Modelling Uncertain Implication Rules in Evidence Theory

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Abstract— **Often knowledge from human experts is expressed in the form of uncertain implication rules, such as "if A then B" with a certain degree of confidence. Therefore, it is important to include such a representation of knowledge into the process of reasoning under uncertainty. Incorporating inference rules based on the material implication of propositional logic into the evidence theory framework (e.g. Dempster-Shafer, Smets, Dezert-Smarandache theory etc.) is conceptually simple, which is not the case with classical probability theory. In this paper, a transformation for converting uncertain implication rules into an evidence theory framework will be presented and it will be shown that it satisfies the main properties of logical implication, namely reflexivity, transitivity and contrapositivity.**

Keywords: uncertain implication rules, evidence theory, Dempster-Shafer, logical axioms.

I. Introduction

Often knowledge from human experts is expressed in the form of uncertain implication rules, such as "if A then B" with a certain degree of confidence. There are essentially two types of uncertain implication rules. An implication rule "if A then B", which holds with a probability p such that $p \in [\alpha, 1]$ and $0 \leq \alpha \leq 1$, will be referred to as *1dof* uncertain implication rule. $p \in [\alpha, 1]$ means that the implication rule is believed true with probability α and $1 - \alpha$ represents the uncertainty. The term *1dof* means that uncertainty can be represented by means of a single parameter (1 degree of freedom), i.e. α. Conversely, an implication rule 'if A then B", which holds with a probability p such that $p \in [\alpha, \beta]$, with $0 \leq \alpha \leq \beta \leq 1$ will be referred to as *2dof* uncertain implication rule. $p \in [\alpha, \beta]$ means that the implication rule is believed true with degree α, false with degree $1 - \beta$ and uncertain with degree $\beta - \alpha$. Note that, in the special case $\beta = 1$, the *2dof* uncertain implication rules reduce to *1dof* rules.

Since human experts represent their knowledge in the form of uncertain implication rules [1]–[4], it is important to include such a representation of knowledge into the process of reasoning under uncertainty. Incorporating inference rules based on the material implication of propositional logic into evidence theory is conceptually simple, which is not the case with classical probability theory. Assuming that the probability of the implication "if A then B" is equal to the conditional probability of B given A leads to the trivialization results of Lewis, i.e., the probability can then only take three values [5]. Under this assumption, probability theory degenerates into a useless theory. In the evidence theory framework, on the contrary, one can assume the equality between the implication and the conditional belief and still avoid the trivialization [3].

An implication rule can be expressed in evidence theory by means of a Basic Belief Assignment (BBA) function using the principle of minimum commitment [3] and its instantiation referred to as the ballooning extension [3]. Transformations to convert *1dof* and *2dof* uncertain implication rules into the evidence theory framework have been presented in [3] and, respectively, [6].

In this paper, it is shown that the transformation in [6] for *2dof* uncertain implication rules does not satisfy two main properties of logical implication, namely *transitivity* and *contrapositivity*. A new transformation for converting *2dof* uncertain implication rules into BBA functions is presented and it is proved that, in contrast, this transformation satisfies transitivity and contrapositivity. The fulfilment of these properties guarantees that the transformation is coherent, in the sense that the logical axioms continue to hold also in the BBA function domain and, thus, that the transformation does not alter the information. Maintaining coherence of information is a fundamental requirement in any data fusion process. Notice that, since the Dezert-Smarandache theory includes the Dempster-Shafer theory as a particular case and it reduces to the Dempster-Shafer theory when the exclusivity of all elements in the frame is assumed [7], the transformation to convert uncertain implication rules into BBA functions can also be used in the Dezert-Smarandache theory framework.

II. Background on Evidence theory

Let Θ_x denote the set of possible values (the *frame*) of a variable x. When modelling real world problems we often deal with many interrelated variables and the resulting joint frame is multi-dimensional. All the variables considered throughout the paper have finite frames. Let $\mathbf{D}$ denote a set of variables. The Cartesian product $\Theta_{\mathbf{D}} \stackrel{\triangle}{=} \times\{\Theta_x : x \in \mathbf{D}\}$ is the frame of $\mathbf{D}$. The elements of $\Theta_{\mathbf{D}}$ are called the *configurations* of $\mathbf{D}$.

The beliefs [8] about the true value of $\mathbf{D}$ are expressed on the subsets of $\Theta_{\mathbf{D}}$. The basic belief assignment (BBA) $m^{\mathbf{D}}$ on domain $\mathbf{D}$ is a *multivariate* belief function which assigns to every subset A of $\Theta_{\mathbf{D}}$ a value in $[0,1]$, i.e. $m^{\mathbf{D}}: 2^{\Theta_{\mathbf{D}}} \rightarrow [0,1]$. The following condition is assumed to be satisfied: $\sum_{A\subseteq\Theta_{\mathbf{D}}} m^{\mathbf{D}}(A) = 1$. The subsets A such that $m^{\mathbf{D}}(A) > 0$ are referred to as the focal elements of the BBA. The state of complete ignorance about the set of variables $\mathbf{D}$ is represented by a *vacuous* BBA defined as $m^{\mathbf{D}}(A) = 1$ if $A = \Theta_{\mathbf{D}}$ and zero otherwise.

Three basic operations on multivariate belief functions are of interest here: *vacuous extension*, *marginalization* and *combination* [8].

Vacuous extension of a BBA defined on domain $\mathbf{D}'$, to a larger domain $\mathbf{D} \supseteq \mathbf{D}'$ is defined as [8]

$$m^{\mathbf{D}'\uparrow\mathbf{D}}(B) = \begin{cases} m^{\mathbf{D}'}(A) & \text{if } B = A^{\mathbf{D}'\uparrow\mathbf{D}} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Marginalization is a projection of a BBA defined on $\mathbf{D}$ into a BBA defined on a coarser domain $\mathbf{D}' \subseteq \mathbf{D}$:

$$m^{\mathbf{D}\downarrow\mathbf{D}'}(A) = \sum_{B:B^{\downarrow\mathbf{D}'}=A} m^{\mathbf{D}}(B) \tag{2}$$

Combination of two BBAs $m_1^{\mathbf{D}_1}$ and $m_2^{\mathbf{D}_2}$ is carried out using the Dempster's rule of combination [8]. However, before applying the Dempster's rule, the vacuous extension of both $m_1^{\mathbf{D}_1}$ and $m_2^{\mathbf{D}_2}$ to the joint domain $\mathbf{D} = \mathbf{D}_1 \cup \mathbf{D}_2$ is needed. The result will be a BBA $m_{12}^{\mathbf{D}}$, defined on domain $\mathbf{D}$, as follows:

$$\begin{aligned} [m_1^{\mathbf{D}_1} \oplus m_2^{\mathbf{D}_2}](A) &= m_{12}^{\mathbf{D}} \\ &= \alpha \sum_{B,C:B^{\uparrow\mathbf{D}}\cap C^{\uparrow\mathbf{D}}=A} m_1^{\mathbf{D}_1}(B)\cdot m_2^{\mathbf{D}_2}(C) \end{aligned} \tag{3}$$

where α is the normalisation constant.

III. Implication Rule

Suppose there are two disjoint domains, $\mathbf{D}_1$ and $\mathbf{D}_2$, with associated frames $\Theta_{\mathbf{D}_1}$ and $\Theta_{\mathbf{D}_2}$ respectively. Formally, an implication rule is an expression of the form

$$A \;\Rightarrow\; B \tag{4}$$

where $A \subseteq \Theta_{\mathbf{D}_1}$ and $B \subseteq \Theta_{\mathbf{D}_2}$. The rule $A \Rightarrow B$ means that every expression that satisfies A also satisfies B. In words, B is true for any expression that satisfies A, i.e. the following truth table holds

A	B	$A \Rightarrow B$
1	1	1
1	0	0
0	1	1
0	0	1

(5)

The logical implication has two important properties:

- **reflexivity:** $A \Rightarrow A$ is always true (tautology);
- **transitivity:** if $A \Rightarrow B$ and $B \Rightarrow C$, then $A \Rightarrow C$.

From the below table of truth (6), it can be seen that $A \Rightarrow B$ is logically equivalent to $\bar{B} \Rightarrow \bar{A}$. In the sequel, this property will be referred to as *contrapositivity*.

A	B	$\bar{A}$	$\bar{B}$	$A \Rightarrow B$	$\bar{B} \Rightarrow \bar{A}$
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

(6)

An uncertain implication rule is an implication rule (4) which is true in a certain percentage of cases, i.e. with a probability (confidence) $p \in [\alpha, 1]$, with $0 \leq \alpha \leq 1$, for *1dof* uncertain implication rules or $p \in [\alpha, \beta]$, with $0 \leq \alpha \leq \beta \leq 1$, for *2dof* uncertain implication rules.

IV. Implication Rules in Evidence Theory

Evidence theory can manage uncertain implication rules in a direct way, i.e. there is a one-to-one mapping between uncertain implication rules and BBA functions. Transformations for converting *1dof* and *2dof* uncertain implication rules into the evidence theory framework have been presented in [3] and, respectively, [1], [6]. In the following sections, these transfomations will be discussed in detail. It will be shown that the transformation presented in [3] is consistent with the axioms of logical implication, i.e. *reflexivity*, *transitivity* and *contrapositivity*. However, *transitivity* and *contrapositivity* properties are not satisfied by the transformation discussed in [1], [6] for *2dof* uncertain implication rules. A new transformation for modelling *2dof* uncertain implication rules within evidence theory will be presented and it will be shown that it satisfies all the axioms of logical implication.

It is important that these axioms be satisfied for the following reasons. The fulfillment of these properties guarantees that the transformation is coherent, in the sense that the logical axioms continue to hold also in the BBA function domain. The transitivity property guarantees that the following procedures are equivalent:

- exploit the semantics of propositional logic to infer that $A \Rightarrow B$ and $B \Rightarrow C$ imply $A \Rightarrow C$ and, thus, convert $A \Rightarrow C$ into a BBA function;
- convert both $A \Rightarrow B$ and $B \Rightarrow C$ into BBA functions and then exploit Dempster's rule of combination to infer that the two transformed BBA functions imply $A \Rightarrow C$.

A. 1dof implication rule

A *1dof* implication rule can be expressed as a BBA function consisting of 2 focal sets on the joint domain $\mathbf{D}_1 \cup \mathbf{D}_2$ [3], [9]:

$$m^{\mathbf{D}_1\cup\mathbf{D}_2}(C) = \begin{cases} \alpha, & \text{if } C = (A \times B) \cup (\overline{A} \times \Theta_{\mathbf{D}_2}) \\ 1-\alpha, & \text{if } C = \Theta_{\mathbf{D}_1\cup\mathbf{D}_2} \end{cases} \tag{7}$$

where $\overline{A}$ is the complement of A in $\Theta_{\mathbf{D}_1}$. The BBA in (7) can be derived in a straightforward way from the table of truth (5). In fact, by definition, $A \Rightarrow B$ is true when:

1) A and B are both true;
2) A is false and B is true or false.

Therefore, the mass α is assigned to the elements of the joint frame $\Theta_{\mathbf{D}_1\cup\mathbf{D}_2}$ which correspond to the previous listed cases, i.e $A\times B$ (A and B are both true) and, respectively, $\overline{A}\times\Theta_{\mathbf{D}_2}$ (A false and B does not matter). The rest of the mass, $1-\alpha$, is given to the universe set.

Example 1: - Let $\mathbf{D}_1 = \{x\}$, $\mathbf{D}_2 = \{y\}$, $\Theta_x = \{x_1, x_2, x_3\}$, $\Theta_y = \{y_1, y_2, y_3\}$, $A = \{x_1, x_2\}$ and $B = \{y_2\}$. Then the BBA representation of the rule $A \Rightarrow B$ with confidence $p \in [\alpha, 1]$ is given by:

$$\begin{aligned} m^{\{x,y\}} \quad & (\{(x_1\, y_2), (x_2\, y_2), (x_3\, y_1), (x_3\, y_2), (x_3\, y_3)\}) = \alpha \\ m^{\{x,y\}} \quad & (\{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), \\ & (x_2, y_3), (x_3, y_1), (x_3, y_2), (x_3, y_3)\}) = 1-\alpha. \end{aligned} \tag{8}$$

■

It can be proved that (7) satisfies the reflexivity, transitivity and contrapositivity properties of implication rules. A proof will be given later in the section. Here only the reflexivity property is discussed. Rule $A \Rightarrow A$ is a tautology, that is a proposition which is always true and so does not carry any additional information. The correct way to represent this statement in the evidence theory framework should be a vacuous belief function.

Theorem 1: - The rule (7) satisfies the reflexivity property.

Proof: - Considering $D_1 = D_2$ and applying (7) to $A \Rightarrow A$, one gets

$$m^{\mathbf{D}_1\cup\mathbf{D}_2}(C) = \begin{cases} \alpha, & \text{if } C = (A\times A)\cup(\overline{A}\times\Theta_{\mathbf{D}_2}) \\ 1-\alpha, & \text{if } C = \Theta_{\mathbf{D}_1}\times\Theta_{\mathbf{D}_2} \end{cases} \tag{9}$$

Being $D_2 = D_1$, it results that $\Theta_{\mathbf{D}_2} = \Theta_{\mathbf{D}_1} = A\cup\overline{A}$ and thus

$$(A\times A)\cup(\overline{A}\times\Theta_{\mathbf{D}_2}) = (A\times A)\cup(\overline{A}\times A)\cup(\overline{A}\times\overline{A})$$

Marginalizing the BBA (9) defined in the domain $\mathbf{D}_1\cup\mathbf{D}_2$ to the domain D_1 (or equivalently to the domain D_2), one gets

$$m^{\mathbf{D}_1}(C) = m^{\mathbf{D}_2}(C) = \begin{cases} 1, & \textit{if } C = \Theta_{\mathbf{D}_1} = \Theta_{\mathbf{D}_2} \\ 0, & \textit{otherwise} \end{cases} \tag{10}$$

which is a vacuous belief function on the frame $\Theta_{\mathbf{D}_1} = \Theta_{\mathbf{D}_2}$. Therefore, the reflexivity property is satisfied. ■

B. *2dof implication rule*

The transformation of an implication rule into a BBA is more complicated for *2dof* uncertain implication rules. In [6], a transformation to represent this kind of uncertain implication rules within the evidence theory framework has been presented. This representation is defined in the following way:

$$m^{\mathbf{D}_1\cup\mathbf{D}_2}(C) = \begin{cases} \alpha, & \text{if } C = (A\times B)\cup(\overline{A}\times\Theta_{\mathbf{D}_2}) \\ 1-\beta, & \text{if } C = (A\times\overline{B})\cup(\overline{A}\times\Theta_{\mathbf{D}_2}) \\ \beta-\alpha, & \text{if } C = \Theta_{\mathbf{D}_1\cup\mathbf{D}_2} \end{cases} \tag{11}$$

where $\overline{A}$ is the complement of A in $\Theta_{\mathbf{D}_1}$, and accordingly $\overline{B}$ is the complement of B in $\Theta_{\mathbf{D}_2}$. Note that the mass $1-\beta$ has been assigned to $(A\times\overline{B})\cup(\overline{A}\times\Theta_{\mathbf{D}_2})$.

Example 2: - Consider again example 1. In the case $p \in [\alpha, \beta]$, applying (11), the following BBA can be obtained:

$$\begin{aligned} m^{\{x,y\}} \quad & (\{(x_1\, y_2), (x_2\, y_2), (x_3\, y_1), (x_3\, y_2), (x_3\, y_3)\}) = \alpha \\ m^{\{x,y\}} \quad & (\{(x_1\, y_1), (x_1, y_3), (x_2\, y_1), (x_2, y_3), (x_3\, y_1), \\ & (x_3\, y_2), (x_3\, y_3)\}) = 1-\beta \\ m^{\{x,y\}} \quad & (\{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), \\ & (x_2, y_3), (x_3, y_1), (x_3, y_2), (x_3, y_3)\}) = \beta-\alpha. \end{aligned} \tag{12}$$

■

However, as it will be proved in the sequel [10], this transformation does not satisfy transitivity and contrapositivity properties. In this paper, a new transformation, referred to hereafter as *new rule*, is introduced and it will be proved that it fulfills transitivity and contrapositivity [1]. The *new rule* [10] is defined as follows:

$$m^{\mathbf{D}_1\cup\mathbf{D}_2}(C) = \begin{cases} \alpha, & \text{if } C = (A\times B)\cup(\overline{A}\times\Theta_{\mathbf{D}_2}) \\ 1-\beta, & \text{if } C = (\overline{A}\times\overline{B})\cup(A\times\Theta_{\mathbf{D}_2}) \\ \beta-\alpha, & \text{if } C = \Theta_{\mathbf{D}_1\cup\mathbf{D}_2} \end{cases} \tag{13}$$

In this case, the mass $1-\beta$ has been assigned to $C = (\overline{A}\times\overline{B})\cup(A\times\Theta_{\mathbf{D}_2})$ instead of $(A\times\overline{B})\cup(\overline{A}\times\Theta_{\mathbf{D}_2})$ as indicated in (11).

Example 3: - Consider again example 1. Applying (13), the following BBA is obtained:

$$\begin{aligned} m^{\{x,y\}} \quad & (\{(x_1\, y_2), (x_2\, y_2), (x_3\, y_1), (x_3\, y_2), (x_3\, y_3)\}) = \alpha \\ m^{\{x,y\}} \quad & (\{(x_1\, y_1), (x_1, y_2), (x_1\, y_3), (x_2, y_1), (x_2\, y_2), \\ & (x_2\, y_3), (x_3\, y_1), (x_3\, y_3)\}) = 1-\beta \\ m^{\{x,y\}} \quad & (\{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), \\ & (x_2, y_3), (x_3, y_1), (x_3, y_2), (x_3, y_3)\}) = \beta-\alpha. \end{aligned} \tag{14}$$

■

Note that the rule (11) and the *new rule* differ only in the second focal set, i.e. the element with mass $1-\beta$.

Theorem 2: - The *new rule* (13) satisfies the contrapositivity property, while rule (11) does not.

Proof - The contrapositivity property is satisfied if the two BBA functions obtained by transforming the implications $A \Rightarrow B$ and, respectively, $\bar{B} \Rightarrow \bar{A}$ are equal. The BBAs relative to the implication $A \Rightarrow B$ obtained by applying the rule (11) and the *new rule* are given in (11) and, respectively, (13). Conversely, applying the two rules to the implication $\bar{B} \Rightarrow \bar{A}$, one gets the BBAs defined below:

	focal sets	
mass	**rule (11)**	**new rule**
α	$(\bar{B}\times\bar{A})\cup(B\times\Theta_{\mathbf{D}_1})$	$(\bar{B}\times\bar{A})\cup(B\times\Theta_{\mathbf{D}_1})$
$1-\beta$	$(\overline{B}\times A)\cup(B\times\Theta_{\mathbf{D}_1})$	$(B\times A)\cup(\overline{B}\times\Theta_{\mathbf{D}_1})$
$\beta-\alpha$	$\Theta_{\mathbf{D}_2\cup\mathbf{D}_1}$	$\Theta_{\mathbf{D}_2\cup\mathbf{D}_1}$

[1] The reflexivity property is also trivially satisfied

Consider the first focal set $(\bar{B} \times \bar{A}) \cup (B \times \mathbf{\Theta}_{\mathbf{D}_1})$; since $\mathbf{\Theta}_{\mathbf{D}_1} = \overline{A} \cup A$ and $\mathbf{\Theta}_{\mathbf{D}_2} = \overline{B} \cup B$, it follows that

$$\begin{aligned}(\bar{B} \times \bar{A}) \cup (B \times \mathbf{\Theta}_{\mathbf{D}_1}) &= (\bar{B} \times \bar{A}) \cup (B \times \overline{A}) \cup (B \times A) \\ &= (\mathbf{\Theta}_{\mathbf{D}_2} \times \bar{A}) \cup (B \times A).\end{aligned}$$

By reversing the order of the variables, one gets:

$$(\mathbf{\Theta}_{\mathbf{D}_2} \times \bar{A}) \cup (B \times A) \rightarrow (A \times B) \cup (\overline{A} \times \mathbf{\Theta}_{\mathbf{D}_2})$$

which is equal to the first focal set of the BBAs in (11) and, respectively, (13). Contrapositivity is thus satisfied for the first focal set of both rules. Considering the second focal set and the rule (11), one gets:

$$(\overline{B} \times A) \cup (B \times \mathbf{\Theta}_{\mathbf{D}_1}) = (\overline{B} \times A) \cup (B \times \overline{A}) \cup (B \times A) \tag{15}$$

Reversing the order of the variables, one gets $(A \times \overline{B}) \cup (\overline{A} \times B) \cup (A \times B)$ which is different from the second focal element in (11), i.e. $(A \times \overline{B}) \cup (\overline{A} \times B) \cup (\overline{A} \times \overline{B})$. The rule (11) does not satisfy the contrapositivity property. Conversely, if the *new rule* is considered, one gets

$$\begin{aligned}(B \times A) \cup (\overline{B} \times \mathbf{\Theta}_{\mathbf{D}_1}) &= (B \times A) \cup (\overline{B} \times \overline{A}) \cup (\overline{B} \times A) \\ &= (\overline{B} \times \overline{A}) \cup (\mathbf{\Theta}_{\mathbf{D}_2} \times A).\end{aligned}$$

In this case, by reversing variable order, one has $(\overline{A} \times \overline{B}) \cup (A \times \mathbf{\Theta}_{\mathbf{D}_2})$ which coincides with the second focal set in (13). Finally, by reversing the variables order, it can be seen that also the third focal set $\mathbf{\Theta}_{\mathbf{D}_2 \cup \mathbf{D}_1}$ is equivalent to $\mathbf{\Theta}_{\mathbf{D}_1 \cup \mathbf{D}_2}$ and, thus, contrapositivity is satisfied by the new rule. ∎

It can be seen that, since the first and third focal sets of the *new rule* (13) coincide with the focal sets of the rule (7), also the transformation for the 1dof implication rule satisfies contrapositivity.

Theorem 3: - The *new rule* satisfies the transitivity property, while rule (11) does not.

Proof - The transitivity property means that $A \Rightarrow B$ and $B \Rightarrow C$ implies $A \Rightarrow C$ where $A \subseteq \mathbf{\Theta}_{\mathbf{D}_1}$, $B \subseteq \mathbf{\Theta}_{\mathbf{D}_2}$ and $C \subseteq \mathbf{\Theta}_{\mathbf{D}_3}$. In terms of BBA functions, we state that the transitivity property is satisfied if

$$(m_1^{\mathbf{D}_1 \cup \mathbf{D}_2} \oplus m_2^{\mathbf{D}_2 \cup \mathbf{D}_3})^{\downarrow \{\mathbf{D}_1 \cup \mathbf{D}_3\}} = m_3^{\mathbf{D}_1 \cup \mathbf{D}_3} \tag{16}$$

where: $m_1^{\mathbf{D}_1 \cup \mathbf{D}_2}$ is the BBA obtained from $A \Rightarrow B$ with $p \in [\alpha_1, \beta_1]$; $m_2^{\mathbf{D}_2 \cup \mathbf{D}_3}$ is the BBA obtained from $B \Rightarrow C$ with $p \in [\alpha_2, \beta_2]$; $m_3^{\mathbf{D}_1 \cup \mathbf{D}_3}$ is the BBA obtained from $A \Rightarrow C$ with $p \in [\alpha_3, \beta_3]$. The objective is to prove that *new rule* satisfies the transitivity property. From (13), it turns out that:

$$m_1^{\mathbf{D}_1 \cup \mathbf{D}_2}(E) = \begin{cases} \alpha_1 & E = (A \times B) \cup (\overline{A} \times \mathbf{\Theta}_{\mathbf{D}_2}) \\ 1 - \beta_1 & E = (\overline{A} \times \overline{B}) \cup (A \times \mathbf{\Theta}_{\mathbf{D}_2}) \\ \beta_1 - \alpha_1 & E = \mathbf{\Theta}_{\mathbf{D}_1 \cup \mathbf{D}_2} \end{cases} \tag{17}$$

$$m_2^{\mathbf{D}_2 \cup \mathbf{D}_3}(E) = \begin{cases} \alpha_2 & E = (B \times C) \cup (\overline{B} \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ 1 - \beta_2 & E = (\overline{B} \times \overline{C}) \cup (B \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ \beta_2 - \alpha_2 & E = \mathbf{\Theta}_{\mathbf{D}_2 \cup \mathbf{D}_3} \end{cases} \tag{18}$$

$$m_3^{\mathbf{D}_1 \cup \mathbf{D}_3}(E) = \begin{cases} \alpha_3 & E = (A \times C) \cup (\overline{A} \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ 1 - \beta_3 & E = (\overline{A} \times \overline{C}) \cup (A \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ \beta_3 - \alpha_3 & E = \mathbf{\Theta}_{\mathbf{D}_1 \cup \mathbf{D}_3} \end{cases} \tag{19}$$

The BBAs $m^{\mathbf{D}_1 \cup \mathbf{D}_2}$ and $m^{\mathbf{D}_2 \cup \mathbf{D}_3}$ can be combined by applying Dempster's combination rule (3). Since the combination of belief functions can only be carried out on the same frame, before applying Dempster's rule, the BBAs $m^{\mathbf{D}_1 \cup \mathbf{D}_2}$ and $m^{\mathbf{D}_2 \cup \mathbf{D}_3}$ must be extended to the common frame $\mathbf{\Theta}_{\mathbf{D}_1 \cup \mathbf{D}_2 \cup \mathbf{D}_3}$. Applying the vacuous extension operator defined in (1), one gets

$$m_1^{\mathbf{D}_1 \cup \mathbf{D}_2 \cup \mathbf{D}_3}(E) = \begin{cases} \alpha_1 & E = (A \times B \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ & \quad \cup\ (\overline{A} \times \mathbf{\Theta}_{\mathbf{D}_2} \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ 1 - \beta_1 & E = (\overline{A} \times \overline{B} \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ & \quad \cup\ (A \times \mathbf{\Theta}_{\mathbf{D}_2} \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ \beta_1 - \alpha_1 & E = \mathbf{\Theta}_{\mathbf{D}_1 \cup \mathbf{D}_2 \cup \mathbf{D}_3} \end{cases} \tag{20}$$

$$m_2^{\mathbf{D}_1 \cup \mathbf{D}_2 \cup \mathbf{D}_3}(E) = \begin{cases} \alpha_2 & E = (\mathbf{\Theta}_{\mathbf{D}_1} \times B \times C) \\ & \quad \cup\ (\mathbf{\Theta}_{\mathbf{D}_1} \times \overline{B} \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ 1 - \beta_2 & E = (\mathbf{\Theta}_{\mathbf{D}_1} \times \overline{B} \times \overline{C}) \\ & \quad \cup\ (\mathbf{\Theta}_{\mathbf{D}_1} \times B \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ \beta_2 - \alpha_2 & E = \mathbf{\Theta}_{\mathbf{D}_1 \cup \mathbf{D}_2 \cup \mathbf{D}_3} \end{cases} \tag{21}$$

Exploiting the fact that $\mathbf{\Theta}_{\mathbf{D}_1} = A \cup \overline{A}$ and $\mathbf{\Theta}_{\mathbf{D}_3} = C \cup \overline{C}$, the result of the combination of the above BBAs on the frame $\mathbf{\Theta}_{\mathbf{D}_1 \cup \mathbf{D}_2 \cup \mathbf{D}_3}$ is given in table I. Finally, by applying the marginalization operator defined in (2), the BBA $m_1 \oplus m_2$ can be marginalized to the frame $\mathbf{\Theta}_{\mathbf{D}_1 \cup \mathbf{D}_3}$; the resulting BBA is shown in table II . Note that $(m_1 \oplus m_2)^{\downarrow \mathbf{D}_1 \cup \mathbf{D}_3}$ has the same focal sets of m_3 in (19), i.e.

$$\begin{aligned}(A \times C) \cup (\overline{A} \times C) \cup (\overline{A} \times \overline{C}) &= (A \times C) \cup (\overline{A} \times \mathbf{\Theta}_{\mathbf{D}_3}) \\ (A \times C) \cup (A \times \overline{C}) \cup (\overline{A} \times \overline{C}) &= (\overline{A} \times \overline{C}) \cup (A \times \mathbf{\Theta}_{\mathbf{D}_3})\end{aligned} \tag{22}$$

Further, note that $(m_1 \oplus m_2)^{\downarrow \mathbf{D}_1 \cup \mathbf{D}_3}$ is equal to m_3 if

$$\alpha_3 = \alpha_1 \alpha_2, \quad 1 - \beta_3 = (1 - \beta_1)(1 - \beta_2) \tag{23}$$

When the knowledge on $A \Rightarrow C$ is only derived by the knowledge on $A \Rightarrow B$ and $B \Rightarrow C$, these conditions are always satisfied. In fact, in this case, the probability interval $[\alpha_3, \beta_3]$ for $A \Rightarrow C$ is chosen only according to the probability intervals $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$. Since the probability that $A \Rightarrow B$ and $B \Rightarrow C$ are both true is $\alpha_1 \alpha_2$ and the probability that they are both false is $(1 - \beta_1)(1 - \beta_2)$, it follows that $\alpha_3 = \alpha_1 \alpha_2$ and $1 - \beta_3 = (1 - \beta_1)(1 - \beta_2)$. Thus, (23) is always satisfied in this case. Conversely, it can be proved in a similar way that the rule (11) does not satisfy the condition (22). In fact, in this case, from (11) it turns out that:

$$m_1^{\mathbf{D}_1 \cup \mathbf{D}_2}(E) = \begin{cases} \alpha_1 & E = (A \times B) \cup (\overline{A} \times \mathbf{\Theta}_{\mathbf{D}_2}) \\ 1 - \beta_1 & E = (A \times \overline{B}) \cup (\overline{A} \times \mathbf{\Theta}_{\mathbf{D}_2}) \\ \beta_1 - \alpha_1 & E = \mathbf{\Theta}_{\mathbf{D}_1 \cup \mathbf{D}_2} \end{cases} \tag{24}$$

TABLE I

RESULT OF THE COMBINATION OF THE BBAS m_1 AND m_2 ON THE FRAME $\Theta_{\mathbf{D}_1\cup\mathbf{D}_2\cup\mathbf{D}_3}$ - BASED ON RULE (13)

$$m_1 \oplus m_2 = \begin{cases} \alpha_1\alpha_2 & (A\times B\times C)\cup(\overline{A}\times B\times C)\cup(\overline{A}\times\overline{B}\times C)\cup(\overline{A}\times\overline{B}\times\overline{C}) \\ (1-\beta_1)(1-\beta_2) & (A\times B\times C)\cup(A\times B\times\overline{C})\cup(A\times\overline{B}\times\overline{C})\cup(\overline{A}\times\overline{B}\times\overline{C}) \\ \alpha_1(1-\beta_2) & (A\times B\times C)\cup(A\times B\times\overline{C})\cup(\overline{A}\times B\times C)\cup(\overline{A}\times B\times\overline{C}) \\ & \cup(\overline{A}\times\overline{B}\times\overline{C}) \\ (1-\beta_1)\alpha_2 & (A\times B\times C)\cup(A\times\overline{B}\times C)\cup(A\times\overline{B}\times\overline{C})\cup(\overline{A}\times\overline{B}\times C) \\ & \cup(\overline{A}\times\overline{B}\times\overline{C}) \\ (\beta_1-\alpha_1)\alpha_2 & (A\times B\times C)\cup(A\times\overline{B}\times C)\cup(A\times\overline{B}\times\overline{C})\cup(\overline{A}\times B\times C) \\ & \cup(\overline{A}\times\overline{B}\times C)\cup(\overline{A}\times\overline{B}\times\overline{C}) \\ (\beta_2-\alpha_2)\alpha_1 & (A\times B\times C)\cup(A\times B\times\overline{C})\cup(\overline{A}\times B\times C)\cup(\overline{A}\times B\times\overline{C}) \\ & \cup(\overline{A}\times\overline{B}\times C)\cup(\overline{A}\times\overline{B}\times\overline{C}) \\ (\beta_2-\alpha_2)(1-\beta_1) & (A\times B\times C)\cup(A\times B\times\overline{C})\cup(A\times\overline{B}\times C)\cup(A\times\overline{B}\times\overline{C}) \\ & \cup(\overline{A}\times\overline{B}\times C)\cup(\overline{A}\times\overline{B}\times\overline{C}) \\ (\beta_1-\alpha_1)(1-\beta_2) & (A\times B\times C)\cup(A\times B\times\overline{C})\cup(A\times\overline{B}\times\overline{C})\cup(\overline{A}\times B\times C) \\ & \cup(\overline{A}\times B\times\overline{C})\cup(\overline{A}\times\overline{B}\times\overline{C}) \\ (\beta_1-\alpha_1)(\beta_2-\alpha_2) & \Theta_{\mathbf{D}_1\cup\mathbf{D}_2\cup\mathbf{D}_3} \end{cases}$$

TABLE II

RESULT OF THE MARGINALIZATION OF $m_1 \oplus m_2$ TO $\Theta_{\mathbf{D}_1\cup\mathbf{D}_3}$ - BASED ON RULE (13)

$$(m_1 \oplus m_2)^{\downarrow\mathbf{D}_1\cup\mathbf{D}_3} = \begin{cases} \alpha_1\alpha_2 & (A\times C)\cup(\overline{A}\times C)\cup(\overline{A}\times\overline{C}) \\ (1-\beta_1)(1-\beta_2) & (A\times C)\cup(A\times\overline{C})\cup(\overline{A}\times\overline{C}) \\ \beta_1+\beta_2-\alpha_1\alpha_2-\beta_1\beta_2 & \Theta_{\mathbf{D}_1\cup\mathbf{D}_3} \end{cases}$$

$$m_2^{\mathbf{D}_2\cup\mathbf{D}_3}(E) = \begin{cases} \alpha_2 & E=(B\times C)\cup(\overline{B}\times\Theta_{\mathbf{D}_3}) \\ 1-\beta_2 & E=(B\times\overline{C})\cup(\overline{B}\times\Theta_{\mathbf{D}_3}) \\ \beta_2-\alpha_2 & E=\Theta_{\mathbf{D}_2\cup\mathbf{D}_3} \end{cases} \tag{25}$$

$$m_3^{\mathbf{D}_1\cup\mathbf{D}_3}(E) = \begin{cases} \alpha_3 & E=(A\times C)\cup(\overline{A}\times\Theta_{\mathbf{D}_3}) \\ 1-\beta_3 & E=(A\times\overline{C})\cup(\overline{A}\times\Theta_{\mathbf{D}_3}) \\ \beta_3-\alpha_3 & E=\Theta_{\mathbf{D}_1\cup\mathbf{D}_3} \end{cases} \tag{26}$$

Again, by applying the Dempster's combination rule (3), the BBAs $m^{\mathbf{D}_1\cup\mathbf{D}_2}$ and $m^{\mathbf{D}_2\cup\mathbf{D}_3}$ can be combined on the extended domain $\mathbf{D}_1\cup\mathbf{D}_2\cup\mathbf{D}_3$, i.e.

$$m_1^{\mathbf{D}_1\cup\mathbf{D}_2\cup\mathbf{D}_3}(E) = \begin{cases} \alpha_1 & E = (A\times B\times\Theta_{\mathbf{D}_3}) \\ & \quad\cup\ (\overline{A}\times\Theta_{\mathbf{D}_2}\times\Theta_{\mathbf{D}_3}) \\ 1-\beta_1 & E = (A\times\overline{B}\times\Theta_{\mathbf{D}_3}) \\ & \quad\cup\ (\overline{A}\times\Theta_{\mathbf{D}_2}\times\Theta_{\mathbf{D}_3}) \\ \beta_1-\alpha_1 & E = \Theta_{\mathbf{D}_1\cup\mathbf{D}_2\cup\mathbf{D}_3} \end{cases} \tag{27}$$

$$m_2^{\mathbf{D}_1\cup\mathbf{D}_2\cup\mathbf{D}_3}(E) = \begin{cases} \alpha_2 & E = (\Theta_{\mathbf{D}_1}\times B\times C) \\ & \quad\cup\ (\Theta_{\mathbf{D}_1}\times\overline{B}\times\Theta_{\mathbf{D}_3}) \\ 1-\beta_2 & E = (\Theta_{\mathbf{D}_1}\times B\times\overline{C}) \\ & \quad\cup\ (\Theta_{\mathbf{D}_1}\times\overline{B}\times\Theta_{\mathbf{D}_3}) \\ \beta_2-\alpha_2 & E = \Theta_{\mathbf{D}_1\cup\mathbf{D}_2\cup\mathbf{D}_3} \end{cases} \tag{28}$$

The result of the combination of the above BBAs on the frame $\Theta_{\mathbf{D}_1\cup\mathbf{D}_2\cup\mathbf{D}_3}$ is given in table III. Finally, the result of the marginalization of the BBA $m_1\oplus m_2$ to the frame $\Theta_{\mathbf{D}_1\cup\mathbf{D}_3}$ is shown in table IV. From table IV, it can be noticed that $(m_1\oplus m_2)^{\downarrow\mathbf{D}_1\cup\mathbf{D}_3}$ has not the same focal sets of m_3 in (26). Therefore, if the rule (11) is used to transform the implications $A\Rightarrow B$ and $B\Rightarrow C$ into two BBA functions, then the transitivity property is not preserved by the Dempster's rule, i.e. $m_1\oplus m_2$ does not have the same focal sets of the BBA m_3 obtained by applying the rule (11) to $A\Rightarrow C$. Thus the *new rule* (13) satisfies the transitivity property, while the rule (11) does not. Also in this case, it can be noted that, since the first and third focal sets of the new rule (13) coincide with the focal sets of the rule (7), the transformation for 1dof implication rule satisfies the transitivity property. ■

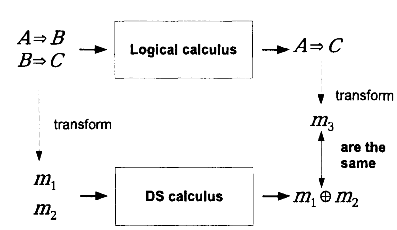

Fig. 1. Transformation (13) preserves logical axioms

TABLE III

RESULT OF THE COMBINATION OF THE BBAS m_1 AND m_2 ON THE FRAME $\mathbf{\Theta_{D_1 \cup D_2 \cup D_3}}$ - BASED ON RULE (11)

$$
m_1 \oplus m_2 = \begin{cases}
\alpha_1\alpha_2 & (A \times B \times C) \cup (\overline{A} \times B \times C) \cup (\overline{A} \times \overline{B} \times C) \cup (\overline{A} \times \overline{B} \times \overline{C}) \\
(1-\beta_1)(1-\beta_2) & (A \times \overline{B} \times C) \cup (A \times \overline{B} \times \overline{C}) \cup (\overline{A} \times B \times \overline{C}) \cup (\overline{A} \times \overline{B} \times C) \\
 & \cup (\overline{A} \times \overline{B} \times \overline{C}) \\
\alpha_1(1-\beta_2) & (A \times B \times \overline{C}) \cup (\overline{A} \times B \times \overline{C}) \cup (\overline{A} \times \overline{B} \times C) \cup (\overline{A} \times \overline{B} \times \overline{C}) \\
(1-\beta_1)\alpha_2 & (A \times \overline{B} \times C) \cup (A \times \overline{B} \times \overline{C}) \cup \cup (\overline{A} \times B \times C) \cup (\overline{A} \times \overline{B} \times C) \\
 & \cup (\overline{A} \times \overline{B} \times \overline{C}) \\
(\beta_1-\alpha_1)\alpha_2 & (A \times B \times C) \cup (\overline{A} \times B \times C) \cup (A \times \overline{B} \times C) \cup (\overline{A} \times \overline{B} \times C) \\
 & \cup (A \times \overline{B} \times \overline{C}) \cup (\overline{A} \times \overline{B} \times \overline{C}) \\
(\beta_2-\alpha_2)\alpha_1 & (A \times B \times C) \cup (A \times B \times \overline{C}) \cup (\overline{A} \times B \times C) \cup (\overline{A} \times B \times \overline{C}) \\
 & \cup (\overline{A} \times \overline{B} \times C) \cup (\overline{A} \times \overline{B} \times \overline{C}) \\
(\beta_2-\alpha_2)(1-\beta_1) & (A \times \overline{B} \times C) \cup (A \times \overline{B} \times \overline{C}) \cup (\overline{A} \times B \times C) \cup (\overline{A} \times B \times \overline{C}) \\
 & \cup (\overline{A} \times \overline{B} \times C) \cup (\overline{A} \times \overline{B} \times \overline{C}) \\
(\beta_1-\alpha_1)(1-\beta_2) & (A \times B \times \overline{C}) \cup (\overline{A} \times B \times \overline{C}) \cup (A \times \overline{B} \times C) \cup (A \times \overline{B} \times \overline{C}) \\
 & \cup (\overline{A} \times \overline{B} \times C) \cup (\overline{A} \times \overline{B} \times \overline{C}) \\
(\beta_1-\alpha_1)(\beta_2-\alpha_2) & \mathbf{\Theta_{D_1 \cup D_2 \cup D_3}}
\end{cases}
$$

TABLE IV

RESULT OF THE MARGINALIZATION OF $m_1 \oplus m_2$ TO $\mathbf{\Theta_{D_1 \cup D_3}}$ - BASED ON RULE (11)

$$
(m_1 \oplus m_2)^{\downarrow \mathbf{D_1 \cup D_3}} = \begin{cases}
\alpha_1\alpha_2 & (A \times C) \cup (\overline{A} \times C) \cup (\overline{A} \times \overline{C}) \\
1-\alpha_1\alpha_2 & \mathbf{\Theta_{D_1 \cup D_3}}
\end{cases}
$$

Summarizing, when transformation (13) is exploited to transform uncertain implication rules, logical and evidence calculus can equivalently be used to combine pieces of information. Figure 1 provides a pictorial representation of the equivalence between logical and evidence theory calculus.

C. A practical application

In this section, a simple example will be used to illustrate the application of (13) to a problem of decision-making under uncertainty. An officer from a civil protection agency must predict the chance of a flood occurring in the next few hours in a given area. He knows that it will be cloudy with a confidence of 0.7 and that:

1) if it is cloudy then it will rain strongly with a confidence between 0.6 and 0.8;
2) if it rains strongly then it will flood with a confidence between 0.5 and 0.9.

This problem is defined in the frame $\Theta_V = \{(r,c,f), (r,\bar{c},f), (\bar{r},c,f), (\bar{r},\bar{c},f), (r,c,\bar{f}), (r,\bar{c},\bar{f}), (\bar{r},c,\bar{f}), (\bar{r},\bar{c},\bar{f})\}$ where: $r = rain$, $\bar{r} = noRain$, $c = cloudy$, $\bar{c} = noCloudy$, $f = flood$ and $\bar{f} = noFlood$. He can model this knowledge with three BBA functions. The statement "cloudy with a confidence of 0.7" can be expressed by:

$$m_1(c) = 0.7, \quad m_1(c,\bar{c}) = 0.3$$

The rest of the knowledge is represented by:

$$
\begin{array}{rcl}
\textit{cloudy} & \Rightarrow & \textit{rain with a confidence between 0.6 and 0.8} \\
\textit{rain} & \Rightarrow & \textit{flood with a confidence between 0.5 and 0.9}
\end{array} \tag{29}
$$

Using the formula given in (13), these rules can be converted into the following BBA functions.

$$
\begin{aligned}
m_2((r,c),(r,\bar{c}),(\bar{r},\bar{c})) &= 0.6 \\
m_2((r,c),(\bar{r},c),(\bar{r},\bar{c})) &= 0.2 \\
m_2((r,c),(\bar{r},c),(r,\bar{c}),(\bar{r},\bar{c})) &= 0.2 \\
m_3((r,f),(\bar{r},f),(\bar{r},\bar{f})) &= 0.5 \\
m_3((r,f),(r,\bar{f}),(\bar{r},\bar{f})) &= 0.1 \\
m_3((r,f),(\bar{r},f),(r,\bar{f}),(\bar{r},\bar{f})) &= 0.4
\end{aligned}
$$

Before combining the previous BBAs via Dempster's rule, they must be extended to the common frame Θ_V. The result of the combination of the two BBAs is reported in table V. The resulting BBA $m_2 \oplus m_3$ is then combined with m_1 and the result is shown in table VI. Finally, marginalizing the BBA in table VI to the frame $\Theta_f = \{f, \bar{f}\}$ of the variable of interest "flood", one gets

$$m(f) = 0.21, \quad m(f,\bar{f}) = 0.79 \tag{30}$$

The BBA (30) can equivalently be obtained by exploiting the transitive property, the equivalence between logical calculus and Dempster's combination rule. In fact, by applying the transitive property to the two implications in (29), one gets

$$\textit{cloudy} \quad \Rightarrow \quad \textit{flood with a confidence between 0.3 and 0.98}$$

which can be transformed into the following BBA

$$
\begin{aligned}
m_4((c,f),(\bar{c},f),(\bar{c},\bar{f})) &= 0.3 \\
m_4((c,f),(c,\bar{f}),(\bar{c},\bar{f})) &= 0.02 \\
m_4((c,f),(\bar{c},f),(c,\bar{f}),(\bar{c},\bar{f})) &= 0.68
\end{aligned}
$$

TABLE V

RESULT OF THE COMBINATION OF $m_2 \oplus m_3$

mass	*focal*	*mass*	*focal*
0.3	$(r,c,f),(r,\bar{c},f),(\bar{r},\bar{c},f)(\bar{r},\bar{c},\bar{f})$	0.24	$(r,c,f),(r,\bar{c},f),(\bar{r},\bar{c},f),(r,c,\bar{f}),(r,\bar{c},\bar{f}),(\bar{r},\bar{c},\bar{f})$
0.02	$(r,c,f),(r,c,\bar{f}),(\bar{r},c,\bar{f})(\bar{r},\bar{c},\bar{f})$	0.08	$(r,c,f),(\bar{r},c,f),(\bar{r},\bar{c},f),(r,c,\bar{f}),(\bar{r},c,\bar{f}),(\bar{r},\bar{c},\bar{f})$
0.06	$(r,c,f),(r,\bar{c},f),(r,c,\bar{f}),(r,\bar{c},\bar{f}),(\bar{r},\bar{c},\bar{f})$	0.1	$(r,c,f),(\bar{r},c,f),(\bar{r},c,\bar{f}),(r,\bar{c},f),(\bar{r},\bar{c},f),(\bar{r},\bar{c},\bar{f})$
0.1	$(r,c,f),(\bar{r},c,f),(\bar{r},\bar{c},f),(\bar{r},c,\bar{f}),(\bar{r},\bar{c},\bar{f})$	0.02	$(r,c,f),(r,c,\bar{f}),(\bar{r},c,\bar{f}),(r,\bar{c},f),(r,\bar{c},\bar{f}),(\bar{r},\bar{c},\bar{f})$
0.08	Θ_V		

TABLE VI

RESULT OF THE COMBINATION OF $m_1 \oplus (m_2 \oplus m_3)$

mass	*focal*	*mass*	*focal*
0.21	(r,c,f)	0.018	$(r,c,f),(r,\bar{c},f),(r,c,\bar{f}),(r,\bar{c},\bar{f}),(\bar{r},\bar{c},\bar{f})$
0.21	$(r,c,f),(r,c,\bar{f})$	0.03	$(r,c,f),(\bar{r},c,f),(\bar{r},\bar{c},f),(\bar{r},c,\bar{f}),(\bar{r},\bar{c},\bar{f})$
0.028	$(r,c,f),(r,c,\bar{f}),(\bar{r},c,\bar{f})$	0.072	$(r,c,f),(r,\bar{c},f),(\bar{r},\bar{c},f),(r,c,\bar{f}),(r,\bar{c},\bar{f}),(\bar{r},\bar{c},\bar{f})$
0.14	$(r,c,f),(\bar{r},c,f),(\bar{r},c,\bar{f})$	0.024	$(r,c,f),(\bar{r},c,f),(\bar{r},\bar{c},f),(r,c,\bar{f}),(\bar{r},c,\bar{f}),(\bar{r},\bar{c},\bar{f})$
0.112	$(r,c,f),(\bar{r},c,f),(r,c,\bar{f}),(\bar{r},c,\bar{f})$	0.03	$(r,c,f),(\bar{r},c,f),(\bar{r},c,\bar{f}),(r,\bar{c},f),(\bar{r},\bar{c},f),(\bar{r},\bar{c},\bar{f})$
0.09	$(r,c,f),(r,\bar{c},f),(\bar{r},\bar{c},f)(\bar{r},\bar{c},\bar{f})$	0.006	$(r,c,f),(r,c,\bar{f}),(\bar{r},c,\bar{f}),(r,\bar{c},f),(r,\bar{c},\bar{f}),(\bar{r},\bar{c},\bar{f})$
0.006	$(r,c,f),(r,c,\bar{f}),(\bar{r},c,\bar{f})(\bar{r},\bar{c},\bar{f})$	0.024	Θ_V

The result of the combination of m_1 and m_4 on the frame Θ_V is given by

$$m_1 \oplus m_4 = \begin{cases} 0.21 & (r,c,f),(\bar{r},c,\bar{f}) \\ 0.49 & (r,c,f),(\bar{r},c,f),(r,c,\bar{f}),(\bar{r},c,\bar{f}) \\ 0.09 & (r,c,f),(r,\bar{c},f),(\bar{r},\bar{c},f),(\bar{r},c,f), \\ & (r,\bar{c},\bar{f}),(\bar{r},\bar{c},\bar{f}) \\ 0.006 & (r,c,f),(\bar{r},c,f),(r,\bar{c},\bar{f}),(r,c,\bar{f}), \\ & (\bar{r},c,\bar{f}),(\bar{r},\bar{c},\bar{f}) \\ 0.204 & \Theta_V \end{cases} \tag{31}$$

Marginalizing the BBA in (31) to Θ_f, the BBA (30) is obtained again. This is just a very simple example; these techniques can, however, be applied to much more complex decision-making problems [1]. The major difficulty in this kind of applications is to quantify the knowledge, from human experts and other sources of information, in precise numerical values. Since humans usually represent the knowledge in terms of uncertain implication rules, the transformation (13) can become very useful.

V. CONCLUSIONS

In this paper, a transformation for converting *2dof* uncertain implication rules within the evidence theory framework has been presented. Further, it has been proven that this transformation is coherent, in the sense that the logical axioms continue to hold in the BBA function domain as well. Therefore, this transformation rule guarantees that the fused information obtained either by combining single pieces of information expressed in terms of uncertain implication rules by applying the semantics of propositional logic or by transforming these pieces of information using the equivalent BBA functions and then combining them by means of evidence theory calculus, will produce identical results.

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