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A new non-specificity measure in evidence theory based on belief intervals

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Abstract In the theory of belief functions, the measure of uncertainty is an important concept, which is used for representing some types of uncertainty incorporated in bodies of evidence such as the discord and the non-specificity. For the non-specificity part, some traditional measures use for reference the Hartley measure in classical set theory; other traditional measures use the simple and heuristic function for joint use of mass assignments and the cardinality of focal elements. In this paper, a new non-specificity measure is proposed using lengths of belief intervals, which represent the degree of imprecision. Therefore, it has more intuitive physical meaning. It can be proved that our new measure can be rewritten in a general form for the non-specificity. Our new measure is also proved to be a strict non-specificity measure with some desired properties. Numerical examples, simulations, the related analyses and proofs are provided to show the characteristics and good properties of the new non-specificity definition. An example of an application of the new non-specificity measure is also presented.

1. Introduction

The theory of belief functions[1] is an important tool for uncertainty modeling and reasoning. It can distinguish the 'unknown' and the 'imprecision' and provides a method for fusing different evidences by using the commutative and associative Dempster's rule of combination. The theory of belief functions has been widely used in the fields of information fusion,[2] pattern classification,[3–5] and multiple attribute decision making,[6,7] etc. Some modified or extended frameworks including the transferable belief model (TBM)[8] and Dezert-Smarandache theory (DSmT)[9] were also proposed by researchers in the past decades.

The measure of uncertainty[10–12] is very crucial in all kinds of theories of uncertainty. The concept of uncertainty is intricately connected to the concept of information. Therefore, to describe the uncertainty, measures in information theory are often used for reference. E.g., in probability theory, the

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Shannon entropy[13] is developed. In fuzzy set theory[14] and its related applications,[15,16] some entropy-alike measures also are proposed to represent the uncertainty.[17] Also, in the theory of belief functions, many entropy-alike measures are proposed such as the ambiguity measure (AM),[10] the aggregated uncertainty(AU) measure[18] to measure the total uncertainty in a basic belief assignment (BBA). Actually, in a BBA, there are two types of uncertainty.[10] One is the discord (or randomness or conflict). Another is the non-specificity. They can be unified under the term ambiguity.

For the discord part, many Shannon entropy-alike measures were introduced by researchers.[19] Non-specificity[10,18,20–22] means two or more alternatives are left unspecified, which represents a degree of imprecision. It only focuses on those focal elements with cardinality larger than 1. Non-specificity is a distinctive uncertainty type in the theory of belief functions when compared with the probability theory. Therefore, in this paper, we focus on the non-specificity part. There are also some non-specificity measures proposed.[21,23,24] The most typical one[23] is a generalization of the Hartley measure,[25] which is originally for the classical set theory. In probability theory, there is only discord (or randomness or conflict).[10] Other available non-specificity measures[21,24] use the simple and heuristic function for joint use of mass assignments and the cardinality of focal elements.

In this paper, we aim to design a new non-specificity measure without using the measure of classical set theory or using heuristic joint use of mass assignments and the cardinality of the focal elements, but to design using intuitive physical explanations of the uncertainty in the theory of belief functions. As aforementioned, the non-specificity actually represents a kind of imprecision. In the theory of belief functions, the precision is often modeled by lengths of the belief intervals. The mean of the belief intervals' lengths for all singletons is defined as the non-specificity. Therefore, our new definition can be considered as an averaging imprecision of different singletons. Furthermore, the new measure can be rewritten to a general form of non-specificity measure and it has several desired properties for uncertainty measures.

The rest of this paper is organized as follows. In Section 2, the essentials of the theory of belief functions are introduced. Some available uncertainty measures, especially the non-specificity measures in the theory of belief functions are briefly introduced in Section 3. In Section 4, a novel non-specificity measure is proposed. Some desired properties are provided together with related proofs. In Section 5, we use some numerical examples and simulations to show the rationality of the proposed new non-specificity measure, where the comparisons between the available measures and the new one are provided. Also, an example of the application of the new non-specificity measure is given in Section 6. Section 7 concludes this paper.

2. Basics of the theory of belief functions

In the theory of belief functions, also called Dempster–Shafer evidence theory (DST)[1], the basic concept is the frame of discernment (FOD), which is a discrete and finite set. The elements in FOD are mutually exclusive and exhaustive. Given an FOD Θ, on its power set 2^{Θ}, a BBA $m:2^{\Theta} \rightarrow [0, 1]$ can be defined satisfying

$$\sum_{A \subseteq \Theta} m(A) = 1, \quad \text{and } m(\emptyset) = 0 \tag{1}$$

A BBA is also called a mass function. All the A with $m(A) > 0$ are called focal elements of a BBA $m(\cdot)$. The set of all the focal elements denoted by F and their corresponding mass assignments m constitute a body of evidence (BOE): (F, m). Based on the definition of BBA in Eq. (1), the belief function (Bel) and the plausibility function (Pl) are defined for any $A \subseteq \Theta$ as follows[1]:

$$\begin{cases} \mathrm{Bel}(A) = \sum\limits_{B \subseteq \Theta, B \subseteq A} m(B) \\ \mathrm{Pl}(A) = \sum\limits_{B \subseteq \Theta, B \cap A \neq \emptyset} m(B) \end{cases} \tag{2}$$

The belief function and plausibility function can be interpreted as a lower and a upper bound of the probability $P(A)$, respectively, i.e., $P(A) \in [\mathrm{Bel}(A), \mathrm{Pl}(A)]$, which is a belief interval of the focal element A. The length of the belief interval $\mathrm{Len}(A) = \mathrm{Pl}(A) - \mathrm{Bel}(A)$ represents the degree of imprecision for A.[1]

The mass assignment for the total set Θ, i.e., $m(\Theta)$ represents the degree of ignorance (or unknown) for a given BBA m. Therefore, the theory of belief functions can discriminate the "imprecision" and the "ignorance".

In the theory of belief functions, independent BBAs (m_1, m_2) are combined using Dempster's rule of combination as follows[1]:

$$m(A) = \begin{cases} 0 & A = \emptyset \\ \dfrac{\sum\limits_{A_i \cap B_j = A} m_1(A_i) m_2(B_j)}{1 - K} & A \neq \emptyset \end{cases} \tag{3}$$

where A_i and B_j denote the focal element of m_1 and m_2, respectively. K denotes the conflict coefficient between m_1 and m_2. Note that the BBAs to be combined using Dempster's rule should be independent. The research related to the dependent BBAs can found in Ref.[26] Dempster's rule of combination is both associative and commutative. There exist many other alternative combination rules, see details in Refs.[9,27]

DST has been argued for its drawbacks in past decades.[28–31] Some modified or improved frameworks were also proposed including the TBM[8] and DSmT.[9]

3. Uncertainty measures in the theory of belief functions

There are various kinds of uncertainty,[10] e.g., the fuzziness, randomness (or discord or conflict), non-specificity, which can be represented and processed by different types of uncertainty theories.[10,11] In the theory of belief functions, a BBA has two types of uncertainty, i.e., the discord and the non-specificity, hence ambiguity.[10] Many uncertainty measures were proposed for the discord, the non-specificity, and the total uncertainty (including both two parts).

3.1. Measures for discord in the theory of belief functions

Measures for discord is to depict the randomness (or discord or conflict) in a BOE. Available measures for discord in the theory of belief functions are listed below. Although with different names, they are all for the discord part of the uncertainty in the theory of belief functions.

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(1) Confusion measure (1982)
The confusion measure is proposed by Hohle[32] as

$$\mathrm{Conf}(m) = -\sum_{A\subseteq\Theta} m(A)\log_2(\mathrm{Bel}(A)) \tag{4}$$

(2) Dissonance measure (1983)
The dissonance measure is proposed by Yager[21] as

$$\mathrm{Disso}(m) = -\sum_{A\subseteq\Theta} m(A)log_2(\mathrm{Pl}(A)) \tag{5}$$

(3) Discord measure (1990)
The discord measure Disc is proposed by Klir and Ramer[33] as

$$\mathrm{Disc}(m) = -\sum_{A\subseteq\Theta} m(A)log_2\left[1-\sum_{B\subseteq\Theta} m(B)\frac{|B-A|}{|B|}\right] \tag{6}$$

(4) Strife measure (1992)
The strife measure Strif is proposed by Klir and Parviz[34]

$$\mathrm{Strif}(m) = -\sum_{A\subseteq\Theta} m(A)log_2\left[1-\sum_{B\subseteq\Theta} m(B)\frac{|A-B|}{|A|}\right] \tag{7}$$

As we can see, they are all Shannon entropy-alike measure. The differences and relationships between the measures above can be found in Refs.[19,34]

3.2. Measures for non-specificity in the theory of belief functions

Non-specificity[20,21,23,35] means two or more alternatives are left unspecified. It represents a degree of imprecision and only focus on those focal elements with cardinality larger than one. Non-specificity is a distinctive uncertainty type in the belief functions framework when compared with the probabilistic framework. So, the non-specificity is mainly concerned here. The available non-specificity measures are as follows.

(1) Dubois & Prade's non-specificity[23]

$$\mathrm{NS}^{\mathrm{DP}}(m) = \sum_{A\subseteq\Theta} m(A)\log_2|A| \tag{8}$$

It is a generalized Hartley measure[25] from the classical set theory to the belief functions framework. When the BBA $m(\cdot)$ is a Bayesian BBA, i.e., it only has singleton focal elements, it reaches the minimum value 0. When BBA $m(\cdot)$ is a vacuous BBA, i.e., $m(\Theta)=1$, it reaches the maximum value $\log_2(|\Theta|)$. In fact, due to $\log_2 1=0$, the mass assignments of singletons are nuisances in the computation of $\mathrm{NS}^{\mathrm{DP}}$. This definition was proved to have the uniqueness by Ramer[22], that is, it satisfies all the expected requirements of the non-specificity measure.[20,22]

(2) Yager's specificity[21]

$$S^{\mathrm{Y}}(m) = \sum_{A\subseteq\Theta}\frac{m(A)}{|A|} \tag{9}$$

The maximum value is 1 (when the BBA is Bayesian); the minimum value is $1/|\Theta|$ (when the BBA is vacuous). One can use $1-S^{\mathrm{Y}}(m)$ to denote the non-specificity. Actually, here the mass assignments of singletons are involved in the computation.

(3) Korner's non-specificity[24]

$$\mathrm{NS}^{\mathrm{K}}(m) = \sum_{A\subseteq\Theta} m(A)\cdot|A| \tag{10}$$

The maximum value is $|\Theta|$ (vacuous BBA); the minimum value is 1 (Bayesian BBA). Actually, here the mass assignments of singletons are involved in the computation.

(4) In Korner's work, a general form of the non-specificity (or specificity) measure is proposed as

$$\mathrm{NS}^{f}(m) = \sum_{A\subseteq\Theta} m(A)\cdot f(|A|) \tag{11}$$

As referred in Ref.[24], if a measure satisfies Eq. (11), it is a non-specificity measure (function).

3.3. Measures for total uncertainty in the theory of belief functions

(1) AU[18]

$$\begin{cases}\mathrm{AU}(m) = \max(-\sum_{\theta\in\Theta}p_\theta\log_2 p_\theta)\ \text{ s.t.}\\ p_\theta\in[0,1],\quad \forall\theta\in\Theta\\ \sum_{\theta\in\Theta}p_\theta = 1\\ \mathrm{Bel}(A)\leqslant\sum_{\theta\in A}p_\theta\leqslant 1-\mathrm{Bel}(\bar{A}),\quad \forall A\subseteq\Theta\end{cases} \tag{12}$$

It is also called the "upper entropy". AU is an aggregated total uncertainty (ATU) measure, which can capture both non-specificity and discord. AU satisfies all the requirements for uncertainty measure including probability consistency, set consistency, value range, subadditivity and additivity for the joint BBA in Cartesian space.[35]

(2) AM

$$\mathrm{AM}(m) = -\sum_{\theta\in\Theta}\mathrm{BetP}_m(\theta)\log_2(\mathrm{BetP}_m(\theta)) \tag{13}$$

where $\mathrm{BetP}_m(\theta)=\sum_{\theta\in B\subseteq\Theta}m(B)/|B|,\ \forall A\subseteq\Theta$ is the pignistic probability[8] of a BBA. AM does not satisfy the sub-additivity which has been pointed out by Klir and Lewis.[36] Moreover in the work of Abellan and Masegosa[35], AM has been proved to be logically non-monotonic under some circumstances.

Note that non-specificity can also be defined in the framework of fuzzy sets[37] or intuitionistic fuzzy sets.[38] Here what we are concerned is the non-specificity in the theory of belief functions.

4. Novel non-specificity measure based on the length of the belief intervals

As we can see in the previous section, traditional non-specificity measures are either the generalization of the Hartley measure in classical set theory, or the one heuristically built from the joint use of the cardinality and the mass assignment of the BBA. We do not prefer such expedient ways and aim to design a new non-specificity measure directly using the intuitive concept of uncertainty in the framework of belief functions. The new measure is introduced below.

4.1. Definition of new non-specificity measure in the theory of belief functions

Non-specificity is in fact a kind of imprecision for different propositions in FOD. In the framework of belief functions, the degree of imprecision for each proposition A is represented by the length of the corresponding focal element's belief interval [Bel(A), Pl(A)]. Given an FOD $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, the belief interval for each singleton $\{\theta_i\}$, i.e., [Bel($\{\theta_i\}$), Pl($\{\theta_i\}$)] can be obtained together with n belief intervals' lengths Len (i) = Pl($\{\theta_i\}$) − Bel($\{\theta_i\}$). We define the mean of all the n belief intervals' lengths as the new non-specificity as follows.

$$\mathrm{NS}^{\mathrm{BI}}(m) = \frac{1}{n}\sum_{i=1}^{n}(\mathrm{Pl}(\{\theta_i\}) - \mathrm{Bel}(\{\theta_i\})) \tag{14}$$

Here, BI denotes the belief interval. $\mathrm{NS}^{\mathrm{BI}}(m)$ represents the averaging imprecision in m, i.e., the non-specificity. To avoid the redundant use of the imprecision for each singleton, here we only use the belief intervals of singletons.

Since

$$\begin{cases} \mathrm{Bel}(\theta_i) = m(\{\theta_i\}) \\ \mathrm{Pl}(\theta_i) = m(\{\theta_i\}) + \sum\limits_{\substack{j=1 \\ i<j}}^{n} m(\{\theta_i, \theta_j\}) \\ \qquad + \sum\limits_{\substack{j=1 \\ i<j}}^{n}\sum\limits_{\substack{k=1 \\ j<k}}^{n} m(\{\theta_i, \theta_j, \theta_k\}) + \cdots + m(\Theta) \end{cases} \tag{15}$$

then, $\forall i = 1, 2, \ldots, n$,

$$\mathrm{Pl}(\{\theta_i\}) - \mathrm{Bel}(\{\theta_i\}) = \sum_{\substack{j=1 \\ i<j}}^{n} m(\{\theta_i, \theta_j\}) + \sum_{\substack{j=1 \\ i<j}}^{n}\sum_{\substack{k=1 \\ j<k}}^{n} m(\{\theta_i, \theta_j, \theta_k\}) + \cdots + m(\Theta) \tag{16}$$

Therefore, the non-specificity definition in Eq. (14) can be rewritten as

$$\begin{aligned} \mathrm{NS}^{\mathrm{BI}}(m) &= \frac{1}{n}\sum_{i=1}^{n}(\mathrm{Pl}(\{\theta_i\}) - \mathrm{Bel}(\{\theta_i\})) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left[\sum_{\substack{j=1 \\ i<j}}^{n} m(\{\theta_i, \theta_j\}) + \sum_{\substack{j=1 \\ i<j}}^{n}\sum_{\substack{k=1 \\ j<k}}^{n} m(\{\theta_i, \theta_j, \theta_k\}) + \cdots + m(\Theta)\right] \\ &= \frac{1}{n}\left[2\sum_{\substack{i,j=1 \\ i<j}}^{n} m(\{\theta_i, \theta_j\}) + 3\sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} m(\{\theta_i, \theta_j, \theta_k\}) + \cdots + n \cdot m(\Theta)\right] \\ &= \frac{1}{n}\left[2\sum_{\substack{A \subseteq \Theta \\ |A|=2}} m(A) + 3\sum_{\substack{A \subseteq \Theta \\ |A|=3}} m(A) + \cdots + n \cdot m(\Theta)\right] \\ &= \sum_{\substack{A \subseteq \Theta \\ |A|>1}} m(A) \cdot \frac{|A|}{n} \end{aligned} \tag{17}$$

So, it is actually the weighted summation of the normalized cardinality size of the focal elements except for singletons, where the weights are their mass assignments. That is to say in computation of our belief interval-based non-specificity, there is no need to calculate the belief intervals but to just follow the final step in Eq. (17) with simple multiplication and summation operations. Eq. (17) can be further rewritten as

$$\mathrm{NS}^{\mathrm{BI}}(m) = \sum_{A \subseteq \Theta} m(A) \cdot \frac{|A|}{n}(1 - \delta(|A| - 1)) \tag{18}$$

where $\delta(\cdot)$ is the Dirac delta function defined as

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{19}$$

Eq. (18) satisfies the general form in Eq. (11) if

$$f(|A|) = \frac{|A|}{n}(1 - \delta(|A| - 1)) \tag{20}$$

So, the definition in Eq. (14) is a non-specificity measure. According to Eqs. (14) and (18), obviously, $\mathrm{NS}^{\mathrm{BI}}(m)$ reaches its minimum value 0, when m is a Bayesian BBA; $\mathrm{NS}^{\mathrm{BI}}(m)$ reaches its maximum value 1, when m is a vacuous BBA. It should be noted that our new measure expressed by Eq. (17) has two differences with that in NS^{K}, although the Eqs. (10) and (17) have closely similar expressions. The first difference is that in our new definition in Eq. (17), singletons are not involved in computation. Actually, the mass assignments are canceled in the calculation of the singletons' belief intervals' lengths. However, in NS^{K}, mass assignments of singletons are used. The second difference is that, our definition has a normalization factor n while NS^{K} has no such a factor.

Note that $\mathrm{NS}^{\mathrm{BI}}(m)$ has many desired properties for an uncertainty measure as analyzed in the next subsection.

4.2. Desired properties of new non-specificity measure

(1) Range
As aforementioned, $\mathrm{NS}^{\mathrm{BI}}(m)$ reaches its minimum value 0, when m is a Bayesian BBA; $\mathrm{NS}^{\mathrm{BI}}(m)$ reaches its maximum value 1, when m is a vacuous BBA. This means that a Bayesian BBA corresponds to a maximally precise statement, while a BBA expressing total ignorance represents the most non-specific (or the most imprecise) statement on the FOD.[20]

(2) Monotonicity
For $(\mathcal{F}_1, m_1) \subseteq (\mathcal{F}_2, m_2)$, i.e., $\forall A \in \mathcal{P}(\Theta) : \mathrm{Bel}_1(A) \geqslant \mathrm{Bel}_2(A)$, $\mathrm{Pl}_1(A) \leqslant \mathrm{Pl}_2(A)$ or $\forall A \in \mathcal{P}(\Theta) : [\mathrm{Bel}_1(A), \mathrm{Pl}_1(A)] \subseteq [\mathrm{Bel}_2(A), \mathrm{Pl}_2(A)]$, if a non-specificity measure NS satisfies $\mathrm{NS}(m_1) \leqslant \mathrm{NS}(m_2)$, then the property of Monotonicity[20] holds. This means that a non-specificity measure in the belief functions theory must not decrease the total quantity of uncertainty in situations where there is a clear decrease in information (increment of uncertainty).

Our new non-specificity measure in Eq. (14) satisfies the monotonicity. See the proof below.

Proof. If $\forall A \in \mathcal{P}(\Theta) : \mathrm{Bel}_1(A) \geqslant \mathrm{Bel}_2(A),\ \mathrm{Pl}_1(A) \leqslant \mathrm{Pl}_2(A)$, then there exists

$$\forall \theta_i \in \Theta : \mathrm{Pl}_1(\{\theta_i\}) \leqslant \mathrm{Pl}_2(\{\theta_i\}), \mathrm{Bel}_1(\{\theta_i\}) \geqslant \mathrm{Bel}_2(\{\theta_i\}).$$

Therefore,

$$\mathrm{Len}_1(i) = \mathrm{Pl}_1(\{\theta_i\}) - \mathrm{Bel}_1(\{\theta_i\}) \leqslant \mathrm{Len}_2(i) = \mathrm{Pl}_2(\{\theta_i\}) - \mathrm{Bel}_2(\{\theta_i\}) \Rightarrow \frac{1}{n}\sum_i \mathrm{Len}_1(i) \leqslant \frac{1}{n}\sum_i \mathrm{Len}_2(i) \Rightarrow \mathrm{NS}^{\mathrm{BI}}(m_1) \leqslant \mathrm{NS}^{\mathrm{BI}}(m_2)$$

End of Proof □

(3) SymmetryIf two BBAs m_1, m_2 assign the same summation of mass assignment values to focal elements with the same cardinality, then the non-specificity values of the two BBAs are equal. This is called symmetry.[22]

Our new non-specificity measure in Eq. (14) satisfies the symmetry. See the proof below.

Proof. According to Eq.(17), $\mathrm{NS}^{\mathrm{BI}}(m) = \frac{1}{n}\sum_{i=1}^{n}(\mathrm{Pl}(\{\theta_i\}) - \mathrm{Bel}(\{\theta_i\})) = \frac{1}{n}\left[2\sum_{\substack{A\subseteq\Theta\\|A|=2}} m(A) + 3\sum_{\substack{A\subseteq\Theta\\|A|=3}} m(A) + \cdots + n\cdot m(\Theta)\right]$

If $\sum_{A\subseteq\Theta,|A|=a} m_1(A) = \sum_{A\subseteq\Theta,|A|=a} m_2(A)$, $\forall a = 2,3,\ldots,n$, obviously, $\mathrm{NS}^{\mathrm{BI}}(m_1) = \mathrm{NS}^{\mathrm{BI}}(m_2)$.

End of Proof □

(4) Multiplicativity for joint BBAA joint BBA[10] $m : \mathcal{P}(\Theta_X \times \Theta_Y) \to [0,1]$ is a BBA defined on the Cartesian product of two sets (two distinct FODs) Θ_X with cardinality n_{Θ_X} and Θ_X with cardinality n_{Θ_Y}, where $\mathcal{P}(\Theta_X \times \Theta_Y)$ is the power set of $\Theta_X \times \Theta_Y$. Suppose that $\mathcal{F}$ is a set of focal elements of the joint BBA on the joint FOD $\Theta_X \times \Theta_Y$, and $S \in \mathcal{F}$.

The projections of S on to Θ_X is denoted by $S_x = \{x \in \Theta_X | (x, y) \in S, \forall x \in \Theta_X\}$. The projections of S on to Θ_Y is denoted by $S_y = \{y \in \Theta_Y | (x, y) \in S, \forall y \in \Theta_Y\}$. Then, the marginal BBAs can be defined as[10]

$$\begin{cases} m_{\Theta_X}(A) = \sum_{S|S_x=A} m(R) \quad \forall A \subseteq \Theta_X \\ m_{\Theta_Y}(A) = \sum_{S|S_y=A} m(R) \quad \forall A \subseteq \Theta_Y \end{cases} \tag{21}$$

If $m(\cdot) = m_{\Theta_X} \times m_{\Theta_Y}$ is a joint BBA on $\Theta_X \times \Theta_Y$, and two independent marginal BBAs are $m_{\Theta_X}(\cdot)$ and $m_{\Theta_Y}(\cdot)$ then

$$\mathrm{NS}(m) = \mathrm{NS}(m_{\Theta_X}) \cdot \mathrm{NS}(m_{\Theta_Y}) \tag{22}$$

This is called the property of multiplicativity.

Our new non-specificity measure in Eq. (14) satisfies the multiplicativity.[20] See the proof below.

Proof.

$$\begin{aligned} \mathrm{NS}^{\mathrm{BI}}(m) &= \sum_{A,B} m(A\times B)\frac{|A\times B|}{n_{\Theta_X}\cdot n_{\Theta_Y}} \\ &= \sum_{A,B} m_{\Theta_X}(A) m_{\Theta_Y}(B) \cdot \frac{|A|\cdot|B|}{n_{\Theta_X}\cdot n_{\Theta_Y}} \\ &= \sum_A m_{\Theta_X}(A)\frac{|A|}{n_{\Theta_X}} \cdot \sum_B m_{\Theta_Y}(B)\frac{|B|}{n_{\Theta_Y}} \\ &= \mathrm{NS}^{\mathrm{BI}}(m_{\Theta_X}) \cdot \mathrm{NS}^{\mathrm{BI}}(m_{\Theta_Y}) \end{aligned} \tag{23}$$

End of Proof □

(5) Sub-multiplicativity for joint BBAIf $m(\cdot) = m_{\Theta_X} \times m_{\Theta_Y}$ is a joint BBA on $\Theta_X \times \Theta_Y$, and its marginal BBAs $m_{\Theta_X}(\cdot)$ and $m_{\Theta_Y}(\cdot)$, which are unknown to be independent or not, then, $\mathrm{NS}(m) \leqslant \mathrm{NS}(m_{\Theta_X}) \cdot \mathrm{NS}(m_{\Theta_Y})$. This is called the property of sub-multiplicativity. The " = " holds only when $m_{\Theta_X}(\cdot)$ and $m_{\Theta_Y}(\cdot)$ are independent.

Note that the physical meaning of sub-multiplicativity is in essential the conservation of information, i.e., the amount of uncertainty in a joint BBA is no greater than the total amount of uncertainty of its corresponding marginal BBAs. The equation holds if and only if the corresponding marginal BBAs are independent, i.e., there is not correlated part. If two marginal BBAs are dependent, then the double counting uncertainty amount should be removed, therefore, the total amount of uncertainty in the joint BBA is larger than the total amount in marginal BBAs.

Proof. Suppose that S is the focal element in Cartesian space $\Theta_X \times \Theta_Y$; proj(S; Θ_X) = S_x and proj (S; Θ_Y) = S_y represent the projections of S on Θ_X and Θ_Y, respectively.

The non-specificity $\mathrm{NS}^{\mathrm{BI}}$ of two marginal BBA $m_{\Theta_X}(\cdot)$ and $m_{\Theta_Y}(\cdot)$ are

$$\mathrm{NS}^{\mathrm{BI}}(m_{\Theta_X}) = \sum_{\substack{A\subseteq\Theta_X\\|A|>1}} m_{\Theta_X}(A)\frac{|A|}{n_{\Theta_X}} = \sum_{\substack{A\subseteq\Theta_X\\|A|>1}} \left(\sum_{A=\mathrm{proj}(S;\Theta_X)=S_x} m(S)\right)\frac{|A|}{n_{\Theta_X}}$$

$$\mathrm{NS}^{\mathrm{BI}}(m_{\Theta_Y}) = \sum_{\substack{B\subseteq\Theta_Y\\|B|>1}} m_{\Theta_Y}(B)\frac{|B|}{n_{\Theta_Y}} = \sum_{\substack{B\subseteq\Theta_Y\\|B|>1}} \left(\sum_{B=\mathrm{proj}(S;\Theta_Y)=S_y} m(S)\right)\frac{|B|}{n_{\Theta_Y}}$$

The multiplication of the $\mathrm{NS}^{\mathrm{BI}}$ for two marginal BBAs are

$$\begin{aligned} \mathrm{NS}^{\mathrm{BI}}(m_{\Theta_X}) \cdot \mathrm{NS}^{\mathrm{BI}}(m_{\Theta_Y}) &= \sum_{\substack{A\subseteq\Theta_X\\|A|>1}} \left(\sum_{A=S_x} m(S)\right)\frac{|A|}{n_{\Theta_X}} \cdot \sum_{\substack{B\subseteq\Theta_Y\\|B|>1}} \left(\sum_{B=S_y} m(S)\right)\frac{|B|}{n_{\Theta_Y}} \\ &= \sum_{\substack{S\subseteq\Theta_X\times\Theta_Y\\|S_x|>1}} m(S)\frac{|S_x|}{n_{\Theta_X}} \cdot \sum_{\substack{S\subseteq\Theta_X\times\Theta_Y\\|S_y|>1}} m(S)\frac{|S_y|}{n_{\Theta_Y}} \\ &= \sum_{\substack{S\subseteq\Theta_X\times\Theta_Y\\|S_x|>1,|S_y|>1}} m(S)\frac{|S_x|\cdot|S_y|}{n_{\Theta_X}\cdot n_{\Theta_Y}} \\ &\geqslant \sum_{\substack{S\subseteq\Theta_X\times\Theta_Y\\|S_x|>1,|S_y|>1}} m(S)\frac{|S|}{n_{\Theta_X}\cdot n_{\Theta_Y}} \\ &= \mathrm{NS}^{\mathrm{BI}}(m_{\Theta_X} \times m_{\Theta_Y}) \end{aligned}$$

End of Proof □

It should be noted that Dubois & Prade's non-specificity in Eq. (8) also satisfies all the properties including the monotonicity, the symmetry, the additivity (which is the counter-part of the multiplicativity here), and the sub-additivity (which is the counter-part of the sub-multiplicativity here).[20] Here we provide detailed explanations for the additivity and sub-additivity.

If $m(\cdot)$ is a joint BBA on $\Theta_X \times \Theta_Y$, and the associated marginal BBAs $m_{\Theta_X}(\cdot)$ and $m_{\Theta_Y}(\cdot)$ are independent, then $\mathrm{NS}^{\mathrm{DP}}(m) = \mathrm{NS}^{\mathrm{DP}}(m_{\Theta_X}) + \mathrm{NS}^{\mathrm{DP}}(m_{\Theta_Y})$ because of log function involved in the definition of NS measure in Eq. (8). This is called the property of additivity. The additivity is in essential equivalent to the multiplicativity, i.e., they both describe the relationship between the non-specificity of a joint BBA and their corresponding independent marginal BBAs.

If the marginal BBAs $m_{\Theta_X}(\cdot)$ and $m_{\Theta_Y}(\cdot)$ are unknown to be independent or not, then $\mathrm{NS}^{\mathrm{DP}}(m) \leqslant \mathrm{NS}^{\mathrm{DP}}(m_{\Theta_X}) + \mathrm{NS}^{\mathrm{DP}}(m_{\Theta_Y})$. This is called the property of sub-additivity. The sub-additivity is in essential equivalent to the sub-multiplicativity.

Therefore, both Dubois & Prade's non-specificity in Eq. (8) and our new measure in Eq. (14) strictly satisfy all the requirements (properties) of a non-specificity measure. Not all the requirements or properties can be satisfied for Yager's definition and Kornor's definition. Our new definition in Eq. (14) and Dubois & Prade's definition in Eq. (8) can both be used as a strict non-specificity measure. Our new measure can be a good alternative of the traditional strict $\mathrm{NS}^{\mathrm{DP}}$. We want to emphasize that the theoretic consistency is an very important issue when one defines some measure under a given theoretic framework. Our proposed measure keeps the consistency, thus, it is not the generalization of any measure in other frameworks. In the next section, some illustrative examples and simulations are provided to show the rationality of our new non-specificity measure.

5. Illustrative examples and simulations

5.1. Example 1

Suppose that the FOD $\Theta = \{\theta_1, \theta_2, \theta_3\}$. A BBA defined on Θ is $m(A) = 1$, $\forall A = \Theta$. At each step, $m(\Theta)$ has a decrease of $\Delta = 0.05$ and each singleton mass $m(\{\theta_i\})$, $i = 1, 2, 3$ has an increase of $\Delta/3$. At the final step, $m(\Theta)$ becomes zero and $m(\{\theta_i\}) = 1/3$, $i = 1, 2, 3$. At each step, Dubois & Prade's non-specificity ($\mathrm{NS}^{\mathrm{DP}}$), Yager's non-specificity (NS^{Y}), Kornor's non-specificity (NS^{K}), our proposed belief interval based non-specificity ($\mathrm{NS}^{\mathrm{BI}}$), total uncertainty measure AU and AM (they also include the non-specificity part according to their definitions) are calculated. The changes of these uncertainty values at different steps are illustrated in Fig. 1.

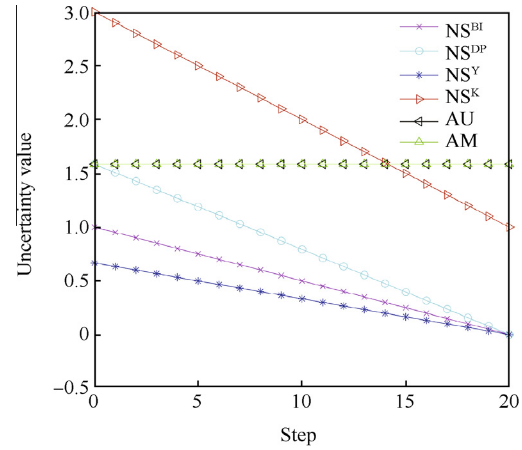

Fig. 1 Change of different non-specificity measures and total uncertainty measures in Example 1.

As shown in Fig. 1, all non-specificity measures decrease with the increase of the mass assignments for singletons and the decrease of the mass assignment for Θ. All the non-specificity measures compared here reach their minimum value when m becomes a Bayesian BBA, and they reach their maximum value when m is a vacuous BBA. They all provide rational behavior. Our proposed $\mathrm{NS}^{\mathrm{BI}} \in [0, 1]$ has natural normalization. Such a value range is more preferred as an uncertainty measure. Both of the minimum values of NS^{Y} and $\mathrm{NS}^{\mathrm{DP}}$ are 0. Therefore, by using normalization factor, they can have the value range of [0, 1]. However, the minimum value of NS^{K} is 1 but not 0, which is not preferred.

For the total uncertainty measure AU and AM, they never change in the whole procedure (at their maximum value). Although AU and AM declare that they can also depict the non-specificity part in the total uncertainty, they cannot discriminate the changes of BBAs at each step. This is because they are defined based on some probabilistic transformation from BBAs. In this example, for the probabilistic transformation used in AM and AU, the results are always a uniformly distributed probability mass function (p.m.f.) $P(\theta_i) = 1/3$, $i = 1, 2, 3$, therefore AM and AU will never change here. According to our opinion, it is not judicious to define total uncertainty measure in the theory of belief functions by using for reference the uncertainty measure in probability framework, i.e., Shannon entropy. It should be better not to switch the framework but to directly design in the framework of belief functions. This is also our concerns and motivations for the design of belief interval based non-specificity measure.

5.2. Example 2

Suppose that the FOD $\Theta = \{\theta_1, \theta_2, \theta_3\}$. A BBA defined on Θ is $m(A) = 1$, $\forall A = \Theta$. At each step, $m(\Theta)$ has a decrease of $\Delta = 0.05$ and one singleton mass $m(\{\theta_1\})$ has an increase of Δ. In the final step, $m(\Theta)$ becomes zero and $m(\{\theta_1\}) = 1$. The changes of these uncertainty values (including $\mathrm{NS}^{\mathrm{DP}}$, NS^{Y}, NS^{K}, $\mathrm{NS}^{\mathrm{BI}}$, AU and AM) at different steps are illustrated in Fig. 2.

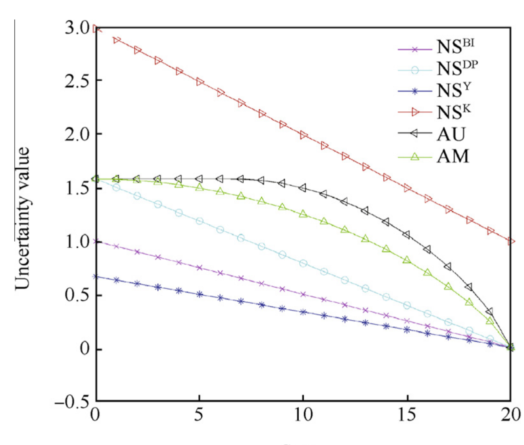

Fig. 2 Change of different non-specificity measures and total uncertainty measures in Example 2.

At the first step, the BBA is a vacuous one, and at the last step, the BBA is a categorical one. So, each uncertainty measure changes from their maximum value to the minimum value. In this example, AM and AU can also bring intuitive results because at each step the probabilistic transformation results are not always uniformly distributed p.m.f.

Although in this simple case, all compared measures perform well, it should be noted that some traditional measures will bring counter-intuitive behaviors as shown in the following examples.

5.3. Example 3

Suppose that the FOD is $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$. A BBA defined on Θ is $m(A) = 1$, $\forall A = \{\theta_1, \theta_2\}$. At each step, $m(A)$ has a decrease of $\Delta = 0.05$, and the other focal elements with cardinality of 2 (including $\{\theta_1, \theta_3\}$, $\{\theta_1, \theta_4\}$, $\{\theta_2, \theta_3\}$, $\{\theta_2, \theta_4\}$, $\{\theta_3, \theta_4\}$) increase $\Delta/5 = 0.01$. At the final step, $m(B) = 1/6$, $\forall |B| = 2$. The changes of these uncertainty values (including $\mathrm{NS}^{\mathrm{DP}}$, NS^{Y}, NS^{K}, $\mathrm{NS}^{\mathrm{BI}}$, AU and AM*)* at different steps are illustrated in Fig. 3.

Since in the whole procedure, all the focal elements' cardinality is 2, all the non-specificity measures do not change. The total uncertainty AM and AU increase at each step and reach their maximum value finally. This is because that the probabilistic transformations of the BBAs gradually approach a uniformly distributed p.m.f. with the change of the BBA at each step.

5.4. Example 4

Suppose that the FOD $\Theta = \{\theta_1, \theta_2\}$. A BBA defined on Θ is $m(\{\theta_1\}) = a$, $m(\{\theta_2\}) = b$, $m(\Theta) = 1 - a - b$, where $a, b \in [0, 0.5]$. We calculate all the uncertainty values (including $\mathrm{NS}^{\mathrm{DP}}$, NS^{Y}, NS^{K}, $\mathrm{NS}^{\mathrm{BI}}$, AU and AM). The change of different uncertainty measures with the change of a and b is illustrated in Fig. 4.

As shown in Fig. 4, the values of $\mathrm{NS}^{\mathrm{DP}}$ and $\mathrm{NS}^{\mathrm{BI}}$ are the same, because $\log_2|A| = |A|/2$, $\forall |A| = 2$. All non-specificity measures reach their maximum values when $a = b = 0$, i.e., $m(\Theta) = 1$, and reach their minimum values when

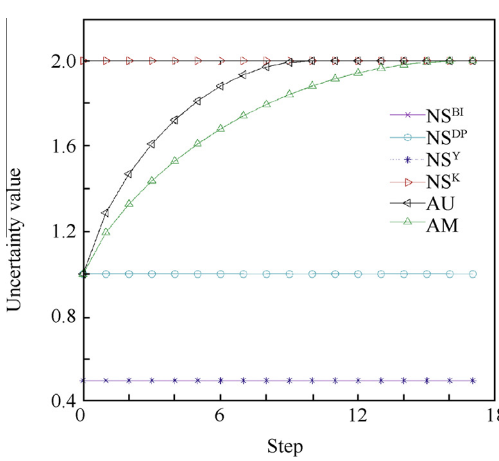

Fig. 3 Change of different non-specificity measures and total uncertainty measures in Example 3.

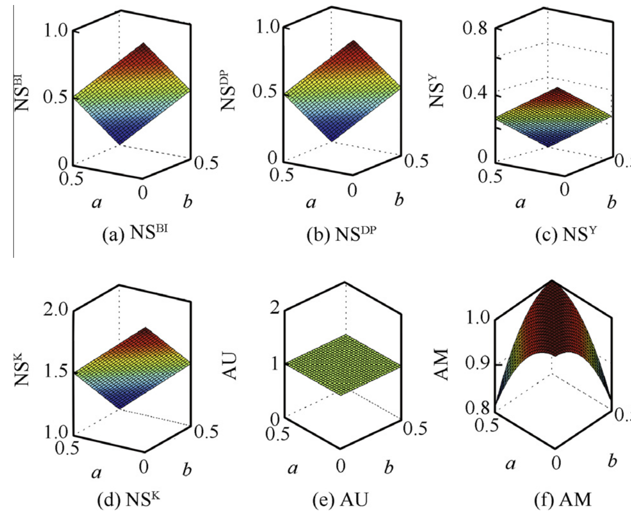

Fig. 4 Change of different non-specificity measures and total uncertainty measures in Example 4.

$a = b = 0.5$, i.e., $m(\Theta) = 0$. Since AU tries to find a p.m.f. with maximum Shannon entropy, and the uniformly distributed $P(\theta_1) = P(\theta_2) = 0.5$ always satisfies the constraints above (because $a, b \in [0, 0.5]$), no matter how a and b change, therefore, $P(\theta_1) = P(\theta_2) = 0.5$ is always picked up when calculating AU and thus, AU always equals $\log_2 2 = 1$. AM reaches its maximum value when $a = b$, because for $a = b$, the corresponding pignistic probability of the BBA is uniformly distributed.

5.5. Example 5

This example is used for reference from.[10] Given the size of the FOD $|\Theta| = 5$. Randomly generate 10 BBAs with k $(1 \leqslant k \leqslant 31)$ focal elements according to algorithm[39] in Table 1. Here, we set the number of focal elements to a fixed value 15.

These generated 10 BBAs ($m_1, m_2, \ldots, m_{10}$) are then combined one by one using Dempster's rule of combination, respectively. At each combination instant t: $m_{t+1}^{\mathrm{DS}} = m_{t+1} \oplus_{DS} m_t^{\mathrm{DS}}$, where $t = 1, 2, \ldots, 9$, and we start with $m_t^{\mathrm{DS}} = m_1$. The values for $\mathrm{NS}^{\mathrm{DP}}$, NS^{Y}, NS^{K}, $\mathrm{NS}^{\mathrm{BI}}$, AU and AM of the combination result at t are calculated for m_t^{DS}, $t = 1, 2, \ldots, 10$. The whole procedure is repeated 100

Table 1 Algorithm: Random generation of BBA.

Algorithm: Random generation of BBA
Input: Θ: Frame of discernment;
$N_{\max}$: Maximum number of focal elements
Output: m: a BBA
Generate the power set of Θ: $\mathcal{P}(\Theta)$
Generate a random permutation of $\mathcal{P}(\Theta) \rightarrow \mathcal{R}(\Theta)$;
Generate a integer between 1 and $N_{\max} \rightarrow k$;
FOR each First k elements of $\mathcal{R}(\Theta)$ do
Generate a value within $[0, 1] \rightarrow \mathrm{mv}(i), i = 1, 2, \ldots, k$;
END
Normalize the vector $\mathbf{mv} = [\mathrm{mv}(1), \mathrm{mv}(2), \ldots, \mathrm{mv}(k)] \rightarrow \mathbf{mv}'$
$m(A_j) = \mathbf{mv}'(j),\ j = 1, 2, \ldots, k$

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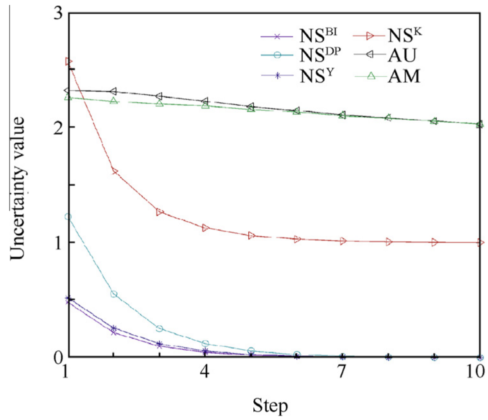

Fig. 5 Change of different non-specificity measures and total uncertainty measures in Example 5.

times and the average values of different uncertainty measures at different instants are shown in Fig. 5.

As shown in Fig. 5, all the uncertainty measures compared here decrease with the increase of the combination steps. This makes sense, because it is intuitive that the uncertainty decreases in the information fusion procedure like the evidence combination. The non-specificity measures drop faster and more significantly than the total uncertainty measures. This is because that in the evidence combination based on Dempster's rule, the focal elements are split into focal elements with smaller cardinality.

As we can see in the above examples, our new proposed belief interval-based non-specificity measure is rational and effective in representing the non-specificity in BBAs.

6. Application of belief interval-based non-specificity measure

Uncertainty measures including the non-specificity measure have be used in many applications such as the weighted evidence combination.[40,41] Here we provide an example of using our new non-specificity in feature evaluation for pattern recognition to further show the rationality of the proposed measure.

We artificially generate three classes of samples. Each class has 100 samples. Each sample has 3 dimensions. In each class, each dimension of the samples is Gaussian distributed with different mean and standard deviation (Std) values as illustrated in Fig. 6 and Table 2.

Table 2 Gaussian distribution parameters of the samples.

Features		Class 1	Class 2	Class 3
Feature 1	Mean	0	4.0	5.5
	Std	1	1.2	1.2
Feature 2	Mean	0	5.0	8.0
	Std	1	1.2	1.2
Feature 3	Mean	5	5.0	5.0
	Std	1	1.1	1.1

As we can see in Fig. 6 and Table 2, the class discrimination capability of Feature 2 is the best, because the three Gaussian probability density functions (PDFs) are quite well separated; that of Feature 3 is the worst, and that of the Feature 1 is in the middle. This can also be verified by using the discrimination criterion as follows.

$$J = \frac{\mathrm{tr}(\boldsymbol{S}_\mathrm{w})}{\mathrm{tr}(\boldsymbol{S}_\mathrm{b})} \tag{24}$$

where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. Suppose that there are C classes and each class C_i has N_i samples. The degree of inner-class cohesion $\boldsymbol{S}_\mathrm{w}$ and the degree of inter-class separability $\boldsymbol{S}_\mathrm{b}$ are as follows.[42]

$$\begin{cases} \boldsymbol{S}_\mathrm{w} = \sum_{i=1}^{C} P(C_i) E\left[\left(\boldsymbol{X} - \frac{1}{N_i}\sum_{\boldsymbol{X}\in C_i}\boldsymbol{X}\right)\left(\boldsymbol{X} - \frac{1}{N_i}\sum_{\boldsymbol{X}\in C_i}\boldsymbol{X}\right)^\mathrm{T}\right] \\ \boldsymbol{S}_\mathrm{b} = \sum_{i=1}^{C} P(C_i)\left(\frac{1}{N_i}\sum_{\boldsymbol{X}\in C_i}\boldsymbol{X} - \boldsymbol{M}\right)\left(\frac{1}{N_i}\sum_{\boldsymbol{X}\in C_i}\boldsymbol{X} - \boldsymbol{M}\right)^\mathrm{T} \end{cases} \tag{25}$$

where $\boldsymbol{X}$ is feature(s) of a sample and

$$\boldsymbol{M} = \frac{1}{C}\sum_{i=1}^{C}\left(\frac{1}{N_i}\sum_{\boldsymbol{X}\in C_i}\boldsymbol{X}\right) \tag{26}$$

is the mean of all the classes' centroids. If J of some feature (or set of features) in Eq. (23) is smaller, then such a feature (or set of features) is more crisp and discriminable.

For our artificially generated samples $J(1) = 0.2557$, $J(2) = 0.1013$, $J(3) = 326.8135$, which means that Feature 2 is the best, Feature 3 is the worst, and Feature 1 is in the middle.

First, we use the following way[43] to generate BBAs for each sample x_q on different feature $i \in \{1, 2, 3\}$.

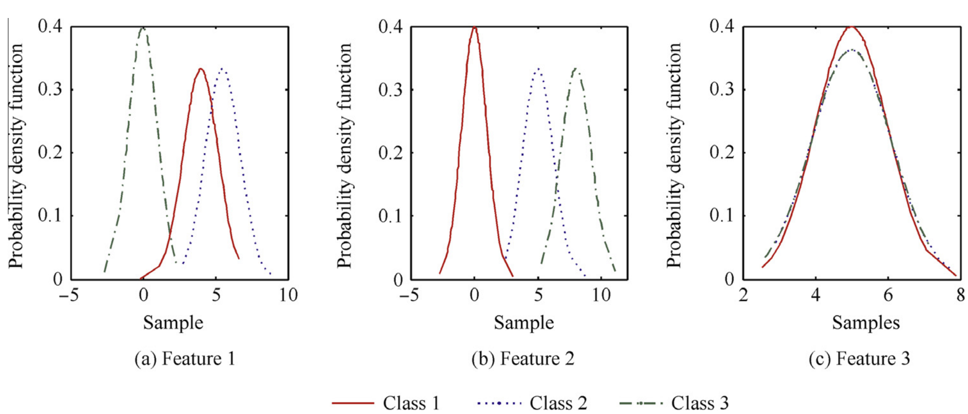

Fig. 6 Probability density function of different features of the samples belonging to three classes.

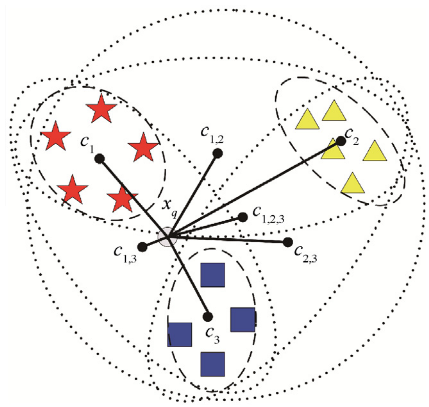

Fig. 7 Illustration of BBA generation.

$$m^i_{x_q}(A_j) = \frac{|A_j|^{-\alpha/(\beta-1)} d_{qj}^{-2/(\beta-1)}}{\sum_{A_k \neq \varnothing} |A_k|^{-\alpha/(\beta-1)} d_{qk}^{-2/(\beta-1)} + \delta^{-2/(\beta-1)}} \tag{27}$$

where d_{qj} denotes the distance between the query sample x_q and the class A_j. Parameters $\alpha = 1$, $\beta = 2$ as suggested in Ref.[43] It should be noted that in the original form in Ref.[43], there exists the mass assignment for the emptyset $\varnothing$ representing the possibility of x_q to be an outlier. In this paper, we only concern the closed-world assumption, i.e., there is no mass assignment for $\varnothing$. Here we give an illustrative example in Fig. 7.

In Fig. 7, three different colors represent three different classes. c_1 denotes the centroid of samples in Class 1; $c_{1,2}$ denotes the centroid of samples in Class 1 and Class 2; $c_{1,2,3}$ denotes the centroid of samples in Class 1, 2, and 3. Calculate the distance d between x_q and those centroids of single classes and compound classes. Then according to Eq. (26), the BBA can be generated.

Second, we calculate NS^{BI} for all the BBAs generated. Then calculate the average values of NS^{BI} for different feature i as

$$\text{mean\_NS}^{BI}(i) = \sum_{x_q \in \{C_1, C_2, C_3\}} NS^{BI}(m^i_{x_q}) \tag{28}$$

The averaging non-specificity of a feature is larger, then it is more discriminable.

For our artificially generated samples illustrated in Fig. 6 and Table 2,

$$\text{mean\_NS}^{BI}(1) = 0.3447, \text{mean\_NS}^{BI}(2) = 0.3088, \text{mean\_NS}^{BI}(3) = 0.4285.$$

This is consistent with the intuition and the feature evaluation based on the discrimination criterion in Eq. (23).

7. Conclusions

A novel strict non-specificity measure in the theory of belief functions is proposed with several desired properties. It should be noted that the new measure is defined directly in the framework of belief functions. There is no need to switch (and thus lose information) from belief functions to the classical probabilistic framework. Numerical examples, simulations, and the application of the new measure are also provided, which show that the new measure can well measure the non-specificity in a BBA and can be effectively used in applications such as feature evaluation.

In future work, we attempt to apply our new measure in other applications such as the weighted evidence combination, etc. We will also research on the other part of uncertainty, i.e., the discord, and the total uncertainty directly in the framework of belief functions (not transforming to the probability framework). There are already some related tentative related works on this.[44–46]

Acknowledgements

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