SMARANDACHE GROUPOIDS

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Abstract:
In this paper we study the concept of Smarandache Groupoids, subgroupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache-Bol groupoids and Strong Bol groupoids and obtain many interesting results about them.

Keywords:
Smarandache groupoid, Smarandache subgroupoid, Smarandache ideal of a Smarandache groupoid, Smarandache semi-normal groupoid, Smarandache normal groupoid, Smarandache semi conjugate subgroupoids, Smarandache Bol groupoid, Smarandache Moufang groupoid

Definition [1]: A groupoid (G, ∗) is a non-empty set, closed with respect to an operation ∗ (in general ∗ need not to be associative)..

Definition 1: A *Smarandache Groupoid* G is a groupoid which has a proper subset $S \subset G$ which is a semigroup under the operation of G.

<u>*Example 1:*</u> Let (G, ∗) be a groupoid on modulo 6 integers. G = {0, 1, 2, 3, 4, 5} is given by the following table:

∗	0	1	2	3	4	5
0	0	3	0	3	0	3
1	1	4	1	4	1	4
2	2	5	2	5	2	5
3	3	0	3	0	3	0
4	4	1	4	1	4	1
5	5	2	5	2	5	2

Clearly S_1 = {0, 3}, S_2 = {1, 4} and S_3 = {2, 5} are semigroups of G. So (G, ∗) is a Smarandache groupoid.

<u>*Example 2:*</u> Let G = {0,1,2,3,4,5,6,7,8,9} be the set of integers modulo 10. Define an operation ∗ on G by choosing a pair (1, 5) such that a ∗ b = 1a + 5b (mod 10) for all a, b ∈ G.

The groupoid is given by the following table.

$*$	0	1	2	3	4	5	6	7	8	9
0	0	5	0	5	0	5	0	5	0	5
1	1	6	1	6	1	6	1	6	1	6
2	2	7	2	7	2	7	2	7	2	7
3	3	8	3	8	3	8	3	8	3	8
4	4	9	4	9	4	9	4	9	4	9
5	5	0	5	0	5	0	5	0	5	0
6	6	1	6	1	6	1	6	1	6	1
7	7	2	7	2	7	2	7	2	7	2
8	8	3	8	3	8	3	8	3	8	3
9	9	4	9	4	9	4	9	4	9	4

Clearly $S_1 = \{0, 5\}$, $S_2 = \{1, 6\}$, $S_3 = \{2, 7\}$, $S_4 = \{3, 8\}$and $S_5 = \{4, 9\}$ are semigroups under the operation $*$. Thus $\{G, *, (1, 5)\}$ is a Smarandache groupoid.

Theorem 2: Let $Z_{2p} = \{0, 1, 2, ..., 2p -1\}$. Define $*$ on Z_{2p} for a, b $\in Z_{2p}$ by $a * b = 1a + pb \pmod{2p}$. $\{Z_{2p}, *, (1,p)\}$ is a Smarandache groupoid.

Proof: Under the operation $*$ defined on Z_{2p} we see $S_1 = \{0, p\}$, $S_2 = \{1, p+1\}$, $S_3 = \{2, p + 2\}$ ... $S_p = \{p - 1, 2p - 1\}$ are semigroups under the operation $*$. Hence $\{Z_{2p}, *, (1,p)\}$ is a Smarandache groupoid.

<u>Example 3</u>: Take $Z_6 = \{0, 1, 2, 3, 4, 5\}$. $(2, 5) = (m, n)$. For a, b $\in Z_6$ define $a * b = ma + nb \pmod 6$. The groupoid is given by the following table:

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	2	1	0	5	4	3
2	4	3	2	1	0	5
3	0	5	4	3	2	1
4	2	1	0	5	4	3
5	4	3	2	1	0	5

Every singleton is an idempotent semigroup of Z_6.

Theorem 3: Let $Z_{2p} = \{0, 1, 2, ..., p-1\}$. Define $*$ on Z_{2p} by $a * b = 2a + (2p-1)b \pmod{2p}$ for a, b $\in Z_{2p}$. Then $\{Z_{2p}, *, (2, 2p - 1)\}$ is a Smarandache groupoid.

Proof: Under the operation $*$ defined on Z_{2p} we see that every element of Z_{2p} is idempotent, therefore every element forms a singleton semigroup. Hence the claim.

<u>Example 4</u>: Consider $Z_6 = \{Z_6, *, (4, 5)\}$ given by the following table:

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	4	3	2	1	0	5
2	2	1	0	5	4	3
3	0	5	4	3	2	1
4	4	3	2	1	0	5
5	2	1	0	5	4	3

{3} is a semigroup. Hence $*$ is a Smarandache groupoid. It is easily verified that Z_6 is a Smarandache groupoid as $\{Z_6, *, (4, 5)\}$ has an idempotent semigroup {3} under $*$.

Theorem 4: Let $Z_{2p} = \{0, 1, 2, ..., 2p-1\}$ be the set of integers modulo 2p. Define $*$ on $a, b \in Z_{2p}$ by $a(2p - 2) + b(2p - 1) \pmod{2p}$. Then $\{Z_{2p}, *, (2p - 2, 2p - 1)\}$ is a Smarandache groupoid.

Proof: $Z_{2p} = \{0, 1, 2, ..., 2p - 1\}$. Take $(2p - 2, 2p - 1) = 1$ from Z_{2p}. For $a, b \in Z_p$ define $a * b = (2p - 2) a + b(2p - 1) \pmod{2p}$. Clearly for $a = b = p$ we have $(2p-2)p + (2p - 1)p = p \pmod{2p}$. Hence {p} is an idempotent semigroup of Z_{2p}. So $\{Z_{2p}, *, (2p - 2, 2p - 1)\}$ is a Smarandache groupoid.

Definition 5: Let $(G, *)$ be a Smarandache groupoid. A non-empty subset H of G is said to be a Smarandache Subgroupoid if H contains a proper subset $K \subset H$ such that K is a semigroup under the operation $*$.

Theorem 6: Not every subgroupoid of a Smarandache groupoid S is in general a Smarandache subgroupoid of S.

Proof: By an example.
Let $Z_6 = \{0, 1, 2, 3, 4, 5\} \pmod 6$. Take $(t, u) = (4, 5) = 1$. For $a, b \in Z_6$ define $*$ on Z_6 by $a * b = at + bu \pmod 6$ given by the following table:

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	4	3	2	1	0	5
2	2	1	0	5	4	3
3	0	5	4	3	2	1
4	4	3	2	1	0	5
5	2	1	0	5	4	3

Clearly $\{Z_6, *, (4, 5)\}$ is a Smarandache groupoid for it contains {0, 3} as a semigroup. But this groupoid has the following subgroupoids:

$A_1 = \{0, 2, 4\}$ and $A_2 = \{1, 3, 5\}$. A_1 has no non-trivial semigroup ($\{0\}$ is a trivial semigroup). But A_2 has a non-trivial semigroup, viz. $\{3\}$. Hence the claim.

Theorem 7: If a groupoid contains a Smarandache subgroupoid then the groupoid is a Smarandache groupoid.

Proof: Let G be a groupoid and $H \subset G$ be a Smarandache subgroupoid, that is H contains a proper subset $P \subset H$ such that P is a semigroup. So $P \subset G$ and P is a semigroup. Hence G is a Smarandache groupoid.

Definition 8:
i) A *Smarandache Left Ideal A of the Smarandache Groupoid* G satisfies the following conditions:
1. A is a Smarandache subgroupoid
2. For all $x \in G$, and $a \in A$, $xa \in A$.

ii) Similarly, one defines a *Smarandache Right Ideal*.
iii) If A is both a Smarandache right and left ideals then A is a *Smarandache Ideal*. We take $\{0\}$ as a trivial Smarandache ideal.

Example 5: Let $\{Z_6, *, (4, 5)\}$ be a Smarandache groupoid. $A = \{1, 3, 5\}$ is a Smarandache subgroupoid and A is Smarandache left ideal and not a Smarandache right ideal. Easy to verify.

Theorem 9: Let G be a groupoid. An ideal of G in general is not a Smarandache ideal of G even if G is a Smarandache groupoid.

Proof: By an example. Consider the groupoid $G = \{Z_6, *, (2, 4)\}$ given by the following table.

*	0	1	2	3	4	5
0	0	4	2	0	4	2
1	2	0	4	2	0	4
2	4	2	0	4	2	0
3	0	4	2	0	4	2
4	2	0	4	2	0	4
5	4	2	0	4	2	0

Clearly G is a Smarandache groupoid for $\{0, 3\}$ is a semigroup of G. Now, $\{0,4,2\}$ is an ideal of G but is not a Smarandache ideal as $\{0,4,2\}$ is not a Smarandache subgroupoid.

Definition 10: Let G be a Smarandache groupoid and V be a Smarandache subgroupoid of G. We say V is a *Smarandache semi-normal subgroupoid* if:
1. $aV = X$ for all $a \in G$.
2. $Va = Y$ for all $a \in G$.

where either X or Y is a Smarandache subgroupoid of G but X and Y are both subgroupoids.

Example 6: Consider the groupoid G = {Z_6, ∗, (4, 5)} given by the table.

∗	0	1	2	3	4	5
0	0	5	4	3	2	1
1	4	3	2	1	0	5
2	2	1	0	5	4	3
3	0	5	4	3	2	1
4	4	3	2	1	0	5
5	2	1	0	5	4	3

Clearly G is a Smarandache groupoid as {3} is a semigroup. Take A = {1, 3, 5}. A is also a Smarandache subgroupoid. Now aA = A is a Smarandache groupoid. Aa = {0, 2, 4}. {0, 2, 4} is not a Smarandache subgroupoid of G. Hence A is a Smarandache semi-normal subgroupoid.

Definition 11: Let A be a Smarandache groupoid and V be a Smarandache subgroupoid. V is said to be *Smarandache normal subgroupoid* if aV = X and Va = Y where both X and Y are Smarandache subgroupoids of G.

Theorem 12: Every Smarandache normal subgroupoid is a Smarandache semi-normal subgroupoid, and not conversely.

Proof: By the definitions 10 and 11, we see every Smarandache normal subgroupoid is Smarandache semi-normal subgroupoid. We prove the converse by an example. In Example 6 we see A is a Smarandache semi-normal subgroupoid but not a normal subgroupoid as Aa = {0, 2, 4} is only a subgroupoid and not a Smarandache subgroupoid.

Example 7: Let G = {Z_8, ∗, (2, 6)} be a groupoid given by the following table:

∗	0	1	2	3	4	5	6	7
0	0	6	4	2	0	6	4	2
1	2	0	6	4	2	0	6	4
2	4	2	0	6	4	2	0	6
3	6	4	2	0	6	4	2	0
4	0	6	4	2	0	6	4	2
5	2	0	6	4	2	0	6	4
6	4	2	0	6	4	2	0	6
7	6	4	2	0	6	4	2	0

Clearly G is a Smarandache groupoid for {0, 4} is a semigroup of G. A = {0, 2, 4, 6} is a Smarandache subgroupoid. Clearly Aa = A and aA = A for all a ∈ G. So A is a Smarandache normal subgroupoid of G.

Definition 13: Let G be a Smarandache groupoid H and P be subgroupoids of G we say *H and P are Smarandache semi-conjugate subgroupoids of G* if:

1. H and P are Smarandache subgroupoids
2. H = xP or Px or P = xH or Hx, for some $x \in$ G.

Definition 14: Let G be a Smarandache groupoid. H and P be subgroupoids of G. We say *H and P are Smarandache conjugate subgroupoids of G* if:

1. H and P are Smarandache subgroupoids
2. H = xP or Px, for some $x \in$ G.
3. P = xH or Hx, for some $x \in$ G.

Example 8: Consider the groupoid G = $\{Z_{12}, *, (1, 3)\}$ which is given by the following table:

*	0	1	2	3	4	5	6	7	8	9	10	11
0	0	3	6	9	0	3	6	9	0	3	6	9
1	1	4	7	10	1	4	7	10	1	4	7	10
2	2	5	8	11	2	5	8	11	2	5	8	11
3	3	6	9	0	3	6	9	0	3	6	9	0
4	4	7	10	1	4	7	10	1	4	7	10	1
5	5	8	11	2	5	8	11	2	5	8	11	2
6	6	9	0	3	6	9	0	3	6	9	0	3
7	7	10	1	4	7	10	1	4	7	10	1	4
8	8	11	2	5	8	11	2	5	8	11	2	5
9	9	0	3	6	9	0	3	6	9	0	3	6
10	10	1	4	7	10	1	4	7	10	1	4	7
11	11	2	5	8	11	2	5	8	11	2	5	8

Clearly G is a Smarandache groupoid for {0, 6} is a semigroup of G. Let A_1 = {0,3,6,9} and A_2= {2, 5, 8, 11} be two subgroupoids. Clearly A_1 and A_2 are Smarandache subgroups of G as {0, 6} and {2, 8} are semigroups of A_1 and A_2 respectively.

Now:

$$A_1 = 3\{2, 5, 8, 11\} = 3A_2$$
$$= \{0, 3, 6, 9\}$$

and similarly:

$$A_2 = 2\{0, 3, 6, 9\} = 2A_1.$$

Hence A_1 and A_2 are conjugate Smarandache subgroupoids of G.

Definition 15: Let $G_1, G_2, G_3, \ldots, G_n$ be n groupoids. We say $G = G_1 \times G_2 \times \ldots \times G_n$ is a *Smarandache direct product of groupoids* if G has a proper subset H of G which is a semigroup under the operations of G. It is important to note that each G_i need not be a Smarandache groupoid for in that case G will be obviously a Smarandache groupoid. Hence we take any set of n groupoids and find the direct product.

Definition 16: Let (G, ∗) and (G', o) be any two Smarandache groupoids. A map ϕ from (G, ∗) to (G', o) is said to be a *Smarandache groupoid homomorphism* if $\phi(a * b) = \phi(a) \text{ o } \phi(b)$ for all $a, b \in A$.

We say the *Smarandache groupoid* homomorphism is an *isomorphism* if ϕ is an isomorphism.

Definition 17: Let G be a Smarandache groupoid. We say G is a *Smarandache commutative groupoid* if there is a proper subset A of G which is a commutative semigroup under the operations of G.

Definition 18: Let G be Smarandache groupoid. We say G is *Smarandache inner commutative groupoid* if every semigroup contained in every Smarandache subgroupoid of G is commutative.

Theorem 19: Every Smarandache inner commutative groupoid G is a Smarandache commutative groupoid and not conversely.

Proof: By the very definitions 18 and 19 we see if G is a Smarandache inner commutative groupoid then G is Smarandache commutative groupoid.
To prove the converse we prove it by an example. Let $Z_2 = \{0, 1\}$ be integers modulo 2. Consider set of all 2×2 matrices with entries from $Z_2 = (0, 1)$ denote it by $M_{2\times 2}$. $M_{2\times 2} = \left\{ \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{matrix} \right\}$.

$M_{2\times 2}$ is made into a groupoid by for $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ in $M_{2\times 2}$.

$$\begin{aligned} A \text{ o } B &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \text{o} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_3 + a_2b_1 \pmod 2 & a_1b_4 + a_2b_2 \pmod 2 \\ a_3b_3 + a_4b_1 \pmod 2 & a_3b_4 + a_4b_2 \pmod 2 \end{pmatrix} \end{aligned}$$

Clearly $(M_{2\times 2}, \text{o})$ is a Smarandache groupoid for $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{o} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

So $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{o} \right\}$ is a semigroup.

Now consider $A_1 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, o\right\}$ is a Smarandache subgroupoid but A_1 is non-commutative Smarandache groupoid for A_1 contains a non-commutative semigroup S. $S = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, o\right\}$ such that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} o \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} o \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So $(M_{2\times 2}, o)$ is a Smarandache commutative groupoid but not a Smarandache inner commutative groupoid.

Definition 20: A groupoid G is said to be a *Moufang groupoid* if for every x, y, z in G we have (xy)(zx) = (x(yz))x.

Definition 21: A Smarandache groupoid (G, ∗) is said to be *Smarandache-Moufang groupoid* if there exists $H \subset G$ such that H is a Smarandache subgroupoid satisfying the Moufang identity: (xy)(zx) = (x(yz)x) for all x, y, z in H.

Definition 22: Let S be a Smarandache groupoid. If every Smarandache subgroupoid H of S satisfies the Moufang identity for all x, y, z in H then S is a *Smarandache Strong Moufang groupoid*.

Theorem 23: Every Smarandache Strong Moufang groupoid is a Smarandache Moufang groupoid and not conversely.

Proof: Every Strong Smarandache Moufang groupoid is a Smarandache Moufang groupoid. The proof of the converse can be proved by constructing examples.

Definition 24: A groupoid G is said to be a *Bol groupoid* if ((xy)z)y = x((yz))y for all x, y, z $\in$ G.

Definition 25: Let G be a groupoid. G is said to be a *Smarandache-Bol groupoid* if G has a subgroupoid H of G such that H is a Smarandache subgroupoid and satisfies the identity ((xy)z)y = x((yz)y) for all x, y, z in H.

Definition 26: Let G be a groupoid. We say G is a *Smarandache Strong Bol groupoid* if every Smarandache subgroupoid of G is a Bol groupoid.

Theorem 27: Every Smarandache Strong Bol groupoid is a Smarandache Bol groupoid and the converse is not true.

Proof: Obvious.

Theorem 28: Let $Z_n = \{0, 1, 2, ..., n-1\}$ be the set of integers modulo n. Let $G = \{Z_n, *, (t, u)\}$ be a Smarandache groupoid. G is a Smarandache Bol groupoid if $t^3 = t \pmod{n}$ and $u^2 = u \pmod{n}$.

Proof: Easy to verify.

Example 9: Let $G = \{Z_6, *, (2,3)\}$ defined by the following table:

*	0	1	2	3	4	5
0	0	3	0	3	0	3
1	2	5	2	5	2	5
2	4	1	4	1	4	1
3	0	3	0	3	0	3
4	2	5	2	5	2	5
5	4	1	4	1	4	1

{0, 3} is a Smarandache subgroupoid and since $2^3 = 2 \pmod{6}$ and $3^2 = 3 \pmod{6}$ we see G is a Smarandache Bol groupoid.

Problem 2: Let $Z_n = \{0, 1, 2, ..., n-1\}$ be the ring of integers modulo n. $G = \{Z_n, *, (t,u)\}$ be a groupoid. Find conditions on n, t and u so that G:

1. is a Smarandache groupoid.
2. has Smarandache semi-normal subgroupoids.
3. has Smarandache normal subgroupoids.
4. is Smarandache commutative.
5. is Smarandache inner commutative.
6. is a Smarandache-Bol groupoid.
7. is a Smarandache Strong Bol groupoid.
8. is a Smarandache-Moufang groupoid.
9. is a Smarandache-Strong-Moufang groupoid.
10. has always a pair of Smarandache conjugate subgroupoids.

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