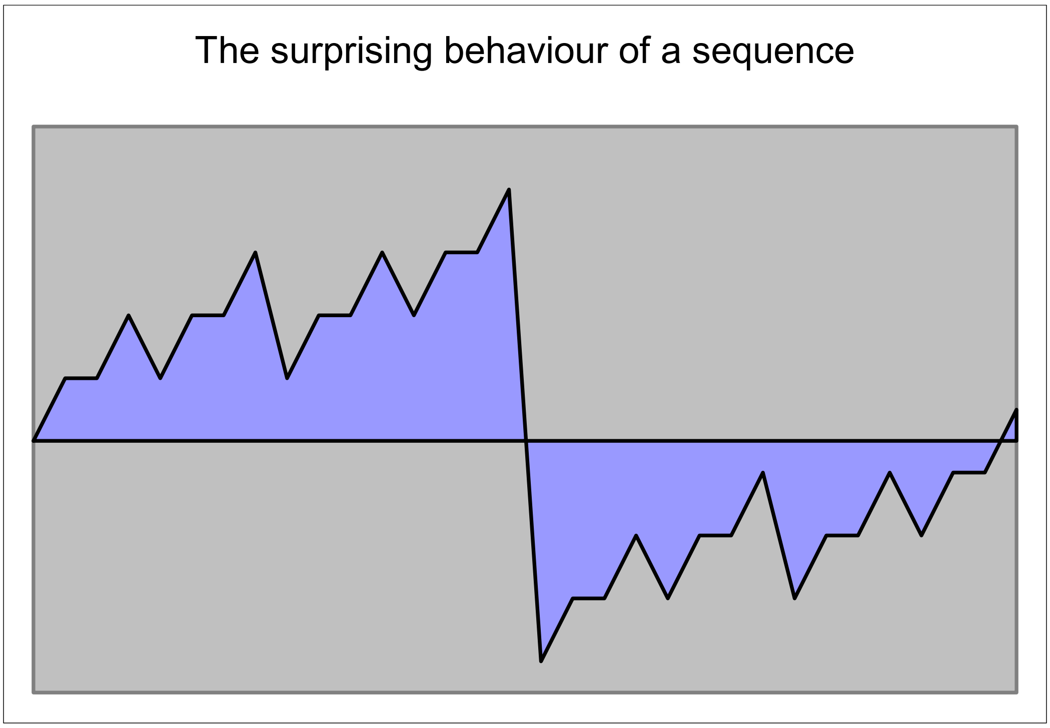
Mainly Natural Numbers

- a few elementary studies on Smarandache sequences and other number problems

Henry Ibstedt



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The diagram on the cover illustrates the Smarandache partial perfect additive sequence. It has an amusing oscillating behaviour, it does not form loops and has no terminating value. Its definition is simply

$$a_1=a_2=1,\ a_{2k+1}=a_{k+1}-1,\ a_{2k+2}=a_{k+1}+1$$

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Preface

This book consists of a selection of papers most of which were produced during the period 1999-2002. They have been inspired by questions raised in recent articles in current Mathematics journals and in Florentin Smarandache's wellknown publication *Only Problems, Not Solutions.*

All topics are independent of one another and can be read separately. Findings are illustrated with diagrams and tables. The latter have been kept to a minimum as it is often not the numbers but the general behaviour and pattern of numbers that matters. One of the facinations with number problems is that they are often easy to formulate but hard to solve – if ever, and if one finds a solution, new questions present themselves and one may end up having more new questions than questions answered.

In many practical as well as theoretical processes we repeat the same action on an object again and again to obtain a final result or sustain a certain state. An interesting case is when we do not know what the result will be after a large number of repetitive actions - iterations. In this book a number of problems are about iterations. In many cases computer simulation is followed by analysis leading to conclusions or conjectures. The process of iterations has also been dealt with in the authors first book *Surfing on the Ocean of Numbers* with some applications in the second book *Computer Analysis of Number Sequences.*

A brief summary will now be given about the contents of each chapter of the book:

Chapter 1: This is in response to the question: Which is the smallest integer that can be expressed as a sum of consecutive integers in a given number of ways? The examination of this question leads to a few interesting conclusions.

Chapter II deals with interesting alterating iterations of the Smarandache function and the Euler ϕ-function (the number of natural numbers less than n and having no divisor in common with n) An important question concerning the Smarandache function is resolved and an important link to the famous Fermat numbers is established. This work has been reviewed in Zentralblatt für Mathematik, Germany.

Chapter III is of a similar nature to that of chapter II. It deals with the alternating iteration of the Smarandache function and the sum of divisors function (σ-function). Some light is thrown on loops and invariants resulting from this iteration. Interesting results are found but the results produce new and very intriguing questions.

Chapter IV. An interesting iteration question was posed in the book *Unsolved Questions in Number Theory* (first edition) by R.K. Guy. Why does the repetitive application of the recursion formula $x_n=(1+x_0+x_1+\ldots x_{n-1})/n$ with $x_0=1$ produce natural numbers for n=1,2, … 42 but not for n= 43. An explanation to this was given by the author and published in Fibonacci Quarterly in 1990 and was later referred to in the second edition of R.K. Guy's book. In of this book I show an iteration sequence which produces integers for the first 600 iterations but not for the 601st which produces a decimal fraction. This is the only article which is based on work prior to 1999.

Chapter V. In the previous chapters iterations have lead to loops or invariants. The Smarandache partial perfect additive sequence has a very simple definition: $a_1=a_2=1$, $a_{2k+1}=a_{k+1}-1$, $a_{2k+2}=a_{k+1}+1$. It does not form loops and it does not have a terminating value. It has an amusing oscillating behavior which is illustrated on the cover of this book.

Chapter VI. The classical definition of continued fractions was transformed to one involving Smarandache sequences by Jose Castillo. In this article proof is given for the fact that Smarandache general continued fractions built with positive integer Smarandache sequences having only a finite number of terms equal to 1 is convergent. This study, like several others from my earlier books, has been reviewed in the Zentralblatt für Mathematik, Germany.

Chapter VII. A k-k additive relationship involves the Smarandache function S(n) which is defined as the smallest integer such that S(n)! is divisible by n. A sequence of function values S(n), S(n+1)+ … +S(n+2k-1) satisfies a k-k additive relationship if S(n)+S(n+1)+ …+S(n+k-1)=S(n+k)+S(n+k+1)+

…+S(n+2k-1). The analysis of these types of relations leads to the conclusion that there are infinitely many 2-2 additive relations and that k-k relations exist for large values of k. Only the first two solutions contain composite numbers. An interesting observation is the great involvement of prime twins in the 2-2 relations.

Chapter VIII. An analysis of the number of relations of the type S(n)-S(n+1)=S(n+2)-S(n+3) for $n<10^8$ where S(n) is the Smarandache function leads to the plausible conclusion that there are infinitely many of those. Like in the case of additive relationships there is a great involvement of prime twins and composite number solutions are rare – only 6 were found.

Chapter IX. Concatenation is a sophisticated word for putting two words together to form one. The words *book* and *mark* are concatenated to form the word *bookmark.* Identical words like "abcd" are concatenated to form infinite chains like "abcdabcdabcdabcdabcd…". This is partitioned in various way, for example like this

abc|dabcdabcda|bcdabcd…

and the properties of the extracted word|dabcdabcda| is then studied. The analysis of concatenations is applied to number sequences and many interesting properties are found. In particular a number of questions raised on the Smarandache deconstructive sequence are resolved.

Chapter X. In the study of a number sequence it was found that the terms often had a factor 333667. We are here dealing with a sequence whos terms grow to thousands of digits. No explanation was attempted in the article were this was found. This intriguing fact and several others are dealt with in this study, where, in a way, the concatenation process is reversed and divisibilty properties studied. The most preoccupying questions in relation to divisibility have always focussed on primality. In the articles in this book other divisibilty properties are often brought into focus.

Finally I express my sincere thanks to Dr. Minh Perez for his support for this book. Last but not least I thank my dear wife Anne-Marie for her patience with me when I am in my world of numbers.

March 2003
Henry Ibstedt

Contents

I. An Integer as a Sum of Consecutive Integers

Abstract: This is a simple study of expressions of positive integers as sums of consecutive integers. In the first part proof is given for the fact that N can be expressed in exactly d(L)-1 ways as a sum of consecutive integers, L is the largest odd factor of N and d(L) is the number of divisors of L. In the second part answer is given to the question: Which is the smallest integer that can be expressed as a sum of consecutive integers in n ways.

1. Introduction

There is a remarkable similarity between the four definitions given below. The first is the well known Smarandache Function. The second function was defined by K. Kashihara and was elaborated on in his book *Comments and Topics on Smarandache Notions and Problems* [1]. This function and the Smarandache Ceil Function were also examined in the author's book *Surfing on the Ocean of Numbers* [2]. These three functions have in common that they aim to answer the question which is the smallest positive integer N which possesses a certain property pertaining to a given integer n. It is possible to pose a large number of questions of this nature.

1) **The Smarandache Function S(n):**
   S(n)=N where N is the smallest positive integer which divides n!.

2) **The Pseudo-Smarandache Function Z(n):**
   Z(n)=N where N is the smallest positive integer such that 1+2+…+N is divisible by n.

3) **The Smarandache Ceil Function of order k, $S_k(n)$:**
   $S_k(n)$=N where N is he smallest positive integer for which n divides N^k.

4) **The n-way consecutive integer representation R(n):**
   R(n)=N where N is the smallest positive integer which can be represented as a sum of consecutive integer is n ways.

There may be many positive integers which can be represented as a sum of positive integers in n distinct ways - but which is the smallest of them? This article gives the answer to this question. In the study of R(n) it is found that the arithmetic function d(n), the number of divisors of n, plays an important role.

2. Questions and Conclusions

Question 1: In how many ways n can a given positive integer N be expressed as the sum of consecutive positive integers?

Let the first term in a sequence of consecutive integers be Q and the number terms in the sequence be M. We have N=Q+(Q+1)+ … +(Q+M-1) where M>1.

(1) $$N = \frac{M(2Q + M - 1)}{2}$$

or

(2) $$Q = \frac{N}{M} - \frac{M-1}{2}$$

For a given positive integer N the number of sequences n is equal to the number of positive integer solutions to (2) in respect of Q. Let us write $N=L\cdot 2^s$ and $M=m\cdot 2^k$ where L and m are odd integers. Furthermore we express L as a product of any of its factors $L=m_1m_2$. We will now consider the following cases:

1. s=0, k=0
2. s=0, k≠0
3. s≠0, k=0
4. s≠0, k≠0

<u>Case 1. s=0, k=0.</u>

Equation (2) takes the form

(3) $$Q = \frac{m_1m_2}{m} - \frac{m-1}{2}$$

Obviously we must have m≠1 and m≠L (=N).

For $m=m_1$ we have $Q>0$ when $m_2-(m_1-1)/2>0$ or $m_1<2m_2+1$. Since m_1 and m_2 are odd, the latter inequality is equivalent to $m_1<2m_2$ or, since $m_2=N/m_1$, $m_1 < \sqrt{2N}$.

We conclude that a factor m ($\neq 1$ and $\neq N$) of N (odd) for which $m < \sqrt{2N}$ gives a solution for Q when M=m is inserted in equation (2).

<u>Case 2. s=0, k≠0.</u>

Since N is odd we see form (2) that we must have k=1. With M=2m equation (2) takes the form

(4) $$Q = \frac{m_1 m_2}{2m} - \frac{2m-1}{2}$$

For m=1 (M=2) we find Q=(N-1)2 which corresponds to the obvious solution $\frac{N-1}{2} + \frac{N+1}{2} = N$.

Since we can have no solution for m=N we now consider $m=m_2$ ($\neq 1$, $\neq N$). We find $Q=(m_1-2m_2+1)/2$. $Q>0$ when $m_1>2m_2-1$ or, since m_1 and m_2 are odd, $m_1>2m_2$ Since $m_1m_2=N$, $m_2=N/m_1$ we find $m > \sqrt{2N}$.

We conclude that a factor m ($\neq 1$ and $\neq N$) of N (odd) for which $m > \sqrt{2N}$ gives a solution for Q when M=2m is inserted in equation (2).

The number of divisors of N is known as the function d(N). Since all factors of N except 1 and N provide solutions to (2) while M=2, which is not a factor of N, also provides a solution (2) we find that the number of solutions n to (2) when N is odd is

(5) $$n=d(N)-1$$

<u>Case 3. s≠0, k=0.</u>

Equation (2) takes the form

(6) $$Q = \frac{1}{2}(\frac{m_1 m_2}{m} 2^{s+1} - m + 1)$$

Q≥1 requires $m^2 < L \cdot 2^{s+1}$. We distinguish three cases:

Case 3.2.	k=0, $m=m_1$.	Q≥1 for $m_1 < m_2 2^{s+1}$ with a solution for Q when $M=m_1$.
Case 3.3.	k=0, $m=m_1 m_2$.	Q≥1 for $L<2^{s+1}$ with a solution for Q when M=L.

Case 4. s≠0, k≠0.

Equation (2) takes the form

(7) $$Q = \frac{1}{2}(\frac{m_1 m_2}{m} 2^{s-k+1} - m \cdot 2^k + 1)$$

Q is an integer if and only if m divides L and $2^{s-k+1}=1$. The latter gives k=s+1. Q≥1 gives

(8) $$Q = \frac{1}{2}(\frac{m_1 m_2}{m} + 1) - m \cdot 2^s \geq 1$$

Again we distinguish three cases:

Case 4.1.	k=s+1, m=1.	Q≥1 for $L>2^{s+1}$ with a solution for Q when $M=2^{s+1}$.
Case 4.2.	k=s+1, $m=m_2$	Q≥1 for $m_1>m_2 2^{s+1}$ with a solution for Q when $M=m_2 2^{s+1}$.
Case 4.3.	k=s+1, m=L	Q≥1 for $1-L \cdot 2^s \geq 1$. No solution

Since all factors of L except 1 provide solutions to (2) we find that the number of solutions n to (2) when N is even is

(9) n=d(L)-1

Conclusions:

- The number of sequences of consecutive positive integers by which a positive integer $N=L\cdot 2^s$,where $L\equiv 1 \pmod 2$, can be represented is $n=d(L)-1$.
- N=L no matter how large we make s.
- When $L<2^s$ the values of M which produce integer values of Q are odd, i.e. N can in this case only be represented by sequences of consecutive integers with an odd number of terms.
- There are solutions for all positive integers L except for L=1, which means that $N=2^s$ are the only positive integers which cannot be expressed as the sum of consecutive integers.
- $N=P\cdot 2^s$ has only one representation which has a different number of terms (<p) for different s until $2^{s+1}>P$ when the number of terms will be p and remain constant for all larger s.

A few examples are given in table 1.

Table 1. The number of sequences for L=105 is 7 and is independent of s in $N=L\cdot 2^s$.

N= 105	s=0	N= 210	s=1	N= 3360	s=5 L> 2^{s+1}	N= 6720	s=6 L< 2^{s+1}
Q	M	Q	M	Q	M	Q	M
34	3	69	3	1119	3	2239	3
19	5	40	5	670	5	1342	5
12	7	27	7	477	7	957	7
1	14	7	15	217	15	441	15
6	10	1	20	150	21	310	21
15	6	12	12	79	35	175	35
52	2	51	4	21	64	12	105

Question 2: Which is the smallest positive integer N which can be represented as a sum of consecutive positive integers in n different ways.

We can now construct the smallest positive integer R(n)=N which can be represented in n ways as the sum of consecutive integers. As we have already seen this smallest integer is necessarily odd and satisfies n=d(N)-1.

With the representation $N = p_1^{\alpha_1} p_2^{\alpha_2} ... p_j^{\alpha_j}$ we have

$$d(N)=(\alpha_1+1)(\alpha_2+1)\ldots(\alpha_j+1)$$

or

(10) $n+1=(\alpha_1+1)(\alpha_2+1)\ldots(\alpha_j+1)$

The first step is therefore to factorize n+1 and arrange the factors (α_1+1), (α_2+1) … (α_j+1) in descending order. Let us first assume that $\alpha_1>\alpha_2> \ldots >\alpha_j$ then, remembering that N must be odd, the smallest N with the largest number of divisors is

$$R(n)= N = 3^{\alpha_1} 5^{\alpha_2} 7^{\alpha_3} ... p_j^{\alpha_j}$$

where the primes are assigned to the exponents in ascending order starting with $p_1=3$. Every factor in (10) corresponds to a different prime even if there are factors which are equal.

```
Example:     n   = 269
             n+1= 2·3³·5 = 5 ·3 ·3 ·3 ·2
             R(n)= 3⁴·5²·7²·11²·13=156080925
```

When n is even it is seen from (10) that α_1, α_2, … α_j must all be even. In other words the smallest positive integer which can be represented as a sum of consecutive integers in a given number of ways must be a square. It is therefore not surprising that even values of n in general generate larger smallest N than odd values of n. For example, the smallest integer that can be represented as a sum of integers in 100 ways is $N=3^{100}$, which is a 48-digit integer, while the smallest integer that can be expressed as a sum of integer in 99 ways is only a 7-digit integer, namely 3898125.

Conclusions:

- 3 is always a factor of the smallest integer that can be represented as a sum of consecutive integers in n ways.
- The smallest positive integer which can be represented as a sum of consecutive integers in given even number of ways must be a square.

Table 2. The smallest integer R(n) which can be represented in n ways as a sum of consecutive positive integers.

n	R(n)	R(n) in factor form
1	3	3
2	9	3^2
3	15	$3\cdot5$
4	81	3^4
5	45	$3^2\cdot5$
6	729	3^6
7	105	$3\cdot5\cdot7$
8	225	3^25^2
9	405	3^45
10	59049	3^{10}
11	315	$3^25\cdot7$
12	531441	3^{12}

References:

1. K. Kashihara, *Comments and Topics on Smarandache Notions and Problems,* Erhus University Press.
2. Henry Ibstedt, *Surfing on the Ocean of Numbers,* Erhus University Press, 1997.

II. Alternating Iterations of the Euler ϕ-Function and the Pseudo-Smarandache Function

Abstract: This study originates from questions posed on alternating iterations involving the Pseudo-Smarandache function Z(n) and the Euler function $\phi(n)$. An important part of the study is a formal proof of the fact that Z(n)<n for all $n \neq 2^k$ ($k \geq 0$). Interesting questions have been resolved through the surprising involvement of Fermat numbers.

1. The behaviour of the Pseudo-Smarandache function

Definition of the Smarandache pseudo function Z(n): Z(n) is the smallest positive integer m such that 1+2+…+m is divisible by n.

Adding up the arithmetical series results in an alternative and more useful formulation: For a given integer n , Z(n) equals the smallest positive integer m such that m(m+1)/2n is an integer. Some properties and values of this function are given in [1], which also contains an effective computer algorithm for calculation of Z(n). The following properties are evident from the definition:

1. Z(1)=1
2. Z(2)=3
3. For any odd prime p, $Z(p^k)=p^k-1$ for $k \geq 1$
4. For $n=2^k$, $k \geq 1$, $Z(2^k)=2^{k+1}-1$

We note that Z(n)=n for n=1 and that Z(n)>n for $n=2^k$ when $k \geq 1$. Are there other values of n for which $Z(n) \geq n$? No, there are none, but to my knowledge no proof has been given. Before presenting the proof it might be useful to see some elementary results and calculations on Z(n). Explicit calculations of $Z(3 \cdot 2^k)$ and $Z(5 \cdot 2^k)$ have been carried out by Charles Ashbacher [2]. For k>0:

$$Z(5\cdot 2^k)=\begin{cases} 2^{k+2} & \text{if } k\equiv 0 \pmod 4 \\ 2^{k+1} & \text{if } k\equiv 1 \pmod 4 \\ 2^{k+2}-1 & \text{if } k\equiv 2 \pmod 4 \\ 2^{k+1}-1 & \text{if } k\equiv 3 \pmod 4 \end{cases}$$

A specific remark was made in each case that $Z(n)<n$.

In this study we will prove that $Z(n)<n$ for all $n\neq 2^k$, $k\geq 0$, but before doing so we will continue to study $Z(a\cdot 2^k)$, a odd and $k>0$. In particular we will carry out
a specific calculation for $n=7\cdot 2^k$.

We look for the smallest integer m for which $\frac{m(m+1)}{7\cdot 2^{k+1}}$ is integer. We distinguish two cases:

Case 1:	Case 2:
$m=7x$	$m=2^{k+1}y$
$m+1=2^{k+1}y$	$m+1=7x$
Eliminating m results in	
$2^{k+1}y-1=7x$	$2^{k+1}y+1=7x$
$2^{k+1}y\equiv 1 \pmod 7$	$2^{k+1}y\equiv -1 \pmod 7$
Since $2^3\equiv 1 \pmod 3$ we have	
If $k\equiv -1 \pmod 3$ then	
$y\equiv 1 \pmod 7$; $m=2^{k+1}-1$	$y\equiv 8$ mod 7); $m=2^{k+1}\cdot 8=2^{k+4}$
If $k\equiv 0 \pmod 3$ then	
$2y\equiv 1 \pmod 7$, $y=4$; $m=2^{k+1}\cdot 4-1=2^{k+3}-1$	$y\equiv 3 \pmod 7$; $m=3\cdot 2^{k+1}$
If $k\equiv 1 \pmod 3$ then	
$4y\equiv 1 \pmod 7$, $y=2$; $m=2^{k+1}\cdot 2-1=2^{k+2}-1$	$y\equiv 5 \pmod 7$; $m=5\cdot 2^{k+1}$

By choosing the smallest m in each case we find:

$$Z(7\cdot 2^k)=\begin{cases} 2^{k+1}-1 & \text{if } k\equiv -1 \pmod 3 \\ 3\cdot 2^{k+1} & \text{if } k\equiv 0 \pmod 3 \\ 2^{k+2}-1 & \text{if } k\equiv 1 \pmod 3 \end{cases}$$

Again we note that $Z(n)<n$.

In a study of alternating iterations [3] it is stated that apart from when $n=2^k$ $(k\geq 0)$ Z(n) is <u>at most n</u>. If it ever happened that Z(n)=n for n>1 then the iterations of Z(n) would arrive at an invariant, i.e. Z(…Z(n)…)=n. This can not happen, therefore it is important to prove the following theorem.

Theorem: $Z(n)<n$ for all $n\neq 2^k$, $k\geq 0$.

Proof: Write n in the form $n=a\cdot 2^k$, where a is odd and k>0. Consider the following four cases:

1. $a\cdot 2^{k+1} | m$
2. $a\cdot 2^{k+1} | (m+1)$
3. $a | m$ and $2^{k+1} | (m+1)$
4. $2^{k+1} | m$ and $a | (m+1)$

If a is composite we could list more cases but this is not important as we will achieve our goal by finding m so that $Z(n)\leq m<n$ (where we will have Z(n)=m in case a is prime)

Cases 1 and 2:
Case 1 is excluded in favor of case 2 which would give $m= a\cdot 2^{k+1}-1>n$. We will see that also case 2 can be excluded in favor of cases 3 and 4.
Cases 3 and 4: In case 3 we write m=ax. We then require $2^{k+1} | (ax+1)$, which means that we are looking for solutions to the congruence

(1) $\quad ax \equiv -1 \pmod{2^{k+1}}$

In case 4 we write m+1=ax and require $2^{k+1} | (ax-1)$. This corresponds to the congruence

(2) $\quad ax \equiv 1 \pmod{2^{k+1}}$

If $x=x_1$ is a solution to one of the congruencies in the interval $2^k <x< 2^{k+1}$ then $2^{k+1}-x_1$ is a solution to the other congruence which lies in the interval $0 <x< 2^k$. So we have m=ax or m=ax-1 with $0<x<2^k$, i.e. m<n exists so that m(m+1)/2 is divisible by n when a>1 in $n=a\cdot 2^k$. If a is a prime number then we also have Z(n)=m<n. If $a=a_1\cdot a_2$ then $Z(n) \leq m$ which is a fortiori less than n.

Let's illustrate the last statement by a numerical example. Take n=70 =5·7·2. An effective algorithm for calculation of Z(n) [1] gives Z(70)=20. Solving our two congruencies results in:

$35x \equiv -1 \pmod 4$ Solution x=1 for which m=35

$35x \equiv 1 \pmod 4$ Solution x=3 for which m=104

From these solutions we chose m=35 which is less than n=70. However, here we arrive at an even smaller solution Z(70)=20 because we do not need to require both a_1 and a_2 to divide one or the other of m and m+1.

II. Iterating the Pseudo-Smarandache Function

The theorem proved in the previous section assures that an iteration of the pseudo-Smarandache function does not result in an invariant, i.e. Z(n)≠n is true for n≠1. On iteration the function will leap to a higher value only when $n=2^k$. It can only go into a loop (or cycle) if after one or more iterations it returns to 2^k. Up to $n=2^{28}$ this does not happen and a statistical view on the results displayed in diagram 1 makes it reasonable to conjecture that it never happens. Each row in diagram 1 corresponds to a sequence of iterations starting on $n=2^k$ finishing on the final value 2. The largest number of iterations required for this was 24 and occurred for $n=2^{14}$ which also had the largest numbers of leaps form 2^j to $2^{j+1}-1$. Leaps are represented by ↑ in diagram 1. For $n=2^{11}$ and 2^{12} the iterations are monotonously decreasing.

III. Iterating the Euler ϕ Function

The function ϕ(n) is defined for n>1 as the number of positive integers less than and prime to n. The analytical expression is given by

$$\phi(n) = n\prod_{p|n}(1-\frac{1}{p})$$

For n expressed in the form $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}$ it is often useful to express ϕ(n) in the form

$$\phi(n) = p_1^{\alpha_1 - 1}(p_1 - 1)p_2^{\alpha_2 - 1}(p_2 - 1) \cdot \ldots \cdot p_r^{\alpha_r - 1}(p_r - 1)$$

It is obvious from the definition that ϕ(n)<n for all n>1. Applying the ϕ function to ϕ(n) we will have ϕ(ϕ(n))< ϕ(n). After a number of such iterations

k/j	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2
2																											↑
3																										↑	↑
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27		↑																								↑	↑
28	↑																							↑		↑	↑

Diagram 1.

the end result will of course be 1. It is what this chain of iterations looks like which is interesting and which will be studied here. For convenience we will write $\phi_2(n)$ for $\phi(\phi(n))$. $\phi_k(n)$ stands for the k^{th} iteration. To begin with we will look at the iteration of a few prime powers.

$\phi(2^{\alpha})=2^{\alpha-1}$, $\phi_k(2^{\alpha})=2^{\alpha-k}$, …. $\phi_{\alpha}(2^{\alpha})=1$.

$\phi(3^{\alpha})=3^{\alpha-1}\cdot 2$, $\phi_2(3^{\alpha})=3^{\alpha-2}\cdot 2$, …. $\phi_k(3^{\alpha})=3^{\alpha-k}\cdot 2$ for $k\leq\alpha$.

In particular $\phi_{\alpha}(3^{\alpha})=2$.

Proceeding in the same way we will write down $\phi_k(p^{\alpha})$, $\phi_{\alpha}(p^{\alpha})$ and first first occurrence of an iteration result which consists purely of a power of 2.

$\phi_k(5^{\alpha})=5^{\alpha-k}\cdot 2^{k+1}$, $k\leq\alpha$ $\phi_{\alpha}(5^{\alpha})=2^{\alpha+1}$.

$\phi_k(7^{\alpha})=7^{\alpha-k}\cdot 3\cdot 2^{k}$, $k\leq\alpha$ $\phi_{\alpha}(7^{\alpha})=3\cdot 2^{\alpha}$ $\phi_{\alpha+1}(7^{\alpha})=2^{\alpha}$.

$\phi_k(11^{\alpha})=11^{\alpha-k}\cdot 5\cdot 2^{2k-1}$, $k\leq\alpha$ $\phi_{\alpha}(11^{\alpha})=5\cdot 2^{2\alpha-1}$ $\phi_{\alpha+1}(11^{\alpha})=2^{2\alpha}$.

$\phi_k(13^{\alpha})=13^{\alpha-k}\cdot 3\cdot 2^{2k}$, $k\leq\alpha$ $\phi_{\alpha}(13^{\alpha})=3\cdot 2^{2\alpha}$ $\phi_{\alpha+1}(13^{\alpha})=2^{2\alpha}$.

$\phi_k(17^{\alpha})=17^{\alpha-k}\cdot 2^{3k+1}$, $k\leq\alpha$ $\phi_{\alpha}(17^{\alpha})=2^{3\alpha+1}$.

$\phi_k(19^{\alpha})=19^{\alpha-k}\cdot 3^{k+1}\cdot 2^{k}$, $k\leq\alpha$ $\phi_{\alpha}(19^{\alpha})=3^{\alpha+1}\cdot 2^{\alpha}$ $\phi_{2\alpha+1}(19^{\alpha})=2^{\alpha}$.

$\phi_k(23^{\alpha})=23^{\alpha-k}\cdot 11\cdot 5\cdot 2^{3k-4}$, $k\leq\alpha$ $\phi_{\alpha}(23^{\alpha})=11\cdot 5\cdot 2^{3\alpha-4}$ $\phi_{\alpha+2}(23^{\alpha})=2^{3\alpha-1}$.

The characteristic tail of descending powers of 2 applies also to the iterations of composite integers and plays an important role in the alternating Z-ϕ iterations which will be subject of the next section.

IV. The alternating iteration of the Euler ϕ function followed by the Smarandache Z function.

Charles Ashbacher [3] found that the alternating iteration $Z(\ldots(\phi(Z(\phi(n)))\ldots)$ ends in 2-cycles of which he found the following four[1]:

[1] It should be noted that 2, 8, 128 and 32768 can be obtained as iteration results only through iterations of the type $\phi(\ldots(Z(\phi(n)))\ldots)$ whereas the "complete" iterations $Z(\ldots(\phi(Z(\phi(n)))\ldots)$ lead to the invariants 3, 15, 255, 65535. Consequently we note that for example $Z(\phi(8))=7$ not 15, i.e. 8 does not belong to its own cycle.

2-cycle	First Instance
2 - 3	$3=2^2-1$
8 - 15	$15=2^4-1$
128 - 255	$255=2^8-1$
32768 - 65535	$65535=2^{16}-1$

The following questions were posed:

1) Does the Z-ϕ sequence always reduce to a 2-cycle of the form $2^{2^r-1} \leftrightarrow 2^{2^r}-1$ for r≥1?

2) Does any additional patterns always appear first for $n = 2^{2^r} - 1$?

Theorem: The alternating iteration Z(…(ϕ(Z(ϕ(n)))…) ultimately leads to one of the following five 2-cycles: 2 -3, 8 - 15, 128 - 255, 32768 - 65535, 2147483648 - 4294967295.

Proof:

Since $\phi(n)<n$ for all n>1 and $Z(n)<n$ for all $n\neq 2^k$ ($k\geq 0$) any cycle must have a number of the form 2^k at the lower end and $Z(2^k)=2^{k+1}-1$ at the upper end of the cycle. In order to have a <u>2-cycle</u> we must find a solution to the equation

$$\phi(2^{k+1}-1)=2^k$$

If $2^{k+1}-1$ were a prime $\phi(2^{k+1}-1)$ would be $2^{k+1}-2$ which solves the equation only when k=1. A necessary condition is therefore that $2^{k+1}-1$ is composite, $2^{k+1}-1=f_1\cdot f_2\cdot\ldots\cdot f_i\cdot\ldots\cdot f_r$ and that the factors are such that $\phi(f_i)=2^{u_i}$ for $1\leq i\leq r$. But this means that each factor f_i must be a prime number of the form $2^{u_i}+1$. This leads us to consider

$$q(r)= (2-1)(2+1)(2^2+1)(2^4+1)(2^8+1) \ldots (2^{2^{r-1}}+1)$$

or

$$q(r)= (2^{2^r}-1)$$

Numbers of the form $F_r=2^{2^r}+1$ are known as Fermat numbers. The first five of these are prime numbers

$F_0=3, F_1=5, F_2=17, F_3=257, F_4=65537$

while $F_5=641\cdot 6700417$ as well as F_6 , F_7 , F_8 , F_9 , F_{10} and F_{11} are all known to be composite.

Table 1. Iteration of p^6. A horizontal line marks where the rest of the iterated values consist of descending powers of 2

#	p=2	p=3	p=5	p=7	p=11	p=13	p=17	p=19	p=23
1	32	486	12500	100842	1610510	4455516	22717712	44569782	141599546
2	16	162	5000	28812	585640	1370928	10690688	14074668	61565020
3	8	54	2000	8232	212960	421824	5030912	4444632	21413920
4	4	18	800	2352	77440	129792	2367488	1403568	7448320
5	2	6	320	672	28160	39936	1114112	443232	2590720
6		2	128	192	10240	12288	524288	139968	901120
7			64	64	4096	4096	262144	46656	327680
8			32	32	2048	2048	131072	15552	131072
9			16	16	1024	1024	65536	5184	65536
10			8	8	512	512	32768	1728	32768
11			4	4	256	256	16384	576	16384
12			2	2	128	128	8192	192	8192
13					64	64	4096	64	4096
14					32	32	2048	32	2048
15					16	16	1024	16	1024
16					8	8	512	8	512
17					4	4	256	4	256
18					2	2	128	2	128
19							64		64
20							32		32
21							16		16
22							8		8
23							4		4
24							2		2

From this we see that

(3) $$\phi(2^{2^r}-1)=\phi(q(r))=\phi(F_0)\phi(F_1)\cdot\ldots\cdot\phi(F_{r-1})=2\cdot 2^2\cdot\ldots\cdot 2^{2^{r-1}}=2^{1+2+2^2+2^3+\ldots+2^{r-1}}=2^{2^r-1}$$

for r=1, 2, 3, 4 5 but breaks down for r=6 (because F_5 is composite) and consequently also for r>6.

Evaluating (3) for r=1,2,3,4,5 gives the complete table of expressions for the five 2-cycles.

Cycle #	2-cycle	Equiv.expression
1	$2 \leftrightarrow 3$	$2 \leftrightarrow 2^2-1$
2	$8 \leftrightarrow 15$	$2^3 \leftrightarrow 2^4-1$
3	$128 \leftrightarrow 255$	$2^7 \leftrightarrow 2^8-1$
4	$32768 \leftrightarrow 65535$	$2^{15} \leftrightarrow 2^{16}-1$
5	$2147483648 \leftrightarrow 4294967295$	$2^{31} \leftrightarrow 2^{32}-1$

The answers to the two questions are implicit in the above theorem.

1) The Z-ϕ sequence always reduces to a 2-cycle of the form $2^{2^r-1} \leftrightarrow 2^{2^r}-1$ for r≥1.
2) Only five patterns exist and they always appear first for $n = 2^{2^r} - 1$, r=1,2,3,4,5.

A statistical survey of the frequency of the different 2-cycles, displayed in table 2, indicates that the lower cycles are favored when the initiating numbers grow larger. Cycle #4 could have appeared in the third interval but as can be seen it is generally scarcely represented. Prohibitive computer execution times made it impossible to systematically examine an interval were cycle #5 members can be assumed to exist. However, apart from the "founding member" 2147483648 ↔ 4294967295 a few individual members were calculated by solving the equation:

$$Z(\phi(n)=2^{32}-1$$

The result is shown in table 3.

Table 2. The distribution of cycles for a few intervals of length 1000.

Interval	Cycle #1	Cycle #2	Cycle #3	Cycle #4
$3 \le n \le 1002$	572	358	70	-
$10001 \le n \le 11000$	651	159	190	-
$100001 \le n \le 101000$	759	100	141	0
$1000001 \le n \le 1001000$	822	75	86	17
$10000001 \le n \le 100001000$	831	42	64	63
$100000001 \le n \le 1000001000$	812	52	43	93

Table 3. A few members of the cycle #5 family.

n	$\phi(n)$	$Z(\phi(n))$	$\phi(Z(\phi(n)))$
38655885321	25770196992	4294967295	2147483648
107377459225	85900656640	4294967295	2147483648
966397133025	515403939840	4294967295	2147483648
1241283428641	1168248930304	4294967295	2147483648
11171550857769	7009493581824	4294967295	2147483648
31032085716025	23364978606080	4294967295	2147483648
279288771444225	140189871636480	4294967295	2147483648
283686952174081	282578800082944	4294967295	2147483648
2553182569566729	1695472800497664	4294967295	2147483648
7092173804352025	5651576001658880	4294967295	2147483648
63829564239168225	33909456009953280	4294967295	2147483648
81985529178309409	76861433622560768	4294967295	2147483648
2049638229457735225	1537228672451215360	4294967295	2147483648

References:

1. H. Ibstedt, *Surfing on the Ocean of Numbers,* Erhus University Press, 1997.
2. Charles Ashbacher, *Pluckings From the Tree of Smarandache Sequences and Functions,* American Research Press, 1998.
3. Charles Ashbacher, On Iterations That Alternate the Pseudo-Smarandache and Classic Functions of Number Theory, *Smarandache Notions Journal, Vol. 11, No 1-2-3.*

III. Alternating Iterations of the Sum of Divisors Function and the Pseudo-Smarandache Function

Abstract: This study is an extension of work done by Charles Ashbacher[3]. Iteration results have been re-defined in terms of invariants and loops. Further empirical studies and analysis of results have helped throw light on a few intriguing questions.

1. Introduction

The following definition forms the basis of Ashbacher's study: For n>1, the Zσ sequence is the alternating iteration of the Sum of Divisors Function σ followed by the Pseudo-Smarandache function Z.

The Zσ sequence originated by n creates a cycle. Ashbacher identified four 2 cycles and one 12 cycle. These are listed in table 1.

Table 1. Iteration cycles C_1 - C_5.

n	C_k	Cycle
2	C_1	$3 \leftrightarrow 2$
$3 \le n \le 15$	C_2	$24 \leftrightarrow 15$
n=16	C_3	$31 \to 32 \to 63 \to 104 \to 64 \to 127 \to 126 \to 312 \to 143 \to 168 \to 48 \to 124$
$17 \le n \le 19$	C_2	$24 \leftrightarrow 15$
n=20	C_3	$42 \leftrightarrow 20$
n=21	C_3	$31 \to 32 \to 63 \to 104 \to 64 \to 127 \to 126 \to 312 \to 143 \to 168 \to 48 \to 124$
$22 \le n \le 24$	C_2	$24 \leftrightarrow 15$
n=25	C_3	$31 \to 32 \to 63 \to 104 \to 64 \to 127 \to 126 \to 312 \to 143 \to 168 \to 48 \to 124$
n=26	C_3	$42 \leftrightarrow 20$
…		
n=381	C_5	$1023 \leftrightarrow 1536$

The search for new cycles was continued up to n=552,000. No new ones were found. This lead Ashbacher to pose the following questions

1) Is there another cycle generated by the Zσ sequence?

2) Is there an infinite number of numbers n that generate the two cycle 4 42 $\leftrightarrow$ 20?
3) Are there any other numbers n that generate the two cycle 2 $\leftrightarrow$ 3?
4) Is there a pattern to the first appearance of a new cycle?

Ashbacher concludes his article by stating that these problems have only been touched upon and encourages others to further explore these problems.

2. An extended study of the Zσ iteration

It is amazing that hundred thousands of integers subject to a fairly simple iteration process all end up with final results that can be described by a few small integers. This merits a closer analysis. In an earlier study of iterations [2] the author classified iteration results in terms of invariants, loops and divergents. Applying the iteration to a member of a loop produces another member of the same loop. The cycles described in the previous section are not loops. The members of a cycle are not generated by the same process, half of them are generated by Z(σ(Z(…σ(n)…))) while the other half is generated by (σ(Z(…σ(n)…)), i.e. we are considering two different operators. This leads to a situation were the iteration process applied to a member of a cycle may generate a member of another cycle as described in table 2.

Table 2. A Zσ iteration applied to an element belonging to one cycle may generate an element belonging to another cycle .

	C_1		C_2		C_3		C_4												C_5	
n	2	3	15	24	20	42	31	32	63	104	64	127	126	312	143	168	48	124	1023	1536
σ(n)	3	4	24	60	42	96	32	63	104	210	127	128	312	840	168	480	124	224	1536	4092
Z(σ(n))	2	7	15	15	20	63	63	27	64	20	126	255	143	224	48	255	31	63	1023	495
σ(Z(σ(n)))		8						40				…		504		…				936
Z(σ(Z(σ(n))))		15						15				15		63		15				143
…																				
Generates	C_1	C_2	C_2	C_2	C_3	C_4	C_4	C_2	C_4	C_3	C_4	C_2	C_4	C_4	C_4	C_2	C_4	C_4	C_5	C_4
*=Shift to other cycle		*				*		*		*		*				*				*

This situation makes it impossible to establish a one-to-one correspondence between a number n to which the sequence of iterations is applied and the cycle that it will generate. Henceforth the iteration function will be Z(σ(n)) which will be denoted Zσ(n) while results included in the above cycles originating from σ(Z(…σ(n)…)) will be considered as intermediate elements. This leads to an unambiguous situation which is shown in table 3.

Table3. The Zσ iteration process described in terms of invariants, loops and intermediate elements.

	I_1	I_2	I_3			Loop				I_4
n	2	15	20	31	63	64	126	143	48	1023
Z(σ(n))	2	15	20	63	64	126	143	48	31	1023
Intermediate element	3	24	42	32	104	127	312	168	124	1536

We have four invariants I_1, I_2, I_3 and I_4 and one loop L with six elements. No other invariants or loops exist for $n \le 10^6$. Each number $n \le 10^6$ corresponds to one of the invariants or the loop. The distribution of results of the Zσ iteration has been examined by intervals of size 50000 as shown in table 4. The stability of this distribution is amazing. It deserves a closer look and will help bringing us closer to answers to the four questions posed by Ashbacher.

Question number 3: Are there any other numbers n that generate the two cycle $2 \leftrightarrow 3$? In the framework set for this study this question will reformulated to: Are there any other numbers than n=2 that belongs to the invariant 2?

Theorem: n=2 is the only element for which Z(σ(n))=2.
Proof:
Z(x)=2 has only one solution which is x=3. Z(σ(n))=2 can therefore only occur when σ(n)=3 which has the unique solution n=2.
□

Question number 2: Is there an infinite number of numbers n that generate the two cycle $42 \leftrightarrow 20$?

Conjecture: There are infinitely many numbers n which generate the invariant 20 (I_3).

Support: Although the statistics shown in table 4 only skims the surface of the "ocean of numbers" the number of numbers generating this invariant is as stable as for the other invariants and the loop. To this is added the fact that any number $>10^6$ will either generate a new invariant or loop (highly unlikely) or "catch on to" one of the already existing end results where I_4 will get its share as the iteration "filters through" from 10^6 until it gets locked onto one of the established invariants or the loop.

□

Table 4. Zσ iteration iteration results.

Interval	I_2	I_3	Loop	I_4
3-50000	18824	236	29757	1181
50001-100000	18255	57	30219	1469
100001-150000	17985	49	30307	1659
150001-200000	18129	27	30090	1754
200001-150000	18109	38	30102	1751
250001-300000	18319	33	29730	1918
300001-350000	18207	24	29834	1935
350001-400000	18378	18	29622	1982
400001-450000	18279	21	29645	2055
450001-550000	18182	24	29716	2078
500001-550000	18593	18	29227	2162
550001-600000	18159	19	29651	2171
600001-650000	18596	25	29216	2163
650001-700000	18424	26	29396	2154
700001-750000	18401	20	29409	2170
750001-800000	18391	31	29423	2155
800001-850000	18348	22	29419	2211
850001-900000	18326	15	29338	2321
900001-950000	18271	24	29444	2261
950001-1000000	18517	31	29257	2195
Average	18335	38	29640	1987

Question number 1: Is there another cycle generated by the Zσ sequence?

Discussion:
The search up to $n=10^6$ revealed no new invariants or loops. If another invariant or loop exists it must be initiated by $n>10^6$.

Let N be the value of n up to which the search has been completed. For n=N+1 there are three possibilities:

Possibility 1.

$Z(\sigma(n))\leq N$. In this case continued iteration repeats iterations which have already been done in the complete search up to $n=N$. No new loops or invariants will be found.

Possibility 2.

$Z(\sigma(n))=n$. If this happens then $n=N+1$ is a new invariant. A necessary condition for an invariant is therefore that

(1) $\frac{n(n+1)}{2\sigma(n)} = q$, where q is positive integer.

If in addition **no** m<n exists so that

(2) $\frac{m(m+1)}{2\sigma(n)} = q_1$, q_1 integer, then n is invariant.

There are 111 potential invariant candidates for n up to $3\cdot 10^8$ satisfying the necessary condition (1). Only four of them n = 2, 15, 20 and 1023 satisfied condition (2). It seems that for a given solution to (1) there is always, for $n>N>1023$, a solution to (2) with $m<n$. This is plausible since we know [4] that $\sigma(n)=O(n^{1+\delta})$ for every positive δ which means that $\sigma(n)$ is small compared to $n(n+1)\approx n^2$ for large n.

Example: The largest $n<3\cdot 10^8$ for which (1) is satisfied is n=292,409,999 with $\sigma(292{,}409{,}999)=341145000$ and $292409999\cdot 292410000/(2\cdot 341145000)=$ 125318571. But m=61370000<n exists for which $61370000\cdot 61370001/(2\cdot 341145000)=5520053$, an integer, which means that n is not invariant.

Possibility 3.

$Z(\sigma(n))>N$. This could lead to a new loop or invariant. Let's suppose that a new loop of length $k\geq 2$ is created. All elements of this loop must be greater than N otherwise the iteration sequence will fall below N and end up on a previously known invariant or loop. A necessary condition for a loop is therefore that

(3) $Z(\sigma(n))>n$ and $Z(\sigma(Z(\sigma(n))))\geq n$.

Denoting the k^{th} iteration $(Z\sigma)_k(n)$ we must finally have

(4) $(Z\sigma)_k(n)= (Z\sigma)_j(n)$ for some $k\neq j$, interpreting $(Z\sigma)_0(n)=n$

There isn't much hope for all this to happen since, for large n, already $Z(\sigma(n))>n$ is a scarce event and becomes scarcer as we increase n. A study of the number of incidents where $(Z\sigma)_3(n)>n$ for n<800,000 was made. There are only 86 of them, of these 65 occurred for n<100,000. From n=510,322 to n=800,000 there was not a single incident.

Question number 4: No particular patterns were found.

3. Epilog

In empirical studies of numbers the search for patterns and general behaviors is an interesting and important part. In this iteration study it is amazing that all these numbers, where not even the sky is the limit[2], after a few iterations filter down to end up on one of three invariants or a single loop. The other amazing thing is the relative stability of distribution between the three invariants and the loop with increasing n (see table 4) . When $(Z\sigma)_k(n)$ drops below n it catches on to an integer which has already been iterated and which has therefore already been classified to belong to one of the four terminal events. This in my mind explains the relative stability. In general the end result is obtained after only a few iterations. It is interesting to see that $\sigma(n)$ often assumes the same value for values of n which are fairly close together. Here is an example: $\sigma(n)=3024$ for n=1020, 1056, 1120, 1230, 1284, 1326, 1420, 1430, 1484, 1504, 1506, 1564, 1670, 1724, 1826, 1846, 1886, 2067, 2091, 2255, 2431, 2515, 2761, 2839, 2911, 3023. I may not have brought this subject much further but I hope to have contributed some light reading in the area of recreational mathematics.

References:

1. H. Ibstedt, *Surfing on the Ocean of Numbers,* Erhus University Press, 1997.
2. Charles Ashbacher, *Pluckings From the Tree of Smarandache Sequences and Functions,* American Research Press, 1998.
3. Charles Ashbacher, On Iterations That Alternate the Pseudo-Smarandache and Classic Functions of Number Theory, *Smarandache Notions Journal, Vol. 11, No 1-2-3.*
4. G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers.* Oxford University Press, 1938.

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[2] "Not even the sky is the limit" expresses the same dilemma as the title of the authors book "Surfing on the ocean of numbers". Even with for ever faster computers and better software for handling large numbers empirical studies remain very limited.

IV. Some Sequences of Large Integers

Abstract: In *Unsolved Problems in Number Theory* [1] the question why iteration of the sequence $x_n = (1 + x_0^m + x_1^m + ... + x_{n-1}^m)/n$, n=1,2,.. 42 times resulted in integers but the 43[rd] iteration breaks the integer sequence. This and similar sequences are studied. . A method is designed to examine how far the terms are integers. For one similar sequence the chain of integers is broken for n=601. This mysterious behaviour has been explained by the author [2] and referenced in the second edition of *Unsolved Problems* ..[3]. The present article is a revision and expansion of an earlier study.

1. Introduction

One of the many interesting problems posed in the book *Unsolved Problems in Number Theory* [1] concerns the sequence

$$x_n = (1 + x_0^m + x_1^m + ... + x_{n-1}^m)/n \qquad n=1,2,\ldots$$

or

$$nx_n = x_{n-1}(x_{n-1}^m + n - 1), \quad x_0, m \varepsilon N.$$

It was introduced by Fritz Göbel and has been studied by Lenstra [1] for m=1 and x_1=2 (x_0=1). Lenstra states that x_n is an integer for all n≤42, but x_{43} is not. For m=2 and x_1=2, David Boyd and Alf van der Poorten state that for n≤88 the only possible denominators in x_n are products of powers of 2, 3, 5 and 7. Why do these denominators cause a problem? Is it possible to find even longer sequences of integers by choosing different values for x_1 and m?
The terms in these sequences grow fast. For m=1, x_1=2 the first ten terms are:

2, 3, 5, 10, 28, 154, 3520, 15518880, 267593772160, 160642690122633501504.

If the number of digits in x_n is denoted N(n), then n(11)=43, N(12)=85, N(13)=168, N(14)=334, N(15)=667, N(16)=1332 and N(17)=2661. The last integer in this sequence, x_{42} has approximately 89288343500 digits**.**

The purpose of this study is to find a method of determining the number of integers in the sequence and apply the method for the parameters 1≤m≤10 and

$2 \le x_1 \le 11$. In particular, the problem of Boyd and van der Poorten will be solved. Some explanations will begiven to why some of these sequences are so long. It will be observed and explained why the integer sequences are in general longer for even than for odd values of m.

2. Method.

For given values of x_1 and m consider the equation

(1) $$kx_k = x_{k-1}(x_{k-1}^m + k - 1)$$

where the prime factorization of k is given by

(2) $$k = \prod_{i=1}^{\ell} p_i^{n_i} .$$

Let us assume that x_{k-1} is an integer and expand x_{k-1} and $x_{k-1}^m + k - 1$ in a number system with $G_i = p_i^{t_i}, \quad (t_i > n_i)$ as base.

(3a) $$x_{k-1} = \sum_j a_j G_i^j \quad (0 \le a_j < G_i)$$

and

(3b) $$x_{k-1}^m + k - 1 = \sum_j b_j G_i^j \quad (0 \le b_j < G_i) .$$

Since $x_{k-1} \neq 0$, it is always possible to choose t_I so that $a_0 \neq 0$. With this t_I we have

(4) $$x_{k-1}(x_{k-1}^m + k - 1) = \sum_{j,\ell} a_j b_\ell G_i^{j+\ell} \equiv a_0 b_0 \ (\mathrm{mod}\ G_i).$$

The congruence

(5) $$kx_k \equiv a_0 b_0 \quad (\mathrm{mod}\ G_i)$$

is soluble iff $(k, G_i) | a_0 b_0$, or, in this case, iff $p_i^{n_i} | a_0 b_0$. But, if $p_i^{n_i} | a_0 b_0$, then by (4) we also have

$$p_i^{n_i} \,|\, x_{k-1}(x_{k-1}^m + k - 1).$$

Furthermore, if (5) is soluble for all expansions originating from (2), then it follows that

$$k \mid x_{k-1}(x_{k-1}^{m} + k - 1)$$

and, consequently, that x_k is an integer. The solution x_k (mod G_i) to $kx_k \equiv a_0 b_0 \pmod{G_i}$ is equal to the first term in the expansion of x_k using the equivalent of (3a). The previous procedure is repeated using (3b), (4) and (5) to examine if x_{k+1} (mod G_i) is an integer.

From the computational point of view, the testing is done up to a certain pre-set limit $k=k_{max}$ for consecutive primes p=2, 3, 5, 7, … to $p \le k_{max}$. One of three things will happen:

1. All congruences are soluble modulus G_i for $k \le k_{max}$ for all $p_i \le k_{max}$.

2. $a_0 b_0 = 0$ for a certain set of values $k \le k_{max}$, $p_i \le k_{max}$.

3. The congruence $kx_k \equiv a_0 b_0 \pmod{G_i}$ is soluble for all $k < n < k_{max}$, but not soluble for k=n and $p=p_i$.

In cases 1 and 2 increase k_{max} , respectively t_i in $G_i = p_i^{t_i}$ (if computer facilities permit) and recalculate. In case3, x_n is not an integer, viz. n has been found so that x_k is an integer for k<n but not for k=n.

3. Results

The results from using this method in the 100 cases $1 \le m \le 10$, $2 \le x_1 \le 11$ are shown in Table 1. In particular, it shows that the integer sequence holds up to n=88 for m=2, $x_1=2$ which corresponds to the problem of Boyd and van der Poorten. The longest sequence of integers was found for $x_1=11$, m=2. For these parameters, the 600 first terms are integers, but x_{601} is not. In the 100 cases studied, only 32 different primes occur in the terminating values n. In 7 cases, the integer sequences are broken by values of n which are not primes. In 6 of these, the value of n is 2 times a prime which had terminated other sequences. For $x_1=3$, m=10, the sequence is terminated by $n=2\cdot 13^2$. The prime 239 is involved in terminating 10 of the 100 sequences studied. It occurs 3 times for m=6 and 7 times for m=10. It is seen from the table that integer sequences are in general longer for even than for odd values of m.

Table 1. x_n is the first noninteger term in the sequence defined by

$nx_n = x_{n-1}(x_{n-1}^m + n - 1)$. The table gives n for parameters x_1 and m.

m	x_1=2	x_1=3	x_1=4	x_1=5	x_1=6	x_1=7	x_1=8	x_1=9	x_1=10	x_1=11
1	43	7	17	34	17	17	51	17	7	34
2	89	89	89	89	31	151	79	89	79	601
3	97	17	23	97	149	13	13	83	23	13
4	214	43	139	107	269	107	214	139	251	107
5	19	83	13	19	13	37	13	37	347	19
6	239	191	359	419	127	127	239	191	239	461
7	37	7	23	37	23	37	17	23	7	37
8	79	127	158	79	103	103	163	103	163	79
9	83	31	41	83	71	83	71	23	41	31
10	239	338	139	137	239	239	239	239	239	389

4. A Model to Explain Some Features of the Sequence

The congruence

$$x(k) \equiv \alpha(k) \pmod p, \quad \alpha(k)\ \varepsilon\ \{-1, 0, 1, \ldots, p-2\}$$

studied in a number system with sufficiently large base p^t, is of particular interest when looking at the integer properties of the sequence. Five cases will be studied. They are:

1. $\alpha(k)$ does not belong to cases 2, 3, 4 or 5 below
2. $\alpha(k) = -1$, $p \neq 2$
3. $\alpha(k) = 0$
4. $\alpha(k) = 1$
5. $\alpha(k) = \alpha(k+1)$ and/or $\alpha(k) = \alpha(k-1)$, $\alpha(k) \neq -1, 0, 1$

These cases are mutually exclusive; however, in case 5 there may be more than one sequence of the described type for a given p, for example, for m=10, x_1=7 and p=11, we have $\alpha(k) = 7$ for k = 1, 2, … ,10 and $\alpha(k) = 4$ for k = 11, 12, … , 15. Therefore, when running through the values of k for a given p, it is possible to classify $\alpha(k)$ into states corresponding to case 5. In this model, $\alpha(1)$ appears as a result of creation rather than transition from one state to another but, formally, it will be considered as resulting from transition from a state 0 (k = 0) to the state corresponding to $\alpha(1)$.

The study of transitions from one state to another in the above model is useful in the explaining why there are such long sequences of integers and why they are in general longer for even than for odd m. Table 2 shows the number of transitions of each kind in the 100 cases studied. Let a_r be the number of transitions from state r to state s:

$$A_r = \sum_r a_{rs}, \quad B_s = \sum_s a_{rs}, \quad Q_s = 100 A_s / B_s.$$

(Note that r and s refer to states not rows and columns in table 2.) The transitions for odd and even values of m are treated separately. It is seen that transitions from states 4, 5 and 2 (for even m) are rare. Only between 5% and 14% of all succh states ”created” are ”destroyed” while the corresponding percentage for other transitions range between 85% and 99%. It is the fact that transitions from certain states are rare, which makes some of these integer sequences so long. That transitions from state 2 are rare for even m (11%) and frequent for odd m (99%) make the integer sequences in general longer for even than for odd m. In all the many transitions observed, it was noted that certain types (underscored in table 2) only occurred for values of k divisible by p, while other types never occurred for k divisible by p. Transitions from state 3 all occur for k divisible by p but , unlike the other transitions which occur for k divisible by p, they have a high frquency. Some of the observations made on the model are explained in the remainder of this paper.

Table 2. The number of transitions of each type for odd and even m
($2\nmid m$ = 2 does not divide m)

From state	To state 1		To state 2		To state 3		To state 4		To state 5		A_r	
	$2\nmid m$	$2\mid m$	$2\nmid m$	$2\mid m$	$2\nmid m$	$2\mid m$	$2\nmid m$	$2\mid m$	$2\nmid m$	$2\mid m$	$2\nmid m$	$2\mid m$
0	457	1847	38	40	60	60	55	55	32	69	652	2071
1			220	701	252	791	247	642	75	307	794	2241
2	181	$\underline{55}$			71	$\underline{21}$	39	$\underline{7}$	2	0	293	83
3	$\underline{202}$	$\underline{634}$	$\underline{36}$	$\underline{30}$			$\underline{111}$	$\underline{80}$	$\underline{9}$	$\underline{16}$	358	760
4	$\underline{20}$	$\underline{35}$	$\underline{2}$	$\underline{6}$	$\underline{39}$	$\underline{12}$				$\underline{3}$	61	56
5	$\underline{2}$	$\underline{2}$	$\underline{1}$	$\underline{2}$	0	$\underline{3}$	0	$\underline{2}$	$\underline{2}$	$\underline{11}$	5	20
B_s	872	2573	297	779	422	887	452	786	120	406	2163	5431
Q_s%	92	95	99	11	85	86	14	8	5	5		

Transitions from state 4 and, for even m only, from state 2

It is evident from $kx_k = x_{k-1}(x_{k-1}^m + k - 1)$ that, if $x_{k-1} \equiv \pm 1 \pmod p$ and $(k, p)=1$, then $x_k = \pm 1 \pmod p$. Assume that we arrive at $x_{k-1} \equiv \pm 1 \pmod p$ for $k < p-m$ and $m < p$. We can then write

(5) $\quad x_{p-m-1} \equiv \pm 1 + \alpha p \pmod{p^2}, \; 0 \le \alpha < p$

and

$x_{p-m-1}^{m} \equiv (\pm 1 + \alpha p)^{m} \equiv 1 \pm m\alpha p \pmod{p^2}$ (m even).

Equations (6) and (7) give

$$(p-m)x_{p-m} \equiv \pm(p-m) \pmod{p^2}$$

or, since (p-m,p)=1,

$$x_{p-m} \equiv \pm 1 \pmod{p^2} \quad \text{or} \quad x_k \equiv \pm 1 \pmod{p^2} \text{ for } p-m \le k \le p-1.$$

For k=p, we have

$$px_p \equiv \pm 1(1+p-1) \pmod{p^2}$$

or, after division by p throughout

$$x_p \equiv \pm 1 \pmod{p}.$$

It is now easy to see that $x_k \equiv 1 \pmod{p}$ continues to hold also for k>p. The integer sequence may, however, bebroken for $k=p^2$.

Transitions from state 3

Let us now assume that $x_j \equiv 0 \pmod{p}$ for some j<p. If (j+1,p)=1, it follows that $x_{j+1} \equiv 0 \pmod{p}$ or, generally, $x_k \equiv 0 \pmod{p}$ for $j \le k \le p-1$. For k=p-1, we can write $x_{p-1} \equiv pa \pmod{p^2}$, $0 \le a < p-1$. We then have

$$px_p \equiv pa(p^m a^m + p - 1) \pmod{p^2},$$

from which follows $x_p \equiv -a \pmod{p}$, viz. x_p is an integer; however if a=0, the state is changed.

Transitions from states of type 5

When, for some j<p-1, it happens that $x_j^m \equiv 1 \pmod{p}$, it is easily seen that $x_k \equiv x_j \pmod{p}$ for $j \le k < p$. This implies

$$px_p \equiv x_j(1+p-1) \pmod{p},$$

from which it is seen that x_p may not be congruent to x_j (mod p) but also that x_p is an integer.

References:

1. R.K. Guy, *Unsolved Problems in Number Theory*. New York: Springer-Verlag, 1981, p. 120.
2. H. Ibstedt. Some Sequences of Large Integers, *Fibonacci Quarterly,* 28(1990), 200-203.
3. R.K. Guy, *Unsolved Problems in Number Theory,* Second Edition. New York: Springer-Verlag, 1994, p. 214-215.

V. The Smarandache Partial Perfect Additive Sequence

Abstract: The sequence defined through $a_{2k+1}=a_{k+1}-1$, $a_{2k+2}=a_{k+1}+1$ for $k\geq1$ with $a_1=a_2=1$ is studied in detail. It is proved that the sequence is neither convergent nor periodic - questions which have recently been posed. It is shown that the sequence has an amusing oscillating behavior and that there are terms that approach $\pm\infty$ for a certain type of large indices.

1. Definition

Definition of Smarandache perfect f_p sequence: If f_p is a p-ary relation on $\{a_1, a_2, a_3, \ldots\}$ and $f_p(a_i, a_{i+1}, a_{i+2, \ldots} a_{i+p-1}) = f_p(a_j, a_{j+1}, a_{j+2, \ldots} a_{j+p-1})$ for all a_i, a_j and all $p>1$, then $\{a_n\}$ is called a Smarandache perfect f_p sequence.

If the defining relation is not satisfied for all a_i,a_j or all p then $\{a_n\}$ may qualify as a Smarandache partial perfect f_p sequence.

2. Analysis and Results

The purpose of this note is to answer some questions posed in an article in the Smarandache Notions Journal, vol. 11 [1] on a particular Smarandache partial perfect sequence defined in the following way:

$a_1=a_2=1$

(1) $a_{2k+1}=a_{k+1}-1,\ k\geq1$

(2) $a_{2k+2}=a_{k+1}+1,\ k\geq1$

Adding both sides of the defining equations results in $a_{2k+2}+a_{2k+1}=2a_{k+1}$ which gives

(3) $$\sum_{i=1}^{2n} a_i = 2\sum_{i}^{n} a_i$$

Let n be of the form $n=k\cdot2^m$. The summation formula now takes the form

(4) $$\sum_{i=1}^{k\cdot 2^m} a_i = 2^m\sum_{i}^{k} a_i$$

From this we note the special cases $\sum_{i=1}^{4} a_i = 4$, $\sum_{i=1}^{8} a_i = 8$, …. $\sum_{i=1}^{2^m} a_i = 2^m$.

The author of the article under reference poses the questions:

Is there a general expression of a_n as a function of n?
Is the sequence periodical, or convergent or bounded?

The first 25 terms of this sequence are[3]:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
a_k	1	1	0	2	-1	1	1	3	-2	0	0	2	0	2	2	4	-3	-1	-1	1	-1	1	1	3	-1

It may not be possible to find a general expression for a_n in terms of n. For computational purposes, however, it is helpful to unify the two defining equations by introducing the δ-function defined as follows:

(5) $$\delta(n) = \begin{cases} -1 & \text{if } n \equiv 0 \pmod 2 \\ 1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

The definition of the sequence now takes the form:

$a_1 = a_2 = 1$

(6) $$a_n = a_{(n+\frac{1+\delta(n)}{2})/2} - \delta(n)$$

A translation of this algorithm to computer language was used to calculated the first 3000 terms of this sequence. A feeling for how this sequence behaves may be best conveyed by table 1 of the first 136 terms, where the switching between positive, negative and zero terms have been made explicit.

Before looking at some parts of this calculation let us make a few observations. Although we do not have a general formula for a_n we may extract very interesting information in particular cases. Successive application of (2) to a case where the index is a power of 2 results in:

(7) $$a_{2^m} = a_{2^{m-1}} + 1 = a_{2^{m-2}} + 2 = \ldots = a_2 + m - 1 = m$$

This simple consideration immediately gives the answer to the main question:

The sequence is neither periodic nor convergent.

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[3] The sequence as quoted in the article under reference is erroneous as from the thirteenth term.

We will now consider the difference a_n-a_{n-1} which is calculated using (1) and (2). It is necessary to distinguish between n even and n odd.

1. n=2k, k≥2.

(8) $a_{2k}-a_{2k-1}=2$ (exception: $a_2-a_1=0$)

2. $n=k\cdot 2^m+1$ where k is odd.

(9) $$a_{k\cdot 2^m+1}-a_{k\cdot 2^m}=a_{k\cdot 2^{m-1}+1}-1-a_{k\cdot 2^{m-1}}-1=..a_{k+1}-a_k-2m=$$

$$=\begin{cases}1-2m \text{ if } k=1\\ 2-2m \text{ if } k>1\end{cases}$$

In particular $a_{2k+1}-a_{2k}=0$ if k≥3 is odd.

Table 1. The first terms of the sequence

n	a_n, a_{n+1}, …	n	a_n, a_{n+1}, …
1	1, 1	50	0, 0
3	0	52	2
4	2	53	0
5	-1	54	2, 2, 4
6	1, 1, 3	57	0
9	-2	58	2, 2, 4, 2, 4, 4, 6
10	0, 0	65	-5, -3, -3, -1, -3, -1, -1
12	2	72	1
13	0	73	-3, -1,-1
14	2, 2, 4	76	1
17	-3, -1, -1	77	-1
20	1	78	1, 1, 3
21	-1	81	-3, -1, -1
22	1, 1, 3	84	1
25	-1	85	-1
26	1, 1, 3, 1, 3, 3, 5	86	1, 1, 3
33	-4, -2, -2	89	-1
36	0	90	1, 1, 3, 1, 3, 3, 5
37	-2	97	-3, -1, -1
38	0, 0	100	1
40	2	101	-1
41	-2	102	1, 1, 3
42	0, 0	105	-1
44	2	106	1, 1, 3, 1, 3, 3, 5
45	0	113	-1
46	2, 2, 4	114	1, 1, 3, 1, 3, 3, 5, 1, 3, 3, 5, 3, 5, 5, 7
49	-2	129	-6, -4, -4, -2, -4, -2, -2

The big drop. The sequence shows an interesting behaviour around the index 2^m. We have seen that $a_{2^m} = m$. The next term in the sequence calculated from (9) is m+1-2·m=-m+1. This makes for the spectacular behaviour shown in diagrams 1 and 2. The sequence gradually struggles to get to a peak for $n=2^m$ where it drops to a low and starts working its way up again. There is a great similarity between the oscillating behaviour shown in the two diagrams. In diagram 3 this behaviour is illustrated as it occurs between two successive peaks.

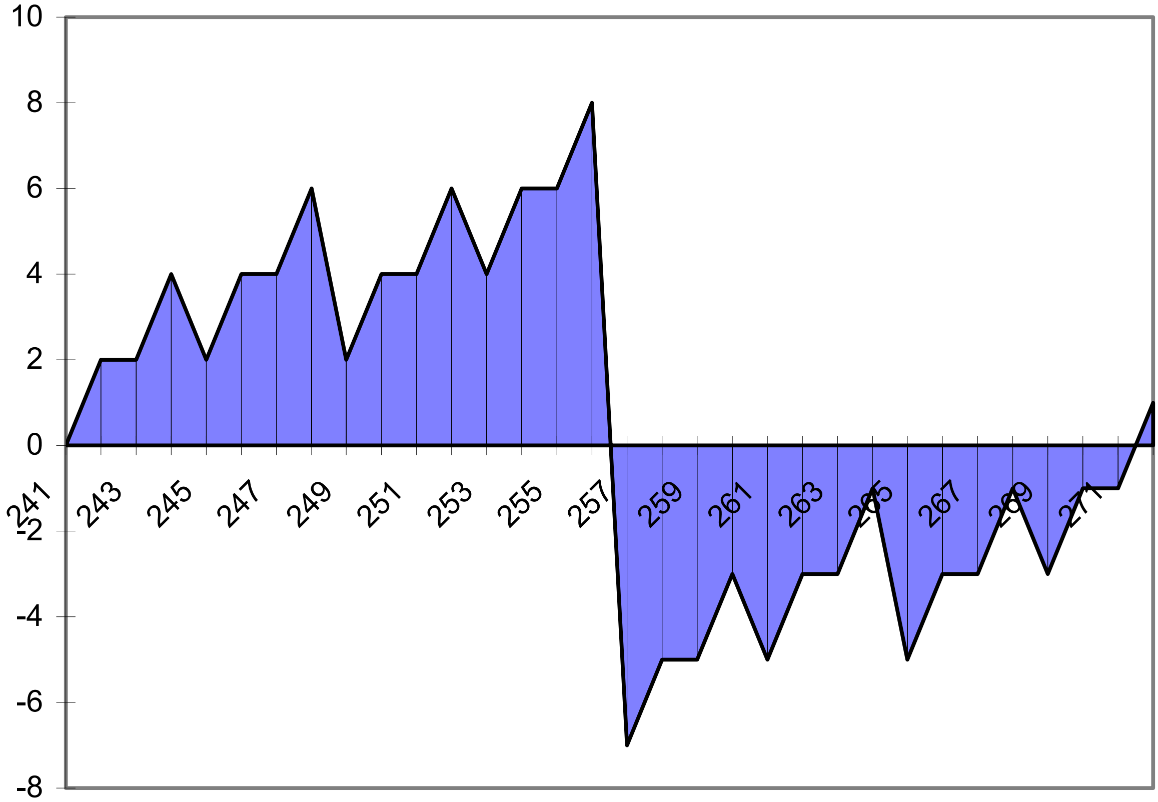

Diagram 1. a_n as a function of n around $n=2^8$ illustrating the "big drop"

When using the defining equations (1) and (2) to calculate elements of the sequence it is necessary to have in memory the values of the elements as far back as half the current index. We are now in a position to generate preceding and proceeding elements to a given element by using formulas based on (8) and (9).

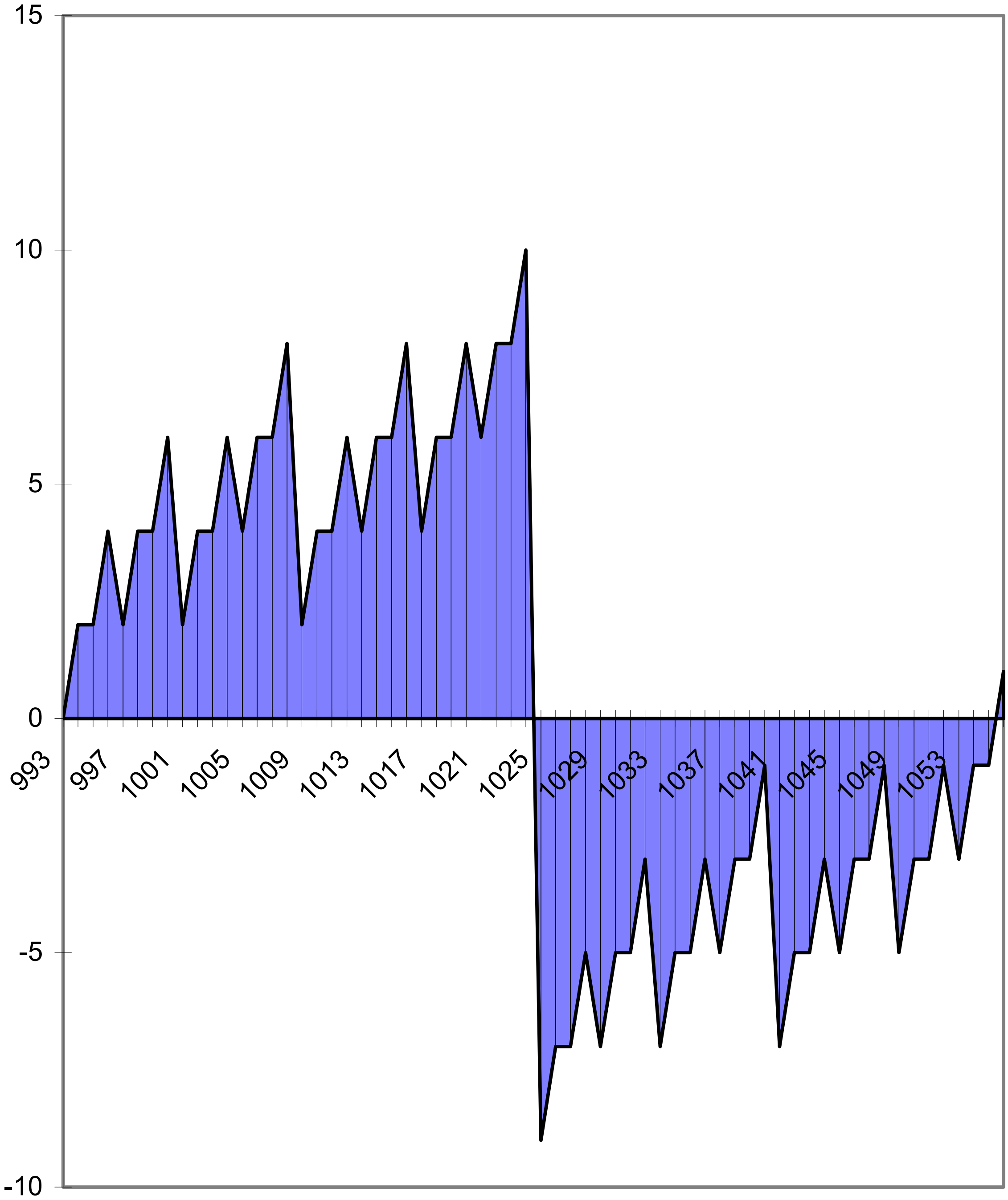

Diagram 2. a_n as a function of n around $n=2^{10}$ illustrating the "big drop"

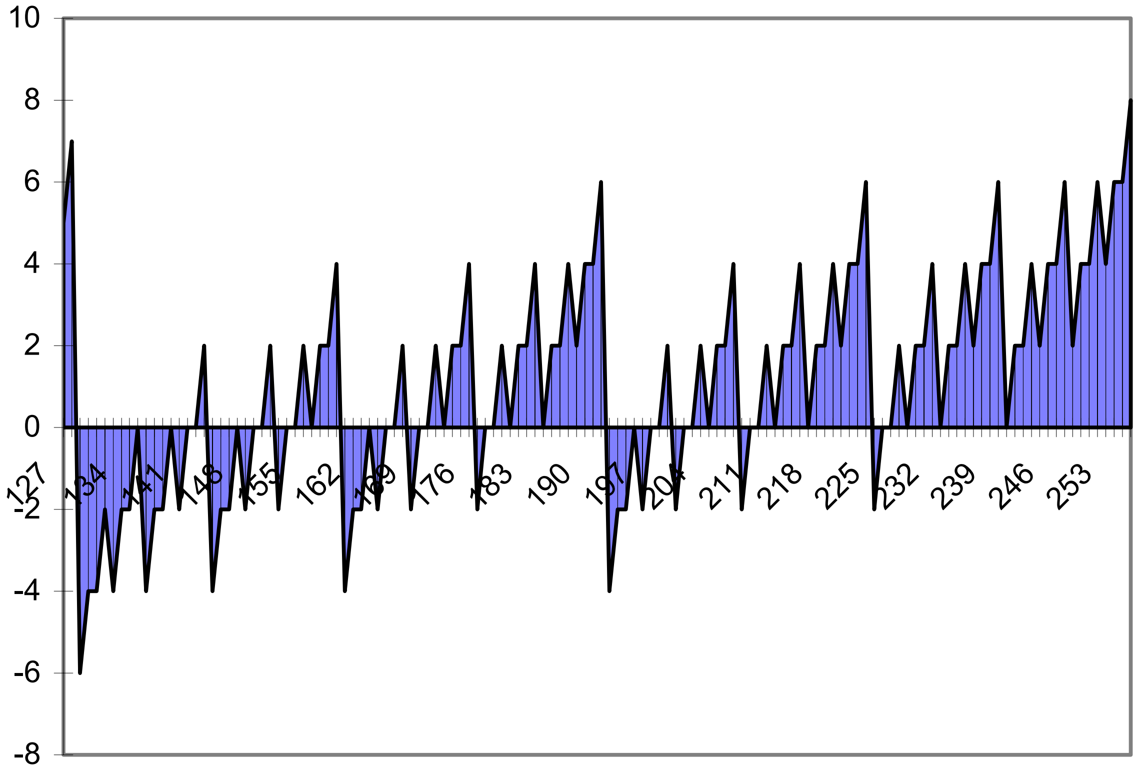

Diagram 3. The oscillating behaviour of the sequence between the peaks for $n=2^7$ and $n=2^8$.

The forward formulas:

$$(10) \qquad a_n = \begin{cases} a_{n-1}+2 & \text{when } n=2k,\ k>1 \\ a_{n-1}+1-2m & \text{when } n=2^m+1 \\ a_{n-1}+2-2m & \text{when } n=k\cdot 2^m+1,\ k>1 \end{cases}$$

Since we know that $a_{2^m} = m$ it will also prove useful to calculate a_n from a_{n+1}.

The reverse formulas:

$$(11) \qquad a_n = \begin{cases} a_{n+1}-2 & \text{when } n=2k-1,\ k>1 \\ a_{n+1}-1+2m & \text{when } n=2^m \\ a_{n+1}-2+2m & \text{when } n=k\cdot 2^m,\ k>1 \end{cases}$$

Finally let's use these formulas to calculate some terms forwards and backwards from one known value say $a_{4096}=12$ ($4096=2^{12}$). It is seen that a_n starts from 0 at n=4001, makes its big drop to -11 for n=4096 and remains negative until n=4001. For an even power of 2 the mounting sequence only has even values and the descending sequence only odd values. For odd powers of 2 it is the other way round.

Table 2. Values of a_n around $n=2^{12}=4096$.

Decending

4095	…4	…3	…2	…1	..80	…9	…8	…7	…6	…5	…4	…3	…2	…1	70	…9 …	4001
10	10	8	10	8	8	6	10	8	8	6	8	6	6	4	10	8 …	0

Ascending

4096	…7	…8	…9	.100	…1	…2	…3	…4	…5	…6	…7	…8	…9	..10	…1	…2 …	4160
12	-11	-9	-9	-7	-9	-7	-7	-5	-9	-7	-7	-5	-7	-5	-5	-3 …	1

References:

1. M. Bencze, Smarandache Relationships and Subsequences, *Smarandache Notions Journal. Vol. 11, No 1-2-3, pgs 79-85.*

VI. Smarandache Continued Fractions

Abstract: The theory of general continued fractions is developed to the extent required in order to calculate Smarandache continued fractions to a given number of decimal places. Proof is given for the fact that Smarandache general continued fractions built with positive integer Smarandache sequences having only a finite number of terms equal to 1 is convergent. A few numerical results are given.

1. Introduction

The definitions of Smarandache continued fractions were given by Jose Castillo in the Smarandache Notions Journal, Vol. 9, No 1-2 [1].

A Smarandache Simple Continued Fraction is a fraction of the form:

$$a(1)+\cfrac{1}{a(2)+\cfrac{1}{a(3)+\cfrac{1}{a(4)+\cfrac{1}{a(5)+\cdots}}}}$$

where a(n), for n≥1, is a Smarandache type Sequence, Sub-Sequence or Function.

Particular attention is given to the Smarandache General Continued Fraction defined as

$$a(1)+\cfrac{b(1)}{a(2)+\cfrac{b(2)}{a(3)+\cfrac{b(3)}{a(4)+\cfrac{b(4)}{a(5)+\cdots}}}}$$

where a(n) and b(n), for n≥1, are Smarandache type Sequences, Sub-Sequences or Functions.

As a particular case the following example is quoted

$$1+\cfrac{1}{12+\cfrac{21}{123+\cfrac{321}{1234+\cfrac{4321}{12345+\cdots}}}}$$

Here 1, 12, 123, 1234, 12345, … is the Smarandache Consecutive Sequences and 1, 21, 321, 4321, 54321, … is the Smarandache Reverse Sequence.

The interest in Castillo's article is focused on the calculation of such fractions and their possible convergens when the number of terms approaches infinity. The theory of simple continued fractions is well known and given in most standard textbooks in Number Theory. A very comprehensive theory of continued fractions, including general continued fractions is found in *Die Lehre von den Kettenbrüchen* [2]. The symbols used to express facts about continued fractions vary a great deal. The symbols which will be used in this article correspond to those used in Hardy and Wright *An Introduction to the Theory of Numbers* [3]. However, only simple continued fractions are treated in the text of Hardy and Wright. Following more or less the same lines the theory of general continued fractions will be developed in the next section as far as needed to provide the necessary tools for calculating Smarandache general continued fractions.

2. General Continued Fractions

The definition given below is an extension of the definition of a simple continued fraction where $r_1=r_2= \ldots =r_n=1$. The theory developed here will apply to simple continued fractions as well by replacing r_k (k=1, 2, …) in formulas by 1 and simply ignoring the reference to r_k when not relevant.

Definition:

We define a finite general continued fraction through

(1) $$C_n = q_0 + \cfrac{r_1}{q_1 + \cfrac{r_2}{q_2 + \cfrac{r_3}{q_3 + \cfrac{r_4}{q_4 + \ldots}}}} = q_0 + \frac{r_1}{q_1 +}\ \frac{r_2}{q_2 +}\ \frac{r_3}{q_3 +}\ \frac{r_3}{q_4 +}\ \ldots\ \frac{r_n}{q_n}$$

where $\{q_0, q_1, q_2, \ldots q_n\}$ and $\{r_1, r_2, r_3 \ldots r_n\}$ are integers which we will assume to be positive.

The formula (1) will usually be expressed in the form

(2) $C_n=[q_0,q_1,q_2,q_3, \ldots q_n,r_1,r_2,r_3 \ldots r_n]$

For a simple continued fraction we would write

(2') $C_n=[q_0,q_1,q_2,q_3, \ldots q_n]$

If we break off the calculation for m≤n we will write

(3) $C_m=[q_0,q_1,q_2,q_3, \ldots q_m,r_1,r_2,r_3 \ldots r_m]$

Equation (3) defines a sequence of finite general continued fractions for m=1, m=2, m=3, … . Each member of this sequence is called a **convergent** to the continued fraction.

Working out the general continued fraction in stages, we shall obviously obtain expressions for its convergents as quotients of two sums, each sum comprising various products formed with $q_0, q_1, q_2, \ldots q_m$ and $r_1, r_2, \ldots r_m$.

If m=1, we obtain the first convergent

(4) $C_1=[q_0,q_1,r_1]= q_0 + \dfrac{r_1}{q_1} = \dfrac{q_0 q_1 + r_1}{q_1}$

For m=2 we have
(5)

$$C_2 = [q_0, q_1, q_2, r_1, r_2] = q_0 + \frac{q_2 r_1}{q_1 q_2 + r_2} = \frac{q_0 q_1 q_2 + q_0 r_2 + q_2 r_1}{q_1 q_2 + r_2}$$

In the intermediate step the value of $q_1 + \frac{r_2}{q_2}$ from the previous calculation has been quoted, putting q_1, q_2 and r_2 in place of $q_{0,}$, q_1 and r_1. We can express this by

(6) $C_2=[q_0,[q_1,q_2,r_2],r_1]$

Proceeding in the same way we obtain for m=3

(7) $$C_3=[q_0, q_1, q_2, q_3, r_1, r_2, r_3] = q_0 + \frac{(q_2 q_3 + r_3) r_1}{q_1 q_2 q_3 + q_1 r_3 + q_3 r_2} =$$
$$\frac{q_0 q_1 q_2 q_3 + q_0 q_1 r_3 + q_0 q_3 r_2 + q_2 q_3 r_1 + r_1 r_3}{q_1 q_2 q_3 + q_1 r_3 + q_3 r_2}$$

or generally

(8) $C_m=[q_0,q_1,\ldots q_{m-2},[q_{m-1},q_m,r_m],r_1,r_2,\ldots r_{m-1}]$

which we can extend to

(9) $C_n=[q_0,q_1,\ldots q_{m-2},[q_{m-1},q_m,\ldots q_n,r_m,\ldots r_n],r_2,r_2,\ldots r_{m-1}]$

Theorem 1:
Let A_m and B_m be defined through

$A_0=q_0$, $A_1=q_0q_1+r_1$, $A_m=q_mA_{m-1}+r_mA_{m-2}$ $(2\leq m\leq n)$

(10) $B_0=1$, $B_1=q_1$, $B_m=q_mB_{m-1}+r_mB_{m-2}$ $(2\leq m\leq n)$

then $C_m=[q_0,q_1,\ldots q_m,r_1,\ldots r_m] = \frac{A_m}{B_m}$, i.e. $\frac{A_m}{B_m}$ is the m*th* convergent to the general continued fraction.

Proof:

The theorem is true for m=0 and m=1as is seen from $[q_0]= \frac{q_0}{1}=\frac{A_0}{B_0}$ and $[q_0,q_1,r_1]= \frac{q_0q_1+r_1}{q_1}=\frac{A_1}{B_1}$. Let us suppose that it is true for a given m<n. We will induce that it is true for m+1.

$$[q_0,q_1,\ldots q_{m+1},r_1,\ldots r_{m+1}] = [q_0,q_1,\ldots q_{m-1},[q_m,q_{m+1},r_{m+1}],r_1,\ldots r_m]$$

$$=\frac{[q_m,q_{m+1},r_{m+1}]A_{m-1}+r_mA_{m-2}}{[q_m,q_{m+1},r_{m+1}]B_{m-1}+r_mB_{m-2}}$$

$$=\frac{(q_m+\frac{r_{m+1}}{q_{m+1}})A_{m-1}+r_mA_{m-2}}{(q_m+\frac{r_{m+1}}{q_{m+1}})B_{m-1}+r_mB_{m-2}}$$

$$=\frac{q_{m+1}(q_mA_{m-1}+r_mA_{m-2})+r_{m+1}A_{m-1}}{q_{m+1}(q_mB_{m-1}+r_mB_{m-2})+r_{m+1}B_{m-1}}$$

$$=\frac{q_{m+1}A_{m-1}+r_{m+1}A_{m-1}}{q_{m+1}B_{m-1}+r_{m+1}B_{m-1}}=\frac{A_{m+1}}{B_{m+1}}$$

□

The recurrence relations (10) provide the basis for an effective computer algorithm for successive calculation of the convergents C_m.

Theorem 2:

(11) $A_mB_{m-1}-B_mA_{m-1}=(-1)^{m-1}\prod_{k=1}^{m} r_k$

Proof: For m=1 we have $A_1B_0-B_1A_0=q_0q_1+r_1-q_0q_1=r_1$.

$$A_mB_{m-1}-B_mA_{m-1}=(q_mA_{m-1}+r_mA_{m-2})B_{m-1}-(q_mB_{m-1}+r_mB_{m-2})A_{m-1}= -r_m(A_{m-1}B_{m-2}-B_{m-1}A_{m-2})$$

By repeating this calculation with m-1, m-2, …, 2 in place of m, we arrive at

$$A_mB_{m-1}-B_mA_{m-1}= \ldots= (A_1B_0-B_1A_0)(-1)^{m-1}\prod_{k=2}^{m} r_k =(-1)^{m-1}\prod_{k=1}^{m} r_k$$

□

Theorem 3:

(12) $A_mB_{m-2}-B_mA_{m-2}=(-1)^m q_m \prod_{k=1}^{m-1} r_k$

Proof:
This theorem follows from theorem 3 by inserting expressions for A_m and B_m

$$A_mB_{m-2}-B_mA_{m-2}=(q_mA_{m-1}+r_mA_{m-2})B_{m-2}-(q_mB_{m-1}+r_mB_{m-2})A_{m-2}=$$
$$q_m(A_{m-1}B_{m-2}-B_{m-1}A_{m-2})=(-1)^m q_m \prod_{k=1}^{m-1} r_k$$

□

Using the symbol $C_m=\frac{A_m}{B_m}$ we can now express important properties of the number sequence C_m, m=1, 2, … , n. In particular we will be interested in what happens to C_n as n approaches infinity.

From (11) we have

(13) $$C_n - C_{n-1} = \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{(-1)^{n-1} \prod_{k=1}^{n} r_k}{B_{n-1}B_n}$$

while (12) gives

(14) $$C_n - C_{n-2} = \frac{A_n}{B_n} - \frac{A_{n-2}}{B_{n-2}} = \frac{(-1)^{n-1} q_n \prod_{k=1}^{n-1} r_k}{B_{n-2}B_n}$$

We will now consider infinite positive integer sequences $\{q_0, q_1, q_2, \ldots.\}$ and $\{r_1, r_2, \ldots\}$ where only a finite number of terms are equal to 1. This is generally the case for Smarandache sequences. We will therefore prove the following important theorem.

Theorem 4:
A general continued fraction for which the sequences $q_0, q_1, q_2, \ldots.$ and $r_1, r_2, \ldots.$ are positive integer sequences with at most a finite number of terms equal to 1 is convergent.

Proof:
We will first show that the product $B_{n-1}B_n$, which is a sum of terms formed by various products of elements from $\{q_1, q_2, \ldots q_n, r_1, r_2, \ldots r_{n-1}\}$, has one term which is a multiple of $\sum_{k=2}^{n} r_k$. Looking at the process by which we calculated C_1, C_2, and C_3, equations 4, 5 and 7, we see how terms with the largest number of factors r_k evolve in numerators and denominators of C_k. This is made explicit in figure 1.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_m	r_1	q_0r_2	r_1r_3	$q_0r_2r_4$	$r_1r_3r_5$	$q_0r_2r_4r_6$	$r_1r_3r_5r_7$	$q_0r_2r_4r_6r_8$
B_m	-	r_2	q_1r_3	r_2r_4	$q_1r_3r_5$	$r_2r_4r_6$	$q_1r_3r_5r_7$	$r_2r_4r_6r_8$

Figure 1. The terms with the largest number of r-factors in numerators A_m and denominators B_m .

As is seen from figure 1 two consecutive denominators B_nB_{n-1} will have a term with $r_2r_3 \ldots r_n$ as factor. This means that the numerator of (13) will not cause C_n-C_{n-1} to diverge. On the other hand $B_{n-1}B_n$ contains the term $(q_1q_2 \ldots q_{n-1})^2q_n$ which approaches ∞ as $n\to\infty$. It follows that $\lim_{n\to\infty}(C_n - C_{n-1}) = 0$.

From (14) we see that

1. If n is odd, say n=2k+1, than $C_{2k+1} < C_{2k-1}$ forming a monotonously decreasing number sequence which is bounded below (positive terms). It therefore has limit.
   $\lim_{k\to\infty} C_{2k+1} = C_1$.
2. If n is even, n=2k, than $C_{2k} > C_{2k-2}$ forming a monotonously increasing number sequence. This sequence has an upper bound because $C_{2k} < C_{2k+1} \to C_1 \text{ as } k \to \infty$. It therefore has limit.
   $\lim_{k\to\infty} C_{2k} = C_2$.
3. Since $\lim_{n\to\infty}(C_n - C_{n-1}) = 0$ we conclude that $C_1=C_2$. Consequently
   $\lim_{n\to\infty} C_n = C$ exists.

□

3. Calculations

A *UBASIC* program has been developed to implement the theory of Smarandache general continued fractions. The same program can be used for classical continued fractions since these correspond to the special case of a general continued fraction where $r_1=r_2= \ldots =r_n=1$.

The complete program used in the calculations is given below. The program applies equally well to simple continued fractions by setting all element of the array R equals to 1.

```
10  point 10
20  dim Q(25),R(25),A25),B25)
30  input "Number of decimal places of accuracy: ";D
40  input "Number of input terms for R (one more for Q) ";N%
50  cls
60  for I%=0 to N%:read Q(I%):next
70  data                                                  'The relevant data q0, q1, ...
80  for I%=1 to N%:read R(I%):next
90  data                                                  'The relevant data for r1, r2, ...
100 print tab(10);"Smarandache Generalized Continued Fraction"
110 print tab(10);"Sequence Q:";
120 for I%=0 to 6:print Q(I%);:next:print "   ETC"
130 print tab(10);"Sequence R:";
140  for I%=1 to 6:print R(I%);:next:print "   ETC"
150  print tab(10);"Number of decimal places of accuracy: ";D
160  A(0)=Q(0):B(0)=1                                     'Initiating  recur. algorithm
170  A(1)=Q(0)*Q(1)+R(1):B(1)=Q(1)
180  Delta=1:M=1                                          'M=loop counter
190  while abs(Delta)>10^(-D)                             'Convergens check
200  inc M
210  A(M)=Q(M)*A(M-1)+R(M)*A(M-2)                         'Recurrence
220  B(M)=Q(M)*B(M-1)+R(M)*B(M-2)
230  Delta=A(M)/B(M)-A(M-1)/B(M-1)                        'Cm-Cm-1
240  wend
250  print tab(10);"An/Bn=";:print using(2,20),A(M)/B(M)  'Cn in decimalform
260  print tab(10);"An/Bn=";:print A(M);"/";B(M)          'Cn in fractional form
270  print tab(10);"Delta=";:print using(2,20),Delta;     'Delta=Last difference
280  print " for n=";M                                    'n=number of iterations
290  print
300  end
```

To illustrate the behaviour of the convergents C_n have been calculated for $q_1=q_2= \ldots= q_n=1$ and $r_1=r_2= \ldots= r_n=10$. The iteration of C_n is stopped when $\Delta_n= |C_n-C_{n-1}|<0.01$. Table 1 shows the result which is illustrated in figure 2. The factor $(-1)^{n-1}$ in (13) produces an oscillating behaviour with diminishing amplitude approaching $\lim_{n\to\infty} C_n = C$.

Table 1. Value of convergents C_n for $q\varepsilon\{1,1,\ldots\}$ and $r\varepsilon\{10,10,\ldots\}$

n	1	2	3	4	5	6	7	8	9
C_n	11	1.91	6.24	2.6	4.84	3.07	4.26	3.35	3.99

Table 1. ctd

n	10	11	12	13	14	15	16	17	18
C_n	3.51	3.85	3.6	3.78	3.65	3.74	3.67	3.72	3.69

Table 1. ctd

n	19	20	21	22
C_n	3.71	3.69	3.71	3.70

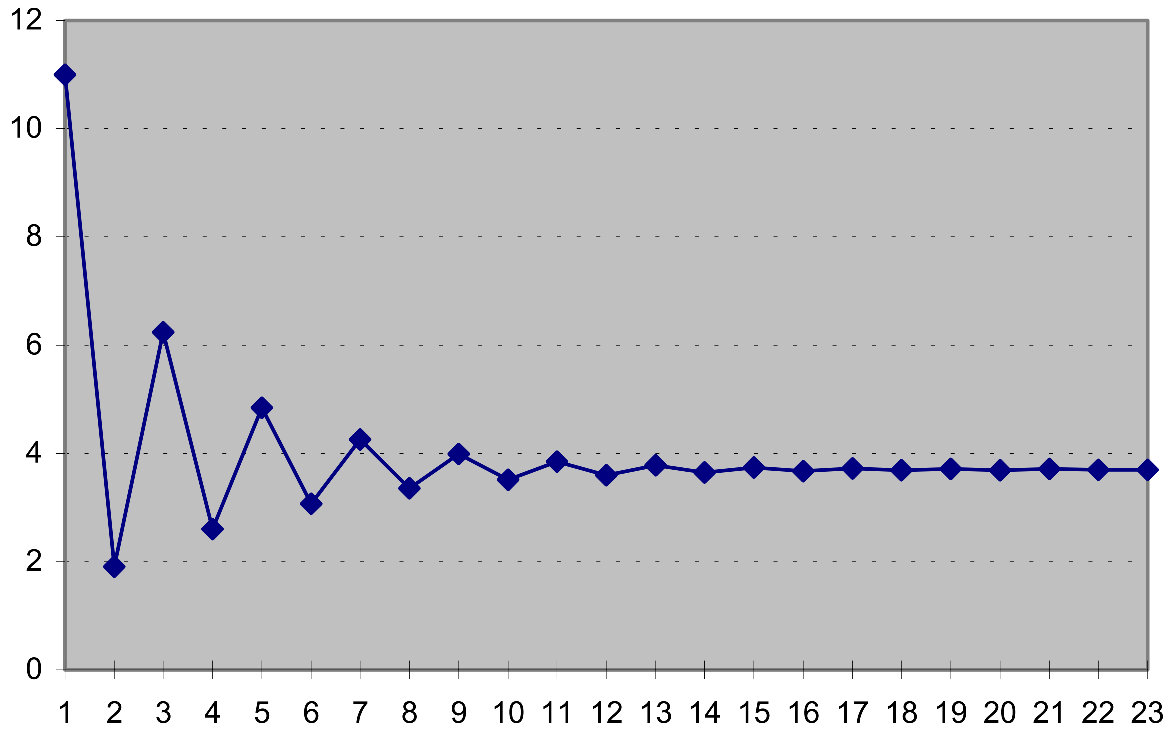

Figure 2. C_n as a function of n

A number of sequences, given below, will be substituted into the recurrence relations (10) and the convergence estimate (13).

$S_1=\{1, 1, 1, \ldots\ldots\}$
$S_2=\{1, 2, 1, 2, 1, 2, \ldots\ldots\}$
$S_3=\{3, 3, 3, 3, 3, 3, \ldots\ldots\}$
Smarandache Consecutive Sequence S_4.

S_4={1, 12, 123, 1234, 12345, 123456, …..}
Smarandache Reverse Sequence S_5.
S_5={1, 21, 321, 4321, 54321, 654321, …..}
CS1={1,1,2,8,9,10,512,513,514,520,521,522,729,730,731,737,738, …
NCS1={1,2,3,4,5,6,7,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,…

The Smarandache CS1 sequence definition: CS1(n) is the smallest number, strictly greater than the previous one (for $n \geq 3$), which is the cubes sum of one or more previous distinct terms of the sequence.
The Smarandache NCS1 sequence definition: NCS1(n) is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of one or more previous distinct terms of the sequence.
These sequences have been randomly chosen form a large number of Smarandache sequences [5].

As expected the last fraction in table 2 converges much slower than the previous one. These general continued fractions are, of course, very artificial as are the sequences on which they are based. As is often the case in empirical number theory it is not the individual figures or numbers which are of interest but the general behaviour of numbers and sequences under certain operations. In the next section we will carry out some experiments with simple continued fractions.

4. Experiments with Simple Continued Fractions

The theory of simple continued fractions is covered in standard textbooks. Without proof we will therefore make use of some of this theory to make some more calculations. We will first make use of the fact that

> *There is a one to one correspondence between irrational numbers and infinite simple continued fractions.*

The approximations given in table 2 expressed as simple continued fractions would therefore show how these are related to the corresponding general continued fractions.

Table 2. Calculation of general continued fractions

Q	R	n	Δ_n	C_n (dec.form)	C_n (fraction)
S_1	S_1	18	$-9\cdot10^{-8}$	1.6180339	$\frac{6765}{4181}$
S_2	S_1	13	$8\cdot10^{-8}$	1.3660254	$\frac{7953}{5822}$
S_2	S_3	22	$-9\cdot10^{-8}$	1.8228756	$\frac{1402652240}{769472267}$
S_4	S_1	2	$-7\cdot10^{-6}$	1.04761	$\frac{7063}{6742}$
		3	$5\cdot10^{-12}$	1.04761198457	$\frac{30519245}{29132203}$
		4	$-2\cdot10^{-20}$	1.047611984579 4017019	$\frac{1657835914708}{1582490405905}$
S_4	S_5	2	$-1\cdot10^{-3}$	1.082	$\frac{540}{499}$
		4	$-7\cdot10^{-10}$	1.082166760	$\frac{8245719435}{7619638429}$
		6	$-1\cdot10^{-19}$	1.082166760514 16702768	$\frac{418939686644589150004}{387130433063328840289}$
S_5	S_1	2	$-7\cdot10^{-6}$	1.04761	$\frac{7063}{6742}$
		3	$5\cdot10^{-12}$	1.04761198457	$\frac{30519245}{29132203}$
		4	$-2\cdot10^{-20}$	1.047611984579 40170194	$\frac{1657835914708}{1582490405905}$
S_5	S_4	2	$-8\cdot10^{-5}$	1.0475	$\frac{2358}{2251}$
		3	$7\cdot10^{-9}$	1.04753443	$\frac{2547455}{2431858}$
		5	$1\cdot10^{-20}$	1.047534436632 36268392	$\frac{60363763803209222}{57624610411155561}$
CS1	NCS 1	6	$-1\cdot10^{-7}$	1.540889	$\frac{1376250}{893153}$
		7	$3\cdot10^{-12}$	1.54088941088	$\frac{1412070090}{916399373}$
		9	$-1\cdot10^{-20}$	1.540889410887 88795255	$\frac{377447939426190}{244954593599743}$
NCS 1	CS1	16	$-5\cdot10^{-5}$	0.6419	$\frac{562791312666017539}{876693583206100846}$

Table 3. Some general continued fractions converted to simple continued fractions

Q	R	C_n (dec.form)	C_n (Simple continued fraction sequence)
S_4	S_5	1.08216676051416702768 (corresponding to 6 terms)	1,12,5,1,6,1,1,1,48,7,2,1,20,2,1,5,1,2,1,1,9,1,1,10,1,1,7,1,3,1,7, 2,1,3,31,1,2,6,38,2 (39 terms)
S_5	S_4	1.04753443663236268392 (corresponding to 5 terms)	1,21,26,1,3,26,10,4,4,19,1,2,2,1,8,8,1,2,3,1, 10,1,2,1,2,3,1,4,1,8 (29 terms)
CS1	NCS1	1.54088941088788795255 (corresponding to 9 terms)	1,1,15,1,1,1,1,2,4,17,1,1,3,13,4,2,2,2,5,1,6,2, 2,9,2,15,1,51 (28 terms)

These sequences show no special regularities. As can be seen from table 3 the number of terms required to reach 20 decimals is much larger than for the corresponding general continued fractions.

A number of Smarandache periodic sequences were explored in the author's book *Computer Analysis of Number Sequences* [6]. An interesting property of simple continued fractions is that

> *A periodic continued fraction is a quadratic surd, i.e. an irrational root of a quadratic equation with integral coefficients.*

In terms of A_n and B_n, which for simple continued fractions are defined through

$A_0=q_0$, $A_1=q_0q_1+1$, $A_n=q_nA_{n-1}+A_{n-2}$

(15) $B_0=1$, $B_1=q_1$, $B_n=q_nB_{n-1}+B_{n-2}$

the quadratic surd is found from the quadratic equation

(16) $B_nx^2+(B_{n-1}-A_n)x-A_{n-1}=0$

where n is the index of the last term in the periodic sequence. The relevant quadratic surd is

(17) $$x = \frac{A_n - B_{n-1} + \sqrt{A_n^2 + B_{n-1}^2 - 2A_nB_{n-1} - 4A_{n-1}B_n}}{2B_n}$$

An example has been chosen from each of the following types of Smarandache periodic sequences:

1. The Smarandache two-digit periodic sequence:

Definition: Let N_k be an integer of at most two digits. N_k' is defined through

$$N_k' = \begin{cases} \text{the reverse of } N_k \text{ if } N_k \text{ is a two digit integer} \\ N_k \cdot 10 \text{ if } N_k \text{ is a one digit integer} \end{cases}$$

N_{k+1} is then determined by

$$N_{k+1} = |N_k - N_k'|$$

The sequence is initiated by an arbitrary two digit integer N_1 with unequal digits.

One such sequence is Q={9, 81, 63, 27, 45}. The corresponding quadratic equation is

$$6210109x^2-55829745x-1242703=0$$

2. The Smarandache Multiplication Periodic Sequence:

Definition: Let $c>1$ be a fixed integer and N_0 and arbitrary positive integer. N_{k+1} is derived from N_k by multiplying each digit x of N_k by c retaining only the last digit of the product cx to become the corresponding digit of N_{k+1}.

For c=3 we have the sequence Q={1, 3, 9, 7} with the corresponding quadratic equation

$$199x^2-235x-37=0$$

3. The Smarandache Mixed Composition Periodic Sequence:

_*Definition.* Let N_0 be a two-digit integer $a_1 \cdot 10+a_0$. If $a_1+a_0<10$ then $b_1= a_1+a_0$ otherwise $b_1= a_1+a_0+1$. $b_0=|a_1-a_0|$. We define $N_1=b_1 \cdot 10+b_0$. N_{k+1} is derived from N_k in the same way.

One of these sequences is Q={18, 97, 72, 95, 54, 91} with the quadratic equation

$$3262583515x^2-58724288064x-645584400=0$$

and the relevant quadratic surd

$$x = \frac{58724288064 + \sqrt{3456967100707577532096}}{6525167030}$$

The above experiments were carried out with *UBASIC* programs. An interesting aspect of this was to check the correctness by converting the quadratic surd back to the periodic sequence.

There are many interesting calculations to carry out in this area. However, this study will finish by this equality between a general continued fraction convergent and a simple continued fraction convergent.

[1,12,123,1234,12345,123456,1234567,*1,21,321,4321,54321,654321*]=
[1,12,5,1,6,1,1,1,48,7,2,1,20,2,1,5,1,2,1,1,9,1,
1,10,1,1,7,1,3,1,7,2,1,3,31,1,2,6,38,2]

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VII. Smarandache k-k Additive Relationships

Abstract: An empirical study of Smarandache k-k additive relationships and related data is tabulated and analyzed. It leads to the conclusion that the number of Smarandache 2-2 additive relations is infinite. It is also shown that Smarandache k-k relations exist for large values of k.

1. Introduction

We recall the definition of the Smarandache function S(n):
Definition: *S(n) is the smallest integer such that S(n)! is divisible by n.*

The sequence of function values starts:

n:	1	2	3	4	5	6	7	8	9	10	…
S(n):	0	2	3	4	5	3	7	4	6	5	…

A table of values of S(n) up to n=4800 is found in Vol. 2-3 of the Smarandache Function Journal [1].

2. Smarandache k-k Additive Relationships

Definition: *A sequence of function values S(n), S(n+1)+ ... +S(n+2k-1) satisfies a k-k additive relationship if*

S(n)+S(n+1)+ ...+S(n+k-1)=S(n+k)+S(n+k+1)+ ...+S(n+2k-1)

or

$$\sum_{j=0}^{k-1} S(n+j) = \sum_{j=k}^{2k-1} S(n+j)$$

A general definition of Smarandache p-q relationships is given by M. Bencze in Vol. 11 of the Smarandache Notions Journal [2]. Bencze gives the following examples of Smarandache 2-2 additive relationships:
S(n)+S(n+1)=S(n+2)+S(n+3)
S(6)+S(7)=S(8)+S(9), 3+7=4+6;
S(7)+S(8)=S(9)+S(10), 7+4=6+5;

S(28)+S(29)=S(30)+S(31), 7+29=5+31.
He asks for others and questions whether there is a finite or infinite number of them. Actually the fourth one is quite far off:
S(114)+S(115)=S(116)+S(117), 19+23=29+13;
The fifth one is even further away:
S(1720)+S(1721)=S(1722)+S(1723), 43+1721=41+1723.
It is interesting to note that this solution is composed to two pairs of prime twins (1721,1723) and (43,41), - one ascending and one descending pair. This is also the case with the third solution found by Bencze.
One example of a Smarandache 3-3 additive relationship is given in the above mentioned article:
S(5)+S(6)+S(7)=S(8)+S(9)+S(10), 5+3+7=4+6+5.
Also in this case the next solution is far away:
S(5182)+S(5183)+S(5184)= S(5185)+S(5186)+S(5187), 2591+73+9= 61+ 2593+19.
To throw some light on these types of relationships an online program for calculation of S(n) [3] was used to tabulate Smarandache k-k additive relationships. Initially the following search limits were set: $n \leq 10^7$; $2 \leq k \leq 26$. For k=2 the search was extended to $n \leq 10^8$. The number of solutions m found in each case is given in table 1 and is displayed graphically in diagram 1 for $3 \leq k \leq 26$. Numerical results for $k \leq 6$ are presented in tables 4 –8, limited because of space requirements.

Table 1. The number m of Smarandache k-k additive solutions for $n<10^7$.

k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
m	158	43	20	8	8	11	5	8	6	5	2	5	7	2	4	8	1	3	4	1	4	6	2	3	2

The first surprising observation - at least to the author of these lines - is that the number of solutions does not drop off radically as we increase k. In fact there are as many 23-23 additive relationships as there are have 10-10 additive relationships and more than the number of 8-8 relations in the search area $n<10^7$. The explanation obviously lies in the distribution of the Smarandache function values, which up n=32000 is displayed in numerical form on page 56 of the Smarandache Function Journal, vol. 2-3 [1]. This study has been extended to $n \leq 10^7$. The result is shown in table 2 and graphically displayed in diagram 2 where the number of values z of S(n) in the intervals $500000y+1 \leq S(n) \leq 500000(y+1)$ is represented for each interval $500000x+1 \leq n \leq 500000(x+1)$ for y=0,1,2,…,18 and x=0,1,2,…,18.The fact

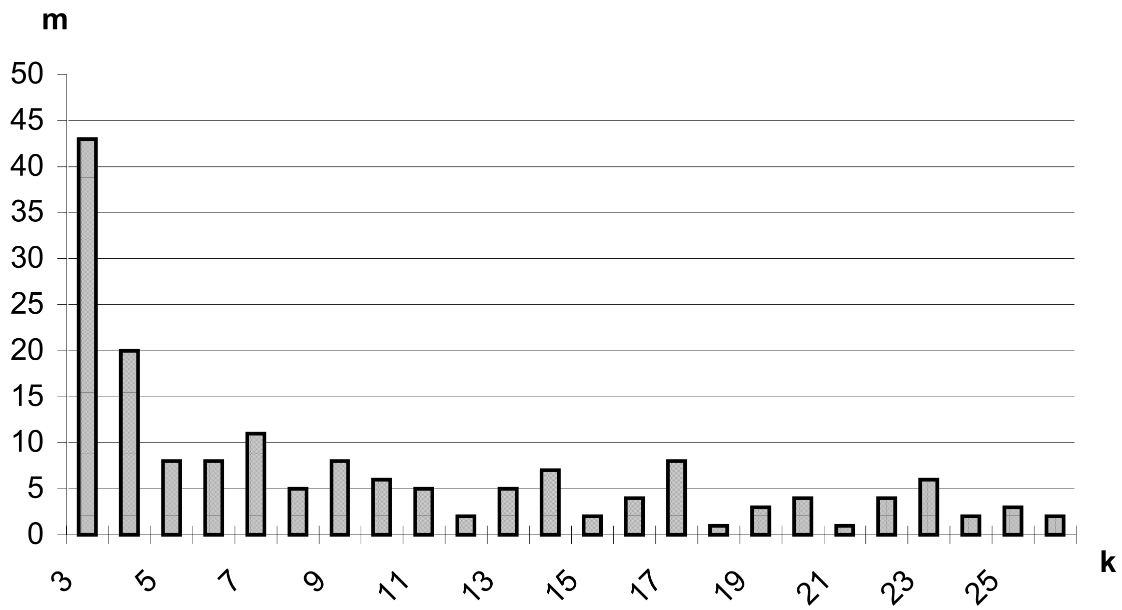

Diagram 1. The number m of Smarandache k-k additive relationships for $n<10^7$ for $3 \le k \le 26$.

that S(p)=p for p prime manifests itself in the line of isolated bars sticking up along the diagonal of the base of the diagram. The next line, which has a gradient = 0.5 ,corresponds to the fact that S(2p)=p. Of course, also the blank squares in the base of the diagram would be filled for n sufficiently large. For the most part, however, the values of S(n) are small compared to n. This corresponds to the large wall running at the back of the diagram. A certain value of S(n) may be repeated a great many times in a given interval. For $n<10^7$ 82% of all values of n correspond to values of S(n) which are smaller than 500000. It is the occurrence of a great number of values of S(n) which are small compared to n that facilitates the occurrence of equal sums of function values when sequences of consecutive values of n are considered. If this argument is as important as I think it is then chances are good that it might be possible to find, say, a Smarandache 50-50 additive relationship. I tried it - there are five of them, see table 9.

Of the 158 solutions to the 2-2 additive relationships for $n<10^7$ 22 are composed of pairs of prime twins. The first of these are marked by * in table 3. Of course there must be one ascending and one descending pair, as in

$$9369317+199=9369319+197$$

A closer look at the 2-2 additive relationships reveals that only the first two contain composite numbers.

Question 1: For a given prime twin pair (p,p+2) what are the chances that p+1 has a prime factor $q \neq 2$ such that q+2 is a factor of p-1 or q-2 a factor of p+3?
Question 2: What percentage of such prime twin pairs satisfy the Smarandache 2-2 additive relationship?
Question 3: Are all the Smarandache 2-2 additive relationships for n>7 entirely composed of primes?

To elucidate these questions a bit further this empirical study was extended in the following directions.

1. All Smarandache 2-2 additive relations up to 10^8 were calculated. There are 481 of which 65 are formed by pairs of prime twins.
2. All Smarandache function values involved in these 2-2 additive relationships for $7<n\leq 10^8$ were prime tested. They are all primes.
3. An analysis of how many of the Smarandache function values for $n<10^8$ are primes, even composite numbers or odd composite numbers respectively was carried out.

The results of this extended search are summarized by intervals in table 3 from which we can make the following observations. The number of composite values of S(n), even as well as odd, are relatively few and decreasing. In the last interval (table 3) there are only 1996 odd composite values. Even so we know that there are infinitely many composite values of S(n), examples $S(p^2)=2p$, $S(p^3)=3p$ for infinitely may primes p. Nevertheless the scarcity of composite values of S(n) explains why all the 2-2 additive relations examined for n>7 are composite.
The number of 2-2 additive relations is of the order of 0.1 % of the number of prime twins. The 2-2 additive relations formed by pairs of prime twins is about 13.5% of the prime twins in the respective intervals.
Although one has to remember that we are still only "surfing on the ocean of numbers" the following conjecture seems safe to make:

y/x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	Sum
19																			31089	31089
18																		31370		31370
17																	31342			31342
16																31516				31516
15															31613					31613
14														31891						31891
13													31908							31908
12												32049								32049
11											32287									32287
10										32565									16271	65130
9									32802								16437	16365		65604
8								32996							16567	16429				65992
7							33334						16761	16573					11153	88971
6						33744					16921	16823				11328	11250	11166		101232
5					34139				17148	16991			11470	11350	11319		8588	8560	8497	136556
4				34778			17453	17325		11641	11604	11533	8730	8723	8683	15614	7014	6931	12788	185531
3			35657		17971	17686	12033	11852	20793	8950	16060	16066	13102	13119	18125	11059	15515	15488	13592	270551
2		36960	18700	30791	21798	28891	22955	28086	23553	27681	23970	27206	24323	26992	24500	26864	24601	26650	24762	495999
1	499999	463040	445643	434431	426092	419679	414225	409741	405704	402172	399158	396323	393706	391352	389193	387190	385253	383470	381848	8208367

Table 2. The number of values z of S(n) in the intervals $500000y+1 \le S(n) \le 500000(y+1)$ is represented for each interval $500000x+1 \le n \le 500000(x+1)$ for y=0,1,2,…,18 and x=0,1,2,…,18.

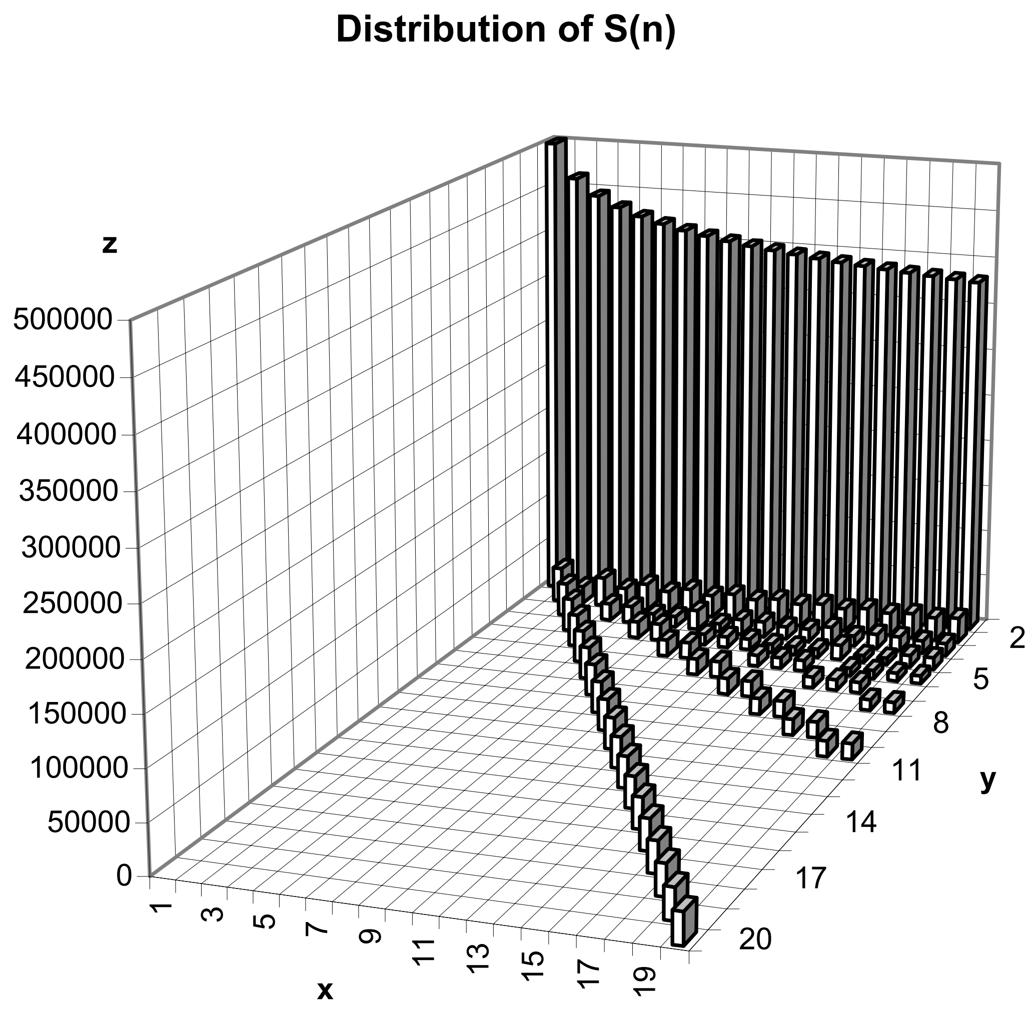

Diagram 2. The distribution of S(n) for $n<10^7$.

Conjecture: The number of Smarandache 2-2 additive relationships is infinite.

What about k>2? Do k-k additive relations exist for all k? If not - which is the largest possible value of k? When they exist, is the number of them infinite or not?

Table 3. Comparison between 2-2 additive relations and other relevant data.

# of → Interval	prime twins	k-k relations	formed by twin pairs	S.function primes	S.function even values	S.function comp values
$n \le 10^7$	58980	158	22	9932747	59037	8215
$10^7 < n \le 2 \cdot 10^7$	48427	59	9	9957779	38023	4198
$2 \cdot 10^7 < n \le 3 \cdot 10^7$	45485	37	4	9963674	32922	3404
$3 \cdot 10^7 < n \le 4 \cdot 10^7$	43861	42	4	9967080	29960	2960
$4 \cdot 10^7 < n \le 5 \cdot 10^7$	42348	40	5	9969366	27962	2672
$5 \cdot 10^7 < n \le 6 \cdot 10^7$	41547	30	2	9971043	26473	2484
$6 \cdot 10^7 < n \le 7 \cdot 10^7$	40908	28	4	9972374	25303	2323
$7 \cdot 10^7 < n \le 8 \cdot 10^7$	39984	41	7	9973482	24327	2191
$8 \cdot 10^7 < n \le 9 \cdot 10^7$	39640	20	4	9974414	23521	2065
$9 \cdot 10^7 < n \le 10^8$	39222	26	4	9975179	22825	1996
Total	440402	481	65	99657140	310355	32508

*Table 4.*The 20 first Smarandache function: 2-2 additive quadruplets

#	n	S(n)	S(n+1)	S(n+2)	S(n+3)	
1	6	3	7	4	6	
2	7	7	4	6	5	
3	28	7	29	5	31	*
4	114	19	23	29	13	
5	1720	43	1721	41	1723	*
6	3538	61	3539	59	3541	*
7	4313	227	719	863	83	
8	8474	223	113	163	173	
9	10300	103	10301	101	10303	*
10	13052	251	229	107	373	
11	15417	571	593	907	257	
12	15688	53	541	523	71	
13	19902	107	1531	311	1327	
14	22194	137	193	179	151	
15	22503	577	97	643	31	
16	24822	197	241	107	331	
17	26413	433	281	587	127	
18	56349	2087	23	1523	587	
19	70964	157	83	137	103	
20	75601	173	367	79	461	

Table 5. The 20 first Smarandache function: 3-3 additive sextets

#	n	S(n)	S(n+1)	S(n+2)	S(n+3)	S(n+4)	S(n+5)
1	5	5	3	7	4	6	5
2	5182	2591	73	9	61	2593	19
3	9855	73	11	9857	53	9859	29
4	10428	79	10429	149	61	163	10433
5	28373	1669	4729	227	3547	1051	2027
6	32589	71	3259	109	97	2963	379
7	83323	859	563	101	683	809	31
8	106488	29	1283	463	461	337	977
9	113409	12601	1031	127	727	4931	8101
10	146572	36643	20939	479	41	9161	48859
11	257474	347	3433	1091	263	3301	1307
12	294742	569	1223	12281	233	8669	5171
13	448137	101	224069	448139	97	448141	224071
14	453250	37	14621	353	1613	13331	67
15	465447	1373	797	6947	107	59	8951
16	831096	97	4643	21871	617	8311	17683
17	1164960	809	1021	1669	673	1283	1543
18	1279039	1279039	571	691	347	1279043	911
19	1348296	56179	2447	499	49937	139	9049
20	1428620	1171	2393	2389	1607	3307	1039

*Table 6.*The 9 first Smarandache function: 4-4 additive octets

#	n	S(n)	S(n+1)	S(n+2)	S(n+3)	S(n+4)	S(n+5)	S(n+6)	S(n+7)
1	23	23	4	10	13	9	7	29	5
2	643	643	23	43	19	647	9	59	13
3	10409	1487	347	359	137	89	127	2083	31
4	44418	673	1033	2221	67	167	1433	617	1777
5	163329	54443	16333	23333	349	701	81667	10889	1201
6	279577	279577	10753	2273	1997	3539	2741	279583	8737
7	323294	1483	3079	10103	1913	5987	10429	61	101
8	368680	709	2903	1429	1699	1511	2731	2221	277
9	857434	8089	769	71453	353	11587	2887	233	65957

Table 7. The 5 first Smarandache function: 5-5 additive relationships

#	n	S(n+1)	S(n+1)	S(n)	S(n+1)	S(n+2)	S(n+3)	S(n+4)	S(n+5)	S(n+6)	S(n+7)
1	13	13	7	5	6	17	6	19	5	7	11
2	570	19	571	13	191	41	23	8	577	34	193
3	1230	41	1231	11	137	617	19	103	1237	619	59
4	392152	49019	392153	9337	733	79	43573	15083	392159	43	463
5	1984525	487	992263	2371	47	1091	797	701	53	2441	992267

Table 8. Smarandache function: 6-6 additive relationships for $n<10^7$

#	n	S(n)	S(n+1)	S(n+2)	S(n+3)	S(n+4)	S(n+5)	S(n+6)	S(n+7)	S(n+8)	S(n+9)	S(n+10)	S(n+11)
1	14	7	5	6	17	6	19	5	7	11	23	4	10
2	158	79	53	8	23	9	163	41	11	83	167	7	26
3	20873	20873	71	167	307	6959	73	20879	29	157	197	6961	227
4	21633	7211	373	4327	601	281	349	7213	541	67	3607	941	773
5	103515	103	3697	1697	71	7963	647	3137	271	643	8627	101	1399
6	132899	10223	443	383	863	14767	449	1399	1303	4583	223	6329	13291
7	368177	661	61363	353	449	3719	9689	1301	46	73637	34	107	1109
8	5559373	5559373	1447	593	15107	3253	643	3323	1193	10837	293	5559383	5387

Table 9. Examples of Smarandache function 50-50 additive relations

	n= 1876		16539		n= 58631		n= 109606		n= 2385965	
S(n)/S(n+51)	67	107	149	313	58631	101	7829	1523	1087	7823
… 48 others …										
S(n+49)/S(n+100)	11	79	29	59	163	839	241	593	727	757

References :

1. H. Ibstedt, The Smarandache Function S(n), *Smarandache Function Journal, Vol. 2-3, No 1, pgs 43-50.*
2. M. Bencze, Smarandache Relationships and Subsequences, *Smarandache Notions Journal, Vol. 11, No 1-2-3, pgs 79-85.*
3. H. Ibstedt, Non-Recursive Sequences, *Computer Analysis of Number Sequences,* American Research Press, 1998.

VIII. Smarandache 2-2 subtractive relationships

Abstract: An analysis of the number of relations of the type S(n)-S(n+1)=S(n+2)-S(n+3) for n<10^8 where S(n) is the Smarandache function leads to the plausible conclusion that there are infinitely many of those.

1. Calculations and Results

A Smarandache 2-2 subtractive relationship is defined by

S(n)-S(n+1)=S(n+2)-S(n+3)

where S(n) denotes the Smarandache function. In an article by Bencze [1] three 2-2 subtractive relationships are given

S(1)-S(2)=S(3)-S(4), 1-2=3-4
S(2)-S(3)=S(4)-S(5), 2-3=4-5
S(49)-S(50)=S(51)-S(52), 14-10=17-13

The first of these solutions must be rejected since S(1)=0 not 1. The question raised in the article is "How many quadruplets verify a Smarandache 2-2 subtractive relationship?"

As in the case of Smarandache 2-2 additive relationships a search was carried for n≤10^8. In all 442 solutions were found. The first 12 of these are shown in table 1.

Table 1. The 12 first 2-2 subtractive relations.

#	n	S(n)	S(n+1)	S(n+3)	S(n+4)
1	2	2	3	4	5
2	40	5	41	7	43
3	49	14	10	17	13
4	107	107	9	109	11
5	2315	463	193	331	61
6	3913	43	103	29	89
7	4157	4157	11	4159	13
8	4170	139	97	149	107
9	11344	709	2269	61	1621
10	11604	967	211	829	73
11	11968	17	11969	19	11971
12	13244	43	883	179	1019

As in the case of 2-2 additive relations there is a great number of solutions formed by pairs of prime twins. There are in all 51 subtractive relations formed by pairs of prime twins for n<10^8.

Table 2. The 15 first subtractive relations formed by pairs of prime twins.

#	n	S(n)	S(n+1)	S(n+3)	S(n+4)
1	40	5	41	7	43
2	4157	4157	11	4159	13
3	11968	17	11969	19	11971
4	19180	137	19181	139	19183
5	666647	666647	197	666649	199
6	895157	895157	137	895159	139
7	1695789	347	101	349	103
8	1995526	71	1995527	73	1995529
9	2007880	101	2007881	103	2007883
10	2272547	2272547	149	2272549	151
11	3198730	1787	3198731	1789	3198733
12	3483088	227	3483089	229	3483091
13	3546268	431	3546269	433	3546271
14	4194917	4194917	197	4194919	199
15	4503640	179	4503641	181	4503643

In the case of 2-2 additive relations only 2 solutions contained composite numbers and these were the first two. This was explained in terms of the distribution Smarandache functions values. For the same reason 2-2 subtractive relations containing composite numbers are also scarce, but there are 6 of them for n<10^8. These are shown in table 3.

It is interesting to note that solutions #3, #5 and #6 have in common with the solutions formed by pairs of prime twins that they are formed by pairs of numbers whose difference is 2. Finally table 4 shows a tabular comparison between the solutions to the 2-2 additive and 2-2 subtractive solutions for n<10^8. The great similarity between these results leads the conclusion: If the conjecture that there are infinitely many 2-2 additive relations is valid then we also have the following conjecture:

Table 3. All 2-2 subtractive relations <10^8 containing composite numbers.

#	n	S(n)	S(n+1)	S(n+3)	S(n+4)
1	2	2	3	4	5
2	49	14	10	17	13
3	107	107	9	109	11
4	530452	202	166	419	383
5	41839378	111	41839379	113	41839381
6	48506848	57	48506849	59	48506851

Table 4. Comparison between 2-2 additive and 2-2 subtractive relations.

	Number of 2-2 additive solutions	Number of 2-2 subtractive solutions
Total number of solutions	481	442
Number formed by pairs of prime twins	65	51
Number containing composite numbers	2	6

Conjecture: There are infinitely many Smarandache 2-2 subtractive relationships.

The tables in this presentation have been abbreviated. For more extensive results see 2].

References:

1. M. Bencze, Smarandache Relationships and Subsequences, *Smarandache Notions Journal, Vol. 11, No 1-2-3, pgs 79-85.*
2. H. Ibstedt, Smarandache k-k additive relationships, *Smarandache Notions Journal, Vol. 12.*

IX. Concatenation Problems

Abstract: This study has been inspired by questions asked by Charles Ashbacher in the *Journal of Recreational Mathematics, vol. 29.2*. It concerns the Smarandache Deconstructive Sequence. This sequence is a special case of a more general concatenation and sequencing procedure which is the subject of this study. Answers are given to the above questions. The properties of this kind of sequences are studied with particular emphasis on the divisibility of their terms by primes.

1. Introduction

In this article the concatenation of a and b is expressed by a_b or simply ab when there can be no misunderstanding. Multiple concatenations like abcabcabc will be expressed by 3(abc).

We consider n different elements (or n objects) arranged (concatenated) one after the other in the following way to form:

$A=a_1a_2 \ldots a_n$.

Infinitely many objects A, which will be referred to as cycles, are concatenated to form the chain:

$B= a_1a_2 \ldots a_n\, a_1a_2 \ldots a_n\, a_1a_2 \ldots a_n\ldots$

B contains identical elements which are at equidistant positions in the chain. Let's write B as

$B=b_1b_2b_3, \ldots b_k\ldots..$ where $b_k=a_j$ when $j\equiv k \pmod n$, $1\leq j\leq n$.

An infinite sequence $C_1, C_2, C_3, \ldots C_k, \ldots.$ is formed by sequentially selecting 1, 2, 3, …k, … elements from the chain B:

$C_1=b_1=a_1$

$C_2=b_2b_3=a_2a_3$

$C_3=b_4b_5b_6=a_4a_5a_6$ (if $n\leq 6$, if $n=5$ we would have $C_3=a_4a_5a_1$)

The number of elements from the chain B used to form first k-1 terms of the sequence C is 1+2+3+ … +k-1=(k-1)k/2. Hence

$$C_k = b_{\frac{(k-1)k}{2}+1}\, b_{\frac{(k-1)k}{2}+2} \cdots b_{\frac{k(k+1)}{2}}$$

However, what is interesting to see is how C_k is expressed in terms of $a_1,\ldots,a_n$. For sufficiently large values of k C_k will be composed of three parts:

The first part $F(k)=a_u\ldots a_n$
The middle part M(k)=AA…A The number of concatenated As depends on k.
The last part $L(k)=a_1a_2\ldots a_w$
Hence

(1) $C_k=F(k)M(k)L(k)$.

The number of elements used to form C_1, C_2, … C_{k-1} is (k-1)k/2. Since the number of elements in A is finite there will be infinitely many terms C_k which have the same first element a_u. u can be determined from $\frac{(k-1)k}{2}+1 \equiv u(\bmod n)$. There can be at most n^2 different combinations to form F(k) and L(k). Let C_j and C_i be two different terms for which F(i)=F(j) and L(i)=L(j). They will then be separated by a number m of complete cycles of length n, i.e.

$$\frac{(j-1)j}{2}-\frac{(i-1)i}{2}=mn$$

Let's write j=i+p and see if p exists so that there is a solution for p which is independent of i.

$$(i+p-1)(i+p)-(i-1)i=2mn$$
$$i^2+2ip+p^2-i-p-i^2+i=2mn$$
$$2ip+p^2-p=2mn$$
$$p^2+p(2i-1)=2mn$$

If n is odd we will put p=n to obtain n+2i-1=2m, or $m=\frac{n+2i-1}{2}$. If n is even we put p=2n to obtain m=2n+2i-1. From this we see that the terms C_k have a peculiar periodic behaviour. The periodicity is p=n for odd n and p=2n for even n. Let's illustrate this for n=4 and n=5 for which the periodicity will be p=8 and p=5 respectively. It is seen from table 1 that the periodicity starts for i=3.

Numerals are chosen as elements to illustrate the case n=5. Let's write i=s+k+pj , where s is the index of the term preceding the first periodical term, k=1,2,…,p is the index of members of the period and j is the number of the period (for convenience the first period is numbered 0). The first part of C_i is denoted B(k) and the last part E(k). C_i is now given by the expression below where q is the number of cycles concatenated between the first part B(k) and the last part E(k).

(2) C_i=B(k)_qA_E(k), where k is determined from i-s≡k (mod p)

Table 1. n=4. A=abcd. B= abcdabcdabcdabcdabcd……

i	C_i	Period #	F(i)	M(i)	L(i)
1	a		a		
2	bc		bc		
3	dab	1	d		ab
4	cdab	1	cd		ab
5	cdabc	1	cd		abc
6	dabcda	1	d	abcd	a
7	bcdabcd	1	bcd	abcd	
8	abcdabcd	1		2(abcd)	
9	abcdabcda	1		2(abcd)	a
10	bcdabcdabc	1	bcd	abcd	abc
11	dabcdabcdab	2	d	2(abcd)	ab
12	cdabcdabcdab	2	cd	2(abcd)	ab
13	cdabcdabcdabc	2	cd	2(abcd)	abc
14	dabcdabcdabcda	2	d	3(abcd)	a
15	bcdabcdabcdabcd	2	bcd	3(abcd)	
16	abcdabcdabcdabcd	2		4(abcd)	
17	abcdabcdabcdabcda	2		4(abcd)	a
18	bcdabcdabcdabcdabc	2	bcd	3(abcd)	abc
19	dabcdabcdabcdabcdab	3	d	4(abcd)	ab
20	cdabcdabcdabcdabcdab	3	cd	4(abcd)	ab

Table 2. n=5. A=12345. B= 123451234512345………

I	C_i	k	q	F(i)/B(k)	M(I)	L(i)/E(k)
1	1			1		
s=2	23			23		
	j=0					
3	451	1	0	45		1
4	2345	2	0	2345		
5	12345	3	1		12345	
6	123451	4	1		12345	1
7	2345123	5	0	2345		123
	j=1					
3+5j	45123451	1	j	45	12345	1
4+5j	234512345	2	j	2345	12345	
5+5j	1234512345	3	j+1		2(12345)	
6+5j	12345123451	4	j+1		2(12345)	1
7+5j	234512345123	5	j	2345	12345	123
	j=2					
3+5j	4512345123451	1	j	45	2(12345)	1
4+5j	23451234512345	2	j	2345	2(12345)	
…						

2. The Smarandache Deconstructive Sequence

The Smarandache Deconstructive Sequence of integers [1] is constructed by sequentially repeating the digits 1-9 in the following way:

1,23,456,7891,23456,789123,4567891,23456789,123456789,123456 7891, …

The sequence was studied in a booklet by Kashihara [2] and a number of questions on this sequence were posed by Ashbacher [3]. In thinking about these questions two observations lead to this study.

1. Why did Smarandache exclude 0 from the integers used to create the sequence? After all 0 is indispensable in all arithmetics most of which can be done using 0 and 1 only.
2. The process used to create the Deconstructive Sequence is a process which applies to any set of objects as has been shown in the introduction.

The periodicity and the general expression for terms in the "generalized deconstructive sequence" shown in the introduction may be the most important results of this study. These results will now be used to examine the questions raised by Ashbacher. It is worth noting that these divisibility questions are dealt with in base 10 although only the nine digits 1,2,3,4,5,6,7,8,9 are used to express numbers. In the last part of this article questions on divisibility will be posed for a deconstructive sequence generated from A="0123456789".

For $i>5$ ($s=5$) any term C_i in the sequence is composed by concatenating a first part B(k), a number q of cycles A="123456789" and a last part E(k), where $i=5+k+9j$, $k=1,2,\ldots 9$, $j\geq 0$, as expressed in (2) and $q=j$ or $j+1$ as shown in table 3.

Members of the Smarandache Deconstructive Sequence are now interpreted as decimal integers. The factorization of B(k) and E(k) is shown in table 3. The last two columns of this table will be useful later in this article.

Together with the factorization of the cycle $A=123456789=3^2\cdot 3607\cdot 3803$ it is now possible to study some divisibility properties of the sequence. We will first find expressions for C_i for each of the 9 values of k. In cases where E(k) exists let's introduce $u=1+[\log_{10}E(k)]$. We also define the function $\delta(j)$ so that $\delta(j)=0$ for $j=0$ and $\delta(j)=1$ for $j>0$. It is possible to construct one algorithm to cover all the nine cases but more functions like $\delta(j)$ would have to be introduced to distinguish between the numerical values of the strings “” (empty string) and “0” which are both evaluated as 0 in computer applications. In order to avoid this four formulas are used

Table 3. Factorization of Smarandache Deconstructive Sequence

i	k	B(k)	q	E(k)	Digit sum	$3\|C_i$?
6+9j	1	789=3·263	j	123=3·41	30+j·45	3
7+9j	2	456789=3·43·3541	j	1	40+j·45	No
8+9j	3	23456789	j		44+j·45	No
9+9j	4		j+1		(j+1)·45	$9\cdot 3^z$ *
10+9j	5		j+1	1	1+(j+1)·45	No
11+9j	6	23456789	j	123=341	50+j·45	No
12+9j	7	456789=3·43·3541	j	$123456=2^6\cdot 3\cdot 643$	60+j·45	3
13+9j	8	789=3·263	j+1	1	25+(j+1)·45	No
14+9j	9	23456789	j	$123456=2^6\cdot 3\cdot 643$	65+j·45	No

*) where z depends on j.

For k=1, 2, 6, 7 and 9:

(3) $$C_{5+k+9j}=E(k)+\delta(j)\cdot A\cdot 10^u\cdot \sum_{r=0}^{j-1} 10^{9r} +B(k)\cdot 10^{9j+u}$$

For k=3:

(4) $$C_{5+k+9j}=\delta(j)\cdot A\cdot \sum_{r=0}^{j-1} 10^{9r} +B(k)\cdot 10^{9j}$$

For k=4:

(5) $$C_{5+k+9j}=A\cdot \sum_{r=0}^{j} 10^{9r}$$

For k=5 and 8:

(6) $$C_{5+k+9j}=E(k)+A\cdot 10^u\cdot \sum_{r=0}^{j} 10^{9r} +B(k)\cdot 10^{9(j+1)+u}$$

Before dealing with the questions posed by Ashbacher we recall the familiar rules: An even number is divisible by 2; a number whose last two digit form a number which is divisible by 4 is divisible by 4. In general we have the following:

Theorem. Let N be an n-digit integer such that $N>2^\alpha$ then N is divisible by 2^α if and only if the number formed by the α last digits of N is divisible by 2^α.

Proof. To begin with we note that

If x divides a and x divides b then x divides (a+b)
If x divides one but not the other of a and b then x does not divide (a+b)
If x does not divides neither a nor b then x may or may not divide (a+b)

Let's write the n-digit number in the form $a\cdot 10^{\alpha}+b$. We then see from the following that $a\cdot 10^{\alpha}$ is divisible by 2^{α}.

$10\equiv 0 \pmod 2$
$100 \equiv 0 \pmod 4$
$1000= 2^3\cdot 5^3\equiv 0 \pmod{2^3}$
…
$10^{\alpha}\equiv 0 \pmod{2^{\alpha}}$
and then
$a\cdot 10^{\alpha}\equiv 0 \pmod{2^{\alpha}}$ independent of a.

Now let b be the number formed by the α last digits of N we then see from the introductory remark that N is divisibe by 2^{α} if and only if the number formed by the α last digits is divisibele by 2^{α}.

Question 1. Does every even element of the Smarandache Deconstructive Sequence contain at least three instances of the prime 2 as a factor?

Question 2. If we form a sequence from the elements of the Smarandache Deconstructive Sequence that end in a 6, do the powers of 2 that divide them form a montonically increasing sequence?

These two questions are related and are dealt with together. From the previous analysis we know that all even elements of the Smarandache Deconstructive Sequence end in a 6. For $i\leq 5$ they are:

$C_3=456=57\cdot 2^3$
$C_5=23456=733\cdot 2^5$

For i>5 they are of the forms:

C_{12+9j} and C_{14+9j} which both end in …789123456.

Examining the numbers formed by the 6, 7 and 8 last digits for divisibility by 2^6, 2^7 and 2^8 respectively we have:

$123456=2^6\cdot 3\cdot 643$

$9123456=2^7 \cdot 149.4673$
89123456 is not divisible by 2^8

From this we conclude that all even Smarandache Deconstructive Sequence elements for $i \geq 12$ are divisible by 2^7 and that no elements in the sequence are divisible by higher powers of 2 than 7.

Answer to Qn 1. Yes

Answer to Qn 2. The sequence is monotonically increasing for $i \leq 12$. For $i \geq 12$ the powers of 2 that divide even elements remain constant=2^7.

Question 3. Let x be the largest integer such that $3^x|i$ and y the largest integer such that $3^y|C_i$. Is it true that x is always equal to y?

From table 3 we se that the only elements C_i of the Smarandache Deconstructive Sequence which are divisible by powers of 3 correspond to i=6+9j, 9+9j, or 12+9j. Furthermore, we see that i=6+9j and C_{6+9j} are divisible by 3 no more no less. The same is true for i=12+9j and C_{12+9j}. So the statement holds in these cases.

From the conguences

$9+9j \equiv 0 \pmod{3^x}$ for the index of the element

and

$45(1+j) \equiv 0 \pmod{3^y}$ for the corresponding element

we conclude that x=y.

*Answer to Qn 3***:** The statement is true. It is interesting to note that, for example the 729 digit number C_{729} is divisible by 729.

Question 4. Are there other patterns of divisibility in this sequence?

A search for other patterns would continue by examining divisibility by the next lower primes 5, 7, 11, … It is obvious from table 3 and the periodicity of the sequence that there are no elements divisible by 5. The algorithms will prove very useful. For each value of k the value of C_i depends on j only. The divisibilty by a prime p is therefore determined by finding out for which values of j and k the congruence $C_i \equiv 0 \pmod{p}$ holds. We evaluate $\sum_{r=0}^{j-1} 10^{9r} = \frac{10^{9j}-1}{10^9-1}$ and introduce $G=10^9-1$. We note that $G=3^4 \cdot 37 \cdot 333667$. From formulas (3) to (6) we now obtain:

For k=1,2,6,7 and 9:

(3′) $C_i \cdot G=10^u \cdot (\delta(j) \cdot A+B(k) \cdot G) \cdot 10^{9j}+E(k) \cdot G-10^u \cdot \delta(j) \cdot A$

For k=3:

(4′) $C_i \cdot G=((\delta(j) \cdot A+B(k) \cdot G) \cdot 10^{9j}-\delta(j) \cdot A$

For k=4:

(5′) $C_i \cdot G=A \cdot 10^{9j}-A$

For k=5 and 8:

(6′) $C_i \cdot G=10^{u+9}(A+B(k) \cdot G) \cdot 10^{9j}+E(k) \cdot G-10^u \cdot A$

The divisibility of C_i by a prime p other tha n 3, 37 and 333667 is therefore determined by solutions for j to the congruences $C_iG \equiv 0 \pmod p$ which are of the form

(7) $a \cdot (10^9)^j+b \equiv 0 \pmod p$

Table 4 shows the results from computer implementation of the congruences. The appearance of elements divisible by a prime p is periodic, the periodicity is given by $j=j_1+m \cdot d$, m=1, 2, 3, … . The first element divisible by p appears for i_1 corresponding to j_1. In general the terms C_i divisible by p are $C_{5+k+9(j_1+md)}$ where d is specific to the prime p and m=1, 2, 3,… .We note from table 4 that d is either equal to p-1 or a divisor of p-1 except for the case p=37 which as we have noted is a factor of A. Indeed this periodicity follows from Euler's extension of Fermat's little theorem because if we write (mod p):

$$a \cdot (10^9)^j+b= a \cdot (10^9)^{j_1+md}+b \equiv a \cdot (10^9)^{j_1}+b \text{ for } d=p-1 \text{ or a divisor of } p-1.$$

Finally we note that the periodicity for p=37 is d=37.

Question: Table 4 indicates some interesting patterns. For instance, the primes 19, 43 and 53 only divides elements corresponding to k=1, 4 or 7 for j<150 which was set as an upper limit for this study. Similarly, the primes 7,11, 41, 73, 79 and 91 only divides elements corresponding to k=4. Is 5 the only prime that cannot divide an element of the Smarandache Deconstructive Sequence?

Table 4. Smarandache Deconstructive Sequence elements divisible by p:

p=7 d=2	k	4
	i_1	18
	j_1	1

p=11 d=2	k	4
	i_1	18
	j_1	1

p=13	k	4	8	9
d=2	i_1	18	22	14
	j_1	1	1	0

p=17	k	1	2	3	4	5	6	7	8	9
d=16	i_1	6	43	44	144	100	101	138	49	95
	j_1	0	4	4	15	10	10	14	4	9

p=19	k	1	4	7
d=2	i_1	15	18	21
	j_1	1	1	1

p=23	k	1	2	3	4	5	6	7	8	9
d=22	i_1	186	196	80	198	118	200	12	184	14
	j_1	20	21	8	21	12	21	0	19	0

p=29	k	1	2	3	4	5	6	7	8	9
d=28	i_1	24	115	197	252	55	137	228	139	113
	j_1	2	12	21	27	5	14	24	14	11

p=31	k	3	4	5
d=5	i_1	26	45	19
	j_1	2	4	1

p=37	k	1	2	3	4	5	6	7	8	9
d=37	i_1	222	124	98	333	235	209	111	13	320
	j_1	24	13	10	36	25	22	11	0	34

p=41	k	4
d=5	i_1	45
	j_1	4

P=43	1	4	7
d=7	33	63	30
	3	6	2

p=47	k	1	2	3	4	5	6	7	8	9
d=46	i_1	150	250	368	414	46	164	264	400	14
	j_1	16	27	40	45	4	17	28	43	0

p=53	k	1	4	7
d=13	i_1	24	117	12
	j_1	2	12	9

p=59	k	1	3	5	6	7	8	9
d=58	i_1	267	413	109	11	255	256	266
	j_1	29	45	11	0	27	27	28

p=61	k	2	4	6
d=20	i_1	79	180	101
	j_1	8	19	10

p=67	k	4	8	9
d=11	i_1	99	67	32
	j_1	10	6	2

p=71	k	1	3	4	5	7
d=35	i_1	114	53	315	262	201
	j_1	12	5	34	28	21

p=73 d=8	k	4
	i_1	72
	j_1	7

p=79 d=13	k	4
	i_1	117
	j_1	12

p=83 d=41	k	1	2	4	6	7	8	9
	i_1	348	133	369	236	21	112	257
	j_1	38	14	40	25	1	11	27

p=89 d=44	k	2	4	6
	i_1	97	396	299
	j_1	10	43	32

p=97 d=32	k	1	2	3	4	5	6	7	8	9
	i_1	87	115	107	288	181	173	201	202	86
	j_1	9	12	11	31	19	18	21	21	8

3. A Deconstructive Sequence generated by the cycle A=0123456789.

Instead of sequentially repeating the digits 1-9 as in the case of the Smarandache Deconstructive Sequence we will use the digits 0-9 to form the corresponding sequence*:*

0,12,345,6789,01234,567890,1234567,89012345,678901234,678901234,5678 9012345,678901234567, …

In this case the cycle has n=10 elements. As we have seen in the introduction the sequence then has a period =2n=20. The periodicity starts for i=8. Table 5 shows how for i>7 any term C_i in the sequence is composed by concatenating a first part B(k), a number q of cycles A="0123456789" and a last part E(k), where i=7+k+20j, k=1,2,…20, j≥0, as expressed in (2) and q=2j, 2j+1 or 2j+2. In the analysis of the sequence it is important to distinguish between the cases where E(k)=0, k=6,11,14,19 and cases where E(k) does not exist, i.e. k=8,12,13,14. In order to cope with this problem we introduce a function u(k) which will at the same time replace the functions δ(j) and $u=1+[\log_{10}E(k)]$ used previously. u(k) is defined as shown in table 5. It is now possible to express C_i in a single formula

$$(8) \qquad C_i = C_{7+k+20j} = E(k) + (A\cdot\sum_{r=0}^{q(k)+2j-1}(10^{10})^r + B(k)\cdot(10^{10})^{q(k)+2j})\cdot 10^{u(k)}$$

The formula for C_i was implemented modulus prime numbers less then 100. The result is shown in table 6 for p≤41. Again we note that the divisibility by a prime p is periodic with a period d which is equal to p-1 or a divisor of p-1, except of p=11 and p=41 which are factors of 10^{10}-1. The cases p=3 and 5 have very simple answers and are not included in table 6.

Table 5. n=10, A=0123456789

i	k	B(k)	q	E(k)	u(k)
8+20j	1	89	2j	012345=3·5·823	6
9+20j	2	6789=3·31·73	2j	01234=2·617	5
10+20j	3	56789=109·521	2j	01234=2·617	5
11+20j	4	56789=109·521	2j	012345=3·5·823	6
12+20j	5	6789=3·31·73	2j	01234567=127·9721	8
13+20j	6	89	2j+1	0	1
14+20j	7	123456789=3^2·3607·3803	2j	01234=2·617	5
15+20j	8	56789=109·521	2j+1		0
16+20j	9		2j+1	012345=3·5·823	6
17+20j	10	6789=3·31·73	2j+1	012=2^2·3	3
18+20j	11	3456789=3·7·97·1697	2j+1	0	1
19+20j	12	123456789=3^2·3607·3803	2j+1		0
20+20j	13		2j+2		0
21+20j	14		2j+2	0	1
22+20j	15	123456789=3^2·3607·3803	2j+1	012=2^2·3	3
23+20j	16	3456789=3·7·97·1697	2j+1	012345=3·5·823	6
24+20j	17	6789=3·31·73	2j+2		0
25+ 20j	18		2j+2	01234=2·617	5
26+20j	19	56789=109·521	2j+2	0	1
27+20j	20	123456789=3^2·3607·3803	2j+1	01234567=127·9721	8

Table 6. Divisibility of the 10-cycle destructive sequence by primes 7≤p≤41

p=7 d=3	k	3	6	7	8	11	12	13	14	15	18	19	20
	i_1	30	13	14	15	38	59	60	61	22	45	46	47
	j_1	1	0	0	0	1	2	2	2	0	1	1	1

p=11 =11	k	1	2	3	4	5	6	7	8	9	10
	i_1	88	9	110	211	132	133	74	35	176	137
	j_1	4	0	5	10	6	6	3	1	8	6
	k	11	12	13	14	15	16	17	18	19	20
	i_1	18	219	220	221	202	83	44	185	146	87
	j_1	0	10	10	10	9	3	1	8	6	3

p=13 d=3	k	2	3	4	12	13	14
	i_1	49	30	11	59	60	61
	j_1	2	1	0	2	2	2

p=17 d=4	k	1	5	10	12	13	14	16
	i_1	48	32	37	79	80	81	43
	j_1	2	1	1	3	3	3	1

p=19 d=9	k	1	2	3	4	5	10	12	13	14	16
	i_1	128	149	90	31	52	117	179	180	181	63
	j_1	6	7	4	1	2	5	8	8	8	2

p=23 d=11	k	1	2	3	4	5	10	12	13	14	16
	i_1	168	149	110	71	52	217	219	220	221	223
	j_1	8	7	5	3	2	10	10	10	10	10

p=29 d=7	k	2	4	10	12	13	14	16
	i_1	129	11	97	139	140	141	43
	j_1	6	0	4	6	6	6	1

p=31	k	3	9	12	13	14	17
d=3	i_1	30	56	59	60	61	64
	j_1	1	2	2	2	2	2

p=37 d=3	k	2	3	4	12	13	14
	i_1	9	30	51	59	60	61
	j_1	0	1	2	2	2	2

p=41 d=41	k	1	2	3	4	5	6	7	8	9	10
	i_1	788	589	410	231	32	353	614	615	436	117
	j_1	39	29	20	11	1	17	30	30	21	5
	k	11	12	13	14	15	16	17	18	19	20
	i_1	678	819	820	821	142	703	384	205	206	467
	j_1	33	40	40	40	6	34	10	9	9	22

References:

1. F. Smarandache, *Only Problems, Not Solutions*, Xiquan Publishing House, Phoenix, Arizona, 1993.
2. K. Kashihara, *Comments and Topics on Smarandache Notions and Problems,* Erhus University Press, Vail, Arizona, 1996.
3. C. Ashbacher, *Some Problems Concerning the Smarandache Deconstructive Sequence,* Journal of Recreational Mathematics, Vol 29, Number 2 – 1998, Baywood Publishing Company,Inc.

X. On a Deconcatenation Problem

Abstract: In a recent study of the *Primality of the Smarandache Symmetric Sequences* Sabin and Tatiana Tabirca [1] observed a very high frequency of the prime factor 333667 in the factorization of the terms of the second order sequence. The question if this primfactor occurs periodically was raised. The odd behaviour of this and a few other primefactors of this sequence will be explained and details of the periodic occurance of this and of several other prime factors will be given.

1. Definition

The n*th* term of the Smarandache symmetric sequence of the second order is defined by S(n)=123…n_n…321 which is to be understood as a concatenation[4] of the first n natural numbers concatenated with a concatenation in reverse order of the n first natural numbers.

2. Factorization and Patterns of Divisibility

The first five terms of the sequence are: 11, 1221, 123321, 12344321, 1234554321.

The number of digits D(n) of S(n) is growing rapidly. It can be found from the formula:

(1) $$D(n) = 2k(n+1) - \frac{2(10^k - 1)}{9}$$ for n in the interval $10^{k-1} \le n < 10^k - 1$

In order to study the repeated occurrance of certain primes S(n) was calculated and partially factorized. The result is shown for n≤100 in table 1.

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[4] In this article the concatenation of a and b is written a_b. Multiplication ab is often made explicit by writing a.b. When there is no reason for misunderstanding the signs "_" and "." are omitted. Several tables contain prime factorizations. Prime factors are given in ascending order, multiplication is expressed by "." and the last factor is followed by ".." if the factorization is incomplete or by Fxxx indicating the number of digits of the last factor. To avoid typing errors all tables are electronically transferred from the calculation program, which is DOS-based to the wordprocessor. All editing has been done either with a spreadsheet program or directly with the text editor. Full page tables have been placed at the end of the article. A non-proportional font has been used to illustrate the placement of digits when this has been found useful.

Table 1. Prime factors of S(n) which are less than 10^8

n	Prime factors of S(n)	n	Prime factors of S(n)
1	11	51	3.37.1847.F180
2	3.11.37	52	F190
3	3.11.37.101	53	3^3.11.43.26539.17341993.F178
4	11.41.101.271	54	3^3.37.41.151.271.347.463.9091.333667.F174
5	3.7.11.13.37.41.271	55	67.F200
6	3.7.11.13.37.239.4649	56	3.11.F204
7	11.73.101.137.239.4649	57	3.31.37.F206
8	3^2.11.37.73.101.137.333667	58	227.9007.20903089.F200
9	3^2.11.37.41.271.9091.333667	59	3.41.97.271.9091.F207
10	F22	60	3.37.3368803.F213
11	3.43.97.548687.F16	61	91719497.F218
12	3.11.31.37.61.92869187.F15	62	3^2.1693.F225
13	109.3391.3631.F24	63	3^2.37.305603.333667.9136499.F213
14	3.41.271.9091.290971.F24	64	11.41.271.9091.F229
15	3.37.661.F37	65	3.839.F238
16	F46	66	3.37.43.F242
17	3.F49	67	11^2.109.467.3023.4755497.F233
18	3^2.37.1301.333667.6038161.87958883.F28	68	3.97.5843.F247
19	41.271.9091.F50	69	3.37.41.271.787.9091.716549.19208653.F232
20	3.11.97.128819.F53	70	F262
21	3.37.983.F61	71	3.F265
22	67.773.F65	72	3^2.31.37.61.163.333667.77696693.F248
23	3.11.7691.F68	73	379.323201.F266
24	3.37.41.43.271.9091.165857.F61	74	3.41^2.43^2.179.271.9091.8912921.F255
25	227.2287.33871.611999.F66	75	3.11.37.443.F276
26	3^3.163.5711.68432503.F70	76	1109.F283
27	3^3.31.37.333667.481549.F74	77	3.10034243.F282
28	146273.608521.F83	78	3.11.37.71.41549.F284
29	3.41.271.9091.F89	79	41.271.9091.F290
30	3.37.5167.F96	80	3.F300
31	11^3.4673.F99	81	3^5.37.333667.4274969.F289
32	3.43.1021.F104	82	F310
33	3.37.881.F109	83	3.20399.5433473.F302
34	11.41.271.9091.F109	84	3.37^2.41.271.9091.F306
35	3^2.3209.F117	85	1783.627041.F313
36	3^2.37.333667.68697367.F110	86	3.11.F324
37	F130	87	3.31.37.43.F324
38	3.1913.12007.58417.597269.63800419.F107	88	67.257.46229.F325
39	3.37.41.271.347.9091.23473.F121	89	3^2.11.41.271.9091.653659.76310887.F314
40	F142	90	3^2.37.244861.333667.F328
41	3.156841.F140	91	173.F343
42	3.11.31.37.61.20070529.F136	92	3.F349
43	71.5087.F148	93	3.37.1637.F348
44	3^2.41.271.9091.1553479.F142	94	41.271.9091.10671481.F343
45	3^2.11.37.43.333667.F151	95	3.43.2833.F356
46	F166	96	3.37.683.F361
47	3.F169	97	11.26974499.F361
48	3.37.173.60373.F165	98	3^2.1299169.F367
49	41.271.929.9091.34613.F162	99	3^2.37.41.271.2767.9091.263273.333667.481417. F347
50	3.167.1789.9923.F172	100	43.47.53.83.683.3533.4919.F367

The computer file containing table 1 is analysed in various ways. Of the 664579 primes which are smaller than 10^7 only 192 occur in the prime factoriztions of S(n) for 1≤n≤200. Of these 192 primes 37 occur more than once. The record holder is 333667, the 28693th prime, which occurs 45 times for 1≤n≤200 while its neighbours 333647 and 333673 do not even occur once.

Table 2. Frequency f of most frequent primes

p	3	333667	37	41	271	9091	11	43	73	53	97	31	47
f	132	45	41	41	41	29	25	24	14	8	7	6	6

Obviously there is something to be explained here. The distribution of the primes 11, 37, 41, 43, 271, 9091 and 333667 is shown in table 3. It is seen that the occurance patterns are different in the intervals 1≤n≤9, 10≤n≤99 and 100≤n≤200. Indeed the last interval is part of the interval 100≤n≤999. It would have been very interesting to include part of the interval 1000≤n≤9999 but as we can see from (1) already S(1000) has 5784 digits.

From the patterns in table 3 we can formulate the occurance of these primes in the intervals 1≤n≤9, 10≤n≤99 and 100≤n≤200, where the formulas for the last interval are indicative. We note, for example, that 11 is not a factor of any term in the interval 100≤n≤999. This indicates that the divisibility patterns for the interval 1000≤n≤9999 and further intervals is a completely open question.
There are other primes which also occur periodically but less frequent.

The frequency of the most frequently occurring primes is shown below in the form $p|S(n_0+d\cdot k)$, where d is the period and k indicates how far the periodicity is valid. In most cases it is not known – this is indicated by ?. As is seen the periodicity property may or may not change when n passes from 10^α to $10^{\alpha+1}$.

Table 4 shows an analysis of the patterns of occurance of the primes in table 1 by interval. Note that we only have observations up to n=200. Nevertheless the interval 100≤n≤999 is used. This will be justified in the further analysis

We note that no terms are divisible by 11 for n>100 in the interval 100≤n≤200 and that no term is divisible by 43 in the interval 1≤n≤9. Another remarkable observation is that the sequence shows exactly the same behaviour for the primes 41 and 271 in the intervals included in the study. Will they show the same behaviour when n≥1000?

Table 3. p|S(n_0+d·k) for k= …

p	n_0	d	k
11	0	1	1,2,…,9
11	9	11	0,1, … ,8
11	12	11	0,1, … ,7
31	12	15	0,1, … ,6
37	2	3	0,5,8
37	3	3	0,2, … ,32
37	99	37	0,2, …, ?
37	122	37	0,2,…,?
41	4	1	0,1
41	9	5	0,1, … ,?
43	11	21	0,1,2,3,4
43	24	21	0,1,2,3
47	100	46	0,1, … ,?
47	105	46	0,1, … ,?
53	100	13	0,1, … .?
271	4	1	0,1
271	9	5	0,1, … ,?
9091	9	5	0,1, … ,19
9091	99	10	0,1, … ,?
333667	8	1	0,1
333667	9	9	0,1, … ,9
333667	99	3	0,1, …,?

***Table 4*. Divisibility patterns**

Interval	p	n	Range for j
1≤n≤9	11	All values of n	
10≤n≤99		12+11j	j=0,1, … ,7
		20+11j	j=0,1, … ,7
100≤n≤999		None	
1≤n≤9	37	2+3j	j=0,1,2
		3+3j	j=0,1,2
10≤n≤99		12+3j	j=0,1,…,28,29
100≤n≤999		122+37j	j=0,1,…,23
		136+37j	j=0,1,…,23
1≤n≤9	41	4+5j	j=0,1
		5	
10≤n≤999		14+5j	j=0,1,…,197
1≤n≤9	43	None	
10≤n≤99		11+21j	j=0,1,3,4
		24+21j	j=0,1,2,3
100≤n≤999		100	
		107+7j	j=0,1,…,127
1≤n≤9	271	4+5j	j=0,1
		5	
10≤n≤999		14+5j	j=0,1,…,197
1≤n≤999	9091	9+5j	j=0,1,…,98
1≤n≤9	333667	8,9	
10≤n≤99		18+9j	j=0,1,…,9
100≤n≤999		102+3j	j=0,1,…,299

3. Explanations

Consider

S(n)=12…n_n…21.

Let p be a divisor of S(n). We will construct a number

(2) N=12…n_0..0_n…21

so that p also divides N. What will be the number of zeros? Before discussing this let's consider the case p=3.

Case 1. p=3.

In the case p=3 we use the familiar rule that a number is divisible by 3 if and only if its digit sum is divisible by 3. In this case we can insert as many zeros as we like in (2) since this does not change the sum of digits. We also note that any integer formed by concatenation of three consecutive integers is divisible by 3, cf a_a+1_a+2, digit sum 3a+3.
It follows that also a_a+1_a+2_a+2_a+1_a is divisible by 3. For a=n+1 we insert this instead of the appropriate number of zeros in (2). This means that if S(n)≡0 (mod 3) then S(n+3)≡0 (mod 3). We have seen that S(2)≡0 (mod 3) and S(3)≡0 (mod 3). By induction it follows that S(2+3j)≡0 (mod 3) for j=1,2,… and S(3j)≡0 (mod 3) for j=1,2,… .

We now return to the general case. S(n) is deconcatenated into two numbers 12…n and n… 21 from which we form the numbers

$A = 12...n \cdot 10^{1+[\log_{10} B]}$ and B=n…21

We note that this is a different way of writing S(n) since indeed A+B=S(n) and that A+B≡0 (mod p). We now form $M=A\cdot 10^s+B$ where we want to determine s so that M≡0 (mod p). We write M in the form $M=A(10^s-1)+A+B$ where A+B can be ignored mod p. We exclude the possibility A≡0 (mod p) which is not interesting. This leaves us with the congruence

$M\equiv A(10^s-1)\equiv 0 \pmod p$

or

$10^s-1\equiv 0 \pmod p$

We are particularly interested in solutions for which

$p \in \{11,37,41,43,271,9091,333667\}$

By the nature of the problem these solutions are periodic. Only the two first values of s are given for each prime.

Table 5. 10^s-1≡0 (mod p)

p	3	11	37	41	43	271	9091	333667
s	1,2	2,4	3,6	5,10	21,42	5,10	10,20	9,18

We note that the result is independent of n. This means that we can use n as a parameter when searching for a sequence

C=n+1_n+2_…n+k_n+k_…n+2_n+1

such that this is also divisible by p and hence can be inserted in place of the zeros to form S(n+k) which then fills the condition S(n+k)≡0 (mod p). Here k is a multiple of s or s/2 in case s is even. This explains the results which we have already obtained in a different way as part of the factorization of S(n) for n≤200, see tables 3 and 4. It remains to explain the periodicity which as we have seen is different in different intervals 10^u≤n≤10^u-1.

This may be best done by using concrete examples. Let us use the sequences starting with n=12 for p=37, n=12 and n=20 for p=11 and n=102 for p=333667. At the same time we will illustrate what we have done above.

Case 2: n=12, p=37. Period=3. Interval: 10≤n≤99.

```
S(n)=   123456789101112____________121110987654321
N=      123456789101112000000000000121110987654321
C=                     131415151413
S(n+k)=123456789101112131415151413121110987654321
```

Let's look at C which carries the explanation to the periodicity. We write C in the form

C=101010101010+30405050403

We know that C≡0 (mod 37). What about 101010101010? Let's write

$101010101010=10+10^3+10^5+\ldots+10^{11}=(10^{12}-1)/9\equiv 0 \pmod{37}$

This congruence mod 37 has already been established in table 5. It follows that also

30405050403≡0 (mod 37)

and that

x·(101010101010)≡0 (mod 37) for x = any integer

Combining these observations we se that

```
232425252423, 333435353433, … 939495959493≡0 (mod 37)
```

Hence the periodicity is explained.

Case 3a: n=12, p=11. Period=11. Interval: 10≤n≤99.

```
S(12)=12_.._12_____________________________________________12_.._21
S(23)=12_.._121314151617181920212223232221201918171615141312_.._21
C=            1314151617181920212223232221201918171615141 3=
C1=           10101010101010101010101010101010101010101010+
C2=            3040506070809101112131312111009080706050403
```

From this we form

```
2·C1+C2=23242526272829303132333323130292827262524 23
```

which is NOT what we wanted, but C1≡0 (mod11) and also C1/10≡0 (mod 11). Hence we form

```
2·C1+C1/10+C2=24252627282930313233343433323130292827262524
```

which is exactly the C-term required to form the next term S(34) of the sequence. For the next term S(45) the C-term is formed by 3·C1+2·C1/10+C2 The process is repeated adding C1+C1/10 to proceed from a C-term to the next until the last term <100, i.e. S(89) is reached.

Case 3b: n=20, p=11. Period=11. Interval: 10≤n≤99.

This case does not differ much from the case n=12. We have

```
S(20)=12_.._20_____________________________________________20_.._21
S(31)=12_.._2021222324252627282930313130292827262524232221 20_.._21
C=            212223242526272829303131302928272625242322 21=
C1=           10101010101010101010101010101010101010101010+
C2=            1020304050607080910111111009080706050403020 1
```

The C-term for S(42) is

```
3·C1+C1/10+C2=32333435363738394041424241403938373635343332
```

In general C=x·C1+(x-1)·C1/10+C2 for x=3,4,5, ..,8. For x=8 the last term S(97) of this sequence is reached.

Case 4: n=102, p=333667. Period=3. Interval: 100≤n≤999.

```
S(102)=12_.._101102__________________102101_.._21
S(105)=12_.._101102103104105105104103102101_.._21
C=                103104105105104103             ≡0 (mod 333667)
C1=               100100100100100100             ≡0 (mod 333667)
C2=                 3004005005004003             ≡0 (mod 333667)
```

Removing 1 or 2 zeros at the end of C1 does not affect the congruence modulus 333667, we have:

```
C1′=                10010010010010010        ≡0 (mod 333667)
C1′′=                1001001001001001        ≡0 (mod 333667)
```

We now form the combinations:

x·C1+y·C1′+z·C1′′+C2≡0 (mod 333667)

This, in my mind, is quite remarkable: All 18-digit integers formed by the concatenation of three consecutive 3-digit integers followed by a concatenation of the same integers in decending order are diivisible by 333667, example 376377378378377376≡0 (mod 333667). As far as the C-terms are concerned all S(n) in the range 100≤n≤999 could be divisible by 333667, but they are not. Why? It is because S(100) and S(101) are not divisible by 333667. Consequently n=100+3k and 101+3k can not be used for insertion of an appropriate C-value as we did in the case of S(102). This completes the explanation of the remarkable fact that every third term S(102+3j) in the range 100≤n≤999 is divisible by 333667.

These three cases have shown what causes the periodicity of the divisibility of the Smarandache symmetric sequence of the second order by primes. The mechanism is the same for the other periodic sequences.

4. Beyond 1000

We have seen that numbers of the type:

10101010…10, 100100100…100, 10001000…1000, etc

play an important role. Such numbers have been factorized and the occurrence of our favorite primes 11, 37, …,333667 have been listed in table 6. In this table a number like 100100100100 has been abbreviated 4(100) or q(E), where q and E are listed in separate columns.

Table 6. Prime factors of q(E) and occurrence of selected primes

q	E	Prime factors <350000	Selected primes
2	10	2.5.101	
3		2.3.5.7.13.37	37
4		2.5.73.101.137	
5		2.5.41.271.9091	41,271,9091
6		2.3.5.7.13.37.101.9901	37,9091
7		2.5.239.4649.	
8		2.5.17.73.101.137.	
9		$2.3^{2}.5.7.13.19.37.52579.333667$	333667
10		2.5.41.101.271.3541.9091.27961	41,271,9091
11		2.5.11.23.4093.8779.21649.	11
12		2.3.5.7.13.37.73.101.137.9901.	37
13		2.5.53.79.859.	
14		2.5.29.101.239.281.4649.	
15		2.3.5.7.13.31.37.41.211.241.271.2161.9091.	37,41,271,9091
16		2.5.17.73.101.137.353.449.641.1409.69857.	
2	10^{2}	$2^{2}.5^{2}.7.11.13$	11
3		$2^{2}.3.5^{2}.333667$	333667
4		$2^{2}.5^{2}.7.11.13.101.9901$	11
5		$2^{2}.5^{2}.31.41.271.$	41,271
6		$2^{2}.3.5^{2}.7.11.13.19.52579.333667$	11,333667
7		$2^{2}.5^{2}.43.239.1933.4649.$	43
8		$2^{2}.5^{2}.7.11.13.73.101.137.9901.$	11,73
9		$2^{2}.3^{2}.5^{2}.757.333667.$	333667
10		$2^{2}.5^{2}.7.11.13.31.41.211.241.271.2161.9091.$	11,41,271,9091
11		$2^{2}.5^{2}.67.21649.$	
12		$2^{2}.3.5^{2}.7.11.13.19.101.9901.52579.333667.$	11,333667
2	10^{3}	$2^{3}.5^{3}.73.137$	
3		$2^{3}.3.5^{3}.7.13.37.9901$	37
4		$2^{3}.5^{3}.17.73.137.$	
5		$2^{3}.5^{3}.41.271.3541.9091.27961$	41,271,9091
6		$2^{3}.3.5^{3}.7.13.37.73.137.9901.$	37
7		$2^{3}.5^{3}.29.239.281.4649.$	
8		$2^{3}.5^{3}.17.73.137.353.449.641.1409.69857.$	
9		$2^{3}.3^{2}.5^{3}.7.13.19.37.9901.52579.333667.$	37,333667
10		$2^{3}.3.5^{3}.41.73.137.271.3541.9091.27961.$	41,271,9091
11		$2^{3}.5^{3}.11.23.89.4093.8779.21649.$	11
2	10^{4}	$2^{4}.5^{4}.11.9091$	11,9091
3		$2^{4}.3.5^{4}.31.37.$	37
4		$2^{4}.5^{4}.11.101.3541.9091.27961$	11,9091
5		$2^{4}.5^{4}.21401.25601.$	
6		$2^{4}.3.5^{4}.7.11.13.31.37.211.241.2161.9091.$	11,37,9091
7		$2^{4}.5^{4}.71.239.4649.123551.$	
8		$2^{4}.5^{4}.11.73.101.137.3541.9091.27961.$	11,9091
9		$2^{4}.3.5^{4}.31.37.238681.333667.$	37,333667
2	10^{5}	$2^{5}.5^{5}.101.9901$	
3		$2^{5}.3.5^{5}.19.52579.333667$	333667
4		$2^{5}.5^{5}.73.101.137.9901..$	
5		$2^{5}.5^{5}.31.41.211.241.271.2161.9091..$	41,271,9091
6		$2^{5}.3.5^{5}.19.101.9901.52579.333667..$	333667
7		$2^{5}.5^{5}.7.43.127.239.1933.2689.4649..$	43
8		$2^{5}.5^{5}.17.73.101.137.9901..$	
9		$2^{5}.3^{2}.5^{5}.19.757.52579.333667..$	333667

Question 1. Does the sequence of terms S(n) divisible by 333667 continue beyond 1000?

Although S(n) was partially factorized only up n=200 we have been able to draw conclusions on divisibility up n=1000. The last term that we have found divisible by 333667 is S(999). Two conditions must be met for there to be a sequence of terms divisible by p=333667 in the interval $1000 \le n \le 9999$.

Condition 1. There must exist a number 10001000…1000 divisible by 333667 to ensure the periodicity as we have seen in our case studies.
In table 7 we find q=9, E=1000. This means that the periodicity will be 9 – if it exists, i.e. condition 1 is met.

Condition 2. There must exist a term S(n) with $n \ge 1000$ divisible by 333667 which will constitute the first term of the sequence.
The last term for n<1000 which is divisible by 333667 is S(999) from which we build

    S(108)=12…999_1000_…_1008_1008_…1000_999-…21

where we deconcatenate 100010011002…10081008…10011000 which is divisible by 333667 and provides the C-term (as introduced in the case studies) needed to generate the sequence, i.e. condition 2 is met.

We conclude that $S(1008+9j) \equiv 0 \pmod{333667}$ for j=0,1,2, … 999. The last term in this sequence is S(9999). From table 7 we see that there could be a sequence with the period 9 in the interval $10000 \le n \le 99999$ and a sequence with period 3 in the interval $100000 \le n \le 999999$. It is not difficult to verify that the above conditions are filled also in these intervals. This means that we have:

$S(1008+9j) \equiv 0 \pmod{333667}$ for j=01,2,…,999, i.e. $10^3 \le n \le 10^4-1$
$S(10008+9j) \equiv 0 \pmod{333667}$ for j=01,2,…,9999, i.e. $10^4 \le n \le 10^5-1$
$S(100002+3j) \equiv 0 \pmod{333667}$ for j=01,2,…,99999, i.e. $10^5 \le n \le 10^6-1$

It is one of the fascinations with large numbers to find such properties. This extraordinary property of the prime 333667 in relation to the Smarandache symmetric sequence probably holds for $n>10^6$. It easy to loose contact with reality when plying with numbers like this. We have $S(999999) \equiv 0 \pmod{333667}$. What does this number S(999999) look like? Applying (1) we find that the number of digits D(999999) of S(999999) is

$$D(999999)=2\cdot 6\cdot 10^6-2\cdot(10^6-)/9=11777778$$

Let's write this number with 80 digits per line, 60 lines per page, using both sides of the paper. We will need 1226 sheets of paper – more that 2 reams!

Question 2. Why is there no sequence of S(n) divisible by 11 in the interval 100≤n≤999?

Condition1. We must have a sequence of the form 100100.. divisible by 11 to ensure the periodicity. As we can see from table 7 the sequence 100100 fills the condition and we would have a periodicity equal to 2 if the next condition is met.

Condition 2. There must exist a term S(n) with n≥100 divisible by 11 which would constitute the first term of the sequence. This time let's use a nice property of the prime 11:

$10^s \equiv (-1)^s \pmod{11}$

Let's deconcatenate the number a_b corresponding to the concatenation of the numbers a and b: We have:

$$a\_b = a \cdot 10^{1+[\log_{10} b]} + b = \begin{cases} -a+b & \text{if } 1+[\log_{10} b] \text{ is odd} \\ a+b & \text{if } 1+[\log_{10} b] \text{ is even} \end{cases}$$

Let's first consider a deconcatenated middle part of S(n) where the concatenation is done with three-digit integers. For convienience I have chosen a concrete example – the generalization should pose no problem

```
273274275275274273≡2-7+3-2+7-4+2-7+5-2+7-5+2-7+4-2+7-3≡0 (mod 11)
+-+-+-+-+-+-+-+-+-
```

It is easy to see that this property holds independent of the length of the sequence above and whether it start on + or -. It is also easy to understand that equivalent results are obtained for other primes although factors other than +1 and –1 will enter into the picture.

We now return to the question of finding the first term of the sequence. We must start from n=97 since S(97) it the last term for which we know that S(n)≡0 (mod 11). We form:

```
9899100101…n_n…1011009998≡2 (mod 11) independent of n<1000.
+-+-+-+-+-… _ …-+-+-+-+-+-
```

This means that S(n)≡2 (mod 11) for 100≤n≤999 and explains why there is no sequence divisible by 11 in this interval.

Question 3. Will there be a sequence divisible by 11 in the interval 1000≤n≤9999?

Condition 1. A sequence 10001000…1000 divisible by 11 exists and would provide a period of 11, se table 6.

Condition 2. We need to find one value n ≥1000 for which S(n)≡0 (mod 11). We have seen that S(999)≡2 (mod 11). We now look at the sequences following S(999). Since S(999)≡2 (mod 9) we need to insert a sequence 10001001..m_m…10011000≡9 (mod 11) so that S(m)≡0 (mod 11). Unfortunately m does not exist as we will see below

```
10001000≡2 (mod 11)
+-+-+-+-
1   1
1000100110011000≡2. (mod 11)
+-+-+-+-+-+-+-+-
1   1   1   1
      1   1
100010011002100210011000≡0 (mod 11)
+-+-+-+-+-+-+-+-+-+-+-+-
1   1   1   1   1   1
      1   2   2   1
10001001100210031003100210011000≡-4≡7 (mod 11)
+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+
1   1   1   1   1   1   1   1
      1   2   3   3   2   1
```

Continuing this way we find that the residues form the period 2,2,0,7,1,4,5,4,1,7,0. We needed a residue to be 9 in order to build sequences divisible by 9. We conclude that S(n) is not divisible by 11 in the interval 1000≤n≤9999.

Trying to do the above analysis with the computer programs used in the early part of this study causes overflow because the large integers involved. However, changing the approach and performing calculations modulus 11 posed no problems. The above method was preferred for clarity of presentation.

5. Epilog

There are many other questions which may be interesting to look into. This is left to the reader. The author's main interest in this has been to develop means by which it is possible to identify some properties of large numbers other than the so frequently asked question as to whether a big number is a prime or not. There are two important ways to generate large numbers which I found particularly interesting – iteration and concatenation. In this article the author has drawn on work done previously, references below. In both these areas very large numbers may be generated for which it may be impossible to find any practical use – the methods are often more important than the results.

References:

1. Tabirca, S. and T., *On Primality of the Smarandache Symmetic Sequences,* Smarandache Notions Journal, Vol. 12, No 1-3 Spring 2001, 114-121.
2. Smarandache F., *Only Problems, Not Solutions*, Xiquan Publ., Pheonix-Chicago, 1993.
3. Ibstedt H. *Surfing on the Ocean of Numbers,* Erhus University Press, Vail, 1997.
4. Ibstedt H, *Some Sequences of Large Integers,* Fibonacci Quarterly, 28(1990), 200-203.

The author treats various topics in ten chapters which can be read independently:

- Which is the smallest integer that can be expressed as a sum of consecutive integers in a given number of ways?

- Alternating iterations of the Smarandache function and the Euler φ-function respectively the sum of divisors function. Some light is thrown on loops and invariants resulting from these iterations. An important question is resolved with the amazing involvement of the famous Fermat numbers.

- One of the problems in R.K. Guy's book "Unsolved questions in Number Theory" is explained. An interesting sequence where the first 600 terms are integers but not the 601st is shown.

- A particularly interesting subject is the Smarandache partial perfect additive sequence, it has a simple definition and a strange behaviour.

- Smarandache general continued fractions are treated in great detail and proof is given for the convergence under specified conditions.

- Smarandache k-k additive relationships as well as subtractive relationships are treated with some observations on the occurrence of prime twins.

- A substantial part is devoted to concatenation and deconcatenation problems. Some divisibilty properties of very large numbers is studied. In particular some questions raised on the Smarandache deconstructive sequence are resolved.

$19.95