Young Bae Jun
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Fuzzy and Neutrosophic Sets in Semigroups

$$
\begin{aligned}
T_{N\cap M}(xy) &= \bigvee \{T_N(xy), T_M(xy)\} \\
&\leq \bigvee \left\{ \bigvee \{T_N(x), T_N(y)\}, \bigvee \{T_M(x), T_M(y)\} \right\} \\
&= \bigvee \left\{ \bigvee \{T_N(x), T_M(x)\}, \bigvee \{T_N(y), T_M(y)\} \right\} \\
&= \bigvee \{T_{N\cap M}(x), T_{N\cap M}(y)\},
\end{aligned}
$$

$$
\begin{aligned}
I_{N\cap M}(xy) &= \bigwedge \{I_N(xy), I_M(xy)\} \\
&\geq \bigwedge \left\{ \bigwedge \{I_N(x), I_N(y)\}, \bigwedge \{I_M(x), I_M(y)\} \right\} \\
&= \bigwedge \left\{ \bigwedge \{I_N(x), I_M(x)\}, \bigwedge \{I_N(y), I_M(y)\} \right\} \\
&= \bigwedge \{I_{N\cap M}(x), I_{N\cap M}(y)\}
\end{aligned}
$$

$$
\begin{aligned}
F_{N\cap M}(xy) &= \bigvee \{F_N(xy), F_M(xy)\} \\
&\leq \bigvee \left\{ \bigvee \{F_N(x), F_N(y)\}, \bigvee \{F_M(x), F_M(y)\} \right\} \\
&= \bigvee \left\{ \bigvee \{F_N(x), F_M(x)\}, \bigvee \{F_N(y), F_M(y)\} \right\} \\
&= \bigvee \{F_{N\cap M}(x), F_{N\cap M}(y)\}
\end{aligned}
$$

$\cdot$	e	a	b	c
e	e	e	e	e
a	e	a	e	a
b	e	e	b	b
c	e	a	b	c

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Foreword

The topics discussed in this book are Int-soft semigroup, Int-soft left (right) ideal, Int-soft (generalized) bi-ideal, Int-soft quasi-ideal, Int-soft interior ideal, Int-soft left (right) duo semigroup, starshaped (∈, ∈∨ qk)-fuzzy set, quasi-starshaped (∈, ∈∨ qk)-fuzzy set, semidetached mapping, semidetached semigroup, (∈, ∈ ∨qk)-fuzzy subsemi-group, (qk, ∈ ∨qk)-fuzzy subsemigroup, (∈, ∈ ∨ qk)-fuzzy subsemigroup, (qk, ∈ ∨ qk)-fuzzy subsemigroup, (∈ ∨ qk, ∈ ∨ qk)-fuzzy subsemigroup, (∈, ∈∨ qkδ)-fuzzy subsemigroup, ∈∨ qkδ -level subsemigroup/bi-ideal, (∈, ∈∨ qkδ)-fuzzy (generalized) bi-ideal, δ-lower (δ-upper) approximation of fuzzy set, δ-lower (δ-upper) rough fuzzy subsemigroup, δ-rough fuzzy subsemigroup, Neutrosophic N -structure, neutrosophic N -subsemigroup, ε-neutrosophic N -subsemigroup, and neutrosophic N -product.

The first chapter, *Characterizations of regular and duo semigroups based on int-soft set theory*, investigates the relations among int-soft semigroup, int-soft (generalized) bi-ideal, int-soft quasi-ideal and int-soft interior ideal. Using int-soft left (right) ideal, an int-soft quasi-ideal is constructed. We show that every int-soft quasi-ideal can be represented as the soft intersection of an int-soft left ideal and an int-soft right ideal. Using int-soft quasi-ideal, an int-soft bi-ideal is established. Conditions for a semigroup to be regular are displayed. The notion of int-soft left (right) duo semigroup is introduced, and left (right) duo semigroup is characterized by int-soft left (right) duo semigroup. Bi-ideal, quasi-ideal and interior ideal are characterized by using (Φ, Ψ)-characteristic soft sets.

The notions of starshaped (∈, ∈ ∨ qk)-fuzzy sets and quasi-starshaped (∈, ∈ ∨ qk)-fuzzy sets are introduced in the second chapter, *Generalizations of starshaped (∈, ∈∨q)-fuzzy sets*, and related properties are investigated. Characterizations of starshaped (∈, ∈ ∨ qk)-fuzzy sets and quasi-starshaped (∈, ∈ ∨ q)-fuzzy sets are discussed. Relations between starshaped (∈, ∈ ∨ qk)-fuzzy sets and quasi-starshaped (∈, ∈∨ qk)-fuzzy sets are investigated.

The notion of semidetached semigroup is introduced the third chapter (*Semidetached semigroups*), and their properties are investigated. Several conditions for a pair of a semigroup and a semidetached mapping to be a semidetached semigroup are provided. The concepts of (∈, ∈ ∨ qk)-fuzzy sub-semigroup, (qk, ∈ ∨ qk)-fuzzy subsemigroup and (∈ ∨ qk, ∈ ∨ qk)-fuzzy subsemigroup are introduced, and relative relations are discussed.

The fourth chapter, *Generalizations of (∈, ∈∨ qk)-fuzzy (generalized) bi-ideals in semigroups*, introduces the notion of (∈, ∈ ∨ qkδ)-fuzzy (generalized) bi-ideals in semigroups, and related properties are investigated. Given a (generalized) bi-ideal, an (∈,∈ ∨ qkδ)-fuzzy (generalized) bi-ideal is constructed. Characterizations of an (∈, ∈ ∨ qkδ)-fuzzy (generalized) bi-ideal are discussed, and shown that an (∈, ∈∨ qkδ)-fuzzy generalized bi-ideal and an (∈, ∈ ∨ qkδ)-fuzzy bi-ideal coincide in regular semigroups. Using a fuzzy set with finite image, an (∈, ∈∨ qkδ)-fuzzy bi-ideal is established.

Lower and upper approximations of fuzzy sets in semigroups are considered in the fifth chapter, *Approximations of fuzzy sets in semigroups*, and several properties are investigated. The notion of rough sets was introduced by Pawlak. This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis. Rough set theory is applied to semigroups and groups, d-algebras, BE-algebras, BCK-algebras and MV-algebras etc.

Finally, in the sixth and last paper, *Neutrosophic N-structures and their applications in semigroups*, the notion of neutrosophic N -structure is introduced, and applied to semigroup. The notions of neutrosophic N-subsemigroup, neutrosophic N-product and ε-neutrosophic N-subsemigroup are introduced, and several properties are investigated. Conditions for neutrosophic N-structure to be neutrosophic N-subsemigroup are provided. Using neutrosophic N-product, characterization of neutrosophic N-subsemigroup is discussed. Relations between neutrosophic N-subsemigroup and ε-neutrosophic N-subsemigroup are discussed. We show that the homomorphic preimage of neutrosophic N-subsemigroup is a neutrosophic N-subsemigroup, and the onto homomorphic image of neutrosophic N - subsemigroup is a neutrosophic N-subsemigroup.

Characterizations of regular and duo semigroups based on int-soft set theory

Abstract Relations among int-soft semigroup, int-soft (generalized) bi-ideal, int-soft quasi-ideal and int-soft interior ideal are investigated. Using int-soft left (right) ideal, an int-soft quasi-ideal is constructed. We show that every int-soft quasi-ideal can be represented as the soft intersection of an int-soft left ideal and an int-soft right ideal. Using int-soft quasi-ideal, an int-soft bi-ideal is established. Conditions for a semigroup to be regular are displayed. The notion of int-soft left (right) duo semigroup is introduced, and left (right) duo semigroup is characterized by int-soft left (right) duo semigroup. Bi-ideal, quasi-ideal and interior ideal are characterized by using (Φ, Ψ)-characteristic soft sets.

Keywords: Int-soft semigroup, Int-soft left (right) ideal, Int-soft (generalized) bi-ideal, Int-soft quasi-ideal, Int-soft interior ideal, Int-soft left (right) duo semigroup.

2010 Mathematics Subject Classification. 06D72, 20M12, 20M99.

1 Introduction

The soft set theory, which is introduced by Molodtsov [13], is a good mathematical model to deal with uncertainty. At present, works on the soft set theory are progressing rapidly. In fact, in the aspect of algebraic structures, the soft set theory has been applied to rings, fields and modules (see [1, 3]), groups (see [2]), semirings (see [6]), BL-algebras (see [15]), BCK/BCI-algebras ([7], [8], [10], [11]), d-algebras (see [9]), Song et al. [14] introduced the notion of int-soft semigroups and int-soft left (resp. right) ideals, and investigated several properties. As a continuation of the paper [14], Jun and Song [12] discussed further properties and characterizations of int-soft left (right) ideals. They introduced the notion of int-soft (generalized) bi-ideals, and provided relations between int-soft generalized bi-ideals and int-soft semigroups. They also considered characterizations of (int-soft) generalized bi-ideals and int-soft bi-ideals. In [5], Dudek and Jun introduced the notion of an int-soft interior, and investigated related properties.

In this paper, we investigate relations among int-soft semigroup, int-soft (generalized) bi-ideal, int-soft quasi-ideal and int-soft interior ideal. Using int-soft left (right) ideal, we construct an int-soft quasi-ideal. We show that every int-soft quasi-ideal can be represented as the soft intersection of an int-soft left ideal and an int-soft right ideal. Using int-soft quasi-ideal, we establish an int-soft bi-ideal. We display conditions for a semigroup to be regular. We introduce the notion of int-soft left (right) duo semigroup, characterize it by int-soft left (right) duo semigroup. We also characterize bi-ideal, quasi-ideal and interior ideal by using (Φ, Ψ)-characteristic soft sets.

2 Preliminaries

Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

A semigroup S is said to be *regular* if for every $x \in S$ there exists $a \in S$ such that $xax = x$,

A nonempty subset A of S is called

- a *subsemigroup* of S if $AA \subseteq A$, that is, $ab \in A$ for all $a, b \in A$,
- a *left* (resp., *right*) *ideal* of S if $SA \subseteq A$ (resp., $AS \subseteq A$), that is, $xa \in A$ (resp., $ax \in A$) for all $x \in S$ and $a \in A$,
- a *two-sided ideal* of S if it is both a left and a right ideal of S,
- a *generalized bi-ideal* of S if $ASA \subseteq A$,
- a *bi-ideal* of S if it is both a semigroup and a generalized bi-ideal of S,
- an *interior ideal* of S if $SAS \subseteq A$.

A semigroup S is said to be

- *left* (resp., *right*) *duo* if every left (resp., right) ideal of S is a two-sided ideal of S,
- *duo* if it is both left and right duo.

A soft set theory is introduced by Molodtsov [13], and Çağman et al. [4] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.1 ([4, 13]). A soft set (α, A) over U is defined to be the set of ordered pairs

$$(\alpha, A) := \{(x, \alpha(x)) : x \in E,\, \alpha(x) \in \mathscr{P}(U)\},$$

where $\alpha : E \to \mathscr{P}(U)$ such that $\alpha(x) = \emptyset$ if $x \notin A$.

The function α is called approximate function of the soft set (α, A). The subscript A in the notation α indicates that α is the approximate function of (α, A).

For any soft sets (α, S) and (β, S) over U, we define

$$(\alpha, S) \,\tilde{\subseteq}\, (\beta, S) \text{ if } \alpha(x) \subseteq \beta(x) \text{ for all } x \in S.$$

The soft union of (α, S) and (β, S) is defined to be the soft set $(\alpha \,\tilde{\cup}\, \beta, S)$ over U in which $\alpha \,\tilde{\cup}\, \beta$ is defined by

$$(\alpha \,\tilde{\cup}\, \beta)(x) = \alpha(x) \cup \beta(x) \text{ for all } x \in S.$$

The soft intersection of (α, S) and (β, S) is defined to be the soft set $(\alpha \,\tilde{\cap}\, \beta, S)$ over U in which $\alpha \,\tilde{\cap}\, \beta$ is defined by

$$(\alpha \,\tilde{\cap}\, \beta)(x) = \alpha(x) \cap \beta(x) \text{ for all } x \in S.$$

The int-soft product of (α, S) and (β, S) is defined to be the soft set $(\alpha \,\tilde{\circ}\, \beta, S)$ over U in which $\alpha \,\tilde{\circ}\, \beta$ is a mapping from S to $\mathscr{P}(U)$ given by

$$(\alpha \,\tilde{\circ}\, \beta)(x) = \begin{cases} \bigcup\limits_{x=yz} \{\alpha(y) \cap \beta(z)\} & \text{if } \exists\, y, z \in S \text{ such that } x = yz \\ \emptyset & \text{otherwise.} \end{cases}$$

3 Int-soft ideals

In what follows, we take $E = S$, as a set of parameters, which is a semigroup unless otherwise specified.

Definition 3.1 ([14]). A soft set (α, S) over U is called an *int-soft semigroup* over U if it satisfies:

$$(\forall x, y \in S)\,(\alpha(x) \cap \alpha(y) \subseteq \alpha(xy)). \tag{3.1}$$

Definition 3.2 ([12]). A soft set (α, S) over U is called an *int-soft generalized bi-ideal* over U if it satisfies:

$$(\forall x, y, z \in S)\,(\alpha(x) \cap \alpha(z) \subseteq \alpha(xyz)). \tag{3.2}$$

If a soft set (α, S) over U is both an int-soft semigroup and an int-soft generalized bi-ideal over U, then we say that (α, S) is an *int-soft bi-ideal* over U.

Definition 3.3 ([14]). A soft set (α, S) over U is called an *int-soft left* (resp., *right*) *ideal* over U if it satisfies:

$$(\forall x, y \in S)\,(\alpha(xy) \supseteq \alpha(y) \text{ (resp., } \alpha(xy) \supseteq \alpha(x))) \,. \tag{3.3}$$

If a soft set (α, S) over U is both an int-soft left ideal and an int-soft right ideal over U, we say that (α, S) is an int-soft two-sided ideal over U.

Obviously, every int-soft left (resp., right) ideal over U is an int-soft semigroup over U. But the converse is not true in general (see [14]).

Definition 3.4. A soft set (α, S) over U is called an *int-soft quasi-ideal* over U if

$$(\alpha \,\tilde{\circ}\, \chi_S, S) \,\tilde{\cap}\, (\chi_S \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\alpha, S) \,. \tag{3.4}$$

Definition 3.5 ([5]). A soft set (α, S) over U is called an *int-soft interior ideal* over U if it satisfies:

$$(\forall a, x, y \in S)\,(\alpha(xay) \supseteq \alpha(a)) \,. \tag{3.5}$$

For a nonempty subset A of S and $\Phi, \Psi \in \mathscr{P}(U)$ with $\Phi \supsetneq \Psi$, define a map $\chi_A^{(\Phi,\Psi)}$ as follows:

$$\chi_A^{(\Phi,\Psi)} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \Phi & \text{if } x \in A, \\ \Psi & \text{otherwise.} \end{cases}$$

Then $\left(\chi_A^{(\Phi,\Psi)}, S\right)$ is a soft set over U, which is called the (Φ, Ψ)-*characteristic soft set*. The soft set $(\chi_S^{(\Phi,\Psi)}, S)$ is called the (Φ, Ψ)-*identity soft set* over U. The (Φ, Ψ)-characteristic soft set with $\Phi = U$ and $\Psi = \emptyset$ is called the *characteristic soft set*, and is denoted by (χ_A, S). The (Φ, Ψ)-identity soft set with $\Phi = U$ and $\Psi = \emptyset$ is called the *identity soft set*, and is denoted by (χ_S, S).

Lemma 3.6. *Let (α, S), (β, S) and (γ, S) be soft sets over U. If $(\alpha, S) \,\tilde{\subseteq}\, (\beta, S)$, then $(\alpha \,\tilde{\circ}\, \gamma, S) \,\tilde{\subseteq}\, (\beta \,\tilde{\circ}\, \gamma, S)$ and $(\gamma \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\gamma \,\tilde{\circ}\, \beta, S)$.*

Proof. For any $x \in S$, if x is expressible as $x = yz$, then

$$\begin{aligned} (\alpha \,\tilde{\circ}\, \gamma)\,(x) &= \bigcup_{x=yz} \{\alpha(y) \cap \gamma(z)\} \\ &\subseteq \bigcup_{x=yz} \{\beta(y) \cap \gamma(z)\} \\ &= (\beta \,\tilde{\circ}\, \gamma)\,(x). \end{aligned}$$

Otherwise implies that $(\alpha \,\tilde{\circ}\, \gamma)(x) = \emptyset = (\beta \,\tilde{\circ}\, \gamma)(x)$. Hence $(\alpha \,\tilde{\circ}\, \gamma, S) \,\tilde{\subseteq}\, (\beta \,\tilde{\circ}\, \gamma, S)$. Similarly, we have $(\gamma \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\gamma \,\tilde{\circ}\, \beta, S)$. □

Theorem 3.7. *Every int-soft quasi-ideal is an int-soft semigroup.*

Proof. Let (α, S) be an int-soft quasi-ideal over U. Since $(\alpha, S) \,\tilde{\subseteq}\, (\chi_S, S)$, it follows from Lemma 3.6 that $(\alpha \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\chi_S \,\tilde{\circ}\, \alpha, S)$ and $(\alpha \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\alpha \,\tilde{\circ}\, \chi_S, S)$. Hence

$$(\alpha \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\chi_S \,\tilde{\circ}\, \alpha, S) \,\tilde{\cap}\, (\alpha \,\tilde{\circ}\, \chi_S, S) \,\tilde{\subseteq}\, (\alpha, S).$$

Therefore (α, S) is an int-soft semigroup over U. □

Theorem 3.8. *Every int-soft quasi-ideal is an int-soft bi-ideal.*

Proof. Let (α, S) be an int-soft quasi-ideal over U. Then (α, S) is an int-soft semigroup by Theorem 3.7, and hence $(\alpha \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\alpha, S)$. Since $(\alpha \,\tilde{\circ}\, \chi_S, S) \,\tilde{\subseteq}\, (\alpha, S)$, we have

$$(\alpha \,\tilde{\circ}\, \chi_S \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\chi_S \,\tilde{\circ}\, \alpha, S)\,. \tag{3.6}$$

Also, since $(\chi_S \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\chi_S, S)$, we have

$$(\alpha \,\tilde{\circ}\, \chi_S \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\alpha \,\tilde{\circ}\, \chi_S, S)\,. \tag{3.7}$$

It follows from (3.4), (3.6) and (3.7) that

$$(\alpha \,\tilde{\circ}\, \chi_S \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\chi_S \,\tilde{\circ}\, \alpha, S) \,\tilde{\cap}\, (\alpha \,\tilde{\circ}\, \chi_S, S) \,\tilde{\subseteq}\, (\alpha, S).$$

Therefore (α, S) is an int-soft bi-ideal over U. □

The converse of Theorem 3.8 is not true in general as seen in the following example.

Example 3.9. Let $S = \{0, 1, 2, 3\}$ be a semigroup with the multiplication table which is appeared in Table 1.
Let (α, S) be a soft set over $U = \mathbb{Z}$ defined as follows:

$$\alpha : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} 2\mathbb{Z} \cup \{1, 3, 5\} & \text{if } x = 0, \\ 4\mathbb{Z} & \text{if } x = 1, \\ 2\mathbb{Z} & \text{if } x = 2, \\ 4\mathbb{N} & \text{if } x = 3. \end{cases}$$

Then (α, S) is an int-soft bi-ideal over U, but it is not an int-soft quasi-ideal over U.

Table 1: Cayley table for the multiplication

	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

Lemma 3.10. *For any soft sets (α, S), (β, S) and (γ, S) over U, we have*

(1) $(\alpha \,\tilde{\circ}\, (\beta \,\tilde{\cup}\, \gamma), S) = ((\alpha \,\tilde{\circ}\, \beta) \,\tilde{\cup}\, (\alpha \,\tilde{\circ}\, \gamma), S)$.

(2) $((\beta \,\tilde{\cup}\, \gamma) \,\tilde{\circ}\, \alpha, S) = ((\beta \,\tilde{\circ}\, \alpha) \,\tilde{\cup}\, (\gamma \,\tilde{\circ}\, \alpha), S)$.

(3) $(\alpha \,\tilde{\circ}\, (\beta \,\tilde{\cap}\, \gamma), S) \,\tilde{\subseteq}\, ((\alpha \,\tilde{\circ}\, \beta) \,\tilde{\cap}\, (\alpha \,\tilde{\circ}\, \gamma), S)$.

(4) $((\beta \,\tilde{\cap}\, \gamma) \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, ((\beta \,\tilde{\circ}\, \alpha) \,\tilde{\cap}\, (\gamma \,\tilde{\circ}\, \alpha), S)$.

Proof. For any $x \in S$, if x is expressible as $x = yz$, then

$$\begin{aligned}
(\alpha \,\tilde{\circ}\, (\beta \,\tilde{\cup}\, \gamma))(x) &= \bigcup_{x=yz} \{\alpha(y) \cap (\beta \,\tilde{\cup}\, \gamma)(z)\} \\
&= \bigcup_{x=yz} \{\alpha(y) \cap (\beta(z) \cup \gamma(z))\} \\
&= \bigcup_{x=yz} \{(\alpha(y) \cap \beta(z)) \cup (\alpha(y) \cap \gamma(z))\} \\
&= \left(\bigcup_{x=yz} \{\alpha(y) \cap \beta(z)\} \right) \cup \left(\bigcup_{x=yz} \{\alpha(y) \cap \gamma(z)\} \right) \\
&= (\alpha \,\tilde{\circ}\, \beta)(x) \cup (\alpha \,\tilde{\circ}\, \gamma)(x) \\
&= ((\alpha \,\tilde{\circ}\, \beta) \,\tilde{\cup}\, (\alpha \,\tilde{\circ}\, \gamma))(x)
\end{aligned}$$

and

$$\begin{aligned}
(\alpha \,\tilde{\circ}\, (\beta \,\tilde{\cap}\, \gamma))(x) &= \bigcup_{x=yz} \{\alpha(y) \cap (\beta \,\tilde{\cap}\, \gamma)(z)\} \\
&= \bigcup_{x=yz} \{\alpha(y) \cap (\beta(z) \cap \gamma(z))\} \\
&= \bigcup_{x=yz} \{(\alpha(y) \cap \beta(z)) \cap (\alpha(y) \cap \gamma(z))\} \\
&\subseteq \left(\bigcup_{x=yz} \{\alpha(y) \cap \beta(z)\} \right) \cap \left(\bigcup_{x=yz} \{\alpha(y) \cap \gamma(z)\} \right) \\
&\subseteq (\alpha \,\tilde{\circ}\, \beta)(x) \cap (\alpha \,\tilde{\circ}\, \gamma)(x) \\
&= ((\alpha \,\tilde{\circ}\, \beta) \,\tilde{\cap}\, (\alpha \,\tilde{\circ}\, \gamma))\,(x).
\end{aligned}$$

Obviously, $(\alpha \,\tilde{\circ}\, (\beta \,\tilde{\cup}\, \gamma))(x) = ((\alpha \,\tilde{\circ}\, \beta) \,\tilde{\cup}\, (\alpha \,\tilde{\circ}\, \gamma))\,(x)$ and

$$(\alpha \,\tilde{\circ}\, (\beta \,\tilde{\cap}\, \gamma))(x) = ((\alpha \,\tilde{\circ}\, \beta) \,\tilde{\cap}\, (\alpha \,\tilde{\circ}\, \gamma))\,(x)$$

if x is not expressible as $x = yz$. Therefore $(\alpha \,\tilde{\circ}\, (\beta \,\tilde{\cup}\, \gamma), S) = ((\alpha \,\tilde{\circ}\, \beta) \,\tilde{\cup}\, (\alpha \,\tilde{\circ}\, \gamma), S)$ and $(\alpha \,\tilde{\circ}\, (\beta \,\tilde{\cap}\, \gamma), S) \,\tilde{\subseteq}\, ((\alpha \,\tilde{\circ}\, \beta) \,\tilde{\cap}\, (\alpha \,\tilde{\circ}\, \gamma), S)$. Similarly we can show that

$$((\beta \,\tilde{\cup}\, \gamma) \,\tilde{\circ}\, \alpha, S) = ((\beta \,\tilde{\circ}\, \alpha) \,\tilde{\cup}\, (\gamma \,\tilde{\circ}\, \alpha), S)$$

and $((\beta \,\tilde{\cap}\, \gamma) \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, ((\beta \,\tilde{\circ}\, \alpha) \,\tilde{\cap}\, (\gamma \,\tilde{\circ}\, \alpha), S)$. □

Lemma 3.11. *If (α, S) is a soft set over U, then*

$$(\alpha \,\tilde{\cup}\, (\chi_S \,\tilde{\circ}\, \alpha), S) \text{ and } (\alpha \,\tilde{\cup}\, (\alpha \,\tilde{\circ}\, \chi_S), S)$$

are an int-soft left ideal and an int-soft right ideal over U respectively.

Proof. Using Lemma 3.10, we have

$$\begin{aligned}
(\chi_S \,\tilde{\circ}\, (\alpha \,\tilde{\cup}\, (\chi_S \,\tilde{\circ}\, \alpha)), S) &= ((\chi_S \,\tilde{\circ}\, \alpha) \,\tilde{\cup}\, (\chi_S \,\tilde{\circ}\, (\chi_S \,\tilde{\circ}\, \alpha)), S) \\
&= ((\chi_S \,\tilde{\circ}\, \alpha) \,\tilde{\cup}\, ((\chi_S \,\tilde{\circ}\, \chi_S) \,\tilde{\circ}\, \alpha), S) \\
&\,\tilde{\subseteq}\, ((\chi_S \,\tilde{\circ}\, \alpha) \,\tilde{\cup}\, (\chi_S \,\tilde{\circ}\, \alpha), S) \\
&= (\chi_S \,\tilde{\circ}\, \alpha, S) \\
&\,\tilde{\subseteq}\, (\alpha \,\tilde{\cup}\, (\chi_S \,\tilde{\circ}\, \alpha), S)\,.
\end{aligned}$$

Hence $(\alpha \,\tilde{\cup}\, (\chi_S \,\tilde{\circ}\, \alpha), S)$ is an int-soft left ideal over U. Similarly, $(\alpha \,\tilde{\cup}\, (\alpha \,\tilde{\circ}\, \chi_S), S)$ is an int-soft right ideal over U. □

Lemma 3.12. *Let (α, S) and (β, S) be an int-soft right ideal and an int-soft left ideal over U respectively. Then $(\alpha \tilde{\cap} \beta, S)$ is an int-soft quasi-ideal over U.*

Proof. Since

$$(((\alpha \tilde{\cap} \beta) \tilde{\circ} \chi_S) \tilde{\cap} (\chi_S \tilde{\circ} (\alpha \tilde{\cap} \beta)), S) \tilde{\subseteq} ((\alpha \tilde{\circ} \chi_S) \tilde{\cap} (\chi_S \tilde{\circ} \beta), S) \tilde{\subseteq} (\alpha \tilde{\cap} \beta, S),$$

we know that $(\alpha \tilde{\cap} \beta, S)$ is an int-soft quasi-ideal over U. □

Theorem 3.13. *Every int-soft quasi-ideal can be represented as the soft intersection of an int-soft left ideal and an int-soft right ideal.*

Proof. Let (α, S) be an int-soft quasi-ideal over U. Then

$$(\alpha \tilde{\cup} (\chi_S \tilde{\circ} \alpha), S) \text{ and } (\alpha \tilde{\cup} (\alpha \tilde{\circ} \chi_S), S)$$

are an int-soft left ideal and an int-soft right ideal over U respectively by Lemma 3.11. Since $(\alpha, S) \tilde{\subseteq} (\alpha \tilde{\cup} (\alpha \tilde{\circ} \chi_S), S)$ and $(\alpha, S) \tilde{\subseteq} (\alpha \tilde{\cup} (\chi_S \tilde{\circ} \alpha), S)$, it follows that

$$\begin{aligned}
(\alpha, S) &\tilde{\subseteq} ((\alpha \tilde{\cup} (\chi_S \tilde{\circ} \alpha)) \tilde{\cap} (\alpha \tilde{\cup} (\alpha \tilde{\circ} \chi_S)), S) \\
&= (((\alpha \tilde{\cup} (\chi_S \tilde{\circ} \alpha)) \tilde{\cap} \alpha) \tilde{\cup} ((\alpha \tilde{\cup} (\chi_S \tilde{\circ} \alpha)) \tilde{\cap} (\alpha \tilde{\circ} \chi_S)), S) \\
&\tilde{\subseteq} ((\alpha \tilde{\cup} ((\alpha \tilde{\cup} (\chi_S \tilde{\circ} \alpha)) \tilde{\cap} (\alpha \tilde{\circ} \chi_S)), S) \\
&= (\alpha \tilde{\cup} ((\alpha \tilde{\cup} (\chi_S \tilde{\circ} \alpha)) \tilde{\cup} ((\chi_S \tilde{\circ} \alpha) \tilde{\cap} (\alpha \tilde{\circ} \chi_S))), S) \\
&\tilde{\subseteq} (\alpha \tilde{\cup} ((\alpha \tilde{\cap} (\alpha \tilde{\circ} \chi_S)) \tilde{\cup} \alpha, S) \\
&\tilde{\subseteq} (\alpha \tilde{\cup} (\alpha \tilde{\cup} \alpha), S) \\
&= (\alpha, S),
\end{aligned}$$

and so that $(\alpha, S) = ((\alpha \tilde{\cup} (\chi_S \tilde{\circ} \alpha)) \tilde{\cap} (\alpha \tilde{\cup} (\alpha \tilde{\circ} \chi_S)), S)$ which is the soft intersection of the int-soft left ideal $(\alpha \tilde{\cup} (\chi_S \tilde{\circ} \alpha), S)$ and the int-soft right ideal $(\alpha \tilde{\cup} (\alpha \tilde{\circ} \chi_S), S)$ over U. □

Theorem 3.14. *Let (α, S) and (β, S) be soft sets over U. If (α, S) is an int-soft quasi-ideal over U, then the soft product $(\alpha \tilde{\circ} \beta, S)$ is an int-soft bi-ideal over U.*

Proof. Assume that (α, S) is an int-soft quasi-ideal over U. Since every int-soft quasi-ideal is an int-soft bi-ideal, we have $(\alpha \tilde{\circ} \chi_S \tilde{\circ} \alpha, S) \tilde{\subseteq} (\alpha, S)$. Hence

$$\begin{aligned}
((\alpha \tilde{\circ} \beta) \tilde{\circ} (\alpha \tilde{\circ} \beta), S) &= ((\alpha \tilde{\circ} \beta \tilde{\circ} \alpha) \tilde{\circ} \beta, S) \\
&\tilde{\subseteq} ((\alpha \tilde{\circ} \chi_S \tilde{\circ} \alpha) \tilde{\circ} \beta, S) \\
&= (\alpha \tilde{\circ} \beta, S)
\end{aligned}$$

and

$$
\begin{aligned}
((\alpha \,\tilde{\circ}\, \beta) \,\tilde{\circ}\, \chi_S \,\tilde{\circ}\, (\alpha \,\tilde{\circ}\, \beta), S) &= ((\alpha \,\tilde{\circ}\, (\beta \,\tilde{\circ}\, \chi_S) \,\tilde{\circ}\, \alpha) \,\tilde{\circ}\, \beta, S) \\
&\tilde{\subseteq} ((\alpha \,\tilde{\circ}\, (\chi_S \,\tilde{\circ}\, \chi_S) \,\tilde{\circ}\, \alpha) \,\tilde{\circ}\, \beta, S) \\
&\tilde{\subseteq} ((\alpha \,\tilde{\circ}\, \chi_S \,\tilde{\circ}\, \alpha) \,\tilde{\circ}\, \beta, S) \\
&\tilde{\subseteq} (\alpha \,\tilde{\circ}\, \beta, S) .
\end{aligned}
$$

Therefore $(\alpha \,\tilde{\circ}\, \beta, S)$ is an int-soft bi-ideal over U. □

Definition 3.15. A semigroup S is said to be *int-soft left* (resp., *right*) *duo* if every int-soft left (resp., right) ideal over U is an int-soft two-sided ideal over U.

If a semigroup S is both int-soft left and int-soft right duo, we say that S is *int-soft duo.*

Lemma 3.16. *For a nonempty subset A of S, the following are equivalent.*

(1) *A is a left (resp., right) ideal of S.*

(2) *The (Φ, Ψ)-characteristic soft set $\left(\chi_A^{(\Phi,\Psi)}, S\right)$ over U is an int-soft left (resp., right) ideal over U.*

Proof. The proof is easy, and hence we omit it. □

Corollary 3.17 ([14]). *For a nonempty subset A of S, the following are equivalent.*

(1) *A is a left (resp., right) ideal of S.*

(2) *The characteristic soft set (χ_A, S) over U is an int-soft left (resp., right) ideal over U.*

Theorem 3.18. *For a semigroup S, the following assertions are equivalent.*

(1) *S is regular.*

(2) *$(\alpha \,\tilde{\cap}\, \beta, S) = (\alpha \,\tilde{\circ}\, \beta, S)$ for every int-soft right ideal (α, S) and every int-soft left ideal (β, S) over U.*

Proof. For the necessity, see [14]. For the sufficiency, assume that (2) is valid. Let A and B be any right ideal and any left ideal of S, respectively. Then obviously $AB \subseteq A \cap B$, and the (Φ, Ψ)-characteristic soft sets $\left(\chi_A^{(\Phi,\Psi)}, S\right)$ and $\left(\chi_B^{(\Phi,\Psi)}, S\right)$ over U are an int-soft right ideal and an int-soft left ideal, respectively, over U by Lemma 3.16. Let $x \in A \cap B$. Then

$$\chi_{AB}^{(\Phi,\Psi)}(x) = (\chi_A^{(\Phi,\Psi)} \tilde{\circ}\, \chi_B^{(\Phi,\Psi)})(x) = (\chi_A^{(\Phi,\Psi)} \tilde{\cap}\, \chi_B^{(\Phi,\Psi)})(x) = \chi_{A\cap B}^{(\Phi,\Psi)}(x) = \Phi$$

and so $x \in AB$ which shows that $A \cap B \subseteq AB$. Hence $A \cap B = AB$, and therefore S is regular. □

Theorem 3.19. *For a regular semigroup S, the following conditions are equivalent.*

(1) *S is left duo.*

(2) *S is int-soft left duo.*

Proof. (1) ⇒ (2). Let (α, S) be an int-soft left ideal over U and let $x, y \in S$. Note that the left ideal Sx is a two-sided ideal of S. It follows from the regularity of S that

$$xy \in (xSx)y \subseteq (Sx)S \subseteq Sx.$$

Thus $xy = ax$ for some $a \in S$. Since (α, S) is an int-soft left ideal over U, we have

$$\alpha(xy) = \alpha(ax) \supseteq \alpha(x).$$

Hence (α, S) is an int-soft right ideal over U and so (α, S) is an int-soft two-sided ideal over U. Therefore S is int-soft left duo.

(2) ⇒ (1). Let A be a left ideal of S. Then the (Φ, Ψ)-characteristic soft set $\left(\chi_A^{(\Phi,\Psi)}, S\right)$ over U is an int-soft left ideal over U by Lemma 3.16. It follows from the assumption that $\left(\chi_A^{(\Phi,\Psi)}, S\right)$ is an int-soft two-sided ideal over U. Therefore A is a two-sided ideal of S by Lemma 3.16. □

Similarly, we have the following theorem.

Theorem 3.20. *For a regular semigroup S, the following conditions are equivalent.*

(1) *S is right duo.*

(2) *S is int-soft right duo.*

Corollary 3.21. *A regular semigroup is duo if and only if it is int-soft duo.*

Theorem 3.22. *For any nonempty subset A of S, the following are equivalent.*

(1) *A is a bi-ideal of S.*

(2) *The* (Φ, Ψ)*-characteristic soft set* $\left(\chi_A^{(\Phi,\Psi)}, S\right)$ *over* U *is an int-soft bi-ideal over* U *for any* $\Phi, \Psi \in \mathscr{P}(U)$ *with* $\Phi \supsetneq \Psi$.

Proof. Assume that A is a bi-ideal of S. Let $\Phi, \Psi \in \mathscr{P}(U)$ with $\Phi \supsetneq \Psi$ and $x, y, z \in S$. If $x, z \in A$, then $\chi_A^{(\Phi,\Psi)}(x) = \Phi = \chi_A^{(\Phi,\Psi)}(z)$, $xz \in AA \subseteq A$ and $xyz \in ASA \subseteq A$. Hence

$$\chi_A^{(\Phi,\Psi)}(xz) = \Phi = \chi_A^{(\Phi,\Psi)}(x) \cap \chi_A^{(\Phi,\Psi)}(z) \tag{3.8}$$

and

$$\chi_A^{(\Phi,\Psi)}(xyz) = \Phi = \chi_A^{(\Phi,\Psi)}(x) \cap \chi_A^{(\Phi,\Psi)}(z). \tag{3.9}$$

If $x \notin A$ or $z \notin A$, then $\chi_A^{(\Phi,\Psi)}(x) = \Psi$ or $\chi_A^{(\Phi,\Psi)}(z) = \Psi$. Hence

$$\chi_A^{(\Phi,\Psi)}(xz) \supseteq \Psi = \chi_A^{(\Phi,\Psi)}(x) \cap \chi_A^{(\Phi,\Psi)}(z) \tag{3.10}$$

and

$$\chi_A^{(\Phi,\Psi)}(xyz) \supseteq \Psi = \chi_A^{(\Phi,\Psi)}(x) \cap \chi_A^{(\Phi,\Psi)}(z). \tag{3.11}$$

Therefore $(\chi_A^{(\Phi,\Psi)}, S)$ is an int-soft bi-ideal over U for any $\Phi, \Psi \in \mathscr{P}(U)$ with $\Phi \supsetneq \Psi$.

Conversely, suppose that the (Φ, Ψ)-characteristic soft set $\left(\chi_A^{(\Phi,\Psi)}, S\right)$ over U is an int-soft bi-ideal over U for any $\Phi, \Psi \in \mathscr{P}(U)$ with $\Phi \supsetneq \Psi$. Let b and a be any elements of AA and ASA, respectively. Then $b = xz$ and $a = xyz$ for some $x, z \in A$ and $y \in S$. Hence

$$\chi_A^{(\Phi,\Psi)}(b) = \chi_A^{(\Phi,\Psi)}(xz) \supseteq \chi_A^{(\Phi,\Psi)}(x) \cap \chi_A^{(\Phi,\Psi)}(z) = \Phi \cap \Phi = \Phi, \tag{3.12}$$

and

$$\chi_A^{(\Phi,\Psi)}(a) = \chi_A^{(\Phi,\Psi)}(xyz) \supseteq \chi_A^{(\Phi,\Psi)}(x) \cap \chi_A^{(\Phi,\Psi)}(z) = \Phi \cap \Phi = \Phi. \tag{3.13}$$

Thus $\chi_A^{(\Phi,\Psi)}(b) = \Phi$ and $\chi_A^{(\Phi,\Psi)}(a) = \Phi$. Hence $b, a \in A$, which shows that $AA \subseteq A$ and $ASA \subseteq A$. Therefore A is a bi-ideal of S. □

Similarly, we have the following theorems.

Theorem 3.23. *For any nonempty subset* A *of* S*, the following are equivalent.*

(1) A *is a quasi-ideal of* S.

(2) *The* (Φ, Ψ)*-characteristic soft set* $\left(\chi_A^{(\Phi,\Psi)}, S\right)$ *over* U *is an int-soft quasi-ideal over* U *for any* $\Phi, \Psi \in \mathscr{P}(U)$ *with* $\Phi \supsetneq \Psi$.

Theorem 3.24. *For any nonempty subset* A *of* S*, the following are equivalent.*

(1) A *is an interior ideal of* S.

(2) *The* (Φ, Ψ)*-characteristic soft set* $\left(\chi_A^{(\Phi,\Psi)}, S\right)$ *over* U *is an int-soft interior ideal over* U *for any* $\Phi, \Psi \in \mathscr{P}(U)$ *with* $\Phi \supsetneq \Psi$.

Theorem 3.25. *For a regular semigroup* S*, the following conditions are equivalent.*

(1) *Every bi-ideal of* S *is a right ideal of* S.

(2) *Every int-soft bi-ideal over* U *is an int-soft right ideal over* U.

Proof. (1) $\Rightarrow$ (2). Let (α, S) be an int-soft bi-ideal over U and let $x, y \in S$. Note that the set xSx is a bi-ideal of S, and so it is a right ideal of S by assumption. The regularity of S implies that

$$xy \in (xSx)S \subseteq xSx,$$

and so there exists $a \in S$ such that $xy = xax$. It follows from (3.2) that

$$\alpha(xy) = \alpha(xax) \supseteq \alpha(x) \cap \alpha(x) = \alpha(x)$$

and so that (α, S) is an int-soft right ideal over U.

(2) $\Rightarrow$ (1). Let A be a bi-ideal of S. Then the (Φ, Ψ)-characteristic soft set $\left(\chi_A^{(\Phi,\Psi)}, S\right)$ is an int-soft bi-ideal over U by Theorem 3.22, and so it is an int-soft right ideal over U by assumption. It follows from Lemma 3.16 that A is a right ideal of S. □

Similarly, we get the following theorem,

Theorem 3.26. *For a regular semigroup* S*, the following conditions are equivalent.*

(1) *Every bi-ideal of* S *is a left ideal of* S.

(2) *Every int-soft bi-ideal over* U *is an int-soft left ideal over* U.

For any two int-soft sets (α, S) and (β, S) over U, we consider the following identity.

$$(\alpha \tilde{\cap} \beta, S) = (\alpha \,\tilde{\circ}\, \beta \,\tilde{\circ}\, \alpha, S)\,. \tag{3.14}$$

Theorem 3.27. *Let S be a regular semigroup. If (α, S) and (β, S) are an int-soft generalized bi-ideal and an int-soft interior ideal, respectively, over U, then the equality (3.14) is valid.*

Proof. Let (α, S) and (β, S) be any int-soft generalized bi-ideal and any int-soft interior ideal, respectively, over U. Then

$$(\alpha \,\tilde{\circ}\, \beta \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\alpha \,\tilde{\circ}\, \chi_S \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\alpha, S)$$

and

$$(\alpha \,\tilde{\circ}\, \beta \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\chi_S \,\tilde{\circ}\, \beta \,\tilde{\circ}\, \chi_S, S) \,\tilde{\subseteq}\, (\beta, S).$$

Thus $(\alpha \,\tilde{\circ}\, \beta \,\tilde{\circ}\, \alpha, S) \,\tilde{\subseteq}\, (\alpha \,\tilde{\cap}\, \beta, S)$. Let $x \in S$. Then there exists $a \in S$ such that $x = xax$ $(= xaxax)$ by the regularity of S. Since (β, S) is an int-soft interior ideal over U, we get

$$\begin{aligned}
(\alpha \,\tilde{\circ}\, \beta \,\tilde{\circ}\, \alpha)(x) &= \bigcup_{x=yz} \{\alpha(y) \cap (\beta \,\tilde{\circ}\, \alpha)(z)\} \\
&\supseteq \alpha(x) \cap (\beta \,\tilde{\circ}\, \alpha)(axax) \\
&= \alpha(x) \cap \left(\bigcup_{axax=pq} \{\beta(p) \cap \alpha(q)\} \right) \\
&\supseteq \alpha(x) \cap (\beta(axa) \cap \alpha(x)) \\
&\supseteq \alpha(x) \cap \beta(x) \\
&= (\alpha \,\tilde{\cap}\, \beta)(x)
\end{aligned}$$

and so $(\alpha \,\tilde{\cap}\, \beta, S) \,\tilde{\subseteq}\, (\alpha \,\tilde{\circ}\, \beta \,\tilde{\circ}\, \alpha, S)$. Therefore $(\alpha \,\tilde{\cap}\, \beta, S) = (\alpha \,\tilde{\circ}\, \beta \,\tilde{\circ}\, \alpha, S)$. □

Corollary 3.28. *Let S be a regular semigroup. If (α, S) and (β, S) are an int-soft bi-ideal and an int-soft interior ideal, respectively, over U, then the equality (3.14) is valid.*

Corollary 3.29. *Let S be a regular semigroup. If (α, S) and (β, S) are an int-soft quasi-ideal and an int-soft interior ideal, respectively, over U, then the equality (3.14) is valid.*

Lemma 3.30 ([14]). *For a semigroup S, the following are equivalent.*

(1) *S is regular.*

(2) *$(\alpha, S) = (\alpha \,\tilde{\circ}\, \chi_S \,\tilde{\circ}\, \alpha, S)$ for every int-soft quasi-ideal (α, S) over U.*

Theorem 3.31. *In a semigroup S, if the equality (3.14) is valid for every int-soft quasi-ideal α and an int-soft two-sided ideal β over U, then S is regular.*

Proof. Note that χ_S is an int-soft two-sided ideal over U. Hence

$$(\alpha, S) = (\alpha \tilde{\cap} \chi_S, S) = (\alpha \tilde{\circ} \chi_S \tilde{\circ} \alpha, S).$$

It follows from Lemma 3.30 that S is regular. □

Theorem 3.32. *If S is a regular semigroup, then $(\alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\alpha \tilde{\circ} \beta, S)$ for all int-soft generalized bi-ideal (α, S) and int-soft left ideal (β, S) over U.*

Proof. Let (α, S) and (β, S) be any int-soft generalized bi-ideal and any int-soft left ideal over U, respectively. For any $x \in S$ there exists $a \in S$ such that $x = xax$ since S is regular. Hence

$$\begin{aligned}(\alpha \tilde{\circ} \beta)(x) &= \bigcup_{x=yz} \{\alpha(y) \cap \beta(z)\} \\ &\supseteq \alpha(x) \cap \beta(ax) \\ &\supseteq \alpha(x) \cap \beta(x) \\ &= (\alpha \tilde{\cap} \beta)(x)\end{aligned}$$

and so $(\alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\alpha \tilde{\circ} \beta, S)$. □

Corollary 3.33. *If S is a regular semigroup, then $(\alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\alpha \tilde{\circ} \beta, S)$ for all int-soft bi-ideal (α, S) and int-soft left ideal (β, S) over U.*

Corollary 3.34. *If S is a regular semigroup, then $(\alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\alpha \tilde{\circ} \beta, S)$ for all int-soft quasi-ideal (α, S) and int-soft left ideal (β, S) over U.*

Lemma 3.35 ([14]). *If (α, S) is an int-soft right ideal over U and (β, S) is an int-soft left ideal over U, then $(\alpha \tilde{\circ} \beta, S) \tilde{\subseteq} (\alpha \tilde{\cap} \beta, S)$.*

Theorem 3.36. *In a semigroup S, if $(\alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\alpha \tilde{\circ} \beta, S)$ for every int-soft quasi-ideal (α, S) and an int-soft left ideal (β, S) over U, then S is regular.*

Proof. Since every int-soft right ideal is an int-soft quasi-ideal, it follows that

$$(\alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\alpha \tilde{\circ} \beta, S)$$

for every int-soft right ideal (α, S) and every int-soft left ideal (β, S) over U. Obviously,

$$(\alpha \tilde{\circ} \beta, S) \tilde{\subseteq} (\alpha \tilde{\cap} \beta, S),$$

and thus $(\alpha \tilde{\circ} \beta, S) = (\alpha \tilde{\cap} \beta, S)$ for every int-soft right ideal (α, S) and every int-soft left ideal (β, S) over U. Therefore S is regular by Theorem 3.18. □

Theorem 3.37. *If S is a regular semigroup, then $(\gamma \tilde{\cap} \alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\gamma \tilde{\circ} \alpha \tilde{\circ} \beta, S)$ for every int-soft right ideal (γ, S), every int-soft generalized bi-ideal (α, S) and every int-soft left ideal (β, S) over U.*

Proof. Let (γ, S), (α, S) and (β, S) be any int-soft right ideal, any int-soft generalized bi-ideal and any int-soft left ideal, respectively, over U. Let $x \in S$. Then there exists $a \in S$ such that $x = xax$ since S is regular. Hence

$$\begin{aligned}
(\gamma \tilde{\circ} \alpha \tilde{\circ} \beta)(x) &= \bigcup_{x=yz} \{\gamma(y) \cap (\alpha \tilde{\circ} \beta)(z)\} \\
&\supseteq \gamma(xa) \cap (\alpha \tilde{\circ} \beta)(x) \\
&= \gamma(x) \cap \left(\bigcup_{x=pq} \{\alpha(p) \cap \beta(q)\} \right) \\
&\supseteq \gamma(x) \cap (\alpha(x) \cap \beta(ax)) \\
&\supseteq \gamma(x) \cap (\alpha(x) \cap \beta(x)) \\
&= (\gamma \tilde{\cap} \alpha \tilde{\cap} \beta)(x),
\end{aligned}$$

and so $(\gamma \tilde{\cap} \alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\gamma \tilde{\circ} \alpha \tilde{\circ} \beta, S)$. □

Corollary 3.38. *If S is a regular semigroup, then $(\gamma \tilde{\cap} \alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\gamma \tilde{\circ} \alpha \tilde{\circ} \beta, S)$ for every int-soft right ideal (γ, S), every int-soft bi-ideal (α, S) and every int-soft left ideal (β, S) over U.*

Corollary 3.39. *If S is a regular semigroup, then $(\gamma \tilde{\cap} \alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\gamma \tilde{\circ} \alpha \tilde{\circ} \beta, S)$ for every int-soft right ideal (γ, S), every int-soft quasi-ideal (α, S) and every int-soft left ideal (β, S) over U.*

Theorem 3.40. *Let (γ, S), (α, S) and (β, S) be soft sets over U in a semigroup S such that*

$$(\gamma \tilde{\cap} \alpha \tilde{\cap} \beta, S) \tilde{\subseteq} (\gamma \tilde{\circ} \alpha \tilde{\circ} \beta, S).$$

If (γ, S) is an int-soft right ideal, (α, S) is an int-soft quasi-ideal and (β, S) is an int-soft left ideal over U, then S is regular.

Proof. Since χ_S is an int-soft quasi-ideal over U, we have

$$(\gamma \tilde{\cap} \beta, S) = (\gamma \tilde{\cap} \chi_S \tilde{\cap} \beta, S) \tilde{\subseteq} (\gamma \tilde{\circ} \chi_S \tilde{\circ} \beta, S) \tilde{\subseteq} (\gamma \tilde{\circ} \beta, S).$$

Clearly, $(\gamma \tilde{\circ} \beta, S) \tilde{\subseteq} (\gamma \tilde{\cap} \beta, S)$. Hence $(\gamma \tilde{\circ} \beta, S) = (\gamma \tilde{\cap} \beta, S)$, and therefore S is regular by Theorem 3.18. □

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Generalizations of starshaped $(\in, \in\vee q)$-fuzzy sets

Abstract The notions of starshaped $(\in, \in \vee q_k)$-fuzzy sets and quasi-starshaped $(\in, \in \vee q_k)$-fuzzy sets are introduced, and related properties are investigated. Characterizations of starshaped $(\in, \in \vee q_k)$-fuzzy sets and quasi-starshaped $(\in, \in \vee q_)$-fuzzy sets are discussed. Relations between starshaped $(\in, \in \vee q_k)$-fuzzy sets and quasi-starshaped $(\in, \in \vee q_k)$-fuzzy sets are investigated.

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1 Introduction

The concept of starshaped fuzzy sets, which are a generalization of convex sets, is introduced by Brown [1], and Diamond defined another type of starshaped fuzzy sets and established some of the basic properties of this family of fuzzy sets in [2]. Brown's starshaped fuzzy sets was called quasi-starshaped fuzzy sets, and its properties are provided in the paper [6]. As a generalization of starshaped fuzzy sets and quasi-starshaped fuzzy sets, Jun et al. [4] used the notion of fuzzy points, and discussed starshaped $(\in, \in \vee q)$-fuzzy sets and quasi-starshaped $(\in, \in \vee q)$-fuzzy sets.

In this paper, we consider more general form than Jun and Song's consideration in the paper [4]. We introduce the concepts of starshaped $(\in, \in \vee q_k)$-fuzzy sets and quasi-starshaped $(\in, \in \vee q_k)$-fuzzy sets, and investigate related properties. We provide characterizations of starshaped $(\in, \in \vee q_k)$-fuzzy sets and quasi-starshaped $(\in, \in \vee q_k)$-fuzzy sets. We provide a condition for a fuzzy set to be a starshaped $(\in, \in \vee q_k)$-fuzzy set. We discuss relations between starshaped $(\in, \in \vee q_k)$-fuzzy sets and quasi-starshaped $(\in, \in \vee q_k)$-fuzzy sets.

2 Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. For $x, y \in \mathbb{R}^n$, the line segment $\overline{xy}$ joining x and y is the set of all points of the form $\alpha x + \beta y$ where $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = 1$. A set $S \subseteq \mathbb{R}^n$ is said to be *starshaped* related to a point $x \in \mathbb{R}^n$ if $\overline{xy} \subseteq S$ for each point $y \in S$. A set $S \subseteq \mathbb{R}^n$ is simply said to be *starshaped* if there exists a point x in $\mathbb{R}^n$ such that S is starshaped relative to it. Note that a star-shaped set is not necessarily convex in the ordinary sense.

A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is called a *starshaped fuzzy set* relative to $y \in \mathbb{R}^n$ (see [6, 7]) if it satisfies:

$$(\forall x \in \mathbb{R}^n)(\forall \delta \in [0,1])\left(\mathcal{A}(\delta(x-y)+y) \geq \mathcal{A}(x)\right). \tag{2.1}$$

A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is called a *quasi-starshaped fuzzy set* relative to $y \in \mathbb{R}^n$ (see [1, 6]) if it satisfies:

$$(\forall x \in \mathbb{R}^n)(\forall \delta \in [0,1])\left(\mathcal{A}(\delta x+(1-\delta)y) \geq \min\{\mathcal{A}(x), \mathcal{A}(y)\}\right). \tag{2.2}$$

A fuzzy set $\mathcal{A}$ in a set X of the form

$$\mathcal{A}(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy set $\mathcal{A}$ in a set X, a fuzzy point x_t is said to

- *contained* in $\mathcal{A}$, denoted by $x_t \in \mathcal{A}$ (see [5]), if $\mathcal{A}(x) \geq t$.
- *be quasi-coincident with* $\mathcal{A}$, denoted by $x_t \, q \, \mathcal{A}$ (see [5]), if $\mathcal{A}(x) + t > 1$.

For a fuzzy point x_t and a fuzzy set $\mathcal{A}$ in a set X, we say that

- $x_t \in\!\vee\, q \, \mathcal{A}$ if $x_t \in \mathcal{A}$ or $x_t \, q \, \mathcal{A}$.

Jun [3] considered the general form of the symbol $x_t \, q \, \mathcal{A}$ as follows: For an arbitrary element k of $[0, 1)$, we say that

- $x_t \, q_k \, \mathcal{A}$ if $\mathcal{A}(x) + t + k > 1$.
- $x_t \in\!\vee\, q_k \, \mathcal{A}$ if $x_t \in \mathcal{A}$ or $x_t \, q_k \, \mathcal{A}$.

3 Starshaped $(\in, \in\vee q_k)$-fuzzy sets

In what follows, let $\mathscr{F}(\mathbb{R}^n)$ and k denote the class of fuzzy sets on $\mathbb{R}^n$ and an arbitrary element of $[0,1)$, respectively, unless otherwise specified.

Definition 3.1 ([4]). A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is called a *starshaped* $(\in, \in\vee q)$*-fuzzy set* relative to $y \in \mathbb{R}^n$ if

$$x_t \in \mathcal{A} \;\Rightarrow\; (\delta(x-y)+y)_t \in\vee\, q\, \mathcal{A} \tag{3.1}$$

for all $x \in \mathbb{R}^n$, $\delta \in [0,1]$ and $t \in (0,1]$.

Definition 3.2. A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is called a *starshaped* $(\in, \in\vee q_k)$*-fuzzy set* relative to $y \in \mathbb{R}^n$ if

$$x_t \in \mathcal{A} \;\Rightarrow\; (\delta(x-y)+y)_t \in\vee\, q_k\, \mathcal{A} \tag{3.2}$$

for all $x \in \mathbb{R}^n$, $\delta \in [0,1]$ and $t \in (0,1]$.

Note that a starshaped $(\in, \in\vee q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$ with $k = 0$ is a starshaped $(\in, \in\vee q)$-fuzzy set relative to $y \in \mathbb{R}^n$.

Example 3.3. The fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R})$ given by

$$\mathcal{A} : \mathbb{R} \to [0,1],\; x \mapsto \begin{cases} 1.25 + x & \text{if } x \in (-1.5, -0.5], \\ 0.25 - x & \text{if } x \in (-0.5, 0], \\ 0.25 + x & \text{if } x \in (0, 0.5], \\ 1.25 - x & \text{if } x \in (0.5, 1.5), \\ 0 & \text{otherwise}, \end{cases}$$

is a starshaped $(\in, \in\vee q_k)$-fuzzy set relative to $y = 0$ with $k = 0.6$.

Obviously, every starshaped $(\in, \in\vee q)$-fuzzy set relative to $y \in \mathbb{R}^n$ is a starshaped $(\in, \in\vee q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$, but the converse is not true. In fact, the starshaped $(\in, \in\vee q_k)$-fuzzy set $\mathcal{A}$ relative to $y = 0$ with $k = 0.6$ in Example 3.3 is not a starshaped $(\in, \in\vee q)$-fuzzy set relative to $y = 0$ since if we take $x = 0.12$, $\delta = 0.9$ and $t = 0.3$, then $x_t \in \mathcal{A}$ and $(\delta x)_t \in \mathcal{A}$, but $(\delta x)_t\, \overline{q}\, \mathcal{A}$. Hence $(\delta x)_t \overline{\in\vee q} \mathcal{A}$.

We provide a condition for a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ to be a starshaped $(\in, \in\vee q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$.

Theorem 3.4. *Given $y \in \mathbb{R}^n$, if a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ satisfies the condition*

$$x_t \in\vee\, q_k\, \mathcal{A} \;\Rightarrow\; (\delta(x-y)+y)_t \in\vee\, q_k\, \mathcal{A} \tag{3.3}$$

for all $x \in \mathbb{R}^n$, $\delta \in [0,1]$ and $t \in (0,1]$, then $\mathcal{A}$ is a starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$.

Proof. Straightforward. □

Corollary 3.5 ([4]). *Given $y \in \mathbb{R}^n$, if a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ satisfies the condition*

$$x_t \in\vee\, q\, \mathcal{A} \;\Rightarrow\; (\delta(x-y)+y)_t \in\vee\, q\, \mathcal{A} \tag{3.4}$$

for all $x \in \mathbb{R}^n$, $\delta \in [0,1]$ and $t \in (0,1]$, then $\mathcal{A}$ is a starshaped $(\in, \in\vee\, q)$-fuzzy set relative to $y \in \mathbb{R}^n$.

We consider characterizations of a starshaped $(\in, \in\vee\, q)$-fuzzy set.

Theorem 3.6. *For a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$, the following are equivalent:*

(1) *$\mathcal{A}$ is a starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$.*

(2) *$\mathcal{A}$ satisfies:*

$$(\forall x \in \mathbb{R}^n)(\forall \delta \in [0,1])\left(\mathcal{A}(\delta(x-y)+y) \geq \min\{\mathcal{A}(x), \tfrac{1-k}{2}\}\right) \tag{3.5}$$

Proof. Assume that $\mathcal{A}$ is a starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $\delta \in [0,1]$. If $\mathcal{A}(x) \geq \frac{1-k}{2}$, then $x_{\frac{1-k}{2}} \in \mathcal{A}$ and so $(\delta(x-y)+y)_{\frac{1-k}{2}} \in\vee\, q_k\, \mathcal{A}$ by (3.2), that is, $\mathcal{A}(\delta(x-y)+y) \geq \frac{1-k}{2}$ or $\mathcal{A}(\delta(x-y)+y) + \frac{1-k}{2} + k > 1$. Thus $\mathcal{A}(\delta(x-y)+y) \geq \frac{1-k}{2}$ since $\mathcal{A}(\delta(x-y)+y) < \frac{1-k}{2}$ induces a contradiction. Consequently, $\mathcal{A}(\delta(x-y)+y) \geq \min\{\mathcal{A}(x), \frac{1-k}{2}\}$ for all $x \in \mathbb{R}^n$ and $\delta \in [0,1]$. Suppose that $\mathcal{A}(x) < \frac{1-k}{2}$. If $\mathcal{A}(\delta(x-y)+y) < \mathcal{A}(x)$, then $\mathcal{A}(\delta(x-y)+y) < t \leq \mathcal{A}(x)$ for some $t \in (0, \frac{1-k}{2})$ and so $x_t \in \mathcal{A}$ but $(\delta(x-y)+y)_t \,\overline{\in}\, \mathcal{A}$. Since $\mathcal{A}(\delta(x-y)+y)+t+k < 1$, we have $(\delta(x-y)+y)_t\, \overline{q_k}\, \mathcal{A}$. Hence $(\delta(x-y)+y)_t\, \overline{\in\vee\, q_k}\, \mathcal{A}$, a contradiction. Thus $\mathcal{A}(\delta(x-y)+y) \geq \mathcal{A}(x)$ and consequently $\mathcal{A}(\delta(x-y)+y) \geq \min\{\mathcal{A}(x), \frac{1-k}{2}\}$ for all $x \in \mathbb{R}^n$ and $\delta \in [0,1]$.

Conversely, assume that a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ satisfies the condition (3.5). Let $x \in \mathbb{R}^n$, $\delta \in [0,1]$ and $t \in (0,1]$ be such that $x_t \in \mathcal{A}$. Then $\mathcal{A}(x) \geq t$. Suppose that $\mathcal{A}(\delta(x-y)+y) < t$. If $\mathcal{A}(x) < \frac{1-k}{2}$, then $\mathcal{A}(\delta(x-y)+y) \geq \min\{\mathcal{A}(x), \frac{1-k}{2}\} = \mathcal{A}(x) \geq t$, a contradiction. Hence $\mathcal{A}(x) \geq \frac{1-k}{2}$, and so

$$\mathcal{A}(\delta(x-y)+y)+t+k > 2\mathcal{A}(\delta(x-y)+y)+k \geq 2\min\{\mathcal{A}(x), \tfrac{1-k}{2}\}+k = 1.$$

Thus $(\delta(x-y)+y)_t \in\vee\, q_k\, \mathcal{A}$. Therefore $\mathcal{A}$ is a starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$. $\square$

Corollary 3.7 ([4]). *A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is a starshaped $(\in, \in\vee\, q)$-fuzzy set relative to $y \in \mathbb{R}^n$ if and only if*

$$(\forall x \in \mathbb{R}^n)(\forall \delta \in [0,1])\,(\mathcal{A}(\delta(x-y)+y) \geq \min\{\mathcal{A}(x), 0.5\}) \tag{3.6}$$

Using Theorem 3.6, we know that if $k < r$ in $[0,1)$, then every starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$ is a starshaped $(\in, \in\vee\, q_r)$-fuzzy set relative to $y \in \mathbb{R}^n$. But the converse is not true. In fact, the starshaped $(\in, \in\vee\, q_{0.6})$-fuzzy set relative to $y = 0$ in Example 3.3 is not a starshaped $(\in, \in\vee\, q_{0.4})$-fuzzy set relative to $y = 0$.

Theorem 3.8. *For a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$, the following are equivalent:*

(1) *$\mathcal{A}$ is a starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$.*

(2) *The nonempty t-level set $U(\mathcal{A}; t)$ of $\mathcal{A}$ is starshaped relative to $y \in \mathbb{R}^n$ for all $t \in (0, \frac{1-k}{2}]$.*

Proof. Assume that $\mathcal{A}$ is a starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$ and let $t \in (0, \frac{1-k}{2}]$ be such that $U(\mathcal{A}; t) \neq \emptyset$. Let $x \in U(\mathcal{A}; t)$. Then $x_t \in \mathcal{A}$, and so

$$\mathcal{A}(\delta(x-y)+y) \geq \min\{\mathcal{A}(x), \tfrac{1-k}{2}\} \geq \min\{t, \tfrac{1-k}{2}\} = t$$

by Theorem 3.6. Hence $\overline{xy} \subseteq U(\mathcal{A}; t)$ for $t \in (0, \frac{1-k}{2}]$. Therefore $U(\mathcal{A}; t)$ is starshaped relative to $y \in \mathbb{R}^n$ for all $t \in (0, \frac{1-k}{2}]$.

Conversely, suppose that the nonempty t-level set $U(\mathcal{A}; t)$ is starshaped relative to $y \in \mathbb{R}^n$ for all $t \in (0, \frac{1-k}{2}]$. For $\delta \in [0,1]$ and $x \in \mathbb{R}^n$, let $\mathcal{A}(x) = t_x$. Then $\overline{xy} \subseteq U(\mathcal{A}; t_x)$, and so

$$\mathcal{A}(\delta(x-y)+y) \geq t_x = \mathcal{A}(x) \geq \min\{\mathcal{A}(x), \tfrac{1-k}{2}\}.$$

It follows from Theorem 3.6 that $\mathcal{A}$ is a starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$. $\square$

Corollary 3.9 ([4]). *A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is a starshaped $(\in, \in\vee\, q)$-fuzzy set relative to $y \in \mathbb{R}^n$ if and only if its nonempty t-level set $U(\mathcal{A}; t)$ is starshaped relative to $y \in \mathbb{R}^n$ for all $t \in (0, 0.5]$.*

Theorem 3.10. *Given a starshaped fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$, the following are equivalent:*

(1) *The nonempty t-level set $U(\mathcal{A};t)$ is a starshaped subset of $\mathbb{R}^n$ relative to $y \in \mathbb{R}^n$ for all $t \in (\frac{1-k}{2}, 1]$.*

(2) *$\mathcal{A}$ satisfies the following condition.*

$$\mathcal{A}(x) \leq \max\{\mathcal{A}(\delta(x-y)+y), \tfrac{1-k}{2}\} \tag{3.7}$$

for all $x \in \mathbb{R}^n$ and $\delta \in [0,1]$.

Proof. Assume that the nonempty t-level set $U(\mathcal{A};t)$ is starshaped relative to $y \in \mathbb{R}^n$ for all $t \in (\frac{1-k}{2}, 1]$. If the condition (3.7) is false, then there exists $a \in \mathbb{R}^n$ such that

$$\mathcal{A}(a) > \max\{\mathcal{A}(\delta(a-y)+y), \tfrac{1-k}{2}\}.$$

Hence $t_a := \mathcal{A}(a) \in (\frac{1-k}{2}, 1]$ and $a \in U(\mathcal{A};t_a)$. But $\mathcal{A}(\delta(a-y)+y) < t_a$ implies that $\overline{ay} \nsubseteq U(\mathcal{A};t_a)$, that is, $U(\mathcal{A};t_a)$ is not a starshaped subset of $\mathbb{R}^n$ relative to $a \in \mathbb{R}^n$. This is a contradiction, and so the condition (3.7) is valid.

Conversely, suppose that $\mathcal{A}$ satisfies the condition (3.7). For any $\delta \in [0,1]$ and $t \in (\frac{1-k}{2}, 1]$, let $x \in U(\mathcal{A};t)$. Using the condition (3.7), we have

$$\max\{\mathcal{A}(\delta(x-y)+y), \tfrac{1-k}{2}\} \geq \mathcal{A}(x) \geq t > \tfrac{1-k}{2}.$$

Thus $\mathcal{A}(\delta(x-y)+y) \geq t$, and hence $\delta(x-y)+y \in U(\mathcal{A};t)$, that is, $\overline{xy} \subseteq U(\mathcal{A};t)$. Therefore the nonempty t-level set $U(\mathcal{A};t)$ is a starshaped subset of $\mathbb{R}^n$ relative to $y \in \mathbb{R}^n$ for all $t \in (\frac{1-k}{2}, 1]$. □

Corollary 3.11 ([4]). *For a starshaped fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$, the nonempty t-level set $U(\mathcal{A};t)$ is a starshaped subset of $\mathbb{R}^n$ relative to $y \in \mathbb{R}^n$ for all $t \in (0.5, 1]$ if and only if $\mathcal{A}$ satisfies the following condition.*

$$\mathcal{A}(x) \leq \max\{\mathcal{A}(\delta(x-y)+y), 0.5\} \tag{3.8}$$

for all $x \in \mathbb{R}^n$ and $\delta \in [0,1]$.

Combining Theorems 3.8 and 3.10, we have a corollary.

Corollary 3.12. *For a starshaped fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$, the nonempty t-level set $U(\mathcal{A};t)$ is a starshaped subset of $\mathbb{R}^n$ relative to $y \in \mathbb{R}^n$ for all $t \in (0,1]$ if and only if $\mathcal{A}$ satisfies two conditions (3.1) and (3.7).*

Theorem 3.13. *Given a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$, the following are equivalent:*

(1) $\mathcal{A}$ *is a starshaped* $(\in, \in\vee q_k)$*-fuzzy set relative to* $y \in \mathbb{R}^n$.

(2) $\mathcal{A}$ *satisfies:*

$$(x+y)_t \in \mathcal{A} \;\Rightarrow\; (\delta x + y)_t \in\vee\, q_k\, \mathcal{A} \tag{3.9}$$

for all $x \in \mathbb{R}^n$, $\delta \in [0,1]$ *and* $t \in (0,1]$.

Proof. Suppose that $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is a starshaped $(\in, \in\vee q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$. Let $\delta \in [0,1]$, $t \in (0,1]$ and $(x+y)_t \in \mathcal{A}$ for every $x \in \mathbb{R}^n$. Then $\mathcal{A}(x+y) \geq t$. Replacing x by $x+y$ in (3.5), we have

$$\begin{aligned}\mathcal{A}(\delta x + y) &= \mathcal{A}(\delta((x+y)-y)+y)\\ &\geq \min\{\mathcal{A}(x+y), \tfrac{1-k}{2}\}\\ &\geq \min\{t, \tfrac{1-k}{2}\}.\end{aligned}$$

If $t \leq \frac{1-k}{2}$, then $\mathcal{A}(\delta x + y) \geq t$ and so $(\delta x + y)_t \in \mathcal{A}$. If $t > \frac{1-k}{2}$, then

$$\mathcal{A}(\delta x + y) + t + k > \tfrac{1-k}{2} + t + k > \tfrac{1-k}{2} + \tfrac{1-k}{2} + k = 1$$

and so $(\delta x + y)_t\, q_k\, \mathcal{A}$. Hence $(\delta x + y)_t \in\vee\, q_k\, \mathcal{A}$.

Conversely, suppose that $\mathcal{A}$ satisfies the condition (3.9). We first show that

$$\mathcal{A}(\delta x + y) \geq \min\{\mathcal{A}(x+y), \tfrac{1-k}{2}\}. \tag{3.10}$$

Assume that $\mathcal{A}(x+y) < \frac{1-k}{2}$. If $\mathcal{A}(\delta x + y) < \mathcal{A}(x+y)$, then $\mathcal{A}(\delta x + y) < t \leq \mathcal{A}(x+y)$ for some $t \in (0, \frac{1-k}{2})$. Hence $(x+y)_t \in \mathcal{A}$ and $(\delta x + y)_t\, \overline{\in}\, \mathcal{A}$. Also, since

$$\mathcal{A}(\delta x + y) + t + k < 2t + k < 1,$$

we get $(\delta x + y)_t\, \overline{q_k}\, \mathcal{A}$. Thus $(\delta x + y)_t\, \overline{\in\vee\, q_k}\, \mathcal{A}$, a contradiction. Hence $\mathcal{A}(\delta x + y) \geq \mathcal{A}(x+y)$. Now, suppose that $\mathcal{A}(x+y) \geq \frac{1-k}{2}$. Then $(x+y)_{\frac{1-k}{2}} \in \mathcal{A}$ and so $(\delta x + y)_{\frac{1-k}{2}} \in\vee\, q_k\, \mathcal{A}$ by (3.9). If $\mathcal{A}(\delta x + y) < \frac{1-k}{2}$, then $(\delta x + y)_{\frac{1-k}{2}}\, \overline{\in}\, \mathcal{A}$ and $\mathcal{A}(\delta x + y) + \frac{1-k}{2} + k < 1$, that is, $(\delta x + y)_{\frac{1-k}{2}}\, \overline{q_k}\, \mathcal{A}$. This is a contradiction, and so $\mathcal{A}(\delta x + y) \geq \frac{1-k}{2}$. Therefore $\mathcal{A}(\delta x + y) \geq \min\{\mathcal{A}(x+y), \frac{1-k}{2}\}$, Now if we replace $x+y$ by x in (3.10), then

$$\begin{aligned}\mathcal{A}(\delta(x-y)+y) = \mathcal{A}(\delta((x+y)-y)+y) &= \mathcal{A}(\delta x + y)\\ &\geq \min\{\mathcal{A}(x+y), \tfrac{1-k}{2}\}\\ &= \min\{\mathcal{A}(x), \tfrac{1-k}{2}\}.\end{aligned}$$

It follows from Theorem 3.6 that $\mathcal{A}$ is a starshaped $(\in, \in\vee q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$. □

Corollary 3.14 ([4]). *A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is a starshaped $(\in, \in\vee q)$-fuzzy set relative to $y \in \mathbb{R}^n$ if and only if it satisfies:*

$$(x+y)_t \in \mathcal{A} \;\Rightarrow\; (\delta x + y)_t \in\vee\, q\, \mathcal{A} \tag{3.11}$$

for all $x \in \mathbb{R}^n$, $\delta \in [0,1]$ and $t \in (0,1]$.

Definition 3.15 ([4]). A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is called a *quasi-starshaped* $(\in, \in\vee q)$-*fuzzy set* relative to $y \in \mathbb{R}^n$ if

$$x_t \in \mathcal{A},\; y_r \in \mathcal{A} \;\Rightarrow\; (\delta x + (1-\delta)y)_{\min\{t,r\}} \in\vee\, q\, \mathcal{A} \tag{3.12}$$

for all $x \in \mathbb{R}^n$, $\delta \in [0,1]$ and $t, r \in (0,1]$.

Definition 3.16. A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is called a *quasi-starshaped* $(\in, \in\vee q_k)$-*fuzzy set* relative to $y \in \mathbb{R}^n$ if

$$x_t \in \mathcal{A},\; y_r \in \mathcal{A} \;\Rightarrow\; (\delta x + (1-\delta)y)_{\min\{t,r\}} \in\vee\, q_k\, \mathcal{A} \tag{3.13}$$

for all $x \in \mathbb{R}^n$, $\delta \in [0,1]$ and $t, r \in (0,1]$.

The quasi-starshaped $(\in, \in\vee q_k)$-fuzzy set $\mathcal{A}$ relative to $y \in \mathbb{R}^n$ with $k = 0$ is a quasi-starshaped $(\in, \in\vee q)$-fuzzy set $\mathcal{A}$ relative to $y \in \mathbb{R}^n$.

Example 3.17. The fuzzy set $\mathcal{A}$ in Example 3.3 is a quasi-starshaped $(\in, \in\vee q_k)$-fuzzy set relative to $y = 0$ with $k = 0.6$.

Example 3.18. The fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R})$ given by

$$\mathcal{A} : \mathbb{R} \to [0,1],\; x \mapsto \begin{cases} 1.75 + x & \text{if } x \in [-1.5, -1), \\ 0.75 & \text{if } x \in [-1, -\sqrt{0.5}) \cup (\sqrt{0.5}, 1], \\ 0.25 + x^2 & \text{if } x \in [-\sqrt{0.5}, \sqrt{0.5}], \\ 1.75 - x & \text{if } x \in (1, 1.5], \\ 0.25 & \text{otherwise} \end{cases}$$

is a quasi-starshaped $(\in, \in\vee q_k)$-fuzzy set relative to $y = 0$ with $k = 0.58$.

We consider characterizations of a quasi-starshaped $(\in, \in\vee q_k)$-fuzzy set.

Theorem 3.19. *For a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$, the following assertions are equivalent:*

(1) *$\mathcal{A}$ is a quasi-starshaped $(\in, \in\vee q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$.*

(2) $\mathcal{A}$ *satisfies:*

$$\mathcal{A}(\delta x + (1-\delta)y) \geq \min\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\} \tag{3.14}$$

for all $x \in \mathbb{R}^n$ *and* $\delta \in [0,1]$.

Proof. Assume that $\mathcal{A}$ is a quasi-starshaped $(\in, \in \vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $\delta \in [0,1]$, and suppose that $\min\{\mathcal{A}(x), \mathcal{A}(y)\} < \frac{1-k}{2}$. If there exists $t \in (0, \frac{1-k}{2})$ such that

$$\mathcal{A}(\delta x + (1-\delta)y) < t \leq \min\{\mathcal{A}(x), \mathcal{A}(y)\},$$

then $x_t \in \mathcal{A}$ and $y_t \in \mathcal{A}$, but

$$(\delta x + (1-\delta)y)_{\min\{t,t\}} = (\delta x + (1-\delta)y)_t \,\overline{\in}\, \mathcal{A}$$

and

$$\mathcal{A}(\delta x + (1-\delta)y) + t + k < 2t + k < 1,$$

that is, $(\delta x + (1-\delta)y)_t \,\overline{q_k}\, \mathcal{A}$. Hence $(\delta x + (1-\delta)y)_t \,\overline{\in \vee\, q_k}\, \mathcal{A}$, a contradiction. Thus

$$\mathcal{A}(\delta x + (1-\delta)y) \geq \min\{\mathcal{A}(x), \mathcal{A}(y)\}$$

Now assume that $\min\{\mathcal{A}(x), \mathcal{A}(y)\} \geq \frac{1-k}{2}$. Then $x_{\frac{1-k}{2}} \in \mathcal{A}$ and $y_{\frac{1-k}{2}} \in \mathcal{A}$, and so

$$(\delta x + (1-\delta)y)_{\frac{1-k}{2}} \in \vee\, q_k\, \mathcal{A},$$

that is, $(\delta x + (1-\delta)y)_{\frac{1-k}{2}} \in \mathcal{A}$ or $(\delta x + (1-\delta)y)_{\frac{1-k}{2}}\, q_k\, \mathcal{A}$ by (3.13). If

$$(\delta x + (1-\delta)y)_{\frac{1-k}{2}} \,\overline{\in}\, \mathcal{A}, \text{ i.e., } \mathcal{A}(\delta x + (1-\delta)y) < \tfrac{1-k}{2}$$

then $\mathcal{A}(\delta x + (1-\delta)y) + \frac{1-k}{2} + k < 1$, i.e., $(\delta x + (1-\delta)y)_{\frac{1-k}{2}} \,\overline{q_k}\, \mathcal{A}$, This is a contradiction. Consequently,

$$\mathcal{A}(\delta x + (1-\delta)y) \geq \tfrac{1-k}{2} \geq \min\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\}$$

for all $x \in \mathbb{R}^n$ and $\delta \in [0,1]$.

Conversely, assume that a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ satisfies the condition (3.14). Let $x \in \mathbb{R}^n$, $\delta \in [0,1]$ and $t, r \in (0,1]$ be such that $x_t \in \mathcal{A}$ and $y_r \in \mathcal{A}$. Then $\mathcal{A}(x) \geq t$ and $\mathcal{A}(y) \geq r$. If $\mathcal{A}(\delta x + (1-\delta)y) < \min\{\mathcal{A}(x), \mathcal{A}(y)\}$, then $\min\{\mathcal{A}(x), \mathcal{A}(y)\} \geq \frac{1-k}{2}$. Otherwise, we have

$$\mathcal{A}(\delta x + (1-\delta)y) \geq \min\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\} \geq \min\{\mathcal{A}(x), \mathcal{A}(y)\},$$

a contradiction. It follows that

$$\begin{aligned}&\mathcal{A}(\delta x + (1-\delta)y) + \min\{t,r\} + k\\&> 2\mathcal{A}(\delta x + (1-\delta)y) + k\\&\geq 2\min\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\} + k = 1\end{aligned}$$

and so that $(\delta x + (1-\delta)y)_{\min\{t,r\}}\, q_k\, \mathcal{A}$. Thus $(\delta x + (1-\delta)y)_{\min\{t,r\}} \in\vee\, q_k \mathcal{A}$, and therefore $\mathcal{A}$ is a quasi-starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$. □

Corollary 3.20 ([4]). *A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is a quasi-starshaped $(\in, \in\vee\, q)$-fuzzy set relative to $y \in \mathbb{R}^n$ if and only if*

$$(\forall x \in \mathbb{R}^n)(\forall \delta \in [0,1])\,(\mathcal{A}(\delta x + (1-\delta)y) \geq \min\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}) \tag{3.15}$$

The following proposition is straightforward by Theorem 3.19.

Proposition 3.21. *For a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$, if $k \in [0,1)$ satisfies:*

$$(\forall x \in \mathbb{R}^n)\left(\mathcal{A}(x) \geq \tfrac{1-k}{2}\right),$$

then $\mathcal{A}$ is a quasi-starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathscr{F}(\mathbb{R}^n)$.

Corollary 3.22. *If a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ satisfies:*

$$(\forall x \in \mathbb{R}^n)\,(\mathcal{A}(x) \geq 0.5),$$

then $\mathcal{A}$ is a quasi-starshaped $(\in, \in\vee\, q)$-fuzzy set relative to $y \in \mathscr{F}(\mathbb{R}^n)$.

Theorem 3.23. *Given a fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$, the following assertions are equivalent:*

(1) *$\mathcal{A}$ is a quasi-starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$.*

(2) *The nonempty t-level set $U(\mathcal{A}; t)$ of $\mathcal{A}$ is starshaped relative to $y \in \mathbb{R}^n$ for all $t \in (0, \min\{\mathcal{A}(y), \frac{1-k}{2}\}]$.*

Proof. Suppose $\mathcal{A}$ is a quasi-starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$. Assume that $U(\mathcal{A}; t) \neq \emptyset$ for every $t \in (0, \min\{\mathcal{A}(y), \frac{1-k}{2}\}]$. Then $y \in U(\mathcal{A}; t)$, that is, $\mathcal{A}(y) \geq t$. If $x \in U(\mathcal{A}; t)$, then $\mathcal{A}(x) \geq t$. It follows from (3.14) that

$$\mathcal{A}(\delta x + (1-\delta)y) \geq \min\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\} \geq \min\{t, \tfrac{1-k}{2}\} = t,$$

that is, $\delta x + (1-\delta)y \in U(\mathcal{A}; t)$. Hence $\overline{xy} \subseteq U(\mathcal{A}; t)$, and so $U(\mathcal{A}; t)$ is starshaped relative to $y \in \mathbb{R}^n$ for all $t \in (0, \min\{\mathcal{A}(y), \frac{1-k}{2}\}]$.

Conversely, suppose the nonempty t-level set $U(\mathcal{A};t)$ is starshaped relative to $y \in \mathbb{R}^n$ for all $t \in (0, \min\{\mathcal{A}(y), \frac{1-k}{2}\}]$. For any $\delta \in [0,1]$ and $x \in \mathbb{R}^n$, let $\mathcal{A}(y) = t_y$ when $\mathcal{A}(y) < \mathcal{A}(x)$. Then $\overline{xy} \subseteq U(\mathcal{A};t_y)$, and so

$$\mathcal{A}(\delta x + (1-\delta)y) \geq \min\{\mathcal{A}(x), \mathcal{A}(y)\} \geq \min\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\}.$$

Similarly, we have

$$\mathcal{A}(\delta x + (1-\delta)y) \geq \min\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\}$$

by putting $\mathcal{A}(x) = t_x$ when $\mathcal{A}(x) \leq \mathcal{A}(y)$. It follows from Theorem 3.19 that $\mathcal{A}$ is a quasi-starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$. □

Corollary 3.24 ([4]). *A fuzzy set $\mathcal{A} \in \mathscr{F}(\mathbb{R}^n)$ is a quasi-starshaped $(\in, \in\vee\, q)$-fuzzy set relative to $y \in \mathbb{R}^n$ if and only if its nonempty t-level set $U(\mathcal{A};t)$ is starshaped relative to $y \in \mathbb{R}^n$ for all $t \in (0, \min\{\mathcal{A}(y), 0.5\}]$.*

Theorem 3.25. *Given $y \in \mathbb{R}^n$, every starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$ is a quasi-starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$.*

Proof. Let $\mathcal{A}$ be a starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$. Taking $\delta = 0$ in (3.5) induces $\mathcal{A}(y) \geq \min\{\mathcal{A}(x), \frac{1-k}{2}\}$ for all $x \in \mathbb{R}^n$. It follows from (3.5) that

$$\mathcal{A}(\delta x + (1-\delta)y) = \mathcal{A}(\delta(x-y)+y) \geq \min\{\mathcal{A}(x), \tfrac{1-k}{2}\} = \min\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\}$$

for all $x \in \mathbb{R}^n$ and $\delta \in [0,1]$. Therefore $\mathcal{A}$ is a quasi-starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$ by Theorem 3.19. □

Corollary 3.26 ([4]). *Given $y \in \mathbb{R}^n$, every starshaped $(\in, \in\vee\, q)$-fuzzy set relative to $y \in \mathbb{R}^n$ is a quasi-starshaped $(\in, \in\vee\, q)$-fuzzy set relative to $y \in \mathbb{R}^n$.*

The converse of Theorem 3.25 is not true in general. In fact, take the quasi-starshaped $(\in, \in\vee\, q_k)$-fuzzy set $\mathcal{A}$ relative to $y = 0$ with $k = 0.58$ in Example 3.18. If we put $x = 0.5$ and $\delta = 0.8$, then $\mathcal{A}(\delta x) < \min\{\mathcal{A}(x), 0.5\}$ and so $\mathcal{A}$ is not a quasi-starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y = 0$ by Corollary 3.7.

Theorem 3.27. *If $\mathcal{A} \in \mathbb{R}^n$ is a quasi-starshaped $(\in, \in\vee\, q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$ with $\mathcal{A}(y) \neq \frac{1-k}{2}$, then the set*

$$A := \{x \in \mathbb{R}^n \mid \mathcal{A}(x) > \tfrac{1-k}{2}\}$$

is starshaped relative to $y \in \mathbb{R}^n$.

Proof. Let $x \in A$. Then $\mathcal{A}(x) > \frac{1-k}{2}$. Take $t_y := \mathcal{A}(y)$ when $\mathcal{A}(x) > \mathcal{A}(y)$. Then, by Theorem 3.23, $U(\mathcal{A}; t_y)$ is starshaped relative to y, and so $\overline{xy} \subseteq U(\mathcal{A}; t_y) \subseteq A$. Similarly, if we take $\mathcal{A}(x) = t_x$ when $\mathcal{A}(x) \leq \mathcal{A}(y)$, then $\overline{xy} \subseteq U(\mathcal{A}; t_x) \subseteq A$. Therefore A is starshaped relative to $y \in \mathbb{R}^n$. □

Corollary 3.28 **([4]).** *If $\mathcal{A} \in \mathbb{R}^n$ is a quasi-starshaped $(\in, \in \vee q)$-fuzzy set relative to $y \in \mathbb{R}^n$ with $\mathcal{A}(y) \neq 0.5$, then the set*

$$A := \{x \in \mathbb{R}^n \mid \mathcal{A}(x) > 0.5\}$$

is starshaped relative $y \in \mathbb{R}^n$.

In Theorem 3.27, the condition $\mathcal{A}(y) \neq \frac{1-k}{2}$ is necessary. In Example 3.18, $\mathcal{A}$ is a quasi-starshaped $(\in, \in \vee q_k)$-fuzzy set relative to $y = 2$ with $k = 0.5$ and $\mathcal{A}(2) = \frac{1-k}{2}$. But the set

$$A = \{x \in \mathbb{R}^n \mid x \in (-1.5, 0)\} \cup \{x \in \mathbb{R}^n \mid x \in (0, 1.5)\}$$

is not starshaped relative to $y = 2$.

Theorem 3.29. *If $\mathcal{A} \in \mathbb{R}^n$ is a quasi-starshaped $(\in, \in \vee q_k)$-fuzzy set relative to $y \in \mathbb{R}^n$ with $\mathcal{A}(y) \neq \frac{1-k}{2}$, then the closure $\overline{A}$ of $A := \{x \in \mathbb{R}^n \mid \mathcal{A}(x) > \frac{1-k}{2}\}$ is starshaped relative to $y \in \mathbb{R}^n$.*

Proof. For any $\delta \in [0, 1]$ and $x_0 \in \overline{A}$, take $a_0 := \delta x_0 + (1 - \delta)y$ in $\mathbb{R}^n$ and let G be a neighborhood of a_0. Since $\mathcal{A}(x) = \delta x + (1 - \delta)y$ is continuous at x, there exists a neighborhood H of x_0 such that if $x \in H$ then $\delta x + (1 - \delta)y \in G$. Since $x_0 \in \overline{A}$, we know that $x \in A \cap H$. Since A is starshaped relative to y by Theorem 3.27, we get $\delta x + (1 - \delta)y \in A \cap G$ and so $\delta x_0 + (1 - \delta)y \in \overline{A}$. Thus $\overline{x_0 y} \subseteq \overline{A}$, and $\overline{A}$ is starshaped relative to $y \in \mathbb{R}^n$. □

Corollary 3.30 **([4]).** *Let $\mathcal{A} \in \mathbb{R}^n$ be a quasi-starshaped $(\in, \in \vee q)$-fuzzy set relative to $y \in \mathbb{R}^n$ with $\mathcal{A}(y) \neq 0.5$. Then the closure $\overline{A}$ of $A := \{x \in \mathbb{R}^n \mid \mathcal{A}(x) > 0.5\}$ is starshaped relative to $y \in \mathbb{R}^n$.*

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Semidetached semigroups

Abstract The notion of semidetached semigroup is introduced, and their properties are investigated. Several conditions for a pair of a semigroup and a semidetached mapping to be a semidetached semigroup are provided. The concepts of $(\overline{\in},\, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup, $(\overline{q_k},\, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup and $(\overline{\in} \vee \overline{q_k},\, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup are introduced, and relative relations are discussed.

Keywords: Semidetached mapping, semidetached semigroup, $(\in,\, \in \vee q_k)$-fuzzy subsemigroup, $(q_k,\, \in \vee q_k)$-fuzzy subsemigroup, $(\overline{\in},\, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup, $(\overline{q_k},\, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup, $(\overline{\in} \vee \overline{q_k},\, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup.

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1 Introduction

Zadeh [28] introduced the concept of a fuzzy set in 1965. Based on the pioneering Zadeh's work, Kuroki introduced fuzzy semigroups and various kinds of fuzzy ideals in semigroups and characterized certain semigroups using those fuzzy ideals (see [15, 16, 17, 18, 19]). Since then the literature of various fuzzy algebraic concepts has been growing very rapidly. In the literature, several authors considered the relationships between the fuzzy sets and semigroups (see [5, 7, 12, 13, 14, 15, 16, 17, 18, 19, 22]). In [23], the idea of *fuzzy point* and its *belongingness to* and *quasi-coincidence with* a fuzzy subset were used to define (α, β)-*fuzzy subgroups,* where $\alpha,\, \beta \in \{\in,\, q,\, \in \vee\, q,\, \in \wedge\, q\}$ and $\alpha \neq\, \in \wedge\, q$. This was further studied in detail by Bhakat [1, 2], Bhakat and Das [3, 4], and Yuan et al. [27]. This notion is applied to semigroups and groups (see [2], [3], [4], [12], [24], [25]), BCK/BCI-algebras (see [6], [8], [9], [10], [21], [29], [30]), and (pseudo-) BL-algebras (see [20], [31]). General form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set is considered by Jun in [11]. Shabir et al. [25] discuss semigroups characterized by $(\in,\, \in \vee\, q_k)$-fuzzy ideals.

In this paper, we introduce the notion of semidetached semigroups, and investigate their properties. We provide several conditions for a pair of a semigroup and a semidetached mapping to be a semidetached semigroup. We also introduced the concepts $(\overline{\in},\, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup, $(\overline{q_k},\, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup and $(\overline{\in} \vee \overline{q_k},\, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup, and investigated relative relations.

2 Preliminaries

Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

Let S be a semigroup. By a *subsemigroup* of S we mean a nonempty subset A of S such that $A^2 \subseteq A$. For the sake of convenience, we may regard the empty set to be a subsemigroup.

A fuzzy set λ in a semigroup S is called a *fuzzy subsemigroup* of S if it satisfies:

$$(\forall x, y \in S)\,(\lambda(xy) \geq \lambda(x) \wedge \lambda(y)). \tag{2.1}$$

For any fuzzy set λ in a set S and any $t \in [0, 1]$, the set

$$U(\lambda; t) = \{x \in S \mid \lambda(x) \geq t\}$$

is called a *level subset* of λ.

A fuzzy set λ in a set S of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{2.2}$$

is said to be a *fuzzy point* with support x and value t and is denoted by (x, t).

For a fuzzy set λ in a set S, a fuzzy point (x, t) is said to

- *contained* in λ, denoted by $(x, t) \in \lambda$ (see [23]), if $\lambda(x) \geq t$.
- *be quasi-coincident with* λ, denoted by $(x, t)\, q\, \lambda$ (see [23]), if $\lambda(x) + t > 1$.
- $(x, t) \in\vee\, q\, \lambda$ if $(x, t) \in \lambda$ or $(x, t)\, q\, \lambda$.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

For any real umbers a and b, we also use $a \vee b$ and $a \wedge b$ instead of $\bigvee\{a, b\}$ and $\bigwedge\{a, b\}$, respectively.

3 Semidetached mappings in semigroups

In what follows, let S denote a semigroup unless otherwise specified.

Jun [11] considered the general form of the symbol $(x,t)\, q\, \lambda$ as follows: For an arbitrary element k of $[0,1)$, we say that

- $(x,t)\, q_k\, \lambda$ if $\lambda(x)+t+k>1$.
- $(x,t) \in\vee\, q_k\, \lambda$ if $(x,t)\in\lambda$ or $(x,t)\, q_k\, \lambda$.

Definition 3.1 ([11]). A fuzzy set λ in S is called an $(\in, \in\vee\, q_k)$-*fuzzy subsemigroup* of S if it satisfies:

$$(\forall x,y\in S)(\forall t_1,t_2\in(0,1])\left((x,t_1)\in\lambda,\ (y,t_2)\in\lambda\ \Rightarrow\ (xy, t_1\wedge t_2)\in\vee\, q_k\,\lambda\right). \tag{3.1}$$

Definition 3.2. Let Ω be a subinterval of $[0,1]$. A mapping $f:\Omega\to\mathcal{P}(S)$ is called a *semidetached mapping* with respect to $t\in\Omega$ (briefly, t-*semidetached mapping* over Ω) if $f(t)$ is a subsemigroup of S.

We say that $f:\Omega\to\mathcal{P}(S)$ is a *semidetached mapping* over Ω if it is t-semidetached mapping with respect to all $t\in\Omega$, and a pair (S,f) is called a *semidetached semigroup* over Ω.

Given a fuzzy set λ in S, consider the following mappings

$$\mathcal{A}_U^{\lambda}:\Omega\to\mathcal{P}(S),\ t\mapsto U(\lambda;t), \tag{3.2}$$

$$\mathcal{A}_{Q_k}^{\lambda}:\Omega\to\mathcal{P}(S),\ t\mapsto\{x\in S\mid (x,t)\, q_k\,\lambda\}, \tag{3.3}$$

$$\mathcal{A}_{\mathcal{E}_k}^{\lambda}:\Omega\to\mathcal{P}(S),\ t\mapsto\{x\in S\mid (x,t)\ \in\vee\, q_k\,\lambda\}. \tag{3.4}$$

Lemma 3.3 ([26]). *A fuzzy set λ is a fuzzy subsemigroup of S if and only if $U(\lambda;t)$ is a subsemigroup of S for all $t\in(0,1]$.*

Theorem 3.4. *A pair $(S,\mathcal{A}_U^{\lambda})$ is a semidetached semigroup over $\Omega=(0,1]$ if and only if λ is a fuzzy subsemigroup of S.*

Proof. Straightforward from Lemma 3.3. □

Theorem 3.5. *If λ is an $(\in,\in)$-fuzzy subsemigroup (or equivalently, λ is a fuzzy subsemigroup) of S, then a pair $\left(S,\mathcal{A}_{Q_k}^{\lambda}\right)$ is a semidetached semigroup over $\Omega=(0,1]$.*

Proof. Let $x, y \in \mathcal{A}^{\lambda}_{Q_k}(t)$ for $t \in \Omega = (0,1]$. Then $(x,t)\, q_k\, \lambda$ and $(y,t)\, q_k\, \lambda$, that is, $\lambda(x) + t + k > 1$ and $\lambda(y) + t + k > 1$. It follows from (2.1) that

$$\begin{aligned}\lambda(xy) + t + k &\geq \bigwedge\{\lambda(x), \lambda(y)\} + t + k \\ &= \bigwedge\{\lambda(x) + t + k, \lambda(y) + t + k\} > 1.\end{aligned}$$

Hence $(xy, t) \in\vee\, q_k\, \lambda$, and so $xy \in \mathcal{A}^{\lambda}_{Q_k}(t)$. Therefore $\mathcal{A}^{\lambda}_{Q_k}(t)$ is a subsemigroup of S. Consequently $\left(S, \mathcal{A}^{\lambda}_{Q_k}\right)$ is a semidetached semigroup over $\Omega = (0,1]$. $\square$

Corollary 3.6. *If λ is an $(\in, \in)$-fuzzy subsemigroup (or equivalently, λ is a fuzzy subsemigroup) of S, then a pair $\left(S, \mathcal{A}^{\lambda}_{Q}\right)$ is a semidetached semigroup over $\Omega = (0,1]$.*

Definition 3.7. A fuzzy set λ in S is called a $(q_k, \in\vee\, q_k)$-*fuzzy subsemigroup* of S if it satisfies:

$$(\forall x, y \in S)(\forall t, r \in (0, \tfrac{1-k}{2}])\, (x_t\, q_k\, \lambda,\ y_r\, q_k\, \lambda \ \Rightarrow\ (xy, t \wedge r) \in\vee\, q_k\, \lambda)\,. \tag{3.5}$$

Theorem 3.8. *Every $(q_k, \in\vee\, q_k)$-fuzzy subsemigroup is an $(\in, \in\vee\, q_k)$-fuzzy subsemigroup.*

Proof. Let λ be a $(q_k, \in\vee\, q_k)$-fuzzy subsemigroup of S. Let $x, y \in S$ and $t, r \in (0,1]$ be such that $(x,t) \in \lambda$ and $(y,r) \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq r$. Suppose that $(xy, t \wedge r)\, \overline{\in\vee\, q_k}\, \lambda$. Then

$$\lambda(xy) < t \wedge r \tag{3.6}$$

$$\lambda(xy) + t \wedge r + k \leq 1. \tag{3.7}$$

It follows that

$$\lambda(xy) < \tfrac{1-k}{2}. \tag{3.8}$$

Combining (3.6) and (3.8), we have

$$\lambda(xy) < \bigwedge\{t, r, \tfrac{1-k}{2}\}$$

and so

$$\begin{aligned}1 - k - \lambda(xy) &> 1 - k - \bigwedge\{t, r, \tfrac{1-k}{2}\} \\ &= \bigvee\{1 - k - t, 1 - k - r, 1 - k - \tfrac{1-k}{2}\} \\ &\geq \bigvee\{1 - k - \lambda(x), 1 - k - \lambda(y), \tfrac{1-k}{2}\}.\end{aligned}$$

Hence there exists $\delta \in (0,1]$ such that

$$1 - k - \lambda(xy) \geq \delta > \bigvee\{1 - k - \lambda(x), 1 - k - \lambda(y), \tfrac{1-k}{2}\}. \tag{3.9}$$

The right inequality in (3.9) implies that $\lambda(x) + \delta + k > 1$ and $\lambda(y) + \delta + k > 1$, that is, $(x, \delta)\, q_k\, \lambda$ and $(y, \delta)\, q_k\, \lambda$. Since λ is a $(q_k, \in\vee\, q_k)$-fuzzy subsemigroup of S, it follows that $(xy, \delta) \in\vee\, q_k\, \lambda$. On the other hand, the left inequality in (3.9) implies that

$$\lambda(xy) + \delta + k \leq 1, \text{ that is, } (xy, \delta)\, \overline{q_k}\, \lambda,$$

and

$$\lambda(xy) \leq 1 - \delta - k < 1 - k - \tfrac{1-k}{2} = \tfrac{1-k}{2} < \delta, \text{ i.e., } (xy, \delta)\, \overline{\in}\, \lambda.$$

Hence $(xy, \delta)\, \overline{\in\vee\, q_k}\, \lambda$, which is a contradiction. Therefore $(xy, t \wedge r) \in\vee\, q_k\, \lambda$, and thus λ is an $(\in, \in\vee\, q_k)$-fuzzy subsemigroup of S. □

Corollary 3.9. *Every $(q, \in\vee\, q)$-fuzzy subsemigroup is an $(\in, \in\vee\, q)$-fuzzy subsemigroup.*

We consider the converse of Theorem 3.8.

Theorem 3.10. *If every fuzzy point has the value t in $(0, \frac{1-k}{2}]$, then every $(\in, \in\vee\, q_k)$-fuzzy subsemigroup is a $(q_k, \in\vee\, q_k)$-fuzzy subsemigroup.*

Proof. Let λ be a $(\in, \in\vee\, q_k)$-fuzzy subsemigroup of S. Let $x, y \in S$ and $t, r \in (0, \frac{1-k}{2}]$ be such that $(x, t)\, q_k\, \lambda$ and $(y, r)\, q_k\, \lambda$. Then $\lambda(x) + t + k > 1$ and $\lambda(y) + r + k > 1$. Since $t, r \in (0, \frac{1-k}{2}]$, it follows that $\lambda(x) > 1 - t - k \geq \frac{1-k}{2} \geq t$ and $\lambda(y) > 1 - r - k \geq \frac{1-k}{2} \geq r$, that is, $(x, t) \in \lambda$ and $(y, r) \in \lambda$. It follows from (3.1) that $(xy, t \wedge r) \in\vee\, q_k\, \lambda$. Therefore λ is a $(q_k, \in\vee\, q_k)$-fuzzy subsemigroup of S. □

Corollary 3.11. *If every fuzzy point has the value t in $(0, 0.5]$, then every $(\in, \in\vee\, q)$-fuzzy subsemigroup is a $(q, \in\vee\, q)$-fuzzy subsemigroup.*

Theorem 3.12. *If $\left(S, \mathcal{A}^{\lambda}_{Q_k}\right)$ is a semidetached semigroup over $\Omega = \left(\frac{1-k}{2}, 1\right]$, then λ satisfies:*

$$(\forall x, y \in S)(\forall t, r \in \Omega)\left((x, t) \in \lambda,\ (y, r) \in \lambda \ \Rightarrow\ (xy, t \vee r)\, q_k\, \lambda\right). \tag{3.10}$$

Proof. Let $x, y \in S$ and $t, r \in \Omega = \left(\frac{1-k}{2}, 1\right]$ be such that $(x, t) \in \lambda$ and $(y, r) \in \lambda$. Then $\lambda(x) \geq t > \frac{1-k}{2}$ and $\lambda(y) \geq r > \frac{1-k}{2}$, which imply that $\lambda(x)+t+k > 1$ and $\lambda(y)+t+k > 1$, that is, $(x, t)\, q_k\, \lambda$ and $(y, r)\, q_k\, \lambda$. It follows that $x, y \in \mathcal{A}^{\lambda}_{Q_k}(t \vee r)$ and $t \vee r \in \left(\frac{1-k}{2}, 1\right]$. Since $\mathcal{A}^{\lambda}_{Q_k}(t \vee r)$ is a subsemigroup of S by assumption, we have $xy \in \mathcal{A}^{\lambda}_{Q_k}(t \vee r)$ and so $(xy, t \vee r)\, q_k\, \lambda$. □

Corollary 3.13. *If $\left(S, \mathcal{A}^{\lambda}_{Q}\right)$ is a semidetached semigroup over $\Omega = (0.5, 1]$, then λ satisfies:*

$$(\forall x, y \in S)(\forall t, r \in \Omega)\left((x, t) \in \lambda,\ (y, r) \in \lambda \ \Rightarrow\ (xy, t \vee r)\, q\, \lambda\right). \tag{3.11}$$

Theorem 3.14. *If $\left(S, \mathcal{A}_{Q_k}^{\lambda}\right)$ is a semidetached semigroup over $\Omega = \left(0, \frac{1-k}{2}\right]$, then λ satisfies:*

$$(\forall x, y \in S)(\forall t, r \in \Omega)\left((x,t)\, q_k\, \lambda,\ (y,r)\, q_k\, \lambda\ \Rightarrow\ (xy, t \vee r) \in \lambda\right). \tag{3.12}$$

Proof. Let $x, y \in S$ and $t, r \in \Omega = \left(0, \frac{1-k}{2}\right]$ be such that $(x,t)\, q_k\, \lambda$ and $(y,r)\, q_k\, \lambda$. Then $x \in \mathcal{A}_{Q_k}^{\lambda}(t)$ and $y \in \mathcal{A}_{Q_k}^{\lambda}(r)$. It follows that $x, y \in \mathcal{A}_{Q_k}^{\lambda}(t \vee r)$ and $t \vee r \in \Omega = \left(0, \frac{1-k}{2}\right]$. Thus $xy \in \mathcal{A}_{Q_k}^{\lambda}(t \vee r)$ since $\mathcal{A}_{Q_k}^{\lambda}(t \vee r)$ is a subsemigroup of S by the assumption. Hence $\lambda(xy) + k + t \vee r > 1$ and so $\lambda(xy) > 1 - k - t \vee r \geq \frac{1-k}{2} \geq t \vee r$. Thus $(xy, t \vee r) \in \lambda$, and (3.12) is valid. □

Corollary 3.15. *If $\left(S, \mathcal{A}_{Q}^{\lambda}\right)$ is a semidetached semigroup over $\Omega = (0, 0.5]$, then λ satisfies:*

$$(\forall x, y \in S)(\forall t, r \in \Omega)\left((x,t)\, q\, \lambda,\ (y,r)\, q\, \lambda\ \Rightarrow\ (xy, t \vee r) \in \lambda\right). \tag{3.13}$$

Theorem 3.16. *If λ is a $(q_k, \in\vee\, q_k)$-fuzzy subsemigroup of S, then $\left(S, \mathcal{A}_{Q_k}^{\lambda}\right)$ is a semidetached semigroup over $\Omega = \left(\frac{1-k}{2}, 1\right]$*

Proof. Let $x, y \in \mathcal{A}_{Q_k}^{\lambda}(t)$ for $t \in \left(\frac{1-k}{2}, 1\right]$. Then $(x,t)\, q_k\, \lambda$ and $(y,t)\, q_k\, \lambda$. Since λ is a $(q_k, \in\vee\, q_k)$-fuzzy subsemigroup of S, we have $(xy, t) \in\vee\, q_k\, \lambda$, that is, $(xy,t) \in \lambda$ or $(xy,t)\, q_k\, \lambda$. If $(xy,t) \in \lambda$, then $\lambda(xy) \geq t > \frac{1-k}{2} > 1 - t - k$ and so $\lambda(xy) + t + k > 1$, i.e., $(xy,t)\, q_k\, \lambda$. Hence $xy \in \mathcal{A}_{Q_k}^{\lambda}(t)$. If $(xy,t)\, q_k\, \lambda$, then $xy \in \mathcal{A}_{Q_k}^{\lambda}(t)$. Therefore $\mathcal{A}_{Q_k}^{\lambda}(t)$ is a subsemigroup of S, and consequently $\left(S, \mathcal{A}_{Q_k}^{\lambda}\right)$ is a semidetached semigroup over $\Omega = \left(\frac{1-k}{2}, 1\right]$. □

Corollary 3.17. *If λ is a $(q, \in\vee\, q)$-fuzzy subsemigroup of S, then $\left(S, \mathcal{A}_{Q}^{\lambda}\right)$ is a semidetached semigroup over $\Omega = (0.5, 1]$*

Theorem 3.18. *For a subsemigroup A of S, let λ be a fuzzy set in S such that*

(1) $\lambda(x) \geq \frac{1-k}{2}$ *for all* $x \in A$,

(2) $\lambda(x) = 0$ *for all* $x \in S \setminus A$.

Then λ is a $(q_k, \in\vee\, q_k)$-fuzzy subsemigroup of S.

Proof. Let $x, y \in S$ and $t, r \in (0, \frac{1-k}{2}]$ be such that $(x,t)\, q_k\, \lambda$ and $(y,r)\, q_k\, \lambda$. Then $\lambda(x) + t + k > 1$ and $\lambda(y) + r + k > 1$, which imply that $\lambda(x) > 1 - t - k \geq \frac{1-k}{2}$ and $\lambda(y) > 1 - r - k \geq \frac{1-k}{2}$. Hence $x \in A$ and $y \in A$. Since A is a subsemigroup of S, we get $xy \in A$ and so $\lambda(xy) \geq \frac{1-k}{2} \geq t \vee r$. Thus $(xy, t \vee r) \in \lambda$, and so $(xy, t \vee r) \in\vee\, q_k\, \lambda$. Therefore λ is a $(q_k, \in\vee\, q_k)$-fuzzy subsemigroup of S. □

Corollary 3.19. *For a subsemigroup A of S, let λ be a fuzzy set in S such that*

(1) $\lambda(x) \geq 0.5$ *for all* $x \in A$,

(2) $\lambda(x) = 0$ *for all* $x \in S \setminus A$.

Then λ *is a* $(q, \in\vee q)$*-fuzzy subsemigroup of* S.

Using Theorems 3.16 and 3.18, we have the following theorem.

Theorem 3.20. *For a subsemigroup* A *of* S*, let* λ *be a fuzzy set in* S *such that*

(1) $\lambda(x) \geq \frac{1-k}{2}$ *for all* $x \in A$,

(2) $\lambda(x) = 0$ *for all* $x \in S \setminus A$.

Then $\left(S, \mathcal{A}^{\lambda}_{Q_k}\right)$ *is a semidetached semigroup over* $\Omega = \left(\frac{1-k}{2}, 1\right]$.

Theorem 3.21. *If* $\left(S, \mathcal{A}^{\lambda}_{\mathcal{E}_k}\right)$ *is a semidetached semigroup over* $\Omega = (0, 1]$*, then* λ *satisfies:*

$$(\forall x, y \in S)(\forall t, r \in \Omega)\left((x, t)\, q_k\, \lambda,\ (y, r)\, q_k\, \lambda \ \Rightarrow\ (xy, t \vee r) \in\vee\, q_k\, \lambda\right). \tag{3.14}$$

Proof. Let $x, y \in S$ and $t, r \in \Omega = (0, 1]$ be such that $(x, t)\, q_k\, \lambda$ and $(y, r)\, q_k\, \lambda$. Then $x \in \mathcal{A}^{\lambda}_{Q_k}(t) \subseteq \mathcal{A}^{\lambda}_{\mathcal{E}_k}(t)$ and $y \in \mathcal{A}^{\lambda}_{Q_k}(r) \subseteq \mathcal{A}^{\lambda}_{\mathcal{E}_k}(r)$. It follows that $x, y \in \mathcal{A}^{\lambda}_{\mathcal{E}_k}(t \vee r)$ and so from the hypothesis that $xy \in \mathcal{A}^{\lambda}_{\mathcal{E}_k}(t \vee r)$. Hence $(xy, t \vee r) \in\vee\, q_k\, \lambda$, and consequently (3.14) is valid. □

Corollary 3.22. *If* $\left(S, \mathcal{A}^{\lambda}_{\mathcal{E}}\right)$ *is a semidetached semigroup over* $\Omega = (0, 1]$*, then* λ *satisfies:*

$$(\forall x, y \in S)(\forall t, r \in \Omega)\left((x, t)\, q\, \lambda,\ (y, r)\, q\, \lambda \ \Rightarrow\ (xy, t \vee r) \in\vee\, q\, \lambda\right). \tag{3.15}$$

Lemma 3.23 ([25])**.** *A fuzzy set* λ *in* S *is an* $(\in, \in\vee\, q_k)$*-fuzzy subsemigroup of* S *if and only if it satisfies:*

$$(\forall x, y \in S)\left(\lambda(xy) \geq \bigwedge\{\lambda(x), \lambda(y), \tfrac{1-k}{2}\}\right). \tag{3.16}$$

Theorem 3.24. *If* λ *is an* $(\in, \in\vee\, q_k)$*-fuzzy subsemigroup of* S*, then* $\left(S, \mathcal{A}^{\lambda}_{Q_k}\right)$ *is a semidetached semigroup over* $\Omega = \left(\frac{1-k}{2}, 1\right]$.

Proof. Let $x, y \in \mathcal{A}^{\lambda}_{Q_k}(t)$ for $t \in \Omega = \left(\frac{1-k}{2}, 1\right]$. Then $(x, t)\, q_k\, \lambda$ and $(y, t)\, q_k\, \lambda$, that is, $\lambda(x) + t + k > 1$ and $\lambda(y) + t + k > 1$. It follows from Lemma 3.23 that

$$\begin{aligned}
\lambda(xy) + t + k &\geq \bigwedge\{\lambda(x), \lambda(y), \tfrac{1-k}{2}\} + t + k \\
&= \bigwedge\{\lambda(x) + t + k, \lambda(y) + t + k, \tfrac{1-k}{2} + t + k\} \\
&> 1.
\end{aligned}$$

Hence $(xy, t)\, q_k\, \lambda$, and so $xy \in \mathcal{A}^{\lambda}_{Q_k}(t)$. Therefore $\mathcal{A}^{\lambda}_{Q_k}(t)$ is a subsemigroup of S for all $t \in \left(\frac{1-k}{2}, 1\right]$, and consequently $\left(S, \mathcal{A}^{\lambda}_{Q_k}\right)$ is a semidetached semigroup over $\Omega = \left(\frac{1-k}{2}, 1\right]$. □

Corollary 3.25. *If λ is an $(\in, \in\vee q)$-fuzzy subsemigroup of S, then $\left(S, \mathcal{A}_Q^\lambda\right)$ is a semidetached semigroup over $\Omega = (0.5, 1]$.*

Theorem 3.26. *If $\left(S, \mathcal{A}_{\mathcal{E}_k}^\lambda\right)$ is a semidetached semigroup over $\Omega = (0, 1]$, then λ is an $(\in, \in\vee q_k)$-fuzzy subsemigroup of S.*

Proof. For a semidetached semigroup $\left(S, \mathcal{A}_{\mathcal{E}_k}^\lambda\right)$ over $\Omega = (0, 1]$, assume that there exists $a, b \in S$ such that

$$\lambda(ab) < \bigwedge\{\lambda(a), \lambda(b), \tfrac{1-k}{2}\} \triangleq t_0.$$

Then $t_0 \in (0, \frac{1-k}{2}]$, $a, b \in U(\lambda; t_0) \subseteq \mathcal{A}_{\mathcal{E}_k}^\lambda(t_0)$, which implies that $ab \in \mathcal{A}_{\mathcal{E}_k}^\lambda(t_0)$. Hence $\lambda(ab) \geq t_0$ or $\lambda(ab)+t_0+k > 1$. This is a contradiction. Thus $\lambda(xy) \geq \bigwedge\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$ for all $x, y \in S$. It follows from Lemma 3.23 that λ is an $(\in, \in\vee q_k)$-fuzzy subsemigroup of S. $\square$

Theorem 3.27. *If λ is an $(\in, \in\vee q_k)$-fuzzy subsemigroup of S, then $\left(S, \mathcal{A}_{\mathcal{E}_k}^\lambda\right)$ is a semidetached semigroup over $\Omega = \left(0, \frac{1-k}{2}\right]$.*

Proof. Let $x, y \in \mathcal{A}_{\mathcal{E}_k}^\lambda(t)$ for $t \in \Omega = \left(0, \frac{1-k}{2}\right]$. Then $(x, t) \in\vee q_k \lambda$ and $(y, t) \in\vee q_k \lambda$. Hence we have the following four cases:

(1) $(x, t) \in \lambda$ and $(y, t) \in \lambda$,

(2) $(x, t) \in \lambda$ and $(y, t)\, q_k \lambda$,

(3) $(x, t)\, q_k \lambda$ and $(y, t) \in \lambda$,

(4) $(x, t)\, q_k \lambda$ and $(y, t)\, q_k \lambda$.

The first case implies that $(xy, t) \in\vee q_k \lambda$ and so $xy \in \mathcal{A}_{\mathcal{E}_k}^\lambda(t)$. For the second case, $(y, t)\, q_k \lambda$ induces $\lambda(y) > 1 - t - k \geq t$, i.e., $(y, t) \in \lambda$. Hence $(xy, t) \in\vee q_k \lambda$ and so $xy \in \mathcal{A}_{\mathcal{E}_k}^\lambda(t)$. Similarly, the third case implies $xy \in \mathcal{A}_{\mathcal{E}_k}^\lambda(t)$. The last case induces $\lambda(x) > 1 - t - k \geq t$ and $\lambda(y) > 1 - t - k \geq t$, that is, $(x, t) \in \lambda$ and $(y, t) \in \lambda$. It follows that $(xy, t) \in\vee q_k \lambda$ and so that $xy \in \mathcal{A}_{\mathcal{E}_k}^\lambda(t)$. Therefore $\mathcal{A}_{\mathcal{E}_k}^\lambda(t)$ is a subsemigroup of S for all $t \in \left(0, \frac{1-k}{2}\right]$. Hence $\left(S, \mathcal{A}_{\mathcal{E}_k}^\lambda\right)$ is a semidetached semigroup over $\Omega = \left(0, \frac{1-k}{2}\right]$. $\square$

Corollary 3.28. *If λ is an $(\in, \in\vee q)$-fuzzy subsemigroup of S, then $\left(S, \mathcal{A}_{\mathcal{E}}^\lambda\right)$ is a semidetached semigroup over $\Omega = (0, 0.5]$.*

Theorem 3.29. *If λ is a $(q_k, \in\vee q_k)$-fuzzy subsemigroup of S, then $\left(S, \mathcal{A}_{\mathcal{E}_k}^\lambda\right)$ is a semidetached semigroup over $\Omega = \left(\frac{1-k}{2}, 1\right]$.*

Proof. Let $x, y \in \mathcal{A}_{\mathcal{E}_k}^\lambda(t)$ for $t \in \Omega = \left(\frac{1-k}{2}, 1\right]$. Then $(x, t) \in\vee q_k \lambda$ and $(y, t) \in\vee q_k \lambda$. Hence we have the following four cases:

(1) $(x,t) \in \lambda$ and $(y,t) \in \lambda$,

(2) $(x,t) \in \lambda$ and $(y,t)\, q_k\, \lambda$,

(3) $(x,t)\, q_k\, \lambda$ and $(y,t) \in \lambda$,

(4) $(x,t)\, q_k\, \lambda$ and $(y,t)\, q_k\, \lambda$.

For the first case, we have $\lambda(x)+t+k \geq 2t+k > 1$ and $\lambda(y)+t+k \geq 2t+k > 1$, that is, $(x,t)\, q_k\, \lambda$ and $(y,t)\, q_k\, \lambda$. Hence $(xy,t) \in\vee\, q_k\, \lambda$, and so $xy \in \mathcal{A}^{\lambda}_{\mathcal{E}_k}(t)$. In the case (2), $(x,t) \in \lambda$ implies $\lambda(x)+t+k \geq 2t+k > 1$, i.e., $(x,t)\, q_k\, \lambda$. Hence $(xy,t) \in\vee\, q_k\, \lambda$, and so $xy \in \mathcal{A}^{\lambda}_{\mathcal{E}_k}(t)$. Similarly, the third case implies $xy \in \mathcal{A}^{\lambda}_{\mathcal{E}_k}(t)$. For the last case, we have $(xy,t) \in\vee\, q_k\, \lambda$, and so $xy \in \mathcal{A}^{\lambda}_{\mathcal{E}_k}(t)$. Consequently, $\mathcal{A}^{\lambda}_{\mathcal{E}_k}(t)$ is a subsemigroup of S for all $t \in \Omega = \left(\frac{1-k}{2}, 1\right]$. Therefore $\left(S, \mathcal{A}^{\lambda}_{\mathcal{E}_k}\right)$ is a semidetached semigroup over $\Omega = \left(\frac{1-k}{2}, 1\right]$. $\square$

Corollary 3.30. *If λ is an $(q, \in\vee q)$-fuzzy subsemigroup of S, then $\left(S, \mathcal{A}^{\lambda}_{\mathcal{E}}\right)$ is a semidetached semigroup over $\Omega = (0.5, 1]$.*

For $\alpha \in \{\in, q_k\}$ and $t \in (0,1]$, we say that $(x,t)\, \overline{\alpha}\, \lambda$ if $(x,t)\, \alpha\, \lambda$ does not hold.

Definition 3.31. A fuzzy set λ in S is called an $(\overline{\in}, \overline{\in}\vee\overline{q_k})$-*fuzzy subsemigroup* of S if it satisfies:

$$(\forall x, y \in S)(\forall t, r \in (0,1])\left((xy, t\wedge r)\,\overline{\in}\,\lambda \;\Rightarrow\; (x,t)\,\overline{\in}\vee\overline{q_k}\,\lambda \;\text{ or }\; (y,r)\,\overline{\in}\vee\overline{q_k}\,\lambda\right). \quad (3.17)$$

An $(\overline{\in}, \overline{\in}\vee\overline{q_k})$-fuzzy subsemigroup with $k=0$ is called an $(\overline{\in}, \overline{\in}\vee\overline{q})$-*fuzzy subsemigroup*. We provide a characterization of an $(\overline{\in}, \overline{\in}\vee\overline{q_k})$-fuzzy subsemigroup.

Theorem 3.32. *A fuzzy set λ in S is an $(\overline{\in}, \overline{\in}\vee\overline{q_k})$-fuzzy subsemigroup of S if and only if the following inequality is valid.*

$$(\forall x, y \in S)\left(\bigvee\{\lambda(xy), \tfrac{1-k}{2}\} \geq \lambda(x)\wedge\lambda(y)\right). \quad (3.18)$$

Proof. Let λ be an $(\overline{\in}, \overline{\in}\vee\overline{q_k})$-fuzzy subsemigroup of S. Assume that (3.18) is not valid. Then there exist $a, b \in S$ such that

$$\bigvee\{\lambda(ab), \tfrac{1-k}{2}\} < \lambda(a)\wedge\lambda(b) \triangleq t.$$

Then $\frac{1-k}{2} < t \leq 1$, $(a,t) \in \lambda$, $(b,t) \in \lambda$ and $(ab,t)\,\overline{\in}\,\lambda$. It follows from (3.17) that $(a,t)\,\overline{q_k}\,\lambda$ or $(b,t)\,\overline{q_k}\,\lambda$. Hence

$$\lambda(a) \geq t \text{ and } \lambda(a)+t+k \leq 1$$

or

$$\lambda(b) \geq t \text{ and } \lambda(b) + t + k \leq 1.$$

In either case, we have $t \leq \frac{1-k}{2}$ which is a contradiction. Therefore

$$\bigvee\{\lambda(xy), \tfrac{1-k}{2}\} \geq \lambda(x) \wedge \lambda(y)$$

for all $x, y \in S$.

Conversely, suppose that (3.18) is valid. Let $(xy, t \wedge r) \,\overline{\in}\, \lambda$ for $x, y \in S$ and $t, r \in (0, 1]$. Then $\lambda(xy) < t \wedge r$. If $\bigvee\{\lambda(xy), \frac{1-k}{2}\} = \lambda(xy)$, then $t \wedge r > \lambda(xy) \geq \lambda(x) \wedge \lambda(y)$ and so $\lambda(x) < t$ or $\lambda(y) < t$. Thus $(x, t) \,\overline{\in}\, \lambda$ or $(y, r) \,\overline{\in}\, \lambda$, which implies that $(x, t) \,\overline{\in} \vee \overline{q_k}\, \lambda$ or $(y, r) \,\overline{\in} \vee \overline{q_k}\, \lambda$. If $\bigvee\{\lambda(xy), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $\lambda(x) \wedge \lambda(y) \leq \frac{1-k}{2}$. Suppose $(x, t) \in \lambda$ or $(y, r) \in \lambda$. Then $t \leq \lambda(x) \leq \frac{1-k}{2}$ or $r \leq \lambda(y) \leq \frac{1-k}{2}$, and so

$$\lambda(x) + t + k \leq \tfrac{1-k}{2} + \tfrac{1-k}{2} + k = 1$$

or

$$\lambda(y) + r + k \leq \tfrac{1-k}{2} + \tfrac{1-k}{2} + k = 1.$$

Hence $(x, t) \,\overline{q_k}\, \lambda$ or $(y, r) \,\overline{q_k}\, \lambda$. Therefore $(x, t) \,\overline{\in} \vee \overline{q_k}\, \lambda$ or $(y, r) \,\overline{\in} \vee \overline{q_k}\, \lambda$. This shows that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S. $\square$

Corollary 3.33. *A fuzzy set λ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subsemigroup of S if and only if the following inequality is valid.*

$$(\forall x, y \in S) \left(\bigvee\{\lambda(xy), 0.5\} \geq \lambda(x) \wedge \lambda(y) \right). \tag{3.19}$$

Theorem 3.34. *A fuzzy set λ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S if and only if $\left(S, \mathcal{A}_U^\lambda\right)$ is a semidetached semigroup over $\Omega = \left(\frac{1-k}{2}, 1\right]$.*

Proof. Assume that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S. Let $x, y \in \mathcal{A}_U^\lambda(t)$ for $t \in \Omega = \left(\frac{1-k}{2}, 1\right]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$. It follows from (3.18) that

$$\bigvee\{\lambda(xy), \tfrac{1-k}{2}\} \geq \lambda(x) \wedge \lambda(y) \geq t.$$

Since $t > \frac{1-k}{2}$, it follows that $\lambda(xy) \geq t$ and so that $xy \in \mathcal{A}_U^\lambda(t)$. Thus $\mathcal{A}_U^\lambda(t)$ is a subsemigroup of S, and $\left(S, \mathcal{A}_U^\lambda\right)$ is a semidetached semigroup over $\Omega = \left(\frac{1-k}{2}, 1\right]$.

Conversely, suppose that $\left(S, \mathcal{A}_U^\lambda\right)$ is a semidetached semigroup over $\Omega = \left(\frac{1-k}{2}, 1\right]$. If (3.18) is not valid, then there exist $a, b \in S$ such that

$$\bigvee\{\lambda(ab), \tfrac{1-k}{2}\} < \lambda(a) \wedge \lambda(b) \triangleq t.$$

Then $t \in (\frac{1-k}{2}, 1]$, $a, b \in \mathcal{A}_U^\lambda(t)$ and $ab \notin \mathcal{A}_U^\lambda(t)$. This is a contradiction, and so (3.18) is valid. Using Theorem 3.32, we know that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S. $\square$

Theorem 3.35. *A fuzzy set λ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S if and only if $\left(S, \mathcal{A}_{Q_k}^\lambda\right)$ is a semidetached semigroup over $\Omega = \left(0, \frac{1-k}{2}\right]$.*

Proof. Assume that $\left(S, \mathcal{A}^{\lambda}_{Q_k}\right)$ is a semidetached semigroup over $\Omega = \left(0, \frac{1-k}{2}\right]$. If (3.18) is not valid, then there exist $a, b \in S$, $t \in \Omega$ and $k \in [0, 1)$ such that

$$\bigvee\{\lambda(ab), \tfrac{1-k}{2}\} + t + k \leq 1 < \lambda(a) \wedge \lambda(b) + t + k.$$

It follows that $(a, t)\, q_k\, \lambda$ and $(b, t)\, q_k\, \lambda$, that is, $a, b \in \mathcal{A}^{\lambda}_{Q_k}(t)$, but $(ab, t)\, \overline{q_k}\, \lambda$, i.e., $ab \notin \mathcal{A}^{\lambda}_{Q_k}(t)$. This is a contradiction, and so (3.18) is valid. Using Theorem 3.32, we know that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S.

Conversely, suppose that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S. Let $x, y \in \mathcal{A}^{\lambda}_{Q_k}(t)$ for $t \in \Omega = \left(0, \frac{1-k}{2}\right]$. Then $(x, t)\, q_k\, \lambda$ and $(y, t)\, q_k\, \lambda$, that is, $\lambda(x) + t + k > 1$ and $\lambda(y) + t + k > 1$. It follows from (3.18) that

$$\bigvee\{\lambda(xy), \tfrac{1-k}{2}\} \geq \lambda(x) \wedge \lambda(y) > 1 - t - k \geq \tfrac{1-k}{2}$$

and so that $\lambda(xy) + t + k > 1$, that is, $xy \in \mathcal{A}^{\lambda}_{Q_k}(t)$. Therefore $\mathcal{A}^{\lambda}_{Q_k}(t)$ is a subsemigroup of S, and $\left(S, \mathcal{A}^{\lambda}_{Q_k}\right)$ is a semidetached semigroup over $\Omega = \left(0, \frac{1-k}{2}\right]$. □

Definition 3.36. A fuzzy set λ in S is called an $(\overline{\in} \vee \overline{q_k}, \overline{\in} \vee \overline{q_k})$-*fuzzy subsemigroup* of S if for all $x, y \in S$ and $t, r \in (0, 1]$,

$$(xy, t \wedge r)\, \overline{\in} \vee \overline{q_k}\, \lambda \;\Rightarrow\; (x, t)\, \overline{\in} \vee \overline{q_k}\, \lambda \text{ or } (y, r)\, \overline{\in} \vee \overline{q_k}\, \lambda. \tag{3.20}$$

Theorem 3.37. *Every $(\overline{\in} \vee \overline{q_k}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup.*

Proof. Let $x, y \in S$ and $t, r \in (0, 1]$ be such that $(xy, t \wedge r)\, \overline{\in}\, \lambda$. Then $(xy, t \wedge r)\, \overline{\in} \vee \overline{q_k}\, \lambda$, and so $(x, t)\, \overline{\in} \vee \overline{q_k}\, \lambda$ or $(y, r)\, \overline{\in} \vee \overline{q_k}\, \lambda$ by (3.20). Therefore λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S. □

Definition 3.38. A fuzzy set λ in S is called a $(\overline{q_k}, \overline{\in} \vee \overline{q_k})$-*fuzzy subsemigroup* of S if for all $x, y \in S$ and $t, r \in (0, 1]$,

$$(xy, t \wedge r)\, \overline{q_k}\, \lambda \;\Rightarrow\; (x, t)\, \overline{\in} \vee \overline{q_k}\, \lambda \text{ or } (y, r)\, \overline{\in} \vee \overline{q_k}\, \lambda. \tag{3.21}$$

Theorem 3.39. *Assume that $t \wedge r \leq \frac{1-k}{2}$ for any $t, r \in (0, 1]$. Then every $(\overline{q_k}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup.*

Proof. Let λ be an $(\overline{q_k}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S. Assume that $(xy, t \wedge r)\, \overline{\in}\, \lambda$ for $x, y \in S$ and $t, r \in (0, 1]$ with $t \wedge r \leq \frac{1-k}{2}$. Then $\lambda(xy) < t \wedge r \leq \frac{1-k}{2}$, and so

$$\lambda(xy) + k + t \wedge r < \tfrac{1-k}{2} + \tfrac{1-k}{2} + k = 1,$$

that is, $(xy, t \wedge r)\, \overline{q_k}\, \lambda$. It follows from (3.21) that $(x, t)\, \overline{\in} \vee \overline{q_k}\, \lambda$ or $(y, r)\, \overline{\in} \vee \overline{q_k}\, \lambda$. Therefore λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S. □

Corollary 3.40. *Assume that* $t \wedge r \leq 0.5$ *for any* $t, r \in (0, 1]$. *Then every* $(\overline{q}, \overline{\in} \vee \overline{q})$*-fuzzy subsemigroup is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy subsemigroup.*

Theorem 3.41. *Assume that* $t \wedge r > \frac{1-k}{2}$ *for any* $t, r \in (0, 1]$. *Then every* $(\overline{\in}, \overline{\in} \vee \overline{q_k})$*-fuzzy subsemigroup is a* $(\overline{q_k}, \overline{\in} \vee \overline{q_k})$*-fuzzy subsemigroup.*

Proof. Let λ be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S. Assume that $(xy, t \wedge r)\, \overline{q_k}\, \lambda$ for $x, y \in S$ and $t, r \in (0, 1]$ with $t \wedge r > \frac{1-k}{2}$. If $(xy, t \wedge r) \in \lambda$, then $\lambda(xy) \geq t \wedge r$ and so

$$\lambda(xy) + k + t \wedge r > \tfrac{1-k}{2} + \tfrac{1-k}{2} + k = 1.$$

Hence $(xy, t \wedge r)\, q_k\, \lambda$, a contradiction. Thus $(xy, t \wedge r)\, \overline{\in}\, \lambda$, which implies from (3.17) that $(x, t)\, \overline{\in} \vee \overline{q_k}\, \lambda$ or $(y, r)\, \overline{\in} \vee \overline{q_k}\, \lambda$. Therefore λ is a $(\overline{q_k}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S. $\square$

Corollary 3.42. *Assume that* $t \wedge r > 0.5$ *for any* $t, r \in (0, 1]$. *Then every* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy subsemigroup is an* $(\overline{q}, \overline{\in} \vee \overline{q})$*-fuzzy subsemigroup.*

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Generalizations of $(\in, \in\vee\, q_k)$-fuzzy (generalized) bi-ideals in semigroups

Abstract The notion of $(\in, \in\vee\, q_k^\delta)$-fuzzy (generalized) bi-ideals in semigroups is introduced, and related properties are investigated. Given a (generalized) bi-ideal, an $(\in, \in\vee\, q_k^\delta)$-fuzzy (generalized) bi-ideal is constructed. Characterizations of an $(\in, \in\vee\, q_k^\delta)$-fuzzy (generalized) bi-ideal are discussed, and shown that an $(\in, \in\vee\, q_k^\delta)$-fuzzy generalized bi-ideal and an $(\in, \in\vee\, q_k^\delta)$-fuzzy bi-ideal coincide in regular semigroups. Using a fuzzy set with finite image, an $(\in, \in\vee\, q_k^\delta)$-fuzzy bi-ideal is established.

Keywords: $(\in, \in\vee\, q_k^\delta)$-fuzzy subsemigroup, $\in\vee\, q_k^\delta$-level subsemigroup/bi-ideal, $(\in, \in\vee\, q_k^\delta)$-fuzzy (generalized) bi-ideal.

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1 Introduction

Fuzzy points are applied to several algebraic structures (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [13], [14], [16], [18], [19], [20] and [21]). As a generalization of fuzzy bi-ideals in semigroups, Kazanci and Yamak [12] introduced $(\in, \in\vee\, q)$-fuzzy bi-ideals in semigroups. Jun et al. [8] considered more general forms of the paper [12], and discussed $(\in, \in\vee\, q_k)$-fuzzy bi-ideals in semigroups.

The aim of this paper is to study the general type of the paper [8]. We introduce the notion of $(\in, \in\vee\, q_k^\delta)$-fuzzy (generalized) bi-ideals in semigroups, and investigate related properties. Given a (generalized) bi-ideal, we construct an $(\in, \in\vee\, q_k^\delta)$-fuzzy (generalized) bi-ideal. We consider characterizations of an $(\in, \in\vee\, q_k^\delta)$-fuzzy (generalized) bi-ideal. We show that an $(\in, \in\vee\, q_k^\delta)$-fuzzy generalized bi-ideal and an $(\in, \in\vee\, q_k^\delta)$-fuzzy bi-ideal coincide in regular semigroups. Using a fuzzy set with finite image, we establish an $(\in, \in\vee\, q_k^\delta)$-fuzzy bi-ideal. We make an $(\in, \in\vee\, q_k^\delta)$-fuzzy bi-ideal generated by a fuzzy set.

2 Preliminaries

Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

Let S be a semigroup. By a *subsemigroup* of S we mean a nonempty subset A of S such that $A^2 \subseteq A$. A nonempty subset A of S is called a *generalized bi-ideal* of S if $ASA \subseteq A$. A nonempty subset A of S is called a *bi-ideal* of S if it is both a generalized bi-ideal and a subsemigroup of S.

For any fuzzy set λ in a set S and any $t \in [0,1]$, the set

$$U(\lambda; t) = \{x \in S \mid \lambda(x) \geq t\}$$

is called a *level subset* of λ.

A fuzzy set λ in a set S of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{2.1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by (x, t).

For a fuzzy set λ in a set S, a fuzzy point (x, t) is said to

- *contained* in λ, denoted by $(x, t) \in \lambda$ (see [15]), if $\lambda(x) \geq t$.
- *be quasi-coincident with* λ, denoted by $(x, t)\, q\, \lambda$ (see [15]), if $\lambda(x) + t > 1$.

For a fuzzy point (x, t) and a fuzzy set λ in a set S, we say that

- $(x, t) \in\vee\, q\, \lambda$ if $(x, t) \in \lambda$ or $(x, t)\, q\, \lambda$.

Jun [7] considered the general form of the symbol $(x, t)\, q\, \lambda$ as follows: For an arbitrary element k of $[0, 1)$, we say that

- $(x, t)\, q_k\, \lambda$ if $\lambda(x) + t + k > 1$.
- $(x, t) \in\vee\, q_k\, \lambda$ if $(x, t) \in \lambda$ or $(x, t)\, q_k\, \lambda$.

Jun et al. [10] considered the general form of the symbol $(x, t)\, q_k\, \lambda$ and $(x, t) \in\vee\, q_k\, \lambda$ as follows: For a fuzzy point (x, t) and a fuzzy set λ in a set S, we say that

- $(x, t)\, q_k^\delta\, \lambda$ if $\lambda(x) + t + k > \delta$,
- $(x, t) \in\vee\, q_k^\delta\, \lambda$ if $(x, t) \in \lambda$ or $(x, t)\, q_k^\delta\, \lambda$

where $k < \delta$ in $[0, 1]$. Obviously, $(x, t)\, q_0^\delta\, \lambda$ implies $(x, t)\, q_k^\delta\, \lambda$.

For any $\alpha \in \{\in, q, \in\vee\, q, \in\wedge\, q, \in\vee\, q_k, \in\vee\, q_k^\delta\}$, we say that

- $(x, t)\, \overline{\alpha}\, \lambda$ if $(x, t)\, \alpha\, \lambda$ does not hold.

3 General types of $(\in, \in\vee q_k)$-fuzzy bi-ideals

In what follows, let S denote a semigroup unless otherwise specified.

Definition 3.1 ([11]). A fuzzy set λ in S is called an $(\alpha, \in\vee q_k^\delta)$-*fuzzy subsemigroup* of S if it satisfies:

$$(x, t_1)\,\alpha\,\lambda,\ (y, t_2)\,\alpha\,\lambda\ \Rightarrow\ (xy, \min\{t_1, t_2\}) \in\vee q_k^\delta\,\lambda \tag{3.1}$$

for all $x, y \in S$ and $t_1, t_2 \in (0, \delta]$ where $\alpha \in \{\in, q_0^\delta\}$.

Definition 3.2. A fuzzy set λ in S is called an $(\alpha, \in\vee q_k^\delta)$-*fuzzy generalized bi-ideal* of S if it satisfies:

$$(x, t_x)\,\alpha\,\lambda,\ (z, t_z)\,\alpha\,\lambda\ \Rightarrow\ (xyz, \min\{t_x, t_z\}) \in\vee q_k^\delta\,\lambda \tag{3.2}$$

for all $x, y, z \in S$ and $t_x, t_z \in (0, \delta]$ where $\alpha \in \{\in, q_0^\delta\}$.

Example 3.3. Consider a semigroup $S = \{a, b, c, d\}$ with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

(1) Let λ be a fuzzy set in S defined by $\lambda(a) = 0.42$, $\lambda(b) = 0.40$, $\lambda(c) = 0.56$, and $\lambda(d) = 0.22$. Then λ is an $(\in, \in\vee q_{0.1}^{0.9})$-fuzzy generalized bi-ideal of S which is also an $(\in, \in\vee q_{0.1}^{0.9})$-fuzzy subsemigroup of S.

(2) Let μ be a fuzzy set in S defined by $\mu(a) = 0.6$, $\mu(b) = 0.3$, $\mu(c) = 0.4$, and $\mu(d) = 0.2$. Then μ is an $(\in, \in\vee q_{0.05}^{0.95})$-fuzzy generalized bi-ideal of S which is not an $(\in, \in\vee q_{0.05}^{0.95})$-fuzzy subsemigroup of S.

Given a generalized bi-ideal A of S and a fuzzy set λ in S, we establish an $(\alpha, \in\vee q_k^\delta)$-fuzzy generalized bi-ideal of S for $\alpha \in \{\in, q_0^\delta\}$.

Theorem 3.4. *Let A be a generalized bi-ideal of S and λ a fuzzy set in S defined by*

$$\lambda(x) = \begin{cases} \varepsilon & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon \geq \frac{\delta - k}{2}$. Then λ is an $(\alpha, \in\vee q_k^\delta)$-fuzzy generalized bi-ideal of S for $\alpha \in \{\in, q_0^\delta\}$.

Proof. Let $x, y, z \in S$ and $t_x, t_z \in (0, \delta]$ be such that $(x, t_x)\, q_0^\delta\, \lambda$ and $(z, t_z)\, q_0^\delta\, \lambda$. Then $\lambda(x) + t_x > \delta$ and $\lambda(z) + t_z > \delta$. If $x \notin A$ or $z \notin A$, then $\lambda(x) = 0$ or $\lambda(z) = 0$. Hence $t_x > \delta$ or $t_z > \delta$ which is a contradiction. Thus $x, z \in A$. Since A is a generalized bi-ideal of S, we have $xyz \in A$ and so $\lambda(xyz) = \varepsilon \geq \frac{\delta-k}{2}$. If $\min\{t_x, t_z\} \leq \frac{\delta-k}{2}$, then $\lambda(xyz) \geq \min\{t_x, t_z\}$ and thus $(xyz, \min\{t_x, t_z\}) \in \lambda$. If $\min\{t_x, t_z\} > \frac{\delta-k}{2}$, then

$$\lambda(xyz) + \min\{t_x, t_z\} + k > \tfrac{\delta-k}{2} + \tfrac{\delta-k}{2} + k = \delta,$$

that is, $(xyz, \min\{t_x, t_z\})\, q_k^\delta\, \lambda$. Therefore $(xyz, \min\{t_x, t_z\}) \in\vee\, q_k^\delta \lambda$. This shows that λ is a $(q_0^\delta, \in\vee\, q_k^\delta)$-fuzzy generalized bi-ideal of S.

Let $x, y, z \in S$ and $t_1, t_2 \in (0, \delta]$ be such that $(x, t_1) \in \lambda$ and $(z, t_2) \in \lambda$. Then $\lambda(x) \geq t_1 > 0$ and $\lambda(z) \geq t_2 > 0$. Thus $\lambda(x) = \varepsilon \geq \frac{\delta-k}{2}$ and $\lambda(z) = \varepsilon \geq \frac{\delta-k}{2}$, which imply that $x, z \in A$. Since A is a generalized bi-ideal of S, we have $xyz \in A$. Hence $\lambda(xyz) = \varepsilon \geq \frac{\delta-k}{2}$. If $\min\{t_1, t_2\} \leq \frac{\delta-k}{2}$, then $\lambda(xyz) \geq \min\{t_1, t_2\}$ and so $(xyz, \min\{t_1, t_2\}) \in \lambda$. If $\min\{t_1, t_2\} > \frac{\delta-k}{2}$, then $\lambda(xyz) + \min\{t_1, t_2\} + k > \frac{\delta-k}{2} + \frac{\delta-k}{2} + k = \delta$ and thus $(xyz, \min\{t_1, t_2\})\, q_k^\delta\, \lambda$. Therefore $(xyz, \min\{t_1, t_2\}) \in\vee\, q_k^\delta \lambda$, and λ is an $(\in, \in\vee\, q_k^\delta)$-fuzzy generalized bi-ideal of S. $\square$

Corollary 3.5 ([17]). *Let A be a generalized bi-ideal of S and λ a fuzzy set in S defined by*

$$\lambda(x) = \begin{cases} \varepsilon & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon \geq \frac{1-k}{2}$. Then λ is an $(\alpha, \in\vee\, q_k)$-fuzzy generalized bi-ideal of S for $\alpha \in \{\in, q\}$.

Corollary 3.6. *Let A be a generalized bi-ideal of S and λ a fuzzy set in S defined by*

$$\lambda(x) = \begin{cases} \varepsilon & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon \geq 0.5$. Then λ is an $(\alpha, \in\vee\, q)$-fuzzy generalized bi-ideal of S for $\alpha \in \{\in, q\}$.

We consider characterizations of an $(\in, \in\vee\, q_k^\delta)$-fuzzy generalized bi-ideal.

Theorem 3.7. *A fuzzy set λ in S is an $(\in, \in\vee\, q_k^\delta)$-fuzzy generalized bi-ideal of S if and only if it satisfies:*

$$(\forall x, y, z \in S)(\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \tfrac{\delta-k}{2}\}). \tag{3.3}$$

Proof. Let λ be an $(\in, \in\vee\, q_k^\delta)$-fuzzy generalized bi-ideal of S. Assume that there exist $a, c \in S$ such that

$$\lambda(abc) < \min\{\lambda(a), \lambda(c), \tfrac{\delta-k}{2}\}$$

for all $b \in S$. If $\min\{\lambda(a), \lambda(c)\} < \frac{\delta-k}{2}$, then $\lambda(abc) < \min\{\lambda(a), \lambda(c)\}$. Hence

$$\lambda(abc) < t \leq \min\{\lambda(a), \lambda(c)\}$$

for some $t \in (0, \delta)$. It follows that $(a, t) \in \lambda$ and $(c, t) \in \lambda$, but $(abc, t)\overline{\in}\lambda$. Moreover, $\lambda(abc) + t < 2t < \delta - k$, and so $(abc, t)\,\overline{q_k^\delta}\,\lambda$. Consequently $(abc, t)\,\overline{\in \vee q_k^\delta}\,\lambda$, this is a contradiction. If $\min\{\lambda(a), \lambda(c)\} \geq \frac{\delta-k}{2}$, then $\lambda(a) \geq \frac{\delta-k}{2}$, $\lambda(c) \geq \frac{\delta-k}{2}$ and $\lambda(abc) < \frac{\delta-k}{2}$. Thus $(a, \frac{\delta-k}{2}) \in \lambda$ and $(c, \frac{\delta-k}{2}) \in \lambda$, but $(abc, \frac{\delta-k}{2})\overline{\in}\lambda$. Also,

$$\lambda(abc) + \tfrac{\delta-k}{2} < \tfrac{\delta-k}{2} + \tfrac{\delta-k}{2} = \delta - k,$$

i.e., $(abc, \frac{\delta-k}{2})\,\overline{q_k^\delta}\,\lambda$. Hence $(abc, \frac{\delta-k}{2})\,\overline{\in \vee q_k^\delta}\,\lambda$, again, a contradiction. Therefore (3.3) is valid.

Conversely, suppose that λ satisfies (3.3). Let $x, y, z \in S$ and $t_1, t_2 \in (0, \delta]$ be such that $(x, t_1) \in \lambda$ and $(z, t_2) \in \lambda$. Then

$$\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \tfrac{\delta-k}{2}\} \geq \min\{t_1, t_2, \tfrac{\delta-k}{2}\}.$$

Assume that $t_1 \leq \frac{\delta-k}{2}$ or $t_2 \leq \frac{\delta-k}{2}$. Then $\lambda(xyz) \geq \min\{t_1, t_2\}$, which implies that $(xyz, \min\{t_1, t_2\}) \in \lambda$. Now, suppose that $t_1 > \frac{\delta-k}{2}$ and $t_2 > \frac{\delta-k}{2}$. Then $\lambda(xyz) \geq \frac{\delta-k}{2}$, and thus

$$\lambda(xyz) + \min\{t_1, t_2\} > \tfrac{\delta-k}{2} + \tfrac{\delta-k}{2} = \delta - k,$$

i.e., $(xyz, \min\{t_1, t_2\})\, q_k^\delta\, \lambda$. Hence $(xyz, \min\{t_1, t_2\}) \in \vee\, q_k^\delta\, \lambda$, and consequently, λ is an $(\in, \in \vee\, q_k^\delta)$-fuzzy generalized bi-ideal of S. □

Theorem 3.8. *For a fuzzy set λ in S, the following are equivalent.*

(1) *λ is an $(\in, \in \vee\, q_k^\delta)$-fuzzy generalized bi-ideal of S.*

(2) *The level subset $U(\lambda; t)$ of λ is a generalized bi-ideal of S for all $t \in (0, \frac{\delta-k}{2}]$.*

Proof. Assume that λ is an $(\in, \in \vee\, q_k^\delta)$-fuzzy generalized bi-ideal of S. Let $t \in (0, \frac{\delta-k}{2}]$, $y \in S$ and $x, z \in U(\lambda; t)$. Then $\lambda(x) \geq t$ and $\lambda(z) \geq t$. It follows from (3.3) that

$$\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \tfrac{\delta-k}{2}\} \geq \min\{t, \tfrac{\delta-k}{2}\} = t$$

so that $xyz \in U(\lambda; t)$. Hence $U(\lambda; t)$ is a generalized bi-ideal of S.

Conversely, suppose that $U(\lambda; t)$ is a generalized bi-ideal of S for all $t \in (0, \frac{\delta-k}{2}]$. If (3.3) is not valid, then there exist $a, b, c \in S$ such that

$$\lambda(abc) < \min\{\lambda(a), \lambda(c), \tfrac{\delta-k}{2}\}$$

and that $\lambda(abc) < t \leq \min\{\lambda(a), \lambda(c), \frac{\delta-k}{2}\}$ for some $t \in (0, 1)$. Then $t \in (0, \frac{\delta-k}{2}]$ and $a, c \in U(\lambda; t)$. Since $U(\lambda; t)$ is a generalized bi-ideal of S, it follows that $abc \in U(\lambda; t)$ so that $\lambda(abc) \geq t$. This is a contradiction. Therefore (3.3) is valid, and λ is an $(\in, \in \vee\, q_k^\delta)$-fuzzy generalized bi-ideal of S by Theorem 3.7. □

Taking $k = 0$ and $\delta = 1$ in Theorem 3.8, we have the following corollary.

Corollary 3.9. *Let λ be a fuzzy set in S. Then λ is an $(\in, \in \vee\, q)$-fuzzy generalized bi-ideal of S if and only if the level subset $U(\lambda; t)$ of λ is a generalized bi-ideal of S for all $t \in (0, 0.5]$.*

Corollary 3.10 ([17]). *For a fuzzy set λ in S, the following are equivalent.*

(1) *λ is an $(\in, \in \vee\, q_k)$-fuzzy generalized bi-ideal of S.*

(2) *The level subset $U(\lambda; t)$ of λ is a generalized bi-ideal of S for all $t \in (0, \frac{1-k}{2}]$.*

Proof. Taking $\delta = 1$ in Theorem 3.8 induces the corollary. □

Definition 3.11. A fuzzy set λ in S is called an $(\in, \in \vee\, q_k^\delta)$-*fuzzy bi-ideal* of S if it is both an $(\in, \in \vee\, q_k^\delta)$-fuzzy subsemigroup and an $(\in, \in \vee\, q_k^\delta)$-fuzzy generalized bi-ideal of S.

An $(\in, \in \vee\, q_k^\delta)$-fuzzy bi-ideal of S with $\delta = 1$ is called an $(\in, \in \vee\, q_k)$-*fuzzy bi-ideal* of S (see [17]), and an $(\in, \in \vee\, q_k)$-fuzzy bi-ideal of S with $k = 0$ is called an $(\in, \in \vee\, q)$-fuzzy bi-ideal of S (see [12]).

Example 3.12. The fuzzy set λ in Example 3.3(1) is an $(\in, \in \vee q_{0.1}^{0.9})$-fuzzy bi-ideal of S.

Combining Theorem 3.4 and [11, Theorem 3.4], we have the following theorem.

Theorem 3.13. *Let A be a bi-ideal of S and λ a fuzzy set in S defined by*

$$\lambda(x) = \begin{cases} \varepsilon & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon \geq \frac{\delta-k}{2}$. Then λ is an $(\alpha, \in \vee\, q_k^\delta)$-fuzzy bi-ideal of S for $\alpha \in \{\in, q_0^\delta\}$.

Corollary 3.14. *Let A be a bi-ideal of S and λ a fuzzy set in S defined by*

$$\lambda(x) = \begin{cases} \varepsilon & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon \geq \frac{1-k}{2}$. Then λ is an $(\alpha, \in \vee\, q_k)$-fuzzy bi-ideal of S for $\alpha \in \{\in, q\}$.

We give characterizations of an $(\in, \in \vee\, q_k^\delta)$-fuzzy bi-ideal.

Theorem 3.15. *A fuzzy set λ in S is an $(\in, \in \vee\, q_k^\delta)$-fuzzy bi-ideal of S if and only if it satisfies (3.3) and*

$$(\forall x, y \in S)(\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\}). \tag{3.4}$$

Proof. It is by Theorem 3.7 and [11, Theorem 3.7]. □

Corollary 3.16. *A fuzzy set λ in S is an $(\in, \in\vee q_k)$-fuzzy bi-ideal of S if and only if it satisfies:*

$$(\forall x, y \in S)(\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \tfrac{1-k}{2}\}), \tag{3.5}$$

$$(\forall x, y, z \in S)(\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \tfrac{1-k}{2}\}). \tag{3.6}$$

Corollary 3.17 ([12]). *A fuzzy set λ in S is an $(\in, \in\vee q)$-fuzzy bi-ideal of S if and only if it satisfies:*

$$(\forall x, y \in S)(\lambda(xy) \geq \min\{\lambda(x), \lambda(y), 0.5\}), \tag{3.7}$$

$$(\forall x, y, z \in S)(\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), 0.5\}). \tag{3.8}$$

Theorem 3.18. *For a fuzzy set λ in S, the following are equivalent.*

(1) *λ is an $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal of S.*

(2) *The level subset $U(\lambda; t)$ of λ is a bi-ideal of S for all $t \in (0, \frac{\delta-k}{2}]$.*

Proof. It is by Theorem 3.8 and [11, Theorem 3.10]. □

Corollary 3.19 ([17]). *For a fuzzy set λ in S, the following are equivalent.*

(1) *λ is an $(\in, \in\vee q_k)$-fuzzy bi-ideal of S.*

(2) *The level subset $U(\lambda; t)$ of λ is a bi-ideal of S for all $t \in (0, \frac{1-k}{2}]$.*

Obviously, every $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal is an $(\in, \in\vee q_k^\delta)$-fuzzy generalized bi-ideal, but the converse is not true in general. In fact, the fuzzy set μ in Example 3.3(2) is an $(\in, \in\vee q_{0.05}^{0.95})$-fuzzy generalized bi-ideal of S which is not an $(\in, \in\vee q_{0.05}^{0.95})$-fuzzy bi-ideal of S.

We now consider conditions for an $(\in, \in\vee q_k^\delta)$-fuzzy generalized bi-ideal to be an $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal.

Theorem 3.20. *In a regular semigroup S, every $(\in, \in\vee q_k^\delta)$-fuzzy generalized bi-ideal is an $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal.*

Proof. Let λ be an $(\in, \in\vee q_k^\delta)$-fuzzy generalized bi-ideal of a regular semigroup S. Let $a, b \in S$. Then $b = bxb$ for some $x \in S$ since S is regular. Hence

$$\lambda(ab) = \lambda(a(bxb)) = \lambda(a(bx)b) \geq \min\{\lambda(a), \lambda(b), \tfrac{\delta-k}{2}\}.$$

This shows that λ is an $(\in, \in\vee q_k^\delta)$-fuzzy subsemigroup of S, and so λ is an $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal of S. □

Corollary 3.21 ([17]). *In a regular semigroup S, every $(\in, \in\vee q_k)$-fuzzy generalized bi-ideal is an $(\in, \in\vee q_k)$-fuzzy bi-ideal.*

Theorem 3.22. *If λ is an $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal of S, then the set*

$$\underline{Q}_k^\delta(\lambda; t) := \{x \in S \mid (x,t)\, \underline{q}_k^\delta\, \lambda\}, \tag{3.9}$$

where $(x,t)\, \underline{q}_k^\delta\, \lambda$ means $(x,t)\, q_k^\delta\, \lambda$ or $\lambda(x)+t+k=\delta$, is a bi-ideal of S for all $t \in (\frac{\delta-k}{2}, 1]$ with $\underline{Q}_k^\delta(\lambda; t) \neq \emptyset$.

Proof. Let $t \in (\frac{\delta-k}{2}, 1]$ be such that $\underline{Q}_k^\delta(\lambda; t) \neq \emptyset$. Let $x, z \in \underline{Q}_k^\delta(\lambda; t)$. Then $\lambda(x)+t+k \geq \delta$ and $\lambda(z)+t+k \geq \delta$. It follows from (3.4) and (3.3) that

$$\lambda(xz) \geq \min\{\lambda(x), \lambda(z), \tfrac{\delta-k}{2}\} \geq \min\{\delta-k-t, \tfrac{\delta-k}{2}\} = \delta-k-t,$$

and

$$\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \tfrac{\delta-k}{2}\} \geq \min\{\delta-k-t, \tfrac{\delta-k}{2}\} = \delta-k-t,$$

that is, $(xz,t)\, \underline{q}_k^\delta\, \lambda$ and $(xyz,t)\, \underline{q}_k^\delta\, \lambda$. Hence $xz, xyz \in \underline{Q}_k^\delta(\lambda; t)$ and therefore $\underline{Q}_k^\delta(\lambda; t)$ is a bi-ideal of S. □

Corollary 3.23. *If λ is an $(\in, \in\vee q_k)$-fuzzy bi-ideal of S, then the set*

$$\underline{Q}_k(\lambda; t) := \{x \in S \mid (x,t)\, \underline{q}_k\, \lambda\}, \tag{3.10}$$

where $(x,t)\, \underline{q}_k\, \lambda$ means $(x,t)\, q_k\, \lambda$ or $\lambda(x)+t+k=1$, is a bi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$ with $\underline{Q}_k(\lambda; t) \neq \emptyset$.

Corollary 3.24. *If λ is an $(\in, \in\vee q)$-fuzzy bi-ideal of S, then the set*

$$\underline{Q}(\lambda; t) := \{x \in S \mid (x,t)\, \underline{q}\, \lambda\}, \tag{3.11}$$

where $(x,t)\, \underline{q}\, \lambda$ means $(x,t)\, q\, \lambda$ or $\lambda(x)+t=1$, is a bi-ideal of S for all $t \in (0.5, 1]$ with $\underline{Q}(\lambda; t) \neq \emptyset$.

Theorem 3.25. *A fuzzy set λ in S is an $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal of S if and only if the set*

$$\underline{U}_k^\delta(\lambda; t) := U(\lambda; t) \cup \underline{Q}_k^\delta(\lambda; t)$$

is a bi-ideal of S for all $t \in (0, \delta]$.

We call $\underline{U}_{q_k}^\delta(\lambda; t)$ an $\in\vee q_k^\delta$*-level bi-ideal* of λ.

Proof. Assume that λ is an $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal of S. Let $x, y \in \underline{U}_k^\delta(\lambda; t)$ for $t \in (0, \delta]$. Then we can consider the following four cases:

(1) $x, y \in U(\lambda; t)$, i.e., $\lambda(x) \geq t$ and $\lambda(y) \geq t$,

(2) $x, y \in \underline{Q}_k^\delta(\lambda; t)$, i.e., $\lambda(x) + t + k \geq \delta$ and $\lambda(y) + t + k \geq \delta$,

(3) $x \in U(\lambda; t)$ and $y \in \underline{Q}_k^\delta(\lambda; t)$, i.e., $\lambda(x) \geq t$ and $\lambda(y) + t + k \geq \delta$,

(4) $x \in \underline{Q}_k^\delta(\lambda; t)$ and $y \in U(\lambda; t)$, i.e., $\lambda(x) + t + k \geq \delta$ and $\lambda(y) \geq t$.

For the case (1), we have

$$\begin{aligned}\lambda(xy) &\geq \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\} \geq \min\{t, \tfrac{\delta-k}{2}\} \\ &= \begin{cases} t & \text{if } t < \frac{\delta-k}{2}, \\ \frac{\delta-k}{2} & \text{if } t \geq \frac{\delta-k}{2}, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\lambda(xay) &\geq \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\} \geq \min\{t, \tfrac{\delta-k}{2}\} \\ &= \begin{cases} t & \text{if } t < \frac{\delta-k}{2}, \\ \frac{\delta-k}{2} & \text{if } t \geq \frac{\delta-k}{2} \end{cases}\end{aligned}$$

for all $a \in S$. Hence $xy \in U(\lambda; t)$ or $\lambda(xy) + t + k \geq \frac{\delta-k}{2} + \frac{\delta-k}{2} + k = \delta$, i.e., $xy \in \underline{Q}_k^\delta(\lambda; t)$. Therefore $xy \in \underline{U}_k^\delta(\lambda; t)$. Similarly, $xay \in \underline{U}_k^\delta(\lambda; t)$. The second case implies that

$$\begin{aligned}\lambda(xy) &\geq \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\} \geq \min\{\delta - k - t, \tfrac{\delta-k}{2}\} \\ &= \begin{cases} \frac{\delta-k}{2} & \text{if } t \leq \frac{\delta-k}{2}, \\ \delta - k - t & \text{if } t > \frac{\delta-k}{2}, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\lambda(xay) &\geq \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\} \geq \min\{\delta - k - t, \tfrac{\delta-k}{2}\} \\ &= \begin{cases} \frac{\delta-k}{2} & \text{if } t \leq \frac{\delta-k}{2}, \\ \delta - k - t & \text{if } t > \frac{\delta-k}{2} \end{cases}\end{aligned}$$

for all $a \in S$. Thus $\lambda(xy) \geq \frac{\delta-k}{2} \geq t$, i.e., $xy \in U(\lambda; t)$ or $\lambda(xy)+t+k \geq \delta-k-t+t+k = \delta$, i.e., $xy \in \underline{Q}_k^\delta(\lambda; t)$. Therefore $xy \in \underline{U}_k^\delta(\lambda; t)$. Similarly, $xay \in \underline{U}_k^\delta(\lambda; t)$. The case (3) induces

$$\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\} \geq \min\{t, \delta - k - t, \tfrac{\delta-k}{2}\}$$

and

$$\lambda(xay) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\} \geq \min\{t, \delta - k - t, \tfrac{\delta-k}{2}\}$$

for all $a \in S$. If $t \leq \frac{\delta-k}{2}$, then $\lambda(xy) \geq \min\{t, \delta - k - t\} = t$ and so $xy \in U(\lambda; t)$. If $t > \frac{\delta-k}{2}$, then $\lambda(xy) \geq \min\{\delta - k - t, \frac{\delta-k}{2}\} = \delta - k - t$ and thus $xy \in \underline{Q}_k^\delta(\lambda; t)$. Therefore

$xy \in \underline{U}_k^\delta(\lambda; t)$. Similarly, $xay \in \underline{U}_k^\delta(\lambda; t)$. The final case is similar to the third case. Consequently, $\underline{U}_k^\delta(\lambda; t)$ is a bi-ideal of S for all $t \in (0, \delta]$.

Conversely, let λ be a fuzzy set in S and $t \in (0, \delta]$ be such that $\underline{U}_k^\delta(\lambda; t)$ is a bi-ideal of S. Assume that there exist $a, b \in S$ such that $\lambda(ab) < \min\{\lambda(a), \lambda(b), \frac{\delta-k}{2}\}$. Then

$$\lambda(ab) < t \leq \min\{\lambda(a), \lambda(b), \tfrac{\delta-k}{2}\}$$

for some $t \in (0, \delta]$. Then $a, b \in U(\lambda; t) \subseteq \underline{U}_k^\delta(\lambda; t)$, which implies that $ab \in \underline{U}_k^\delta(\lambda; t)$. Hence $\lambda(ab) \geq t$ or $\lambda(ab) + t + k > \delta$, a contradiction. Therefore $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ for all $x, y \in S$. Similarly, we obtain $\lambda(xay) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ for all $a, x, y \in S$. Using Theorem 3.15, we conclude that λ is an $(\in, \in\vee\, q_k^\delta)$-fuzzy bi-ideal of S. $\square$

Corollary 3.26. *A fuzzy set λ in S is an $(\in, \in\vee\, q_k)$-fuzzy bi-ideal of S if and only if the set*

$$\underline{U}_k(\lambda; t) := U(\lambda; t) \cup \underline{Q}_k(\lambda; t)$$

is a bi-ideal of S for all $t \in (0, 1]$.

Corollary 3.27. *A fuzzy set λ in S is an $(\in, \in\vee\, q)$-fuzzy bi-ideal of S if and only if the set*

$$\underline{U}(\lambda; t) := U(\lambda; t) \cup \underline{Q}(\lambda; t)$$

is a bi-ideal of S for all $t \in (0, 1]$.

Let λ be a fuzzy set in S. For $\alpha \in \{\in\vee\, q, \in\vee\, q_k, \in\vee\, q_k^\delta\}$, an $(\in, \alpha)$-fuzzy bi-ideal μ in S is said to be an $(\in, \alpha)$-*fuzzy bi-ideal generated* by λ in S if

(i) $\lambda \subseteq \mu$, that is, $\lambda(x) \leq \mu(x)$ for all $x \in S$,

(ii) For any $(\in, \alpha)$-fuzzy bi-ideal γ in S, if $\lambda \subseteq \gamma$ then $\mu \subseteq \gamma$.

Theorem 3.28. *Let λ be a fuzzy set in S with finite image. Define bi-ideals A_i of S as follows:*

$$A_0 = \langle\{x \in S \mid \lambda(x) \geq \tfrac{\delta-k}{2}\}\rangle,$$

$$A_i = \langle A_{i-1} \cup \{x \in S \mid \lambda(x) = \sup\{\lambda(z) \mid z \in S \setminus A_{i-1}\}\rangle$$

for $i = 1, 2, \cdots, n$ where $n < |\mathrm{Im}(\lambda)|$ and $A_n = S$. Let λ^* be a fuzzy set in S defined by

$$\lambda^*(x) = \begin{cases} \lambda(x) & \text{if } x \in A_0, \\ \sup\{\lambda(z) \mid z \in S \setminus A_{i-1}\} & \text{if } x \in A_i \setminus A_{i-1}. \end{cases}$$

Then λ^* is the $(\in, \in\vee\, q_k^\delta)$-fuzzy bi-ideal generated by λ in S.

Proof. Note that the A_i's form a chain

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = S$$

of bi-ideals ending at S. We first show that λ^* is an $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal of S. Let $x, y \in S$. If $x, y \in A_0$, then $xy \in A_0$ and $xay \in A_0$ for all $a \in S$. Hence

$$\lambda^*(xy) = \lambda(xy) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\} = \min\{\lambda^*(x), \lambda^*(y), \tfrac{\delta-k}{2}\}.$$

and

$$\lambda^*(xay) = \lambda(xay) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\} = \min\{\lambda^*(x), \lambda^*(y), \tfrac{\delta-k}{2}\}.$$

Let $x \in A_i \setminus A_{i-1}$ and $y \in A_j \setminus A_{j-1}$. We may assume that $i < j$ without loss of generality. Then $x, y \in A_j$ and so $xy \in A_j$ and $xay \in A_j$ for all $a \in S$. It follows that

$$\begin{aligned}\lambda^*(xy) &\geq \sup\{\lambda(z) \mid z \in S \setminus A_{j-1}\}\\ &\geq \min\{\sup\{\lambda(z) \mid z \in S \setminus A_{i-1}\}, \sup\{\lambda(z) \mid z \in S \setminus A_{j-1}\}, \tfrac{\delta-k}{2}\}\\ &= \min\{\lambda^*(x), \lambda^*(y), \tfrac{\delta-k}{2}\}.\end{aligned}$$

and

$$\begin{aligned}\lambda^*(xay) &\geq \sup\{\lambda(w) \mid w \in S \setminus A_{j-1}\}\\ &\geq \min\{\sup\{\lambda(w) \mid w \in S \setminus A_{i-1}\}, \sup\{\lambda(w) \mid w \in S \setminus A_{j-1}\}, \tfrac{\delta-k}{2}\}\\ &= \min\{\lambda^*(x), \lambda^*(y), \tfrac{\delta-k}{2}\}.\end{aligned}$$

Hence λ^* is an $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal of S whose $\in\vee q_k^\delta$-level bi-ideals are precisely the members of the chain above. Obviously, $\lambda \subseteq \lambda^*$ by the construction of λ^*. Now let μ be any $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal of S such that $\lambda \subseteq \mu$. If $x \in A_0$, then $\lambda^*(x) = \lambda(x) \leq \mu(x)$. Let $\{B_{t_i}\}$ be the class of $\in\vee q_k^\delta$-level bi-ideals of μ in S. Let $x \in A_1 \setminus A_0$. Then $\lambda^*(x) = \sup\{\lambda(z) \mid z \in S \setminus A_0\}$ and $A_1 = \langle K_1 \rangle$ where

$$K_1 = A_0 \cup \{x \in S \mid \lambda(x) = \sup\{\lambda(z) \mid z \in S \setminus A_0\}\}.$$

Let $x \in K_1 \setminus A_0$. Then $\lambda(x) = \sup\{\lambda(z) \mid z \in S \setminus A_0\}$. Since $\lambda \subseteq \mu$, it follows that

$$\sup\{\lambda(z) \mid z \in S \setminus A_0\} \leq \inf\{\mu(x) \mid x \in K_1 \setminus A_0\} \leq \mu(x).$$

Putting $t_{i1} = \inf\{\mu(x) \mid x \in K_1 \setminus A_0\}$, we get $x \in B_{t_{i1}}$ and hence $K_1 \setminus A_0 \subseteq B_{t_{i1}}$. Since $A_0 \subseteq B_{t_{i1}}$, we have $A_1 = \langle K_1 \rangle \subseteq B_{t_{i1}}$. Thus $\mu(x) \geq t_{i1}$ for all $x \in A_1$. Therefore

$$\lambda^*(x) = \sup\{\lambda(z) \mid z \in S \setminus A_0\} \leq t_{i1} \leq \mu(x)$$

for all $x \in A_1 \setminus A_0$. Similarly, we can prove that $\lambda^*(x) \leq \mu(x)$ for all $x \in A_i \setminus A_{i-1}$ where $2 \leq i \leq n$. Consequently, λ^* is the $(\in, \in\vee q_k^\delta)$-fuzzy bi-ideal generated by λ in S. $\square$

Corollary 3.29. *Let λ be a fuzzy set in S with finite image. Define bi-ideals A_i of S as follows:*

$$A_0 = \langle\{x \in S \mid \lambda(x) \geq \tfrac{1-k}{2}\}\rangle,$$

$$A_i = \langle A_{i-1} \cup \{x \in S \mid \lambda(x) = \sup\{\lambda(z) \mid z \in S \setminus A_{i-1}\}\rangle$$

for $i = 1, 2, \cdots, n$ where $n < |\mathrm{Im}(\lambda)|$ and $A_n = S$. Let λ^* be a fuzzy set in S defined by

$$\lambda^*(x) = \begin{cases} \lambda(x) & \text{if } x \in A_0, \\ \sup\{\lambda(z) \mid z \in S \setminus A_{i-1}\} & \text{if } x \in A_i \setminus A_{i-1}. \end{cases}$$

Then λ^* is the $(\in, \in\vee\, q_k)$-fuzzy bi-ideal generated by λ in S.

Corollary 3.30. *Let λ be a fuzzy set in S with finite image. Define bi-ideals A_i of S as follows:*

$$A_0 = \langle\{x \in S \mid \lambda(x) \geq 0.5\}\rangle,$$

$$A_i = \langle A_{i-1} \cup \{x \in S \mid \lambda(x) = \sup\{\lambda(z) \mid z \in S \setminus A_{i-1}\}\rangle$$

for $i = 1, 2, \cdots, n$ where $n < |\mathrm{Im}(\lambda)|$ and $A_n = S$. Let λ^* be a fuzzy set in S defined by

$$\lambda^*(x) = \begin{cases} \lambda(x) & \text{if } x \in A_0, \\ \sup\{\lambda(z) \mid z \in S \setminus A_{i-1}\} & \text{if } x \in A_i \setminus A_{i-1}. \end{cases}$$

Then λ^* is the $(\in, \in\vee\, q)$-fuzzy bi-ideal generated by λ in S.

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Approximations of fuzzy sets in semigroups

Abstract Lower and upper approximations of fuzzy sets in semigroups are considered, and several properties are investigated.

Keywords: δ-lower (δ-upper) approximation of fuzzy set, δ-lower (δ-upper) rough fuzzy subsemigroup, δ-rough fuzzy subsemigroup.

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1 Introduction

The notion of rough sets was introduced by Pawlak in his paper [9]. This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis (see [10]). Rough set theory is applied to semigroups and groups (see [3, 5, 6, 7, 11, 13]), d-algebras (see [1]), BE-algebras (see [2]), BCK-algebras (see [4]) and MV-algebras (see [12]) etc.

In this paper, we investigate some properties of the lower and upper approximations of fuzzy sets with respect to the congruences in semigroups.

2 Preliminaries

Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

Let S be a semigroup. By a *subsemigroup* of S we mean a nonempty subset A of S such that $A^2 \subseteq A$. A nonempty subset A of S is called a *left* (*right*) *ideal* of S if $SA \subseteq A$ ($AS \subseteq A$). A nonempty subset A of S is called an *interior ideal* of S if $SAS \subseteq A$.

For the sake of convenience, we may regard the empty set to be a subsemigroup, a left (right) ideal and an interior ideal.

For fuzzy sets λ and μ in a set S, we say that $\lambda \leq \mu$ if $\lambda(x) \leq \mu(x)$ for all $x \in S$. We define $\lambda \vee \mu$ and $\lambda \wedge \mu$ by $(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}$ and $(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}$, respectively, for all $x \in S$.

For any fuzzy set λ in a set S and any $t \in [0,1]$, the set

$$U(\lambda;t) = \{x \in S \mid \lambda(x) \geq t\}$$

is called a *level subset* of λ. For two fuzzy sets λ and μ in S, the *product* of λ and μ, denoted by $\lambda \circ \mu$, is defined by

$$\lambda \circ \mu : S \to [0,1],\ x \mapsto \sup_{x=yz} \min\{\lambda(y), \mu(z)\}.$$

A fuzzy set λ in a semigroup S is called a *fuzzy subsemigroup* of S if it satisfies:

$$(\forall x, y \in S)\,(\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\})\,. \tag{2.1}$$

A fuzzy set λ in a semigroup S is called a *fuzzy left (right) ideal* of S if it satisfies:

$$(\forall x, y \in S)\,(\lambda(xy) \geq \lambda(y)\ \ (\lambda(xy) \geq \lambda(x)))\,. \tag{2.2}$$

A fuzzy set λ in a semigroup S is called a *fuzzy interior ideal* of S if it satisfies:

$$(\forall x, a, y \in S)\,(\lambda(xay) \geq \lambda(a))\,. \tag{2.3}$$

We refer the reader to the book [8] for further information regarding (fuzzy) semigroups.

3 Approximations of fuzzy sets

In what follows, let S denote a semigroup unless otherwise specified.

By a *congruence* on S (see [6]), we mean an equivalence relation δ on S such that

$$(\forall a, b, x \in S)\,((a,b) \in \delta\ \Rightarrow\ (ax, bx) \in \delta\ \text{ and }\ (xa, xb) \in \delta)\,. \tag{3.1}$$

We denote by $[a]_\delta$ the δ-congruence class containing $a \in S$. Note that if δ is a congruence on S, then

$$(\forall a, b \in S)\,([a]_\delta[b]_\delta \subseteq [ab]_\delta)\,. \tag{3.2}$$

A congruence δ on S is said to be *complete* (see [6]) if it satisfies:

$$(\forall a, b \in S)\,([a]_\delta[b]_\delta = [ab]_\delta)\,. \tag{3.3}$$

For a nonempty subset A of S, the sets

$$\delta_*(A) := \{x \in S \mid [x]_\delta \subseteq A\} \tag{3.4}$$

and

$$\delta^*(A) := \{x \in S \mid [x]_\delta \cap A \neq \emptyset\} \tag{3.5}$$

are called the δ-*lower* and δ-*upper approximations*, respectively, of A (see [6]).

The ordered pair $\delta(A) := (\delta_*(A), \delta^*(A))$ is called a δ-*rough subset* of $2^S \times 2^S$ if $\delta_*(A) \neq \delta^*(A)$.

Proposition 3.1 ([6]). *Let δ and ε be congruences on S and let A and B be subsets of S. Then*

(1) $\delta_*(A) \subseteq A \subseteq \delta^*(A)$,

(2) $\delta^*(A \cup B) = \delta^*(A) \cup \delta^*(B)$,

(3) $\delta_*(A \cap B) = \delta_*(A) \cap \delta_*(B)$,

(4) $A \subseteq B \;\Rightarrow\; \delta_*(A) \subseteq \delta_*(B),\; \delta^*(A) \subseteq \delta^*(B)$,

(5) $\delta_*(A) \cup \delta_*(B) \subseteq \delta_*(A \cup B)$,

(6) $\delta^*(A \cap B) \subseteq \delta^*(A) \cap \delta^*(B)$,

(7) $\varepsilon \subseteq \delta \;\Rightarrow\; \delta_*(A) \subseteq \varepsilon_*(A),\; \varepsilon^*(A) \subseteq \delta^*(A)$,

(8) $\delta^*(A)\delta^*(B) \subseteq \delta^*(AB)$,

(9) *If δ is complete, then* $\delta_*(A)\delta_*(B) \subseteq \delta_*(AB)$,

(10) $(\delta \cap \varepsilon)^*(A) \subseteq \delta^*(A) \cap \varepsilon^*(A)$,

(11) $(\delta \cap \varepsilon)_*(A) = \delta_*(A) \cap \varepsilon_*(A)$.

Definition 3.2 ([2, 11]). Let δ be a congruence on S. Given a fuzzy set λ in S, the fuzzy sets $\delta_*(\lambda)$ and $\delta^*(\lambda)$ are defined as follows:

$$\delta_*(\lambda) : S \to [0,1],\; x \mapsto \inf_{y \in [x]_\delta} \lambda(y)$$

and

$$\delta^*(\lambda) : S \to [0,1],\; x \mapsto \sup_{y \in [x]_\delta} \lambda(y),$$

which are called the δ-*lower* and δ-*upper approximations*, respectively, of λ.

We say that $\delta(\lambda) \triangleq (\delta_*(\lambda), \delta^*(\lambda))$ is a δ-*rough fuzzy set* of λ if $\delta_*(\lambda) \neq \delta^*(\lambda)$.

Theorem 3.3. *Let $f : S \to T$ be an onto homomorphism of semigroups. For a relation δ on T, let*

$$\varepsilon := \{(x, y) \in S \times S \mid (f(x), f(y)) \in \delta\}. \tag{3.6}$$

(1) *If δ is congruence on T, then ε is a congruence on S.*

(2) *If δ is complete and f is one-one, then ε is complete.*

(3) $f(\varepsilon^*(A)) = \delta^*(f(A))$ *for any subset A of S.*

(4) *$f(\varepsilon_*(A)) \subseteq \delta_*(f(A))$ for any subset A of S.*

(5) *If f is one-one, then the equality in (4) is valid.*

Proof. (1) Assume that δ is congruence on T. Obviously, ε is an equivalence relation on S. Let $(a,b) \in \varepsilon$ for $a, b \in S$. Then $(f(a), f(b)) \in \delta$. Since f is onto homomorphism and δ is congruence on T, it follows that

$$(f(ax), f(bx)) = (f(a)f(x), f(b)f(x)) \in \delta$$

and

$$(f(xa), f(xb)) = (f(x)f(a), f(x)f(b)) \in \delta$$

for all $x \in S$. Hence $(ax, bx) \in \varepsilon$ and $(xa, xb) \in \varepsilon$. Therefore ε is a congruence on S.

(2) Suppose that δ is complete and f is one-one. For any $a, b \in S$, let $z \in [ab]_\varepsilon$. Then $(z, ab) \in \varepsilon$ and so $(f(z), f(ab)) \in \delta$. Since δ is complete, it follows that

$$f(z) \in [f(ab)]_\delta = [f(a)f(b)]_\delta = [f(a)]_\delta [f(b)]_\delta,$$

which implies that there exist $x, y \in S$ such that $f(z) = f(x)f(y) = f(xy)$, $f(x) \in [f(a)]_\delta$ and $f(y) \in [f(b)]_\delta$. Since f is one-one, it follows that $z = xy$, $x \in [a]_\varepsilon$ and $y \in [b]_\varepsilon$. Hence $z \in [a]_\varepsilon [b]_\varepsilon$, and so $[ab]_\varepsilon \subseteq [a]_\varepsilon [b]_\varepsilon$. It follows from (3.2) that $[ab]_\varepsilon = [a]_\varepsilon [b]_\varepsilon$, and consequently ε is complete.

(3) Let $y \in f(\varepsilon^*(A))$. Then $f(x) = y$ for some $x \in \varepsilon^*(A)$, and thus $[x]_\varepsilon \cap A \neq \emptyset$, say $a \in [x]_\varepsilon \cap A$. Then $f(a) \in f(A)$ and $(f(a), f(x)) \in \delta$, i.e., $f(a) \in [f(x)]_\delta$. Hence $[f(x)]_\delta \cap f(A) \neq \emptyset$, which implies $y = f(x) \in \delta^*(f(A))$. Therefore $f(\varepsilon^*(A)) \subseteq \delta^*(f(A))$. Now let $y \in \delta^*(f(A))$. Since f is onto, there exists $x \in S$ such that $y = f(x)$. Hence $[f(x)]_\delta \cap f(A) \neq \emptyset$, say $b \in [f(x)]_\delta \cap f(A)$. Then there exists $a \in A$ such that $b = f(a) \in f(A)$ and $f(a) = b \in [f(x)]_\delta$, i.e., $(f(a), f(x)) \in \delta$. Thus $(a, x) \in \varepsilon$ and so $a \in [x]_\varepsilon$. Hence $[x]_\varepsilon \cap A \neq \emptyset$ which implies $x \in \varepsilon^*(A)$. Therefore $y = f(x) \in f(\varepsilon^*(A))$ which shows that $\delta^*(f(A)) \subseteq f(\varepsilon^*(A))$.

(4) If $y \in f(\varepsilon_*(A))$, then $y = f(x)$ for some $x \in \varepsilon_*(A)$. Hence $[x]_\varepsilon \subseteq A$. Now, if $b \in [y]_\delta$, then there exists $a \in S$ such that $f(a) = b \in [y]_\delta = [f(x)]_\delta$. It follows that $a \in [x]_\varepsilon \subseteq A$ and so that $b = f(a) \in f(A)$. Thus $[y]_\delta \subseteq f(A)$, which induces $y \in \delta_*(f(A))$. Hence $f(\varepsilon_*(A)) \subseteq \delta_*(f(A))$.

(5) Assume that f is one-one and let $y \in \delta_*(f(A))$. Then there exists $x \in S$ such that $y = f(x)$ and $[f(x)]_\delta = [y]_\delta \subseteq f(A)$. Let $a \in [x]_\varepsilon$. Then $f(a) \in [f(x)]_\delta \subseteq f(A)$, and so $a \in A$ since f is one-one. Hence $[x]_\varepsilon \subseteq A$, and thus $x \in \varepsilon_*(A)$ which implies that $y = f(x) \in f(\varepsilon_*(A))$. Therefore $\delta_*(f(A)) \subseteq f(\varepsilon_*(A))$. Combing this and (4) induces $f(\varepsilon_*(A)) = \delta_*(f(A))$. □

Theorem 3.4. *Let ε and δ be congruences on S. If λ and μ are fuzzy sets in S, then the following assertions are valid.*

(1) $\delta_*(\lambda) \leq \lambda \leq \delta^*(\lambda)$,

(2) $\delta^*(\lambda \vee \mu) = \delta^*(\lambda) \vee \delta^*(\mu)$,

(3) $\delta_*(\lambda \wedge \mu) = \delta_*(\lambda) \wedge \delta_*(\mu)$,

(4) *If* $\lambda \leq \mu$, *then* $\delta_*(\lambda) \leq \delta_*(\mu)$ *and* $\delta^*(\lambda) \leq \delta^*(\mu)$,

(5) $\delta_*(\lambda) \vee \delta_*(\mu) \leq \delta_*(\lambda \vee \mu)$,

(6) $\delta^*(\lambda \wedge \mu) \leq \delta^*(\lambda) \wedge \delta^*(\mu)$,

(7) *If* $\delta \subseteq \varepsilon$, *then* $\varepsilon_*(\lambda) \leq \delta_*(\lambda)$ *and* $\varepsilon^*(\lambda) \geq \delta^*(\lambda)$,

(8) $(\delta \cap \varepsilon)^*(\lambda) \leq \delta^*(\lambda) \wedge \varepsilon^*(\lambda)$,

(9) $(\delta \cap \varepsilon)_*(\lambda) \geq \delta_*(\lambda) \vee \varepsilon_*(\lambda)$.

Proof. (1) Since $x \in [x]_\delta$ for all $x \in S$, we have

$$\delta_*(\lambda)(x) = \inf_{y\in[x]_\delta} \lambda(y) \leq \lambda(x) \leq \sup_{y\in[x]_\delta} \lambda(y) = \delta^*(\lambda)(x)$$

which proves (1).

(2) For any $x \in S$, we have

$$\begin{aligned} \delta^*(\lambda \vee \mu)(x) &= \sup_{y\in[x]_\delta} (\lambda \vee \mu)(x) = \sup_{y\in[x]_\delta} \max\{\lambda(y), \mu(y)\} \\ &= \max\left\{ \sup_{y\in[x]_\delta} \lambda(y), \sup_{y\in[x]_\delta} \mu(y) \right\} \\ &= \max\{\delta^*(\lambda)(x), \delta^*(\mu)(x)\} \\ &= (\delta^*(\lambda) \vee \delta^*(\mu))(x), \end{aligned}$$

and so $\delta^*(\lambda \vee \mu) = \delta^*(\lambda) \vee \delta^*(\mu)$.

(3) For any $x \in S$, we have

$$\begin{aligned} \delta_*(\lambda \wedge \mu)(x) &= \inf_{y\in[x]_\delta} (\lambda \wedge \mu)(x) = \inf_{y\in[x]_\delta} \min\{\lambda(y), \mu(y)\} \\ &= \min\left\{ \inf_{y\in[x]_\delta} \lambda(y), \inf_{y\in[x]_\delta} \mu(y) \right\} \\ &= \min\{\delta_*(\lambda)(x), \delta_*(\mu)(x)\} \\ &= (\delta_*(\lambda) \wedge \delta_*(\mu))(x), \end{aligned}$$

which shows that $\delta_*(\lambda \wedge \mu) = \delta_*(\lambda) \wedge \delta_*(\mu)$.

(4) Assume that $\lambda \leq \mu$. Then $\lambda \wedge \mu = \lambda$ and $\lambda \vee \mu = \mu$. Using (2) and (3), we have

$$\delta^*(\mu) = \delta^*(\lambda \vee \mu) = \delta^*(\lambda) \vee \delta^*(\mu)$$

and

$$\delta_*(\lambda) = \delta_*(\lambda \wedge \mu) = \delta_*(\lambda) \wedge \delta_*(\mu).$$

Hence $\delta_*(\lambda) \leq \delta_*(\mu)$ and $\delta^*(\lambda) \leq \delta^*(\mu)$.

(5) Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$, it follows from (4) that $\delta_*(\lambda) \leq \delta_*(\lambda \vee \mu)$ and $\delta_*(\mu) \leq \delta_*(\lambda \vee \mu)$. Therefore $\delta_*(\lambda) \vee \delta_*(\mu) \leq \delta_*(\lambda \vee \mu)$.

(6) Since $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$, it follows from (4) that $\delta^*(\lambda \wedge \mu) \leq \delta^*(\lambda)$ and $\delta^*(\lambda \wedge \mu) \leq \delta^*(\mu)$. Thus $\delta^*(\lambda \wedge \mu) \leq \delta^*(\lambda) \wedge \delta^*(\mu)$.

(7) Let $x \in S$. If $\delta \subseteq \varepsilon$, then $[x]_\delta \subseteq [x]_\varepsilon$. Hence

$$\varepsilon_*(\lambda)(x) = \inf_{y \in [x]_\varepsilon} \lambda(y) \leq \inf_{y \in [x]_\delta} \lambda(y) = \delta_*(\lambda)(x)$$

and

$$\varepsilon^*(\lambda)(x) = \sup_{y \in [x]_\varepsilon} \lambda(y) \geq \sup_{y \in [x]_\delta} \lambda(y) = \delta^*(\lambda)(x).$$

Therefore $\varepsilon_*(\lambda) \leq \delta_*(\lambda)$ and $\varepsilon^*(\lambda) \geq \delta^*(\lambda)$.

(8) For any $x \in S$, we get

$$\begin{aligned}(\delta \cap \varepsilon)^*(\lambda)(x) &= \sup_{y \in [x]_{\delta \cap \varepsilon}} \lambda(y) = \sup_{y \in [x]_\delta \cap [x]_\varepsilon} \lambda(y) \\ &\leq \min\left\{ \sup_{y \in [x]_\delta} \lambda(y), \sup_{y \in [x]_\varepsilon} \lambda(y) \right\} \\ &= \min\{\delta^*(\lambda)(x), \varepsilon^*(\lambda)(x)\} \\ &= (\delta^*(\lambda) \wedge \varepsilon^*(\lambda))(x),\end{aligned}$$

and so $(\delta \cap \varepsilon)^*(\lambda) \leq \delta^*(\lambda) \wedge \varepsilon^*(\lambda)$.

(9) For any $x \in S$, we obtain

$$\begin{aligned}(\delta \cap \varepsilon)_*(\lambda)(x) &= \inf_{y \in [x]_{\delta \cap \varepsilon}} \lambda(y) = \inf_{y \in [x]_\delta \cap [x]_\varepsilon} \lambda(y) \\ &\geq \max\left\{ \inf_{y \in [x]_\delta} \lambda(y), \inf_{y \in [x]_\varepsilon} \lambda(y) \right\} \\ &= \max\{\delta_*(\lambda)(x), \varepsilon_*(\lambda)(x)\} \\ &= (\delta_*(\lambda) \vee \varepsilon_*(\lambda))(x),\end{aligned}$$

which shows that $(\delta \cap \varepsilon)_*(\lambda) \geq \delta_*(\lambda) \vee \varepsilon_*(\lambda)$. □

Theorem 3.5. *Let δ be a congruence on S. If λ is a fuzzy set in S, then*

$$U(\delta_*(\lambda);t) = \delta_*(U(\lambda;t)) \text{ and } U(\delta^*(\lambda);t) = \delta^*(U(\lambda;t))$$

for all $t \in (0,1]$.

Proof. For any $t \in (0,1]$ and $x \in S$, we have

$$\begin{aligned} x \in U(\delta_*(\lambda);t) &\Leftrightarrow \delta_*(\lambda)(x) \geq t \\ &\Leftrightarrow \inf_{y \in [x]_\delta} \lambda(y) \geq t \\ &\Leftrightarrow \lambda(y) \geq t \text{ for all } y \in [x]_\delta \\ &\Leftrightarrow y \in U(\lambda;t) \text{ for all } y \in [x]_\delta \\ &\Leftrightarrow [x]_\delta \subseteq U(\lambda;t) \\ &\Leftrightarrow x \in \delta_*(U(\lambda;t)), \end{aligned}$$

and

$$\begin{aligned} x \in U(\delta^*(\lambda);t) &\Leftrightarrow \delta^*(\lambda)(x) \geq t \\ &\Leftrightarrow \sup_{y \in [x]_\delta} \lambda(y) \geq t \\ &\Leftrightarrow \lambda(y) \geq t \text{ for some } y \in [x]_\delta \\ &\Leftrightarrow y \in U(\lambda;t) \text{ for some } y \in [x]_\delta \\ &\Leftrightarrow [x]_\delta \cap U(\lambda;t) \neq \emptyset \\ &\Leftrightarrow x \in \delta^*(U(\lambda;t)). \end{aligned}$$

Therefore $U(\delta_*(\lambda);t) = \delta_*(U(\lambda;t))$ and $U(\delta^*(\lambda);t) = \delta^*(U(\lambda;t))$. □

Theorem 3.6. *Let δ be a congruence on S and let λ and μ be fuzzy sets in S. Then*

(1) $\delta^*(\lambda) \circ \delta^*(\mu) \leq \delta^*(\lambda \circ \mu)$,

(2) $\delta_*(\lambda) \circ \delta_*(\mu) \leq \delta_*(\lambda \circ \mu)$ *if δ is complete.*

Proof. For any $x \in S$, we have

$$\begin{aligned}
(\delta^*(\lambda) \circ \delta^*(\mu))(x) &= \sup_{x=yz} \min\{\delta^*(\lambda)(y), \delta^*(\mu)(z)\} \\
&= \sup_{x=yz} \min\left\{ \sup_{a\in[y]_\delta} \lambda(a),\ \sup_{b\in[z]_\delta} \mu(b) \right\} \\
&= \sup_{x=yz} \left(\sup_{a\in[y]_\delta,\ b\in[z]_\delta} \min\{\lambda(a), \mu(b)\} \right) \\
&\leq \sup_{x=yz} \left(\sup_{ab\in[yz]_\delta} \min\{\lambda(a), \mu(b)\} \right) \\
&= \sup_{ab\in[x]_\delta} \min\{\lambda(a), \mu(b)\} \\
&= \sup_{c\in[x]_\delta,\ c=ab} \min\{\lambda(a), \mu(b)\} \\
&= \sup_{c\in[x]_\delta} \left(\sup_{c=ab} \min\{\lambda(a), \mu(b)\} \right) \\
&= \sup_{c\in[x]_\delta} (\lambda \circ \mu)(c) \\
&= \delta^*(\lambda \circ \mu)(x),
\end{aligned}$$

which shows that $\delta^*(\lambda) \circ \delta^*(\mu) \leq \delta^*(\lambda \circ \mu)$.

Assume that δ is complete and let $x \in S$. Then

$$\begin{aligned}
(\delta_*(\lambda) \circ \delta_*(\mu))(x) &= \sup_{x=yz} \min\{\delta_*(\lambda)(y), \delta_*(\mu)(z)\} \\
&= \sup_{x=yz} \min\left\{ \inf_{a\in[y]_\delta} \lambda(a),\ \inf_{b\in[z]_\delta} \mu(b) \right\} \\
&= \sup_{x=yz} \left(\inf_{a\in[y]_\delta,\ b\in[z]_\delta} \min\{\lambda(a), \mu(b)\} \right) \\
&\leq \sup_{x=yz} \left(\inf_{a\in[y]_\delta,\ b\in[z]_\delta} \sup_{ab=cd} \min\{\lambda(c), \mu(d)\} \right) \\
&= \sup_{x=yz} \left(\inf_{a\in[y]_\delta,\ b\in[z]_\delta} (\lambda \circ \mu)(ab) \right) \\
&= \sup_{x=yz} \left(\inf_{ab\in[yz]_\delta} (\lambda \circ \mu)(ab) \right) \\
&= \sup_{x=yz} \delta_*(\lambda \circ \mu)(yz) \\
&= \delta_*(\lambda \circ \mu)(x).
\end{aligned}$$

Therefore $\delta_*(\lambda) \circ \delta_*(\mu) \leq \delta_*(\lambda \circ \mu)$. □

Definition 3.7. Let δ be a congruence on S. A fuzzy set λ in S is called a *δ-lower* (resp., *δ-upper*) *rough fuzzy subsemigroup* of S if $\delta_*(\lambda)$ (resp., $\delta^*(\lambda)$) is a fuzzy subsemigroup of S.

We say that $\delta(\lambda) \triangleq (\delta_*(\lambda), \delta^*(\lambda))$ is a *δ-rough fuzzy subsemigroup* of S if

(i) $\delta(\lambda)$ is a δ-rough fuzzy set,

(ii) $\delta_*(\lambda)$ and $\delta^*(\lambda)$ are fuzzy subsemigroups of S.

Theorem 3.8. *If δ is a congruence on S, then every fuzzy subsemigroup of S is a δ-upper rough fuzzy subsemigroup of S. Moreover, if δ is a complete congruence on S, then the δ-lower approximation of a fuzzy subsemigroup of S is a fuzzy subsemigroup of S.*

Proof. Let λ be a fuzzy subsemigroup of S. Then $U(\lambda;t)$ is a subsemigroup of S for all $t \in [0,1]$. Using (8) and (4) in Proposition 3.1, we have

$$\delta^*(U(\lambda;t))\delta^*(U(\lambda;t)) \subseteq \delta^*(U(\lambda;t)U(\lambda;t)) \subseteq \delta^*(U(\lambda;t)).$$

It follows from Theorem 3.5 that $U(\delta^*(\lambda);t) = \delta^*(U(\lambda;t))$ is a subsemigroup of S. Therefore $\delta^*(\lambda;t)$ is a fuzzy subsemigroup of S.

Now assume that δ is complete. Using (9) and (4) in Proposition 3.1, we have

$$\delta_*(U(\lambda;t))\delta_*(U(\lambda;t)) \subseteq \delta_*(U(\lambda;t)U(\lambda;t)) \subseteq \delta_*(U(\lambda;t)).$$

Hence, by Theorem 3.5, we know that $U(\delta_*(\lambda);t) = \delta_*(U(\lambda;t))$ is a subsemigroup of S. Therefore $\delta_*(\lambda;t)$ is a fuzzy subsemigroup of S. $\square$

Corollary 3.9. *Let δ be a complete congruence on S and λ a fuzzy set in S such that $\delta(\lambda)$ is a δ-rough fuzzy set. If λ is a fuzzy subsemigroup of S, then $\delta(\lambda) \triangleq (\delta_*(\lambda), \delta^*(\lambda))$ is a δ-rough fuzzy subsemigroup of S.*

Theorem 3.10. *If δ is a congruence on S, then every fuzzy interior ideal of S is a δ-upper rough fuzzy interior ideal of S. Moreover, if δ is a complete congruence on S, then the δ-lower approximation of a fuzzy interior ideal of S is a fuzzy interior of S.*

Proof. Note that a fuzzy set λ in S is a fuzzy interior ideal of S if and only if $U(\lambda;t)$ is an interior ideal of S for all $t \in [0,1]$. Hence the proof is similar to the proof of Theorem 3.8. $\square$

Theorem 3.11. *Let $f : S \to T$ be an onto homomorphism of semigroups. For a relation δ on T, let ε be a relation on S which is given in Theorem 3.3. If the ε-upper approximation of A is a subsemigroup of S, then the δ-upper approximation of $f(A)$ is a subsemigroup of T where A is a subset of S. Also, the converse is valid if f is one-one.*

Proof. Assume that $\varepsilon^*(A)$ is a subsemigroup of S. Let $x, y \in \delta^*(f(A))$. Then $x, y \in f(\varepsilon^*(A))$ by Theorem 3.3(3), and so there exist $a, b \in \varepsilon^*(A)$ such that $f(a) = x$ and $f(b) = y$. Then $ab \in \varepsilon^*(A)$, and thus

$$xy = f(a)f(b) = f(ab) \in f(\varepsilon^*(A)) = \delta^*(f(A)).$$

Hence $\delta^*(f(A))$ is a subsemigroup of T. Now, suppose that f is one-one and $\delta^*(f(A))$ is a subsemigroup of T. Let $x, y \in \varepsilon^*(A)$. Then $f(x), f(y) \in f(\varepsilon^*(A)) = \delta^*(f(A))$, and so

$$f(xy) = f(x)f(y) \in \delta^*(f(A)) = f(\varepsilon^*(A)).$$

Hence there exists $a \in \varepsilon^*(A)$ such that $f(xy) = f(a)$. Since f is one-one, it follows that $[a]_\varepsilon \cap A \neq \emptyset$ and $xy \in [a]_\varepsilon$. Thus $[xy]_\varepsilon \cap A \neq \emptyset$, and so $xy \in \varepsilon^*(A)$. Therefore $\varepsilon^*(A)$ is a subsemigroup of S. □

Theorem 3.12. *Let $f : S \to T$ be an isomorphism of semigroups. For a congruence δ on T, let ε be a relation on S which is given in Theorem 3.3. If the ε-lower approximation of A is a subsemigroup of S, then the δ-lower approximation of $f(A)$ is a subsemigroup of T where A is a subset of S. Also the converse is true if ε is complete.*

Proof. Suppose that $\varepsilon_*(A)$ is a subsemigroup of S. Let $x, y \in \delta_*(f(A))$. Then $x, y \in f(\varepsilon_*(A))$ by Theorem 3.3(5), and thus $x = f(a)$ and $y = f(b)$ for some $a, b \in \varepsilon_*(A)$. Then $ab \in \varepsilon_*(A)$ and

$$xy = f(a)f(b) = f(ab) \in f(\varepsilon_*(A)) = \delta_*(f(A)).$$

Therefore $\delta_*(f(A))$ is a subsemigroup of T.

Conversely, assume that $\delta_*(f(A))$ is a subsemigroup of T and ε is complete. Let $x, y \in \varepsilon_*(A)$. Then $f(x), f(y) \in f(\varepsilon_*(A)) = \delta_*(f(A))$, and so $f(xy) = f(x)f(y) \in \delta_*(f(A))$. It follows that

$$\begin{aligned} f([xy]_\varepsilon) &= f([x]_\varepsilon[y]_\varepsilon) = f([x]_\varepsilon)f([y]_\varepsilon) \\ &= [f(x)]_\delta[f(y)]_\delta \subseteq [f(x)f(y)]_\delta \\ &= [f(xy)]_\delta \subseteq f(A) \end{aligned}$$

and so that $[xy]_\varepsilon \subseteq A$. Thus $xy \in \varepsilon_*(A)$, and $\varepsilon_*(A)$ is a subsemigroup of S. □

Theorem 3.13. *If δ is a congruence on S, then the δ-rough fuzzy set of a fuzzy left ideal is a fuzzy left ideal.*

Proof. Let λ be a fuzzy left ideal of S and let $x, y \in S$. Then

$$\delta^*(\lambda)(xy) = \sup_{z \in [xy]_\delta} \lambda(z) \geq \sup_{b \in [y]_\delta} \lambda(xb) \geq \sup_{b \in [y]_\delta} \lambda(b) = \delta^*(\lambda)(y).$$

Also, we get

$$\delta_*(\lambda)(xy) = \inf_{z\in[xy]_\delta} \lambda(z) \geq \inf_{b\in[y]_\delta} \lambda(xb) \geq \inf_{b\in[y]_\delta} \lambda(b) = \delta_*(\lambda)(y).$$

Hence $\delta(\lambda) \triangleq (\delta_*(\lambda), \delta^*(\lambda))$ is a fuzzy left ideal of S. $\square$

Similarly, we have

Theorem 3.14. *If δ is a congruence on S, then the δ-rough fuzzy set of a fuzzy right ideal is a fuzzy right ideal.*

In the following example, we show that there exists a fuzzy set such that its upper approximation is a fuzzy left ideal, but it is not a fuzzy left ideal.

Example 3.15. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table (Table 1).

Table 1: Cayley table of the operation $\cdot$

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	c	c	c
d	d	c	b	a

Let δ be a congruence on S such that the δ-congruence classes are the subsets $\{a\}$, $\{d\}$ and $\{b, c\}$. Let λ be a fuzzy set in S given by $\lambda(a) = \lambda(c) = \lambda(d) = 0.4$ and $\lambda(b) = 0.8$. Then λ is not a fuzzy left ideal of S since $\lambda(cb) = \lambda(c) = 0.4 < 0.8 = \lambda(b)$. The δ-upper approximation of λ is given as follows: $\delta^*(\lambda)(a) = \delta^*(\lambda)(d) = 0.4$ and $\delta^*(\lambda)(b) = \delta^*(\lambda)(c) = 0.8$. It is routine to verify that $\delta^*(\lambda)$ is a fuzzy left ideal of S.

Theorem 3.16. *Let δ be a congruence on S. If λ is a fuzzy right ideal and μ is a fuzzy left ideal of S, then*

$$\delta^*(\lambda \circ \mu) \leq \delta^*(\lambda) \wedge \delta^*(\mu) \quad \text{and} \quad \delta_*(\lambda \circ \mu) \leq \delta_*(\lambda) \wedge \delta_*(\mu). \tag{3.7}$$

Proof. Let $x \in S$. Then

$$\begin{aligned}
\delta^*(\lambda \circ \mu)(x) &= \sup_{y \in [x]_\delta} (\lambda \circ \mu)(y) \\
&= \sup_{y \in [x]_\delta} \left(\sup_{y=ab} \min\{\lambda(a), \mu(b)\} \right) \\
&\leq \sup_{y \in [x]_\delta} \left(\sup_{y=ab} \min\{\lambda(ab), \mu(ab)\} \right) \\
&= \sup_{y \in [x]_\delta} \min\{\lambda(y), \mu(y)\} \\
&\leq \sup_{a \in [x]_\delta,\ b \in [x]_\delta} \min\{\lambda(a), \mu(b)\} \\
&= \min\left\{ \sup_{a \in [x]_\delta} \lambda(a), \sup_{b \in [x]_\delta} \mu(b) \right\} \\
&= \min\{\delta^*(\lambda)(x), \delta^*(\mu)(x)\} \\
&= \left(\delta^*(\lambda) \wedge \delta^*(\mu)\right)(x),
\end{aligned}$$

and

$$\begin{aligned}
\delta_*(\lambda \circ \mu)(x) &= \inf_{y \in [x]_\delta} (\lambda \circ \mu)(y) \\
&= \inf_{y \in [x]_\delta} \left(\sup_{y=ab} \min\{\lambda(a), \mu(b)\} \right) \\
&\leq \inf_{y \in [x]_\delta} \left(\sup_{y=ab} \min\{\lambda(ab), \mu(ab)\} \right) \\
&= \inf_{y \in [x]_\delta} \min\{\lambda(y), \mu(y)\} \\
&= \min\left\{ \inf_{a \in [x]_\delta} \lambda(a), \inf_{b \in [x]_\delta} \mu(b) \right\} \\
&= \min\{\delta_*(\lambda)(x), \delta_*(\mu)(x)\} \\
&= \left(\delta_*(\lambda) \wedge \delta_*(\mu)\right)(x).
\end{aligned}$$

Therefore $\delta^*(\lambda \circ \mu) \leq \delta^*(\lambda) \wedge \delta^*(\mu)$ and $\delta_*(\lambda \circ \mu) \leq \delta_*(\lambda) \wedge \delta_*(\mu)$. □

Theorem 3.17. *Let δ be a congruence on S and let λ and μ be a fuzzy right ideal and a fuzzy left ideal, respectively, of S. If S is regular, then $\delta^*(\lambda \circ \mu) = \delta^*(\lambda) \wedge \delta^*(\mu)$ and $\delta_*(\lambda \circ \mu) = \delta_*(\lambda) \wedge \delta_*(\mu)$.*

Proof. Let a be any element of S. Then $a = aca$ for some $c \in S$ since S is regular. Hence we have

$$
\begin{aligned}
\delta^*(\lambda \circ \mu)(x) &= \sup_{a \in [x]_\delta} (\lambda \circ \mu)(a) \\
&= \sup_{a \in [x]_\delta} \left(\sup_{a=yz} \min\{\lambda(y), \mu(z)\} \right) \\
&\geq \sup_{a \in [x]_\delta} \min\{\lambda(ac), \mu(a)\} \\
&\geq \sup_{a \in [x]_\delta} \min\{\lambda(a), \mu(a)\} \\
&= \min \left\{ \sup_{a \in [x]_\delta} \lambda(a), \sup_{a \in [x]_\delta} \mu(a) \right\} \\
&= \min\{\delta^*(\lambda)(x), \delta^*(\mu)(x)\} \\
&= \left(\delta^*(\lambda) \wedge \delta^*(\mu)\right)(x)
\end{aligned}
$$

for all $x \in S$. Hence $\delta^*(\lambda \circ \mu) \geq \delta^*(\lambda) \wedge \delta^*(\mu)$. Similarly, we have $\delta_*(\lambda \circ \mu) \geq \delta_*(\lambda) \wedge \delta_*(\mu)$. Therefore $\delta^*(\lambda \circ \mu) = \delta^*(\lambda) \wedge \delta^*(\mu)$ and $\delta_*(\lambda \circ \mu) = \delta_*(\lambda) \wedge \delta_*(\mu)$ by Theorem 3.16. □

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Neutrosophic $\mathcal{N}$-structures and their applications in semigroups

Abstract The notion of neutrosophic $\mathcal{N}$-structure is introduced, and applied it to semigroup. The notions of neutrosophic $\mathcal{N}$-subsemigroup, neutrosophic $\mathcal{N}$-product and ε-neutrosophic $\mathcal{N}$-subsemigroup are introduced, and several properties are investigated. Conditions for neutrosophic $\mathcal{N}$-structure to be neutrosophic $\mathcal{N}$-subsemigroup are provided. Using neutrosophic $\mathcal{N}$-product, characterization of neutrosophic $\mathcal{N}$-subsemigroup is discussed. Relations between neutrosophic $\mathcal{N}$-subsemigroup and ε-neutrosophic $\mathcal{N}$-subsemigroup are discussed. We show that the homomorphic preimage of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup, and the onto homomorphic image of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup.

Keywords: Neutrosophic $\mathcal{N}$-structure, neutrosophic $\mathcal{N}$-subsemigroup, ε-neutrosophic $\mathcal{N}$-subsemigroup, neutrosophic $\mathcal{N}$-product.

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1 Introduction

Zadeh [9] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [2] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components

$$(\text{t, i, f}) = (\text{truth, indeterminacy, falsehood}).$$

For more detail, refer to the site

http://fs.gallup.unm.edu/FlorentinSmarandache.htm.

The concept of neutrosophic set (NS) developed by Smarandache [7] and Smarandache [8] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part (refer to the site
http://fs.gallup.unm.edu/neutrosophy.htm).

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [3] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. This structure is applied to BE-algebra, BCK/BCI-algebra and BCH-algebra etc. (see [1], [3], [4], [5]).

In this paper, we introduce the notion of neutrosophic $\mathcal{N}$-structure and applied it to semigroup. We introduce the notion of neutrosophic $\mathcal{N}$-subsemi-group and investigate several properties. We provide conditions for neutrosophic $\mathcal{N}$-structure to be neutrosophic $\mathcal{N}$-subsemigroup. We define neutrosophic $\mathcal{N}$-product, and give characterization of neutrosophic $\mathcal{N}$-subsemigroup by using neutrosophic $\mathcal{N}$-product. We also introduce ε-neutrosophic subsemigroup, and investigate relations between neutrosophic subsemigroup and ε-neutrosophic subsemigroup. We show that the homomorphic preimage of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemi-group, and the onto homomorphic image of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup.

2 Preliminaries

Let X be a semigroup. Let A and B be subsets of X. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in X \mid a \in A,\ b \in B\}.$$

By a *subsemigroup* of X we mean a nonempty subset A of X such that $A^2 \subseteq A$. We consider the empty set $\emptyset$ is always a subsemigroup of X.

We refer the reader to the book [6] for further information regarding fuzzy semigroups.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

For any real numbers a and b, we also use $a \vee b$ and $a \wedge b$ instead of $\bigvee\{a, b\}$ and $\bigwedge\{a, b\}$, respectively.

3 Neutrosophic $\mathcal{N}$-structures

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X. In what follows, let X denote the nonempty universe of discourse unless otherwise specified.

Definition 3.1. A *neutrosophic $\mathcal{N}$-structure* over X is defined to be the structure

$$X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\} \tag{3.1}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on X.

Note that every neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X satisfies the condition:

$$(\forall x \in X)\,(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)\,.$$

Example 3.2. Consider a universe of discourse $X = \{x, y, z\}$. We know that

$$X_{\mathbf{N}} = \left\{ \frac{x}{(-0.7, -0.5, -0.1)}, \frac{y}{(-0.2, -0.3, -0.4)}, \frac{z}{(-0.3, -0.6, -0.1)} \right\}$$

is a neutrosophic $\mathcal{N}$-structure over X.

Definition 3.3. Let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ and $X_{\mathbf{M}} := \frac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-structures over X. We say that $X_{\mathbf{M}}$ is a *neutrosophic $\mathcal{N}$-substructure* over X, denoted by $X_{\mathbf{N}} \subseteq X_{\mathbf{M}}$, if it satisfies:

$$(\forall x \in X)(T_N(x) \geq T_M(x), I_N(x) \leq I_M(x), F_N(x) \geq F_M(x)).$$

If $X_{\mathbf{N}} \subseteq X_{\mathbf{M}}$ and $X_{\mathbf{M}} \subseteq X_{\mathbf{N}}$, we say that $X_{\mathbf{N}} = X_{\mathbf{M}}$.

Definition 3.4. Let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ and $X_{\mathbf{M}} := \frac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-structures over X.

(1) The *union* of $X_{\mathbf{N}}$ and $X_{\mathbf{M}}$ is defined to be a neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N} \cup \mathbf{M}} = (X; T_{N \cup M}, I_{N \cup M}, F_{N \cup M})$$

where

$T_{N \cup M}(x) = \bigwedge\{T_N(x), T_M(x)\}$, $I_{N \cup M}(x) = \bigvee\{I_N(x), I_M(x)\}$ and
$F_{N \cup M}(x) = \bigwedge\{F_N(x), F_M(x)\}$

for all $x \in X$.

(2) The *intersection* of $X_{\mathbf{N}}$ and $X_{\mathbf{M}}$ is defined to be a neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}\cap\mathbf{M}} = (X; T_{N\cap M}, I_{N\cap M}, F_{N\cap M})$$

where
$T_{N\cap M}(x) = \bigvee\{T_N(x), T_M(x)\}$, $I_{N\cap M}(x) = \bigwedge\{I_N(x), I_M(x)\}$ and
$F_{N\cap M}(x) = \bigvee\{F_N(x), F_M(x)\}$
for all $x \in X$.

Definition 3.5. Given a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ over X, the *complement* of $X_{\mathbf{N}}$ is defined to be a neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}^c} := \frac{X}{(T_{N^c}, I_{N^c}, F_{N^c})}$$

over X where

$$T_{N^c}(x) = -1 - T_N(x),\ I_{N^c}(x) = -1 - I_N(x) \text{ and } F_{N^c}(x) = -1 - F_N(x)$$

for all $x \in X$.

Example 3.6. Let $X = \{a, b, c\}$ be a universe of discourse and let $X_{\mathbf{N}}$ be the neutrosophic $\mathcal{N}$-structure over X in Example 3.2. Let $X_{\mathbf{M}}$ be a neutrosophic $\mathcal{N}$-structure over X which is given by

$$X_{\mathbf{M}} = \left\{ \frac{x}{(-0.3, -0.5, -0.2)}, \frac{y}{(-0.4, -0.2, -0.2)}, \frac{z}{(-0.5, -0.7, -0.8)} \right\}.$$

The union and intersection of $X_{\mathbf{N}}$ and $X_{\mathbf{M}}$ are given as follows respectively:

$$X_{\mathbf{N}\cup\mathbf{M}} = \left\{ \frac{x}{(-0.7, -0.5, -0.2)}, \frac{y}{(-0.4, -0.3, -0.4)}, \frac{z}{(-0.5, -0.7), -0.8)} \right\}$$

and

$$X_{\mathbf{N}\cap\mathbf{M}} = \left\{ \frac{x}{(-0.3, -0.5, -0.1)}, \frac{y}{(-0.2, -0.2, -0.2)}, \frac{z}{(-0.3, -0.6, -0.1)} \right\}.$$

The complement of $X_{\mathbf{N}}$ is given by

$$X_{\mathbf{M}^c} = \left\{ \frac{x}{(-0.7, -0.5, -0.8)}, \frac{y}{(-0.6, -0.8, -0.8)}, \frac{z}{(-0.5, -0.3, -0.2)} \right\}.$$

4 Applications in semigroups

In this section, we take a semigroup X as the universe of discourse unless otherwise specified.

Definition 4.1. A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a *neutrosophic $\mathcal{N}$-subsemigroup* of X if the following condition is valid:

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(xy) \leq \bigvee\{T_N(x), T_N(y)\} \\ I_N(xy) \geq \bigwedge\{I_N(x), I_N(y)\} \\ F_N(xy) \leq \bigvee\{F_N(x), F_N(y)\} \end{array} \right). \tag{4.1}$$

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets.

$$\begin{aligned} T_N^\alpha &:= \{x \in X \mid T_N(x) \leq \alpha\}, \\ I_N^\beta &:= \{x \in X \mid I_N(x) \geq \beta\}, \\ F_N^\gamma &:= \{x \in X \mid F_N(x) \leq \gamma\}. \end{aligned} \tag{4.2}$$

The set

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called a (α, β, γ)-*level set* of $X_{\mathbf{N}}$. Note that

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma.$$

Theorem 4.2. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X, then the (α, β, γ)-level set of $X_{\mathbf{N}}$ is a subsemigroup of X whenever it is nonempty.*

Proof. Assume that $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Let $x, y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Then $T_N(x) \leq \alpha$, $I_N(x) \geq \beta$, $F_N(x) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from (4.1) that
$T_N(xy) \leq \bigvee\{T_N(x), T_N(y)\} \leq \alpha$,
$I_N(xy) \geq \bigwedge\{I_N(x), I_N(y)\} \geq \beta$, and
$F_N(xy) \leq \bigvee\{F_N(x), F_N(y)\} \leq \gamma$.
Hence $xy \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$, and therefore $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is a subsemigroup of X. □

Theorem 4.3. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If T_N^α, I_N^β and F_N^γ are subsemigroups of X, then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X.*

Proof. Assume that there are $a, b \in X$ such that $T_N(ab) > \bigvee\{T_N(a), T_N(b)\}$. Then $T_N(ab) > t_\alpha \geq \bigvee\{T_N(a), T_N(b)\}$ for some $t_\alpha \in [-1, 0)$. Hence $a, b \in T_N^{t_\alpha}$ but $ab \notin T_N^{t_\alpha}$, which is a contradiction. Thus

$$T_N(xy) \leq \bigvee\{T_N(x), T_N(y)\}$$

for all $x, y \in X$. If $I_N(ab) < \bigwedge\{I_N(a), I_N(b)\}$ for some $a, b \in X$, then $a, b \in I_N^{t_\beta}$ and $ab \notin I_N^{t_\beta}$ for $t_\beta := \bigwedge\{I_N(a), I_N(b)\}$. This is a contradiction, and so

$$I_N(xy) \geq \bigwedge\{I_N(x), I_N(y)\}$$

for all $x, y \in X$. Now, suppose that there exist $a, b \in X$ and $t_\gamma \in [-1, 0)$ such that

$$F_N(ab) > t_\gamma \geq \bigvee\{F_N(a), F_N(b)\}.$$

Then $a, b \in F_N^{t_\gamma}$ and $ab \notin F_N^{t_\gamma}$, which is a contradiction. Hence

$$F_N(xy) \leq \bigvee\{F_N(x), F_N(y)\}$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X. □

Theorem 4.4. *The intersection of two neutrosophic $\mathcal{N}$-subsemigroups is also a neutrosophic $\mathcal{N}$-subsemigroup.*

Proof. Let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ and $X_{\mathbf{M}} := \frac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-subsemi-groups of X. For any $x, y \in X$, we have

$$\begin{aligned} T_{N\cap M}(xy) &= \bigvee\{T_N(xy), T_M(xy)\} \\ &\leq \bigvee\left\{\bigvee\{T_N(x), T_N(y)\}, \bigvee\{T_M(x), T_M(y)\}\right\} \\ &= \bigvee\left\{\bigvee\{T_N(x), T_M(x)\}, \bigvee\{T_N(y), T_M(y)\}\right\} \\ &= \bigvee\{T_{N\cap M}(x), T_{N\cap M}(y)\}, \end{aligned}$$

$$\begin{aligned} I_{N\cap M}(xy) &= \bigwedge\{I_N(xy), I_M(xy)\} \\ &\geq \bigwedge\left\{\bigwedge\{I_N(x), I_N(y)\}, \bigwedge\{I_M(x), I_M(y)\}\right\} \\ &= \bigwedge\left\{\bigwedge\{I_N(x), I_M(x)\}, \bigwedge\{I_N(y), I_M(y)\}\right\} \\ &= \bigwedge\{I_{N\cap M}(x), I_{N\cap M}(y)\} \end{aligned}$$

and

$$\begin{aligned} F_{N\cap M}(xy) &= \bigvee\{F_N(xy), F_M(xy)\} \\ &\leq \bigvee\left\{\bigvee\{F_N(x), F_N(y)\}, \bigvee\{F_M(x), F_M(y)\}\right\} \\ &= \bigvee\left\{\bigvee\{F_N(x), F_M(x)\}, \bigvee\{F_N(y), F_M(y)\}\right\} \\ &= \bigvee\{F_{N\cap M}(x), F_{N\cap M}(y)\} \end{aligned}$$

for all $x, y \in X$. Hence $X_{\mathbf{N}\cap\mathbf{M}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X. □

Corollary 4.5. *If* $\{X_{N_i} \mid i \in \mathbb{N}\}$ *is a family of neutrosophic* $\mathcal{N}$*-subsemigroups of* X*, then so is* $X_{\cap N_i}$.

Let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ and $X_{\mathbf{M}} := \frac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-structures over X. The *neutrosophic* $\mathcal{N}$*-product* of $X_{\mathbf{N}}$ and $X_{\mathbf{M}}$ is defined to be a neutrosophic $\mathcal{N}$-structure over X

$$\begin{aligned} X_{\mathbf{N}} \odot X_{\mathbf{M}} &= \frac{X}{T_{N\circ M},\, I_{N\circ M},\, F_{N\circ M}} \\ &= \left\{ \frac{x}{T_{N\circ M}(x),\, I_{N\circ M}(x),\, F_{N\circ M}(x)} \mid x \in X \right\} \end{aligned}$$

where

$$T_{N\circ M}(x) = \begin{cases} \bigwedge\limits_{x=yz} \{T_N(y) \vee T_M(z)\} & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{N\circ M}(x) = \begin{cases} \bigvee\limits_{x=yz} \{I_N(y) \wedge I_M(z)\} & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ -1 & \text{otherwise} \end{cases}$$

and

$$F_{N\circ M}(x) = \begin{cases} \bigwedge\limits_{x=yz} \{F_N(y) \vee F_M(z)\} & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0 & \text{otherwise.} \end{cases}$$

For any $x \in X$, the element $\frac{x}{T_{N\circ M}(x),\, I_{N\circ M}(x),\, F_{N\circ M}(x)}$ is simply denoted by

$$(X_{\mathbf{N}} \odot X_{\mathbf{M}})(x) := (T_{N\circ M}(x),\, I_{N\circ M}(x),\, F_{N\circ M}(x))$$

for the sake of convenience.

Theorem 4.6. *A neutrosophic* $\mathcal{N}$*-structure* $X_{\mathbf{N}}$ *over* X *is a neutrosophic* $\mathcal{N}$*-subsemigroup of* X *if and only if* $X_{\mathbf{N}} \odot X_{\mathbf{N}} \subseteq X_{\mathbf{N}}$.

Proof. Assume that $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X and let $x \in X$. If $x \neq yz$ for all $x, y \in X$, then clearly $X_{\mathbf{N}} \odot X_{\mathbf{N}} \subseteq X_{\mathbf{N}}$. Assume that there exist $a, b \in X$ such that $x = ab$.

$$T_{N\circ N}(x) = \bigwedge_{x=ab} \{T_N(a) \vee T_N(b)\} \geq \bigwedge_{x=ab} T_N(ab) = T_N(x),$$

$$I_{N\circ N}(x) = \bigvee_{x=ab} \{I_N(a) \wedge I_N(b)\} \leq \bigvee_{x=ab} I_N(ab) = I_N(x),$$

and

$$F_{N\circ N}(x) = \bigwedge_{x=ab} \{F_N(a) \vee F_N(b)\} \geq \bigwedge_{x=ab} F_N(ab) = F_N(x).$$

Therefore $X_{\mathbf{N}} \odot X_{\mathbf{N}} \subseteq X_{\mathbf{N}}$.

Conversely, let $X_{\mathbf{N}}$ be any neutrosophic $\mathcal{N}$-structure over X such that $X_{\mathbf{N}} \odot X_{\mathbf{N}} \subseteq X_{\mathbf{N}}$. Let x and y be any elements of X and let $a = xy$. Then

$$T_N(xy) = T_N(a) \leq T_{N\circ N}(a) = \bigwedge_{a=bc} \{T_N(b) \vee T_N(c)\} \leq T_N(x) \vee T_N(y),$$

$$I_N(xy) = I_N(a) \geq I_{N\circ N}(a) = \bigvee_{a=bc} \{I_N(b) \wedge I_N(c)\} \geq I_N(x) \wedge I_N(y),$$

and

$$F_N(xy) = F_N(a) \leq F_{N\circ N}(a) = \bigwedge_{a=bc} \{F_N(b) \vee F_N(c)\} \leq F_N(x) \vee F_N(y).$$

Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X. $\square$

Since $[-1, 0]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 4.7. *If $\{X_{N_i} \mid i \in \mathbb{N}\}$ is a family of neutrosophic $\mathcal{N}$-subsemigroups of X, then $(\{X_{N_i} \mid i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Theorem 4.8. *Let X be a semigroup with identity e and let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over X such that*

$$(\forall x \in X)\,(X_{\mathbf{N}}(e) \geq X_{\mathbf{N}}(x)),$$

that is, $T_N(e) \leq T_N(x)$, $I_N(e) \geq I_N(x)$ and $F_N(e) \leq F_N(x)$ for all $x \in X$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X, then $X_{\mathbf{N}}$ is neutrosophic idempotent, that is, $X_{\mathbf{N}} \odot X_{\mathbf{N}} = X_{\mathbf{N}}$.

Proof. For any $x \in X$, we have

$$T_{N\circ N}(x) = \bigwedge_{x=yz} \{T_N(y) \vee T_N(z)\} \leq T_N(x) \vee T_N(e) = T_N(x),$$

$$I_{N\circ N}(x) = \bigvee_{x=yz} \{I_N(y) \wedge I_N(z)\} \geq I_N(x) \wedge I_N(e) = I_N(x)$$

and

$$F_{N\circ N}(x) = \bigwedge_{x=yz} \{F_N(y) \vee F_N(z)\} \leq F_N(x) \vee F_N(e) = F_N(x).$$

This shows that $X_{\mathbf{N}} \subseteq X_{\mathbf{N}} \odot X_{\mathbf{N}}$. Since $X_{\mathbf{N}} \supseteq X_{\mathbf{N}} \odot X_{\mathbf{N}}$ by Theorem 4.6, we know that $X_{\mathbf{N}}$ is neutrosophic idempotent. $\square$

Definition 4.9. A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called an *ε-neutrosophic $\mathcal{N}$-subsemigroup* of X if the following condition is valid:

$$(\forall x, y \in X) \begin{pmatrix} T_N(xy) \leq \bigvee\{T_N(x), T_N(y), \varepsilon_T\} \\ I_N(xy) \geq \bigwedge\{I_N(x), I_N(y), \varepsilon_I\} \\ F_N(xy) \leq \bigvee\{F_N(x), F_N(y), \varepsilon_F\} \end{pmatrix}. \tag{4.3}$$

where $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [-1, 0]$ such that $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$.

Example 4.10. Let $X = \{e, a, b, c\}$ be a semigroup with the Cayley table which is given in Table 1.

Table 1: Cayley table for the binary operation "·"

$\cdot$	e	a	b	c
e	e	e	e	e
a	e	a	e	a
b	e	e	b	b
c	e	a	b	c

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X which is given as follows:

$$X_{\mathbf{N}} = \left\{ \frac{e}{(-0.4, -0.3, -0.25)}, \frac{a}{(-0.3, -0.5, -0.25)}, \frac{b}{(-0.2, -0.3, -0.2)}, \frac{c}{(-0.1, -0.7, -0.1)} \right\}.$$

Then $X_{\mathbf{N}}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X with $\varepsilon = (-0.4\ -0.2, -0.3)$.

Proposition 4.11. *Let $X_{\mathbf{N}}$ be an ε-neutrosophic $\mathcal{N}$-subsemigroup of X. If $X_{\mathbf{N}}(x) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$, that is, $T_N(x) \geq \varepsilon_T$, $I_N(x) \leq \varepsilon_I$ and $F_N(x) \geq \varepsilon_F$, for all $x \in X$, then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X.*

Proof. Straightforward. □

Theorem 4.12. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X, then the (α, β, γ)-level set of $X_{\mathbf{N}}$ is a subsemigroup of X whenever $(\alpha, \beta, \gamma) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$, that is, $\alpha \geq \varepsilon_T$, $\beta \leq \varepsilon_I$ and $\gamma \geq \varepsilon_F$.*

Proof. Assume that $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Let $x, y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Then $T_N(x) \leq \alpha$, $I_N(x) \geq \beta$, $F_N(x) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from (4.3) that

$T_N(xy) \leq \bigvee\{T_N(x), T_N(y), \varepsilon_T\} \leq \bigvee\{\alpha, \varepsilon_T\} = \alpha$,
$I_N(xy) \geq \bigwedge\{I_N(x), I_N(y), \varepsilon_I\} \geq \bigwedge\{\beta, \varepsilon_I\} = \beta$, and
$F_N(xy) \leq \bigvee\{F_N(x), F_N(y), \varepsilon_F\} \leq \bigvee\{\gamma, \varepsilon_F\} = \gamma$.

Hence $xy \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$, and therefore $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is a subsemigroup of X. □

Theorem 4.13. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If T_N^{α}, I_N^{β} and F_N^{γ} are subsemigroups of X for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [-1, 0]$ with $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$ and $(\alpha, \beta, \gamma) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$, then $X_{\mathbf{N}}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X.*

Proof. Assume that there are $a, b \in X$ such that

$$T_N(ab) > \bigvee\{T_N(a), T_N(b), \varepsilon_T\}.$$

Then $T_N(ab) > t_\alpha \geq \bigvee\{T_N(a), T_N(b), \varepsilon_T\}$ for some $t_\alpha \in [-1, 0)$. It follows that $a, b \in T_N^{t_\alpha}$, $ab \notin T_N^{t_\alpha}$ and $t_\alpha \geq \varepsilon_T$. This is a contradiction since $T_N^{t_\alpha}$ is a subsemigroup of X by hypothesis. Thus

$$T_N(xy) \leq \bigvee\{T_N(x), T_N(y), \varepsilon_T\}$$

for all $x, y \in X$. Suppose that $I_N(ab) < \bigwedge\{I_N(a), I_N(b), \varepsilon_I\}$ for some $a, b \in X$. If we take $t_\beta := \bigwedge\{I_N(a), I_N(b), \varepsilon_I\}$, then $a, b \in I_N^{t_\beta}$, $ab \notin I_N^{t_\beta}$ and $t_\beta \leq \varepsilon_I$. This is a contradiction, and so

$$I_N(xy) \geq \bigwedge\{I_N(x), I_N(y), \varepsilon_I\}$$

for all $x, y \in X$. Now, suppose that there exist $a, b \in X$ and $t_\gamma \in [-1, 0)$ such that

$$F_N(ab) > t_\gamma \geq \bigvee\{F_N(a), F_N(b), \varepsilon_F\}.$$

Then $a, b \in F_N^{t_\gamma}$, $ab \notin F_N^{t_\gamma}$ and $t_\gamma \geq \varepsilon_F$, which is a contradiction. Hence

$$F_N(xy) \leq \bigvee\{F_N(x), F_N(y), \varepsilon_F\}$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X. □

Theorem 4.14. *For any $\varepsilon_T, \varepsilon_I, \varepsilon_F, \delta_T, \delta_I, \delta_F \in [-1, 0]$ with $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$ and $-3 \leq \delta_T + \delta_I + \delta_F \leq 0$, if $X_{\mathbf{N}}$ and $X_{\mathbf{M}}$ are an ε-neutrosophic $\mathcal{N}$-subsemigroup and a δ-neutrosophic $\mathcal{N}$-subsemigroup, respectively, of X, then their intersection is a ξ-neutrosophic $\mathcal{N}$-subsemigroup of X for $\xi := \varepsilon \wedge \delta$, that is, $(\xi_T, \xi_I, \xi_F) = (\varepsilon_T \vee \delta_T, \varepsilon_I \wedge \delta_I, \varepsilon_F \vee \delta_F)$.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
T_{N\cap M}(xy) &= \bigvee \{T_N(xy), T_M(xy)\} \\
&\leq \bigvee \left\{\bigvee\{T_N(x), T_N(y), \varepsilon_T\}, \bigvee\{T_M(x), T_M(y), \delta_T\}\right\} \\
&\leq \bigvee \left\{\bigvee\{T_N(x), T_N(y), \xi_T\}, \bigvee\{T_M(x), T_M(y), \xi_T\}\right\} \\
&= \bigvee \left\{\bigvee\{T_N(x), T_M(x), \xi_T\}, \bigvee\{T_N(y), T_M(y), \xi_T\}\right\} \\
&= \bigvee \left\{\bigvee\{T_N(x), T_M(x)\}, \bigvee\{T_N(y), T_M(y)\}, \xi_T\right\} \\
&= \bigvee \{T_{N\cap M}(x), T_{N\cap M}(y), \xi_T\},
\end{aligned}$$

$$\begin{aligned}
I_{N\cap M}(xy) &= \bigwedge \{I_N(xy), I_M(xy)\} \\
&\geq \bigwedge \left\{\bigwedge\{I_N(x), I_N(y), \varepsilon_I\}, \bigwedge\{I_M(x), I_M(y), \delta_I\}\right\} \\
&\geq \bigwedge \left\{\bigwedge\{I_N(x), I_N(y), \xi_I\}, \bigwedge\{I_M(x), I_M(y), \xi_I\}\right\} \\
&= \bigwedge \left\{\bigwedge\{I_N(x), I_M(x), \xi_I\}, \bigwedge\{I_N(y), I_M(y), \xi_I\}\right\} \\
&= \bigwedge \left\{\bigwedge\{I_N(x), I_M(x)\}, \bigwedge\{I_N(y), I_M(y)\}, \xi_I\right\} \\
&= \bigwedge \{I_{N\cap M}(x), I_{N\cap M}(y), \xi_I\},
\end{aligned}$$

and

$$\begin{aligned}
F_{N\cap M}(xy) &= \bigvee \{F_N(xy), F_M(xy)\} \\
&\leq \bigvee \left\{\bigvee\{F_N(x), F_N(y), \varepsilon_F\}, \bigvee\{F_M(x), F_M(y), \delta_F\}\right\} \\
&\leq \bigvee \left\{\bigvee\{F_N(x), F_N(y), \xi_F\}, \bigvee\{F_M(x), F_M(y), \xi_F\}\right\} \\
&= \bigvee \left\{\bigvee\{F_N(x), F_M(x), \xi_F\}, \bigvee\{F_N(y), F_M(y), \xi_F\}\right\} \\
&= \bigvee \left\{\bigvee\{F_N(x), F_M(x)\}, \bigvee\{F_N(y), F_M(y)\}, \xi_F\right\} \\
&= \bigvee \{F_{N\cap M}(x), F_{N\cap M}(y), \xi_F\}.
\end{aligned}$$

Therefore $X_{N\cap M}$ is a ξ-neutrosophic $\mathcal{N}$-subsemigroup of X. □

Theorem 4.15. *Let $X_{\mathbf{N}}$ be an ε-neutrosophic $\mathcal{N}$-subsemigroup of X. If*

$$\kappa := (\kappa_T, \kappa_I, \kappa_F) = \left(\bigvee_{x\in X}\{T_N(x)\}, \bigwedge_{x\in X}\{I_N(x)\}, \bigvee_{x\in X}\{F_N(x)\}\right),$$

then the set

$$\Omega := \{x \in X \mid T_N(x) \leq \kappa_T \vee \varepsilon_T,\ I_N(x) \geq \kappa_I \wedge \varepsilon_I,\ F_N(x) \leq \kappa_F \vee \varepsilon_F\}$$

is a subsemigroup of X.

Proof. Let $x, y \in \Omega$ for any $x, y \in X$. Then

$T_N(x) \leq \kappa_T \vee \varepsilon_T = \bigvee_{x \in X} \{T_N(x)\} \vee \varepsilon_T,$
$I_N(x) \geq \kappa_I \wedge \varepsilon_I = \bigwedge_{x \in X} \{I_N(x)\} \wedge \varepsilon_I,$
$F_N(x) \leq \kappa_F \vee \varepsilon_F = \bigvee_{x \in X} \{F_N(x)\} \vee \varepsilon_F,$
$T_N(y) \leq \kappa_T \vee \varepsilon_T = \bigvee_{y \in X} \{T_N(y)\} \vee \varepsilon_T,$
$I_N(y) \geq \kappa_I \wedge \varepsilon_I = \bigwedge_{y \in X} \{I_N(y)\} \wedge \varepsilon_I,$
$F_N(y) \leq \kappa_F \vee \varepsilon_F = \bigvee_{y \in X} \{F_N(y)\} \vee \varepsilon_F.$

It follows from (4.3) that

$$\begin{aligned} T_N(xy) &\leq \bigvee \{T_N(x), T_N(y), \varepsilon_T\} \\ &\leq \bigvee \{\kappa_T \vee \varepsilon_T, \kappa_T \vee \varepsilon_T, \varepsilon_T\} \\ &= \kappa_T \vee \varepsilon_T, \end{aligned}$$

$$\begin{aligned} I_N(xy) &\geq \bigwedge \{I_N(x), I_N(y), \varepsilon_I\} \\ &\geq \bigwedge \{\kappa_I \wedge \varepsilon_I, \kappa_I \wedge \varepsilon_I, \varepsilon_I\} \\ &= \kappa_I \wedge \varepsilon_I \end{aligned}$$

and

$$\begin{aligned} F_N(xy) &\leq \bigvee \{F_N(x), F_N(y), \varepsilon_F\} \\ &\leq \bigvee \{\kappa_F \vee \varepsilon_F, \kappa_F \vee \varepsilon_F, \varepsilon_F\} \\ &= \kappa_F \vee \varepsilon_F, \end{aligned}$$

and so that $xy \in \Omega$. Therefore Ω is a subsemigroup of X. □

For a map $f : X \to Y$ of semigroups and a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}} := \frac{Y}{(T_N, I_N, F_N)}$ over Y and $\varepsilon = (\varepsilon_T, \varepsilon_I, \varepsilon_F)$ with $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$, define a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}^{\varepsilon} := \frac{X}{(T_N^{\varepsilon}, I_N^{\varepsilon}, F_N^{\varepsilon})}$ over X by

$T_N^{\varepsilon} : X \to [-1, 0],\ x \mapsto \bigvee \{T_N(f(x)), \varepsilon_T\},$
$F_N^{\varepsilon} : X \to [-1, 0],\ x \mapsto \bigwedge \{I_N(f(x)), \varepsilon_I\},$
$F_N^{\varepsilon} : X \to [-1, 0],\ x \mapsto \bigvee \{F_N(f(x)), \varepsilon_F\}.$

Theorem 4.16. *Let $f : X \to Y$ be a homomorphism of semigroups. If a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}} := \frac{Y}{(T_N, I_N, F_N)}$ over Y is an ε-neutrosophic $\mathcal{N}$-subsemigroup of Y, then $X^{\varepsilon}_{\mathbf{N}} := \frac{X}{(T^{\varepsilon}_N, I^{\varepsilon}_N, F^{\varepsilon}_N)}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
T^{\varepsilon}_N(xy) &= \bigvee \{T_N(f(xy)), \varepsilon_T\} \\
&= \bigvee \{T_N(f(x)f(y)), \varepsilon_T\} \\
&\leq \bigvee \left\{\bigvee\{T_N(f(x)), T_N(f(y)), \varepsilon_T\}, \varepsilon_T\right\} \\
&= \bigvee \left\{\bigvee\{T_N(f(x)), \varepsilon_T\}, \bigvee\{T_N(f(y)), \varepsilon_T\}, \varepsilon_T\right\} \\
&= \bigvee \{T^{\varepsilon}_N(x), T^{\varepsilon}_N(y), \varepsilon_T\},
\end{aligned}$$

$$\begin{aligned}
I^{\varepsilon}_N(xy) &= \bigwedge \{I_N(f(xy)), \varepsilon_I\} \\
&= \bigwedge \{I_N(f(x)f(y)), \varepsilon_I\} \\
&\geq \bigwedge \left\{\bigwedge\{I_N(f(x)), I_N(f(y)), \varepsilon_I\}, \varepsilon_I\right\} \\
&= \bigwedge \left\{\bigwedge\{I_N(f(x)), \varepsilon_I\}, \bigwedge\{I_N(f(y)), \varepsilon_I\}, \varepsilon_I\right\} \\
&= \bigwedge \{I^{\varepsilon}_N(x), I^{\varepsilon}_N(y), \varepsilon_I\},
\end{aligned}$$

and

$$\begin{aligned}
F^{\varepsilon}_N(xy) &= \bigvee \{F_N(f(xy)), \varepsilon_F\} \\
&= \bigvee \{F_N(f(x)f(y)), \varepsilon_F\} \\
&\leq \bigvee \left\{\bigvee\{F_N(f(x)), F_N(f(y)), \varepsilon_F\}, \varepsilon_F\right\} \\
&= \bigvee \left\{\bigvee\{F_N(f(x)), \varepsilon_F\}, \bigvee\{F_N(f(y)), \varepsilon_F\}, \varepsilon_F\right\} \\
&= \bigvee \{F^{\varepsilon}_N(x), F^{\varepsilon}_N(y), \varepsilon_F\}.
\end{aligned}$$

Therefore $X^{\varepsilon}_{\mathbf{N}} := \frac{X}{(T^{\varepsilon}_N, I^{\varepsilon}_N, F^{\varepsilon}_N)}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X. □

Let $f : X \to Y$ be a function of sets. If $Y_{\mathbf{M}} := \frac{Y}{(T_M, I_M, F_M)}$ is a neutrosophic $\mathcal{N}$-structures over Y, then the *preimage* of $Y_{\mathbf{M}}$ under f is defined to be a neutrosophic $\mathcal{N}$-structures

$$f^{-1}(Y_{\mathbf{M}}) = \frac{X}{(f^{-1}(T_M),\ f^{-1}(I_M),\ f^{-1}(F_M))}$$

over X where $f^{-1}(T_M)(x) = T_M(f(x))$, $f^{-1}(I_M)(x) = I_M(f(x))$ and $f^{-1}(F_M)(x) = F_M(f(x))$ for all $x \in X$.

Theorem 4.17. *Let $f : X \to Y$ be a homomorphism of semigroups. If $Y_{\mathbf{M}} := \frac{Y}{(T_M, I_M, F_M)}$ is a neutrosophic $\mathcal{N}$-subsemigroup of Y, then the preimage of $Y_{\mathbf{M}}$ under f is a neutrosophic $\mathcal{N}$-subsemigroup of X.*

Proof. Let

$$f^{-1}(Y_{\mathbf{M}}) = \frac{X}{(f^{-1}(T_M),\ f^{-1}(I_M),\ f^{-1}(F_M))}$$

be the preimage of $Y_{\mathbf{M}}$ under f. For any $x, y \in X$, we have

$$\begin{aligned} f^{-1}(T_M)(xy) &= T_M(f(xy)) = T_M(f(x)f(y)) \\ &\leq \bigvee \{T_M(f(x)), T_M(f(y))\} \\ &= \bigvee \left\{f^{-1}(T_M)(x), f^{-1}(T_M)(y)\right\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(I_M)(xy) &= I_M(f(xy)) = I_M(f(x)f(y)) \\ &\geq \bigwedge \{I_M(f(x)), I_M(f(y))\} \\ &= \bigwedge \left\{f^{-1}(I_M)(x), f^{-1}(I_M)(y)\right\} \end{aligned}$$

and

$$\begin{aligned} f^{-1}(F_M)(xy) &= F_M(f(xy)) = F_M(f(x)f(y)) \\ &\leq \bigvee \{F_M(f(x)), F_M(f(y))\} \\ &= \bigvee \left\{f^{-1}(F_M)(x), f^{-1}(F_M)(y)\right\}. \end{aligned}$$

Therefore $f^{-1}(Y_{\mathbf{M}})$ is a neutrosophic $\mathcal{N}$-subsemigroup of X. □

Let $f : X \to Y$ be an onto function of sets. If $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ is a neutrosophic $\mathcal{N}$-structures over X, then the *image* of $X_{\mathbf{N}}$ under f is defined to be a neutrosophic $\mathcal{N}$-structures

$$f(X_{\mathbf{N}}) = \frac{Y}{(f(T_N),\ f(I_N),\ f(F_N))}$$

over Y where

$$f(T_N)(y) = \bigwedge_{x \in f^{-1}(y)} T_N(x),$$
$$f(I_N)(y) = \bigvee_{x \in f^{-1}(y)} I_N(x),$$
$$f(F_N)(y) = \bigwedge_{x \in f^{-1}(y)} F_N(x).$$

Theorem 4.18. *For an onto homomorphism $f : X \to Y$ of semigroups, let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over X such that*

$$(\forall T \subseteq X)\,(\exists x_0 \in T) \begin{pmatrix} T_N(x_0) = \bigwedge\limits_{z \in T} T_N(z) \\ I_N(x_0) = \bigvee\limits_{z \in T} I_N(z) \\ F_N(x_0) = \bigwedge\limits_{z \in T} F_N(z) \end{pmatrix}. \tag{4.4}$$

If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X, then the image of $X_{\mathbf{N}}$ under f is a neutrosophic $\mathcal{N}$-subsemigroup of Y.

Proof. Let

$$f\left(X_{\mathbf{N}}\right) = \frac{Y}{(f(T_N),\ f(I_N),\ f(F_N))}$$

be the image of $X_{\mathbf{N}}$ under f. Let $a, b \in Y$. Then $f^{-1}(a) \neq \emptyset$ and $f^{-1}(a) \neq \emptyset$ in X, which imply from (4.4) that there are $x_a \in f^{-1}(a)$ and $x_b \in f^{-1}(b)$ such that
$T_N(x_a) = \bigwedge\limits_{z \in f^{-1}(a)} T_N(z)$, $I_N(x_a) = \bigvee\limits_{z \in f^{-1}(a)} I_N(z)$, $F_N(x_a) = \bigwedge\limits_{z \in f^{-1}(a)} F_N(z)$,
$T_N(x_b) = \bigwedge\limits_{w \in f^{-1}(b)} T_N(w)$, $I_N(x_b) = \bigvee\limits_{w \in f^{-1}(b)} I_N(w)$, $F_N(x_b) = \bigwedge\limits_{w \in f^{-1}(b)} F_N(w)$.
Hence

$$\begin{aligned} f(T_N)(ab) &= \bigwedge_{x \in f^{-1}(ab)} T_N(x) \leq T_N(x_a x_b) \\ &\leq \bigvee \{T_N(x_a), T_N(x_b)\} \\ &= \bigvee \left\{ \bigwedge_{z \in f^{-1}(a)} T_N(z), \bigwedge_{w \in f^{-1}(b)} T_N(w) \right\} \\ &= \bigvee \left\{ f(T_N)(a), f(T_N)(b) \right\}, \end{aligned}$$

$$\begin{aligned} f(I_N)(ab) &= \bigvee_{x \in f^{-1}(ab)} I_N(x) \geq I_N(x_a x_b) \\ &\geq \bigwedge \{I_N(x_a), I_N(x_b)\} \\ &= \bigwedge \left\{ \bigvee_{z \in f^{-1}(a)} I_N(z), \bigvee_{w \in f^{-1}(b)} I_N(w) \right\} \\ &= \bigwedge \left\{ f(I_N)(a), f(I_N)(b) \right\}, \end{aligned}$$

and

$$\begin{aligned} f(F_N)(ab) &= \bigwedge_{x \in f^{-1}(ab)} F_N(x) \leq F_N(x_a x_b) \\ &\leq \bigvee \{F_N(x_a), F_N(x_b)\} \\ &= \bigvee \left\{ \bigwedge_{z \in f^{-1}(a)} F_N(z), \bigwedge_{w \in f^{-1}(b)} F_N(w) \right\} \\ &= \bigvee \{f(F_N)(a), f(F_N)(b)\}. \end{aligned}$$

Therefore $f(X_{\mathbf{N}})$ is a neutrosophic $\mathcal{N}$-subsemigroup of Y. □

Conclusions

In order to deal with the negative meaning of information, Jun et al. [3] have introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. The concept of neutrosophic set (NS) has been developed by Smarandache in [7] and [8] as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. In this article, we have introduced the notion of neutrosophic $\mathcal{N}$-structure and applied it to semigroup. We have introduced the notion of neutrosophic $\mathcal{N}$-subsemi-group and investigated several properties. We have provided conditions for neutrosophic $\mathcal{N}$-structure to be neutrosophic $\mathcal{N}$-subsemigroup. We have defined neutrosophic $\mathcal{N}$-product, and gave characterization of neutrosophic $\mathcal{N}$-subsemigroup by using neutrosophic $\mathcal{N}$-product. We also have introduced ε-neutrosophic subsemigroup, and investigated relations between neutrosophic subsemigroup and ε-neutrosophic subsemigroup. We have shown that the homomorphic preimage of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup, and the onto homomorphic image of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup.

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The first chapter, *Characterizations of regular and duo semigroups based on int-soft set theory*, investigates the relations among int-soft semigroup, int-soft (generalized) bi-ideal, int-soft quasi-ideal and int-soft interior ideal. Using int-soft left (right) ideal, an int-soft quasi-ideal is constructed. We show that every int-soft quasi-ideal can be represented as the soft intersection of an int-soft left ideal and an int-soft right ideal. Using int-soft quasi-ideal, an int-soft bi-ideal is established. Conditions for a semigroup to be regular are displayed. The notion of int-soft left (right) duo semigroup is introduced, and left (right) duo semigroup is characterized by int-soft left (right) duo semigroup. Bi-ideal, quasi-ideal and interior ideal are characterized by using (Φ, Ψ)-characteristic soft sets.

The notions of starshaped $(\in, \in \vee qk)$-fuzzy sets and quasi-starshaped $(\in, \in \vee qk)$-fuzzy sets are introduced in the second chapter, *Generalizations of starshaped $(\in, \in\vee q)$-fuzzy sets*, and related properties are investigated. Characterizations of starshaped $(\in, \in \vee qk)$-fuzzy sets and quasi-starshaped $(\in, \in \vee q)$-fuzzy sets are discussed. Relations between starshaped $(\in, \in \vee qk)$-fuzzy sets and quasi-starshaped $(\in, \in\vee qk)$-fuzzy sets are investigated.

The notion of semidetached semigroup is introduced the third chapter (*Semidetached semigroups*), and their properties are investigated. Several conditions for a pair of a semigroup and a semidetached mapping to be a semidetached semigroup are provided. The concepts of $(\in, \in \vee qk)$-fuzzy sub-semigroup, $(qk, \in \vee qk)$-fuzzy subsemigroup and $(\in \vee qk, \in \vee qk)$-fuzzy subsemigroup are introduced, and relative relations are discussed.

The fourth chapter, *Generalizations of $(\in, \in\vee qk)$-fuzzy (generalized) bi-ideals in semigroups*, introduces the notion of $(\in, \in \vee qk\delta)$-fuzzy (generalized) bi-ideals in semigroups, and related properties are investigated. Given a (generalized) bi-ideal, an $(\in,\in \vee qk\delta)$-fuzzy (generalized) bi-ideal is constructed. Characterizations of an $(\in, \in \vee qk\delta)$-fuzzy (generalized) bi-ideal are discussed, and shown that an $(\in, \in\vee qk\delta)$-fuzzy generalized bi-ideal and an $(\in, \in \vee qk\delta)$-fuzzy bi-ideal coincide in regular semigroups. Using a fuzzy set with finite image, an $(\in, \in\vee qk\delta)$-fuzzy bi-ideal is established.

Lower and upper approximations of fuzzy sets in semigroups are considered in the fifth chapter, *Approximations of fuzzy sets in semigroups*, and several properties are investigated. The notion of rough sets was introduced by Pawlak. This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis. Rough set theory is applied to semigroups and groups, d-algebras, BE-algebras, BCK-algebras and MV-algebras etc.

Finally, in the sixth and last paper, *Neutrosophic N-structures and their applications in semigroups*, the notion of neutrosophic N-structure is introduced, and applied to semigroup. The notions of neutrosophic N-subsemigroup, neutrosophic N-product and ε-neutrosophic N-subsemigroup are introduced, and several properties are investigated. Conditions for neutrosophic N-structure to be neutrosophic N-subsemigroup are provided. Using neutrosophic N-product, characterization of neutrosophic N-subsemigroup is discussed. Relations between neutrosophic N-subsemigroup and ε-neutrosophic N-subsemigroup are discussed. We show that the homomorphic preimage of neutrosophic N-subsemigroup is a neutrosophic N-subsemigroup, and the onto homomorphic image of neutrosophic N -subsemigroup is a neutrosophic N-subsemigroup.

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