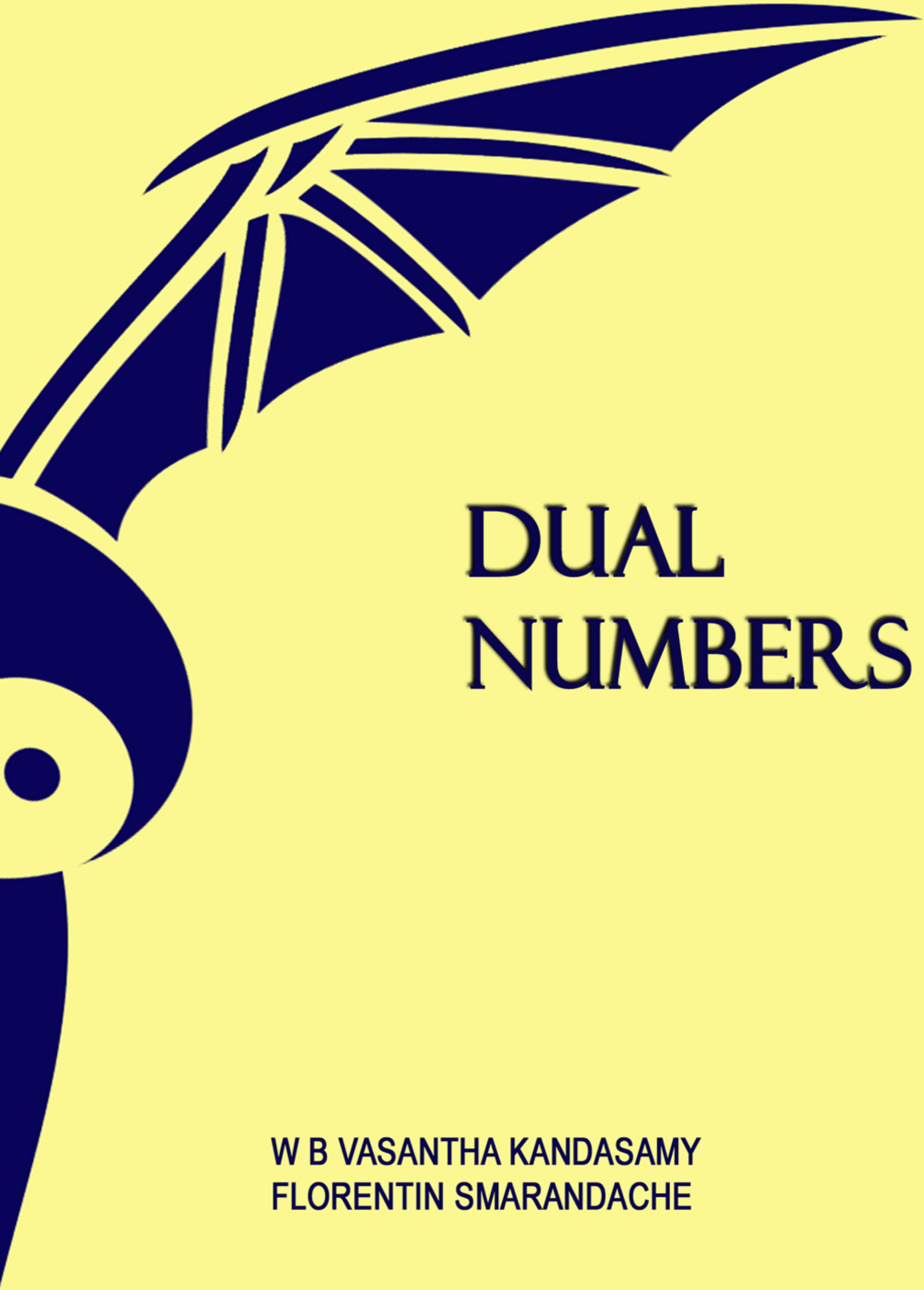

DUAL NUMBERS

W B VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Dual Numbers

W. B. Vasantha Kandasamy
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DEDICATION

WILLIAM KINGDON CLIFFORD
(1845 –1879)

We dedicate this book to
William Kingdon Clifford (1845-1879),
English mathematician and philosopher
who invented dual numbers.
Though he died really young,
he left a profound and lasting impact
on this world.
Remembered today for his work
on Clifford Algebras and
his theory of graphs,
he has been acknowledged
for anticipating and foreshadowing
several modern concepts.
His pioneering ideas on the Space-Theory of Matter
played a significant role in
Albert Einstein's development of the Theory of Relativity.
The authors take pride in remembering WK Clifford, a
mathematician whose work was highly influential.

PREFACE

Dual numbers was first introduced by W.K. Clifford in 1873. This nice concept has lots of applications; to screw systems, modeling plane joint, iterative methods for displacement analysis of spatial mechanisms, inertial force analysis of spatial mechanisms etc.

In this book the authors study dual numbers in a special way. The main aim of this book is to find rich sources of new elements g such that $g^2 = 0$. The main sources of such new elements are from Z_n, n a composite number. We give algebraic structures on them.

This book is organized into six chapters. The final chapter suggests several research level problems. Fifth chapter indicates the applications of dual numbers. The forth chapter introduces the concept of interval dual numbers, we also extend it to the concept of neutrosophic and fuzzy dual numbers. Higher dimensional dual numbers are defined, described and developed in chapter three. Chapter two gives means and methods to construct the new element g such that $g^2 = 0$. The authors feel Z_n (n a composite positive integer) is a rich source for getting new element, the main component of the dual number $x = a + bg$.

We thank Dr. K.Kandasamy for proof reading and being extremely supportive.

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

INTRODUCTION

In this book the authors study more properties about dual numbers. This concept was first defined / described by W.K. Clifford in 1873. Applications of this were studied by A.P. Kotelnikov in 1895.

Here we develop algebraic structures on dual numbers and give means to generate dual numbers. Further higher dimensional dual numbers are defined. We further define interval dual numbers, fuzzy dual numbers, neutrosophic dual numbers and finite complex modulo integer neutrosophic dual numbers.

We give here the references which would be essential to read this book.

In the first place we make use of semigroups, semirings and semivector spaces [17-19]. Also the notion of null semigroup and null rings are used [20]. We also make use of the modulo integers and finite complex modulo integers [15].

Finally the concept of neutrosophic numbers are used [10, 17]. The notion of natural class of intervals and neutrosophic class of intervals and their fuzzy analogue are used for which the reader is requested to refer [17].

However we in this book use the dual number $x = a + bg$ where g is a new element such that $g^2 = 0$ and so all powers are zero, that is $g^3 = g^4 = g^5 = \ldots = g^n = \ldots = 0$ and $0g = g0 = 0$, where a and b are reals for the given x; a and b are uniquely determined pair [2, 8].

We define $x = x_1 + x_2g_1 + x_3g_2$ where x_1, x_2 and x_3 are reals; g_1 and g_2 are new elements such that $g_1^2 = 0$ and $g_2^2 = 0$ with $g_1 \times g_2 = g_2 \times g_1 = 0$ to be a three dimensional dual number.

Suppose $x = x_1 + x_2g_1 + x_3g_2 + \ldots + x_ng_{n-1}$ where x_i's are reals $1 \leq i \leq n$, g_j's are new elements; $1 \leq j \leq n - 1$ such that $g_j^2 = 0$ with $g_jg_i = g_ig_j = 0$ if $i \neq j$; $1 \leq i, j \leq n - 1$ we define x to be a n-dimensional dual number. We generate dual numbers of any desired dimension using $m \times n$ matrices, m and n are finite integers such that $m > 1$ and $n > 1$.

Chapter Two

DUAL NUMBERS

In this chapter we introduce a new notion called general dual number ring and derive a few properties about them. These dual number general ring can be ring of matrices or polynomials.

DEFINITION 2.1: *A commutative ring R is said to be a general dual number commutative ring if every element in R is of the form a + be where a, b $\in$ R and e is a new element such that $e^2 = (0)$ and this e is unique and $e \neq 0$.*

We will first illustrate this situation by some examples.

Example 2.1: Let Z be the ring of integers. G = {{0, 1, 2} $\subseteq Z_4$ where $g_2 = 2$ so that $g_2^2 \equiv 2^2 \equiv 0 \pmod 4$ and $g_1 = 1$} be the semigroup under product, where g_2 is the new element. ZG be the semigroup ring of the semigroup G over the ring Z with a.0 = 0 and a.1 = a for all a $\in$ Z. ZG = {$a + bg_2$ | a, b $\in$ Z and $g_2 \in$ G} is a ring such that $(a + bg_2)^2 = a^2 + b^2 g_2^2 + 2abg_2 = a^2 + 2abg_2$. RS is a general dual number ring.

Example 2.2: Let R be the ring of reals.
$S = \{\{0, 1, 3\} \subseteq Z_9 \mid 3^2 \equiv 0 \pmod 9)$ denote 1 by $h_1 = 1$, 3 by $h_3\}$, h_3 is the new element. S is a semigroup under product. RS be the semigroup ring. $RS = \{a + bh_3 \mid a, b \in R, h_3 \in S, h_3^2 = 0\}$ is a general dual number ring.

Example 2.3: Let Q be the field of rationals.

$T = \{\{0, 1, 4\} \subseteq Z_{16}$ with the notion $t_1 = 1, 4 = t_4\}$; t_4 is the new element of the semigroup under multiplication modulo 16. QT the semigroup ring is a general dual number ring.

These three dual general number rings are of infinite order.

THEOREM 2.1: *Let R be a commutative ring of characteristic zero with unit (that is R = Q or Z or R only).*

$S = \{1, g, 0 \mid g^2 = 0\}$ be a semigroup under product, with g the new element. RS be the semigroup ring. RS is a dual number general ring.

Proof: $RS = \{a + bg \mid a, b \in R\}$, we see every $x \in RS$ is such that $x = a + bg$ with $g^2 = 0$. Further if $x = a + bg$ and $y = c + dg$ then;

$$\begin{aligned} x + y &= (a + bg) + (c + dg) \\ &= (a + c) + (b + d)g \in R. \\ xy &= (a + bg)(c + dg) \\ &= ac + bcg + dag + dbg^2 \\ &= ac + (bc + da)g \in R. \end{aligned}$$

Also
$$\begin{aligned} x^2 &= (a + bg)^2 \\ &= a^2 + b^2 g^2 + 2abg \\ &= a^2 + 2bag. \end{aligned}$$

Thus RS is a ring in which every element is of the form a + bg with $g^2 = 0$ and $a, b \in RS$ as $R \subseteq RS$ and $S \subseteq RS$. Hence RS is a general dual number ring.

We now proceed onto define modulo dual numbers.

DEFINITION 2.2: *Let $x = a + bg$ be such that $a, b \in Z_n \setminus \{0\}$ $(1 < n < \infty)$ be the modulo integers and g be such that $g^2 = 0$, g a new element. We define $x = a + bg$ to be the dual modulo numbers if both a and b are modulo numbers.*

Example 2.4: Let $x = a + bg$ where $g = 10 \pmod{100} \in Z_{100}$ and $a, b \in Z_7 \setminus \{0\}$. Every $x = a + bg$ is such that $g^2 = 100 \pmod{100} \equiv 0$ is a dual modulo number as a and b are modulo integers.

We now define dual general modulo number ring.

DEFINITION 2.3: *Let $F = Z_p$ (p an odd prime) be a finite field. $S = \{0, 1, g \mid 1.g = g.1 = g$ and $g^2 = 0\}$ be a semigroup under product. FS be the semigroup ring of the semigroup S over the field F. Every element $a + bg$ in FS with $a \neq 0$ and $b \neq 0$ in F is such that $(a+bg)^2 = a^2 + 2abg = c+dg$, $c, g \in F$ is a dual modulo element and the ring FS is defined as the general dual modulo integer ring.*

We illustrate this situation by some example.

Example 2.5: Let $F = Z_5$ be the field of characteristic five. $S = \{0, 1, g = 9 \mid 9 \in Z_{81}\}$ be a semigroup under product. The semigroup ring FS is a general dual modulo number ring.

Example 2.6: Let $F = Z_{23}$ be the field of characteristic twenty three. $S = \{0, 1, 4 = g \mid g = 4 \in Z_{16}\}$ be a semigroup under product. FS the semigroup ring is a general dual modulo integer number ring. For every element $a + bg$ $(a, b \in Z_{23} \setminus \{0\})$ is a dual element of FS.

The following results are interesting and important while working with dual numbers.

THEOREM 2.2: *Let S be a general dual number ring. Suppose $a + bg$ and $d + eg$ are elements in S with $a, b, d, e \in S \setminus \{0\}$ and $g^2 = 0$ then the sum of $(a + bg) + (d + eg)$ in general need not be of the form $x + yg$ $(x, y \in S \setminus \{0\})$.*

Proof: We have two cases to deal with

(1) S is an infinite general dual number ring.
(2) S is a finite modulo integer general dual number ring.

Case 1: Suppose S is an infinite general dual ring. We know (S, +) is an abelian group. Thus if $a + bg \in S$ ($a, b \in S \setminus \{0\}$) then $-a + b_1g \in S$ ($-a, b_1 \in S \setminus \{0\}$).

$a_1 + (-bg) \in S$ ($a_1, -b \in S \setminus \{0\}$) and $-a -bg \in S$.

Now $(a + bg) + (-a + b_1g) = (b+b_1)g \in S$ but $(b + b_1)g$ is not of the form $x + yg$.

Likewise we see $a + bg$ and $a_1 + -bg \in S$ and
$(a + bg) + (a_1 - bg) = (a + a_1) \in S$ but $(a + a_1)$ is not in the form $x + yg$.

Finally $(a + bg)$ and $-a -bg$ is in S and their sum is 0. Thus the sum of two dual numbers in general is not be a dual number.

Case (ii): Suppose S be a finite modulo integer general dual number ring.

Let $a + bg$ be in S such that $a, b \in S \setminus \{0\}$.
Consider $(n-1)a + b_1g \in S$, with $b_1 \neq (n-1)b \in S \setminus \{0\}$.

We see $a + bg + (n-1)a + b_1 g$

$$= (a + (n-1)a + (b+b_1)g) \pmod n$$
$$= 0 + (b_1 + b)g \pmod n.$$

Thus $(b_1 + b)g$ is not a dual number.

Likewise for $a + bg$ ($a, b \in S \setminus \{0\}$) take $a_1 + (n-1)\, bg \in S$, we see $a + bg + a_1 + (n-1)bg = (a + a_1) \pmod n$ where $a_1 \neq (n-1)\, a$ and is in $S \setminus \{0\}$.

Hence the claim.

Finally for a + bg we have (n–1)a + (n–1)bg in S is such that a + bg + (n–1)a + (n–1)bg (mod n) = 0. Thus we see sum of two general dual numbers in general is not a dual number.

We will however in the later part of the book develop an algebraic structure different from a ring where the sum of two dual numbers is also a dual number.

Example 2.7: Let F = Q the field of rationals.

S = {0, 1, 8 = g | g = 8 $\in Z_{16}$} be a semigroup under product. FS be the semigroup ring of the semigroup S over the ring F.

Consider 5 + 8g in S, we see –5 + 9g $\in$ S is such that 5 + 8g + (–5 + 9g) = 17g $\in$ S and 17g is not a general dual number.

Likewise 12 + 15g and –3 + (–15g) $\in$ S but their sum 12+(–3) + (15g) + (–15g) = 9 $\in$ S is not a general dual number.

Example 2.8: Let F = Z_{17} be the field of characteristic 17. S = {0, 1, g = 6 | g = 6 $\in Z_{12}$} be a semigroup under product. FS be the semigroup ring. Consider 8 + g and 9 + 3g $\in$ FS; we see their sum (8 + g) + (9 + 3g) (mod 17) = 4g $\in$ FS but 4g is not a general dual number.

THEOREM 2.3: *Let FS be a general dual number semigroup ring where F is a field. Suppose a + bg and x + dg $\in$ FS then (a + bg) (x + dg) is a general dual number if and only if bx + ad $\neq$ 0 (or bx $\neq$ –ad) a, b, x, d $\in F \setminus \{0\}$.*

Proof: Suppose a + bg and x + dg $\in$ FS with (a, b, x, d $\in F \setminus \{0\}$), then (a + bg) (x + dg) = ax + (bx + ad)g.

ax + (bx + ad)g is a general dual number if and only if bx + ad $\neq$ 0 since ax $\neq$ 0 as a, x $\in F \setminus \{0\}$ and F is a field.

Corollary 2.1: If F is not a field but a ring with zero divisors the above result is not valid.

Throughout this chapter we have assumed F to be Z or Q or R or Z_p (p a prime) and never a ring with zero divisors.

Can we define a dual number ring P = {a + bg | a, b ∈ F \ {0}}?

The answer is no for if we assume P to have a ring structure it becomes essential that (P, +) is an abelian group which in turn forces for every a + bg we need c + dg ∈ P such that (a + bg) + (c + dg) = (a + c) + (b + d)g and

(a + bg) (c+dg) = ac + (bc + da)g are in P, but it is natural a + c = 0 and or b + d = 0 can occur likewise bc + da = 0 or ac = 0 can also occur so even if b + d = 0 or bc + da = 0 we have in P the number associated with g viz b is 0 hence P cannot be a dual ring it can only be a general dual ring that is in P we allow a ∈ P and bg ∈ P (a, b ∈ F \ {0}).

However we have algebraic structures P in which every element is of the form a + bg where a and b are different from 0 and g is a new element such that $g^2 = 0$ and that P will not be a ring. This concept will be discussed in this book.

Note we call a + bg to be a dual number and we demand both a and b to be numbers and g such that $g^2 = 0$ where g is a new element. Infact we have infinite collection of general dual number rings which are constructured using semigroups and rings Z or Q or R or Z_p (p a prime).

We now proceed onto define dual number matrix and dual number polynomials.

DEFINITION 2.4: *Let $x = a + bg$ be such that $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ where a_i, b_j ∈ R or Z or Q or Z_p (p a prime), with g a new element such that $g^2 = 0$. We define $x = a + bg$ to be a general dual row vector (matrix) number if $x^2 = c + dg$, c and d are non zero row matrices.*

First we will illustrate this situation by some simple examples.

Example 2.9: Let x = (3, 7, 8, 1, 5) + (9, 1, 0, 2, –1)g where g = 7 ∈ Z_{49}. The row matrix takes its values from Z. We see x^2 = (9, 49, 64, 1, 25) + 2 (27, 7, 0, 2, –5)g = (9, 49, 64, 1, 25) + (54, 14, 0, 4, –10)g (since g is such that $g^2 \equiv 0$ (mod 49)), x is a general row matrix dual number.

Example 2.10: Let x = (3, 5, 1, 2) + (–4, 3, –7, 9)g where g = 3 ∈ Z_9 be a general row matrix dual number.

A natural question would be, can g be a matrix which we have considered. The answer is yet but it will only be a general dual number provided the numbers are from Z or Q or R.

We will only illustrate this situation by some examples.

Example 2.11: Let x = 9 + 8 (3, 6, 0) where (3, 6, 0 ∈ Z_9), 9, 8 ∈ Z. x is a dual number for the new element is such that (3, 6, 0) (3, 6, 0) = (0,0, 0) (mod 9).

Example 2.12: Let x = 3 + 4 (8, 4, 12, 8, 0, 4) where 3, 4 ∈ Z_{17} and 8, 4, 12 ∈ Z_{16}. We see x is only a dual number and x^2 = 9 + 16 (0 0 0 0 0 0) (as $8^2 \equiv 0$ (mod 16), $4^2 \equiv 0$ (mod 16), $12^2 \equiv 0$ (mod 16) and $8^2 \equiv 0$ (mod 16)) + 24 (8, 4, 12, 8, 0, 4) (mod 17); x = 9 + 7 (8, 4, 12, 8, 0, 4) is a general dual modulo number.

A natural question would be can be have for the dual number g to be a column matrix; yes provided we assume that the product is the natural product $\times_n$ on the column matrices.

We will illustrate this by some examples.

Example 2.13: Let $x = 2 + 7 \begin{bmatrix} 3 \\ 6 \\ 0 \\ 6 \\ 6 \end{bmatrix}$; 3, 6 ∈ Z_9 where $\begin{bmatrix} 3 \\ 6 \\ 0 \\ 6 \\ 6 \end{bmatrix}$

is a nilpotent new element of order two.

$$\text{We see } x^2 = \left(2 + 7\begin{bmatrix}3\\6\\0\\6\\6\end{bmatrix}\right)\left(2 + 7\begin{bmatrix}3\\6\\0\\6\\6\end{bmatrix}\right)$$

$$= 4 + 49\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix} + 28\begin{bmatrix}3\\6\\0\\6\\6\end{bmatrix}$$

$$= 4 + 28\begin{bmatrix}3\\6\\0\\6\\6\end{bmatrix}; \text{ here } g = \begin{bmatrix}3\\6\\0\\6\\6\end{bmatrix}$$

is a column matrix with entries from Z_9 and under the natural product $\times_n$.

$$\text{Thus } \begin{bmatrix}3\\6\\0\\6\\6\end{bmatrix} \times \begin{bmatrix}3\\6\\0\\6\\6\end{bmatrix} = \begin{bmatrix}9(\text{mod}\,9)\\36(\text{mod}\,9)\\0(\text{mod}\,9)\\36(\text{mod}\,9)\\36(\text{mod}\,9)\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}.$$

Thus g can be a choosen column matrix which is a new element that gives dual numbers.

Example 2.14: Let

$$x = 10 + (-3)\begin{bmatrix} 8 \\ 4 \\ 12 \\ 8 \\ 4 \end{bmatrix} \text{ with } 10, -3 \in Q \text{ and } 4, 12, 8 \in Z_{16},$$

$$\text{where } g = \begin{bmatrix} 8 \\ 4 \\ 12 \\ 8 \\ 4 \end{bmatrix} \text{ is a new element such that } g^2 = (0).$$

We see $x^2 = 100 - 60 \begin{bmatrix} 8 \\ 4 \\ 12 \\ 8 \\ 4 \end{bmatrix}$ using natural product on column matrices.

Next we can also have g to be a m × n (m ≠ n) rectangular matrix and still we get only a dual number provided $g^2 = (0)$.

We will illustrate this situation by some examples.

Example 2.15: Let

$$x = -3 + \begin{bmatrix} 4 & 0 & 8 \\ 4 & 0 & 4 \\ 0 & 8 & 8 \\ 0 & 4 & 4 \end{bmatrix} \text{ be such that } -3 \in Q \text{ and } 8, 4, 8 \in Z_{16}.$$

$$\begin{bmatrix} 4 & 0 & 8 \\ 4 & 0 & 4 \\ 0 & 8 & 8 \\ 0 & 4 & 4 \end{bmatrix}$$

is the nilpotent element of order two.

We see $x^2 = -9 + \begin{bmatrix} 4 & 0 & 8 \\ 4 & 0 & 4 \\ 0 & 8 & 8 \\ 0 & 4 & 4 \end{bmatrix} \times_n \begin{bmatrix} 4 & 0 & 8 \\ 4 & 0 & 4 \\ 0 & 8 & 8 \\ 0 & 4 & 4 \end{bmatrix} -6 \begin{bmatrix} 4 & 0 & 8 \\ 4 & 0 & 4 \\ 0 & 8 & 8 \\ 0 & 4 & 4 \end{bmatrix}$

$$= -9 + \begin{bmatrix} 16 & 0 & 64 \\ 16 & 0 & 16 \\ 0 & 64 & 64 \\ 0 & 16 & 16 \end{bmatrix} - 6 \begin{bmatrix} 4 & 0 & 8 \\ 4 & 0 & 4 \\ 0 & 8 & 8 \\ 0 & 4 & 4 \end{bmatrix}$$

$$= -9 - 6 \begin{bmatrix} 4 & 0 & 8 \\ 4 & 0 & 4 \\ 0 & 8 & 8 \\ 0 & 4 & 4 \end{bmatrix},$$

we have used natural product to find the value of x^2.

Thus x is a dual number with $g = \begin{bmatrix} 4 & 0 & 8 \\ 4 & 0 & 4 \\ 0 & 8 & 8 \\ 0 & 4 & 4 \end{bmatrix}$.

Example 2.16: Let

$$x = 12 + 5 \begin{bmatrix} 3 & 6 & 0 & 6 \\ 6 & 6 & 6 & 6 \\ 3 & 3 & 3 & 0 \end{bmatrix} \text{ where } 12, 5 \in Z \text{ and } 3, 6 \in Z_9$$

we see x is a dual number and $g = \begin{bmatrix} 3 & 6 & 0 & 6 \\ 6 & 6 & 6 & 6 \\ 3 & 3 & 3 & 0 \end{bmatrix}$ is the new element such that $g^2 = g \times_n g$ under the natural product of matrices.

Finally we can replace the rectangular matrices by square matrices. We can arrive g such that $g^2 = (0)$ is a nilpotent matrix under usual product or a nilpotent matrix under the natural product. We describe both by some examples. However still those numbers will be known as the dual numbers only g is a square matrix with $g^2 = (0)$.

Example 2.17: Let

$$x = 3 + 7 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad g = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$g^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

under usual product. However $g^2 \neq (0)$ under natural product, $\times_n$.

However x is a dual number.

We have nice theorem for square matrices with entries from Q or Z or R.

THEOREM 2.4: *$x = a + b\,(m_{ij})_{n\times n} = a + bA$ where $A = (m_{ij})$ is a dual number if and only if $A \times A = (m_{ij}) \times (m_{ij}) = (0)$ that is if and only if A is a nilpotent matrix under usual product.*

Proof is direct and hence left as an exercise to the reader.

Note however a matrix nilpotent under usual product will not be nilpotent under natural product $\times_n$.

Example 2.18: Let

$$x = 9 + 3\begin{pmatrix} 4 & 8 & 12 \\ 12 & 4 & 0 \\ 8 & 4 & 12 \end{pmatrix}$$

where $4, 8, 12 \in Z_{16}$ we find

$$\begin{pmatrix} 4 & 8 & 12 \\ 12 & 4 & 0 \\ 8 & 4 & 12 \end{pmatrix} \times_n \begin{pmatrix} 4 & 8 & 12 \\ 12 & 4 & 0 \\ 8 & 4 & 12 \end{pmatrix}$$

(under natural product) $= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ so that

$$x^2 = 81 + 54\begin{pmatrix} 4 & 8 & 12 \\ 12 & 4 & 0 \\ 8 & 4 & 12 \end{pmatrix}.$$

Also under the usual matrix product

$$x^2 = \left(9 + 3\begin{pmatrix} 4 & 8 & 12 \\ 12 & 4 & 0 \\ 8 & 4 & 12 \end{pmatrix}\right)^2$$

$$= 81 + 54\begin{pmatrix} 4 & 8 & 12 \\ 12 & 4 & 0 \\ 8 & 4 & 12 \end{pmatrix}.$$

We have to make the study. If a square matrix A is such that $A^2 = (0)$ under usual product $A \times_n A = (0)$

$$\text{However } A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \text{ is such that}$$

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ under usual product;}$$

$$\text{however } A \times_n A \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq (0).$$

The converse that is if A is such that $A \times_n A = (0)$ will $A \times A = (0)$. This is to be studied ?

Now we can have the dual number to be a matrix e such that $e^2 = (0)$ and a + eb is such that $(a+eb)^2 = c + de$ for a, b, c and d reals.

Now e can also be polynomials p(x) with coefficients from Z_n and $e = p(x)$ with $e^2 = (0)$.

For take $p(x) = 3x^3 + 6x^2 + 6$ with $p(x) \in Z_9[x]$ we see $(p(x))^2 = 0$. Thus a+bp(x), a, b $\in$ Z (or Q or R) is such that $(a+bp(x))^2 = c+dp(x)$ and $(p(x))^2 = 0$.

Thus we wish to state in a dual number the new element e can be modulo integers or matrices or polynomials or even intervals or interval matrices.

For take e = [3, 6] $\in N_c(Z_9)$ we see e^2 = [9, 36] (mod 9) = [0, 0] (mod 9).

Thus a + be is a dual number. It is easily seen we can replace the interval by matrices A with interval entries such that $A^2 = (0)$ the product can be natural product in case of column matrices and rectangular matrices.

Interested reader can give examples of them as it is considered to be a matter of routine.

Now we continue the notion of dual matrix number A + Be where A and B are matrices of same order with $e^2 = 0$, such that $(A + Be)^2 = C + De$, D and C matrices of same order.

We illustrate this situation by some examples.

Let

$$x = \begin{bmatrix} 3 & 2 \\ 7 & 0 \\ 1 & 4 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 5 & 0 \\ 8 & 1 \end{bmatrix} g$$

where $g = 5 \in Z_{25}$.

$$\text{We see } x^2 = 2\begin{bmatrix} 3 & 2 \\ 7 & 0 \\ 1 & 4 \\ 5 & 3 \end{bmatrix} \times_n \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 5 & 0 \\ 8 & 1 \end{bmatrix} g$$

$$+ \begin{bmatrix} 3 & 2 \\ 7 & 0 \\ 1 & 4 \\ 5 & 3 \end{bmatrix} \times_n \begin{bmatrix} 3 & 2 \\ 7 & 0 \\ 1 & 4 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 5 & 0 \\ 8 & 1 \end{bmatrix} \times_n \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 5 & 0 \\ 8 & 1 \end{bmatrix} g \times g$$

$$= 2\begin{bmatrix} 6 & 2 \\ 7 & 0 \\ 5 & 0 \\ 40 & 3 \end{bmatrix} g + \begin{bmatrix} 9 & 4 \\ 49 & 0 \\ 1 & 16 \\ 25 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 1 & 4 \\ 25 & 0 \\ 64 & 1 \end{bmatrix} \times 0$$

$$= \begin{bmatrix} 12 & 4 \\ 14 & 0 \\ 10 & 0 \\ 80 & 6 \end{bmatrix} g + \begin{bmatrix} 9 & 4 \\ 49 & 0 \\ 1 & 16 \\ 25 & 9 \end{bmatrix}.$$

This is an example of a dual rectangular matrix number.

Consider

$$y = \begin{bmatrix} 3 & 1 & 4 & 2 \\ 0 & 5 & 3 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} g$$

where $g = 6 \in Z_{12}$ and the matrices take their entries from Z (or Q or R). We see y is a dual rectangular matrix number.

Further

$$y^2 = \begin{bmatrix} 9 & 1 & 16 & 4 \\ 0 & 25 & 9 & 49 \end{bmatrix} + \begin{bmatrix} 42 & 0 & 8 & 8 \\ 0 & 10 & 6 & 42 \end{bmatrix} g$$

as $g^2 = 0 \pmod{12}$.

Example 2.19: Let

$$x = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 4 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 6 & 0 \end{bmatrix} g \text{ where } g = 2 \in Z_4.$$

$$x^2 = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 4 & 5 & 7 \end{bmatrix}^2 + \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 4 & 5 & 7 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 6 & 0 \end{bmatrix} g +$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 6 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 4 & 5 & 7 \end{bmatrix} g + \left(\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 6 & 0 \end{bmatrix}\right)^2 g^2 \ (g^2 = 0)$$

$$= \begin{bmatrix} 29 & 41 & 58 \\ 8 & 11 & 16 \\ 40 & 56 & 79 \end{bmatrix} + \begin{bmatrix} 0 & 49 & 26 \\ 3 & 16 & 5 \\ 15 & 66 & 33 \end{bmatrix} g + \begin{bmatrix} 8 & 11 & 16 \\ 29 & 41 & 58 \\ 0 & 6 & 12 \end{bmatrix} g$$

$$= \begin{bmatrix} 29 & 41 & 58 \\ 8 & 11 & 16 \\ 40 & 56 & 79 \end{bmatrix} + \begin{bmatrix} 8 & 60 & 42 \\ 32 & 57 & 63 \\ 15 & 72 & 45 \end{bmatrix} g.$$

x is a dual square matrix number. We can also using the same x find the square of x using natural product $\times_n$ on matrices;

$$\text{in that case } x^2 = \begin{bmatrix} 9 & 16 & 25 \\ 0 & 1 & 4 \\ 16 & 25 & 49 \end{bmatrix} + \begin{bmatrix} 0 & 8 & 20 \\ 0 & 8 & 20 \\ 0 & 60 & 0 \end{bmatrix} g.$$

We see x^2 under usual product is not equal to x^2 under natural product $\times_n$ of square matrices.

Now we can find the dual polynomial number.

We take $x = p(x) + q(x)\, g$ where $g^2 = 0$ with $p(x), q(x) \in Z[x]$ (or $R[x]$ or $Q[x]$). We see $x^2 = r(x) + s(x)g$ where $r(x), s(x) \in Z[x]$ (or $R[x]$ or $Q[x]$).

We will first illustrate this situation by some examples.

Example 2.20: Let M = $(x^3 + 2x + 1) + (x^2 + 5x + 7)g$ where $g = 10 \in Z_{20}$.

Consider $M^2 = [(x^3 + 2x + 1) + (x^2 + 5x + 7)g)^2$

$= \quad (x^3 + 2x + 1)^2 + (x^3 + 2x + 1)\,(x^2 + 5x + 7)g.$

$= \quad (x^9 + 4x^2 + 1 + 4x^4 + 4x + 2x^3) + (x^5 + 2x^3 + x^2 + 5x^4 + 10x^2 + 5x + 7x^3 + 14x + 7)g$

$= \quad (x^9 + 4x^4 + 2x^3 + 4x^2 + 1) + (x^5 + 5x^4 + 9x^3 + 11x^2 + 19x + 7)g;$

where the polynomials take their entries from Z[x] (or Q[x] or R[x]).

Example 2.21: Let n = $(3x^3 - 4x + 2) + (2x - 7)g$ where g = 8 $\in Z_{16}$.
We see n is a dual polynomial number.

$$n^2 = (3x^3 - 4x + 2)^2 + 2(2x\text{–}7)\,(3x^3 - 4x + 2)g$$
$$= (9x^6 + 16x^2 + 4 + 12x^3 - 24x^4 - 16x) + 2\,(6x^4 - 21x^3 - 8x^2 + 28x + 4x - 14)g.$$

$$= (9x^6 + 12x^3 - 24x^4 + 16x^2 - 16x + 4) + (12x^4 - 42x^3 - 16x^2 + 64x - 28)g$$

is again in the form a = p(x) + q(x)g.

We can replace e by a matrix A such that $A^2 = (0)$ or a polynomial p(x) such that $(p(x))^2 = 0$, still those will be dual matrix number or a dual polynomial number and so on.

Now we proceed onto give algebraic structures on them.

Let us proceed onto define structures on them.

Suppose M = {A + Bg | A, B $\in$ {set of all $1 \times n$ row matrices with entries from R or Q or Z} and $g^2 = 0$}. M is a general dual number row matrix ring.

If A, B is replaced by $m \times 1$ column matrix in M then M is a general dual number column matrix ring under natural product.

If A and B in M is replaced by $m \times n$ ($m \neq n$) matrices then M is a general dual number rectangular matrix ring.

On similar lines we can define general dual number square matrix ring commutative under natural product $\times_n$ and a non commutative ring under usual product $\times$.

We will give examples of these rings in the following.

Example 2.22: Let

P = {A + Bg | A, B $\in$ {set of all $n \times 1$ column matrices with entries from R or Q or Z} and $g^2 = 0$} be a general dual number column matrix ring under natural product $\times_n$.

Example 2.23: Let

$$S = \{A + Bg \mid A, B \in \{\text{set of all } m \times n\ (m \neq n)$$

matrices with entries from R or Q or Z} with $g^2 = 0$} be the general dual number rectangular matrices dual number ring under natural product $\times_n$.

Example 2.24: Let

$$S = \left\{A + Bg \;\middle|\; A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_7 \end{bmatrix};\ a_i, b_j \in Z \text{ (or Q or R)}\right.$$

$$\left. g = 3 \in Z_9,\ 1 \leq i, j \leq 7\right\}$$

be the dual column matrix ring under the usual product $\times_n$.

$$A = \begin{bmatrix} 8 \\ 3 \\ 0 \\ 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ -8 \\ 2 \\ 5 \\ 3 \\ 0 \\ 1 \end{bmatrix} \text{ then } A + Bg = \begin{bmatrix} 8 \\ 3 \\ 0 \\ 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -8 \\ 2 \\ 5 \\ 3 \\ 0 \\ 1 \end{bmatrix} g \in S.$$

$$(A + Bg)^2 = A^2 + B^2g^2 + 2ABg \; (g^2 \equiv 0 \pmod 9)$$

$$= A^2 + 2\, ABg.$$

This way addition and natural multiplication is performed. S is a ring which is commutative.

Example 2.25: Let $S = \{A + Bg \mid A, B \in 3 \times 3$ square matrix with entries from Z or Q or R with $g = 4 \in Z_{16}$, $g^2 = 0 \pmod{16}\}$ be a ring which is commutative. S is a non commutative dual square matrix number ring. If the usual product is replaced by the natural product $\times_n$ then S is a commutative dual square matrix number ring.

Example 2.26: Let $P = \{A + Bg \mid A$ and B are 4×2 matrices with entries from Z or Q or R; with $g = 5 \in Z_{25}$ such that $g^2 \equiv 0 \pmod{25}\}$ be the dual number commutative 4×2 matrix ring.

$$\text{Take } x = \begin{bmatrix} 2 & 0 & 1 & 5 \\ 7 & 3 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 5 & 0 \end{bmatrix} g \text{ in P.}$$

$$\text{We find } x^2 = \left\{ \begin{bmatrix} 2 & 0 & 1 & 5 \\ 7 & 3 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 5 & 0 \end{bmatrix} g \right\}$$

$$= \begin{bmatrix} 2 & 0 & 1 & 5 \\ 7 & 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 5 \\ 7 & 3 & 0 & 2 \end{bmatrix} +$$

$$2\begin{bmatrix} 2 & 0 & 1 & 5 \\ 7 & 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 5 & 0 \end{bmatrix} g +$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 5 & 0 \end{bmatrix} g^2$$

$$= \begin{bmatrix} 4 & 0 & 1 & 25 \\ 49 & 9 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 & 30 \\ 0 & 6 & 0 & 0 \end{bmatrix} g$$

($g^2 = 0$ so the last term is 0). We see P is a dual 4×2 matrix number ring.

Example 2.27: Let T = $\{A + Bg \mid$ A and B are 2×5 rectangular matrices with entries from Z (or Q or R) where $g = 3 \in Z_9$ and $g^2 \equiv 0 \pmod 9\}$ be a dual number rectangular 2×5 matrix ring under natural product $\times_n$.

$$\text{Let } x = \begin{bmatrix} 3 & 6 & 1 & 5 & -1 \\ 2 & -1 & 4 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 \end{bmatrix} g \in P,$$

then

$$x^2 = \begin{bmatrix} 9 & 36 & 1 & 25 & 1 \\ 4 & 1 & 16 & 9 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 24 & 2 & 10 & 0 \\ 0 & -2 & 16 & 6 & 8 \end{bmatrix} g \in P.$$

This is the way dual number rectangular 2×5 matrix ring is obtained.

We have seen in a dual number $a + bg$ with $a, b \in Z$ (or Q or R) can be replaced by $m \times n$ matrices where $m = 1$ or $n = 1$ or $m \neq n$ then we call the dual number to be dual matrix number.

In a similar way we can define dual polynomial number $p(x) + q(x)g$ where g is a new element such that $g^2 \equiv 0$.

We now see how they look.

Example 2.28: Let

$M = \{p(x) + q(x)\, g \mid p(x), q(x) \in Z[x]$ (or $R[x]$ or $Q[x]$) where $g = 4 \in Z_{16}\}$ be a dual polynomial number ring which is commutative.

Take $p = (3x^3 + 5x - 3) + (7x^2 - 8x + 1)g$ in M then

$$p^2 = \{(3x^3 + 5x - 3) + (7x^2 - 8x + 1)g\}^2$$
$$= (3x^3 + 5x - 3)^2 + 2(3x^3 + 5x - 3)(7x^2 - 8x + 1)g + (7x^2 - 8x + 1)^2g^2 \ (g^2 = 0)$$

$$= (9x^6 + 25x^2 + 9 + 30x^4 - 30x - 18x^3) + (42x^5 + 70x^3 - 42x^2 - 48x^4 - 80x^2 + 48x + 6x^3 + 10x - 6)g$$

$$= (9x^6 + 30x^4 - 18x^3 + 25x^2 - 30x + 9) + (42x^5 - 48x^4 + 76x^3 - 122x^2 + 58x - 6)g \text{ is in M.}$$

It is easily verified that M under addition is commutative and associative. Thus M is a dual number polynomial commutative general ring.

Now having seen dual polynomial numbers and dual matrix number we now proceed on to define the new notion of dual numbers which are exclusive of elements a and bg that is only $a + bg$ ($a \neq 0$ and $b \neq 0$).

To this end we first recall the notion of semifields.

A non empty set S with two binary operations + and × is a semifield if the following conditions are true

(i) (S, +) is a commutative monoid under +,
(ii) (S, ×) is a commutative monoid under ×,
(iii) (S, +, ×) is a strict semiring and
(iv) S has no zero divisors, then S is a semifield.

$Z^+ \cup \{0\}$ is a semifield $Q^+ \cup \{0\}$ is a semifield and $R^+ \cup \{0\}$ is a semifiled. Let S be a semifield. Take a g such that $g^2 = 0$.

Consider $P = \{a + bg \mid a, b \in S \setminus \{0\}$ with $g^2 = 0\} \cup \{0\}$ where $S = Z^+ \cup \{0\}$ or $S = Q^+ \cup \{0\}$ or $S = R^+ \cup \{0\}$. P is a dual number semifield.

For a + bg and c + dg we have a + bg + c + dg = a + c + (b + d)g; since $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ is a strict semifield. We see P is also strict and $a + c \neq 0$ and $d + b \neq 0$ so $(a + c) + (b + d)\, g \in P$.

Consider (a + bg) (c + dg)

$$= ac + bcg + dag + bdg^2$$
$$= ac + (bc + da)g \qquad (\because g^2 = 0)$$

We see $ac \neq 0$ and $bc + da \neq 0$ as $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ is a strict semifield.

Thus P is a semifield of dual numbers.

We give examples of this situations.

Example 2.29: Let

$$H = \{a + b \begin{bmatrix} 2 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \mid a, b \in Z^+, 2 \in Z_4\} \cup \{0\}$$

is again the dual number semifield.

Example 2.30: Let

$$M = \{a + b \begin{bmatrix} 4 \\ 0 \\ 8 \\ 12 \end{bmatrix} \mid a, b \in Z^+; 4, 8, 12 \in Z_{16}\} \cup \{0\}$$

is the dual number semifield / semiring.

Example 2.31: Let

$$T = \{a + b\begin{bmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 5 \end{bmatrix} \mid a, b \in Z^+; 5 \in Z_{25}\}$$

be the dual number semifield | semiring

$$\text{Consider } x = 3 + 2\begin{bmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 5 \end{bmatrix} \text{ and } y = 7 + 4\begin{bmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 5 \end{bmatrix} \in T;$$

$$xy = 3 \times 7 + (12 + 14)\begin{bmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 5 \end{bmatrix} + 8\begin{bmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 5 \end{bmatrix}^2$$

$$= 21 + 26\begin{bmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 5 \end{bmatrix} + 8\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 21 + 26\begin{bmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 5 \end{bmatrix} \in T.$$

Thus T is a dual number semiring / semifield.

Example 2.32: Let

$$S = \{a + bg \mid a, b \in R^+, g = 10 \text{ with } g^2 = 0 \pmod{29}\}$$

be the semiring of dual numbers.

THEOREM 2.5: *Let $S = Z^+ \cup \{0\}$ (or $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$) be a semifield. Consider $M = \{a + bg \mid a, b \in S$ with $g^2 = 0\}$, M is a general dual number semiring.*

The proof is direct and hence left as an exercise to the reader.

Next we proceed onto define / describe dual number matrix semiring and dual number polynomial semiring.

We will just describe the dual number row matrix semiring.

Consider $M = \{A + Bg \mid A = (x_1, \ldots, x_n)$ and $B = (y_1, \ldots, y_n)$ where $x_i, y_i \in Z^+$ (or Q^+ or R^+), $1 \leq i \leq n$ and g such that $g^2 = 0\} \cup \{(0, 0, \ldots, 0)\}$. M is the dual number row matrix semiring.

We will give an example or two.

Example 2.33: Let

$$P = \{(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4)g \mid x_i, y_j \in Z^+,$$

$$1 \leq i, j \leq 4, g = 3 \in Z_9\} \cup \{(0, 0, 0, 0)\}$$

be the dual number row matrix semiring / semifield.

Example 2.34: Let

$$P = \{(x_1, x_2, x_3) + (y_1, y_2, y_3) \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} \mid x_i, y_i \in Q^+, 2 \in Z_4;$$

$$1 \leq i \leq 3\} \cup \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

be the dual number row matrix semifield.

Example 2.35: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} g \;\middle|\; b_j, a_i \in R^+, g = 10 \in Z_{20} \right.$$

$$\left. \text{with } 1 \leq i, j \leq 4 \right\} \cup \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

be the dual number column matrix semifield.

Example 2.36: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{10} \end{bmatrix} (12,4,8,0,8,4) \;\middle|\; a_i, b_j \in Q^+, 1 \leq i, j \leq 10; \right.$$

$$\left. 12, 8, 4 \in Z_{16} \right\} \cup \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

is the dual number column matrix semifield under natural product of column matrices.

Example 2.37: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{bmatrix} g \;\middle|\; a_i, b_j \in R^+, \right.$$

$$1 \le i, j \le 6;\ g = 6 \in Z_{12}\} \cup \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\}$$

be a dual number 2 × 3 matrix semifield.

Example 2.38: Let

$$W = \left\{\left.\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ \vdots & \vdots & \vdots \\ b_{13} & b_{14} & b_{15} \end{bmatrix}(g_1, g_2, g_3)\right)\ \right|\ a_i, b_j \in R^+,\right.$$

$$1 \le i, j \le 15,\ g_1 = 4,\ g_2 = 8 \text{ and } g_3 = 12,\ g_1, g_2, g_3 \in Z_{16}\} \cup \{(0)\}$$

be the dual number 5 × 3 matrix semifield under natural product.

Example 2.39: Let

$$M = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} g\ \right|\ a_i, b_j \in Z^+,\ 1 \le i, j \le 9\right.$$

$$\text{with } g^2 = (0);\ g = 10 \in Z_{20}\} \cup \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

be only a dual square matrix number semiring under usual product and a dual number semifield under the natural product $\times_n$. M is a non commutative dual square matrix number semiring.

Example 2.40: Let

$$T = \left\{ \left[\begin{array}{ccccc} a_1 & a_2 & a_3 & \ldots & a_8 \\ a_9 & a_{10} & a_{11} & \ldots & a_{16} \\ a_{17} & a_{18} & a_{19} & \ldots & a_{24} \end{array}\right] + \left[\begin{array}{ccccc} b_1 & b_2 & b_3 & \ldots & b_8 \\ b_9 & b_{10} & b_{11} & \ldots & b_{16} \\ b_{17} & b_{18} & b_{19} & \ldots & b_{24} \end{array}\right] g \right|$$

$a_i, b_j \in R^+, 1 \leq i, j \leq 24, g = 4 \in Z_{16}$, with $g^2 = 0\} \cup$

$$\left\{ \left[\begin{array}{ccccc} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{array}\right] \right\}$$

be a dual number 3×8 matrix semifield under usual product $\times_n$.

Now having seen examples of dual number matrix semifields / semirings now we proceed onto define dual number polynomial semifields.

Let $S = \{p(x) = \sum_{i=0}^{\infty} a_i x^i \mid a_i \in Z^+ \text{ or } R^+ \text{ or } Q^+\} \cup \{0\}$ be a polynomial semifield in the variable x.

Consider

$P = \{p(x) + q(x) g \mid p(x), q(x) \in S$ and g is such that $g^2 = 0\}$; P is a semifield defined as the dual number polynomial semifield.

We will first illustrate this situation by some examples.

Example 2.41: Let

$M = \{p(x) + q(x)g \mid p(x), q(x) \in Z^+[x] \cup \{0\}$ where $g = 5 \in Z_{25}\}$

be the dual polynomial number semifield.

Example 2.42: Let

$$P = \{p(x) + q(x)g \mid p(x), q(x) \in Q^{+}[x] \cup \{0\}$$

$$\text{where } g = \begin{bmatrix} 4 & 8 & 12 \\ 8 & 4 & 12 \\ 12 & 8 & 4 \end{bmatrix}; 4, 8, 12 \in Z_{16}\}$$

be only a dual polynomial number semiring under usual matrix product, but a dual polynomial number semifield under natural product of matrices.

Example 2.43: Let

$$S = \{p(x) + q(x) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

where $p(x), q(x) \in Z^{+}[x] \cup \{0\}$ and $a_i \in \{4, 8, 0, 12\} \subseteq Z_{16}$, $1 \leq i \leq 4\}$

be a dual number polynomial semifield.

$$\text{We see } \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \times_n \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ where } a_i, b_j \in \{4, 8, 12, 0\}$$

$$\subseteq Z_{16}, 1 \leq i, j \leq 4.$$

Example 2.44: Let $S = \{p(x) + q(x) (a_1, a_2, a_3, a_4, a_5, a_6) \mid p(x), q(x) \in Z^{+} [x] \cup \{0\}, a_i \in \{0, 11\} \subseteq Z_{121}, 1 \leq i \leq 6\}$ be a dual polynomial number semifield.

Now having seen examples of dual number general ring and dual number semifield / semiring we now proceed onto define / describe and develop the concept of dual number vector space and dual number semivector space and their Smarandache analogue.

DEFINITION 2.5: *Let*
$V = \{a + be \mid a, b \in Q$; e a new element is such that $e^2 = 0\}$. V is an abelian group under addition. Clearly V is a vector space over the field F = Q. We define V to be a general dual number vector space over F.

One can study the basis, dimension and transformation of these spaces. This can be treated as a matter of routine and we proceed onto give examples of these structures.

Example 2.45: Let $V = \{(a_1, a_2, a_3) \mid a_i = x_i + y_ig$ where $x_i, y_i \in R$, $1 \leq i \leq 3$ and g is a new element such that $g^2 = (0)\}$. V is a dual number vector space over R (also V is a general dual number vector space over Q).

Clearly dimension of V over R is nine.

Take $W = \{(a, a, a) \mid a = x + yg$ where $x, y \in R$, $g^2 = 0\} \subseteq V$. W is a subspace of V called the dual number vector subspace of V over R. However the dimension is different if R in example 2.45 is replaced by Q. We will denote by

$R(g) = \{a + bg \mid a, b \in R$ and $g^2 = 0\}$ and

$Q(g) = \{a + bg \mid a, b \in Q; g^2 = 0\}$. Clearly for the same g we have $Q(g) \subseteqq R(g)$.

With this notation for sake of easy representation we give more examples of general dual number vector spaces.

Example 2.46: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in Q\,(g);\ 1 \leq i \leq 5,\ g^2 = 0 \right\}$$

be a general dual number vector space over Q, the field.

Example 2.47: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \end{bmatrix} \middle| \; a_i \in R\,(e);\ e^2 = 0 \text{ where } 1 \leq i \leq 21 \right\}$$

be a general dual number vector space over the field R or Q.

Example 2.48: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in R(g),\ g = 4 \in Z_{16};\ 1 \leq i \leq 9 \right\}$$

be a general dual number vector space over the field R (or Q).

Take

$$W = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \end{bmatrix} \middle| \; a_i \in R\,(g)\ 1 \leq i \leq 3,\ g = 4 \in Z_{16}, \right.$$

$$\left. \text{with } g^2 = (0) \right\} \subseteq V;$$

W is a general dual number vector subspace of V over R.

Example 2.49: Let

$$T = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} \middle| \; a_i \in R\,(g);\ 1 \le i \le 8;\ g = 3 \in Z_9, \; g^2 = (0) \right\}$$

be a general dual number vector space over the field R.

Take

$$W = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; a_i \in R\,(g);\ 1 \le i \le 4; \; g = 3 \in Z_9,\ g^2 = (0) \right\} \subseteq T;$$

W is a general dual number vector subspace of T over R.

Example 2.50: Let

$$V = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| \; a_i \in Q\,(g) \text{ with } 1 \le i \le 6; \; g = 6 \in Z_{36},\ g^2 = (0) \right\}$$

be a dual number vector space over Q.

Consider

$$P_1 = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; a_1 \in Q\,(g) \right\} \subseteq V,$$

$$P_2 = \left\{ \begin{pmatrix} 0 & a_2 & a_3 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; a_2, a_3 \in Q\,(g) \right\} \subseteq V,$$

$$P_3 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ a_4 & a_5 & 0 \end{pmatrix} \middle| \ a_4, a_5 \in Q\,(g)\right\} \subseteq V \text{ and}$$

$$P_4 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_6 \end{pmatrix} \middle| \ a_6 \in Q\,(g)\right\} \subseteq V$$

be dual number vector subspaces of V over Q.

Clearly $P_i \cap P_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ if $i \neq j$ and $V = P_1 + P_2 + P_3 + P_4$; thus V is the direct sum of dual number vector subspaces of V over Q.

On similar lines we can define pseudo direct sum of dual number vector subspaces if $P_i \cap P_j \neq (0)$ if $i \neq j$ and so on.

This task is simple and hence left as an exercise to the reader.

Example 2.51: Let $M = \{a + bg \mid a, b \in R \text{ with } g = 3 \in Z_9\}$ be a dual number vector space over the field Q. Clearly $W \subseteq M$ is not a direct sum with any other subspace where
$W = \{a + bg \mid a, b \in Q \text{ with } g = 3 \in Z_9\} \subseteq M$ is dual number subspace of M over Q. We see $W \oplus T = M$ is not possible for any $T \subseteq M$.

Example 2.52: Let

$$S = \left\{(a_1, a_2, a_3, a_4) + (b_1, b_2, b_3, b_4)\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \middle| \ a_i, b_j \in Q \text{ and } 2 \in Z_4;\ 1 \leq i, j \leq 4\right\}$$

be a dual number vector space over Q.

Take

$$P_1 = \{(a_1, 0, 0, 0) + (b_1, 0, 0, 0)\begin{bmatrix}2\\0\\2\\0\end{bmatrix} \Bigg| \; a_1, b_1 \in Q\} \subseteq S,$$

$$P_2 = \{(0, a, 0, 0) + (0, b, 0, 0)\begin{bmatrix}2\\0\\2\\0\end{bmatrix} \Bigg| \; a, b \in Q\} \subseteq S,$$

$$P_3 = \{(0, 0, a, 0) + (0, 0, b, 0)\begin{bmatrix}2\\0\\2\\0\end{bmatrix} \Bigg| \; a, b \in Q\} \subseteq S \text{ and}$$

$$P_4 = \{(0, 0, 0, a) + (0, 0, 0, b)\begin{bmatrix}2\\0\\2\\0\end{bmatrix} \Bigg| \; a, b \in Q\} \subseteq S$$

be the dual number vector subspaces of S.

Clearly $P_i \cap P_j = (0)$ if $i \neq j$. Further $P_1 + P_2 + P_3 + P_4 = V$. Take

$$B_1 = \{(a_1, a_2, 0, 0) + (0, 0, b_1, b_2)\begin{bmatrix}2\\0\\2\\0\end{bmatrix} \text{ where } a_i, b_j \in Q;$$

$$1 \leq i, j \leq 2\} \subseteq S,$$

$$B_2 = \{(a_1, 0, a_2, 0) + (b_1, 0, b_2, 0)\begin{bmatrix}2\\0\\2\\0\end{bmatrix} \text{ where } a_i, b_j \in Q;$$

$$1 \leq i, j \leq 2\} \subseteq S,$$

$$B_3 = \{(0, a_1, a_2, 0) + (0, b_1, b_2, 0)\begin{bmatrix}2\\0\\2\\0\end{bmatrix} \text{ where } a_i, b_j \in Q;$$

$$1 \leq i, j \leq 2\} \subseteq S \text{ and}$$

$$B_4 = \{(0, a_1, a_2, a_3) + (0, b_1, b_2, b_3)\begin{bmatrix}2\\0\\2\\0\end{bmatrix} \text{ where } a_i, b_j \in Q;$$

$$1 \leq i, j \leq 3\} \subseteq S$$

be dual number vector subspaces of S. Clearly $B_i \cap B_j \neq (0)$ even $i \neq j$, $1 \leq i, j \leq 4$. We see $S \subseteq B_1 + B_2 + B_3 + B_4$ so S is not a direct sum only a pseudo direct sum.

Example 2.53: Let

$$P = \left\{\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix} + \begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}(x_1, y_1) \,\middle|\, a_i, b_j \in R; x_1 = 3 \text{ and } y_1 = 6;\right.$$

$$3, 6 \subseteq Z_9; 1 \leq i, j \leq 4\}$$

be the dual number vector space over R.

$$M_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{bmatrix} (x, y) \;\middle|\; a_i, b_j \in R; \; x = 3 \text{ and } y = 6 \text{ in } Z_9; \; 1 \leq i, j \leq 2 \right\}$$

and

$$M_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix} (x, y) \;\middle|\; a_i, b_j \in R; \; x = 3 \text{ and } y = 6 \text{ are in } Z_9; \; 1 \leq i, j \leq 2 \right\} \subseteq P$$

are dual number vector subspaces of P such that $M_1 \, . \, M_2 = (0)$ and $M_1 + M_2 = P$ and is only a direct sum of dual number vector subspaces of P.

$(M_1)^{\perp} = M_2$ and $(M_2)^{\perp} = M_1$.

Now we proceed onto define dual number linear algebras. If V is a dual number vector space on which we can define a product then we define V to be a dual number linear algebra.

We will illustrate this by some examples.

Example 2.54: Let V = $\{a + bg \mid a, b \in Q$ with $g = 10 \in Z_{20}\}$ be the dual number linear algebra over Q.

Example 2.55: Let M = $\{(a_1, a_2, \ldots, a_{12}) \mid a_i = x_i + y_i g; \; x_i, y_i \in Q$ with $g = 4 \in Z_{16}; \; 1 \leq i \leq 12\}$ be a dual number linear algebra over Q. Consider

$$P_1 = \{(a_1, a_2, 0, 0, \ldots, 0) \mid a_i = x_i + y_i\, g;\ x_i, y_i \in Q;\ 1 \le i \le 2\} \subseteq M,$$

$$P_2 = \{(0, 0, a_1, a_2, 0, 0, \ldots, 0) \mid a_i = x_i + y_i\, g;\ x_i, y_i \in Q;\ 1 \le i \le 2\} \subseteq M,$$

$$P_3 = \{(0, 0, 0, 0, a_1, a_2, a_3, 0, \ldots, 0) \mid a_i = x_i + y_i\, g;\ x_i, y_i \in Q;\ 1 \le i \le 3\} \subseteq M,$$

$$P_4 = \{(0, 0, 0, 0, 0, 0, 0, a_1, a_2, a_3, 0, 0) \mid a_i = x_i + y_i\, g;\ x_i, y_i \in Q;\ 1 \le i \le 4\} \subseteq M \text{ and}$$

$$P_5 = \{(0, 0, \ldots, 0, a_1, a_2) \mid a_i = x_i + y_i\, g;\ x_i, y_i \in Q;\ 1 \le i \le 2\} \subseteq M$$

be dual number linear subalgebras of M over Q.

Clearly $P_i \cap P_j = (0)$ if $i \neq j$ and $P_1 + P_2 + P_3 + P_4 + P_5 = M$ is a direct sum of dual number linear subalgebras.

Consider

$$B_1 = \{(a_1, a_2, a_3, a_4, 0, 0, \ldots, 0) \mid a_i = x_i + y_i g;\ 1 \le i \le 4\} \subseteq M,$$

$$B_2 = \{(0, 0, 0, 0, a_2, a_3, a_4, 0, \ldots, 0) \mid a_i = x_i + y_i g;\ 1 \le i \le 4\} \subseteq M,$$

$$B_3 = \{(a_1, 0, 0, 0, a_2, 0, 0, 0, a_3, a_4, 0, 0) \mid a_i = x_i + y_i g;\ 1 \le i \le 4\} \subseteq M$$

and

$$B_4 = \{(a_1, 0, 0, 0, a_2, 0, 0, 0, 0, 0, a_3, a_4) \mid a_i = x_i + y_i g;\ 1 \le i \le 4\} \subseteq M$$

be dual number linear subalgebra of M over the field Q.

We see $B_i \cap B_j \neq (0)$ even if $i \neq j$. Further $B_1 + B_2 + B_3 + B_4 \supseteq M$ so M is only a pseudo direct sum of dual number linear subalgebras.

We can as in case of usual vector spaces / linear algebras define basis, dimension, linear dependence and linear transformation of dual number linear algebras with simple but appropriate modifications. We leave all this work as exercise to the reader.

We can also define dual number Smarandache vector spaces and dual number Smarandache linear algebras as in case of usual vector spaces / linear algebras.

We define a dual number vector space V to be a Smarandache dual number vector space if V is defined only over a S-ring S. Similarly a Smarandache dual number linear algebra V only if V is defined over a S-ring.

We will illustrate this situation by some examples.

Example 2.56: Let $V = \{(x_1, x_2)$ where $x_i = a_i + b_ig$ with $a_i, b_i \in Q$; $g = 4 \in Z_{16}$, $1 \leq i \leq 2\}$ be a Smarandache dual number vector space over the S-ring,

$S = \{a + bg \mid a, b \in Q; g = 4 \in Z_{16}\}$.

Infact V is a Smarandache dual number linear algebra over S-ring S.

Example 2.57: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i = x_i + y_i\, g;\, x_i, y_i \in Q; \right.$$

$$\left. g = 6 \in Z_{12};\ 1 \leq i \leq 9 \right\}$$

be a Smarandache dual number non commutative linear algebra over the S-ring, $Q(g) = \{a + bg \mid a, b \in Q, g = 6 \in Z_{12}\}$.

If the usual product in M is replaced by the natural product $\times_n$ then M is a S-dual number linear algebra which is commutative.

The properties of inner product, linear transformation, linear functionals and all other properties follow as a matter of routine. They are left to the reader as a simple exercise.

Now we give examples of vector spaces and linear algebras of dual numbers using the finite field Z_p.

Example 2.58: Let $P = \{a + bg \mid a, b \in Z_{23}, g = 4 \in Z_{16}\}$ be a dual modulo number vector space over the field Z_{23}.

Example 2.59: Let $M = \{(a_1, a_2, a_3, \ldots, a_{10}) \mid a_i = x_i + y_i\, g \in Z_{19}(g)$ with $g = 2 \in Z_4$; $1 \leq i \leq 10\}$ be a dual modulo number vector space over the field Z_{19}. We see M is also a dual modulo number linear algebra over the field Z_{19}. Clearly M has only finite number of elements and is finite dimensional.

Consider $A_1 = \{(a_1, a_2, 0, \ldots, 0) \mid a_1, a_2 \in Z_{19}(g)\} \subseteq M$,

$$A_2 = \{(0, 0, a_1, a_2, 0, \ldots, 0) \mid a_1, a_2 \in Z_{19}(g)\} \subseteq M,$$

$$A_3 = \{(0, 0, 0, 0, a_1, a_2, 0, 0, 0, 0) \mid a_1, a_2 \in Z_{19}(g)\} \subseteq M,$$

$$A_4 = \{(0, \ldots, 0, a_1, a_2, 0, 0) \mid a_1, a_2 \in Z_{19}(g)\} \subseteq M \text{ and}$$

$$A_5 = \{(0, 0, \ldots, 0, a_1, a_2) \mid a_1, a_2 \in Z_{19}(g)\} \subseteq M,$$

dual modulo number vector subspaces of M over Z_{19}.

Clearly $A_i \cap A_j = (0)$ if $i \neq j$ and $M = A_1 + A_2 + A_3 + A_4 + A_5$ that is M is a direct sum of dual modulo number vector subspaces of M.

Example 2.60: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| a_i \in Z_{13}(g);\ g = 8 \in Z_{16};\ 1 \le i \le 12 \right\}$$

be a dual modulo number vector space over the field Z_{13}. T is not a dual modulo number linear algebra. However T is a dual modulo number linear algebra under natural product $\times_n$ over Z_{13}. If we replace Z_{13} the field by $Z_{13}(g)$ the S-ring; T becomes a Smarandache dual modulo number vector space / linear algebra over the S-ring $Z_{13}(g)$.

Example 2.61: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \middle| a_i \in Z_5(g);\ 1 \le i \le 8 \text{ with } g = 4 \in Z_8 \right\}$$

be the dual modulo number vector space (linear algebra under natural product $\times_n$ on matrices) over the field Z_5.
Take

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & a_3 & 0 & 0 \end{bmatrix} \middle| a_i \in Z_5(g);\ 1 \le i \le 3 \right\} \subseteq W,$$

$$P_2 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 \end{bmatrix} \middle| a_i \in Z_5(g);\ 1 \le i \le 3 \right\} \subseteq W,$$

$$P_3 = \left\{ \begin{bmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & 0 \end{bmatrix} \middle| a_i \in Z_5(g);\ 1 \le i \le 3 \right\} \subseteq W,$$

$$P_4 = \left\{\begin{bmatrix} a_1 & 0 & 0 & a_2 \\ 0 & a_3 & 0 & 0 \end{bmatrix} \middle| \; a_i \in Z_5(g);\ 1 \leq i \leq 3\right\} \subseteq W,$$

$$P_5 = \left\{\begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_2 & 0 & a_3 & 0 \end{bmatrix} \middle| \; a_i \in Z_5(g);\ 1 \leq i \leq 3\right\} \subseteq W$$

and

$$P_6 = \left\{\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_3 \end{bmatrix} \middle| \; a_i \in Z_5(g);\ 1 \leq i \leq 3\right\} \subseteq W$$

be dual modulo number vector subspaces (linear subalgebras) of W over the field Z_5.

We see $P_i \cap P_j \neq (0)$ if $i \neq j$ $1 \leq i, j \leq 6$.

Further $W \subseteq P_1 + P_2 + P_3 + P_4 + P_5 + P_6$, thus W is only a pseudo direct sum of dual modulo number vector subspaces of W.

Example 2.62: Let

$$S = \left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix} \middle| \; a_i \in Z_{11}(g);\ 1 \leq i \leq 16\right.$$

$$\left. \text{with } g = 5 \in Z_{258}\right\}$$

be a dual modulo number vector space (non commutative linear algebra under usual product of matrices or commutative linear algebra under natural product $\times_n$ of matrices) over the field Z_{11}.

S has subspaces and S can be written as direct seem and S has subspaces so that S can be written as a pseudo direct sum.

Now having seen dual modulo number vector spaces / linear algebras and S-dual number vector spaces / linear algebras over the S-ring $Z_p(g)$; p a prime, we now proceed onto study the concept of dual number semivector spaces semilinear algebras over semifields.

We just introduce some simple notations:
$Z^+(g) = \{a + bg \mid a, b \in Z^+ \text{ and } g^2 = 0\}$,
$Q^+(g) = \{a + bg \text{ with } a, b \in Q^+ \text{ and } g^2 = 90\}$ and
$R^+(g) = \{a + bg \text{ with } a, b \in R^+ \text{ and } g^2 = 0\}$.

Clearly we can just adjoin 0 with these sets. These set with adjoined '0' becomes dual number semifield.

Using these semifields we just describe how semivector spaces of dual number over the semifields is constructed.

Let $V = \{a + bg \mid a, b \in R^+, g^2 = 0\} \cup \{0\}$ be defined as a dual number semivector space over the semifield $R^+ \cup \{0\}$ (or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$).

We will give examples of them.

Example 2.63: Let $M = \{(a_1, a_2, a_3, a_4)$ where $a_i = x_i + y_i g$ with $x_i, y_i \in r+$ $g = 4 \in Z_{16}$, $1 \leq i \leq 4\} \cup \{(0, 0, 0, 0)\}$ be a dual number semivector space over the semifield $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$ or $Z^+ \cup \{0\}$.

Example 2.64: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| a_i = x_i + y_i g;\ x_i, y_i \in Q^+, g = 3 \in Z_9;\ 1 \leq i \leq 10 \right\}$$

be a dual modulo number semivector space over the semifield $Z^+ \cup \{0\}$.

Example 2.65: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{bmatrix} g \;\middle|\; a_i, b_j \in Q^+; \right.$$

$$\left. 1 \leq i, j \leq 6, g^2 = 0 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

be a dual modulo integer semivector space over the semifield $Q^+ \cup \{0\}$.

Example 2.66: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \;\middle|\; a_i \in Q^+(g); 1 \leq i \leq 30; g = 9 \in Z_{81} \right\}$$

be the dual modulo integer semivector space over the semifield $S = Q^+ \cup \{0\}$.

Example 2.67: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{61} & a_{62} & a_{63} & a_{64} \end{bmatrix} \;\middle|\; a_i \in Z^+ (g); 1 \leq i \leq 64; \right.$$

$$\left. g = 4 \in Z_{16} \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

be the dual modulo integer semivector space over the semifield $S = Z^+ \cup \{0\}$.

Example 2.68: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} g \,\middle|\, a_i, b_j \in Q^+; \right.$$

$$\left. 1 \leq i, j \leq 4, g = 2 \in Z_4 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

be a dual modulo number semivector space over the semifield $S = Q^+ \cup \{0\}$.

We see M is a dual modulo number semilinear algebra over the semifield $Q^+ \cup \{0\}$ which is non commutative.

However if we use the natural product $\times_n$ on matrices M will turn out to be a commutative dual number semilinear algebra over the semifield $S = Q^+ \cup \{0\}$.

Example 2.69: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \,\middle|\, a_i \in Q^+(g); 1 \leq i \leq 9, \right.$$

$$\left. g = 12 \in Z_{24} \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

be a dual number semilinear algebra over the semifield $S = Q^+ \cup \{0\}$. T is non commutative under usual product and commutative under natural product $\times_n$.

Example 2.70: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \; a_i \in R^+(g);\ 1 \leq i \leq 8 \text{ and } g = 3 \in Z_9 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

be a dual number semilinear algebra under natural product $\times_n$ over the semifield $R^+ \cup \{0\}$ (or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$). M is a strong dual number semilinear algebra if M is defined over $R^+(g) \cup \{0\}$.

Interested reader can derive all properties associated with semivector spaces / linear algebras even in case of dual number semivector spaces / linear algebras.

Let

$$V = \{p(x) + q(x)\, g \mid p(x), q(x) \in R^+ [x] \cup \{0\},\ g = 4 \in Z_{16}\}$$

be a dual number polynomial semivector space over the semifield $Z^+ \cup \{0\}$.

We can define in this way both dual number polynomial semivector spaces as well as semilinear algebras over $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$.

Study of these also is a matter of routine and hence is left as an exercise to the reader.

Chapter Three

HIGHER DIMENSIONAL DUAL NUMBERS

In this chapter we for the first time define the notion of components of dual numbers and their algebraic structures. Also we define higher dimensional concept of dual numbers we give several interesting properties and discuss / describe some features about their algebraic structures.

Suppose a + bg is a dual number we call g the dual component of the dual number. In case of a + bg we have only one dual component viz g. We have studied in chapter two for a given dual component the properties enjoyed by the collection of dual numbers a+bg for a, b $\in$ Q (or R or Z_n or Z or $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$) and for varying structures of a and b as matrices or polynomials.

Now we define a three dimensional dual number as follows.

DEFINITION 3.1: *Let $a + bg_1 + cg_2$ where $g_1 \neq g_2$ with a, b, c, g_1 and g_2 are non zeros such that $g_1^2 = 0$, $g_2^2 = 0$ and $g_1g_2 = g_2g_1 = (0)$ then we define $a + bg_1 + cg_2$ to be a three dimensional dual number if $(a + bg_1 + cg_2)^2$ is of the form $x + yg_1 + zg_2$ (x, y, z all reals).*

We first give some examples before we proceed to work with them.

Example 3.1: Let $s = a + bg_1 + cg_2$ where $g_1 = 4$ and $g_2 = 8$, $g_1, g_2 \in Z_{16}$ a, b, c $\in R \setminus \{0\}$. Consider

$$\begin{aligned} s^2 &= (a + bg_1 + cg_2)^2 \\ &= a^2 + b^2.0 + c^2.0 + 2abg_1 + 2acg_2 + 2bcg_1\, g_2 \\ &= a^2 + 2abg_1 + 2acg_2\,. \end{aligned}$$

We see s^2 is again in the same form that is as that of s and s is a three dimensional dual number.

Suppose $x = a + bg_1 + cg_2$ and $y = t + ug_1 + vg_2$ with a, b, c, t, u, v $\in R \setminus \{0\}$. Consider $xy = at + (bt + ua)\, g_1 + (ct + av)g_2$ using the fact $g_1\, g_2 = g_2\, g_1 = 0$ and $g_1^2 = g_2^2 = 0$. Further xy will be a three dimensional dual number only if $bt + ua \neq 0$ and $ct + av \neq 0$.

Example 3.2: Let $x = a + bg_1 + cg_2$ where a, b, c $\in R \setminus \{0\}$ and $g_1 = (3, 6, 6, 0)$ and $g_2 = (6, 6, 0, 6)$ with $3, 0, 6 \in Z_9$.

We see x is also a three dimensional dual number row matrix.

Example 3.3: Let

$x = a + bg_1 + cg_2$ where a, b, c $\in R \setminus \{0\}$ with

$$g_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ 2 \end{bmatrix} \text{ and } g_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} \text{ where } 2 \in Z_4.$$

$$g_1 \times_n g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad g_1 \times_n g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } g_2 \times_n g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus x is a three dimensional dual column matrix number. We can also have $m \times n$ matrix three dimensional number.

We give only illustration of them.

Example 3.4: Let

$$x = a + bg_1 + cg_2 \text{ where } a, b, c \in R \text{ and } g_1 = \begin{bmatrix} 8 & 0 & 4 & 0 \\ 4 & 4 & 4 & 4 \\ 0 & 8 & 0 & 8 \end{bmatrix}$$

and

$$g_2 = \begin{bmatrix} 4 & 8 & 4 & 8 \\ 0 & 8 & 0 & 8 \\ 4 & 0 & 4 & 0 \end{bmatrix} \text{ where } \{0, 8, 4\} \subseteq Z_{16}.$$

$$\text{Clearly } g_1^2 = g_2^2 = g_1 g_2 = g_2 g_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

x is a three dimensional dual number of matrices.

Example 3.5: Let

$$y = a + bg_1 + cg_2 \text{ where } a, b, c \in R \text{ and } g_1 = \begin{bmatrix} 3 & 6 & 3 \\ 6 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix} \text{ and}$$

$$g_2 = \begin{bmatrix} 6 & 6 & 6 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \text{ with } \{0, 3, 6\} \subseteq Z_9.$$

y is a three dimensional dual number of matrices.

We see these three dimensional dual numbers enjoy some algebraic structure.

We give algebraic structures to the elements which contribute to the dual numbers and some more additional properties.

Consider if $x = a + bg_1 + cg_2$ is a three dimensional dual number then we see $g_1^2 = 0$, $g_2^2 = 0$ and $g_1 g_2 = g_2 g_1 = 0$.

Thus $S = \{0, g_1, g_2\}$ is a semigroup we define S is a zero square semigroup.

We will first provide some examples of them.

Example 3.6: Let $S = \{0, 3, 6\} \subseteq Z_9$, S is a zero square semigroup under multiplication modulo 9.

Example 3.7: Let $S = \{0, 4, 8\} \subseteq Z_{16}$, S is a zero square semigroup, for $4^2 \equiv 0 \pmod{16}$, $8^2 \equiv 0 \pmod{16}$ and
$4.8 = 8.4 \equiv 0 \pmod{16}$.

Example 3.8: Let $S = \{0, 5, 10\} \subseteq Z_{25}$ be a zero square semigroup under product $\times$, modulo 25.

Example 3.9: Let $S = \{0, 6, 12\} \subseteq Z_{36}$ be a zero square semigroup under product modulo 36.

Now we see most of these zero square semigroups are not closed under addition modulo n.

Now can we extend three dimensional dual number to four dimensional dual number?

Consider $x = a + bg_1 + cg_2 + dg_3$ where a, b, c, d $\in$ R and g_1, g_2 and g_3 are three distinct elements such that $g_1^2 = 0$, $g_2^2 = 0$, $g_3^2 = 0$ and $g_i g_j = g_j g_i = 0$, $1 \leq i, j \leq 3$.

We call x a four dimensional dual number.

We will illustrate this situation by some simple examples.

Example 3.10: Let $x = a + bg_1 + cg_2 + dg_3$ where a, b, c, d $\in$ R and $g_1 = 4$, $g_2 = 8$ and $g_3 = 12$; $\{0, 4, 8, 12\} \subseteq Z_{16}$. x is a four dimensional dual number.

Example 3.11: Let $x = a + bg_1 + cg_2 + dg_3$ where a, b, c, d $\in$ R and $g_1 = 8$, $g_2 = 16$ and $g_3 = 24$ with $\{0, 8, 16, 24\} \subseteq Z_{32}$. x is a four dimensional dual number.

Example 3.12: Let $x = a + bg_1 + cg_2 + dg_3$ where a, b, c, d $\in$ Q and $g_1 = 9$, $g_2 = 18$ and $g_3 = 27$ with $\{0, g_1, g_2, g_3\} \subseteq Z_{81}$. x is a four dimensional dual number.

We see we can give algebraic structures to the four dimensional dual numbers.

THEOREM 3.1: *Let $x = a + bg_1 + dg_2 + cg_3$ with a, b, c, d $\in$ R, $g_1^2 = g_2^2 = g_3^2 = 0$ and $g_ig_j = g_j g_i = 0$, $1 \leq i, j \leq 3$ be a four dimensional dual number. $S = \{0, g_1, g_2, g_3\}$ is a zero square semigroup of order 4 under multiplication.*

THEOREM 3.2: *Let $A = \{a + bg_1 + dg_2 + cg_3 \mid a, b, c, d \in R;$ $g_i^2 = g_ig_j = 0$, j and i = 1, 2, 3$\}$. A is a semigroup under product.*

Proofs of the above two theorems are simple and direct and hence left as an exercise to the reader.

We can extend the notion to n-dimensional dual numbers, $x = a_1 + a_2g_1 + a_3g_2 + \ldots + a_ng_{n-1}$ is a n-dimensional dual number if $g_ig_j = 0$, j, i = 1, 2, …, n–1, i is also equal to j and $a_i \in R$; $1 \leq i \leq n$. We call the $\{0, g_1, \ldots, g_{n-1}\}$ as the n-dual tuple.

We will illustrate this situation by some examples.

Example 3.13: Let $x = a_1 + a_2g_1 + \ldots + a_3g_2 + \ldots + a_{16}\, g_{15}$ with $a_i \in R$, $1 \leq i \leq 16$ and $g_j \in \{16, 32, 48, 64, 80, 96, 112, 128, 144, 160, 176, 192, 208, 224, 240\} \subseteq Z_28$, x is a 16 dimensional dual number and S = {0, 16, 32, 48, 64, 80, …, 208, 224, 240} is a zero square semigroup.

In view of this we have the following theorem.

THEOREM 3.3: *Let $x = a_1 + a_2g_1 + \ldots + a_pg_{p-1}$, with $p = 2^{n/2}$, $a_j \in R$; $1 \leq j \leq p$ and $g_i \in \{m, 2m, 3m, 2^n–2^{n/2}\} \subseteq Z_{2^n}$, where $m = 2^{n/2}$ and n an even number be a $2^{n/2}$ dimensional dual number $1 \leq i \leq p–1$; Then the set $\{2^{n/2} = m, 2m, \ldots, m(2^{n/2}–1) = 2^{n/2}\,(2^{n/2} - 1)\}$ is a zero square semigroup under product.*

Proof: Given $x = a_1 + a_2g_1 + \ldots + a_pg_{p-1}$ is a $2^{n/2}$ dimensional dual number. Clearly $g_ig_j = 0$ (mod 2n), $i \neq j$ and $g_i^2 = 0$ (mod 2^n); $1 \leq i, j \leq 2^{n/2} - 1$. Hence $S = \{0, g_1, g_2, \ldots, g_{p-1}\}$ is a semigroup of order $2^{n/2}$ under multiplication modulo 2^n and S is a zero square semigroup which is a null semigroup.

THEOREM 3.4: *Let $x = a_1 + a_2g_1 + \ldots + a_pg_{p-1}$ be a $p = 2^{n/2}$ dimensional dual number, n even and $\{g_1, g_2, \ldots, g_{p-1}\} \subseteq Z_{2^n}$, then $S = \{0, g_1, \ldots, g_{p-1}\}$ is a commutative zero square ring which is a null ring.*

Proof is direct for more about zero square rings please refer [20].

We will illustrate this situation by some examples.

Example 3.14: Let $x = a_1 + a_2g_1 + \ldots + a_pg_{p-1}$ with $a_i \in R$, $1 \leq i \leq p$, $n = 2^5$ and $g_1 = 2^5$ $g_2 = 2.2^5 = 2^6$, …, $g_{p-1} = 2^5 (2^5-1)$. Then x is a 32 dimensional dual number and $S = \{g_1, \ldots, g_{p-1}, 0\}$ is a commutative null semigroup and a null ring.

Example 3.15: Let

$x = a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 + a_8g_7$ with $a_i \in R$, $g_1 = 8$, $g_2 = 16$ and $g_3 = 24$, $g_4 = 32$, $g_5 = 40$, $g_6 = 48$ and $g_7 = 56$ in Z_{2^6}. x is a eight dimensional dual number and $S = \{0, 8, 16, 24, 32, 40, 48, 56\} \subseteq Z_{2^6}$ is a null semigroup and a null ring.

We see what are the elements in Z_{2n} when n is odd.

Consider the modulo integer $Z_{2^5} = \{0, 1, 2, \ldots, 31\}$. The zero square elements of Z_{2^5} are 8, 16 and 24 in Z_{2^5} such that they are zero square elements as well as elements of a null ring.

So in view of this we have the following theorem.

THEOREM 3.5: *Let $x = a_1 + a_2g_1 + \ldots + a_pg_{p-1}$ with $a_i \in R$; $1 \leq i \leq p$ and $\{g_1, g_2, \ldots, g_{p-1}\} \subseteq Z_{2^n}$ where n is odd then $p = 2^{(n-1)/2} - 1$; be a p dimensional dual number. $S = \{0, g_1, \ldots, g_{p-1}\} \subseteq Z_{2^n}$ is a null semigroup as well as null ring but the number of elements in S is $2^{(n-1)/2}$.*

Now we work for any Z_n, $n = p^t$ or n a composite number.

Example 3.16: Let $Z_{27} = \{0, 1, 2, \ldots, 26\}$ be a semigroup. Consider $\{9, 18\} \subseteq Z_{27}$; $x = a_1 + a_29 + a_318$ is a 3 dimensional dual number and $a_1, a_2, a_3 \in R$.

Example 3.17: Let $Z_9 = \{0, 1, 2, \ldots, 8\}$ be a semigroup. Consider $\{3, 6\} \subseteq Z_9$, $x = a_1 + a_23 + a_36$ is a 3-dimensional dual number, $a_i \in R$ or Q; $1 \leq i \leq 3$.

Example 3.18: Let $Z_{81} = \{0, 1, 2, \ldots, 80\}$ be a semigroup under product modulo 81. Take

$S = \{0, 9, 18, 27. 36, 45, 54, 63, 72\} \subseteq Z_{81}$; S is a null semigroup as well as a null ring.

Take $S = \{0, 9, 18, 27, 36, 45, 54, 63, 72\} \subseteq Z_{81}$; S is a null semigroup as well as null ring.

Take $x = a_1 + a_2 9 + a_3 18 + \ldots + a_9 72$; $a_i \in R$ or Q, $1 \leq i \leq 9$ is a 9-dimensional dual number.

Inview of this we can say the following.

THEOREM 3.6: *Suppose $Z_{p^2} = \{0, 1, 2, \ldots, p^2 - 1\}$ be a ring, then $S = \{p, 2p, \ldots, (p-1) p\} \subseteq Z_{p^2}$ is such that*

$x = a_1 + a_2 p + a_3 2p + \ldots + a_p(p-1)p$ is a p dimensional dual number and $S \cup \{0\}$ is a null ring.

Corollary 3.1: If $Z_{p^3} = \{0, 1, 2, \ldots, p^3 - 1\}$ is taken, then also the null subring of Z_{p^3} has only p elements and it can contribute only to a p-dimensional dual number.

Corollary 3.2: If $Z_{p^4} = \{0, 1, 2, \ldots, p^4 - 1\}$ is considered then the null subring $S = \{0, p^2, 2p^2, \ldots, (p^2-1) p^2\} \subseteq Z_{p^4}$ and S can contribute to a p^2 dimensional dual number.

Corollary 3.3: If Z_{p^n} is taken n even or odd the null subring has $p^{n/2}$ elements and this can contribute to $p^{n/2}$ dimensional dual number.

We will illustrate this above situations by some examples.

Example 3.19: Let $S = \{0, 5, 10, 15, 20\} \subseteq Z_{5^2}$ be a null subsemigroup of Z_{5^2}. $\alpha = a_1 + a_2 5 + a_3 10 + a_4 15 + a_5 20$ is a 5-dimensional dual number.

Example 3.20: Let S = {0, 25, 50, 75, 100} $\subseteq Z_{5^3}$ be a null subsemigroup of Z_{5^3}. $\alpha = a_1 + a_2 25 + a_3 50 + a_4 75 + a_5 100$ is a 5-dimensional dual number.

Example 3.21: Let S = {0, 25, 50, 75, 100, 125, 150, 175, 200, 225, 250, 275, 300, 325, 350, 375, 400, 425, 450, 475, 500, 525, 550, 575, 600} $\subseteq Z_{5^4}$ be a null subsemigroup of Z_{5^4}.

$\alpha = a_1 + a_2 25 + a_3 50 + \ldots + a_{25} 600$ is a 25-dimensional dual number, $a_i \in$ R (or Q) $1 \leq i \leq 50$.

Now we see some more examples.

Example 3.22: Let S = {0, 7, 14, 21, 28, 35, 42} $\subseteq Z_{7^2}$ be a null subsemigroup of Z_{7^2} and

$\alpha = a_1 + a_2 7 + a_3 14 + a_4 21 + a_5 28 + a_6 35 + a_7 42$ is a 7 dimensional dual number $a_i \in$ Q or R; $1 \leq i \leq 7$.

Example 3.23: Let Z_6 = {0, 1, 2, 3, 4, 5} be a semigroup under product modulo 6.

Z_6 has no nilpotent element hence Z_6 has no nontrivial null subsemigroup.

Example 3.24: Let Z_{10} = {0, 1, 2, 3, 4, 5,. 6, 7, 8, 9} be a semigroup. Z_{10} has no nilpotent element, hence has no nontrivial null subsemigroup.

Example 3.25: Let Z_{14} = {0, 1, 2, …, 13} be a semigroup, Z_{14} has no nilpotent element and non trivial null subsemigroup.

In view of this we have the following theorem.

THEOREM 3.7: *Let Z_{2p} = {0, 1, 2, 3, …, 2p–1} be a semigroup under multiplication modulo 2p, p an odd prime.*

(i) Z_{2p} has no nilpotent elements.

(ii) Z_{2p} has no non trivial null subsemigroup.

Proof follows using simple numbers theoretic techniques and the fact $2p = n^2$ for any n where p is an odd prime.

THEOREM 3.8: *Let $Z_{pq} = \{0, 1, 2, 3, \ldots, pq-1\}$ be a semigroup under product, p and q two distinct primes.*

(i) Z_{pq} has no nilpotent elements.
(ii) Z_{pq} has no nontrivial null subsemigroup.

Proof follows from the simple fact $pq = n^2$ is impossible for any n.

Example 3.26: Let $Z_{21} = \{0, 1, 2, \ldots, 20\}$ be a semigroup under product modulo 21. Clearly Z_{21} has no nilpotent element of order two that is we have no $x \in Z_{21}$ such that $x^2 = 21$.

THEOREM 3.9: *Let $Z_n = \{0, 1, 2, \ldots, n-1\}$ where $n = p_1, p_2, \ldots, p_t$ where p_i's distinct primes $1 \leq i \leq t$ be a semigroup under multiplication modulo n.*

(i) Z_n has no nilpotent element of order two.
(ii) Z_n has no nontrivial null subsemigroup.

Proof follows from the simple fact $x^2 = n = p_1\, p_2 \ldots p_n$ is impossible for any $x \in Z_n$.

Example 3.27: Let $S = Z_{12} = \{0, 1, 2, 3, \ldots, 11\}$ be a semigroup $6 \in Z_{12}$ is such that $6^2 = 0 \pmod{12}$. $T = \{0, 6\} \subseteq Z_{12}$ is a null subsemigroup. $x = a_1 + a_2 6$ is a dual number for all $a_1, a_2 \in R$ (or Q).

Example 3.28: Let $S = Z_{20} = \{0, 1, 2, \ldots, 19\}$ be a semigroup. $10 \in Z_{20}$ is such that $10^2 = 0 \pmod{20}$ is a nilpotent element of S. $T = \{0, 10\} \subseteq Z_{20}$ is a null subsemigroup of S. $x = a_1 = a_2 10$ is a dual number for all $a_1, a_2 \in R$ (or Q).

Example 3.29: Let $S = Z_{63} = \{0, 1, 2, \ldots, 62\}$ be a semigroup under product modulo 63. Consider $21, 42 \in S$, clearly $21^2 = 0 \pmod{63}$ and $(42)^2 = 0 \pmod{63}$ both are nilpotent elements. Further $21 \times 42 \equiv 0 \pmod{63}$.

Thus $T = \{0, 21, 42\} \subseteq Z_{63}$ is a nontrivial null subsemigroup of S. $x = a_1 + a_2 21 + a_3 42$ is a 3 dimensional dual number.

Inview of this we have the following theorem.

THEOREM 3.10: *Let $Z_n = \{0, 1, 2, 3, \ldots, n-1\}$ be the ring of integers modulo n, where $n = p^2q$ $(p \neq q)$ then*

(i) Z_n has nilpotent elements.
(ii) Z_n has non trivial null subsemigroup of order $p-1$
(iii) $x = a_1 + a_2 g_1 + \ldots + a_p\, g_{p-1}$ is a p dimensional dual number where $S = \{0, g_1, \ldots, g_{p-1}\}$.

The proof is direct by using simple number theoretic methods. Thus we see Z_n for varying n we get the t-dimensional dual numbers.

Now we can also get a t-dimensional dual number from a single dual number a + bg where a, b $\in$ R and $g^2 = 0$ in the following way. Take

$S = \{(g, 0, g, 0, \ldots, g)\ (g, g, g, \ldots, g), (0, g, 0, \ldots, 0) \ldots (0, 0, \ldots, g)\}$, S is a dual number set. $S \cup \{(0, 0, 0, \ldots, 0)\}$ is a semigroup which is a null semigroup.

Thus we can get any desired dimensional dual number using a + bg with $g^2 = 0$ and a, b $\in$ R.

We can also take

$$S = \left\{ \begin{bmatrix} g \\ g \\ \vdots \\ g \end{bmatrix}, \begin{bmatrix} 0 \\ g \\ \vdots \\ g \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ g \\ g \\ \vdots \\ g \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\},$$

S is a null semigroup under natural product $\times_n$.

Likewise if

$$S = \left\{ \begin{bmatrix} g & g & \dots & g \\ g & g & \dots & g \\ \vdots & \vdots & & \vdots \\ g & g & \dots & g \end{bmatrix}, \begin{bmatrix} 0 & g & \dots & g \\ g & g & \dots & g \\ \vdots & \vdots & & \vdots \\ g & g & \dots & g \end{bmatrix}, \begin{bmatrix} 0 & 0 & g & \dots & g \\ g & g & g & \dots & g \\ \vdots & \vdots & \vdots & & \vdots \\ g & g & g & \dots & g \end{bmatrix}, \dots, \right.$$

$$\left. \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & g \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \right\}$$

be a $m \times p$ matrices ($m \neq p$) with the entries from the set $T = \{0, g \mid g^2 = 0\}$. Then S is a null semigroup under the natural product $\times_n$ of matrices.

Infact using any n distinct elements from S (n a desired number $n \leq o(S)\}$ we can get a n dimensional dual number.

We can use m = p, that is square matrices that will also be a null semigroup both under natural product $\times_n$ as well as under the usual product $\times$.

Thus we have the concept of null semigroups of any desired dimension dual numbers.

We will give examples of them.

Example 3.30: Let S = {(0, 3, 3, 3, 3, 3), (0, 0, 3, 3, 3, 3), (0, 0, 0, 3, 3, 3), (0, 3, 0, 0, 0, 0), (3, 0, 0, 0, 0, 0), (0,0,3,0,0,0), (0, 0, 0, 3, 0, 0), (0, 0, 0, 0, 0, 3), (3,3,0, …,0), …, (0,0,0,0,3,3), …, (3, 3, 3, 3, 3, 3)} ∪ {(0, 0, 0, 0, 0, 0)} be a null semigroup.

Order of S is 63. Thus using any n distinct elements from S we can get n-dimensional dual numbers ($n \leq 63$).

(Here $3 \in Z_9$ we see $3^2 \equiv 0 \pmod 9$)

$x = a_1 + a_2x_1 + \ldots + a_nx_{n-1}$ where $x_i \in S$ $x_i \neq x_j$ if $i \neq j$; $1 \leq i, j \leq n-1 = 63$ $a_i \in R$ or Q; $1 \leq i \leq n$ is a n-dimensional dual number.

Example 3.31: Let

$$S = \left\{ \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \\ 4 \\ 4 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

be a null semigroup under the natural product $\times_n$ where $4 \in Z_{16}$; $4^2 \equiv 0 \pmod{16}$.

Clearly $o(S) = 31$. Thus using elements of S we can get a maximum of 31-dimensional dual number. Also we can get any desired t-dimensional dual number $1 < t \leq 31$.

Example 3.32: Let

$$S = \left\{ \begin{bmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \right\} \cup$$

$$\left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

(where $5 \in Z_{25}$, $5^2 \equiv 0 \pmod{25}$) be a null semigroup of order $1 + {}_{16}C_2 + \ldots + {}_{16}C_{15} + 1 = 2^{16}$.

Clearly taking any t-elements from S (t of them distinct) we have a t-dimensional dual number ($t \leq 2^{16} - 1$) as $o(S) = 2^{16} - 1$.

Example 3.33: Let

$$S = \left\{ \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

be a null semigroup under natural product $\times_n$. $o(S) = 2^{18} - 1$ where $2 \in Z_4$ and $2^2 \equiv 0 \pmod 4$.

We can using S get any m-dimensional dual number $m \leq 2^{18} - 1$.

Now we can also get more than $2^n–1$ elements in S by the following way which is illustrated by examples.

Example 3.34: Let

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \; x_i \in \{0, 3, 6\} \subseteq Z_9;\ 1 \leq i \leq 4;\ 3^2 \equiv 0 \pmod 9), \right.$$

$$\left. 6^2 \equiv 0 \pmod 9) \text{ and } 3.6 = 0 \pmod 9) \right\}$$

be a null semigroup.

$$S \setminus \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$ is such that any n-distinct elements from S will contribute to a n-dimensional dual number.

Example 3.35: Let S = {[x_1, x_2, x_3] | $x_i \in \{0, 4, 8\} \subseteq Z_{16}$; $1 \leq i \leq 3$; $4^2 \equiv 0 \pmod{16}$, $8^2 \equiv 0 \pmod{16}$ and 4.8 = 0 (mod 16)} be a null semigroup of order $3^3 = 27$. S \ {(0 0 0)} gives 26 elements.

Taking any t distinct elements from S \ {(0 0 0)} say $y_1, y_2, \ldots, y_t$; $y_i \in S \setminus \{(0\ 0\ 0)\}$, $1 \leq i \leq t \leq 26$ we have
$a_1 + a_2y_1 + \ldots + a_{27}\, y_{26}$ or $a_1 + a_2y_1 + \ldots + a_{t+1}\, y_t$ gives a t-dimensional dual number.

Likewise we can use columns also.

Example 3.36: Let

$$S = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \middle| \; y_i \in \{0, 4, 8\} \subseteq Z_{16};\ 1 \leq i \leq 4;\ 4^2 \equiv 0 \pmod{16}, \right.$$

$$\left. 8^2 \equiv 0 \pmod{16} \text{ and } 4.8 = 0 \pmod{16} \right\}$$

be a null semigroup under the natural product $\times_n$ of column matrices. We get 3^4 elements. Hence we can have a t- dimensional dual number $2 \leq t \leq 3^4 - 1$.

Thus we see a major role is played by the modulo integers and matrix theory in constructing n-dimensional dual numbers.

We define now the notion of finite complex modulo integer dual number.

DEFINITION 3.2: *Let $x = a + bs$ where $s = (m + ni_F)$; $m, n \in Z_t$ and $i_F^2 = t-1$; $2 \leq t < \infty$; $a, b \in R$ or Q or Z. If $s^2 = 0$ then we call x to be a dual finite complex modulo integer of dimension two.*

We first give some simple examples.

Example 3.37: Let $x = a + bs$ where $s = 1 + i_F \in C(Z_2)$; $i_F^2 = 1$, $a, b \in Q$ or Z or R. We see $s^2 = (1 + i_F)^2 = 0 \pmod 2$. Thus x is a dual finite complex modulo integer of dimension two.

Example 3.38: Let $x = a + bs$ where $s = (2 + 2i_F) \in C(Z_4)$; $a, b \in R$, x is a dual complex modulo integer of dimension two.

Example 3.39: Let $x = a + bs$ where $s = (6 + 6i_F) \in C(Z_{12})$; $a, b \in Q$, be a dual complex modulo integer of dimension two.

Example 3.40: Let $x = a + bs$ $a, b \in R$ and $s = (12 + 12i_F) \in C(Z_{16})$ be a dual finite complex modulo integer of dimension two.

We can have any number of examples.
We have the following results.

THEOREM 3.11: *Let $S = \{a + bs \mid s \in C(Z_n)$ with $s^2 = 0 \pmod n$ $s = (t + ui_F)$; $t, u \in Z_n \setminus \{0\}$; $a, b \in Q\}$; S is a general dual complex finite modulo integer ring.*

Proof is straight forward and hence left as an exercise to the reader.

Example 3.41: Let

$S = \{a + bs \mid a, b \in Q;\ s = (4 + 4i_F) \in C\ (Z_{16})\}$ be the general dual finite complex number ring. We see S under addition as an abelian group with $0 + 0s$ as its additive identity and (S, ×) is a commutative semigroup. Thus (S, +, ×) is a general ring of dual complex modulo integers.

We use the term general to make sure Q ⊆ S; and we have in a + bs (a can be 0 or b can be zero); that is elements of S need not be always of the form with a + bs it can be Q or Qs both proper subsets of S.

We have infinitely many general dual finite complex modulo integer rings.

We can also find infinitely many such structures using semigroup rings.

We will briefly describe this notion.

Let S = {C (Z_n), ×} where n is a composite and C(Z_n) has atleast one non trivial nilpotent element of order two of the form a + bi_F with a, b ∈ $Z_n \setminus \{0\}$. Clearly S is a commutative semigroup with unit.

Let F = R (or Q or Z) be the ring. FS be the semigroup ring of the semigroup S over the ring F.

We will illustrate this situation by some examples.

Example 3.42: Let S = C(Z_4) be the complex semigroup. F = Q be the field of rationals. FS the semigroup ring of the semigroup S over the field F.

Consider P = {a + bs | s = (2 + $2i_F$) ∈ S, a, b ∈ Q = F} ⊆ FS.

P is a subsemigroup of FS and infact P is a general dual finite complex modulo integer subring of FS. Further it is interesting to note that P is an ideal of FS.

Example 3.43: Let S = C (Z_{12}) be the complex modulo integer semigroup. Z be the ring of integers. ZS the semigroup ring. Take P = {a + bs | s = 6 + $6i_F$ ∈ S = C(Z_{12})} ⊆ ZS; P is again a general dual complex modulo integer subring as well as an ideal.

Inview of this we have the following theorem.

THEOREM 3.12: *Let S = $(C(Z_n), \times)$ where n is even be a complex modulo integer semigroup. Q be the field of rationals. QS be the semigroup ring of the semigroup S over the ring Q.*

Consider

$$P = \{a + bs \mid s = \frac{(n+1)}{2} + \left(\frac{n+1}{2}\right) i_F;\ a, b \in Q\} \subseteq QS;$$

P is a general dual finite complex number subring of QS.

The proof is direct and hence is left as an exercise to the reader.

Example 3.44: Let $C(Z_9)$ = S be the complex modulo integer semigroup. Z be the ring of integers. ZS be the semigroup ring. Take P = $\{a + bs \mid s = 3 + 3i_F \in C(Z_9),\ a, b \in Z\} \subseteq ZS$. Clearly P is a subring which is a general finite dual complex modulo integer ring.

Example 3.45: Let $C(Z_9)$ = S be the complex semigroup of finite integers.

Take $x = a + bs_1 + cs_2$ where a, b, c $\in$ Z and $s_1 = 3 + 3i_F$ and $s_2 = 6 + 6i_F$ are in $C(Z_9)$. Clearly $x^2 = A + Bs_1 + Cs_2$ A, B, C $\in$ Z and $s_1^2 = 0 \pmod 9$ and $s_2^2 = 0 \pmod 9$ and $s_1 \times s_2 = (0) \pmod 9$.

We call x a three dimensional dual finite complex modulo integer. Thus we can build higher dimensional finite complex dual modulo integers also.

We will illustrate this situation only by examples as the definition is a matter of routine.

Example 3.46: Let C (Z_{16}) = S be a semigroup of complex modulo integers. Take

P = $\{a_1 + a_2s_1 + a_3s_2 + a_4s_3 \mid a_i \in Q; 1 \leq i \leq 4$ $s_1 = 4 + 4i_F$, $s_2 = 8 + 8i_F$ and $s_3 = 12 + 12i_F$ are in $C(Z_{16})$ with $i_F^2 = 15\}$; P is a ring called the four dimensional dual complex modulo integer ring.

The reader is left with the task of verifying this claim in the above example.

Example 3.47: Let $C(Z_8)$ = S be a semigroup of complex modulo integers. Consider $s = 4 + 4i_F \in S$; we see $s^2 \equiv 0 \pmod 8$.

Now let

P = {(s, s, s, s), (0, s, s, s), (s, 0, s, s), (s, s, s, 0), (s,s,0,s), (s,s,0,0), (s,0,s,0), (s,0,0,s), (0,0,s,s), (0,s, s,0), (0,s,0,s), (s,0,0,0), (0,s,0,0), (0,0,s,0) (0,0,0,s), (0,0,0,0)}; P is a semigroup under product. Infact P is a null semigroup. Further P is a ring under + and × modulo 8. P is also a null ring.

Let G = $\{a_1 + a_2g_1 + \ldots + a_{16}\, g_{15} \mid a_i \in Q; 1 \leq i \leq 16, g_j \in P \setminus \{0\}$ and g_j's are distinct $1 \leq j \leq 15\}$.

G is again a ring which is a general ring of finite dual complex modulo integers of all dimensions less than or equal to 15.

Thus this method of constructing nilpotent elements helps one to make any desired dimensional dual complex numbers.

We will illustrate this situation by more examples.

Example 3.48: Let S = C (Z_4) be a semigroup under product. Consider $g = 2+2i_F \in S$; clearly $g^2 = 0 \pmod 4$.

$$\text{Now let P} = \left\{ \begin{bmatrix} g \\ g \\ g \\ g \\ g \\ g \end{bmatrix}, \begin{bmatrix} 0 \\ g \\ g \\ g \\ g \\ g \end{bmatrix}, \ldots, \begin{bmatrix} g \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\};$$

P is a semigroup under natural product $\times_n$. Infact (P, $\times_n$) is a null semigroup of complex modulo integers.

Also (P, $\times_n$, +) is a null ring of complex modulo integers we see the number of elements in P is $2^6 = 64$.

We can get

$$M = \{a_1 + a_2\, g_1 + \ldots + a_{64}\, g_{63}\} \mid a_i \in Q;\ 1 \leq i \leq 64;$$

$$g_j \in P \setminus \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right.;\ 1 \leq j \leq 63 \text{ and } g_j\text{'s are all distinct}\}.$$

M is a collection of all t-dimensional dual complex modulo integers with $1 \leq t \leq 63$ and infact (P, +, $\times$) is a general dual complex modulo integer ring.

Example 3.49: Let S = C (Z_{26}) be a complex finite modulo integer semigroup. Let $g = 13 + 13i_F \in S$; $i_F^2 = 25$. a + bg with a, b $\in$ Q is a finite complex dual number.

Construct W =

$$\left\{\begin{bmatrix} g & g & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & g \end{bmatrix}, \begin{bmatrix} 0 & g & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & g \end{bmatrix}, \ldots, \begin{bmatrix} g & g & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & 0 \end{bmatrix},\right.$$

$$\begin{bmatrix} 0 & 0 & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & g \end{bmatrix}, \ldots, \begin{bmatrix} g & g & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & g \\ g & g & g & g \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g & 0 & g \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g & g \end{bmatrix},$$

$$\left.\begin{bmatrix} g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right|$$

$$g = 13 + 13i_F \in C(Z_{26});\ i_F^2 = 25\};$$

W is a null semigroup under the natural product $\times_n$ of matrices. Further (W, +, $\times_n$) is a null ring of order of W is 2^{20}.

Using W we can construct finite integer ring of dual number of t-dimension, $2 \leq t \leq 2^{20} - 1$.

$$V = \{a_1 + a_2p_1 + \ldots + a_{2^{20}}p_{(2^{20}-1)} \mid a_i \in Q;$$

$$p_j\text{'s are distinct and } p_j \in W \setminus \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\}$$

$$1 \leq i \leq 2^{20} \text{ and } 1 \leq j \leq 2^{20} - 1\}$$

is a general t-dimensional dual finite complex modulo integer ring $1 \leq t \leq 2^{20} - 1$.

Example 3.50: Let S = C (Z_{25}) be a finite complex modulo integer ring / semigroup.

Consider $g = 5 + 5i_F \in S$ where $i_F^2 = 24$.

Clearly $g^2 = 0 \pmod{25}$.

Take P =

$$\left\{\begin{bmatrix} g & g & g \\ g & g & g \\ g & g & g \end{bmatrix}, \begin{bmatrix} 0 & g & g \\ g & g & g \\ g & g & g \end{bmatrix}, \begin{bmatrix} g & 0 & g \\ g & g & g \\ g & g & g \end{bmatrix}, \ldots, \begin{bmatrix} g & g & g \\ g & g & g \\ g & 0 & g \end{bmatrix},\right.$$

$$\begin{bmatrix} g & g & g \\ g & g & g \\ g & g & 0 \end{bmatrix}, \ldots, \begin{bmatrix} g & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & g \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & g \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Bigg|$$

$g = 5+5i_F\}$ be a ring of finite complex modulo integers.

Suppose $V = \{a_1 + a_2x_1 + \ldots + a_{2^9}x_{(2^9-1)} \mid a_i \in Q$; x_j's are distinct and $x_j \in P \setminus \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$ $1 \leq i \leq 2^9$; $1 \leq j \leq 2^9 - 1\}$;

V is a general dual complex modulo integer ring with varying t-dimensional dual complex modulo integers $1 \leq t \leq 2^9 - 1$.

Thus using a single finite complex modulo integer we can construct any desired dimensional dual complex modulo integers.

Now we wish to show we can also use more than one complex modulo integer which contributes to dual numbers in a complex null complex modulo integer semigroup and construct dual complex modulo number of any desired dimension.

We will only illustrate this situation by some simple examples.

Example 3.51: Let $S = C(Z_{16})$ be a semigroup of complex modulo integers. Consider $g_1 = 4+4i_F$, $g_2 = 8+8i_F$ and $g_3 = 12 + 12i_F$ in $C(Z_{16}) = S$. We see $x = a_1 + a_2g_1 + a_3g_2 + a_4 g_3$ for $a_i \in Q$ or Z or R; $1 \leq i \leq 4$ is a 4-dimensional dual complex finite modulo number.

We can use this and construct any desired dimensional dual complex modulo finite integer as follows:

Let

$V = \{(t_1, t_2, \ldots, t_{10}) \mid t_i \in \{0, g_1, g_2, g_3\} \subseteq C(Z_{16})\ 1 \leq i \leq 10\}$, V is a null semigroup under product. Infact a null ring. Construct $P = \{a_1 + a_2s_1 + \ldots + a_{o(v)}s_{o(v)-1} \mid a_i \in Q$ or Z or R $1 \leq i \leq o(V)$ and $s_j \in V \setminus \{(0)\}$, s_j's are distinct and $1 \leq j \leq o(v) - 1\}$, P gives various dimensional dual finite complex modulo numbers and P is a ring.

Likewise we can in P replace the row matrices by column matrices and adopt the natural product of matrices $\times_n$ or by $m \times n$ ($m \neq n$) matrices with once again the natural product $\times_n$ on matrices or square matrices with entries from the set $\{0, g_1, g_2, g_3\}$ and with usual matrix product and get using the same ring different dimensional finite complex modulo dual numbers.

For instance if

$x = 5 + 9\ (g_1, g_2, g_1, g_2, g_1, 0, 0, 0, 0, 0)$; $g_1 = 4 + 4i_F$ and $g_2 = 8 + 8i_F$ is a 2-dimensional dual finite complex modulo number.

Take

$$y = -12 + 7\begin{bmatrix} g_2 \\ g_2 \\ 0 \\ g_2 \\ g_1 \end{bmatrix} + 3\begin{bmatrix} g_3 \\ 0 \\ g_3 \\ 0 \\ g_1 \end{bmatrix}; \ g_2 = 8 + 8i_F,\ g_1 = 4 + 4i_F \text{ and}$$

$$g_3 = 12 + 12i_F \text{ in } C\ (Z_{16}),$$

y is a three dimensional finite complex modulo dual numbers.

Consider

$$z = 7-2\begin{bmatrix} g_1 & g_2 & g_3 & g_2 \\ 0 & g_1 & 0 & g_1 \\ g_3 & 0 & g_2 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & g_1 & g_1 & g_2 \\ g_2 & 0 & g_2 & g_3 \\ g_3 & g_3 & g_3 & 0 \end{bmatrix} +$$

$$19\begin{bmatrix} g_1 & g_1 & g_1 & g_1 \\ 0 & 0 & 0 & 0 \\ g_2 & g_2 & g_2 & g_2 \end{bmatrix};$$

z is a four dimensional finite complex modulo dual number.

Let

$$p = -90 + 24\begin{bmatrix} g_1 & g_2 & g_3 \\ 0 & g_1 & g_1 \\ 0 & 0 & g_2 \end{bmatrix} + 5\begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_1 & 0 \\ 0 & 0 & g_2 \end{bmatrix}$$

$$-7\begin{bmatrix} g_2 & g_2 & g_2 \\ 0 & 0 & 0 \\ g_1 & g_1 & g_1 \end{bmatrix} + 13\begin{bmatrix} 0 & 0 & g_2 \\ 0 & g_2 & 0 \\ g_2 & 0 & 0 \end{bmatrix} + 17\begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_1 & g_2 \\ g_3 & 0 & g_1 \end{bmatrix}$$

$$-40\begin{bmatrix} g_1 & 0 & g_3 \\ g_1 & 0 & g_3 \\ g_1 & 0 & g_3 \end{bmatrix}$$

where $g_1 = 4+ 4i_F$, $g_2 = 8 + 8i_F$ and $g_3 = 12 + 12i_F$ in $C(Z_{16})$.

P is a 7-dimensional dual finite complex modulo integer.

Interested readers can construct on similar lines any finite dimensional dual complex modulo finite numbers.

Example 3.52: Let S = $C(Z_{64})$ be the finite complex modulo integer.

Take $g_1 = 8 + 8i_F$, $g_2 = 16 + 16i_F$, $g_3 = 8 + 16i_F$, $g_4 = 16 + 8i_F$, $g_5 = 24 + 24i_F$, $g_6 = 24 + 8i_F$, $g_7 = 24 + 16i_F$, $g_8 = 16 + 24i_F$, $g_9 = 8 + 24i_F$, $g_{10} = 32 + 32i_F$, $g_{11} = 8 + 32i_F$, $g_{12} = 32 + 8i_F$, $g_{13} = 32 + 24i_F$, $g_{14} = 24 + 32i_F$, $g_{15} = 32 + 16i_F$, $g_{16} = 16 + 32i_F$, $g_{17} = 40 + 40i_F$, $g_{18} = 40 + 8i_F$, $g_{19} = 8 + 40i_F$, $g_{20} = 16 + 40i_F$, $g_{21} = 40 + 16i_F$ and so on.

We can build several dual complex finite modulo integers.

We can use these g_i's in square matrices or column matrices or row matrices or rectangular matrices and the net result will lead to a desired dimensional dual finite complex numbers.

Inview of this we propose the simple computational problem.

Problem: Let $C(Z_n) = S$ suppose $n = t^2$, then $t + ti_F$, $2t + 2ti_F$, …., $(t–1)t + (t–1)ti_F$ are complex numbers which are nilpotent of order two. Also $rt + sti_F$, $1 \leq r, s \leq (t–1)$ are also nilpotent finite complex number of order two.

Find the total number of such nilpotent finite complex numbers of order two in S and prove such a collection with zero added to it is a null subsemigroup of S.

We will illustrate this situation by some simple examples.

Example 3.53: Let $S = C(Z_9)$ be the finite complex modulo integer. The set of all nilpotent complex numbers of S are $V = \{3 + 3i_F, 6 + 6i_F, 6 + 3i_F, 3+6i_F\} \cup \{0\}$. V is a null semigroup infact a null ring. The number of non zero nilpotent elements of order two in S is four.

Example 3.54: Let $S = C(Z_{81})$ be the semigroup. The non zero nilpotent elements of S are as follows:

$V = \{9 + 9i_F, 18 + 18i_F, 27 + 27i_F, 36 + 36i_F, …, 72 + 72i_F, …, 72 + 63i_F\} \cup \{0\}$. The number of elements in V is 64.

Using these 64 elements in matrices we can get several dual finite complex modulo integers of any desired dimension.

Next we study the concept of dual semirings which are finite complex modulo numbers.

Let $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ be semirings.

Let $x = a + bg$ where $a, b \in Z^+ \cup \{0\}$ (or $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$) and $g \in C(Z_n)$, $g = x + yi_F$ with $i_F^2 = n-1$, $x, y \in Z_n \setminus \{0\}$ and $g^2 = 0 \pmod n$.

We see $x^2 = A + Bg$ ($A, B \in Z^+ \cup \{0\}$ (or $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$)); x is defined as the dual complex modulo integer.

We will just give one or two examples.

Example 3.55: Let $S = C(Z_{25})$ be a semigroup of complex modulo integers. Consider $x = 5 + 5i_F$ (or $y = 5 + 10i_F$ or $10 + 5i_F$ (or $y = 5 + 10i_F$ or $10 + 5i_F$ or $10 + 10i_F$).

Let $p = 3 + 4x$; p is a two dimensional dual finite complex modulo integer.

$$p^2 = (3 + 4x)^2 = 9 + 16x^2 + 24x$$
$$= 9 + 24x.$$

is again a two dimensional dual finite complex modulo integer.

Suppose
$s = 3 + 4\,(5 + 5i_F) + 7\,(10 + 5i_F) + 8\,(5+10i_F) + 12\,(10+10i_F)$ we see s is a 5 dimensional dual finite complex modulo integer.

Example 3.56: Let $S = C\,(Z_8)$ we see $g = 4 + 4i_F$ is such that $x = 9 + 8g$ is a two dimensional dual finite complex number $9, 8 \in Z^+ \cup \{0\}$.

Example 3.57: Let

$$M = \{a + bg \mid a, b \in Z^+ \cup \{0\},\ g = 8 + 8i_F \in C(Z_{16})\}$$

be the collection of all two dimensional dual complex finite modulo number and $Z^+ \cup \{0\}$ be the semifield;

(i) M is a semigroup under addition with identity 0.
(ii) (M, $\times$) is a semigroup under product with 1 as the multiplicative identity.
(iii) (M, +, $\times$) is a strict semiring which is not a semifield as $ag \times bg = 0$ even if $a \neq 0$ and $b \neq 0$.

However M will be defined as the general dual two dimensional finite complex modulo integer semiring.

Example 3.58: Let
$P = \{a + bg \mid a, b \in Z^+, g = 6 + 6i_F \in C(Z_9) \cup \{0\}$. (P, $\times$, +) be a semifield of two dimensional finite complex modulo dual numbers. The same is true if Z^+ is replaced by R^+ or Q^+.

Example 3.59: Let
$S = \{a + bg \mid a, b \in Q^+, g = 2 + 2i_F \in C(Z_4)\} \cup \{0\}$, (S, $\times$, +) be a semifield of two dimensional dual finite complex modulo integers.

Example 3.60: Let $S = \{a_1 + a_2g_1 + a_3g_2 \mid a_1, a_2, a_3 \in R^+$, $g_1 = 3+3i_F$ and $g^2 = 6 + 3i_F \in C(Z_9)\} \cup \{0\}$. $\{S,+,\times\}$ be semifield of three dimensional finite complex modulo dual integers.

Example 3.61: Let $M = \{a_1 + a_2g_1 + a_3g_2 + a_4 g_3 + a_5g_4 + a_6g_5 \mid a_1, a_2, a_3, a_4, a_5, a_6 \in Z^+, g_1 = 8 + 8i_F, g_2 = 8 + 12i_F, g_3 = 12 + 8i_F, g_4 = 12 + 12i_F$ and $g_5 = 4 + 4i_F; g_i \in C(Z_{16}); 1 \leq i \leq 5\} \cup \{0\}$. M is a 6 dimensional dual complex finite modulo integer semifield.

Example 3.62: Let $P = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 + a_8g_7 \mid a_i \in Q^+; 1 \leq i \leq 8; g_1 = 6 + 6i_F, g_2 = 12 + 12i_F, g_3 = 6 + 12i_F, g_4 = 12 + 6i_F, g_5 = 18 + 18i_F, g_6 = 18 + 12i_F$ and $g_7 = 12 + 18i_F; g_i \in C(Z_{36}); 1 \leq i \leq 7\} \cup \{0\}$. P be the dual finite complex modulo integer semifield.

Example 3.63: Let P = $\{a_1 + a_2g_1 \mid a_1, a_2 \in R^+, g_1 = 13 + 13i_F \in C(Z_{26}),\ g_1^2 \equiv 0 \pmod{26}\} \cup \{0\}$ be a dual semifield of complex modulo integer or semifield of dual complex modulo integers.

Example 3.64: Let

$$P = \left\{ \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \\ g_5 & g_6 \end{bmatrix} \middle| g_i \in \{0, 3+3i_F, 6+6i_F\} \subseteq C(Z_9), 1 \leq i \leq 6 \right\}.$$

Take

V = $\{a_1 + a_2x_1 + a_3x_2 + a_4x_3 + a_5x_4 + a_6x_5 + a_7x_6 + a_8x_7 + a_9x_8$ where $a_i \in Q^+ \cup \{0\}$, $1 \leq i \leq 9$, $x_j \in P \setminus \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$; $1 \leq j \leq o(P) - 1$ and x_j are distinct$\}$. V is a dual semiring of finite complex modulo integers.

Example 3.65: Let W = $\left\{ \begin{bmatrix} g_1 & g_2 & g_3 & g_4 & g_5 \\ g_6 & g_7 & g_8 & g_9 & g_{10} \end{bmatrix} \middle| g_i \in S = \{0, 12 + 12i_F, 8 + 8i_F, 4 + 4i_F, 4 + 12i_F, 8 + 4i_F, 4 + 8i_F, 12 + 4i_F, 8 + 12i_F, 12 + 8i_F\} \subseteq C(Z_{16}); 1 \leq i \leq 10 \right\}$.

Take P = $\{a_1 + a_2g_1 + a_2g_2 + \ldots + a_{27}\, g_{26} \mid a_i \in R^+, 1 \leq i \leq 27;$

g_j's are distinct and $g_j \in S \setminus \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}; 1 \leq j \leq 26\}$;

P is a semifield of 27 dimensional dual complex modulo integer. If we permit $a_i \in R^+ \cup \{0\}$ we see P is only a general semiring and it can contain dual finite complex numbers of dimension less than or equal to 27. Further $R^+ \cup \{0\} \subseteq P$ and $R^+ g_i \cup \{0\} \subseteq P$; $g_i \in S$.

Example 3.66: Let P = $\left\{\begin{bmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 \end{bmatrix}\right.$ where $g_i \in \{5 + 5i_F$, $10+10i_F$, $20+2i_F$, $15+15i_F$, $5 + 10i_F$, $5 + 20i_F$, $5 + 15i_F$, $10+5i_F$, $10+20i_F$, $10+15i_F$, $15+5i_F$, $15+10i_F$, $15+20i_F$, $20+5i_F$, $20+10i_F$, $20+15i_F$, $0\} \subseteq C(Z_{25})$; $1 \leq i \leq 9\}$.

Consider S = $\{a_1 + a_2x_1 + a_3x_2 + \ldots + a_{15}x_{14}$ where $a_i \in R^+ \cup \{0\}$, $1 \leq i \leq 15$, $x_j \in P$ x_j's distinct; $1 \leq j \leq 14\}$, S is a 15 dimensional or less; dual semiring complex modulo integers.

Now we just see how these get the vector space structure of dual finite complex modulo integers and semivector space of dual finite complex modulo integers.

Let V = $\{a_1 + a_2g_1 + \ldots + a_ng_{n-1} \mid a_i \in R, 1 \leq i \leq n, g_j \in S \subseteq C(Z_m)$ with $g_j^2 = 0 \pmod m$ and each g_j is distinct and different from zero. Also $g_pg_j = 0$ if $p \neq j$, $1 \leq p, j \leq n-1\}$.

It is easily verified V is a vector space over R (or Q). Further we can define product on V so that V is a linear algebra. We call V as a dual finite complex modulo vector integer space / linear algebra over R (or Q).

We now provide some examples of them.

Example 3.67: Let

$$P = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 \mid a_i \in Q,$$

$$1 \leq i \leq 6 \text{ and } g_j \in \left\{\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 & x_9 & x_{10} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \end{bmatrix} \right| x_t \in S$$

$$\{0, 3+3i_F, 6 + 6i_F, 3 + 6i_F, 6 + 3i_F\} \subseteq C(Z_9);$$
$$1 \leq j \leq 5, 1 \leq t \leq 15\};$$

P is a vector space over Q.

Infact P is a dual complex finite modulo integer vector space over Q. If we define the natural product $\times_n$ of matrices on g_i's then P is a dual complex finite modulo integer linear algebra over Q.

Interested reader can study the notion of subspaces, dimension, basis, linear transformation, linear operator and linear functionals on these vector spaces / linear algebra which is considered as a matter of routine. So we do not proceed to explain them. Further the concept of direct sum and pseudo direct sum of subspaces is direct and interested reader can study them.

Example 3.68: Let V = $\{a_1 + a_2g_1 + a_3g_2 + \ldots + a_8g_7 \mid a_i \in Q$; $1 \leq i \leq 8$ and $g_p \in \{(t_1, t_2, t_3, t_4) \mid t_j \in \{6 + 6i_F, 12+12i_F, 24+24i_F, 18+18i_F, 30+30i_F, 6+12i_F, 6+24i_F, 6+30i_F, 12+6i_F, 12+24i_F, 12+18i_F, 18+6i_F, 18+12i_F, 18+24i_F, 24+6i_F, 24+12i_F, 24+18i_F, 0\} \subseteq C(Z_{36})$; $1 \leq j \leq 4$ and g_p's are distinct $1 \leq p \leq 7\}$ be a dual finite complex modulo integer vector space over the field Q. Clearly under product, V is also a dual complex modulo integer linear algebra over Q.

Example 3.69: Let V = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 \mid a_i \in R$;

$1 \leq i \leq 5$; $g_j \in \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right.$ where $x_t \in \{7 + 7i_F, 7+14i_F, 14+14i_F,$

$14+7i_F, 21+21i_F, 7+21i_F, 21+7i_F, 14+21i_F, 21+14i_F, 28+28i_F, 28+7i_F, 28+14i_F, 28+21i_F, 7+28i_F, 14+28i_F, 21+28i_F, 35+35i_F, 35+7i_F, 35+14i_F, 35+21i_F, 35+28i_F, 7+35i_F, 14+35i_F, 21+35i_F, 28+35i_F, 42+42i_F, 42+7i_F, 42+14i_F, 42+21i_F, 42+28i_F, 7+42i_F, 42+35i_F, 14+42i_F, 21+42i_F, 28+42i_F, 35+42i_F, 0\} \subseteq C(Z_{49})$; g_j's are distinct, $1 \leq j \leq 4$; $1 \leq t \leq 5\}$ be a dual finite complex modulo integer vector space over Q. Infact V is a dual finite complex modulo integer linear algebra if we can define on g_i's the natural product $\times_n$ of matrices.

Next we just proceed onto indicate by examples dual finite complex modulo integer semivector space / semilinear algebra.

Example 3.70: Let

$$V = \{a + bg \mid a, b \in R+, g = 3+3i_F \in C(Z_9)\}$$

be a semivector space over the semifield $Q^+ \cup \{0\}$ (or $R^+ \cup \{0\}$ or $Z^+ \cup \{0\}$). V is a dual two dimensional complex modulo integer semivector over the semifield.

Example 3.71: Let

$$V = \{a + bg \mid a, b \in R^+ \cup \{0\}, g = (x_1, x_2, x_3, x_4)$$

$$= \{(3i_F + 3, 6 + 6i_F, 3+6i_F, 6+3i_F)\ x_i \in C(Z_9);\ 1 \leq i \leq 4\}$$

be a two dimension complex modulo integer dual semivector space over $Q^+ \cup \{0\}$.

Example 3.72: Let $M = \{a + bg_1 + cg_2 \mid a, b, c \in R^+, g_1 = 3+3i_F$ and $g_2 = 6 + 3i_F$; $g_1, g_2 \in C(Z_9)\}$ be a dual three dimensional finite complex modulo number semivector space over the semifield $Q^+ \cup \{0\}$. Infact M is also a three dimensional dual finite complex modulo integer semilinear algebra over $Q^+ \cup \{0\}$.

Example 3.73: Let

$$S = \{a + bg_1 + cg_2 + dg_3 + eg_4 \mid a, b, c, d, e \in Q^+;$$

$$g_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ where } x_i \in T$$

$= \{4 + 4i_F, 8+8i_F, 12+12i_F, 4+8i_F, 4+12i_F, 8+4i_F, 8+12i_F,$

$12+4i_F, 12+8i_F, 0\} \subseteq C(Z_{16}); 1 \leq i \leq 4\}$

be a five dimensional dual semivector space of finite complex modulo numbers over the semifield F = $Q^+ \cup \{0\}$. If on S we define product and natural product of the column matrices we cannot get a dual t-dimensional semilinear algebra over the semifield F = $Q^+ \cup \{0\}$, for if

$$x = 3+2\begin{bmatrix} 4+4i_F \\ 0 \\ 0 \\ 4+12i_F \end{bmatrix} + \begin{bmatrix} 8+8i_F \\ 0 \\ 4+8i_F \\ 0 \end{bmatrix}$$

$$+\frac{3}{4}\begin{bmatrix} 4+12i_F \\ 4+8i_F \\ 4+4i_F \\ 0 \end{bmatrix} + \frac{7}{12}\begin{bmatrix} 0 \\ 8+8i_F \\ 12+4i_F \\ 8+12i_F \end{bmatrix}$$

and

$$y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4+8i_F \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 4+4i_F \\ 0 \\ 8+4i_F \\ 0 \end{bmatrix} + 8\begin{bmatrix} 12+12i_F \\ 12+8i_F \\ 0 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 8+12i_F \\ 0 \\ 0 \\ 8+4i_F \end{bmatrix} + 1$$

be in S.

To find $x \times_n y$, $x \times_n y = 3 + 2\begin{bmatrix} 4+4i_F \\ 0 \\ 0 \\ 4+12i_F \end{bmatrix} +$

$$\begin{bmatrix} 8+8i_F \\ 0 \\ 4+8i_F \\ 0 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 4+12i_F \\ 4+8i_F \\ 4+4i_F \\ 0 \end{bmatrix} + \frac{7}{12}\begin{bmatrix} 0 \\ 8+8i_F \\ 12+4i_F \\ 8+12i_F \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 0 \\ 4+8i_F \end{bmatrix} +$$

$$\frac{9}{2}\begin{bmatrix} 4+4i_F \\ 0 \\ 8+4i_F \\ 0 \end{bmatrix} + 24\begin{bmatrix} 12+12i_F \\ 12+8i_F \\ 0 \\ 0 \end{bmatrix} + 15\begin{bmatrix} 8+12i_F \\ 0 \\ 0 \\ 8+4i_F \end{bmatrix} \text{ is not in S.}$$

In order to make this into a semilinear algebra we make the following changes;

We make

$$S = \left\{ a_1 + \sum_{i=2}^{n} a_i g_{i-1} \;\middle|\; \text{for all } g_j;\ 1 \leq j \leq o(T) \right\}$$

a dual semilinear algebra of finite complex modulo integer of dimension less than o(T). Thus we can define any desired dimension dual semilinear algebra / semivector space of complex modulo finite integers.

Interested reader can study subspaces of semivector spaces, semilinear transformation, semilinear operator and semilinear functions. Study of all things related with semivector spaces is a matter of routine and hence left as an exercise to the reader.

Chapter Four

DUAL INTERVAL NUMBERS AND INTERVAL DUAL NUMBERS

In this chapter we for the first time introduce the concept of dual interval numbers and interval dual numbers and study their properties.

Let S = {a + bg | a, b $\in$ r and g is such that $g^2 = 0$}. S is a general two dimensional dual number ring.

Consider $I_c(S)$ = {[x, y] | x, y $\in$ S}; we define $I_c(S)$ as the general dual number intervals or dual interval numbers. Here these types of intervals are present in $I_c(S)$.

[a, b + cg] (a, b, c $\in$ R \ {0}; $g^2 = 0$) or [b + cg, a] a, b and c in R \ {0} or [ag + b, cg] where a, b, c $\in$ R \ {0} or [cg, b + ag] or [cg, a]; a, c $\in$ R \ {0} or [a, cg] or [x + yg, t + sg];
t, s, x, y $\in$ R \ {0} or [ag, bg] or [a, b].

We just show how addition and product are defined on these intervals of dual numbers.

Suppose x = [3g, 8g] and y = [5 – g, 3+2g] $\in$ I (S).
x + y = [3g, 8g] + [5–g, 3+2g] = [5 + 2g, 3+10g].

Now x $\times$ y = [3g, 8g] $\times$ [5 – g, 3 + 2g]
= [3g $\times$ 5–g, 8g (3+2g)]
= [15g, 24g] ($\because$ $g^2 = 0$).

Take x = [–3g, 7g] and y = [8g, 4g] now x + y = [5g, 11g]
and x–y = [–11g, 3g] and x $\times$ y = [0, 0] = 0.

Take x = [3g – 1, 2+4g] and y = [7, 8g];
now x + y = [3g + 6, 2 + 12g] and xy = [21g – 7, 16g]

Let x = [3, –4] and y = [7g, 4g] $\in$ $I_c(S)$.
x $\times$ y = [21g, –16g] and x + y = [3 + 7g, –4 + 4g] $\in$ $I_c(S)$.

It is easily verified $I_c(S)$ is closed both under '+' and $\times$. Infact ($I_c(S)$, +) is an additive abelian group and ($I_c(S)$, $\times$) is only a semigroup under $\times$.

Thus ($I_c(S)$, +, $\times$) can easily be verified to form a commutative ring with divisors of zero.

We can replace in S, R by Q or Z and still the results will be true. $I_c(S)$ is easily seen to be the natural class of intervals of dual numbers of dimension two.

We will first illustrate this situation by some examples.

Example 4.1: Let S = {a + bg | g = 5 $\in$ Z_{25}, a, b $\in$ Q} be the general two dimensional dual number ring.

$I_c(S)$ = {[a + bg, c + dg] | a + bg and c + dg $\in$ S} be the natural class of closed intervals of dual numbers.

$I_c(S)$ is a ring with zero divisors.

$K = \{[ag, bg] \mid a, b \in R,\ g = 5 \in Z_{25}\} \subseteq I_c(S)$ is a subring which is also an ideal of $I_c(S)$. Infact K is a null subring of $I_c(S)$.

Further $T = \{[a, b] \mid a, b \in R\} \subseteq I_c(S)$ is a subring of $I_c(S)$, but is not a null subring, but T has zero divisors and T does not contain any nontrivial nilpotent element.

Take $P = \{[a + bg, cg] \mid a + bg, cg \in I_c(S)\} \subseteq I_c(S)$; P is a subring as well as an ideal of $I_c(S)$, for in P the second term cg will continue to be dg for some $d \in R$.

Consider $M = \{[a, b+cg] \mid a, b, c \in R\} \subseteq I_c(S)$, M is only a subring and not an ideal of $I_c(S)$.

Likewise $N = \{[a + bg, c]\ a, b, c \in R\} \subseteq I_c(S)$ is only a subring and not an ideal of $I_c(S)$. Thus $I_c(S)$ has ideals as well subrings which are not ideals. However $I_c(S)$ has units and zero divisors.

Example 4.2: Let $M = \{a + bg \mid g = 4 \in Z_{16}$ and $a, b \in Z\}$ be the two dimensional general dual ring.

$I_c(M) = \{[a + bg, c + dg] \mid a, b, c, d \in Z\}$ be the closed intervals of dual numbers. $I_c(M)$ is the ring of natural class of closed intervals of dual numbers.

$I_c(M)$ has ideals, subrings which are not ideals, null subring, zero divisors and no nontrivial units only $[1, 1]$, $[-1, 1]$, $[1, -1]$ and $[-1, -1]$ are units in $I_c(M)$. It is but a matter of routine to see that we can replace the closed intervals by open intervals; $I_o(S)$ (or $I_o(M)$) will denote the natural class of open intervals of dual numbers, all results true for $I_c(S)$ (or $I_c(M)$) will hold good.

Similarly

$I_{oc}(S)$ (or $I_{co}(M)$) $= \{(a + bg, c + dg] \mid a, b, c, d \in R\}$

is the natural class of open-closed intervals of dual numbers.

$I_o(M) = \{(a + bg, c + dg) \mid a, b, c, d \in Z\}$ is the natural class of open intervals of dual numbers.

Finally $I_{co}(M) = \{[a + bg, c + dg) \mid a, b, c, d \in Z\}$ is the natural class of closed-open intervals of dual numbers.

All these structures $I_o(S)$, $I_{oc}(S)$ and $I_{co}(S)$ are commutative rings with zero divisors and units. They contain ideals, subrings which are not ideals and null subsemirings.

Interested reader can study more properties and those results are a matter of routine. Now we can define three dimensional / n-dimensional natural class of open (closed, open-closed, closed-open) intervals of dual numbers which is treated as a matter of routine.

We will only illustrate this situation by some simple examples.

Example 4.3: Let
$S = \{a + bg_1 + cg_2 \mid a, b, c \in R$ (or Q or Z) $g_1 = 4$, $g_2 = 8 \in Z_{16}\}$ be the collection of three dimensional dual numbers where $a, b, c \in R \setminus \{0\}$ even if one of b or c is zero we get two dimensional dual number. If $a = 0$ we get a nilpotent element of order two.

So if we give any operation S we see S will contain R, nilpotent elements of the form $ag_1 + bg_2$ and nilpotent elements of the form ag_1 (bg_2) or $a + bg_1$ (or $a + bg_2$). Thus we define (S, +, ×) as a general dual number ring of dimension three.

Now using this S we define
$I_c(S) = \{[a + bg_1 + cg_2, x + yg_1 + zg_2]$ where $a, b, c, x, y, z \in R$ (or Q or Z) and $g_1^2 = 0$, $g_2^2 = 0$, $g_1 g_2 = g_2g_1 = 0\}$ as the natural class of three dimensional closed intervals of dual numbers.

Clearly ($I_c(S)$, +, ×) is a ring known as the general ring of three dimensional intervals of dual numbers. We see $I_c(S)$

contains subrings which are null rings as well as $I_c(S)$ has ideals and subrings which are not ideals.

Take $P = \{[ag_1, bg_2] \mid a, b \in R\} \subseteq I_c(S)$, P is an ideal as well as null subring of $I_c(S)$.

Consider
$T = \{[a_1g_1 + a_2g_2, a_3g_1 + a_4g_2] \mid a_i \in R, 1 \leq i \leq 4\} \subseteq I_c(S)$ is again an ideal which is a null subring of $I_c(S)$.

Let $P = \{[a, 0] \mid a \in R\} \subseteq I_c(S)$; P is only a subring and is not an ideal.

Consider $S = \{[0, ag_1] \mid a \in R\} \subseteq I_c(S)$, S is an ideal as well as a null subring of $I_c(S)$ we see $P.S = \{0\}$, for every $x \in S$; every $y \in P$ is such that $x.y = 0$.

Now $L = \{[ag_2, 0] \mid a \in R\} \subseteq I_c(S)$ is an ideal of $I_c(S)$. L is also a null subring. Thus $I_c(S)$ has several null subrings.

Now we can replace the closed interval $I_c(S)$ by $I_o(S)$ or $I_{co}(S)$ or $I_{oc}(S)$; still all results continue to be true.

Now we can define any desired dimensional closed intervals of dual numbers or (open intervals or closed-open intervals or open-closed intervals) which is direct and hence left as an exercise to the reader.

We will illustrate this situation by some examples.

Example 4.4: Let $I_c(S) = \{[a_1 + a_2g_1 + a_3g_2 + a_4g_5 + a_5g_4 + a_6g_5 + a_7g_6 + a_8g_7 + a_9g_8 + a_{10}g_9, b_1 + b_2g_1 + b_3g_2 + b_4g_3 + b_5g_4 + b_6g_5 + b_7g_6 + b_8g_7 + b_9g_8 + b_{10}g_9] \mid a_i, b_i \in R$ and $\{g_1 = 4 + 4i_F, g_2 = 8 + 8i_F, g_3 = 12 + 12i_F, g_4 = 4 + 8i_F, g_5 = 4 + 12i_F, g_6 = 8 + 4i_F, g_7 = 8 + 12i_F, g_8 = 12 + 4i_F$ and $g_9 = 12 + 8i_F\} \in C(Z_{16}); 1 \leq i \leq 10\}$ be collection of ten dimensional closed intervals of dual numbers.

We see $I_c(S)$ is a general ring of ten dimensional closed intervals of dual numbers. Clearly $I_c(S)$ contains closed intervals of dual numbers of all dimension less than or equal to ten. This $I_c(S)$ has several null subrings which are ideals and also subrings which are not ideals. Further $I_c(S)$ has zero divisors.

Example 4.5: Let $I_o(S) = \{(a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4, b_1 + b_2g_1 + b_3g_2 + b_4g_3 + b_5g_4) \mid a_i\ b_j \in R;\ 1 \le i, j \le 5$ and $g_1 = 3 + 3i_F$, $g_2 = 6 + 6i_F$, $g_3 = 3 + 6i_F$ and $g_4 = 6 + 3i_F \in C\,(Z_9)\}$ be the general ring of five dimensional open intervals of dual numbers. This has null subrings, ideals and subrings which are not ideals. $I_o(S)$ contains $N_o(R)$ also as a subring.

We will also call all t-dimensional dual numbers ($t > 2$) as extended dual numbers.

Now we can also give coordinate representation of them. If $x = a + bg$ where $a, b \in R$ and g is such that $g^2 = 0$ then we know x is represented as (a, bg) likewise if $x = a_1 + a_2g_1 + a_3g_2$ then this will be represented as (a_1, a_2g_1, a_3g_2), thus t-dimensional dual number $y = a_1 + a_2g_1 + a_3g_2 + \ldots + a_tg_{t-1}$ by $(a_1, a_2g_1, \ldots, a_tg_{t-1})$, a t-tuple.

This way of representation will certainly find its applications. We see if $y = a_1 + a_2g_1 + a_3g_2 + \ldots + a_tg_{t-1}$ then $y^2 = A_1 + A_2g_1 + A_3g_2 + \ldots + A_tg_{t-1}$ where $A_i \in R;\ 1 \le i \le t$. Thus $S = \{g_1, g_2, \ldots, g_{t-1}, 0\}$ forms a null semigroup under product.

Thus we can for instance imagine an interval.

$$\begin{aligned} A &= [a_1 + a_2g_1, b_1 + b_2g_1] \\ &= [(a_1, a_2g_1), (b_1, b_2g_1)] \\ &= [(a_1, b_1), (a_2g_1, b_2g_1)] \\ &= [a_1, b_1] + [a_2, b_2]\, g_1. \end{aligned}$$

Thus we see we get A = $[a_1, b_1] + [a_2, b_2]g_2$, which we define as dual interval coefficient numbers or interval dual numbers.

Thus $S_c(I) = \{[a_1, b_1] + [a_2, b_2]\, g_1 \mid [a_i, b_i] \in N_c(R),\ g_1^2 = 0\}$ is defined as the dual interval coefficient numbers.

If $[a_1, b_1] + [a_2, b_2]g_1 = x$ and $y = [c_1, d_1] + [c_2, d_2]\, g_1$, then $x + y = [a_1 + c_1, b_1 + d_1] + [a_2 + c_2, b_2 + d_2]g_1$ and

$$\begin{aligned} xy &= [a_1, b_1]\,[c_1, d_1] + [a_2, b_2]\,[c_1, d_1]\, g_1 + \\ &\quad [a_1, b_1]\,[c_2, d_2]g_1 + 0 \qquad\qquad (\because\ g_1^2 = 0) \\ &= [a_1c_1, b_1d_1] + [a_2c_1 + a_1c_2, b_2d_1 + b_1d_2]g \in S(I) \end{aligned}$$

is again a dual interval coefficient number.

We see $(S_c(I), +, \times)$ is a commutative ring of dual closed interval coefficient numbers. We can define $S_o(I)$, $S_{oc}(I)$ and $S_{co}(I)$ in a similar way which is left for the reader as an exercise as it is a matter of routine.

We can also define any t-dimensional dual closed (open or open-closed or closed - open) interval coefficient numbers.

We will just illustrate all these situations by some examples.

Example 4.6: Let

$I_o(S) = \{(a_1, b_1), (a_2, b_2)g_1, (a_3, b_3)g_2, (a_4, b_4)g_3\}$

$= \{(a_1 + a_2g_1 + a_3g_2 + a_4g_3,\ b_1 + b_2g_1 + b_3g_2 + b_4g_3)\}$ be of dimension four interval coefficient dual number.

$I_o(S) = \{(a_1, b_1) + (a_2, b_2)g_1 + (a_3, b_3)g_2 + (a_4, b_4)g_3\}$.

Thus $(I_o(S), +, \times)$ is again a ring called the dimension four interval coefficient dual number general ring.

Example 4.7: Let $I_{oc}(S) = \{(a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4, b_1 + b_2g_1 + b_3g_2 + b_4g_3 + b_5g_4] \mid a_i, b_j \in R; 1 \leq i, j \leq 5$ and $g_1 = 8 + 8i_F, g_2 = 4 + 4i_F, g_3 = 12 + 12i_F, g_4 = 12 + 8i_F \in C(Z_{16})\}$

$= \{(a_1, b_1] + (a_2, b_2] g_1 + (a_3, b_3]g_2 + (a_4, b_4]g_3 + (a_5, b_5] g_4 \mid (a_i, b_i] \in N_{oc}(R); 1 \leq i \leq 5; g_1 = 8 + 8i_F, g_2 = 4 + 4i_F, g_3 = 12 + 12i_F, g_4 = 12 + 8i_F \in C(Z_{10})\}$.

$\{(a_1, b_1], (a_2, b_2]g_1, (a_3, b_3]g_2, (a_4, b_4]g_3, (a_5, b_5]g_4) \mid (a_i, b_i] \in N_{oc}(R); 1 \leq i \leq 5$ and $g_1 = 8 + 8i_F, g_2 = 4 + 4i_F, g_3 = 12 + 12i_F$ and $g_4 = 12 + 8i_F \in C(Z_{16})\}$ be the five dimension open-closed interval coefficient dual numbers.

Clearly $(I_{oc}(S), +, \times)$ is a commutative ring with unit and also $I_{oc}(S)$ has zero divisors and no units different from units of the form $(a, b] \in N_{oc}(R) \subseteq I_{oc}(S)$ where $S = \{(a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_5, b_1 + b_2g_1 + b_3g_2 + b_4g_3 + b_5g_5] \mid a_i, b_j \in R; 1 \leq i, j \leq 5; g_1 = 8 + 8i_F, g_2 = (4 + 4i_F), g_3 = 12 + 12i_F$ and $g_4 = 12 + 8i_F \in C(Z_{16})\}$ is only a general ring of open-closed interval coefficient dual numbers.

Clearly $N_{oc}(R) \subseteq I_{oc}(S)$ and $N_{oc}(R)$ is just a subring and not an ideal. Also $(R, 0] \subseteq I_{oc}(S)$ is again a subring. Several properties of $I_{oc}(S)$ can be studied and it is a matter of routine so it is left as an exercise to the reader.

Example 4.8: Let $I_{co}(S) = \{[a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6, b_1 + b_2g_1 + b_3g_2 + b_4g_3 + b_5g_4 + b_6g_5 + b_7g_6) \mid a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6, b_1 + b_2g_1 + b_3g_2 + b_4g_3 + b_5g_4 + b_6g_5 + b_7g_6 \in S = \{$all 7-dimensional dual numbers with $a_i, b_j \in Q, 1 \leq i, j \leq 6$ and $g_1 = 7, g_2 = 14, g_3 = 21, g_4 = 28, g_5 = 35, g_6 = 42 \in Z_{49}\}\}$. $(I_{co}(S), +, \times)$ is a ring.

This ring has ideals, for take $J = \{[a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6, 0) \mid a_i \in Q; 1 \leq i \leq 7$ and $g_j \in \{7, 14, 21, 28, 35, 42\} \subseteq Z_{49}$, g_j's distinct and $1 \leq j \leq 6\} \subseteq I_{co}(S)$ and $I = \{[0, b_1 + b_2g_1 + b_3g_2 + b_4g_3 + b_5g_4 + b_6g_5 + b_7g_6) \mid b_j \in Q,$

$1 \leq j \leq 7$ and g_i's $\in \{7, 14, 21, 28, 35, 42\} \subseteq Z_{49}$ and g_i's are distinct$\} \subseteq I_{co}(S)$, clearly J and I are subrings which are ideals and $I + J = I_{co}(S)$ with $I.J = I \cap J = (0)$.

Consider
$P = \{[a_1 + a_2g_1, b_1 + b_2g_1) \mid a_i \in Q; 1 \leq i \leq 2\ g_1 = 7 \in Z_{49}\} \subseteq I_{co}(S)$; P is only a subring of $I_{co}(S)$ and is not an ideal of $I_{co}(S)$.

Likewise $M = \{[a_1 + a_2g_1 + a_3g_2 + a_4g_3, b_1 + b_2g_4 + b_3g_5) \mid a_i, b_j \in Q, 1 \leq i \leq 4$ and $1 \leq j \leq 3; g_1 = 7, g_2, 14, g_3 = 21, g_4 = 28$ and $g_5 = 35 \in Z_{49}\} \subseteq I_{co}(S)$ is only a subring and not an ideal of $I_{co}(S)$. We see $I_{co}(S)$ has subrings which are not ideals and also zero divisors. Several other interesting properties about $I_{co}(S)$ can be derived by the reader.

Further we can define the notion of positive intervals and the dual interval coefficients semirings and semifields.

Recall $I_{co}(R^+ \cup \{0\}) = \{[a, b) \mid a, b \in R^+ \cup \{0\}\}$ is the collection of all positive intervals and $I_{co}(R^+ \cup \{0\})$ is a semigroup under '+' and $(I_{co}(R^+ \cup \{0\}), \times)$ is a semigroup under $\times$. Infact $(I_{co}(R^+ \cup \{0\}), \times)$ has zero divisors and units. Thus $\{I_{co}(R^+ \cup \{0\}), +, \times\}$ is only a semiring which is a strict semiring but is not a semifield as it contains zero divisors.

These properties continue to hold good if $I_{co}(R^+ \cup \{0\})$ is replaced by open closed intervals.

$I_{oc}(R^+ \cup \{0\}) = \{(a, b] \mid a, b \in R^+ \cup \{0\}\}$ or closed intervals $I_c(R^+ \cup \{0\}) = \{[a, b] \mid a, b \in R^+ \cup \{0\}\}$. Also the above results hold good if $R^+ \cup \{0\}$ is replaced by $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$. All these structures are only strict semirings and not semifields. These semirings have semiideals also subsemirings which are not semiideals.

First we will illustrate this situation by some examples.

Example 4.9: Let $I_c(Q^+ \cup \{0\}) = \{[a, b] \mid a, b \in Q^+ \cup \{0\}\}$ = natural class of positive rational intervals. Clearly
$J = \{[a, 0] \mid a \in Q^+ \cup \{0\}\} \subseteq I_c(Q^+ \cup \{0\})$is a subsemiring of $I_c(Q^+ \cup \{0\})$ which is a semiideal of $I_c(Q^+ \cup \{0\})$.

Likewise $I = \{[0, a] \mid a \in Q^+ \cup \{0\}\} \subseteq I_c(Q^+ \cup \{0\})$ is a subsemiring which is a semiideal of $I_c(Q^+ \cup \{0\})$.

Further $I \cap J = (0)$ and $I + J = I_c(Q^+ \cup \{0\})$. Consider $P = \{[a, b] \mid a, b \in Z^+ \cup \{0\} \subseteq Q^+ \cup \{0\}\} \subseteq I_c(Q^+ \cup \{0\})$; P is only a subsemiring of $I_c(Q^+ \cup \{0\})$ and is not a semiideal of $I_c(Q^+ \cup \{0\})$. Thus $I_c(Q^+ \cup \{0\})$ has subsemirings which are not semiideals. Also $I_c(Q^+ \cup \{0\})$ has zero divisors and units.

Example 4.10: Let
$M = \{[a, b] + [c, d]g_1 \mid a, b, c, d \in Q^+ \cup \{0\}, g_1 = 4 \in Z_{16}\}$; M is a semiring which is not a semifield. M has both semiideals and subsemirings which are not semiideals. M will be known as the general semiring of dual interval coefficient numbers.
Take
$P = \{[a, 0] + [0, b]g_1 \mid a, b \in Q^+ \cup \{0\}, g_1 = 4 \in Z_{16}\} \subseteq M$ is a semisubring of M. Suppose
$T = \{[a, 0] + [0, b]g_1 \mid a, b \in Z^+ \cup \{0\}, g_1 = 4 \in Z_{16}\} \subseteq M$; then T is only a subsemiring and not a semiideal of M.

For consider $[5, 0] + [0, 8]g_1$ in T.

Suppose $[3/2, 6] + [6, 3/7]\, g_1 \in M$; consider

$$([5, 0] + [0, 8]g_1)\,([3/2, 6] + [6, 3/7]g_1)$$
$$= [15/2, 0] + [0, 48]g_1 + [30, 0]g_1 + (0)$$
$$= [15/2, 0] + [30, 48]g_1 \notin T.$$

Also if $[6, 0] + [0, 3]g_1 \in P$ and $[8, 4] + [7, 1]g_1 \in M$.

We see $([6, 0] + [0, 3]g_1) \times ([8, 4] + [7, 1]g_1)$
$= [48, 0] + [0, 12]g_1 + [42, 0]g_1 + (0)$
$= [48, 0] + [42, 12]\, g_1 \notin P$. Hence the claim.

However it is interesting to observe that T is a subsemiring of the subsemiring P and is not a semiideal of P. Thus we have subsemirings in M which are not semiideals. Consider
$W = \{[a, 0] + [b, 0]g_1 \mid a, b \in Q^+ \cup \{0\}, g_1 = 4 \in Z_{16}\} \subseteq M$, W is a subsemiring as well as a semiideal of M.

Suppose
$N = \{[a, 0] + [b, 0]g_1 \mid a, b \in 3Z^+ \cup \{0\}, g_1 = 4 \in Z_{16}\} \subseteq M$;
N is only a subsemiring and is not a semiideal of M.

Take
$V = \{[0, a] + [0, b]g_1 \mid a, b \in Q^+ \cup \{0\}, g_1 = 4 \in Z_{16}\} \subseteq M$;
V is a subsemiring as well as a semiideal of M.

Further $V \cap W = (0) = VW$ and $V + W = M$. We see V and W are orthogonal semiideals which are also known as the orthogonal semiideals with dual interval coefficient numbers.

We do not have fields from the rings of interval coefficients dual numbers but we can have semifields with interval coefficient dual numbers.

We will illustrate this situation by some examples.

Example 4.11: Let
$P = \{[a, b] + [c, d]g_1 \mid a, b, c, d \in Q^+ \text{ and } g_1 = 6 \in Z_{12}\} \cup \{0\}$; be a semiring of natural rational interval coefficient of dual numbers. P is a strict semiring and $\{P, +, \times\}$ is a semifield known as the semifield of natural rational interval coefficient semifield.

Example 4.12: Let
$S = \{[a, b) + [c, d)g_1 \mid a, b, c, d \in Z^+, g_1 = 20 \in Z_{100}\} \cup \{0\}$ be the semifield of natural closed-open interval coefficient dual numbers.

This has subsemirings which are not semifields.

Consider
$P = \{[a, b) + [c, d)\, g_1 \mid a, b, c, d \in 5Z^+, g_1 = 20 \in Z_{100}\} \cup \{0\} \subseteq S$,
P is only a strict semiring which has no zero divisors or is a semidomain.

Example 4.13: Let
$P = \{[a_1, b_1]+[a_2, b_2]g_1+[a_3, b_3]g_2+[a_4, b_4]\, g_3 \mid a_i, b_j \in Q^+ \cup \{0\}, 1 \leq i, j \leq 4;\ g_1 = 4, g_2 = 8 \text{ and } g_3 = 12 \in Z_{16}\}$ be a semiring of

dual number of closed interval coefficient of dimension four. P has zero divisors, units, ideals and semisubrings which are not ideals. Infact
$I = \{[a,0] + [a_2,0]g_1 + [a_3,0]g_2 + [a_4,0]g_3 \mid a_i \in Q^+ \cup \{0\}, 1 \leq i \leq 4;$ $g_1 = 4$, $g_2 = 8$ and $g_3 = 12 \in Z_{16}\} \subseteq P$ is a semiideal of P.

$J = \{[0, b_1] + [0, b_2] + [0, b_3]g_2 + [0, b_4]g_3 \mid b_j \in Q^+ \cup \{0\};$ $1 \leq j \leq 4$, $g_1 = 4$, $g_2 = 8$ and $g_3 = 12 \in Z_{16}\} \subseteq P$ is also a semiideal of P with $I + J = P$ and $I \cap J = (0)$.

In view of this we have the following theorem.

THEOREM 4.1: *Let $S = \{[a_1, b_1] + [a_2, b_2]g_1 + \ldots + [a_n, b_n]g_{n-1} \mid [a_i, b_i] \in N_c(R^+ \cup \{0\})$; $1 \leq i \leq n$ and $g_i \in T$ where $T \cup \{0\}$ is a null semigroup and $g_i^2 = 0$, $g_i\, g_j = g_j\, g_i = 0$ if $i \neq j$; $1 \leq i, j \leq n-1\}$ be a dual semiring of closed natural interval coefficient numbers then S has two disjoint semiideals I and J such that*

$$I + J = S \text{ and}$$
$$I \cap J = I.\, J = \{0\}.$$

Proof: Follows from the simple fact if we take
$I = \{[a_1, 0] + [a_2, 0]g_1 + \ldots + [a_n, 0]g_{n-1} \mid a_i \in R^+ \cup \{0\}, g_j \in T;$ $1 \leq i \leq n$ and $1 \leq j \leq n-1\} \subseteq S$ and

$J = \{[0, b_1] + [0, b_2]g_1 + \ldots + [0, b_n]g_{n-1}$ where $b_i \in R^+ \cup \{0\}$, $g_j \in T$; $1 \leq i \leq n$ and $1 \leq j \leq n-1\} \subseteq S$ then I and J are semiideals such that $I \cap J = (0)$ and $I + J = S$.

Now we prove the following theorem.

THEOREM 4.2: *Let $S = \{[a_1, b_1] + [a_2, b_2]g_1 + \ldots + [a_n, b_n]g_{n-1} \mid [a_i, b_i] \in N_c(Q^+)$ $1 \leq i \leq n$ and $g_j \in T = \{n, 2n, \ldots, (n-1)n\} \subseteq Z_{n^2}$ $(g_j^2 = 0$ $g_pg_j = 0)$ $1 \leq j, p \leq n-1\} \cup \{0\}$ be a semifield of closed natural interval coefficient dual numbers. P has semiinterval domains and no zero divisors.*

Proof: $P = \{[a_1, b_1] + [a_2, b_2]\, g_1 + \ldots + [a_n, b_n]g_{n-1} \mid [a_i, b_i] \in N_c(3Z^+)$; $1 \leq i \leq n$ and $g_j \in T\} \subseteq S$ is a semiinterval domain.

Replace $N_c(3Z^+)$ by $N_c(mZ^+)$; $2 \leq m < \infty$ we get infinite number of semiintegral domains.

Note: If Q^+ is replaced by R^+ or Z^+ the claims in the theorem hold good.

Example 4.14: Let M = $\{[a_1, b_1] + [a_2, b_2]g_1 + [a_3, b_3]g_2 + \ldots + [a_9, b_9]g_8 \mid a_i, b_i \in Q^+; 1 \leq i \leq 9$ and $g_j \in T = \{9, 18, 27, 36, 45, 54, 63, 72\} \subseteq Z_{81}$; g_j's distinct; $1 \leq j \leq 8\} \cup \{0\}$; be a semifield of closed natural of interval coefficients of dual numbers.

Take P = $\{[a_1, b_1] + [a_2, b_2]g_1 + \ldots + [a_9, b_9]g_8 \mid [a_j, b_j] \in N_c(5Z^+); 1 \leq j \leq 9; g_p \in T; 1 \leq p \leq 8\} \cup \{0\} \subseteq M$; P is only a semiintegral domain of M and not a subsemifield.

Take S = $\{[a_1, b_1] + [a_2, b_2]g_1 + \ldots + [a_9, b_9]\ g_8 \mid [a_i, b_i] \in N_c(Q^+)), 1 \leq i \leq 9, g_j \in T, 1 \leq j \leq 8\} \cup \{0\} \subseteq M$, S is a subsemifield of M. Infact M has infinitely many semiintegral domains but only one subsemifield. However if Q^+ is replaced by R^+ then M will have two subsemifield and infinitely many semi integral domains.

Now we wish to bring in some relation between the vector spaces and semivector spaces.

We see if V is a vector space over R say
V = $\{(a_1, \ldots, a_n) \mid a_i \in R; 1 \leq i \leq n\}$ is a vector space over R.

Consider
W = $\{(a_1, \ldots, a_n) \mid a_i \in R^+ \cup \{0\}, 1 \leq i \leq n\} \subseteq V$; W is a semivector space over $R^+ \cup \{0\}$ and infact the positive cone of V.

This result will always be true in case of vector space V, of dual numbers with interval (natural) coefficient from R or Q; if only positive natural class of intervals are taken as coefficient and then $W \subseteq V$ is a positive cone of dual numbers with interval coefficients or it is semivector space over $R^+ \cup \{0\}$.

We will just illustrate this fact from some examples.

Example 4.15: Let

$M = \{[a_1, b_1] + [a_2, b_2]g_1 + \ldots + [a_{10}, b_{10}]g_9 \mid [a_i, b_i] \in N_c(R)$; $1 \leq i \leq 10$, $g_j \in T = \{10, 20, 30, 40, 50, 60, 70, 80, 90\} \subseteq Z_{100}$, g_j's distinct $1 \leq j \leq 9\}$ be a vector space of dual numbers with coefficients from natural class of interval $N_c(R)$ over the field R. Consider $W = \{[a_1, b_1] + [a_2, b_2]g_1 + \ldots + [a_{10}, b_{10}]g_9 \mid [a_i, b_i] \in N_c(R^+ \cup \{0\})$; $g_j \in T \subseteq Z_{10}$, g_j's are distinct $1 \leq i \leq 10$ and $1 \leq j \leq 9\} \subseteq M$; clearly W is a semivector space of dual numbers with coefficients from N_c $(R^+ \cup \{0\})$ over the semifield $R^+ \cup \{0\}$. Infact W is a positive cone of M.

Inview of this we have the following theorem.

THEOREM 4.3: *Let*
$V = \{[a_1, b_1] + [a_2, b_2]g_1 + \ldots + [a_n, b_n]g_{n-1} \mid [a_i, b_i] \in N_c(R)$ (or $N_c(Q)$) $1 \leq i \leq n$; $g_j \in T = \{n, 2n, 3n, \ldots, (n-1)n\} \subseteq Z_{n^2}$; g_j's distinct; $1 \leq j \leq n-1\}$ be vector space of interval coefficient dual numbers over R (or Q).

$W = \{[a_1, b_1] + [a_2, b_2]g_1 + \ldots + [a_n, b_n]g_{n-1} \mid [a_i, b_i] \in N_c$ $(R^+ \cup \{0\})$ (or N_c $(Q^+ \cup \{0\})$; $1 \leq i \leq n$; $g_j \in T$, g_j's distinct $1 \leq j \leq n-1\} \subseteq V$ is a semivector space of interval coefficient dual numbers from $N_c(R^+ \cup \{0\})$ (or $N_c(Q^+ \cup \{0\})$) over the semifield $R^+ \cup \{0\}$ (or $Q^+ \cup \{0\}$). W is always a positive cone of V.

Proof is straight forward and hence left as an exercise to the reader.

Now we proceed onto illustrate different types of dual numbers with coefficients from interval matrices or matrices with interval entries. The definition of this concept is a matter routine and hence is left as exercise to the reader.

Example 4.16: Let

$P = \{([a_1^1, b_1^1], [a_2^1, b_2^1], [a_3^1, b_3^1]) + ([a_1^2, b_1^2] + [a_2^2, b_2^2] + [a_3^2, b_3^2])g_1 + ([a_1^3, b_1^3], [a_2^3, b_2^3], [a_3^3, b_3^3])g_2 + ([a_1^4, b_1^4], [a_2^4, b_2^4],$

$[a_3^4, b_3^4])g_3 \mid [a_i^t, b_i^t] \in N_c\ (R^+ \cup \{0\});\ 1 \le i \le 3,\ 1 \le t \le 4$ and $g_1 = 4,\ g_2 = 8$ and $g_3 = 16 \in Z_{16}\}$ be a semivector space of interval matrix coefficient dual numbers over the semifield $R^+ \cup \{0\}$ (or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$).

Example 4.17: Let

$$S = \left\{ \begin{bmatrix} [a_1^1, b_1^1] \\ [a_2^1, b_2^1] \\ [a_3^1, b_3^1] \\ [a_4^1, b_4^1] \\ [a_5^1, b_5^1] \end{bmatrix} + \begin{bmatrix} [a_1^2, b_1^2] \\ [a_2^2, b_2^2] \\ [a_3^2, b_3^2] \\ [a_4^2, b_4^2] \\ [a_5^2, b_5^2] \end{bmatrix} g_1 + \begin{bmatrix} [a_1^3, b_1^3] \\ [a_2^3, b_2^3] \\ [a_3^3, b_3^3] \\ [a_4^3, b_4^3] \\ [a_5^3, b_5^3] \end{bmatrix} g_2 \;\middle|\; [a_i^t, b_i^t] \in \right.$$

$$\left. N_c(Z^+ \cup \{0\});\ 1 \le i \le 5 \text{ and } 1 \le t \le 3,\ g_1 = 3 \text{ and } g_2 = 6 \in Z_9 \right\}$$

be a semivector space of column interval matrix coefficient of dual numbers over the semifield $Z^+ \cup \{0\}$.

Example 4.18: Let

$$P = \left\{ \begin{bmatrix} [a_1^1, b_1^1] & [a_2^1, b_2^1] \\ [a_3^1, b_3^1] & [a_4^1, b_4^1] \\ [a_5^1, b_5^1] & [a_6^1, b_6^1] \\ [a_7^1, b_7^1] & [a_8^1, b_8^1] \end{bmatrix} + \begin{bmatrix} [a_1^2, b_1^2] & [a_2^2, b_2^2] \\ [a_3^2, b_3^2] & [a_4^2, b_4^2] \\ [a_5^2, b_5^2] & [a_6^2, b_6^2] \\ [a_7^2, b_7^2] & [a_8^2, b_8^2] \end{bmatrix} g_1 \;\middle|\; [a_i^t, b_i^t] \in \right.$$

$$\left. N_c(Q^+ \cup \{0\});\ 1 \le i \le 8,\ t = 1, 2 \text{ and } g_1 = 12 \in Z_{24} \right\}$$

be a semivector space of dual numbers with interval matrix coefficients over the semifield $Q^+ \cup \{0\}$ (or $Z^+ \cup \{0\}$ we can define natural product $\times_n$ on matrices so that P is also a semilinear algebra.

We now can rewrite P as follows:

$$P = \begin{bmatrix} [a_1^1 + a_1^2, (b_1^1 + b_1^2)g_1] & [a_2^1 + a_2^2 (b_2^1 + b_2^2)g_1] \\ [a_3^1 + a_3^2, (b_3^1 + b_3^2)g_1] & [a_4^1 + a_4^2 (b_4^1 + b_4^2)g_1] \\ [a_5^1 + a_5^2, (b_5^1 + b_5^2)g_1] & [a_6^1 + a_6^2 (b_6^1 + b_6^2)g_1] \\ [a_7^1 + a_7^2, (b_7^1 + b_7^2)g_1] & [a_8^1 + a_8^2 (b_8^1 + b_8^2)g_1] \end{bmatrix}$$

$$= \begin{bmatrix} [a_1^1 + a_1^2 + (b_1^1 + b_1^2)g_1, a_2^1 + a_2^2 (b_2^1 + b_2^2)g_1] \\ [a_3^1 + a_3^2, (b_3^1 + b_3^2)g_1, a_4^1 + a_4^2 (b_4^1 + b_4^2)g_1] \\ [a_5^1 + a_5^2, (b_5^1 + b_5^2)g_1, a_6^1 + a_6^2 (b_6^1 + b_6^2)g_1] \\ [a_7^1 + a_7^2, (b_7^1 + b_7^2)g_1, a_8^1 + a_8^2 (b_8^1 + b_8^2)g_1] \end{bmatrix}$$

is the matrix with interval coefficient dual numbers as its entries.

Thus we can go from interval matrix structure to coefficient interval matrix structure and vice-versa. However it is interesting to note that P under natural product $\times_n$ of matrices is a semilinear algebra of interval dual numbers.

Example 4.19: Let $T = \left\{ \begin{bmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_4^1 & a_5^1 & a_6^1 \\ a_7^1 & a_8^1 & a_9^1 \end{bmatrix} + \begin{bmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_4^2 & a_5^2 & a_6^2 \\ a_7^2 & a_8^2 & a_9^2 \end{bmatrix} g_1 + \right.$

$\begin{bmatrix} a_1^3 & a_2^3 & a_3^3 \\ a_4^3 & a_5^3 & a_6^3 \\ a_7^3 & a_8^3 & a_9^3 \end{bmatrix} g_2$ where $g_1 = 2$ and $g_2 = 2 + 2i_F \in C(Z_4)$ and

$a_i^t = [x_i^t, y_i^t] \in N_c(R^+ \cup \{0\})$; $t = 1, 2, 3$ and $1 \leq i \leq 9\}$ be a semivector space of matrix interval coefficient dual numbers over the semifield $R^+ \cup \{0\}$.

Now if usual product of matrices is taken then T is a semilinear algebra which is non commutative. If on T for the coefficient matrices natural product $\times_n$ is defined then T is commutative semilinear algebra of matrix coefficients dual numbers.

We can also rewrite T as

$$T = \left\{ \begin{bmatrix} a_1^1 + a_1^2 g_1 + a_1^3 g_2 & a_2^1 + a_2^2 g_1 + a_2^3 g_2 & a_3^1 + a_3^2 g_1 + a_3^3 g_2 \\ a_4^1 + a_4^2 g_1 + a_4^3 g_2 & a_5^1 + a_5^2 g_1 + a_5^3 g_2 & a_6^1 + a_6^2 g_1 + a_6^3 g_2 \\ a_7^1 + a_7^2 g_1 + a_7^3 g_2 & a_8^1 + a_8^2 g_1 + a_8^2 g_2 & a_9^1 + a_9^2 g_1 + a_9^3 g_2 \end{bmatrix} \right.$$

where $a_i^t = [x_i^t, y_i^t] \in N_c(R^+ \cup \{0\})$; $t = 1, 2, 3$ and $1 \leq i \leq 9\}$, T is defined as the semivector space of interval coefficient dual number matrices over the semifield $R^+ \cup \{0\}$. Further T is a semilinear algebra, commutative or otherwise depending on the operation defined on the matrices.

We can also derive all properties associated with these semivector spaces. Infact all these semivector spaces can be realized as the positive cone of the appropriate vector space of interval coefficient dual number matrices. We can also use interval dual numbers $[g_i, g_j]$ where $g_i, g_j \in S$; S a null semigroup under product we have the concept of n-tuple intervals, that is

$[(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)]$ where $x_i, y_i \in S$. S a null semigroup $1 \leq i, j \leq n$.

Also we have $\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_t \end{bmatrix} \right)$ where $a_i, b_j \in S$;

S a null semigroup $1 \leq i, j \leq t$.

Further we can have

$$\left(\begin{bmatrix} g_1 & g_2 & \cdots & g_m \\ g_{m+1} & g_{m+2} & \cdots & g_{2m} \\ \vdots & \vdots & & \vdots \\ g_{(p-1)m+1} & g_{(p-1)m+2} & \cdots & g_{pm} \end{bmatrix}, \begin{bmatrix} b_1 & b_2 & \cdots & b_m \\ b_{m+1} & b_{m+2} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{(p-1)m+1} & b_{(p-1)m+2} & \cdots & b_{pm} \end{bmatrix}\right)$$

where $g_i, b_j \in S$, S a null semigroup, $1 \leq i, j \leq pm$ and so on.

We will just illustrate all these situations by some examples.

Example 4.20: Let W = $\{([g_1, g_2, g_3, g_4], [h_1, h_2, h_3, h_4]) \mid g_i, h_j \in S = \{5, 10, 15, 20, 0\} \subseteq Z_{25}, 1 \leq i, j \leq 4\}$ be a row matrix interval of nilpotent elements of order two. Clearly W is a null semigroup under product.

For if x = ([5, 0, 10, 0], [15, 20, 5, 10]) and y = ([10, 5, 10, 5], [0, 10, 0, 15]) are in W then x × y = (0); easy to verify.

Infact (W, +, ×) is a ring defined as the null ring.

Using the elements of the null ring W we can build vector spaces of interval dual numbers. For if we take
X = $\{a+bw \mid a, b \in R$ and $w=([5, 10, 16, 20], [10, 5, 0, 0]) \in W\}$, it is easily verified X is a semigroup under product and X is not closed under addition.

However if Y = $\{a_1 + \sum_{w_i \in W \setminus \{(0)\}} a_i w_i \mid a_i, a_i \in R; w_i \in W \setminus \{0\}, 1 \leq i \leq |W| - 1\}$ then Y is a group under addition and Y is a vector space of matrix interval dual numbers over the field R. Infact Y is a linear algebra over R.

Example 4.21: Let

$$V = \left\{ x_1 + x_2 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + x_3 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + x_4 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right| x_1, x_2, x_3, x_4 \in Q,$$

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \text{ where } 2 \in Z_4\}$$

be a vector space of dual column matrix numbers over the field Q.

Example 4.22: Let

$$M = \{x_1 + x_2 \left(\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right) + x_3 \left(\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right) + x_4 \left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right) \Bigg|$$

$$x_i \in R;\ 1 \leq i \leq 4, \text{ and } 2 \in Z_4\}$$

be a vector space of dual matrix interval numbers over Q or R.

Infact by defining natural product we see M is a linear algebra over Q or R.

Example 4.23: Let

$$P = \{x_1 + x_2 \left(\begin{bmatrix} 2+2i_F & 2 & 2i_F \\ 0 & 2i_F & 0 \\ 2 & 0 & 2+2i_F \end{bmatrix}, \begin{bmatrix} 0 & 2 & 2i_F \\ 2 & 2i_F & 0 \\ 2 & 2+2i_F & 2i_F \end{bmatrix} \right) +$$

$$x_3\left(\begin{bmatrix} 2i_F & 2+2i_F & 0 \\ 0 & 2 & 2+2i_F \\ 2 & 0 & 2i_F \end{bmatrix}, \begin{bmatrix} 2 & 2i_F & 2+2i_F \\ 0 & 2 & 0 \\ 2i_F & 0 & 2i_F \end{bmatrix}\right) +$$

$$x_4\left(\begin{bmatrix} 2 & 2i_F & 2i_F \\ 0 & 2+2i_F & 2+2i_F \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2+2i_F & 2+2i_F \\ 0 & 2 & 0 \\ 2i_F & 0 & 2i_F \end{bmatrix}\right) \Bigg| \; x_i \in Q;$$

$1 \leq i \leq 4$ and $\{2, 2+2i_F, 2i_F, 0\} \subseteq C(Z_4)\}$ is a vector space of interval matrix dual numbers over Q.

Infact each interval is a nilpotent of order two. We see P is a linear algebra over Q.

Example 4.24: Let

$$W = \{x_1 + x_2\left(\begin{bmatrix} 5 & 5i_F \\ 0 & 0 \\ 5i_F & 5 \end{bmatrix}, \begin{bmatrix} 0 & 5+5i_F \\ 5 & 5i_F \\ 5+5i_F & 0 \end{bmatrix}\right) +$$

$$x_3\left(\begin{bmatrix} 0 & 5i_F \\ 0 & 0 \\ 5i_F & 5 \end{bmatrix}, \begin{bmatrix} 5 & 5+5i_F \\ 5 & 5i_F \\ 5+5i_F & 0 \end{bmatrix}\right) +$$

$$x_4\left(\begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \Bigg| \; x_1, x_2, x_3, x_4 \in Q \text{ and}$$

$$\{0, 5, 5i_F, 5+5i_F\} \subseteq C(Z_{10})\}$$

be a linear algebra of interval matrix dual numbers under natural product over Q.

We can also define the notion of semivector spaces of interval matrix dual numbers over semifield.

We will only give examples of them as the definition a matter of routine.

Example 4.25: Let S = {0, 6, 12, 18, 24, 30} $\subseteq Z_{36}$ be the null semigroup. Suppose

P = $\{((a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)) \mid a_i, b_j \in S, 1 \leq i, j \leq 4\}$; P is also a null semigroup under product.

Let M = $\{x_1 + x_2 g_1 + x_3\, g_2 + \ldots + x_{o(p)}\, g_{o(p)-1} \mid x_i \in Z^+ \cup \{0\}$; $1 \leq i \leq o(P)$ and $g_j \in P \setminus \{0\}$ with each g_j distinct and $1 \leq j \leq o(P) - 1\}$. It is easily verified M is a semivector space of interval dual numbers over the semifield $Z^+ \cup \{0\}$.

Further M is a semilinear algebra of interval dual numbers over the semifield $Z^+ \cup \{0\}$.

Example 4.26: Let S = $\{0, 7, 14, 21, 28, 35, 42, 7i_F, 14i_F, 21i_F, 28i_F, 35i_F, 42i_F, 7+7i_F, 7+14i_F, 7+21i_F, 7+28i_F, 7+35i_F, 7+42i_F, 14+7i_F, 14+14i_F, \ldots, 35+42i_F, \ldots, 42+42i_F\} \subseteq C(Z_{49})$; be a null semigroup under multiplication modulo 49.

$$V = \{x_1 + x_2 \left(\begin{pmatrix} a_1^1 & a_2^1 & a_3^1 & a_4^1 \\ a_5^1 & a_6^1 & a_7^1 & a_8^1 \end{pmatrix}, \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 & b_4^1 \\ b_5^1 & b_6^1 & b_7^1 & b_8^1 \end{pmatrix} \right) + \ldots +$$

$$x_{16} \left(\begin{pmatrix} a_1^{15} & a_2^{15} & a_3^{15} & a_4^{15} \\ a_5^{15} & a_6^{15} & a_7^{15} & a_8^{15} \end{pmatrix}, \begin{pmatrix} b_1^{15} & b_2^{15} & b_3^{15} & b_4^{15} \\ b_5^{15} & b_6^{15} & b_7^{15} & b_8^{15} \end{pmatrix} \right)$$

where $x_i \in Q^+ \cup \{0\}$ and $a_p^t, b_j^t \in S$; $1 \leq t \leq 15$ and $1 \leq p, j \leq 8$ and $1 \leq i \leq 16\}$

is a semivector space of interval matrix dual numbers over the semifield $Q^+ \cup \{0\}$. Infact this is the positive cone of the vector space if $x_i \in Q$ instead of $Q^+ \cup \{0\}$ and defined over the field Q instead of semifield $Q^+ \cup \{0\}$.

Further this V has subsemispaces. It is a matter of routine to find a basis, semilinear operator on V and semilinear functionals from V to $Q^+ \cup \{0\}$.

Example 4.27: Let

$$M = \left\{x_1 + x_2 \left(\begin{bmatrix} 10 & 10+10i_F \\ 0 & 10i_F \\ 10+10i_F & 0 \\ 10i_F & 10 \\ 10 & 10i_F+10 \end{bmatrix}, \begin{bmatrix} 10i_F & 10 \\ 10 & 10i_F \\ 10+10i_F & 0 \\ 0 & 10+10i_F \\ 10i_F & 10i_F \end{bmatrix} \right) \right|$$

$10,\ 10+10i_F,\ 10i_F \in C(Z_{20}),\ x_1, x_2 \in Z^+ \cup \{0\}\}$
be a semilinear algebra of interval dual number of dimension two over the semifield $Z^+ \cup \{0\}$.

Clearly $a = x_1 + x_2g_1$ where

$$g_1 = \left(\begin{bmatrix} 10 & 10+10i_F \\ 0 & 10i_F \\ 10+10i_F & 0 \\ 10i_F & 10 \\ 10 & 10i_F+10 \end{bmatrix}, \begin{bmatrix} 10i_F & 10 \\ 10 & 10i_F \\ 10+10i_F & 0 \\ 0 & 10+10i_F \\ 10i_F & 10i_F \end{bmatrix} \right)$$

is such that $g_1^2 = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$ and $a_2 = X_1 + X_2\, g_1$ where

$X_1, X_2 \in Z^+ \cup \{0\}$.

For if $a = 7 + 3g_1$ then

$a^2 = (7 + 3g_1)^2 = 49 + 9.0 + 2 \times 7.3 \times g_1 = 49 + 42g_1$ is again in the same form.

Interested reader can find subsemispaces, bases, linear operators on M.

Example 4.28: Let $V = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid (x_1, x_2, x_3, x_4)$

$$\in Q^+ \cup \{0\};\ g_1 = \begin{bmatrix} 4 & 8 & 4 \\ 4 & 4 & 8 \\ 12 & 12 & 4 \end{bmatrix},\ g_2 = \begin{bmatrix} 8 & 0 & 8 \\ 8 & 8 & 0 \\ 8 & 8 & 8 \end{bmatrix} \text{ and}$$

$$g_3 = \begin{bmatrix} 12 & 8 & 12 \\ 12 & 12 & 8 \\ 4 & 4 & 12 \end{bmatrix} \text{ and } 4, 8, 12 \in Z_{16}\}$$

be a semivector space of interval dual numbers over the semifield $Q^+ \cup \{0\}$.

Clearly $g_1^2 = (0)$, $g_2^2 = (0)$ and $g_3^2 = (0)$, also $g_1g_2 = (0)$, $g_3g_2 = (0)$ and $g_1g_3 = (0)$.

It is easily verified V is also a dual semilinear algebra over the semifield $Q^+ \cup \{0\}$.

Example 4.29: Let

$$M = \left\{x_1 + x_2 \begin{bmatrix} 2 & 2+2i_F & 0 & 2i_F & 2 \\ 2i_F & 2 & 0 & 2+2i_F & 2i_F \\ 0 & 2i_F & 2+2i_F & 0 & 2 \end{bmatrix} \right| 2, 2i_F,$$

$$2+2i_F \in C(Z_4),\ x_1, x_2 \in Q^+ \cup \{0\}\}$$

be a semivector space of matrix dual numbers over the semifield $Z^+ \cup \{0\}$. M is also a semilinear algebra.

We can also define polynomials of null semigroup. We first describe this concept.

Consider

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{4, 8, 12, 0\} \subseteq Z_{16} \right\}$$

a collection of polynomials in the variable x. We see (P, +) is an abelian group and (P, ×) is a semigroup. Infact P is a null semigroup.

Suppose S = $\{a + bg \mid a, b \in Q\ g \in P$ (g fixed)$\}$; S is a general ring of dual polynomials.

It is pertinent to mention here g can be finite or infinite in both cases a + bg is a dual number.

Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{2i_F, 2, 2i_F + 2, 0\} \subseteq C(Z_4) \right\}$$

be the polynomials with coefficients from $\{2i_F, 2, 2+2i_F, 0\} \subseteq C(Z_4)$. P is again a null semigroup as well as null ring. Several properties can be built using this structure.

Again S = $\{a + bg \mid g \in P$ and $a, b \in Z\}$ is a dual polynomial general ring of dimension two.

Suppose X = $\{a_1 + a_2g_1 + a_3g_2 + \ldots + a_ng_{n-1} \mid g_i \in P \setminus \{0\}$, g_i's distinct, $1 \leq i \leq n-1$ and $a_j \in Q$; $1 \leq j \leq n\}$. X is not a general dual polynomial ring of dimension n. For we see X is not closed under addition. However X is closed under product and is a commutative semigroup.

Consider

$$S = \left\{ a_0 + \sum_{i=0}^{\infty} a_i x^i \,\middle|\, g_j \in P,\ a_0, a_i \in Q \right\},$$

S is a general ring of infinite dimensional dual polynomial numbers.

Further every s in S \ Q is a dual number.

Only by this method we get infinite dimensional dual number.

It is pertinent to mention here that Q or R or Z can also be replaced by C = {a + bi | a, b ∈ R} and still all the results mentioned about dual numbers will hold good. As it not a new theory but a usual outcome we have not made a special mention of it.

Now we can define semirings of dual polynomials numbers. Here it is very important to note that we cannot define semirings using complex numbers for they will not be proper semirings.

Now we can define semirings by replacing Q by $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$ in the examples or to be more specific in

$$S = \left\{ a_0 + \sum_{i=0}^{\infty} a_i g_i \,\middle|\, g_j \in P;\ a_0, a_i \in Q^+ \cup \{0\} \text{ or } R^+ \cup \{0\} \text{ or } \right.$$

$$Z^+ \cup \{0\},\ g_j\text{'s distinct and } g_j \in P = \sum a_i x_i \mid a_i \in$$

$$\{\text{null semigroup}\} \subseteq C(Z_n) \text{ or } Z_n\}.$$

The properties associated to infinite dimensional semirings of dual polynomial numbers can be studied and it is left as an exercise to the reader.

Construction of vector spaces / linear algebras as well as semivector spaces / semilinear algebras are a matter of routine and this is also left as an exercise to the reader.

However authors felt the book would not be complete if study / definition of dual neutrosophic numbers is not defined. So we now proceed onto define the new notion of neutrosophic dual numbers for the first time.

Let x = a+bg if g contains a neutrosophic part or only a neutrosophic number such that $g = t + vI$, $t, v \in Z_n$ and $g^2 = (0)$ then $x^2 = A + Bg$ with $A, B, a, b \in R$ or C or Q or Z.

We define x as a dual neutrosophic number. We describe this situation by some simple examples.

Example 4.30: Let
$V = \{a + bg \mid g = 2 + 2I \in \langle Z_4 \cup I\rangle, a, b \in Q\}$ be the collection of all dual neutrosophic numbers. V is a general neutrosophic dual number ring.

Example 4.31: Let
$M = \{a + bg \mid g = 4 + 4I \in \langle Z_{16} \cup I\rangle, a, b \in Z\}$ be the collection of all dual neutrosophic numbers. M is a general neutrosophic dual number ring.

Example 4.32: Let $\langle Z_4 \cup I\rangle$ be the neutrosophic modulo integer ring. Take $P = \{2I, 2I+2, 2, 0\} \subseteq \langle Z_4 \cup I\rangle$, P is null semigroup, defined as the neutrosophic null semigroup.

Example 4.33: Let $\langle Z_{16} \cup I\rangle$ be the neutrosophic modulo integer ring.

Consider $S = \{0, 4, 4I, 4+4I, 8I, 8, 8+8I, 4+8I, 8+4I, 12, 12I, 12+12I, 4+12I, 8+12I, 12+4I, 12+8I\} \subseteq \langle Z_{16} \cup I\rangle$ is also a null semigroup.

Any element $g \in S$ with $a + bg$, $a, b \in Q$ gives a general neutrosphic dual number ring for varying a and b in Q.

Now as in case of usual dual numbers we can in case of neutrosophic dual numbers also define several dimensions in them. We first record x = a + bg with a, b $\in$ Q (or Z or R) and $g^2 = 0$ is a two dimensional neutrosophic number. Consider

x = $a_1 + a_2g_1 + a_3g_2$ where g_1 and g_2 are two distinct neutrosophic numbers such that $g_1^2 = 0$, $g_2^2 = 0$ and g_1, $g_2 = g_2 g_1 = 0$; $a_1, a_2, a_3 \in$ Q (or Z or R); we define x to be a three dimensional neutrosophic number.

Finally suppose

S = {0, 3I, 6I, 3, 6, 3+3I, 6+6I, 3+6I, 6+3I} $\subseteq \langle Z_9 \cup I\rangle$; S is a null semigroup of neutrosophic numbers and we can use this S to get maximum number of neutrosophic dual number of dimension / cardinality of S.

Consider $x = a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 + a_8g_7 + a_9g_8$; $g_i \in S \setminus \{0\}$, g_i's are distinct, $a_j \in$ Q (or Z or C or R); $1 \leq j \leq 9$, $1 \leq i \leq 8$.

Thus we get a neutrosophic dual number of dimension nine which is the cardinality of S.

Let M = {0, 6, 6I, 6+6I, 12, 12I, 12+12I, 6+12I, 12+6I, 18, 18I, 18+18I, 18+6I, 18+12I, 6+18I, 12+18I, 24, 24I, 24+24I, 24+6I, 24+12I, 24+18I, 6+24I, 12+24I, 18+24I, 30, 30I, 30+30I, 30+6I, 30+2I, 30+18I, 30+24I, 6+30I, 18+30I, 12+30I, 24+30I} $\subseteq \langle Z_{36} \cup I\rangle$ be a null semigroup of neutrosophic numbers. Clearly o(M); that is cardinality of M is 36 we can using M get maximum or atmost a general ring of dimension 36 of dual neutrosophic numbers.

In view of this we can have the following two theorems.

THEOREM 4.4: *Let $\langle Z_{n^2} \cup I\rangle = P$ be a neutrosophic ring of integers modulo n; $n \geq 2$. Then S = {collection of all numbers x in P which are such that $x^2 = 0$ and $x.y = 0$ for every pair x, y in*

P} $\subseteq \langle Z_{n^2} \cup I\rangle = P$*, is null semigroup of neutrosophic numbers of order* n^2*. Further S is a null ring.*

The proof is left as an exercise to the reader as it is direct.

THEOREM 4.5: *Let* $M = \{a_1 + a_2g_1 + \ldots + a_{o(S)}\, g_{o(S)-1} \mid a_i \in Q$ *(or Z or R or C);* $1 \leq i \leq o(S)$ *and*

$g_i \in S$ = *{all neutrosophic numbers x from* $\langle Z_{n^2} \cup I\rangle$ *such that* $x^2 = 0$*; xy = 0 for every pair x, y* $\in \langle Z_{n^2} \cup I\rangle\}\}$ *that is, S is the null semigroup of* $\langle Z_{n^2} \cup I\rangle$*. M is a o(S) dimensional general neutrosophic dual number ring.*

This proof is also simple and direct and hence is left as an exercise to the reader.

Now we give one or two examples of general neutrosophic dual number ring.

Example 4.34: Let

$S = \{0, 3, 3I, 3+3I, 6, 6I, 6+6I, 3+6I, 6+3I\} \subseteq \langle Z_9 \cup I\rangle$ be a null semigroup.

$P = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 + a_8g_7 + a_9g_8 \mid a_i \in Q; 1 \leq i \leq 9, g_j \in S \setminus \{0\}$ and g_j's are distinct $1 \leq j \leq 8\}$ be a nine dimensional neutrosophic dual number general ring.

Example 4.35: Consider

$S = \{0, 6, 6I, 6+6I, 12, 12I, 12+12I, 6+12I, 12+6I\} \subseteq \langle Z_{18} \cup I\rangle$, S is a null semigroup. Take

$P = \{a_1 + a_2g_1 + a_3g_2 + \ldots + a_9g_8 \mid a_i \in Z; 1 \leq i \leq 9, g_j \in S\setminus \{0\};$ g_j's are distinct, $1 \leq j \leq 8\}$, P is a dual neutrosophic number general ring of dimension nine.

Example 4.36: Let $S = \{0, 12, 12I, 24, 24I, 36, 36I, 12+12I, 12+24I, 12+36I, 24+12I, 24+36I, 36+12I, 36+24I, 36+36I, 24+24I\} \subseteq \langle Z_{48} \cup I\rangle$ be the null neutrosophic semigroup. Consider $P = \{a_1 + a_2g_1 + \ldots + a_{16}\, g_{15} \mid a_i \in Q, 1 \leq i \leq 16, g_j \in$

$S \setminus \{0\}$ and g_j's are distinct, $1 \leq j \leq 15\}$ be the 16 dimensional neutrosophic dual number general ring.

Now having seen general dual number neutrosophic rings we leave it as an exercise to the reader to find all the properties associated with such rings.

We can also define semirings of neutrosophic dual numbers which is a matter of routine and hence left as an exercise to the reader. However we substance this by examples.

Example 4.37: Let S = {0, 4, 4I, 8, 8I, 12, 12I, 12+12I, 4+4I, 8+8I, 4+12I, 4+8I, 8+4I, 8+12I, 12+4I, 12+8I} $\subseteq \langle Z_{16} \cup I\rangle$ be the null semigroup.

Consider $A = \{a_1 + a_2g \mid a_1, a_2 \in R^+ \cup \{0\}, g = 4+4I\}$; A is a two dimensional semiring of neutrosophic dual numbers. Infact S is a strict semigroup.

Consider $B = \{a_1 + a_2g_1 + a_3g_2 \mid a_i \in Q^+ \cup \{0\}, 1 \leq i \leq 3, g_1 = 4+4I$ and $g_2 = 12+8I\}$, B is a three dimensional semiring of neutrosophic dual numbers.

Take $C = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 \mid a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 6, g_1 = 12+8I, g_2 = 8+8I, g_3 = 8+12I, g_4 = 8+4I, g_5 = 4+8I\}$, C is a six dimensional semiring of neutrosophic dual numbers.

Using S we can obtain a maximum of 16 dimensional semiring of neutrosophic dual numbers.

We can also build also vector spaces / linear algebras of neutrosophic dual numbers. Further the notion of semivector spaces / semilinear algebras using neutrosophic dual numbers is a matter of routine.

We will only illustrate these situations by some simple examples.

Example 4.38: Let V = $\left\{\begin{bmatrix} a_1^1 + a_2^1 g_1 & a_1^2 + a_2^2 g_1 \\ a_1^3 + a_2^3 g_1 & a_1^4 + a_2^4 g_1 \end{bmatrix}\right.$ where $a_i^t \in Q$; $1 \leq t \leq 4$ and I = 1, 2 $g_1 = 4 + 4I \in \langle Z_{16} \cup I\rangle\}$ be the collection of all 2×2 matrices with dual neutrosophic number V is a vector space over the field Q; called the dual neutrosophic number vector space. Infact V is a non commutative linear algebra of dual neutrosophic numbers over Q.

If instead of usual product natural product $\times_n$ is used then V will be a commutative dual neutrosophic number linear algebra over Q.

Take W = $\left\{\begin{bmatrix} a_1^1 + a_2^1 g_1 & a_1^2 + a_2^2 g_1 \\ a_1^3 + a_2^3 g_1 & a_1^4 + a_2^4 g_1 \end{bmatrix}\right.$ where $a_i^t \in Q^+ \cup \{0\}$;

$1 \leq t \leq 4$ and i = 1, 2, $g_1 = 4 + 4I \in \langle Z_{16} \cup I\rangle\} \subseteq V$; W is the positive cone of V and infact W is a semivector space (as well as semilinear algebra).

Example 4.39: Let P = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_j \in \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 \mid x_i \in R,\ 1 \leq i \leq 5\ g_1 = 3+3I,\ g_2 = 3I,\ g_3 = 3+6I,\ g_4 = 6+3I \in \langle Z_9 \cup I\rangle\};\ 1 \leq j \leq 5\}$ be the collection of all neutrosophic dual numbers of dimension five.

P is a vector space (as well as a linear algebra) over the field Q or R of neutrosophic dual number.

If M = $\{(b_1, b_2, b_3, b_4, b_5) \mid b_j \in \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 \mid x_i \in R^+ \cup \{0\};\ 1 \leq i \leq 5$ and $g_1 = 3+3I,\ g_2 = 3I,\ g_3 = 3+6I,\ g_4 = 6+3I \in \langle Z_9 \cup I\rangle,\ 1 \leq j \leq 5\} \subseteq P$, M is a positive cone of P as well as M is a semivector space (as well as semilinear algebra of dual neutrosophic numbers) over the semifield $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$.

Example 4.40: Let $S = \left\{\begin{bmatrix} a_1 & a_6 \\ a_2 & a_7 \\ a_3 & a_8 \\ a_4 & a_9 \\ a_5 & a_{10} \end{bmatrix}\right.$ where $a_i \in \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5 + x_7g_6 + x_8g_7 + x_9g_8 \mid x_j \in Q; 1 \leq j \leq 9$ and $g_1 = 6 + 6I$, $6+12I = g_2$, $g_3 = 12I$, $g_4 = 12+12I$, $g_5 = 6I$, $g_6 = 6$, $g_7 = 12$, $g_8 = 12+6I \in \langle Z_{36} \cup I\rangle$, $1 \leq i \leq 10\}\}$ be a vector space (linear algebra under natural product $\times_n$) of neutrosophic dual numbers.

$P = \left\{\begin{bmatrix} a_1 & a_6 \\ a_2 & a_7 \\ a_3 & a_8 \\ a_4 & a_9 \\ a_5 & a_{10} \end{bmatrix}\right|$ $a_i \in \{x_1 + x_2g_1 + x_3g_2 + \ldots + x_9g_8 \mid x_j \in Q^+ \cup \{0\}$, $1 \leq j \leq 9$, $g_1 = 6 + 6I$, $g_2 = 12I+6$, $g_3 = 12I$, $g_4 = 12+12I$, $g_5 = 6I$, $g_6 = 6$, $g_7 = 12$ and $g_8 = 12+6I \in \langle Z_{36} \cup I\rangle$; $1 \leq i \leq 10\}\} \subseteq S$ is a positive cone of S as well as the semivector space (semilinear algebra under natural product of matrices $\times_n$) over the semifield $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$, that is P is also known as the dual neutrosophic number semivector space over the semifield.

Example 4.41: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right|$$

$a_i \in T = \{x_1 + x_2g_1 + \ldots + x_{24}g_{23} + x_{25}\, g_{24} \mid x_j \in Q; 1 \leq j \leq 25$ and $g_1 = 7$, $g_2 = 7I$, $g_3 = 7+7I$, $g_4 = 14$, $g_5 = 14I$, $g_6 = 14+14I$, $g_7 = 7+14I$, $g_8 = 14+7I$, $g_9 = 21$, $g_{10} = 21I$, $g_{11} = 21+21I$,

$g_{12} = 21+7I$, $g_{13} = 21 + 14I$, $g_{14} = 7+21I$, $g_{15} = 14+21I$, $g_{16} = 28$, $g_{17} = 28I$, $g_{18} = 28+28I$, $g_{19} = 7+28I$, $g_{20} = 14+28I$, $g_{21} = 21+28I$, $g_{22} = 28+7I$, $g_{23} = 28+14I$, $g_{24} = 28 + 21I\} \subseteq \langle Z_{49} \cup I\rangle$; $1 \leq i \leq 16\}$ be a vector space of neutrosophic dual numbers over the field Q and infact a neutrosophic dual number linear algebra over the field Q if natural product $\times_n$ on matrices is defined.

$$\text{Infact } P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in T; \right.$$

$x_j \in Q^+ \cup \{0\}$; $1 \leq i \leq 16$, $1 \leq j \leq 25\} \subseteq S$ is a positive cone of S and infact a semivector space (semilinear algebra under natural product of matrices $\times_n$) over the semifield $Q^+ \cup \{0\}$.

Infact we can using these neutrosophic dual number extend the notion to neutrosophic complex modulo integer numbers. We will first illustrate this situation by some examples before we proceed to describe them generally.

Let $S = \{2I, 2i_F, 2, 0, 2+2I, 2+2i_F, 2I + 2i_F, 2 + 2i_F, 2I, 2i_FI, 2+2i_FI, 2i_F + 2i_FI, 2+2I+2i_FI, 2i_F+2I+2i_FI, 2+2i_F+2I+2i_FI\} \subseteq C(\langle Z_4 \cup I\rangle)$; it is easily verified S is null semigroup under product. Also (S, +) is group under addition and S is a null ring contained in $C(\langle Z_4 \cup I\rangle)$. Now thus in general all complex neutrosophic modulo integers in $C(\langle Z_{n^2} \cup I\rangle)$ contain a semigroup S of order $n^2 \times n^2 = n^4$, distinct nilpotent elements of order two. Infact S is a null semigroup as well as S is a null ring contained in $C(\langle Z_{n^2} \cup I\rangle)$.

Also using Z_n, n a composite number we have nilpotent elements in $C(\langle Z_n \cup I\rangle)$.

We will illustrate by an example or two.

Example 4.42: Let $M = \{12, 0, 12i_F, 12I, 12+12I, 12+12i_F, 12+12i_F+12I, 12i_FI, 12I+12i_F, 12+12i_FI, 12I+12i_FI, 12i_F+12i_FI, 12I+12i_F+12i_FI, 12I+12+12i_FI, 12i_F+12+12i_FI, 12+12I+12i_F+$

$12i_FI\} \subseteq C(\langle Z_{24} \cup I\rangle)$ is a null subsemigroup of $C(\langle Z_{24} \cup I\rangle)$ as well as null semigroup.

Using any $g_1 \in M$ we get $a_1 + a_2g_1$ to be a two dimensional complex neutrosophic number provided.

$$g_1 = 12 + 12i_FI \text{ or}$$
$$g_1 = 12I + 12i_F \text{ or}$$
$$g_1 = 12+12i_F+12I \text{ or}$$
$$g_1 = 12+12i_F+12I+12Ii_F;$$

i.e., g_1 should have a neutrosophic term as well as a finite complex number term i_F.

Suppose
$P = \{a_1 + a_2g_1 \mid a_1, a_2 \in Q, g_1 \in 12 + 12i_F + 12I + 12i_FI\}$ then P is defined as the general complex neutrosophic dual number ring of dimension two.

If $A = \{a_1 + a_2g_1 + a_3g_2 + \ldots + a_{16}g_{15} \mid g_i \in M \setminus \{0\}$; g_i's distinct, $a_j \in Q$ $1 \leq j \leq 16\}$, then A is defined as the 16-dimensional neutrosophic complex modulo integer general ring.

Infact A contains a proper subset which is only a finite complex modulo integer dual number general ring of dimension four; that is $B = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3$ where $g_1 = 12i_F$; $g_2 = 12+12i_F$, $12 = g_3$ $a_i \in Q$, $1 \leq i \leq 4\} \subseteq A$ of dimension three.

On similar line we have $C = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_i \in Q$, $1 \leq i \leq 4$ $g_1 = 12$, $g_2 = 12I$, and $g_3 = 12+12I \in M\} \subseteq A$; C is also a dimension four neutrosophic dual general ring.

Example 4.43: Let $M = \{6, 6i_F, 6I, 0, 6 + 6I, 6 + 6i_F, 6I + 6i_F, 6 + 6i_F + 6I, 6i_FI, 6 + 6i_FI, 6 + 6i_F + 6I + 6i_FI, 6 + 6i_F + 6I, 6 + 6i_FI + 6i_F, 6i_FI + 6i_F, 6i_FI + 6I, 6i_F + 6I + 6i_F\} \subseteq C(\langle Z_{12} \cup I\rangle)$ be a null semigroup.

Let $V = \{a_1 + a_2g_1 + \ldots + a_{36}g_{35} \mid g_i$'s are distinct and belong to $M \setminus \{0\}$; $1 \leq i \leq 35$ and a_j's are in Q; $1 \leq j \leq 36\}$ be a general ring of neutrosophic complex dual numbers.

M has four types of subrings viz. subrings which are general subrings of neutrosophic complex dual numbers, subrings which are general subrings of neutrosophic dual numbers, subrings which are general subrings of complex modulo integer dual numbers modulo integer dual numbers and subrings which are just dual numbers.

Here X = $\{a + bg_1 \mid g_1 = 6 \in C(\langle Z_{12} \cup I\rangle); a, b \in Q\} \subseteq V$ is the general subring of dual numbers.

Take Y = $\{a + bg_1 \mid g_1 = 6+6I \in C(\langle Z_{12} \cup I\rangle), a, b \in Q\} \subseteq V$; Y is a general subring of neutrosophic dual numbers. Take W = $\{a + bg_1 + cg_2 \mid g_1 = 6i_F$ and $g_2 = 6 + 6i_F$ in $C(\langle Z_{12} \cup I\rangle)$; a, b, c $\in Q\} \subseteq V$ is the general subring of complex modulo integer dual numbers.

Finally B = $\{a + bg_1 \mid g_1 = 6 + 6i_F + 6I + 6i_FI \in C(\langle Z_{12} \cup I\rangle)$, a, b $\in Q\} \subseteq V$ is the general subring of neutrosophic complex modulo integer dual numbers.

Inview of this we have the following theorem.

THEOREM 4.6: *Let*

S = $\{a_1 + a_2g_1 + \ldots + a_tg_{t-1} \mid a_i \in Q;\ 1 \leq i \leq t,\ g \in T = \{$all nilpotent elements of order two from $C(\langle Z_n \cup I\rangle)\}$; g_j's distinct and $g_j \neq 0$. $o(T) = t\}$ be a general ring of complex neutrosophic modulo integer dual numbers.

Then S has four types of subrings say S_1, S_2, S_3 and S_4 where S_1 is just a general subring of dual numbers of S.

S_2 is the general subring of neutrosophic dual numbers of S, S_3 is the general subring of complex modulo integer dual numbers of S and S_4 is the general subring of complex modulo integer neutrosophic dual numbers of S.

The proof is direct and straight forward hence left as an exercise to the reader.

Example 4.44: Let S = {3, 0, 6, $3i_F$, $6i_F$, 3I, 6I, $3i_FI$, $6i_FI$, 6 + 3I, 6 + $3Ii_F$, 6 + $3i_F$, 3 + 3I, 3 + $3i_F$, 3 + $3i_FI$, 3 + 6I, 3 + $6Ii_F$, 3 + $6i_F$, 6 + $6i_F$, 6 + 6I, 6 + $6i_FI$, 3 + $3i_F$ + 3I, …, 3 + $6i_F$ + $6i_FI$ + 3I, …, 6I + 6 + $6i_F$ + $6Ii_F$} $\subseteq C(\langle Z_9 \cup I\rangle)$.

We see S is a null semigroup. S has four types of null subsemigroups. Take S_1 = {0, 3, 6} $\subseteq$ S; S_1 is a subsemigroup which is a null subsemigroup.

S_2 = {0, 3+3I, 6+6I, 3I, 6I, 3, 6, 3+6I, 6+3I} $\subseteq$ S is again a subsemigroup which is a null subsemigroup of S.

S_3 = {0, 3, $3i_F$, $6i_F$, 6, 3+$3i_F$, 6+$6i_F$, 3+$6i_F$, 6+$3i_F$} $\subseteq$ S is again a subsemigroup which is a null subsemigroup of S.

Take S_4 = {0, 3 + $3i_F$ + 3I + $3i_FI$, 6+$6i_F$+6I + $6i_F$} $\subseteq$ S is again a null subsemigroup of S. Hence the claim.

S_1 will be known as modulo integer null subsemigroup S_2 will be known as complex modulo integer null subsemigroup, S_3 will be known as the neutrosophic modulo integer null subsemigroup and S_4 will be known as the complex modulo integer neutrosophic subsemigroup of S.

Inview of this we have the following theorem the proof of which is left as an exercise to the reader.

THEOREM 4.7: *Let S = {0, all elements x in $C(\langle Z_n \cup I\rangle)$ such that $x^2 = 0$} $\subseteq C(\langle Z_n \cup I\rangle)$.*

(i) S is a null neutrosophic complex modulo integer semigroup contained in $C(\langle Z_n \cup I\rangle)$
(ii) S has four types of null subsemigroups viz. $S_1 \underset{\neq}{\subset} S$ is just the null semigroup of modulo integers, $S_2 \underset{\neq}{\subset} S$ is the null complex modulo integer subsemigroup of S. $S_3 \underset{\neq}{\subset} S$ is the null neutrosophic subsemigroup of S.

$S_4 \underset{\neq}{\subset} S$ is the null neutrosophic complex modulo integer subsemigroup of S.

However it can be easily verified by the interested reader that S is a null ring of complex modulo integer neutrosophic numbers. Using this null semigroup S we can construct all rings of complex neutrosophic modulo integers of dimension less than or equal to o(S).

Infact these null semigroups of complex neutrosophic modulo integers can be used to build complex neutrosophic modulo dual number general ring.

Further we can define the notion of semirings by restricting the real number in the dual number to be positive.

We will give some examples of them.

Example 4.45: Let S = {6, 6I, 0, 6+6I, $6i_F$, $6+6i_F$, $6I+6i_F$, 6+6I, $6i_F$, $6i_FI$, $6+6i_FI$, …, $6+6i_F + 6I + 6i_FI$} $\subseteq C(\langle Z_{12} \cup I\rangle)$ be a null semigroup of neutrosophic complex modulo integer. Take $P = \{a + bg_1 \mid g_1 = 6 + 6i_FI \in C(\langle Z_{12} \cup I\rangle)$ with $a, b \in Z^+ \cup \{0\}\}$; P is a dual neutrosophic complex modulo integer semiring.

Consider
$B = \{a + bg_1 \mid g_1 = 6 + 6I + 6i_F + 6i_FI,\ a, b \in Q^+ \cup \{0\}\}$, B is again a dual neutrosophic complex modulo integer semiring.

Example 4.46: Let S = {$4 + 4i_F$, 4, 0, $4i_F$, 4I, $4i_FI$, $8+8i_F$, 8, $8i_F$, $8i_F$, 8+8I, 8I, $8i_FI$, $8+8i_F+8I$, …, $12+12i_F$, 12, $12i_F$, 12I, $12i_FI$, …, $12+12i_F + 12I + 12i_FI$} $\subseteq C(\langle Z_{16} \cup I\rangle)$ be the null semigroup of complex modulo integer neutrosophic nilpotent elements of order two.

Consider $P_1 = \{a_1 + a_2g_1 + a_3g_2 \mid a_i \in Q^+ \cup \{0\},\ 1 \leq i \leq 3,\ g_1 = 6 + 6i_F + 6I + 6i_FI$ and $g_2 = 12 + 12i_F + 12I + 12Ii_F \in S\}$; P is a three dimensional general ring of complex neutrosophic dual numbers.

T = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 \mid a_i \in Z^+ \cup \{0\}$, $1 \leq i \leq 7$, $g_1 = 8i_FI$, $g_2 = 4 + 4i_FI$, $g_3 = 12I + 12i_F + 12$, $g_4 = 4 + 4i_F + 4I$, $g_5 = 8+8i_FI + 8I$ and $g_6 = 12+12i_FI + 8i_F+4I \in S\}$; T is a general ring of seven dimensional neutrosophic complex modulo integer dual numbers.

We can also build vector spaces / semivector spaces of neutrosophic complex modulo integer dual number over field / semifields respectively.

Here we supply a few examples.

Example 4.47: Let S = $\{0, 2, 2i_F, 2I, 2+2i_F, 2i_FI, 2+2i_FI, 2+2I, 2+2i_F+2I, 2+2i_F+2Ii_F, 2+2i_FI+2I, 2I+2i_F+2i_FI, 2+2i_F+2I+2i_FI, 2I+2i_F, 2I+2i_FI, 2i_F+2i_FI\} \subseteq C(\langle Z_4 \cup I\rangle)$ be the null semigroup of complex neutrosophic modulo integers. Consider

V = $\{a_1 + a_2g_1 + a_3g_2 + \ldots + a_{16}g_{15} \mid a_i \in Q; 1 \leq i \leq 16, g_j \in S \setminus \{0\}$, g_j's are distinct $1 \leq j \leq 15\}$ be a vector space of 16 dimensional complex neutrosophic modulo integer dual numbers over the field Q.

If W = $\{a_1 + a_2g_1 + \ldots + a_{16}g_{15} \mid a_i \in Q^+ \cup \{0\}; 1 \leq i \leq 16, g_j \in S \setminus \{0\}$; g_j's are distinct $1 \leq j \leq 15\} \subseteq V$, W is a semivector space over the semifield $Q^+\cup\{0\}$ (or $Z^+\cup\{0\}$) of complex neutrosophic modulo dual numbers and if W defined over $Q^+ \cup \{0\}$; W is also the positive cone of V.

Example 4.48: Let S = $\{0, 5, 5I, 5i_F, 5i_FI, 5 + 5i_F, 5 + 5I, 5 + 5i_FI, 5i_F + 5I, 5i_F + 5i_FI, 5I + 5i_FI, 5 + 5I + 5i_F, 5 + 5I + 5i_FI, 5 + 5i_F + 5i_FI, 5i_F + 5I + 5i_FI, 5 + 5I + 5i_F + 5i_FI\} \subseteq C(\langle Z_{10} \cup I\rangle)$ be a null semigroup of complex modulo neutrosophic integers.

Consider W = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 + a_8g_7 \mid a_i \in R, 1 \leq i \leq 7; g_j \in S \setminus \{0\}$, g_j's distinct, $1 \leq j \leq 7$ (g_j's fixed elements in $S \setminus \{0\}\}$ be a vector space of eight dimensional neutrosophic complex modulo integers of dual numbers over the field Q or R.

We see if V = $\{a_1 + a_2g_1 + a_3g_2 + \ldots + a_8g_7 \mid a_i \in R^+ \cup \{0\}$, $1 \leq i \leq 8$ and g_j's as in W; $1 \leq j \leq 7\} \subseteq W$, V is a positive cone of V as well as V is a semivector space over $R^+ \cup \{0\}$ of eight dimensional neutrosophic complex dual numbers.

Interested reader can construct several examples, study subspaces in these vector spaces find basis, linear operators and linear functionals. All these are considered as a matter of routine.

Now we can also define other types of vector spaces and rings using complex neutrosophic finite modulo dual numbers.

Let S = $\{0, 5, 5i_F, 5I, 5Ii_F, 5 + 5i_F, 5 + 5I, 5 + 5Ii_F, 5I + 5i_F, \ldots, 5 + 5i_F + 5I + 5Ii_F\} \subseteq C(\langle Z_{10} \cup I\rangle)$ be a null semigroup of nilpotent finite complex neutrosophic numbers.

Let P = $\{(a_{ij})_{5\times5} \mid a_{ij} \in S; 1 \leq i, j \leq 5\}$. P is again a null semigroup of finite complex neutrosophic numbers. Consider V = $\{a_1 + a_2g_1 + \ldots + a_tg_{t-1} \mid a_i \in Q, 1 \leq i \leq t$ and $g_j \in P \setminus \{0\}$ (g_j are distinct and $t \leq o(p)$); V is a vector space of t-dimensional neutrosophic finite complex modulo matrix of dual numbers.

Let S be as before.

Consider $T = \left\{\begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \\ \vdots & \vdots \\ g_9 & g_{10} \end{bmatrix} \middle| \; g_i \in S; 1 \leq i \leq 10\right\}$.

Suppose

M = $\{a_1 + a_2g_1 + a_3g_2 + \ldots + a_tg_{t-1} \mid a_i \in Q; 1 \leq i \leq t$ and g_j's are distinct, $g_j \in T \setminus \{(0)\}$ $1 \leq j \leq t-1$ $o(T) = T\}$, T is a t-dimensional vector space of neutrosophic finite complex numbers of dual numbers. Likewise we can build vector spaces and interested reader can construct several of them.

Now we can using the same S mentioned earlier build polynomials with coefficients from S.

$P = \left\{\sum_{i=0}^{\infty} g_i x^i \,\middle|\, g_i \in S;\ g_j\text{'s are distinct}\right\}$. P is also a null semigroup. Now let

V = $\{a_1 + a_2g_1 + \ldots + a_rg_{r-1} \mid g_i \in P$ are polynomials with coefficient from S, $1 \leq i \leq r-1$ and $a_j \in Q$, $1 \leq j \leq r\}$ $(1 \leq r \leq \infty)$ (P will be known as the neutrosophic complex modulo integer coefficient polynomial null semigroup). V is the r dimensional neutrosophic complex modulo dual polynomial numbers of vector space over the field Q.

Now we proceed onto describe fuzzy dual numbers.

Let $S = \{0, 4, 8, 12\} \subseteq Z_{16}$ consider $x_1 + x_2g$ where $g \in S$ and $x_1, x_2 \in [0, 1]$, $x_1 + x_2g$ is defined as the fuzzy dual number. $V = \{x_1 + x_2g \mid x_1, x_2 \in [0, 1]\ g = 12\}$ is a fuzzy dual number semigroup under product. For consider $0.2 + 0.4(12) = x$, $x^2 = (0.2)^2 + (0.4)^2\,(12)^2 + 0.2 \times 0.4 \times 12 = 0.04 + 0.08.12 + 0$ is again a fuzzy dual number.

However we cannot define addition on V; we can define only min $\{x_1 + x_2g, y_1 + y_2g\} = \min\{x_1, y_1\} + \min\{x_2, y_2\}$ g or

max $\{x_1 + x_2g, y_1+y_2g\} = \max\{x_1, y_1\} + \min\{x_2, y_2\}g$. Thus '+' can be replaced by max or min.

We can construct semirings and semivector spaces of fuzzy dual numbers.

We can consider $S = \{2i_F, 2, 0, 2+2i_F\} \subseteq C(Z_4)$.

If $x = a_1 + a_2g_1 = 2i_F = g_i \in S$ ($a_1, a_2 \in [0, 1]$) then we define x to be a complex finite modulo integer fuzzy dual number or fuzzy complex finite modulo integer dual number.

Further if we take $S = \{0, 2i_F, 2I, 2i_FI, 2, 2+2i_F, 2+2I, 2+2i_FI, 2i_F+2I, 2+2i_F+2I, \ldots, 2+2i_F+2I + 2i_FI\} \subseteq C(\langle Z_4 \cup I\rangle)$; we see $x = a_1 + a_2g_1$ with $g_1 \in S$ and $g_1 = 2i_F + 2i_FI + 2I + 2$ and $a_1, a_2 \in [0, 1]$ is a fuzzy neutrosophic finite complex modulo integer dual number.

Only a few properties can be true in case of fuzzy dual neutrosophic complex number as it is not possible to define a group structure on them with respect to '+'. Even if we define max or min it can only be a semigroup so only semiring or semivector structures can be defined on them.

Interested reader can develop these properties as it is direct and easy by using appropriate changes.

Finally we wish to state we can in an analogous way develop intervals of dual numbers using intervals of complex modulo dual numbers or neutrosophic complex modulo dual numbers or fuzzy dual numbers or fuzzy complex modulo dual numbers or fuzzy neutrosophic complex modulo integers or fuzzy neutrosophic dual numbers, their matrix and polynomial analogues.

The development of all these concepts are direct and simple and with appropriate modifications can be derived / studied by any interested reader.

Chapter Five

APPLICATIONS OF THESE NEW TYPES OF DUAL NUMBERS

Dual number were first discovered / defined by W.K. Clifford in 1873. This concept was first applied by A.P. Kolelnikor in 1895 to mechanics. It is unfortunate that his original paper published in Annals of Imperial University of Kazan in (1895) was destroyed during Russian revolution. Scientist have applied this concept in various fields like modeling plane joint, in an iterative method for displacement analysis of spatial mechanisms [4-7], Yang in inertial force analysis of spatial mechanisms, Sugimoto and Duffy in screw systems, Wohlhart in computational Kinematics, Duffy in the analysis of Mechanisms in Robot Manipulators (1980), Y.L. Gu and J.Y.S. Luh, in Robotics (1987), I.S. Fisher in velocity analysis of mechanisms with ball Joints (2003); joints with manufacturing tolerances, computer aided analysis and optimization of mechanical system dynamics and so on [3, 7-9, 12, 21] .

The authors in this book have introduced the new notion of finite complex modulo integer dual number, neutrosophic dual number and finite complex modulo integer neutrosophic dual number. Study of these concepts and constructing algebraic structures using these concepts are carried out. These new structures will certainly find applications in due course of time. Finally the concept of fuzzy dual number are defined and these can find applications in fuzzy models.

Now we have presently given t-dimensional: dual numbers, neutrosophic dual number, finite complex modulo integer dual number, and interval dual numbers.

Certainly all these concepts will find applications in the related fields of applications of dual numbers.

Chapter Six

SUGGESTED PROBLEMS

In this chapter we suggest around 116 problems of which some are simple, some are difficult and some of them are research problems.

1. Give an example of a general dual number ring, which is of finite order and has zero divisors and idempotents.

2. Prove $S = \{a + bg \mid g = (0, 3, 6, 3)$ where $6, 3, 0 \in Z_9$ with $a, b \in Z\}$ is a general dual number ring.

   (i) Find subrings of S.
   (ii) Can S have ideals?
   (iii) Can S have zero divisors?

3. Give some interesting properties and applications of general dual number ring.

4. Let S = $\{a + bg \mid a, b \in Z_{23},\ g = \begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \\ 6 \\ 0 \end{bmatrix}$ with $3, 6, 0 \in Z_9\}$

   under natural product $\times_n$ on g that is $g \times_n g = (0)$ be the general dual number ring.

   (i) Prove S is of finite order.
   (ii) Does S have subrings which are not ideals?
   (iii) Can S have units?
   (iv) Can S have idempotents?

5. Let $S = \{a + bg \mid a, b \in Z_{23},\ g = 5 \in Z_{25}\}$ be a dual modulo number ring.

   (i) Does S have subring?
   (ii) Can S have ideals?
   (iii) Prove S cannot have unit for all $a + bg$ where $a, b \in Z_{23}$.
   (iv) Find the number of elements in S.
   (v) Can S have idempotents?

6. Is $P = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \ a_i = x_i + y_i g \text{ where } x_i, y_i \in Z_{16};\ 1 \leq i \leq 4;\right.$

   $g = 8 \in Z_{16}\}$, a dual number modulo integer ring?

7. Is

   $M = \left\{\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} g \ \middle| \ g=8 \in Z_{16},\ x_i, y_i \in Z_{19}, 1 \leq i \leq 19\right\}$,

   a dual number modulo integer ring?

(i) Find subrings of M which are not ideals.
(ii) Can M have S-ideals?
(iii) Can M have zero divisors?

8. Let P = $\{(a_1, a_2, a_3, a_4, a_5, a_6) + (b_1, b_2, b_3, b_4, b_5, b_6)g \mid a_i, b_j \in Z_{11};\ 1 \leq i, j \leq 6;\ g = 2 \in Z_4\}$ be a dual number modulo integer ring.

   (i) Find subrings and ideals of P.
   (ii) Can P have ideals I and J such that $I \cap J = (0)$?

9. Let

$$S = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \\ b_7 & b_8 \end{bmatrix} g \ \right| \ a_i, b_j \in Z_{17}, 1 \leq i, j \leq 8, \right.$$

   $g = 6 \in Z_{12}\}$ be a dual number modulo integer ring under natural product $\times_n$.

   (i) Can S have ideals?
   (ii) Can S have units?
   (iii) Is S a Smarandache ring?

10. Let T = $\{p(x) + q(x)g \mid p(x), q(x) \in q(x) \text{ and } g = 3 \in Z_9\}$ be a dual number polynomial ring.

   (i) Is T commutative?
   (ii) Is T a S-ring?
   (iii) Can T have S-ideals?
   (iv) Is T a principal ideal domain?

11. Let Q be the field. S = $\{0, 1, 4\} \subseteq Z_{16}$ be a subsemigroup of the semigroup $(Z_{16}, \times)$. QS be the semigroup ring of the semigroup S over the ring Q.

   (i) Prove QS is a dual number general ring.

(ii) Can QS have ideals which are dual number ideals?
(iii) Does QS contain subrings H which are not ideals and H a dual number general ring?

12. Let M = $Z_{11}S$ where S = {0, 1, 10} $\subseteq Z_{20}$ be a semigroup under multiplication modulo 20.

(i) Can M have zero divisors?
(ii) Find the number of elements in M.
(iii) Can M have units?
(iv) Can M have ideals?

13. Let $P = \left\{ a + b\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \, a, b \in Z_5, a_i = 2, i=1, 2, 3, 4; 2 \in Z_4 \right\}$

be the dual number modulo integer ring.

(i) Find the number of elements in P.
(ii) Can P be a S-ring?
(iii) Can P have S-zero divisors?

14. Obtain some nice applications of dual number modulo integer ring.

15. Let V = {p(x) + q(x)g | p(x), q(x) $\in$ Z [x], g= 4 $\in Z_{16}$} be a dual number polynomial general ring.

(i) Find ideals in V.
(ii) Can V have subrings which are not ideals?
(iii) Give any other interesting property enjoyed by V.

16. Let
P = {p(x) + q(x)g | p(x), q(x) are in $Z_7[x]$ and g = 2 $\in Z_4$} be the dual modulo number ring.

(i) Find ideals if any in P.
(ii) Is P an integral domain?
(iii) Can P have zero divisors?

17. Describe some applications of the dual number semifield $Q(g) = \{a+bg \mid a, b \in Q \text{ and } g^2 = 0\}$.

18. Let $T = \{(a_1, a_2, a_3) \mid a_i = x_i + y_ig$ with $g = 4 \in Z_{16}$; $x_i, y_i \in Q^+$; $1 \leq i \leq 3\} \cup (0, 0, 0)$ be the dual row matrix number semiring.

(i) Can T have zero divisors?
(ii) Can T have units?

19. Let $M = \{(a_1, a_2, \ldots, a_{10}) + (b_1, b_2, \ldots, b_{10})g \mid a_i, b_j \in Z^+$, $1 \leq i, j \leq 10$; $g = 10 \in Z_{20}\} \cup \{(0\ 0\ \ldots\ 0)\}$ be a dual row matrix number semiring. Give any two striking properties about M.

20. Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} g \;\middle|\; a_i, b_j \in Z^+; 1 \leq i, j \leq 4, \right.$$

$g = 6 \in Z_{12}\}$ be a dual square matrix number semiring.

(i) Is P a semifield?
(ii) Is P commutative under usual matrix multiplication?
(iii) Can P have ideals?
(iv) Can P have S-units and S zero divisors?

21. Give some nice applications of dual number square matrix commutative ring.

22. Compare the dual number matrix ring with dual number matrix semiring both under usual product.

23. Obtain some special features enjoyed by dual number polynomial semirings.

24. Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} g \;\middle|\; a_i, b_j \in Z_{23}, 1 \leq i, j \leq 3; g = 4 \in Z_{16} \right\}$$

be a dual number matrix ring.

(i) Find the number of elements in M.
(ii) Can M have zero divisors?
(iii) Does M have subrings which are not ideals?

25. Let S = $\{p(x) + q(x)g \mid p(x), q(x) \in Z_{11}, g = 2 \in Z_4\}$ be a dual number polynomial general ring. Prove or disprove S is an integral domain.

26. Let $L = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_{11} \\ a_{12} & a_{13} & \ldots & a_{22} \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & \ldots & b_{11} \\ b_{12} & b_{13} & \ldots & b_{22} \end{pmatrix} g \;\middle|\; \right.$

$\left. a_i, b_j \in Q^+ \cup \{0\}, 1 \leq i, j \leq 22, g = 4 \in Z_{16} \right\}$ be a dual number matrix semiring. Can L have zero divisors?

27. Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} g \;\middle|\; a_i, b_j \in Q, 1 \leq i, j \leq 3 \text{ and } g = 4 \in Z_{16} \right\}$$

be a dual number matrix vector space over Q.

(i) Find the dimension of V over Q.
(ii) Find a basis for V.

(iii) Write V as a direct sum.
(iv) Write V as a pseudo direct sum.
(v) Can $V = W \oplus W^{\perp}$ where W is a dual number matrix vector subspace of V and $W^{\perp}$ the orthogonal space of W?

28. Let $M = \{p(x) + q(x)g \mid p(x), q(x) \in R[x], g = 2 \in Z_4\}$ be a dual number polynomial linear algebra over the field R.

(i) Find a basis of M.
(ii) Is M finite dimensional?
(iii) Find a linear operator T on M such that T^{-1} does not exist.
(iv) Define a linear functional $f : M \to R$ and find the algebraic structure of $L_R(M) = \{f : M \to R\}$.

29. Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} g \;\middle|\; a_i, b_j \in Z_{13}, 1 \le i, \right.$$

$j \le 9$; $g = 3 \in Z_9\}$ be a dual number non commutative linear algebra over the field Z_{13}.

(i) Find the number of elements in P.
(ii) Find a basis of P over Z_{13}.
(iii) Write P as a direct sum.
(iv) Write P as a pseudo direct sum.
(v) Find the algebraic structure enjoyed by $\mathrm{Hom}_{Z_{13}}(P, P)$.

30. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} g \;\middle|\; g = 3 \in Z_9, a_i, b_j \in Q^{+} \cup \{0\} \right\}$$

be dual number matrix semilinear algebra over the semifield $Z^+ \cup \{0\}$.

(i) Is V infinite dimensional?
(ii) If the semifield $Z^+ \cup \{0\}$ is replaced by $Q^+ \cup \{0\}$ will V be finite dimensional?
(iii) Find a basis of V, V as a semivector space over $Q^+ \cup \{0\}$.
(iv) Write V as a direct sum of semivector subspaces.
(v) Find $\text{Hom}_{Q^+ \cup \{0\}}(V, V)$.

31. Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in Z^+(g), a_i = x_i + y_i g \text{ with} \right.$$

$g = 4 \in Z_{16}$ $1 \leq i \leq 9\}$ be a dual number semivector space over $Z^+ \cup \{0\}$.

(Note : $\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ is a dual number matrix as

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{pmatrix} g \Bigg).$$

(i) Find dimension of S over $Z^+ \cup \{0\}$.
(ii) Find the algebraic structure enjoyed by $L_{Z^+ \cup \{0\}}(S, Z^+ \cup \{0\})$
(iii) Suppose $Z^+ \cup \{0\}$ is replaced by $Z^+(g) \cup \{0\}$ then mention the special features enjoyed by S over $Z^+(g) \cup \{0\}$.

32. Let $L = \{(a_1, a_2, a_3, a_4, a_5) = (x_1, x_2, x_3, x_4, x_5) + (x_1, x_2, x_3, x_4, x_5)g \mid x_i, y_j \in Q^+ \cup \{0\}; g = 11 \in Z_{121}, 1 \leq i, j \leq 5\}$ be a dual number row matrix semilinear algebra over the semifield $Q^+ (g) \cup \{0\} = F$.

    (i) Find a basis of L over $Q^+ (g) \cup \{0\}$.
    (ii) Write L as a direct sum of subspaces.
    (iii) Find the algebraic structure enjoyed by
        a) $Hom_F (L, L)$ b) $Hom_F (L, F)$.

33. Give an example of a 7-dimensional dual number.

34. Let $V = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_i \in R; 1 \leq i \leq 4;$

$$g_i \in \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \middle| \; x_i \in \{4 + 4i_F, 8+8i_F, 12+12i_F, 4+8i_F, 4+12i_F, \right.$$

$8+12i_F, 8+4i_F, 12+8i_F, 12+4i_F, 0\} \subseteq C(Z_{16}); 1 \leq i \leq 5\} = G$.

    (i) Is V a dual number ring?
    (ii) What is the maximum number of elements T can have?
    (iii) Is V a vector space over R?
    (iv) Can V ever be a linear algebra over R?

35. Let $T = \{(x_1, x_2, x_3, x_4) \mid x_i \in \{0, 6, 12, 18, 24, 30\} \subseteq Z_{36}, 1 \leq i \leq 4\}$.

    (i) Is T a null ring?
    (ii) Can T be a null semigroup?
    (iii) What is the cardinality of T?
    (iv) If $V = \{a_1 + a_2g_1 + \ldots + a_o(T) \, g_{o(T)-1} \mid a_i \in Q; g_j \in T; g_j$'s distinct, $1 \leq i \leq o(T)$ and $1 \leq j \leq o_{(T)-1}\}$. Is V a o(T)-dimensional dual number linear algebra over Q?

(v) Is V a general null ring?

36. Obtain some special properties enjoyed by a n-dimensional null ring.

37. Find the number of elements x in Z_{148} such that $x^2 \equiv 0 \bmod (148)$.

38. Let

$$T = \left\{ \left[\begin{matrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \end{matrix}\right] \middle| \; g_i \in \{9, 18, 27, 36, 45, 54, 63, 72, 0\} \subseteq Z_{81};\ 1 \leq i \leq 6 \right\}.$$

(i) What is the algebraic structure enjoyed by T?
(ii) Find the number of elements in T.
(iii) Sg = {a + bg | g ∈ T, g fixed element a, b ∈ R}. Is Sg a general dual number ring?
(iv) How many three dimensional dual number rings can be constructed using T?
(v) Can Sg have proper subrings which are not ideals?
(vi) Does Sg contain ideals?
(vii) Can S = {a + bg | g ∈ T; a, b ∈ R} be a ring or a semigroup?

39. Let

$$M = \left\{ \left[\begin{matrix} g_1 & g_2 \\ g_3 & g_4 \\ g_5 & g_6 \\ g_7 & g_8 \end{matrix}\right] \middle| \; g_i \in \{11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 0\} \subseteq Z_{121};\ 1 \leq i \leq 8 \right\}.$$

(i) Find the number of elements in M.
(ii) If $S = \{a_1 + a_2m_1 + \ldots + a_tm_{t-1} \mid a_i \in Q, m_j \in M;\ 1 \leq j \leq t-1,.\ 1 \leq i \leq t\}$.

a) What is the value of t so that S is a ring?
b) Is S a null ring?
c) Can S have zero divisors?
d) Is S a general dual number ring for $t = 5$?
e) What value of t, will S be a general dual number ring under natural product?

(iii) Can S be a dual number linear algebra for all values of t?

(iv) Can S be a dual number vector space for all values of $t-1$, $1 \leq t \leq o(M) - 1$.

40. Obtain some special properties enjoyed by zero square semigroup.

41. What is the speciality of the semigroup ring, QS where S is a null semigroup?

42. Can QS in problem 41 have subrings which are not ideals?

43. What are the special properties enjoyed by the null ring?

44. Can a dual general ring be a null ring?

45. Suppose $S = Z_{48}$. Can S have elements which are nilpotent of order two?

46. (For what values of n) Z_n (n not a prime) be free from nilpotent elements of order 2?

47. Prove for any given n one can construct a n-dimensional dual number.

48. What are the special features enjoyed by dual number vector spaces?

49. Let $V = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 \mid a_i \in Q \setminus \{0\}, 1 \leq i \leq 5, g_1 = 3 + 3i_F, g_2 = 6+6i_F, g_3 = 6+3i_F$ and $g_4 = 3+6i_F$ $g_j \in C(Z_9); 1 \leq j \leq 4\}$ be a dual number vector space over the field Q.

(i) Find a basis of V over Q.
(ii) What is the dimension of V over Q?
(iii) Can V be written as a direct sum of subspaces?
(iv) Find Hom_Q (V,V).
(v) Can V be a linear algebra?

50. Let P = $\{a + bg_1 + cg_2 \mid a, c, b \in Q; g_1 = 4 + 4i_F$ and $g_2 = 8 + 8i_F, g_1, g_2 \in C(Z_{16})\}$ be a dual number vector space of finite complex modulo integers over Q.

(i) Find a basis for P.
(ii) Write P as a direct sum of subspaces.
(iii) Find the algebraic structures enjoyed by Hom_Q (P, P).
(iv) What is the algebraic structure associated with L (P, Q)?

51. Let M =

$$\left\{ a_1 + a_2\begin{bmatrix} 3+3i_F \\ 6+6i_F \\ 0 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 3+3i_F \\ 6+6i_F \end{bmatrix} + a_4\begin{bmatrix} 0 \\ 3+3i_F \\ 6+6i_F \\ 0 \end{bmatrix} + a_5\begin{bmatrix} 0 \\ 3+3i_F \\ 0 \\ 6+6i_F \end{bmatrix} + \right.$$

$$\left. a_6\begin{bmatrix} 6+6i_F \\ 0 \\ 3+3i_F \\ 0 \end{bmatrix} + a_7\begin{bmatrix} 6+6i_F \\ 3+3i_F \\ 0 \\ 0 \end{bmatrix} \right| a_i \in Q; 1 \le i \le 7, 6 + 6i_F,$$

$3+3i_F \in C(Z_9)\}$ be a dual number vector space of finite complex modulo integers over Q.

(i) Find dimension of M over Q.
(ii) Find a basis of M over Q.
(iii) Find Hom_Q (M, M).
(iv) Can M be a linear algebra?

52. Obtain some interesting properties about dual number semivector spaces / semilinear algebras.

53. Let T = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_i \in R^+, 1 \leq i \leq 4, g_1 = 3+3i_F, g_2 = 6+6i_F, g_3 = 3+6i_F, g_i \in C(Z_9); 1 \leq i \leq 3\} \cup \{0\}$ be a semivector space of finite complex modulo integer dual numbers over the semifield $F = R^+ \cup \{0\}$.

(i) What is the dimension of T over F?
(ii) Find a basis of T over F.
(iii) Can T have more than one basis?
(iv) Find $Hom_F(T,T)$.
(v) Find the algebraic structure enjoyed by L(T, F).

54. Let M =

$$\left\{ a_1 + a_2\begin{bmatrix} 2+2i_F \\ 2+2i_F \\ 2+2i_F \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 2+2i_F \end{bmatrix} + a_4\begin{bmatrix} 2+2i_F \\ 0 \\ 0 \end{bmatrix} + a_5\begin{bmatrix} 0 \\ 2+2i_F \\ 0 \end{bmatrix} + \right.$$

$$\left. a_6\begin{bmatrix} 2+2i_F \\ 2+2i_F \\ 0 \end{bmatrix} + a_7\begin{bmatrix} 2+2i_F \\ 0 \\ 2+2i_F \end{bmatrix} + a_8\begin{bmatrix} 0 \\ 2+2i_F \\ 2+2i_F \end{bmatrix} \right| \; 2+2i_F \in C(Z_4);$$

$a_i \in Q^+ \cup \{0\}, \; 1 \leq i \leq 8\}$ be a dual number complex modulo integer semilinear algebra over the semifield $F = Q^+ \cup \{0\}$.

(i) Find a basis of M as a semilinear algebra?
(ii) Find a basis of M, M treated as a semivector space.
(iii) Write M as a direct sum of semilinear subalgebras.

55. Prove or disprove the set S of nilpotent elements of order two in $C(Z_n)$ (n a composite number) is a semigroup.

56. Can S in problem (55) be a subring?

57. Can S in problem (55) be an ideal?

58. Let $C(Z_{p^2}) = S$ be the semigroup of complex modulo integer. Q be the field of rationals, QS be the semigroup ring of S over Q.

    (i) Can QS have general dual number complex modulo integer subring?
    (ii) Can QS have dual number finite complex modulo integers?
    (iii) Give some interesting properties enjoyed by QS.

59. Let S = {(s, 0, 0), (s, s, s), (0, s, s), (s, 0, s), (s, s, 0), (0, 0, 0), (0, 0, s), (0, s, 0) | $s = 2 + 2i_F \in C(Z_4)$} be the null semigroup.
    Consider $T = \{a_1 + a_2g_1 + \ldots + a_8g_7 \mid g_i \in S \setminus \{(0, 0, 0)\}, 1 \leq i \leq 7$ and $a_j \in Q^+$, g_i's are distinct$\} \cup \{0\}$.

    (i) Is T a dual number finite complex modulo integer semivector space over the semifield $F = Q^+ \cup \{0\}$?
    (ii) Is T finite dimensional?
    (iii) Find a basis of T over F.
    (iv) Is T a dual finite complex modulo integer semilinear algebra over $F = Q^+ \cup \{0\}$.
    (v) Study all the four problems (i) to (iv) if $F = Q^+ \cup \{0\}$ is replaced by $K = Z^+ \cup \{0\}$.

60. Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in T = \{0, 3+3i_F, 6 + 6i_F, 3+6i_F, 6+3i_F\} \subseteq C(Z_9) \right\}.$$

    (i) Find the number of distinct elements in S.
    (ii) Is S a null semigroup under natural product $\times_n$ of matrices?

(iii) FS where FS = $\left\{\sum t_i s_i \,\middle|\, t_i \in Z^+ \cup \{0\} \text{ and } s_i \in S\right\}$. Is FS a dual number finite complex modulo integer semiring?

(iv) Can FS be a semifield?

61. Compare between a dual number vector space and dual number semivector space.

62. Can any other natural number other than modulo integers and finite complex modulo integers and matrices built of them contribute to dual number (that is the nilpotent element of order two in a + bg with $g^2 = 0$, a, b $\in$ R).

63. Enumerate some interesting properties enjoyed by t-dimensional semivector spaces of complex modulo finite integers.

64. Give some interesting properties of dual interval numbers.

65. Can dual interval number find application in dynamic analysis of mechanisms?

66. Enumerate the special features enjoyed by dual neutrosophic numbers.

67. Can the dual neutrosophic numbers be a field?

68. Can t-dimensional dual numbers be useful in engineering applications?

69. Prove dual interval number can form a ring.

70. Give an example of a 5-dimensional dual interval numbers.

71. Prove we can define t-dimensional dual interval number vector space V over a field R or Q. Does the positive cone of V, a semivector space over $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$?

72. Give an example of a semiring of dual interval numbers.

73. Let

$$W = \left\{ \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \\ b_7 & b_8 \\ b_9 & b_{10} \\ b_{11} & b_{12} \end{bmatrix} g \right) \middle| \; a_i, b_j \in Q, \right.$$

$g = 2 + 2i_F \in C(Z_4)$, $1 \leq i, j \leq 12\}$ be a vector space of dual number over the field Q.

(i) Find the positive cone of W.
(ii) Can W have subspaces?
(iii) Can W be written as a direct sum of subspaces?
(iv) What is the dimension of W over Q?
(v) If natural product $\times_n$ be defined on W will W be a linear algebra?
(vi) Will W as a linear algebra have a different basis?

74. Let

$M = \{([a_1, b_1], [a_2, b_2], [a_3, b_3])$ where $a_i, b_j \in \{x_1 + x_2g_1 + x_3g_2 \mid x_t \in Q;\ 1 \leq t \leq 3,\ g_1 = 4$ and $g_2 = 8$ in $Z_{16}\};\ 1 \leq i, j \leq 3\}$ be a vector space of three dimensional dual interval numbers over Q.

(i) Find a basis for V over Q.
(ii) Can V be made into a linear algebra?
(iii) Find $T : V \to V$ so that T^{-1} does not exist.
(iv) Find $f : V \to Q$ so that f is a linear functional.

75. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \right.$ $a_i \in [x_1 + x_2g_1 + x_3g_2 +$ $x_4g_3, y_1 + y_2g_1 + y_3g_2 + y_4g_3] \mid x_i, y_j \in R, 1 \leq i, j \leq 4$ and $g_1 = 8$, $g_2 = 12$ and $g_3 = 4$ where $g_1, g_2, g_3 \in Z_{16}\}$ be the collection of 3×5 matrices with entries from the closed interval of four dimensional dual numbers.

(i) Can S be a vector space over R?
(ii) Can S be given a general commutative ring structure?
(iii) Will S be a linear algebra if natural product $\times_n$ is defined on it?

76. Let P = $\{[a, b) + [c, d)g \mid g = 12 + 12i_F \in C(Z_{24}), a, b, c, d \in R^+ \cup \{0\}\}$.

(i) Is P a dual number semiring?
(ii) Is P a strict semiring?
(iii) Can P be a semifield?
(iv) Can P have semiideals?
(v) Can P have subsemirings which are not semiideals?

77. Derive some interesting properties about interval dual number semirings of t-dimension.

78. What are the special and striking features enjoyed by t-dimensional dual number interval vector spaces?

79. Give an example of a t-dimensional dual number interval vector space which is not a linear algebra.

80. Compare dual number semivector spaces and a dual number vector space. Does a vector space always contain a semivector space?

81. Let $P = \left\{ \begin{bmatrix} [a_1, b_1] \\ [a_2, b_2] \\ \vdots \\ [a_{10}, b_{10}] \end{bmatrix} \middle| \; a_i, b_j \in \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid g_1 = 4 + 4i_F,\ g_2 = 8 + 8i_F \text{ and } g_3 = 12 + 12i_F \in C(Z_{16}),\ x_t \in Q;\ 1 \le t \le 4\};\ 1 \le i, j \le 10 \right\}$.

(i) Is P a general interval dual number ring?
(ii) Is P a vector space of matrix interval dual numbers over Q?

82. Let V =

$$\left\{ \begin{bmatrix} [a_1^1, b_1^1] \\ [a_2^1, b_2^1] \\ \vdots \\ [a_{10}^1, b_{10}^1] \end{bmatrix} + \begin{bmatrix} [a_1^2, b_1^2] \\ [a_2^2, b_2^2] \\ \vdots \\ [a_{10}^2, b_{10}^2] \end{bmatrix} g_1 + \begin{bmatrix} [a_1^3, b_1^3] \\ [a_2^3, b_2^3] \\ \vdots \\ [a_{10}^3, b_{10}^3] \end{bmatrix} g_2 + \begin{bmatrix} [a_1^4, b_1^4] \\ [a_2^4, b_2^4] \\ \vdots \\ [a_{10}^4, b_{10}^4] \end{bmatrix} g_3 \;\middle|\; a_i^j, b_j^t \in Q;\ 1 \le i, j \le 10,\ t = 1, 2, 3, 4,\ g_1 = 4 + 4i_F,\ g_2 = 8 + 8i_F \text{ and } g_3 = 12 + 12i_F \in C(Z_{16}) \right\}.$$

(i) Is V a matrix interval coefficient dual number ring?
(ii) Can V be a linear algebra / vector space over Q?
(iii) Is P in problem 81 isomorphic to V as rings / vector spaces?

83. Give some nice applications of matrix of interval coefficient dual numbers?

84. For problem 82 find the algebraic structure of $\text{Hom}_Q(V,V)$, V as a vector space.

85. Let $P = C(Z_{42})$ be the finite complex modulo integer ring.

(i) Find the set S of all nilpotent elements of order two.
(ii) Does S form a null ring?

(iii) Using S find the maximum dimension of the general ring of dual numbers that can be constructed.

86. Find the null semigroup S_1 of $C(Z_{273})$.

87. Find the null semigroup S_2 of Z_{273}.

88. Compare S_1 and S_2 given in problems 86 and 87.

89. Can $C(Z_n)$, n any arbitrary large composite number have two distinct null semigroups S_1 and S_2 such that $S_1 \not\subseteq S_2$ and $S_2 \not\subseteq S_1$.

90. Find the null semigroup of $C(Z_{16}) \times Z_9$.

91. Describe some special features enjoyed by neutrosophic dual numbers.

92. Find the null semigroup S contained in $\langle Z_{81} \cup I\rangle$. Using S construct neutrosophic dual number ring.

93. Let M = $\{x_1 + x_2g_1 + x_3g_2 + \ldots + x_{10}g_9 \mid x_i \in Q; 1 \leq i \leq 10$ and $g_1 = 4 + 4I$, $g_2 = 4 + 8I$, $g_3 = 8 + 4I$, $g_4 = 8 + 8I$, $g_5 = 12 + 12I$, $g_6 = 12 + 4I$, $g_7 = 12 + 8I$, $g_8 = 4+12I$ and $g_9 = 8 + 12I$ in $\langle Z_{16} \cup I\rangle\}$ be a neutrosophic dual number general ring.

    (i) Does M have ideals?
    (ii) Find subring in M.
    (iii) Can M have zero divisors?

94. Obtain some applications of complex neutrosophic finite modulo dual numbers.

95. Let S = $\{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid x_i \in Q, 1 \leq i \leq 4$, $g_1 = 4 + 4i_F, +4I+4i_FI$, $g_2 = 8+8i_F+8I+8i_FI$ and $g_3 = 12 + 12i_F + 12I + 12i_F \in C\langle Z_{16} \cup I\rangle\}$ be the dual neutrosophic complex modulo integer number general ring.

    (i) Find subring of S which are not ideals of S.

(ii) Can S have ideals?
(iii) Can S have zero divisors?
(iv) Is S a Smarandache ring?

96. Let T =

$$\left\{ x_1 + x_2\begin{bmatrix} 5+5I & 5I \\ 5 & 0 \end{bmatrix} + x_3\begin{bmatrix} 5 & 5I \\ 5+5I & 0 \end{bmatrix} + x_3\begin{bmatrix} 0 & 0 \\ 5+5I & 5 \end{bmatrix} + \right.$$

$$\left. x_4\begin{bmatrix} 5+5I & 5I \\ 5I & 5 \end{bmatrix} + x_5\begin{bmatrix} 5 & 5I \\ 0 & 5 \end{bmatrix} + x_6\begin{bmatrix} 5I & 0 \\ 5I & 0 \end{bmatrix} \right| x_i \in R,$$

$1 \le i \le 6$, 5, 5I, 5+5I $\in \langle Z_{10} \cup I\rangle\}$ be a general ring of neutrosophic dual numbers.

(i) Find subrings of T.
(ii) Find ideals in T.
(iii) Does T have a maximal ideal?
(iv) Can T have a minimal ideal?

97. Let W =

$$\left\{ x_1 + x_2\begin{bmatrix} [a_1^1, a_2^1] \\ [a_1^2, a_2^2] \\ [a_1^3, a_2^3] \\ [a_1^4, a_2^4] \end{bmatrix} + x_3\begin{bmatrix} [b_1^1, b_2^1] \\ [a_1^2, b_2^2] \\ [b_1^3, b_2^3] \\ [b_1^4, b_2^4] \end{bmatrix} + x_4\begin{bmatrix} [c_1^1, c_2^1] \\ [c_1^2, c_2^2] \\ [c_1^3, c_2^3] \\ [c_1^4, c_2^4] \end{bmatrix} g_2 \right|$$

$a_i^t, b_j^t, c_k^t \in \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_3 + a_7g_6 + a_8g_7 \mid g_1 = 6 + 6I, g_2 = 6, g_3 = 6I, g_4 = 6 + 6i_F + 6I, g_5 = 6i_F, g_6 = 6 + 6i_F$ and $g_7 = 6i_F+6I\} \subseteq C\langle Z_{12} \cup I\rangle$, $a_p \in Q$; $1 \le p \le 8$, $1 \le t \le 4$, $1 \le i, j \le 4\}$ be a neutrosophic finite complex modulo integer dual interval number general vector space over Q.

(i) Is W a linear algebra?
(ii) Find a basis for W

(iii) Write W as a direct sum of subspaces.
(iv) Can we define positive cone of W?

98. What are the interesting properties of t-dimensional neutrosophic complex modulo integer dual number semilinear algebras?

99. Give an example of a semivector space of neutrosophic complex modulo integer dual numbers which is not a semilinear algebra.

100. Find the null semigroup of $C\langle Z_{45} \cup I\rangle$.

101. Can $C(\langle Z_{19} \cup I\rangle)$ have null semigroup?

102. Find the null semigroup of $C\langle Z_{(19)^2} \cup I\rangle$.

103. Find the order of the null semigroup of $C\langle Z_{n^2} \cup I\rangle$.

104. Let $M = \{x_1 + x_2g \mid x_1, x_2 \in [0, 1]$ and $g = 8 \in Z_{16}\}$ be the fuzzy dual number.

(i) What is the algebraic structure that can be given to M?
(ii) Can M be a fuzzy dual number general ring?
(iii) Can M be a fuzzy dual number vector space of the field $\{0, 1\} = Z_2$?

105. Obtain some interesting properties enjoyed by fuzzy dual numbers.

106. Derive some interesting results about fuzzy interval dual number.

107. Study the properties of fuzzy finite complex modulo integer dual numbers.

108. Let $V = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + \ldots + a_9g_8$ where $a_j \in [0, 1]$; $1 \leq j \leq 9$ and $g_1 = 3 + 3i_F$, $g_2 = 3$, $g_3 = 3i_F$,

$g_4 = 6 + 6i_F$, $g_5 = 6i_F$, $g_6 = 6$, $g_7 = 6 + 3i_F$ and $g_8 = 3 + 6i_F \in C(Z_9)\}$ be a fuzzy complex modulo integer semigroup under product.

(i) Can V have subsemigroups?
(ii) Can V have null subsemigroups?
(iii) Can V have ideals?
(iv) Is every null subsemigroup an ideal?

109. Study the properties enjoyed by fuzzy neutrosophic dual numbers.

110. Let $S = \{a_1 + a_2g_1 + \ldots + a_9g_8, a_i \in [0, 1], 1 \leq i \leq 9, g_1 = 8I, g_2 = 16I, g_3 = 8, g_4 = 16, g_5 = 8 + 8I, g_6 = 8 + 16I, g_7 = 16 + 8I$ and $g_8 = 16 + 16I \in \langle Z_{32} \cup I\rangle\}$ be a semigroup of fuzzy neutrosophic dual numbers.
(i) Define min and max on S. Will {S, min, max} be a semiring?
(ii) Find the null subsemigroup of S under product.
(iii) Can {S, min} have dual number properties?

111. Give an example of a interval fuzzy neutrosophic dual number semigroup?

112. Does a fuzzy neutrosophic complex modulo integer dual number semiring exist?

113. Does a fuzzy interval dual number semiring exist? Justify your claim.

114. Find the fuzzy integer neutrosophic semigroup using the null semigroup of $\langle Z_{12} \cup I\rangle$.

115. Can these dual number fuzzy matrices be applied / used in fuzzy models?

116. Enumerate any other interesting property associated with fuzzy neutrosophic complex modulo integer interval matrices of dual numbers.

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.
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The concept of dual numbers was introduced in 1873 by W.K.Clifford. In this book, the authors build higher dimenional dual numbers, interval dual numbers and impose some algebraic structures on them. The S-vector space of dual numbers bulit over a Smarandache dual ring can have eigen values and eigen vectors to be dual numbers. Complex modulo integer dual numbers and neutrosophic dual numbers are also introduced. The notion of fuzzy dual numbers can play a vital role in fuzzy models. Some research level problems are also proposed here.

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