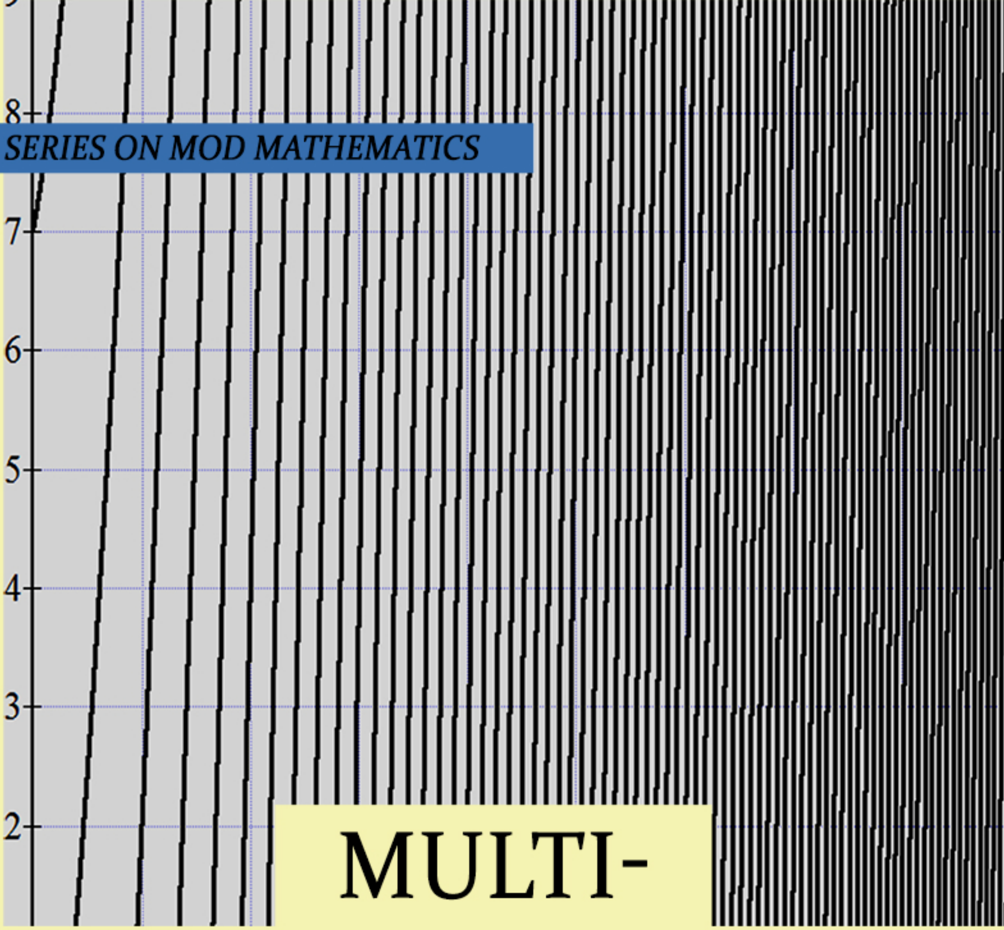

SERIES ON MOD MATHEMATICS

MULTI-DIMENSIONAL MOD PLANES

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EuropaNova
2015

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ISBN-13: 978-1-59973-365-4
EAN: 9781599733654

Printed in the United States of America

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PREFACE

In this book authors name the interval [0, m); $2 \leq m \leq \infty$ as mod interval. We have studied several properties about them but only here on wards in this book and forthcoming books the interval [0, m) will be termed as the mod real interval, [0, m)I as mod neutrosophic interval, [0,m)g; $g^2 = 0$ as mod dual number interval, [0, m)h; $h^2 = h$ as mod special dual like number interval and [0, m)k, $k^2 = (m - 1)$ k as mod special quasi dual number interval. However there is only one real interval (∞, ∞) but there are infinitely many mod real intervals [0, m); $2 \leq m \leq \infty$. The mod complex modulo finite integer interval (0, m) i_F; $i_F^2 = (m - 1)$ does not satisfy any nice properly as that interval is not closed under product .

Here we define mod transformations and discuss several interesting features about them. So chapter one of this book serves the purpose of only recalling these properties. Various properties of mod interval transformation of varying types are discussed. Next the mod planes are introduced mainly to make this book a self contained one. Finally the new notion of higher and mixed multidimensional mod planes are introduced and their properties analysed. Several problems are discussed some of which are open conjectures.

We wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

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Chapter One

RENAMING OF CERTAIN MATHEMATICAL STRUCTURES AS MOD STRUCTURES

The main aim of this book is to define small sets which more or less depicts the real line or natural number line or any other as MOD. So by MOD mathematical structure we mean they are relatively very small in comparison with the existing one.

When these structures somewhat represent the larger one we call them as semiMOD structures or deficit MOD structures or demi MOD structures.

We will proceed onto describe and develop them.

We know Z_n the set of modulo integer, $2 \leq n < \infty$ has several nice algebraic structures.

Z_n's are defined as the semiMOD pseudo intervals.

The map from $\eta_z : Z \to Z_n$ is such that

$\eta_z(1) = 1$, $\eta_z(2) = 2$, … $\eta_z(n - 1) = n - 1$,

$\eta_z(n) = 0$ and so on.

Thus $\eta_z(m) = m$ if $0 \leq m \leq n - 1$ and if $m = n$ then $\eta_z(m) = 0$

if $m > n$ then $\frac{m}{n} = t + \frac{r}{n}$ then $\eta_z(m) = r$.

Thus we have infinite number of elements in Z mapped on to single point.

Further $\eta_z(-t) = m - t$ if $t < m$ and if $t > m$ then

$$\frac{-t}{m} = -s - \frac{r}{m} \text{ so that } \eta_z(t) = m - r.$$

We will first illustrate this situation by an example or two.

Example 1.1: Let Z_7 be the ring of modulo integers.

$\eta_z : Z \rightarrow Z_7$ be a MOD transformation

$\eta_z(x) = x$ if $x \in \{0, 1, 2, \ldots, 6\}$;

$\eta_z(9) = 2 = \eta_z(7 + 2) = 0 + 2$ as $\eta_z(7) = 0$.

$\eta_z(-3) = 4$, $\eta_z(-15) = 6$ and so on.

We call the map η_z of the semi pseudo MOD interval map or semi MOD pseudo interval or demi MOD pseudo interval map or deficit MOD pseudo interval map.

η_z is a semi pseudo MOD interval transformation.

$\eta_z : Z \rightarrow Z_{15}$ be a demi pseudo MOD interval transformation.

$\eta_z(m) = m$ if; $0 \leq m \leq 15$.

$\eta_z(t) = \eta(15n + s) = s$ where $1 \leq s < \infty$ if t is negative,

$\eta_z(t) = \eta(15n - s) = 15 - s$.

Now Z_n is called the demi MOD pseudo interval. η_z is the demi MOD pseudo interval transformation.

Clearly we have a $T_z : Z_n \to Z$ is such that T_z is a pseudo mapping.

$T_z(x) = nt + x; \quad 0 \leq t < \infty.$

T_z is not even a map under the standard definition of the map or function from the domain space Z_n to range space Z. Hence T_z is a MOD pseudo transformation.

This is the first time we map one to many elements. Infact one can define or make Z into n equivalence classes and say map $0 \leq t \leq n$ into the corresponding equivalence classes of Z but not into Z; or one has to divide Z into equivalence classes and then define

$$T_z : Z_n \to Z.$$

Thus this is not a usual function/map. Such functions or maps are in abundant.

This will be proved soon in the following chapters.

We call Z_n the MOD dual special pseudo interval.

Infact $(Z_n, +)$ is the semi MOD group and $\{Z_n, \times\}$ is the semi MOD semigroup.

Finally $\{Z_n, +, \times\}$ is a semi MOD ring. $\{Z_n, +, \times\}$ is semi MOD field if n = p is a prime.

Here we are not interested to study in this direction.

For the limitations of this book is to introduce certain MOD notions of intervals, planes and spaces.

However the term MOD is used to represent the study is made on minimized or reduced or made small the existing big structures with maximum care taken to see that the properties of the big structure are not changed to large extent.

Now $\eta_z : Z \to Z_n$ is a MOD transformation of integers to the semi MOD pseudo structures (interval).

We see for some sake of uniformity and to derive a sort of link we define these well known structures also as MOD structure for they are derived from the infinite line or the integer set Z.

While trying to get from Q this MOD collection we call it as MOD interval.

For we shall denote the MOD rational interval by $[0, m)_q$ where the suffix q distinguishes it from the semi open interval [0, m) of R reals defined in chapter II of this book.

$\eta_q : Q \to [0, m)_q$; a MOD transformation of rationals to semi MOD pseudo rational interval does not exist or cannot be defined as s/t where t divides m is not defined in $[0, m)_q$.

Let us for concreteness take $[0, 9)_q$ and let $\eta_q : Q \to [0, 9)_q$ (Recall $[0, m)_q$ contains all rationals lying between 0 and 9 of course 9 is not included).

For if x = $\frac{429}{5} \in Q$ then $\eta_q\left(\frac{429}{5}\right) = \frac{24}{5} \in [0, 9)_q$

$$\eta_q\left(\frac{-608}{7}\right) = \frac{22}{7} \in [0, 9)_q$$

$$\eta_q\left(\frac{45}{13}\right) = \frac{45}{13} \in [0, 9)_q$$

$\eta_q\left(\frac{67}{17}\right) = \frac{67}{17} \in [0, 9)_q$. $\eta_q\left(\frac{5}{9}\right)$ is not defined.

This is the way the pseudo semi MOD rational transformation takes place but is not well defined as it is only selective.

Let $\eta_q : Q \to [0, 12)_q$

$[0, 12)_q$ = {all elements from the rational between 0 and 12; 12 not included in $[0, 12)_q$}. Let $\frac{254}{3} \in Q$

$\eta_q\left(\frac{254}{3}\right) = \frac{2}{3} \notin [0, 12)_q$.

$\eta_q(-483) = 9 \in [0, 12)_q$.

$$\eta_q\left(\frac{427}{11}\right) = \frac{9}{11} \in [0, 12)_q.$$

Thus we also have MOD rational intervals.

We plan to work with MOD real intervals in chapter II of this book.

Now $Z_m i_F = \{ai_F \mid a \in Z_m\}$; $i_F^2 = m - 1$.

We call $Z_m i_F = \{ai_F \mid a \in Z_m\}$ and is defined as the MOD integer semi complex number.

We will illustrate this by some examples.

Example 1.2: Let $Z_{10} i_F = \{ai_F \mid a \in Z_{10}\}$ be the MOD integer semi complex number. Thus $Z_{10} i_F = \{i_F, 2i_F, 3i_F, 4i_F, 5i_F, 6i_F, 7i_F, 8i_F, 9i_F, 0\}$.

We see $Z_{10}i_F$ is an abelian group under +.

$\{Z_{10}i_F, \times\}$ is not even closed. So $\{Z_{10}i_F, +\}$ is an abelian group.

We can map from $Zi = \{ai \mid a \in Z\}$; $i^2 = -1$ into $Z_{10}i_F$, $i_F^2 = 9$.

We can have a MOD pseudo map from Zi to $Z_{10}i_F$ in the following way.

$\eta_{zi} : Zi \to Z_{10}i_F$;

$\eta_{zi}(49i) = 9i_F$ first $\eta_{zi}(i) = i_F$,

$\eta_{zi}(-93i) = 7i_F$, $\eta_{zi}(28i) = i_F$,

$\eta_{zi}(90) = 0$, $\eta_{zi}(-90) = 0$.

Let us take the semi MOD complex modulo integer interval Z_mi_F.

Consider $\eta_{zi} : Zi \to Z_{26}i_F$; we show how the MOD demi integer complex interval transformation takes place.

$\eta_{zi}(167i) = 11i_F$, $\eta_{zi}(-427i) = 15i_F$, $\eta_{zi}(22i) = 22i$,

$\eta_{zi}(-16i) = 10i_F$ and $\eta_{zi}(178i) = 22i_F$.

Let $\eta_{zi} : Zi \to Z_{13}i_F$ is defined as follows.

$\eta(12i) = 12i_F$, $\eta(27i) = i_F$, $\eta(-12i) = i_F$, $\eta(-25i) = i_F$.

Thus we see 27i, –12i and –25i $\in$ Zi are mapped on to the same element in $Z_{13}i_F$.

Infact we have infinitely many elements in Zi which are mapped on to a single element of $Z_{13}i_F$. This is true for all Z_mi_F.

We will show by some more examples.

Let us consider $Z_{15}i_F$.

$\eta_{zi} : Zi \rightarrow Z_{15}i_F$ is such that $\eta_{zi}(n \times 15i_F) = 0$ for all $n \in Z^+$.

$\eta_{zi}(16i_F) = \eta_{zi}(31i_F) = \eta_{zi}(46i_F) = \eta_{zi}(65i_F) = \eta_{zi}(76i_F) = \eta_{zi}(91i_F) = \ldots = i_F$.

In general $\eta_{zi}(n15i_F + i_F) = i_F$ for all $n \in Z^+$.

Thus infinitely many points in Zi are mapped on to a single point in $Z_m i_F$.

Infact we can say all those points which are mapped to a single point are periodic.

This is shown in the above example.

$$\eta_{zi}(17i) = \eta_{zi}(2i) = \eta_{zi}(32i) = \eta_{zi}(47i) = \ldots$$

$$= \eta_{zi}(n15i + 2i) = 2i_F \text{ for } n \in Z^+.$$

Another natural question would be what happens when elements in Zi are negative.

We will illustrate the situation before we generalize.

$\eta_{zi}(-14i) = i_F$, $\eta_{zi}(-29i) = i_F$, $\eta_{zi}(-44i) = i_F$ and so on.

$\eta_{zi}(-15ni - 14i) = i_F$ for $n \in Z^+$.

Thus all these elements are periodic and are mapped on to the single element i_F.

Consider $\eta_{zi}(-13i) = 2i_F = \eta_{zi}(-28i) = \eta_{zi}(-43i)$ and so on.

Thus for negative values in Zi is an infinite number of periodic elements and are mapped on to a single element in Z_ni_F.

Hence the mapping is a nice mapping done in a regular periodic and in a systematic way.

Now having seen the properties enjoyed by Z_ni_F and its relation to Zi the integer imaginary line we now proceed on to study can we get back Zi given Z_ni_F in a unique way.

The answer is yes.

First this is illustrated by some examples.

Example 1.3: Let Z_7i_F be the complex modulo integers $\{0, i_F, 2i_F, 3i_F, \ldots, 6i_F\}$.

Can we get back Zi? The answer is yes.

For all elements mapped in this way by $\eta_{Z_7i_F} : Z_7i_F \to Z$ are.

We shall denote them by $[0], [i_F], [2i_F], \ldots, [6i_F]$.

$\eta_{Z_7i_F}([0]) = \{0, n7i, n \in Z\}$.

$\eta_{Z_7i_F}([i_F) = \{0, n7i + i, n_17i - 6i, n_1 \in Z^-, n \in Z^+\}$.

$\eta_{Z_7i_F}([2i_F]) = \{n7i + 2i, 0, n_17i - 5i, n_1 \in Z^-, n \in Z^+\}$

likewise for other terms in Z_7i_F.

Thus the map is visualized in a different way further this is not a function in the usual sense.

This way of mapping is mainly done for in case of $[0, m)_qi_F$; $x \in [0, m)_q$ are only rationals is a subcollection from Q and $[0, m)_qi_F$; $x \in [0, m)_q$ and $x \in R$ are to be defined and they cannot be conveniently put into this form as in case of Z_mi_F.

Now having seen a special pseudo transformation which by all means is not a function in the usual form.

We now proceed on to define functions $\eta_{q\,i_F} : Q \to [0, m)_q i_F$ and its corresponding pseudo maps which are not usual functions also known as MOD interval transformation and pseudo interval transformation respectively.

Example 1.4: Let $[0, 9)_q i_F$ be the MOD rational complex modulo integer interval with $i_F^2 = 8$.

Now $\eta_{q\,i_F} : Qi \to [0, 9)_q i_F$; $5i \in Qi$ is such that

$\eta_{q\,i_F}(5i) = 5i_F$. $\eta_{q\,i_F}(i) = i_F$. $\eta_{q\,i_F}(7/8i) = 7/8i_F$.

$$\eta_{q\,i_F}\left(\frac{426i}{5}\right) = 2\,\frac{1}{5}\,i_F \text{ and so on.}$$

Clearly the interval $[0, m)_q i_F$ contains all rationals of the form p/q; q ≠ 0 and p and q integers q < m.

Let $\eta_{q\,i_F}\left(-\frac{478i}{7}\right) = 3\frac{5}{7}\,i_F$.

$$\eta_{q\,i_F}\left(-\frac{7i}{13}\right) = \frac{6}{13}\,i_F \text{ and so on.}$$

But for $\frac{216}{9}\,i \in Qi$ is not defined in $[0, 9)_q i_F$. All elements ti/s where s = 3 or 6 is also not defined in $[0, 9)_q i_F$.

Thus, this way of operation of MOD transformation from Qi to $[0, m)_q i_F$ is not well extended.

Example 1.5: Let $[0, 18)_q i_F$ be the MOD complex finite modulo integer interval.

$\eta_{q i_F}$: Qi → $[0, 18)_q i_F$ is the MOD complex interval transformation.

$$\eta_{q i_F}(i) = i_F,\ \eta_{q i_F}(5i) = 5i_F,\ \eta_{q i_F}(-10i) = 8i_F,$$

$$\eta_{q i_F}(486i) = 0,\ \eta_{q i_F}(-216i) = 0,$$

$$\eta_{q i_F}(-219i) = 15i_F,\ \eta_{q i_F}\left(\frac{49}{2}i\right)$$ is not defined as 2 divides 18 and so on.

Infact the MOD rational complex modulo integer transformation is not possible as $\frac{1}{18}i$ has no element in $[0, 18)_q i_F$ or $\frac{1}{t}i$ where t-divides 18.

Example 1.6: Let $[0, 11)_q i_F$ be the MOD rational complex number interval $\eta_{q i_F}\left(\frac{3}{4}i\right) = \frac{3}{4}i_F$, $\eta_{q i_F}(i) = i_F$,

$$\eta_{q i_F}\left(\frac{483}{8}i\right) = 5\frac{3}{8}i_F$$

$$\eta_{q i_F}\left(\frac{-587}{4}i\right) = 7\frac{1}{4}i_F \text{ and so on.}$$

Now what is the algebraic structure enjoyed by $[0, m)_q i_F$?

Clearly $[0, m)_q i_F$ is a group under + and it is not even closed under the product operation.

We will illustrate this situation by an example or two.

Example 1.7: $G = [0, 6)_q i_F$ is a MOD complex rational number group. $0 \in G$ and $0 + \alpha = \alpha + 0$ for all $\alpha \in G$.

Let $x = 0.85 i_F \in G$ then $y = 5.15 i_F \in G$ is such that $y + x = x + y = 0$.

Let $x = \frac{92 i_F}{100}$ and $y = \frac{7 i_F}{10} \notin G$ as

$$(100, 6) \neq 1 \text{ and } (10, 6) \neq 1.$$

In general $\frac{1}{100}$ or $\frac{1}{10}$ is not even defined in G.

However for all practicalities we assume $[0, m)_q i_F$ contains p/q $q \neq 0$, $0 < q < m$, p and q integers and if pi and qi $\in$ Qi yet $pi \times qi = -pq$ is not in Qi so product is not defined even in Qi.

Thus if $x = \frac{478}{11} i \in Qi$ then we know $\eta_i(x)$ is in $[0, 6)_q i_F$.

Hence our study reveals the interval $[0, m)_q$ or $[0, m)_q i_F$ has no proper transformation.

So at this juncture it is left as a open conjecture how $[0, m)_q$ or $[0, m)_q i_F$ can have at least a partially defined MOD transformation.

Thus from now onwards we can have MOD intervals $[0, m) \subseteq [0, \infty)$; only reals and we do not in this book venture to study $[0, m)_q$ or $[0, m)_q i_F$.

However this study is left open for any researcher.

Let us consider $[0, m)$, $m < \infty$, $x \in [0, m)$ are reals from the real line $[0, m)$. Then we can map $R \to [0, m)_r$ in a nice way.

For all elements have decimal representation so one need not go for the concept p/q; $q \neq 0$, $q \neq m$. p/q = some real decimal, here $[0, m)_r$ is also denoted by $[0, m)$.

For 1/3 in R is 0.3333…3 and is defined in [0, 3) whereas 1/3 is not defined in $[0, 3)_q$.

Let us consider $[0, m) \subset [0, \infty)$, real interval and any $x \in [0, m)$ is in $[0, \infty)$ and vice versa.

For if $x = m \cdot a_1a_2\ldots a_n \in [0, \infty)$ then $0 \cdot a_1a_2\ldots a_n \in [0, m)$.

Thus there exists a MOD real interval transformation unlike the improperly defined MOD rational interval transformation $[0, m)_q$.

We will first illustrate this by some examples.

Example 1.8: Let [0,12) be the MOD real interval.

We define the MOD real transformation from

$$R = (-\infty, \infty) \to [0, 12).$$

$\eta_r : (-\infty, \infty) \to [0, 12)$ as follows:

$\eta_r(9.870813) = 9.870813$,

$\eta_r(-1.301152) = 10.698848 \in [0, 12)$,

$\eta_r(13.0925) = 1.0925 \in [0, 12)$,

$\eta_r(6.3815) = 5.6185 \in [0, 12)$,

$\eta_r(-46.072) = 1.928 \in [0, 12)$ and

$\eta_r(-18.4) = 5.6 \in [0, 12)$.

This is the way the real MOD transformation from the real line $(-\infty, \infty)$ to $[0, 12)$ is carried out.

Example 1.9: Let $[0, 7)$ be the real MOD interval.

$\eta_r : (-\infty, \infty) \to [0, 7)$ is defined by

$\eta_r(6.31048) = 6.31048 \in [0, 7)$

$\eta_r(10.481) = 3.481 \in [0, 7)$

$\eta_r(-16.5) = 4.5 \in [0, 7)$

$\eta_r(-14.00681) = 6.99319 \in [0, 7)$

$\eta_r(35.483) = 0.483 \in [0, 7)$ and so on.

Example 1.10: Let $[0, 16)$ be the real MOD interval.

$\eta_r : (-\infty, \infty) \to [0, 16)$

$\eta_r(48) = 0$,

$\eta_r(-64) = 0$,

$\eta_r(32.092) = 0.092$,

$\eta_r(-32.092) = 31.908$,

$\eta_r(43.48973) = 11.48973$ and

$\eta_r(-43.48973) = 4.51027$ are all in $[0, 16)$.

Thus the MOD real interval transformation is well defined.

Now we proceed on to define for the complex interval $(-\infty i, \infty i)$ to $[0, m)i_F$ where $0 < m < \infty$ and $i_F^2 = m - 1$ is defined as the MOD complex finite interval.

We will illustrate this situation by some examples.

$[0, 5)i_F$, $i_F^2 = 4$ is the complex finite interval.

$[0, 17)i_F$, $i_F^2 = 16$ is again the complex finite interval.

We will describe the MOD complex transformation.

This is illustrated by the following example.

Example 1.11: Let $[0, 15)i_F$, $i_F^2 = 14$ be the MOD complex interval.

Define $\eta : (-\infty i, \infty i) \to [0, 15)i_F$ as follows:

$\eta(8i) = 8i_F$, $\eta(21i) = 6i_F$, $\eta(12i) = 12i_F$, $\eta(-10i) = 5i_F$ and

$\eta(-16i) = 14i_F$, $\eta_r(30.378i) = 0.378i_F$ and so on.

Now we will describe some more examples.

Example 1.12: Let $[0, 10)i_F$ is the MOD complex interval.

$$\eta : (-\infty i, \infty i) \to [0, 10)i_F$$

$$\eta(3.585i) = 3.585i_F$$

$$\eta(0.345908i) = 0.345908i_F$$

$\eta(-3.585i) = 6.415i_F$

$\eta(-0.345908i) = 9.654092i_F$.

This is the way the MOD complex interval transformation.

Clearly $[0, m)i_F$; $0 < m < 1$ is a group under addition; however $[0, m)i_F$ is not even closed under product.

We will illustrate this by some examples.

Example 1.13: Let $[0, 12)i_F$ be the MOD complex interval.

$\eta : (-\infty i, \infty i) \to [0, 12)i_F$

$\eta(25i) = i_F$,

$\eta(-43i) = 5i_F$,

$\eta(50i) = 10i_F$,

$\eta(-50i) = 2i_F$,

$\eta(-25i) = 11i_F$ and so on.

However 0 acts as the additive identity of $[0, 12)i_F$.

Forever $x \in [0, 12)i_F$ we have a unique

$y \in [0, 12)i_F$ such that $x + y = 0$.

Let $x = 8.3i_F$ and $y = 4.7i_F \in [0, 12)i_F$. $x + y = i_F$.

Let $x = 9.3i_F \in [0, 12)i_F$ there exists a

$y = 2.7i_F$ such that $9.3i_F + 2.7i_F = 0$.

Example 1.14: Let $[0, 7)i_F$ be the complex interval; $i_F^2 = 6$.

Let $x = 27.8i \in (-\infty i, \infty i)$.

$\eta(x) = 6.8i_F$;

Let $x = 0.8806i \in (-\infty i, \infty i)$

$\eta(x) = 0.8806i$.

Let $x = -27.8i \in (-\infty i, \infty i)$; $\eta(x) = 0.2i_F$,

$\eta(-0.8806i) = 6.1194i_F$ and so on.

This is the way MOD complex interval transformation are carried out.

Next we study the MOD dual number interval $[0, m)g$, $g^2 = 0$.

$(-\infty g, \infty g) = \{ag \mid a \in (-\infty, \infty), g^2 = 0\}$ is the dual number interval.

When $[0, m)g = \{ag \mid a \in [0, m), g^2 = 0\}$ is the MOD dual number interval.

Now we give some examples.

Example 1.15: Let $[0, 10)g = \{ag \mid a \in [0, 10), g^2 = 0\}$ be the MOD dual number interval.

Consider $\eta_g : (-\infty g, \infty g) \to [0, 10)g$, is defined as follows :

$\eta_g(108g) = 8g$

$\eta_g(-108g) = 2g$

$\eta_g(0.258g) = 0.258g$

$\eta_g(-0.258g) = 9.742g$

$\eta_g(22.483g) = 2.483g$ and so on.

Example 1.16: Let $\eta_g : (-\infty g, \infty g) \to [0, 9)g$ be defined by

$\eta_g(148.3g) = 4.3g$,

$\eta_g(-148.3g) = 4.7g$,

$\eta_g(5.318g) = 5.318g$

$\eta_g(-5.318g) = 3.682g$

$\eta_g(0.748g) = 0.748g$ and so on.

Example 1.17: Let $[0, 4)g$ be the MOD dual number interval.

$\eta_g : (-\infty g, \infty g) \to [0, 4)g$ is defined by

$\eta_g(428g) = 0$,

$\eta_g(408g) = 0$,

$\eta_g(-4g) = 0$,

$\eta_g(-13g) = 3g$,

$\eta_g(-23g) = g$ and so on.

Similarly $\eta_g(0.8994g) = 0.8994g$ and

$\eta_g(-0.8994g) = 3.1006g$.

Example 1.18: Let [0, 9)g be the MOD dual number interval.

$\eta_g : (-\infty g, \infty g) \to [0, 9)g$ is defined as follows:

$\eta_g(4.8732g) = 4.8732g$

$\eta_g(-4.8732g) = 4.1268g$

$\eta_g(12.37g) = 3.37g$

$\eta_g(-12.37g) = 5.63g$ and so on.

Infact $\eta_g(n9g + tg) = tg$ where $0 \leq t < 9$

$\eta_g(-n9g - tg) = 9g - tg$ for all $n \in Z^+$.

$\eta(4.5g) = 4.5g$ $\eta(-4.5g) = 4.5g$ and so on.

Example 1.19: Let [0, 42)g be the MOD dual number interval. The MOD dual number interval transformation;

$\eta_g : (-\infty g, \infty g) \to [0, 42)g$ is defined by $\eta_g(43g) = g$,

$\eta_g(-10g) = 32g$, $\eta_g(15g) = 15g$, $\eta_g(2.74g) = 2.74g$,

$\eta_g(-2.74g) = 39.26g$ and so on.

Here also infinitely elements which are periodic are mapped on to same element of [0, m)g.

Infact though [0, m)g itself is infinite still infinite number of elements in $(-\infty g, \infty g)$ are mapped on to a single element in [0, m)g.

Example 1.20: Let G = [0, 8)g be a MOD group under addition. For $0 \in$ G serves as the additive identity.

Infact for every $x \in G$ there exists a unique $y \in G$ such that $x + y = 0$.

Let $x = 3.1592g$ and $y = 7.5384g \in G$. $x + y = 3.1592g + 7.5384g = 2.6976g \in [0, 8)g$.

Further $0.348g + 6.12g = 6.468g \in [0, 8)g$.

Now $\{[0, 8)g, +\}$ is a MOD abelian group of infinite order.

Example 1.21: Let $G = \{[0, 5)g, +\}$ be the MOD dual number interval group of infinite order.

Let $x = 3.0075892g \in G$, $y = 1.9924108g$ in G is such that $x + y = 0$.

Thus G, is an abelian MOD group of infinite order under addition modulo 5.

This MOD dual number group has MOD subgroups of both finite and infinite order.

Example 1.22: Let $S = \{[0, 24)g, g^2 = 0, +\}$ be the MOD dual number group under addition module 24.

We see $B_1 = \{Z_{24}g, +\}$ is a MOD subgroup of G of finite order.

$B_2 = \{0, 2g, 4g, 6g, 22g\} \subseteq S$ is also a MOD subgroup of finite order under +.

$B_3 = \{0, 12g\}$ is a MOD subgroup of order two.

$B_4 = \{0, 6g, 12g, 18g\}$ is a MOD subgroup of order four.

$B_5 = \{0, 8g, 16g\} \subseteq S$ is again a MOD subgroup of order 3.

$B_6 = \{0, 4g, 8g, 12g, 16g$ and $20g\}$ is again a MOD subgroup of order 6.

$B_7 = \{0, 3g, 6g, 9g, 12g, 15g, 18g, 21g\} \subseteq S$ is a MOD subgroup of order eight.

Thus S has several MOD subgroups of finite order.

Example 1.23: Let $S = \{[0, 7)g, g^2 = 0, +\}$ be a MOD group of infinite order. $\{0, 3.5g\} \subseteq S$ is a MOD subgroup of order two.

$\{0, 1.75g, 3.50g, 5.25g\}$ is a MOD subgroup of order four and so on.

So $[0, m)g$ has MOD subgroups of finite order even though m may be prime or a non prime.

Always $[0, m)g$ has a MOD subgroup of order two, four, five, eight, ten and so on. Depending on m it will have MOD subgroups of order 3, 6, 9, 11, 13 and so on.

If in $[0, m)$, m is a multiple of 3 then it will have MOD subgroups of order 3 and 6.

Likewise one can find conditions for the MOD dual number group $[0, m)g$ to have finite MOD subgroups of desired order.

Next we find $[0, m)g$ the MOD dual number interval is closed under the product $\times$ and the operation $\times$ is associative on $[0, m)g$ as $g^2 = 0$.

DEFINITION 1.1: *Let $S = \{[0, m)g \mid g^2 = 0, 0 < m < \infty, \times\}$ be the MOD dual number semigroup. Since $ag \times bg = 0$ for all $ag, bg \in [0, m)g$. S is defined as the MOD zero square dual number semigroup of infinite order.*

Examples of this situation is given in the following.

Example 1.24: Let S = $\{[0, 7)g, g^2 = 0, \times\}$ is the MOD zero square dual number semigroup which is commutative and is of infinite order.

For a = rg and b = sg; rg, sg $\in$ [0, 7)g we have ab = 0 as $g^2 = 0$.

Infact S has MOD dual number subsemigroups of order two, three, four and so on. Further any subset of S is a MOD dual number subsemigroup of S.

A = $\{0, 5.02g, 2.7981g\} \subseteq$ S is such that A is a MOD dual number subsemigroup of order three.

B = $\{0, 0.3389102g\} \subseteq$ S is a MOD subsemigroup of order two and so on.

Study in this direction is interesting for every subset with zero of S is also an ideal of S. Thus every dual number MOD subsemigroup finite or infinite order is also a MOD dual number ideal of S.

THEOREM 1.1: *Let S = $\{[0, m)g, g^2 = 0, \times\}$ be a MOD zero square dual number semigroup.*

i) Every subset together with zero of S finite or infinite order is a dual number MOD subsemigroup of S.
ii) Every subset of S with zero finite or infinite order is a dual number MOD ideal of S.

Proof. Directly follows from the fact for x, y $\in$ S. $x \times y = 0$.

Hence the claim.

Example 1.25: Let S = $\{[0, 12)g, g^2 = 0, \times\}$ be the MOD dual number semigroup.

Every pair A = $\{0, x \mid x \in [0, 12)g \setminus \{0\}\}$ is a MOD dual number subsemigroup of order two.

Every MOD dual subsemigroup of S must contain zero as $x \times y = 0$ for every $x, y \in [0, 12)g$.

Example 1.26: Let $S = \{[0, 15)g, g^2 = 0, \times\}$ be the MOD dual number semigroup.

Every subset A of S with $0 \in A$ is a MOD dual number subsemigroup as well as an ideal an of S.

Clearly a subset of S which does not contain the zero element is not a dual number subsemigroup.

Let $A = \{0.778g, 4.238g, 11.5g, 14.006g\} \subseteq S$; clearly A is only a subset of S and not a dual number subsemigroup of S as $0 \notin A$.

Now having defined the notion of MOD dual number semigroup we proceed on to define MOD dual number ring.

In the first place we wish to keep on record that these MOD dual number semigroups are never monoids as

$1 \notin S = \{[0, m)g, g^2 = 0, \times\}, 0 < m < \infty$.

DEFINITION 1.2: *Let $S = \{[0, m)g, +, \times, g^2 = 0\}$ be a MOD dual number ring which has no unit. Infact S is a dual number ring of infinite order which is commutative.*

We will first illustrate this situation by some examples.

Example 1.27: Let $R = \{[0, 9)g, g^2 = 0, +, \times\}$ be the MOD dual number ring of infinite order. Every subset with zero of R is not a MOD dual number subring of R.

For if $x = 8.312g$ and $y = 4.095g$ then $xy = 0 = x^2 = y^2$.

By $x + y = 3.407g \notin \{x, y, 0\} = A$.

Hence the claim.

Further $x + x = 8.312g + 8.312g = 7.624g \notin A$.

Similarly $y + y = 4.095g + 4.095g = 8.19g \notin A$.

Thus we are forced to find under what condition a subset $\{A, +, \times\}$ will be a subring of MOD dual numbers.

If $\{A, +\}$ is a MOD dual number subgroup then $\{A, +, \times\}$ is a MOD dual number subring.

We will give examples of them.

$S_1 = \{Z_9g, +, \times\}$ is a zero square MOD dual number ring. Clearly S_1 is associative.

Infact R is itself an associative ring that is $a \times (b \times c) = (a \times b) \times c = 0$ for every $a, b, c \in R$; as R is a zero square ring.

$S_2 = \{0, 4.5g\} \subseteq R$ is again a zero square MOD dual number subring. Clearly both S_1 and S_2 are also ideals of R.

Thus every finite MOD dual subring of R is also a MOD ideal of R.

$S_3 = \{0, 3g, 6g\} \subseteq R$ is a MOD subring of dual numbers as well as ideal of R.

Thus these class of zero square rings of infinite order behaves in a very different and interesting way.

Example 1.28: Let $R = \{[0, 18)g, g^2 = 0, +, \times\}$ be the MOD dual number ring. Consider

$$S_1 = \{0, 9g\},$$

$S_2 = \{0, 6g, 12g\}$,

$S_3 = \{0, 3g, 6g, 9g, 12g, 15g\}$,

$S_4 = \{0, 2g, 4g, 6g, 8g, 10g, 12g, 14g, 16g\}$,

$S_5 = \{0, 4.50g, 9g, 13.50g\}$,

$S_6 = \{0, 2.25g, 4.50g, 6.75g, 9g, 11.25g, 13.50g, 15.75g\}$

are all subgroups of R which are also MOD subrings of dual numbers. These are also MOD ideals of R.

$S_7 = \{0, 1.8g, 3.6g, 5.4g, 9g, 7.2g, 10.8g, 12.6g, 14.4g, 16.2g\} \subseteq R$

is a MOD dual number subgroup as well as MOD ideal of R.

$S_8 = \{0, 1.5g, 3g, 4.5g, 6g, 7.5g, 9g, 10.5g, 12g, 13.5g, 15g, 16.5g\} \subseteq R.$

$S_9 = \{0, 1.125g, 2.250g, 3.375g, 4.500g, 5.625g, 6.750g, 7.875g, 9g, 10.125g, 11.250g, 12.375g, 13.5g, 14.625g, 15.750g, 16.875g\} \subseteq R$

is again an ideal of order 16 however R is of infinite order.

Infact R has infinite number of finite order MOD dual number subrings which are ideals.

Example 1.29: Let $R = \{[0, 10)g, g^2 = 0, +, \times\}$ be the MOD dual number ring which is a zero square ring of infinite order.

$S_1 = \{0, 5g\}$,

$S_2 = \{0, 2g, 4g, 6g, 8g\}$,

$S_3 = \{0, 2.5g, 5g, 7.50g\}$,

$S_4 = \{0, 1.25g, 2.5g, 3.75g, 5g, 6.25g, 7.5g, 8.75g\}$,

$S_5 = \{Z_{10}g\}$,

$S_6 = \{0, 0.5g, 1g, 1.5g, 2g, 2.5g, 3g, 3.5g, 4g, 4.5g, 5g, 5.5g, 6g, 6.5g, 7g, 7.5g, 8g, 8.5g, 9g, 9.5g\}$,

$S_7 = \{0, 0.625g, 1.25g, 1.875g, 2.5g, 3.125g, 3.750g, 4.375g, 5g, 5.625g, 6.25g, 6.875g, 7.5g, 8.125g, 8.750g, 9.375g\}$

are all subrings of R which are also MOD ideals of finite order in R.

Infact R has infinitely many finite MOD dual number subrings which are zero square subrings of R.

Example 1.30: Let $R = \{[0, 7)g, +, \times\}$ be the MOD dual number zero square ring.

Let $S_1 = \{0, 3.5g\}$,

$S_2 = \{0, 1.75g, 3.50g, 5.25g\}$,

$S_3 = \{0, 1.4g, 2.8g, 4.2g, 5.6g\}$,

$S_4 = \{0, 0.875g, 1.750g, 2.625g, 3.500g, 4.375g, 5.250, 6.125g\}$

are some of the finite order MOD dual number subgroups which are also ideals of R.

Now having seen both MOD dual number ideals of MOD dual number rings and MOD dual number semigroups we proceed on to define, describe and develop the notion of MOD special dual like number interval

$[0, m)h = \{ah \mid a \in [0, m) \text{ and } h^2 = h\}$ is defined as the MOD special dual like number interval.

Clearly $[0, m)h$ is of infinite order.

$x \in 8.7h$ and $y = 7h \in [0, m)h$ then $60.9h \pmod m \in [0, m)h$.

We will illustrate this by some examples.

Let $[0, 8)h$ be the MOD special dual like interval.

Let $4h \in [0, 8)h$; $(4h)^2 = 0$.

Let $0.9h$ and $4h \in [0, 8)h$, $3.6h \in [0, 8)h$.

$4h + 4h = 0$. $4h + 0.9h = 4.9h$.

This is the way both the operations + and × are performed on this MOD special dual like number interval.

Now we see if $(-\infty h, \infty h)$ is the infinite real special dual like number interval where $h^2 = h$ then we have the MOD special dual like number interval transformations.

$\eta_h : (-\infty h, \infty h) \rightarrow [0, 8)h$ as follows:

$\eta_h(83.5h) = 3.5h$, $\eta_h(h) = h$, $\eta_h(-10.5h) = 5.5h$,

$\eta_h(8h) = 0 = \eta_h(16h) = \eta_h(-8h) = \eta_h(-16h)$

$= \eta_h(-24h) = \eta_h(-32h) = \ldots = \eta_h(n8h)$; $n \in Z$.

Thus we see infinite number of points of $(-\infty h, \infty h)$ are mapped on to a single point in $[0, 8)h$.

Let us consider $x = -48.5h \in (-\infty h, \infty h)$; $\eta_h(x) = (7.5h)$ and so on.

$\eta_h(n8h + th) = th$ for all $t \in [0, m)$.

$\eta_h(n8h - th) = mh - th$ for all $th \in [0, m)h$. t is positive value. Only –t is negative.

Thus we see η_h maps infinitely many periodic elements on to a single element. This is the way η_h the MOD special dual like number interval transformation functions.

Now $\eta_h : (-\infty h, \infty h) \to [0, m)h$. $h^2 = h$, $1 \leq m < \infty$.

This map is unique depending only on them.

Once the m is fixed $\eta_h : (-\infty h, \infty h) \to [0, m)h$ is unique.

Example 1.31: Let $\eta_h : (-\infty h, \infty h) \to [0, 9)h$ be the MOD special dual like number transformation.

$$\eta_h(27h) = 0,$$

$$\eta_h(49h) = 4h,$$

$$\eta_h(9h) = 0,$$

$$\eta_h(-25h) = 2h.$$

Example 1.32: Let $\eta_h : (-\infty h, \infty h) \to [0, 12)h$ be the MOD special dual like number transformation.

$$\eta_h(27h) = 3h,$$

$$\eta_h(49h) = h,$$

$$\eta_h(9h) = 9h,$$

$$\eta_h(-25h) = 2h.$$

We see η_h is dependent on [0, m)h clear from the above two examples.

Now we see the algebraic structure enjoyed by [0, m)h.

DEFINITION 1.3: *Let $[0, m)h = \{ah \mid a \in [0, m), h^2 = h\}$ be the MOD special dual like number interval $G = \{[0, m)h, +\}$ is defined as a MOD special dual like number interval group of infinite order.*

DEFINITION 1.4: *$S = [0, m)h = \{a \in [0, m), h^2 = h\}$ is a semigroup under $\times$ and it is defined as a MOD special dual like number interval semigroup.*

We will give examples of this situation.

Example 1.33: Let $M = \{[0, 10)h, h^2 = h, +\}$ be MOD special dual number interval group of infinite order.

M has MOD special dual like number interval subgroups of finite order.

$S_1 = \{Z_{10}h, +\}$,

$S_2 = \{0, 5h, +\}$,

$S_3 = \{0, 2h, 4h, 6h, 8h, +\}$,

$S_4 = \{0, 2.5h, 5h, 7.5h\}$,

$S_5 = \{0, 1.25h, 2.5h, 3.75h, 5h, 6.25h, 7.50h, 8.75h\}$

and so on are all finite MOD dual like number finite subgroups of M.

Now we see some more example.

Example 1.34: Let M = $\{[0, 11)h, h^2 = h, +\}$ be the MOD special dual like number interval group of infinite order.

$S_1 = \{Z_{11}h, +\}$,

$S_2 = \{0, 5.5h\}$,

$S_3 = \{0, 2.75h, 5.50h, 8.75h\}$,

$S_4 = \{0, 1.375h, 2.750h, 4.125h, 5.500h, 6.875h, 8.250h, 9.625h\}$,

$S_5 = \{0, 1.1h, 2.2h, 3.3h, 4.4h, 5.5h, 6.6h, 7.7h, 8.8h, 9.9h\}$

are some of MOD special dual like number interval finite subgroups of M.

Example 1.35: Let S = $\{[0, 12)h, h^2 = h, +\}$ be the MOD special dual like number group.

$P_1 = \{Z_{12}h\}$,

$P_2 = \{0, 6h\}$,

$P_3 = \{0, 4h, 8h\}$,

$P_4 = \{0, 3h, 6h, 9h\}$,

$P_5 = \{0, 2h, 4h, 6h, 8h, 10h\}$,

$P_6 = \{0, 2.4h, 4.8h, 7.2h, 9.6h\}$

and so on are all MOD special dual like number subgroups of S of finite order.

Next we proceed on to study the notion of MOD special dual like number semigroups.

Example 1.36: Let B = $\{[0, 9)h, h^2 = h, \times\}$ be the MOD special dual like number semigroup.

Let $x = 0.72h$ and $y = 0.5h \in B$.

$x \times y = 0.72h \times 0.5h = 0.36h \in B$.

$x^2 = 0.5184h$

$y^2 = 0.25h \in B$.

This is the way product is performed on B. Clearly $h \in B$ acts as the multiplicative identity;

for if $a = xh$ then $xh \times h = xh = a$ as $h^2 = h$.

Now B is a commutative monoid of infinite order.

Next we proceed on to show $B = \{[0, m)h, \times\}$ can have zero divisors.

For if $B = \{[0, 24)h, h^2 = h, \times\}$ be the semigroup.

Let $x = 12h$ and $y = 2h \in B$, $x \times y = 0$.

Thus B has several zero divisors.

$x = 3h$, $y = 8h \in B$ is such that $xy = 0$ and so on.

Can B have ideals? the answer is yes.

For take $P_1 = \{0, 12h\} \subseteq B$ is only a subsemigroup and not an ideal of B.

$P_2 = \{0, 4h, 12h, 8h, 16h, 20h\} \subseteq B$ is only a subsemigroup and not an ideal.

For if $x = 1.1h$ then $x \times 4h = 4.4h \notin P_2$ hence the claim.

Let $x = 0.9h \in B$ and $y = 20h \in P_2$ then $x \times y = 18h \notin P_2$.

Let $P_3 = \{0, 2h, 4h, \ldots, 22h\}$ be the subsemigroup of B.

Let $x = 0.5h \in B$ and $y = 2h$ in P_3 then $y \times x = h \notin P_3$.

Thus P_3 is not an ideal of B.

Let $P_4 = \{0, 3h, 6h, 9h, 12h, 15h, 18h, 21h\}$ is only a subsemigroup and not an ideal of B.

Take $x = 3h \in P_4$, $y = 1.5h \in B$, $x \times y = 4.5h \notin P_4$.

Thus P_4 is not an ideal of B.

Now having seen subsemigroups which are not ideals it remains as a challenging problem to find ideals in B.

Will they be only of infinite order or of finite order and so on?

Example 1.37: Let $B = \{[0, 9)h, h^2 = h, \times\}$ be the MOD special dual like number monoid of infinite order. B has several subsemirings of finite order.

However we are not able to find any ideals of B.

In view of this the following is left as an open conjecture.

Conjecture 1.1: Let $B = \{[0, m)h, h^2 = h, \times\}$ be the MOD special dual like number commutative monoid of infinite order.

i. Find nontrivial ideals of B.
ii. Can ideals of B be of finite order?

Example 1.38: Let B = {[0, 3)h, h^2 = h, ×} be the MOD special dual like number monoid of infinite order.

Let P_1 = {0, 1.5h} be a subsemigroup and not an ideal.

P_2 = {0, 0.75h, 1.50h, 2.25h} is again a MOD subsemigroup and not an ideal of B.

Now having seen examples of MOD special dual like number we proceed onto define MOD special dual like number interval pseudo rings.

DEFINITION 1.5: *Let R = {[0, m)h, +, ×, h^2 = h}; R is defined as the pseudo MOD special dual like number interval pseudo ring.*

Clearly h serves as the multiplicative identity of R. We call R as a pseudo ring as the distributive law is not true.

For a, b, c ∈ R we have a × (b + c) ≠ a × b + a × c in general.

We will illustrate this situation by some examples.

Example 1.39: Let R = {[0, 5)h, h^2 = h, +, ×} be the MOD special dual like number interval pseudo ring.

Consider x = 0.35h, y = 4.15h and z = 0.85h ∈ R.

$$x \times (y + z) = x \times (4.15h + 0.85h)$$

$$= x \times 0$$

$$= 0 \quad \ldots \quad \text{I}$$

Consider x × y + x × z

$$= 0.35h \times 4.15h + 0.35h \times 0.85h$$

$$= 1.4525h + 0.2975h \quad \ldots \quad \text{II}$$

I and II are distinct so the distributive law is not true in R general.

Example 1.40: Let R = {[0, 6)h | h^2 = h, +, ×} be the MOD special dual like number interval pseudo ring. x = 0.3h, y = 4.8h and z = 1.2h ∈ R is such that

$$x \times (y + z) = 0 \text{ (as } y + z = 0) \qquad \ldots \qquad \text{I}$$

$$\begin{aligned} x \times y + x \times z &= 0.3h \times 4.8h + 0.3h \times 1.2h \\ &= 1.44h + 0.36h \\ &= 1.8h \qquad \ldots \qquad \text{II} \end{aligned}$$

Clearly I and II are distinct so the ring is only pseudo and does not satisfy the distributive law.

Infact R has nontrivial zero divisors. h is the unit of R. x = 5h is such that x^2 = h

x = 3h and y = 2h in R is such that

x × y = 0, x = 3h and y = 4h in R is such that

x × y = 3h × 4h = 0.

Now {Z_6h, +, ×} is a subring which is not an ideal of R.

Similarly R has several subrings which are not ideals.

Example 1.41: Let M = {[0, 7)h, h^2 = h, +, ×} be the MOD special dual like number interval pseudo ring.

M has zero divisors. M has subring however we are not in position to find ideals in these pseudo ring.

We leave it as a open problem to find MOD ideals of special dual like number intervals.

Conjecture 1.2: Let R = {[0, m)h, h^2 = h, +, ×} be the MOD special dual like number interval pseudo ring.

i. Can R have ideals of finite order?
ii. Find nontrivial ideals of R.

Now having seen pseudo ring of special dual like number intervals we next proceed on to discuss about the special quasi dual number intervals (–∞k, ∞k) where $k^2 = -k$.

Now [0, m)k = {ak | a ∈ [0, m), k^2 = (m – 1)k} is defined as the MOD special quasi dual number interval.

We define MOD special quasi dual number interval and the MOD special quasi dual number interval transformation.

Let η_k : (–∞k, ∞k) → [0, m)k where $k^2 = -k$ in (–∞k, ∞k) and k^2 = (m – 1)k; in case of interval [0, m)k. $\eta_k(k) = k$.

We will illustrate this by some examples.

Example 1.42: Let (–∞k, ∞k) be the real special quasi dual numbers. [0, 5)k be the special quasi dual number interval.

Define η_k : (–∞k, ∞k) → [0, 5)k

$\eta_k(4.8222k) = 4.8222k$

$\eta_k(-4.8222k) = 0.1778k$

$\eta_k(15.889352k) = 0.889352k$

$\eta_k(-15.889352k) = 4.11065k$

$\eta_k(108.3k) = 3.3k$

$\eta_k(-108.3k) = 1.7k$.

$\eta_k(5k) = \eta_k(10k) = \eta_k(15k) = \eta_k(45k) = 0$ thus several; infact infinitely many elements are mapped onto a single element.

$\eta_k(5k + 0.3k) = \eta_k(10k + 0.3k) = \eta_k(15.3k) = \eta_k(20.3k) = \eta_k(25.3k) = \eta_k(30.3k) = \eta_k(35.3k) = \eta_k(40.3k) = \eta_k(45.3k) = \ldots = \eta_k(5nk + 0.3k) = 0.3k$.

Thus infinitely many numbers $(-\infty k, \infty k)$ is mapped on to a single element.

$\eta_k(6.35k) = 1.35k$,

$\eta_k(-6.35k) = 3.65k$ and so on.

Example 1.43: Let $\eta_k : (-\infty k, \infty k) \to [0, 11)k$, $k^2 = -k$ in real and $k^2 = 10k$ in $[0, 11)k$.

$\eta_k(-\infty k, \infty k) \to [0, 11)k$.

$\eta_k(10k) = 10k$,

$\eta_k(-48k) = 3k$,

$\eta_k(48k) = 8k$,

$\eta_k(-110k) = 0$, $\eta_k(49k) = 5k$ and so on.

Now as in case of all other MOD interval transformations this η_k is the same as that of other MOD transformation.

Now we proceed on to define algebraic structures on $[0, m)k$, $k^2 = (m - 1)k$.

DEFINITION 1.6: *Let B = {[0, m)k / k^2 = (m – 1)k, +} be the group under addition modulo m. B is an infinite commutative MOD special quasi dual number interval group.*

We will give examples of them.

Example 1.44: Let G = {[0, 8)k, k^2 = 7k, +} be the MOD special quasi dual like number interval group. G has several subgroups of finite order.

B_1 = {0, 2k, 4k, 6k},

B_2 = {0, 4k},

B_3 = {0, 0.8k, 1.6k, 2.4k, 3.2k, 4k, 4.8k, 5.6k, 6.4k, 7.2k},

B_4 = {0, 1.6k, 3.2k, 4.8k, 6.4k},

B_5 = {k, 2k, …, 7k, 0},

B_6 = {0, 0.5k, 1k, 1.5k, 2k, 2.5k, 3k, 3.5k, 4k, 4.5k, 5k, 5.5k, 6k, 6.5k, 7k, 7.5k},

B_7 = {0, 0.25k, 0.5k, 0.75k, k, 1.25k, 1.50k, 1.75k, 2k, 2.25k, 2.5k, 2.75k, 3k, 3.25k, 3.5k, 3.75k, …, 7k, 7.25k, 7.5k, 7.75k}

and so on are all MOD subgroups of finite order.

We can have several such MOD subgroups of finite order.

Example 1.45: Let G = {[0, 15)k, k^2 = 14k, +} be the MOD special quasi dual number interval group.

Let $P_1 = Z_{15}k$,

P_2 = {0, 3k, 6k, 9k, 12k},

$P_3 = \{0, 5k, 10k\}$,

$P_4 = \{0, 7.5k\}$,

$P_5 = \{0, 3.75k, 7.50k, 11.25k\}$,

$P_6 = \{0, 2.5k, 5k, 7.5k, 10k, 12.5k\}$,

$P_7 = \{0, 1.875k, 3.750k, 5.625k, 7.5k, 9.375k, 11.250k, 13.125k\}$,

$P_8 = \{0, 1.5k, 3k, 4.5k, 6k, 7.5k, 9k, 10.5k, 12k, 13.5k\}$,

$P_9 = \{0, 1.25k, 2.5k, 3.75k, 5k, 6.25k, 7.5k, 8.75k, 10k, 11.25k, 12.5k, 13.75k\}$ and

$P_{10} = \{0, 0.9375k, 1.8750k, 2.8125k, 3.7500k, 4.6875k, 5.6250k, 6.5625k, 7.500k, 8.4375k, 9.3750k, 10.3125k, 11.2500k, 12.1875, 13.1250, 14.0625k\}$

are all some of the MOD subgroups of infinite order.

Infact G has infinite number of finite MOD subgroups.

Example 1.46: Let $G = \{[0, 24)k, k^2 = 23k, +\}$ be the MOD special quasi dual number interval group of infinite order. G has several finite order MOD subgroups.

Infact $H_1 = \langle 0.1k \rangle$ is a MOD subgroup of G generated by 0.1k.

Likewise $H_2 = \langle 0.01k \rangle$ is again a MOD subgroup of G.

$H_3 = \{\langle 0.001k \rangle\}$ is again a MOD subgroup of G.

Likewise we can have several MOD subgroups of finite order in G.

Next we proceed on to describe and define the notion of MOD special quasi dual number interval semigroup of infinite order under product $\times$.

DEFINITION 1.7: *Let $S = \{[0, m)k, k^2 = (m - 1)k, \times\}$ be the MOD special quasi dual number interval semigroup of infinite order.*

However S is a commutative MOD semigroup or to be more precise S is a MOD special quasi dual number interval semigroup of infinite order.

We will illustrate this situation by some examples.

Example 1.47: Let $S = \{[0, 5)k, k^2 = 4k, \times\}$ be the MOD special quasi dual number interval semigroup.

$P_1 = \{Z_5k, \times\}$ is a subsemigroup of finite order.

Can S have any other subsemigroup of finite order?

Let $P_2 = \langle 0.1k \rangle$; clearly P_2 is an infinite order MOD subsemigroup of S.

$P_3 = \langle 0.01k \rangle$ is again an infinite order MOD subsemigroup of S.

Study in this direction is innovative and interesting.

The following is left as an open problem.

Problem 1.1: Let $S = \{[0, m)k, k^2 = (m-1)k, \times\}$ be the MOD special quasi dual number interval semigroup. If m is a prime can S has more than one finite order subsemigroup got from S.

Example 1.48: Let $S = \{[0, 11)k, k^2 = 10k, \times\}$ be the MOD special quasi dual number interval semigroup.

$P_1 = Z_{11}k$ is a MOD subsemigroup of order 11.

$P_2 = \{10k\}$ is a MOD subsemigroup of S.

$P_3 = \{10k, k\}$ is also a MOD subsemigroup of S.

$P_4 = \{0, 10k\}$ is a MOD subsemigroup of order two.

Interested reader can study both finite and infinite order MOD subsemigroups of S.

Clearly the MOD special quasi dual number interval semigroup has zero divisors.

THEOREM 1.2: *Let $S = \{[0, m)k, k^2 = (m - 1)k, \times\}$ be the MOD special quasi dual number semigroup. If m is a composite number and Z_mk has large number of subsemigroups then S has many MOD special quasi dual number subsemigroups of finite order.*

Proof of the theorem is direct and hence left as an exercise to the reader. Finding idempotents of S is a challenging problem.

Next we proceed on to define the notion of MOD special quasi dual number interval pseudo rings.

DEFINITION 1.8: *$R = \{[0, m)k, k^2 = (m - 1)k, +, \times\}$ is defined as the MOD special quasi dual number interval pseudo ring as the operation of + over $\times$ is not distributive.*

That is $a \times (b + c) \neq a \times b + a \times c$ in general for $a, b, c \in R$.

We will illustrate this by some examples.

Example 1.49: Let $R = \{[0, 9)k, k^2 = 8k, +, \times\}$ be the MOD special quasi dual number pseudo ring.

Let $x = 8.3k$, $y = 6.107k$ and $z = 0.5k \in R$;

$$x \times (y + z) = 8.3k(6.107k + 0.5k)$$

$$= 8.3k \times 6.607k$$

$$= 0.8381k \times 8$$

$$= 6.7048k \qquad \ldots \text{ I}$$

Consider

$$x \times y + x \times z = 8.3k \times 6.107k + 8.3k \times 0.5k$$

$$= 5.6881 \times 8k + 4.15 \times 8k$$

$$= 0.5048k + 6.2k$$

$$= 6.7048k \qquad \ldots \text{ II}$$

For this triple the distributive law is true.

Consider $x = 8.012k$, $y = 4.03k$ and $z = 4.97k \in R$

$$x \times (y + z) = 8.012k \times (4.03k + 4.97k)$$

$$= 8.012k \times 0$$

$$= 0 \qquad \ldots \text{ I}$$

Consider

$$x \times y + x \times z = 8.012k \times 4.03k + 8.012k \times 4.97k$$

$$= 5.28836 \times 8k + 3.81964 \times 8k$$

$$= 6.30688k + 3.55712k$$

$$= 0.864k \qquad \ldots \text{ II}$$

I and II are distinct so for this triple the distributive law is not true. Infact there are infinitely many such triples such that the distributive law is not true.

Now we have seen the pseudo rings of MOD special quasi dual number are non distributive but is of infinite order.

Certainly this MOD pseudo rings have zero divisors, units and idempotents solely depending on the m of the MOD interval [0, m)k.

Thus we have six types of MOD intervals, reals [0, m), complex modulo integer [0, m)i_F, neutrosophic MOD interval [0, m)I, dual number MOD interval [0, m)g, $g^2 = 0$, special dual like number MOD interval [0, m)h, $h^2 = h$ special quasi dual number MOD interval [0, m)k, $k^2 = (m – 1)k$.

On all these structures MOD groups, MOD semigroups and MOD pseudo rings is defined and developed.

Only in case of MOD special dual number interval [0, m)g, $g^2 = 0$ we see both the semigroups and rings are zero square semigroups and rings respectively.

Only in this case the distributive law is also true.

We will illustrate matrices and supermatrices built using these structures by examples.

Example 1.50: Let A = $\{(x_1, x_2, x_3, x_4) \mid x_i \in [0, 6); +, 1 \leq i \leq 4\}$ be the collection of all row matrices.

We can define on A two types of operations + and ×.

Under + A is a group defined as the MOD real row matrix group.

A is commutative and is of infinite order.

Example 1.51: Let

$$B = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_i \in [0, 17); +, 1 \leq i \leq 7\}$$

be the collection of all 1×7 row matrices.

B is a MOD real row matrix group.

Example 1.52: Let

$$C = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right| a_i \in [0, 15); 1 \leq i \leq 6, + \right\}$$

be a MOD real column matrix interval group.

Example 1.53: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{10} \end{bmatrix} \right| a_i \in [0, 23), 1 \leq i \leq 10, + \right\}$$

be the MOD real interval column matrix group of infinite order.

Example 1.54: Let

$$V = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \leq i \leq 7, +\right\}$$

be the MOD real interval column matrix group of infinite order.

Example 1.55: Let

$$V = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 45); 1 \leq i \leq 12, +\right\}$$

be the MOD real interval square matrix group of infinite order.

Example 1.56: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 15); 1 \leq i \leq 12, +\right\}$$

be the MOD real matrix interval group of infinite order.

Example 1.57: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_5 & a_9 \\ a_2 & a_6 & a_{10} \\ a_3 & a_7 & a_{11} \\ a_4 & a_8 & a_{12} \end{bmatrix} \middle| \, a_i \in [0, 14); 1 \leq i \leq 12; + \right\}$$

be the MOD real number interval group of infinite order.

This has subgroups of both finite order and infinite order.

Example 1.58: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \, a_i \in [0, 47); 1 \leq i \leq 18, + \right\}$$

be the MOD real interval matrix group.

This has at least ${}_{18}C_1 + {}_{18}C_2 + \ldots + {}_{18}C_{18}$ number of finite subgroups.

Study of using other MOD intervals is considered as a matter of routine and left as an exercise to the reader however one or two examples to illustrate this is given.

Example 1.59: Let

$$W = \{(x_1, x_2, \ldots,, x_7) \mid x_i \in [0, 9)i_F; 1 \leq i \leq 7, i_F^2 = 8, +\}$$

be the MOD complex modulo integer row matrix group.

W is of infinite order has subgroups of finite and infinite order.

Example 1.60: Let

$$V = \left\{ \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix} \middle| \; x_i \in [0, 11)i_F;\ i_F^2 = 10,\ 1 \leq i \leq 6, +\right\}$$

be the MOD complex modulo integer matrix group. V has subgroups of both finite and infinite order.

Example 1.61: Let

$$X = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_i \in [0, 43)i_F;\ i_F^2 = 42,\ 1 \leq i \leq 15, +\right\}$$

be the MOD complex modulo integer interval group. X is of infinite order.

Example 1.62: Let

$$Y = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_6 \\ a_7 & a_8 & \dots & a_{12} \\ a_{13} & a_{14} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{24} \end{bmatrix} \middle| \; a_i \in [0, 15)i_F;\ i_F^2 = 14,\ 1 \leq i \leq 24, +\right\}$$

be the MOD complex modulo integer interval group.

Y is of infinite order.

Example 1.63: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{19} & a_{20} \end{bmatrix} \middle| \; a_i \in [0, 25)i_F;\ i_F^2 = 24,\ 1 \leq i \leq 20; + \right\}$$

be the MOD complex modulo integer interval group of infinite order.

Example 1.64: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in [0, 43)i_F,\ i_F^2 = 42,\ 1 \leq i \leq 16, + \right\}$$

be the MOD complex modulo integer interval group of infinite order.

Example 1.65: Let

$$M = \{(a_1, a_2, \ldots, a_9) \mid a_i \in [0, 18)I;\ I^2 = I,\ 1 \leq i \leq 9, +\}$$

be the MOD neutrosophic modulo integer group.

Example 1.66: Let

$$X = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \end{bmatrix} \middle| \; a_i \in [0, 98)I;\ I^2 = I,\ 1 \leq i \leq 18, + \right\}$$

be the MOD neutrosophic modulo integer interval group.

Example 1.67: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 14)I;\ I^2 = I,\ 1 \le i \le 12, + \right\}$$

be the MOD modulo integer neutrosophic matrix group.

Example 1.68: Let

$$A = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 31)I;\ I^2 = I,\ 1 \le i \le 9; + \right\}$$

be the MOD modulo integer neutrosophic matrix group.

Example 1.69: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_i \in [0, 14)I;\ I^2 = I,\ 1 \le i \le 15, + \right\}$$

be the MOD modulo integer neutrosophic interval matrix group.

Example 1.70: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_5 & a_9 \\ a_2 & a_6 & a_{10} \\ a_3 & a_7 & a_{11} \\ a_4 & a_8 & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 10)I; I^2 = I, 1 \leq i \leq 12, +\right\}$$

be the MOD modulo integer neutrosophic interval group.

Example 1.71: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in [0, 17)g, g^2 = 0, 1 \leq i \leq 6, +\right\}$$

be the MOD modulo integer dual number interval matrix group.

Example 1.72: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in [0, 18)g, g^2 = 0, 1 \leq i \leq 6, +\right\}$$

be the MOD dual number interval matrix group.

Example 1.73: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \; a_i \in [0, 17)g, g^2 = 0; 1 \leq i \leq 18, + \right\}$$

be the MOD dual number interval matrix group.

Example 1.74: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \end{bmatrix} \middle| \; a_i \in [0, 29)g, g^2 = 0; 1 \leq i \leq 27, + \right\}$$

be the MOD dual number group under +.

Example 1.75: Let

$$M = \{[a_1 \ldots a_{10}] \mid a_i \in [0, 19)h; h^2 = h, 1 \leq i \leq 10; +\}$$

be the MOD special dual like number interval matrix group under +.

Example 1.76: Let

$$T = \{(a_1 \ldots a_{19}) \mid a_i \in [0, 43)h, h^2 = h; 1 \leq i \leq 19, +\}$$

be the MOD special dual like number interval matrix group under +.

Example 1.77: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right| a_i \in [0, 14)h;\ h^2 = h; \right.$$

$$1 \leq i \leq 16, +\}$$

be the MOD special dual like number interval matrix group.

Example 1.78: Let

$$T = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{15} \end{bmatrix} \right| a_i \in [0, 178)h,\ h^2 = h;\ 1 \leq i \leq 15, + \right\}$$

be the MOD special dual like number interval matrix group.

Example 1.79: Let

$$S = \{(a_1 \ldots a_9) \mid a_i \in [0, 43)k;\ k^2 = 42k;\ 1 \leq i \leq 9; +\}$$

be the MOD special quasi dual number interval matrix group.

Example 1.80: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in [0, 18)k,\ k^2 = 17k,\ 1 \leq i \leq 9, + \right\}$$

be the MOD special quasi dual number interval matrix group.

M is infinite and has subgroups of finite order.

Example 1.81: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle| \; a_i \in [0, 11)k, \right.$$

$$\left. k^2 = 10k, 1 \leq i \leq 24, + \right\}$$

be the MOD special quasi dual number interval matrix group.

M has at least $n({}_{24}C_1 + {}_{24}C_2 + \ldots + {}_{24}C_{24})$ number of subgroups where n is the number of finite subgroups of T.

Next we proceed on to give examples of MOD matrix interval semigroups using these 6 types of MOD intervals.

Example 1.82: Let

$$B = \{(x_1, x_2, x_3) \mid x_i \in [0, 24); 1 \leq i \leq 3; \times\}$$

be the MOD real interval matrix semigroup.

B is commutative and is of infinite order.

B has infinite number of zero divisors.

B has also idempotents and (1, 1, 1) acts as the multiplicative identity.

Example 1.83: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in [0, 23); 1 \leq i \leq 5, \times_n \right\}$$

be the MOD real interval matrix semigroup.

M has infinite number of zero divisors, units and idempotents.

Example 1.84: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_5 \\ a_3 & a_4 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 108); 1 \leq i \leq 9, \times_n \right\}$$

be the MOD real interval matrix commutative semigroup.

P has infinitely many zero divisors and several subsemigroups.

If in the above example $\times_n$ is replaced by $\times$ the usual matrix product then P is not a commutative semigroup.

P is only a MOD real interval matrix non commutative monoid of infinite order.

Example 1.85: Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{11} \\ a_{13} & a_{14} \end{bmatrix} \middle| \; a_i \in [0, 17); 1 \le i \le 14, \times_n \right\}$$

be the MOD real matrix interval semigroup.

W has infinite number of zero divisors.

However W has only finite number of idempotents and units.

Next we proceed on to give examples to show MOD interval matrix finite complex modulo integer semigroups does not exist.

Example 1.86: Let

$$W = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 7)i_F;\ i_F^2 = 6; 1 \le i \le 5, \times\}$$

be the MOD interval matrix finite complex modulo integer, W is not even closed under $\times$.

Example 1.87: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in [0, 8)i_F;\; i_F^2 = 7,\; 1 \leq i \leq 6, \times_n \right\}$$

cannot be the MOD complex modulo integer interval semigroup as it is not defined.

Example 1.88: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 19)i_F,\; i_F^2 = 18;\; 1 \leq i \leq 9, \times_n \right\}$$

be the MOD complex modulo integer interval which is not closed.

Example 1.89: Let

$$M = \{(a_1, a_2, \ldots, a_9) \text{ where } a_i \in [0, 15)I;\; 1 \leq i \leq 9, \times;\; I^2 = I\}$$

be the MOD neutrosophic interval semigroup of infinite order.

(I I I I …I) acts as the identity.

So M is a commutative monoid of infinite order.

Example 1.90: Let

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{13} \end{bmatrix} \middle| a_i \in [0, 42)I;\ 1 \leq i \leq 13,\ I^2 = I, \times_n \right\}$$

be the MOD neutrosophic interval monoid of infinite order.

$$\begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} = \text{Id is the identity element of W; } |W| = \infty.$$

Example 1.91: Let

$$V = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| a_i \in [0, 45)I;\ I^2 = I,\ 1 \leq i \leq 4, \times_n \right\}$$

be the MOD neutrosophic interval matrix semigroup. V is the commutative monoid.

$$\text{Id} = \begin{pmatrix} I & I \\ I & I \end{pmatrix} \text{ is the identity element of V.}$$

If in V the natural product $\times_n$ is replaced by $\times$ then V is a non commutative monoid and $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I_{2 \times 2}$ is the identity of V.

In both cases V has zero divisors, units and idempotents.

Example 1.92: Let

$$B = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \right| a_i \in [0, 15)I; I^2 = I, 1 \leq i \leq 12, \times_n \right\}$$

be the MOD neutrosophic interval commutative semigroup (infact monoid) of infinite order.

$$Id = \begin{bmatrix} I & I \\ I & I \\ I & I \\ I & I \\ I & I \\ I & I \end{bmatrix} \text{ is the identity of B.}$$

Example 1.93: Let

$$T = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \right| a_i \in [0, 13)I; I^2 = I, 1 \leq i \leq 24, \times_n \right\}$$

be the MOD neutrosophic interval semigroup (monoid) of T.

T is of infinite order and is commutative.

$$Id = \begin{bmatrix} I & I & I & I \\ I & I & I & I \\ I & I & I & I \\ I & I & I & I \\ I & I & I & I \\ I & I & I & I \end{bmatrix} \text{ is the identity of T.}$$

T has infinite number of zero divisors, finite number of units and idempotents.

T has many subsemigroups of finite order as well as of infinite order.

Example 1.94: Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_i \in [0, 20)g, g^2 = 0; 1 \leq i \leq 9, \times_n \right\}$$

be the MOD dual number interval semigroup.

B is not a monoid. Infact B is a zero square semigroup. B has both finite and infinite subsemigroups.

Example 1.95: Let

$$W = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \mid a_i \in [0, 11)g, g^2 = 0;$$
$$1 \leq i \leq 8, \times\}$$

be the MOD dual number interval semigroup of infinite order.

W is commutative and W is a zero square semigroup. Infact every subsemigroup is also an ideal of W.

Example 1.96: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 15)g, g^2 = 0; 1 \leq i \leq 9, \times_n \right\}$$

be the MOD dual number interval semigroup of infinite order. S is a zero square semigroup and has no unit. S is commutative.

If $\times_n$, the natural product is replaced by $\times$ then also S is commutative as S is a zero square semigroup.

Example 1.97: Let

$$P = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mid a_i \in [0, 9)h; h^2 = h; 1 \leq i \leq 7, \times\}$$

be the MOD special dual like number semigroup of infinite order.

Id = (h, h, h, h, h, h) $\in$ P acts as the identity. So P is infact a commutative monoid of infinite order.

Example 1.98: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i \in [0, 15)h, h^2 = h, \times_n; 1 \leq i \leq 10 \right\}$$

be the MOD special dual like number interval semigroup.

$$\text{Id} = \begin{bmatrix} h \\ h \\ \vdots \\ h \end{bmatrix} \text{ is the identity element of M for and } X \in M,$$

$$X \times_n \text{Id} = \text{Id} \times_n X = X.$$

Infact M is an infinite interval commutative monoid.

We will give a few more examples of this structure.

Example 1.99: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| a_i \in [0, 18)h;\ h^2 = h;\ 1 \leq i \leq 10, \times_n \right\}$$

be the MOD special dual like number interval semigroup of infinite order.

Infact M is only a monoid $\text{Id} = \begin{bmatrix} h & h \\ h & h \\ h & h \\ h & h \\ h & h \end{bmatrix} \in M$ is the identity element of M.

Example 1.100: Let

$$P = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) \mid a_i \in [0, 19)k,\ k^2 = 18k, \times_n;\ 1 \leq i \leq 9\}$$

be the MOD special quasi dual number interval semigroup of infinite order.

Id = (18k, 18k, …, 18k) acts as the identity of P. So P is the monoid of infinite order.

Example 1.101: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in [0, 48)k, k^2 = 47k, 1 \leq i \leq 6, \times_n \right\}$$

be the MOD special quasi dual number interval semigroup of infinite order.

$$Id = \begin{bmatrix} 47k \\ 47k \\ 47k \\ 47k \\ 47k \end{bmatrix} \in M \text{ acts as the identity of M.}$$

$$\text{Let } A = \begin{bmatrix} 8k \\ 10k \\ 46k \\ 0 \\ 11k \end{bmatrix} \in M.$$

$$A \times_n Id = \begin{bmatrix} 8k \\ 10k \\ 46k \\ 0 \\ 11k \end{bmatrix} \times_n \begin{bmatrix} 47k \\ 47k \\ 47k \\ 47k \\ 47k \end{bmatrix} = \begin{bmatrix} 8k \\ 10k \\ 46k \\ 0 \\ 11k \end{bmatrix} = A.$$

Example 1.102: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \right| a_i \in [0, 12)k, \right.$$

$$k^2 = 11k, 1 \leq i \leq 24, \times_n \}$$

be the MOD special quasi dual number interval semigroup of infinite order. M is infact a commutative monoid.

$$Id = \begin{bmatrix} 11k & 11k & 11k & 11k & 11k & 11k \\ 11k & 11k & 11k & 11k & 11k & 11k \\ 11k & 11k & 11k & 11k & 11k & 11k \\ 11k & 11k & 11k & 11k & 11k & 11k \end{bmatrix} \text{ is the identity of M.}$$

$A \times_n Id = Id \times_n A = A$ for all $A \in M$.

Example 1.103: Let

$$D = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in [0, 24)k, k^2 = 23k, 1 \leq i \leq 9, \times_n \right\}$$

be the MOD commutative special quasi dual number interval semigroup (monoid).

$$\text{Id} = \begin{bmatrix} 23k & 23k & 23k \\ 23k & 23k & 23k \\ 23k & 23k & 23k \end{bmatrix} \text{ is the identity of D.}$$

If in D we replace by × then D is a non commutative monoid

$$\text{with } I_{3\times3} = \begin{bmatrix} 23k & 0 & 0 \\ 0 & 23k & 0 \\ 0 & 0 & 23k \end{bmatrix}.$$

Thus for all A ∈ D, $A \times I_{3\times3} = I_{3\times3} \times A = A$.

Example 1.104: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle|\; a_i \in [0, 6)k, k^2 = 5k, 1 \leq i \leq 12; \times_n \right\}$$

be a MOD commutative special quasi dual number interval semigroup.

$$\text{Infact Id} = \begin{bmatrix} 5k & 5k \\ 5k & 5k \\ 5k & 5k \\ 5k & 5k \\ 5k & 5k \\ 5k & 5k \end{bmatrix} \text{ is the identity element of B.}$$

So B is a commutative monoid of infinite order.

Next we just give examples of MOD interval pseudo rings using all the 6 MOD intervals.

Example 1.105: Let

$$M = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_i \in [0, 8); 1 \leq i \leq 7, +, \times\}$$

be a MOD real interval pseudo ring.

M is commutative and is of infinite order. Id = (1, 1, 1, 1, 1, 1, 1) in M acts as the identity element of M.

M has several pseudo subring of infinite order which are ideals.

Infact M has subrings of finite order which are not ideals of M.

Example 1.106: Let

$$T = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in [0, 17); 1 \leq i \leq 5, +, \times_n\right\}$$

be the MOD real interval commutative pseudo ring with identity;

$$Id = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in T.$$

This T has pseudo ideals of infinite order and all subrings of finite order are not ideals of T.

Example 1.107: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in [0, 144);\ 1 \leq i \leq 4, +, \times_n\right\}$$

be the MOD real interval pseudo ring which is commutative and

has Id = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is the identity M is of infinite order.

Now if in M $\times_n$ is replaced by $\times$; then M is a MOD real interval pseudo ring which is non commutative.

$I_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in$ M is the identity of M under the usual product $\times$.

Example 1.108: Let

$$T = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \middle| \; a_i \in [0, 45);\ 1 \leq i \leq 24, +, \times_n\right\}$$

be the MOD real interval pseudo ring. T has infinite number of zero divisors.

This ring has only infinite order subrings to be ideals.

However all infinite order subrings of T are not ideals of T.

For

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ a_7 & a_8 & a_9 \\ 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in Z_{45}, a_i \in [0, 45), \right.$$

$$4 \leq i \leq 12\}$$

is an infinite order pseudo subring which is not an ideal of T.

However none of the finite subrings are ideal of T.

$$L = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in [0, 45), +, \times_n \right\}$$

is a MOD interval pseudo subring of T.

L is an ideal and |L| is infinite.

Example 1.109: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| a_i \in [0, 16); 1 \leq i \leq 16, +, \times_n \right\}$$

be the MOD real interval pseudo ring.

W is commutative and is of infinite order.

$$\text{Id} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in W \text{ acts as the identity of W.}$$

W is pseudo ring with identity. W has ideal of infinite order.

None of the pseudo subrings of finite order of W is an ideal of W.

Now if $\times_n$ the natural product is replaced by the usual product then W is a non commutative MOD real interval pseudo ring.

$$I_{4 \times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ acts as the identity of W.}$$

Example 1.110: Let

$$M = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in [0, 16); 1 \leq i \leq 6, +, \times\}$$

be the MOD real modulo integer interval pseudo ring.

M is a commutative pseudo ring with Id = (1, 1, 1, 1, 1, 1). M has zero divisors.

M has ideals all of which are of infinite order. M has subrings of finite order which are not ideals of M.

All ideals of M are of infinite order.

Example 1.111: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in [0, 43); 1 \leq i \leq 6; +, \times_n \right\}$$

be the MOD real integer interval pseudo ring which is commutative and has no identity.

V has pseudo ideals of infinite order.

$$P = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in [0, 43), +, \times_n \right\} \subseteq V \text{ is an ideal of V.}$$

V is of infinite order. V has ${}_6C_1 + {}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5$ are all ideals of V.

Similarly V has ${}_6C_1 + {}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5 + {}_6C_6$ number of subrings.

Example 1.112: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 13); +, \times_n\right\}$$

be the MOD real interval pseudo ring. M has infinite number of zero divisors.

Only M has finite MOD subrings which are not ideals. M has ideals of infinite order given by

$$P_1 = \left\{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in [0, 13); +, \times_n\right\},$$

$$P_2 = \left\{\begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_2 \in [0, 13), +, \times_n\right\}$$

and so on all are pseudo ideals of M.

$$P_3 = \left\{\begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_3 \in [0, 13), +, \times_n\right\}, \ldots,$$

$$P_9 = \left\{\left[\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_9 \end{matrix}\right] \middle| a_9 \in [0, 13), +, \times_n\right\}, \ldots,$$

$$P_{1, 2, \ldots, 8} = \left\{\left[\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 0 \end{matrix}\right] \middle| a_i \in [0, 13); 1 \leq i \leq 8, +, \times_n\right\}$$

and so on.

$$P_{2, \ldots, 9} = \left\{\left[\begin{matrix} 1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{matrix}\right] \middle| a_i \in [0, 13); 2 \leq i \leq 9, +, \times_n\right\}$$

are all pseudo ideals of M.

Thus M has ${}_9C_1 + {}_9C_2 + {}_9C_3 + \ldots + {}_9C_8$ number of pseudo ideals all of them are of infinite order.

Example 1.113: Let

$$L = \{(a_1, a_2, \ldots, a_6) \mid a_i \in [0, 15)I; I^2 = I, 1 \leq i \leq 6, +, \times\}$$

be the MOD neutrosophic interval pseudo ring of infinite order.

L has infinite number of zero divisors.

L has finite subrings and they are not ideals. L has infinite order pseudo subrings which are ideals.

$$M_1 = \{(a_1, 0, \ldots, 0) \mid a_1 \in [0, 15)I, +, \times\},$$

$$M_2 = \{(0, a_2, 0, \ldots, 0) \mid a_2 \in [0, 15)I, +, \times\},$$

$$M_3 = \{(0, 0, a_3, 0, 0, 0) \mid a_3 \in [0, 15)I, +, \times\}, \ldots,$$

$M_{1, 2, \ldots, 5} = \{(a_1, a_2, a_3, a_4, a_5, 0) \mid a_i \in [0, 15)I; 1 \leq i \leq 5\}, \ldots,$

$M_{2, 3, \ldots, 6} = \{(0, a_2, a_3, a_4, a_5, a_6) \mid a_i \in [0, 15)I; 2 \leq i \leq 6, +, \times\}$ are all pseudo subrings which are all pseudo ideals of L.

Example 1.114: Let

$$M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \, a_i \in [0, 160)I; I^2 = I, 1 \leq i \leq 5; +, \times_n\right\}$$

be the MOD neutrosophic interval pseudo ring. M has infinite number of zero divisors.

M has ${}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4$ number of MOD neutrosophic interval pseudo ideals.

Example 1.115: Let

$$M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix} \middle| \, a_i \in [0, 14)I, I^2 = I, 1 \leq i \leq 8, +, \times_n\right\}$$

be the MOD neutrosophic interval pseudo ring. M has infinite number of zero divisors.

M has finite number of finite subrings. M has pseudo ideals of infinite order.

All pseudo ideals can only be of infinite order.

Example 1.116: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 15)I; I^2 = I, 1 \leq i \leq 9; +, \times_n \right\}$$

be the MOD neutrosophic interval pseudo ring. P has infinite number of zero divisors.

P has subrings of finite order which are not ideals. All pseudo ideals of P are of infinite order.

Example 1.117: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_i \in [0, 16)I; I^2 = I, 1 \leq i \leq 15, +, \times_n \right\}$$

be the MOD neutrosophic interval pseudo ring.

M has infinite number of zero divisors. M has finite number of idempotents and so on.

Example 1.118: Let

$$B = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in [0, 10)g, g^2 = 0; 1 \leq i \leq 6; +, \times\}$$

be the MOD dual number interval ring.

Clearly these rings are zero square rings so the distributive law is true. $|B| = \infty$.

B has ideals of finite order. B has infinite number of ideals.

B also has infinite number of subrings all of which are only ideals.

Example 1.119: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \; a_i \in [0, 40)g, g^2 = 0; 1 \leq i \leq 7, +, \times_n \right\}$$

be the MOD dual number interval ring which is not pseudo.

M has infinite number of ideals both of finite and infinite order. M is a zero square ring.

Example 1.120: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \middle| \; a_i \in [0, 12)g, g^2 = 0; 1 \leq i \leq 30, +, \times_n \right\}$$

be the MOD dual number ring which is not pseudo.

Every subset of M with zero matrix is again an ideal as well as subring.

In view of all these we have the following theorem.

THEOREM 1.3: *Let $S = \{A = (m_{ij}) \mid m_{ij} \in [0, m)g;\ g^2 = 0;\ 1 \leq i \leq n$ and $1 \leq j \leq s, +, \times_n\}$ be a $n \times s$ matrix MOD dual number ring.*

i. *S is a zero square ring.*
ii. *All subsets P which are groups under + are subrings as well as ideals.*
iii. *S has infinite number of ideals.*
iv. *All subrings of S are ideals of S.*

Proof. Follows from the fact S is a zero square ring.

Now we proceed on to describe the MOD special dual like number pseudo ring by examples.

Example 1.121: Let

$$M = \{(a_1, \ldots, a_7) \mid a_i \in [0, 12)h, h^2 = h; 1 \leq i \leq 7, +, \times\}$$

be the MOD special dual like number interval pseudo ring. M is of infinite order.

M has infinite number of zero divisors.

M has subrings of finite order which are not ideals. M has pseudo ideals which are of infinite order.

Let $P_1 = \{(a_1, 0, \ldots, 0) \mid a_1 \in [0, 12)h, h^2 = h, +, \times\}$,

$P_2 = \{(0, a_2, 0, \ldots, 0) \mid a_2 \in [0, 12)h; h^2 = h, +, \times\}$,

$P_3 = \{(0, 0, a_3, 0, \ldots, 0) \mid a_3 \in [0, 12)h, h^2 = h, +, \times\}$ and so on.

$P_7 = \{(0, 0, \ldots, 0, a_7) \mid a_7 \in [0, 12)h, h^2 = h, \times, +\}$,

$P_{1,2} = \{(a_1, a_2, 0, \ldots, 0) \mid a_1, a_2 \in [0, 12)h; h^2 = h, +, \times\}, \ldots,$

$P_{1,7} = \{(a_1, 0, 0, 0, 0, 0, a_7) \mid a_1, a_7 \in [0, 12)h; h^2 = h, +, \times\}$,

$$P_{1,2,3} = \{(a_1, a_2, a_3, 0, \ldots, 0) \mid a_1, a_2, a_3 \in [0, 12)h; h^2 = h, +, \times\}, \ldots,$$

$P_{1,6,7} = \{(a_1, 0, \ldots, 0, a_6, a_7) \mid a_1, a_6, a_7 \in [0, 12)h;$
$h^2 = h, +, \times\}$ and so on.

$P_{5,6,7} = \{(0, 0, 0, 0, a_5, a_6, a_7) \mid a_5, a_6, a_7 \in [0, 12)h,$
$h^2 = h, +, \times\},$

$P_{1,2,3,4} = \{(a_1, a_2, a_3, a_4, 0, 0, 0) \mid a_i \in [0, 12)h,$
$h^2 = h, +, \times; 1 \leq i \leq 4\}$

and so on are all pseudo subsemirings which are also pseudo ideals of M.

M has at least $5({}_7C_1 + {}_7C_2 + \ldots + {}_7C_7)$ number of finite subrings which are not ideals of M. M has at least ${}_7C_1 + {}_7C_2 + \ldots + {}_7C_6$ number of pseudo subrings of infinite order which are ideals.

Infact M has more than ${}_7C_1 + {}_7C_2 + \ldots + {}_7C_6$ number of pseudo infinite subsemirings which are not ideals of M.

Example 1.122: Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in [0, 17)h, h^2 = h, 1 \leq i \leq 6, +, \times_n \right\}$$

be the MOD special dual like number interval pseudo rings. B has infinite number of zero divisors.

B also has finite number of finite order subrings which are not ideals.

$$M_1 = \{\begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1 \in Z_{17}h, h^2 = h, +, \times_n\}$$ is only a subring of

finite order and not an ideal.

$$M_2 = \{\begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1 \in [0, 17)h, h^2 = h, +, \times_n\}$$

is pseudo subsemiring as well as pseudo ideal of B.

Let $$T_1 = \{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1 \in Z_{17}h \text{ and } a_2 \in [0, 17)h, h^2 = h, +, \times_n\}$$

be the MOD pseudo subsemiring of B of infinite order.

Clearly T_1 is not an ideal. So B has many pseudo subsemirings of infinite order which are not ideals of B.

Example 1.123: Let

$$D = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in [0, 12)h, h^2 = h, 1 \leq i \leq 9, +, \times_n\right\}$$

be the MOD special dual like number pseudo interval ring of infinite order.

D has pseudo ideals all of which are of infinite order.

D has subrings which are not ideals of finite order.

D has infinite pseudo subrings which are not ideals.

$$M_1 = \{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_1 \in [0, 12)h, h^2 = h, +, \times_n\}$$

is a pseudo subring which is also an ideal of D.

$$M_2 = \{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_1 \in Z_{12}h; h^2 = h, +, \times_n\}$$

is a subring which is not a pseudo ideal and M_2 is of finite order.

$$M_3 = \{\begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_1 \in Z_{12}h, a_2 \in [0, 12)h; h^2 = h, +, \times_n\}$$

is a pseudo subring. But M_3 is not a pseudo ideal only a pseudo subring.

Infact D has many infinite order pseudo subrings which are not pseudo ideals.

However all matrix MOD special dual like number interval pseudo rings has MOD pseudo ideals all of them are of infinite order but has finite subrings as well as infinite order pseudo subrings which are not pseudo ideals.

Next we wish to state a MOD special dual like number interval pseudo ring is a Smarandache ring if Z_mI (Z_mi_F or Z_nk or Z_nh) is a Smarandache ring.

This is true for all MOD pseudo rings except the MOD special dual number interval ring.

$R = \{[0, m)g, g^2 = 0, +, \times\}$ for they are in the first place not pseudo ring for they satisfy the distributive law and secondly they are such that $ab = 0$ for any $a, b \in [0, m)g$ that is R is a zero square ring.

Finally we just give a few illustrations of MOD special quasi dual number interval pseudo rings.

$B = \{[0, m)k \mid k^2 = (m – 1)k, +, \times\}$.

Example 1.124: Let

$$B = \{(x_1, x_2, x_3) \mid x_i \in [0, 9)k, k^2 = 8k, 1 \leq i \leq 3, +, \times\}$$

be the MOD special quasi dual number pseudo ring.

B has zero divisors. Clearly (8k, 8k, 8k) acts as the identity of matrix ring with entries from Z_9k.

For if $x = (3k, 5k, 7k) \in B$

$$\begin{aligned} x \times (8k, 8k, 8k) &= (24 \times k^2, 40k^2, 56k^2) \\ &= (6 \times 8k, 4 \times 8k, 2 \times 8k) \\ &= (3k, 5k, 7k) \\ &= x \in B. \end{aligned}$$

Thus we see for B contains both subrings of finite order as well as infinite pseudo subrings which are pseudo ideals.

However $P = \{(x_1, x_2, 0) \mid x_1 \in Z_9k$ and $x_2 \in [0, 9)k, +, \times\}$ is only a pseudo subring of infinite order but is not an ideal.

For if $A = (0.2k, 0.9k, 6k) \in P$ then for $x = (7k, 0.8k, 0) \in P$.

$$Ax = (0.2k, 0.9k, 6k) \times (7k, 0.8k, 0)$$

$$= (1.4 \times 8k, 0.72k \times 8, 0)$$

$$= (2.2k, 5.76k, 0) \notin P.$$

Thus P is only a pseudo subring and not an ideal of B.

Hence there are pseudo infinite subrings which are not pseudo ideals.

Example 1.125: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \;\middle|\; a_i \in [0, 14)k, k^2 = 13k, +, \times_n \right\}$$

be the MOD special quasi dual number interval pseudo ring. M has zero divisors.

Finding idempotents is a difficult task.

M has at least $2({}_6C_1 + \ldots + {}_6C_6)$ number of finite subrings and has at least ${}_6C_1 + {}_6C_2 + \ldots + {}_6C_5$ number of infinite subrings which are ideals.

M has many infinite order subrings which are not ideals.

For instance

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \, a_1 \in [0, 14)k, a_2 \in Z_{14}k \text{ and } a_3 \in \{0, 7k\}, +, \times_n \right\} \subseteq M$$

is an infinite pseudo subring of M but is not a pseudo ideal of M.

$$\text{For if } T = \begin{bmatrix} 0.3k \\ 2.5k \\ 0.5k \\ 0 \\ 0 \\ 0 \end{bmatrix} \in M \text{ and } S = \begin{bmatrix} 0.115k \\ 8k \\ 7k \\ 0 \\ 0 \\ 0 \end{bmatrix} \in P.$$

$$\text{Clearly } T \times_n S = \begin{bmatrix} 0.3k \\ 2.5k \\ 0.5k \\ 0 \\ 0 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 0.115k \\ 8k \\ 7k \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.0345 \times 13k \\ 6 \times 13k \\ 7k \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4485k \\ 8k \\ 3.5k \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin P.$$

Hence P is only a pseudo subsemiring and not an ideal of M.

Example 1.126: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 27)k; k^2 = 26k, 1 \leq i \leq 9, +, \times \right\}$$

be a MOD non commutative special quasi dual number interval pseudo ring.

W has zero divisors, units and idempotents. W has both finite and infinite order subrings.

Infact W has right ideals which are not left ideals and vice versa.

However W with '$\times_n$' the natural product is commutative.

Example 1.127: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} \end{bmatrix} \middle| \right.$$

$$\left. a_i \in [0, 40)k, k^2 = 39k, +; \times_n, 1 \leq i \leq 21 \right\}$$

be the MOD special quasi dual number matrix interval pseudo ring.

W has several subrings of finite order which are not pseudo.

W has several pseudo subrings of infinite order which are not ideals.

W has at least ${}_{21}C_1 + {}_{21}C_2 + \ldots + {}_{21}C_{20}$ number of pseudo subrings of infinite order which are ideals of W.

Study of these happens to be an interesting problem.

However we leave the following as an open conjecture.

Conjecture 1.3: Let W = {s × t matrices with entries from $[0, m)k$, $k^2 = (m - 1)k$, +, $\times_n$} be the MOD special quasi dual number interval pseudo ring of matrices.

i. Can W has idempotents where the entries are from $[0, m)k \setminus \{0\}$?

ii. Can W have S–idempotents?

Example 1.128: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \middle| \; a_i \in [0, 15)k, k^2 = 14k, +, \times_n, 1 \leq i \leq 14 \right\}$$

be the MOD special quasi dual number interval matrix pseudo ring.

$$Id = \begin{bmatrix} 14k & 14k \\ 14k & 14k \\ 14k & 14k \\ 14k & 14k \\ 14k & 14k \\ 14k & 14k \\ 14k & 14k \end{bmatrix}$$ acts as the unit for a section of the set S.

For take $A = \begin{bmatrix} 0.5k & 11.2k \\ 5.3k & 4.5k \\ 3.1k & 5.7k \\ 6.8k & 7.5k \\ 9k & 8.3k \\ 12k & 13.1k \\ 14.2k & 14k \end{bmatrix} \in S$.

$$A \times_n Id = \begin{bmatrix} 0.5k & 11.2k \\ 5.3k & 4.5k \\ 3.1k & 5.7k \\ 6.8k & 7.5k \\ 9k & 8.3k \\ 12k & 13.1k \\ 14.2k & 14k \end{bmatrix} \times_n \begin{bmatrix} 14k & 14k \\ 14k & 14k \\ 14k & 14k \\ 14k & 14k \\ 14k & 14k \\ 14k & 14k \\ 14k & 14k \end{bmatrix}$$

$$= \begin{bmatrix} 7\times 14k & 6.8\times 14k \\ 14.2\times 14k & 3\times 14k \\ 13.4\times 14k & 4.8\times 14k \\ 5.2\times 14k & 14\times 7k \\ 6\times 14k & 11.2\times 14k \\ 3\times 14k & 3.4\times 14k \\ 3.8\times 14k & 14k \end{bmatrix} = \begin{bmatrix} 8k & 5.2k \\ 3.8k & 12k \\ 7.6k & 7.2k \\ 12.8k & 8k \\ 9k & 6.8k \\ 12k & 2.8k \\ 8.2k & 14k \end{bmatrix} \neq A.$$

However Id acts as the identity for a finite subring of S given by

$$P = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{13} & a_{14} \end{bmatrix} \mid a_i \in Z_{15}k, k^2 = 14k, 1 \leq i \leq 14, +, \times_n\}.$$

So we can call Id as the restricted non pseudo identity of S.

Example 1.129: Let

$$B = \left\{\begin{bmatrix} a_1 & a_4 & a_7 & a_{10} \\ a_2 & a_5 & a_8 & a_{11} \\ a_3 & a_6 & a_9 & a_{12} \end{bmatrix} \middle| \, a_i \in [0, 17)k, k^2 = 16k, 1 \leq i \leq 12, +, \times_n\right\}$$

be the MOD special quasi dual number interval matrix pseudo ring of infinite order.

B has zero divisors. B has no unit.

However B has subrings of finite order which are not pseudo.

B has at least ${}_{12}C_1 + \ldots + {}_{12}C_{12}$ number finite subrings which are not pseudo.

Study of these pseudo rings is innovative.

Finding idempotents happens to be a challenging problem.

Several problems are suggested in the last chapter of this book.

Chapter Two

PROPERTIES OF MOD INTERVAL TRANSFORMATION OF VARYING TYPES

In this chapter we describe the MOD intervals of different types. In the first place any interval that has the capacity to represent the real line $(-\infty, \infty)$ will be known as the MOD real interval. Infact there are several such different types of MOD intervals like MOD complex modulo integer interval, MOD neutrosophic interval, MOD dual number interval, MOD special dual like number interval and MOD special quasi dual number interval.

These intervals for a given interval are infinitely many. Each of these will be represented by examples as well as by the MOD theory behind it.

For MOD theory in general makes the infinite real number continuum $(-\infty, \infty)$ into a MOD semi open interval $[0, m)$; $1 \leq m \leq \infty$.

For we see such MOD real semi open intervals which represent the real line $(-\infty, \infty)$ are infinite in number for m takes all values in $Z^+ \setminus \{\infty\}$.

Hence we define the notion of MOD real interval transformation. $I_{\eta}^{r} : (-\infty, \infty) \to [0, m)$.

In the first place MOD real interval transformation maps infinite number of points on to a singleton point.

This is done by the MOD real interval transformation periodically.

This is represented by the following example.

For instance [0, 12) be the semi open interval.

η_I^r $(-\infty, \infty) \to [0, 12)$ is defined as follows:

η_I^r (12) = 0,

η_I^r (24) = 0,

η_I^r (36) = 0 and so on.

η_I^r (12.7) = 0.7,

η_I^r (24.7) = 0.7 and so on.

Thus periodically these are mapped on a fixed point.

Likewise η_I^r (6.9) = 6.9,

η_I^r (18.9) = 6.9,

η_I^r (30.9) = 6.9 and so on.

This is the way infinite number of periodic points are mapped on to a single point.

Let $\eta_I^r : (-\infty, \infty) \to [0, 7)$

$$\eta_I^r\ (7) = 0 = \eta_I^r\,(14) = \eta_I^r\,(21) = \ldots \infty$$

$$\eta_I^r\ (9.8) = 2.8,\ \eta_I^r\,(16.8) = 2.8 \text{ and so on.}$$

Throughout our discussions in this book a interval [0, m) $1 < m < \infty$ is nothing but the MOD interval of real interval $(-\infty, \infty)$.

Since it is a open interval we demand out MOD interval to be semi open interval.

Further if $x \in (-\infty, \infty)$ the MOD real interval transformation $I_\eta^r : (-\infty, \infty) \to [0, m)$ is defined as

$$I_\eta^r\,(x) = \begin{cases} x & \text{if } 0 < x < m \\ 0 & \text{if } x = \pm m \\ r & \text{if } x = s + \dfrac{r}{m} \\ m - x & \text{if } x \text{ is negative less than } m \\ m - t & \text{if } x \text{ is negative and } x = n + \dfrac{t}{m} \end{cases}$$

This is the way MOD interval transformation is defined.

We will just illustrate this situation by an example or two.

Example 2.1: Let [0, 9) be the MOD interval.

$I_\eta^r : (-\infty, \infty) \to [0, 9)$ is defined for these number in the following.

$$I_\eta^r\ (6.03371) = 6.03371$$

$$I_\eta^r\ (27.6215) = 0.6215$$

$$I_\eta^r\ (-216.824) = 8.176$$

$$I_\eta^r\ (-7.321) = 1.679$$

Thus $I_\eta^r : (-\infty, \infty) \to [0, 9)$. It is important to note that in general there are infinite number of points in $(-\infty, \infty)$ which is mapped onto the interval $[0, 9)$ as this interval can represent $(-\infty, \infty)$ we choose to call this interval as MOD interval.

Another advantage is that we have infinitely many MOD intervals.

Infact the interval [0, 1] a fuzzy interval will be called as the fuzzy MOD interval if we take the semi open interval [0, 1). The main reason for using $[0, m)$; $1 \leq m < \infty$.

The main advantage of using MOD interval is that we have infinitely many MOD intervals and as per the need of the problem one can choose any of the intervals.

This flexibility is not true in case of $(-\infty, \infty)$ for we have one and only one real line.

Further any entry in $(-\infty, \infty)$ can be easily and precisely represented in the MOD interval $[0, m)$.

Further all points in $(-\infty, \infty)$ which are mapped onto the same point of $[0, m)$ will be called as equally represented points or similarly represented points or as points which enjoy the same MOD status.

So these points will be MOD similar in the MOD interval $[0, m)$.

We will describe them by an example or two. So we have by this transformation converted infinite number of points to a single point by clearly stating these points enjoy the same MOD status or they are similar on the MOD interval.

Such study is not only new but is innovative; so the infinite line $(-\infty, \infty)$ is mapped to infinite continuum $[0, m)$ without much losing the property of the line.

We see any practical problem will have its solution only to be finite so it is infact a waste to work with the infinite line $(-\infty, \infty)$.

Further it is pertinent that most of the values which are accepted are only positive values; this also pertains to practical problems.

So in this MOD intervals $[0, m)$; $1 \leq m < \infty$ we see no negative solution is possible. So we see there are several advantages of using these MOD intervals.

They are more practical, periodic and feasible apart from saving one from working on a very large interval $(-\infty, \infty)$.

Further as there are infinite number of MOD intervals one can choose any appropriate one which is needed.

Certainly use of these MOD intervals can save both time and economy.

Further they are highly recommended for storage because of the periodicity they enjoy and the compactness of these intervals as well as better security.

Thus [0, m); these real MOD intervals is the best alternative replacement for the real interval $(-\infty, \infty)$.

The advantages of using them are:

i. There are many such MOD real intervals so one has more choice.

ii. Periodic elements which are infinite in number are mapped onto single element. This condenses the interval very largely.

iii. Since the negative part of the interval has no place in practical and real world problems we think use of MOD real intervals will be a boon to any researchers.

iv. Certainly use of MOD intervals in the storage systems will save certainly time and money and give more security.

v. It is time a change should take place in the world ruled by MOD technology so these visualization or a realization of the real interval $(-\infty, \infty)$ by $[0, m)$ for $1 \leq m < \infty$ is nothing but MODnizing the interval and appropriate m can be chosen depending on the problem and its need.

We will supply a few examples of them before we proceed onto define other types of MOD intervals.

Example 2.2: Let [0, 8) be the MOD real interval the real MOD interval transformation.

$$I_{\eta}^{r} : (-\infty, \infty) \rightarrow [0, 8)$$

I_η^r (6.8) = 6.8,

I_η^r (16) = 0,

I_η^r (29.3) = 5.3,

I_η^r (–5.2) = 2.8 and

I_η^r (–19.8) = 4.2.

Thus all 8m (m ∈ Z) are mapped by I_η^r to zero of the MOD interval.

Infact [0, 8) is an interval which represent every element of (–∞, ∞).

However we cannot say given a x ∈ [0, 8) can we find uniquely the corresponding element in (–∞, ∞).

Infact we can only give periodic values of the corresponding x which are infinite in number.

We have at present no method or means to fix the preimage of x ∈ [0, 8) to a unique element in (–∞, ∞).

This is true for any [0, m); 1 ≤ m < ∞. However when one has ventured to work out with a particular value of m one knows the actual transformations which are important to them.

As invariably practical problems do not need the negative part of the real line one can to some extent fix the periodic repeating values of (–∞, ∞) mapped to a single value in [0, m).

Finally as the concept of negative numbers happens to be a very difficult one for the understanding of small children the replacement of $(-\infty, \infty)$ by $[0, m)$ may be a boon to them.

Such a paradigm of shift can benefit the education system.

Example 2.3: Let [0, 11) be the MOD real interval of $(-\infty, \infty)$ got by the MOD real interval transformation.

$$I_\eta^r : (-\infty, \infty) \to [0, 11)$$

by $I_\eta^r\ (-7) = 4$, $I_\eta^r\ (10) = 10$, $I_\eta^r\ (-18) = 4 = I_\eta^r\ (-7) =$

$I_\eta^r\ (4) = I_\eta^r\ (15) = I_\eta^r\ (26) = I_\eta^r\ (-29)$ and so on.

Now having seen the new notion of MOD real interval we now proceed onto define MOD complex interval.

For this interval $(-\infty i, \infty i)$ is defined as the complex interval and $i^2 = -1$.

Now we call $[0, mi_F)$ to be the complex MOD modulo integer interval. $2 \leq m < \infty$ and $i_F^2 = m - 1$.

Clearly $[0, 3i_F) = \{ai_F \mid a \in [0, 3);\ i_F^2 = 2\}$ is a complex MOD modulo integer interval.

$[0, 6i_F) = \{ai_F \mid a \in [0, 6);\ i_F^2 = 5\}$ is again a complex MOD modulo integer interval.

We have infinitely many such complex MOD modulo integer intervals.

Now we denote by I^c_η the MOD complex transformation of $(-\infty i, \infty i) \to [0, mi_F)$; $i_F^2 = m - 1$ and $i^2 = -1$.

For easy and simple understanding we will first illustrate this situation by some examples.

Example 2.4: Let $I^c_\eta : (-\infty i, \infty i) \to [0, 12i_F)$ be the MOD complex modulo integer interval transformation.

Define I^c_η (25i) = i_F. Clearly I^c_η (i) = i_F in all MOD complex modulo integer interval transformations.

$$I^c_\eta(-49.8i) = (10.2i_F).$$

$$I^c_\eta(12i) = 0, \qquad I^c_\eta(-48i) = 0,$$

$$I^c_\eta(-1.8i) = 10.2i_F \text{ and so on.}$$

It is clearly seen more than one point is mapped onto the same point.

Infact we have infinitely many such points in $(-\infty i, \infty i)$ which are mapped onto the same point in $[0, mi_F)$.

Thus the MOD complex interval transformation I^c_η also maps periodically infinitely many points in $(-\infty i, \infty i)$ into single point in $[0, mi_F)$.

Example 2.5: Let $I^c_\eta : (-\infty i, \infty i) \to [0, 14i_F)$ be the MOD complex interval transformation.

$$I^c_\eta(13.8i) = 13.8i_F, \; I^c_\eta(24.7) = 10.7i_F,$$

$$I^c_\eta\,(-14.3i) = 13.7i_F \text{ and } I^c_\eta\,(-27.8i) = 0.2i_F.$$

This is the way MOD complex interval transformation is carried out.

Example 2.6: Let $I^c_\eta : (-\infty i, \infty i) \to [0, 23i_F)$ is the MOD complex interval transformation.

$$\text{Define } I^c_\eta\,(42.8i) = 19.8i_F$$

$$I^c_\eta\,(14.384i) = 14.384i_F.$$

$$I^c_\eta\,(-10.75i) = 12.25i_F \text{ and so on.}$$

Thus MOD complex interval transformation.

Next we will proceed onto define other intervals before we give some properties enjoyed by $[0, mi_F)$.

i. The MOD interval complex modular transformation maps infinite number of elements in $(-\infty i, \infty i)$ to a single elements in $[0, mi_F)$. Infact this occurs periodically.

   However we do not have a unique means of retrieving the preimages of elements in $[0, mi_F)$.

ii. The MOD interval reduces the problem of leaving $(-\infty i, 0)$ and making $[0, \infty i)$ onto a infinite interval but small interval depending on m. This is compact that is why we choose to define this as MOD complex modulo integer interval.

iii. Another interesting feature about MOD complex modular integer interval is that we have infinite number of such intervals like $[0, mi_F)$, $2 \leq m < \infty$.

iv. The infinite number of MOD complex modulo integer intervals will help in working in better adaptation for an appropriate $[0, mi_F)$ can be done $i_F^2 = m - 1$, $2 \leq m < \infty$.

Next we define the notion of MOD neutrosophic intervals $[0, mI)$ where $I^2 = 0$; $2 \leq m < \infty$.

Clearly $(-\infty i, \infty i)$ is the neutrosophic real interval.

We call $[0, mI)$ as the MOD neutrosophic interval where $1 \leq m < \infty$.

We will first illustrate this situation by examples.

Example 2.7: Let $[0, 5I)$ be the MOD neutrosophic interval $[0, 5I) = \{aI \mid a \in [0, 5)\}$.

Clearly $[0, 5I)$ is of infinite order yet compact called as the MOD neutrosophic interval.

Example 2.8: Let $[0, 7I)$ be the MOD neutrosophic interval $[0, 7I) = \{aI \mid a \in [0, 7)\}$.

Clearly $[0, 7I)$ is of infinite order yet compact called as the MOD neutrosophic interval.

Example 2.9: Let $[0, 20I) = \{aI \mid a \in [0, 20)\}$ be the MOD neutrosophic interval.

Now we show how the MOD neutrosophic interval transformation I_η^I from $(-\infty I, \infty I)$ to $[0, mI)$; $1 \leq m < \infty$ is defined.

$$I_\eta^I (I) = I.$$

First we will illustrate this situation by some examples.

Example 2.10: Let [0, 6I) be the MOD neutrosophic interval.

Define $I_\eta^I : (-\infty I, \infty I) \to [0, 6I)$ as follows:

$$I_\eta^I (8I) = 2I,$$

$$I_\eta^I (3.7I) = 3.7I,$$

$$I_\eta^I (-5I) = I,$$

$$I_\eta^I (27.342I) = 3.342I,$$

$$I_\eta^I (-14.45I) = 3.55,$$

$$I_\eta^I (-42.7I) = 5.3I \text{ and so on.}$$

This is the way the MOD neutrosophic interval transformation is carried out.

Example 2.11: Let [0, 10I) be the MOD neutrosophic interval.

Define $I_\eta^I : (-\infty I, \infty I) \to [0, 10I)$ as follows:

$$I_\eta^I (15.723I) = 5.723I$$

$$I_\eta^I (-24.62I) = 5.38I$$

$$I_\eta^I (40I) = 0,\ I_\eta^I (-40I) = 0,$$

$$I_\eta^I\,(250I) = 0,\ I_\eta^I\,(-470I) = 0,$$

$$I_\eta^I\,(-7.5I) = 2.5I,\ I_\eta^I\,(-22I) = 8I \text{ and so on.}$$

Example 2.12: Let [0, 13I) be the MOD neutrosophic interval.

Let I_η^I : $(-\infty I, \infty I) \to [0, 13I)$ define $I_\eta^I\,(4.3I) = 4.3I$, $I_\eta^I\,(16.5I) = 3.5I$, $I_\eta^I\,(-19.7I) = 6.3I$, $I_\eta^I\,(-4.32I) = 8.68I$ and so on.

Infact infinite number of points in $(-\infty I, \infty I)$ is mapped onto a single point in [0, 13I).

Example 2.13: Let [0, 4I) be the MOD neutrosophic interval $I_\eta^I : (-\infty I, \infty I) \to [0, 4I)$ is defined as follows:

$$I_\eta^I\,(10.34I) = 2.34I,\ I_\eta^I\,(-10.34I) = 1.66I,\ I_\eta^I\,(48.3I) = 0.3I,$$

$$I_\eta^I\,(-48.3I) = 3.7I \text{ and so on.}$$

It can be easily verified that infinite number of elements in $(-\infty I, \infty I)$ into a single element of [0, 4I).

Now having seen the definitions of MOD neutrosophic intervals and MOD neutrosophic interval transformations we now proceed onto enumerate a few of its properties associated with them.

i. The real neutrosophic interval $(-\infty I, \infty I)$ is unique but however we have infinite number of MOD neutrosophic intervals.

ii. However this flexibility of choosing from the infinite number of MOD neutrosophic intervals helps one to choose the appropriate one.

iii. Further instead of working with the large $(-\infty I, \infty I)$ interval we can work with a compact small MOD neutrosophic interval.

iv. This use of MOD neutrosophic interval $[0, mI)$ $1 < m < \infty$ can make one save $(-\infty I, 0)$ for negative numbers are very difficult.

v. Infact the MOD neutrosophic interval transformation maps infinite number of elements into the interval $[0, mI)$; $1 \leq m < \infty$.

Next we proceed onto define, describe and develop the notion of MOD dual number intervals. Consider the dual number interval $(-\infty g, \infty g)$ where $g^2 = 0$.

The MOD dual number interval

$[0, mg) = \{ag \mid a \in [0, m); g^2 = 0\}$; $1 \leq m < \infty$ are infinite in number.

We will first illustrate this situation by some examples.

Example 2.14: Let $[0, 5g) = \{ag \mid a \in [0, 5), g^2 = 0\}$ be the MOD dual number interval.

Example 2.15: Let $[0, 6g) = \{ag \mid a \in [0, 6); g^2 = 0\}$ be the MOD dual number interval.

Example 2.16: Let $[0, 12g) = \{ag \mid a \in [0, 12), g^2 = 0\}$ be the MOD dual number interval.

We will show how the MOD dual number interval is mapped onto dual number interval

$$(-\infty g, \infty g) = \{ag \mid a \in (-\infty, \infty); g^2 = 0\}.$$

We define the MOD dual number transformation I_η^d from $(-\infty g, \infty g) \to [0, mg)$ as follows:

$I_\eta^d : (-\infty g, \infty g) \to [0, 15g)$ is defined by $I_\eta^d(g) = g$.

$I_\eta^d(19.6g) = 4.6g$, $I_\eta^d(-10.7g) = 4.3g$, $I_\eta^d(17.82g) = 2.82g$, $I_\eta^d(-48.37g) = 11.63g$, $I_\eta^d(15g) = 0$, $I_\eta^d(30g) = 0$, $I_\eta^d(60g) = 0$ and so on.

$I_\eta^d(34.6g) = 4.6g$, $I_\eta^d(49.6g) = 4.6g$, $I_\eta^d(64.6g) = 4.6g$ and so on.

This is the way the MOD dual number transformation is defined.

Example 2.17: Let $I_\eta^d : (-\infty g, \infty g) \to [0, 12g)$.

$$I_\eta^d(13.7g) = 1.7g,\ I_\eta^d(20g) = 8g,\ I_\eta^d(-5g) = 7g,$$

$$I_\eta^d(10.2g) = 10.2g,\ I_\eta^d(-30.7g) = 5.3g \text{ and so on.}$$

Example 2.18: Define $I_\eta^d : (-\infty g, \infty g) \to [0, 11g)$ by

$$I_\eta^d(121g) = 0,\ I_\eta^d(22g) = 0,\ I_\eta^d(-11g) = 0,\ I_\eta^d(-44g) = 0,$$

$$I_\eta^d(66g) = 0,\ I_\eta^d(15g) = 4g,\ I_\eta^d(26g) = 4g,\ I_\eta^d(37g) = 4g$$

and so on.

Example 2.19: Let $I_\eta^d : (-\infty g, \infty g) \to [0, 8g)$ be the MOD dual number interval transformation.

$$I_\eta^d(4g) = 4g,\ I_\eta^d(8g) = 0,\ I_\eta^d(-6g) = 2g,\ I_\eta^d(14g) = 6g,$$

$$I_\eta^d(-148g) = 4g,\ I_\eta^d(19g) = 3g \text{ and so on.}$$

Example 2.20: Let $I_\eta^d : (-\infty g, \infty g) \to [0, 53g)$ be defined by

$$I_\eta^d(50g) = 50g,\ I_\eta^d(0.38g) = 0.38g,\ I_\eta^d(6.3g) = 6.3g,$$

$$I_\eta^d(-14g) = 39g \text{ and so on.}$$

$I_\eta^d(106g) = 0$. Thus infinite number of points in $(-\infty g, \infty g)$ is mapped onto a single point in the MOD dual number interval.

However for a point in $[0, mg)$ we cannot uniquely get the corresponding point in $(-\infty g, \infty g)$.

This is one of the flexibility for we can choose an appropriate one need as the expected solution of the problem.

Now having seen MOD dual number intervals $[0, mg)$; $g^2 = 0$; $1 \leq m < \infty$ we now proceed onto define special dual like number MOD intervals.

We know the special dual like number interval is $(-\infty h, \infty h)$ where $h^2 = h$.

Now as in case of dual numbers we have infinite number of special dual like number intervals

$$[0, mh);\ h^2 = h \text{ and } 1 \leq m < \infty.$$

We will first illustrate this situation by some examples.

Example 2.21: Let $[0, 8h)$; $h^2 = h$ be the MOD special dual like number interval.

Example 2.22: Let [0, 12h); $h^2 = h$ be the special MOD dual like number interval.

Example 2.23: Let [0, 18h); $h^2 = h$ be the special dual like MOD number interval.

Infact we have infinitely many MOD special dual like number intervals but only one special dual like number real interval $(-\infty h, \infty h)$; $h^2 = h$.

Now as in case of other intervals we in case of MOD special dual like number intervals also obtain MOD special dual like number MOD interval transformation.

$$I_\eta^{dl} : (-\infty h, \infty h) \to [0, mh);\ h^2 = h;\ \text{here } I_\eta^{dl}(h) = h.$$

We will first illustrate this situation by some examples.

Example 2.24: Let [0, 9h); $h^2 = h$ be the MOD special dual like interval.

$$I_\eta^{dl} : (-\infty h, \infty h) \to [0, 9h) \text{ be defined as } I_\eta^{dl}(h) = h.$$

$$I_\eta^{dl}(8.3h) = 8.3h,\ I_\eta^{dl}(9.34h) = 0.34h,$$

$$I_\eta^{dl}(18.34h) = 0.34h,\ I_\eta^{dl}(-8.66h) = 0.34h \text{ and so on.}$$

Infact infinitely many points in $(-\infty h, \infty h)$ are mapped onto a single point in $[0, mh)$, $h^2 = h$; $1 \leq m < \infty$.

Example 2.25: Let [0, 19h); $h^2 = h$ be the MOD special dual like number plane. $I_\eta^{dl} : (-\infty h, \infty h) \to [0, 19h)$ is defined by $I_\eta^{dl}(25h) = 6h$

$$I_\eta^{dl}(25h) = 6h$$

$$I_\eta^{dl}(-10h) = 9h,\ I_\eta^{dl}(17h) = 17h \text{ and so on.}$$

Thus periodically infinite number of points are mapped onto the single point for instance $(n19 + 0.558)h$ for varying $-\infty < n < \infty$ is mapped by $I_\eta^{dl}((n19 + 0.558)h) = 0.558h$.

Thus these points $(19n + 0.558)h$ for varying n; $-\infty < n < \infty$ are called as periodic points.

These periodic points are infinite and are mapped to 0.558h.

Likewise every point in $(-\infty h, \infty h)$ is a point which is periodic.

Infact if k is a decimal lying in (0, 1) then all $(nm + k)h \in (-\infty h, \infty h)$ for $n \in Z$ are mapped by the MOD interval special dual like number transformation into k.

We call the collection $(nm + k)h \in (-\infty h, \infty h)$ for a fixed k in (0, 1) as the infinite periodic elements of $(-\infty h, \infty h)$ which are mapped by I_η^{dl} into $k \in [0, mh)$.

Since the number of k's in [0, 1) is infinite so we have infinite number of periodic elements which are mapped into a single element in [0, mh).

Thus at every stage we wish to call these MOD special dual like intervals as they are small but non bounded in comparison with $(-\infty h, \infty h)$.

Next we proceed onto the special quasi dual number real interval $(-\infty k, \infty k)$; $k^2 = -k$.

We see $[0, mk)$; $k^2 = (m - 1)k$. $1 \leq m < \infty$; is the MOD special quasi dual number.

Define I_{η}^{dq} : (–∞k, ∞k) → [0, mk) by

I_{η}^{dq} (k) = k, I_{η}^{dq} (mt.k) = 0 for all t ∈ Z.

There are infinite number of elements in (–∞k, ∞k) which is mapped onto the zero element of [0, mk).

I_{η}^{dq} ((mt + r)k)

= rk ∈ [0, mk); r ∈ (0, m).

Consider [0, 6k); $k^2 = 5k$.

Define I_{η}^{dq} (15.032k) = 3.032k ∈ [0, 6k).

I_{η}^{dq} (–4.5k) = 1.5k, I_{η}^{dq} (27.05k) = 3.05k,

I_{η}^{dq} (7.5k) = 1.5k, I_{η}^{dq} (13.5k) = 1.5k,

I_{η}^{dq} (25.5k) = 1.5k, I_{η}^{dq} (43.5k) = 1.5k and so on.

Let I_{η}^{dq} : (–∞k, ∞k) → [0, 23k) functions as follows:

I_{η}^{dq} (40.3k) = 17.3k;

I_{η}^{dq} (–3k) = 20k;

I_{η}^{dq} (43k) = 20k,

I_{η}^{dq} (66k) = 20k, we see several elements in (–∞k, ∞k) onto a single element of [0, 23k).

Thus all special quasi dual number interval transformation I_η^{dq} maps periodically infinite number of elements into a single element in [0, mk).

Further we cannot as in case of other elements map elements in [0, mk) into a single element in (–∞k, ∞k). The reverse transformation is not that easy.

Let [0, 10k); $k^2 = 9k$ be the MOD special quasi dual number interval. Let $x = 8.009k \in (-\infty k, \infty k)$ we see

$$I_\eta^{dq} = 8.009k.$$

$$I_\eta^{dq}\ (48.64k) = 8.64k,\ I_\eta^{dq}\ (0.98k) = 0.98k,$$

$$I_\eta^{dq}\ (-9.2k) = 0.8k,$$

$$I_\eta^{dq}\ (12.53k) = 2.53k,$$

$$I_\eta^{dq}\ (-43.42k) = 6.58k \text{ and so on.}$$

This is the way the MOD special quasi interval transformation is carried out.

Example 2.26: Let [0, 12k); $k^2 = 11k$ be the MOD special quasi dual number interval.

$$I_\eta^{dq} : (-\infty k, \infty k) \to [0, 12k) \text{ is defined as follows:}$$

$$I_\eta^{dq}\ (49.372k) = 1.372k,\ I_\eta^{dq}\ (10k) = 10k,$$

$$I_\eta^{dq}\ (-7k) = 5k \text{ and } I_\eta^{dq}\ (-243.7k) = 8.3k.$$

This is the way the MOD special quasi interval transformation is carried out.

Infact we have infinitely many periodic elements in $(-\infty k, \infty k)$ mapped onto $[0, 12k)$.

However the mapping of $T_\eta^{dq} : [0, 12k) \to (-\infty k, \infty k)$ is an one to one embedding which we call as the trivial map.

If we map $T_\eta^{dq}(x) = (12n + x)k$ for every $x \in [0, 12k)$ and $n \in (-\infty, \infty)$ then certainly it is not the usual map for in no place an element from domain space is mapped onto infinite number of elements in $(-\infty k, \infty k)$ we can this as special pseudo map.

For if $[0, 5k)$; $k^2 = 4k$ is the special quasi dual number interval then $T_\eta^{dq}(x) = 5tk + x$ this is true for all $x \in [0, 5k)$; $t \in (-\infty, \infty)$.

For instance $x = 4.5k$ is in $[0, 5k)$ then

$$T_\eta^{dq}(4.5k) = (5t + 4.5)k \; t \in (-\infty, \infty).$$

So for the first time we define such a map.

We see $I_\eta^{dq} \; T_\eta^{dq} = T_\eta^{dq} \text{ o } I_\eta^{dq} = I.$

For instance if instead $\eta: (-\infty, \infty) \to [0, m)$ then
$\eta_T : [0, m) \to (-\infty, \infty)$ such that every $x \in [0, m)$ is mapped onto $mt + x$, $t \in (-\infty, \infty)$.

$\eta_T \circ \eta = \eta \circ \eta_T$ = identity map.

Similarly for $\eta_I : (-\infty I, \infty I) \to [0, mI)$ we have T^I such that $T^I: [0, mI) \to (-\infty I, \infty I)$ with

$T^I(x) = (mt + x)$; $m \in (-\infty, \infty)$ so x is mapped onto infinite number of points.

However T^I is not a map. Similarly for η_c we have T^c and so on.

These are not maps but a special type of pseudo functions which maps a single point in $[0, mi_F)$ into infinitely many elements in $(-\infty i, \infty i)$.

Chapter Three

INTRODUCTION TO MOD PLANES

In this chapter we for the sake of completeness recall the new notion of MOD planes introduced in [24]. This is done for two reasons

(i) as the subject is very new and

(ii) if they are recalled the reader can readily know those concepts without any difficulty.

We just recall them in the following.

DEFINITION 3.1: *Let $R_n(m) = \{(a, b) / a, b \in [0, m)\}$; we define $R_n(m)$ as the MOD real plane built or using the interval $[0, m)$.*

We see the whole of the real plane $R \times R$ can be mapped onto the real MOD plane $R_n(m)$.

This is done by the transformation η: $R \times R \rightarrow R_n(m)$.

$$\text{by } \eta(a, b) = \begin{cases} (0,0) & \text{if } a = ms \text{ and } b = mt \\ (0,t) & \text{if } a = ms \text{ and } \dfrac{b}{m} = v + \dfrac{t}{m} \\ (u,0) & \text{if } \dfrac{a}{m} = s + \dfrac{u}{m} \text{ and } b = mr \\ (r,s) & \text{if } \dfrac{a}{m} = t + \dfrac{r}{m} \text{ and } \dfrac{b}{m} = u + \dfrac{s}{m} \\ (a,b) & \text{if } a < m \text{ and } b < m \end{cases}$$

The transformation η is such that several pair in R or infinite number of pairs in R are mapped onto a single point in $R_n(m)$.

We have called this as real MOD transformation of R ×R to $R_n(m)$.

We will illustrate this situation by some examples.

Example 3.1: Let $R_n(3) = \{(a, b) \mid a, b \in [0,3)\}$ be the MOD plane on the MOD interval [0,3).

Clearly if $(23.7, 84.52) \in R \times R$ then by the MOD transformation $\eta : R \to R_n(3)$.

We get $\eta(23.7, 84.52) = (2.7, 0.52) \in R_n(3)$.

If $x = (-5.37, 2.103) \in R \times R$ then $\eta(x) = \{(0.63, 2.103)\}$.

This is the way MOD transformation from the real plane $R \times R$ to the MOD plane $R_n(3)$ is carried out.

The MOD real plane on the MOD interval [0, 3) is as given Figure 3.1.

Example 3.2: Let $R_n(8) = \{(a, b) \mid a, b \in [0, 8)\}$ be the real or Euclid MOD plane on the MOD interval [0, 8).

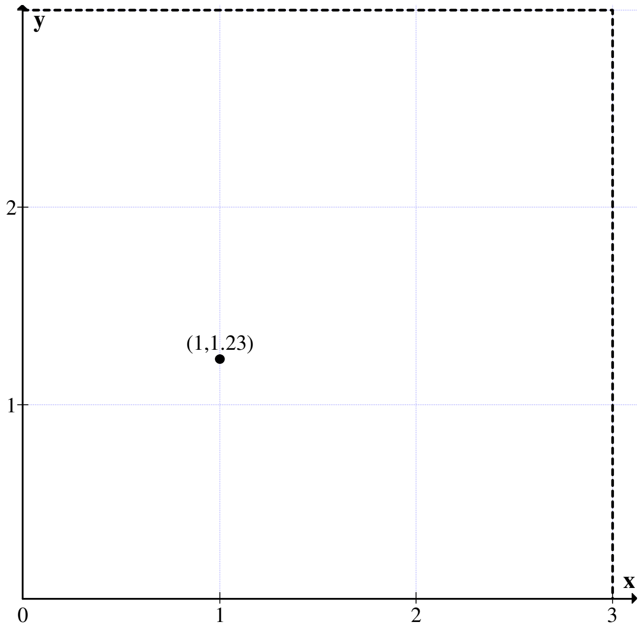

Figure 3.1

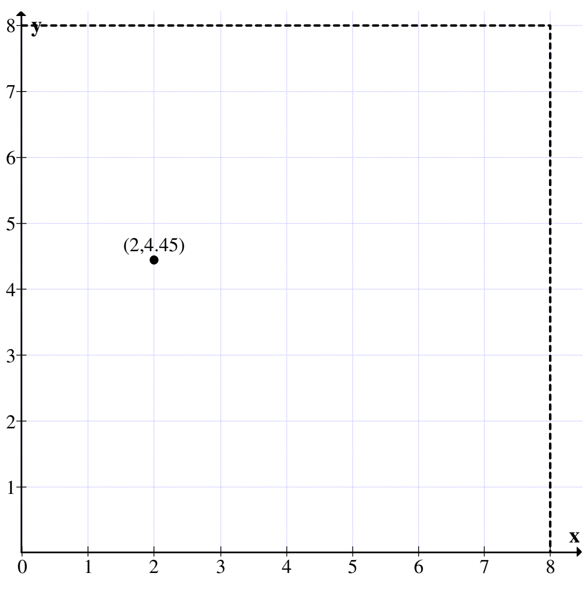

Figure 3.2

Let x = (27.003, 4.092) ∈ R × R.
Now the MOD transformation of x is as follows.

$$\eta(x) = (3.003, 4.092) \in R_n(8).$$

We give the MOD plane $R_n(8)$ in Figure 3.2.

Example 3.3: Let $R_n(13)$ be the real MOD plane on the MOD interval [0, 13).

Let us define $\eta : R \times R \to R_n(13)$ by $\eta((18.021, 16.0079)) = (5.021, 3.0079) \in R_n(13)$.

The MOD Euclid plane is as follows:

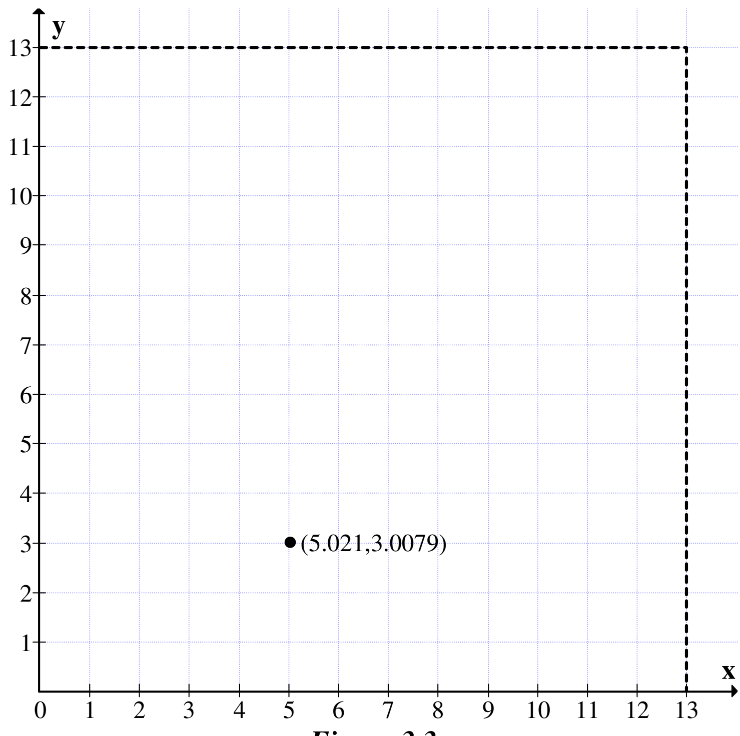

Figure 3.3

Example 3.4: Let $R_n(10)$ be the MOD real plane. Let η be a map from $R \times R \to R_n(10)$.

Take x = (98.503, –5.082) $\in$ R $\times$ R.

η(x) = (8.503, 4.918) $\in$ $R_n(10)$. η is a MOD real transformation.
Now for properties of MOD real or euclid planes refer [24].

Next we proceed onto recall the definition of the notion of MOD complex plane and the MOD complex transformation from C to $C_n(m)$ defined by $\eta_c : C \to C_n(m)$ where

$C_n(m) = \{a + bi_F = (a, b) \mid i_F^2 = m – 1, a, b \in [0, m)\}$ is the complex modulo integer MOD plane defined in [24].

Examples if them will be provided before the complex MOD transformation is defined.

Example 3.5: Let $C_n(15) = \{a + bi_F = (a, b),\ i_F^2 = 14\}$ be the MOD complex modulo integer plane.

Example 3.6: Let $C_n(13) = \{a + bi_F = (a, b);\ i_F^2 = 12\}$ be another complex modulo MOD integer plane.

Example 3.7: Let $C_n(48) = \{a + bi_F = (a, b);\ i_F^2 = 147\}$ be a complex modulo integer MOD plane.

It is important to keep on record that we have infinite number of complex modulo integer planes, just like we have infinite number of real or euclid MOD integer planes.

Now we just give the diagrammatic representation of them.

Example 3.8: Let $C_n(6)$ be the complex modulo MOD integer plane given by Figure 3.4.

Example 3.9: Let $C_n(11) = \{a + bi_F = (a, b);\ i_F^2 = 10\}$ be the complex finite modulo integer MOD plane.
This is given in Figure 3.5.

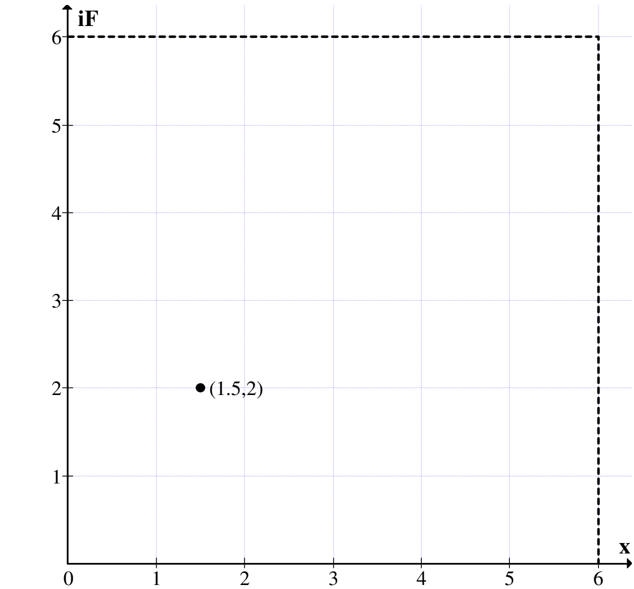

Figure 3.4

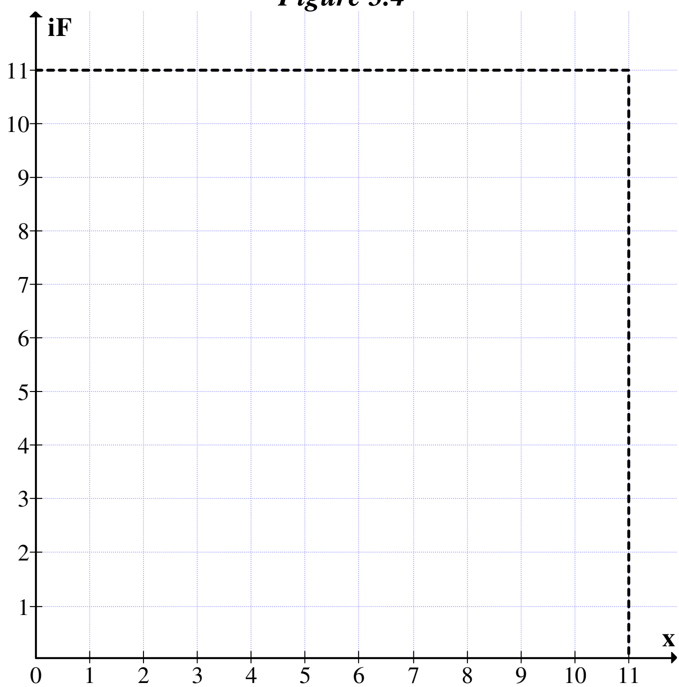

Figure 3.5

Example 3.10: Let $C_n(9) = \{a + bi_F = (a, b);\ i_F^2 = 8\}$ be the complex finite modulo integer MOD plane.

The graph or the MOD plane $C_n(9)$ is as follows:

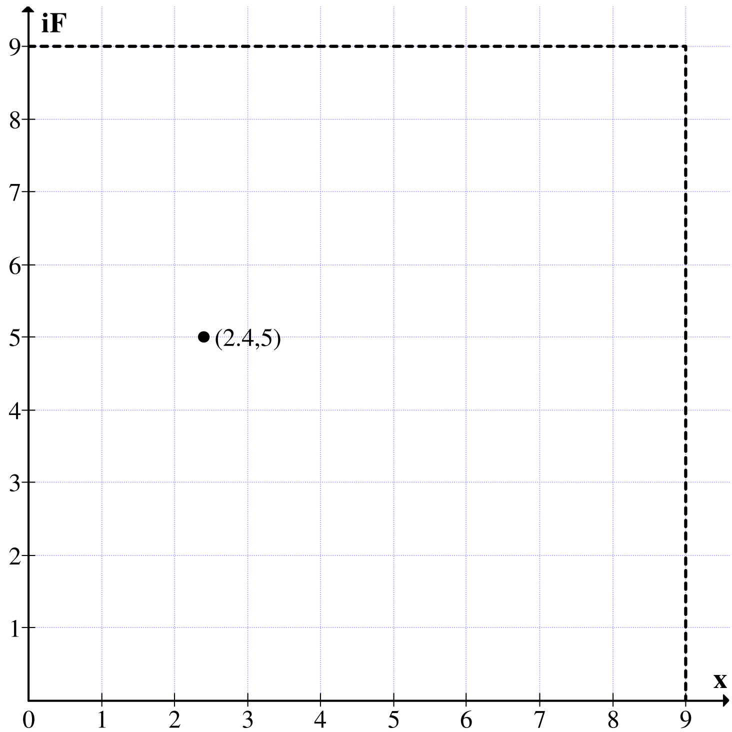

Figure 3.6

Now we give a few illustrations of MOD complex modulo integer transformation.

Let $C_n(16) = \{a + bi_F = (a, b);\ i_F^2 = 15\}$.

Let $x + iy = 23.5 + 19.312i \in C$.

$\eta_c(i) = i_F$ is always fixed.

$$\eta_c(x + iy) = (7.5, 3.312i_F) = 7.5 + 3.312i_F \in C_n(16).$$

This is the way the MOD complex transformation is carried out.

Let $x_1 + iy_1 = 39.5 + 51.312i_F \in C$.

$$\eta_c (x_1 + iy_1) = 7.5 + 3.312i_F \in C_n(16).$$

Thus both $x + iy$ and $x_1 + iy_1 \in C$ are mapped onto the same point in $C_n(16)$; however both $x + iy$ and $x_1 + iy_1$ are distinct.

Let $C_n(7) = \{a + bi_F = (a, b);\ i_F^2 = 6\}$ be the complex modulo integer MOD plane.

The plane is represented in the following:

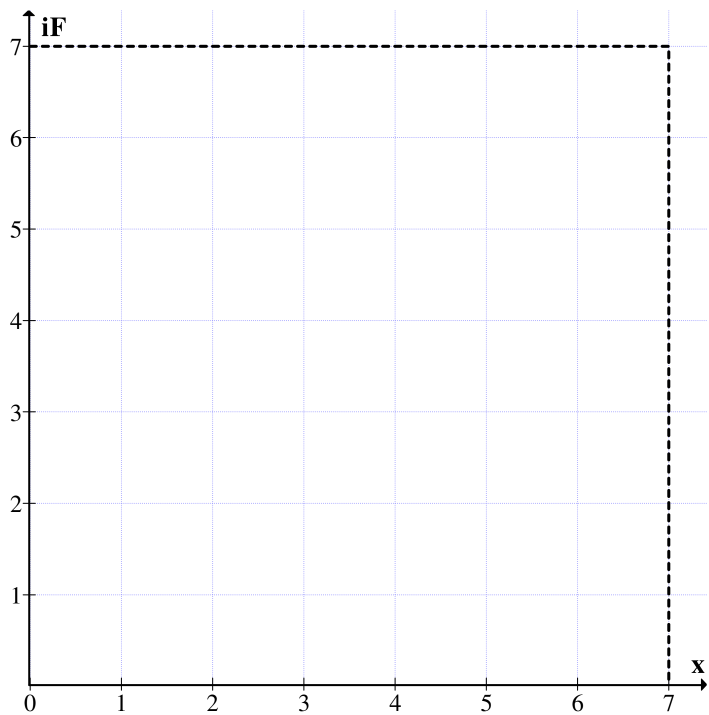

Figure 3.7

Let $(-40.331, 17.5i) \in C$ that is $z = -40.331 + 17.5i_F$

$\eta_c(z) = (1.669, 3.5i_F) \in C_n(7)$.

Let $z_1 = x_1 + iy_1 = 28.0092 + 49.003$

$\eta_c(z_1) = (0.0092, 0.003)$.

This is the way the MOD complex transformation is carried out.

Let $z_2 = x_2 + iy_2 = (x_2, y_2)$; $(40.0032, -15.312) \in C$.

$\eta_c(z_2) = (5.0032, 5.688) = 5.0032 + 5.688i_F \in C_n(7)$.

Next we give some examples of some more transformation.

Let $C_n(20) = \{a + bi_F | a, b \in [0, 20); i_F^2 = 19\}$ be the MOD complex finite modulo integer plane.

We show how MOD complex transformation η_c from C to $C_n(20)$ is defined $\eta_c : C \to C_n(20)$ defined in the way.

Let $z = 40 + 60i \in C$

$$\eta_c(z) = (0, 0).$$

Let $z_1 = -68 + 70i \in C$; $\eta_c(z_1) = (12, 10) = 12 + 10i_F$.

Let $z_2 = -48 + 5.32i \in C$. $\eta_c(z_2) = (12, 5.32) = 12 + 5.32i_F$

Let $z_3 = -3.75-4.205i \in C$.

$\eta_c(z_3) = (16.25, 15.795) = 16.25 + 15.795i_F \in C_n(20)$.

Let $z_4 = 6.853 + 7.432i \in C$.

$\eta_c(z_4) = 6.853 + 7.432i_F \in C_n(20)$.

This is the way the MOD complex transformation takes place.

Now for more about MOD complex plane and MOD complex transformation refer [24-5].

Next we proceed onto recall the notion of neutrosophic MOD planes.

$R_n^I(m) = \{a + bI \mid a, b \in R, I^2 = I, a, b \in [0, m)\}$ is the neutrosophic MOD plane built using the interval [0,m).

For more about this refer [24-5].

However for the sake of self containment a few examples of neutrosophic MOD planes and the MOD transformation will be provided for the reader.

Example 3.11: Let $R_n^I(5) = \{a + bI \mid a, b \in [0, 5), I^2 = I\}$ is the MOD neutrosophic plane built on the MOD interval [0, 5).

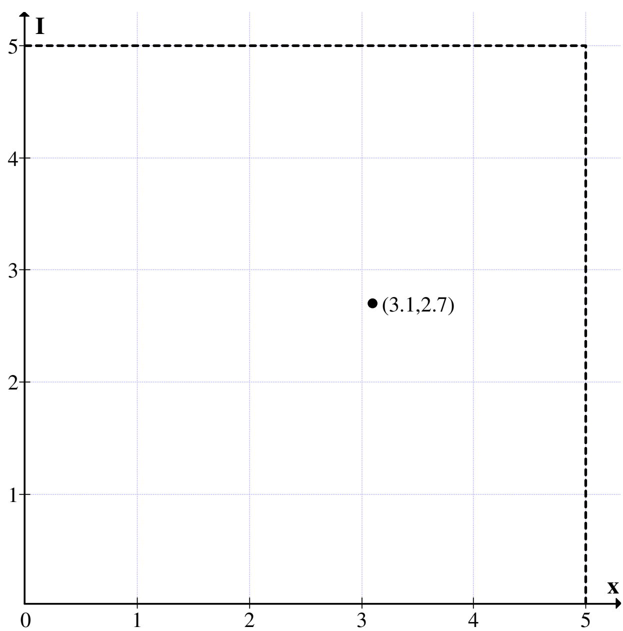

Figure 3.8

If x = 3.1 + 2.7I then x in the above plane is marked.

Example 3.12: Let $R_n^I(10) = \{a + bI|\ a, b \in [0, 10), I^2 = I\}$ be the MOD real neutrosophic plane built on [0, 10).

The graphic representation is as follows.

We plot x = 9.3 + 2.7I, y = 5.7 + 3I and z = 6.8 + 4.6I in this MOD plane.

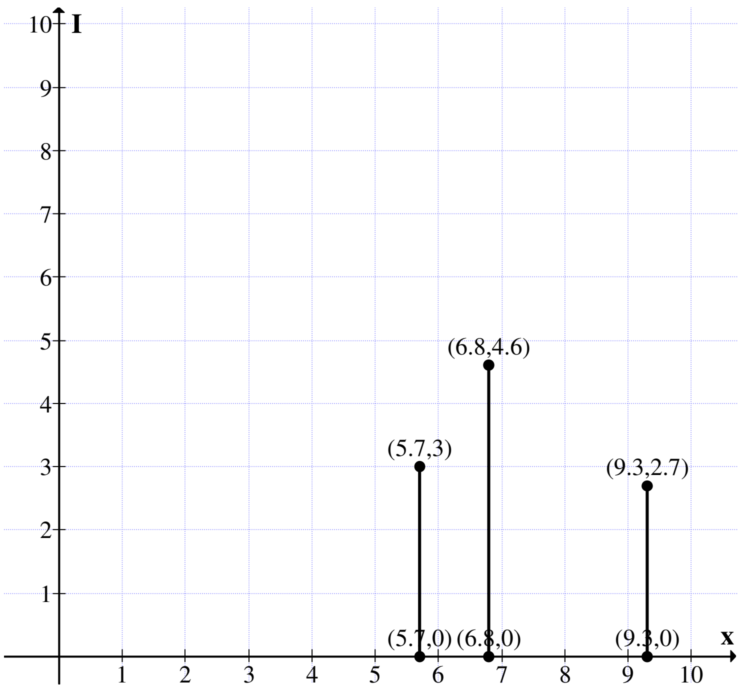

Figure 3.9

Example 3.13: Let

$$R_n^I(13) = \{a + bI \mid a, b \in [0, 13), I^2 = I\}$$

be the MOD neutrosophic plane built on [0, 13).

Let us place the point

$x = (5, 5I) = 5 + 5I$, $y = 10.5 + 3.1I$ and $z = 2.7 + 4.2I$

in Figure 3.10.

Now having defined and illustrated by examples of MOD neutrosophic planes we just show how the MOD neutrosophic transformation is performed.

Example 3.14: Let

$$R_n^I(10) = \{a + bI \mid a, b \in [0, 10), I^2 = I\}$$

be the MOD neutrosophic plane.

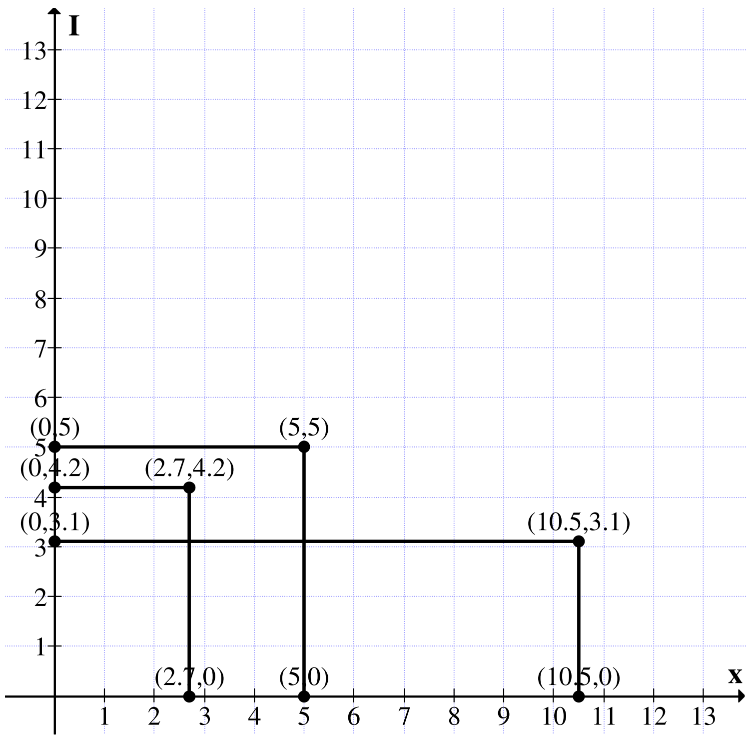

Figure 3.10

Let $\eta_I : \langle R \cup I\rangle \to R_n^I(10)$ be defined as follows.

Let

$$x = 27.33 + 43.42I \in \langle R \cup I\rangle$$

$$\eta_I(x) = 7.33 + 3.42I = (7.33, 3.42) \in R_n^I(10).$$

Let

$$y = -40.37 + 62.34I \in \langle R \cup I\rangle$$

$$\eta_I(y) = 9.63 + 2.34I \in R_n^I(10)$$

η_I is a MOD neutrosophic transformation of $\langle R \cup I\rangle$ to $R_n^I(10)$. $\eta_I(I) = I$ always.

Consider

$$z = -20.37 + 42.34I \in \langle R \cup I\rangle$$

$$\eta_I(z) = 9.63 + 2.34I \in R_n^I(10).$$

This is the way operation is performed. Thus it is clear z and y in the neutrosophic plane $\langle R \cup I\rangle$ are distinct but however z and y are identical in the MOD neutrosophic plane $R_n^I(10)$. Thus infinite number of points in $\langle R \cup I\rangle$ are mapped onto a single point in the MOD neutrosophic plane $R_n^I(10)$. Thus η is a many to one transformation from $\langle R \cup I\rangle$ to $R_n^I(10)$.

However the entire neutrosophic real plane is mapped onto the one quadrant but infinite compact neutrosophic MOD plane.

Example 3.15: Let

$$R_n^I(7) = \{a + bI \mid a, b \in [0, 7), I^2 = I\}$$

be the MODneutrosophic plane on the MOD interval [0, 7).

Let

$$x = 48 + 0.7I \in \langle R \cup I\rangle.$$

Using the MOD transformation $\eta_I : \langle R \cup I\rangle \to R_n^I(7)$

$$\eta_I(4.8 + 0.7I) = 6 + 0.7I \in R_n^I(7)$$

Let

$$y = -42 + 8.5I \in \langle R \cup I\rangle; \eta_I(y) = (0 + 1.5I) \in R_n^I(7).$$

Let

$$z = 17.5 + 14I \in \langle R \cup I\rangle. \eta_I(z) = 3.5 + 0I \in R_n^I(7)$$

Let

$$v = -10.8 - 16.5I \in \langle R \cup I\rangle. \eta_I(v) = 3.2 + 4.5I \in R_n^I(7)$$

This is the way the MOD neutrosophic transformation are performed from $\langle R \cup I\rangle \to R_n^I(7)$.

Example 3.16: Let $R_n^I(6)$ be the MOD neutrosophic plane on the MOD interval [0, 6).

Let

$$x = 12.7 + 30.5I \in \langle R \cup I\rangle;$$

$$\eta_I : \langle R \cup I\rangle \to R_n^I(6)$$

$$\eta_I(12.7 +\_ 30.5I) = 0.7 + 0.5I \in R_n^I(6).$$

Thus this MOD neutrosophic transformation can map and will map infinite number of elements in $\langle R \cup I\rangle$ in $R_n^I(6)$.

For

$$y = 24.7 + 42.5I \in \langle R \cup I\rangle$$

is mapped by η_I to 0.7 + 0.5I.

Let

$$z = 6.7 + 12.5I \in \langle R \cup I\rangle$$

is mapped by η_I to 0.7 + 0.5I and so on.

Thus this MOD neutrosophic transformation η_I maps infinite number of coordinates in $\langle R \cup I \rangle$ onto the MOD plane $R_n^I(m)$ built on the MOD interval [0, m).

Thus this property will be exploited. Further the advantage of using MOD neutrosophic plane is that instead of working on the four quadrant of the usual real neutrosophic plane $\langle R \cup I \rangle$; we can work on the single compact quadrant yet infinite.

The following figure suggest why the concept of MOD is advantageous over $\langle R \cup I \rangle$.

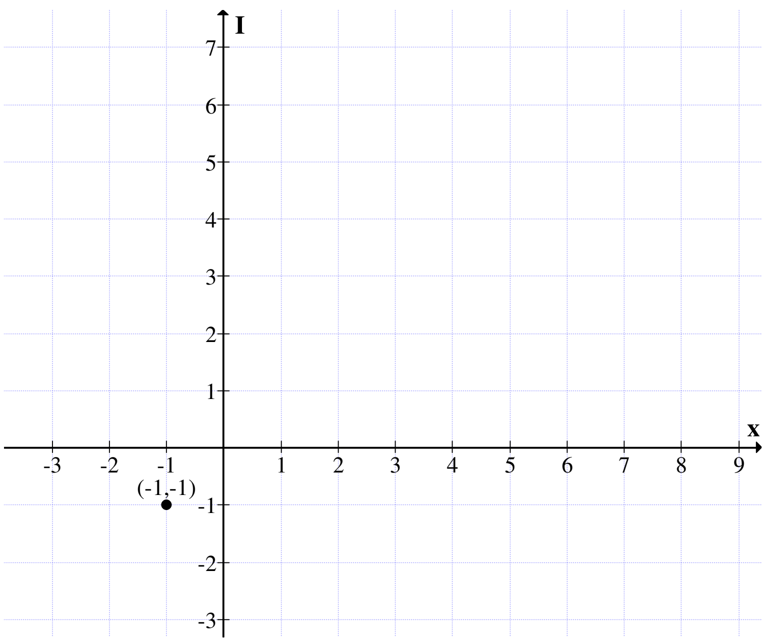

Figure 3.11

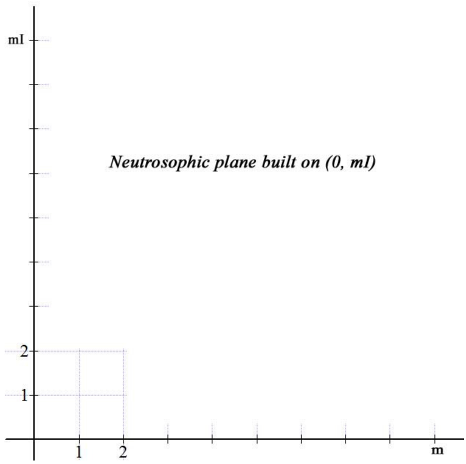

Figure 3.12

Now having seen the notion of MOD neutrosophic plane we proceed onto describe and develop MOD dual number plane and their generalizations.

We will illustrate this situation by some examples.

Example 3.17: Let

$$R_n(5)(g) = \{a + bg \mid a, b \in [0, 5)\ g^2 = 0\}$$

be the MOD dual number plane built using the interval [0, 5).

$$R_n(5)g = \{(a, b) = a + bg\}$$

is also another way of representing a MOD dual number plane.

The advantage of using MOD dual number plane is instead of working with four quadrant we work compactly on only one quadrant.

This is illustrated by the following graphs.

$$R(g) = \{a + bg = (a, b) \mid a, b \in R, g^2 = 0\}$$

is the real dual number plane which has four quadrants.

The work in this direction can be had from [24].

Further it is pertinent to keep on record that we can develop this dual number plane easily to n dimensional dual number plane.

However at this juncture we use only a two dimension plane.

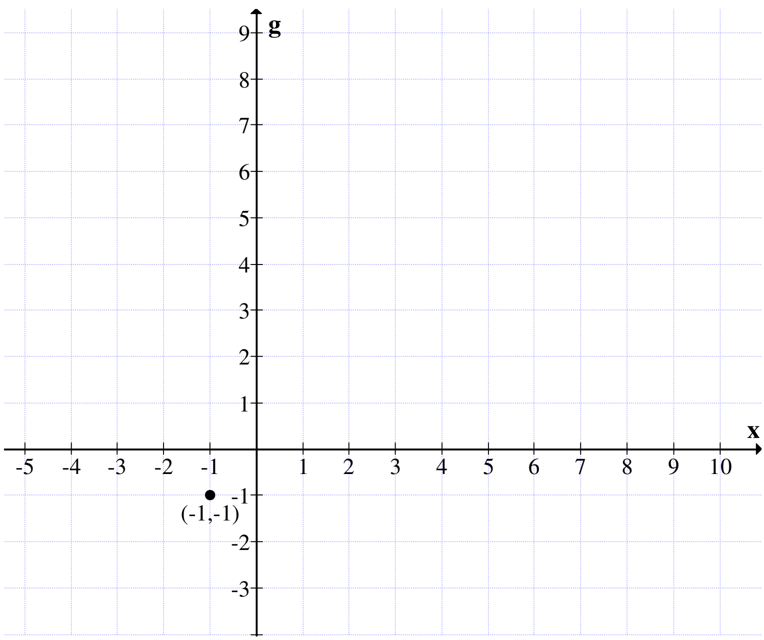

Figure 3.13

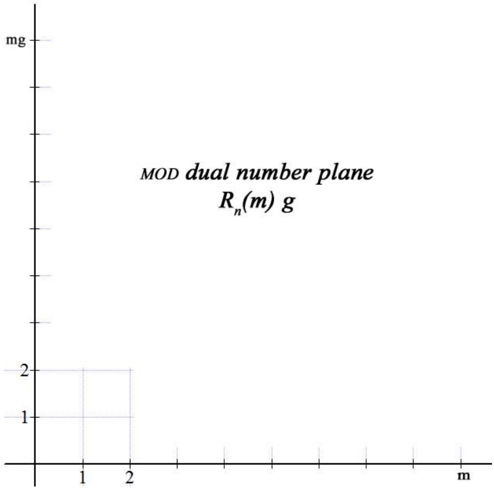

Figure 3.14

Example 3.18: Let $R_n(8)(g) = \{a + bg = (a, b) \mid a, b \in [0, 8), g^2 = 0\}$ be the MOD dual number plane built on the interval [0, 8).

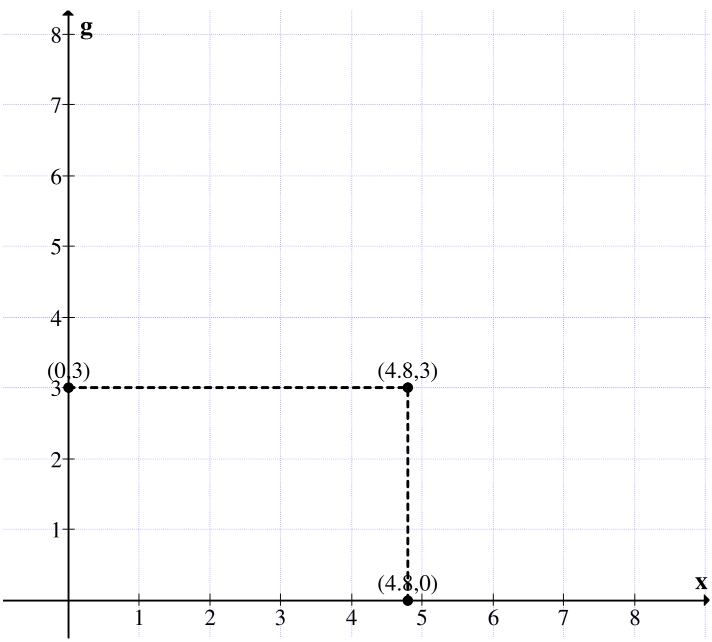

Figure 3.15

Next we proceed onto describe and develop these MOD dual number planes and the MOD dual transformations from R(g) to $R_n(m)(g)$.

Example 3.19: Let $R_n(9)g$ be the dual number MOD plane. The graph of the plane is given in Figure 3.16.

If x = 5.5 + 6g then it is plotted as above.

Example 3.20: Let $R_n(6)g$ be the MOD dual number plane built using [0, 6).

The plane of $R_n(6)g$ and points in $R_n(6)g$ are given in the Figure 3.17.

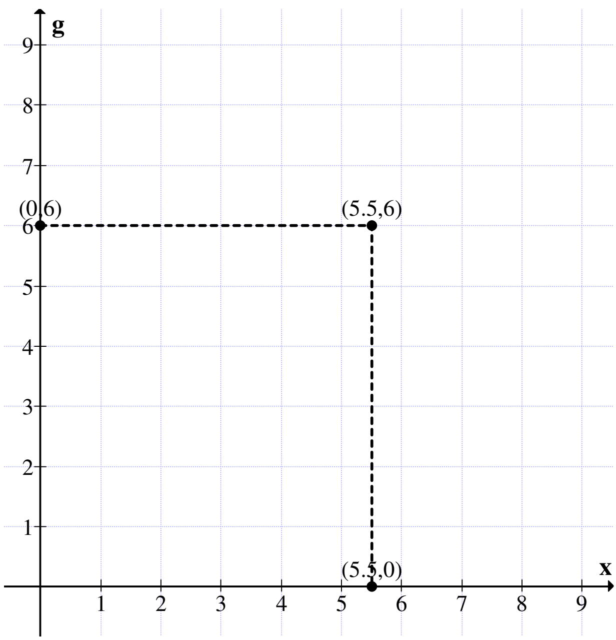

Figure 3.16

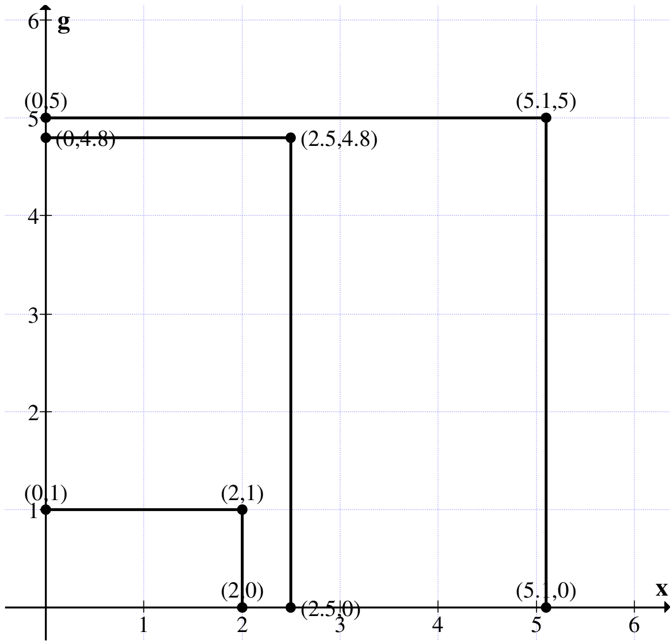

Figure 3.17

Let $x_1 = 2.5 + 4.8g$, $x_2 = 5.1 + 5g$, $x_3 = g + 2 \in R_n(16)$.

The graph of them is as above.

Now having seen how the representation in the MOD dual number plane looks we now proceed onto describe the MOD dual number transformations η from R(g) to $R_n(m)g$ in the following.

Recall

$$R(g) = \{a + bg| \ a, b \in R, g^2 = 0\}$$

and

$$R_n(m)g = \{a + bg| \ a, b \in [0, m), g^2 = 0\}.$$

Let $x = 13 + 24g \in R(g)$ to find the point x in $R_n(9)g$.

$$\eta_g(g) = g$$

and

$$\begin{aligned} \eta_g(g) &= \eta_g(x) = \eta_g(13 + 24g) \\ &= (4 + 6g) \end{aligned}$$

$= \quad (4, 6) \in R_n(g).$

Let $x_1 = 18 + 45g \in R(g)$

$\eta_g(x_1) = 0.$

Let $x_2 = -46 + 29g \in R(g)$

$\eta_g(x_2) = (8, 2) = 8 + 2g.$

Let $x_3 = -49.23 + 8.34g \in R(g)$

$\eta_g(x_3) = (4.77, 8.34) = 4.77 + 8.34g \in R_n(g).$

Now having seen the examples of MOD dual number planes and the MOD dual number transformation we proceed onto describe the MOD special dual like number planes.

For more about this concept please refer [24-5].

$R_n(m)(h)$ or $R_n(m)(g_1) = \{a + bg_1 \mid a, b \in [0, m),\ g_1^2 = g_1\}$ is defined as the special MOD dual like number plane [17].

Infact with varying m; $1 < m < \infty$ we have infinitely many such planes. We will illustrate this by some examples.

Example 3.21: Let

$$R_n(10)g_1 = \{a + bg_1 = (a, b) \mid g_1^2 = g_1, a, b \in [0, 10)\}$$

be the MOD special dual like number plane.

Let $y = 3.5 + 7g_1$ and $x = 9.6 + 2.7g_1 \in R_n(10)g.$

The graph of its given below:

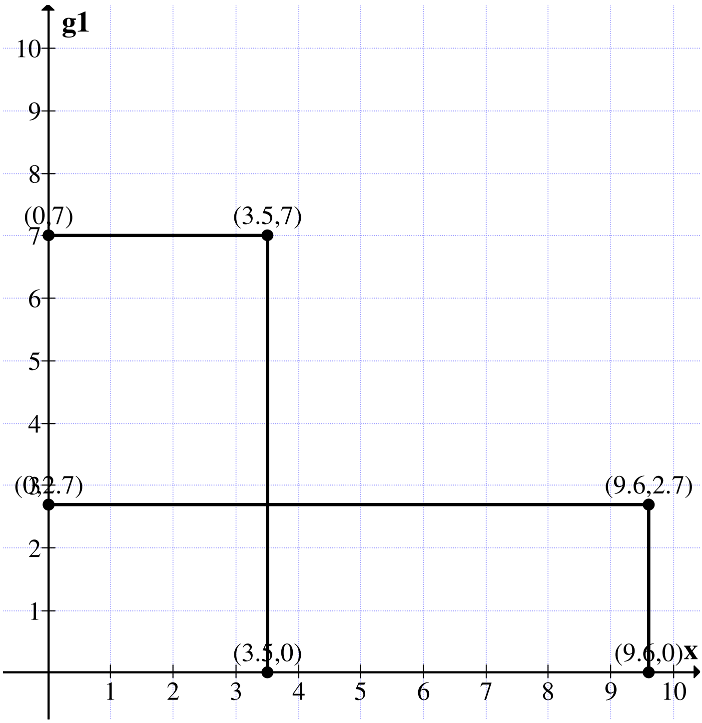

Figure 3.18

Example 3.22: Let

$$R_n(11)g_1 = \{(a, b) = a + bg_1 \mid a, b \in [0, 11),\ g_1^2 = g_1\}$$

be the special dual like number MOD plane on the MOD interval [0, 11).

Let

$$x = 9.5 + 2.9g_1,$$
$$y = 7.6 + 3.4g_1$$

and

$$z = 1.8 + 5g_1 \in R_n(11)g_1.$$

This graph is given below:

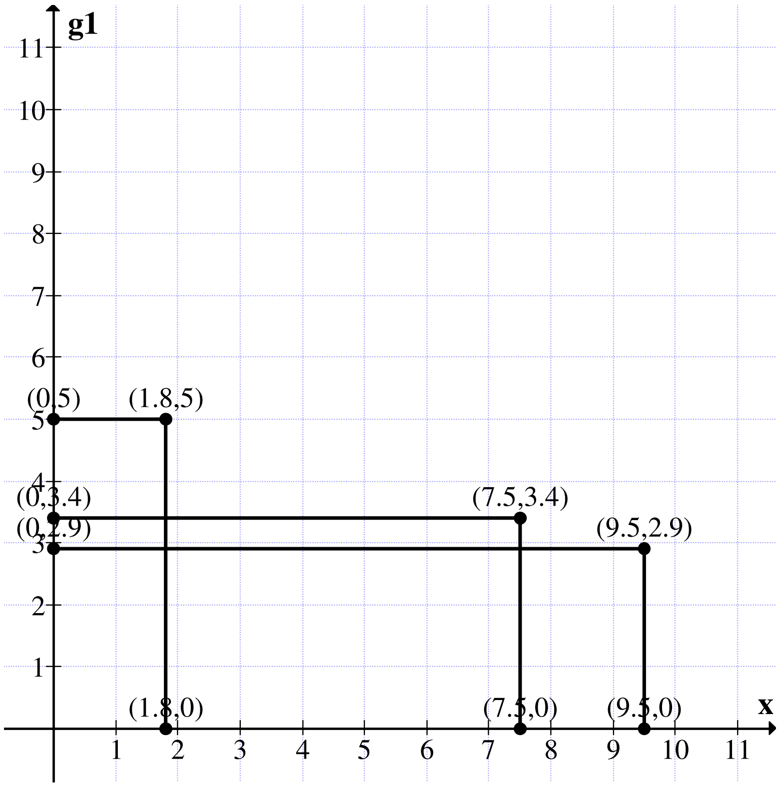

Figure 3.19

Example 3.23: Let

$$R_n(12)g_1 = \{(a, b) = a + bg_1 \mid a, b \in [0, 12),\ g_1^2 = g_1\}$$

be the special MOD dual like number plane.

We see we can fix or plot the given points on this plane.

Let

$$x = 10.2 + 7.5g_1,$$

$$y = 8 + 6g_1$$

and

$$z = 4.3 + 2g_1 \in R_n(12)g_1.$$

The graph is as follows:

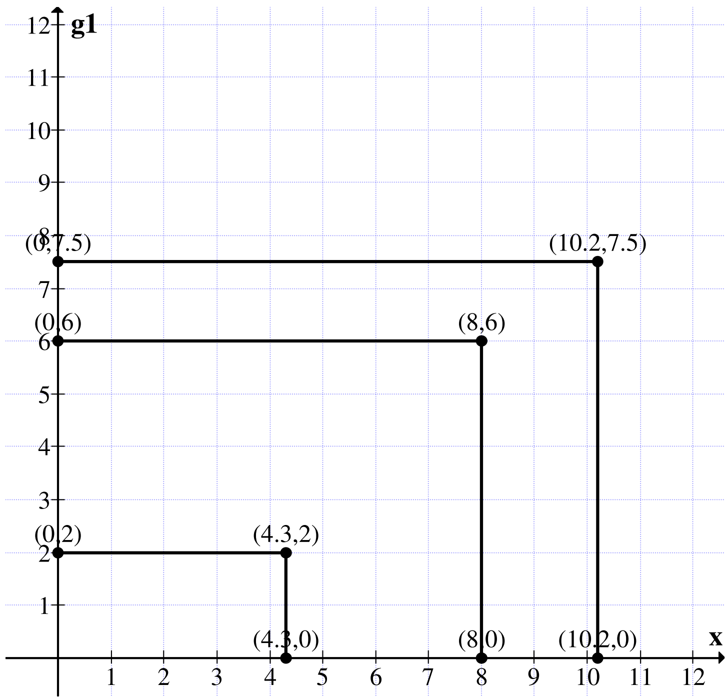

Figure 3.20

Next we proceed onto describe the MOD special dual like number transformation.

$$\eta_{g_1} : R(g_1) \rightarrow R_n(m)\, g_1.$$

$$\eta_{g_1}(g_1) = g_1 \text{ for any } a + bg_1 = x$$

$$\eta_{g_1}(x) = (d, u).$$

$$\text{Here } \frac{a}{m} = c + \frac{d}{m} \text{ and } \frac{b}{m} = t + \frac{u}{m}$$

We will illustrate the various values of $x = a + bg_1$ in $R(g_1)$.

Let $x = 14 + 12.9g_1 \in R(g_1)$ to find the special quasi dual MOD transformation $\eta_{g_1} : R(g_1) \to R_n(10)g_1$;

$$\eta_{g_1}(x) = 4 + 2.9g_1 \in R_n(10)g_1$$

Let

$$y = 3 + 4.337g_1 \in R(g_1)$$

$$\eta_{g_1}(y) = 3 + 4.337g_1 \in R_n(10)g_1.$$

Let

$$z = 24 + 22.9g_1 \in R(g_1)$$

$$\eta_{g_1}(z) = 4 + 2.9g_1 \in R_n(10)g_1.$$

Let

$$u = 32 + 16.34g_1 \in R(g_1)$$

$$\eta_{g_1}(u) = 2 + 6.34g_1 \in R_n(10)g_1.$$

Let

$$v = 34 + 42.9g_1 \in R(g_1)$$

$$\eta_{g_1}(v) = 4 + 2.9g_1 \in R_n(10)g_1.$$

We see x, z and v in $R(g_1)$ are mapped onto the same point $4 + 2.9g_1$

Let $x = 19.72 + 4.331g_1 \in R(g_1)$ now to find where x is mapped in the MOD special dual like number plane $R_n(5)g_1$.

$$\eta_{g_1}(x) = (4.72, 4.331) = 4.72 + 4.331g_1 \in R_n(5)g_1.$$

Let

$$y = 14.72 + 14.331g_1 \in R(g_1)$$

$$\eta_{g_1}(y) = 4.72 + 4.331g_1 \in R_n(5)g_1.$$

We see $x \neq y$ in $R(g_1)$ but $\eta_{g_1}(x) = \eta_{g_1}(y)$.

Thus we see several points can be mapped onto single point.

Infact we have infinite number of points in the special dual like number plane $R(g_1)$ mapped onto single point.

Thus the entire special dual like number plane $R(g_1)$ is mapped onto the MOD special dual like number plane $R_n(5)g_1$ that is the all elements from all the four quadrants of $R(g_1)$ is mapped onto the $R_n(5)g_1$ which is only a single quadrant MOD plane.

Thus this MOD plane is not only considered compact but one has a choice to choose any number and get the plane without leaving a single point in the special dual like number plane $R(g_1)$.

Further we have infinite number of such MOD special dual like number planes.

Let us consider the MOD special dual like number plane $R_n(12)g_1$.

Now we show how $R(g_1)$ is transformed into $R_n(12)g_1$ using MOD special dual like number transformations

$$\eta_{g_1} : R(g_1) \to R_n(12)g_1.$$

Let us consider the element $x = 49.8 + 205.6g_1 \in R(g_1)$. $\eta_{g_1}(g_1) = g_1$ and $\eta_{g_1}(x) = 1.8 + 1.6g_1 \in R_n(12)g_1$

$$\eta_{g_1}(y) = \eta_{g_1}(-10.6 + 72g_1) = 1.4 \in R_n(12)g_1$$

$$\eta_{g_1}(75.3 + 108g_1) = (3.3 + 0) = 3.3 \in R_n(12)g_1.$$

$$\eta_{g_1}(96 + 84g_1) = 0 \in R_n(12)g_1.$$

$$\eta_{g_1}(5.6663 + 7.22215g_1) = 5.6663 + 7.22215g_1 \in R_n(12)g_1.$$

This is the way MOD special dual like number transformation is carried out from $R(g_1)$ onto $R_n(12)(g_1)$.

Next we consider the special quasi dual number MOD planes using the special quasi dual numbers.

Recall $R(g_2) = \{a + bg_2 \mid a, b \in R;\ g_2^2 = -g_2\}$ is defined as the real special quasi dual number plane.

However $R(g_2) = \{a + bg_2 = (a, b) \mid a, b \in R;\ g_2^2 = -g_2\}$.

Now we proceed onto give the notion of special MOD quasi dual number planes.

$R_n(m)g_2 = \{a + bg_2 \mid a, b \in [0, m),\ g_2^2 = (m-1)g_2\} = \{(a, b) \mid a, b \in [0, m);\ g_2^2 = (m-1)g_2\}$ is defined as the special MOD quasi dual number plane defined on the MOD interval $[0, m)$.

We will first illustrate this situation by some examples.

Example 3.24: Let

$$R_n(10)g_2 = \{a + bg_2 = (a, b) \mid g_2^2 = 9g_2,\ a, b \in [0, 10)\}$$

be the MOD special quasi dual number plane built on the MOD interval $[0, 10)$.

Example 3.25: Let

$$R_n(7)g_2 = \{a + bg_2 = (a, b) \mid a, b \in [0, 7);\ g_2^2 = 6g_2\}$$

is the MOD special quasi dual number plane built on the MOD interval $[0, 7)$.

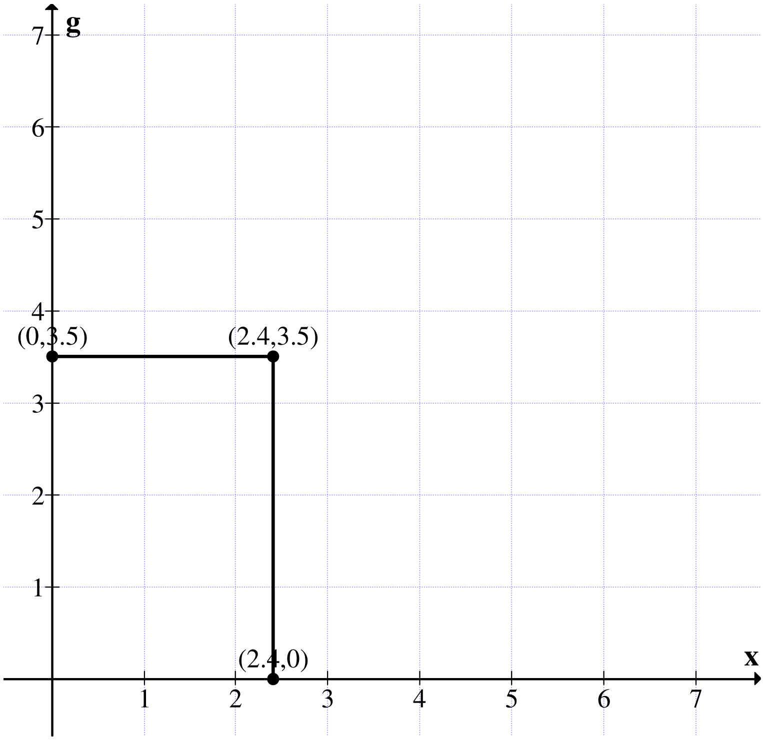

Figure 3.21

Example 3.26: Let

$$R_n(12)g_2 = \{a + bg_2 = (a, b) \mid a, b \in [0, 12);\ g_2^2 = 11g_2\}$$

be the special MOD quasi dual number plane using the interval [0, 12).

Let $x = 10 + 3.2g_2$ and $y = 5.5 + 11g_2 \in R_n(12)g_2$.

The graph of them is as follows:

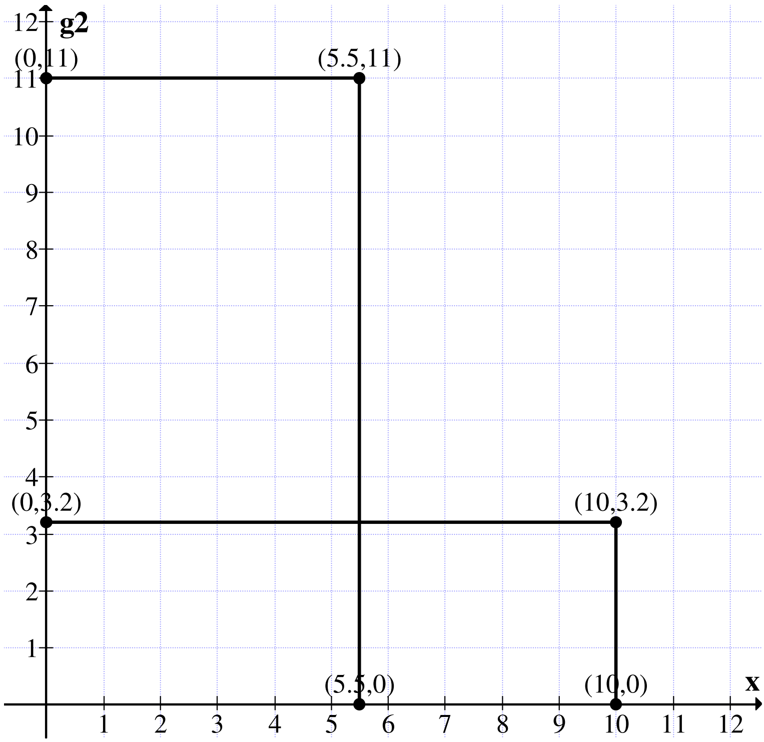

Figure 3.22

Example 3.27: Let

$$R_n(18)g_2 = \{a + bg_2 = (a, b) \mid a, b \in [0, 18);\ g_2^2 = 17g_2\}$$

be the MOD special quasi dual number plane built on the interval [0, 18).

We next proceed onto show how the MOD special quasi dual number transformation from $R(g_2)$ to $R_n(m)g_2$ is performed.

Let us take m = 18.

Suppose $\eta_{g_2} : R(g_2) \to R_n(m)g_2$ we map $\eta_{g_2}(g_2) = g_2$.

Now if $x = 49 + 106g_2$ then $\eta_{g_2}(x) = 13 + 16g_2 \in R_n(m)g_2$.

Let $y = 13 + 34g_2 \in R(g_2)$

$\eta_{g_2}(y) = 13 + 16g_2 \in R_n(m)g_2$.

Next let $z = 31 + 52g_2 \in R(g_2)$

$\eta_{g_2}(z) = 13 + 16g_2 \in R_n(m)g_2$.

Clearly $x, y, z \in R(g_2)$ are all distinct, however
$\eta_{g_2}(x) = \eta_{g_2}(y) = \eta_{g_2}(z) = 13 + 16g_2$.

Thus η the MOD special quasi dual number transformation maps several elements onto a single element, infact η_{g_2} maps infinite number of elements in $R(g_2)$ onto a single element in $R_n(m)g_2$.

This is one of the vital properties enjoyed by MOD transformations in general and MOD quasi special dual numbers transformation in particular.

Now having seen all these planes and their structure we now proceed onto define multi dimensional MOD planes of very many type.

Chapter Four

HIGHER AND MIXED MULTIDIMENSIONAL MOD PLANES

In this chapter we introduce the notion higher dimensional MOD planes of different types. We have 6 types of different higher or multidimensional planes apart from the infinite number of mixed multidimensional planes.

We will define, describe and develop these new multi structure planes in this chapter.

We know $R^n = \{R \times \ldots \times R\}$ is a n-dimensional real space;

$R^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in R; 1 \leq i \leq n\}$.

R^n is a commutative ring with respect to + and $\times$.

Similarly

$C^n = \{(C_1, C_2, \ldots, C_n) \mid C_i \in C$ the complex plane $i = 1, 2, 3, \ldots, n\}$ is the multi or n dimensional complex plane.

x = {(5 + 2i, –3 + 4i, …, 10 + 5i)} be any specific element in C^n with usual + and ×; C^n is also a ring called the multidimensional complex ring or space.

Likewise $(\langle R \cup I\rangle)^m = \{(b_1, \ldots, b_m) \mid b_i \in (R \cup I) = \{a + bI \mid a, b \in R; I^2 = I\}\ 1 \le i \le m\}$ $m \in N$ is the multidimensional or m–dimensional neutrosophic space.

Clearly $(\langle R \cup I\rangle)^n$ under usual + and × is a ring.

Next $[R(g)]^p = \{(a_1 + b_1g, \ldots, a_p + b_pg)$ where $a_i, b_i \in R$; $1 \le i \le p$, $g^2 = 0\}$ is defined as the p-dimensional or multidimensional dual number space; $p \in N$.

$[R(g)]^p$ is a ring under usual + and ×.

Now $(R(h))^t = \{(a_1 + b_1h, a_2 + b_2h, \ldots, a_t + b_th) \mid a_i, b_i \in R$; $1 \le i \le t$; $h^2 = h\}$ is defined as the t-dimensional or multidimensional special dual like number space, $t \in N$.

Clearly $(R(h))^t$ under + and × is a ring.

$(R(k))^m = \{(a_1 + b_1k, a_2 + b_2k, \ldots, a_m + b_mk) \mid a_i, b_i \in R$; $1 \le i \le m$, $k^2 = (m - 1)k\}$ is a multidimensional special quasi dual number space.

$(R(k))^m$ is a ring with respect to + and ×.

We will first illustrate this situation by some examples.

Example 4.1: Let

$$R^5 = \{(a_1, a_2, \ldots, a_5) \text{ where } a_i \in R; 1 \le i \le 5, +, \times\}$$

be a ring which is a multidimensional space.

$x = (0.5, \sqrt{3}, –5.2, \sqrt{7}, 8)$ and

$y = (43, -10.8, 11, \sqrt{17}, 4.331)$ are elements of R.

$x + y$ and $x \times y \in R^5$ it is not difficult to verify.

Example 4.2: Let

$C^7 = \{(a_1, a_2, \ldots, a_7) \mid a_i \in C; 1 \leq i \leq 7, +, \times\}$ be a ring.

$x = (0.5i, 8 + 3i, 4.5i, 7, 0, 3i, -10 - 4i)$ and

$y = (0, 3i, 2, 4 - 2i, -7 + 14i, 10 - 2i, 0) \in C^7$

$x + y = (0.5i, 8 + 6i, 6 - 5i, 11 - 2i, -7 + 14i, 10 + i, -10 - 4i)$ and

$x \times y = (0, 24i - 9, 8 - 10i, 28 - 14i, 30i + 6, 0) \in C^7$.

This is the way operations are performed on C^7.

Example 4.3: Let

$$\langle R \cup I\rangle^4 = \{(a_1, a_2, a_3, a_4) \mid a_i \in R \cup I; 1 \leq i \leq 4, +, \times\}$$

be the multidimensional neutrosophic ring.

Let $x = (5 + 2I, -7I, 4, 10 - I)$ and

$y = (0, 8 - 4I, -11 + 5I, -4I) \in (\langle R \cup I\rangle)^4$.

$x + y = (5 + 2I, 8 - 11I, -7 + 5I, 10 - 5I)$,

$x \times y = (0, -28I, -44 + 20I, -36I) \in \langle R \cup I\rangle$.

This is the way operations are performed on $\langle R \cup I\rangle^4$.

Example 4.4: Let

$$(R(g))^3 = \{(a_1, a_2, a_3) \mid a_i \in R(g), 1 \leq i \leq 3, +, \times\}$$

be the multidimensional dual number ring.

Let $x = (5 - 3g, -2 + g, 0)$ and

$y = (4 + 2g, 5 - g, 7 + 3g) \in (R(g))^3$.

$x + y = (9 - g, 3, 7 + 3g)$.

$x \times y = (20 - 2g, -10 + 7g, 0) \in (R(g))^3$.

This is the way operations are performed.

Example 4.5: Let

$$[R(h)]^5 = \{(a_1, a_2, \ldots, a_6) \mid a_i \in R(h), h^2 = h; 1 \leq i \leq 6, +, \times\}$$

be multidimensional special dual like number ring.

Let $x = (0, 3 + h, 0, 4 - 2h, 0, -6 + h)$ and

$y = (7 + 8h, 0, 6 + 5h, 3h, 8 + 4h, 2) \in (R(h))^5$.

$x + y = (7 + 8h, 3 + h, 6 + 5h, 4 + h, 8 + 4h, -4 + h)$ and

$x \times y = (0, 0, 0, 6h, 0, -12 + 2h) \in (R(h))^6$.

This is the way the operations + and $\times$ are carried out in $(R(h))^6$.

Example 4.6: Let

$$(R(k))^4 = \{(b_1, b_2, b_3, b_4) \mid b_i \in R(k); 1 \leq i \leq 4, k^2 = -k, +, \times\}$$

be the multidimensional special quasi dual number plane.

Let $x = (7 - k, 8 + 3k, 4k, -5 - 4k)$ and

$y = (8k, 4 + 2k, 8 + 5k, 3k) \in (R(k))^4$.

$x + y = (7 + 7k, 12 + 5k, 8 + 9k, -5 - k)$ and

$x \times y = (64k, 32 + 22k, 12k, 27k) \in (R(k))^4$.

This is the way operations are performed on $(R(k))^4$.

Next we proceed onto define mixed multidimensional spaces.

We will first illustrate this situation by some examples.

Example 4.7: Let

$R \times C \times \langle R \cup I\rangle \times C \times R(g) = \{(a, b, c, d, e) \mid a \in R^2, b, d \in C, c \in \langle R \cup I\rangle$ and $e \in R(g); g^2 = 0, I^2 = I, i = -1\} = M$;

M is defined as the mixed multidimensional space.

We see how '+' and '×' operations are performed on M. Let

$$x = (5, 3 - 4i, -10 + 4I, -2 + I, 3 + 4g)$$

and

$$y = (-2, 6 + 3i, 3 + I, 4i, -2 - g) \in M.$$

$x + y = (3, 9 - i, -7 + 5I, -2 + 5i, 1 + 3g) \in M$

$x \times y = (-10, 30 - 15i, -30 + 6I, -8i - 4, -6 - 12g) \in M$.

This is the way operations are performed on the mixed multidimensional spaces.

Example 4.8: Let

$S = C \times C \times C \times R(g) \times R(g) \times R(k) \times \langle R \cup I\rangle = \{(a_1, a_2, a_3, a_5, a_6, a_7) \mid a_i \in C; 1 \leq i \leq 3, a_4, a_5 \in R(g); g^2 = 0, a_6 \in R(k)$ and $a_7 \in \langle R \cup I\rangle, I^2 = I, k^2 = -k, +, \times\}$ be the mixed multi dimensional space.

Now if

$x = (-3 + 2i, 7 - I, -4 - I, 2g + 1, 5g, 10k - 4, 8 + 2I)$ and

$y = (6 - i, 2i, 5, 4g, 8g, 3k, 4) \in S;$

$x + y = (3 + i, 7 + i, 1 - i, 6g + 1, 9g, 13k - 4, 12 + 2I)$ and

$x \times y = (-16i + 15i, 14i + 2, -20 - 5i, 4g, 0, -42k, 32 + 8I) \in S.$

This is the way operations are carried out on S.

Example 4.9: Let

$N = R \times R \times R(k) \times R(k) \times R(h) \times R(h) \times R(g) \times C = \{(a, b, c, d, e, f, g, h) \mid a, b \in R, c, d \in R(k), k^2 = -k, e, f \in R(h), h^2 = h, g_1 \in R(g)\ g^2 = 0$ and $h \in C\}$ be the mixed multidimensional space.

Now if

$x = (5, -2, 3 + 8k, 4 - k, h - 3, 4h + 1, 8g, 10 - i)$ and

$y = (-7, 8, 5k, 3 + 2k, h, 4h + 1, g, 10 + i) \in N.$

$x + y = (-2, 6, 3 + 13k, 7 + k, 2h - 3, 8h + 2, 9g, 20);$

$x \times y = (-35, -16, 25k, 12 + 7k, -2h, 1 + 24h, 101) \in N.$

This is the way sum and product operations are performed on the mixed multidimensional spaces.

Example 4.10: Let

B = $\{C \times R \times R(g) \times C \times R(h) \times R \times C\} = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mid a_1, a_4, a_7 \in C, a_2, a_6 \in R, a_3 \in R(g); g^2 = 0, a_5 \in R(h); h^2 = h, + \times\}$ be the mixed multidimensional space.

Clearly for $x = (10 + i, 7, 8g + 2, 3i, 5 + h, 3, 2 + 4i)$ and

$y = (3, -1, 5g, 6 + i, h, 6, 0) \in B$.

$x + y = (13 + i, 6, 13g + 2, 6 + 4i, 5 + 2h, 9, 2 + 4i)$ and

$x \times y = (30 + 3i, -7, 10g, 18i - 3, 6h, 18, 0)$ are in B.

This is the way operations are performed.

Now we can just define a mixed multidimensional space.

DEFINITION 4.1: *Let $M = \{P_1 \times P_2 \times \ldots \times P_n\}$ where $P_i \in \{C, R, R(g), g^2 = 0; R(h), h^2 = h; R(k); k^2 = -k, \langle R \cup I\rangle\}$; $1 \leq i \leq n$ be the mixed multidimensional space. M is a ring under usual + and ×.*

Now it is important to note that all these mixed multidimensional spaces are vector space of multidimension over the reals R.

However they are in general not mixed multivector spaces over $\langle R \cup I\rangle$, R(g), $g^2 = 0$; R(h); $h^2 = h$; R(k) = $k^2 = -k$ and so on.

Example 4.11: Let

$$M = \{(R \times R(g) \times C \times R(h) \times C) \mid g^2 = 0, h^2 = h; +\}$$

be the mixed multidimensional vector space over R; however M is not a mixed multidimensional vector spaces over R(g) or R(h) or C.

Example 4.12: Let

$$M = \{(C \times C \times C \times \langle R \cup I\rangle \times \langle R \cup I\rangle \times R(g) \times R(h)) \quad g^2 = 0, h^2 = h, I^2 = I, +\}$$

be a mixed multidimensional vector space over R.

M is not a mixed multidimensional vector space over C or R(g) or R(h) or $\langle R \cup I\rangle$.

Example 4.13: Let

$V = \{(R(g) \times R(h) \times R(k) \times R(g) \times \langle R \cup I\rangle \times C \times C \times R(h)) \mid g^2 = 0, h^2 = h, I^2 = I, k^2 = -k, i^2 = -1, +\}$ be mixed multi dimensional vector space only over R.

Clearly V is not a mixed multidimensional vector space over R(h) or R(g) or C or $\langle R \cup I\rangle$ or R(k).

Now we proceed onto define, describe and develop multidimensional MOD planes and MOD mixed multidimensional planes.

First we will describe them by examples.

Example 4.14: Let

$$M = [0, m) \times [0, m) \times [0, m) \times \ldots \times [0, m) = \{(a_1, a_2, \ldots, a_6) \mid a_i \in [0, m), 1 \leq i \leq 6\}$$

be the 6 dimensional MOD real plane.

For take m = 10

x = (9.2, 0.63, 8.42, 1.8, 4.8, 0) $\in$ M, we can add them under +, M is a group.

Example 4.15: Let

$$N = \{(a_1, a_2, a_3) \mid a_i \in [0, 7); 1 \leq i \leq 3, +\}$$

be a MOD real multidimensional structure.

Let x = (6.03, 0.78, 1.05) and

y = (3.1, 4.1, 3.1) $\in$ N.

x + y = (2.13, 4.88, 4.15) and

x $\times$ y = (4.693, 3.198, 3.255) $\in$ N.

This is the way sum and $\times$ is performed on N.

Example 4.16: Let

$$T = \{(b_1, b_2, \ldots, b_8) \mid b_i \in [0, 12); 1 \leq i \leq 8, +\}$$

be a MOD real multidimensional plane under + is a group.

Let x = (10.3, 7.5, 0.71, 2.05, 1.8, 0.21, 6.32, 4.52) $\in$ T

x + x = (8.6, 3, 14.2, 4.1, 3.6, 0.42, 0.64, 9.04) $\in$ T.

x – x = (0, 0, 0, 0, 0, 0, 0, 0) $\in$ T.

x $\times$ x = (10.09, 8.25, 0.5041, 4.2025, 3.24, 0.0441, 3.9424, 8.4304) $\in$ T.

However under product T is closed.

Example 4.17: Let P = [0, 10) × [0, 9) × [0, 24) × [0, 13) × [0, 7) = {(x_1, x_2, x_3, x_4, x_5) | $x_1 \in$ [0, 10), $x_2 \in$ [0, 9), $x_3 \in$ [0, 24), $x_4 \in$ [0, 13) and $x_5 \in$ [0, 7), +} be a MOD mixed real multidimensional space.

We use the term mixed real that is the MOD intervals are different but all are real and intervals are different.

This is not possible in case of real plane only in case of MOD real intervals the variety is very different.

Let x = (7.2, 4.5, 13.6, 12.1, 6.2) and

y = (8.5, 6.7, 21.7, 10.7, 4.3) $\in$ P.

x + y = (5.7, 2.2, 11.3, 9.8, 3.5) $\in$ P.

x × y = (1.2, 3.15, 7.12, 12. 47, 5.66) $\in$ P.

Example 4.18: Let

$$M = \{([0, 10) \times [0, 5) \times [0, 10) \times [0, 6) \times [0, 6) \times [0, 5) \times [0, 2) \times [0, 4))\}$$

be a MOD real mixed multidimensional space. The sum and × are defined on M.

It is pertinent to keep on record that only in case of MOD intervals we can

(1) have the concept of mixed multi real dimensional space. This is not possible using reals.

(2) we have infinitely many such MOD mixed real multidimensional spaces.

However these spaces are groups under +. Under × they are only semigroups.

Further using both the operations + and × does not satisfy the distributive law only they are MOD multi mixed dimensional spaces are only MOD multidimensional pseudo rings.

We will illustrate them by examples.

Example 4.19: Let

$$V = \{([0, 7) \times [0, 4) \times [0, 12) \times [0, 10) \times [0, 6) \times [0, 5) \times [0, 4)), +, \times\}$$

be the MOD real mixed multidimensional pseudo ring.

Let $x = (3.7, 0.32, 6.1, 9.5, 5.2, 4.3, 3.1)$,

$y = (0.8, 0.7, 4.1, 0.2, 1.432, 3.2, 1.5)$ and

$z = (4.2, 3.2, 10.6, 9.3, 4.3, 4.1, 3.1) \in V$.

Consider $(x \times y) \times z$

$= (2.96, 0.224, 1.01, 1.9, 1.3996, 3.76, 0.65) \times (4.2, 3.2, 10.6, 9.3, 4.3, 4.1, 3.1)$

$= (5.432, 0.7168, 7.67, 0.01828, 0.416, 2.015)$ … I

Consider $x \times (y \times z)$

$= (3.7, 0.32, 6.1, 9.5, 5.2, 4.3, 3.1) \times (3.36, 2.24, 7.46, 1.86, 0.1189, 3.12, 0.65)$

$= (5.432, 0.7168, 1.506, 7.67, 0.01828, 0.416, 2.015)$ … II

Clearly I and II are equal. Hence certainly the operation $\times$ is associative.

However $x \times (y + z) \neq x \times y + x \times z$ for $x, y, z \in V$.

$x \times (5, 3.9, 2.7, 9.5, 5.723, 2.3, 0.6)$

$$= (4.5, 1.248, 4.47, 0.25, 5.7596, 4.7596, 1.61) \quad \ldots I$$

$x \times y + x \times z$

$$= (3.7, 0.32, 6.1, 9.5, 5.2, 4.3, 3.1) \times (0.8, 0.7, 4.1, 0.2, 1.423, 3.2, 1.5) + (3.7, 0.32, 6.1, 9.5, 5.2, 4.3, 3.1) \times (4.2, 3.2, 10.6, 9.3, 4.3, 4.1, 3.1)$$

$$= (2.96, 0.224, 1.01, 1.9, 1.3996, 3.76, 0.65) + (1.54, 1.024, 4.66, 8.35, 4.36, 2.63, 1.61)$$

$$= (4.50, 1.248, 5.67, 0.25, 5.7596, 1.39, 2.26) \quad \ldots II$$

I and II are distinct so the ring is only a pseudo ring. So only the MOD mixed real multidimensional space is a pseudo ring.

Example 4.20: Let W = $\{([0, 14) \times [0, 10) \times [0, 14) \times [0, 5) \times [0, 80) \times [0, 5) \times [0, 10) \times [0, 23)), +, \times\}$ be a MOD real mixed multidimensional space pseudo ring.

Next we proceed on to define MOD multidimensional complex space.

DEFINITION 4.2: *Let M = $(C_n(m) \times C_n(m) \times \ldots \times C_n(m))$ - t-times; i_F^2 = m – 1} be the MOD multidimensional complex modulo integer space.*

We will give examples of them.

Example 4.21: Let

$$V = (C_n(10) \times C_n(10) \times C_n(10) \times C_n(10))$$

be the MOD multidimensional complex modulo integer space.

We see $x = (9 + 2i_F, 3 + 4i_F, 6i_F, 7) \in V$ and

$y = (0.33 + 6.01i_F, 7.201 + 4.533i_F, 0.00073i_F, 0.3312) \in V$.

Clearly V under + is a MOD multidimensional complex modulo integer group and (V, $\times$) is a MOD complex multidimensional semigroup.

Example 4.22: Let

$$A = \{(C_n(7) \times C_n(7) \times C_n(7) \times C_n(7) \times C_n(7)), +\}$$

be the MOD multidimensional complex modulo integer group.

For if $x = (2 + 3i_F, 0.334i_F, 2.701, 0.5 + 1.5i_F, 6 + 2i_F)$ and

$y = (0.3 + 0.772i_F, 4 + 3i_F, 5.11i_F + 2, 6.5 + 4.5i_F, 1 + 5i_F) \in A$, then

$$x + y = (2.3 + 3.772i_F, 4 + 3.334i_F, 5.11i_F + 4.701, 6i_F, 0) \in A.$$

This is the way the operation '+' is performed on A.

Example 4.23: Let

$$B = \{(C_n(10) \times C_n(10) \times C_n(10) \times C_n(10) \times C_n(10) \times C_n(10)), \times\}$$

be the MOD multidimensional complex modulo integer semigroup.

Let $x = (0.112 + 5i_F, 4 + 2i_F, 0.7 + 0.6i_F, 4.25i_F, 0.1114, 0)$ and

$y = (0, 2 + 7i_F, 0.8 + 0.7i_F, 6.115i_F, 4.2 + 3i_F, 7 + 0.8889i_F) \in B.$

$x \times y = (0, 4 + 2i_F, 4.34 + 0.97i_F, 3.89875i_F, 0.46788 + 0.3342, 0) \in B.$

This is the way product operation is performed on the MOD multidimensional complex modulo integer semigroup.

Example 4.24: Let

$$M = \{(C_n(5) \times C_n(5) \times C_n(5)), \times\}$$

be the MOD multidimensional complex modulo integer semigroup.

Next we proceed on to define MOD mixed multidimensional complex modulo integer space.

DEFINITION 4.3: *$B = \{(C_n(m_1) \times C_n(m_2) \times C_n(m_3) \times \ldots \times C_n(m_s))$ atleast some m_i's are distinct $i_F^2 = (m_i - 1)$ for $i = 1, 2, \ldots, s\}$ is defined as the MOD mixed multidimensional complex modulo integer space.*

We will give some examples of them.

Example 4.25: Let

$$T = \{(C_n(7) \times C_n(10) \times C_n(12) \times C_n(7) \times C_n(5))\}$$

be the MOD mixed multidimensional complex modulo integer space.

Example 4.26: Let

$$W = \{(C_n(11) \times C_n(14) \times C_n(8) \times C_n(11) \times C_n(14) \times C_n(9))\}$$

be a MOD mixed multidimensional complex modulo integer space.

We can on these spaces define + and × operation.

Under + operation the MOD mixed multidimensional complex modulo integer group and under × the MOD mixed complex modulo integer multidimensional space is a semigroup.

We will illustrate this situation by some examples.

Example 4.27: Let

$$M = \{(C_n(10) \times C_n(7) \times C_n(5) \times C_n(4) \times C_n(2)), +\}$$

be the MOD mixed multidimensional complex modulo integer group.

Let $x = (3 + 7i_F, 4i_F + 3, 2 + 2.5i_F, 3i_F, 1 + i_F)$ and

$y = (2 + 3i_F, 4, 2i_F, 0.2, i_F) \in M$.

$x + y = (5, 4i_F, 2 + 4.5i_F, 3i_F + 0.2, 1)$.

This is the way '+' operation is performed on M.

Example 4.28: Let

$$T = \{(C_n(5) \times C_n(7) \times C_n(6) \times C_n(11)), +\}$$

be the MOD mixed multidimensional modulo integer group. T has subgroups of finite order.

T also has subgroups of infinite order.

Example 4.29: Let

$$W = \{(C_n(4) \times C_n(6) \times C_n(8) \times C_n(10)), \times\}$$

be the MOD mixed multidimensional complex modulo integer semigroup.

Let $x = (3 + 0.2i_F, 0.5 + 4i_F, 6 + 2i_F, 7.1 + 3i_F)$ and

$y = (2.1 + 0.4i_F, 0.7 + 0.6i_F, 3 + 0.7i_F, 5 + 2i_F) \in W$.

$x \times y = (2.54 + 1.62i_F, 0.35 + 3.1i_F, 3.8 + 2.2i_F, 0.3 + 2.04i_F) \in W$.

This is the way product operation is performed on W.

Example 4.30: Let

$$V = \{(C_n(17) \times C_n(12) \times C_n(15) \times C_n(40)), \times\}$$

be the MOD mixed multidimensional complex modulo integer semigroup.

Infact we have infinitely many MOD mixed multidimensional complex number spaces.

However we cannot get any mixed multidimensional complex spaces we have only multidimensional complex spaces using C.

Next we can define MOD mixed multidimensional complex modulo integer pseudo rings.

DEFINITION 4.4: *Let* $R = \{(C_n(m_1) \times C_n(m_2) \times \ldots \times C_n(m_t)) \mid i_F^2 = (m_i - 1);\ 1 \leq i \leq t, +, \times\}$ *be the MOD mixed multidimensional complex modulo integer space. R under the operations + and × is the MOD mixed multidimensional complex modulo integer pseudo ring.*

We see if $m_1 = m_2 = \ldots = m_t = m$ then R is just a MOD multidimensional complex modulo integer pseudo ring.

We will give examples of them.

Example 4.31: Let

$$W = \{(C_n(7) \times C_n(9) \times C_n(11) \times C_n(13)); +, \times\}$$

be the MOD mixed multidimensional complex modulo integer pseudo ring.

Let $x = (3 + 0.5i_F, 0.4 + 7i_F, 10 + 2.5i_F, 1.2 + 10i_F)$ and

$y = (0.7 + 4i_F, 6 + 0.5i_F, 0.7 + 2i_F, 5 + 0.3i_F)$ and

$z = (3.3 + 3i_F, 4 + 2i_F, 2 + 10i_F, 7 + 2i_F) \in W$.

$x + y = (3.7 + 4.5i_F, 6.4 + 7.5i_F, 10.7 + 4.5i_F, 6.2 + 10.3i_F)$ and

$x \times y = (0.1 + 5.35i_F, 3.4 + 6.2i_F, 2 + 10.75i_F, 3 + 11.36i_F) \in W$

Consider $x \times (y + z) = x \times y + x \times z$

$= (0.1 + 5.35i_F, 3.4 + 6.2i_F, 2 + 10.75i_F, 3 + 11.36i_F) + (4.9 + 3.65i_F, 5.6 + 1.8i_F, 6 + 6i_F, 1.4 + 7.4i_F)$

$= (5 + 2i_F, 8i_F, 8 + 7.75i_F, 4.4 + 0.76i_F)$ … I

$x \times (y + z)$
$= (3 + 0.5i_F, 0.4 + 7i_F, 10 + 2.5i_F, 1.2 + 10i_{1F}) \times (4, 1 + 2.5i_F, 2.7 + i_F, 12 + 2.3i_F)$

$= (5 + 2i_F, 5.4 + 8i_F, 6 + 5.75i_F, 4 + 5.76i_F)$ … II

Clearly I and II are distinct hence the distributive law in general is not true.

Thus W is a MOD mixed multidimensional complex modulo integer pseudo ring.

Example 4.32: Let

$$V = \{(C_n(12) \times C_n(45) \times C_n(24) \times C_n(20) \times C_n(24) \times C_n(12)), +, \times\}$$

be the MOD mixed multidimensional complex modulo integer pseudo ring.

Clearly the MOD multidimensional complex modulo integer vector space over $C_n(m)$; however the MOD mixed multidimensional complex modulo integer group is not a vector space for they are mixed.

Next we proceed on to define these concepts using MOD multidimensional neutrosophic structures.

DEFINITION 4.5: *Let* $V = \{(R_n^I(m) \times \ldots R_n^I(m) \times R_n^I(m) \times R_n^I(m))\}$ *be the MOD multidimensional neutrosophic space.*

We will first give examples of them.

Example 4.33: Let

$$M = \{(R_n^I(8) \times R_n^I(8) \times R_n^I(8) \times R_n^I(8))\}$$

be the MOD multidimensional neutrosophic space.

Example 4.34: Let

$$V = \{(R_n^I(20) \times R_n^I(20) \times R_n^I(20) \times R_n^I(20) \times R_n^I(20))\}$$

be the multidimensional neutrosophic space.

We define algebraic structure on them.

DEFINITION 4.6: *Let*

$$V = \{((R_n^I(m) \times R_n^I(m) \times \ldots \times R_n^I(m)), +\}$$

be the MOD multidimensional neutrosophic group.

We will examples of them.

Example 4.35: Let

$$V = \{(R_n^I(9) \times R_n^I(9) \times R_n^I(9) \times R_n^I(9)), +\}$$

be the MOD multidimensional neutrosophic group.

For if x = (3 + 4I, 2.5 + 3.2I, 7.6 + 4I, 8.5 + 3.5I) and

y = (6 + 3.5I, 8 + 6.3I, 4 + 3.5I, 7 + 4.5I) $\in$ V.

x + y = (7.5I, 1.5 + 0.5I, 2.6 + 7.5I, 6.5 + 8I) $\in$ V.

This is the way '+' operation is performed on the MOD multidimensional neutrosophic group.

Example 4.36: Let

$$M = \{(R_n^I(10) \times R_n^I(10) \times R_n^I(10) \times R_n^I(10) \times R_n^I(10), +\}$$

be the MOD multidimensional neutrosophic group.

Next we proceed on to define the MOD multidimensional neutrosophic semigroup.

DEFINITION 4.7: *Let W = $\{(R_n^I(m) \times \ldots \times R_n^I(m)), \times\}$ be the MOD neutrosophic multidimensional semigroup.*

We give examples of them.

Example 4.37: Let

$$M = \{ (R_n^I(12) \times R_n^I(12) \times R_n^I(12) \times R_n^I(12) \times R_n^I(12)),\ \times\}$$

be the MOD multidimensional neutrosophic semigroup.

Let x = (3 + 4.5I, 7.5I, 8.3, 10 + 5I, 6I) and

y = (8, 10 + 4.8I, 11 + 0.3I, 7I, 6 + 7.5I) $\in$ M.

x $\times$ y = (0, 3I, 7.3 + 2.49I, 9I, 9I) $\in$ M.

This is the way product operation $\times$ will be performed.

Example 4.38: Let

$$M = \{ R_n^I(10) \times R_n^I(10) \times R_n^I(10) \times R_n^I(10) \times R_n^I(10) \times R_n^I(10) \times R_n^I(10), \times\}$$

be the MOD multidimensional neutrosophic semigroup.

Let x = (6 + 3I, 7 + 4I, 1.8I, 4.5, 1.6 + 0.5I, 0.3 + 0.5I, 8I) and

y = (3I, 0.4I, 6 + 5I, 7 + 2I, 5, 6I, 4 + 3.5I) $\in$ M;

x $\times$ y = (8 + 9I, 4.11I, 9.8I, 9I + 1.5, 8 + 2.5I, 4.8I, 0) $\in$ M.

This is the way product operations are performed on MOD multidimensional neutrosophic semigroup.

Next we proceed on to define MOD multidimensional neutrosophic pseudo ring.

DEFINITION 4.8: *Let*

$$V = \{ (R_n^I(m) \times R_n^I(m) \times \ldots \times R_n^I(m)),\ +,\ \times\};$$

V is defined as the MOD multidimensional neutrosophic pseudo ring. V is called as the pseudo ring as the distributive law is not true.

We will illustrate this situation by some examples.

Example 4.39: Let

$$M = \{ (R_n^I(5) \times R_n^I(5) \times R_n^I(5)), +, \times\}$$

be the MOD multidimensional neutrosophic pseudo ring.

Let x = (0.3 + 4I, 0.8I, 0.9),

y = (0.2I, 0.8 + 2.5I, 4.2 + 0.6I) and

z = (3, 4.2 + 2.5I, 0.8 + 4.4I) $\in$ M.

We find x + y, x $\times$ y, xy + xz and x(y + z) in the following.

x + y = (0.3 + 4.2I, 0.8 + 3.3I, 0.1 + 0.6I) $\in$ M.

x $\times$ y = (0.86I, 2.64I, 3.78 + 0.54I) $\in$ M.

Consider xy + xz

= (0.86I, 2.64, 3.78 + 0.54I) + (0.9 + 2I, 0.36I, 0.72 + 3.96I)

= (0.9 + 2.86I, 3I, 4.5 + 4.5I) …I

x(y + z) = (0.3 + 4I, 0.8I, 0.9) $\times$ (3 + 0.2I, 0, 0)

= (0.9 + 2.86I, 0, 0) …II

Clearly I and II are distinct so the MOD multidimensional neutrosophic pseudo ring does not satisfy the distributive law.

Example 4.40: Let

$$S = \{ (R_n^I(18) \times R_n^I(18) \times R_n^I(18) \times R_n^I(18) \times R_n^I(18)), +, \times\}$$

be the MOD multidimensional neutrosophic pseudo ring.

Next we proceed on to define the notion of MOD mixed multidimensional neutrosophic structures and illustrate them by examples.

DEFINITION 4.9: *Let*

$$S = \{ (R_n^I(m_1) \times R_n^I(m_2) \times \ldots \times R_n^I(m_t));$$

atleast some of the m_i's are distinct $I^2 = I$}; S is defined as the MOD mixed multidimensional neutrosophic space.

We will give one or two examples of them.

Examples 4.41: Let

$$M = \{ (R_n^I(12) \times R_n^I(10) \times R_n^I(16) \times R_n^I(4)) \}$$

be the MOD mixed multidimensional neutrosophic space.

Example 4.42: Let

$$T = \{(R_n^I(7) \times R_n^I(97) \times R_n^I(14) \times R_n^I(14) \times R_n^I(7) \times R_n^I(14))\}$$

be the MOD mixed multidimensional neutrosophic space.

Now we define operations on MOD mixed multidimensional neutrosophic spaces.

DEFINITION 4.10: *Let*

$$P = \{ (R_n^I(m_1) \times R_n^I(m_2) \times \ldots \times R_n^I(m_t)),\ +\}.$$

P is defined as the MOD *mixed multidimensional neutrosophic group.*

We will give examples of them.

Example 4.43: Let

$$P = \{ (R_n^I(10) \times R_n^I(8) \times R_n^I(15) \times R_n^I(20)), +\}$$

be the MOD mixed multidimensional neutrosophic group.

Let x = (8 + 4.5I, 7 + 0.8I, 10.5 + 4I, 16.8I) and

y = (7 + 8.5I, 5 + 4.9I, 12 + 9I, 12 + 8I) $\in$ P.

x + y = (5 + 3I, 4 + 5.7I, 7.5 + 13I, 12 + 4.8I) $\in$ P.

This is the way + operation is performed on P.

Example 4.44: Let

$$T = \{ (R_n^I(12) \times R_n^I(6) \times R_n^I(12) \times R_n^I(9) \times R_n^I(11)),\ +\}$$

be the MOD mixed multidimensional neutrosophic group.

Next we proceed on to define MOD mixed multidimensional neutrosophic group.

DEFINITION 4.11: Let

$$W = \{ (R_n^I(m_1) \times R_n^I(m_2) \times \ldots \times R_n^I(m_r));\ \times\};$$

W under the product × is defined as the MOD *mixed multidimensional neutrosophic semigroup.*

Clearly W is of finite order and W has infinite zero divisors.

Infact all ideals in W are of infinite order. W has subsemigroups of finite and infinite order which are not ideals of W.

We will illustrate this situation by some examples.

Example 4.45: Let

$$S = \{ (R_n^I(10) \times R_n^I(15) \times R_n^I(7) \times R_n^I(6)),\ \times\}$$

be the MOD neutrosophic mixed multidimensional semigroup.

Let x = (6 + 7.5I, 1.8 + 6I, 4 + 3I, 3.5 + 0.5I) and

y = (5I, 10I, 0.5I, 5I) $\in$ S.

x $\times$ y = (7.5I, 3I, 3.5I, 2I) $\in$ S.

This is the way the product operation is performed on S.

$P_1 = \{(R_n^I(10) \times \{0\} \times \{0\} \times \{0\}), \times\}$ is a subsemigroup of infinite order. Infact an ideal of S.

Let $P_2 = \{(\{0\} \times \langle Z_{15} \cup I\rangle \times \{0\} \times \{0\});\ \times\} \subseteq S$ is a subsemigroup of finite order and is not an ideal of S.

$P_3 = \{(\langle Z_{10} \cup I\rangle \times \{0\} \times \{0\} \times R_n^I(6)),\ \times\}$ is a subsemigroup of S of infinite order. Clearly P_3 is not an ideal of S.

Thus S has subsemigroups of infinite order which are not ideal of S.

Next we proceed on to give one more example.

Example 4.46: Let

$$M = \{(R_n^I(12) \times R_n^I(12) \times R_n^I(19) \times R_n^I(10) \times R_n^I(5) \times R_n^I(2)), \times\}$$

be the MOD mixed multidimensional neutrosophic semigroup of infinite order.

$P_1 = \{(\{0\} \times \{0\} \times R_n^I(19) \times \{0\} \times \{0\} \times R_n^I(2)), \times\} \subseteq M$ is the MOD mixed multidimensional neutrosophic subsemigroup which is also an ideal of M.

Let $P_2 = \{(\{0\} \times \{0\} \times Z_{19} \times \{0\} \times \{0\} \times \{0\}), \times\} \subseteq M$ is a finite subsemigroup of M which is not an ideal of M.

Let $P_3 = \{(R_n^I(12) \times \{0\} \times \{0\} \times \{0\} \times \langle Z_5 \cup I\rangle \times \{0\}), \times\} \subseteq M$ be a subsemigroup of M of infinite order.

Clearly P_3 is not an ideal of M only a subsemigroup.

Next we proceed on to describe and develop the notion of MOD mixed multidimensional neutrosophic pseudo rings.

DEFINITION 4.10: *Let $M = \{(R_n^I(m_1) \times R_n^I(m_2) \times \ldots \times R_n^I(m_s))$; m_i's are distinct $1 \leq i \leq s$, $+$, $\times\}$. M be defined as the MOD neutrosophic mixed multidimensional pseudo ring.*

Clearly M is an infinite commutative pseudo ring.

We will give examples of them.

Example 4.47: Let

$$M = \{(R_n^I(10) \times R_n^I(7) \times R_n^I(4) \times R_n^I(10));\ +, \times\}$$

be the MOD mixed multidimensional neutrosophic pseudo rings.

Clearly M has infinite number of zero divisors, finite number of units and idempotents.

M has subrings of finite order as well as pseudo subrings of infinite order which are not ideals. However all ideals of M are of infinite order.

To this effect we will give pseudo subrings of infinite order which are not pseudo ideals.

Consider $P_1 = \{(R_n^I(10) \times \{0\} \times \{0\} \times \{0\})\} \subseteq M$ is a pseudo subring of infinite order which is also a pseudo ideal.

Consider $P_2 = \{(R_n^I(10) \times \{0\} \times \{0\} \times Z_{10}), +, \times\} \subseteq M$ is a pseudo subring of infinite order which is not an ideal of M.

$P_3 = \{(\langle Z_{10} \cup I\rangle \times R_n^I(7) \times Z_4 \times \{0\}); +, \times\} \subseteq M$ is a pseudo subring of infinite order which is not a pseudo ideal of M.

Clearly M has atleast ${}_4C_1 + {}_4C_2 + {}_4C_3$ number of ideals; however M has several pseudo subrings of infinite order which are not pseudo ideals.

M also has subrings of finite order.

For $T_1 = \{(Z_{10} \times \{0\} \times \{0\} \times \{0\}), +, \times\} \subseteq M$,

$T_2 = \{(\langle Z_{10} \cup I\rangle \times \{0\} \times \{0\} \times \{0\}), +, \times\} \subseteq M$,

$T_3 = \{(Z_{10} \times \{\langle Z_7 \cup I\rangle\} \times \{0\} \times \{0\}), +, \times\} \subseteq M$ and so on are some of the subrings of finite order which are not pseudo.

Example 4.48: Let

$$N = \{ (R_n^I(5) \times R_n^I(2) \times R_n^I(4) \times R_n^I(6)),\ +, \times\}$$

be the MOD mixed multidimensional neutrosophic pseudo ring.

N has (1,1,1,1) to be unit. N has infinite number of zero divisors. N has several idempotents.

$P_1 = \{(R_n^I(5) \times \{0\} \times \{0\} \times R_n^I(6), +, \times)\} \subseteq N$ is a pseudo ideal.

$P_2 = \{(Z_5 \times (\langle Z_2 \cup I \rangle) \times \{0\} \times \{0\}); +, \times\} \subseteq N$ is a subring of finite order not pseudo.

$P_3 = \{(Z_5 \times R_n^I(2) \times \{0\} \times \{0\}); +, \times\} \subseteq N$ is a pseudo subring of infinite order which is not an ideal. However all MOD multidimensional neutrosophic groups.

Under + say $V = \{(R_n^I(m) \times R_n^I(m) \times \ldots \times R_n^I(m)); +\}$ are MOD neutrosophic vector spaces over Z_m if m is a prime or MOD S-pseudo neutrosophic vector spaces over $\langle Z_m \cup I \rangle$ or just MOD S-pseudo neutrosophic vector spaces over $R_n^I(m)$.

Further it is pertinent to keep on record that MOD mixed multidimensional neutrosophic groups under + in general are not MOD vector spaces and cannot also be made into MOD neutrosophic vector spaces as one common base field or S-ring cannot be determined.

Next we proceed onto study MOD multidimensional dual number spaces and MOD mixed multidimensional dual number spaces.

DEFINITION 4.12: *Let*

$$W = \{(R_n(m)g \times R_n(m)g \times \ldots \times R_n(m)g);\ g^2 = 0,\ t\text{-times}\}.$$

W is defined as the MOD multidimensional dual number space.

$W = \{(a_1 + b_1g, a_2 + b_2g, \ldots, a_t + b_tg); g^2 = 0, a_i\ b_i \in [0, m); 1 \leq i \leq t\}$.

W is a finite collection.

First we will give examples of them.

Example 4.49: $M = \{(R_n(10)g \times R_n(10)g \times R_n(10)g \times R_n(10)g);\ g^2 = 0\} = \{(a_1 + b_1g, a_2 + b_2g, a_3 + b_3g, a_4 + b_4g) \mid a_i\ b_i \in [0, 10)\ g^2 = 0, 1 \leq i \leq 4\}$ be the MOD multidimensional dual number space.

$x = (5 + 2.3g, 3.7 + 4.72g, 6.3 + 4.103g, 0.72 + 5.007g) \in M$.

Example 4.50: Let $B = \{(R_n(19)g \times R_n(19)g \times R_n(19)g \times R_n(19)g \times R_n(19)g \times R_n(19)g);\ g^2 = 0\}$

$= \{(a_1 + b_1g, a_2 + b_2g, a_3 + b_3g, a_4 + b_4g, a_5 + b_5g, a_6 + b_6g) \mid g^2 = 0, a_i, b_i \in [0, 19); 1 \leq i \leq 6\}$

We can define algebraic operations + and × on these MOD multidimensional dual number spaces.

DEFINITION 4.13: *Let*

$$P = \{(R_n(m)g, R_n(m)g, \ldots, R_n(m)g) \,/\, g^2 = 0;\ 2 \leq m < \infty;\ +\};$$

P is a group under addition. P is defined as the MOD multidimensional dual number group. P is an abelian group of infinite order.

We will give examples of them.

Example 4.51: Let

$$M = \{(R_n(11)g \times R_n(11)g \times R_n(11)g \times R_n(11)g) \mid g^2 = 0; +\}$$

be the MOD multidimensional dual number group M is a MOD dual number vector space over Z_{11}.

However M is only a Smarandache MOD dual number vector space over $(\langle Z_{11} \cup g\rangle)$ or $R_n(11)$ or $R_n(11)(g)$.

Study of properties of enjoyed by these MOD vector spaces is a matter of routine and hence left as an exercise to the reader.

Example 4.52: Let

$$M = \{(R_n(12)g \times R_n(12)g \times R_n(12)g \times R_n(12)g \times R_n(12)g) \mid g^2 = 0, +\}$$

be a S-pseudo MOD multidimensional dual number vector space are the S-ring Z_{12} or $\langle Z_{12} \cup g\rangle$ or $[0, 12)$.

Study of these properties are also interesting and work in this direction is left as an exercise for the reader.

Example 4.53: Let

$$T = \{(R_n(9)g \times R_n(9)g \times R_n(9)g \times R_n(9)g);\ g^2 = 0, \times\}$$

be the MOD multidimensional dual number semigroup.

We see T has subsemigroups which are zero square subsemigroups.

Let $x = (4 + 3g, 2.5g, 7 + 3.5g, 2 + 4g)$ and

$y = (0.3 + 4g, 4 + 2g, g + 1, 6.5 + 0.2g) \in T$.

$x \times y = \{(1.2 + 7.9g, g, 7 + 1.5g, 4 + 8.4g)\} \in T$

Let $a = (5g, 7.881g, 0.4438g, 1.6g)$ and

$b = (7g, 6.534g, 0.785g, 0.093g) \in T$

$a \times b = (0, 0, 0, 0)$.

Thus if $B = \{(a_1g, a_2g, a_3g, a_4g) \mid a_i \in [0, 9);\ 1 \leq i \leq 4, g^2 = 0\} \subseteq T$ is such that B is a subsemigroup and infact B is an ideal.

B is infact a zero square semigroup.

Example 4.54: Let

$$S = \{(R_n(12)g \times R_n(12)g \times R_n(12)g \times R_n(12)g \times R_n(12)g), g^2 = 0, \times\}$$

be the MOD multidimensional dual number semigroup.

S has MOD ideals which are of infinite order.

Infact S has subsemigroups of finite order none of which are ideals.

Infact S has subsemigroups of infinite order which are also not ideals only subsemigroups.

Let $W = \{(R_n(12)g \times \{0\} \times \{0\} \times Z_{12} \times \langle Z_{12} \cup g\rangle); g^2 = 0\}$ be only a subsemigroup of S and is not ideal of S.

Now we just show $V = \{(Z_{12} \times \{0, 6\} \times \{0, 3, 6, 9\} \times \{0, 4g, 8g\} \times \langle Z_{12} \cup g\rangle) \mid g^2 = 0, \times\} \subseteq S$ is only a subsemigroup of finite order.

Next we proceed on to define the notion of MOD multidimensional dual number pseudo ring.

DEFINITION 4.14: *Let $R = \{(R_n(m)g \times R_n(m)g \times \ldots \times R_n(m)g) \mid g^2 = 0;\ 2 \leq m < \infty, +, \times\}$ be defined as the MOD multidimensional dual number pseudo ring.*

Clearly R is of infinite order commutative but does not satisfy the distributive laws.

All pseudo ideals in R are of infinite order. However R has finite subrings which are not pseudo. Infact R has infinite number of zero divisors.

We will illustrate this situation by some examples.

Example 4.55: Let

$$P = \{(R_n(10)g \times R_n(10)g \times R_n(10g) \times R_n(10)g) \mid g^2 = 0, +, \times\}$$

be the MOD multidimensional dual number pseudo ring.

Let $T_1 = \{(Z_{10} \times Z_{10} \times Z_{10} \times \{0\}), +, \times\} \subseteq P$ be the MOD multidimensional dual number subring of finite order.

$T_2 = \{(R_n(10)g \times \{0\} \times \{0\} \times \{0\} \times \{0\}); +, \times\}$ is a MOD multidimensional dual number pseudo subring which is a pseudo ideal of P.

Let $W_1 = \{(\{0\} \times R_n(10)g \times \{0\} \times \{0\} \times Z_{10}); g^2 = 0, +, \times\} \subseteq P$ be the MOD multidimensional dual number pseudo subring of infinite order which is not a pseudo ideal of P.

Example 4.56: Let

$$M = \{(R_n(17)g \times R_n(17)g \times R_n(17)g \times R_n(17)g); g^2 = 0, +, \times\}$$

be the MOD multidimensional dual number pseudo ring.

$P_1 = \{(R_n(17)g \times \{0\} \times \{0\} \times \{0\}), +, \times\}$ is the MOD multidimensional pseudo ideal.

$P_2 = \{(R_n(7) \times \{0\} \times \{0\} \times \{0\}) \mid +, \times\}$ is only a MOD multidimensional pseudo subring of infinite order but is not a pseudo ideal of M.

$P_3 = \{(Z_7 \times \langle Z_7 \cup g\rangle \times \{0\} \times \{0\}), +, \times\}$ is a MOD multidimensional subring of finite order which is not a pseudo ring.

Thus these MOD multidimensional dual number pseudo rings have subrings which are not pseudo.

We see next it is a matter of routine to build MOD mixed multidimensional dual number groups, semigroups and pseudo rings using MOD mixed multidimensional dual number spaces.

This will be illustrated by an example or two.

Example 4.57: Let

$$M = \{(R_n(5)g \times R_n(10)g \times R_n(12)g \times R_n(9)g);\ g^2 = 0, +\}$$

be the MOD mixed multidimensional dual number space group.

Clearly M is of infinite order but M has subgroups of both finite and infinite order.

$T_1 = \{(R_n(15)g \times \{0\} \times \{0\} \times \{0\}) \mid g^2 = 0, +\} \subseteq M$ is a subgroup of M of infinite order.

$T_2 = \{(Z_5 \times Z_{10} \times Z_{12} \times Z_9);\ +, \times\} \subseteq M$ is a subgroup of finite order.

Example 4.58: Let

$$B = \{(R_n(11)g \times R_n(12)g \times R_n(10)g \times R_n(11)g \times R_n(24)g \times R_n(12)g) \mid g^2 = 0, +\}$$

be the MOD mixed multidimensional dual number space group. B has both subgroups of finite and infinite order.

Example 4.59: Let

$$M = \{(R_n(10)g \times R_n(6)g \times R_n(2)g \times R_n(15)g) \mid g^2 = 0, \times\}$$

be the MOD mixed multidimensional dual number semigroup. M is of infinite order and is commutative.

All ideals of M are of infinite order but has subsemigroups of both finite and infinite order.

Infact M has infinite order subsemigroups which are not ideals.

M has infinite number of zero divisors.

$N_1 = \{(Z_{10} \times Z_6 \times \{0\} \times \{0\}), \times\} \subseteq M$ is a subsemigroup of finite order.

$N_2 = \{(R_6(10)g \times \{0\} \times \{0\} \times \{0\}) \mid g^2 = 0, \times\} \subseteq M$ is an infinite subsemigroup which is an ideal of M.

$N_3 = \{(R_n(10) \times \langle Z_6 \cup g\rangle \times \{0\} \times \{0\}); g^2 = 0, \times\}$ is the subsemigroup of infinite order but is not ideal of M.

Example 4.60: Let

$$M = \{(R_n(18)g \times R_n(40)g \times R_n(12)g \times R_n(7)g \times R_n(492)g) \mid g^2 = 0, \times\}$$

be the MOD mixed multidimensional dual number semigroup.

$N_1 = \{(Z_8 \times \{0\} \times \{0\} \times \{0\} \times Z_{492}) \times\}$ be the MOD subsemigroup of finite order which is not an ideal of M.

Next we proceed on to give example of MOD mixed multidimensional dual number pseudo ring.

Example 4.61: Let

$$M = \{(R_n(16)g \times R_n(7)g \times R_n(22)g \times R_n(45)g \times R_n(11)g \times R_n(7)g) \mid g^2 = 0, +, \times\}$$

be the MOD mixed multidimensional dual number space pseudo ring.

M has subrings of finite order which are not pseudo. All subrings of infinite order are pseudo. However all pseudo ideals are of infinite order. But M has pseudo subrings of infinite order are not ideals.

$P_1 = \{(Z_{16} \times \{0\} \times \{0\} \times \{0\} \times R_n(11) \times Z_7), +, \times\} \subseteq M$ is a pseudo subring of infinite order but is not an ideal of P_1.

$P_2 = \{(\{0\} \times R_n(7)g \times \{0\} \times \{0\} \times \{0\}), g^2 = 0, +, \times\}$ is an pseudo ideal of M.

$P_3 = \{(\{0\} \times \{0\} \times R_n(22) \times \{0\} \times \{0\} \times \{0\}), +, \times\}$ is a pseudo subring of M of infinite order but is not an ideal of M.

Example 4.62: Let

$$S = \{(R_n(11)g \times R_n(18)g \times R_n(17)g); g^2 = 0, +, \times\}$$

be the MOD mixed multidimensional dual number space pseudo ring.

S has infinite order pseudo subring which are not pseudo ideals.

S has subrings of finite order which is not pseudo.

Next we proceed on to give examples of MOD multidimensional special dual like number space groups, semigroups and pseudo rings.

Example 4.63: Let

$$S = \{(R_n(10)h \times R_n(12)h \times R_n(40)h \times R_n(12)h); h^2 = h, +\}$$

be the MOD mixed multidimensional special dual like number space group. S has both finite and infinite order subgroups.

$V_1 = \{(Z_{10} \times \langle Z_{12} \cup h\rangle \times \{0\} \times \{0\}) \mid h^2 = h, +\}$ is a finite order subgroup.

$V_2 = \{(R_n(10)h \times \{0\} \times \{0\} \times \{0\}), h^2 = h, +\}$ is an infinite order subgroup.

Example 4.64: Let

$$M = \{(R_n(16)h \times R_n(18)h \times R_n(19)h \times R_n(47)h \times R_n(27)h);\ h^2 = h, +\}$$

be the MOD mixed multidimensional special dual like number group of infinite order.

M has subgroups of both of finite and infinite order.

Example 4.65: Let

$$T = \{(R_n(10)h \times R_n(10)h \times R_n(10)h);\ h^2 = h, +\}$$

be the MOD multidimensional special dual like number space group. T has both subgroups of finite and infinite order.

Example 4.66: Let

$$N = \{(R_n(20)h \times R_n(20)h \times R_n(20)h \times R_n(20)h \times R_n(20)h);\ h^2 = h, +\}$$

be the MOD multidimensional special dual like number space group. N has both subgroups of finite order as well as subgroups of infinite order.

Example 4.67: Let

$$P = \{(R_n(11)h \times R_n(11)h \times R_n(11)h \times R_n(11)h);\ h^2 = h, \times\}$$

be the MOD multidimensional special dual like number semigroup.

P has subsemigroups of finite order.

$B_1 = \{(Z_{11} \times Z_{11} \times \{0\} \times \{0\}), \times\}$ is a subsemigroup of finite order which is not an ideal.

$B_2 = \{(R_n(11)h \times \{0\} \times \{0\} \times\{0\}) \times\}$ is a subsemigroup of infinite order which is an ideal.

$B_3 = \{(R_n(11) \times R_n(11) \times \{0\} \times \{0\}), \times\}$ is a subsemigroup of infinite order which is not an ideal.

Example 4.68: Let

$$M = \{(R_n(10)h \times R_n(18)h \times R_n(128)h);\ h^2 = h, \times\}$$

be the MOD mixed multidimensional special quasi dual like space semigroup.

$N_1 = \{(R_n(10) \times \{0\} \times\{0\}) \times\}$ is the infinite order subsemigroup.

$N_2 = \{(Z_{10} \times \{0\} \times Z_{128}), \times\}$ is finite order subsemigroup.

$N_3 = \{(\{0\} \times \{0\} \times R_n(128)h),\ h^2 = h, \times\}$ is an infinite order subsemigroup which is an ideal of M.

Example 4.69: Let

$$S = \{(R_n(20)h \times R_n(45)h \times R_n(20)h \times R_n(2)h \times R_n(4)h);\ h^2 = h, \times\}$$

be the MOD mixed multidimensional special dual like number semigroup. S has ideals all of which are of infinite order.

S has subsemigroups of finite order which are not ideals. S has subsemigroups of infinite order which are not ideals.

Example 4.70: Let

$$S = \{(R_n(10)h \times R_n(10)h \times R_n(10)h \times R_n(10)h);\ h^2 = h, +, \times\}$$

be the MOD multidimensional special quasi dual like number space pseudo ring.

S has subrings of finite order which are not pseudo. S has subrings of infinite order which are pseudo. Some pseudo subrings of infinite order are not ideals.

$P_1 = \{(Z_{10} \times \{0\} \times \{0\} \times \{0\}), +, \times\}$ is a subring of finite order which are not pseudo.

$P_2 = \{(R_n(10) \times \{0\} \times \{0\} \times Z_{10}), +, \times\}$ is a pseudo subring of infinite order but is not an ideal of S.

However

$$P_3 = \{(\{0\} \times R_n(10)h \times R_n(10)h \times \{0\}); h^2 = h, +, \times\}$$

is a multidimensional special quasi dual number space pseudo subring which is also of infinite order and is a pseudo ideal.

Example 4.71: Let

$$M = \{(R_n(8)h \times R_n(10)h \times R_n(13)h \times R_n(16)h); h^2 = h, +, \times\}$$

be a MOD mixed multidimensional special dual number space pseudo ring.

M has both infinite and finite order subrings which are not ideals.

$T_1 = \{(Z_8 \times \{0\} \times \{0\} \times Z_{16}); +, \times\}$ is a subring of finite order which is not pseudo.

$T_2 = \{(R_n(8) \times R_n(10) \times \{0\} \times \{0\}), +, \times\}$ is a subring of infinite order which is pseudo but is not an ideal.

$T_3 = \{(R_n(8)h \times R_n(10)h \times \{0\} \times \{0\}); h^2 = h, +, \times\}$ is a pseudo subring of infinite order which is a pseudo ideal.

Now we just give a few examples of MOD multidimensional special quasi dual number space groups, semigroups and pseudo rings.

We will also give examples of MOD mixed multidimensional special quasi dual number space groups, semigroups and pseudo rings.

Example 4.72: Let

$$M = \{(R_n(12)k \times R_n(12)k \times R_n(12)k \times R_n(12)k);\ k^2 = 11k\}$$

be the MOD multidimensional special quasi dual number space.

Example 4.73: Let

$$N = \{(R_n(10)k_1 \times R_n(15)k_2 \times R_n(27)k_3 \times R_n(9)k_4),\ k_1^2 = 9k_1,\ k_2^2 = 14k_2,\ k_3^2 = 26k_3,\ k_4^2 = 8k_4\}$$

be the MOD mixed multidimensional special quasi dual number space.

Example 4.74: Let

$$P = \{(R_n(8)k \times R_n(8)k \times R_n(8)k \times R_n(8)k);\ k^2 = 7k, +\}$$

be the MOD multidimensional special quasi dual number space group.

P is of infinite order. P has subgroups of both finite and infinite order.

Example 4.75: Let

$$M = \{(R_n(9)k_1 \times R_n(20)k_2) \mid k_1^2 = 8k_1 \text{ and } k_2^2 = 19k_2, +\}$$

be the MOD mixed multidimensional special quasi dual number space group.

$P_1 = \{Z_9 \times Z_{20}\}$ is a subgroup of finite order.

$P_2 = \{R_n(9)k_1 \times \{0\}\}$ is a subgroup of infinite order.

M has several subgroups of finite order as well as subgroups of infinite order.

Example 4.76: Let

$$S = \{(R_n(10)k \times R_n(10)k \times R_n(10)k),\ k^2 = 9k, \times\}$$

be the MOD multidimensional special quasi dual number space semigroup.

All ideals of S are of infinite order. However S has subsemigroups of infinite order which are not ideal.

Finally all subsemigroups of finite order are not ideals.

$P_1 = \{(R_n(10)k \times \{0\} \times \{0\})\}$ is an ideal of S.

$P_2 = \{(R_n(10) \times R_n(10) \times \{0\})\}$ is a subsemigroup of infinite order and is not an ideal of S.

$P_3 = \{(Z_{10} \times Z_{10} \times Z_{10}) \mid \times\}$ is a subsemigroup of finite order.

Example 4.77: Let

$$B = \{(R_n(12)k_1 \times R_n(13)k_2 \times R_n(15)k_3 \times R_n(24)k_4) \mid k_1^2 = 11k_1,\ k_2^2 = 12k_2,\ k_3^2 = 14k_3,\ k_4^2 = 23k_4, \times\}$$

be the MOD mixed multidimensional special quasi dual number space semigroup.

$B_1 = \{Z_{12} \times Z_{13} \times Z_{15} \times Z_{24}, \times\}$ is a subsemigroup of finite order.

$B_2 = \{R_n(12) \times R_n(13) \times R_n(15) \times R_n(24), \times\}$ is a MOD subsemigroup of infinite order which is not an ideal.

$B_3 = \{(R_n(12)k_1 \times \{0\} \times \{0\} \times \{0\}), \times\}$ is a MOD subsemigroup of infinite order which is an ideal of B.

Example 4.78: Let

$$W = \{(R_n(13)k \times R_n(13)k \times R_n(13)k \times R_n(13)k)\ k^2 = 12k, +, \times\}$$

be the MOD multidimensional special quasi dual number space pseudo ring.

$P_1 = \{Z_{13} \times Z_{13} \times \{0\} \times \{0\}, +, \times\}$ is a finite subring which is not pseudo.

$P_2 = \{(R_n(13)k \times \{0\} \times \{0\} \times \{0\});\ k^2 = 12k, +, \times\}$ is a pseudo subring of infinite order which is also a pseudo ideal.

$P_3 = \{(R_n(13) \times R_n(13) \times \{0\} \times \{0\})\ +, \times\}$ is a pseudo subring of infinite order which is not an ideal of W.

Example 4.79: Let

$$M = \{(R_n(12)k_1 \times R_n(12)k_2 \times R_n(12)k_3),\ k_1^2 = 11k_1,$$
$$k_2^2 = 26k_2 \text{ and } k_3^2 = 48k_3, +, \times\}$$

be the MOD mixed multidimensional special quasi dual number pseudo ring.

$V_1 = \{R_{12} \times R_{27} \times R_{49};\ +, \times\}$ is a subring of finite order which is not pseudo.

$V_2 = \{(R_n(12) \times R_n(27) \times \{0\}), +, \times\}$ is a subring of infinite order which is pseudo but is not an ideal.

$V_3 = \{(R_3(12)k_1 \times \{0\} \times \{0\}),\ k_1^2 = 11k_1, +, \times\}$ is a pseudo subring of infinite order which is also a pseudo ideal of M.

Now we proceed on to develop the notion of MOD mixed multidimensional spaces and groups, semigroups and pseudo rings built using them.

Example 4.80: Let

$$B = \{(R_n(10) \times R_n^I(12) \times R_n(15) \times R_n(7)g);\ g^2 = 0,\ I^2 = I\}$$

be the MOD mixed multidimensional space.

Example 4.81: Let

$$B = \{(R_n(8)k \times R_n(9)h \times R_n^I(14) \times R_n(10)g \times C_n(20) \times R_n(17))\ g^2 = 0,\ k^2 = 7k,\ h^2 = h,\ I^2 = I,\ i_F^2 = 19\}$$

be the MOD mixed multidimensional space.

Example 4.82: Let

$$M = \{(R_n(7) \times R_n^I(20) \times R_n(5)), +\}$$

be the MOD mixed multidimensional space group. M has subgroups of both finite and infinite order.

$P_1 = \{(Z_7 \times \{0\} \times Z_5), +\}$ is a subgroup of finite order.

$P_2 = \{(R_n(7) \times \{0\} \times R_n(5)), +\}$ is a subgroup of infinite order.

Example 4.83: Let

$$M = \{(R_n(7) \times R_n^I(20) \times R_n(15)g \times R_n(17)h \times C_n(20) \times R_n(7));\ +\}$$

be a MOD mixed multidimensional group.

Example 4.84: Let

$$V = \{(R_n(5)h \times R_n(20)g \times R_n^I(40)),\ h^2 = h,\ g^2 = 0,\ I^2 = I, \times\}$$

be the MOD mixed multidimensional semigroup.

Clearly V is of infinite order. All ideals of V are of infinite order. There are subsemigroups of finite order as well as infinite order which are not ideals.

$W_1 = \{(Z_5 \times Z_{20} \times Z_{40}); \times\}$ is a subsemigroup of finite order. Clearly W_1 is not an ideal of V.

$W_2 = \{(R_n(5) \times R_n(20) \times \{0\}), \times\}$ is again a subsemigroup of infinite order which is not an ideal of V.

$W_3 = \{(\{0\} \times \{0\} \times R_n^I(40)), \times\}$ is a subsemigroup of infinite order which is also an ideal of V.

Example 4.85: Let

$$M = \{(R_n(5) \times R_n(10) \times R_n^I(5) \times C_n(10) \times R_n(10)g \times R_n(10)h)\ h^2 = h,\ g^2 = 0,\ I^2 = 0,\ i_F^2 = 9, \times\}$$

be the MOD mixed multidimensional space semigroup. M has infinite number of zero divisors, some finite number of units and idempotents.

However M has subsemigroups of both finite and infinite order which are not ideals. But all ideals of M are of infinite order.

Example 4.86: Let

$$B = \{(R_n^I(7) \times R_n(10) \times R_n(5)g \times R_n(5)h \times R_n(10)k \times C_n(10)) \mid I^2 = I,\ g^2 = 0,\ h^2 = h,\ k^2 = 9k,\ i_F^2 = 9, +, \times\}$$

be the MOD mixed multidimensional pseudo ring.

B is of infinite order. All ideals of B are of infinite order and are pseudo. B has subrings of finite order which are not ideals.

B has subrings of infinite order which are not pseudo ideals but only pseudo subrings.

Let $V_1 = \{(Z_7 \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times C(Z_{10})), +, \times\}$ is a subring of B of finite order. Clearly V_1 is not an ideal of B.

$V_2 = \{(\{0\} \times Z_{10} \times Z_5 \times \{0\} \times \{0\} \times \{0\}), +, \times\}$ is again a subring of B and not an ideal of B.

$V_3 = \{(R_n(7) \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times R_n(10)), +, \times\}$ is a subring of B and is infact a pseudo subring of B but V_3 is not a pseudo ideal of B.

Let $V_4 = \{(\{0\} \times \{0\} \times R_n(5) \times R_n(5) \times R_n(10) \times \{0\}), +, \times\}$ is a pseudo subring of B of infinite order but is not a pseudo ideal of B.

Let $V_5 = \{(R_n^I(7) \times R_n(10) \times \{0\} \times \{0\} \times \{0\} \times \{0\}), +, \times\}$ is a pseudo subring of B of infinite order which is also a pseudo ideal of B.

Let $V_6 = \{(\{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times C_n(10)), +, \times\}$ be the pseudo subring of infinite order which is also a pseudo ideal of B.

Example 4.87: Let

$$T = \{(R_n(7)h \times R_n(7)g \times R_n^I(7) \times C_n(7) \times R_n(7)k) \mid h^2 = h,\ g^2 = 0,\ k^2 = -k;\ I^2 = I,\ i_F^2 = 6, +, \times\}$$

be the MOD mixed multidimensional pseudo ring.

Clearly T can be realized as the MOD mixed multidimensional pseudo linear algebra over the field Z_7.

In view of all these we make the following theorem.

THEOREM 4.5: *Let $V = \{(R_n(m)k \times R_n(m) \times R_n(m)g \times R_n(m)h \times R_n(m)k \times C_n(m)) / g^2 = 0, h^2 = h, k^2 = (m-1)k, i_F^2 = m - 1, +, \times\}$ be the MOD mixed multidimensional pseudo ring.*

V is a MOD mixed multidimensional pseudo linear algebra over Z_m if and only if m is a prime V is a MOD mixed multidimensional S-pseudo linear algebra if m is a non prime and a Z_m is a S-ring.

Proof is direct and hence left as an exercise to the reader.

Thus we can have MOD mixed multidimensional pseudo rings.

Example 4.88: Let

$$M = \{(R_n(5) \times R_n(7) \times R_n(8)g \times R_n(11)h \times R_n(5)) \mid g^2 = 0, h^2 = h, +, \times\}$$

be the MOD mixed multidimensional pseudo ring.

Let $x = (0.7, 5, 2 + 0.8g, 10 + 4h, 3)$ and

$y = (4, 2, 4g, 5h, 2) \in M$

$x \times y = (2.8, 3, 0, 4h, 4) \in M$

Clearly M is not mixed multidimensional vector space or pseudo linear algebra over any common field or S–ring.

We can work in this direction to get many other interesting properties about these newly constructed MOD structures.

Chapter Five

SUGGESTED PROBLEMS

In this chapter we suggest a few problems for the reader. Some of these problems are at research level and some are simple and innovative.

1. Can we say use of MOD real intervals $[0, m)$; $2 \leq m < \infty$ instead of $(-\infty, \infty)$ in appropriate problems are advantageous?

2. Can one claim use of MOD real intervals in storage system will save time and economy?

3. Specify the advantages of using MOD real intervals $[0, m)$ in the place of $(-\infty, \infty)$; $2 \leq m < \infty$.

4. Can we say MOD real intervals is capable of better results?

5. What are the advantages of making the negatives into the MOD interval $[0, m)$?

6. Show the MOD real interval transformation is a periodic one mapping periodically infinite number of points to a single point in $[0, m)$.

7. Can we say [0, 1) is the MOD fuzzy interval which can depict $(-\infty, \infty)$?

8. What is the major difference between the use of fuzzy interval [0, 1] and the MOD fuzzy interval [0, 1)?

9. Show the MOD interval [0, 9) can be got as a multidimensional projection of $(-\infty, \infty)$ to [0, 9).

   Can we have an inverse of $I_\eta: (-\infty, \infty) \to [0, 9)$?

10. Show in problem 9
    $I_\eta (9t + s) = s$ where $s \in [0, 9)$ and $t \in (-\infty, \infty)$.

11. In problem 9 if $t \in Z$ will map $I_\eta: (-\infty, \infty) \to [0, 9)$ complete?

    Justify your claim.

12. Let $I_\eta : (-\infty, \infty) \to [0, 18)$ be the MOD real transformation.

    i. What are the special features enjoyed by I_η?
    ii. What is zeros of I_η? Is it nontrivial?
    iii. When is $I_\eta(m) = 0$?
       Show $|I_\eta(m)| = \infty$.
    iv. Can I_η have a T_η such that $I_\eta \circ T_\eta = T_\eta \circ I_\eta$?
    v. Can T_η be a map or a pseudo special type of map?
    vi. Show a point in [0, 18) is mapped by T_η into infinitely many points in $(-\infty, \infty)$.
    vii. Enumerate all the special features enjoyed by T_η.
    viii. Prove T_η is not the classical map or function but such study is also needed.

13. What makes $T_\eta : [0, m) \to (-\infty, \infty)$ essential?

14. Prove maps like $T_\eta : [0, m) \to (-\infty, \infty)$ are also well defined in their own way.

15. Obtain any other special properties of MOD special pseudo functions,

$$T_\eta : [0, m) \to (-\infty, \infty).$$

16. Give the MOD map from $\eta_z : Z_n \to Z$.

    Enumerate the properties enjoyed by η_z.

17. Show S = {[0, m), ×} is a MOD non associative semigroup of infinite order.

18. Prove S = {[0, 15), ×} can have zero divisors.

    i. Can S have idempotents?
    ii. Does S contain S–idempotents?
    iii. Can S have S–zero divisors?
    iv. Can S have infinite number of zero divisors?
    v. Can S have S–units?
    vi. Can S have infinite number of units?
    vii. Can S have ideals of finite order?
    viii. Can S have subsemigroups of finite order?
    ix. Can S have S-ideals?

19. Let S_1 = {[0, 23), ×} be the MOD real interval non associative semigroup.

    Study questions i to ix of problem 18 for this S_1.

20. Let S_2 = {[0, 24), ×} be the real MOD semigroup.

    Study questions i to ix of problem 18 for this S_2.

21. Let R = {[0, 7), ×} be the real MOD semigroup.

    Prove R has a triple which is non associative.

22. Let G = {[0, 9), +} be a MOD real interval group.

i. Prove G is of infinite order.
ii. Prove G has subgroups of infinite order.
iii. Find 3 subgroups of infinite order in G.
iv. Find 5 subgroups of finite order.

23. Let G = {[0, 12), +} be a MOD real interval group.

Study i to iv of questions of problem 22 for this G.

24. Let R = {[0, 5), +, ×} be the MOD real interval pseudo ring.

i. Prove R is non associative.
ii. Can R have zero divisors?
iii. Can R have S–zero divisors?
iv. Find all units in R.
v. Can R have idempotents?
vi. Can R have S–units?
vii. Can R have S–ideals?
viii. Can R have S–subrings of infinite order which are not ideals?
ix. Can R have ideals of finite order?
x. Find two ideals of infinite order in S.

25. Let R_1 = {[0, 25), +, ×} be the MOD interval pseudo ring.

i. Find all zero divisors in R_1.
ii. Can R_1 have S-zero divisors?
iii. Can R_1 have idempotents?
iv. Can R_1 have S-idempotents?
v. Can R_1 have units?
vi. Is it possible for R_1 to have S-units?
vii. Find all finite subrings of R_1.
viii. Can R_1 have ideals of finite order?
ix. Find all pseudo subrings of infinite order.

26. Let M = {[0, 17), +, ×} be the MOD real pseudo ring.

Study questions i to ix of problem 25 for this M.

27. Enumerate all the special features enjoyed by MOD real pseudo rings.

28. Prove using the MOD complex finite modulo integer $[0, 24)i_F$ one cannot build MOD complex semigroups or MOD complex pseudo rings?

29. Enumerate the special features enjoyed by the MOD complex modulo integer group. $B = \{[0, 12)i_F, +\}$.

30. Develop all the special features enjoyed by $S=\{[0, m)i_F, +\}$.

31. Study all the properties enjoyed by $P = \{[0, 13)i_F, +\}$.

32. Let $M = \{[0, 24)I| \ I^2 = I, +\}$ be the MOD neutrosophic interval group.

    i. Find all subgroups of finite order in M.
    ii. Find all subgroups of infinite order.
    iii. Find any other special feature enjoyed by M.

33. Let $V = \{[0, 47)I \mid I^2 = I; +\}$ be the MOD neutrosophic interval group.

    Study questions i to iii of problem 32 for this V.

34. Let $T = \{[0, 24)I, I^2 = I, \times\}$ be the MOD interval neutrosophic semigroup.

    i. Can T have zero divisors?
    ii. Can T have idempotents?
    iii. Can T have S-zero divisors?
    iv. Can T have S-idempotents?
    v. Can T have units?
    vi. Is I the unit of T?
    vii. Can T have S-units?
    viii. Is T a Smarandache semigroup?

ix. Prove T has finite order subsemigroups.
x. Prove T has ideals all of them are only of infinite order.

35. Let S = $\{[0, 28)I; I^2 = I, \times\}$ be the MOD neutrosophic interval semigroup.

    i. Is S a S-semigroup?
    ii. Can S have S-zero divisors?
    iii. Can S have zero divisors which are not S-zero divisors?
    iv. Can S have S-ideals?
    v. Does S have ideals which are not S-ideals?
    vi. Can S have S-units?
    vii. Can S have subsemigroups which of finite order which is not an ideal?

36. Let P = $\{[0, 45)I; I^2 = I, \times\}$ be the MOD neutrosophic interval semigroup.

    Study questions i to vii of problem 35 for this P.

37. Let L = $\{[0, 492)I; I^2 = I, \times\}$ be the MOD neutrosophic interval semigroup.

    Study questions i to vii of problem 35 for this L.

38. Let R = $\{[0, 12)I, I^2 = I, +, \times\}$ be the MOD neutrosophic interval pseudo ring.

    i. Is R a S-pseudo ring?
    ii. Can R have ideals of finite order?
    iii. Can R have S-subrings of finite order?
    iv. Can R have S-ideals?
    v. Can R have zero divisors?
    vi. Is R a ring with unit?
    vii. Can R have S-units?
    viii. Can R have idempotents?
    ix. Can R have S-idempotents?
    x. Can R have S-zero divisors?
    xi. Can R have ideals which are not S-ideals?

39. Let M = $\{[0, 17)I, I^2 = I, +, \times\}$ be the MOD neutrosophic interval pseudo ring.

    Study questions i to xi of problem 38 for this M.

40. Let B = $\{[0, 48)I; I^2 = I, +, \times\}$ be the MOD neutrosophic interval pseudo ring.

    Study questions i to xi of problem 38 for this B.

41. Let M = $\{[0, 9)g; g^2 = 0, +\}$ be the MOD dual number interval group.

    i. Prove M is of infinite order.
    ii. Can M have finite order subgroup?
    iii. Can M have subgroups of infinite order?

42. H = $\{[0, 13)g, g^2 = 0, +\}$ be the MOD dual number interval group.

    Study questions i to iii of problem 41 for this H.

43. Let S = $\{[0, 18)g, g^2 = 0, +\}$ be the MOD dual number interval group.

    Study questions i to iii of problem 41 for this S.

44. Let T = $\{[0, 19)g, g^2 = 0, \times\}$ be the MOD dual number interval semigroup.

    i. Find zero divisors if any in T.
    ii. Can T have S-zero divisors?
    iii. Is T a monoid?
    iv. Can T have ideals of finite order?
    v. Can T have subsemigroups of finite order?
    vi. Can T have S-idempotents?
    vii. Can T have idempotents which are not S-idempotents?

viii. Is T a S-semigroup?
ix. Can T have S-ideals?

45. Let S = $\{[0, 48)g|\ g^2 = 0, \times\}$ be the MOD dual semigroup.

    i. Study questions i to iii of problem 41 for this S.
    ii. Prove questions vi, vii and viii of problem 43 are not true.

46. Let W = $\{[0, 24)g|\ g^2 = 0, +, \times\}$ be the MOD dual number ring.

    i. Can W have units?
    ii. Is W a zero square ring?
    iii. Find all properties enjoyed by W.
    iv. Can W have finite order subrings?
    v. Can W have ideals of finite order?
    vi. Obtain some special features enjoyed by W.
    vii. Can W have subrings which are not ideals? Justify.

47. Let S = $\{[0, 12)g|\ g^2 = 0, +, \times\}$ be the MOD dual number pseudo ring.

    Study questions i to vii of problem 46 for this S.

48. Let M = $\{[0, 49)g,\ g^2 = 0, +, \times\}$ be the MOD dual number interval ring.

    i. Can M be a S-pseudo ring?
    ii. Study questions i to vii of problem 46 for this S.

49. Let T = $\{[0, 45)h,\ h^2 = h, \times\}$ be the MOD special dual like number semigroup.

    i. Show T has finite order subgroups.
    ii. Is T a monoid?
    iii. What is the unit element of T?
    iv. Is h the unit element of T?
    v. Can T have ideals of finite order?

vi. Is every infinite subsemigroup an ideal?
vii. Is T a S-semigroup?
viii. Can T have S-idempotents?
ix. Can T have S-unit?
x. Obtain any other special feature enjoyed by T.

50. Let M = {[0, 45)h, h^2 = h, +} be the MOD special dual like number group.

    Obtain all the special features enjoyed by M.

51. If [0, 45)h in 50 is replaced by [0, 23)h will these groups enjoy different properties.

52. Let V = {[0, 48)h, h^2 = h, ×} be the MOD special dual like number semigroup.

    Study questions i to x of problem 49 for this V.

53. Let S = {[0, 47)h, h^2 = h, ×} be the MOD special dual like number interval semigroup.

    Study questions i to x of problem 49 for this S.

54. Let X = {[0, 144)h, h^2 = h, +, ×} be the MOD special dual like number interval pseudo ring.

    i. Is X a S-ring?
    ii. Can S have S-ideals?
    iii. Does S contain units?
    iv. Can S has ideals of finite order?
    v. Can S have subrings of finite order?
    vi. Can S have S-units?
    vii. Can S have S-idempotents?
    viii. Can S have zero divisors which are not S-zero divisors?
    ix. Obtain some very special features associated with S.

55. Let H = {[0, 43)h, h^2 = h, +, ×} be the MOD special dual like number interval pseudo ring.

Study questions i to ix of problem 54 for this H.

56. Let Y = $\{[0, 192)h, h^2 = h, +, \times\}$ be the MOD special dual like number interval pseudo ring.

    Study questions i to ix of problem 54 for this Y.

57. Let Z = $\{[0, 12)k, k^2 = 11k, +, \times\}$ be the MOD special quasi dual number interval pseudo ring.

    Study questions i to ix of problem 54 for this Z.

58. Let M = $\{[0, 25)k, k^2 = 24k, +, \times\}$ be the MOD quasi dual number interval pseudo ring.

    Study questions i to ix of problem 54 for this Z.

59. Let V = $\{[0, 48)k, k^2 = 47k, +, \times\}$ be the MOD quasi dual number interval pseudo ring.

    Study questions i to ix of problem 54 for this V.

60. Let Z = $\{[0, 28)k, k^2 = k, \times\}$ be the MOD quasi dual number interval semigroup.

    i. Can Z have S-zero divisors?
    ii. Is Z a monoid?
    iii. Is Z a S-semigroup?
    iv. Find idempotents if any in Z.
    v. Can Z have S-idempotents?
    vi. Can Z have finite order subsemigroups?
    vii. Can ideals of Z be of finite order?
    viii. Find ideals of Z.
    ix. Can Z have a subsemigroup of infinite order which is not an ideal of Z?
    x. Obtain any other special property enjoyed by Z.

61. Let P = {[0, 17)k, k^2 = 16k, ×} be the MOD special quasi dual number semigroup.

    Study questions i to x of problem 60 for this P.

62. Let S = {[0, 24)k, k^2 = 23k, ×} be the MOD special quasi dual number interval semigroup.

    Study questions i to x of problem 60 for this S.

63. Let V = {(x_1, x_2, x_3, x_4) | x_i ∈ [0, 15); 1 ≤ i ≤ 4, +} be the MOD real matrix group.

    i. Prove V has subgroups of finite order.
    ii. Prove V has subgroups of infinite order.
    iii. Give any other interesting property associated with V.

64. Let W = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \right|$ a_i ∈ [0.97);

    1 ≤ i ≤ 24, +} be the MOD real number interval group.

    Study questions i to iii of problem 63 for this W.

65. Let M = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i \in [0, 10)\ X_n\ 1 \leq i \leq 10 \right\}$ be the MOD real matrix interval semigroup.

    i. Prove M has infinite number of zero divisors.
    ii. Can M have S-zero divisors?
    iii. Can M have S-units?
    iv. Can M have infinite number of idempotents?
    v. Does M contain S-idempotents?
    vi. Find subsemigroup of finite order in M.
    vii. Can M have ideals of finite order?
    viii. Find all ideals of M.
    ix. Obtain any other special property enjoyed by M.
    x. Can M be a S-semigroup?

66. Let P = $\left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{bmatrix} \middle| \; a_i \in [0, 23),\ 1 \leq i \leq 40,\ \times_n \right\}$

    be the MOD real interval matrix semigroup.

    Study questions i to x of problem 65 for this P.

67. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \; a_i \in [0, 28); 1 \leq i \leq 18, \times_n\right\}$

    be the MOD real matrix interval semigroup.

    Study questions i to x of problem 65 for this M.

68. Let B = $\{(a_1, a_2, \ldots, a_{10}) \mid a_i \in [0, 19); 1 \leq i \leq 10, +, \times\}$ be the MOD real matrix interval pseudo ring.

    i. Is B a S-ring?
    ii. Can B have ideals of finite order?
    iii. Can B have S-ideals?
    iv. Can B have subrings of finite order?
    v. Does B contain zero divisors which are not S-zero divisors?
    vi. Does B contain S-idempotents?
    vii. Can B have S-units?
    viii. Find any other special feature enjoyed by B.
    ix. Prove B has pseudo subrings of infinite order which are not ideals.
    x. If $P_1 = \{(a_1, 0 \ldots 0) \mid a_i \in [0, 19), +, \times\} \subseteq B$. Find the quotient pseudo ring B/P_1.

69. Let X = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 44); 1 \leq i \leq 9, +, \times_n\right\}$

    be the MOD real interval matrix pseudo ring.

    i. Study questions i to x of problem 68.

ii. If the natural product $\times_n$ is replaced by $\times$ show $\times$ is non commutative MOD pseudo ring.

Study questions i to x of problem 68 for this non commutative structure.

70. Let $S = \{(a_1, a_2, \ldots, a_6) \mid a_i \in [0, 9)i_F; 1 \leq i \leq 6, +\}$ be the MOD complex interval group.

i. Find subgroups of infinite order in S.
ii. Find all subgroups of finite order in S.

71. Let $A = \{\begin{pmatrix} a_1 & \ldots & a_{10} \\ a_{11} & \ldots & a_{20} \end{pmatrix} \mid a_i \in [0, 125)i_F; 1 \leq i \leq 20, +\}$

be the MOD complex interval matrix group.

Study questions i and ii of problem 70 for this A.

72. Can on $[0, m)i_F$ one can build semigroups under $\times$?

Justify your claim.

73. Can on $[0, m)i_F$ one can build interval pseudo ring?

Justify your claim.

74. Let $M = \{(a_1, a_2, \ldots, a_{20}) \mid a_i \in [0, 27)I; 1 \leq i \leq 20, +\}$ be the MOD neutrosophic interval matrix group.

Study questions i and ii of problem 70 for this M.

75. Let B = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \right.$ | $a_i \in [0, 23)I$; $1 \le i \le 7$, +} be the MOD neutrosophic interval matrix group.

Study questions i and ii of problem 70 for this B.

76. Let X = $\left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \\ a_{22} & a_{23} & \ldots & a_{28} \end{bmatrix} \right.$ | $a_i \in [0, 43)I$, $1 \le i \le 28$, $\times_n$} be the MOD neutrosophic interval matrix semigroup.

i. Prove X is commutative and is of infinite order.
ii. Prove X has infinite number of zero divisors.
iii. Can X have S-zero divisors?
iv. Can X have S-units?
v. Can X have finite subsemigroups which are ideals?
vi. Can X have ideals of finite order?
vii. Prove X has infinite subsemigroups of infinite order which are not ideals.
viii. Is X a S-semigroup?
ix. Can X have S-ideals?
x. Enumerate other special features enjoyed by X.

77. Let B = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \right|$ $a_i \in [0, 24)I$; $1 \leq i \leq 10, \times_n\}$ be the

    MOD neutrosophic interval matrix semigroup.

    Study questions i to x of problem 76 for this B.

78. Let V = $\{(a_1, a_2, \ldots, a_9)$ where $a_i \in [0, 24)I$; $1 \leq i \leq 9, +, \times\}$ be the MOD neutrosophic matrix interval pseudo ring.

    i. Prove V is of infinite order.
    ii. Can V have zero divisors?
    iii. Can V have idempotents?
    iv. Can V have S-zero divisors?
    v. Can V have S-idempotents?
    vi. Prove V has no subrings of finite order.
    vii. Prove V has subrings of infinite order which are not ideals.
    viii. Prove all pseudo ideals are of infinite order.
    ix. Can V have S-ideals?
    x. Prove V has S-subrings which are not S-ideals.
    xi. Find all important and interesting properties associated with V.

79. Let V = $\left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \right.$ | where $a_i \in [0, 35)I$;

$1 \leq i \leq 40$, $+$, $\times_n\}$ be the MOD neutrosophic interval matrix pseudo ring.

Study questions of i to xi of problem 78 for this V.

80. Let S = $\left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \\ a_{28} & a_{29} & \dots & a_{36} \end{bmatrix} \right.$ | $a_i \in [0, 32)I$;

$1 \leq i \leq 36$, $\times_n$, $+\}$ be the MOD neutrosophic interval matrix pseudo ring.

Study questions i to xi of problem 78 for this S.

81. Let M = $\{(a_1 \dots a_7) \mid a_i \in [0, 24)g, g^2 = 0, 1 \leq i \leq 7, +\}$ be a MOD dual number group.

    i. Find all finite subgroups of M.

    ii. Find all infinite order subgroups of M.

82. Let $M_1 = \{\begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \\ a_{28} & a_{29} & \dots & a_{36} \\ a_{37} & a_{38} & \dots & a_{45} \end{bmatrix} \mid a_i \in [0, 43)g, g^2 = 0,$

$1 \leq i \leq 45, +\}$ be the MOD dual number group.

Study questions i and ii of problem 81 for this M_1.

83. Let $V = \{(a_1, a_2, \dots, a_9) \mid a_i \in [0, 12)g, g^2 = 0; 1 \leq i \leq 9, \times\}$ be the MOD dual number matrix interval semigroup.

   i. Prove V is a zero square semigroup.
   ii. Prove every subset X in V which contains (0, 0, …, 0) is a MOD ideal of V.
   iii. Prove V has subsemigroups of order 2, 3 and so on upto ∞.

84. Let $W = \{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \mid a_i \in [0, 43)g, g^2 = 0, 1 \leq i \leq 12, \times_n\}$ be the

MOD dual number matrix interval semigroup.

Study questions i to iii of problem 83 for this W.

85. Let S = { $\begin{pmatrix} a_1 & a_2 & \dots & a_{12} \\ a_{13} & a_{14} & \dots & a_{24} \end{pmatrix}$ | $a_i \in [0, 42)g$, $g^2 = 0$, $1 \leq i \leq 24$, $\times_n$} be the MOD dual number matrix interval semigroup.

    Study questions i to iii of problem 83 for this S.

86. Let
    V = {$(a_1, \dots, a_{18})$ | $a_i \in [0, 40)g$, $g^2 = 0$, $1 \leq i \leq 18$, +, $\times_n$} be the MOD dual number matrix interval pseudo ring.

    i. V is a zero divisor ring.
    ii. V has pseudo ideals of finite order as well as infinite order ideals.
    iii. V has infinite number of pseudo subrings all of which are ideals.
    iv. If W is a subring of V then W is a pseudo ideal. Prove.

87. Let T = { $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}$ | $a_i \in [0, 21)g$, $g^2 = 0$, $1 \leq i \leq 10$, +, $\times_n$} be the MOD dual number matrix interval pseudo ring.

    Study questions i to iv of problem 86 for this T.

88. Let M = { $\begin{pmatrix} a_1 & a_2 & \dots & a_8 \\ a_9 & a_{10} & \dots & a_{16} \\ a_{17} & a_{18} & \dots & a_{24} \end{pmatrix}$ | $a_i \in [0, 16)h$, $h^2 = h$, +, $1 \leq i \leq 24$} be the MOD special dual like number interval matrix group.

i. Prove M is of infinite order.
ii. Prove M has subgroups of both finite and infinite order.

89. Let W = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{36} \end{bmatrix} \right| a_i \in [0, 43)h, h^2 = h,$

$1 \leq i \leq 36, +\}$ be the MOD special dual like number interval matrix group.

Study questions i and ii of problem 88 for this W.

90. Let Z = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{21} & a_{22} \end{bmatrix} \right| a_i \in [0, 25)h, h^2 = h; 1 \leq i \leq 22, +\}$

be the MOD special dual like number interval matrix group.

Study questions i and ii of problem 88 for this Z.

91. Let B = $\left\{\begin{pmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \end{pmatrix} \right| a_i \in [0, 23)h, h^2 = h,$

$1 \leq i \leq 14, \times_n\}$ be the MOD dual number like matrix interval semigroup.

i. Show B is of infinite order.
ii. Can B have ideals of finite order?
iii. Can B have S-ideals?

iv. Is B a S-semigroup?
v. Can B have subsemigroups of finite order?
vi. Can B have zero divisors which are not S–zero divisors?
vii. Can B have S-units?
viii. Show B has non trivial idempotents.
ix. Can B have infinite number of S-idempotents?
x. Does B has infinite order subsemigroups which are not ideals?

92. Let $T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \middle| a_i \in [0, 48)h; h^2 = h; \right.$

$1 \leq i \leq 21, \times_n\}$ be the MOD special dual like number interval matrix semigroup.

Study questions i to x of problem 91 for this T.

93. Let $R = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \end{pmatrix} \middle| a_i \in [0, 13)h, h^2 = h; \right.$

$1 \leq i \leq 27, \times_n\}$be the MOD special dual like number matrix interval semigroup.

Study questions i to x of problem 91 for this R.

94. Let M = $\{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \mid a_i \in [0, 14)h, h^2 = h, 1 \leq i \leq 4, +, \times_n\}$

be the MOD special dual like number matrix interval pseudo ring.

i. Prove M is of infinite order.
ii. Can M have S-zero divisors?
iii. Prove M has infinite number of zero divisors.
iv. Can M have S–idempotents?
v. If I is a pseudo ideal of M, should I be of infinite order.
vi. Show M has finite order subrings.
vii. Show M has pseudo subrings of infinite order which are not pseudo ideals.
viii. Can M have S-unit?
ix. Is M a S-pseudo ring?
x. Does M contain S-subrings which are not S–ideals?

95. Let Z = $\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} \mid a_i \in [0, 15)h, h^2 = h; 1 \leq i \leq 8, +, \times_n\}$

be the MOD special dual like number interval matrix pseudo ring.

Study questions i to x of problem 94 for this Z.

96. Let $R = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right.$ | $a_i \in [0, 40)h, h^2 = h;$

$1 \leq i \leq 16, +, \times\}$ be the MOD special dual like number interval matrix pseudo ring.

i. Show R is a non commutative pseudo ring.
ii. Study questions i to x of problem 94 for this R.
iii. Can R have right pseudo ideals which are not pseudo left ideals and vice versa?

97. Let $T = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{pmatrix} \right.$ | $a_i \in [0, 4)h, h^2 = h,$

$1 \leq i \leq 40, +, \times_n\}$ be the MOD special dual like number matrix interval pseudo ring.

Study questions i to x of problem 94 for this T.

98. Let
$V = \{(x_1, x_2, \ldots, x_{12}) \mid x_i \in [0, 12)k, k^2 = 11k; 1 \leq i \leq 12, +\}$ be the MOD special quasi dual number matrix interval group.

i. Show V is of infinite order.
ii. Show V has subgroups of infinite order.
iii. Show V has subgroups of finite order.
iv. Obtain any other special feature enjoyed by V.

99. Let M = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_9 \\ a_{10} & a_{11} & a_{12} & \dots & a_{18} \\ a_{19} & a_{20} & a_{21} & \dots & a_{27} \\ a_{28} & a_{29} & a_{30} & \dots & a_{36} \\ a_{37} & a_{38} & a_{39} & \dots & a_{45} \\ a_{46} & a_{47} & a_{48} & \dots & a_{54} \end{bmatrix} \right.$ | $a_i \in [0, 45)k$,

$k^2 = 44k$; $1 \leq i \leq 54$, +} be the MOD special quasi dual number interval matrix group.

Study questions i to iv of problem 98 for this M.

100. Let
V = {$(a_1, a_2, \dots, a_9)$ | $a_i \in [0, 25)k$, $k^2 = 24k$; $1 \leq i \leq 9$, $\times_n$} be the MOD special quasi dual number interval matrix semigroup.

i. Prove V is of infinite order.
ii. Prove V has infinite number of zero divisors.
iii. Prove V has infinite number of S-zero divisors.
iv. Prove V has ideals of infinite order.
v. Can V have ideals of finite order?
vi. Can V have infinite order subsemigroups which are not ideals of V?
vii. Prove/disprove V cannot have S–pseudo ideals.

101. Let M = $\left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \\ a_{41} & a_{42} & \dots & a_{50} \end{bmatrix} \right.$ | $a_i \in [0, 412)k$,

$k^2 = 411k$, $1 \leq i \leq 50$, $\times_n$} be the MOD special quasi dual number interval matrix semigroup.

Study questions i to vii of problem 100 for this M.

102. Let W = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \right.$ | $a_i \in [0, 24)k$, $k^2 = 23k$, $1 \leq i \leq 25$, $\times\}$ be the MOD special quasi dual number interval matrix semigroup.

    i. Prove W is a non commutative MOD semigroup.
    ii. Give 2 right ideals of W which are not left ideals of W and vice versa.
    iii. Study questions i to vii of problem 100 for this W.

103. Let
    M = $\{(a_1, a_2, \ldots, a_9) \mid a_i \in [0, 26)k$, $k^2 = 25k$, $1 \leq i \leq 9$, +, $\times\}$ be the MOD special quasi dual number matrix interval pseudo ring.

    i. Prove M has subrings of finite order.
    ii. Show M has zero divisors.
    iii. Can M infinite number of S-zero divisors?
    iv. Show all MOD pseudo ideals of M are of infinite order.
    v. Show M has MOD pseudo subring of infinite order which are not pseudo ideals.
    vi. Can M have idempotents?
    vii. Does M contain S-units?
    viii. Is M a S-pseudo ring?
    ix. Can M have S-pseudo ideals?
    x. What are the special properties enjoyed by M?

104. Let $S = \{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \mid a_i \in [0, 20)k; k^2 = 19k, 1 \leq i \leq 12, +,$ $\times_n\}$ be the special quasi dual number matrix interval pseudo ring.

Study questions i to x of problem 103 for this S.

105. Let $R = \{\begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \\ a_{28} & a_{29} & \ldots & a_{36} \\ a_{37} & a_{38} & \ldots & a_{45} \end{bmatrix}$ where $a_i \in [0, 155)k,$ $k^2 = 154k, 1 \leq i \leq 45, +, \times_n\}$ be the special quasi dual number matrix interval pseudo ring.

Study questions i to x of problem 103 for this R.

106. Let $P = \{\begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \\ a_{22} & a_{23} & \ldots & a_{28} \\ a_{29} & a_{30} & \ldots & a_{35} \\ a_{36} & a_{37} & \ldots & a_{42} \\ a_{43} & a_{44} & \ldots & a_{49} \end{bmatrix}$ where $a_i \in [0, 16)k,$ $k^2 = 15k, 1 \leq i \leq 49, +, \times\}$ be the MOD special quasi dual number matrix interval MOD pseudo ring.

i. Prove P is non commutative.
ii. Study questions i to x of problem 103 for this P.
iii. Give 3 examples of right pseudo ideals of P which are not left pseudo ideals.
iv. Give 2 examples of left pseudo ideals of P which are not right pseudo ideals of P.
v. Can P have left zero divisors which are not right zero divisors?
vi. Can P have right units which are not left units and vice versa?

107. Let $M_1 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in [0, 19)k, k^2 = 18k,\right.$

$1 \leq i \leq 9, +, \times\}$ be the MOD special quasi dual number matrix interval pseudo ring.

Study questions i to iv of problem 106 for this M_1.

108. Let $W = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} \end{bmatrix} \middle| a_i \in [0, 423)k,\right.$

$k^2 = 422k, 1 \leq i \leq 28, +, \times_n\}$ be the MOD special quasi dual number matrix interval pseudo ring.

Study questions i to x of problem 103 for this W.

109. Let $(-\infty, \infty)$ be the real line. Define a MOD real interval transformation from $(-\infty, \infty)$ to $[0, 17)$.

$\eta : (-\infty, \infty) \to [0, 17)$.

i. Is ∞ a function?
ii. Can we take about kernel of η?
iii. Is it possible to define a map from $\eta^{-1} = [0, 17) \to (-\infty, \infty)$?
iv. If η^{-1} a usual map?
v. Can $\eta \circ \eta^{-1} = \eta^{-1} \circ \eta = \text{Id}$ (Identity)?
vi. Obtain any other special feature enjoyed by these MOD real interval transformation.

110. Let $\eta : (-\infty, \infty) \to [0, 43)$ be the MOD real transformation.

Study questions i to vi of problem 109 for this map.

111. Let $\eta_I : (-\infty I, \infty I) \to [0, 45)I$ be the MOD real interval neutrosophic transformation.

Study questions i to vi of problem 109 for this η_I.

112. Let $\eta_I : (-\infty I, \infty I) \to [0, 23)I$ be the MOD real neutrosophic interval transformation.

Study questions i to vi of problem 109 for this η_I.

113. Let $\eta_{i_F} : (-\infty i, \infty i) \to [0, 24)i_F$ be the MOD complex modulo integer interval transformation.

Study questions i to vi of problem 109 for this η_{i_F}.

114. Let $\eta_{i_F} : (-\infty i, \infty i) \to [0, 43)i_F$ be the MOD complex modulo integer interval transformation.

Study questions i to vi of problem 109 for this ηi_F.

115. Let $\eta_g : (-\infty g, \infty g) \to [0, 18)g$, $(g^2 = 0)$ be the MOD dual number interval transformation.

Study questions i to vi of problem 108 for this η_g.

116. Let $\eta_g : (-\infty g, \infty g) \to [0, 29)g$, $(g^2 = 0)$ be the MOD dual number interval transformation.

Study questions i to vi of problem 109 for this η_g.

117. Let $\eta_h : (-\infty h, \infty h) \to [0, 144)h$, $(h^2 = h)$ be the MOD special dual like number interval transformation.

Study questions i to vi of problem 109 for this η_h.

118. Let $\eta_h : (-\infty h, \infty h) \to [0, 13)h$, $(h^2 = h)$ be the MOD special dual like number interval transformation.

Study questions i to vi of problem 109 for this η_h.

119. Let $\eta_k : (-\infty k, \infty k) \to [0, 248)k$; $(k^2 = 247k)$ be the MOD special quasi dual number transformation.

Study questions i to vi of problem 109 for this η_k.

120. Let $\eta_k : (-\infty k, \infty k) \to [0, 13)k$; $k^2 = 13k$ be the MOD special quasi dual number transformation.

Study questions i to vi of problem 109 for this η_k.

121. Let $R_n(24)$ be the MOD real plane $\eta : R \times R \to R_n(24)$ be the MOD real plane transformation.

i. Study the natural properties of a map enjoyed by η.
ii. Can η^{-1} exist is it unique through not a map?
iii. What are the difference between η^{-1} and a usual map?
iv. Obtain any other special features enjoyed by η and η^{-1}.
v. Show $\eta \circ \eta^{-1} = \eta^{-1} \circ \eta$ = identity.

122. Let $\eta : R \times R \rightarrow R_n(47)$ be the MOD real plane transformation.

Study questions i to v of problem 121 for this η.

123. Let $\eta_C : C \rightarrow C_n(29)$ be the MOD complex plane transformation.

Study questions i to v of problem 121 for this η_C.

124. Let $\eta_C : C \rightarrow C_n(48)$ be the MOD complex plane transformation.

Study questions i to v of problem 121 for this η_C.

125. Let $\eta_I : \langle R \cup I\rangle \rightarrow R_n^I(53)$ be the MOD neutrosophic plane transformation.

Study questions i to v of problem 121 for this η_I.

126. Let $\eta_I : \langle R \cup I\rangle \rightarrow R_n^I(448)$ be the MOD neutrosophic plane transformation.

Study questions i to v of problem 121 for this η_I.

127. Let $\eta_g : R(g) \rightarrow R_n(15)g$; $g^2 = 0$ be the MOD dual number plane transformation.

Study questions i to v of problem 121 for this η_g.

128. Let $\eta_g : R(g) \rightarrow R_n(43)g$, $g^2 = 0$ be the MOD dual number plane transformation.

Study questions i to v of problem 121 for this η_g.

129. Let $\eta_h : R(h) \rightarrow R_n(24)h$, $h^2 = h$ be the MOD special dual like number plane transformation.

Study questions i to v of problem 121 for this η_h.

130. Let $\eta_h : R(h) \to R_n(59)h$, $h^2 = h$ be the MOD special dual like number plane transformation.

Study questions i to v of problem 121 for this η_h.

131. Let $\eta_k : R(k) \to [0, 43)k$; $k^2 = 42k$ be the MOD special quasi dual number plane transformation.

Study questions i to v of problem 121 for this η_k.

132. Let $\eta_k : R(k) \to [0, 6)k$, $k^2 = 5k$; be the MOD special quasi dual number plane transformation.

Study questions i to v of problem 121 for this η_k.

133. Let $M = \{(R \times R(g) \times R(h) \times R(k)) \mid g^2 = 0, h^2 = h$ and $k^2 = -k, i^2 = -1, +\}$ be the mixed multidimensional group.

Study all interesting properties associated with M.

134. Study M in problem 133 if + is replaced by $\times$ for the semigroup.

135. Let $N = \{(C \times R(h) \times R(g) \times R(k)); h^2 = h, g^2 = 0$ and $k^2 = -k, \times\}$ be the mixed multidimensional semigroup.

i. Prove N is a commutative monoid of infinite order.
ii. Prove N has infinite number of zero divisors.
iii. Can N has S-zero divisors?
iv. Find all zero divisors which are not S-zero divisors.
v. Can N have S-idempotents?
vi. Does N contain infinite number of idempotents?
vii. Show N have infinite number of subsemigroups.

viii. Can N have finite order subsemigroups?
ix. Can N contain ideals of finite order?
x. Prove all ideals of N are of infinite order.
xi. Obtain any other interesting property enjoyed by N.

136. Let W = $\{(R \times R(g) \times R \times R(h) \times R(k) \times C \times C) \mid g^2 = 0, h^2 = h, k^2 = -k, i^2 = -1, \times\}$ be the mixed multi dimensional semigroup.

Study questions i to xi of problem 135 for this W.

137. Let M = $\{(R(g) \times R(g) \times C \times C \times R(h) \times R(h) \times R(k) \times R(k)) \mid g^2 = 0, h^2 = h$ and $k^2 = -k, \times\}$ be the mixed multidimensional semigroup.

Study questions i to xi of problem 135 for this M.

138. Let P = $\{(R(g) \times R(h) \times C \times C(g) \times C(h)) \mid h^2 = h$ and $g^2 = 0, +\}$ be the mixed multidimensional group. P is a mixed multidimensional vector space over R or Q.

i. Is P a finite dimensional vector space over R?
ii. Find atleast 3 sets of basis for P over R.
iii. Find subspaces of P over R(and Q).
iv. Define a linear operator on P.
v. If V = $Hom_R(P, P)$ find dimension of V over R.
vi. Is V $\cong$ P?
vii. Obtain any other interesting property associated with P.
viii. Give a linear transformation in which Ker is a non empty subspace of P.
ix. Give a nontrivial linear transformation which is one to one.
x. Obtain any other property associated with V.
xi. Is projection from W = $\{(R(g) \times \{0\} \times \{0\} \times \{0\} \times \{0\})\}$ (be a subspace of P) to P well defined.
xii. Write P as a direct sum of subspaces.

139. Let $S = \{(R(h) \times C(h) \times R \times C \times R(g) \times R(k)) \mid g^2 = 0, h^2 = h$ and $k^2 = -k, i^2 = 1, +\}$ be the mixed multidimensional vector space over R (or Q).

Study questions i to xii of problem 138 for this S over R (and Q).

140. Let $M = \{(R(h) \times R \times R(k) \times C \times C(h) \times R(g)), +, \times\}$ be the mixed multidimensional ring.

i. Prove M is commutative.
ii. Prove M has infinite number of zero divisors.
iii. Can M have S-zero divisors?
iv. Is M a S-ring?
v. Can M have S-ideals?
vi. Does M have ideals which are not S–ideals?
vii. Can M have S-idempotents?
viii. Can M have subrings of finite order?
ix. Obtain any other special property enjoyed by M.
x. Does M contain S-subrings?

141. Let $S = \{(C(g) \times C(h) \times \langle R \cup I\rangle \times R(g) \times \langle C \cup I\rangle \times R(h)) \mid g^2 = 0, h^2 = h, I^2 = I, i^2 = -1, +, \times\}$ be the mixed multidimensional ring.

Study questions i to x of problem 140 for this S.

142. Let S be as in problem 141. S is a mixed multidimensional linear algebra over R.

i. What is dimension of S over R?
ii. Is S finite dimensional?
iii. Find subspaces of S over R so that S can be written as a direct sum.
iv. Find Hom(S, S) = V.
v. What is the algebraic structure enjoyed by V?
vi. Find Hom(S, R) = W.
vii. Find the algebraic structure enjoyed by W.
viii. Is $W \cong S$?

143. Let $V = \{(R_n(m) \times R_n(m) \times R_n(m)) \mid m = 8, \times\}$ be the MOD multidimensional semigroup.

i. Can V have subgroups of finite order?
ii. Can ideals of V be of finite order?
iii. Can V have S-units?
iv. Can V have S-idempotents?
v. Can V have S-zero divisors?
vi. Is V a S-semigroup?
vii. Can V have S-ideals?
viii. Does V contain S-subsemigroups which are not S-ideals?
ix. Obtain any other special feature enjoyed by this MOD multidimensional semigroup.

144. Let $S = \{(C_n(10) \times R_n(5) \times R_n(12)g \times R_n(11))\ g^2 = 0, \times\}$ be MOD mixed multidimensional semigroup.

Study properties i to ix of problem 143 for this S.

145. Let
$B = \{(R_n(7)g \times R_n(7)g \times R_n(7)g \times R_n(7)g) \mid g^2 = 0, \times\}$ be the MOD multidimensional semigroup.

Study properties i to ix of problem 143 for this B.

146. Let
$P = \{(C_n(10) \times C_n(10) \times C_n(10) \times C_n(10));\ i_F^2 = 9, \times\}$ be the MOD multidimensional complex modulo integer semigroup.

Study questions i to ix of problem 143 for this P.

147. Let $M = \{(R_n(9)h \times R_n(9)h \times R_n(9)h) \mid h^2 = h, \times\}$ be the MOD multidimensional special dual like number space semigroup.

Study questions i to ix of problem 143 for this M.

148. Let $S = \{(C_n(10) \times R_n(5) \times R_n(7)g \times R_n(5)h); \times\}$ be the MOD mixed multidimensional space semigroup.

Study questions i to ix of problem 143 for this S.

149. Let $P = \{(R_n(10)g \times R_n(11)h \times R_n(12)k \times C_n(10) \times C_n(11)), g^2 = 0, h^2 = h, k^2 = 11k, \times\}$ be the MOD mixed multidimensional space semigroup.

Study questions i to ix of problem 143 for this P.

150. Let $M = \{(R_n(5) \times R_n(5)h \times C_n(5) \times R_n(5)k) \mid h^2 = h, k^2 = 4k, +\}$ be the MOD mixed multidimensional space group. M is a MOD mixed multidimensional vector space over Z_5.

i. What is the dimension of M over Z_5?
ii. Is M an infinite dimensional mixed multidimensional vector space over Z_5?
iii. Find subspaces of V.
iv. Does there exist a subspace of finite dimension over Z_5?
v. Find Hom(V, V) = W.
vi. What is the algebraic property enjoyed by W?
vii. Obtain any other special features associated with W.

151. Let $S = \{(C_n(7) \times C_n(5) \times C_n(10) \times R_n(7) \times R_n(5)) +, \times\}$ be the MOD mixed multidimensional space pseudo ring.

i. S is a S-ring.
ii. Prove S has infinite number of zero divisors.
iii. Prove S has only finite number of units.
iv. Can S have S-units?
v. Does S contain S-ideals?
vi. Can S have ideals of finite order?
vii. Prove S has S-subrings?
viii. Can S have S-idempotents?

ix. Let I be an ideal of S-find S/I.
x. Obtain any other property associated with S.

152. Let M = $\{(R_n(7) \times R_n(5)g \times R_n(12) \times C_n(8) \times R_n(8)h)$ $g^2 = 0$, $h^2 = h$, $i_F^2 = 7$, $+, \times\}$ be the MOD mixed multidimensional space pseudo ring.

Study questions i to x of problem 151 for this M.

153. Let W = $\{(C_n(7) \times C_n(7) \times C_n(7) \times C_n(7)), +, \times\}$ be the MOD multidimensional space pseudo ring.

Study questions i to x of problem 151 for this W.

154. Let B = $\{(C_n(10) \times R_n(7) \times R_n(5)g \times R_n(12) \times R_n(17)h) \mid h^2 = h$, $g^2 = 0$, $i_F^2 = 9$, $+, \times\}$ be the MOD mixed multidimensional space pseudo ring.

Study questions i to x of problem 151 for this B.

155. Let V = $\{(C_n(11) \times R_n(11) \times R_n(11)h \times R_n(11)g \times R_n(11)k) \mid k^2 = 10k$, $g^2 = 0$, $h^2 = h$, $+, \times\}$ be the MOD mixed multidimensional pseudo ring.

i. Prove V is a MOD mixed multidimensional pseudo linear algebra over Z_{11}.
ii. Prove V is a S-MOD mixed multidimensional pseudo linear algebra over [0, 11).
iii. Is V finite dimensional over [0, 11)?
iv. Obtain any other interesting property about V.

156. Mention some of the special features enjoyed by MOD multidimensional pseudo ring.

157. Distinguish between the MOD mixed multidimensional pseudo ring and MOD multidimensional pseudo ring.

158. Mention all the innovating properties enjoyed by MOD multidimensional pseudo linear algebras over [0, m).

159. Enumerate the special features enjoyed by MOD mixed multidimensional pseudo linear algebras over Z_p. (p a prime).

160. If Z_p in problem 159 is replaced by [0, p) study the related structures.

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.
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The main purpose of this book is to define and develop the notion of multi-dimensional MOD planes. Here, several interesting features enjoyed by these multi-dimensional MOD planes are studied and analyzed.

Interesting problems are proposed for the reader.

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ISBN 978-1-59973-365-4

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