Alliance -based evidential reasoning approach with unknown evidence weights

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Abstract

In the evidential reasoning approach of decision theory, different evidence weights can generate different combined results. Consequently, evidence weights can significantly influence solutions. In terms of the "psychology of economic man," decision-makers may tend to seek similar pieces of evidence to support their own evidence and thereby form alliances. In this paper, we extend the concept of evidential reasoning (ER) to evidential reasoning based on alliances (ERBA) to obtain the weights of evidence. In the main concept of ERBA, pieces of evidence that are easy for decision-makers to negotiate are classified in the same group or "alliance." On the other hand, if the pieces of evidence are not easy to negotiate, they are classified in different alliances. In this study, two negotiation optimization models were developed to provide relative importance weights based on intra- and inter-alliance evidence features. The proposed models enable weighted evidence to be combined using the ER rule. Experimental results showed that the proposed approach is rational and effective.

Keywords: evidential reasoning; alliance; negotiation; decision-making; evidence weights.

1. Introduction

Evidence theory (Dempster, 1967; Shafer, 1976) was originally developed by Arthur P. Dempster and later extended and refined by Glenn Shafer. It is thus sometimes referred to as the Dempster–Shafer (DS) theory of evidence. DS theory is a powerful and flexible mathematical tool for addressing imprecise and uncertain information. Hence, it has been employed in areas such as expert systems (Beynon et al., 2001) , uncertainty reasoning (Jones et al., 2002), pattern classification (Denoeux et al., 2004), fault diagnosis and detection (Fan and Zuo, 2006), information fusion (Telmoudi and Chakhar, 2006), multiple attribute decision analysis (Xu et al., 2006), image processing (Huber, 2001), risk analysis (Deng et al., 2011), e-commerce security (Zhang et al., 2012)

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, financial applications (Dymova et al., 2012; Dymova et al., 2016), and water distribution systems (Bazargan-Lari, 2014).

Meanwhile, decision theory, or the theory of choice, is the study of the reasoning behind a decision-maker’s choice. It is used to solve problems involving selection from a finite number of choices. Decision theory literature has an extensive history. However, many studies published in this area, such as works on the analytic hierarchy process (AHP), assume complete information in the decision-making process. That is, the methods assume that decision-makers are fully aware of their specific preferences. In cases in which data for assessing alternatives against criteria are partially or completely unavailable, or the decision-maker’s knowledge of an alternative evaluation is insufficient, the decision-makers are more likely to use uncertain assessment information.

DS theory, on the other hand, is well suited to handling uncertainty. It is particularly useful for dealing with uncertain subjective judgments when multiple pieces of evidence must be simultaneously considered. An evidential reasoning (ER) approach based on both decision theory and DS theory was thus proposed by Yang and Singh (Yang and Singh, 1994). In the past two decades, the original ER approach has been extensively developed to solve multi-attribute decision making (MADM) problems with uncertainties, including fuzzy evaluation grades, interval evaluation grades, fuzzy belief structures, interval belief degrees, dynamic belief degrees, partially ordered preferences under uncertainty, and unknown attribute weights in various values and preference judgments (Yang, 2001; Yang and Xu, 2002a; Yang and Xu, 2002b; Huynh et al., 2006; Yang et al., 2006; Guo et al., 2009; Wang et al., 2006; Hu et al., 2010; Fu and Chin, 2014).

Furthermore, the ER approach and its extensions have been widely applied to MADM problems in business performance assessment (Yang et al., 2001), pre-qualification of construction contractors (Sönmez et al., 2002), environmental impact assessment (Wang et al., 2006), organizational self-assessment (Siow, 2001), safety analysis (Wang and Yang, 2001; Liu et al., 2004), bridge condition assessment (Wang and Elhag, 2007), behavior prediction (Zhou et al., 2010), fault prediction (Si et al., 2011) , risk analysis (Tang et al., 2011), job offering (Mahmud et al., 2013), software selection (Chin and Fu, 2014) , and group decision analysis (Fu et al., 2014) .

In the above ER approaches, the assessment information for each criterion is regarded as a piece of evidence; the criterion weight provides the evidence weight. The residual support remains uncommitted because the evidence weight is assigned to any singleton proposition and the universal set proposition, which contains all elements of a proposition. This specific assignment can differentiate between ignorance and residual support, while reflecting the relative importance of other evidence. As Xu (2009) contended, this specificity enables the ER approach to solve counterintuitive problems in which conflicting pieces of evidence are combined using Dempster’s rule.

However, most existing ER approaches have made significant advancements in solving MADM problems with different decision scenarios based on the original ER algorithm. In this algorithm, it is assumed that local ignorance exists in none of the evidence. While this assumption is reasonable when solving MADM problems, it is difficult to apply to other domains.

To expand the range of ER applications, Yang and Xu (2013) relinquished this assumption and generalized the ER algorithm into a new weighted ER rule that accounts for both global and local ignorance. In addition, they further expanded the ER rule to combine multiple pieces of evidence according to their weights and degrees of reliability. These advancements have considerably enriched ER theory. It is said that the importance of a piece of evidence reflects a decision-maker's preferences over the evidence. The importance is thus subjective; it depends on the agent making the judgment when using the evidence.

Nevertheless, none of the above ER approaches explain how to determine the relative importance of the evidence weight. Moreover, the individual behavior of the decision-maker is not considered. Because the results are interest-driven, it is impossible for the decision-maker to have an unbiased, isolated perspective regarding the evidence. In certain decision-making situations, the determining agent seeks similar pieces of evidence to support their own evidence, thereby forming an alliance of evidence. We therefore propose the "evidential reasoning" approach (ERBA), which is based on alliances.

The remainder of this paper is organized as follows. Section 2 introduces the relevant concepts of DS theory, evidential reasoning, and the pignistic probability distance. Section 3 describes the importance of evidence weight. Section 4 details the development of the ERBA approach, and Section 5 analyzes the rationality of the proposed approach. Section 6 concludes the paper.

2. Preliminaries

In this section, we introduce some prior knowledge regarding DS theory, the evidential reasoning algorithm, and the pignistic probability distance, which is used as the basis for later discussions.

2.1. DS theory

DS theory is viewed as a generalization of probability theory that can handle multiple possible propositions, e.g., sets of propositions. Let $\Theta = \{H_1, \cdots, H_N\}$ be a collectively exhaustive and mutually exclusive set of propositions. It is called the frame of discernment. Three important functions exist in DS theory: the *basic probability assignment* function (bpa or *m*), *belief* function (*Bel*), and *plausibility* function (*Pl*). These functions are defined below.

Definition 1 (Dempster, 1967) **.** A basic belief assignment (bba) (also called a belief structure) is a function, $m: 2^{\Theta} \to [0,1]$ (called the *basic probability assignment* (bpa) in Shafer's original definition), which satisfies

$$\begin{cases} m(\phi) = 0, \\ \sum_{A \subseteq \Theta} m(A) = 1 (0 \le m(A) \le 1), \end{cases}$$

where ϕ is an empty set, *A* is a subset of Θ, and 2^{Θ} is the power set of Θ, which consists of all the subsets of Θ, i.e., $2^{\Theta} = \{\phi, \{H_1\}, \cdots, \{H_N\}, \{H_1, H_2\}, \cdots, \{H_1, H_N\}, \cdots, \Theta\}$. The function $m(A)$ is the bpa of *A*. It measures the belief explicitly assigned to *A* and represents how strongly the evidence supports *A*. Each subset A is called a focal element of *m*. All related focal elements are collectively called the body of evidence.

Definition 2 (Dempster, 1967) **.** Both the belief function and plausibility function can be derived from the basic probability assignment, $m(A)$, as

$$Bel(A) = \sum\nolimits_{B \subseteq A} m(B) \quad \text{and} \quad Pl(A) = \sum\nolimits_{B \cap A \ne \phi} m(B).$$

In this definition, $Bel(A)$ represents the sum of the basic probability masses assigned to A, whereas $Pl(A)$ represents the sum of possible basic probability masses that could be assigned to A.

The core concept of DS theory is Dempster's rule, by which pieces of evidence from different sources are combined or aggregated. This rule assumes that information sources are independent. It thus uses the so-called orthogonal sum to combine multiple belief structures as $m_1 \oplus m_2 \cdots \oplus m_n$, where $\oplus$ represents the combination operator. With multiple bbas, $m_1, m_2, \cdots, m_n$, Dempster's rule is defined as

$$[m_1 \oplus m_2 \cdots \oplus m_n](\theta) = \begin{cases} 0, \quad \theta = \phi, \\ \dfrac{\sum_{A_1 \cap A_2 \cdots \cap A_n = \theta, A_1, A_2, \cdots, A_n \subseteq \Theta} m_1(A_1) m_2(A_2) \cdots m_n(A_n)}{\sum_{A_1 \cap A_2 \cdots \cap A_n \ne \phi, A_1, A_2, \cdots, A_n \subseteq \Theta} m_1(A_1) m_2(A_2) \cdots m_n(A_n)} & , \theta \ne \phi, \end{cases}$$

under the condition that $\sum_{A_1 \cap A_2 \cdots \cap A_n \ne \phi} m_1(A_1) m_2(A_2) \cdots m_n(A_n) \ne 0$.

2.2. Evidential reasoning rule

Yang and Xu (2013) extended DS theory and proposed a refined ER rule that proceeds along the following steps. First, the original evidence is transformed into modified evidence using relative weights as follows:

$$
\begin{cases}
M_i(\theta) = w_i m_i(\theta), \quad i = 1,2,\cdots,n; \theta \subset \Theta, \\
M_i(\Theta) = \overline{M}_i(\Theta) + M_i(\Theta) = 1 - w_i \sum_{\theta \subset \Theta} m_i(\theta), i = 1,2,\cdots,n, \\
M_i(\Theta) = w_i(1 - \sum_{\theta \subset \Theta} m_i(\theta)), \quad i = 1,2,\cdots,n, \\
\overline{M}_i(\Theta) = 1 - w_i, \quad i = 1,2,\cdots,n.
\end{cases}
\tag{1}
$$

Note that the probability mass assigned to the whole set, $M_i(\Theta)$, which denotes the degree of ignorance, is split into two parts: $\bar{M}_i(\Theta)$, caused by the relative importance of evidence, and $\tilde{M}_i(\Theta)$, caused by the incompleteness of evidence.

Second, the belief degrees of the modified pieces of evidence are combined into aggregated belief degrees using the following analytical formulas:

$$
\begin{cases}
M_1 \oplus M_2 \cdots \oplus M_L(\theta) = K^{-1} \cdot \sum_{A_1 \cap A_2 \cdots \cap A_L = \theta} M_1(A_1) M_2(A_2) \cdots M_L(A_L), \theta \subset \Theta, \\
M_1 \oplus M_2 \cdots \oplus M_L(\Theta) = K^{-1} \cdot M_1(\Theta) M_2(\Theta) \cdots M_L(\Theta), \\
\overline{M}_1 \oplus \overline{M}_2 \cdots \oplus \overline{M}_L(\Theta) = K^{-1} \cdot \overline{M}_1(\Theta) \overline{M}_2(\Theta) \cdots \overline{M}_L(\Theta), \\
M_1 \oplus M_2 \cdots \oplus M_L(\Theta) = M_1 \oplus M_2 \cdots \oplus M_L(\Theta) - \overline{M}_1 \oplus \overline{M}_2 \cdots \oplus \overline{M}_L(\Theta), \\
K = \sum_{A_1 \cap A_2 \cdots \cap A_L \neq \phi} M_1(A_1) M_2(A_2) \cdots M_L(A_L),
\end{cases}
\tag{2}
$$

where K is the normalization factor and ϕ is an empty set. Finally, the aggregated belief degrees are normalized as

$$
\begin{cases}
M_{\theta,1,2,\cdots,n} = \dfrac{M_1 \oplus M_2 \cdots \oplus M_L(\theta)}{1 - (\overline{M}_1 \oplus \overline{M}_2 \cdots \oplus \overline{M}_L(\Theta))}, \theta \subset \Theta, \\
M_{\Theta,1,2,\cdots,n} = \dfrac{M_1 \oplus M_2 \cdots \oplus M_L(\Theta)}{1 - (\overline{M}_1 \oplus \overline{M}_2 \cdots \oplus \overline{M}_L(\Theta))}.
\end{cases}
\tag{3}
$$

2.3. Pignistic probability distance

Let m be a bba on Θ. Its associated pignistic probability function (Smets, 1994) $BetP_m : \rightarrow [0,1]$ is defined as

$$
BetP_m(\{\theta_j\}) = \sum\nolimits_{\theta_j \in \theta \subseteq \Theta} \frac{m(\theta)}{|\theta|}, \tag{4}
$$

where $|\theta|$ is the cardinality of subset θ. The function $BetP_m$ can be extended as a function on 2^{Θ} as

$$
BetP_m(\theta) = \sum\nolimits_{\theta_j \in \theta \subseteq \Theta} BetP_m(\{\theta_j\}), \tag{5}
$$

For example, suppose that bba m is constructed as

$$m: m\{a\}=0.3, m\{a,b\}=0.2, m\{b,c\}=0.2, m\{a,b,c\}=0.3,$$

where $\Theta=\{a,b,c\}$ is the frame of evidence discernment. According to Eq. (4), the pignistic probability is

$$BetP_m(a)=m(a)+\frac{1}{2}m(a,b)+\frac{1}{3}m(a,b,c)=0.5,$$

$$BetP_m(b)=\frac{1}{2}m(a,b)+\frac{1}{2}m(b,c)+\frac{1}{3}m(a,b,c)=0.3,$$

$$BetP_m(c)=\frac{1}{2}m(b,c)+\frac{1}{3}m(a,b,c)=0.2.$$

For

$$\sum_{\theta_j\in\theta\subseteq\Theta}\frac{m(\theta)}{|\theta|}=\sum_{\theta_j\in\theta\subset\Theta}\frac{m(\theta)}{|\theta|}+\frac{m(\Theta)}{|\Theta|}=\sum_{\theta_j\in\theta\subset\Theta}\frac{m(\theta)}{|\theta|}+\frac{1-\sum_{\theta\subset\Theta}m(\theta)}{|\Theta|}=$$
$$\sum_{\theta_j\in\theta\subset\Theta}\frac{m(\theta)}{|\theta|}-\frac{\sum_{\theta\subset\Theta}m(\theta)}{|\Theta|}+\frac{1}{|\Theta|},$$

Eq. (4) can be rewritten as

$$BetP_m(\{\theta_j\})=(\sum_{\theta_j\in\theta\subseteq\Theta}\frac{m(\theta)}{|\theta|}-\frac{\sum\limits_{\theta\subset\Theta}m(\theta)}{|\Theta|})+\frac{1}{|\Theta|}. \tag{6}$$

Let M be the result obtained by modifying m with the relative weight w using Eq. (1). Then,

$$BetP_M(\{\theta_j\})=\sum_{\theta_j\in\theta\subseteq\Theta}\frac{M(\theta)}{|\theta|}=\sum_{\theta_j\in\theta\subset\Theta}\frac{M(\theta)}{|\theta|}+\frac{M(\Theta)}{|\Theta|}=\sum_{\theta_j\in\theta\subset\Theta}\frac{w\cdot m(\theta)}{|\theta|}+\frac{1-w\cdot\sum\limits_{\theta\subset\Theta}m(\theta)}{|\Theta|}).$$

Thus, the associated pignistic probability function $BetP_M$ can be expressed as

$$BetP_M(\{\theta_j\})=w(\sum_{\theta_j\in\theta\subseteq\Theta}\frac{m(\theta)}{|\theta|}-\frac{\sum\limits_{\theta\subset\Theta}m(\theta)}{|\Theta|})+\frac{1}{|\Theta|}. \tag{7}$$

The pignistic probability distance is a reasonable measure of the evidence distance (Liu, 2006) . It is defined as follows.

Definition 3 (Liu, 2006) Let m_i and m_j be two bbas on frame Θ, and let $BetP_{m_i}$ and $BetP_{m_j}$ be the corresponding pignistic transformations. Then,

$$difBetP_{m_i}^{m_j}=\max_{A\subseteq\Theta}(\left|BetP_{m_i}(A)-BetP_{m_j}(A)\right|) \tag{8}$$

is called the pignistic probability distance of the two bbas.

The value $\left|BetP_{m_i}(A)-BetP_{m_j}(A)\right|$ is the difference between the belief degrees of *A* from the two bbas, m_i and m_j. Thus, the pignistic probability distance is the maximum extent of the differences between belief degrees to all subsets from two evidence sources.

For example, suppose that two bbas, m_1 and m_2, are constructed as

$$m_1 : m_1\{1,2,3,4\} = 1;$$

$$m_2 : m_2\{4,5\} = 0.8, m_2\{3,6\} = 0.2,$$

where $\Theta = \{1,2,3,4,5,6\}$ is the frame of evidence discernment.

According to Eq. (8), A is any nonempty subset of Θ; i.e., $A = \{1\},\cdots,\{6\},\{1,2\},\cdots,\{5,6\}, \cdots,\{1,2,3,4,5,6\}$. There are $2^{|\Theta|}-1$ such subsets. Thus, to calculate $difBetP_{m_1}^{m_2}$ in the above example, we must compute the value of $\left|BetP_{m_1}(A) - BetP_{m_2}(A)\right|$ a total of 63 times. After comparing the computed results, we find that $\left|BetP_{m_1}(A) - BetP_{m_2}(A)\right|$ is maximized when $A = \{4,5,6\}$. Thus, according to Eq. (8), we have

$$difBetP_{m_1}^{m_2} = \max_{A\subseteq\Theta}\left(\left|BetP_{m_1}(A) - BetP_{m_2}(A)\right|\right) = \left|BetP_{m_1}(\{4,5,6\}) - BetP_{m_2}(4,5,6)\right| = \left|0.25-0.9\right|$$

$= 0.65.$ We now simplify Eq. (8) and make it more intuitive.

Proposition 1. Let $difBetP_{m_i}^{m_j}$ be the pignistic probability distance of two bbas, m_i and m_j. Then,

$$difBetP_{m_i}^{m_j} = \frac{1}{2}\sum_{\chi\in\Theta}\left|BetP_{m_i}(\chi) - BetP_{m_j}(\chi)\right|, \tag{9}$$

where Θ is the frame of evidence discernment.

The proof of this proposition is provided in the appendix. The computational process of Eq. (9) is simple and convenient. According to Eq. (9), the computed result for the above example is

$$difBetP_{m_1}^{m_2} = \frac{1}{2}(|0.25-0|+|0.25-0|+|0.25-0.1|+|0.25-0.4|+|0-0.4|+|0-0.1|) = 0.65.$$

Obviously, the two results given above are the same. Nonetheless, it is much easier to compute the pignistic probability distance using Eq. (9) than Eq. (8).

According to the definition of the pignistic probability distance, we can state the following proposition.

Proposition 2. Let $difBetP_{m_i}^{m_j}$ be the pignistic probability distance of two bbas m_i and m_j. Then, $0 \le difBetP_{m_i}^{m_j} \le 1$.

The proof of this proposition is provided in the appendix.

3. Evidence weight

In real-world decision making, using Dempster's rule will lead to unreasonable and counter-intuitive results when conflicting pieces of evidence are combined, as illustrated by Zadeh (1986). The following example highlights the problem.

Example 1. Suppose that two bbas, m_1 and m_2 are constructed as

$$m_1 : m_1(A) = 0.99, m_1(B) = 0.01, m_1(C) = 0.00; \quad m_2 : m_2(A) = 0.00, m_2(B) = 0.01, m_2(C) = 0.99.$$

According to Dempster's rule, the combined results are

$$m_{12} : m_{12}(A) = 0.00, m_{12}(B) = 1.00, m_{12}(C) = 0.00.$$

This indicates that, if any piece of evidence does not support a proposition, the proposition will be completely ruled out. Obviously, such a result contradicts human intuition; nevertheless, disputes exist about solving this counterintuitive problem. Some researchers believe the problem is caused by Dempster's rule and the combined rule should be modified (Lefevre, et al., 2009; Smarandache and Dezert, 2009; Yager, 1987; Yamada, 2008). Others argue that the problem is due to the data sources, and each piece of evidence should be modified by an evidence weight (Smarandache, et al., 2010; Martin, et al., 2008; Murphy, 2000; Han, et al., 2013).

In fact, the concept of evidence weights was first analyzed in a direct and intuitive manner in Shafer's book (Shafer, 1976). Specifically, evidence weights were translated into simple support functions. Haenni (2002) argued that it is more reasonable to modify evidence, regardless of whether it is from the viewpoint of engineering practices, mathematics, or philosophical logic. In many real-world applications, not all pieces of evidence are reliable, and unreliable evidence may lead to incorrect decisions. Thus, it is necessary to preprocess evidence according to its weight in the combination process (Yang and Xu, 2013). For convenience, methods based on the modification of original evidence are defined as discounting combined methods.

To provide a more reasonable combined result, Yang and Xu (2013) reanalyzed the importance of evidence weight and examined several typical discounting combined methods. Consequently, they proposed the ER method. The ER method can successfully solve the counterintuitive problem described above when combining multiple pieces of evidence. If we assume that the relative weights of m_1 and m_2 are equal, i.e., $w_1 = 0.5, w_2 = 0.5$, the combined results in the above example are $m_{12}^{'} : m_{12}^{'}(A) = 0.495, m_{12}^{'}(B) = 0.01, m_{12}^{'}(C) = 0.495$.

By comparing m_{12} with $m_{12}^{'}$, it is clear that the combined results are counterintuitive according to Dempster's rule and reasonable according to the ER algorithm. However, the combined results are different when the pieces of evidence are combined with different weights, as shown in Figure 1. It can be concluded that the combined results produced by the ER method are largely influenced by the evidence weight. Unfortunately, this method only describes the evidence combination process; it does not provide a measure for confirming whether the applied weights are reasonable.

To date, several methods have been proposed to determine appropriate evidence weights. These can be categorized into two groups. The first adopts the decision-making concept of "placing the overall benefit above all else" (Chen and Wang, 2014; Wang, et al., 2006). It determines the evidence weights to minimize the differences in the overall modified evidence. As long as a piece of evidence is conducive to this goal, it will be assigned a relatively larger weight; otherwise, it will be assigned a smaller weight.

The second group adopts the "majority rules" decision-making concept (Ye, et al., 2006; Deng, et al., 2004; Guo and Li, 2011; Lu, et al., 2008). Accordingly, the closer a piece of evidence is to the majority of all other evidence, the larger is the weight that will be assigned. Otherwise, a smaller weight will be assigned. These research results can be used to determine the evidence weights in the ER approach. However, they do not consider the personal preferences of the decision-makers. Therefore, it is necessary to improve the original ER approach.

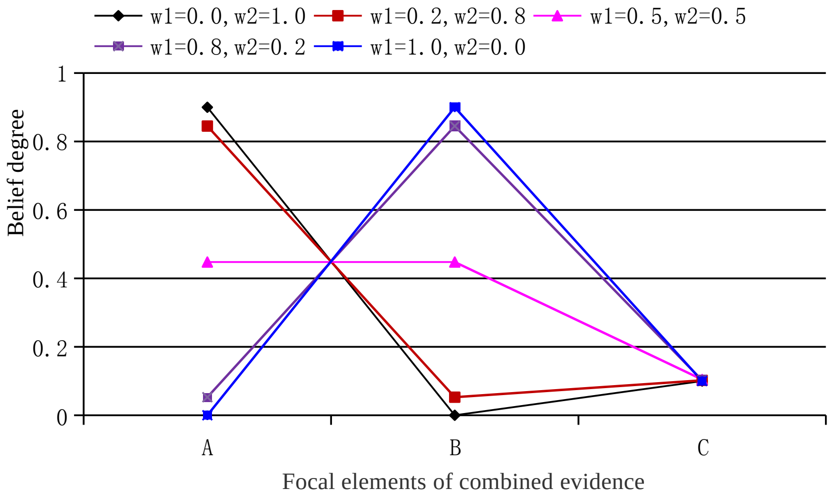

Fig. 1: Combined results generated using the ER algorithm

4. ERBA approach

To confirm the determination of evidence weights that are most reasonable, we extend the ER algorithm to the ERBA approach. In the latter approach, all pieces of evidence are classified into different alliances. We develop optimization models for intra- and inter-alliance negotiation. These models provide the relative weights for modifying the pieces of evidence so they can be more reasonably combined.

4.1. Classifying evidence into different alliances

Because the results are interest-driven, it is impossible for the decision-maker to have an unbiased, isolated perspective on the evidence. In terms of their personal benefit (individual interest), decision-makers desire to obtain more similar pieces of evidence to support their own evidence. That is, they strive for all pieces of evidence modified by relative weights to be as close as possible to their own evidence. In terms of their common benefit (overall interest), decision-makers reduce this discrepancy by modifying the evidence with relative weights. In other words, the decision-makers' common interest is to minimize the discrepancy of the modified evidence. However, the personal benefit of the decision-maker is often contradictory to the overall group interest. Thus, it is necessary to solve the problem of making the correct decision. An effective means of solving this problem is to ensure that the individual interests of the group are satisfied as much as possible when they are consistent.

In contrast, when a conflict exists in the individual interests, the overall interest should be satisfied as much as possible. Therefore, when the discrepancy among pieces of evidence is within some tolerance value range of the decision-maker, the pieces of evidence are classified into a single group and form an alliance. On the other hand, when the discrepancy among pieces of evidence exceeds the tolerance value of the decision-maker, the pieces of evidence are classified into different groups and form other alliances. Because all pieces of evidence within an alliance are similar, the interests of each decision-maker within the alliance are similar. Therefore, the "individual interest" strategy is used to negotiate the respective evidence weights. Conversely, inter-alliance evidence tends to have relatively larger discrepancies, indicating a conflict of overall interest between alliances. In this case, the "overall interest" strategy is adopted to negotiate the respective evidence weights.

Let ε be the tolerance threshold of evidence discrepancy (i.e., the proximity of evidence items in terms of similarity, defined here as "distance") for decision-makers. According to Proposition 2, the maximum value of the pignistic probability distance is 1. Thus, the value of ε is between 0 and 1. Because there is no existing "absolute meaningful threshold" of discrepancy tolerance that can satisfy all the decision-makers, the choice of ε is determined by all included decision-makers, or a subset of them in practical applications. In general, the closer ε is to 1, the higher is the discrepancy tolerance.

When the evidence of two different decision-makers is sufficiently close, i.e., $difBetP \le \varepsilon$, the implication is that the decision-makers have common interests. Thus, the evidence is classified into the same alliance. In contrast, if the distance satisfies $difBetP > \varepsilon$, the decision-makers are assumed to have few common interests. Thus, the evidence is classified into different alliances. The steps involved in computing the evidence alliances can be described as follows. Assume there are N pieces of evidence, $m_1, m_2, \cdots, m_N$. Let each piece of evidence be classified into a single group to form $\{m_1\},\{m_2\},\cdots,\{m_N\}$. Note that "{}" is the symbol of a group. For example, {m_1, m_2} means that m_1, m_2 are in the same group.

Step 1: Identify the number of groups. If the number of groups is greater than 1, proceed to Step 2; otherwise, proceed to Step 4.

Step 2: Calculate the distance between all pairs of groups, and identify the pair with the minimum distance (note: the distance value between two groups is equal to the maximum value of the distance between each piece of evidence in one group to a piece of evidence in the other group). If the calculated distance between these two groups is less than or equal to ε, proceed to Step 3; otherwise, proceed to Step 4.

Step 3: Merge the two groups into a single group, reduce the number of groups by 1, and return to Step 1.

Step 4: End the process and obtain the possible combinations of evidence, i.e., the evidence alliances.

The following example illustrates the detailed evidence alliance process.

Example 2. Suppose that two bbas, m_1 and m_2, are constructed as

$$m_1: m_1(A)=0.40, m_1(B)=0.40, m_1(C)=0.20;$$
$$m_2: m_2(A)=0.35, m_2(B)=0.45, m_2(C)=0.20;$$
$$m_3: m_3(A)=0.20, m_3(B)=0.70, m_3(C)=0.10;$$
$$m_4: m_4(A)=0.00, m_4(B)=0.80, m_4(C)=0.20;$$
$$m_5: m_5(A)=0.80, m_5(B)=0.10, m_5(C)=0.10.$$

Let the tolerance value of the decision-maker after negotiation be $\varepsilon=0.25$; thus, the initial grouping is $\{m_1\},\{m_2\},\{m_3\},\{m_4\},\{m_5\}$. Among the five groups, those that are the minimum distance apart are $\{m_1\}$ and $\{m_2\}$. The distance between these groups is 0.1. As $\varepsilon>0.1$, $\{m_1\}$ and $\{m_2\}$ are merged into a single group, and the grouping becomes $\{m_1,m_2\},\{m_3\},\{m_4\},\{m_5\}$. Among the remaining four groups, $\{m_3\}$ and $\{m_4\}$ are closest. As their distance is 0.2 and $\varepsilon>0.2$, they are merged into one group, and the grouping becomes $\{m_1,m_2\},\{m_3,m_4\},\{m_5\}$. Of these three groups, $\{m_1,m_2\}$ and $\{m_3,m_4\}$ have the smallest distance of 0.40; however, as $\varepsilon<0.40$, the two groups cannot be merged. At this point, the merging process ends, and the final evidence alliances are $\{m_1,m_2\},\{m_3,m_4\},\{m_5\}$.

4.2. Optimization model for intra-alliance negotiation

In terms of the "psychology of economic man," the decision-maker's individual interest is to make all pieces of evidence as close as possible to his/her own by modifying them with relative weights. In other words, the individual interest of the decision-maker who gives the i^{th} piece of evidence in the l^{th} alliance is to minimize the distance,

$$\sum_{k=1}^{N^{(l)}} difBetP_{m_i^{(l)}}^{M_k^{(l)}},$$

where $m_i^{(l)}$ is the i^{th} piece of evidence in the l^{th} alliance, and $M_k^{(l)}$ is the evidence produced by modifying $m_i^{(l)}$ with the negotiation-based weight, $w_i^{(l)}$. To maximize the individual interests of all included decision-makers, the negotiation objective is to minimize in the negotiation process:

$$\sum_{i=1}^{N^{(l)}} \sum_{k=1}^{N^{(l)}} difBetP_{m_i^{(l)}}^{M_k^{(l)}}. \tag{10}$$

The decision-maker is only willing to negotiate with others when the sum of the distances between the modified evidence and his own following negotiation is not greater than that without negotiation. Thus, the constraint condition

$$\sum_{k=1}^{N^{(l)}} difBetP_{m_i^{(l)}}^{\overline{M}_k^{(l)}} \geq \sum_{k=1}^{N^{(l)}} difBetP_{m_i^{(l)}}^{M_k^{(l)}} \tag{11}$$

must be satisfied in the negotiation process, where $\overline{M}_k^{(l)}$ is the evidence produced by modifying $m_i^{(l)}$ with the average weight $1/N$.

In reality, the decision-makers may not be willing to negotiate with each other. In other words, some of them may play an active role in all negotiations, whereas others may engage in partial negotiation or no negotiation. Let $N^{(l)}$ be the number of pieces of evidence in the l^{th} alliance, let $a_i^{(l)}$ $(1 \leq i \leq N^{(l)})$ be the negotiation part of the weight given by the decision-maker of the i^{th} piece of evidence in the l^{th} alliance, and let $1/N$ be the average weight. Thus, the second constraint

$$\left| w_i^{(l)} - 1/N \right| \leq a_i^{(l)}, i \in \{1, 2, \ldots, N^{(l)}\} \tag{12}$$

must be satisfied in the negotiation process. Here, the value of $a_i^{(l)}$ ($a_i^{(l)} \leq 1/N$) is determined by the personal preferences of the decision-makers. Higher values of $a_i^{(l)}$ imply that the decision-makers are more willing to negotiate. If $a_i^{(l)} = 0$, the decision-maker will not negotiate, whereas if $a_i^{(l)} = 1/N$, the decision-maker will consider all parts of the original weight in the negotiation.

In addition, the relative weights must be normalized. Thus, the third constraint,

$$\sum_{i=1}^{N^{(l)}} w_i^{(l)} = 1, \quad w_1^{(l)}, w_2^{(l)}, \ldots, w_{N^{(l)}}^{(l)} \geq 0, \tag{13}$$

must be satisfied in the negotiation process.

From Eqs. (10) to (13), we obtain the following definition.

Definition 4 (Intra-alliance negotiation model). Suppose that $m_i^{(l)}$ is the i^{th} piece of evidence in the l^{th} alliance, $M_k^{(l)}$ is the evidence produced by modifying $m_i^{(l)}$ with negotiation-based weight $w_i^{(l)}$, and $\overline{M}_k^{(l)}$ is the evidence produced by modifying $m_i^{(l)}$ with the average weight $1/N$. The optimization model for intra-alliance negotiation is then defined as

$$\min \sum_{i=1}^{N^{(l)}} \sum_{k=1}^{N^{(l)}} difBetP_{m_i^{(l)}}^{M_k^{(l)}}$$

$$s.t. \begin{cases} \sum_{k=1}^{N^{(l)}} difBetP_{m_i^{(l)}}^{\overline{M}_k^{(l)}} \ge \sum_{k=1}^{N^{(l)}} difBetP_{m_i^{(l)}}^{M_k^{(l)}}, \quad i \in \{1, 2, \ldots, N^{(l)}\}, \\ \left| w_i^{(l)} - 1/N \right| \le a_i^{(l)}, \quad i \in \{1, 2, \ldots, N^{(l)}\}, \\ \sum_{i=1}^{N^{(l)}} w_i^{(l)} = 1, \quad w_1^{(l)}, w_2^{(l)}, \ldots, w_{N^{(l)}}^{(l)} \ge 0, \end{cases} \tag{14}$$

where $l = 1, 2, \cdots, R$. Here, R is the total number of alliances and $N^{(l)}$ is the number of pieces of evidence in the l^{th} alliance. Note that the variables to be optimized are $w_1^{(l)}, w_2^{(l)}, \cdots, w_{N^{(l)}}^{(l)}$ and the value of $a_i^{(l)} (a_i^{(l)} \le 1/N)$ is subjective; i.e., it is determined by the personal preferences of the decision-makers.

According to Eq. (9), the model in (14) can be expressed as follows:

$$\min \frac{1}{2} \sum_{i=1}^{N^{(l)}} \sum_{k=1}^{N^{(l)}} \sum_{j=1}^{|\Theta|} | BetP_{m_i^{(l)}}(\theta_j) - BetP_{M_k^{(l)}}(\theta_j) |$$

$$s.t. \begin{cases} \sum_{k=1}^{N^{(l)}} \sum_{j=1}^{|\Theta|} \left| BetP_{m_i^{(l)}}(\theta_j) - BetP_{\overline{M}_k^{(l)}}(\theta_j) \right| \ge \sum_{k=1}^{N^{(l)}} \sum_{j=1}^{|\Theta|} \left| BetP_{m_i^{(l)}}(\theta_j) - BetP_{M_k^{(l)}}(\theta_j) \right|, i \in \{1, 2, \ldots, N^{(l)}\}, \theta_j \in \Theta \\ \left| w_i^{(l)} - 1/N \right| \le a_i^{(l)}, \quad i \in \{1, 2, \ldots, N^{(l)}\}, \\ \sum_{i=1}^{N^{(l)}} w_i^{(l)} = 1, \quad w_1^{(l)}, w_2^{(l)}, \ldots, w_{N^{(l)}}^{(l)} \ge 0, \end{cases} \tag{15}$$

where $BetP_{m_i^{(l)}}$, $BetP_{\overline{M}_i^{(l)}}$, and $BetP_{M_i^{(l)}}$ are the results of three pignistic transformations from bbas $m_i^{(l)}$, $\overline{M}_i^{(l)}$, and $M_i^{(l)}$, respectively.

According to Eqs. (6) and (7), model (15) can be expressed as follows:

$$\min \frac{1}{2} \sum_{i=1}^{N^{(l)}} \sum_{k=1}^{N^{(l)}} \sum_{j=1}^{|\Theta|} | Z_{ij}^{(l)} - Z_{kj}^{(l)} w_{kj}^{(l)} |$$

$$
s.t.\begin{cases}\sum_{k=1}^{N^{(l)}}\sum_{j=1}^{|\Theta|}\left|z_{ij}^{(l)}-z_{kj}^{(l)}*1/N^{(l)}\right|\geq\sum_{k=1}^{N^{(l)}}\sum_{j=1}^{|\Theta|}\left|z_{ij}^{(l)}-z_{kj}^{(l)}w_k^{(l)}\right|,\ i\in\{1,2,\ldots,N^{(l)}\},\\ \left|w_i^{(l)}-1/N\right|\leq a_i^{(l)},\quad i\in\{1,2,\ldots,N^{(l)}\},\\ \sum_{i=1}^{N^{(l)}}w_i^{(l)}=1,w_1^{(l)},w_2^{(l)},\ldots,w_{N^{(l)}}^{(l)}\geq 0.\end{cases}\tag{16}
$$

Here, $z_{ij}^{(l)}=\sum_{\theta_j\in\theta\subset\Theta}\frac{m_{\theta,i}^{(l)}}{|\theta|}-\frac{\sum_{\theta\subset\Theta}m_{\theta,i}^{(l)}}{|\Theta|})$ and $z_{kj}^{(l)}=\sum_{\theta_j\in\theta\subset\Theta}\frac{m_{\theta,k}^{(l)}}{|\theta|}-\frac{\sum_{\theta\subset\Theta}m_{\theta,k}^{(l)}}{|\Theta|})$.

Model (16) can be simplified to

$$
\min\sum_{i=1}^{N^{(l)}}\sum_{k=1}^{N^{(l)}}\sum_{j=1}^{|\Theta|}(Z_{ij}^{(l)}-Z_{kj}^{(l)}w_{kj}^{(l)})^2
$$

$$
s.t.\begin{cases}\sum_{k=1}^{N^{(l)}}\sum_{j=1}^{|\Theta|}(z_{ij}^{(l)}-z_{kj}^{(l)}*1/N^{(l)})^2\geq\sum_{k=1}^{N^{(l)}}\sum_{j=1}^{|\Theta|}(z_{ij}^{(l)}-z_{kj}^{(l)}w_k^{(l)})^2,\ i\in\{1,2,\ldots,N^{(l)}\},\\ \left|w_i^{(l)}-1/N\right|\leq a_i^{(l)},\quad i\in\{1,2,\ldots,N^{(l)}\},\\ \sum_{i=1}^{N^{(l)}}w_i^{(l)}=1,\quad w_1^{(l)},w_2^{(l)},\ldots,w_{N^{(l)}}^{(l)}\geq 0.\end{cases}\tag{17}
$$

Such nonlinear programming models can be easily implemented using existing optimization software packages, such as LINGO or MATLAB Optimization Toolbox. From the above model, we can obtain the negotiation weights, $w_1^{(l)},w_2^{(l)},\ldots,w_{R^{(l)}}^{(l)}$. The pieces of evidence $m_1^{(l)},m_2^{(l)},\ldots,m_{R^{(l)}}^{(l)}$ can then be combined into the aggregated evidence $m^{(l)}$ using Eqs. (1) to (3).

4.3. Optimization model for inter-alliance negotiation

Unlike the intra-alliance evidence, inter-alliance evidence has a large discrepancy, indicating that the individual interests of the decision-makers are in conflict. It is crucial that the decision-makers reduce this discrepancy by modifying the evidence with relative weights. In other words, the decision-makers' common interest is to minimize the discrepancy of the modified evidence. To maximize the common interests of all the included decision-makers, the negotiation objective is to minimize

$$
\sum_{i=1}^{R}\sum_{k=1}^{R}difBetP_{M^{(i)}}^{M^{(k)}}\tag{18}
$$

in the negotiation process. Here, $M^{(i)}$ and $M^{(k)}$ represent the evidence produced by modifying $m^{(i)}$ and $m^{(k)}$ with negotiation-based weights $\xi^{(i)}$ and $\xi^{(k)}$, respectively.

In reality, the decision-makers may not be willing to negotiate with each other. In other words, some of them may play an active part in all negotiations, whereas others may engage in partial negotiation or no negotiation. It follows that the negotiation-based weights are restricted by the personal preferences of the decision-makers.

Let $b^{(l)}$ be the original weight, which is the ratio of the number of pieces of evidence of the l^{th} alliance to that of all of the alliances. It can be shown that

$$b^{(l)} = \frac{N^{(l)}}{\sum_{i=1}^{R} N^{(i)}} \quad l = 1, 2, \cdots, R,$$

where $N^{(l)}$ is the number of pieces of evidence of the l^{th} alliance and R is the total number of alliances. Let $a^{(l)}$ be the negotiation part of the weight given by the decision-makers of the l^{th} alliance. Thus, the first constraint

$$\left|\xi^{(l)} - b^{(l)}\right| \le a^{(l)} \tag{19}$$

must be satisfied in the negotiation process. Here, the value of $a^{(l)}$ ($a^{(l)} \le b^{(l)}$) is determined by the personal preferences of the decision-makers. Higher values of $a^{(l)}$ imply that the decision-makers are more willing to negotiate. If $a^{(l)} = 0$, the decision-maker will not participate in the negotiation, whereas if $a^{(l)} = b^{(l)}$, the decision-maker will consider all parts of the original weight in the negotiation.

In addition, the relative weights must be normalized. Thus, the second constraint,

$$\sum_{i=1}^{R} \xi^{(i)} = 1, \xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(R)} \ge 0 \quad , \tag{20}$$

must be satisfied in the negotiation process.

From Eqs. (18) to (20), we obtain the following definition.

Definition 5 (Inter-alliance negotiation model). Suppose that $M^{(i)}$ and $M^{(k)}$ represent the evidence produced by modifying $m^{(i)}$ and $m^{(k)}$ with negotiation-based weights $\xi^{(i)}$ and $\xi^{(k)}$, respectively. The optimization model for inter-alliance negotiation is then defined as follows:

$$\begin{aligned} &\min \sum_{i=1}^{R} \sum_{k=1}^{R} difBetP_{M^{(i)}}^{M^{(k)}} \\ &s.t. \begin{cases} \left|\xi^{(l)} - b^{(l)}\right| \le a^{(l)}, l \in L, L = \{1, 2, \cdots, R\}, \\ \sum_{i=1}^{R} \xi^{(i)} = 1, \xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(R)} \ge 0, \end{cases} \end{aligned} \tag{21}$$

where R is the total number of alliances. Note that the variables to be optimized are $\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(R)}$.

According to Eq. (7), model (21) can be expressed as

$$\min \frac{1}{2}\sum_{i=1}^{R}\sum_{k=1}^{R}\sum_{j=1}^{|\Theta|}\left|\xi^{(i)}z_j^{(i)} - \xi^{(k)}z_j^{(k)}\right|$$

$$s.t.\begin{cases}\left|\xi^{(l)} - b^{(l)}\right| \le a^{(l)}, l \in L, \\ \sum_{i=1}^{R}\xi^{(i)} = 1, \xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(R)} \ge 0,\end{cases} \tag{22}$$

where $z_j^{(i)} = \sum_{\theta_j \in \theta \subset \Theta} \frac{m_\theta^{(i)}}{|\theta|} - \frac{\sum_{\theta\subset\Theta} m_\theta^{(i)}}{|\Theta|}$ and $z_j^{(k)} = \sum_{\theta_j \in \theta \subset \Theta} \frac{m_\theta^{(k)}}{|\theta|} - \frac{\sum_{\theta\subset\Theta} m_\theta^{(k)}}{|\Theta|}$.

The model in (22) can be simplified to

$$\min \sum_{i=1}^{R}\sum_{k=1}^{R}\sum_{j=1}^{|\Theta|}(\xi^{(i)}z_j^{(i)} - \xi^{(k)}z_j^{(k)})^2$$

$$s.t.\begin{cases}\left|\xi^{(l)} - b^{(l)}\right| \le a^{(l)}, l \in L, \\ \sum_{i=1}^{R}\xi^{(i)} = 1, \xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(R)} \ge 0.\end{cases} \tag{23}$$

This nonlinear programming model can again be implemented using LINGO or MATLAB Optimization Toolbox. From model (23), the relative weights $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(R)}$ can be obtained, and the pieces of evidence can be combined into the aggregated evidence m using Eqs. (1) to (3).

5. Experiments

The ERBA approach was established on the basis of the optimization models of intra- and inter-alliance negotiation. Thus, it was necessary to analyze the rationality of the two optimization models. Numerical and simulation studies were therefore conducted to verify the features of the ERBA approach.

As discussed in Section 4, the closer ε is to 1, the higher is the discrepancy tolerance. When $\varepsilon = 1$, any evidence discrepancy (distance) is acceptable to decision-makers. In this special case, all the pieces of evidence are classified in the same alliance. The two optimization models are then reduced to the following model for intra-alliance negotiation:

$$\min \sum_{i=1}^{N}\sum_{k=1}^{N} difBetP_{m_i}^{M_k}$$

$$s.t.\begin{cases}\sum_{k=1}^{N} difBetP_{m_i}^{\overline{M}_k} \ge \sum_{k=1}^{N} difBetP_{m_i}^{M_k}, i = 1, 2, \cdots, N, \\ \left|w_i - 1/N\right| \le a_i, \quad i = 1, 2, \cdots, N, \\ \sum_{i=1}^{N} w_i = 1.\end{cases} \tag{24}$$

When $\varepsilon = 0$, no evidence discrepancy is acceptable to decision-makers. In this case, each piece of evidence is classified in a single alliance, and the two optimization models are reduced to

$$\min \sum_{i=1}^{N}\sum_{k=1}^{N} difBetP_{M^{(i)}}^{M^{(k)}}$$
$$s.t.\begin{cases} \left|\xi^{(i)} - 1/N\right| \le a^{(i)}, i \in 1,2,\cdots,N, \\ \sum_{i=1}^{N}\xi^{(i)} = 1, \xi^{(1)},\xi^{(2)},\cdots,\xi^{(N)} \ge 0. \end{cases} \tag{25}$$

Here, recall that $M^{(i)}, M^{(k)}$ denote the evidence produced by modifying $m^{(i)}, m^{(k)}$ with negotiation-based weights $\xi^{(i)}, \xi^{(k)}$, respectively.

The two negotiation models have the following advantages in common. (1) Prior to negotiation, the average weight of each piece of evidence is considered the original weight, thereby reflecting fairness among all pieces of evidence. In addition, based on their own risk preferences, decision-makers can assign different negotiation factors, such as full participation, partial participation, or non-participation in the negotiation. This process fully respects the will of each decision-maker. (2) The objectives and constraints of both models are straightforward, which ensures they are acceptable to the decision-makers. Moreover, existing software, such as LINGO and MATLAB, greatly enhances the practicality of this method.

The unique advantages of each negotiation model are as follows. Model (24) uses the decision-making concept of "placing individual interest above all else." It assigns negotiated weights based on the goal of minimizing the difference between the modified evidence and the decision-maker's own evidence. As long as the evidence is conducive to this goal, it is assigned a larger weight, and vice versa. Model (25) adopts the decision-making concept of "placing overall interest above all else." It assigns evidence weights based on the goal of minimizing the overall distance between the modified evidence. Evidence that is in favor of this goal is assigned a greater weight, and vice versa. In other words, model (24) considers the "individual interest" of the decision-maker, rather than the "overall interest" of the group, whereas the opposite is true for model (25).

We now present definitions that are used for subsequent discussion. Let $\overline{M}_i$ be the modified evidence of m_i, which was adjusted according to the original weight, and let M_i be the modified evidence of m_i, which was adjusted according to the negotiation weight. The distance from m_i to the modified evidence $\overline{M}_1, \overline{M}_2, \cdots, \overline{M}_N$ is then $\sum_{j=1}^{N} difBetP_{m_i}^{\overline{M}_j}$, and the distance to the modified evidence $M_1, M_2, \cdots, M_N$ is $\sum_{j=1}^{N} difBetP_{m_i}^{M_j}$. The discrepancy between the two distances can be written as

$$\Delta d_i = \sum_{j=1}^{N} difBetP_{m_i}^{\overline{M}_j} - \sum_{j=1}^{N} difBetP_{m_i}^{M_j} .$$

Evidently, when considering the "individual interest," each decision-maker will desire that all modified pieces of evidence are closer to the decision-maker's own original evidence. Thus, a higher value of Δd_i implies that the decision-maker is more willing to negotiate. A value of $\Delta d_i \geq 0$ after negotiation indicates an increase in "individual interest" for the i^{th} decision-maker, and vice versa. However, from the viewpoint of the "overall interest," the decision-makers desire to minimize the difference between the overall modified evidence, i.e., minimize the sum of the distances among all of modified pieces of evidence, which can be defined as

$$\Delta D = \sum_{i=1}^{N} \sum_{j=1}^{N} difBetP_{M_i}^{M_j} .$$

The following example illustrates the advantages of both negotiation models.

Example 3: Consider a frame of discernment $\Theta = \{A, B, C\}$ in which $A, B,$ and C are mutually exclusive and collectively exhaustive. Eight experts, $e_1, e_2, \cdots, e_8$, provide eight bbas of independent evidence $s_1, s_2, \cdots, s_8$, respectively, as listed in Table 1.

Table 1: Eight pieces of independent evidence

	A	B	C	A,B	A,C	B,C	Θ
s_1	0.5	0.1	0.1	0.2			0.1
s_2	0.5	0.2		0.2			0.1
s_3		0.7				0.3	
s_4	0.1	0.3	0.2	0.2			0.2
s_5		0.7	0.1		0.2		
s_6	0.1	0.3	0.2	0.2			0.2
s_7	0.5	0.1	0.1		0.3		
s_8	0.5		0.2		0.3		

To ensure fairness, the original weight of each piece of evidence is set as the average weight, i.e., $1/8$. Let the five vectors of the negotiation factors provided by the decision-makers be $\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4, \mathrm{a}_5$. These vectors are listed below:

a_1 : 0.00, 0.00, 0.00, 0.00, 0.00, 0.00, 0.00, 0.00; a_2 : 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08;
a_3 : 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.10, 0.11; a_4 : 0.10, 0.09, 0.08, 0.07, 0.09, 0.10, 0.11, 0.12;

a_5 : 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125;

where a_1 indicates that none of the original weights is included in the negotiations; a_2, a_3, a_4 indicate that only some of the original weights are included in the negotiations; and a_5 indicates that all of the original weights are included in the negotiations.

(1) Results comparison with different negotiation factors

First, the evidence weights with different negotiation factors were calculated according to models (24) and (25). The ER algorithm was then used to combine the evidence. The results of this combination are shown in Figures 2 and 3, where it is evident that the combination results given by different negotiation factors are different. For example, according to Figure 2, with negotiation vectors a_1, a_2, a_3, the belief degree of focal element B is higher than that of focal element A. However, with negotiation vectors a_4, a_5, the belief degree of focal element A is higher than that of focal element B. Thus, the combination results are related not only to each piece of evidence, but also to the negotiation factors provided by each decision-maker.

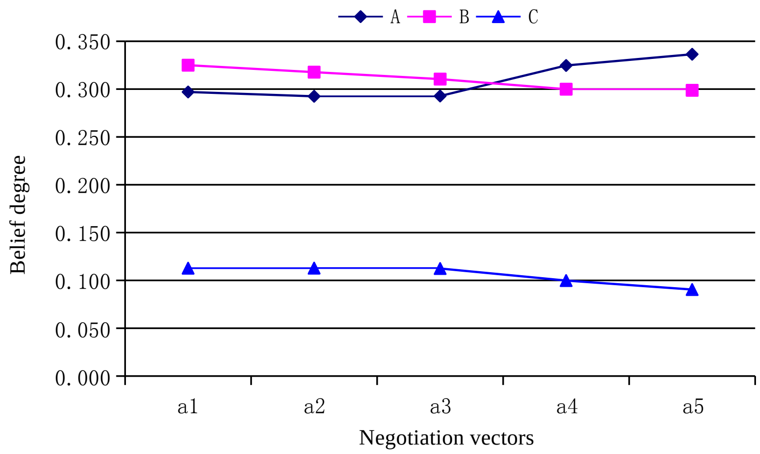

Fig. 2: Comparison of combination results with different negotiation factors using model (24)

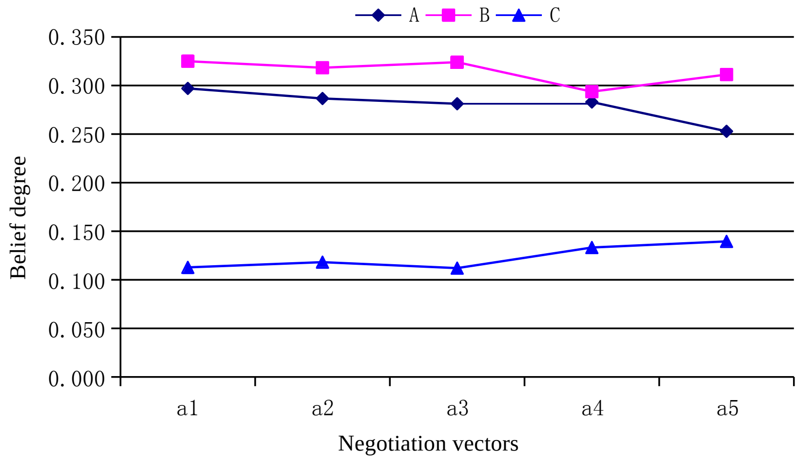

Fig. 3: Comparison of combination results with different negotiation factors using model (25)

(2) Analysis of individual interests of decision-makers with different negotiation factors

We used models (24) and (25) to calculate the negotiation weights of evidence with different negotiation factors, a_1, a_2, a_3, a_4, a_5. We then computed the difference in distances between the evidence, $\Delta d_i (i = 1, 2, \cdots, 8)$. The results are shown in Figures 4 and 5. According to Figure 4, with negotiation factors a_2, a_3, a_4, a_5, the difference in the distance between each piece of evidence is greater than that of a_1, indicating that the negotiation weight calculated according to model (24) increased the "individual interest" of each decision-maker.

However, according to Figure 5, with negotiation factors a_2, a_3, a_4, a_5, the difference in the distance between certain pieces of evidence may be greater than or less than it is for a_1. This indicates that, although the negotiation weights calculated according to model (25) increase the "individual interest" of some decision-makers, they also decrease the "individual interest" of others. Therefore, in terms of the decision-makers' "individual interest," model (24) is preferable to model (25).

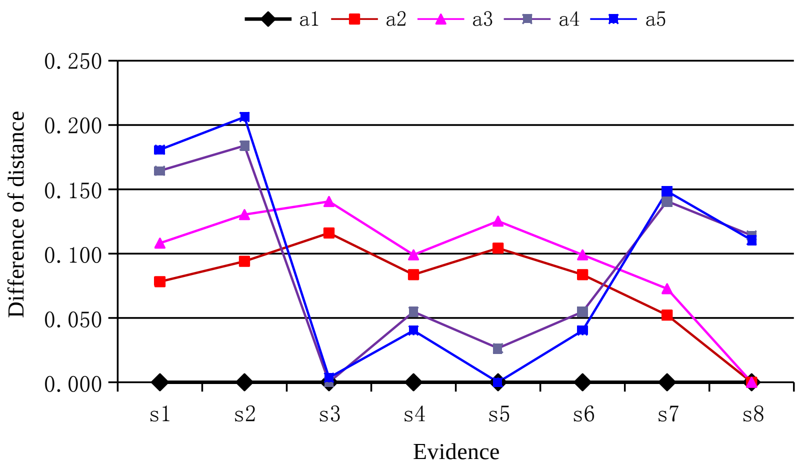

Fig. 4: Comparison of differences in evidence distance with different negotiation factors using model (24)

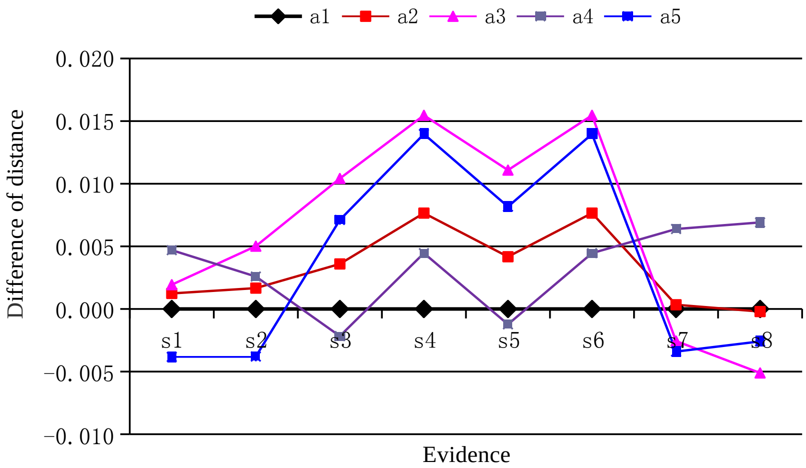

Fig. 5: Comparison of differences of evidence distance with different negotiation factors using model (25)

(3) Analysis of overall interest of decision-makers with different negotiation factors

We used models (24) and (25) to calculate the negotiation weight with different negotiation factors a_1, a_2, a_3, a_4, a_5. We then computed the total distance between all pieces of modified evidence, denoted by ΔD. The results are shown in Figure 6. According to Figure 6, the total distance ΔD calculated according to model (25) with each negotiation factor is always below that of model (24). This indicates that model (25) is more conducive to the "overall interest" of the decision-makers than model (24).

In addition, as the evidence included in the negotiations increases, i.e., $a_1 < a_2 < a_3 < a_4 < a_5$, the total distance ΔD calculated according to model (25) decreases. Thus, the greater the amount of evidence is that comprises the negotiations, the smaller the total distance is according to model (25), and the higher the overall interest is of the decision-makers. However, the total distance according to model (24) does not show a decreasing trend. In some cases, the greater the amount of evidence is that comprises the negotiations, i.e., $a_3 < a_4 < a_5$, the larger is the total distance, which evidently does not satisfy the "overall interest" of the decision-makers. Therefore, in terms of the "overall interest" of the decision-makers, model (25) is preferable to model (24).

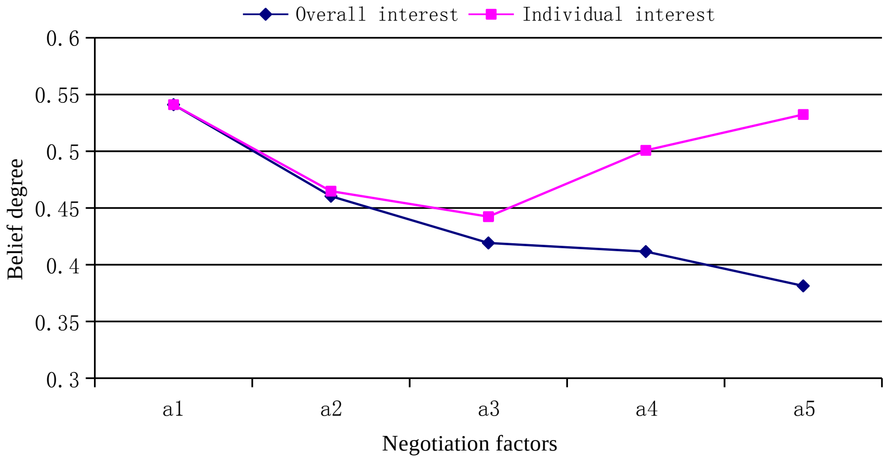

Fig. 6: Comparison of total distance of modified evidence with different negotiation factors using models (24) and (25)

We now consider a typical example (Ye, et al., 2006) for comparative analysis to illustrate the feasibility and rationality of the proposed method.

Example 4: Consider a frame of discernment $\Theta = \{A, B, C\}$, where A, B, and C are mutually exclusive and collectively exhaustive. Five pieces of independent evidence, $m_1, m_2, \cdots, m_5$ are represented by the five bpas listed in Table 2.

Table 2: Bpa value of each piece of evidence

	m_1	m_2	m_3	m_4	m_5
A	0.5	0.0	0.55	0.55	0.55
B	0.2	0.9	0.10	0.10	0.10
C	0.3	0.1	0.35	0.35	0.35

(1) Comparison of combination results of different methods

After calculating the weights of evidence with the existing methods, the five pieces of evidence in Table 1 were combined according to the ER combination rule. The results are presented in Table 3. In Table 2, m_2 has the largest range of values. According to Chen's method (Chen and Wang, 2014) and Wang's method (Wang, et al., 2006) , its relative weight is calculated as zero; thus, its impact on the combination of results is completely eliminated.

However, based on Ye's method (Ye, et al., 2006) , Deng's method (Deng, et al., 2004) , Guo's method (Guo and Li, 2011) , and Lu's method (Lu, et al., 2008) , m_2 is assigned a non-zero weight that is lower than the average. Regardless of the number of pieces of evidence, Table 3 clearly indicates that the belief degree of focal element A is greatest when calculated according to the methods proposed in (Chen and Wang, 2014; Wang, et al., 2006) , followed by the calculations based on the methods proposed in (Ye, et al., 2006; Deng, et al., 2004; Guo and Li, 2011; Lu, et al., 2008). It is smallest when calculated according to the equal weights method.

Table 3: Combination results of different combination methods

	m_1, m_2, m_3	m_1, m_2, m_3, m_4	m_1, m_2, m_3, m_4, m_5
Average weight method	(A,0.358); (B,0.407); (C,0.234)	(A,0.410); (B,0.311); (C,0.279)	(A,0.450); (B,0.264); (C,0.286)
Chen's method	(A,0.539); (B,0.156); (C,0.305)	(A,0.554); (B,0.133); (C,0.313)	(A,0.562); (B,0.120); (C,0.318)
Wang's method	(A,0.539); (B,0.156); (C,0.305)	(A,0.554); (B,0.133); (C,0.313)	(A,0.562); (B,0.120); (C,0.318)
Ye's method	(A,0.362); (B,0.402); (C,0.236)	(A,0.413); (B,0.305); (C,0.281)	(A,0.451); (B,0.262); (C,0.287)
Deng's method	(A,0.465); (B,0.289); (C,0.246)	(A,0.501); (B,0.201); (C,0.298)	(A,0.522); (B,0.167); (C,0.310)
Guo's method	(A,0.465); (B,0.289); (C,0.246)	(A,0.501); (B,0.201); (C,0.298)	(A,0.522); (B,0.167); (C,0.310)
Lu's method	(A,0.458); (B,0.300); (C,0.242)	(A,0.508); (B,0.192); (C,0.300)	(A,0.535); (B,0.151); (C,0.315)

To obtain the different combinations of results using the proposed method under different alliance scenarios, the tolerance level of the evidence distance ε was classified into four ranges: $\varepsilon_1 : \varepsilon = 0$, $\varepsilon_2 : \varepsilon \in (0.0, 0.1]$, $\varepsilon_3 : \varepsilon \in (0.1, 0.8]$, and $\varepsilon_4 : \varepsilon \in (0.8, 1.0]$. The evidence alliances acquired under these ranges are presented in Table 4, where items in parentheses represent intra-alliance evidence. Suppose that each decision-maker is willing to participate in the negotiation with their original evidence weight. The combination results of the proposed method with different ε ranges are listed in Table 5.

At tolerance level ε_1, there is only one piece of evidence within each alliance. In this situation, our method is reduced to that of (Chen and Wang, 2014; Wang, et al., 2006); hence, the combination results are the same. At tolerance levels ε_2 and ε_3, the inter-alliance evidence adopts the decision-making concept of “placing the overall benefit above all else,” which reduces the evidence difference between alliances. Therefore, the calculated belief degree of focal element *A* is greater than that from the methods of (Ye, et al., 2006; Deng, et al., 2004; Guo and Li, 2011; Lu, et al., 2008). Finally, at tolerance level ε_4, all the evidence belongs to the same alliance. Using the proposed method, only the “individual interest” of each decision-maker is considered in this situation. Hence, the influence of m_2, which has a relatively larger difference, is not diminished. Therefore, as in Table 5, the calculated belief degree of focal element *A* is smaller than with tolerance levels of ε_1, ε_2, or ε_3.

Table 4: Evidence alliances based on different tolerance values of evidence distance

	m_1, m_2, m_3	m_1, m_2, m_3, m_4	m_1, m_2, m_3, m_4, m_5
ε_1	$\{m_1\},\{m_2\},\{m_3\}$	$\{m_1\},\{m_2\},\{m_3\},\{m_4\}$	$\{m_1\},\{m_2\},\{m_3\},\{m_4\},\{m_5\}$
ε_2	$\{m_1\},\{m_2\},\{m_3\}$	$\{m_1\},\{m_2\},\{m_3, m_4\}$	$\{m_1\},\{m_2\},\{m_3, m_4, m_5\}$
ε_3	$\{m_1, m_3\},\{m_2\}$	$\{m_1, m_3, m_4\},\{m_2\}$	$\{m_1, m_3, m_4, m_5\},\{m_2\}$
ε_4	$\{m_1, m_2, m_3\}$	$\{m_1, m_2, m_3, m_4\}$	$\{m_1, m_2, m_3, m_4, m_5\}$

Table 5: Combination results based on different tolerance values of evidence distance

	m_1, m_2, m_3	m_1, m_2, m_3, m_4	m_1, m_2, m_3, m_4, m_5
ε_1	(A,0.539); (B,0.156); (C,0.305)	(A,0.554); (B,0.133); (C,0.313)	(A,0.562); (B,0.120); (C,0.318)
ε_2	(A,0.539); (B,0.156); (C,0.305)	(A,0.544); (B,0.156); (C,0.300)	(A,0.545); (B,0.156); (C,0.299)
ε_3	(A,0.552); (B,0.133); (C,0.315)	(A,0.565); (B,0.112); (C,0.323)	(A,0.576); (B,0.092); (C,0.332)
ε_4	(A,0.387); (B,0.358); (C,0.256)	(A,0.431); (B,0.298); (C,0.271)	(A,0.459); (B,0.258); (C,0.283)

(2) Analysis of interests of decision-makers with different methods

The computed results of the distance differences $\Delta d_i (i = 1, 2, \cdots, 5)$ obtained using the existing methods to calculate the weights of evidence are shown in Figure 7. The total distance of the modified evidence ΔD is shown in Figure 8. Figure 7 illustrates that some of the distance differences calculated according to the methods in (Chen and Wang, 2014; Wang, et al., 2006; Ye, et al., 2006; Deng, et al., 2004; Guo and Li, 2011; Lu, et al., 2008) are situated above the X-axis, whereas others are below it. This suggests that the results obtained using these methods cannot satisfy the “individual interest” of all the decision-makers.

As shown in Figure 8, the total distance of modified evidence ΔD is minimized when it is calculated according to the methods in (Chen and Wang, 2014; Wang, et al., 2006). These results are based on the decision-making concept. The methods in (Chen and Wang, 2014; Wang, et al., 2006) adopt the decision-making concept of “placing overall benefit above all else,” and target the minimization of differences in the modified evidence. However, the methods in (Ye, et al., 2006; Deng, et al., 2004; Guo and Li, 2011; Lu, et al., 2008) adopt the “majority rules” decision-making concept, whereby the closer a piece of evidence is to the majority of all other pieces of evidence, the larger is the weight assigned to it, and vice versa.

Based on the above analyses, the methods in (Chen and Wang, 2014; Wang, et al., 2006) considered the “overall interest” but not the “individual interest” of the decision-makers, whereas the methods in (Ye, et al., 2006; Deng, et al., 2004; Guo and Li, 2011; Lu, et al., 2008) considered neither the “overall interest” nor the “individual interest” of the decision-makers.

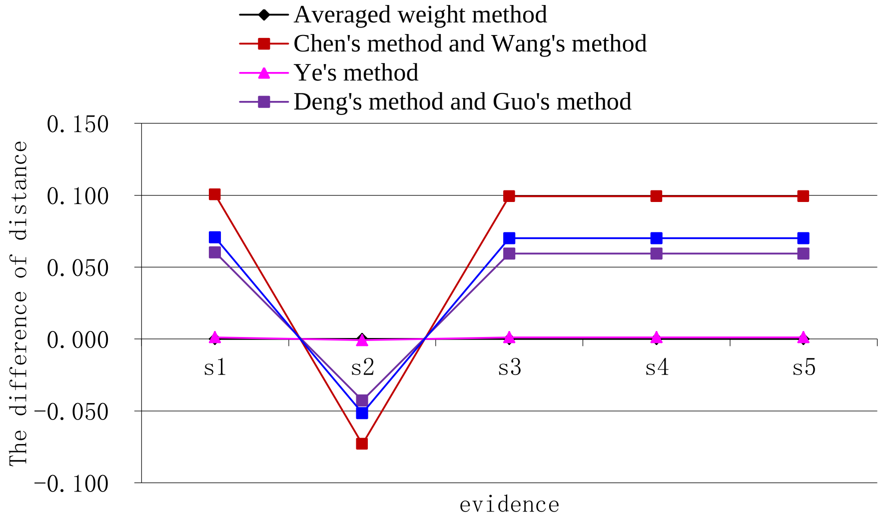

Fig. 7: Comparison of differences of distance between original evidence and modified evidence calculated using different methods

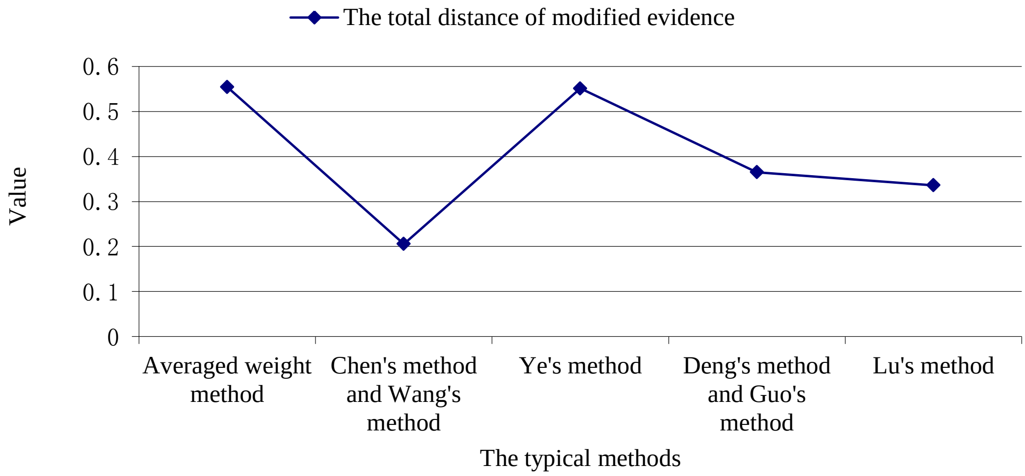

Fig. 8: Total distance of modified evidence calculated using different methods

The computed results of the distance differences $\Delta d_i (i = 1, 2, \cdots, 5)$ obtained using the ERBA method to calculate the weights of evidence with different tolerance levels are shown in Figure 9. The total distance of the modified evidence ΔD is shown in Figure 10. Under tolerance level ε_1, the ERBA method is reduced to the methods proposed in (Chen and Wang, 2014; Wang, et al., 2006), which consider only the "overall interest" of the decision-makers. As apparent in Figure 10, these give the smallest total distance of modified evidence ΔD with tolerance level ε_1. Under tolerance level ε_4, the ERBA method considers only the "individual interest." As shown in Figure 9, the distance differences of evidence m_1, m_3, m_4, and m_5 lie above the X-axis, whereas that of evidence m_2 lies below it with tolerance level $\varepsilon_1, \varepsilon_2$, or ε_3.

However, all the differences in evidence distance lie above the X-axis with the tolerance level ε_4. Under tolerance levels ε_2 or ε_3, the ERBA method considers the "individual interest" of the decision-makers when dealing with intra-alliance evidence, whereas the "overall interest" is considered when dealing with inter-alliance evidence. In other words, it considers not only "individual interest," but also the "overall interest" of the decision-makers.

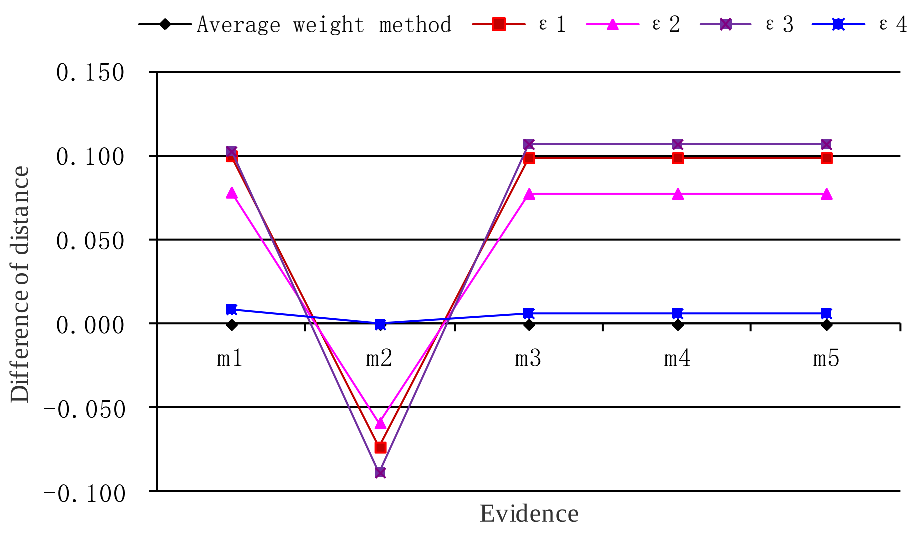

Fig. 9: Comparison of differences of distance between original evidence and modified evidence calculated with different tolerance conditions

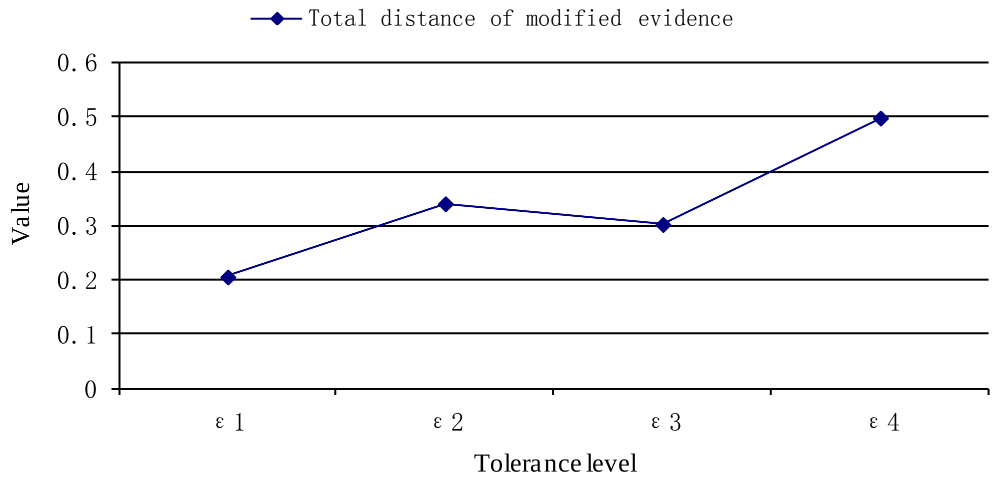

Fig. 10: Comparison of total distance values of modified evidence with different tolerance conditions

Next, we consider an application example for comparative analysis. In this example, the steps involved in computing the evidence alliances have been provided and the advantages of the proposed method are further verified.

Example 5: Eighteen judging panels $s_1, s_2, ..., s_{18}$ were invited to nominate the winner of a singing contest from among three different singers, *A*, *B*, and *C*. Each panel consisted of ten people, and each person could cast one vote. The voting information is given in Table 6. The numerical values in the i^{th} row of the table represent the voter turnout for panel s_i. For example, the first row indicates that five of the ten people in panel s_1 voted for *A*, one person voted for *B*, one person voted for *C*, two people could not decide between *A* and *B* (and are therefore considered to have voted for *A* or *B*), and one person abstained from voting, which is represented as a vote for *A*, *B*, or *C* (denoted by Θ).

Table 6: Voting information of media juries

	A	B	C	A,B	A,C	B,C	Θ
s_1	0.5	0.1	0.1	0.2			0.1
s_2	0.5	0.1				0.2	0.2
s_3		0.6	0.2	0.2			
s_4	0.9			0.1			
s_5	0.4		0.1		0.2		0.3
s_6	0.8	0.1	0.1				

	A	B	C	A,B	A,C	B,C	Θ
s_7	0.4	0.1			0.3		0.2
s_8	0.9				0.1		
s_9	0.2	0.6	0.2				
s_{10}	0.5	0.1	0.1	0.2			0.1
s_{11}	0.8	0.1	0.1				
s_{12}	0.4	0.1			0.3		0.2
s_{13}	0.9				0.1		
s_{14}	0.2	0.6	0.2				
s_{15}	0.5	0.1				0.2	0.2
s_{16}		0.6	0.2	0.2			
s_{17}	0.9			0.1			
s_{18}	0.4		0.1		0.2		0.3

In Table 6, the sum of each row, i.e., the total voting information of each judging panel, is 1, which satisfies the condition for constructing the bba of the evidence stated in Definition 1. In this practical application, the proposed method nominates a winner as follows.

1) Classification of evidence into different alliances.

Let 0.3 be the threshold of evidence distance tolerance for the judging panels. If the distance of evidence given by the panels satisfies $difBetP \leq 0.3$, then they are classified into the same alliance. Otherwise, they are classified into different alliances. Thus, the pieces of evidence given by $s_1, s_2, s_5, s_7, s_{10}, s_{12}, s_{15}, s_{18}$ are classified into the first alliance, the pieces of evidence given by $s_4, s_6, s_8, s_{11}, s_{13}, s_{17}$ are classified into the second alliance, and the pieces of evidence given by s_3, s_9, s_{14}, s_{16} are classified into the third alliance. The classified evidence is listed in Table 7, where $m_i^{(j)}$ represents the i^{th} piece of evidence in the j^{th} alliance.

Table 7: Pieces of evidence in different alliances

	A	B	C	A,B	A,C	B,C	Θ
$m_1^{(1)}$	0.5	0.1	0.1	0.2			0.1
$m_2^{(1)}$	0.5	0.1				0.2	0.2

	A	B	C	A,B	A,C	B,C	Θ
$m_3^{(1)}$	0.4		0.1		0.2		0.3
$m_4^{(1)}$	0.4	0.1			0.3		0.2
$m_5^{(1)}$	0.5	0.1	0.1	0.2			0.1
$m_6^{(1)}$	0.4	0.1			0.3		0.2
$m_7^{(1)}$	0.5	0.1				0.2	0.2
$m_8^{(1)}$	0.4		0.1		0.2		0.3
$m_1^{(2)}$	0.9			0.1			
$m_2^{(2)}$	0.8	0.1	0.1				
$m_3^{(2)}$	0.9				0.1		
$m_4^{(2)}$	0.8	0.1	0.1				
$m_5^{(2)}$	0.9				0.1		
$m_6^{(2)}$	0.9			0.1			
$m_1^{(3)}$		0.6	0.2	0.2			
$m_2^{(3)}$	0.2	0.6	0.2				
$m_3^{(3)}$	0.2	0.6	0.2				
$m_4^{(3)}$		0.6	0.2	0.2			

2) Calculating the relative weights of intra-alliances and combining the evidence

Let $w_i^{(j)}$ represent the relative weight of the i^{th} piece of evidence in the j^{th} alliance. According to optimization model (13), the evidence weights in the three alliances are as follows:

$$(w_1^{(1)}, w_2^{(1)}, w_3^{(1)}, w_4^{(1)}, w_5^{(1)}, w_6^{(1)}, w_7^{(1)}, w_8^{(1)}) = (0.149, 0.085, 0.119, 0.148, 0.149, 0.148, 0.085, 0.119),$$

$$(w_1^{(2)}, w_2^{(2)}, w_3^{(2)}, w_4^{(2)}, w_5^{(2)}, w_6^{(2)}) = (0.185, 0.129, 0.185, 0.185, 0.129, 0.185),$$

$$(w_1^{(3)}, w_2^{(3)}, w_3^{(3)}, w_4^{(3)}) = (0.25, 0.25, 0.25, 0.25).$$

The pieces of evidence in the intra-alliances can then be combined into the aggregated evidence $m^{(1)}, m^{(2)}, m^{(3)}$ using Eqs. (1) to (3). The combined results are given in Table 8.

Table 8: Combined evidence of intra-alliances

	A	B	C	A,B	A,C	B,C	Θ

	A	B	C	A,B	A,C	B,C	Θ
$m^{(1)}$	0.54	0.07	0.06	0.05	0.11	0.03	0.14
$m^{(2)}$	0.91	0.02	0.04		0.03		
$m^{(3)}$	0.08	0.63	0.19	0.11			

3) Calculating the relative weights of inter-alliances and combining the evidence

According to Eq. (14), the original weights among the three alliances are

$b^{(1)} = 8/18, b^{(2)} = 6/18, b^{(3)} = 4/18$.

Suppose that the three negotiation factors given by the decision-makers are

$a^{(1)} = 0.2, a^{(2)} = 0.2, a^{(3)} = 0.2$.

According to optimization model (15), the relative weights are then

$\xi^{(1)} = 0.644, \xi^{(2)} = 0.299, \xi^{(3)} = 0.057$.

The evidence $m^{(1)}, m^{(2)}, m^{(3)}$ can be combined using Eqs. (1) to (3). The combined result is given in Table 9, which indicates that the basic probability assignment of *A* is 0.664, i.e., singer *A* is the winner. From Table 6, we can easily conclude that the average belief assigned to *A* is greater than the belief assigned to *B* or *C*. Thus, it is clear that the combined result obtained by the ERBA approach is consistent with intuitive judgment.

Table 9: Combined evidence of inter-alliances

	A	B	C	A,B	A,C	B,C	Θ
m	0.664	0.068	0.052	0.034	0.077	0.016	0.087

The ERBA approach considers both the evidence difference and the subjectivity of the decision-makers from the viewpoint of negotiation.

Let the negotiation factors of the decision-makers be given by

$X: a^{(1)} = 0, a^{(2)} = 0, a^{(3)} = 0$,

$Y: a^{(1)} = 0.1, a^{(2)} = 0.1, a^{(3)} = 0.1$,

$Z: a^{(1)} = 0.2, a^{(2)} = 0.2, a^{(3)} = 0.2$.

The combined results obtained by the ERBA approach are shown in Fig. 11. The figure indicates that the combined results corresponding to different negotiation factors are different, i.e., they are associated with the personal preferences of the decision-makers.

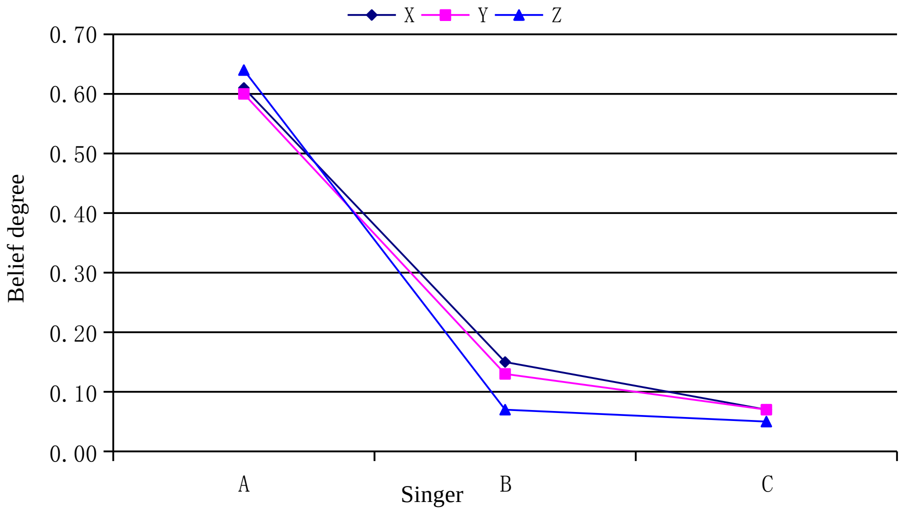

Fig. 11: Combined results produced by different negotiation factors

4) Analysis of interests of decision-makers with different methods

Let the tolerance level of the evidence distance be classified into three cases: $\varepsilon = 0$, $\varepsilon = 0.3$, and $\varepsilon = 1$. To obtain the different decision-makers' interests under these tolerance levels in the proposed approach, we supposed that all decision-makers are willing to participate in the negotiation with their original evidence weight. After calculating the weights of evidence with the existing methods, the eighteen pieces of evidence in Table 6 were combined according to the ER combination rule.

The results are presented in Table 10. The computed results of the distance differences $\Delta d_i (i = 1, 2, \cdots, 5)$ obtained using the existing methods to calculate the weights of evidence are shown in Figure 12, and the total distance of the modified evidence ΔD is shown in Figure 13.

Table 10: Combination results of different combination methods

	A	B	C	A,B	A,C	B,C	Θ
Average weight method	0.58	0.16	0.07	0.04	0.05	0.02	0.07
Chen's method and Wang's method	0.62	0.09	0.05	0.04	0.07	0.03	0.09
Ye's method	0.66	0.11	0.05	0.01	0.07	0.02	0.07
Deng's method and Guo's method	0.60	0.15	0.07	0.04	0.05	0.02	0.07
Lu's method	0.61	0.14	0.06	0.05	0.05	0.02	0.07
ERBA ($\varepsilon = 1$)	0.60	0.19	0.07	0.07	0.01	0.03	0.04
ERBA ($\varepsilon = 0$)	0.62	0.09	0.05	0.04	0.07	0.03	0.09
ERBA ($\varepsilon = 0.3$)	0.68	0.05	0.04	0.03	0.09	0.02	0.09

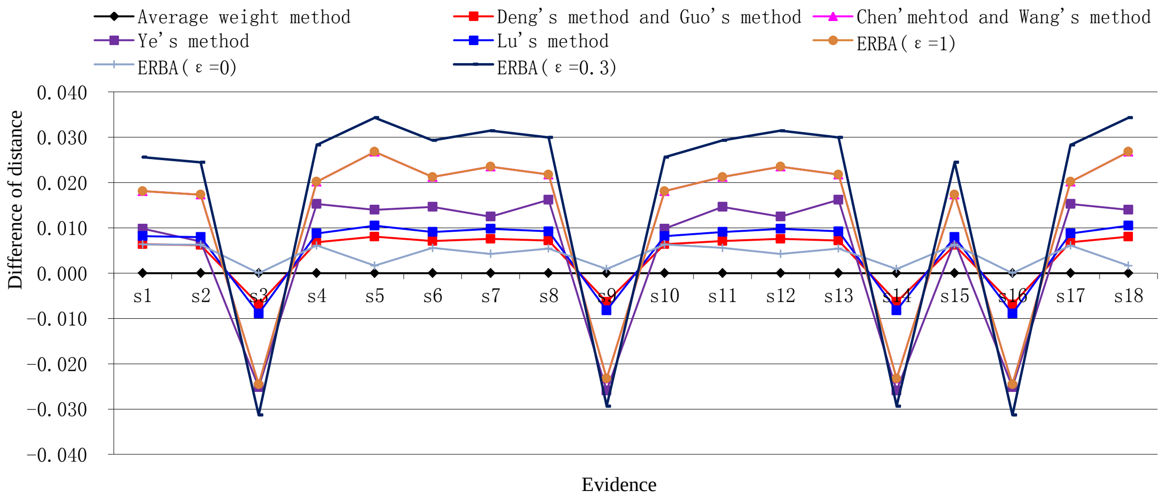

Fig.12: Comparison of differences of distance between original evidence and modified evidence calculated using different methods

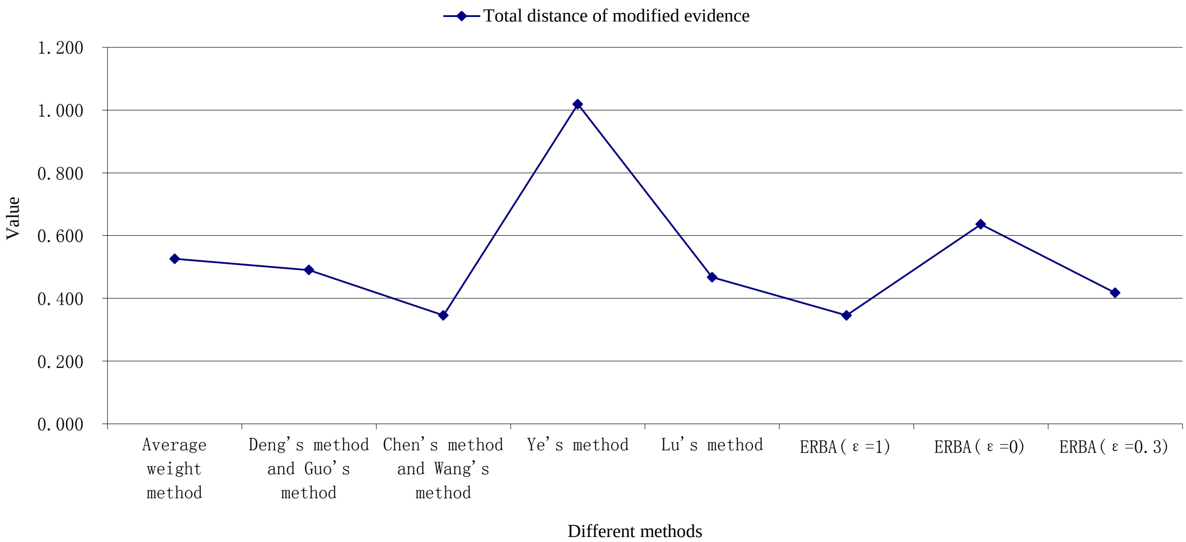

Fig. 13: Total distance of modified evidence calculated using different methods

Table 6 illustrates that the belief in *A* produced by the existing methods is similar to that given by the ERBA method, i.e., it is insignificantly differentiated. However, only the ERBA method considers both the "individual interest" and "overall interest" of the decision-makers. Under each tolerance level $\varepsilon\,(0 \le \varepsilon \le 1)$, the ERBA method considers the "individual interest" of the decision-makers when dealing with intra-alliance evidence, whereas the "overall interest" is considered when dealing with inter-alliance evidence. In this example, under tolerance level $\varepsilon = 0.3$, it considers the "individual interest" in the intra-alliance evidence given by judging panels s_1, s_2, s_5, s_7, s_{10}, s_{12}, and s_{18}; judging panels s_4, s_6, s_8, s_{11}, s_{13}, and s_{17}; and judging panels s_3, s_9, s_{14}, s_{16} s_3, s_9, s_{14}, and s_{16}. On the other hand, the "overall interest" is considered when dealing with inter-alliance evidence. In the special case of tolerance level $\varepsilon = 0$, the ERBA method is reduced to the methods proposed in (Chen and Wang, 2014; Wang, et al., 2006), which consider only the "overall interest" of the decision-makers. As shown in Figure 13, these give the smallest total distance of modified evidence ΔD with $\varepsilon = 0$. Under a tolerance level of $\varepsilon = 1$, the ERBA method considers only the "individual interest." As presented in Figure 12, all the differences in evidence distance lie above the X-axis at this tolerance level. However, the distance differences of evidence given by s_3, s_9, s_{14}, and s_{16} lie above the X-axis, whereas those of evidence s_1, s_2, s_4, s_5, s_6, s_7, s_8, s_{10}, s_{11}, s_{12}, s_{13}, and s_{15} lie below this axis when using the other methods.

Based on the above analyses, we can additionally conclude that the methods in (Chen and Wang, 2014; Wang, et al., 2006) consider the "overall interest" but not the "individual interest" of the decision-makers. Moreover, the methods in (Ye, et al., 2006; Deng, et al., 2004; Guo and Li, 2011; Lu, et al., 2008) consider neither the "overall interest" nor the "individual interest" of the decision-makers, whereas the ERBA method considers not only "individual interest" but also the "overall interest" of the decision-makers.

6. Conclusions

In this paper, we proposed an evidential reasoning theorem based on a refined approach, namely, evidential reasoning based on alliances. This approach considers alliances in dealing with the needs of decision-makers. The main concept of the ERBA approach is that pieces of evidence that are easy to negotiate for decision-makers are classified into the same alliance; otherwise, they are classified into different alliances. To obtain more reasonable results, two optimization models of negotiation were developed with the aim of providing relative importance weights, thus allowing the weighted evidence to be combined with the evidential reasoning rule of the combination. Unlike previous models, the models developed in this paper consider both conflicts of evidence and decision-maker subjectivity. Experimental results showed that the proposed approach is rational and effective.

Appendix

Proposition 1. Let $difBetP_{m_i}^{m_j}$ be the pignistic probability distance of two bbas, m_i and m_j. Then,

$$difBetP_{m_i}^{m_j} = \frac{1}{2}\sum_{\chi \in \Theta} \left| BetP_{m_i}(\chi) - BetP_{m_j}(\chi) \right|.$$

Proof: $\exists\, A^*(A^* \subseteq \Theta)$ satisfying the following equation:

$$|\, BetP_{m_i}(A^*) - BetP_{m_j}(A^*)\,| = \max_{A \subseteq \Theta}\left(\left| BetP_{m_i}(A) - BetP_{m_j}(A) \right|\right).$$

a) If $BetP_{m_i}(A^*) - BetP_{m_j}(A^*) > 0$, then

$$BetP_{m_i}(A^*) - BetP_{m_j}(A^*) = \max_{A \subseteq \Theta}(\left|BetP_{m_i}(A) - BetP_{m_j}(A)\right|). \quad \text{(A1)}$$

According to Eq. (2), we have

$$BetP_{m_i}(A^*) - BetP_{m_j}(A^*) = \sum\nolimits_{\theta \in A^*}(BetP_{m_i}(\theta) - BetP_{m_j}(\theta)). \quad \text{(A2)}$$

$$\therefore \ \max_{A \subseteq \Theta}(\left|BetP_{m_i}(A) - BetP_{m_j}(A)\right|) = \sum\nolimits_{\theta \in A^*}(BetP_{m_i}(\theta) - BetP_{m_j}(\theta)). \quad \text{(A3)}$$

$\therefore \quad \forall \chi \in A^*, \ \ BetP_{m_i}(\chi) - BetP_{m_j}(\chi) \geq 0$,

$\forall \chi \in \overline{A^*}$, $BetP_{m_i}(\chi) - BetP_{m_j}(\chi) \leq 0$ (where $\overline{A^*}$ is the complement set of A^*).

That is,

$\forall \chi \in A^*$, $BetP_{m_j}(\chi) - BetP_{m_i}(\chi) \leq 0$,

$\forall \chi \in \overline{A^*}$, $BetP_{m_j}(\chi) - BetP_{m_i}(\chi) \geq 0$.

$$\therefore \ BetP_{m_j}(\overline{A^*}) - BetP_{m_i}(\overline{A^*}) = \max_{A \subseteq \Theta}(\left|BetP_{m_i}(A) - BetP_{m_j}(A)\right|. \quad \text{(A4)}$$

According to Eqs. (A1) and (A4), the formula can be expressed as follows:

$\therefore \ BetP_{m_i}(A^*) - BetP_{m_j}(A^*) + BetP_{m_j}(\overline{A^*}) - BetP_{m_i}(\overline{A^*}) = 2\max_{A \subseteq \Theta}(\left|BetP_{m_i}(A) - BetP_{m_j}(A)\right|$.

$\sum\nolimits_{\chi \in A^*}(BetP_{m_i}(\chi) - BetP_{m_j}(\chi)) + \sum\nolimits_{\chi \in \overline{A^*}}(BetP_{m_j}(\chi) - BetP_{m_i}(\chi)) = 2\max_{A \subseteq \Theta}(\left|BetP_{m_i}(A) - BetP_{m_j}(A)\right|)$.

$\because A^* \cup \overline{A^*} = \Theta$,

$$\therefore \max_{A \subseteq \Theta}(\left|BetP_{m_i}(A) - BetP_{m_j}(A)\right|) = \frac{1}{2}\sum_{\chi \in \Theta}\left|BetP_{m_i}(\chi) - BetP_{m_j}(\chi)\right|. \quad \text{(A5)}$$

b) If $BetP_{m_i}(A^*) - BetP_{m_j}(A^*) > 0$, then using a similar argument, the formula can be expressed as follows:

$$\max_{A \subseteq \Theta}(\left|BetP_{m_i}(A) - BetP_{m_j}(A)\right|) = \frac{1}{2}\sum_{\chi \in \Theta}\left|BetP_{m_i}(\chi) - BetP_{m_j}(\chi)\right|. \quad \text{(A6)}$$

According to Eqs. (A5) and (A6), the proposition can be directly obtained.

Proposition 2. Let $difBetP_{m_i}^{m_j}$ be the pignistic probability distance of two bbas, m_i and m_j. Then,

$0 \leq difBetP_{m_i}^{m_j} \leq 1$.

Proof: According to Eq. (4), we have

$Betp_{m_i}(\{H_1\})$

$= m_i(\{H_1\}) +$

$m_i(\{H_1,H_2\})/2+m_i(\{H_1,H_3\})/2+\cdots+m_i(\{H_1,H_N\})/2+$

$m_i(\{H_1,H_2,H_3\})/3+m_i(\{H_1,H_2,H_4\})/3+\cdots+m_i(\{H_1,H_{N-1},H_N\})/3+$

$\cdots+$

$$m_i(\{H_1,H_2,\cdots,H_N\})/N, \quad \text{(B1)}$$

$Betp_{m_i}(\{H_2\})$

$=m_i(\{H_2\})+$

$m_i(\{H_2,H_1\})/2+m_i(\{H_2,H_3\})/2+\cdots+m_i(\{H_2,H_N\})/2+$

$m_i(\{H_2,H_3,H_1\})/3+m_i(\{H_2,H_3,H_4\})/3+\cdots+m_i(\{H_2,H_{N-1},H_N\})/3+$

$\cdots+$

$$m_i(\{H_1,H_2,\cdots,H_N\})/N, \quad \text{(B2)}$$

$\vdots$

$Betp_{m_i}(\{H_N\})$

$=m_i(\{H_N\})+$

$m_i(\{H_N,H_1\})/2+m_i(\{H_N,H_3\})/2+\cdots+m_i(\{H_N,H_{N-1}\})/2+$

$m_i(\{H_N,H_1,H_2\})/3+m_i(\{H_N,H_1,H_3\})/3+\cdots+m_i(\{H_N,H_{N-1},H_{N-2}\})/3+$

$\cdots+$

$$m_i(\{H_1,H_2,\cdots,H_N\})/N. \quad \text{(BN)}$$

According to Eqs. (B1), (B2), $\cdots$, and (BN), we have

$Betp_{m_i}(\{H_1\})+Betp_{m_i}(\{H_2\})+\cdots+Betp_{m_i}(\{H_N\})$

$=m_i(\{H_1\})+m_i(\{H_2\})+\cdots\ m_i(\{H_N\})+$

$m_i(\{H_1,H_2\})+m_i(\{H_1,H_3\})+\cdots+m_i(\{H_{N-1},H_N\})+$

$m_i(\{H_1,H_2,H_3\})+m_i(\{H_1,H_2,H_4\})+\cdots+m_i(\{H_{N-2},H_{N-1},H_N\})+$

$\cdots+$

$m_i(\{H_1,H_2,\cdots,H_N\})$

$=\sum_{A\subseteq\Theta} m_i(A)$

$\because\ \sum_{A\subseteq\Theta} m_i(A)=1$

$\therefore\ Betp_{m_i}(\{H_1\})+Betp_{m_i}(\{H_2\})+\cdots+Betp_{m_i}(\{H_N\})=1$

$\therefore\ \sum_{w\in\Theta} Betp_{m_i}(w)=1$

$\because\ 0\le\sum_{w\in A} Betp_{m_i}(w)\le\sum_{w\in\Theta} Betp_{m_i}(w)$

$\therefore\ 0 \le \sum_{w \in A} Betp_{m_i}(w) \le 1$

According to Eq. (4), we have

$Betp_{m_i}(A) = \sum_{w \in A} Betp_{m_i}(w)$

$\therefore\ 0 \le Betp_{m_i}(A) \le 1$

In the same way, we have

$0 \le Betp_{m_j}(A) \le 1$

$\therefore\ -1 \le Betp_{m_i}(A) - Betp_{m_j}(A) \le 1$

Thus, Eq. (4) gives

$difBetP_{m_i}^{m_j} = \max_{A \subseteq \Theta} \left| BetP_{m_i}(A) - BetP_{m_j}(A) \right|$

$\therefore\ 0 \le difBetP_{m_i}^{m_j} \le 1$, and the proposition can be directly obtained.

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