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# On the combination and normalization of conflicting interval-valued belief structures

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ABSTRACT

Dempster-Shafer theory (DST) or evidence theory has significant advantages in the fields of information aggregation and decision analysis. In this paper, in order to overcome the counter-intuitive behavior or specificity changes caused by evidence theory, the evidential reasoning (ER) rule which handles the weight and reliability of evidence in an appropriate way, is generalized to deal with the combination of conflicting interval-valued belief structures (IBSs). Specifically, an optimization model of pignistic probability distance is established from the global perspective to provide the relative weights for interval evidence so that the modified interval evidence can be reasonably combined, and then a modified interval evidence combination approach is proposed which is based on ER rule. The method can lead to a rational combination of conflicting interval evidence, which is also a development of Yang's ER rule. Numerical examples are provided to indicate that the proposed method is not only suitable for combining conflict-free interval evidence, but can also suitably combine conflicting interval evidence. At last, a case study is conducted on the actual pattern recognition problem to illustrate the applicability of the proposed method and the potential in dealing with the combination of conflicting interval evidence.

## 1. Introduction

The Dempster-Shafer theory (DST) firstly developed by Dempster (1967) and later extended and refined by Shafer (1976), is a general framework for reasoning with uncertainty. Yager and Alajlan (2015) have introduced a Dempster-Shafer belief structure. It provides a formal mathematical framework for representing various types of uncertain information, which can be used for decision-making under uncertainty. As one of the leading theories for modeling uncertainty in imprecise situations (Silva & de Almeida-Filho, 2016); DST is used for several purposes like target recognition (Dou, Sun, & Lin, 2014); stochastic modeling (Li, Wang, & Chen, 2017); safety analysis (Zhang, Ding, Wu, & Skibniewski, 2017); global positioning system (Aggarwal, Bhatt, Devabhaktuni, & Bhattacharya, 2013); localization in wireless sensor networks (Elkin, Kumarasiri, Rawat, & Devabhaktuni, 2017), environmental impact assessment (EIA) (Wang & Yang, 2006); stock portfolio selection (Mitra Thakur, Bhattacharyya, & Sarkar Mondal, 2018) and voice activity detection (Park & Chang, 2018).

The original DST was developed for combination of precise (crisp) belief degrees or belief structures. However, due to the uncertainty of decision makers' (DMs) subjective judgments, linguistic ambiguity and the lack of information, probability masses assigned to propositions can be uncertain or imprecise. For example, when diagnosing and reasoning disease, a doctor may be unable to give a precise judgment about the disease if he/she cannot definitely confirm his/her diagnosis. In this situation, the belief degree expressed in the form of interval number rather than a crisp number may be easier for him/her. In the problem of group decision-making (GDM), belief degrees may be provided by different DMs or experts, although these belief degrees can be synthesized to get a precise point estimate, it will inevitably lead to information loss. Thus, the use of interval-valued belief structures (IBSs) is an ideal choice, which can not only preserve the views of different DMs or experts, but also express the uncertainty about the opinions of DMs or experts.

IBSs, as an extension of belief structures in DST, are developed for better exploitation of uncertain and imprecise information (Song, Wang, Lei, & Yue, 2016). There have been many studies devoted to extend DST to IBSs. Interested readers are referred to Lee and Zhu

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(1992), Denoeux (1999, 2000), Yager (2001), Wang et al. (2006, 2007), Sevastianov, Dymova, and Bartosiewicz (2012), Song, Wang, Lei, and Xue (2014) and Chen and Wang (2014) for details. However, the problems for combining and normalizing of IBSs have not been fully resolved. Existing approaches are mainly divided into two categories: one type is based on interval arithmetic operations (Lee & Zhu, 1992; Sevastianov et al., 2012; Song et al., 2014; Yager, 2001) and the other type is based on programming models (Chen & Wang, 2014; Denoeux, 1999, 2000; Wang et al., 2006, 2007).

Wang et al. (2006, 2007) reinvestigated most of the existing representative methods (Denoeux, 1999, 2000; Lee & Zhu, 1992; Yager, 2001) and pointed out the shortcomings of these methods, and they can provide true intervals of the combination result by the optimality approach (Song et al., 2014). Furthermore, they extended evidential reasoning (ER) approach, which was developed to support multiple attribute decision analysis (MADA) problems, to combine interval uncertainty information including interval data and interval belief degrees. However, using their approaches may lead to counter-intuitive behavior or change the specificity of the original interval evidence in some cases. Since these two methods are based on the DST and ER, respectively. Within the DST framework, the residual support is assigned to the frame of discernment. This specific assignment does not differentiate between ignorance and the residual support, whilst the former is an intrinsic property of the evidence and the latter reflects its extrinsic feature related to its relative importance compared with other evidence (Yang & Xu, 2013). This indiscrimination changes the specificity of interval evidence, even if all pieces of interval evidence do not have any global ignorance before combination their combined results for the frame of discernment will still have global ignorance. And in the framework of ER, the residual support can only be redistributed to single propositions and the frame of discernment, depending upon what propositions other interval evidence supports.

As the latest research work on interval evidence combination and normalization in recent years, Sevastianov et al. (2012) proposed a new framework for rule-base ER in the interval setting and applied to diagnosing type 2 diabetes; unfortunately, this method is not suitable for conflicting interval evidence combination. Song et al. (2014) developed a novel combination approach which is based on the operation on intuitionistic fuzzy set, elicited by DST. However, similar to Sevastianov's method (Sevastianov et al., 2012), the method is suitable for conflicting-free interval evidence combination and information loss occurs during the combination process. Chen and Wang (2014) studied the issues of combination and normalization of conflicting interval evidence, but, the Chen's method is one type of evidence discounting combination method in the framework of DST, and the specificity of the interval evidence will be changed by using Chen's method.

It can be seen from the above review of the literatures about the interval evidence combination that these representative and the latest methods are mainly based on the frameworks of DST and ER, which focus mainly on conflicting-free interval evidence combination. Although these methods have excellent performance for conflicting-free interval evidence combination, the combination results would be counter-intuitive or irrational when the interval evidence encountered conflict, especially a high conflict among them. Therefore, it is necessary to propose a new method for interval evidence combination that considers conflicting, which forms the motivation of this paper.

In this paper, the ER rule is the further development of DST and the ER approach (Yang & Xu, 2013); which considers both evidence weights and reliabilities in a more general framework to tackle with the problems of combination and normalization of conflicting IBSs.

The main contributions of the paper can be summarized as follows:

(1) A means based on pignistic probability distance is established from the global perspective to objectively determine the weights of interval evidence.
(2) A more general theoretical framework for the combination and normalization of interval evidence is constructed. The method can effectively combine conflicting or conflicting-free interval evidence in a more reasonable way.
(3) The proposed method based on ER rule can effectively overcome the counter-intuitive behavior or specificity changes within the framework of DST, which can be seen as an extension of Yang's ER rule.
(4) The ER rule is applied to the actual pattern recognition problem, and the robustness of the proposed method is further verified.

The rest of the paper is organized as follows. Section 2 gives a brief introduction of DST and investigates the counter-intuitive results of interval evidence combination. In Section 3, we introduce the relevant conceptions of the ER rule and show its advantages for combining IBSs. In Section 4, the preliminary details of the normalization of IBSs are described. Furthermore, we provide a method for determining the relative weights of interval evidence and establish a general optimization model for combining and normalizing IBSs. Numerical examples are provided in Section 5 to demonstrate its advantages of the proposed method, Section 6 presents a case study to illustrate our proposed approach and Section 7 concludes the paper with a summary.

## 2. Counter-intuitive results of combining interval evidence

In this section, we will briefly introduce the relevant concepts of the DST as the basis for subsequent discussions. Moreover, the counter-intuitive behavior of interval evidence combination in the framework of DST is investigated.

### 2.1. DST

The DST is defined on a finite nonempty set of $N$ mutually exclusive and exhaustive hypotheses. This set is known as the frame of discernment (FOD) and is denoted by $\Theta$. Further, $2^{\Theta}$ is the power set of $\Theta$, contains all possible propositions of the elements in $\Theta$, and can be denoted as: $2^{\Theta} = \{\varnothing, \{\theta_1\}, \dots, \{\theta_N\}, \{\theta_1, \theta_2\}, \dots, \{\theta_1, \theta_N\}, \dots, \Theta\}$. The core definitions in DST are described as follows.

**Definition 1** (*Dempster, 1967*). Let $\Theta$ be the frame of discernment, then the basic probability assignment (BPA) is a function $m$: $2^{\Theta} \to [0, 1]$, which satisfies the two following conditions:

$$\begin{cases} m(\varnothing) = 0, \\ \sum_{A \subseteq \Theta} m(A) = 1, \end{cases} \tag{1}$$

where $\varnothing$ denotes an empty set and the value $m(A)$ taken by $m$ is termed the basic probability mass of $A$. Each subset $A \subseteq \Theta$ with $m(A) > 0$ is referred to as a focal element of $m$.

**Definition 2** (*Dempster, 1967*). The belief and plausibility functions are defined as follows:

$$\begin{aligned} Bel(A) &= \textstyle\sum_{B \subseteq A} m(B), \\ Pl(A) &= \textstyle\sum_{A \cap B \neq \varnothing} m(B). \end{aligned} \tag{2}$$

The core of the DST is Dempster's rule of combination by which evidence from different sources are combined. This rule assumes that information sources are independent and employs the orthogonal sum to combine multiple pieces of evidence. Assuming $m_1, m_2, \dots, m_n$ denote different BPAs derived from multiple independent pieces of evidence and their orthogonal sum $m = m_1 \oplus m_2 \cdots \oplus m_n$, where $\oplus$ represents the combination operator. With two BPAs, Dempster's combination rule is defined as

$$[m_1 \oplus m_2](C) = \begin{cases} 0, \ C = \varnothing, \\ \frac{\sum_{A \cap B = C} m_1(A) m_2(B)}{1 - \sum_{A \cap B = \varnothing} m_1(A) m_2(B)}, \ C \neq \varnothing, \end{cases} \tag{3}$$

where $A$ and $B$ are both focal elements and $[m_1 \oplus m_2](C)$ is a BPA. The denominator, $1 - \sum_{A\cap B=\varnothing} m_1(A)m_2(B)$, is denoted by $k$ and called the normalization factor. $\sum_{A\cap B=\varnothing} m_1(A)m_2(B)$ is called the degree of conflict and measures the conflict between the pieces of evidence. The division by $k$ is called normalization.

**Definition 3** (*Liu, 2006*). A pignistic probability distribution function is a mapping of $m$ from $\Theta$ to [0,1], defined as follows:

$$BetP_m(\{\theta_j\}) = \sum_{\theta\subseteq\Theta,\theta_j\in\theta} \frac{1}{|\theta|}\frac{m(\theta)}{1-m(\varnothing)},\ m(\varnothing)\neq 1, \tag{4}$$

where $|\theta|$ is the number of propositions in $\theta$. The pignistic probability function $BetP_m$ can be extended as a function on the power set of the frame of discernment $\Theta$, which is

$$BetP_m(\theta) = \sum_{\theta_j\in\theta} BetP_m(\{\theta_j\}). \tag{5}$$

The transformation from BPA $m$ to pignistic probability function $BetP_m$ is known as a pignistic transformation. When $m(\varnothing)=0$, $\frac{m(\theta)}{1-m(\varnothing)}$ is degraded to $m(\theta)$.

### 2.2. Counter-intuitive results of combining conflicting IBSs

In the actual decision-making process, the use of Dempster's combination rule to combine conflicting evidence may produce counter-intuitive results (Deng, Han, Dezert, Deng, & Yu, 2016; Wang, Xiao, Deng, Fe, & Deng, 2016; Zadeh, 1986). We illustrate this problem with the following example.

**Example 1.** Let $\Theta$ be the frame of discernment with three propositions $\{H_1,\ H_2,\ H_3\}$. Suppose that two BPAs, $m_1$ and $m_2$, are constructed as

$m_1(H_1)=0.98,\ m_1(H_2)=0.02,\ m_1(H_3)=0.00,$

$m_2(H_1)=0.00,\ m_2(H_2)=0.02,\ m_2(H_3)=0.98.$

According to Definition 2, the degree of conflict $\sum_{A\cap B=\varnothing} m_1(A)m_2(B)=0.9996$, which indicates that two BPAs, $m_1$ and $m_2$ are in high conflict with each other. Using Eq. (3) to combine evidence $m_1$ and $m_2$, the result is

$m_{12}(H_1)=0.00,\ m_{12}(H_2)=1.00,\ m_{12}(H_3)=0.00.$

Obviously, the above result means that if either pieces of evidence $m_1$ and $m_2$ does not support proposition $H_1$ or $H_3$, then proposition $H_1$ or $H_3$ will no longer be supported, no matter how strongly the other piece of evidence supports proposition $H_1$ or $H_3$, this result contradicts people's intuition.

In fact, combining conflicting interval evidence using Dempster's combination rule is also likely to have counter-intuitive behavior. We illustrate this by the following example.

**Example 2.** Let $\Theta$ be the frame of discernment with three propositions $\{H_1,\ H_2,\ H_3\}$. Suppose that two normalized IBSs $m_1$ and $m_2$, are constructed as:

$m_1(H_1)=[0.98,\ 0.99],\ m_1(H_2)=[0.01,\ 0.02],\ m_1(H_3)=[0,\ 0].$

$m_2(H_1)=[0,\ 0],\ m_2(H_2)=[0.01,\ 0.02],\ m_2(H_3)=[0.98,\ 0.99].$

The final combined results can be achieved through the computational process described in their published works (Chen & Wang, 2014; Sevastianov et al., 2012; Song et al., 2014; Wang, Yang, Xu, & Chin, 2007). The combined results obtained by these methods are presented in Table 1.

**Table 1**
Combination results obtained by different methods.

| | Wang's method | Sevastianov's method | Song's method | Chen's method |
|---|---|---|---|---|
| $m_{12}=(H_1)$ | [0, 0] | [0, 0] | [0, 0] | [0, 0] |
| $m_{12}=(H_2)$ | [1, 1] | [0.40, 1.60] | [1, 1] | [1, 1] |
| $m_{12}=(H_3)$ | [0, 0] | [0, 0] | [0, 0] | [0, 0] |

The interval-valued probability mass of proposition $H_1$ in interval evidence $m_1$ is approximate to 1, the interval-valued probability mass of proposition $H_2$ in interval evidence $m_1$ is approximate to 0, and the interval-valued probability mass of proposition $H_3$ in interval evidence $m_1$ is equal to 0. However, the interval-valued probability mass of proposition $H_1$ in interval evidence $m_2$ is equal to 0, the interval-valued probability mass of proposition $H_2$ in interval evidence $m_2$ is approximate to 0, and the interval-valued probability mass of proposition $H_3$ in interval evidence $m_2$ is approximate to 1. Therefore, interval evidence $m_1$ and $m_2$ are in conflict with each other.

The interval-valued probability masses of proposition $H_2$ in interval evidence $m_1$ and $m_2$ are both approximate to 0. However, the interval-valued probability mass of proposition $H_2$ in their combined result using the four methods are all approximate to 1, which is obviously counter-intuitive.

It can be seen from the above example that the existing methods cannot truly solve the problem of interval evidence combination, especially the conflicting interval evidence combination. Therefore, it is necessary to continue to study the problem.

## 3. ER rule solution for counter-intuitive behavior

In this section, the conceptions of belief distribution, weighted belief distribution (WBD), and the ER rule are briefly reviewed, and the counter-intuitive results of combining IBSs mentioned in the previous section were reinvestigated.

**Definition 4** (*Yang & Xu, 2013*). Suppose $p_{\theta,i}$ is the belief degree for proposition $\theta$ by evidence $e_i$, with $0\leqslant p_{\theta,i}\leqslant 1(\theta\subseteq\Theta)$, then $e_i$ can be profiled as the following distribution:

$$e_i = \left\{(\theta,\ p_{\theta,i}),\ \forall\ \theta\subseteq\Theta,\ \sum_{\theta\subseteq\Theta} p_{\theta,i}=1\right\} \tag{6}$$

**Definition 5** (*Yang & Xu, 2013*). Suppose $w_i$ $(0\leqslant w_i\leqslant 1)$ is the weight of evidence $e_i$, the weighted belief degree for $e_i$ defined in the ER rule with evidence weight are given as follows:

$$m_{\theta,i} = m_i(\theta) = \begin{cases} 0,\ \ \theta=\varnothing, \\ w_i p_{\theta,i},\ \theta\subseteq\Theta,\ \theta\neq\varnothing, \\ 1-w_i,\ \theta=P(\Theta), \end{cases} \tag{7}$$

where $w_i=0$ means that it is not important at all and $w_i=1$ signifies that it is the most important. $1-w_i$ is assigned to the power set of the frame of discernment instead of any single subset. So $1-w_i$ is attached to $P(\Theta)$ that allows it to be redistributed to all propositions in the power set of the frame of discernment because $\theta\cap P(\Theta)=\theta$. The term $m_{\theta,i}$ are generated as basic probability mass for $\theta$ from evidence $e_i$. Obviously, the specificity of the belief degree does not change. The WBD of $e_i$, denoted by $m_i$, is constructed from Eqs. (6) and (7) as follows:

$$m_i = \{(\theta,\ m_{\theta,i}),\ \ \forall\ \theta\subseteq\Theta;\ (P(\Theta),\ m_{P(\Theta),i})\} \tag{8}$$

**Definition 6** (*Yang & Xu, 2013*). The WBD is then extended to consider both weight and reliability of evidence. Let $w_i$ is the weight of evidence $e_i$ defined in Eq. (6) with $0\leqslant w_i\leqslant 1$ and $r_i$ is the reliability of evidence $e_i$ with $0\leqslant r_i\leqslant 1$and $r_i=0$ and 1 standing for "not reliable at all" and "fully reliable" respectively. The basic probability masses for evidence $e_i$ are then assigned as follows

$$\tilde{m}_{\theta,i} = \begin{cases} 0,\ \ \theta=\varnothing, \\ \tilde{w}_i p_{\theta,i},\ \theta\subseteq\Theta,\ \theta\neq\varnothing, \\ 1-\tilde{w}_i,\ \theta=P(\Theta), \end{cases} \tag{9}$$

where

$$\widetilde{w}_i = \frac{w_i}{1 + w_i - r_i}, \tag{10}$$

Eq. (9) is called weighted belief distribution with reliability (WBDR). $\widetilde{w}_i$ can be seen as a comprehensive coefficient to adjust both $w_i$ and $r_i$ of evidence $e_i$. $1 - \widetilde{w}_i$ is the residual support for evidence $e_i$ from $w_i$ and $r_i$. So evidence $e_i$ can be denoted by Eq. (11) which is just a generalization of Eq. (8).

$$m_i = \{(\theta, \widetilde{m}_{\theta,i}),\ \forall\,\theta \subseteq \Theta; (P(\Theta), \widetilde{m}_{P(\Theta),i})\}. \tag{11}$$

It is clear that $\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,i}+\widetilde{m}_{p(\Theta),i} = 1$.The ER rule (Yang & Xu, 2013) which considers both evidence weights and reliabilities in a coherent framework is generalized from the ER approach (Yang & Xu, 2002; Yang, 2001). Since ER approach only considers the weights of evidence, so we call it ER rule with weights.

**Definition 7** (*Yang & Xu, 2013*). Suppose there are $L$ pieces of independent evidence denoted by Eq. (6) to be combined which are discounted by Eqs. (9)–(11). Let $e(i)$ be the fusion of the first $i$ pieces of discounted evidence and we will have the orthogonal sum on the first $i$ WBDRs as follows:

$$\hat{m}_{\theta,e(i)} = [(1 - \widetilde{w}_i)\widetilde{m}_{\theta,e(i-1)} + \widetilde{m}_{P(\Theta),e(i-1)}\widetilde{m}_{\theta,i}] + \sum_{B\cap C=\theta} \widetilde{m}_{B,e(i-1)}\widetilde{m}_{C,i},$$

$$\forall\,\theta \subseteq \Theta, \tag{12}$$

$$\hat{m}_{P(\Theta),e(i)} = (1 - \widetilde{w}_i)\widetilde{m}_{P(\Theta),e(i-1)}, \tag{13}$$

$$\widetilde{m}_{\theta,e(i)} = k\cdot\hat{m}_{\theta,e(i)} = \frac{\hat{m}_{\theta,e(i)}}{\sum_{D\subseteq\Theta} \hat{m}_{D,e(i)}+\hat{m}_{P(\Theta),e(i)}},\ \ \forall\,\theta \subseteq \Theta \tag{14}$$

$$\widetilde{m}_{P(\Theta),e(i)} = k\cdot\hat{m}_{P(\Theta),e(i)} = \frac{\hat{m}_{P(\Theta),e(i)}}{\sum_{D\subseteq\Theta} \hat{m}_{D,e(i)}+\hat{m}_{P(\Theta),e(i)}}. \tag{15}$$

$\widetilde{w}_i$ is generated from the weight and reliability of evidence $e_i (i = 1, 2, ...,L)$ by Eq. (10). $\widetilde{m}_{\theta,e(1)} = \widetilde{m}_{\theta,1}$, $\widetilde{m}_{P(\Theta),e(1)} = \widetilde{m}_{P(\Theta),1}$ with $0 \leqslant \widetilde{m}_{\theta,e(i)} \leqslant 1$ and $0 \leqslant \widetilde{m}_{P(\Theta),e(i)} \leqslant 1$. Eq. (14) is the probability mass supports proposition $\theta$ for the combined WBDR of $e(i)$ after normalization while Eq. (12) is the non-normalized probability mass of the first $i$ pieces of evidence after $i - 1$ times of orthogonal sum operation recursively on Eq. (11). Eqs. (13) and (15) represent the non-normalized and normalized residual support for $e(i)$ respectively. After $L - 1$ times of calculation, the combined normalized probability mass for all the $L$ pieces of evidence can be obtained that is denoted by $\widetilde{m}_{\theta,e(L)} (\theta \subseteq \Theta)$, and the combined normalized residual support for $e(L)$ is also obtained as $\widetilde{m}_{P(\Theta),e(L)}$. The final combined belief degree is then generated as follows:

$$p_{\theta,e(L)} = \frac{\widetilde{m}_{\theta,e(L)}}{1 - \widetilde{m}_{P(\Theta),e(L)}} = \frac{\hat{m}_{\theta,e(L)}}{\sum_{D\subseteq\Theta} \hat{m}_{D,e(L)}},\ \ \forall\,\theta \subseteq \Theta. \tag{16}$$

In the actual decision-making problems, if $r_i$ denotes the reliability of information provided from evidence $e_i$, then $1 - r_i$ refers to the unreliability of evidence $e_i$. The unreliability of assessment information we get may be due to the problematic assessment data of the used method or equipment. When $r_i < 1$, the information provided from evidence $e_i$ is not fully reliable which means we could not completely believe the assessment information from evidence $e_i$ to the degree of $1 - r_i$.In order to verify the validity and rationality of ER rule to eliminate counter-intuitive behavior, we will reinvestigate the examples in the above section. The effectiveness of ER rule is demonstrated below by examples in Section 2. Without loss of generality, assume that each piece of precise evidence or interval evidence is equally weighted, namely, $w_1 = w_2 = 0.5$ and the reliability of each piece of precise evidence or interval evidence is also equal to 0.5, namely, $r_1 = r_2 = 0.5$.

**Example 3** (*Example 1 revisited*). By Eqs. (9)–(11), we can get two WBDRs, respectively, representing $m_1$ and $m_2$ as:

$\widetilde{m}_1(H_1) = 0.49$, $\widetilde{m}_1(H_2) = 0.01$, $\widetilde{m}_1(H_3) = 0.00$, $\widetilde{m}_1(P(\Theta)) = 0.50$,

$\widetilde{m}_2(H_1) = 0.00$, $\widetilde{m}_2(H_2) = 0.01$, $\widetilde{m}_2(H_3) = 0.49$, $\widetilde{m}_2(P(\Theta)) = 0.50$.

According to Eqs. (12)–(15), the combined result of the two WBDRs is $\widetilde{m}'_{e(2)}(H_1) = 0.327$, $\widetilde{m}'_{e(2)}(H_2) = 0.013$, $\widetilde{m}'_{e(2)}(H_3) = 0.327$, $\widetilde{m}'_{e(2)}(\Theta) = 0.000$, $\widetilde{m}'_{e(2)}(P(\Theta)) = 0.333$.By comparing $\widetilde{m}'_{e(2)}$ with $m_{12}$, obviously, the combination result obtained by using Dempster's combination rule is counter-intuitive, but the combination result obtained by using ER rule is intuitive.

**Example 4** (*Example 2 revisited*). Since IBSs $m_1$ and $m_2$ are two normalized IBSs, the optimization models can be applied to solve these two normalized IBSs combination problem, as in Wang's method (Wang et al., 2007). Combining these two normalized IBSs, we have the following formulas for the non-normalized probability masses:

$$\hat{m}_{\theta,e(2)} = [(1 - \widetilde{w}_2)\widetilde{m}_{\theta,e(1)} + \widetilde{m}_{P(\Theta),e(1)}\widetilde{m}_{\theta,2}] + \sum_{B\cap C=\theta} \widetilde{m}_{B,e(1)}\widetilde{m}_{C,2},\ \ \theta \subseteq \Theta,$$

$$\hat{m}_{P(\Theta),e(2)} = (1 - \widetilde{w}_2)\widetilde{m}_{P(\Theta),e(1)}.$$

The normalized probability masses will be given by

$$\widetilde{m}_{\theta,e(2)} = k\cdot\hat{m}_{\theta,e(2)} = \frac{\hat{m}_{\theta,e(2)}}{\sum_{D\subseteq\Theta} \hat{m}_{D,e(2)}+\hat{m}_{P(\Theta),e(2)}},\ \ \forall\,\theta \subseteq \Theta,$$

$$\widetilde{m}_{P(\Theta),e(2)} = k\cdot\hat{m}_{P(\Theta),e(2)} = \frac{\hat{m}_{P(\Theta),e(2)}}{\sum_{D\subseteq\Theta} \hat{m}_{D,e(2)}+\hat{m}_{P(\Theta),e(2)}}.$$

In order to determine the normalized probability mass interval for each proposition, we need to solve the following pairs of optimization models:

$$\text{Max/Min } \widetilde{m}_{\theta,e(2)} = \frac{\hat{m}_{\theta,e(2)}}{\sum_{D\subseteq\Theta} \hat{m}_{D,e(2)} + \hat{m}_{P(\Theta),e(2)}}$$

s.t. $\hat{m}_{\theta,e(2)} = [(1 - \widetilde{w}_2)\widetilde{m}_{\theta,e(1)} + \widetilde{m}_{P(\Theta),e(1)}\widetilde{m}_{\theta,2}] + \sum_{B\cap C=\theta} \widetilde{m}_{B,e(1)}\widetilde{m}_{C,2}$, $\theta \subseteq \Theta$,

$$\begin{gathered}
\hat{m}_{P(\Theta),e(2)} = (1 - \widetilde{w}_2)\widetilde{m}_{P(\Theta),e(1)},\\
\widetilde{w}_1 = 0.5,\\
\widetilde{w}_2 = 0.5,\\
\widetilde{m}_1(H_1) = \widetilde{w}_1 m_1(H_1),\\
\widetilde{m}_1(H_2) = \widetilde{w}_1 m_1(H_2),\\
\widetilde{m}_1(H_3) = \widetilde{w}_1 m_1(H_3),\\
\widetilde{m}_1(P(\Theta)) = \widetilde{w}_1 m_1(P(\Theta)),\\
\widetilde{m}_2(H_1) = \widetilde{w}_2 m_2(H_1),\\
\widetilde{m}_2(H_2) = \widetilde{w}_2 m_2(H_2),\\
\widetilde{m}_2(H_3) = \widetilde{w}_2 m_2(H_3),\\
\widetilde{m}_2(P(\Theta)) = \widetilde{w}_2 m_2(P(\Theta)),\\
m_1(H_1) + m_1(H_2) + m_1(H_3) = 1,\\
m_2(H_1) + m_2(H_2) + m_2(H_3) = 1,\\
0.98 \leqslant m_1(H_1) \leqslant 0.99,\\
0.01 \leqslant m_1(H_2) \leqslant 0.02,\\
0 \leqslant m_1(H_3) \leqslant 0,\\
0 \leqslant m_2(H_1) \leqslant 0,\\
0.01 \leqslant m_2(H_2) \leqslant 0.02,\\
0.98 \leqslant m_2(H_3) \leqslant 0.99.
\end{gathered}$$

With the help of LINGO software package and solving the above pair of models for $\theta \subseteq \Theta$, respectively, we get the normalized probability mass intervals as follows:

$\widetilde{m}''_{e(2)}(H_1) = [0.327, 0.330]$, $\widetilde{m}''_{e(2)}(H_2) = [0.007, 0.013]$, $\widetilde{m}''_{e(2)}(H_3) = [0.327, 0.330]$, $\widetilde{m}''_{e(2)}(\Theta) = [0,\ 0]$, $\widetilde{m}''_{e(2)}(P(\Theta)) = [0.333,\ 0.333]$, where the interval for $\widetilde{m}''_{e(2)}(P(\Theta))$ is obtained by solving the models with $\hat{m}_{P(\Theta),e(2)}$ used as the numerator of the objective function.

According to the combined result, $\widetilde{m}''_{e(2)}$, it is obviously that the combined result is counter-intuitive using existing approaches but conforms to intuitive judgment according to ER rule. The above examples indicate that the ER rule is a good option to the problem of the interval evidence combination, which can generate intuitive and reasonable combination results than existing methods.

## 4. ER rule with IBSs

We first introduce the following definitions, which are based on the published works of Denoeux (1999) and Wang et al. (2006, 2007). In addition, a method based on pignistic probability distance for determining the weights of interval evidence, is presented. A more general optimal model for combining IBSs based on the ER rule is also proposed.

### 4.1. The definition of IBSs

**Definition 8** (*Denoeux, 1999*). Let $H = \{H_1, ..., H_N\}$ be the frame of discernment, $F_1, ..., F_n$ be $n$ subsets of $H$ and $[a_i,\ b_i]$ be $n$ intervals with $0 \leqslant a_i \leqslant b_i \leqslant 1$ $(i = 1, ..., n)$. An IBS is the belief structure on $H$ such that

(1) $a_i \leqslant m(F_i) \leqslant b_i$, *where* $0 \leqslant a_i \leqslant b_i \leqslant 1$ *for* $i = 1, ..., n$;
(2) $\sum_{i=1}^{n} a_i \leqslant 1$ *and* $\sum_{i=1}^{n} b_i \geqslant 1$;
(3) $m(A) = 0,\ \forall\ A \notin \{F_1, ..., F_n\}$.

**Definition 9** (*Wang, Yang, Xu, & Chin, 2006; Wang et al., 2007*). Let $m$ be a valid IBS with interval-valued probability masses $a_i \leqslant m(F_i) \leqslant b_i$, $i = 1, ..., n$. If $\sum_{j=1}^{n} b_j - (b_i - a_i) \geqslant 1$ and $\sum_{j=1}^{n} a_j + (b_i - a_i) \leqslant 1$, $\forall\ i \in \{1, ..., n\}$, then $m$ is called a normalized IBS.

**Definition 10** (*Denoeux, 1999*). Let $m$ be a normalized IBS on $H$ with interval-valued probability masses $a_i \leqslant m(F_i) \leqslant b_i$, $i = 1, ..., n$. For $A \subseteq H$, the belief function (Bel) and the plausibility function (Pl) of $A$ are both the closed intervals defined respectively by

$$\mathrm{Bel}_m(A) = [\mathrm{Bel}_m^-(A),\ \mathrm{Bel}_m^+(A)] \tag{17}$$

$$\mathrm{Pl}_m(A) = [\mathrm{Pl}_m^-(A),\ \mathrm{Pl}_m^+(A)] \tag{18}$$

where

$$\mathrm{Bel}_m^-(A) = \min \sum_{F_i \subseteq A} m(F_i) = \max\left[\sum_{F_i \subseteq A} a_i, \left(1 - \sum_{F_i \not\subseteq A} b_j\right)\right], \tag{19}$$

$$\mathrm{Bel}_m^+(A) = \max \sum_{F_i \subseteq A} m(F_i) = \min\left[\sum_{F_i \subseteq A} b_i, \left(1 - \sum_{F_i \not\subseteq A} a_j\right)\right], \tag{20}$$

$$\mathrm{Pl}_m^-(A) = \min \sum_{F_i \cap A \neq \Phi} m(F_i) = \max\left[\sum_{F_i \cap A \neq \Phi} a_i, \left(1 - \sum_{F_i \cap A = \Phi} b_j\right)\right], \tag{21}$$

$$\mathrm{Pl}_m^+(A) = \max \sum_{F_i \cap A \neq \Phi} m(F_i) = \min\left[\sum_{F_i \cap A \neq \Phi} b_i, \left(1 - \sum_{F_i \cap A = \Phi} a_j\right)\right]. \tag{22}$$

**Definition 11** (*Song et al., 2014*). Let $m$ be a normalized IBS on $H = \{H_1, ..., H_N\}$, with interval-valued probability masses $a_i \leqslant m(F_i) \leqslant b_i$ for $i = 1, ..., n$. $m'$ denotes a Bayesian belief structure elicited by $m$. The probability masses of $m'$ are interval values, defined by:

$$m'(H_j) = BetP(H_j) = [BetP^-(H_j),\ BetP^+(H_j)], \tag{23}$$

where $BetP^-(H_j) = \sum_{A_j \in F_i} \frac{a_i}{|F_i|}$, $BetP^+(H_j) = \min\left[1,\ \sum_{A_j \in F_i} \frac{b_i}{|F_i|}\right]$, $i = 1, ..., n$, $j = 1, ..., N$.

### 4.2. Determination of interval evidence weights

To overcome the counter-intuitive behavior in the process of evidence combination, some researchers argue that the results are caused by the evidence combination rules themselves, which should be improved (Lefevre, Colot, & Vannoorenberghe, 2002; Smarandache & Dezert, 2006; Yager, 1987; Yamada, 2008); in contrast, others believe that the results due to the evidence itself, which should be modified by weight (Han, Deng, & Han, 2013; Martin, Jousselme, & Osswald, 2008; Murphy, 2000; Smarandache & Dezert, 2010). Haenni (2002) believes that it is a reasonable practice to modify the original evidence, no matter from the viewpoint of philosophical logic, mathematics, or engineering practices. In fact, the original ER rule was only developed for evidence combination, which did not provide a method for determining evidence weights. However, different evidence weights will result in different combination results. Therefore, it is necessary to provide a method to determine the weights for interval evidence so that the results of the interval evidence combination are more reasonable and meaningful.

In this section, we provide an objectively method from the perspective of minimizing the overall discrepancies to determine the weights for pieces of interval evidence. The method is built based on pignistic probability distance, which assists decision-making problems which use DST and provides a reasonable measure of the difference between the evidence (Smets & Kennes, 1994).

**Definition 12** (*Liu, 2006*). Let $\Theta$ be the frame of discernment, $m_i$ and $m_j$ be two BPAs on frame $\Theta$, and $BetP_{m_i}$, $BetP_{m_j}$ are their pignistic probability function respectively. The pignistic probability distance between two BPAs is defined as follows:

$$difBetP_{m_i}^{m_j} = \max_{A \subseteq \Theta}(|BetP_{m_i}(A) - BetP_{m_j}(A)|). \tag{24}$$

**Proposition 1** (*Chen, Wang, Shi, Zhang, & Lin, 2017*). *Let* $difBetP_{m_i}^{m_j}$ *be the pignistic probability distance of two BPAs,* $m_i$ *and* $m_j$*. Then,*

$$difBetP_{m_i}^{m_j} = \frac{1}{2}\sum_{\chi \in \Theta} |BetP_{m_i}(\chi) - BetP_{m_j}(\chi)|\ \ 0 \leqslant difBetP_{m_i}^{m_j} \leqslant 1. \tag{25}$$

*Although Eq. (25) is equivalent to Eq. (24), the calculation process of the former is simpler. For example, suppose that two BPAs,* $m_1$ *and* $m_2$ *on the frame of discernment* $\Theta = \{1, 2, 3, 4, 5\}$ *are constructed as*

$m_1 = \{1, 2, 3, 4, 5\} = 1,$

$m_2 = \{3, 4\} = 0.6,\ m_2 = \{1, 2\} = 0.4.$

*According to Eq. (25), the result is*
$difBetP_{m_1}^{m_2} = \frac{1}{2}(|0.20 - 0.20| + |0.20 - 0.20| + |0.20 - 0.30| + |0.20 - 0.30| + |0.20 - 0|) = 0.20.$
*For* $BetP_{M_i}(\{\theta_j\}) = \sum_{\theta \subseteq \Theta, \theta_j \in \theta} \frac{M_{\theta,i}}{|\theta|} = \sum_{\theta \subset \Theta, \theta_j \in \theta} \frac{M_{\theta,i}}{|\theta|} + \frac{M_i(\Theta)}{|\Theta|} = \sum_{\theta \subset \Theta, \theta_j \in \theta} \frac{w_i m_{\theta,i}}{|\theta|} + \frac{1 - w_i \sum_{\theta \subset \Theta} m_{\theta,i}}{|\Theta|} = w_i\left(\sum_{\theta \subset \Theta, \theta_j \in \theta} \frac{m_{\theta,i}}{|\theta|} - \frac{\sum_{\theta \subset \Theta} m_{\theta,i}}{|\Theta|}\right) + \frac{1}{|\Theta|}$*.Values are assigned:*

$$\alpha_{ij} = \sum_{\theta \subset \Theta, \theta_j \in \theta} \frac{m_{\theta,i}}{|\theta|} - \frac{\sum_{\theta \subset \Theta} m_{\theta,i}}{|\Theta|} \text{ and } \alpha_{kj} = \sum_{\theta \subset \Theta, \theta_j \in \theta} \frac{m_{\theta,k}}{|\theta|} - \frac{\sum_{\theta \subset \Theta} m_{\theta,k}}{|\Theta|} \tag{26}$$

*Then:* $|BetP_{M_i}(\theta_j) - BetP_{M_k}(\theta_j)| = |\alpha_{ij} w_i - \alpha_{kj} w_k|.$

**Proposition 2.** *When the evidence are interval evidence with IBSs, the pignistic probability distance,* $difBetP_{M_i}^{M_k}$*, between two modified interval belief structures* $M_i$ *and* $M_k$ *is denoted as follow:*

$$difBetP_{M_i}^{M_k} = \frac{1}{2}\sum_{j=1}^{|\Theta|} |\alpha_{ij} w_i - \alpha_{kj} w_k|$$
$$s.\ t.\ \begin{cases} \sum_{\theta \subseteq \Theta} m_{\theta,i} = 1,\ \ i = 1, 2, ...., n, \\ m_{\theta,i}^- \leqslant m_{\theta,i} \leqslant m_{\theta,i}^+,\ \ \forall\ \theta \subseteq \Theta, \\ w_1 + w_2 + ... + w_n = 1,\ \ w_1, w_2, ..., w_n \geqslant 0. \end{cases} \tag{27}$$

*Due to differences in data format, determining the weights of interval evidence is more difficult than precise (crisp) evidence. This study minimizes the discrepancies among pieces of modified interval evidence based on a*

**Table 2**
The BPAs.

| | $A$ | $B$ | $C$ |
|---|---|---|---|
| $m_1$ | [0.50 ,0.60] | [0.10, 0.20] | [0.30, 0.40] |
| $m_2$ | [0.00, 0.00] | [0.90, 0.95] | [0.05, 0.10] |
| $m_3$ | [0.55, 0.60] | [0.15, 0.20] | [0.25, 0.30] |
| $m_4$ | [0.55, 0.60] | [0.15, 0.20] | [0.25, 0.30] |
| $m_5$ | [0.55, 0.60] | [0.15, 0.20] | [0.25, 0.30] |

*global perspective modeling approach, in order to minimize the conflict among them, therefore, a model can be constructed as follows*:

$$Min\ difBetP = \min \sum_{i=1}^{n}\sum_{k=1}^{n} difBetP_{M_i}^{M_k}$$
$$s.\ t.\ \begin{cases} \sum_{\theta\subseteq\Theta} m_{\theta,i} = 1,\ \ i = 1,\,2,\,....,n, \\ m_{\theta,i}^{-} \leqslant m_{\theta,i} \leqslant m_{\theta,i}^{+},\ \ \forall\ \theta\subseteq\Theta, \\ w_1 + w_2 + ... + w_n = 1,\ \ w_1, w_2, ..., w_n \geqslant 0. \end{cases} \tag{28}$$

*In order to solve the model conveniently, Eq. (28) can be converted into Eq. (29)*:

$$Min\ difBetP = \min \sum_{j=1}^{|\Theta|}\sum_{i=1}^{n}\sum_{k=1}^{n} (\alpha_{ij}w_i - \alpha_{kj}w_k)^2$$
$$s.\ t.\ \begin{cases} \sum_{\theta\subseteq\Theta} m_{\theta,i} = 1,\ \ i = 1,\,2,\,....,n, \\ m_{\theta,i}^{-} \leqslant m_{\theta,i} \leqslant m_{\theta,i}^{+},\ \ \forall\ \theta\subseteq\Theta, \\ w_1 + w_2 + ... + w_n = 1,\ \ w_1, w_2, ..., w_n \geqslant 0. \end{cases} \tag{29}$$

*The objective function of Eq. (29) is a convex function, and the constraint conditions are equality constraints. Therefore, Eq. (29) satisfies the conditions that the nonlinear programming model has a global optimal solution. The weight vector of the modified interval evidence $w = (w_1, w_2, ..., w_n)^T$ can be obtained with the help of LINGO or MATLAB.*

### 4.3. New optimal model for interval evidence combination

To overcome the drawbacks of the previous approaches, an optimization model based on the ER rule for combining and normalizing interval evidence is built. The model is constructed as follows:

$$\begin{aligned}
&\text{Max/Min } p_{\theta,e(i)} = \frac{\tilde{m}_{\theta,e(i)}}{1-\tilde{m}_{P(\Theta),e(i)}} \\
&\text{s.t. } \hat{m}_{\theta,e(i)} = [(1-\tilde{w}_i)\tilde{m}_{\theta,e(i-1)} + \tilde{m}_{P(\Theta),e(i-1)}\tilde{m}_{\theta,i}] \\
&\qquad + \textstyle\sum_{B\cap C=\theta} \tilde{m}_{B,e(i-1)}\tilde{m}_{C,i},\ \ \forall\ \theta\subseteq\Theta, \\
&\hat{m}_{P(\Theta),e(i)} = (1-\tilde{w}_i)\tilde{m}_{P(\Theta),e(i-1)}, \\
&\tilde{m}_{\theta,e(i)} = k\cdot\hat{m}_{\theta,e(i)} = \frac{\hat{m}_{\theta,e(i)}}{\sum_{D\subseteq\Theta}\hat{m}_{D,e(i)} + \hat{m}_{P(\Theta),e(i)}},\ \ \forall\ \theta\subseteq\Theta, \\
&\tilde{m}_{P(\Theta),e(i)} = k\cdot\hat{m}_{P(\Theta),e(i)} = \frac{\hat{m}_{P(\Theta),e(i)}}{\sum_{D\subseteq\Theta}\hat{m}_{D,e(i)} + \hat{m}_{P(\Theta),e(i)}}, \\
&\tilde{w}_i = \frac{w_i}{1+w_i-r_i}, \\
&\textstyle\sum_{\theta\subseteq\Theta} m_{\theta,i} = 1,\ \ i = 1,\,2,\,...,n, \\
&m_{\theta,i}^{-} \leqslant m_{\theta,i} \leqslant m_{\theta,i}^{+},\ \ \forall\ \theta\subseteq\Theta, \\
&0 \leqslant r_i \leqslant 1,\ \ i = 1,\,2,\,...,n, \\
&w_1 + w_2 + ... + w_n = 1,\ \ w_1, w_2, ..., w_n \geqslant 0.
\end{aligned} \tag{30}$$

The objective functions of above models indicate the respective maximum and minimum final combined belief degree with respect to proposition $\theta$ by $n$ pieces of interval evidence $e(i)$. $w_i$ $(i = 1, 2, ..., n)$ is generated from the weight of $e(i)$ by Eq. (29). $0 \leqslant \tilde{m}_{\theta,e(i)}, \tilde{m}_{p(\Theta),e(i)} \leqslant 1$ and $\sum_{\theta\subseteq\Theta} \tilde{m}_{\theta,e(i)} + \tilde{m}_{p(\Theta),e(i)} = 1$ for $i = 2, ..., n$. $\tilde{m}_{\theta,e(i)}$ is the non-normalized interval-valued probability mass of the $n$ pieces of interval evidence after $n-1$ times of orthogonal sum operation recursively, with $\tilde{m}_{\theta,e(1)} = \tilde{m}_{\theta,1}$ and $\tilde{m}_{P(\Theta),e(1)} = \tilde{m}_{P(\Theta),1}$. It is worth noting that the above optimization models consider both the combination and normalization problems in the recursive combination process in the same time, and the residual support is not assigned to the frame of discernment or one of its specific propositions in advance, but to any subset of the power set. In addition, the probability masses must be normalized within the framework of ER rule.

## 5. Comparison studies

In this section, two numerical examples with conflicting and normalized random IBSs coming from Chen's paper (Chen & Wang, 2014) are given respectively to demonstrate the effectiveness and rationality of the proposed method by comparing with Wang's method (Wang et al., 2007); Sevastianov's method (Sevastianov et al., 2012); Song's method (Song et al., 2014); Chen's method (Chen & Wang, 2014), and

**Table 3**
Combination results obtained by different methods.

| | | $A$ | $B$ | $C$ | $\Theta$ | $P(\Theta)$ |
|---|---|---|---|---|---|---|
| $m_{12}$ | Wang's method | [0, 0] | [0.692, 0.927] | [0.073, 0.308] | [0, 0] | – |
| | Sevastianov's method | [0, 0] | [0.537, 1.134] | [0.090, 0.239] | [0, 0] | – |
| | Song's method | [0, 0] | [0.922, 0.933] | [0.067, 0.078] | [0, 0] | – |
| | Chen's method | [0.435, 0.551] | [0.139, 0.265] | [0.270, 0.381] | [0.022, 0.031] | – |
| | Yang's method | – | – | – | – | – |
| | Proposed method | [0.500, 0.600] | [0.100, 0.200] | [0.300, 0.400] | [0, 0] | [0, 0] |
| $m_{123}$ | Wang's method | [0, 0] | [0.529, 0.910] | [0.090, 0.471] | [0, 0] | – |
| | Sevastianov's method | [0, 0] | [0.401, 1.130] | [0.112, 0.357] | [0, 0] | – |
| | Song's method | [0, 0] | [0.958, 0.959] | [0.041, 0.042] | [0, 0] | – |
| | Chen's method | [0.641, 0.773] | [0.055, 0.150] | [0.172, 0.297] | [0, 0] | – |
| | Yang's method | – | – | – | – | – |
| | Proposed method | [0.671, 0.800] | [0.033, 0.103] | [0.167, 0.293] | [0, 0] | [0, 0] |
| $m_{1234}$ | Wang's method | [0, 0] | [0.360, 0.890] | [0.110, 0.640] | [0, 0] | – |
| | Sevastianov's method | [0, 0] | [0.286, 1.073] | [0.132, 0.508] | [0, 0] | – |
| | Song's method | [0, 0] | [0.980, 0.980] | [0.020, 0.020] | [0, 0] | – |
| | Chen's method | [0.782, 0.900] | [0.016, 0.068] | [0.083, 0.198] | [0, 0] | – |
| | Yang's method | – | – | – | – | – |
| | Proposed method | [0.798, 0.911] | [0.009, 0.045] | [0.079, 0.190] | [0, 0] | [0, 0] |
| $m_{12345}$ | Wang's method | [0, 0] | [0.220, 0.866] | [0.134, 0.780] | [0, 0] | – |
| | Sevastianov's method | [0, 0] | [0.194, 0.969] | [0.149, 0.688] | [0, 0] | – |
| | Song's method | [0, 0] | [0.991, 0.991] | [0.009, 0.009] | [0, 0] | – |
| | Chen's method | [0.873, 0.959] | [0.004, 0.027] | [0.037, 0.120] | [0, 0] | – |
| | Yang's method | – | – | – | – | – |
| | Proposed method | [0.882, 0.963] | [0.003, 0.018] | [0.035, 0.114] | [0, 0] | [0, 0] |

**Table 4**
Belief structures of random interval evidence.

| | $A$ | $A, B$ | $B$ | $B, C$ | $C$ |
|---|---|---|---|---|---|
| $m_1$ | [0.284, 0.289] | [0.219, 0.224] | [0.118, 0.123] | [0.309, 0.314] | [0.055, 0.060] |
| $m_2$ | [0.182, 0.187] | [0.330, 0.335] | [0.143, 0.148] | [0.209, 0.214] | [0.121, 0.126] |
| $m_3$ | [0.442, 0.447] | [0.208, 0.213] | [0.006, 0.011] | [0.106, 0.111] | [0.223, 0.228] |
| $m_4$ | [0.195, 0.200] | [0.008, 0.013] | [0.308, 0.313] | [0.120, 0.125] | [0.354, 0.359] |

Yang's method (Yang & Xu, 2013).

**Example 5.** Let $\Theta$ be the frame of discernment with three propositions $\{A, B, C\}$. Five pieces of interval evidence with IBSs on $\Theta$ are listed in Table 2.

It can be concluded that these five pieces of interval evidence satisfy the conditions of Definition 9, so, they are all normalized. We assume that all pieces of interval evidence are completely reliable, namely, $r_1 = r_2 = r_3 = r_4 = r_5 = 1$. The results of combing these five pieces of interval evidence by different combination methods are detailed in Table 3.

As shown in Table 2, it is evident that four pieces of interval evidence $m_1$, $m_3$, $m_4$, and $m_5$ mainly support proposition $A$. However, interval evidence $m_2$ strongly supports proposition $B$, which is a very different piece of interval evidence. Thus, interval evidence $m_2$ is in conflict with the others. According to Wang's method (Wang et al., 2007), the combined results means that if interval evidence $m_2$ does not support proposition $A$, then proposition $A$ will no longer be supported, no matter how strongly others pieces of interval evidence $m_1$, $m_3$, $m_4$, and $m_5$ support proposition $A$, it is obviously counter-intuitive. The reason is that Wang's approach is in the framework of DST, which provides a process for combining two pieces of non-compensatory interval evidence, in the sense that if either of them completely opposes a proposition, the proposition will not be supported at all, no matter how strongly it may be supported by the other piece of interval evidence. In this example, $m_2(A) = [0.00, 0.00]$, meaning that proposition $A$ is not supported by interval evidence $m_2$ at all, the numerator of Eq. (3) will be zero. In other words, in Wang's approach, a proposition will be supported only if both pieces of interval evidence each support it to some degrees.

Both Sevastianov's method (Sevastianov et al., 2012) and Song's method (Song et al., 2014) are based on interval arithmetic operations. The combined results of these two methods are similar to Wang's method (Wang et al., 2007); namely, if interval evidence $m_2$ does not support proposition $A$, then proposition $A$ will no longer be supported, no matter how strongly others pieces of interval evidence $m_1$, $m_3$, $m_4$, and $m_5$ support proposition $A$. It is obviously counter-intuitive.

Similar to Wang's method (Wang et al., 2007), the Chen's method (Chen & Wang, 2014) is one type of evidence discounting combination method in the framework of DST. However, the specificity of the interval evidence will be changed by using Chen's method (Chen & Wang, 2014). In order to explain this problem more clearly, we re-examine the data in Table 2, the first two pieces of interval evidence $m_1$ and $m_2$ do not have any global ignorance before combination, but after using Chen's method (Chen & Wang, 2014) there is $m(\Theta) = [0.022, 0.031]$ as shown in the fifth row and the sixth column of Table 3, it is obviously irrational. The global ignorance generated by Chen's method (Chen & Wang, 2014) is entirely due to the use of discounting method in the process of interval evidence combination; which will inevitably change the specificity of the interval evidence (Yang & Xu, 2013). In addition, Yang's method (Yang & Xu, 2013) was originally used to solve the precise evidence combination, so it is invalid to directly solve the interval evidence combination problem.

The proposed method determines the interval evidence weights from the perspective of global to minimize overall discrepancies. If a piece of interval evidence is close to most other pieces of interval evidence, then it will get a relatively large weight, and vice versa. In this example, interval evidence $m_2$ is in conflict with the others. Therefore, based on the Eq. (29), the interval evidence weights $w = (w_1, w_2, w_3, w_4, w_5) = (0.3023, 0, 0.2326, 0.2326, 0.2326)$, where, for the second evidence, the relative weight of $m_2$ is $w_2 = 0$, which reduces its interference on the combination results, thus improving the accuracy of the method in this paper. At the same time, from Table 3, it can be seen that the existing methods have deviated from the reasonable fusion result when the first two pieces of interval evidence or the first three pieces of interval evidence are combined. Because, when the first two pieces of interval evidence or the first three pieces of interval evidence are combined, the combined probability mass interval of proposition $A$ is [0,0], but the interval evidence $m_1$ supports proposition $A$ with interval-valued probability mass [0.50,0.60] before the

**Table 5**
Combination results obtained by different methods.

| | | $A$ | $A, B$ | $B$ | $B, C$ | $C$ | $P(\Theta)$ |
|---|---|---|---|---|---|---|---|
| $m_{12}$ | Wang's method | [0.265, 0.276] | [0.103, 0.107] | [0.440, 0.452] | [0.092, 0.097] | [0.080, 0.086] | – |
| | Sevastianov's method | [0.266, 0.276] | [0.104, 0.108] | [0.436, 0.457] | [0.093, 0.096] | [0.080, 0.086] | – |
| | Song's method | [0.240, 0.240] | [0.356, 0.357] | [0.067, 0.067] | [0.311, 0.311] | [0.255, 0.256] | – |
| | Chen's method | [0.260, 0.270] | [0.118, 0.123] | [0.420, 0.432] | [0.100, 0.104] | [0.082, 0.089] | – |
| | Yang's method | – | – | – | – | – | – |
| | Proposed method | [0.164, 0.169] | [0.164, 0.167] | [0.139, 0.144] | [0.151, 0.154] | [0.060, 0.064] | [0, 0] |
| $m_{123}$ | Wang's method | [0.450, 0.472] | [0.044, 0.047] | [0.360, 0.380] | [0.020, 0.022] | [0.095, 0.106] | – |
| | Sevastianov's method | [0.450, 0.475] | [0.044, 0.047] | [0.352, 0.387] | [0.020, 0.020] | [0.097, 0.105] | – |
| | Song's method | [0.423, 0.423] | [0.272, 0.272] | [0.002, 0.002] | [0.111, 0.111] | [0.192, 0.192] | – |
| | Chen's method | [0.390, 0.408] | [0.071, 0.076] | [0.371, 0.389] | [0.046, 0.050] | [0.094, 0.103] | – |
| | Yang's method | – | – | – | – | – | – |
| | Proposed method | [0.191, 0.196] | [0.162, 0.165] | [0.140, 0.145] | [0.143, 0.146] | [0.071, 0.075] | [0, 0] |
| $m_{1234}$ | Wang's method | [0.281, 0.311] | [0.001, 0.002] | [0.523, 0.552] | [0.007, 0.008] | [0.147, 0.167] | – |
| | Sevastianov's method | [0.282, 0.312] | [0.001, 0.002] | [0.504, 0.570] | [0.007, 0.008] | [0.149, 0.166] | – |
| | Song's method | [0.524, 0.524] | [0.017, 0.017] | [0.003, 0.003] | [0.059, 0.059] | [0.397, 0.397] | – |
| | Chen's method | [0.306, 0.329] | [0.032, 0.035] | [0.469, 0.493] | [0.028, 0.031] | [0.130, 0.144] | – |
| | Yang's method | – | – | – | – | – | – |
| | Proposed method | [0.192, 0.197] | [0.149, 0.152] | [0.162, 0.168] | [0.138, 0.141] | [0.089, 0.093] | [0, 0] |

**Table 6**
BPAs determined by 6 sensors.

| | $A$ | $B$ | $C$ | $A, C$ | $B, C$ | $\Theta$ |
|---|---|---|---|---|---|---|
| $m_1$ | [0.2, 0.3] | [0.0, 0.1] | [0.0, 0.4] | [0.1, 0.4] | [0.0, 0.1] | [0.0, 0.0] |
| $m_2$ | [0.2, 0.4] | [0.0, 0.1] | [0.0, 0.3] | [0.1, 0.5] | [0.0, 0.1] | [0.0, 0.1] |
| $m_3$ | [0.0, 0.2] | [0.7, 0.8] | [0.0, 0.4] | [0.1, 0.5] | [0.0, 0.1] | [0.0, 0.0] |
| $m_4$ | [0.0, 0.3] | [0.7, 0.8] | [0.0, 0.3] | [0.2, 0.5] | [0.0, 0.0] | [0.0, 0.1] |
| $m_5$ | [0.2, 0.4] | [0.0, 0.1] | [0.1, 0.4] | [0.1, 0.5] | [0.0, 0.1] | [0.0, 0.0] |
| $m_6$ | [0.2, 0.3] | [0.0, 0.1] | [0.0, 0.4] | [0.2, 0.5] | [0.0, 0.0] | [0.0, 0.1] |

combination, although the interval evidence $m_2$ does not support proposition $A$ at all. The proposed method has shown that proposition $A$ has the highest interval-valued probability masses when the first two pieces of interval evidence or the first three pieces of interval evidence are combined, and the combined result of proposition $A$ is consistent with the final fusion result. Therefore, in the case of existing conflicts situations in interval evidence, this method is different from the existing methods, and the counter-intuitive results can be suppressed. The method is still reasonable, effective, and has high convergence.

**Example 6.** Let $\Theta$ be the frame of discernment with three propositions $\{A, B, C\}$. The BPA of each piece of interval evidence is the normalized random intervals number between 0 and 1. Four pieces of interval evidence with IBSs on $\Theta$ are listed in Table 4.

It can be seen from Table 4 that there is no obvious conflict among the four pieces of random interval evidence. Assuming that the reliability of interval evidence is $r_1 = r_2 = r_3 = r_4 = 0.5$, the combined results in Table 5 are basically consistent and are all reasonable (conform to intuitive judgment) except for Song's method (Song et al., 2014), whose combined results degenerate to deterministic values (the minimum and maximum values of the interval probability mass for each proposition are equal), it is obviously irrational. In addition, Yang's method (Yang & Xu, 2013) to solve this problem is invalid. According to the Eq. (29), the relative weights of pieces of interval evidence $w = (w_1, w_2, w_3, w_4) = (0.4031, 0.4527, 0.0744, 0.0698)$, thus, it can be seen that the distribution of four pieces of random interval evidence is relatively balanced and there is low conflict among them.

At the same time, when the first two pieces of interval evidence or the first three pieces of interval evidence are combined, the proposed method has indicated that the BPA values of proposition $A$, $B$, and $C$ are compared as: $A > B > C$, which is consistent with the final fusion result. Therefore, in the case that there is no conflict or low conflict among pieces of interval evidence, the fusion results of this method are also reasonable and effective, and has high convergence.

## 6. Case study

In modern warfare, the extensive use of target recognition system and missile weapons has formed a complex, changeable and serious threat of electronic countermeasures environment. Under these conditions, electronic reconnaissance plays an increasingly important role in the war. Target recognition system is an important research direction in electronic reconnaissance, which uses the reflection (or secondary scattering) of electromagnetic wave from the target to find the target and determine its position. With the development of sensor technology,

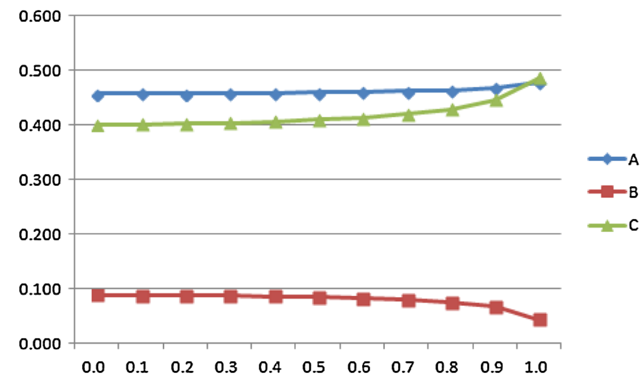


**Fig. 1a.** Changes of combined belief degrees with $r_{s_1}$.

the task of sensor is not only to measure the distance, azimuth and elevation angle of the target, but also to measure the speed of the target and to get more information about the target from the echo of the target. Then the target can be recognized and classified by using this information, so that the enemy missiles can be recognized and aimed at the target before it enters the range threatening our security, and take different action measures against the target characteristics to minimize their losses and protect themselves to the maximum extent, and make themselves in a dominant position in modern warfare.

In a military exercise at a military base, suppose that a real target is detected by an automatic target discernment system with multi-sensor. In this multisensor-based target recognition system, there are totally three types of targets $A$, $B$ and $C$, which constitute the frame of discernment $\Theta = \{A, B, C\}$, and assume the real target is $A$. There are six different sensors including charge coupled device (CCD) ($S_1$), complementary metal oxide semiconductor (CMOS) ($S_2$), electronic support measures (ESM) ($S_3$), electronic countermeasures (ECM) ($S_4$), electronic counter-countermeasures (ECCM) ($S_5$) and audio sensor system (AES) ($S_6$). From six different sensors, the system has acquired six pieces of interval evidence listed as follows:

### 6.1. Construction of the IBSs

Considering that in the actual pattern recognition problem, much of the information provided by the target recognition system may be incomplete, inaccurate or unreliable. As shown in Table 6, the information provided by $m_1$, $m_2$, $m_3$, $m_4$, $m_5$ and $m_6$ is expressed by six IBSs, and it can be concluded that these six pieces of interval evidence satisfy the conditions of Definition 9, so, they are all normalized. In this example, we assume that the reliabilities of the six sensors are $r_{s_1} = r_{s_2} = 0.4$, $r_{s_3} = r_{s_4} = 0.6$, and $r_{s_5} = r_{s_6} = 0.9$, respectively.

### 6.2. Determination of interval evidence weights

As shown in Table 6, it is evident that four pieces of interval evidence $m_1$, $m_2$, $m_5$ and $m_6$ mainly support proposition $A$. However, interval evidence $m_3$ and $m_4$ strongly support proposition $B$. Thus, interval evidence $m_3$ and $m_4$ are in conflict with the others. Therefore, based on the Eq. (29), the interval evidence weights $w = (w_1, w_2, w_3, w_4, w_5, w_6) = (0.1827, 0.2351, 0.0840, 0.0824, 0.2024, 0.2134)$,

**Table 7**
Combination results obtained by the proposed method.

| | $A$ | $B$ | $C$ | $A, C$ | $B, C$ | $\Theta$ | $P(\Theta)$ |
|---|---|---|---|---|---|---|---|
| $m_{12}$ | [0.088, 0.169] | [0.000, 0.042] | [0.023, 0.167] | [0.040, 0.194] | [0.000, 0.041] | [0.000, 0.022] | [0.555, 0.569] |
| $m_{123}$ | [0.078, 0.185] | [0.072, 0.136] | [0.021, 0.183] | [0.045, 0.202] | [0.000, 0.046] | [0.000, 0.019] | [0.484, 0.498] |
| $m_{1234}$ | [0.071, 0.186] | [0.135, 0.218] | [0.020, 0.185] | [0.058, 0.207] | [0.000, 0.041] | [0.000, 0.026] | [0.424, 0.440] |
| $m_{12345}$ | [0.140, 0.418] | [0.055, 0.171] | [0.072, 0.404] | [0.061, 0.283] | [0.000, 0.056] | [0.000, 0.011] | [0.169, 0.187] |
| $m_{123456}$ | [0.172, 0.590] | [0.022, 0.128] | [0.068, 0.575] | [0.060, 0.235] | [0.000, 0.029] | [0.000, 0.022] | [0.065, 0.084] |

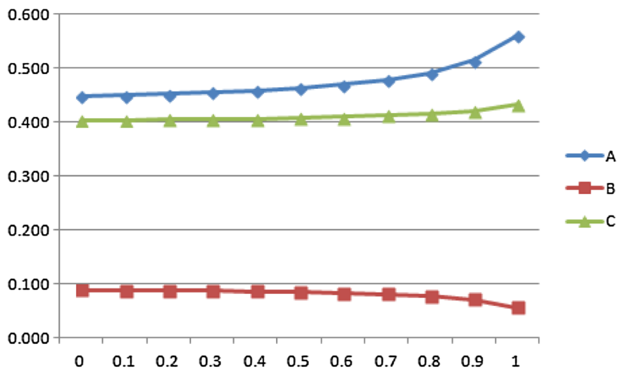


**Fig. 1b.** Changes of combined belief degrees with $r_{s_2}$.

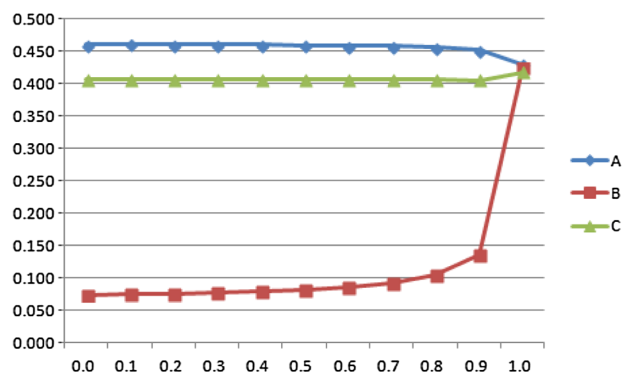


**Fig. 1c.** Changes of combined belief degrees with$r_{s_3}$.

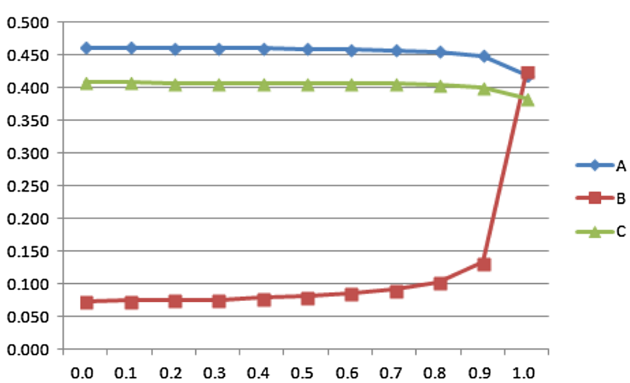


**Fig. 1d.** Changes of combined belief degrees with $r_{s_4}$.

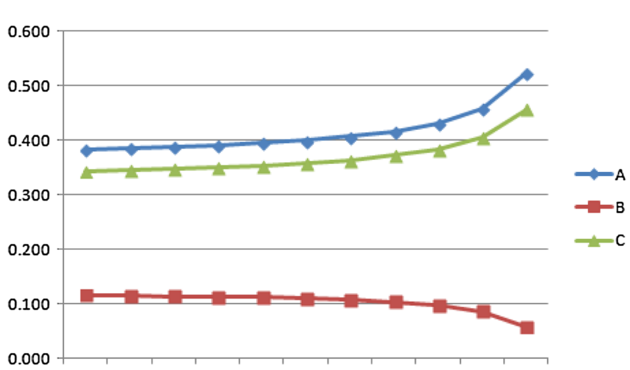


**Fig. 1e.** Changes of combined belief degrees with $r_{s_5}$.

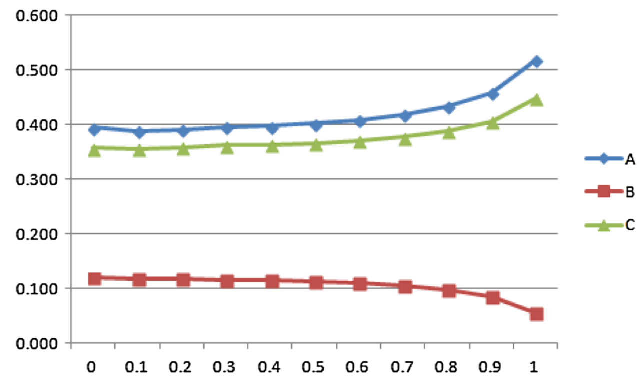


**Fig. 1f.** Changes of combined belief degrees with $r_{s_6}$.

where, for the third and fourth piece of interval evidence, the relative weights of $m_3$ and $m_4$ are $w_3 = 0.0840$ and $w_4 = 0.0824$, respectively, which reduces its interference on the combination results, thus improving the accuracy of the pattern recognition problem.

### 6.3. Generating the aggregated belief degrees

After the generation of weights and reliabilities for all pieces of interval evidence, the results of combing these six pieces of interval evidence by using Eq. (30) are detailed in Table 7. The combined results in Table 7 are all reasonable and conform to intuitive judgment. When the first two pieces of interval evidence are combined, the combination results obviously support proposition $A$. When the third and fourth pieces of interval evidences are combined, the combination results gradually turn to support proposition $B$. When the fifth and sixth pieces of interval evidences are combined, the combination results obviously begin to support proposition $A$. Therefore, this proposed method has a good ability to combine conflicting interval evidence. According to Definition 11, the final combination results can be transformed to a Bayesian belief structure as: $m'_{123456}(A) = [0.202, 0.715]$, $m'_{123456}(B) = [0.022, 0.150]$, $m'_{123456}(C) = [0.098, 0.714]$. It's easy to find that the transformed Bayesian belief structure is an interval-valued Bayesian belief structure. For comparison purposes, we take the average values of the interval-valued probability masses as: $\bar{m}'_{123456}(A) = 0.459$, $\bar{m}'_{123456}(B) = 0.086$, $\bar{m}'_{123456}(C) = 0.406$. The results show that the target identified by the system should be $A$.

### 6.4. Sensitivity analysis

To test the robustness of the methods proposed above in this paper, sensitivity analysis is to be carried out with respect to the reliability of interval evidence. In order to facilitate the analysis, we transform the final combination results into Bayesian belief structures, in the case of the weights of interval evidence are calculated according to Eq. (29), and then take the average values of the combined interval-valued probability masses.

We analyze the change of the combined belief degrees with respect to the reliability of the interval evidence. We could get the changes of the combined average probability masses on $A$, $B$ and $C$ with respect to the reliability of interval evidence as shown in Figs. 1a-1f. Here, the horizontal and vertical axes refer to the value of interval evidence reliability and the combined average probability masses respectively.

Fig. 1a shows the changes of the combined average probability masses on $A$, $B$ and $C$ with respect to $r_{s_1}$. From Fig. 1a, we can see that with the increase of the value assigned to $r_{s_1}$, the combined average probability masses assigned to $A$ and $C$ also increase gradually while the assessment to $B$ decreases. It is easy to be interpreted because the value of $r_{s_1}$ varies from 0 to 1 in the aggregation process, which makes comprehensive coefficient $\tilde{w}_1 = 0.1827/(1 + 0.1827 - r_{s_1})$ larger and larger. When $r_{s_1} = 1$, the other parameters remain the same, the combined average probability masses on $A$, $B$ and $C$ is 0.478, 0.044 and 0.486 respectively. It can be concluded that all the three curves are not sensitive to the changes of $r_{s_1}$. Besides, the three curves in Figs. 1b, 1e and 1f are similar to those in Fig. 1a, respectively, and it indicates that they have similar characteristics.

Figs. 1c and 1d show the changes of the combined average probability masses on $A$, $B$ and $C$ with respect to $r_{s_3}$ and $r_{s_4}$, respectively. It is obvious that $r_{s_3}$ and $r_{s_4}$ almost do not influence the results, and the results are similar to the judgments of others interval evidence whose weights are much larger than $w_3$ and $w_4$. From the above analysis, we can see that weight of interval evidence actually dominates the final result.

## 7. Conclusions

Interval information is common in practical decision-making problem. In this paper, the ER rule is extended to tackle with IBSs combination problem considering the weights and reliabilities of interval evidence. The purpose of this paper is to construct a more general method for interval evidence combination. Firstly, an optimization model of pignistic probability distance is established to determine the relative weights of interval evidence from the global perspective, and then a modified interval evidence combination approach is proposed which is based on ER rule. Secondly, compared with others existing approaches through numerical examples indicated that the proposed method is not only suitable for combining conflict-free interval evidence, but can also suitably combine conflicting interval evidence. Finally, the extension of the ER rule in this paper can contribute to widen its applications in pattern recognition problem. Further researches may include two aspects. First, the proposed approach will be further developed to areas with linguistic information such as GDM problem and MADA problem (Zhang, Wang, Chen, Chu, & Chen, 2018); where the key point is how to combine the assessments of DMs with different weights and reliabilities in an acceptable consistent way. Second, the consensus is an essential issue in the GDM (Zhang, Dong, Chiclana, & Yu, 2019); and how to solve the consensus problem in GDM should also consider their different backgrounds and expertise.

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