0, 1], I_{BL}:V x V →[0, 1], I_{BU}:V x V →[0, 1] and F_{BL}:V x V →[0,1], F_{BU}:V x V →[0, 1] are such that

$$T_{BL}(\{v_i, v_j\}) \leq \min\,[T_{AL}(v_i), T_{AL}(v_j)],$$

$$T_{BU}(\{v_i, v_j\}) \leq \min\,[T_{AU}(v_i), T_{AU}(v_j)],$$

$$I_{BL}(\{v_i, v_j\}) \geq \max[I_{BL}(v_i), I_{BL}(v_j)],$$

$$I_{BU}(\{v_i, v_j\}) \geq \max[I_{BU}(v_i), I_{BU}(v_j)],$$

$$\mathrm{F_{BL}}(\{\mathrm{v_i}, \mathrm{v_j}\}) \geq \max[\mathrm{F_{BL}}(\mathrm{v_i}), \mathrm{F_{BL}}(\mathrm{v_j})],$$

$$\mathrm{F_{BU}}(\{\mathrm{v_i}, \mathrm{v_j}\}) \geq \max[\mathrm{F_{BU}}(\mathrm{v_i}), \mathrm{F_{BU}}(\mathrm{v_j})],$$

denote the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in$ E respectively, where

$$0 \leq T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \leq 3,$$

for all $\{v_i, v_j\} \in$ E (i, j = 1, 2,…, n).

We call A the interval valued neutrosophic vertex set of V, B the interval valued neutrosophic edge set of E, respectively. Note that B is a symmetric interval valued neutrosophic relation on A. We use the notation (v_i, v_j) for an element of E. Thus, G = (A, B) is a interval valued neutrosophic graph of G^*= (V, E) if —

$$T_{BL}(v_i, v_j) \leq \min\,[T_{AL}(v_i), T_{AL}(v_j)],$$

$$T_{BU}(v_i, v_j) \leq \min\,[T_{AU}(v_i), T_{AU}(v_j)],$$

$$I_{BL}(v_i, v_j) \geq \max[I_{BL}(v_i), I_{BL}(v_j)],$$

$$I_{BU}(v_i, v_j) \geq \max[I_{BU}(v_i), I_{BU}(v_j)],$$

$$F_{BL}(v_i, v_j) \geq \max[F_{BL}(v_i), F_{BL}(v_j)],$$

$$F_{BU}(v_i, v_j) \geq \max[F_{BU}(v_i), F_{BU}(v_j)], \quad \text{for all } (v_i, v_j) \in \text{E}.$$

Hereafter, we use the notation *xy* for (x, y) an element of E.

3 Strong Interval Valued Neutrosophic Graph

Throught this paper, we denote G^* = (V, E) a crisp graph, and G = (A, B) an interval valued neutrosophic graph.

Definition 3.1

An interval valued neutrosophic graph G = (A, B) is called strong interval valued neutrosophic graph if

$T_{BL}(xy)$= min $(T_{AL}(x), T_{AL}(y))$, $I_{BL}(xy)$ =max $(I_{AL}(x), I_{AL}(y))$ and $F_{BL}(xy)$ =max $(F_{AL}(x), F_{AL}(y))$

$T_{BU}(xy)$= min $(T_{AU}(x), T_{AU}(y))$, $I_{BU}(xy)$ = max $(I_{AU}(x), I_{AU}(y))$ and $F_{BU}(xy)$ =max $(F_{AU}(x), F_{AU}(y))$ such that

$0 \le T_{BU}(x,y)) + I_{BU}(x,y)) + F_{BU}(x,y)) \le 3$, for all $x, y \in$ E.

Example 3.2

Figure 1 is an example for IVNG, G=(A, B) defined on a graph G^* = (V, E) such that V = {x, y, z}, E = {xy, yz, zx}, A is an interval valued neutrosophic set of V.

A={<x, [0.5, 0.7], [0.2, 0.3], [0.1, 0.3]>,<y, [0.6, 0.7], [0.2, 0.4], [0.1, 0.3]>, <z, [0.4, 0.6], [0.1, 0.3], [0.2, 0.4],>},

B={< xy, [0.3, 0.6], [0.2, 0.4], [0.2, 0.4]>, <yz, [0.3, 0.5], [0.2, 0.3], [0.2, 0.4]>, <xz, [0.3, 0.5], [0.1, 0.5], [0.2, 0.4]>}.

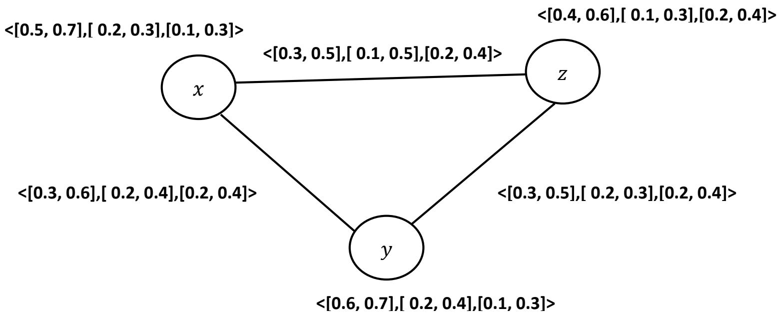

Figure 3. Interval valued neutrosophic graph

Example 3.2

Figure 2 is a SIVNG G = (A, B), where

A={<x, [0.5, 0.7], [0.1, 0.4], [0.1, 0.3]>, <y, [0.6, 0.7],[0.2, 0.3], [0.1, 0.3]>, <z, [0.4, 0.6],[0.2, 0.3], [0.2, 0.4],>},

B={< xy, [0,5, 0.7], [0.2 0.4], [0.1, 0.3]>, <yz, [0.4, 0.6],[0.2, 0.3], [0.2, 0.4]>, <xz, [0.4, 0.6], [0.2, 0.4], [0.2, 0.4]>}

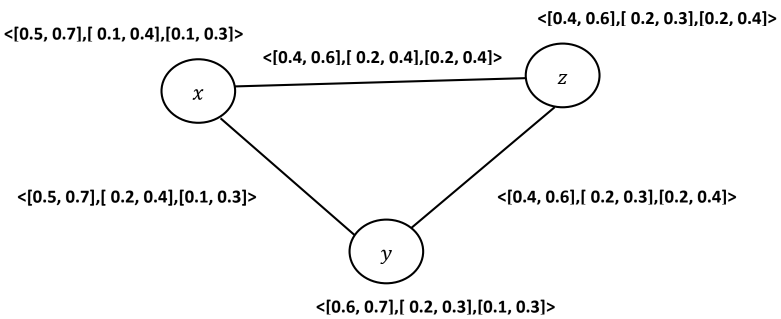

Figure 4. Strong Interval valued neutrosophic graph.

Proposition 3.3

A strong interval valued neutrosophic graph is the generalization of strong interval valued fuzzy graph.

Proof

Suppose G=(V, E) is a strong interval valued neutrosophic graph. Then, by setting the indeterminacy-membership and falsity-membership values of vertex set and edge set equals to zero, the strong interval valued neutrosophic graph is reduced to strong interval valued fuzzy graph.

Proposition 3.4

A strong interval valued neutrosophic graph is the generalization of strong interval valued intuitionistic fuzzy graph.

Proof

Suppose G=(V, E) is a strong interval valued neutrosophic graph. Then by setting the indeterminacy-membership values of vertex set and edge set equals to zero reduces the strong interval valued neutrosophic graph to strong interval valued intuitionistic fuzzy graph.

Proposition 3.5

A strong interval valued neutrosophic graph is the generalization of strong intuitionistic fuzzy graph.

Proof

Suppose G=(V, E) is a strong interval valued neutrosophic graph. Then by setting the indeterminacy-membership, upper truth-membership and upper falsity-membership values of vertex set and edge set equals to zero reduces the strong interval valued neutrosophic graph to strong intuitionistic fuzzy graph.

Proposition 3.6

A strong interval valued neutrosophic graph is the generalization of strong single neutrosophic graph.

Proof

Suppose G = (V, E) is a strong interval valued neutrosophic graph. Then by setting the upper truth-membership equals lower truth-membership, upper indeterminacy-membership equals lower indeterminacy-membership and upper falsity-membership equals lower falsity-membership values of vertex set and edge set reduces the strong interval valued neutrosophic graph to strong single valued neutrosophic graph.

Definition 3.7

Let A_1 and A_2 be interval-valued neutrosophic subsets of V_1 and V_2 respectively. Let B_1 and B_2 interval-valued neutrosophic subsets of E_1 and E_2 respectively. The Cartesian product of two SIVNGs G_1 and G_2 is denoted by $G_1 \times G_2 = (A_1 \times A_2$, $B_1 \times B_2$) and is defined as follows:

1) $(T_{A_1L} \times T_{A_2L})\ (x_1, x_2) = \min\ (T_{A_1L}(x_1), T_{A_2L}(x_2))$
$(T_{A_1U} \times T_{A_2U})\ (x_1, x_2) = \min\ (T_{A_1U}(x_1), T_{A_2U}(x_2))$
$(I_{A_1L} \times I_{A_2L})\ (x_1, x_2) = \max\ (I_{A_1L}(x_1), I_{A_2L}(x_2))$
$(I_{A_1U} \times I_{A_2U})\ (x_1, x_2) = \max\ (I_{A_1U}(x_1), I_{A_2U}(x_2))$
$(F_{A_1L} \times F_{A_2L})\ (x_1, x_2) = \max\ (F_{A_1L}(x_1), F_{A_2L}(x_2))$
$(F_{A_1U} \times F_{A_2U})\ (x_1, x_2) = \max\ (F_{A_1U}(x_1), F_{A_2U}(x_2))$ for all $(x_1, x_2) \in V$

2) $(T_{B_1L} \times T_{B_2L})\ ((x, x_2)(x, y_2)) = \min\ (T_{A_1L}(x), T_{B_2L}(x_2y_2))$
$(T_{B_1U} \times T_{B_2U})\ ((x, x_2)(x, y_2)) = \min\ (T_{A_1U}(x), T_{B_2U}(x_2y_2))$
$(I_{B_1L} \times I_{B_2L})\ ((x, x_2)(x, y_2)) = \max\ (I_{A_1L}(x), I_{B_2L}(x_2y_2))$
$(I_{B_1U} \times I_{B_2U})\ ((x, x_2)(x, y_2)) = \max\ (I_{A_1U}(x), I_{B_2U}(x_2y_2))$
$(F_{B_1L} \times F_{B_2L})\ ((x, x_2)\ (x, y_2)) = \max\ (F_{A_1L}(x), F_{B_2L}(x_2y_2))$
$(F_{B_1U} \times F_{B_2U})\ ((x, x_2)(x, y_2)) = \max(F_{A_1U}(x), F_{B_2U}(x_2y_2))\ \forall\ x \in V_1$ and $\forall\ x_2y_2 \in E_2$

3) $(T_{B_1L} \times T_{B_2L})\ ((x_1, z)\ (y_1, z)) = \min\ (T_{B_1L}(x_1y_1), T_{A_2L}(z))$
$(T_{B_1U} \times T_{B_2U})\ ((x_1, z)\ (y_1, z)) = \min\ (T_{B_1U}(x_1y_1), T_{A_2U}(z))$
$(\mathrm{I}_{B_1L} \times \mathrm{I}_{B_2L})\ ((x_1, z)\ (y_1, z)) = \max\ (\mathrm{I}_{B_1L}(x_1y_1), \mathrm{I}_{A_2L}(z))$
$(I_{B_1U} \times I_{B_2U})\ ((x_1, z)\ (y_1, z)) = \max\ (I_{B_1U}(x_1y_1), I_{A_2U}(z))$
$(F_{B_1L} \times F_{B_2L})\ ((x_1, z)\ (y_1, z)) = \max\ (F_{B_1L}(x_1y_1), F_{A_2L}(z))$
$(F_{B_1U} \times F_{B_2U})\ ((x_1, z)\ (y_1, z)) = \max\ (F_{B_1U}(x_1y_1), F_{A_2U}(z))\ \forall\ z \in V_2$ and $\forall\ x_1y_1 \in E_1$

Proposition 3.7

If G_1 and G_2 are the strong interval valued neutrosophic graphs, then the cartesian product G_1x G_2 is a strong interval valued neutrosophic graph.

Proof

Let G_1 and G_2 are SIVNGs, there exist $x_i, y_i \in E_i$, i= 1, 2 such that

$$T_{B_iL}(x_i, y_i) = \min\ (T_{A_iL}(x_i), T_{A_iL}(y_i)),\ \mathrm{i} = 1, 2.$$

$T_{B_iU}(x_i, y_i)$ = min $(T_{A_iU}(x_i), T_{A_iU}(y_i))$, i =1, 2.
$I_{B_iL}(x_i, y_i)$ = max $(I_{A_iL}(x_i), I_{A_iL}(y_i))$, i =1, 2.
$I_{B_iU}(x_i, y_i)$ = max $(I_{A_iU}(x_i), I_{A_iU}(y_i))$, i =1, 2.
$F_{B_iL}(x_i, y_i)$ = max $(F_{A_iL}(x_i), F_{A_iL}(y_i))$, i =1, 2.
$F_{B_iU}(x_i, y_i)$ = max $(F_{A_iU}(x_i), F_{A_iU}(y_i))$, i =1, 2.

Let E= $\{(x, x_2)\ (x, y_2)\ /x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)\ (y_1, z)\ /z \in V_2, x_1y_1 \in E_1\}$.

Consider, $(x, x_2)\ (x, y_2) \in E$, we have

$(T_{B_1L} \times T_{B_2L})\ ((x, x_2)\ (x, y_2))$ = min $(T_{A_1L}(x),\ T_{B_2L}(x_2y_2))$

= min $(T_{A_1L}(x), T_{A_2L}(x_2), T_{A_2L}(y_2))$

Similarly,

$(T_{B_1U} \times T_{B_2U})\ ((x, x_2)\ (x, y_2))$ = min $(T_{A_1U}(x), T_{B_2U}(x_2y_2))$

= min $(T_{A_1U}(x), T_{A_2U}(x_2), T_{A_2U}(y_2))$

$(T_{A_1L} \times T_{A_2L})\ (x_1, x_2)$ = min $(T_{A_1L}(x_1), T_{A_2L}(x_2))$

$(T_{A_1U} \times T_{A_2U})\ (x_1, x_2)$ = min $(T_{A_1U}(x_1), T_{A_2U}(x_2))$

$(T_{A_1L} \times T_{A_2L})\ (x_1, y_2)$ = min $(T_{A_1L}(x_1), T_{A_2L}(y_2))$

$(T_{A_1U} \times T_{A_2U})\ (x_1, y_2)$ = min $(T_{A_1U}(x_1), T_{A_2U}(y_2))$

Min $((T_{A_1U} \times T_{A_2U})\ (x, x_2), (T_{A_1U} \times T_{A_2U})\ (x, y_2))$

= min (min $(T_{A_1U}(x), T_{A_2U}(x_2))$, min $(T_{A_1U}(x), T_{A_2U}(y_2)))$

= min $(T_{A_1U}(x), T_{A_2U}(x_2), T_{A_1U}(y_2))$

Hence

$(T_{B_1L} \times T_{B_2L})\ ((x, x_2)\ (x, y_2))$ = min $((T_{A_1L} \times T_{A_2L})\ (x, x_2), (T_{A_1L} \times T_{A_2L})\ (x, y_2))$

$(T_{B_1U} \times T_{B_2U})\ ((x, x_2)\ (x, y_2))$ = min $((T_{A_1U} \times T_{A_2U})\ (x, x_2), (T_{A_1U} \times T_{A_2U})\ (x ,y_2))$.

Similarly, we can show that —

$(I_{B_1L} \times I_{B_2L})\ ((x, x_2)\ (x, y_2))$ = max $((I_{A_1L} \times I_{A_2L})\ (x, x_2), (I_{A_1L} \times I_{A_2L})\ (x, y_2))$

$(I_{B_1U} \times I_{B_2U})\ ((x, x_2)\ (x, y_2))$ = max $((I_{A_1U} \times I_{A_2U})\ (x, x_2), (I_{A_1U} \times I_{A_2U})\ (x, y_2))$.

And also

$(F_{B_1L} \times F_{B_2L})$ $((x, x_2)$ $(x, y_2))$ = max $((F_{A_1L} \times F_{A_2L})$ (x, x_2), $(F_{A_1L} \times F_{A_2L})$ $(x, y_2))$

$(F_{B_1U} \times F_{B_2U})$ $((x, x_2)$ $(x, y_2))$ = max $((F_{A_1U} \times F_{A_2U})$ (x, x_2), $(F_{A_1U} \times F_{A_2U})$ $(x, y_2))$.

Hence, G_1x G_2 strong interval valued neutrosophic graph. This completes the proof.

Proposition 3.8

If G_1 x G_2 is strong interval valued neutrosophic graph, then at least G_1 or G_2 must be strong.

Proof

Let G_1 and G_2 be no strong interval valued neutrosophic graphs; there exists $x_i, y_i \in E_i$, I = 1, 2, such that

$T_{B_iL}(x_i, y_i) <$ min $(T_{A_iL}(x_i), T_{A_iL}(y_i))$, i =1, 2.
$T_{B_iU}(x_i, y_i) <$ min $(T_{A_iU}(x_i), T_{A_iU}(y_i))$, i =1, 2.
$I_{B_iL}(x_i, y_i) >$ max $(I_{A_iL}(x_i), I_{A_iL}(y_i))$, i =1, 2.
$I_{B_iU}(x_i, y_i) >$ max $(I_{A_iU}(x_i), I_{A_iU}(y_i))$, i =1, 2.
$F_{B_iL}(x_i, y_i) >$ max $(F_{A_iL}(x_i), F_{A_iL}(y_i))$, i =1, 2.
F_{B_iU} $(x_i, y_i) >$ max $(F_{A_iU}(x_i), F_{A_iU}(y_i))$, i =1, 2.

Let E= $\{(x, x_2)$ (x, y_2) $/x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)$ (y_1, z) $/z \in V_2, x_1y_1 \in E_1\}$

Consider, (x, x_2) $(x, y_2) \in E$, we have

$(T_{B_1L} \times T_{B_2L})$ $((x, x_2)$ $(x, y_2))$ = min $(T_{A_1L}(x),$ $T_{B_2L}(x_2y_2))$

$<$ min $(T_{A_1L}(x), T_{A_2L}(x_2), T_{A_2L}(y_2))$.

Similarly —

$(T_{B_1U} \times T_{B_2U})$ $((x, x_2)$ $(x, y_2))$ = min $(T_{A_1U}(x), T_{B_2U}(x_2y_2))$

$<$ min $(T_{A_1U}(x), T_{A_2U}(x_2), T_{A_2U}(y_2))$

$(T_{A_1L} \times T_{A_2L})$ (x_1, x_2) = min $(T_{A_1L}(x_1), T_{A_2L}(x_2))$

$(T_{A_1U} \times T_{A_2U})$ (x_1, x_2) = min $(T_{A_1U}(x_1), T_{A_2U}(x_2))$

$(T_{A_1L} \times T_{A_2L})$ (x_1, y_2) = min $(T_{A_1L}(x_1), T_{A_2L}(y_2))$

$(T_{A_1U} \times T_{A_2U})$ (x_1, y_2) = min $(T_{A_1U}(x_1), T_{A_2U}(y_2))$

min $((T_{A_1U} \times T_{A_2U})$ (x, x_2), $(T_{A_1U} \times T_{A_2U})$ $(x, y_2))$

= min (min $(T_{A_1U}(x), T_{A_2U}(x_2))$, min $(T_{A_1U}(x), T_{A_2U}(y_2)))$

= min $(T_{A_1U}(x), T_{A_2U}(x_2), T_{A_1U}(y_2))$.

Hence

$(T_{B_1L} \times T_{B_2L})$ $((x, x_2)$ $(x, y_2)) <$ min $((T_{A_1L} \times T_{A_2L})$ (x, x_2), $(T_{A_1L} \times T_{A_2L})$ $(x, y_2))$,

$(T_{B_1U} \times T_{B_2U})$ $((x, x_2)$ $(x, y_2)) <$ min $((T_{A_1U} \times T_{A_2U})$ (x, x_2), $(T_{A_1U} \times T_{A_2U})$ $(x ,y_2))$.

Similarly, we can show that

$(I_{B_1L} \times I_{B_2L})$ $((x, x_2)$ $(x, y_2)) >$ max $((I_{A_1L} \times I_{A_2L})$ (x, x_2), $(I_{A_1L} \times I_{A_2L})$ $(x, y_2))$,

$(I_{B_1U} \times I_{B_2U})$ $((x, x_2)$ $(x, y_2)) >$ max $((I_{A_1U} \times I_{A_2U})$ (x, x_2), $(I_{A_1U} \times I_{A_2U})$ $(x, y_2))$.

And also

$(F_{B_1L} \times F_{B_2L})$ $((x, x_2)$ $(x, y_2)) >$ max $((F_{A_1L} \times F_{A_2L})$ (x, x_2), $(F_{A_1L} \times F_{A_2L})$ $(x, y_2))$,

$(F_{B_1U} \times F_{B_2U})$ $((x, x_2)$ $(x, y_2)) >$ max $((F_{A_1U} \times F_{A_2U})$ (x, x_2), $(F_{A_1U} \times F_{A_2U})$ $(x, y_2))$.

Hence, G_1 x G_2 is not strong interval valued neutrosophic graph, which is a contradiction. This completes the proof.

Remark 3.9

If G_1 is a SIVNG and G_2 is not a SIVNG, then G_1 x G_2 is need not be an SIVNG.

Example 3.10

Let G_1= (A_1,B_1) be a SIVNG, where A_1= {< a, [0.6, 0.7], [0.2, 0.5], [0.1, 0.3]> ,< b, [0.6, 0.7], [0.2, 0.5], [0.1, 0.3]>} and B_1= {< ab, [0.6, 0.7], [0.2, 0.5], [0.1, 0.3]>}

G_1

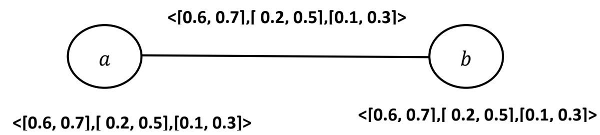

Figure 5. Interval valued neutrosophic G_1.

G_2= (A_2,B_2) is not a SIVNG, where A_2= {< c, [0.4, 0.6], [0.2, 0.4], [0.1, 0.3]>, < d, [0.4, 0.6], [0.1,0.3], [0.2, 0.4] >} and B_2= < cd, [0.3,0.5], [0.1, 0.2], [0.3, 0.5]>}.

G_2

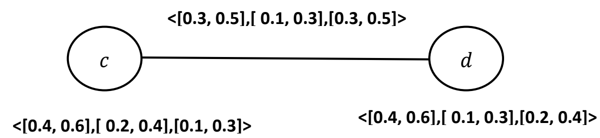

Figure 6. Interval valued neutrosophic G_2.

G_1x G_2= (A_1x A_2, B_1x B_2) is not a SIVNG, where

A_1x A_2= {< (a, c), [0.4, 0.6],[0.2, 0.3], [0.2, 0.4]>, < (a, d), [0.4, 0.6], [0.2, 0.3], [0.2, 0.4]>, < (b, c), [0.4, 0.6], [0.2, 0.6], [0.2, 0.4]>, < (b, d), [0.4, 0.6], [0.3, 0.4], [0.2, 0.4]>},

B_1x B_2= {< ((a, c), (a, d)), [0.3, 0.5], [0.3, 0.5], [0.3, 0.5] >, < ((a, c), (b, c)), [0.4, 0.6], [0.1, 0.4], [0.3, 0.4] >, < ((b, c), (b, d)), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >, < ((a, d), (b, d)),[0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >}. In this example, G_1 is a SIVNG and $G_2$is not a SIVNG, then G_1x $G_2$is not a SIVNG.

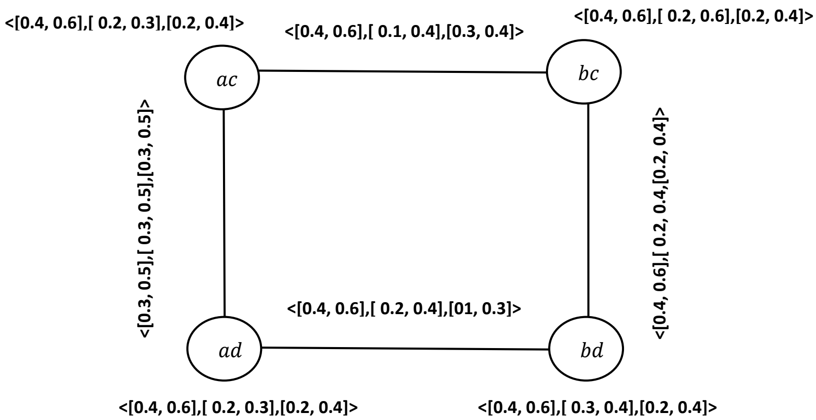

Figure 7. Cartesian product G_1x G_2

Example 3.11

Let G_1= (A_1, B_1) be a SIVNG, where A_1= {< a, [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >, < b, [0.4, 0.6], [0.2, 0.4], [0.1, 0.3]>} and B_1={< ab, [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >,

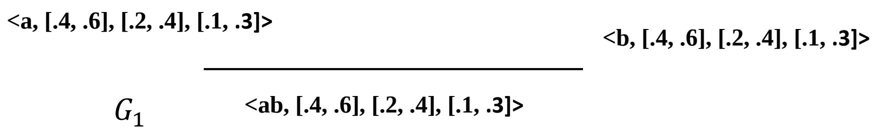

Figure 8. Interval valued neutrosophic G_1.

G_2= (A_2, B_2) is not a SIVIFG, where A_2= {< c, [0.6, 0.7], [0.1, 0.3], [0.1, 0.3] >, < d, [0.6, 0.7], [0.1, 0.3], [0.2, 0.4] >} and B_2= {< cd, [0.5, 0.6], [0.2, 0.4], [0.2, 0.4]>},

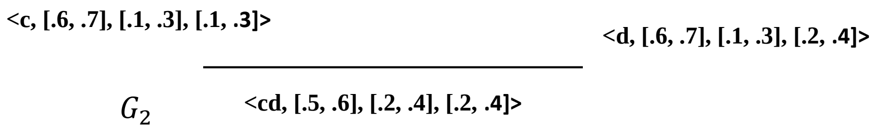

Figure 9. Interval valued neutrosophic G_2.

G_1x G_2= (A_1x A_2, B_1x B_2) is a SIVNG, where

A_1x A_2= {< (a, c), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >, < (a, d), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >, < (b, c), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >, < (b, d), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >} and

B_1x B_2= {< ((a, c), (a, d)), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >, < ((a, c), (b, c)), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >, < ((b, c), (b, d)), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >, < ((a, d), (b, d)), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4]>}. In this example, $G_1$is a SIVNG and $G_2$is not a SIVNG, then G_1x $G_2$is a SIVNG.

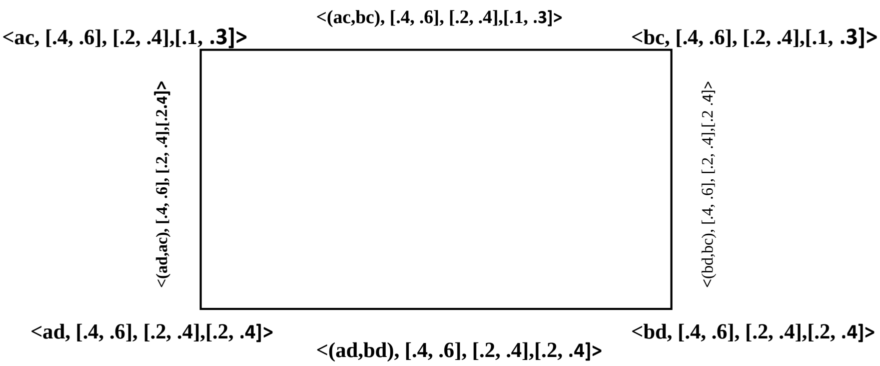

Figure 10. Cartesian product

Proposition 3.12

Let G_1be a strong interval valued neutrosophic graph. Then for any interval valued neutrosophic graph G_2,G_1x G_2 is strong interval valued neutrosophic graph iff

$T_{A_1L}(x_1) \leq T_{B_1L}(x_2y_2)$, $I_{A_1L}(x_1) \geq I_{B_1L}(x_2y_2)$ and $F_{A_1L}(x_1) \geq F_{B_1L}(x_2y_2)$,

$T_{A_1U}(x_1) \leq T_{B_1U}(x_2y_2)$, $I_{A_1U}(x_1) \geq I_{B_1U}(x_2y_2)$ and $F_{A_1U}(x_1) \geq F_{B_1U}(x_2y_2)$, $\forall\, x_1 \in V_1$,$x_2y_2 \in E_2$.

Definition 3.13

Let $A_1$and A_2be interval valued neutrosophic subsets of $V_1$and V_2respectively. Let $B_1$and B_2 interval-valued neutrosophic subsets of E_1 and E_2 respectively. The *composition* of two strong interval valued neutrosophic graphs G_1 and $G_2$is denoted by $G_1[\,G_2]$ = $(A_1 \circ A_2, B_1 \circ B_2\,)$ and is defined as follows

1) $(T_{A_1L} \circ T_{A_2L})\,(x_1, x_2)$ = min $(T_{A_1L}(x_1), T_{A_2L}(x_2))$
$(T_{A_1U} \circ T_{A_2U})\,(x_1, x_2)$ = min $(T_{A_1U}(x_1), T_{A_2U}(x_2))$
$(I_{A_1L} \circ I_{A_2L})\,(x_1, x_2)$ = max $(I_{A_1L}(x_1), I_{A_2L}(x_2))$
$(I_{A_1U} \circ I_{A_2U})\,(x_1, x_2)$ = max $(I_{A_1U}(x_1), I_{A_2U}(x_2))$

$(F_{A_1L} \circ F_{A_2L})\ (x_1, x_2) = \max\ (F_{A_1L}(x_1), F_{A_2L}(x_2))$
$(F_{A_1U} \circ F_{A_2U})\ (x_1, x_2) = \max\ (F_{A_1U}(x_1), F_{A_2U}(x_2))\ \ \forall\ x_1 \in V_1, x_2 \in V_2$

2) $(T_{B_1L} \circ T_{B_2L})\ ((x, x_2)(x, y_2)) = \min\ (T_{A_1L}(x), T_{B_2L}(x_2y_2))$
$(T_{B_1U} \circ T_{B_2U})\ ((x, x_2)(x, y_2)) = \min\ (T_{A_1U}(x), T_{B_2U}(x_2y_2))$
$(I_{B_1L} \circ I_{B_2L})\ ((x, x_2)(x, y_2)) = \max\ (I_{A_1L}(x), I_{B_2L}(x_2y_2))$
$(I_{B_1U} \circ I_{B_2U})\ ((x, x_2)(x, y_2)) = \max\ (I_{A_1U}(x), I_{B_2U}(x_2y_2))$
$(F_{B_1L} \circ F_{B_2L})\ ((x, x_2)\ (x, y_2)) = \max\ (F_{A_1L}(x), F_{B_2L}(x_2y_2))$
$(F_{B_1U} \circ F_{B_2U})\ ((x, x_2)(x, y_2)) = \max\ (F_{A_1U}(x), F_{B_2U}(x_2y_2))\ \forall\ x \in V_1, \forall\ x_2y_2 \in E_2$

3) $(T_{B_1L} \circ T_{B_2L})\ ((x_1, z)\ (y_1, z)) = \min\ (T_{B_1L}(x_1y_1), T_{A_2L}(z))$
$(T_{B_1U} \circ T_{B_2U})\ ((x_1, z)\ (y_1, z)) = \min\ (T_{B_1U}(x_1y_1), T_{A_2U}(z))$
$(I_{B_1L} \circ I_{B_2L})\ ((x_1, z)\ (y_1, z)) = \max\ (I_{B_1L}(x_1y_1), I_{A_2L}(z))$
$(I_{B_1U} \circ I_{B_2U})\ ((x_1, z)\ (y_1, z)) = \max\ (I_{B_1U}(x_1y_1), I_{A_2U}(z))$
$(F_{B_1L} \circ F_{B_2L})\ ((x_1, z)\ (y_1, z)) = \max\ (F_{B_1L}(x_1y_1), F_{A_2L}(z))$
$(F_{B_1U} \circ F_{B_2U})\ ((x_1, z)\ (y_1, z)) = \max\ (F_{B_1U}(x_1y_1), F_{A_2U}(z))\ \forall\ z \in V_2, \forall\ x_1y_1 \in E_1$

4) $(T_{B_1L} \circ T_{B_2L})\ ((x_1, x_2)\ (y_1, y_2)) = \min\ (T_{A_2L}(x_2), T_{A_2L}(y_2), T_{B_1L}(x_1y_1))$
$(T_{B_1U} \circ T_{B_2U})\ ((x_1, x_2)\ (y_1, y_2)) = \min\ (T_{A_2U}(x_2), T_{A_2U}(y_2), T_{B_1U}(x_1y_1))$
$(I_{B_1L} \circ I_{B_2L})\ ((x_1, x_2)\ (y_1, y_2)) = \max(I_{A_2L}(x_2), I_{A_2L}(y_2), I_{B_1L}(x_1y_1))$
$(I_{B_1U} \circ I_{B_2U})\ ((x_1, x_2)\ (y_1, y_2)) = \max\ (I_{A_2U}(x_2), I_{A_2U}(y_2), I_{B_1U}(x_1y_1))$
$(F_{B_1L} \circ F_{B_2L})\ ((x_1, x_2)\ (y_1, y_2)) = \max(F_{A_2L}(x_2), F_{A_2L}(y_2), F_{B_1L}(x_1y_1))$
$(F_{B_1U} \circ F_{B_2U})((x_1, x_2)\ (y_1, y_2)) = \max\ (F_{A_2U}(x_2), F_{A_2U}(y_2), F_{B_1U}(x_1y_1))$
$\forall (x_1, x_2)(y_1, y_2) \in E^0$ -E , where $E^0 = E \cup \{(x_1, x_2)\ (y_1, y_2)\ | x_1y_1 \in E_1, x_2 \neq y_2\}$.

The following propositions are stated without their proof.

Proposition 3.14

If G_1 and G_2 are the strong interval valued neutrosophic graphs, then the composition $G_1[\ G_2]$ is a strong interval valued neutrosophic graph.

Proposition 3.15

If $G_1[\ G_2]$ is strong interval valued neutrosophic graphs, then at least composition G_1 or G_2 must be strong.

Example 3.16

Let G_1= (A_1,B_1) be a SIVNG, where A_1= {< a, [0.6, 0.7], [0.2, 0.3], [0.1, 0.3], > ,< b, [0.6, 0.7], [0.2, 0.3], [0.1, 0.3] >} and B_1= {< ab, [0.6, 0.7], [0.2, 0.3], [0.1, 0.3]>}.

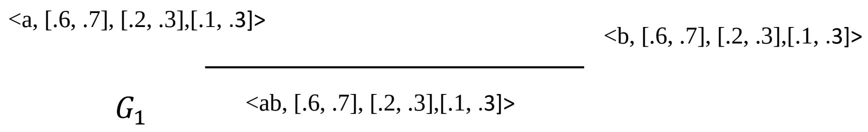

Figure 11. Interval valued neutrosophic G_1.

G_2= (A_2,B_2) is not a SIVNG, where A_2= {< c, [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >, < d, [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >} and B_2=<cd, [0.3, 0.5], [0.2, 0.5], [0.3, 0.5]>}.

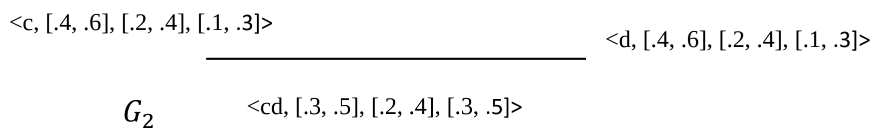

Figure 12. Interval valued neutrosophic G_2.

$G_1[G_2]$= (A_1oA_2, B_1oB_2) is not a SIVNG, where

A_1oA_2= {< (a,c), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >, < (a,d), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >, < (b, c), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >, < (b, d), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >},

B_1oB_2= {< ((a, c), (a, d)), [0.3, 0.5], [0.2, 0.4], [0.3, 0.5] >, < ((a, c), (b, c)), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3]>, < ((b, c), (b, d)),[0.3, 0.5], [0.2, 0.4], [0.3, 0.5] >, < ((a, d), (b, d)), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >, < ((a, c), (b, d)), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >, < ((a, d), (b, c)), [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] >}. In this example, G_1 is a SIVNG and $G_2$is not a SIVNG, then $G_1[G_2]$ is not a SIVNG.

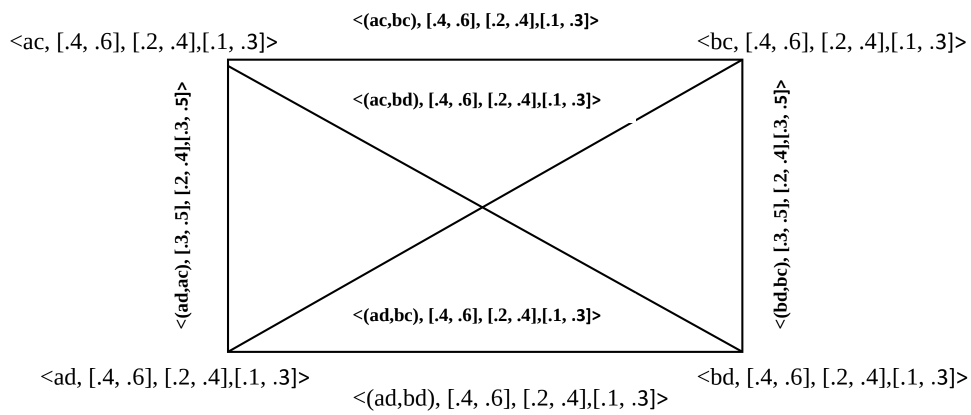

Figure 13. Composition

Example 3.17

Let G_1= (A_1,B_1) be a SIVNG, where A_1= {< a, [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >, < b, [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >}and B_1={<ab, [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >}.

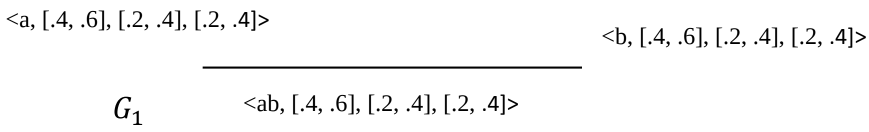

Figure 14. Interval valued neutrosophic G_1.

G_2= (A_2,B_2) is not a SIVNG, where A_2= {< c, [0.6, 0.7], [0.1, 0.3], [0.1, 0.3] >, <d, [0.6, 0.7], [0.2, 0.4], [0.1, 0.3] >} and B_2={< c d, [0.5, 0.6], [0.2, 0.4], [0.2, 0.4] >}.

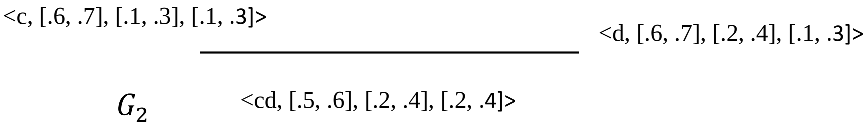

Figure 15. Interval valued neutrosophic G_2.

$G_1[G_2]$ = (A_1oA_2, B_1 oB_2) is a SIVNG, where

A_1oA_2= {< (a, c), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >, < (a, d), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >, < (b, c), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >, < (b, d), [0.4, 0.6], [0.2, 0.4],[0.2, 0.4] >} and

B_1 oB_2= {< ((a, c), (a, d)), [0.4, 0.6], [0.2, 0.4] [0.2, 0.4] >, < ((a, c), (b, c)), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >, < ((b, c), (b, d)), [0.4, 0.6], [0,2, 0.4] [0.2, 0.4] >, < ((a, d), (b, d)), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4] >, < ((a, c), (b, d)), [0.4, 0.6], [0.2, 0.4] [0.2, 0.4]>, < ((a, d), (b, c)), [0.4, 0.6], [0.2, 0.4], [0.2, 0.4]>}. In this example, $G_1$is an SIVIFG and $G_2$is not a SIVNG, then $G_1[G_2]$ is a SIVNG.

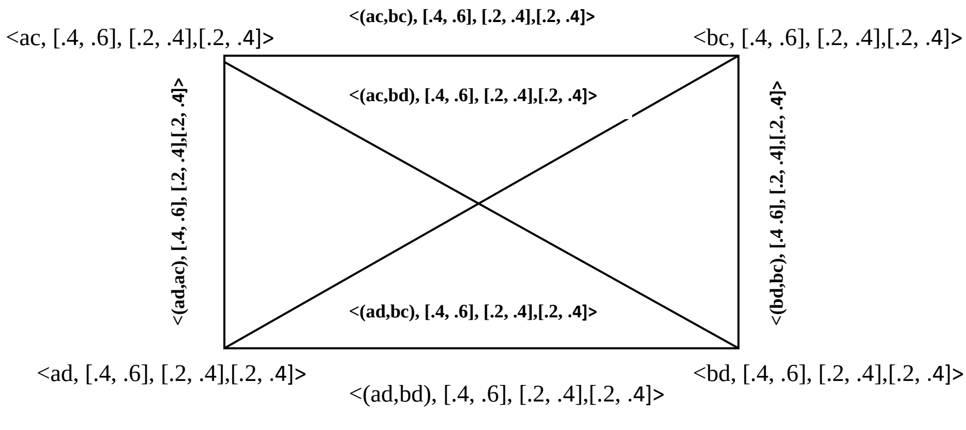

Figure 16. Composition of G_1 and G_2.

Proposition 3.18

Let G_1be a strong interval valued neutrosophic graph. Then for any interval valued neutrosophic graph G_2,$G_1[G_2]$ is strong interval valued neutrosophic graph iff —

$T_{A_1L}(x_1) \leq T_{B_1L}(x_2y_2)$, $I_{A_1L}(x_1) \geq I_{B_1L}(x_2y_2)$ and $F_{A_1L}(x_1) \geq F_{B_1L}(x_2y_2)$,

$T_{A_1U}(x_1) \leq T_{B_1U}(x_2y_2)$, $I_{A_1U}(x_1) \geq I_{B_1U}(x_2y_2)$ and $F_{A_1U}(x_1) \geq F_{B_1U}(x_2y_2)$, $\forall\, x_1 \in V_1$,$x_2y_2 \in E_2$.

Definition 3.19

Let $A_1$and A_2 be interval valued neutrosophic subsets of $V_1$and V_2 respectively. Let $B_1$and B_1 interval valued neutrosophic subsets of E_1 and E_2 respectively. The join of two strong interval valued neutrosophic graphs G_1 and G_2 is denoted by $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ and is defined as follows

1) $$(T_{A_1L} + T_{A_2L})(x) = \begin{cases} (T_{A_1L} \cup T_{A_2L})(x) & \text{if } x \in V_1 \cup V_2 \\ T_{A_1L}(x) & \text{if } x \in V_1 \\ T_{A_2L}(x) & \text{if } x \in V_2 \end{cases}$$

$$(T_{A_1U} + T_{A_2U})(x) = \begin{cases} (T_{A_1U} \cup T_{A_2U})(x) & \text{if } x \in V_1 \cup V_2 \\ T_{A_1U}(x) & \text{if } x \in V_1 \\ T_{A_2U}(x) & \text{if } x \in V_2 \end{cases}$$

$$(I_{A_1L} + I_{A_2L})\,(x) = \begin{cases} (I_{A_1L} \cap I_{A_2L})(x) & \text{if } x \in V_1 \cup V_2 \\ I_{A_1L}(x) & \text{if } x \in V_1 \\ I_{A_2L}(x) & \text{if } x \in V_2 \end{cases}$$

$$(I_{A_1U} + I_{A_2U})\,(x) = \begin{cases} (I_{A_1U} \cap I_{A_2U})(x) & \text{if } x \in V_1 \cup V_2 \\ I_{A_1U}(x) & \text{if } x \in V_1 \\ I_{A_2U}(x) & \text{if } x \in V_2 \end{cases}$$

$$(F_{A_1L} + F_{A_2L})\,(x) = \begin{cases} (F_{A_1L} \cap F_{A_2L})(x) & \text{if } x \in V_1 \cup V_2 \\ F_{A_1L}(x) & \text{if } x \in V_1 \\ F_{A_2L}(x) & \text{if } x \in V_2 \end{cases}$$

$$(F_{A_1U} + F_{A_2U})\,(x) = \begin{cases} (F_{A_1U} \cap F_{A_2U})(x) & \text{if } x \in V_1 \cup V_2 \\ F_{A_1U}(x) & \text{if } x \in V_1 \\ F_{A_2U}(x) & \text{if } x \in V_2 \end{cases}$$

2) $$(T_{B_1L} + T_{B_2L})\,(x\mathrm{y}) = \begin{cases} (T_{B_1L} \cup T_{B_2L})(\mathrm{xy}) & \text{if } xy \in E_1 \cup E_2 \\ T_{B_1L}(xy) & \text{if } xy \in E_1 \\ T_{B_2L}(xy) & \text{if } xy \in E_2 \end{cases}$$

$$(T_{B_1U} + T_{B_2U})\,(x\mathrm{y}) = \begin{cases} (T_{B_1U} \cup T_{B_2U})(\mathrm{xy}) & \text{if } xy \in E_1 \cup E_2 \\ T_{B_1U}(xy) & \text{if } xy \in E_1 \\ T_{B_2U}(xy) & \text{if } xy \in E_2 \end{cases}$$

$$(I_{B_1L} + I_{B_2L})\,(xy) = \begin{cases} (I_{B_1L} \cap I_{B_2L})(\mathrm{xy}) & \text{if } xy \in E_1 \cup E_2 \\ I_{B_1L}(xy) & \text{if } xy \in E_1 \\ I_{B_2L}(xy) & \text{if } xy \in E_2 \end{cases}$$

$$(I_{B_1U} + I_{B_2U})\,(xy) = \begin{cases} (I_{B_1U} \cap I_{B_2U})(\mathrm{xy}) & \text{if } xy \in E_1 \cup E_2 \\ I_{B_1U}(xy) & \text{if } xy \in E_1 \\ I_{B_2U}(xy) & \text{if } xy \in E_2 \end{cases}$$

$$(F_{B_1L} + F_{B_2L})\,(xy) = \begin{cases} (F_{B_1L} \cap F_{B_2L})(\mathrm{xy}) & \text{if } xy \in E_1 \cup E_2 \\ F_{B_1L}(xy) & \text{if } xy \in E_1 \\ F_{B_2L}(xy) & \text{if } xy \in E_2 \end{cases}$$

$$(F_{B_1U} + F_{B_2U})\,(x\,y) = \begin{cases} (F_{B_1U} \cap F_{B_2U})(\mathrm{xy}) & \text{if } xy \in E_1 \cup E_2 \\ F_{B_1U}(xy) & \text{if } xy \in E_1 \\ F_{B_2U}(xy) & \text{if } xy \in E_2 \end{cases}$$

3) $(T_{B_1L} + T_{B_2L})\,(x\,y) = \min\,(T_{B_1L}(x), T_{B_2L}(x))$
$(T_{B_1U} + T_{B_2U})\,(xy) = \min\,(T_{B_1U}(x), T_{B_2U}(x))$
$(I_{B_1L} + I_{B_2L})\,(xy) = \max\,(I_{B_1L}(x), I_{B_2L}(x))$
$(I_{B_1U} + I_{B_2U})\,(x\,y) = \max\,(I_{B_1U}(x), I_{B_2U}(x)$
$(F_{B_1L} + F_{B_2L})\,(xy) = \max\,(F_{B_1L}(x), F_{B_2L}(x))$

$(F_{B_1U} + F_{B_2U})\,(xy) = \max\,(F_{B_1U}(x), F_{B_2U}(x))$ if $xy \in E'$, where E' is the set of all edges joining the nodes of V_1 and V_2 and where we assume $V_1 \cap V_2 = \emptyset$.

4 Conclusion

Interval valued neutrosophic set is a generalization of fuzzy set and intuitionistic fuzzy set, interval valued fuzzy set, interval valued intuitionstic fuzzy set and single valued neutrosophic set. Interval valued neutrosophic model gives more precisions, flexibility and compatibility to the system as compared to the classical, fuzzy, intuitionistic fuzzy and single valued neutrosophic models. In this paper, we have discussed a subclass of interval valued neutrosophic graph called strong interval valued neutrosophic graph, and we have introduced some operations, such as Cartesian product, composition and join of two strong interval valued neutrosophic graph, with proofs. In future studies, we plan to extend our research to regular interval valued neutrosophic graphs and irregular interval valued neutrosophic graphs.

References

[1] A. V. Devadoss, A. Rajkumar & N. J. P .Praveena. *A Study on Miracles through Holy Bible using Neutrosophic Cognitive Maps (NCMS)*, in "International Journal of Computer Applications", 69(3) (2013).

[2] A. Mohamed Ismayil and A. Mohamed Ali. *On Strong Interval-Valued Intuitionistic Fuzzy*, Graph International Journal of Fuzzy Mathematics and Systems. Volume 4, Number 2, 2014, pp. 161-168.

[3] A. Aydoğdu. *On Similarity and Entropy of Single Valued Neutrosophic Sets*, Gen. Math. Notes, Vol. 29, No. 1, July 2015, pp. 67-74.

[4] A. Q. Ansari, R. Biswas & S. Aggarwal. *Neutrosophic classifier: An extension of fuzzy classifier*. Elsevier - Applied Soft Computing, 13, 2013, pp. 563-573, http://dx.doi.org/10.1016/j.asoc.2012.08.002.

[5] A. Q. Ansari, R.Biswas & S. Aggarwal. Poster Presentation. *Neutrosophication of Fuzzy Models*, IEEE Workshop On Computational Intelligence: Theories, Applications and Future Directions, 14th July 2013.

[6] A. Q. Ansari, R. Biswas & S. Aggarwal. *Extension to fuzzy logic representation: Moving towards neutrosophic logic - A new laboratory rat*, Fuzzy Systems (FUZZ), 2013 IEEE International Conference, 1–8,DOI:10.1109/FUZZ-IEEE.2013.6622412.

[7] A. Nagoor Gani, M. Basheer Ahamed, *Order and Size in Fuzzy Graphs*, Bulletin of Pure and Applied Sciences, Vol 22E (No.1) (2003) 145-148.

[8] A. Nagoor Gani. A and S. Shajitha Begum. *Degree, Order and Size in Intuitionistic Fuzzy Graphs*, International Journal of Algorithms, Computing and Mathematics, (3)3, 2010.

[9] A. Nagoor Gani and S.R Latha. *On Irregular Fuzzy Graphs*, Applied Mathematical Sciences, Vol. 6, 2012, no. 11, pp. 517-523.

[10] F. Smarandache. *Refined Literal Indeterminacy and the Multiplication Law of Sub-Indeterminacies*, Neutrosophic Sets and Systems, Vol. 9, 58- 63, 2015,

[11] F. Smarandache. *Types of Neutrosophic Graphs and neutrosophic Algebraic Structures together with their Applications in Technology*, seminar, Universitatea Transilvania din Brasov, Facultatea de Design de Produs si Mediu, Brasov, Romania 06 June 2015.

[12] F. Smarandache, *Symbolic Neutrosophic Theory*, Europanova, 195 p., 2015.

[13] F. Smarandache. *Neutrosophic set - a generalization of the intuitionistic fuzzy set*, Granular Computing, 2006 IEEE International Conference, 38 – 42,2006, DOI: 10.1109/GRC.2006.1635754.

[14] F. Smarandache, *A geometric interpretation of the neutrosophic set — A generalization of the intuitionistic fuzzy set,* Granular Computing (GrC), 2011 IEEE International Conference, pp. 602 – 606, 2011, DOI 10.1109/GRC.2011.6122665.

[15] Gaurav Garg, Kanika Bhutani, Megha Kumar and Swati Aggarwal. *Hybrid model for medical diagnosis using Neutrosophic Cognitive Maps with Genetic Algorithms*, FUZZ-IEEE 2015, IEEE International conference on fuzzy systems.

[16] H. Wang, Y. Zhang, R. Sunderraman. *Truth-value based interval neutrosophic sets*, Granular Computing, IEEE International Conference, (2005) 274 - 277 Vol. 1, 2005, DOI: 10.1109/ GRC.2005.1547284.

[17] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman. *Single valued Neutrosophic Sets*, Multispace and Multistructure 4, 2010, pp. 410-413.

[18] H. Y. Zhang , J. Q. Wang , X. H. Chen. *Interval neutrosophic sets and their application in multicriteria decision making problems*, The Scientific World Journal ,DOI:10.1155/2014/ 645953, 2014 .

[19] H. Y. Zhang , P. Ji, J. Q. Wang & X. H. Chen. *An Improved Weighted Correlation Coefficient Based on Integrated Weight for Interval Neutrosophic Sets and its Application in Multi-criteria Decision-making Problems*, International Journal of Computational Intelligence Systems, Vol. 8, Issue 6, 2015, DOI:10.1080/18756891. 2015.1099917

[20] H, Zhang, J .Wang, X. Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets, Neural Computing and Applications, pp. 1-13, 2015.

[21] I. Deli, M. Ali, F. Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems, Advanced Mechatronic Systems (ICAMechS), International Conference, 2015, pp. 249 – 254.

[22] I. Turksen. *Interval valued fuzzy sets based on normal forms*, Fuzzy Sets and Systems, vol. 20, 1986, pp. 191-210.

[23] J. Ye. *Vector similarity measures of simplified neutrosophic sets and their application in multicriteria decision making*, International Journal of Fuzzy Systems, Vol. 16, No. 2, p. 204-211, 2014.

[24] J. Ye. *Single-Valued Neutrosophic Minimum Spanning Tree and Its Clustering Method*, Journal of Intelligent Systems 23(3), pp. 311–324, 2014.

[25] J. Ye. *Similarity measures between interval neutrosophic sets and their applications in Multi-criteria decision-making*, Journal of Intelligent and Fuzzy Systems, 26, pp. 165-17, 2014.

[26] J. Ye. *Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute decision making*, Journal of Intelligent & Fuzzy Systems (27) 2231-2241, 2014.

[27] K. Atanassov. *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, vol. 20, p. 87-96, 1986.

[28] K. Atanassov and G. Gargov. *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, vol. 31, 1989, pp. 343-349.

[29] K. Atanassov. *Intuitionistic fuzzy sets: theory and applications*. Physica, New York, 1999.

[30] L. Zadeh. *Fuzzy sets*, Inform and Control, 8(1965), 338-353

[31] P. Bhattacharya. *Some remarks on fuzzy graphs*, Pattern Recognition Letters 6: 297-302, 1987.

[32] P. Liu , L. Shi. *The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multiple attribute decision making*, Neural Computing and Applications, 26 (2) (2015) 457-471.

[33] R. Parvathi, M. G. Karunambigai. *Intuitionistic Fuzzy Graphs*, Computational Intelligence, Theory and applications, International Conference in Germany, Sept 18 -20, 2006.

[34] R. Rıdvan, A. Küçük. *Subsethood measure for single valued neutrosophic sets*, Journal of Intelligent & Fuzzy Systems, vol. 29, no. 2, pp. 525-530, 2015,DOI: 10.3233/IFS-141304.

[35] R. Şahin. *Cross-entropy measure on interval neutrosophic sets and its applications in multicriteria decision making*, Neural Computing and Applications, pp 1-11 (2015).

[36] S. Aggarwal, R. Biswas, A. Q. Ansari, *Neutrosophic modeling and control*, Computer and Communication Technology (ICCCT), International Conference, 2010, pp. 718 – 723, DOI: 10.1109/ ICCCT.2010.5640435

[37] S. Broumi, F. Smarandache, *New distance and similarity measures of interval neutrosophic sets*, Information Fusion (FUSION), 2014 IEEE 17th International Conference, 2014,p 1 – 7.

[38] S. Broumi, F. Smarandache, *Single valued neutrosophic trapezoid linguistic aggregation operators based multi-attribute decision making*, Bulletin of Pure & Applied Sciences- Mathematics and Statistics,(2014)135-155, DOI : 10.5958/2320-3226.2014.00006.X.

[39] S.Broumi, M. Talea, F. Smarandache. *Single Valued Neutrosophic Graphs: Degree, Order and Size*, (2016) submitted

[40] S.Broumi, M. Talea, A. Bakali, F. Smarandache, *Single Valued Neutrosophic Graphs*, Journal of new theory , N 10(2016) 86-101

[41] S.Broumi, M. Talea, A. Bakali, F. Smarandache. *Interval Valued Neutrosophic Graphs*, Critical Review (2016) in press.

[42] S.Broumi, M. Talea, A. Bakali, F. Smarandache. *Operations on Interval Valued Neutrosophic Graphs*, (2016) submitted.

[43] Y. Hai-Long, G. She, Yanhonge, L. Xiuwu. *On single valued neutrosophic relations*, Journal of Intelligent & Fuzzy Systems, vol. Preprint, no. Preprint, pp. 1-12, 2015.

[44] W. B. Vasantha Kandasamy, F. Smarandache. *Fuzzy Cognitive Maps and Neutrosophic Congtive Maps*, 2013.

[45] W. B. Vasantha Kandasamy, K. Ilanthenral, Florentin Smarandache. *Neutrosophic Graphs: A New Dimension to Graph Theory*, Kindle Edition, 2015.

[46] W.B. Vasantha Kandasamy and F. Smarandache. *Analysis of social aspects of migrant laborers living with HIV/AIDS using Fuzzy Theory and Neutrosophic Cognitive Maps*, Xiquan, Phoenix, 2004.

[47] F. Smarandache, *Interval-Valued Neutrosophic Oversets, Neutrosophic Understes, and Neutrosophic Offsets*, International Journal of Science and Engineering Investigations, Vol. 5, Issue 54, 1-4, July 2016; http://fs.gallup.unm.edu/IV-Neutrosophic-Overset-Underset-Offset.pdf

[48] Florentin Smarandache, *Neutrosophy. Neutrosophic Probability, Set, and Logic*, Amer. Res. Press, Rehoboth, USA, 105 p., 1998; http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition); reviewed in Zentralblatt für Mathematik (Berlin, Germany), https://zbmath.org/?q=an:01273000

The Concept of Neutrosophic Less Than or Equal To: A New Insight in Unconstrained Geometric Programming

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Abstract

In this paper, we introduce the concept of *neutrosophic less than* or *equal to*. The neutrosophy considers every idea $<\mathrm{A}>$ together with its opposite or negation $<\mathrm{antiA}>$ and with their spectrum of neutralities $<\mathrm{neutA}>$ in between them (i.e. notions or ideas supporting neither $<\mathrm{A}>$ nor $<\mathrm{antiA}>$). The $<\mathrm{neutA}>$ and $<\mathrm{antiA}>$ ideas together are referred to as $<\mathrm{nonA}>$. Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic) [5]. In neutrosophic logic, a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$. Another purpose of this article is to explain the mathematical theory of *neutrosophic geometric programming* (the unconstrained posynomial case). It is necessary to work in fuzzy neutrosophic space $\mathrm{FN_s} = [0,1] \cup [0, \mathrm{nI}], \mathrm{n} \in [0,1]$. The theory stated in this article aims to be a complementary theory of *neutrosophic geometric programming*.

Keywords

Neutrosophic Less Than or Equal To, Geometric Programming (GP), Signomial Geometric Programming (SGP), Fuzzy Geometric Programming (FGP), Neutrosophic Geometric Programming (NGP), Neutrosophic Function in Geometric Programming.

1 Introduction

The classical Geometric Programming (GP) is an optimization technique developed for solving a class of non-linear optimization problems in engineering design. GP technique has its origins in Zener's work (1961). Zener tried a new approach to solve a class of unconstrained non-linear optimization problems, where the terms of the objective function were posynomials. To solve these problems, he used the well-known arithmetic-geometric mean inequality (i.e. the arithmetic mean is greater than or equal to the geometric mean). Because of this, the approach came to be known as GP technique. Zener used this technique to solve only problems where the number of posynomial terms of the objective function was one more than the number of variables, and the function was not subject to any constraints. Later on (1962), Duffin extended the use of this technique to solve problems where the number of posynomial terms in the objective function is arbitrary. Peterson (1967), together with Zener and Duffin, extended the use of this technique to solve problems which also include the inequality constraints in the form of posynomials. As well, Passy and Wilde (1967) extended this technique further to solve problems in which some of the posynomial terms have negative coefficients. Duffin (1970) condensed the posynomial functions to a monomial form (by a logarithmic transformation, it became linear), and particularly showed that a "duality gap" function could not occur in geometric programming. Further, Duffin and Peterson (1972) pointed out that each of those posynomial programs GP can be reformulated so that every constraint function becomes posy-/bi-nomial, including at most two posynomial terms, where posynomial programming - with posy-/mo-nomial objective and constraint functions - is synonymous with linear programming.

As geometric programming became a widely used optimization technique, it was desirable that an efficient and highly flexible method of solutions were available. As the complexity of prototype geometric programs to be solved increased, several considerations became important. Canonically, the degree of problem difficulty and the inactive constraints reported an algorithm capable of dealing with these considerations. Consequently, McNamara (1976) proposed a solution procedure for geometric programming involving the formulation of an augmented problem that possessed zero degree of difficulty.

Accordingly, several algorithms have been proposed for solving GP (1980's). Such algorithms are somewhat more effective and reliable when they are applied to a convex problem, and also avoid difficulties with derivative singularities, as variables raised to fractional powers approach zero, since logs of such variables will approach $-\infty$, and large negative lower bounds should be placed on those variables.

In the 1990's, a strong interest in interior point (IP) algorithms has spawned several (IP) algorithms for GP. Rajgopal and Bricker (2002) produced an efficient procedure for solving posynomial geometric programming. The procedure, which used the concept of condensation, was embedded within an algorithm for a more general (signomial) GP problem. The constraint structure of the reformulation provides insight into why this algorithm is successful in avoiding all of the computational problems, traditionally associated with dual-based algorithms.

Li and Tsai (2005) proposed a technique for treating (positive, zero or negative) variables in SGP. Most existing methods of global optimization for SGP actually compute an approximate optimal solution of a linear or convex relaxation of the original problem. However, these approaches may sometimes provide an infeasible solution, or might form the true optimum to overcome these limitations.

A robust solution algorithm is proposed for global algorithm optimization of SGP by Shen, Ma and Chen (2008). This algorithm guarantees adequately to obtain a robust optimal solution which is feasible and close to the actual optimal solution, and is also stable under small perturbations of the constraints [6].

In the past 20 years, FGP has developed extensively. In 2002, B. Y. Cao published the first monography of fuzzy geometric programming as applied optimization. A large number of FGP applications have been discovered in a wide variety of scientific and non-scientific fields, since FGP is superior to classical GP in dealing with issues in fields like power system, environmental engineering, postal services, economical analysis, transportation, inventory theory; and so more to be discovered.

Arguably, fuzzy geometric programming potentially becomes a ubiquitous optimization technology, the same as fuzzy linear programming, fuzzy objective programming, and fuzzy quadratic programming [2].

This work is the first attempt to formulate the neutrosophic posynomial geometric programming (the simplest case, i.e. the unconstrained case). A previous work investigated the maximum and the minimum solutions to the neutrosophic relational GP [7,8].

2 Neutrosophic Less than or Equal To

In order to understand the concept of neutrosophic less than or equal to in optimization, we begin with some preliminaries which serve the subject.

Definition 2.1

Let X be the set of all fuzzy neutrosophic variable vectors x_i , $i = 1,2, \dots, m$, i.e. $X = \{(x_1, x_2, \dots, x_m)^{\mathrm{T}} \mid x_i \in \mathrm{FN_s}\}$. The function $\mathrm{g}(x) \colon X \ \rightarrow \ \mathrm{R} \cup \mathrm{I}$ is said to be the neutrosophic GP function of x, where $\mathrm{g}(x) = \sum_{\mathrm{k}=1}^{\mathrm{J}} \mathrm{c_k} \prod_{\mathrm{l}=1}^{\mathrm{m}} \mathrm{x_l}^{\gamma_{\mathrm{kl}}}$, $\mathrm{c_k} \geq 0$ are constants, γ_{kl} - are arbitrary real numbers.

Definition 2.2

Let $\mathrm{g}(x)$ be any linear or non-linear neutrosophic function, and let $\mathrm{A_0}$ be the neutrosophic set for all functions $\mathrm{g}(x)$ that are neutrosophically less than or equal to 1.

$$\begin{aligned}\mathrm{A_0} &= \{\, \mathrm{g}(x) < ₦ 1, x_i \in \mathrm{FN_s}\} \\ &= \left\{\, \mathrm{g}(x) < 1, \quad \mathrm{anti}\big(\, \mathrm{g}(x)\big) > 1, \quad \mathrm{neut}\big(\, \mathrm{g}(x)\big) = 1, x_i \in \mathrm{FN_s}\right\}.\end{aligned}$$

Definition 2.3

Let $\mathrm{g}(x)$ be any linear or non-linear neutrosophic function, where $x_i \in [0,1] \cup [0, \mathrm{nI}]$ and $x = (x_1, x_2, \dots, x_m)^{\mathrm{T}}$ a m-dimensional fuzzy neutrosophic variable vector.

We have the inequality

$$\mathrm{g}(x) < ₦ \ 1 \tag{1}$$

where " $<$ ₦" denotes the neutrosophied version for " $\leq$ " with the linguistic interpretation being "less than (the original claimed), greater than (the anti-claim of the original less than), equal (neither the original claim, nor the anti-claim)".

The inequality (1) can be redefined as follows:

$$\left.\begin{array}{r}\mathrm{g(x)} < 1 \\ \mathrm{anti}\ (\mathrm{g(x)}) > 1 \\ \mathrm{neut}(\ \mathrm{g(x)}) = 1\end{array}\right\} \tag{2}$$

Definition 2.4

Let $\mathrm{A_0}$ be the set of all neutrosophic non-linear functions that are neutrosophically less than or equal to 1.

$$\begin{aligned}\mathrm{A_0} &= \{\, \mathrm{g}(x) < ₦ \ 1, x_i \in \mathrm{FN_s}\} \\ &= \left\{\, \mathrm{g}(x) < 1, \quad \mathrm{anti}\big(\, \mathrm{g}(x)\big) > 1, \quad \mathrm{neut}\big(\, \mathrm{g}(x)\big) = 1, x_i \in \mathrm{FN_s}\right\}.\end{aligned}$$

It is significant to define the following membership functions:

$$\mu_{\mathrm{A_o}}\big(\, \mathrm{g(x)}\big) = \begin{cases} 1 & 0 \leq \mathrm{g(x)} \leq 1 \\ \left(\mathrm{e}^{\frac{-1}{\mathrm{d_o}}(\mathrm{g(x)}-1)} + \mathrm{e}^{\frac{-1}{\mathrm{d_o}}(\mathrm{anti}(\,\mathrm{g(x)})-1)} - 1\right), & 1 < \mathrm{g(x)} \leq 1 - \mathrm{d_o} \ln 0.5 \end{cases} \tag{3}$$

$$\mu_{A_o}(\text{anti}(\,g(x))) = \begin{cases} 0 & 0 \le g(x) \le 1 \\ \left(1 - e^{\frac{-1}{d_o}(\text{anti}(\,g(x))-1)} - e^{\frac{-1}{d_o}(g(x)-1)}\right), & 1 - d_o \ln 0.5 \le g(x) \le 1 + d_o \end{cases} \quad (4)$$

It is clear that $\mu_{A_o}(\text{neut}(\,g(x)))$ consists of intersection the following functions:

$$e^{\frac{-1}{d_o}(g(x)-1)}, \quad 1 - e^{\frac{-1}{d_o}(\text{anti}(\,g(x))-1)}$$

i.e.

$$\mu_{A_o}(\text{neut}(\,g(x))) = \begin{cases} 1 - e^{\frac{-1}{d_o}(\text{anti}(\,g(x))-1)} & 1 \le g(x) \le 1 - d_o \ln 0.5 \\ e^{\frac{-1}{d_o}(g(x)-1)} & 1 - d_o \ln 0.5 < g(x) \le 1 + d_o \end{cases} \quad (5)$$

Note that $d_o > 0$ is a constant expressing a limit of the admissible violation of the neutrosophic non-linear function $g(x)$ [3].

2.1 The relationship between $g(x)$, anti $g(x)$ in NGP

1. At

$$1 < g(x) \le 1 - d_o \ln 0.5$$
$$\mu_{A_o}(\,g(x)\,) > \mu_{A_o}(\text{anti}(\,g(x)\,) \qquad \text{(see } \textit{Figure 1}\text{)}$$
$$e^{\frac{-1}{d_o}(g(x)-1)} > 1 - e^{\frac{-1}{d_o}(\text{anti}(\,g(x)\,)-1)}$$
$$e^{\frac{-1}{d_o}(\text{anti}(\,g(x)\,)-1)} > 1 - e^{\frac{-1}{d_o}(g(x)-1)}$$
$$\frac{-1}{d_o}(\text{anti}(\,g(x)\,) - 1) > \ln(1 - e^{\frac{-1}{d_o}(g(x)-1)})$$
$$\text{anti}(\,g(x)\,) < 1 - d_o \ \ln(1 - e^{\frac{-1}{d_o}(g(x)-1)})$$

2. Again at

$$1 - d_o \ln 0.5 < g(x) \le 1 + d_o$$
$$\mu_{A_o}\big(\,g(x)\big) < \mu_{A_o}(\text{anti}(\,g(x)))$$
$$\therefore \ \text{anti}(\,g(x)\,) > 1 - d_o \ \ln(1 - e^{\frac{-1}{d_o}(g(x)-1)})$$

3 Neutrosophic Geometric Programming (the unconstrained case)

Geometric programming is a relative method for solving a class of non-linear programming problems. It was developed by Duffin, Peterson, and Zener (1967) [4]. It is used to minimize functions that are in the form of posynomials, subject to constraints of the same type.

Inspired by Zadeh's fuzzy sets theory, fuzzy geometric programming emerged from the combination of fuzzy sets theory with geometric programming.

Fuzzy geometric programming was originated by B.Y. Cao in the Proceedings of the second IFSA conferences (Tokyo, 1987) [1].

In this work, the neutrosophic geometric programming (the unconstrained case) was established where the models were built in the form of posynomials.

Definition 3.1

Let

$$\left.\begin{array}{l} \overset{N}{\text{(P)}} \quad \overset{N}{\min}\ g(x) \\ \quad x_i \in FN_s \end{array}\right\}. \tag{6}$$

The neutrosophic unconstrained posynomial geometric programming, where $x = (x_1, x_2, \dots, x_m)^{\mathrm{T}}$ is a m-dimensional fuzzy neutrosophic variable vector, "T" represents a transpose symbol, and $g(x) = \sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}$ is a neutrosophic posynomial GP function of x, $c_k \geq 0$ a constant, γ_{kl} an arbitrary real number, $g(x) < ₦\, z \rightarrow \overset{N}{\min} g(x)$; the objective function $g(x)$ can be written as a minimizing goal in order to consider z as an upper bound; z is an expectation value of the objective function $g(x)$, "$<₦$" denotes the neutrosophied version of "$\leq$" with the linguistic interpretation (see Definition 2.3), and $d_o > 0$ denotes a flexible index of $g(x)$.

Note that the above program is undefined and has no solution in the case of $\gamma_{kl} < 0$ with some x_l's taking indeterminacy value, for example,

$$\overset{N}{\min} \qquad g(x) = 2x_1^{-.2} x_2^{.3} x_4^{1.5} + 7x_1^{3} x_2^{-.5} x_3,$$

where $x_i \in FN_s, i = 1,2,3,4$.

This program is not defined at $x = (.2I, .3, .25, I)^{\mathrm{T}}$, $g(x) = 2(.2I)^{-.2}(.3)^{.3} I^{1.5} + 7(.2I)^3(.3)^{-.5}(.25)$ is undefined at $x_1 = .2I$ with $\gamma_1 = -0.2$.

Definition 3.2

Let A_0 be the set of all neutrosophic non-linear functions $g(x)$ that are neutrosophically less than or equal to z, i.e.

$$A_0 = \{\, g(x) < ₦\, z, x_i \in FN_s\}.$$

The membership functions of $g(x)$ and anti$(g(x))$ are:

$$\mu_{A_o}(g(x)) = \begin{cases} 1 & 0 \leq g(x) \leq z \\ \left(e^{\frac{-1}{d_o}(g(x)-z)} + e^{\frac{-1}{d_o}(\text{anti}\,(g(x))-z)} - 1\right), & z < g(x) \leq z - d_o \ln 0.5 \end{cases} \tag{7}$$

$$\mu_{A_o}(\text{anti}(g(x))) = \begin{cases} 0 & 0 \leq g(x) \leq z \\ \left(1 - e^{\frac{-1}{d_o}(\text{anti}\,(g(x))-z)} - e^{\frac{-1}{d_o}(g(x)-z)}\right), & z - d_o \ln 0.5 \leq g(x) \leq z + d_o \end{cases} \tag{8}$$

Eq. (6) can be changed into

$$g(x) < ₦\ z, \qquad x = (x_1, x_2, \dots, x_m), x_i \in FN_s \tag{9}$$

The above program can be redefined as follow:

$$\left.\begin{array}{l} g(x) < z \\ \text{anti}(\,g(x)) > z \\ \text{neut}(\,g(x)) = z \\ x = (x_1, x_2, \dots, x_m), x_i \in FN_s \end{array}\right\} \tag{10}$$

It is clear that $\mu_{A_o}(\text{neut}(\,g(x)))$ consists from the intersection of the following functions:

$$e^{\frac{-1}{d_o}(g(x)-z)} \quad \& \quad 1 - e^{\frac{-1}{d_o}(\text{anti}(g(x))-z)}$$

$$\mu_{A_o}(\text{neut}(\,g(x))) = \begin{cases} 1 - e^{\frac{-1}{d_o}(\text{anti}(\,g(x))-z)} & z \le g(x) \le z - d_o \ln 0.5 \\ e^{\frac{-1}{d_o}(g(x)-z)} & z - d_o \ln 0.5 < g(x) \le z + d_o \end{cases} \tag{11}$$

Definition 3.3

Let $\tilde{N}$ be a fuzzy neutrosophic set defined on $[0,1] \cup [0, nI], n \in [0,1]$; if there exists a fuzzy neutrosophic optimal point set A_o^* of $g(x)$ such that

$$\tilde{N}(x) = \begin{array}{c} \min\{\mu(\text{neut}\, g(x))\} \\ x = (x_1, x_2, \dots, x_m), x_i \in FN_s \end{array} \tag{12}$$

$$\tilde{N}(x) = e^{\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right)} \wedge \; 1 - e^{\frac{-1}{d_o}\left(\text{anti}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)},$$

then $\max \tilde{N}(x)$ is said to be a neutrosophic geometric programming (the unconstrained case) with respect to $\tilde{N}(x)$ of $g(x)$.

Definition 3.4

Let x^* be an optimal solution to $\tilde{N}(x)$, i.e.

$$\tilde{N}(x^*) = \max \tilde{N}(x) \;, x = (x_1, x_2, \dots, x_m), x_i \in FN_s\,, \tag{13}$$

and the fuzzy neutrosophic set $\tilde{N}$ satisfying (12) is a fuzzy neutrosophic decision in (9).

Theorem 3.1

The maximum of $\tilde{N}(x)$ is equivalent to the program:

$$\left.\begin{array}{l} \max \alpha \\ g(x) < z - d_o \ln \alpha \\ \text{anti}\, g(x) > z - d_o \ln(1-\alpha) \\ x = (x_1, x_2, \dots, x_m), x_i \in FN_s\,, d_o > 0 \end{array}\right\} \tag{14}$$

Proof

It is known by definition (3.4) that x^* satisfied eq. (12), called an optimal solution to (9). Again, x^* bears the similar level for $g(x)$, $\text{anti}(g(x))$ & $\text{neut}(g(x))$. Particularly, x^* is a solution to neutrosophic posynomial geometric programming (6) at $\tilde{N}(x^*) = 1$. However, when $g(x) < z$ and $\text{anti}(g(x)) > z$, there exists

$$\tilde{N}(x) = e^{\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right)} \wedge \; 1 - e^{\frac{-1}{d_o}\left(\text{anti}\,\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)},$$

given $\alpha = \tilde{N}(x)$. Now, $\forall\, \alpha \in FN_s$; it is clear that

$$e^{\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right)} \geq \alpha \tag{15}$$

$$1 - e^{\frac{-1}{d_o}\left(\text{anti}\,\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)} \geq \alpha \tag{16}$$

From (15), we have

$$\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right) \geq \ln \alpha$$

$$g(x) = \left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) \leq z - d_o \ln \alpha\,. \tag{17}$$

From (16), we have

$$1 - \alpha \geq e^{\frac{-1}{d_o}\left(\text{anti}\,\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)}$$

$$\rightarrow \text{anti}\,\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z \geq -d_o \ln(1-\alpha) \tag{18}$$

$$\text{anti}\,\big(g(x)\big) \geq z - d_o \ln(1-\alpha).$$

Note that, for the equality in (17) & (18), it is exactly equal to neut $g(x)$.

Therefore, the maximization of $\tilde{N}(x)$ is equivalent to (14) for arbitrary $\alpha \in FN_s$, and the theorem holds.

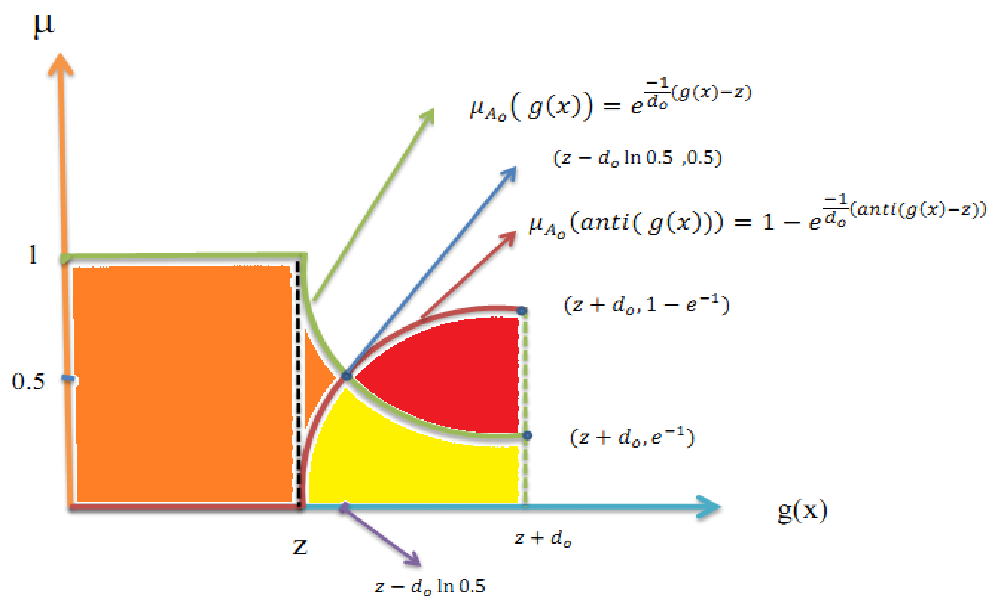

Figure 1. The orange color means the region covered by $\mu_{A_o}(\,g(x))$, the red color means the region covered by $\mu_{A_o}(\text{anti}(\,g(x)))$, and the yellow color means the region covered by $\mu_{A_o}(\text{neut}(\,g(x)))$.

4 Conclusion

The innovative concept and procedure explained in this article suit to the neutrosophic GP. A neutrosophic less than or equal to form can be completely turned into classical less than, greater than and equal forms. The feasible region for unconstrained neutrosophic GP can be determined by a fuzzy neutrosophic optimal point set in the fuzzy neutrosophic decision region $\tilde{N}(x^*)$.

5 References

[1] B.Y. Cao, *Fuzzy Geometric Programming*, Dordrecht: Springer Science Business Media, 2002.

[2] B.Y. Cao & J. H. Yang, *Advances in Fuzzy Geometric Programming*, Berlin-Heidelberg: Springer Verlag, 2007.

[3] D. Dubois, H. Prade, *Fuzzy Sets and Systems. Theory and Applications*, New York: Academic Press, 1980.

[4] R. J. Duffin, E. L. Peterson, C. Zener. *Geometric Programming. Theory and Application*, New York: John Wiley and Sons, 1967.

[5] V. Kandasamy & F. Smarandache, *Fuzzy Relational Maps and Neutrosophic Relational Maps*, Rehoboth: American Research Press, 2004.

[6] H. E. Khalid, A. Y. Al-Bayati, *Investigation in the Sensitivity Analysis of Generalized Geometric Programming Problems*, PhD Thesis, The Council of the College of Computers Sciences and Mathematics University of Mosul, 2010.

[7] H. E. Khalid, *An Original Notion to Find Maximal Solution in the Fuzzy Neutrosophic Relation Equations (FNRE) with Geometric Programming (GP)*, in "Neutrosophic Sets and Systems", vol. 7, 2015, pp. 3-7.

[8] H. E. Khalid, *The Novel Attempt for Finding Minimum Solution in Fuzzy Neutrosophic Relational Geometric Programming (FNRGP) with (max , min) Composition*, in "Neutrosophic Sets and Systems", vol. 11, 2016, pp. 107-112.

[9] F. Smarandache, *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics*, 168 p., Pons Editions, Bruxelles, Belgique, 2016, https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf

Multi-Criteria Decision Making Method for *n*-wise Criteria Comparisons and Inconsistent Problems

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Abstract

The purpose of this paper is to present an alternative of a hybrid method based on Saaty's Analytical Hierarchy Process and on the Technique for Order Preference by using the Similarity to Ideal Solution method (AHP-TOPSIS) and, based on the AHP and its use of pairwise comparisons, to extend it to a new method called α-D MCDM-TOPSIS (α-Discounting Method for multi-criteria decision making-TOPSIS). The new method overcomes the limits of AHP, which works only for pairwise comparisons of criteria, to any-wise (*n*-wise) comparisons, with crisp coefficients or with interval-valued coefficients. An extended MCMD method (called Extended α-D MCDM) of α-D MCDM, introduced by Smarandache to solve decision making problems, is developed. α-D MCDM-TOPSIS and Extended α-D MCDM are verified on several examples, to demonstrate how they work with consistent, weak inconsistent or strong inconsistent problems. Finally, we discuss and compare all methods.

Keyword

Decision making, Extended α-D MCDM, Consistency, Inconsistency, n-wise criteria comparisons, AHP TOPSIS.

1 Introduction

Many economic, social or technological problems have been widely discussed and resolved in recent years by multi-criteria decision making methods [8]. However, the quantity of data, the complexity of the modern world and the recent technological advances have made obviously that MCDM methods are more challenging than ever, hence the necessity of developing other methods, able to give quality solutions.

Among MCDM methods, the most often used to improve the reliability of the decision making process is the combined method AHP-TOPSIS [12], [3], [2], [8], [10], [11] and [4].

AHP-TOPSIS is indeed a useful MCDM method to resolve difficult decision making problems and to select the best of the alternatives. Its applications are significant [8]: a support for management and planning of flight mission at NASA [12]; a method to study how the traffic congestion of urban roads is evaluated [3]; choosing logistics service provider in the mobile phone industry domains [10]; summarizing an e-SCM performance for management of supply chain [11]; evaluating faculty performance in engineering education, or sharing capacity assessment knowledge of supply chain [4].

Our paper is organized as follows. In the next section (*Section 2*), a literature survey for consistency problems is given. *Section 3* and *Section 4* focus on AHP-TOPSIS, and on the proposed α-D MCDM-TOPSIS model, respectively. The proposed method is tested on consistent, weak inconsistent and strong inconsistent examples (in *Section 5*). AHP method employed to rank the preferences is considered in *Section 6*. An extended α-D MCDM is introduced in *Section 7*, and it is shown how it can be applied for ranking preferences. We discuss developments via the use of an example to compare all methods. Finally, we draw conclusions and envisage some perspectives.

2 Comparison of characteristics between AHP and α-D MCDM: Consistency

2.1 A brief overview of Analytic Hierarchy Process (AHP)

AHP, introduced by the Saaty [6], is one of the most complete methods of multi-criteria decision making technique, determining the weights of criteria and ranking alternatives. The use of AHP only, or its hybrid use with other methods, proved its capacity to solve MCDM problems and to be a popular technique for determining weights — see more than a thousand references in [9]. Besides the performance of AHP and its added value at both levels, theoretical and practical, this method functions only if the problem is perfectly consistent, which is rarely checked in real MCDM problems.

2.2 Description of α-D MCDM

α-D MCDM (α-Discounting Method for Multi-Criteria Decision Making) was introduced by Smarandache — see [7]. The new method overcomes the limits of AHP, which work only for pairwise comparisons of criteria, expanding to any-wise (n-wise) comparisons.

Smarandache used the homogeneous linear mathematical equations to express the relationship between criteria with crisp coefficients or with interval-valued coefficients also for non-linear equations, with crisp coefficients or with interval-valued coefficients.

The two aims of α-D MCDM method were: firstly, to transform the equations of each criterion with respect to other criteria that has only a null solution into a linear homogeneous system having a non-null solution by multiplying each criteria of the right hand by non-null positive parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$; secondly, to apply the "Fairness Principle" on the general solution of the above system by discounting each parameter by the same value $(\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k)$.

2.2.1 α-D MCDM method

The general idea of the α-D MCDM is to transform any MCDM inconsistent problem (in which AHP does not work) to a MCDM consistent problem, by discounting each coefficient by the same percentage.

Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$, with $n \geq 2$, is a set of criteria, and let's construct a linear homogeneous system of equations.

Each criterion C_i can be expressed as linear homogeneous equation, or as non-linear equation, with crisp coefficients or with interval-valued coefficients of other criteria $C_1, \cdots C_j \cdots C_n$ —

$$C_i = f(C_1, \cdots C_j \cdots C_n).$$

Consequently, a comparisons matrix associated to this linear homogeneous system is constructed.

To determine the weights w_i of the criteria, we solve the previous system.

The α-D MCDM method is not designed to rank preferences P_i based on C_i criteria, as AHP method does, but to determine only the weights of criteria in any type of problems (consistent, inconsistent).

AHP as cited above is a complete method designed to calculate the weights of criteria C_i and to rank the preferences P_i. In addition, when the AHP is used with TOPSIS, or other MCDM method, we just benefit from the part of weight calculation criteria and we use TOPSIS to rank preferences — or other MCDM methods.

The same for α-D MCDM: firstly, it is just used to calculate the weight of criteria, that will be used later by TOPSIS to rank preferences, and, secondly, the α-D MCDM is extended to a complete method, in order to rank the preferences.

Therefore, we use α-D MCDM for calculating the weight of criteria C_i and not to rank P_i preferences.

We have —

$$C_i = f(\{C\} \setminus C_i).$$

Then, criteria C_i is a linear equation of C_j such as —

$$C_i = \sum_{j=1 j \neq i}^{n} x_{ij} C_j .$$

So, the comparisons criteria matrix has the number of criteria by rows and columns (rows number n = number of criteria, and columns number m = number of equations). In the result, we have a square matrix ($n = m$), consequently we can calculate the determinant of this matrix. At this point, we have an $n \times n$ linear homogeneous system and its associated matrix —

$$\begin{cases} x_{1,1}w_1 + x_{1,2}w_2 + \cdots + x_{1,n}w_n = 0 \\ \vdots \\ x_{n,1}w_1 + x_{m,2}w_2 + \cdots + x_{n,n}w_n = 0 \end{cases}$$

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}.$$

The difference between AHP and α-D MCDM is the ability of the latter to work with consistent and inconsistent problems, and if the problem is inconsistent, α-D MCDM method transforms it in a consistent problem, while AHP is unable to do that, managing strictly consistent problems.

In the following, we deal with the relationship between determinant of matrix and consistency, and the parameterization of system by α_i, in order to get a consistent problem.

Property 1

— If $det(X) = 0$, the system has a solution (i.e. MCDM problem is consistent).
— If $det(X) \neq 0$, the system has only the null solution (i.e. MCDM problem is inconsistent).

If the problem is inconsistent, then one constructs the parameterized matrix, denoted $X(\alpha)$, by parameterizing the right-hand in order to get $\det(X(\alpha)) = 0$ and use Fairness principe (set equal parameters to all criteria $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k > 0$). To get priority vector, one resolves the new system obtained and set 1 to secondary variable, then normalize the vector by dividing the sum of all components.

2.3 Consistency of decision making problems

In this section, we discuss the consistency of the MCDM problems for both methods (α-D MCDM and AHP).

For resolving a linear system of equations, in mathematics we use raw operations, such as substitution, interchange, … .

Definition 1 [7]

Applying any substitution raw operations on two equations, if it does not influence the system consistency and there is an agreement of all equations, we say that the linear system of equations (of the linear MCDM problem) is consistent.

Definition 2 [7]

Applying any substitution raw operations on two equations, if equation result is in disagreement with another, we say that the linear system of equations (of the linear MCDM problem) is weakly consistent.

Definition 3 [7]

Applying any substitution raw operations on two equations, if equation result is in opposition with another, we say that the linear system of equations (of the linear MCDM problem) is strongly inconsistent.

2.4 Consistency

AHP provides the decision maker with a way of examining the consistency of entries in a pairwise comparison matrix; the problem of accepting/rejecting matrices has been largely discussed [5], [1], [13], especially regarding the relation between the consistency and the scale used to represent the decision maker's judgments. AHP is too restrictive when the size of the matrix increases, and when order n of judgment matrix is large; the satisfying consistency is more difficult to be met [5], [1].

This problem may become a very difficult one when the decision maker is not perfectly consistent, moreover, it seems impossible (AHP does not work) when there are not pairwise comparisons, but all kind of comparisons between criteria, such as $n-$wise, because there is set a strict consistency condition in the AHP, in order to keep the rationality of preference intensities between compared elements.

In addition, the inconsistency exists in all judgments [5]; comparing three alternatives — or more, it is possible that inconsistency exists when there are more than 25 percent of the $3-by-3$ reciprocal matrices with a consistency ratio less than or equal to ten percent. Consequently, as the matrix size increases, the percentage of inconsistency decreases dramatically [1], [5].

Furthermore, the AHP method sets a consistency ratio (CR) threshold ($CR(X) > 0.1$), which should not be exceeded, by examining the inconsistency of the pairwise comparison matrix, but this requirement for the Saaty's matrix is not achievable in the real situations.

In order to overcome this deficiency, instead of the AHP we suggest employing an α-D MCDM, which is very natural and more suitable for the linguistic descriptions of the Saaty's scale and, as a result of it, it is easier to reach this requirement in the real situations.

Moreover, the attractiveness of α-D MCDM is due to its potential to overcome limits of AHP, which works only for pairwise comparisons of criteria, expanding to n-wise (with $n \geq 2$) comparisons, with crisp coefficients or with interval-valued coefficients. Therefore, α-D MCDM method works for inconsistent, weak inconsistent and strong inconsistent problems.

As previously shown, in α-D MCDM method, in order to transform a inconsistent MCDM problem to a consistent problem — we multiply each criteria of the right hand by non-null positive parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$ and we use "Fairness Principle" assigning to each parameter the same value ($\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k$).

Property 2

In α-D MCDM (and Fairness-Principle for coefficients α_i), the parameter α (or $\frac{1}{\alpha}$) signifies the degree of consistency and β ($\beta = f(\alpha)$) represents the degree of inconsistency.

— If $0 < \alpha < 1$, then α and $\beta = 1 - \alpha$ represent the degree of consistency and the degree of inconsistency, respectively, of the decision-making problem.

— If $\alpha > 1$, then $\frac{1}{\alpha}$ and $\beta = 1 - \frac{1}{\alpha}$ represent the degree of consistency and the degree of inconsistency, respectively, of the decision-making problem.

Property 3

In AHP method, RI — consistency index, CR — consistency ratio and λ (λ_{max} largest) — the eigenvalue of the $(X)_{n \times n}$ pairwise comparison matrix.

— We say that MCDM problem is consistent (pairwise comparison matrix X is consistent), if $Rank(X) = 1$ and $\lambda = n$ (ideal case).
— We say that MCDM problem is consistent too (pairwise comparison matrix X is consistent), if consistency ratio $CR(X) \leq 0.1$, $CR(X) = \frac{CI(X)}{RI(X)}$, where $CI(X) = \frac{\lambda_{max} - n}{n-1}$, and RI values are given (simulation parameter).
— If $CR(X) \leq 0.1$, the MCDM problem is inconsistent and the pairwise comparison matrix should be improved.

Characteristics	**AHP**	α **-D MCDM**
Weight elicitation	Pairwise comparison	n -wise comparison ($n \geq 2$)
Number of attributes accommodated	7 ± 2	Large inputs
Consistent problems	Provided	Not provided and α-D MCDM gives same result as AHP
Weakly inconsistent problems	Does not work	Justifiable results
Strongly inconsistent problems	Does not work	Justifiable results

Table 1: Comparison of characteristics of both methods (AHP, α-D MCDM)

3 Description of data structure decision problems under consideration

Taking into account that pertinent data is frequently very high-priced to collect, we can't change real life problems to obtain a specific form of data. In addition, information from real world certainly includes imperfection — such as uncertainty, conflict, etc.

Hence, the choice of the MCDM method is based, firstly, on the structure of decision problem considered, secondly, on the types of data that can be obtained, and, finally, on the capability to get accurate results. For this reason, we detail the different types of all data structure decision problem, for example:

— If decision matrix illustrates the importance of alternatives with respect of criteria, the pairwise (or n-wise) comparison can't be used directly in the hybrid AHP-TOPSIS approach. Firstly, priority weights for criteria are calculated using AHP technique, and then the alternatives are prioritized using TOPSIS approach.

The derivation of weights is a central step in eliciting the decision-maker's preferences, but the hybrid AHP-TOPSIS method is more difficult to be met: on one hand, AHP does not work in inconsistent problems, on the other hand it cannot be employed for the n-wise comparisons criteria cases.

The problem can be abstracted as how to derive weights for a set of activities according to their impact on the situation and the objective of decisions to be made.

Hence, this study will extend AHP-TOPSIS to a MCDM to fit real world. A complete and efficient procedure for decision making will then be provided. The developed model has been analyzed to select the best alternative using α-D MCDM and the technique for order preference by similarity to ideal solution (TOPSIS) as a hybrid approach.

Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$ is a set of Criteria, with $n \geq 2$, and $A = \{A_1, A_2, \cdots, A_m\}$ is the set of Preferences (Alternatives), with $m \geq 1$.

$$\begin{array}{ccc} \begin{array}{ccc} C_1 & & C_n \\ \downarrow & \cdots & \downarrow \end{array} & & \\ \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} & \begin{array}{c} \leftarrow C_1 \\ \vdots \\ \leftarrow C_n \end{array} \end{array}$$

	C_1	C_2	$\cdots$	C_n
	w_1	w_2	$\cdots$	w_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Table 2: Decision matrix

If the data cannot be obtained directly to construct the decision matrix $A=(a_{ij})$ above, we should have, for each criteria C_i, a pairwise (or n-wise) comparison matrix of the preferences (not just for the criterion).

The comparison matrix of the preferences gives the relative importance (b_{ij}) of each alternative A_i compared with another A_j with respect to criterion C_k.

As mentioned, the comparison matrices of the preferences should be given, but for comparing the results, we will demonstrate how we can obtain it from decision matrix.

For each criterion C_k, the comparison matrix of the preferences is defined by $B_k=(b_{ij})$ such as $b_{ij}=\frac{a_{ik}}{a_{jk}}$, with $i=1,2\cdots,n$ (for each criterion a comparison matrix of preferences, consequently n comparisons preferences matrices will be constructed).

C_i	A_1	A_2	$\cdots$	A_n
A_1	b_{11}	b_{12}	$\cdots$	b_{1m}
A_2	b_{21}	b_{22}	$\cdots$	b_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	b_{m1}	b_{m2}	$\cdots$	b_{mm}

Table 3: Relative importance of alternative comparison matrix

Hence, we need to construct n (number of criteria) matrices with pairwise or n-wise comparisons of size $m*m$ each, with m the number of preferences, in these cases we can use AHP or α-D MCDM.

In this case, AHP method is used both to calculate the weights of criteria and to ranking preferences by calculate the priority.

AHP being more difficult to be met, we will extend α-D MCDM to work for the calculation of the weights criteria and ranking preferences.

4 AHP-TOPSIS method

In the real word decisions problems (*case 1, Section 3*) we have multiple preferences and diverse criteria. The MCDM problem can be summarized as it follows:

— Calculate weights w_i of criteria C_i;

— Rank preferences (alternatives) A_i.

Let us assume there are n criteria and their pairwise relative importance is x_{ij}.

TOPSIS assumes that we have n alternatives (preferences) $A_i (i = 1, 2, \cdots, m)$ and n attributes/criteria $C_j (j = 1, 2, \cdots, n)$ and comparison matrix a_{ij} of preference i with respect to criterion j.

The AHP-TOPSIS method is described in the following steps:

Step 4.1. *Construct decision matrix denoted by* $A = (a_{ij})_{m \times n}$

	C_1	C_2	$\cdots$	C_n
	w_1	w_2	$\cdots$	w_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Table 4: Decision matrix

Step 4.2. *Determine weights* (w_j) *of each criterion using AHP Method*, where

$$\sum_{j=1}^{n} w_j = 1, \; j = 1, 2, \cdots, n.$$

Step 4.2.1. *Build a pairwise comparison matrix of criteria*

The pairwise comparison of criterion i with respect to criterion j gives a square matrix $(X)_{n\times n} = (x_{ij})$ where x_{ij} represents the relative importance of criterion i over the criterion j. In the matrix, $x_{ij} = 1$ when $i = j$ and $x_{ij} = 1/x_{ji}$. So, we get a $n \times n$ pairwise comparison matrix $(X)_{n\times n}$.

Step 4.2.2. *Find the relative normalized weight* (w_j) *of each criterion* defined by following formula —

$$w_j = \frac{\prod_{j=1}^{n} (x_{ij})^{1/n}}{\sum \prod_{j=1}^{n} (x_{ij})^{1/n}}$$

Then, get w_i weight of the i^{th} criterion.

Step 4.2.3. *Calculate matrix* X_3 *and* X_4 — such that $X_3 = X_1 \times X_2$ and where $X_4 = X_3 / X_2$ —

$$X2 = [w_1, w_2, \cdots, w_j]^T .$$

Step 4.2.4. *Find the largest eigenvalue of pairwise comparison matrix*

For simplified calculus, the largest eigenvalue of pairwise comparison matrix is the average of X_4. Furthermore, according to the Perron-Frobenius theorem, principal eigenvalue λ_{max} always exists for the Saaty's matrix and it holds $\lambda_{max} \geq n$; for fully consistent matrix $\lambda_{max} = n$.

Consistency check is then performed to ensure that the evaluation of the pair-wise comparison matrix is reasonable and acceptable.

Step 4.2.5. *Determine the consistency ratio* (CR)

After calculation consistency ratio (RC) using equation (eq.), and in order to verify the consistency of the matrix that is considered to be consistent if CR is less than threshold and not otherwise, according to Saaty and search. At this point, we have the weights of criteria and if the consistency is checked, we will be using TOPSIS to rank preferences.

Step 4.3. *Normalize decision matrix*

The normalize decision matrix is obtained, which is given here with r_{ij}

$$r_{ij} = a_{ij} / \left(\sum_{i=1}^{m} a_{ij}^2 \right)^{0.5} ; \ j = 1, 2, \cdots, n; i = 1, 2 \cdots, m .$$

Step 4.4. *Calculate the weighted decision matrix*

Weighting each column of obtained matrix by its associated weight.

$$v_{ij} = w_j r_{ij}; \ j = 1, 2, \cdots, n; i = 1, 2 \cdots, m.$$

Step 4.5. *Determine the positive ideal solution (PIS) and negative ideal solution (NIS)*

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \left\{ \left(max_i \left\{ v_{ij} \mid j \in B \right\} \right), \left(min_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\};$$

$$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \left\{ \left(min_i \left\{ v_{ij} \mid j \in B \right\} \right), \left(max_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\}.$$

The benefit and cost solutions are represented by B and C respectively.

Step 4.6. *Calculate the distance measure for each alternative from the PIS and NIS*

The distance measure for each alternative from the PIS is —

$$S_i^+ = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5} ; i = 1, 2 \cdots, m$$

Also, the distance measure for each alternative from the NIS is —

$$S_i^- = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5} ; i = 1, 2 \cdots, m$$

Step 4.7. *Determine the values of relative closeness measure*

For each alternative we calculate the relative closeness measure as it follows:

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)} ; i = 1, 2 \cdots, m .$$

Rank alternatives set according to the order of relative closeness measure values T_i.

5 α-D MCDM-TOPSIS method

The MCDM problem description is the same as the one used in AHP-TOPSIS method (*Section 4*), but in this case we have n-wise comparisons matrix of criteria. Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$, with $n \geq 2$, and $\{A_1, A_2, \cdots, A_m\}$, with $m \geq 1$, are a set of criteria and a set of preferences, respectively. Let us assume that each criterion C_i is a linear homogeneous equation of the other criteria $C_1, C_2, \cdots, C_n$:

$$C_i = f(\{C\} \setminus C_i).$$

The α-D MCDM-TOPSIS method is described in the following steps:

Step 5.1. *Construct decision matrix denoted by* $A = (a_{ij})_{m \times n}$

	C_1	C_2	$\cdots$	C_n
	w_1	w_2	$\cdots$	w_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Table 5: Decision matrix

Step 5.2. *Determine weights (w_i) of each criterion using α-D MCDM Method*

Step 5.2.1. *Using α-D MCDM to determine the importance weight (w_i) of the criteria*, where —

$$\sum_{j=1}^{n} w_j = 1,\ j = 1, 2, \cdots, n.$$

Step 5.2.2. *Build a system of equations and its associated matrix*

To construct linear system of equations, each criterion C_i is expressed as a linear equation of C_j such as —

$$C_i = \sum_{j=1\, j\neq i}^{n} x_{ij} C_j$$

Consequently, we have a system of *n* linear equations (one equation of each criterion) with *n* variables (variable w_i is weight of criterion).

$$\begin{cases} x_{1,1}w_1 + x_{1,2}w_2 + \cdots + x_{1,n}w_n = 0 \\ \vdots \\ x_{n,1}w_1 + x_{m,2}w_2 + \cdots + x_{n,n}w_n = 0 \end{cases}$$

In mathematics, each linear system can be associated to a matrix, in this case, denoted by $X = (x_{ij})$, $1 \le i \le n$ and $1 \le j \le n$ where —

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}.$$

Step 5.2.3. *Solve system of equation using whose associated matrix*

Solve the system of equation; the different cases are discussed in *Property 1* in that we compute the determinant of X (find strictly positive solution $w_i > 0$).

Solving this homogeneous linear system, in different cases above the general solution that we set as a solution vector —

$$S = [s_1, s_2, \cdots, s_n]$$

and set 1 to secondary variable, we get —

$$W = [w_1, w_2, \cdots, w_n].$$

Dividing each vector element on sum of all components of vector to get normalized vector, where —

$$w_j = \frac{s_j}{\sum_{k=1}^{n} s_k}; i = 1, 2 \cdots, n.$$

Step 5.2.4. *Build a pairwise comparison matrix of criteria*

The pairwise comparison of criterion i with respect to criterion j gives a square matrix $(X)_{n\times n} = (x_{ij})$ where x_{ij} represents the relative importance of criterion i over the criterion j. In the matrix, $x_{ij} = 1$ when $i = j$ and $x_{ij} = 1/x_{ji}$. So we get a $n \times n$ pairwise comparison matrix $(X)_{n\times n}$.

Step 5.2.5. *Find the relative normalized weight (w_j) of each criterion* defined by the following formula —

$$w_j = \frac{\prod_{j=1}^{n} (x_{ij})^{1/n}}{\sum \prod_{j=1}^{n} (x_{ij})^{1/n}}.$$

Then, get w_i weight of the i^{th} criterion.

Step 5.2.6. *Calculate matrix X_3 and X_4* — such that $X_3 = X_1 \times X_2$ and where $X_4 = X_3/X_2$ —

$$X2 = [w_1, w_2, \cdots, w_j]^T.$$

Step 5.2.7. *Find the largest eigenvalue of pairwise comparison matrix*

For simplifying the calculus, the largest eigenvalue of pairwise comparison matrix is the average of X_4. Furthermore, according to the Perron-Frobenius theorem, principal eigenvalue λ_{max} always exists for the Saaty's matrix and it holds $\lambda_{max} \geq n$; for fully consistent matrix $\lambda_{max} = n$.

Consistency check is then performed to ensure that the evaluation of the pair-wise comparison matrix is reasonable and acceptable.

Step 5.2.8. *Determine the consistency ratio* (CR)

After calculating consistency ratio (RC) using equation (eq.), and in order to verify the consistency of the matrix that is considered to be consistent if CR is less than threshold and not otherwise, according to Saaty and search.

At this point, we have the weights of criteria and if the consistency is checked, we will be using TOPSIS to rank preferences.

Step 5.3. *Normalize decision matrix*

The normalized decision matrix is obtained, which is given here with r_{ij}

$$r_{ij} = a_{ij} / \left(\sum_{i=1}^{m} a_{ij}^2 \right)^{0.5} ;\ j = 1, 2, \cdots, n;\ i = 1, 2 \cdots, m\,.$$

Step 5.4. *Calculate the weighted decision matrix*

Weighting each column of obtained matrix by its associated weight —

$$v_{ij} = w_j r_{ij};\ j = 1, 2, \cdots, n;\ i = 1, 2 \cdots, m.$$

Step 5.5. *Determine the positive ideal solution (PIS) and negative ideal solution (NIS)*

$$A^{+} = (v_1^{+}, v_2^{+}, \cdots, v_n^{+}) = \left\{ \left(max_i \left\{ v_{ij} \mid j \in B \right\} \right), \left(min_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\};$$

$$A^{-} = (v_1^{-}, v_2^{-}, \cdots, v_n^{-}) = \left\{ \left(min_i \left\{ v_{ij} \mid j \in B \right\} \right), \left(max_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\}.$$

The benefit and cost solutions are represented by B and C, respectively.

Step 5.6. *Calculate the distance measure for each alternative from the PIS and NIS*

The distance measure for each alternative from the PIS is —

$$S_i^{+} = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^{+})^2 \right\}^{0.5} ;\ i = 1, 2 \cdots, m\,.$$

Also, the distance measure for each alternative from the NIS is —

$$S_i^{-} = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^{-})^2 \right\}^{0.5} ;\ i = 1, 2 \cdots, m\,.$$

Step 5.7. *Determine the values of relative closeness measure*

For each alternative, we calculate the relative closeness measure as it follows:

$$T_i = \frac{S_i^{-}}{(S_i^{+} + S_i^{-})};\ i = 1, 2 \cdots, m\,.$$

Rank alternatives set according to the order of relative closeness measure values T_i.

6 Numerical examples

We examine a numerical example in which a synthetic evaluation desire to rank four alternatives A_1, A_2, A_3 and A_4 with respect to three benefit attribute C_1, C_2 and C_3.

	C_1	C_2	C_3
	w_1	w_2	w_3
A_1	7	9	9
A_2	8	7	8
A_3	9	6	8
A_4	6	7	8

Table 6: Decision matrix

In the examples below, we use α-D MCDM and AHP (if it works) to calculate the weights of the criteria w_1, w_2 and w_3. After we used TOPSIS to rank the four alternatives, the decision matrix (*Table 6*) is used for the three following examples.

6.1 Consistent Example 1

We use the α-D MCDM. Let the Set of Criteria be $\{C_1, C_2, C_3\}$ with $w_1 = w(C_1) = x$ and $w_3 = w(C_3) = z$.

Let us consider the system of equations associated to MCDM problem and its associated matrix.

$$\begin{cases} x = & 4y \\ y = & 3z \\ z = & \dfrac{x}{12} \end{cases} \quad X1 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 3 \\ \dfrac{1}{12} & 0 & 1 \end{pmatrix}.$$

We calculate $\det(X)$ (in this case, equal = 0); then MCDM problem is consistent; we solve the system; we get the following solution —

$$S = \begin{bmatrix} 12z & 3z & z \end{bmatrix}.$$

Setting 1 to secondary variable, the general solution becomes —

$$S = \begin{bmatrix} 12 & 3 & 1 \end{bmatrix},$$

and normalizing the vector (dividing by sum=12+3+1), the weights vector is:

$$W = \begin{bmatrix} \frac{12}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

Using AHP, we get the same result.

The pairwise comparison matrix of criteria is:

$$X1 = \begin{pmatrix} 1 & 4 & 12 \\ \frac{1}{4} & 1 & 3 \\ \frac{1}{12} & \frac{1}{3} & 1 \end{pmatrix},$$

whose maximum eigenvalue is $\lambda_{max} = 3$ and its corresponding normalized eigenvector (Perron-Frobenius vector) is —

$$W = \begin{bmatrix} \frac{12}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

We use TOPSIS to rank the four alternatives.

a_{ij}^2	C_1	C_2	C_3
w_i	12/16	3/16	1/16
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 7: Calculate (a_{ij}^2) for each column

r_{ij}	C_1	C_2	C_3
w_i	12/16	3/16	1/16
A_1	0.4616	0.6138	0.5447
A_2	0.5275	0.4774	0.4842
A_3	0.5934	0.4092	0.4842
A_4	0.3956	0.4774	0.4842
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 8: Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

v_{ij}	C_1	C_2	C_3
w_i	12/16	3/16	1/16
A_1	0.3462	0.1151	0.0340
A_2	0.3956	0.0895	0.0303
A_3	0.4451	0.0767	0.0303
A_4	0.2967	0.0895	0.0303
v_{max}	0.4451	0.1151	0.0340
v_{min}	0.2967	0.0767	0.0303

Table 9: Multiply each column by w_j to get v_{ij}

Alternative	S_i^+	S_i^-	T_i	Rank
A_1	0.0989	0.0627	0.3880	3
A_2	0.0558	0.0997	0.6412	2
A_3	0.0385	0.1484	0.7938	1
A_4	0.1506	0.0128	0.0783	4

Table 10: The separation measure values and the final rankings for decision matrix (*Table 4*) using AHP-TOPSIS and α-D MCDM-TOPSIS

Table 10 presents the rank of alternatives (A_1, A_2, A_3, A_4) and separation measure values of each alternative from the PIS and from NIS in which the weighted values are calculated by AHP or α-D MCDM. Both methods, AHP and α-D MCDM with Fairness Principle, give the same weights as proven above methods together give same result in consistent problem.

6.2 Weak inconsistent Example 2 where AHP does not work

Let us consider another example investigated by [7] for which AHP does not work (i.e. AHP-TOPSIS does not work too); we use the α-D MCDM to calculate the weights values and ranking the four alternatives by TOPSIS (see *Table 14*).

Let the Set of Criteria be $\{C_1, C_2, C_3\}$ with $w_1 = w(C_1) = x$ and $w_3 = w(C_3) = z$. Let us consider the system of equations associated to MCDM problem and its associated matrix.

$$\begin{cases} x = & 2y + 3z \\ y = & \dfrac{x}{2} \\ z = & \dfrac{x}{3} \end{cases}$$

$$X1 = \begin{pmatrix} 1 & -2 & -3 \\ \frac{-1}{2} & 1 & 0 \\ \frac{-1}{3} & 0 & 1 \end{pmatrix}.$$

The solution of this system is $x = y = z = 0$; be the sum of weights = 1, then this solution is not acceptable.

Parameterizing the right-hand side coefficient of each equation by α_i we get:

$$\begin{cases} x = 2\alpha_1 y + 3\alpha_2 z \\ y = \dfrac{\alpha_3 x}{2} \\ z = \dfrac{\alpha_4 x}{3} \end{cases}$$

We solve the system and we get the following solution —

$$S = \begin{cases} y = \frac{\alpha_3 x}{2} \\ z = \frac{\alpha_4 x}{3} \end{cases} \text{ or } S = \begin{bmatrix} x & \dfrac{\alpha_3 x}{2} & \dfrac{\alpha_4 x}{3} \end{bmatrix}.$$

Setting 1 to secondary variable, the general solution becomes —

$$S = \begin{bmatrix} 1 & \dfrac{\alpha_3}{2} & \dfrac{\alpha_4}{3} \end{bmatrix}.$$

Applying Fairness Principle, then replacing $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha$, whence $\alpha = \dfrac{\sqrt{2}}{2}$.

$$S = \begin{bmatrix} 1 & \dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{2}}{6} \end{bmatrix}.$$

Normalizing vector (dividing by sum), the weights vector is:

$$W = [0.62923 \quad 0.22246 \quad 0.14831].$$

TOPSIS is used to rank the four alternative: application of TOPSIS method is in the same manner as in the previous example (the four alternatives (Ai) are ranked in the following *Table 14*).

a_{ij}^2	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 11: Calculate (a_{ij}^2) for each column

r_{ij}	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	0.4616	0.6138	0.5447
A_2	0.5275	0.4774	0.4842
A_3	0.5934	0.4092	0.4842
A_4	0.3956	0.4774	0.4842
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 12: Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

v_{ij}	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	0.2904	0.1365	0.0808
A_2	0.3319	0.1062	0.0718
A_3	0.3734	0.0910	0.0718
A_4	0.2489	0.1062	0.0718
v_{max}	0.3734	0.1365	0.0808
v_{min}	0.2489	0.0910	0.0718

Table 13: Multiply each column by w_j to get v_{ij}

Alternative	S_i^+	S_i^-	T_i	Rank
A_1	0.0830	0.0622	0.4286	3
A_2	0.0522	0.0844	0.6178	2
A_3	0.0464	0.1245	0.7285	1
A_4	0.1284	0.0152	0.1057	4

Table 14: The separation measure values and the final rankings for decision matrix (*Table 4*) using α -D MCDM-TOPSIS

6.3 Jean Dezert's strong inconsistent Example 3

Smarandache [7] introduced a Jean Dezert's Strong Inconsistent example. Let us consider the system of equations associated to MCDM problem and its associated matrix.

$$X = \begin{pmatrix} 1 & 9 & \frac{1}{9} \\ \frac{1}{9} & 1 & 9 \\ 9 & \frac{1}{9} & 1 \end{pmatrix} \qquad \begin{cases} x = & 9y, x > y \\ x = & \frac{1}{9}z, x < z \\ y = & 9z, y > z \end{cases}$$

We follow the same process as in the example above to get the general solution:

$$W = \begin{bmatrix} \frac{1}{6643} & \frac{81}{6643} & \frac{6561}{6643} \end{bmatrix}.$$

We use TOPSIS to rank the four alternatives.

a_{ij}^2	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 15: Calculate (a_{ij}^2) for each column

r_{ij}	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	0.503	0.699	0.623
A_2	0.574	0.543	0.553
A_3	0.646	0.466	0.553
A_4	0.431	0.543	0.553
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 16: Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

v_{ij}	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	0.0001	0.0075	0.5380
A_2	0.0001	0.0058	0.4782
A_3	0.0001	0.0050	0.4782
A_4	0.0001	0.0058	0.4782
v_{max}	0.0001	0.0075	0.5380
v_{min}	0.0001	0.0050	0.4782

Table 17: Multiply each column by w_j to get v_{ij}

Alternative	S_i^+	S_i^-	T_i	Rank
A_1	0.0000	0.0598	0.999668	1
A_1	0.0598	0.0008	0.013719	2
A_1	0.0598	0.0000	0.000497	4
A_1	0.0598	0.0008	0.013715	3

Table 18: The separation measure values and the final rankings
for decision matrix (*Table 4*) using α-D MCDM-TOPSIS

7 AHP method

As we proved in *Section 3. Description of data structure decision problems under consideration*, AHP method can be used in the second case, in which data is structured as pairwise comparisons of matrices of preferences.

Let us assume that X is the comparison matrix of criteria; for each criterion C_k $(k = 1, 2, \cdots, n)$ we have a comparison matrix of preferences B_k and the consistency condition is perfect.

We use AHP to determine the importance weight (w_i) of the criteria.

We apply again AHP method for each comparison matrices of preferences B_k to determine the maximum eigenvalue and its associate eigenvector (same that is used to determine the weights of criteria).

For each matrix B_k (associated to C_k), we calculate the λ_{max} and priority vector (eigenvector).

C_i	A_1	A_2	$\cdots$	A_n	Priority vector
A_1	a_{11}	a_{12}	$\cdots$	a_{1m}	p_{1i}
A_2	a_{21}	a_{22}	$\cdots$	a_{2m}	p_{2i}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mm}	p_{mi}

Table 19: Comparison matrix of the preferences with priority vector in latest column

We get a decision matrix (different from the decision matrix above), formed using priority vectors, in which the entries of the decision matrix are p_{ij}, and not a_{ij}.

	C_1	C_2	$\cdots$	C_n
w_i	w_1	w_2	$\cdots$	w_n
A_1	p_{11}	p_{12}	$\cdots$	p_{1n}
A_2	p_{21}	p_{22}	$\cdots$	p_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}

Table 20: Decision matrix of priority of preferences

The last step of AHP method is to rank the preferences using the following formula —

$$\sum_{j=1}^{n} w_j p_{ij}.$$

	C_1	C_2	$\cdots$	C_n	R_i
w_i	w_1	w_2	$\cdots$	w_n	
A_1	p_{11}	p_{12}	$\cdots$	p_{1n}	$\sum_{j=1}^{n} w_j p_{1j}$
A_2	p_{21}	p_{22}	$\cdots$	p_{2n}	$\sum_{j=1}^{n} w_j p_{2j}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}	$\sum_{j=1}^{n} w_j p_{mj}$

Table 21: Ranking decision matrix

7.1 Numerical examples

Let us consider the three numerical examples (*Section 6*) and the decision matrix mentioned above.

For weak inconsistent and strong inconsistent examples, AHP does not work, as proved above, consequently the AHP method will be applied on consistent example 1.

The weights of criteria are calculated by using AHP (consistent example 1, *Section 6*).

As mentioned above (case 2, *Section 3*), for applying AHP we need to construct three pairwise comparisons matrices of size $4*4$ each.

C_1	A_1	A_2	A_3	A_4
A_1	1	7/8	7/9	7/6
A_2	8/7	1	8/9	8/6
A_3	9/7	7/8	1	9/6
A_4	6/7	7/8	6/9	1

C_2	A_1	A_2	A_3	A_4
A_1	1	9/7	9/6	9/7
A_2	7/9	1	7/6	1
A_3	6/9	6/7	1	6/7
A_4	7/9	1	7/6	1

C_3	A_1	A_2	A_3	A_4
A_1	1	9/8	9/8	9/8
A_2	8/9	1	1	1
A_3	8/9	1	1	1
A_4	8/9	1	1	1

Table 22: Relative importance comparison matrix of alternatives for each criteria

We apply AHP method on three pairwise comparison matrices to rank the preferences, and calculate priority (i.e., normalized eigenvector), consistency index (CI) and consistency ratio (CR) for each matrix.

The eigenvalue, consistency index (CI) and consistency ratio (CR) for each matrix are: C1 — (CR =0, CI=0, λ = 4), C2 — (CR =0, CI=0, λ = 4) and C3 — (CR =0, CI=0, λ = 4).

The priority vectors for three matrices are listed respectively in *Table 23*.

C_1	A_1	A_2	A_3	A_4	Priority vector
A_1	1	7/8	7/9	7/6	0.2333
A_2	8/7	1	8/9	8/6	0.2667
A_3	9/7	7/8	1	9/6	0.3000
A_4	6/7	7/8	6/9	1	0.2000
C_2	A_1	A_2	A_3	A_4	Priority vector
A_1	1	9/7	9/6	9/7	0.3103

A_2	7/9	1	7/6	1	0.2414
A_3	6/9	6/7	1	6/7	0.2069
A_4	7/9	1	7/6	1	0.2414

C_3	A_1	A_2	A_3	A_4	Priority vector
A_1	1	9/8	9/8	9/8	0.2727
A_2	8/9	1	1	1	0.2424
A_3	8/9	1	1	1	0.2424
A_4	8/9	1	1	1	0.2424

Table 23: Relative importance comparison matrix of alternatives for each criteria

The last step of AHP method is applying it to rank the preferences using the following formula $\sum_{j=1}^{n} w_j p_{ij}$ (resulted as listed in the last column of the matrix above).

	C_1	C_2	C_3	$\sum_{j=1}^{n} w_j p_{ij}$	Rank
w_i	12/16	3/16	1/16		
A_1	0.2333	0.3103	0.2727	0.2502	3
A_2	0.2667	0.2414	0.2424	0.2604	2
A_3	0.3000	0.2069	0.2424	0.2789	1
A_4	0.2000	0.2414	0.2424	0.2104	4

Table 24: Decision matrix of priority of preferences and its ranking

8 Extended α-D MCDM

α-D MCDM introduced by Smarandache is not designed to rank preferences, but only for generating the weights for preferences or criteria, based on their n-wise matrix comparison. Hence, we proposed above a new method, called α-D MCDM-TOPSIS, employed to calculate the criteria weights for pairwise comparison matrices and for n-wise comparison matrices of criteria, in which α-D MCDM is used for calculate criteria weights, and TOPSIS — to rank the preferences.

In this section, we do not focus on criteria weights problem of α-D MCDM, as discussed above and calculated for the three examples, but we propose an extension of α-D MCDM benefiting the skills of α-D MCDM to calculate maximum eigenvalue and its associate eigenvector of n-wise comparison matrix, in order to apply it again on α-wise matrices of preferences.

An extended α-D MCDM can be described as it follows:

Let us consider the second case of data structure decision problem (*Section 3*) — for each criterion C_i corresponds a pairwise (or n-wise) comparison matrix of preferences and criteria.

Step 8.1 *We use α-D MCDM to calculate the weight (w_i) of the criteria*

Step 8.2 *We apply again α-D MCDM method* for each comparison matrices of preferences B_k to determine the maximum eigenvalue and its associate eigenvector (the same that is used to determine the weights of criteria).

For each matrix B_k (associated to C_k), we calculate the λ_{max} and priority vector (eigenvector).

We get —

C_i	A_1	A_2	…	A_n	Priority vector
A_1	a_{11}	a_{12}	…	a_{1m}	p_{1i}
A_2	a_{21}	a_{22}	…	a_{2m}	p_{2i}
⋮	⋮	⋮	⋮	⋮	⋮
A_m	a_{m1}	a_{m2}	…	a_{mm}	p_{mi}

Table 25: Comparison matrix of the preferences with priority vector in latest column

Step 8.3 *We get a decision matrix* (different from the decision matrix above), formed using priority vectors, in which the entries of the decision matrix are p_{ij}, and not a_{ij}.

	C_1	C_2	…	C_n
w_i	W_1	W_2	…	W_n
A_1	p_{11}	p_{12}	…	p_{1n}
A_2	p_{21}	p_{22}	…	p_{2n}
⋮	⋮	⋮	⋮	⋮
A_m	p_{m1}	p_{m2}	…	p_{mn}

Table 26: Decision matrix of priority of preferences

Step 8.4 *We employ the following formula (simple additive weighting) to rank the preferences* —

$$\sum_{j=1}^{n} w_j p_{ij}.$$

	C_1	C_2	$\cdots$	C_n	R_i
w_i	w_1	w_2	$\cdots$	w_n	
A_1	p_{11}	p_{12}	$\cdots$	p_{1n}	$\sum_{j=1}^{n} w_j p_{1j}$
A_2	p_{21}	p_{22}	$\cdots$	p_{2n}	$\sum_{j=1}^{n} w_j p_{2j}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}	$\sum_{j=1}^{n} w_j p_{mj}$

Table 27: Ranking decision matrix

8.1 Numerical examples

Let us consider the three examples mentioned above (numerical consistent, weak inconsistent and strong inconsistent examples (*Section 6*).

We do not repeat the calculation of weights criteria by using α-D MCDM, because it was already done in *Section 6*, and we got the following priority vectors of criteria:

Consistent example 1 —

$$W = \begin{bmatrix} \frac{12}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

Weak inconsistent example 2 —

$$W = \begin{bmatrix} 0.62923 & 0.22246 & 0.14831 \end{bmatrix}.$$

Strong inconsistent example 3 —

$$W = \begin{bmatrix} \frac{1}{6643} & \frac{81}{6643} & \frac{6561}{6643} \end{bmatrix}.$$

We construct three comparisons matrices of size $4*4$ each (or three linear homogeneous systems), based on decision matrix of *Section 6*, and apply extended α-D MCDM to the three examples.

Let

$$m(A_1) = x \text{ , } m(A_2) = y \text{ , } m(A_3) = z \text{ and } m(A_4) = t.$$

For criteria C_1 —

- A_2 is 8 seventh as important as A_1 ,

- A_3 is 9 seventh as important as A_1 ,
- A_4 is 6 seventh as important as A_1 .

The linear homogeneous system associated is —

$$\begin{cases} y \quad \frac{8}{7}x \\ z \quad \frac{9}{7}x \\ t \quad \frac{6}{7}x \end{cases}$$

and its general solution is —

$$W = \begin{bmatrix} \frac{7}{30} & \frac{8}{30} & \frac{9}{30} & \frac{6}{30} \end{bmatrix}.$$

For criteria C_2 —

- A_1 is 9 sixth as important as A_3 ,
- A_3 is 7 sixth as important as A_3 ,
- A_4 is 7 sixth as important as A_3 .

The linear homogeneous system associated is —

$$\begin{cases} x \quad \frac{9}{6}z \\ y \quad \frac{7}{6}z \\ t \quad \frac{7}{6}z \end{cases}$$

and its general solution is —

$$W = \begin{bmatrix} \frac{9}{29} & \frac{7}{29} & \frac{6}{29} & \frac{7}{29} \end{bmatrix}.$$

For criteria C_3 —

- A_1 is 9 eighth as important as A_2 ,
- A_2 , A_3 and A_4 have the same importance.

The associated linear homogeneous system is —

$$\begin{cases} x \quad \frac{9}{8}y \\ y \quad z \\ t \quad z \\ \end{cases}$$

and its general solution is —

$$W = \begin{bmatrix} \frac{9}{33} & \frac{8}{33} & \frac{8}{33} & \frac{8}{33} \end{bmatrix}.$$

The results of extended α-D MCDM are summarized in the *Table 24*:

Consistent example 1 —

	C_1	C_2	C_3	R_i	*Rank*
w_i	*12/16*	*3/16*	*1/16*		
A_1	*7/30*	*9/29*	*9/33*	*0.2502*	*3*
A_2	*8/30*	*7/29*	*8/33*	*0.2604*	*2*
A_3	*9/30*	*6/29*	*8/33*	*0.2789*	*1*
A_4	*6/30*	*7/29*	*8/33*	*0.2104*	*4*

Weak inconsistent example 2 —

	C_1	C_2	C_3	R_i	*Rank*
w_i	*0.62923*	*0.22246*	*0.14831*		
A_1	*7/30*	*9/29*	*9/33*	*0.2563*	*3*
A_2	*8/30*	*7/29*	*8/33*	*0.2574*	*2*
A_3	*9/30*	*6/29*	*8/33*	*0.2707*	*1*
A_4	*6/30*	*7/29*	*8/33*	*0.2155*	*4*

Strong inconsistent example 3 —

	C_1	C_2	C_3	R_i	*Rank*
w_i	*1/6643*	*81/6643*	*6561/6643*		
A_1	*7/30*	*9/29*	*9/33*	*0.27318*	*1*
A_2	*8/30*	*7/29*	*8/33*	*0.24242*	*2*
A_3	*9/30*	*6/29*	*8/33*	*0.24200*	*4*
A_4	*6/30*	*7/29*	*8/33*	*0.24241*	*3*

Table 24: Decision matrix of priority of preferences and its ranking using Extended -DMCDM

Example	Alternative	AHP-TOPSIS		α-DMCDM-TOPSIS		AHP		Extended α-DMCDM	
Consistent example 1	A_1	*0.3880*	*3*	*0.3880*	*3*	*0.2502*	*3*	*0.2502*	*3*
	A_2	*0.6412*	*2*	*0.6412*	*2*	*0.2604*	*2*	*0.2604*	*2*
	A_3	*0.7938*	*1*	*0.7938*	*1*	*0.2789*	*1*	*0.2789*	*1*
	A_4	*0.0783*	*4*	*0.0783*	*4*	*0.2104*	*4*	*0.2104*	*4*
Weak Inconsistent Example 2	A_1	*Does not works*		*0.3880*	*3*	*Does not works*		*0.2563*	*3*
	A_2			*0.6412*	*2*			*0.2574*	*2*
	A_3			*0.7938*	*1*			*0.2707*	*1*
	A_4			*0.0783*	*4*			*0.2155*	*4*
Strong Inconsistent Example 3	A_1	*Does not works*		*0.999668*	*1*	*Does not works*		*0.27318*	*1*
	A_2			*0.013719*	*2*			*0.24242*	*2*
	A_3			*0.000497*	*4*			*0.24200*	*4*
	A_4			*0.013715*	*3*			*0.24241*	*3*

Table 25: Summary of the results of three examples of all methods

For the three examples presented in this paper, the *Table 25*, summarizing all results of all methods, illustrates that the AHP and AHP-TOPSIS methods work just for the first example, in which criteria and alternatives are consistent in their pairwise comparisons. Our proposed methods — Extended α-D MCDM and α-D MCDM-TOPSIS — work not only for consistent example 1, giving the same results as AHP and AHP-TOPSIS methods, but also for weak inconsistent and strong inconsistent examples.

In the example 1, it is recorded that the alternative A3 has the first rank with the value (0.2789), the alternative A2 gets second rank with the coefficient value (0.2604), the alternative A1 realizes following rank with value (0.2502) and the alternative A4 lowers rank with the coefficient (0.2104), by using the AHP and our proposed method Extended α-DMCDM. Results indicate that all considered alternatives have near score values, for example 0.0685 ((A3)0.2789 - (A4) 0.2104). As a difference between the first and the latest ranking alternative, it is not sufficient to make a founded decision making, hence that can have a strong impact in practice to choose the best alternatives.

The results claimed that AHP-TOPSIS and our α-D MCDM-TOPSIS methods preserves the ranking order of the alternatives and overcome the near score values problem. By using AHP-TOPSIS and our α-D MCDM-TOPSIS methods, the score value of A3 was changed from 0.2789 to 0.7938, the score value of A2 was changed from 0.2604 to 0.6412, and the score value of A4 was changed from 0.2104 to 0.0783.

The bigger differences between the score values of alternatives 0.7155 ((A3) 0.7938 - (A4) 0.0783) is also subject to gain additional insights.

In the two last examples (weak inconsistent and strong inconsistent), one sees that the importance of discounting in our approaches suggest that they can be used to solve real-life problems in which criteria are not only pairwise, but n-wise comparisons, and the problems are not perfectly consistent. It is however worth to note that the ranking order of the four alternatives obtained by both methods is similar, but score values are slightly different. Both Extended α-D MCDM and α-D MCDM-TOPSIS methods allow taking into consideration any numbers of alternatives and any weights of criteria.

9 Conclusions

We have proposed two multi-criteria decision making methods, Extended α-D MCDM and α-D MCDM-TOPSIS models that allow to work for consistent and inconsistent MCDM problems. In addition, three examples have demonstrated that the α-D MCDM-TOPSIS model is efficient and robust.

Our approaches, Extended α-D MCDM and α-D MCDM-TOPSIS, give the same result as AHP-TOPSIS and AHP in consistent MCDM problems and elements of decision matrix are pairwise comparisons, but for weak inconsistent and strong inconsistent MCDM problems in which AHP and AHP-TOPSIS are limited and unable, our proposed methods — Extended α-D MCDM and α-D MCDM-TOPSIS — give justifiable results.

Furthermore, our proposed approaches — α-D MCDM-TOPSIS and Extended α-D MCDM — can be used to solve real-life problems in which criteria are not only pairwise, but n-wise comparisons, and the problems are not perfectly consistent.

References

[1] Alonso, J.A., Lamata, M. *Consistency in the analytic hierarchy process: a new approach*. In "International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems" 14 (4), 445–459, 2006.

[2] Ghosh, D.N. *Analytic hierarchy process and TOPSIS method to evaluate faculty performance in engineering education*. Dipendra Nath Ghosh et al., UNIASCIT 1 (2), 63–70, 2011.

[3] Jayanta, A., Guptaa, P., Gargb, S.K., Khana, M. *TOPSIS-AHP based approach for selection of reverse logistics service provider: A case study of mobile phone industry*. Procedia Engineering 97, Elsevier 97, 2147–2156, 2014. DOI: 10.1016/j.proeng.2014.12.458.

[4] Maheshwarkar, M., Sohani, N. *Combined AHP-TOPSIS based approach for the evaluation of knowledge sharing capabilities of supply chain partners*. Management Science and Engineering 7(3), 27–32, 2013. DOI: 10.3968/j.mse.1913035X20130703.2573.

[5] Pelaez, J.I., Lamata, M.T. *New measure of consistency for positive reciprocal matrices*. In "International Journal Computers and Mathematics with Applications", Elsevier 46, 1839–1845, 2003. DOI: lO.l016/SO898-1221(03) 00407-3.

[6] Saaty, T. *Fundamentals of decision making and priority theory with the AHP*. RWS Publications, Pittsburgh, PA, U.S.A., 1994.

[7] Smarandache, F.: *α-discounting method for multi-criteria decision making (α-D MCDM).* Fusion 2010 International Conference, Scotland, 2010; http://fs.gallup.unm.edu/alpha-DiscountingMCDM-book.pdf

[8] Tavana, M., Hatami-Marbini, A. *A group AHP-TOPSIS framework for human spaceflight mission planning at NASA*. In "Expert Systems with Applications" 38, 13, 588–13,603, 2011. DOI: 10.1016/j.eswa.2011.04. 108.

[9] Triantaphyllou, E., Mann, S.H. *Using the analytic hierarchy process for decision making in engineering application: dome challenges*. In "International Journal of Industrial Engineering: Applications and Practice" 2 (1), 35–44, 1995.

[10] Tyagia, M., Kumarb, P., Kumarc, D. *A hybrid approach using AHP-TOPSIS for analyzing e-SCM performance*. 12th Global Congress On Manufac-turing and Management Engineering, Elsevier 97, 2195–2203, 2014.

[11] Wu, M. *TOPSIS-AHP simulation model and its application to supply chain management*. In "World Journal of Modelling and Simulation", 3 (3), 196–201, 2007.

[12] Yu, J., Wang, L., Gong, X. *Study on the status evaluation of urban road intersections traffic congestion based on AHP-TOPSIS model*. 13th COTA International Conference of Transportation Professionals, CICTP 2013 96, 609–616, 2013.

[13] *Weak consistency: A new approach to consistency in the Saaty's analytic hierarchy process*. In "Mathematica", 52 (2), 71–83.

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