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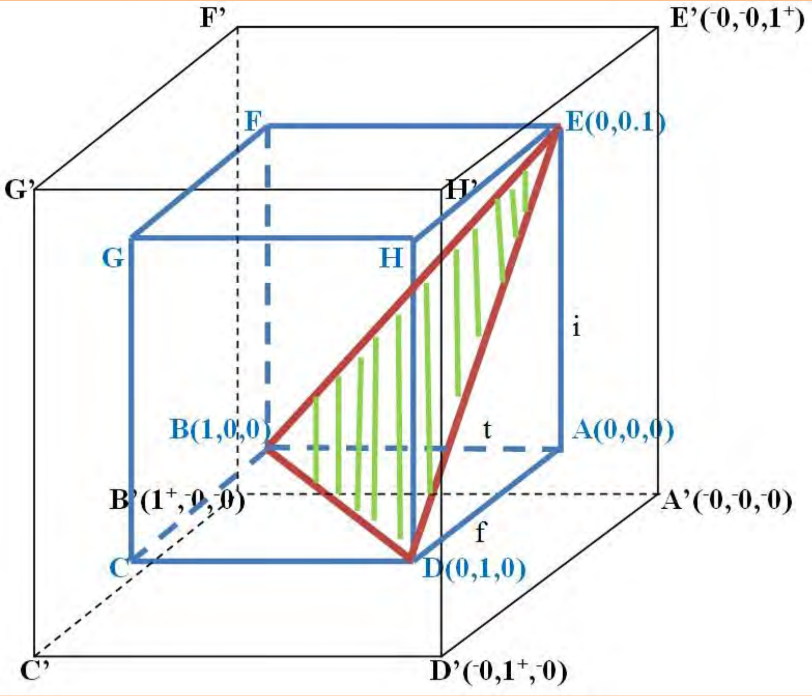

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*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Neutrosophic Integer Programming Problem

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Abstract. In this paper, we introduce the integer programming in neutrosophic environment, by considering coffecients of problem as a triangulare neutrosophic numbers. The degrees of acceptance, indeterminacy and rejection of objectives are simultaneously considered. The Neutrosophic Integer Programming Problem (NIP) is transformed into a crisp programming model, using truth membership (T), indeterminacy membership (I), and falsity membership (F) functions as well as single valued triangular neutrosophic numbers. To measure the efficiency of the model, we solved several numerical examples.

Keywords: Neutrosophic; integer programming; single valued triangular neutrosophic number.

1 Introduction

In linear programming models, decision variables are allowed to be fractional. For example, it is reasonable to accept a solution giving an hourly production of automobiles at $64\frac{1}{2}$, if the model were based upon average hourly production. However, fractional solutions are not realistic in many situations and to deal with this matter, integer programming problems are introduced. We can define integer programming problem as a linear programming problem with integer restrictions on decision variables. When some, but not all decision variables are restricted to be integer, this problem called a mixed integer problem and when all decision variables are integers, it's a pure integer program. Integer programming plays an important role in supporting managerial decisions. In integer programming problems the decision maker may not be able to specify the objective function and/or constraints functions precisely. In 1995, Smarandache [1-3] introduce neutrosophy which is the study of neutralities as an extension of dialectics. Neutrosophic is the derivative of neutrosophy and it includes neutrosophic set, neutrosophic probability, neutrosophic statistics and neutrosophic logic. Neutrosophic theory means neutrosophy applied in many fields of sciences, in order to solve problems related to indeterminacy. Although intuitionistic fuzzy sets can only handle incomplete information not indeterminate, the neutrosophic set can handle both incomplete and indeterminate information.[4] Neutrosophic sets characterized by three independent degrees as in Fig.1., namely truth-membership degree (T), indeterminacy-membership degree(I), and falsity-membership degree (F), where T,I,F are standard or non-standard subsets of $]^-0, 1^+[$. The decision makers in neutrosophic set want to increase the degree of truth-membership and decrease the degree of indeterminacy and falsity membership.

The structure of the paper is as follows: the next section is a preliminary discussion; the third section describes the formulation of integer programing problem using the proposed model; the fourth section presents some illustrative examples to put on view how the approach can be applied; the last section summarizes the conclusions and gives an outlook for future research.

2 Some Preliminaries

2.1 Neutrosophic Set [4]

Let X be a space of points (objects) and $x\in X$. A neutrosophic set A in X is defined by a truth-membership function (x), an indeterminacy-membership function (x) and a falsity-membership function $F_A(x)$. (x), $I(x)$ and $F(x)$ are real standard or real nonstandard subsets of $]0^-,1^+[$. That is $T_A(x){:}X\to]0^-,1^+[$, $I_A(x){:}X\to]0^-,1^+[$ and $F_A(x){:}X\to]0^-,1^+[$. There is no restriction on the sum of (x), (x) and $F_A(x)$, so

$$0^-\leq \sup(x)\leq \sup I_A(x)\leq F_A(x)\leq 3^+.$$

2.2 Single Valued Neutrosophic Sets (SVNS) [3-4]

Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form $A=\{\langle x, T(x), I_A(x),F_A(x)\rangle{:}x\in X\}$, where $T_A(x){:}X\to[0,1]$, $I_A(x){:}X\to[0,1]$ and $F_A(x){:}X\to[0,1]$ with $0\leq T_A(x)+ I_A(x)+F_A(x)\leq 3$ for all $x\in X$. The intervals $T(x)$,

$I(x)$ and $F_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

In the following, we write SVN numbers instead of single valued neutrosophic numbers. For convenience, a SVN number is denoted by $A=(a,b,c)$, where $a,b,c\in[0,1]$ and $a+b+c\leq 3$.

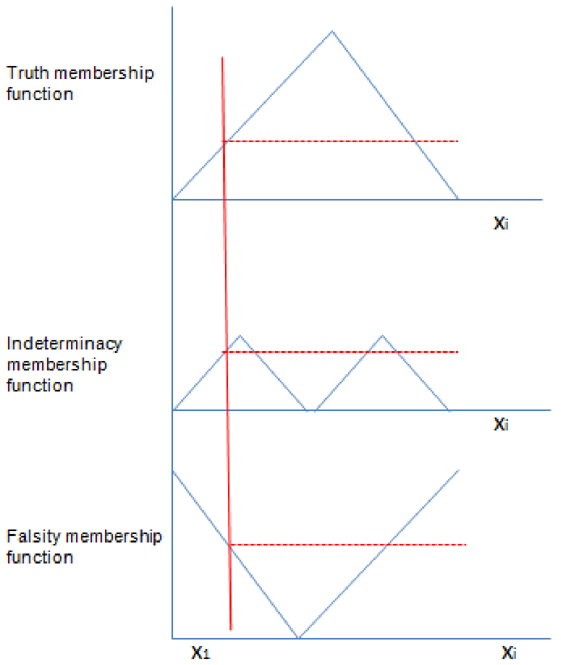

Figure 1: Neutrosophication process

2.3 Complement [5]

The complement of a single valued neutrosophic set A is denoted by $_C(A)$ and is defined by

$$T_c(A)(x) = F(A)(x),$$
$$I_c(A)(x) = 1 - I(A)(x),$$
$$F_c(A)(x) = T(A)(x) \quad \text{for all } x \text{ in } X$$

2.4 Union [5]

The union of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = AUB, whose truth-membership, indeterminacy membership and falsity-membership functions are given by

$$T(C)(x) = max\ (T(A)(x), T(B)(x)),$$
$$I(C)(x) = max\ (I(A)(x), I(B)(x)),$$
$$F(C)(x) = min((A)(x), F(B)(x)) \ \text{for all } x \text{ in } X$$

2.5 Intersection [5]

The intersection of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = A∩B, whose truth-membership, indeterminacy membership and falsity-membership functions are given by

$$T(C)(x) = min\ (T(A)(x), T(B)(x)),$$
$$I(C)(x) = min\ (I(A)(x), I(B)(x)),$$
$$F(C)(x) = max((A)(x), F(B)(x)) \ \text{for all } x \text{ in } X$$

3 Neutrosophic Integer Programming Problems

Integer programming problem with neutrosophic coefficients (NIPP) is defined as the following:

Maximize $Z = \sum_{j=1}^{n} \tilde{c}_j x_j$

Subject to

$$\sum_{j=1}^{n} a_{ij}^{\sim n} x_j \leq b_i \quad i = 1, \dots, m, \tag{1}$$
$$x_j \geq 0, \qquad j = 1, \dots n,$$
$$x_j \quad \text{integer for } \ j \in \{0,1, \dots n\}.$$

Where $\tilde{c}_j$, $a_{ij}^{\sim n}$ are neutrosophic numbres.

The single valued neutrosophic number ($a_{ij}^{\sim n}$) is donated by A=(a,b,c) where a,b,c ∈ [0,1] And a,b,c ≤ 3

The truth- membership function of neutrosophic number $a_{ij}^{\sim n}$ is defined as:

$$T\, a_{ij}^{\sim n}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2 \\ \frac{a_2-x}{a_3-a_2} & a_2 \leq x \leq a_3 \\ 0 & otherwise \end{cases} \tag{2}$$

The indeterminacy- membership function of *neutrosophic number* a_{ij}^{n} is defined as:

$$I\, a_{ij}^{\sim n}(x) = \begin{cases} \frac{x-b_1}{b_2-b_1} & b_1 \leq x \leq b_2 \\ \frac{b_2-x}{b_3-b_2} & b_2 \leq x \leq b_3 \\ 0 & otherwise \end{cases} \tag{3}$$

And its falsity- membership function of *neutrosophic number* $a_{ij}^{\sim n}$ is defined as:

$$F\, a_{ij}^{\sim n}(x) = \begin{cases} \frac{x-C_1}{C_2-C_1} & C_1 \leq x \leq C_2 \\ \frac{b_2-x}{b_3-b_2} & C_2 \leq x \leq C_3 \\ 1 & otherwise \end{cases} \tag{4}$$

Then we find the maximum and minimum values of the objective function for truth-membership, indeterminacand falsity membership as follows:

$f^{max} = \max\{f(x_i^*)\}$ and $f^{min} = \min\{f(x_i^*)\}$ where $1 \leq i \leq k$

$f_{min}^{F} = f_{min}^{T}$ and $f_{max}^{F} = f_{max}^{T} - R(f_{max}^{T} - f_{min}^{T})$

$f^I_{max}=f^I_{max}$ and $f^I_{min}=f^I_{min} - S(f^T_{max} - f^T_{min})$

Where R ,S are predetermined real number in (0,1)

The truth membership, indeterminacy membership, falsity membership of objective function as follows:

$$T^f(x) = \begin{cases} 1 & if \ \ f \le f^{min} \\ \frac{f^{max}-f(x)}{f^{max}-f^{min}} & if \ \ f^{min} < f(x) \le f^{max} \\ 0 & if \ f(x) \ > f^{max} \end{cases} \quad (5)$$

$$I^f(x) = \begin{cases} 0 & if \ \ f \le f^{min} \\ \frac{f(x)-f^{max}}{f^{max}-f^{min}} & if \ \ f^{min} < f(x) \le f^{max} \\ 0 & if \ f(x) \ > f^{max} \end{cases} \quad (6)$$

$$F^f(x) = \begin{cases} 0 & if \ \ f \le f^{min} \\ \frac{f(x)-f^{min}}{f^{max}-f^{min}} & if \ \ f^{min} < f(x) \le f^{max} \\ 1 & if \ f(x) \ > f^{max} \end{cases} \quad (7)$$

The neutrosophic set of the j^{th} decision variable x_j is defined as:

$$T^{(x)}_{x_j} = \begin{cases} 1 & if \ \ x_j \le 0 \\ \frac{d_j-x_j}{d_j} & if \ \ 0 < x_j \le d_j \\ 0 & if \ \ x_j \ > d_j \end{cases} \quad (8)$$

$$F^{(x)}_{x_j} = \begin{cases} 0 & if \ \ x_j \le 0 \\ \frac{x_j}{d_j+b_j} & if \ \ 0 < x_j \le d_j \\ 1 & if \ \ x_j \ > d_j \end{cases} \quad (9)$$

$$I^{(x)}_j = \begin{cases} 0 & if \ \ x_j \le 0 \\ \frac{x_j-d_j}{d_j+b_j} & if \ \ 0 < x_j \le d_j \\ 0 & if \ \ x_j \ > d_j \end{cases} \quad (10)$$

Where d_j, b_j are integer numbers.

4 Neutrosophic Optimization Model of integer programming problem

In our neutrosophic model we want to maximize the degree of acceptance and minimize the degree of rejection and indeterminacy of the neutrosophic objective function and constraints. Neutrosophic optimization model can be defined as:

$$\begin{aligned} & maxT_{(x)} \\ & minF_{(x)} \\ & minI_{(x)} \\ & Subject\ to \\ & T_{(X)} \ge F_{(x)} \\ & T_{(X)} \ge I_{(x)} \\ & 0 \le T_{(X)} + I_{(x)} + F_{(x)} \le 3 \\ & T_{(X)}, \quad I_{(X)}, \quad F_{(X)} \ge 0 \\ & x \ge 0 \ \text{, integer.} \end{aligned} \quad (11)$$

Where $T_{(x)}. F_{(x)}, I_{(x)}$ denotes the degree of acceptance, rejection and indeterminacy of x respectively.

The above problem is equivalent to the following:

$$\begin{aligned} & max\ \alpha, \ min\ \beta\ , min\ \theta \\ & Subject\ to \\ & \alpha \le T_{(x)} \\ & \beta \le F_{(x)} \\ & \theta \le I_{(x)} \\ & \alpha \ge \beta \\ & \alpha \ge \theta \\ & 0 \le \ \alpha + \ \beta + \ \theta \le 3 \\ & x \ge 0 \ \text{, integer.} \end{aligned} \quad (12)$$

Where α denotes the minimal acceptable degree, β denote the maximal degree of rejection and θ denote maximal degree of indeterminacy.

The neutrosophic optimization model can be changed into the following optimization model:

$$\begin{aligned} & max(\alpha - \ \beta - \ \theta) \\ & Subject\ to \\ & \alpha \le T_{(x)} \\ & \beta \ge F_{(x)} \\ & \theta \ge I_{(x)} \\ & \alpha \ge \beta \\ & \alpha \ge \theta \\ & 0 \le \ \alpha + \ \beta + \ \theta \le 3 \\ & \alpha, \beta, \theta \ge 0 \\ & x \ge 0 \ \text{, integer.} \end{aligned} \quad (13)$$

The previous model can be written as:

$$\begin{aligned} & min\ (1- \ \alpha)\ \beta\ \theta \\ & Subject\ to \\ & \alpha \le T_{(x)} \\ & \beta \ge F_{(x)} \\ & \theta \ge I_{(x)} \\ & \alpha \ge \beta \\ & \alpha \ge \theta \\ & 0 \le \ \alpha + \ \beta + \ \theta \le 3 \\ & x \ge 0 \ \text{, integer.} \end{aligned} \quad (14)$$

5 The Algorithms for Solving Neutrosophic integer Programming Problem (NIPP)

5.1 Neutrosophic Cutting Plane Algorithm

Step 1: Convert neutrosophic integer programming problem to its crisp model by using the following method:
By defining a method to compare any two single valued triangular neutrosophic numbers which is based on the score function and the accuracy function. Let $\tilde{a} = \langle (a_1, b_1, c_1), w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ be a single valued triangular neutrosophic number, then

$$S(\tilde{a}) = \frac{1}{16}[a + b + c]\times(2 + \mu_{\tilde{a}} - v_{\tilde{a}} - \lambda_{\tilde{a}}) \quad (15)$$

and

$$A(\tilde{a}) = \frac{1}{16}[a + b + c]\times(2 + \mu_{\tilde{a}} - v_{\tilde{a}} + \lambda_{\tilde{a}}) \quad (16)$$

is called the score and accuracy degrees of $\tilde{a}$, respectively. The neutrosophic integer programming NIP can be represented by crisp programming model using truth membership, indeterminacy membership, and falsity membership functions and the score and accuracy degrees of ã, at equations (15) or (16).

Step 2: Create the decision set which include the highest degree of truth-membership and the least degree of falsity and indeterminacy memberships.

Step 3: Solve the problem as a linear programming problem and ignore integrality.

Step 4: If the optimal solution is integer, then it's right. Otherwise, go to the next step.

Step 5: Generate a constraint which is satisfied by all integer solutions and add this constraint to the problem.

Step 6: Go to step 1.

5.2 Neutrosophic Branch and Bound Algorithm

Step 1: Convert neutrosophic integer programming problem to its crisp model by using the following method:
By defining a method to compare any two single valued triangular neutrosophic numbers which is based on the score function and the accuracy function. Let $\tilde{a} = \langle (a_1, b_1, c_1), w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ be a single valued triangular neutrosophic number, then

$$S(\tilde{a}) = \frac{1}{16}[a + b + c]\times(2 + \mu_{\tilde{a}} - v_{\tilde{a}} - \lambda_{\tilde{a}}) \quad (15)$$

and

$$A(\tilde{a}) = \frac{1}{16}[a + b + c]\times(2 + \mu_{\tilde{a}} - v_{\tilde{a}} + \lambda_{\tilde{a}}) \quad (16)$$

is called the score and accuracy degrees of $\tilde{a}$, respectively. The neutrosophic integer programming NIP can be represented by crisp programming model using truth membership, indeterminacy membership, and falsity membership functions and the score and accuracy degrees of ã, at equations (15) or (16).

Step 2: Create the decision set which include the highest degree of truth-membership and the least degree of falsity and indeterminacy memberships.

Step 3: At the first node let the solution of linear programming model with integer restriction as an upper bound and the rounded-down integer solution as a lower bound.

Step 4: For branching process, we select the variable with the largest fractional part. Two constrains are obtained after the branching process, one for≤ and the other is ≥ constraint.

Step 5: Create two nodes for the two new constraints.

Step 6: Solve the model again, after adding new constraints at each node.

Step 7: The optimal integer solution has been reached, if the feasible integer solution has the largest upper bound value of any ending node. Otherwise return to step 4.

The previous algorithm is for a maximization model. For a minimization model, the solution of linear programming problem with integer restrictions are rounded up and upper and lower bounds are reversed.

6 Numerical Examples

To measure the efficiency of our proposed model we solved many numerical examples.

6.1 Illustrative Example #1

$$max \quad \tilde{5}x_1 + \tilde{3}x_2$$

$$subject\ to \quad \begin{aligned} \tilde{4}x_1 + \tilde{3}x_2 &\le \widetilde{12} \\ \tilde{1}x_1 + \tilde{3}x_2 &\le \tilde{6} \\ x_1, x_2 &\ge 0\ and\ integer \end{aligned}$$

where

$\tilde{5} = \langle (4,5,6), 0.8, 0.6, 0.4 \rangle$
$\tilde{3} = \langle (2.5,3,3.5), 0.75, 0.5, 0.3 \rangle$
$\tilde{4} = \langle (3.5,4,4.1), 1, 0.5, 0.0 \rangle$
$\tilde{3} = \langle (2.5,3,3.5), 0.75, 0.5, 0.25 \rangle$
$\tilde{1} = \langle (0,1,2), 1, 0.5, 0 \rangle$
$\tilde{3} = \langle (2.8,3,3.2), 0.75, 0.5, 0.25 \rangle$
$\widetilde{12} = \langle (11,12,13), 1, 0.5, 0 \rangle$
$\tilde{6} = \langle (5.5,6,7.5), 0.8, 0.6, 0.4 \rangle$

Then the neutrosophic model converted to the crisp model by using Eq.15 , Eq.16.as follows :

$$\max \quad 5.6875x_1 + 3.5968x_2$$
$$\text{subject to} \quad \begin{array}{l} 4.3125x_1 + 3.625x_2 \leq 14.375 \\ 0.2815x_1 + 3.925x_2 \leq 7.6375 \\ x_1, x_2 \geq 0 \text{ and integer} \end{array}$$

The optimal solution of the problem is $x^* = (3,0)$ with optimal objective value 17.06250.

6.2 Illustrative Example #2

$$max \quad \widetilde{25}x_1 + \widetilde{48}x_2$$
$$subject\ to \quad \begin{array}{l} 15x_1 + 30x_2 \leq 45000 \\ 24x_1 + 6x_2 \leq 24000 \\ 21x_1 + 14x_2 \leq 28000 \\ x_1, x_2 \geq 0 \text{ and integer} \end{array}$$

where

$\widetilde{25} = \langle (19,25,33), 0.8,0.5,0 \rangle$;

$\widetilde{48} = \langle (44,48,54), 0.9,0.5,0 \rangle$

Then the neutrosophic model converted to the crisp model as :

$$\max \quad 27.8875x1 + 55.3x_2$$
$$\text{subject to} \quad \begin{array}{l} 15x_1 + 30x_2 \leq 45000 \\ 24x_1 + 6x_2 \leq 24000 \\ 21x_1 + 14x_2 \leq 28000 \\ x_1, x_2 \geq 0 \text{ and integer} \end{array}$$

The optimal solution of the problem is $x^* = (500,1250)$ with optimal objective value 83068.75.

7 Conclusions and Future Work

In this paper, we proposed an integer programming model based on neutrosophic environment, simultaneously considering the degrees of acceptance, indeterminacy and rejection of objectives, by proposed model for solving neutrosophic integer programming problems (NIPP). In the model, we maximize the degrees of acceptance and minimize indeterminacy and rejection of objectives. NIPP was transformed into a crisp programming model using truth membership, indeterminacy membership, falsity membership and score functions. We also give numerical examples to show the efficiency of the proposed method. Future research directs to studying the duality theory of integer programming problems based on Neutrosophic.

References

[1] Smarandache, F. "A Unifying Field in Logics:Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability: Neutrosophic Logic.Neutrosophy, Neutrosophic Set, Neutrosophic Probability. Infinite Study, 2005.

[2] I. M. Hezam, M. Abdel-Baset, F. Smarandache 2015 Taylor Series Approximation to Solve Neutrosophic Multiobjective Programming Problem Neutrosophic Sets and Systems An International Journal in Information Science and Engineering Vol.10 pp.39-45.

[3] Abdel-Baset, M., Hezam, I. M., & Smarandache, F. (2016). Neutrosophic Goal Programming. Neutrosophic Sets & Systems, 11.

[4] Smarandache, F. "A Geometric Interpretation of the Neutrosophic Set-A Generalization of the Intuitionistic Fuzzy Set." arXiv preprint math/0404520(2004).

[5] R. Şahin, and Muhammed Y. "A Multi-criteria neutrosophic group decision making metod based TOPSIS for supplier selection." arXiv preprint arXiv:1412.5077 (2014).

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Multi-Objective Structural Design Optimization using Neutrosophic Goal Programming Technique

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Abstract: This paper develops a multi-objective Neutrosophic Goal Optimization (NSGO) technique for optimizing the design of three bar truss structure with multiple objectives subject to a specified set of constraints. In this optimum design formulation, the objective functions are weight and deflection; the design variables are the cross-sections of the bar; the constraints are the stress in member. The classical three bar truss structure is presented here in to demonstrate the efficiency of the neutrosophic goal programming approach. The model is numerically illustrated by generalized NSGO technique with different aggregation method. The result shows that the Neutrosophic Goal Optimization technique is very efficient in finding the best optimal solutions.

Keywords: Neutrosophic Set, Single Valued Neutrosophic Set, Generalized Neutrosophic Goal Programming, Arithmetic Aggregation, Geometric Aggregation, Structural Optimization.

1 Introduction

The research area of optimal structural design has been receiving increasing attention from both academia and industry over the past four decades in order to improve structural performance and to reduce design costs. In the real world, uncertainty or vagueness is prevalent in the Engineering Computations. In the context of structural design the uncertainty is connected with lack of accurate data of design factors. This tendency has been changing due to the increase in the use of fuzzy mathematical algorithm for dealing with such kind of problems.

Fuzzy set (FS) theory has long been introduced to deal with inexact and imprecise data by Zadeh [1], Later on the fuzzy set theory was used by Bellman and Zadeh [2] to the decision making problem. A few work has been done as an application of fuzzy set theory on structural design. Several researchers like Wang et al. [3] first applied α-cut method to structural designs where various design levels α were used to solve the non-linear problems. In this regard ,a generalized fuzzy number has been used Dey et al. [4] in context of a non-linear structural design optimization. Dey et al. [5] used basic t-norm based fuzzy optimization technique for optimization of structure and Dey et al. [6] developed parameterized t-norm based fuzzy optimization method for optimum structural design.

In such extension, Intuitionistic fuzzy set which is one of the generalizations of fuzzy set theory and was characterized by a membership, a non- membership and a hesitancy function was first introduced by Atanassov [21] (IFS). In fuzzy set theory the degree of acceptance is only considered but in case of IFS it is characterized by degree of membership and non-membership in such a way that their sum is less or equal to one. Dey et al. [7] solved two bar truss non-linear problem by using intuitionistic fuzzy optimization problem.Again Dey et al. [8] used intuitionistic fuzzy optimization technique to solve multi objective structural design. R-x Liang et al. [9] applied interdependent inputs of single valued trapezoidal neutrosophic information on Multi-criteria group decision making problem. P Ji et al. [10], S Yu et al. [11] did so many research study on application based neutosophic sets and intuitionistic linguistic number. Z-p Tian et al. [12] Simplified neutrosophic linguistic multi-criteria group decision-making approach to green product development. Again J-j Peng et al. [13] introduced multi-valued neutrosophic qualitative flexible approach based on likelihood for multi-criteria decision-making problems. Also, H Zhang et. al. [22] investigates a case study on a novel decision support model for satisfactory restaurants utilizing social information. P Ji et al. [14] developed a projection-based TODIM method under multi-valued neutrosophic environments and its application in personnel selection.Intuitionistic fuzzy sets consider both truth and falsity membership and can only handle incomplete information but not the information which is connected with indeterminacy or inconsistency.

In neutrosophic sets indeterminacy or inconsistency is quantified explicitly by indeterminacy membership function. Neutrosophic Set (NS), introduced by Smarandache [15] was characterized by truth, falsity and indeterminacy membership so that in case of single valued NS set their sum is less or equal to three. In early [17] Charnes and Cooper first introduced Goal programming problem for a linear model. Usually conflicting goal are presented in a multi-objective goal programming problem. Dey et al. [16] used intuitionistic goal programming on nonlinear structural model. This is the first time NSGO technique is in application to multi-objective structural design. Usually objective goals of existing structural model are considered to be deterministic and a fixed quantity. In a situation, the decision maker can be doubtful with regard to accomplishment of the goal. The DM may include the idea of truth, indeterminacy and falsity bound on objectives goal.The goal may have a target value with degree of truth,indeterminacy as well as degree of falsity.Precisely ,we can say a human being that express degree of truth membership of a given element in a fuzzy set,truth and falsity membership in a intuitionistic fuzzy set,very often does not express the corresponding degree of falsity membership as complement to 3. This fact seems to take the objective goal as a neutrosophic set. The present study investigates computational algorithm for solving multi-objective structural problem by single valued generalized NSGO technique. The results are compared numerically for different aggregation method of NSGO technique. From our numerical result, it has been seen the best result obtained for geometric aggregation method for NSGO technique in the perspective of structural optimization technique.

2 Multi-objective structural model

In the design problem of the structure i.e. lightest weight of the structure and minimum deflection of the loaded joint that satisfies all stress constraints in members of the structure. In truss structure system, the basic parameters (including allowable stress,etc.) are known and the optimization's target is that identify the optimal bar truss cross-section area so that the structure is of the smallest total weight with minimum nodes displacement in a given load conditions .

The multi-objective structural model can be expressed as

$$\text{Minimize } WT(A)$$

(1)

$$\text{minimize } \delta(A)$$

$$\text{subject to } \sigma(A) \le [\sigma]$$

$$A^{\min} \le A \le A^{\max}$$

where $A = [A_1, A_2, ..., A_n]^T$ are the design variables for the cross section, n is the group number of design variables for the cross section bar , $WT(A) = \sum_{i=1}^{n} \rho_i A_i L_i$ is the total weight of the structure , $\delta(A)$ is the deflection of the loaded joint ,where L_i, A_i and ρ_i are the bar length, cross section area and density of the i^{th} group bars respectively. $\sigma(A)$ is the stress constraint and $[\sigma]$ is allowable stress of the group bars under various conditions, $A^{\min}$ and $A^{\max}$ are the lower and upper bounds of cross section area A respectively.

3 Mathematical preliminaries

3.1 Fuzzy set

Let X be a fixed set. A fuzzy set A set of X is an object having the form $\tilde{A} = \{(x, T_A(x)) : x \in X\}$ where the function $T_A : X \to [0,1]$ defined the truth membership of the element $x \in X$ to the set A .

3.2 Intuitionistic fuzzy set

Let a set X be fixed. An intuitionistic fuzzy set or IFS $\tilde{A}^i$ in X is an object of the form

$$\tilde{A}^i = \{< X, T_A(x), F_A(x) > | x \in X\} \text{ where}$$

$$T_A : X \to [0,1] \text{ and } F_A : X \to [0,1]$$

define the truth membership and falsity membership respectively, for every element of $x \in X$ $0 \le T_A + F_A \le 1$.

3.3 Neutrosophic set

Let a set X be a space of points (objects) and $x \in X$ .A neutrosophic set $\tilde{A}^n$ in X is defined by a truth membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity membership function $F_A(x)$,and denoted by $\tilde{A}^n = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\}$.

$T_A(x)$ $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$ .That is $T_A(x) : X \to]0^-, 1^+[$, $I_A(x) : X \to]0^-, 1^+[$, and $F_A(x) : X \to]0^-, 1^+[$, . There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ so $0^- \le \sup T_A(x) + I_A(x) + \sup F_A(x) \le 3^+$.

3.4 Single valued neutrosophic set

Let a set X be the universe of discourse. A single valued neutrosophic set $\tilde{A}^n$ over X is an object having the form $\tilde{A}^n = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\}$ where $T_A : X \to [0,1]$, $I_A : X \to [0,1]$, and $F_A : X \to [0,1]$ with $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ for all $x \in X$.

3.5 Complement of neutrosophic Set

Complement of a single valued neutrosophic set A is denoted by $c(A)$ and is defined by $T_{c(A)}(x) = F_A(x)$, $I_{c(A)}(x) = 1 - F_A(x)$, $F_{c(A)}(x) = T_A(x)$

3.6 Union of neutrosophic sets

The union of two single valued neutrosophic sets A and B is a single valued neutrosophic set C , written as $C = A \cup B$,whose truth membership, indeterminacy-membership and falsity-membership functions are given by

$T_{c(A)}(x) = \max(T_A(x), T_B(x))$,

$I_{c(A)}(x) = \max(I_A(x), I_B(x))$,

$F_{c(A)}(x) = \min(F_A(x), F_B(x))$ for all $x \in X$.

3.7 Intersection of neutrosophic sets

The intersection of two single valued neutrosophic sets A and B is a single valued neutrosophic set C , written as $C = A \cap B$,whose truth membership, indeterminacy-membership and falsity-membership functions are given by

$T_{c(A)}(x) = \min(T_A(x), T_B(x))$,

$I_{c(A)}(x) = \min(I_A(x), I_B(x))$,

$F_{c(A)}(x) = \max(F_A(x), F_B(x))$ for all $x \in X$.

4 Mathematical analysis

4.1 Neutrosophic Goal Programming

Neutrosophic Goal Programming problem is an extension of intuitionistic fuzzy as well as fuzzy goal programming problem in which the degree of indeterminacy of objective(s) and constraints are considered with degree of truth and falsity membership degree.

Goal programming can be written as

Find

$$x = (x_1, x_2, ..., x_n)^T \quad (1)$$

to achieve:

$z_i = t_i \;\; i = 1, 2, ..., k$

Subject to $x \in X$ where t_i are scalars and represent the target achievement levels of the objective functions that the decision maker wishes to attain provided, X is feasible set of constraints.

The nonlinear goal programming problem can be written as

Find

$$x = (x_1, x_2, ..., x_n)^T \quad (2)$$

So as to

Minimize z_i with target value t_i ,acceptance tolerance a_i ,indeterminacy tolerance d_i rejection tolerance c_i

$x \in X$

$g_j(x) \le b_j$, $j = 1, 2, ..., m$

$x_i \ge 0, \; i = 1, 2, ..., n$

This neutrosophic goal programming can be transformed into crisp programming and can be transformed into crisp programming problem model by maximizing the degree of truth and indeterminacy and minimizing the degree of falsity of neutrosophic objectives and constraints. In the above problem (2), multiple objectives are considered as neutrosophic with some relaxed target. This representation demonstrates that decision maker (DM) is not sure about minimum value of $z_i, i = 1, 2, .., k$. DM has some illusive ideas of some optimum values of $z_i, i = 1, 2, .., k$. Hence it is quite natural to have desirable values violating the set target. Then question arises that how much bigger the optimum values may be .DM has also specified it with the use of tolerances. The tolerances are set in such a manner that the sum of truth, indeterminacy and falsity membership of objectives $z_i, i = 1, 2, .., k$ will lie between 0 and 3 . Let us consider the following theorem on membership function:

Theorem 1.

For a generalized neutrosophic goal programming problem (2)

The sum of truth, indeterminacy and falsity membership function will lie between 0 and $w_1 + w_2 + w_3$

Proof:

Let the truth, indeterminacy and falsity membership functions be defined as membership functions

$$T_i^{w_1}(z_i) = \begin{cases} w_1 & if \;\; z_i \le t_i \\ w_1\left(\dfrac{t_i + a_i - z_i}{a_i}\right) & if \;\; t_i \le z_i \le t_i + a_i \\ 0 & if \;\; z_i \ge t_i + a_i \end{cases}$$

$$I_i^{w_2}(z_i) = \begin{cases} 0 & if \;\; z_i \le t_i \\ w_2\left(\dfrac{z_i - t_i}{d_i}\right) & if \;\; t_i \le z_i \le t_i + a_i \\ w_2\left(\dfrac{t_i + a_i - z_i}{a_i - d_i}\right) & if \;\; t_i + d_i \le z_i \le t_i + a_i \\ 0 & if \;\; z_i \ge t_i + a_i \end{cases}$$

$$F_i^{w_3}(z_i)=\begin{cases}0 & if\ \ z_i\le t_i\\ w_3\left(\dfrac{z_i-t_i}{c_i}\right) & if\ \ t_i\le z_i\le t_i+c_i\\ w_3 & if\ \ z_i\ge t_i+c_i\end{cases}$$

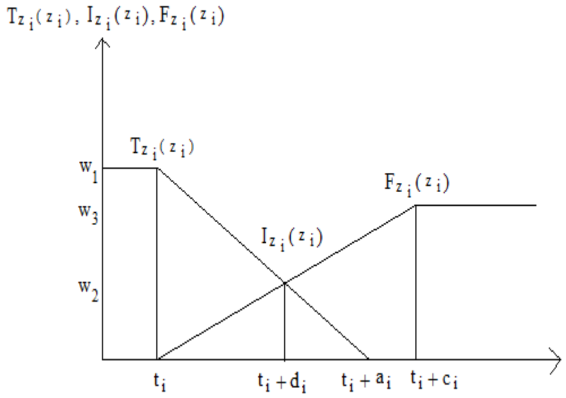

Fig. 1. Truth membership, Indeterminacy membership and Falsity membership function of z_i

From Fig. (1) and definition of generalized single valued neutrosophic set, it is clear that:

$0\le T_{z_i}(z_i)\le w_1$, $0\le I_{z_i}(z_i)\le w_2$ and $0\le F_{z_i}(z_i)\le w_1$

when $(z_i)\le t_i$

$T_{z_i}(z_i)=w_1$ and $I_{z_i}(z_i)=0$ and $F_{z_i}(z_i)=0$

Therefore $T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)=w_1\le w_1+w_2+w_3$

and $w_1\ge 0$ implies that $T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)\ge 0$

when $z_i\in(t_i,t_i+a_i)$ from fig (A) we see that $T_{z_i}(z_i)$ and $F_{z_i}(z_i)$ intersects each other and the point whose coordinate is (t_i+d_i,d_ic_i),

where $d_i=\dfrac{w_1}{\dfrac{w_1}{a_i}+\dfrac{w_2}{c_i}}$.

Now in the interval $z_i\in(t_i,t_i+d_i)$ we see that

$$T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)=w_2\left(\frac{z_i-t_i}{d_i}\right)\le w_2\le w_1+w_2+w_3$$

Again, in the interval $z_i\in(t_i+d_i,t_i+a_i)$ we see that

$$T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)=w_2\left(\frac{t_i+a_i-z_i}{a_i-d_i}\right)\le w_2\le w_1+w_2+w_3.$$

Also, for $t_i\le z_i\le t_i+a_i$

when $z_i\ge t_i$, $T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)>w_2\ge 0$ and

$T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)>w_1\ge 0$ and when

$z_i\le t_i+a_i$, $T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)\le w_1\dfrac{a_i}{c_i}<w_1\le w_1+w_2$

(as $\dfrac{a_i}{c_i}\le 1$).

In the interval $z_i\in(t_i+a_i,t_i+c_i]$

when $z_i>t_i+a_i$, $T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)>w_2\dfrac{a_i}{c_i}>w_2\ge 0$

(as $\dfrac{a_i}{c_i}\le 1$)

and when

$z_i\le t_i+c_i$, $T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)\le w_1\le w_1+w_2+w_3$

for $z_i>t_i+c_i$,

$T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)=w_3\le w_1+w_2+w_3$

and as $w_2\ge 0$, $T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)\ge 0$.

Therefore, combining all the cases we get

$0\le T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)\le w_1+w_2+w_3$

Hence the proof.

4.2. Solution Procedure of Neutrosophic Goal Programming Technique

In fuzzy goal programming, Zimmermann [18] has given a concept of considering all membership functions greater than a single value α which is to be maximized. Previously many researcher like Bharti and Singh [20], Parvathi and Malathi [19] have followed him in intutionistic fuzzy optimization. Along with the variable α and β, γ is optimized in neutrosophic goal programming problem.

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming problem (2) can be formulated as:

$$Maximize\ T_{z_i}(z_i),\ i=1,2,....,k \qquad (3)$$

$$Maximize\ I_{z_i}(z_i),\ i=1,2,....,k$$

$$Minimize\ F_{z_i}(z_i),\ i=1,2,....,k$$

Subject to

$$0\le T_{z_i}(z_i)+I_{z_i}(z_i)+F_{z_i}(z_i)\le w_1+w_2+w_3,\ i=1,2,....,k$$

$$T_{z_i}(z_i)\ge 0, I_{z_i}(z_i)\ge 0, F_{z_i}(z_i)\ I=1,2,...,k$$

$$T_{z_i}(z_i)\ge I_{z_i}(z_i),\ I=1,2,....,k$$

$$T_{z_i}(z_i)\ge F_{z_i}(z_i),\ i=1,2,...,k$$

$$0\le w_1+w_2+w_3\le 3$$

$$w_1,w_2,w_3\in[0,1]$$

$$g_j(x)\le b_j,\ \ j=1,2,.....,m$$

$$x_i\ge 0,\ i=1,2,.....,n$$

Now the decision set $\tilde{D}^n$, a conjunction of Neutrosophic objectives and constraints is defined:

$$\tilde{D}^n=\left(\bigcap_{i=1}^{k}\tilde{z}_i^n\right)\bigcap\left(\bigcap_{j=1}^{q}\tilde{g}_j^n\right)=\left\{\left(x,T_{\tilde{D}^n}(x)\right),I_{\tilde{D}^n}(x),F_{\tilde{D}^n}(x)\right\}$$

$$\text{Here } \alpha=T_{\tilde{D}^n}(x)=\min\left\{\begin{array}{l}T_{\tilde{z}_1^n}(x),T_{\tilde{z}_2^n}(x),T_{\tilde{z}_3^n}(x),\ldots\ldots.T_{\tilde{z}_p^n}(x);\\T_{\tilde{g}_1^n}(x),T_{\tilde{g}_2^n}(x),T_{\tilde{g}_3^n}(x),\ldots\ldots.T_{\tilde{g}_q^n}(x)\end{array}\right\}\quad \text{for all } x\in X$$

$$\gamma=I_{\tilde{D}^n}(x)=\min\left\{\begin{array}{l}I_{\tilde{z}_1^n}(x),I_{\tilde{z}_2^n}(x),I_{\tilde{z}_3^n}(x),\ldots\ldots.I_{\tilde{z}_p^n}(x);\\I_{\tilde{g}_1^n}(x),I_{\tilde{g}_2^n}(x),I_{\tilde{g}_3^n}(x),\ldots\ldots.I_{\tilde{g}_q^n}(x)\end{array}\right\}\quad \text{for all } x\in X$$

$$\beta=F_{\tilde{D}^n}(x)=\min\left\{\begin{array}{l}F_{\tilde{z}_1^n}(x),F_{\tilde{z}_2^n}(x),F_{\tilde{z}_3^n}(x),\ldots\ldots.F_{\tilde{z}_p^n}(x);\\F_{\tilde{g}_1^n}(x),F_{\tilde{g}_2^n}(x),F_{\tilde{g}_3^n}(x),\ldots\ldots.F_{\tilde{g}_q^n}(x)\end{array}\right\}\quad \text{for all } x\in X$$

where $T_{\tilde{D}^n}(x),I_{\tilde{D}^n}(x),F_{\tilde{D}^n}(x)$ are truth-membership function, indeterminacy membership function,falsity membership function of neutrosophic decision set respectively .Now using the neutrosophic optimization, problem (2) is transformed to the non-linear programming problem as

$$\text{Maximize } \alpha, \text{Maximize } \gamma, \text{Minimize } \beta \qquad (4)$$

$$z_i\le t_i+a_i\left(1-\frac{\alpha}{w_1}\right), i=1,2,\ldots,k$$

$$z_i\ge t_i+\frac{d_i}{w_2}\gamma, i=1,2,\ldots,k$$

$$z_i\le t_i+a_i-\frac{\gamma}{w_2}\left(a_i-d_i\right), i=1,2,\ldots,k$$

$$z_i\le t_i+\frac{c_i}{w_3}\beta, i=1,2,\ldots,k$$

$$z_i\le t_i, i=1,2,\ldots..,k$$

$$0\le\alpha+\beta+\gamma\le w_1+w_2+w_3;$$

$$\alpha\in[0,w_1],\gamma\in[0,w_2],\beta\in[0,w_3];$$

$$w_1\in[0,1],w_2\in[0,1],w_3\in[0,1];$$

$$0\le w_1+w_2+w_3\le 3.$$

Now, based on arithmetic aggregation operator above problem can be formulated as

$$\text{Minimize }\left\{\frac{(1-\alpha)+\beta+(1-\gamma)}{3}\right\} \qquad (5)$$

Subjected to the same constraint as (4).
With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming, based on geometric aggregation operator can be formulated as:

$$\text{Minimize } \sqrt[3]{(1-\alpha)\beta(1-\gamma)} \qquad (6)$$

Subjected to the same constraint as (4).
Now this non-linear programming problem (4 or 5 or 6) can be easily solved by an appropriate mathematical programming to give solution of multi-objective non-linear programming problem (1) by generalized neutrosophic goal optimization approach.

5. Solution of Multi-Objective Structural Optimization Problem (MOSOP) by Generalized Neutrosophic Goal Programming Technique

The multi-objective neutrosophic fuzzy structural model can be expressed as :

Minimize $WT(A)$ with target value WT_0 ,truth tolerance a_{WT} ,indeterminacy tolerance d_{WT} and rejection tolerance c_{WT} (7)

minimize $\delta(A)$ with target value δ_0 ,truth tolerance a_{δ_0} ,indeterminacy tolerance d_{δ_0} and rejection tolerance c_{δ_0}

subject to $\sigma(A)\le[\sigma]$

$A^{\min}\le A\le A^{\max}$

where $A=[A_1,A_2,\ldots.,A_n]^T$ are the design variables for the cross section, n is the group number of design variables for the cross section bar.
To solve this problem we first calculate truth, indeterminacy and falsity membership function of objective as follows:

$$T_{WT}^{w_1}(WT(A))=\begin{cases}w_1 & \text{if } WT(A)\le WT_0\\ w_1\left(\dfrac{WT_0+a_{WT}-WT(A)}{a_{WT}}\right) & \text{if } WT_0\le WT(A)\le WT_0+a_{WT}\\ 0 & \text{if } WT(A)\ge WT_0+a_{WT}\end{cases}$$

$$I_{WT(A)}^{w_2}(WT(A))=\begin{cases}0 & \text{if } WT(A)\le WT_0\\ w_2\left(\dfrac{WT(A)-WT_0}{d_{WT}}\right) & \text{if } WT_0\le WT(A)\le WT_0+a_{WT}\\ w_2\left(\dfrac{WT_0+a_{WT}-WT(A)}{a_{WT}-d_{WT}}\right) & \text{if } WT_0+d_{WT}\le WT(A)\le WT_0+a_{WT}\\ 0 & \text{if } WT(A)\ge WT_0+a_{WT}\end{cases}$$

where $d_{WT}=\dfrac{w_1}{\dfrac{w_1}{a_{WT}}+\dfrac{w_2}{c_{WT}}}$

$$F_{WT(A)}^{w_3}\left(WT(A)\right)=\begin{cases}0 & if\ WT(A)\le WT_0\\ w_3\left(\dfrac{WT(A)-WT_0}{c_{WT}}\right) & if\ WT_0\le WT(A)\le WT_0+c_{WT}\\ w_3 & if\ WT(A)\ge WT_0+c_{WT}\end{cases}$$

and

$$T_{\delta(A)}^{w_1}\left(\delta(A)\right)=\begin{cases}w_1 & if\ \delta(A)\le\delta_0\\ w_1\left(\dfrac{\delta_0+a_{\delta_0}-\delta(A)}{a_{\delta_0}}\right) & if\ \delta_0\le\delta(A)\le\delta_0+a_{\delta_0}\\ 0 & if\ \delta(A)\ge\delta_0+a_{\delta_0}\end{cases}$$

$$I_{\delta(A)}^{w_2}\left(\delta(A)\right)=\begin{cases}0 & if\ \delta(A)\le\delta_0\\ w_2\left(\dfrac{\delta(A)-\delta_0}{d_\delta}\right) & if\ \delta_0\le\delta(A)\le\delta_0+a_\delta\\ w_2\left(\dfrac{\delta_0+a_\delta-WT(A)}{a_\delta-d_\delta}\right) & if\ \delta_0+d_\delta\le\delta(A)\le\delta_0+a_\delta\\ 0 & if\ \delta(A)\ge\delta_0+a_\delta\end{cases}$$

$$d_\delta=\frac{w_1}{\dfrac{w_1}{a_\delta}+\dfrac{w_2}{c_\delta}}$$

$$F_{\delta(A)}^{w_3}\left(\delta(A)\right)=\begin{cases}0 & if\ \delta(A)\le\delta_0\\ w_3\left(\dfrac{\delta(A)-\delta_0}{c_\delta}\right) & if\ \delta_0\le\delta(A)\le\delta_0+c_\delta\\ w_3 & if\ \delta(A)\ge\delta_0+c_\delta\end{cases}$$

According to generalized neutrosophic goal optimization technique using truth, indeterminacy and falsity membership function, MOSOP (7) can be formulated as:

Model I

$$Maximize\ \alpha, Maximize\ \gamma, Minimize\ \beta \tag{8}$$

$$WT(A)\le WT_0+a_{WT}\left(1-\frac{\alpha}{w_1}\right),$$

$$WT(A)\ge WT_0+\frac{d_{WT}}{w_2}\gamma,$$

$$WT(A)\le WT_0+a_{WT}-\frac{\gamma}{w_2}\left(a_{WT}-d_{WT}\right),$$

$$WT(A)\le WT_0+\frac{c_{WT}}{w_3}\beta,$$

$$WT(A)\le WT_0,$$

$$\delta(A)\le\delta_0+a_\delta\left(1-\frac{\alpha}{w_1}\right),$$

$$\delta(A)\ge\delta_0+\frac{d_\delta}{w_2}\gamma,$$

$$\delta(A)\le\delta_0+a_\delta-\frac{\gamma}{w_2}\left(a_\delta-d_\delta\right),$$

$$\delta(A)\le\delta_0+\frac{c_\delta}{w_3}\beta,\ \delta(A)\le\delta_0,$$

$$0\le\alpha+\beta+\gamma\le w_1+w_2+w_3;$$

$$\alpha\in[0,w_1],\gamma\in[0,w_2],\beta\in[0,w_3];$$

$$w_1\in[0,1],w_2\in[0,1],w_3\in[0,1];$$

$$0\le w_1+w_2+w_3\le 3;$$

$$g_j(x)\le b_j,\ j=1,2,.....,m$$

$$x_j\ge 0,\ \ j=1,2,....,n$$

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming based on arithmetic aggregation operator can be formulated as:

Model II

$$Minimize\left\{\frac{(1-\alpha)+\beta+(1-\gamma)}{3}\right\} \tag{9}$$

Subjected to the same constraint as (8)

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming based on geometric aggregation operator can be formulated as:

Model -III

$$Minimize\ \sqrt[3]{(1-\alpha)\beta(1-\gamma)} \tag{10}$$

Subjected to the same constraint as (8)

Now these non-linear programming Model-I, II, III can be easily solved through an appropriate mathematical programming to give solution of multi-objective non-linear programming problem (7) by generalized neutrosophic goal optimization approach.

6 Numerical illustration

A well-known three bar planer truss is considered in Fig.2 to minimize weight of the structure $WT(A_1,A_2)$ and minimize the deflection $\delta(A_1,A_2)$ at a loading point of a statistically loaded three bar planer truss subject to stress constraints on each of the truss members.

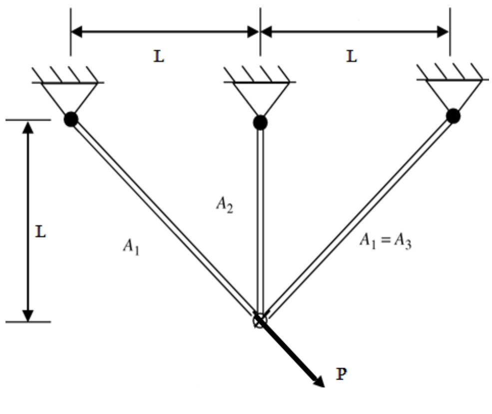

Fig. 2 Design of three bar planar truss

The multi-objective optimization problem can be stated as follows:

$$Minimize \quad WT(A_1,A_2)=\rho L\left(2\sqrt{2}A_1+A_2\right) \qquad (11)$$

$$Minimize \quad \delta(A_1,A_2)=\frac{PL}{E\left(A_1+\sqrt{2}A_2\right)}$$

Subject to

$$\sigma_1(A_1,A_2)=\frac{P\left(\sqrt{2}A_1+A_2\right)}{\left(\sqrt{2}A_1^2+2A_1A_2\right)}\le\left[\sigma_1^T\right];$$

$$\sigma_2(A_1,A_2)=\frac{P}{\left(A_1+\sqrt{2}A_2\right)}\le\left[\sigma_2^T\right];$$

$$\sigma_3(A_1,A_2)=\frac{PA_2}{\left(\sqrt{2}A_1^2+2A_1A_2\right)}\le\left[\sigma_3^C\right];$$

$$A_i^{\min}\le A_i\le A_i^{\max} \quad i=1,2$$

where $P=$ applied load ; $\rho=$ material density ; $L=$ length ; $E=$ Young's modulus ; $A_1=$ Cross section of bar-1 and bar-3; $A_2=$ Cross section of bar-2; δ is deflection of loaded joint. $\lfloor\sigma_1^T\rfloor$ and $\lfloor\sigma_2^T\rfloor$ are maximum allowable tensile stress for bar 1 and bar 2 respectively, σ_3^C is maximum allowable compressive stress for bar 3.The input data is given in table1.

This multi objective structural model can be expressed as neutrosophic fuzzy model as

$Minimize \quad WT(A_1,A_2)=\rho L\left(2\sqrt{2}A_1+A_2\right)$ with target value $4\times10^2 KN$ truth tolerance $2\times10^2 KN$ indeterminacy tolerance $\frac{w_1}{0.5w_1+0.22w_2}\times10^2 KN$ and rejection tolerance $4.5\times10^2 KN$ (12)

$Minimize \quad \delta(A_1,A_2)=\frac{PL}{E\left(A_1+\sqrt{2}A_2\right)}$ with target value $2.5\times10^{-7}m$,truth tolerance $2.5\times10^{-7}m$,indeterminacy tolerance $\frac{w_1}{0.4w_1+0.22w_2}\times10^{-7}m$ and rejection tolerance $4.5\times10^{-7}m$

Subject to

$$\sigma_1(A_1,A_2)=\frac{P\left(\sqrt{2}A_1+A_2\right)}{\left(\sqrt{2}A_1^2+2A_1A_2\right)}\le\left[\sigma_1^T\right];$$

$$\sigma_2(A_1,A_2)=\frac{P}{\left(A_1+\sqrt{2}A_2\right)}\le\left[\sigma_2^T\right];$$

$$\sigma_3(A_1,A_2)=\frac{PA_2}{\left(\sqrt{2}A_1^2+2A_1A_2\right)}\le\left[\sigma_3^C\right];$$

$$A_i^{\min}\le A_i\le A_i^{\max} \quad i=1,2$$

According to generalized neutrosophic goal optimization technique using truth, indeterminacy and falsity membership function ,MOSOP (12) can be formulated as:

Model I

$$Maximize\ \alpha, Maximize\ \gamma, Minimize\ \beta \qquad (13)$$

$$\left(2\sqrt{2}A_1+A_2\right)\le 4+2\left(1-\frac{\alpha}{w_1}\right),$$

$$\left(2\sqrt{2}A_1+A_2\right)\ge 4+\frac{w_1}{w_2\left(0.5w_1+0.22w_2\right)}\gamma,$$

$$\left(2\sqrt{2}A_1+A_2\right)\le 4+2-\frac{\gamma}{w_2}\left(2-\frac{w_1}{\left(0.5w_1+0.22w_2\right)}\right),$$

$$\left(2\sqrt{2}A_1+A_2\right)\le 4+\frac{4.5}{w_3}\beta,$$

$$\left(2\sqrt{2}A_1+A_2\right)\le 4,$$

$$\frac{20}{\left(A_1+\sqrt{2}A_2\right)}\le 2.5+2.5\left(1-\frac{\alpha}{w_1}\right),$$

$$\frac{20}{\left(A_1+\sqrt{2}A_2\right)}\ge 2.5+\frac{w_1}{w_2\left(0.4w_1+0.22w_2\right)}\gamma,$$

$$\frac{20}{\left(A_1+\sqrt{2}A_2\right)}\le 2.5+2.5-\frac{\gamma}{w_2}\left(2.5-\frac{w_1}{\left(0.4w_1+0.22w_2\right)}\right),$$

$$\frac{20}{\left(A_1+\sqrt{2}A_2\right)}\le 2.5+\frac{4.5}{w_3}\beta,$$

$$\frac{20}{\left(A_1+\sqrt{2}A_2\right)}\le 2.5,$$

$$0\le\alpha+\beta+\gamma\le w_1+w_2+w_3;$$

$$\alpha\in[0,w_1],\gamma\in[0,w_2],\beta\in[0,w_3];$$

$$w_1\in[0,1],w_2\in[0,1],w_3\in[0,1];$$

$$0\le w_1+w_2+w_3\le 3;$$

$$\frac{20\left(\sqrt{2}A_1+A_2\right)}{\left(\sqrt{2}A_1^2+2A_1A_2\right)}\le 20;$$

$$\frac{20}{\left(A_1+\sqrt{2}A_2\right)}\le 20;$$

$$\frac{20A_2}{\left(\sqrt{2}A_1^2+2A_1A_2\right)}\leq 15;$$

$$0.1\leq A_i\leq 5 \quad i=1,2$$

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming problem (12) based on arithmetic aggregation operator can be formulated as:

Model II

$$Minimize\left\{\frac{(1-\alpha)+\beta+(1-\gamma)}{3}\right\} \quad (14)$$

Subjected to the same constraint as (13)

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming problem (12) based on geometric aggregation operator can be formulated as:

Model III

$$Minimize\ \sqrt[3]{(1-\alpha)\beta(1-\gamma)} \quad (15)$$

Subjected to the same constraint as (13)

The above problem can be formulated using Model I, II, III and can be easily solved by an appropriate mathematical programming to give solution of multi-objective non-linear programming problem (12) by generalized neutrosophic goal optimization approach and the results are shown in the table 2.

Again, value of membership function in GNGP technique for MOSOP (11) based on different Aggregation is given in Table 3.

Table 1: Input data for crisp model (11)

Applied load P (KN)	Volume density ρ (KN/m^3)	Length L (m)	Maximum allowable tensile stress $\lfloor\sigma^T\rfloor$ (KN/m^2)	Maximum allowable compressive stress $\lfloor\sigma^C\rfloor$ (KN/m^2)	Young's modulus E (KN/m^2)	A_i^{min} and A_i^{max} of cross section of bars $(10^{-4}m^2)$
20	100	1	20	15	2×10^7	$A_1^{min}=0.1$ $A_1^{max}=5$ $A_2^{min}=0.1$ $A_2^{max}=5$

Table 2: Comparison of GNGP solution of MOSOP (11) based on different Aggregation

Methods	A_1 $\times10^{-4}m^2$	A_2 $\times10^{-4}m^2$	$WT(A_1,A_2)$ $\times10^2 KN$	$\delta(A_1,A_2)$ $\times10^{-7}m$
Generalized Fuzzy Goal programming(GFGP) $w_1=0.15$	0.5392616	4.474738	6	2.912270
Generalized Intuitionistic Fuzzy Goal programming(GIFGP) $w_1=0.15$ $w_3=0.8$	0.5392619	4.474737	6	2.912270
Generalized Neutrosophic Goal programming (GNGP) $w_1=0.4, w_2=0.3, w_3=0.7$	5	0.4321463	4.904282	3.564332
Generalized Intuitionistic Fuzzy optimization (GIFGP) based on Arithmetic Aggregation $w_1=0.15, w_3=0.8$	0.5392619	4.474737	6	2.912270

Generalized Neutosophic optimization (GNGP) based on Arithmetic Aggregation $w_1 = 0.4, w_2 = 0.3, w_3 = 0.7$	5	0.4321468	4.904282	3.564333
Generalized Intuitionistic Fuzzy optimization (GIFGP) based on Geometric Aggregation $w_1 = 0.15, w_3 = 0.8$	0.5727008	2.380158	4	5.077751
Generalized Neutosophic optimization (GNGP) based on Geometric Aggregation $w_1 = 0.4, w_2 = 0.3, w_3 = 0.7$	5	1.109954	4.462428	3.044273

Here we get best solutions for the different value of w_1, w_2, w_3 in geometric aggregation method for objective functions. From Table 2 it is clear that Neutrosophic Optimization technique is more fruitful in optimization of weight compare to fuzzy and intuitionistic fuzzy optimization technique.

Moreover it has been seen that more desired value is obtain in geometric aggregation method compare to arithmetic aggregation method in intuitionistic as well as neutrosophic environment in perspective of structural engineering.

Table 3: Value of membership function in GNGP technique for MOSOP (11) based on different Aggregation

Methods	$\alpha^*, \beta^*, \gamma^*$	Sum of Truth, Indeterminacy and Falsity Membership Function
Neutrosophic Goal programming (GNGP) $w_1 = 0.4, w_2 = 0.3, w_3 = 0.7$	$\alpha^* = .1814422$ $\beta^* = .2191435$ $\gamma^* = .6013477$	$T_{WT}\left(WT\left(A_1, A_2\right)\right) + I_{WT}\left(WT\left(A_1, A_2\right)\right) + F_{WT}\left(WT\left(A_1, A_2\right)\right)$ $= .2191435 + .1804043 + .1406661 = .5402139$ $T_\delta\left(\delta\left(A_1, A_2\right)\right) + I_\delta\left(\delta\left(A_1, A_2\right)\right) + F_\delta\left(\delta\left(A_1, A_2\right)\right)$ $= .2297068 + .1804043 + .1655628 = .5756739$
Generalized Neutosophic optimization (GNGP) based on Arithmetic Aggregation $w_1 = 0.4, w_2 = 0.3, w_3 = 0.7$	$\alpha^* = .2191435$ $\beta^* = .2191435$ $\gamma^* = .6013480$	$T_{WT}\left(WT\left(A_1, A_2\right)\right) + I_{WT}\left(WT\left(A_1, A_2\right)\right) + F_{WT}\left(WT\left(A_1, A_2\right)\right)$ $= .2191435 + .1804044 + .1406662 = .5402141$ $T_\delta\left(\delta\left(A_1, A_2\right)\right) + I_\delta\left(\delta\left(A_1, A_2\right)\right) + F_\delta\left(\delta\left(A_1, A_2\right)\right)$ $= .2297068 + .1804044 + .1655629 = .5756741$
Generalized Neutosophic optimization (GNGP) based on Geometric Aggregation $w_1 = 0.4, w_2 = 0.3, w_2 = 0.7$	$\alpha^* = .3075145$ $\beta^* = .3075145$ $\gamma^* = .3075145$	$T_{WT}\left(WT\left(A_1, A_2\right)\right) + I_{WT}\left(WT\left(A_1, A_2\right)\right) + F_{WT}\left(WT\left(A_1, A_2\right)\right)$ $= .3075145 + .0922543 + .07193320 = .471702$ $T_\delta\left(\delta\left(A_1, A_2\right)\right) + I_\delta\left(\delta\left(A_1, A_2\right)\right) + F_\delta\left(\delta\left(A_1, A_2\right)\right)$ $= .3129163 + .09225434 + .08466475 = .48983539$

From the above table it is clear that all the objective functions attained their goals as well as restriction of truth, indeterminacy and falsity membership function in neutrosophic goal programming problem based on different aggregation operator.

The sum of truth,indeterminacy and falsity membership function for each objective is less than sum of gradiation $\left(w_1 + w_2 + w_3\right)$. Hence the criteria of generalized neutrosophic set is satisfied.

7. Conclusions

The research study investigates that neutrosophic goal programming can be utilized to optimize a nonlinear structural problem. . The results obtained for different aggregation method of the undertaken problem show that the best result is achieved using geometric aggregation method. The concept of neutrosophic optimization technique allows one to define a degree of truth membership, which is not a complement of degree of falsity; rather, they are independent with degree of indeterminacy. As we have considered a non-linear three bar truss design problem and find out minimum weight of the structure as well as minimum deflection of loaded joint, the results of this study may lead to the development of effective neutrosophic technique for solving other model of nonlinear programming problem in different field.

References

[1] Zadeh, L.A. Fuzzy set. Information and Control, vol. 8, Issue 3. (1965), 338-353.

[2] Bellman, R. E., &Zadeh, L. A., Decision-making in a fuzzy environment. *Management science*, Vol. 17, Issue.4, (1970). B-141.

[3] Wang, G.Y. & Wang, W.Q., Fuzzy optimum design of structure. *Engineering Optimization*, Vol. 8, (1985), 291-300.

[4] Dey, S., & Roy, T. K. ,A Fuzzy programming Technique for Solving Multi-objective Structural Problem. *International Journal of Engineering and Manufacturing*, Vol.4,Issue.5, (2014), 24.

[5] Dey, S., & Roy, T., Optimum shape design of structural model with imprecise coefficient by parametric geometric programming. *Decision Science Letters*, Vol. 4, Issue.3, (2015), 407-418.

.[6] Dey, S., & Roy, T. K. ,Multi-objective structural design problem optimization using parameterized t-norm based fuzzy optimization programming Technique. *Journal of Intelligent and Fuzzy Systems*, Vol.30, Issue.2, (2016),971-982.

[7] Dey, S., & Roy, T. K., Optimized solution of two bar truss design using intuitionistic fuzzy optimization technique. *International Journal of Information Engineering and Electronic Business*, Vol.6, Issue.4, (2014), 45.

[8] Dey, S., & Roy, T. K., Multi-objective structural optimization using fuzzy and intuitionistic fuzzy optimization technique. *International Journal of Intelligent systems and applications*, Vol.7, Issue 5, (2015), 57.

[9] Liang, R-x., Wang, J., Li, L.,.Multi-criteria group decision making method based on interdependent inputs of single valued trapezoidal neutrosophic information, Neural Computing and Applications, DOI: 10.1007/s00521-016-2672-2, (2016).

[10] Ji, P., Wang, J., Zhang, H. Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third party logistics, Neural Computing and Applications, DOI: 10.1007/s00521-016-2660-6, (2016).

[11] Yu ,S., Wang, J., Wang, J-q., An extended TODIM approach with intuitionistic linguistic numbers, International Transactions in Operational Research, DOI: 10.1111/itor.12363, (2016).

[12] Tian ,Z-p., Wang ,J., Wang ,J-q., Zhang ,H-y., Simplified neutrosophic linguistic multi-criteria group decision-making approach to green product development, Group Decision and Negotiation, DOI: 10.1007/s10726-016-9479-5, (2016)..

[13] Peng, J-j., Wang, J-q., Yang, W.-E. A multi-valued neutrosophic qualitative flexible approach based on likelihood for multi-criteria decision-making problems, International Journal of Systems Science, Vol. 48, Issue 2, (2017), 425-435.

[14] Zhang, H., Ji, P., Wang, J., Chen, X., A novel decision support model for satisfactory restaurants utilizing social information: A case study of TripAdvisor.com, Tourism Management, 59: (2017),281-297

[15] Smarandache, F., Neutrosophy, neutrosophic probability, set and logic, Amer. Res. Press, Rehoboth, USA, (1998) ,105.

[16] Dey, S., & Roy, T. K., Intuitionistic Fuzzy Goal Programming Technique for Solving Non-Linear Multi-objective Structural Problem. *Journal of Fuzzy Set Valued Analysis*, Vol.3 (2015), 179-193.

[17] Charns, A., & Cooper, R. ,Management Models and Industrial Application of Linear Programming. (1961).

[18] Zimmermann, H.L., Fuzzy Programming and Linear Programming with several Objective Function. *Fuzzy Sets and Systems*, Vol. 1, (1978), 45-55.

[19] Parvathi, R., & Malathi, C. Intuitionistic Fuzzy Linear Programming Problems. *World Applied Science Journal*, Vol. 17, Issue.2, (2012), 1802-1807.

[20] Bharti, S.K., & Singh , S. R. Solving Multi-objective Linear Programming Problems using Intuitionistic Fuzzy Optimization Method: A Comparative Study. *International Journal of Modelling and applications*, Vol. 4, Issue 1, (2014), 10-16.

[21] Atanassov, K. T. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, Vol. 20, Issue 1, (1986), 87-96.

[22] Ji, P., Zhang, H-y., Wang, J-q. A projection-based TODIM method under multi-valued neutrosophic environments and its application in personnel selection, Neural Computing and Applications, DOI: 10.1007/s00521-016-2436-z,2016.

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Topological Manifold Space via Neutrosophic Crisp Set Theory

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Abstract. In this paper, we introduce and study a neutrosophic crisp manifold as a new topological structure of manifold via neutrosophic crisp set. Therefore, we study some new topological concepts and some metric distances on a neutrosophic crisp manifold.

Keywords: neutrosophic crisp manifold, neutrosophic crisp coordinate chart, neutrosophic crisp Haussdorff, neutrosophic crisp countable, neutrosophic crisp basis, neutrosophic crisp Homeomorphism, neutrosophic locally compact.

1 Introduction

Neutrosophics found their places into contemporary research; we have introduced the notions of neutrosophic crisp sets, neutrosophic crisp point and neutrosophic topology on crisp sets.

We presented some new topological concepts and properties on neutrosophic crisp topology. A manifold is a topological space that is locally Euclidean and around every point there is a neighborhood that is topologically the same as the open unit in R^n.

The aim of this paper is to build a new manifold topological structure called neutrosophic crisp manifold as a generalization of manifold topological space by neutrosophic crisp point and neutrosophic crisp topology and present some new topological concepts on a neutrosophic crisp manifold space.

Also, we study some metric distances on a neutrosophic crisp manifold.

The paper is structured as follows: in Section 2, we introduce preliminary definitions of the neutrosophic crisp point and neutrosophic crisp topology; in Section 3, some new topological concepts on neutrosophic crisp topology are presented and defined; in Section 4, we propose some topological concepts on neutrosophic crisp manifold space; Section 5 introduces some metric distances on a neutrosophic crisp manifold. Finally, our future work is presented in conclusion.

2 Terminologies [1, 2, 4]

We recollect some relevant basic preliminaries.

Definition 2.1:

Let $A =< A_1, A_2, A_3 >$ be a neutrosophic crisp set on a set X, then

$p =< \{p_1\}, \{p_2\}, \{p_3\} >$, $p_1 \neq p_2 \neq p_3 \in X$ is called a neutrosophic crisp point.

A NCP $p =< \{p_1\}, \{p_2\}, \{p_3\} >$ belongs to a neutrosophic crisp set

$A =< A_1, A_2, A_3 >$ of X denoted by $p \in A$ if it defined by: $\{p_1\} \subseteq A_1, \{p_2\} \subseteq A_2$ and $\{p_3\} \subseteq A_3$.

Definition 2.2:

A neutrosophic crisp topology (NCT) on a non empty set X is a family of Γ of neutrosophic crisp subsets in X satisfying the following axioms:

i. $\phi_N, X_N \in \Gamma$

ii. $A_1 \cap A_2 \in \Gamma$ $for\ any$ $A_1, A_2 \in \Gamma$

iii. $\cup A_j \in \Gamma \, \forall \{A_j \, j \in J\} \subseteq \Gamma$

Then (X, Γ) is called a neutrosophic crisp topological space (NCTS) in X and the elements in Γ are called neutrosophic crisp open sets (NCOSs).

3 Neutrosophic Crisp Topological Manifold Spaces [2, 5, 4, 7]

We present and study the following new topological concepts about the new neutrosophic crisp topological manifold Space.

Definition 3.1:

A neutrosophic crisp topological space (X, Γ) is a neutrosophic crisp Haussdorff (NCH) if for each two neutrosophic crisp points p $=<\{p_1\},\{p_2\},\{p_3\}>$ and q $=<\{q_1\},\{q_2\},\{q_3\}>$ in X such that $p \neq q$ there exist neutrosophic crisp open sets U $=<u_1,u_2,u_3>$ and V $=<v_1,v_2,v_3>$ such that p in U, q in V and $U \cap V = \phi_N$.

Definition 3.2:

β is collection of neutrosophic crisp open sets in (X, Γ) is said to be neutrosophic crisp base of neutrosophic crisp topology (NCT) if $\Gamma_{NC} = \cup$ β.

Definition 3.3:

Neutrosophic crisp topology (X, Γ) is countable if it has neutrosophic crisp countable basis for neutrosophic crisp topology, i.e. there exist a countable collection of neutrosophic crisp open set $\{U_\alpha\}_{\alpha\in N} =<u_{11},u_{12},u_{13}>,<u_{21},u_{22},u_{23}>,\ldots\ldots,<u_{n1},u_{n2},u_{n3}>$ such that for any neutrosophic crisp open set U containing a crisp neutrosophic point p in U, there exist a $\beta \in N$ such that $p \in U_\beta \subseteq U$.

Definition 3.4:

Neutrosophic crisp homeomorphism is a bijective mapping f of NCTs (X, Γ_1) onto NCTs (Y, Γ_2) is called a neutrosophic crisp homeomorphism if it is neutrosophic crisp continuous and neutrosophic crisp open.

Definition 3.5:

Neutrosophic crisp topology is neutrosophic crisp Locally Euclidean of dimension n if for each neutrosophic crisp point p $=<\{p_1\},\{p_2\},\{p_3\}>$ in X, there exist a neutrosophic crisp open set U $=<u_1,u_2,u_3>$ and a map $\phi: U \to R^n$ such that $\phi: U \to \phi(U)$ *which is* $\phi(U) =<\phi(u_1),\phi(u_2),\phi(u_3)>$ is a homeomorphism; in particular $\phi(U)$ is neutrosophic crisp open set of R^n .

We define a neutrosophic crisp topological manifold (NCM) as follows:

Definition 3.6:

(NCM) is a neutrosophic crisp topological manifold space if the following conditions together satisfied

1. (NCM) is satisfying neutrosophic crisp topology axioms.
2. (NCM) is neutrosophic crisp Haussdorff.
3. (NCM) is countable neutrosophic crisp topology.
4. (NCM) is neutrosophic crisp Locally Euclidean of dimension n.

We give the terminology $(M_{NC})^n$ to mean that it is a neutrosophic crisp manifold of dimension n.

The following graph represents the neutrosophic crisp topological manifold space as a generalization of topological manifold space:

Topological Manifold
↙ ↓ ↘
Haussdorff Second Countable Locally Euclidean
↓ ↓ ↓
Neutrosophic crisp Haussdorff Neutrosophic crisp Countable
Neutrosophic crisp Locally Euclidean
↘ ↓ ↙
Neutrosophic Crisp Topological Manifold

Figure 3.1 A graph of generalization of topological manifold space

4 Some New Topological Concepts on NCM Space [2, 3, 4, 6, 8]

The neutrosophic crisp set U and map $\phi(U)$ in the Definition 3.5 of neutrosophic crisp Locally Euclidean is called a neutrosophic crisp coordinate chart.

Definition 4.1:

A neutrosophic crisp coordinate chart on $(M_{NC})^n$ is a pair $(U, \phi(U))$where U in $(M_{NC})^n$ is open and $\phi: U \to \phi(U) \subseteq R^n$ is a neutrosophic crisp homeomorphism, and then the neutrosophic crisp set U is called a neutrosophic crisp coordinate domain or a neutrosophic crisp coordinate neighborhood.

A neutrosophic crisp coordinate chart $(U, \phi(U))$ is centered at p if

$\phi(p) = 0$ *where*

a neutrosophic crisp *coordinate ball* $\phi(U)$

is a ball in R^n.

Definition 4.1.1:

A Ball in neutrosophic crisp topology is an open ball (r, ϵ, p) , r is radius

$0 \leq r \leq 1$, $0 < \epsilon < r$ *and* p *is NCP*.

Theorem 4.1:

Every NCM has a countable basis of *coordinate ball*.

Theorem 4.2:

In $(M_{NC})^n$ every neutrosophic crisp point $p = (<\{p_1\},\{p_2\},\{p_3\}>) \in (M_{NC})^n$ is contained in neutrosophic coordinate ball centered at p if:

$(\phi^{-1}(\phi(p)), \phi(\phi^{-1}(\phi(p))))$

and then if we compose ϕ with a translating we must get $p = \phi(p) = 0$.

Proof: Since $(M_{NC})^n$ neutrosophic crisp Locally Euclidean, p must be contained in a coordinate chart$(U, \phi(U))$. Since $\phi(U)$ is a neutrosophic crisp open set containing$\phi(p)$, by the NCT of R^n there must be an open ball B containing $\phi(p)$ and contained in$\phi(U)$. The appropriate coordinate ball is$(\phi^{-1}(\phi(p)), \phi(\phi^{-1}(\phi(p))))$. Compose ϕ with a translation taking $\phi(p)$ to 0 , then $p = \phi(p) = 0$, we have completed the proof.

Theorem 4. 3:

The neutrosophic crisp graph $G(f)$ of a continuous function$f: U \to R^k$,
where U *is neutrosophic crisp set in* R^n , is NCM.

$$G(f) = \{(p, f(p))\ in\ R^n \times R^k : p\ \ NCPin\ U\}$$

Proof: Obvious.

Example: Spheres are NCM. An n-sphere is defined as:
$S^n = \{p\ NCP\ in\ R^{n+1}: |p|^2 = \sqrt[2]{{p_1}^2 + {p_2}^2 + {p_3}^2} = 1\}$.

Definition 4.2:

Every neutrosophic crisp point p has a neutrosophic crisp neighborhood point p_{NCbd} contained in an open ball B.

Definition 4.3:

Here come the basic definitions first.
Let (X, Γ) be a NCTS.

a) If a family$\{< G_{i1}, G_{i2}, G_{i3} >: i \in J\}$ of NCOSs in X satisfies the condition $\cup \{< G_{i1}, G_{i2}, G_{i3} >: i \in J\} = X_N$ then it is called a neutrosophic open cover of X.
b) A finite subfamily of an open cover $\{< G_{i1}, G_{i2}, G_{i3} >: i \in J\}$ on X, which is also a neutrosophic open cover of X is called a neutrosophic finite subcover $\{ < G_{i1}, G_{i2}, G_{i3} >: i \in J\}$.
c) A family $\{< K_{i1}, K_{i2}, K_{i3} >: i \in J\}$ of NCOSs in X satisfies the finite intersection property [FIP] iff every finite subfamily $\{< K_{i1}, K_{i2}, K_{i3} >: i = 1, 2, \dots\dots, n\}$ of the family satisfies the condition: $\cap \{< K_{i1}, K_{i2}, K_{i3} >: i \in J\} \neq \phi_N$.
d) A NCTS (X, Γ) is called a neutrosophic crisp compact iff each crisp neutrosophic open cover of X has a finite subcover.

Corollary:

A NCTS (X, Γ) is a neutrosophic crisp compact iff every family $\{ < G_{i1}, G_{i2}, G_{i3} >: i \in J\}$ of NCCS in X having the FIP has non-empty intersection.

Definition 4.4:

Every neutrosophic point has a neutrosophic neighborhood contained in a neutrosophic compact set is called neutrosophic locally compact set.

Corollary:

Every NCM is neutrosophic locally compact set.

5 Some Metric Distances on a Neutrosophic Crisp Manifold [10, 9]

5.1. Haussdorff Distance between Two Neutrosophic Crisp Sets on NCM:
Let $A =< A_1, A_2, A_3 >$ and $B =< B_1, B_2, B_3 >$ two neutrosophic crisp sets on NCM then the Haussdorff distance between A and B is

$$d_H(A, B) = sup(d(A_i, B_j), d(B_j, A_i))$$

$$d(A_i, B_j) = inf|A_i - B_j| \quad , \forall\ i, j \in J$$

5.2. Modified Haussdorff Distance between Two Neutrosophic Crisp Sets on NCM:
Let $A =< A_1, A_2, A_3 >$ and $B =< B_1, B_2, B_3 >$ two neutrosophic crisp sets on NCM then the Haussdorff distance between A and B is

$$d_H(A, B) = \frac{1}{n}[sup(d(A_i, B_j), d(B_j, A_i))], n\ is\ number\ of\ NCPs$$

$$d(A_i, B_j) = inf|A_i - B_j| \quad , \forall\ i, j \in J.$$

Conclusion and Future Work

In this paper, we introduced and studied the neutrosophic crisp manifold as a new topological structure of manifold via neutrosophic crisp set, and some new topological concepts on a neutrosophic crisp manifold space via neutrosophic crisp set, and also some metric distances on a neutrosophic crisp manifold. Future work will approach neutrosophic fuzzy manifold, a new topological structure of manifold via neutrosophic fuzzy set, and some new topological concepts on a neutrosophic fuzzy manifold space via neutrosophic fuzzy set.

References

1. A. A Salama. Neutrosophic Crisp Points & Neutrosophic Crisp Ideals Neutrosophic Sets and Systems, Vol. 1, pp. 50-54, 2013.
2. A.A Salama, F. Smarandache, V. Kroumov. Neutrosophic Crisp Sets & Neutrosophic Crisp Topological Spaces, Neutrosophic Sets and Systems, Vol. 2, pp. 25-30, 2014.
3. A. A Salama, S. Broumi, F. Smarandache. Neutrosophic Crisp Open Sets and Neutrosophic Crisp Continuity via Neutrosophic Crisp Ideals, I.j. Information Engineering and Electronic Business, Vol. 6, pp.1-8, 2014.

4. A. A Salama and F. Smarandache. Neutrosophic Crisp Set Theory. USA Book, Educational. Education Publishing 1313 Chesapeake, Avenue, Coiumbus, Ohio, 43212, USA, 2015.
5. A. A Salama, F. Smarandache, S. A Alblowi. New Neutrosophic Crisp Topological Concepts, Neutrosophic Sets and Systems, Vol. 4, pp. 50-54, 2014.
6. Chow, Bennett, Glickenstein, et al. The Ricci flow: techniques and applications, American Mathematical Society, No. 135, 2010.
7. Lee, John M. Smooth Manifolds Introduction to Smooth Manifolds, Springer New York, pp.1-29, 2003.
8. Jenny Wilson. Manifolds WOMP, 2012.
9. M., and Jain, A. A Modified Haussdorff Distance for Object Matching Dubuisson,Vol.1, pp. 566–568, 1994.
10. Haussdorff, F. Grundzge der Mengenlehre, Leipzig: Veit and Company, 1914.

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Neutrosophic Graphs of Finite Groups

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Abstract: Let G be a finite multiplicative group with identity e and $N(G)$ be the Neutrosophic group with indeterminate I. We denote by $Ne(G, I)$, the Neutrosophic graph of G, $N(G)$ and I. In this paper, we study the graph $Ne(G, I)$ and its properties. Among the results, it is shown that for any finite multiplicative group G, $Ne(G, I)$ is a connected graph of diameter less than or equal to 2. Moreover, for finite group G, we obtain a formula for enumerating basic Neutrosophic triangles in $Ne(G, I)$. Furthermore, for every finite groups G and G', we show that $G \cong G'$ if and only if $N(G) \cong N(G')$, and if $N(G) \cong N(G')$, then $Ne(G, I) \cong Ne(G', I')$.

Keywords: Indeterminacy; Finite Multiplicative group; Neutrosophic Group; Basic Neutrosophic triangle; Neutrosophic group and graph isomorphism.

1 Introduction

Most of the real world problems in the fields of philosophy, physics, statistics, finance, robotics, design theory, coding theory, knot theory, engineering, and information science contain subtle uncertainty and inconsistent, which causes complexity and difficulty in solving these problems. Conventional methods failed to handle and estimate uncertainty in the real world problems with near tendency of the exact value. The determinacy of uncertainty in the real world problems have been great challenge for the scientific community, technological people, and quality control of products in the industry for several years. However, different models or methods were presented systematically to estimate the uncertainty of the problems by various incorporated computational systems and algebraic systems. To estimate the uncertainty in any system of the real world problems, first attempt was made by the Lotfi A Zadesh [1] with help of Fuzzy set theory in 1965. Fuzzy set theory is very powerful technique to deal and describe the behavior of the systems but it is very difficult to define exactly. Fuzzy set theory helps us to reduce the errors of failures in modeling and different fields of life. In order to define system exactly, by using Fuzzy set theory many authors were modified, developed and generalized the basic theories of classical algebra and modern algebra. Along with Fuzzy set theory there are other different theories have been study the properties of uncertainties in the real world problems, such as probability theory, intuitionistic Fuzzy set theory, rough set theory, paradoxist set theory [2-5]. Finally, all above theories contributed to explained uncertainty and inconsistency up to certain extent in real world problems. None of the above theories were not studied the properties of indeterminacy of the real world problems in our daily life. To analyze and determine the existence of indeterminacy in various real world problems, the author Smarandache [6] introduced philosophical theory such as Neutrosophic theory in 1990.

Neutrosophic theory is a specific branch of philosophy, which investigates percentage of Truthfulness, falsehood and neutrality of the real world problem. It is a generalization of Fuzzy set theory and intuitionistic Fuzzy set theory. This theory is considered as complete representation of a mathematical model of a real world problem. Consequently, if uncertainty is involved in a problem we use Fuzzy set theory, and if indertminancy is involved in a problem we essential Neutrosophic theory.

Kandasamy and Smarandache [7] introduced the philosophical algebraic structures, in particular, Neutrosophic algebraic structures with illustrations and examples in 2006 and initiated the new way for the emergence of a new class of structures, namely, Neutrosophic groupoids, Neutrosophic groups, Neutrosophic rings etc. According to these authors, the Neutrosophic algebraic structures N(I) was a nice composition of indeterminate I and the elements of a given algebraic structure $(N, *)$. In particular, the new algebraic structure $(N(I), *)$ is called Neutrosophic algebraic structure which is generated by N and I.

In [8], Agboola and others have studied some properties of Neutrosophic group and subgroup. Neutrosophic group denoted by $(N(G), \cdot)$ and defined by $N(G) = \langle G \cup I \rangle$, where G is a group with respect to multiplication. These authors also shown that all Neutrosophic groups generated by the Neutrosophic element I and any group isomorphic to Klein 4-group are Lagrange Neutrosophic groups.

Recent research in Neutrosophic algebra has concerned developing a graphical representation of the elements of a given finite Neutrosophic set, and then graph theoretically developing and analyzing the depiction to research Neutrosophic algebraic conclusions about the finite Neutrosophic set. The most well-known of these models is the Neutrosophic graph of Neutrosophic set, first it was introduced by Kandasamy and Smarandache [9].

Recently, the authors Kandasamy and Smarandache in [9-10] have introduced Neutrosophic graphs, Neutrosophic edge graphs and Neutrosophic vertex graphs, respectively. If the edge values are from the set $\langle G \cup I \rangle$ they will termed as Neutrosophic graphs, and a Neutrosophic graph is a graph in which at least one edge is indeterminacy. Let $V(G)$ be the set of all vertices of G. If the edge set $E(G)$, where at least one of the edges of G is an indeterminate one. Then we call such graphs as a Neutrosophic edge graphs. Further, a Neutrosophic vertex graph G_N is a graph G with finite non empty set $V_N = V_N(G)$ of $p-$points where at least one of the point in $V_N(G)$ is indeterminate vertex. Here $V_N(G) = V(G) + N$, where $V(G)$ are vertices of the graph G and N the non empty set of vertices which are indeterminate.

In the present paper, indeterminacy of the real world problems are expressed as mathematical model in the form of new algebraic structure $(GI, \cdot)$, and its properties are studied in second section, where G is finite group with respect to multiplication and I indeterminacy of the real world problems.

In the third section, to find the relation between G, I and N (G) we introduced Neutrosophic graph $Ne(G, I)$ of the Neutrosophic group $(N(G), \cdot)$, by studying its important concrete properties of these graphs.

In the fourth section, we introduced basic Neutrosophic triangles in the graph $Ne(G, I)$ and obtained a formula for enumerating basic Neutrosophic triangles in $Ne(G, I)$ to understand the internal mutual relations between the elements in G, I and N (G).

In the last section, all finite isomorphic groups G and G' such that $N(G) \cong N(G')$ and $Ne(G, I) \cong Ne(G', I')$ are characterized with examples.

Throughout this paper, all groups are assumed to be finite multiplicative groups with identity e. Let $N(G)$ be a Neutrosophic group generated by G and I. For classical theorems and notations in algebra and Neutrosophic algebra, the interest reader is refereed to [11] and [8].

Let X be a graph with vertex set $V(X)$ and edge set $E(X)$. The cardinality of $V(X)$ and $E(X)$ are denoted by $|V(X)|$ and $|E(X)|$, which are order and size of X, respectively. If X is connected, then there exist a path between any two vertices in X. We denote by K_n the complete graph of order n. Let $u \in V(X)$. Then degree of u, $\deg(u)$ in X is the number of edges incident at u. If $\deg(u) = 1$ then the vertex u is called pendent. The girth of X is the length of smallest cycle in X. The girth of X is infinite if X has no cycle. Let $d(x, y)$ be the length of the shortest path from two vertices x and y in X, and the diameter of X denoted by

$$Diam(X) = \max\{d(x, y) : x, y \in V(X)\}.$$

For further details about graph theory the reader should see [12].

2 Basic Properties of Neutrosophic set and GI

This section will present a few basic concepts of Neutrosophic set and Neutrosophic group that will then be used repeatedly in further sections, and it will introduce a convenient notations. A few illustrations and examples will appear in later sections.

Neutrosophic set is a mathematical tool for handling real world problems involving imprecise, inconsistent data and indeterminacy; also it generalizes the concept of the classic set, fuzzy set, rough set etc. According to authors Vasantha Kandasamy and Smarandache, the Neutrosophic set is a nice composition of an algebraic set and indeterminate element of the real world problem.

Let N be a non-empty set and I be an indeterminate. Then the set $N(I)=\langle N\cup I\rangle$ is called a Neutrosophic set generated by N and I .If '$\cdot$' is usual multiplication in N , then I has the following axioms.

1. $0\cdot I=0$
2. $1\cdot I=I=I\cdot 1$
3. $I^2=I$
4. $a\cdot I=I\cdot a$, for every $a\in N$.
5. I^{-1} does not exist.

For the definition, notation and basic properties of Neutrosophic group, we refer the reader to Agbool [8]. As treated in [8], we shall denote the finite Neutrosophic group by $N(G)$ for a group G .

Definition 2.1 Let G be any finite group with respect to multiplication. Then the set GI defined as $GI=\{gI:g\in G\}=\{Ig:g\in G\}$.

Definition 2.2 If a map f from a finite nonempty set S into a finite nonempty set S' is both one-one and onto then there exist a map g from S' into S that is also one-one and onto. In this case we say that the two sets are equivalent, and, abstractly speaking, these sets can be regarded as the same cardinality. We write $S\sim S'$ whenever there is a one-one map of a set S onto S'.

Two finite rings R and R' are equivalent if there is a one-one correspondence between R and R'. We write $R\sim R'$.

Definition 2.3 Let G be any finite group with respect to multiplication and let $N(G)=\langle G\cup I\rangle$ .Then $(N(G),\cdot)$ is called a Neutrosophic group generated by G and I under the binary operation '$\cdot$'on G . The Neutrosophic group $N(G)$ has the following properties.

1. $N(G)$ is not a group.
2. $G\subset N(G)$.
3. $GI\subset N(G)$.
4. $N(G)$ is a specific composition of G and I .

Lemma 2.4 Let G be any finite group with respect to multiplication and $I^2=I$. Then $G\sim GI$. In particular, $|G|=|GI|$.

Proof. For any finite group G , we have $G\neq GI$ and $GI\not\subset G$. Now define a map $f:G\to GI$ by the relation $f(a)=aI$ for every $a\in I$. Let $a,b\in G$. Then
$a=b \Leftrightarrow a-b=0 \Leftrightarrow (a-b)I=0I \Leftrightarrow aI=bI \Leftrightarrow f(a)=f(b)$ .This shows that f is a well defined one-one function. Further, we have

$$\text{Range}(f)=\{f(a)\in GI:a\in G\}$$
$$=\{aI\in GI:a\in G\}=GI.$$

This show that for every $aI\in GI$ at least one $a\in G$ such that $f(a)=aI$.

Therefore, $f:G\to GI$ is one-one correspondence and consequently a bijective function. Hence $G\sim GI$.

Lemma 2.5 Let G be any finite group with respect to multiplication and let $N(G)=\langle G\cup I\rangle$. Then the order of $N(G)$ is $2|G|$.

Proof: We have $GI=\{gI:g\in G\}$. Obviously, $GI\not\subset G$ and $G\not\subset GI$ but $GI\subset N(G)$. It is clear that $N(G)$ is the disjoint union of G and GI .That is,

$$N(G)=G\cup GI \text{ and } G\cap GI=\phi.$$

Therefore, $|N(G)|=|G|+|GI|=2|G|$, since $|G|=|GI|$.

Lemma 2.6 The set GI is not Neutrosophic group with respect to multiplication of group G .

Proof: It is obvious, since $GI\neq\langle G\cup I\rangle$.

Lemma 2.7 The elements in GI satisfies the following properties,

1. $e\cdot gI=gI$
2. $(gI)^2=g^2I$
3. $\underbrace{gI\cdot gI\ldots\cdot gI}_{n\text{ terms}}=g^nI$ for all positive integers n .
4. $(gI)^{-1}$ does not exist, since I^{-1} does not exist.
5. $gI=g'I\Leftrightarrow g=g'$.

Proof: Directly follows from the results of the group $(N(G),\cdot)$.

Theorem 2.8 The structure $(GI, \cdot)$ is a monoid under the operation $(aI)(bI) = abI$ for all a, b in the group $(G, \cdot)$ and $I^2 = I$.

Proof: We know that $GI = \{gI : g \in G\}$. Let aI , bI and cI be any three elements in GI . Then the binary operation $(aI)(bI) = abI$ in $(GI, \cdot)$ satisfies the following axioms.

1. $abI \in GI \Rightarrow (aI)(bI) \in GI$.
2. $[(aI)(bI)](cI) = [(ab)I](cI)$
   $= [(ab)c)]I = [a(bc))]I = aI[(bI)(cI)]$
3. Let e be the identity element in $(G, \cdot)$.Then $eI = I = Ie$ and $I(aI) = aI^2 = aI = (aI)I$.

Remark 2.9 The structure $(GI, \cdot)$ is never a group because I^{-1} does not exist.

Here we obtain lower bounds and upper bounds of the order of the Neutrosophic group $N(G)$. Moreover, these bounds are sharp.

Theorem 2.10 Let G be a finite group with respect to multiplication. Then,

$$1 \le |G| \le n \Leftrightarrow 2 \le |N(G)| \le 2n.$$

Proof. We have,

$|G| = 1 \Leftrightarrow G = \{e\} \Leftrightarrow N(G) = G \cup GI = \{e, I\}$

$\Leftrightarrow |N(G)| = 2$. This is one extreme of the required inequality. For other extreme, by the Lemma [2.4],

$|G| > 1 \Leftrightarrow |GI| > 1$

$\Leftrightarrow |G| + |GI| > 2$ and $|G| + |GI|$ is not odd

$\Leftrightarrow |G| + |GI|$ is even.

$\Leftrightarrow |N(G)| = |G| + |GI| = 2n$.

Hence, the theorem.

3 Basic Properties of Neutrosophic Graph

In this section, our aim is to introduce the notion and definition of Neutrosophic graph of finite Neutrosophic group with respect to multiplication and study on its basic and specific properties such as connectedness, completeness, bipartite, order, size, number of pendent vertices, girth and diameter.

Definition 3.1 A graph $Ne(G, I)$ associated with Neutrosophic group $(N(G), \cdot)$ is undirected simple graph whose vertex set is $N(G)$ and two vertices x and y in $N(G)$ if and only if xy is either x or y .

Theorem 3.2 For any group $(G, \cdot)$, the Neutrosophic graph $Ne(G, I)$ is connected.

Proof: Let e be the identity element in G .Then $e \in N(G)$, since $G \subset N(G)$. Further, $xe = x$, for every $x \neq e$ in $N(G)$. It is clear that the vertex e is adjacent to all other vertices of the graph $Ne(G, I)$. Hence $Ne(G, I)$ is connected.

Theorem 3.3 Let $|G| > 1$.Then the graph has at least one cycle of length 3.

Proof: Since $|G| > 1$ implies that $|N(G)| \ge 4$. So there is at least one vertex gI of $N(G)$ such that gI is adjacent to the vertices e and I in $Ne(G, I)$,since $eI = I$, $I(gI) = gI^2 = gI$ and $(gI)e = geI = gI$. Hence we have the cycle $e - I - gI - e$ of length 3, where $g \neq e$.

Example 3.4 Since

$$N(G_{10}) = \{2, 4, 6, 8, 2I, 4I, 6I, 8I\}$$

is the Neutrosophic group of the group $G_{10} = \{2, 4, 6, 8\}$ with respect to multiplication modulo 10, where $e = 6$ .The Neutrosophic graph $Ne(G_{10}, I)$ contains three cycles of length 3, which are listed below.

$$C_1 : 6 - I - 2I - 6,$$
$$C_2 : 6 - I - 4I - 6,$$
$$C_3 = 6 - I - 8I - 8.$$

Theorem 3.5 The Neutrosophic graph $Ne(G, I)$ is complete if and only if $|G| = 1$.

Proof: **Necessity.** Suppose that $Ne(G, I)$ is complete. If possible assume that $|G| > 1$, then $|N(G)| \ge 4$. So without loss of generality we may assume that $|N(G)| = 4$ and clearly the vertices

$e, g, I, gI \in V(Ne(G, I))$ .Therefore the vertex g is not adjacent to the vertex I in $Ne(G, I)$, since $gI \neq g$ or I for each $g \neq e$ in G, this contradicts our assumption that $Ne(G, I)$ is complete. It follows that $|N(G)|$ cannot be four. Further, if $|N(G)| > 4$, then obviously we arrive a contradiction. So our assumption is wrong , and hence $|G| = 1$.

Sufficient. Suppose that $|G| = 1$. Then, trivially $|N(G)| = 2$ .Therefore, $Ne(G, I) \cong K_2$, since $eI = I$. Hence, $Ne(G, I)$ is a complete graph.

Recall that $|V(Ne(G, I))|$ is the order and $|E(Ne(G, I))|$ is the size of the Neutrosophic graph $Ne(G, I)$. But,

$$|V(Ne(G, I))| = |N(G)| = 2|G|$$

and the following theorem shows that the size of $Ne(G, I)$.

Theorem 3.6 The size of Neutrosophic graph $Ne(G, I)$ is $3|G| - 2$.

Proof: By the definition of Neutrosophic graph, $Ne(G, I)$ contains $2(|G|-1)^2$ non adjacent pairs. But the number of combinations of any two distinct pairs from $N(G)$ is $\binom{|N(G)|}{2}$. Hence the total number of adjacent pairs in $Ne(G, I)$ is

$$|E(Ne(G, I))| = \binom{|N(G)|}{2} - 2(|G|-1)^2$$

$$= 3|G| - 2.$$

Theorem 3.7 [11] The size of a simple complete graph of order n is $\frac{1}{2}n(n-1)$.

Corollary 3.8 The Neutrosophic graph $Ne(G, I)$, $|G| > 1$ is never complete.

Proof: Suppose on contrary that $Ne(G, I)$, $|G| > 1$ is complete. Then, by the Theorem [3.7], the total number of edges in $Ne(G, I)$ is $\frac{1}{2}(2|G|(2|G|-1)) = |G|(2|G|-1)$, but in view of Theorem [3.6], we arrived a contradiction to the completeness of $Ne(G, I)$.

Theorem 3.9 The graph $Ne(G, I)$ has exactly $|G|-1$ pendent vertices.

Proof: Since $N(G) = G \cup GI$ and $G \cap GI = \phi$. Let $x \in N(G)$. Then either $x \in G$ or $x \in GI$. Now consider the following cases on GI and G , respectively.

Case 1. If $x \in GI$, then $x = gI$ for $g \in G$. But $xI = (gI)I = gI^2 = gI = x$ and $ex = egI = gI = x$. This implies that the vertex x is adjacent to both the vertices e and I in $N(G)$ .Hence $\deg(x) \neq 1$ for every $x \in GI$.

Case 2. If $x \in G$, then $ex = x$,for every $x \neq e$ and $egI = gI$, for every $gI \in GI$. Therefore $\deg(e) = |N(G)| - 1 \neq 1$. Now show that $\deg(x) = 1$, for every $x \neq e$ in G . Suppose, $\deg(x) > 1$, for every $x \neq e$ in G. Then there exist another vertex $y \neq e$ in G such that either $xy = x$ or y, this is not possible in G, because G is a finite multiplication group. Thus $\deg(x) = 1$, for $x \neq e$ in G.

From case (1) and (2), we found the degree of each non identity vertex in G is 1. This shows that each and every non identity element in G is a pendent vertex in $Ne(G, I)$. Hence, the total number of pendent vertices in $Ne(G, I)$ is $|G| - 1$.

The following result shows that $Ne(G, I)$ is never a traversal graph.

Corollary 3.10 Let $|G| > 1$. Then $Ne(G, I)$ is never Eulerian and never Hamiltonian.

Proof. It is obvious from the Theorem [3.9].

Theorem 3.11 [11] A simple graph is bipartite if and only it does not have any odd cycle.

Theorem 3.12 The Neutrosophic graph $Ne(G, I)$, $|G| > 1$ is never bipartite.

Proof. Assume that $|G|>1$. Suppose, $Ne(G, I)$ is a bipartite graph. Then there exist a bipartition (G, GI), since $N(G)=G\cup GI$ and $G\cap GI=\phi$. But $e\in G$ and $I\in GI$, where $e\neq I$. So there exist at least one vertex gI in $Ne(G, I)$ such that $e-I-gI-e$ is an odd cycle of length 3 because $eI=I,\ \ I(gI)=gI$ and $(gI)e=gI$.
This violates the condition of the Theorem [3.11].
Hence $Ne(G, I)$ is not a bipartite graph.

Theorem 3.13 The girth of a Neutrosophic graph is 3.

Proof. In view of Theorem [3.3], for $|G|>1$, we always have a cycle $e-I-gI-e$ of length 3, for each $g\neq e$ in G, which is smallest in $Ne(G, I)$.
This completes the proof.

Remark 3.14 Let G be a finite group with respect to multiplication. Then $gir\,(Ne(G, I))=\infty$ if $|G|=1$, since $Ne(G, I)$ is acyclic graph if and only if $|G|=1$.

Theorem 3.15 $\text{Diam}\,(Ne(G, I))\leq 2$.

Proof. Let G be a finite group with respect to multiplication. Then we consider the following two cases.

Case 1 Suppose $|G|=1$. The graph $Ne(G, I)\cong K_2$. It follows that $Ne(G, I)$ is complete, so $\text{diam}\,(Ne(G, I))=1$.

Case 2 Suppose $|G|>1$. Then the vertex e is adjacent to every vertex of $Ne(G, I)$. However the vertex aI is not adjacent to bI for all $a\neq b$ in G, so $d(aI, bI)>1$. But in $Ne(G, I)$, there always exist a path $aI-I-bI$, since $(aI)I=aI$ and $I(bI)=bI$, which gives $d(aI, bI)=2$, for every $aI\,, bI\in N(G)$.
Hence, both the cases conclude that:
$\text{Diam}\,(Ne(G, I))\leq 2$.

4 Enumeration of basic Neutrosophic triangles in $Ne(G, I)$

Since $Ne(G, I)$ is triangle free graph for $|G|=1$, we will consider $|G|>1$ in this section.

Let us denote a traingle by (x, y, z) in $Ne(G, I)$ with vertices x, y and z. Without loss of generality we may assume that our triangles (e, I, gI) have vertices e, I and gI, where $g\neq e$ in G. These triangles are called basic Neutrosophic triangles in $Ne(G, I)$, which are defined as follows.

Definition 4.1 A triangle in the graph $Ne(G, I)$ is said to be basic Neutrosophic if it has the common vertices e and I .The set of all basic Neutrosophic triangles in $Ne(G, I)$ denoted by

$$T_{eI}=\{(e, I, gI): g\neq e \text{ in } G\}.$$

A triangle (x, y, z) in $Ne(G, I)$ is called non-basic Neutrosophic if $(x, y, z)\notin T_{eI}$.

The following short table illustrates some finite Neutrosophic graphs and their total number of basic Neutrosophic triangles.

$Ne(G, I)$	$Ne(Z_p^*, I)$	$Ne(C_n, I)$	$Ne(G_{2p}, I)$	$Ne(V_4, I)$
$\lvert T_{eI}\rvert$	$p-2$	$n-1$	$p-2$	3

where $Z_p^*=Z_p-\{0\}$ is a group with respect to multiplication modulo p , a prime,
$C_n=\{1, g, g^2,..., g^{n-1}: g^n=1\}$
is a cyclic group generated by g with respect to multiplication,
$G_{2p}=\{0, 2, 4,..., 2(p-1)\}$
is a group with respect to multiplication modulo *2p* and
$V_4=\{e, a, b, c: a^2=b^2=c^2=e\}$
is a Klein 4-group.

Before we continue, it is important to note that the multiplicative identity e may differ from group to group. However, for simplicity sake we will continue to notate that $e=1$, and we leave it to reader to understand from context of the group for e.

The following results give information about enumeration of basic and non-basic Neutrosophic triangles in the graph $Ne(G, I)$.

First we begin a lemma, which gives a formula for enumerating the number of Neutrosophic triangles in $Ne(G, I)$ corresponding to fixed elements e and I in the Neutrosophic set $N(G)$.

This is useful for finding the total number of non-basic Neutrosophic triangles in $Ne(G, I)$.

Theorem 4.2 Let $|G| > 1$. Then the total number of basic Neutrosophic triangles in $Ne(G, I)$ is $|T_{eI}| = |G| - 1$.

Proof. Since $N(G) = G \cup GI$ and $G \cap GI = \phi$. It is clear that $e \neq I$. For any $aI \in GI$, the traid $(e, I, aI) \in T_{eI} \Leftrightarrow (e, I), (e, aI),$ and (I, aI) are edges in $Ne(G, I)$

$$\Leftrightarrow eI = I,\ e(aI) = aI, I(aI) = aI$$
$$\Leftrightarrow I, aI \in GI \text{ , where } a \neq e \text{ in } G.$$

That is, for fixed vertices e, I and for each $aI \in GI$, the traid (e, I, aI) exists in $Ne(G, I)$. Further, for any vertex $a \in G$, the vertices e, I and a does not form a triangle in $Ne(G, I)$ because (I, a) is not an edge in $Ne(G, I)$, since $aI \neq a$ or I for all $a \neq e$. So that the total number of triangles having common verities e and I in $Ne(G, I)$ is

$$|T_{eI}| = |N(G)| - (|G| + 1)$$
$$= 2|G| - (|G| + 1) = |G| - 1.$$

Theorem 4.3 The total number of non-basic Neutrosophic triangles in $Ne(G, I)$ is zero.

Proof. Suppose that two vertices either x, y or y, z or z, x are not equal to e and I.

Then the traid (x, y, z) is a non-basic triangle in

$$Ne(G, I) \Leftrightarrow (x, y, z) \notin T_{eI}$$
$$\Leftrightarrow xy = x,\ yz = y \text{ and } zx = z$$
$$\Leftrightarrow \text{either } xyzx = x \text{ or } yzxy = y$$
$$\text{or } zxyz = z.$$

This is not possible in the Neutrosophic group $N(G)$. Thus there is no any non-basic triangle in the graph $Ne(G, I)$, and hence the total number of non-basic Neutrosophic triangles in $Ne(G, I)$ is zero.

In view of Theorems [3.9] and [4.2], the following theorem is obvious.

Theorem 4.4 The total number of pendent vertices and basic Neutrosophic triangles in $Ne(G, I)$ is same, which is equal to $|G| - 1$.

5 Isomorphic properties of Neutrosophic groups and graphs

In this section we consider important concepts known as isomorphism of groups and Neutrosophic groups. But the notion of isomorphism is common to all aspects of modern algebra [14] and Neutrosophic algebra. An isomorphism of groups and Neutrosophic groups are maps which preserves operations and structures. More precisely we have the following definitions which we make for finite groups and Neutrosophic finite groups.

Definition.5.1 Two finite groups G and G' are said to be isomorphic if there is a one-one correspondence $f : G \to G'$ such that $f(ab) = f(a) f(b)$ for all $a, b \in G$ and we write $G \cong G'$.

Now we proceed on to define isomorphism of finite Neutrosophic groups with distinct indeterminate, which can be defined over distinct groups with same binary operation. We can establish two main results.

1. Two groups are isomorphic and their Neutrosophic groups are also isomorphic.
2. If two Neutrosophic groups are isomorphic, then their Neutrosophic graphs are also isomorphic.

Definition 5.2 Let $(G, \cdot)$ and $(G', \cdot)$ be two finite groups and let $I \neq I'$ be two indeterminates of two distinct real world problems. The Neutrosophic groups $N(G) = (\langle G \cup I \rangle, \cdot)$ and $N(G') = (\langle G' \cup I' \rangle, \cdot)$ are isomorphic if there exist a group isomorphism φ from G onto G' such that $\varphi(I) = I'$ and we write $N(G) \cong N(G')$.

Definition 5.3 [13] If there is a one-one mapping $a \leftrightarrow a'$ of the elements of a group G onto those a group G' and if $a \leftrightarrow a'$ and $b \leftrightarrow b'$ implies $ab \leftrightarrow a'b'$, then we say that G and G' are isomorphic and write $G \cong G'$. If we put $a' = f(a)$ and $b' = f(b)$ for $a, b \in G$, then $f : G \to G'$ is a bijection satisfying $f(ab) = a'b' = f(a) f(b)$.

Lemma 5.4 $G \cong G' \Leftrightarrow N(G) \cong N(G')$.

Proof. Necessity. Suppose $G \cong G'$. Then there exist a group isomorphism φ from G onto G' such that $\varphi(a) = a'$ for every $a \in G$ and $a' \in G'$. By the definition [12], the relation says that φ sends ab onto $a'b'$, where $a' = \varphi(a)$ and $b' = \varphi(b)$ are the elements of

G' one-one corresponding to the elements a, b in G. We will prove that $N(G)\cong N(G')$. For this we define a map $f: N(G)\to N(G')$ by the relation $f(G)=G'$, $f(I)=I'$ and $f(GI)=G'I'$.

Suppose $x,\ y\in N(G)$.

Then either $x,\ y\in G$ or $x,\ y\in GI$. Now consider the following two cases.

Case 1 Suppose $x,\ y\in G$.

Then $x\leftrightarrow x'$ and $y\leftrightarrow y'$.

Trivially, $f(x)=x'=\varphi(x)$, for every $x\in G$ and $x'\in G'$, since $G\cong G'$. Thus, $N(G)\cong N(G')$.

Case 2 Suppose $x,\ y\in GI$.

Then $x=aI$ and $y=bI$ for $a, b\in G$. Obviously, f is one-one correspondence between $N(G)$ and $N(G')$, since $G\cong G'$ and $f(I)=I'$. Further,

$$
\begin{aligned}
f(xy) &= f\left((aI)(bI)\right)\\
&= f(abI)=a'b'I',\\
&\quad \text{since } f(GI)=G'I'\\
&=(a'I')(b'I')\\
&= f(aI)f(bI) = f(x)f(y).
\end{aligned}
$$

Thus f is a Neutrosophic group isomorphism from $N(G)$ onto $N(G')$, and hence $N(G)\cong N(G')$.

Sufficiency. It is similar to necessity, because $\langle G\cup I\rangle\cong\langle G'\cup I'\rangle$ implies that $G\cong G'$ and $GI\cong G'I'$ under the mapping $a\leftrightarrow a'$ and $aI\leftrightarrow a'I'$, respectively.

Theorem 5.5 If $G\cong G'$, then $Ne\left(G,\ I\right)\cong Ne\left(G',\ I'\right)$, where $I\neq I'$.

But converse is not true.

Proof. Suppose $N(G)=\langle G\cup I\rangle$ and $N(G')=\langle G'\cup I'\rangle$ be two different Neutrosophic groups generated by G, I and G', I', respectively.

Let φ be an isomorphism from G onto G'. Then φ is one-one correspondence between the graphs $Ne\left(G,\ I\right)$ and $Ne\left(G',\ I'\right)$ under the relation $\varphi(x)=x'$ for every $x\in N(G)$ and $x'\in N(G')$. Further to show that φ preserves the adjacency. For this let x and y be any two vertices of the graph $Ne\left(G,\ I\right)$, then x, $y\in N(G)$. This implies that

$$
\begin{aligned}
(x,\ y)\in E\left(Ne(G,\ I)\right) &\Leftrightarrow xy=x\\
&\Leftrightarrow \varphi(x)\varphi(y)=\varphi(x) \ \Leftrightarrow x'y'=x'\\
&\Leftrightarrow (x',\ y')\in E\left(Ne(G',\ I')\right).
\end{aligned}
$$

Hence, G and G' are adjacent in $Ne\left(G',\ I'\right)$. similarly, φ maps non-adjacent vertices to non-adjacent vertices. Thus, φ is a Neutrosophic graph isomorphism from $Ne\left(G,\ I\right)$ onto $Ne\left(G',\ I'\right)$, that is, $Ne\left(G,\ I\right)\cong Ne\left(G',\ I'\right)$.

The converse of the Theorem [5.5] is not true, in general. Let $G=V_4$ and let $G'=Z_5^*$. Clearly, $Ne\left(G,\ I\right)\cong Ne\left(G',\ I'\right)$, but V_4 is not isomorphic to Z_5^*.

This is illustrated in the following figure.

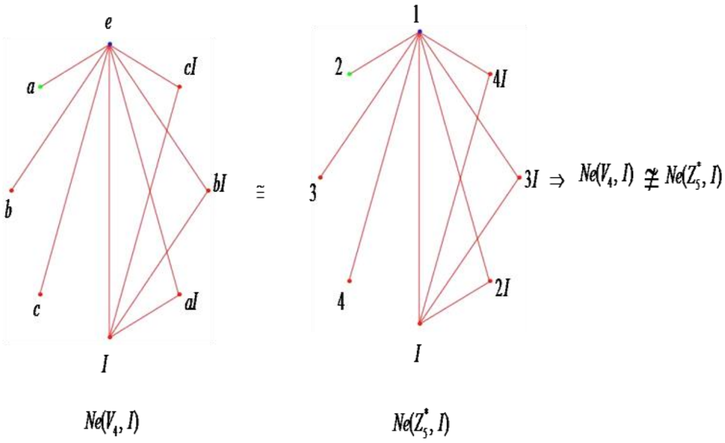

Figure Neutrosophic graphs of V_4 and Z_5^*

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References

[1] L.A. Zadeh, Fuzzy sets, Inform and Control. 8 (1965), 338–353.
[2] E. T. Jaynes, G. Larry Bretthorst. Probability Theory: The Logic of Science. Cambridge University Press(2003).
[3] K. Atanassov. More on intuitionistic fuzzy sets, Fuzzy Sets and Systems 33 (1989), 37–46.
[4] Z. Pawlak: Rough sets, International Journal of Computer and Information Sciences, 11(1982) 341-356.

[5] Florentin Smarandache. A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic. Americal research press, Rehoboth (1999).
[6] Florentin Smarandache, Unmatter, mss. Archives Vâlcea (1980).
[7] W.B. Vasantha Kandasamy, Florentin Smarandache. Basic Neutrosophic Algebraic Structures and their Applications to Fuzzy and Neutrosophic Models, Hexis, Church Rock (2004).
[8] A.A.A. Agboola, A.O. Akwu, Y.T. Oyebo, Neutrosophic Groups and Neutrosophic Subgroups, Int. J. of Math. Comb.3 (2012)1-9.
[9] W. B. Vasantha Kandasamy, Ilanthenral K, Florentin Smarandache . Neutrosophic Graphs: A New Dimension to Graph Theory. EuropaNova (2015).
[10] W.B. Vasantha Kandasamy, F. Smarandache, Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures, Hexis, Phoenix, Arizona (2006).
[11] David Joyner. Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical. Toys. 2nd edition, JHU Press (2008).
[12] I.Vitaly, I. Voloshin. Introduction to graph theory, Nova Science Publishers. Inc. New York (2009).
[13] Kiyosi Itô.: Encyclopaedic Dictionary of Mathematics, Volume 1. The MIT Press. 2nd edition (2000).
[14] Linda Gilbert. Elements of Modern Algebra. 8th edition Cengage Learning, (2014).

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A New Similarity Measure Based on Falsity Value between Single Valued Neutrosophic Sets Based on the Centroid Points of Transformed Single Valued Neutrosophic Numbers with Applications to Pattern Recognition

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Abstract. In this paper, we propose some transformations based on the centroid points between single valued neutrosophic numbers. We introduce these transformations according to truth, indeterminacy and falsity value of single valued neutrosophic numbers. We propose a new similarity measure based on falsity value between single valued neutrosophic sets. Then we prove some properties on new similarity measure based on falsity value between single valued neutrosophic sets. Furthermore, we propose similarity measure based on falsity value between single valued neutrosophic sets based on the centroid points of transformed single valued neutrosophic numbers. We also apply the proposed similarity measure between single valued neutrosophic sets to deal with pattern recognition problems.

Keywords: Neutrosophic sets, Single Valued Neutrosophic Numbers, Centroid Points.

1 Introduction

In [1] Atanassov introduced a concept of intuitionistic sets based on the concepts of fuzzy sets [2]. In [3] Smarandache introduced a concept of neutrosophic sets which is characterized by truth function, indeterminacy function and falsity function, where the functions are completely independent. Neutrosophic set has been a mathematical tool for handling problems involving imprecise, indeterminant and inconsistent data; such as cluster analysis, pattern recognition, medical diagnosis and decision making.In [4] Smarandache et.al introduced a concept of single valued neutrosophic sets. Recently few researchers have been dealing with single valued neutrosophic sets [5-10].

The concept of similarity is fundamentally important in almost every scientific field. Many methods have been proposed for measuring the degree of similarity between intuitionistic fuzzy sets [11-15]. Furthermore, in [13-15] methods have been proposed for measuring the degree of similarity between intuitionistic fuzzy sets based on transformed techniques for pattern recognition. But those methods are unsuitable for dealing with the similarity measures of neutrosophic sets since intuitionistic sets are characterized by only a membership function and a non-membership function. Few researchers dealt with similarity measures for neutrosophic sets [16-22]. Recently, Jun [18] discussed similarity measures on internalneutrosophic sets, Majumdar et al.[17] discussed similarity and entropy of neutrosophic sets, Broumi et.al.[16]discussed several similarity measures of neutrosophic sets, Ye [9] discussed single-valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine, Deli et.al.[10] discussed multiple criteria decision making method on single valued bipolar neutrosophic set based on correlation coefficient similarity measure, Ulucay et.al. [21] discussed Jaccard vector similarity measure of bipolar neutrosophic set based on multi-criteria decision making and Ulucay et.al.[22] discussed similarity measure of bipolar neutrosophic sets and their application to multiple criteria decision making.

In this paper, we propose methods to transform between single valued neutrosophic numbers based on centroid points. Here, as single valued neutrosophic sets are made up of three functions, to make the transformation functions be applicable to all single valued neutrosophic numbers, we divide them into four according to their truth, indeterminacy and falsity values. While grouping according to the truth values, we take into account whether the truth values are greater or smaller than the indeterminancy and falsity values. Similarly, while grouping according to the indeterminancy/falsity values, we examine the indeterminancy/falsity values and their greatness or smallness with respect to their remaining two values. We also propose a new method to measure the degree of similarity based on falsity values between single valued neutrosophic sets. Then we prove some properties of new similarity measure based on falsity value between single valued neutrosophic sets. When we take this measure with respect to truth or indeterminancy we show that it does not satisfy one of the conditions of similarity measure. We also apply the proposed new similarity measures based on falsity value between single valued neutrosophic sets to deal with pattern recognition problems. Later, we define the method based on falsity value to measure the degree of similarity between single valued neutrosophic set based on centroid points of transformed single valued neutrosophic numbers and the similarity measure based on falsity value between single valued neutrosophic sets.

In section 2, we briefly review some concepts of single valued neutrosophic sets [4] and property of similarity measure between single valued neutrosophic sets. In section 3, we define transformations between the single valued neutrosophic numbers based on centroid points. In section 4, we define the new similarity measures based on falsity value between single valued neutrosophic sets and we prove some properties of new similarity measure between single valued neutroshopic sets. We also apply the proposed method to deal with pattern recognition problems. In section 5, we define the method to measure the degree of similarity based on falsity value between single valued neutrosophicset based on the centroid point of transformed single valued neutrosophic number and we apply the measure to deal with pattern recognition problems. Also we compare the traditional and new methods in pattern recognition problems.

2 Preliminaries

Definition 2.1[3] Let U be a universe of discourse. The neutrosophic set A is an object having the farm $A = \{\langle x: T_{A(x)}, I_{A(x)}, F_{A(x)}\rangle, x \in U\}$ where the functions $T, I, F: U \to]^{-}0, 1^{+}[$ respectively the degree of membership, the degree of indeterminacy and degree of nonmembership of the element $x \in U$ to the set A with the condition:

$$0^{-} \leq T_{A(x)} + I_{A(x)} + F_{A(x)} \leq 3^{+}$$

Definition 2.2 [4] Let U be a universe of discourse.The single valued neutrosophic set A is an object having the farm $A = \{\langle x: T_{A(x)}, I_{A(x)}, F_{A(x)}\rangle, x \in U\}$ where the functions $T, I, F: U \to [0,1]$respectively the degree of membership, the degree of indeterminacy and degree of nonmembership of the element $x \in U$ to the set A with the condition:

$$0 \leq T_{A(x)} + I_{A(x)} + F_{A(x)} \leq 3$$

For convenience we can simply use x = (T,I,F) to represent an element x in SVNS, and element x can be called a single valued neutrosophic number.

Definition 2.3 [4] A single valued neutrosophic set A is equal to another single valued neutrosophic set B, $A = B$ if$\forall x \in U$,

$$T_{A(x)} = T_{B(x)},\ I_{A(x)} = I_{B(x)},\ F_{A(x)} = F_{B(x)}.$$

Definition 2.4[4] A single valued neutrosophic set A is contained in another single valued neutrosophic set B , $A \subseteq B$ if $\forall x \in U$,

$$T_{A(x)} \leq T_{B(x)},\ I_{A(x)} \leq I_{B(x)},\ F_{A(x)} \geq F_{B(x)}.$$

Definition 2.5[16] (Axiom of similarity measure)

A mapping$S(A,B): NS_{(x)} \times NS_{(x)} \to [0,1]$, where $NS_{(x)}$ denotes the set of all NS in$x = \{x_1, \ldots, x_n\}$,is said to be the degree of similarity between A and B if it satisfies the following conditions:

sP_1) $0 \leq S(A,B) \leq 1$

sp_2) $S(A,B) = 1$ if and only if $A = B$, $\forall A, B \in NS$

sP_3) $S(A,B) = S(B,A)$

sp_4) If $A \subseteq B \subseteq C$ for all $A, B, C \in NS$, then $S(A,B) \geq S(A,C)$and $S(B,C) \geq S(A,C)$.

3 The Transformation Techniques between Single Valued Neutrosophic Numbers

In this section, we propose transformation techniques between a single valued neutrosophic number $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ and a single valued neutrosophic number $C_{(x_i)}$. Here $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ denote the single valued neutrosophic numbers to represent an element x_i in the single valued neutrosophic set A, and $C_{A(x_i)}$ is the center of a triangle (SLK) which was obtained by the transformation on the three-dimensional $Z - Y - M$ plane.

First we transform single valued neutrosophic numbers according to their distinct T_A, I_A, F_A values in three parts.

3.1 Transformation According to the Truth Value

In this section, we group the single valued neutrosophic numbers after the examination of their truth values T_A's greatness or smallness against I_A and F_A values. We will shift the $T_{A(x_i)}$ and $F_{A(x_i)}$ values on the Z – axis and $T_{A(x_i)}$ and $I_{A(x_i)}$ values on the Y – axis onto each other. We take the $F_{A(x_i)}$ value on the M – axis. The shifting on the Z and Y planes are made such that we shift the smaller value to the difference of the greater value and 2, as shown in the below figures.

1. **First Group**

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$, if

$$T_{A(x_i)} \leq F_{A(x_i)}$$

and

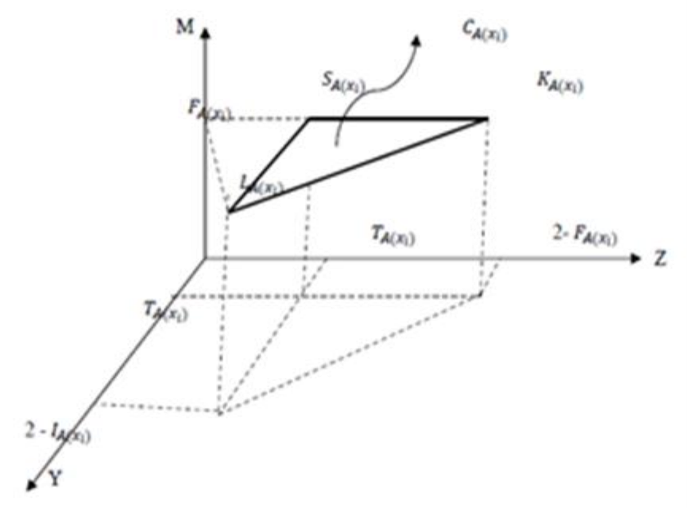
$$T_{A(x_i)} \leq I_{A(x_i)},$$

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(T_{A(x_i)}, T_{A(x_i)}, F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(2 - F_{A(x_i)},\ T_{A(x_i)},\ F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(T_{A(x_i)},\ 2 - I_{A(x_i)},\ F_{A(x_i)}\right).$$



Here, as

$$T_{C_{A(x_i)}} = T_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

$$I_{C_{A(x_i)}} = T_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

and

$$F_{C_{A(x_i)}} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

2. **Second Group**

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$, if

$$T_{A(x_i)} \geq F_{A(x_i)}$$

and

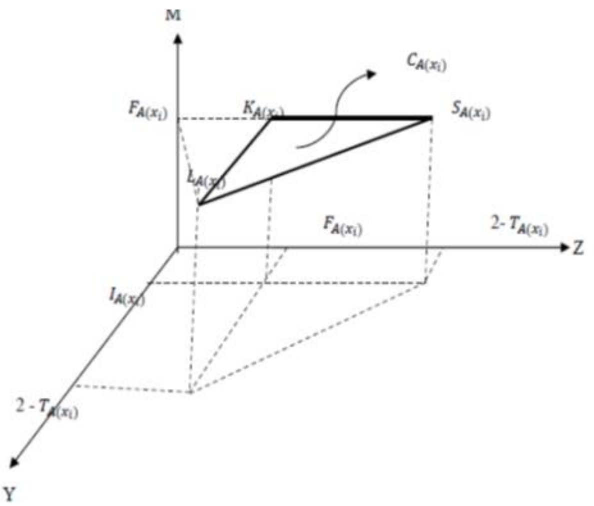
$$T_{A(x_i)} \geq I_{A(x_i)},$$

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$S_{A(x_i)} = \left(F_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\right)$

$L_{A(x_i)} = \left(F_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)}\right)$

$K_{A(x_i)} = \left(2 - T_{A(x_i)}, I_{A(x_i)}, F_{a(x_i)}\right)$.



Here, as

$$T_{C_A(x_i)} = F_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2F_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2I_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - T_{A(x_i)} + 2F_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2I_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

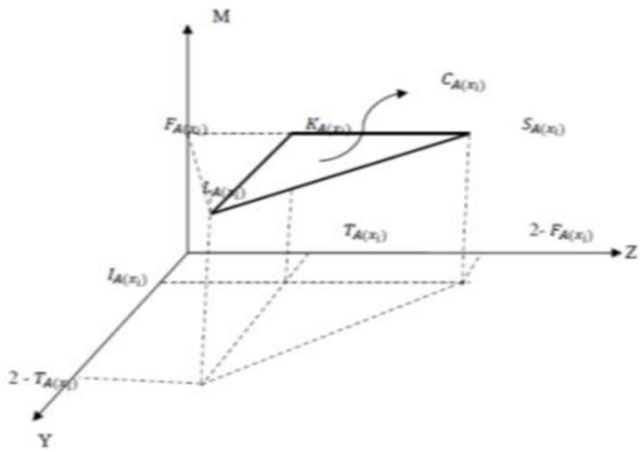
3. Third Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if $I_{A(x_i)} \le T_{A(x_i)} \le F_{A(x_i)}$, as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$S_{A(x_i)} = \left(T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\right)$

$L_{A(x_i)} = \left(T_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)}\right)$

$K_{A(x_i)} = \left(2 - F_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\right)$.



Here, as

$$T_{C_A(x_i)} = T_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2T_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2I_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2T_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2I_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

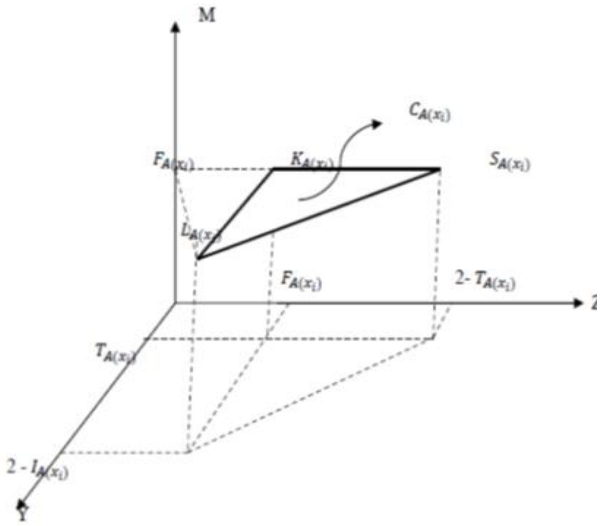
4. Fourth Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if $F_{A(x_i)} \le T_{A(x_i)} \le I_{A(x_i)}$, as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$S_{A(x_i)} = \left(F_{A(x_i)},\ T_{A(x_i)},\ F_{A(x_i)}\right)$

$L_{A(x_i)} = \left(F_{A(x_i)},\ 2 - T_{A(x_i)},\ F_{A(x_i)}\right)$

$K_{A(x_i)} = \left(2 - T_{A(x_i)},\ T_{A(x_i)},\ F_{A(x_i)}\right)$.



Example3.1.1 Transform the following single valued neutrosophic numbers according to their truth values.

$\langle 0.2,\ 0.5,\ 0.7\rangle$, $\langle 0.9,\ 0.4,\ 0.5\rangle$, $\langle 0.3,\ 0.2,\ 0.5\rangle$, $\langle 0.3,\ 0.2,\ 0.4\rangle$.

$i.$ $\langle 0.2,\ 0.5,\ 0.7\rangle$ single valued neutrosophic number belongs to the first group.

The center is calculated by the formula,$C_{A(x_i)} = \left(\frac{2- F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2- I_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right)$

and we have$C_{A(x)} = \langle 0.566,\ 0.633,\ 0.7\rangle$.

$ii.$ $\langle 0.9,\ 0.4,\ 0.5\rangle$single valued neutrosophic number is in the second group.

The center for the values of the second group is, $C_{A(x_i)} = \left(\frac{2- T_{A(x_i)} + 2\,F_{A(x_i)}}{3}, \frac{2- T_{A(x_i)} + 2I_{A(x_i)}}{3}, F_{A(x_i)}\right)$

and for $\langle 0.9,\ 0.4,\ 0.5\rangle$,$C_{A(x)} = \langle 0.7,\ 0.633,\ 0.5\rangle$.

$iii.$ $\langle 0.3,\ 0.2,\ 0.5\rangle$single valued neutrosophic number belongs to the third group.

The formula for the center of $\langle 0.3,\ 0.2,\ 0.5\rangle$ is $C_{A(x_i)} = \left(\frac{2- F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2- T_{A(x_i)} + 2I_{A(x_i)}}{3}, F_{A(x_i)}\right)$and therefore we have $C_{A(x)} = \langle 0.7,\ 0.7,\ 0.5\rangle$.

$iv.$ $\langle 0.3,\ 0.2,\ 0.4\rangle$single valued neutrosophic number is in the third group and the center is calculated to be $C_{A(x)} = \langle 0.733,\ 0.7,\ 0.4\rangle$.

Corollary 3.1.2 The corners of the triangles obtained using the above method need not be single valued neutrosophic number but by definition, trivially their centers are.

Note 3.1.3 As for the single valued neutrosophic number$\langle 1,$ ber$\langle 1,\ 1,\ 1\rangle$ there does not exist any transformable triangle in the above four groups, we take its transformation equal to itself.

Corollary 3.1.4 If $F_{A(x_i)} = T_{A(x_i)} = I_{A(x_i)}$ the transformation gives the same center in all four groups. Also, if $T_{A(x_i)} = I_{A(x_i)} \leq F_{A(x_i)}$, then the center in the first group is equal to the one in the third group and if $F_{A(x_i)} \leq T_{A(x_i)} = I_{A(x_i)}$, the center in the second group is equal to the center in the fourth group. Similarly, if $T_{A(x_i)} = F_{A(x_i)} \leq I_{A(x_i)}$, then the center in the first group is equal to the center in the fourth group and if $I_{A(x_i)} \leq T_{A(x_i)} = F_{A(x_i)}$, the center in the second group is equal to the one in the third group.

3.2Transformation According to the Indeterminancy Value

In this section, we group the single valued neutrosophic numbers after the examination of their indeterminancy values I_A's greatness or smallness against T_A and$F_A$values. We will shift the $I_{A(x_i)}$ and $F_{A(x_i)}$ values on the Z – axis and $T_{A(x_i)}$ and $I_{A(x_i)}$values on the Y – axis onto each other. We take the $F_{A(x_i)}$value on the M – axis. The shifting on the Z and Y planes are made such that we shift the smaller value to the difference of the greater value and 2, as shown in the below figures.

1. **First Group**

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if

$$I_{A(x_i)} \leq F_{A(x_i)}$$

and

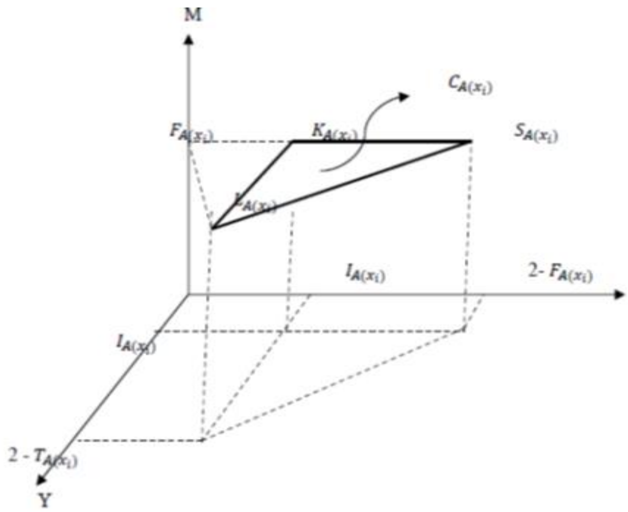
$$I_{A(x_i)} \leq F_{A(x_i)} \ ,$$

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(I_{A(x_i)}, \quad I_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(2 - F_{A(x_i)}, \ I_{A(x_i)}, \ F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(I_{A(x_i)}, \ 2 - T_{A(x_i)}, \ F_{A(x_i)}\right) .$$



We transformed the single valued neutrosophic number $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$into the center of the SKL triangle, namely $C_{A(x_i)}$. Here, as

$$T_{C_{A(x_i)}} = I_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\, I_{A(x_i)}}{3}$$

$$I_{C_{A(x_i)}} = T_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2\, I_{A(x_i)}}{3}$$

and

$$F_{C_{A(x_i)}} = F_{A(x_i)} \ ,$$

we have

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\, I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2\, I_{A(x_i)}}{3}, F_{A(x_i)}\right) .$$

2. Second Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$, if

$$I_{A(x_i)} \geq F_{A(x_i)}$$

and

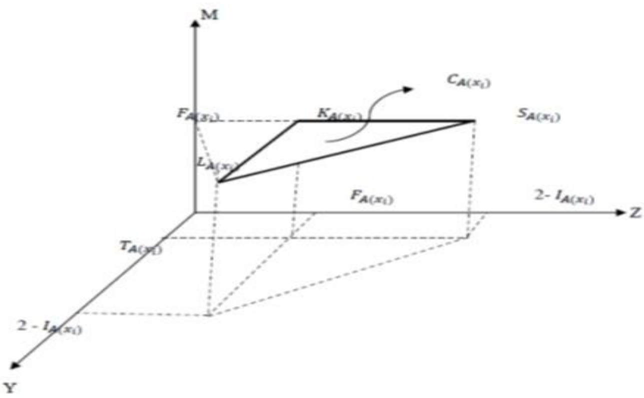
$$I_{A(x_i)} \geq F_{A(x_i)} \ ,$$

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(F_{A(x_i)}, \quad T_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(\ F_{A(x_i)}, 2 - I_{A(x_i)}, \ F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - I_{A(x_i)}, \ T_{A(x_i)}, \ F_{A(x_i)}\right) .$$



Here, as

$$T_{C_{A(x_i)}} = F_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2\, F_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = T_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - I_{A(x_i)} + 2\,F_{A(x_i)}}{3}, \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

3. Third Group

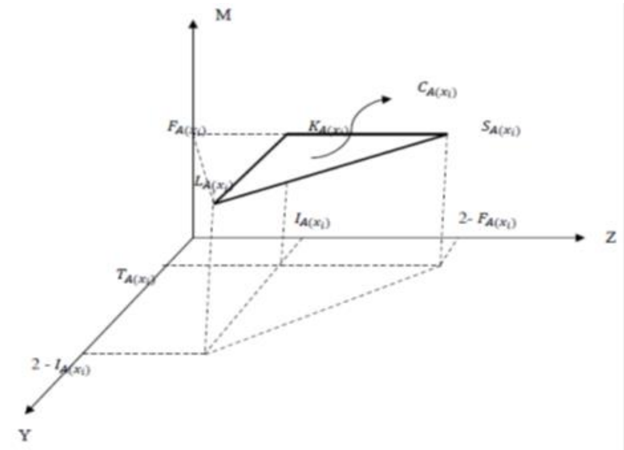
For the single valued neutrosophic number $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if $\mathrm{T}_{A(x_i)} \leq \mathrm{I}_{A(x_i)} \leq \mathrm{F}_{A(x_i)}$,

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(I_{A(x_i)}, \quad T_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(I_{A(x_i)}, 2 - I_{A(x_i)}, F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - F_{A(x_i)}, T_{A(x_i)}, F_{A(x_i)}\right).$$



Here as

$$T_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = T_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}, \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

4. Fourth Group

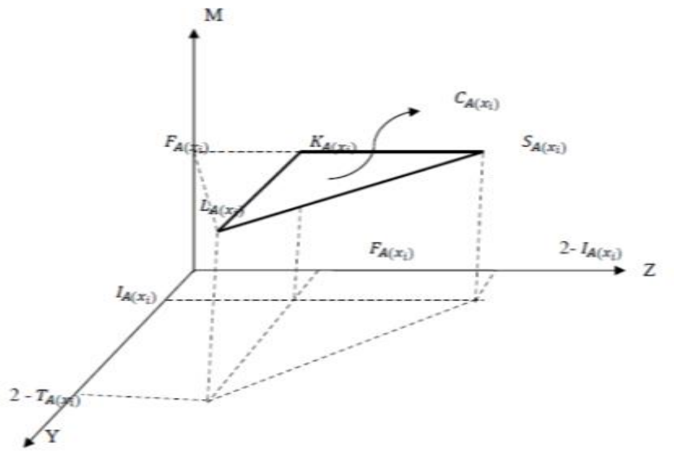
For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if $\mathrm{F}_{A(x_i)} \leq \mathrm{I}_{A(x_i)} \leq \mathrm{T}_{A(x_i)}$,

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic numbers $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(F_{A(x_i)}, \quad I_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(F_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - I_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\right).$$



Here, as

$$T_{C_A(x_i)} = F_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2F_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2I_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - I_{A(x_i)} + 2F_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2I_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

Example3.2.1:Transform the single neutrosophic numbers of Example 3.1.3 ,

$\langle 0.2, 0.5, 0.7\rangle$, $\langle 0.9, 0.4, 0.5\rangle$, $\langle 0.3, 0.2, 0.5\rangle$, $\langle 0.3, 0.2, 0.4\rangle$according to their indeterminancy values.

i. $\langle 0.2, 0.5, 0.7\rangle$ single valued neutrosophic number is in the third group. The center is given by the formula

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2I_{A(x_i)}}{3}, \frac{2 - I_{A(x_i)} + 2T_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

and so $C_{A(x)} = \langle 0.766, 0.633, 0.7\rangle$.

ii. $\langle 0.9, 0.4, 0.5\rangle$ single valued neutrosophic number is in the first group.

By

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2I_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

we have $C_{A(x)} = \langle 0.733, 0.633, 0.5\rangle$.

iii. $\langle 0.3, 0.2, 0.5\rangle$single valued neutrosophic number belongs to the first group and the center is

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2I_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

so,$C_{A(x)} = \langle 0.633, 0.9, 0.5\rangle$.

iv. $\langle 0.3, 0.2, 0.4\rangle$single valued neutrosophic number is in the first group.
Using

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2I_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

we have $C_{A(x)} = \langle 0.666, 0.7, 0.4\rangle$.

Corollary 3.2.2 The corners of the triangles obtained using the above method need not be single valued neutrosophic numbers but by definition, trivially their centers are.

Note 3.2.3 As for the single valued neutrosophic number $\langle 1, 1, 1\rangle$ there does not exist any transformable triangle in the above four groups, we take its transformation equal to itself.

Corollary 3.2.4 If $F_{A(x_i)} = T_{A(x_i)} = I_{A(x_i)}$,the transformation gives the same center in all four groups. Also if $T_{A(x_i)} = I_{A(x_i)} \le F_{A(x_i)}$, then the center in the first group is equal to the center in the third group, and if $F_{A(x_i)} \le T_{A(x_i)} = I_{A(x_i)}$, then the center in the second group is the same as the one in the fiurth group. Similarly, if$F_{A(x_i)} = I_{A(x_i)} \le T_{A(x_i)}$, then the center in the first group is equal to the one in the fourth and in the case that $T_{A(x_i)} \le F_{A(x_i)} = I_{A(x_i)}$,the center in the second group is equal to the center in the third.

3.3 Transformation According to the Falsity Value

In this section, we group the single valued neutrosophic numbers after the examination of their indeterminancy values F_A's greatness or smallness against I_A and F_A values. We will shift the $I_{A(x_i)}$and $F_{A(x_i)}$values on the Z – axis and $T_{A(x_i)}$ and $F_{A(x_i)}$values on the Y – axis onto each other. We take the $F_{A(x_i)}$value on the M – axis. The shifting on the Z and Y planes are made such that we shift the smaller value to the difference of the greater value and 2, as shown in the below figures.

1. First Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$, if

$$F_{A(x_i)} \leq T_{A(x_i)}$$

and

$$F_{A(x_i)} \leq I_{A(x_i)},$$

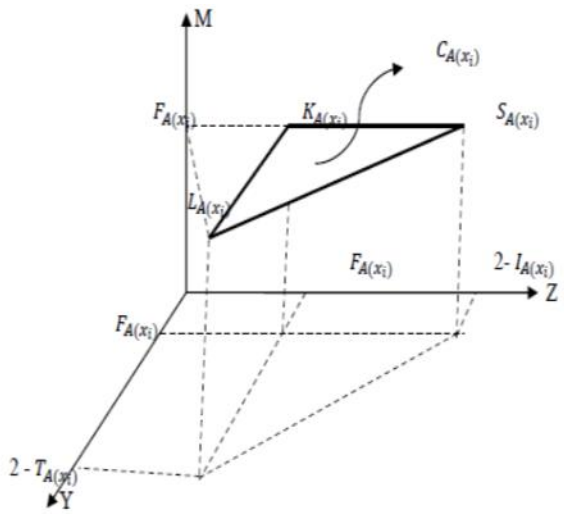
then

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(F_{A(x_i)}, \quad F_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(2 - I_{A(x_i)}, F_{A(x_i)}, F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(F_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)}\right).$$



Here, as

$$T_{C_A(x_i)} = F_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2F_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = F_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2F_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we get

$$C_{A(x_i)} = \left(\frac{2 - I_{A(x_i)} + 2F_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2F_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

2. Second Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$, if

$$F_{A(x_i)} \geq T_{A(x_i)}$$

and

$$F_{A(x_i)} \geq I_{A(x_i)},$$

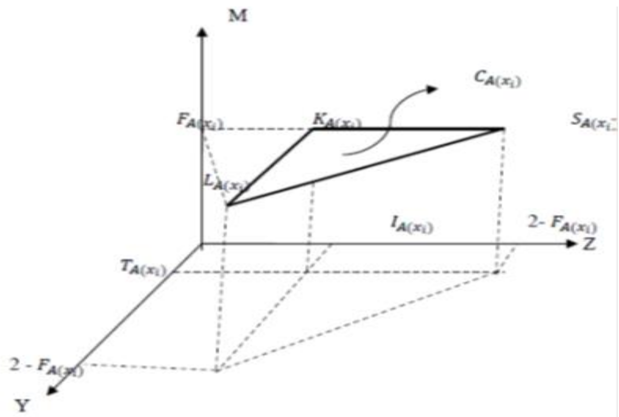
then

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic numbers $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(I_{A(x_i)}, \quad T_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(I_{A(x_i)}, 2 - F_{A(x_i)}, F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - F_{A(x_i)}, T_{A(x_i)}, F_{A(x_i)}\right).$$



Here, as

$$T_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2I_{A(x_i)}}{3}$$

$$I_{C_{A(x_i)}} = T_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

and

$$F_{C_{A(x_i)}} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

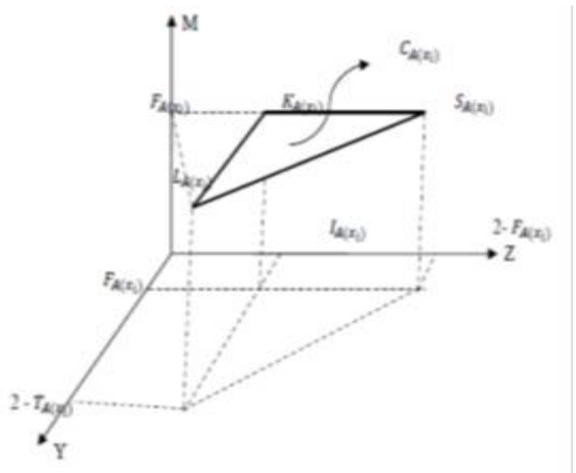
3. Third Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if $I_{A(x_i)} \le F_{A(x_i)} \le T_{A(x_i)}$ then as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic numbers $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(I_{A(x_i)}, \quad F_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(I_{A(x_i)}, 2 - T_{A(x_i)}, \ F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - F_{A(x_i)}, \ F_{A(x_i)}, \ F_{A(x_i)}\right).$$



Here, as

$$T_{C_{A(x_i)}} = I_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}$$

$$I_{C_{A(x_i)}} = F_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2\,F_{A(x_i)}}{3}$$

and

$$F_{C_{A(x_i)}} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2\,F_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

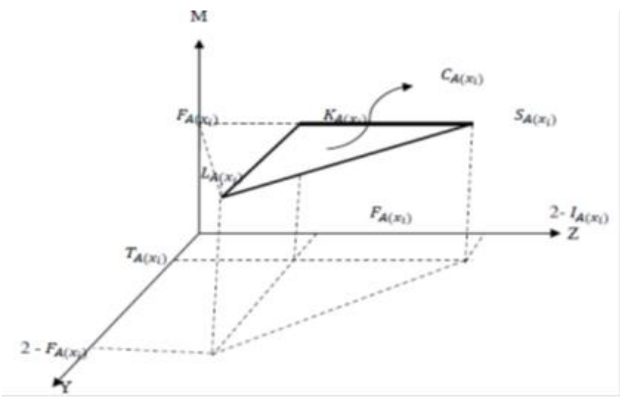
4. Fourth Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if $T_{A(x_i)} \le F_{A(x_i)} \le I_{A(x_i)}$, then as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic numbers $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(F_{A(x_i)}, \quad T_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(F_{A(x_i)}, 2 - F_{A(x_i)}, \ F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - I_{A(x_i)}, \ T_{A(x_i)}, \ F_{A(x_i)}\right).$$



Example 3.3.1: Transform the single neutrosophic numbers of Example 3.1.3.

$\langle 0.2, 0.5, 0.7\rangle$, $\langle 0.9, 0.4, 0.5\rangle$, $\langle 0.3, 0.2, 0.5\rangle$, $\langle 0.3, 0.2, 0.4\rangle$according to their falsity values.

i. $\langle 0.2, 0.5, 0.7\rangle$ single valued neutrosophic number belongs to the second group. So, the center is

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

and we get $C_{A(x)} = \langle 0.766, 0.7, 0.7\rangle$.

ii. $\langle 0.9, 0.4, 0.5\rangle$ single valued neutrosophic number is in the third group. Using the formula

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2\,F_{A(x_i)}}{3}, F_{A(x_i)}\right)$$

we see that$C_{A(x)} = \langle 0.766, 0.7, 0.5\rangle$.

iii. $\langle 0.3, 0.2, 0.5\rangle$single valued neutrosophic number is in the second group. As

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

the center of the triangle is $C_{A(x)} = \langle 0.633, 0.7, 0.5\rangle$.

iv. $\langle 0.3, 0.2, 0.4\rangle$single valued neutrosophic number belongs to the second group.

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

and so we have $C_{A(x)} = \langle 0.666, 0.733, 0.4\rangle$.

Corollary 3.3.2 The corners of the triangles obtained using the above method need not be single valued neutrosophic numbers but by definition, trivially their centers are single valued neutrosophic values.

Note 3.3.3 As for the single valued neutrosophic ber$\langle 1, 1, 1\rangle$ there does not exist any transformable triangle in the above four groups, we take its transformation equal to itself.

Corollary 3.3.4 If $F_{A(x_i)} = T_{A(x_i)} = I_{A(x_i)}$, the transformation gives the same center in all four groups. Also, if $T_{A(x_i)} = F_{A(x_i)} \leq I_{A(x_i)}$, then the center in the first group is equal to the one in the fourth group, and if $I_{A(x_i)} \leq T_{A(x_i)} = F_{A(x_i)}$, then the center in the second group is the same as the center in the third. Similarly, if $I_{A(x_i)} = F_{A(x_i)} \leq T_{A(x_i)}$, then the centers in the first and third groups are same and lastly, if $T_{A(x_i)} \leq I_{A(x_i)} = F_{A(x_i)}$, then the center in the second group is equal to the one in the fourth group.

4. A New Similarity Measure Based on Falsity Value Between Single Valued Neutrosophic Sets

In this section, we propose a new similarity measure based on falsity value between single valued neutrosophic sets.

Definition 4.1 Let A and B two single valued neutrosophic sets in $x = \{x_1, x_2, \dots, x_n\}$.
Let $A = \{\langle x, T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle\}$

and

$B = \{\langle x, T_{B(x_i)}, I_{B(x_i)}, F_{B(x_i)}\rangle\}$.

The similarity measure based on falsity value between the neutrosophic numbers $A(x_i)$ and $B(x_i)$ is given by

$$S(A_{(x_i)}, B_{(x_i)}) = 1 - \left(\frac{\left|2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})\right|}{9} + \frac{\left|2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})\right|}{9} + \frac{3\left|(F_{A(x_i)} - F_{B(x_i)})\right|}{9}\right).$$

Here, we use the values

$$2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(T_{A(x_i)} - T_{B(x_i)}\right),$$

$$2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(I_{A(x_i)} - I_{B(x_i)}\right),$$

$$2\left(F_{A(x_i)} - F_{B(x_i)}\right) + \left(F_{A(x_i)} - F_{B(x_i)}\right) = 3\left(F_{A(x_i)} - F_{B(x_i)}\right).$$

Since we use the falsity values $F_{A(x_i)}$ in all these three values, we name this formula as “similarity measure based on falsity value between single valued neutrosophic numbers”.

Property4.2 :$0 \leq S(A_{(x_i)}, B_{(x_i)}) \leq 1$.

Proof: By the definition of Single valued neutrosophic numbers, as

$$0 \leq T_{A(x_i)}, T_{B(x_i)}, I_{A(x_i)}, I_{B(x_i)}, F_{A(x_i)}, F_{B(x_i)} \leq 1,$$

we have

$$0 \leq 2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(T_{A(x_i)}, T_{B(x_i)}\right) \leq 3$$

$$0 \leq 2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(I_{A(x_i)}, I_{B(x_i)}\right) \leq 3$$

and

$$0 \leq 3\left(F_{A(x_i)}, F_{B(x_i)}\right) \leq 3 .$$

So,

$$0 \leq 1 - \left(\frac{\left|2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})\right|}{9} + \frac{\left|2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})\right|}{9} + \frac{3\left|(F_{A(x_i)} - F_{B(x_i)})\right|}{9}\right) \leq 1 .$$

Therefore, $0 \leq S\left(A_{(x_i)}, B_{(x_i)}\right) \leq 1.$

Property 4.3: $S(A_{(x_i)}, B_{(x_i)}) = 1 \Leftrightarrow A_{(x_i)} = B_{(x_i)}$

Proof.i) First we show $A_{(xi)} = B_{(xi)}$ when $S\left(A_{(x_i)}, B_{(x_i)}\right) = 1$.

Let $\left(A_{(x_i)}, B_{(x_i)}\right) = 1$.

$$S\left(A_{(x_i)}, B_{(x_i)}\right) = 1 - \left(\frac{\left|2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})\right|}{9} + \frac{\left|2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})\right|}{9} + \frac{3\left|(F_{A(x_i)} - F_{B(x_i)})\right|}{9}\right)$$

$$= 1$$

and thus,

$$\left(\frac{\left|2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})\right|}{9} + \frac{\left|2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})\right|}{9} + \frac{3\left|(F_{A(x_i)} - F_{B(x_i)})\right|}{9}\right) = 0$$

So,

$$\left|(F_{A(x_i)} - F_{B(x_i)})\right| = 0,$$

$$\left|2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})\right| = 0,$$

and

$$\left|2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})\right| = 0 .$$

As $\left|(F_{A(x_i)} - F_{B(x_i)})\right| = 0$, then $F_{A(x_i)} = F_{B(x_i)}$.

If $F_{A(x_i)} = F_{B(x_i)}$,

$$\left|2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(T_{A(x_i)} - T_{B(x_i)}\right)\right| = 0$$

and

$$T_{A(x_i)} = T_{B(x_i)}.$$

When $F_{A(x_i)} = F_{B(x_i)}$,

$$\left|2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})\right| = 0$$

and

$$I_{A(x_i)} = I_{B(x_i)}$$

Therefore, if $\left(A_{(x_i)}, B_{(x_i)}\right) = 1$, then by Definition 2.3, $A_{(x_i)} = B_{(x_i)}$.

ii) Now we show if $A_{(x_i)} = B_{(x_i)}$, then $S\left(A_{(x_i)}, B_{(x_i)}\right) = 1$. Let $A_{(x_i)} = B_{(x_i)}$. By Definition 2.3 ,

$$T_{A(x_i)} = T_{B(x_i)}, \; I_{A(x_i)} = I_{B(x_i)}, \; F_{A(x_i)} = F_{B(x_i)}$$

and we have

$$T_{A(x_i)} - T_{B(x_i)} = 0, \; I_{A(x_i)} - I_{B(x_i)} = 0, \; F_{A(x_i)} - F_{B(x_i)} = 0 .$$

So,

$$S(A_{(x_i)},B_{(x_i)}) = 1-\left(\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})|}{9}+\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})|}{9}+\frac{3|(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})|}{9}\right)$$

$$= 1-\frac{0}{9} = 1 .$$

Property4.4 : $S(A_{(x_i)},B_{(x_i)}) = S(B_{(x_i)},A_{(x_i)})$.

Proof:

$$S(A_{(x_i)},B_{(x_i)}) = 1-\left(\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})|}{9}+\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})|}{9}+\frac{3|(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})|}{9}\right)$$

$$= 1-\left(\frac{|2(-(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)}))-(-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)}))|}{9}+\frac{|2((-\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)}))-(-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)}))|}{9}+\frac{3|-(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})|}{9}\right)$$

$$= 1-\left(\frac{|2(\mathrm{F}_{\mathrm{B}(x_i)}-\mathrm{F}_{\mathrm{A}(x_i)})-(\mathrm{T}_{\mathrm{B}(x_i)}-\mathrm{T}_{\mathrm{A}(x_i)})|}{9}+\frac{|2(\mathrm{F}_{\mathrm{B}(x_i)}-\mathrm{F}_{\mathrm{A}(x_i)})-(\mathrm{I}_{\mathrm{B}(x_i)}-\mathrm{I}_{\mathrm{A}(x_i)})|}{9}+\frac{3|(\mathrm{F}_{\mathrm{B}(x_i)}-\mathrm{F}_{\mathrm{A}(x_i)})|}{9}\right)$$

$$= S(B_{(x_i)},A_{(x_i)}).$$

Property 4.5 : If $A \subseteq B \subseteq C$,

i) $S(A_{(x_i)},B_{(x_i)}) \geq S(A_{(x_i)},C_{(x_i)})$

ii) $S(B_{(x_i)},C_{(x_i)}) \geq S(A_{(x_i)},C_{(x_i)})$

Proof:

i) By the single valued neutrosophic set property, if $A \subseteq B \subseteq C$, then

$$\mathrm{T}_{\mathrm{A}(x_i)} \leq \mathrm{T}_{\mathrm{B}(x_i)} \leq \mathrm{T}_{\mathrm{C}(x_i)},$$

$$\mathrm{I}_{\mathrm{A}(x_i)} \leq \mathrm{I}_{\mathrm{B}(x_i)} \leq \mathrm{I}_{\mathrm{C}(x_i)},$$

$$\mathrm{F}_{\mathrm{A}(x_i)} \geq \mathrm{F}_{\mathrm{B}(x_i)} \geq \mathrm{F}_{\mathrm{C}(x_i)} .$$

So,

$$\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)} \leq 0,$$

$$\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)} \leq 0,$$

$$\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)} \geq 0 \qquad (1)$$

$$\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{C}(x_i)} \leq 0,$$

$$\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{C}(x_i)} \leq 0,$$

$$\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{C}(x_i)} \geq 0 \qquad (2)$$

$$\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)} \geq \mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{C}(x_i)},$$

$$\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)} \geq \mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{C}(x_i)},$$

$$\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)} \leq \mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{C}(x_i)} \qquad (3)$$

Using (1), we have

$$2(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)}) \geq 0$$

$$2(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)}) \geq 0$$

and

$$3(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)}) \geq 0 .$$

Thus, we get

$$S(A_{(x_i)},B_{(x_i)}) = 1-\left(\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})|}{9}+\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})|}{9}+\frac{3|(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})|}{9}\right)$$

$$= 1-\frac{7(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})}{9} . \qquad (4)$$

Similarly, by (2), we have

$$S(A_{(x_i)},C_{(x_i)})=1-\left(\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{C(x_i)})-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{C}(x_i)})|}{9}+\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{C}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{C}(x_i)})|}{9}+\frac{3|(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{C}(x_i)})|}{9}\right)$$

$$=1-\frac{7(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{C}(x_i)})-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{C}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{C}(x_i)})}{9}\ . (5)$$

Using (4) and (5) together, we get

$$S(A_{(x_i)},B_{(x_i)})-S(A_{(x_i)},C_{(x_i)})$$

$$=1-\frac{7(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})}{9}$$

$$-1+\frac{7(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})}{9}$$

$$=\frac{7(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})}{9}-\frac{(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})}{9}-\frac{(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})}{9}$$

$$+\frac{7(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{C}(x_i)})}{9}-\frac{(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{C}(x_i)})}{9}-\frac{(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{C}(x_i)})}{9}$$

$$=\frac{7(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})}{9}+\frac{7(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{C}(x_i)})}{9}-\frac{(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})}{9}$$

$$-\frac{(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{C}(x_i)})}{9}-\frac{(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})}{9}-\frac{(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{C}(x_i)})}{9}$$

by (1) and (3),

$$\frac{7(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})}{9}+\frac{7(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{C}(x_i)})}{9}\geq 0,$$

$$-\frac{(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})}{9}-\frac{(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{C}(x_i)})}{9}\geq 0,$$

$$-\frac{(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})}{9}-\frac{(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{C}(x_i)})}{9}\geq 0$$

and therefore

$$S(A_{(x_i)},B_{(x_i)})-S(A_{(x_i)},C_{(x_i)})\geq 0$$

and

$$S(A_{(x_i)},B_{(x_i)})\geq S(A_{(x_i)},C_{(x_i)})\ .$$

ii. The proof of the latter part can be similarly done as the first part.

Corollary 4.6 : Suppose we make similar definitions to Definition 4.1, but this time based on truth values or indeterminancy values. If we define a truth based similarity measure, or namely,

$$S(A_{(x_i)},B_{(x_i)})=1-\left(\frac{|2(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})-(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})|}{9}+\frac{|2(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})|}{9}+\frac{3|(\mathrm{T}_{\mathrm{A}(x_i)}-T_{\mathrm{B}(x_i)})|}{9}\right),$$

or if we define a measure based on indeterminancy values like

$$S(A_{(x_i)},B_{(x_i)})=1-\left(\frac{|2(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})-(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})|}{9}+\frac{|2(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})-(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})|}{9}+\frac{3|(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})|}{9}\right)$$

these two definitions don’t provide the conditions of Property 4.5 . For instance, for the truth value

$$S(A_{(x_i)},B_{(x_i)})=1-\left(\frac{|2(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})-(\mathrm{F}_{\mathrm{A}(x_i)}-\mathrm{F}_{\mathrm{B}(x_i)})|}{9}+\frac{|2(\mathrm{T}_{\mathrm{A}(x_i)}-\mathrm{T}_{\mathrm{B}(x_i)})-(\mathrm{I}_{\mathrm{A}(x_i)}-\mathrm{I}_{\mathrm{B}(x_i)})|}{9}+\frac{3|(\mathrm{T}_{\mathrm{A}(x_i)}-T_{\mathrm{B}(x_i)})|}{9}\right),$$

when we take the single valued neutrosophic numbers $\mathrm{A}_{(x)}=\langle 0,0.1,0\rangle$, $\mathrm{B}_{(x)}=\langle 1,0.2,0\rangle$ and $\mathrm{C}_{(x)}=\langle 1,0.3,0\rangle$, we see $S(A_{(x)},B_{(x)})=0.233$ and $S(A_{(x)},C_{(x)})=0.244$. This contradicts with the results of Property 4.5.

Similarly, for the indeterminancy values,

$$S(A_{(x_i)}, B_{(x_i)}) = 1 - \Bigg(\frac{|2(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})|}{9} + \frac{|2(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)}) - (\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})|}{9} + \frac{3|(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})|}{9}\Bigg)$$

if we take the single valued neurosophic numbers $\mathrm{A}_{(x)} = \langle 0.1, 0, 1\rangle$, $\mathrm{B}_{(x)} = \langle 0.2, 1, 1\rangle$and $\mathrm{C}_{(x)} = \langle 0.3, 1, 1\rangle$, we have $S(A_{(x)}, B_{(x)}) = 0.233$ and $S(A_{(x)}, C_{(x)}) = 0.244$.

These results show that the definition 4.1 is only valid for the measure based on falsity values.

Defintion 4.7 As

$$S(A_{(x_i)}, B_{(x_i)}) = 1 - \Bigg(\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})|}{9} + \frac{|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})|}{9} + \frac{3|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})|}{9}\Bigg),$$

The similarity measure based on the falsity value between two single valued neutrosophic sets A and B is;

$$\mathrm{S}_{\mathrm{NS}}(A, B) = \sum_{i=1}^{n} \left(w_i \times \mathrm{S}\left(\mathrm{A}_{(x_i)}, \mathrm{B}_{(x_i)}\right)\right).$$

Here,$\mathrm{S}_{\mathrm{NS}}(A, B) \in [0,1]$and w_i's are the weights of the x_i's with the property $\sum_{i=1}^{n} w_i = 1$. Also,

$$A = \{\langle x: T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle\},$$

$$B = \{\langle x: T_{B(x_i)}, I_{B(x_i)}, F_{B(x_i)}\rangle\}.$$

Example4.8 Let us consider three patterns$\mathrm{P}_1, \mathrm{P}_2$, P_3 represerted by single valued neutrosophic sets $\widetilde{\mathrm{P}_1}$ and $\widetilde{\mathrm{P}_2}$in $X = \{x_1, x_2\}$ respectively, where $\widetilde{\mathrm{P}_1} = \{\langle \mathrm{x}_1, 0.2, 0.5, 0.7\rangle, \langle \mathrm{x}_2, 0.9, 0.4, 0.5\rangle\}$ and $\widetilde{\mathrm{P}_2} = \{\langle \mathrm{x}_1, 0.3, 0.2, 0.5\rangle, \langle \mathrm{x}_2, 0.3, 0.2, 0.4\rangle\}$. We want to classify an unknown pattern represented by a single valued neutrosophic set $\tilde{Q}$ in $X = \{x_1, x_2\}$ into one of the patterns $\widetilde{\mathrm{P}_1}$, $\widetilde{\mathrm{P}_2}$;where $\tilde{Q} = \{\langle \mathrm{x}_1, 0.4, 0.4, 0.1\rangle, \langle \mathrm{x}_2, 0.6, 0.2, 0.3\rangle\}$.

Let w_i be the weight of element w_i , where $w_i = \frac{1}{2}$ $1 \leq i \leq 2$,

$$\mathrm{S}_{\mathrm{NS}}(\widetilde{P_1}, \tilde{Q}) = 0.711$$

and

$$\mathrm{S}_{\mathrm{NS}}(\widetilde{P_1}, \tilde{Q}) = 0.772 .$$

We can see that $\mathrm{S}_{\mathrm{NS}}(\widetilde{P_2}, \tilde{Q})$is the largest value amongthe values of $\mathrm{S}_{\mathrm{NS}}(\widetilde{P_1}, \tilde{Q})$ and $\mathrm{S}_{\mathrm{NS}}(\widetilde{P_2}, \tilde{Q})$.

Therefore, the unknown pattern represented by single valued neutrosophic set$\tilde{Q}$ should be classified into the pattern P_2.

5. A New Similarity Measure Based on Falsity Measure Between Neutrosophic Sets Based on the Centroid Points of Transformed Single Valued Neutrosophic Numbers

In this section, we propose a new similarity measure based on falsity value between single valued neutrosophic sets based on the centroid points of transformed single valued neutrosophic numbers.

Definition5.1:

$$S(A_{(x_i)}, B_{(x_i)}) = 1 - \Bigg(\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})|}{9} + \frac{|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})|}{9} + \frac{3|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})|}{9}\Bigg),$$

Taking the similarity measure as defined in the fourth section, and letting $\mathrm{C}_{\mathrm{A}(x_i)}$ and$\mathrm{C}_{\mathrm{B}(x_i)}$be the centers of the triangles obtained by the transformation of $\mathrm{A}_{(x_i)}$and $\mathrm{B}_{(x_i)}$in the third section respectively,the similarity measure based on falsity value between single valued neutrosophic sets A and B based on the centroid points of transformed single valued neutrosophic numbers is

$$\mathrm{S}_{\mathrm{NSC}}(A, B) = \sum_{i=1}^{n} \left(w_i \mathrm{x} \mathrm{S}(\mathrm{C}_{\mathrm{A(xi)}}, \mathrm{C}_{\mathrm{B(xi)}})\right),$$

where

$$A = \{x: \langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle\},$$
$$B = \{x: \langle T_{B(x_i)}, I_{B(x_i)}, F_{B(x_i)} \rangle\}.$$

Here again, w_i's are the weights of the x_i's with the property $\sum_{i=1}^{n} w_i = 1$.

Example5.2: Let us consider two patterns P_1 and P_2 represented by single valued neutrosophic sets $\widetilde{P_1}$, $\widetilde{P_2}$ in $X = \{x_1, x_2\}$respectively in Example 4.8,where

$$\widetilde{P_1} = \{\langle x_1, 0.2, 0.5, 0.7\rangle, \langle x_2, 0.9, 0.4, 0.5\rangle\}$$

and

$$\widetilde{P_2} = \{\langle x_1, 0.3, 0.2, 0.5\rangle, \langle x_2, 0.3, 0.2, 0.4\rangle\}.$$

We want to classify an unknown pattern represented by single valued neutrosophic set $\tilde{Q}$ in $X = \{x_1, x_2\}$ into one of the patterns $\widetilde{P_1}$, $\widetilde{P_2}$, where

$$\tilde{Q} = \{\langle x_1, 0.4, 0.4, 0.1\rangle, \langle x_2, 0.6, 0.2, 0.3\rangle\}.$$

We make the classification using the measure in Definition 5.1, namely

$$S_{NSC}(A, B) = \sum_{i=1}^{n} \left(w_i \times S\left(C_{A(xi)}, C_{B(xi)}\right)\right).$$

Also we find the $C_{A(xi)}$, $C_{B(xi)}$ centers according to the truth values.

Let w_i be the weight of elementx_i, $w_i = \frac{1}{2}$; $1 \le i \le 2$.

$\widetilde{P_1}x_1 = \langle 0.2,\ 0.5,\ 0.7\rangle$transformed based on falsity value in Example 3.1.1

$$C_{\widetilde{P_1}x_1} = (0.566, 0.633, 0.7)$$

$\widetilde{P_1}x_2 = \langle 0.9,\ 0.4,\ 0.5\rangle$ transformed based on falsity value in Example 3.1.1

$$C_{\widetilde{P_1}x_2} = (0.7, 0.633, 0.5)$$

$\widetilde{P_2}x_1 = \langle 0.3,\ 0.2,\ 0.5\rangle$ transformed based on falsity value in Example 3.1.1

$$C_{\widetilde{P_2}x_1} = (0.7, 0.7, 0.5)$$

$\widetilde{P_2}x_2 = \langle 0.3,\ 0.2,\ 0.4\rangle$ transformed based on falsity value in Example 3.1.1

$$C_{\widetilde{P_2}x_2} = (0.733, 0.7, 0.4)$$

$\tilde{Q}_{x_1} = \langle x_1, 0.4, 0.4, 0.1\rangle$ transformed based on falsity value in Section 3.1

$C_{\tilde{Q}x_1} = \langle 0.6,\ 0.8,\ 0.1\rangle$(second group)

$\tilde{Q}_{x_2} = \langle x_2, 0.6,\ 0.2,\ 0.3\rangle$transformed based on truth falsity in Section 3.1

$C_{\tilde{Q}x_2} = \langle 0.666,\ 0.6,\ 0.3\rangle$(second group)

$$S_{NSC}(\widetilde{P_1}, \tilde{Q}) = 0.67592$$

$$S_{NSC}(\widetilde{P_2}, \tilde{Q}) = 0.80927$$

Therefore, the unknown patternQ,represented by a single valued neutrosophic set based on truth value is classified into pattern P_2.

Example5.3 : Let us consider two patterns P_1 and P_2 of example 4.8, represented by single valued neutrosophic sets $\widetilde{P_1}$, $\widetilde{P_2}$, in $X = \{x_1, x_2\}$ respectively, where

$$\widetilde{P_1} = \{\langle x_1, 0.2, 0.5, 0.7\rangle, \langle x_2, 0.9, 0.4, 0.5\rangle\}$$

and

$$\widetilde{P_2} = \{\langle x_1, 0.3, 0.2, 0.5\rangle, \langle x_2, 0.3, 0.2, 0.4\rangle\}.$$

We want to classify an unknown pattern represented by the single valued neutrosophic set $\tilde{Q}$ in $X = \{x_1, x_2\}$ into one of the patterns $\widetilde{P_1}$, $\widetilde{P_2}$,where

$$\tilde{Q} = \{\langle x_1, 0.4, 0.4, 0.1\rangle, \langle x_2, 0.6, 0.2, 0.3\rangle\}.$$

We make the classification using the measure in Definition 5.1, namely

$$S_{NSC}(A, B) = \sum_{i=1}^{n} \left(w_i x S\left(C_{A(xi)}, C_{B(xi)}\right)\right).$$

Also we find the $C_{A(xi)}$, $C_{B(xi)}$ centers according to the indeterminacy values.

Let w_i be the weight of elementx_i, $w_i = \frac{1}{2}$; $1 \le i \le 2$.

$\widetilde{P_1}x_1 = \langle 0.2,\ 0.5,\ 0.7\rangle$ transformed based on falsity value in Example 3.2.1

$$C_{\widetilde{P_1}x_1} = (0.766, 0.633, 0.7)$$

$\widetilde{P_1}x_2 = \langle 0.9,\ 0.4,\ 0.5\rangle$ transformed based on falsity value in Example 3.2.1

$$C_{\widetilde{P_1}x_2} = (0.766, 0.633, 0.5)$$

$\widetilde{P_2}x_1 = \langle 0.3,\ 0.2,\ 0.5\rangle$ transformed based on falsity value in Example 3.2.1

$$C_{\widetilde{P_2}x_1} = (0.633, 0.9, 0.5)$$

$\widetilde{P_2}x_2 = \langle 0.3,\ 0.2,\ 0.4\rangle$ transformed based on falsity value in Example 3.2.1

$$C_{\widetilde{P_2}x_2} = (0.666, 0.7, 0.4)$$

$\tilde{Q}_{x_1} = \langle x_1, 0.4, 0.4, 0.1\rangle$ transformed based on falsity value in Section 3.2

$C_{\tilde{Q}x_1} = \langle 0.6,\ 0.8,\ 0.1\rangle$(second group)

$\tilde{Q}_{x_2} = \langle x_2, 0.6,\ 0.2,\ 0.3\rangle$transformed based on truth falsity in Section 3.2

$C_{\tilde{Q}x_2} = \langle 0.7,\ 0.666,\ 0.3\rangle$ (first group)

$$S_{NSC}(\widetilde{P_1}, \tilde{Q}) = 0.67592$$

$$S_{NSC}(\widetilde{P_2}, \tilde{Q}) = 0.80927$$

Therefore, the unknown patternQ, represented by a single valued neutrosophic set based on indeterminacy value is classified into pattern P_2.

Example5.4: Let us consider in example 4.8, two patterns P_1 and P_2 represented by single valued neutrosophic sets $\widetilde{P_1}$, $\widetilde{P_2}$ in $X = \{x_1, x_2\}$ respectively ,where

$$\widetilde{P_1} = \{\langle x_1, 0.2, 0.5, 0.7\rangle, \langle x_2, 0.9, 0.4, 0.5\rangle\}$$

and

$$\widetilde{P_2} = \{\langle x_1, 0.3, 0.2, 0.5\rangle, \langle x_2, 0.3, 0.2, 0.4\rangle\}.$$

We want to classify an unknown pattern represented by single valued neutrosophic set $\tilde{Q}$ in$x = \{x_1\ , x_2\}$ into one of the patterns $\widetilde{P_1}$, $\widetilde{P_2}$, where

$$\tilde{Q} = \{\langle x_1, 0.4, 0.4, 0.1\rangle, \langle x_2, 0.6, 0.2, 0.3\rangle\}.$$

We make the classification using the measure in Definition 5.1, namely

$$S_{NSC}(A, B\ ) = \sum_{i=1}^{n} \left(w_i x S\left(C_{A(xi)}\ , C_{B(xi)}\right)\right).$$

Also we find the $C_{A(xi)}$, $C_{B(xi)}$ centers according to the falsity values.

Let w_i be the weight of elementx_i, $w_i = \frac{1}{2}$; $1 \le i \le 2$.

$\widetilde{P_1}x_1 = \langle 0.2,\ 0.5,\ 0.7\rangle$transformed based on falsity value in Example 3.3.1

$$C_{\widetilde{P_1}x_1} = (0.766, 0.7, 0.7)$$

$\widetilde{P_1}x_2 = \langle 0.9,\ 0.4,\ 0.5\rangle$ transformed based on falsity value in Example 3.3.1

$$C_{\widetilde{P_1}x_2} = (0.766, 0.7, 0.5)$$

$\widetilde{P_2}x_1 = \langle 0.3,\ 0.2,\ 0.5\rangle$ transformed based on falsity value in Example 3.3.1

$$C_{\widetilde{P_2}x_1} = (0.633, 0.7, 0.5)$$

$\widetilde{P_2}x_2 = \langle 0.3,\ 0.2,\ 0.4\rangle$ transformed based on falsity value in Example 3.3.1

$$C_{\widetilde{P_2}x_2} = (0.666, 0.733, 0.4)$$

$\tilde{Q}_{x_1} = \langle x_1, 0.4, 0.4, 0.1\rangle$ transformed based on falsity value in Section 3.3

$C_{\tilde{Q}x_1} = \langle 0.6,\ 0.6,\ 0.1\rangle$(first group)

$\tilde{Q}_{x_2} = \langle x_2, 0.6,\ 0.2,\ 0.3\rangle$transformed based on truthfalsity in Section 3.3

$C_{\tilde{Q}x_2} = \langle 0.7,\ 0.666,\ 0.3\rangle$ (third group)

$$S_{NSC}(\widetilde{P_1}, \tilde{Q}) = 0.7091$$

$$S_{NSC}(\widetilde{P_2}, \tilde{Q}) = 0.8148$$

Therefore, the unknown pattern Q, represented by a single valued neutrosophic set based on falsity value is classified into pattern P_2.

In Example 5.2, Example 5.3 and Example 5.4, all measures according to truth, indeterminancy and falsity values give the same exact result.

Conclusion

In this study, we propose methods to transform between single valued neutrosophic numbers based on centroid points. We also propose a new method to measure the degree of similarity based on falsity values between single valued neutrosophic sets. Then we prove some properties of new similarity measure based on falsity value between single valued neutrosophic sets. When we take this measure with respect to truth or indeterminancy we show that it does not satisfy one of the conditions of similarity measure. We also apply the proposed new similarity measures based on falsity value between single valued neutrosophic sets to deal with pattern recognition problems.

References

[1] K. Atanassov,Intuitionistic fuzzy sets, Fuzzy Sets Syst. (1986) 20:87–96

[2] L. A. Zadeh, Fuzzy sets, Inf Control, (1965), 8:338–353

[3] F. Smarandache, A Unifying Field in logics, Neutrosophy: NeutrosophicProbability , Set and Logic, American Research Press (1998)

[4] H. Wang, F. Smarandache, Y. Q. Zhang, Sunderraman R, Single valued neutrosophic sets. Multispace Multistructure (2010) 4:410–413.

[5] J. Ye, Improved correlation coefficients of single valued neutrosophic sets and interval neutrosophic sets for multiple attribute decision making,J Intell Fuzzy Syst (2014) 27:24532462.

[6] S. Broumi, F. Smarandache, M. Talea , A. Bakali, An introduction to bipolar single valued neutrosophic graph theory. Appl Mech Mater (2016) 841:184–191.

[7] S. Broumi, F. Smarandache, M. Talea , A. Bakali, On bipolar single valued neutrosophic graphs. J New Theory (2016) 11:84–102.

[8] S. Broumi, F. Smarandache, M. Talea , A. Bakali, Single valued neutrosophic graphs. J New Theory (2016) 10:86–101.

[9]J. Ye, Single-valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine. Soft Comput 1–9. (2015) doi:10.1007/ s00500-015-1818-y.

[10]I. Deli, Y. A. Subas, Multiple criteria decision making method on single valued bipolar neutrosophic set based on correlation coefficient similarity measure. In: International conference on mathematics and mathematics education (ICMME2016), (2016), Frat University, May 12–14, Elazg, Turkey

[11] G. Beliakov, M.Pagola, T. Wilkin,Vector valued similarity measures for Atanassov's intuitionistic fuzzy sets. Inf.Sci(2014)

[12]L.Baccour, A. M. Alini,R. I. John,similarity measures for intuitionistic fuzzy set: State of the art, J. Intell, Fuzzy syst. 24(1) (2013) 37-49

[13] S. M. Chen, C.H. Chang, A novel similarity measure between Atanassov's intuitionistic fuzzy sets based on the transformation techniques with applications to patternrecognition, Inf. Sci. 291 (2015) 96 - 114

[14] S. M. Chen, C. H. Chang,T. C. Lan, A novel similarity measure between intuitionistic fuzzy sets based on the centroid points of transformed fuzzy numbers with applications to pattern recognition, Inf.Sci. 343-344 (2016) 15-40

[15] S.H. Ceng, S.M. Chen, T. C. Lan, A new similarity measure between intuitionistic fuzzy set for pattorn recognition based on the centroid points of transformed fuzzy number , in Proceedings of 2015 IEEE International conference on Systems man on Cybernetics , Hong Kong , (2015) pp 2244-2249.

[16] S. Broumi, F. Smarandache, Several similarity measures of NeutrosophicSets ,Neutrosophic Sets And Systems , 1(2013) 54-62

[17] P.Majumdar, S. K. Samanta, On similarity and entropy of neutrosophic sets, J. Intell ,Fuzzy Systems,26(2014) 1245-1252

[18] Y. Jun, similarity measures between interval neutrosophic sets and their applications in multicriteria decision making , Journal of intelligent and Fuzzy Systems , 2013 DOI:10.3233/IFS-120727

[19] A. A. Salama, S. A. AL-Blowi, Correlation of Neutrosophic data , International Refereed Journal of Engineering and Science (IRJES) ISSN , Volume 1, Issue 2, (2012) 39-43

[20] Y. Jun, Multicriteria decision-making using the correlation coefficient under single - valued neutrosophic environment International of Journal of General System 2013, 42 (4) 386-394

[21] M. Sahin, I. Deli, I, and V. Ulucay, Jaccard vector similarity measure of bipolar neutrosophic set based on multi-criteria decision making. In: International conference on natural science and engineering (ICNASE'16),(2016) March 19–20, Kilis

[22] Ulucay,V., Deli, I, andM. Sahin Similarity measure of bipolar neutrosophic sets and their application to multiple criteria decision making, Neural Comput & Applic, DOI 10. 1007/S00521-016-2479-1 (2016)1-10

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Multi-Criteria Assignment Techniques in Multi-Dimensional Neutrosophic Soft Set Theory

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Abstract: In this paper, we have introduced a new concept of multi-dimensional neutrosophic soft sets together with various operations, properties and theorems on them. Then we have proposed an algorithm named $2-DNS$ based on our proposed two-dimensional neutrosophic soft set for solving neutrosophic multi-criteria assignment problems with multiple decision makers. At last, we have applied the $2-DNS$ Algorithm for solving neutrosophic multi-criteria assignment problem in medical science to evaluate the effectiveness of different modalities of treatment of a disease.

Keywords: Assignment, Neutrosophic Multi-Criteria, Multi-Dimensional Neutrosophic Soft Set, $2-DNS$ Algorithm, Application.

1 Introduction

Most of the recent mathematical methods meant for formal modeling,reasoning and computing are crisp, accurate and deterministic in nature. But in ground reality, crisp data is not always the part and parcel of the problems encountered in different fields like economics, engineering, social science, medical science, environment etc. As a consequence various theories viz. theory of probability, theory of fuzzy sets introduced by Zadeh [1], theory of intuitionistic fuzzy sets by Atanassov[2], theory of vague sets by Gau[3], theory of interval mathematics by Gorzalczany[4], theory of rough sets by Pawlak[5] have been evolved in process. But difficulties present in all these theories have been shown by Molodtsov [6]. The cause of these problems is possibly related to the inadequacy of the parametrization tool of the theories. As a result Molodtsov proposed the concept of soft theory as a new mathematical tool for solving the uncertainties which is free from the above difficulties. Maji et al. [7, 8] have further done various research works on soft set theory. For presence of vagueness Maji et al.[9, 10] have introduced the concept of Fuzzy Soft Set. Then Mitra Basu et al. [14] proposed the mean potentiality approach to get a balanced solution of a fuzzy soft set based decision making problem.

But the intuitionistic fuzzy sets can only handle the incomplete information considering both the truth-membership (or simply membership) and falsity-membership (or non-membership) values. It does not handle the indeterminate and inconsistent information which exists in belief system. Smarandache [13] introduced the concept of **neutrosophic set(NS)** which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. He showed that NS is a generalization of the classical sets, conventional fuzzy sets, Intuitionistic Fuzzy Sets (IFS) and Interval Valued Fuzzy Sets (IVFS). Then considering the fact that the parameters or criteria (which are words or sentences) are mostly neutrosophic set, Maji [11, 12] has combined the concept of soft set and neutrosophic set to make the mathematical model **neutrosophic soft set** and also given an algorithm to solve a decision making problem. But till now there does not exist any method for solving neutrosophic soft set based assignment problem.

In several real life situations we are encountered with a type of problem which includes in assigning men to offices, jobs to machines, classes in a school to rooms, drivers to trucks, delivery trucks to different routs or problems to different research teams etc in which the assignees depend on some criteria which posses varying degree of efficiency, called cost or effectiveness. The basic assumption of this type of problem is that one person can perform one job at a time. An assignment plan is optimal if it is able to optimize all criteria. Now if such problem contains only one criterion then it can be solved by well known Hungarian method introduced by Kuhn[15]. In case of such problems containing more than one criterion, i.e., for multi-criteria assignment problems De et al [16] have proposed a solution methodology. Kar et al[17] have proposed two different methods for solving a neutrosophic multi-criteria assignment problem.

Till date these all research work have concentrated on multiple criteria assignment problems containing only one decision maker, i.e., all the criteria matrices are determined or observed by only one decision maker. But there may be such type of multiple criteria assignment problems in which the criteria be neutrosophic in nature and the degree of efficiency of the criteria are determined by more than one decision makers according to their own opinions. There does not exist any procedure to solve neutrosophic multi-criteria assignment problem with multiple decision makers or in other words there is a demand to come a methodology to solve multi-criteria assignment problems in the parlance of neutrosophic soft set theory.

In this paper we have first introduced the concept of neutrosophic multi-criteria assignment problem(NMCAP) with multiple decision makers. Then we have proposed the new concept of multi-dimensional neutrosophic soft sets along with few operations, properties and theorems on them. Moreover an algorithm named $2-DNS$ has been developed based on two-dimensional neutrosophic soft set for solving NMCAP with more than one decision maker. At last we have applied the $2-DNS$ Algorithm for solving neutrosophic multi-criteria assignment problem in medical science to evaluate the effectiveness of different modalities of treatment of a disease.

2 Preliminaries

2.1 Definition: Soft Set [6]

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the set of all subsets of U. Let $A \subseteq E$. Then a pair (F, A) is called a **soft set** over U, where F is a mapping given by, $F: A \rightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U.

2.2 Definition: NOT Set of a Set of Parameters [9]

Let $E = \{e_1, e_2, e_3, ..., e_n\}$ be a set of parameters.

The NOT set of E denoted by $|E$ is defined by

$$|E = \{\ |e_1,\ |e_2,\ |e_3, ...,\ |e_n\}, where\ |e_i = note_i, \forall i.$$

The operator not of an object, say k, is denoted by $|k$ and is defined as the negation of the object; e.g., let we have the object k = beautiful, then $|k$ i.e., not k means k is not beautiful.

2.3 Definition: Neutrosophic Set [13]

A **neutrosophic set** A on the universe of discourse X is defined as

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\},$$

$$where\ T, I, F: X \rightarrow]^-0, 1^+[$$

and $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$; T, I, F are called neutrosophic components.

"Neutrosophic" etymologically comes from "neutro-sophy" (French *neutre* < Latin *neuter*, neutral and Greek *sophia*, skill/wisdom) which means knowledge of neutral thought.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. The non-standard finite numbers $1^+ = 1 + \delta$, where 1 is the standard part and δ is the non-standard part and $^-0 = 0\delta$, where 0 is its standard part and δ is non-standard part. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$.

Any element neutrosophically belongs to any set, due to the percentages of truth/indeterminacy/falsity involved, which varies between 0 and 1 or even less than 0 or greater than 1.

Thus $x(0.5, 0.2, 0.3)$ belongs to A (which means, with a probability of 50 percent x is in A, with a probability of 30 percent x is not in A and the rest is undecidable); or $y(0,0,1)$ belongs to A (which normally means y is not for sure in A); or $z(0,1,0)$ belongs to A (which means one does know absolutely nothing about z 's affiliation with A); here $0.5+0.2+0.3=1$; thus A **is a NS and an IFS too**.

The subsets representing the appurtenance, indeterminacy and falsity may overlap, say the element $z(0.30, 0.51, 0.28)$ and in this case $0.30+0.51+0.28>1$; then **B is a NS but is not**

an IFS; we can call it paraconsistent set (from paraconsistent logic, which deals with paraconsistent information).

Or, another example, say the element $z(0.1, 0.3, 0.4)$ belongs to the set C, and here $0.1+0.3+0.4<1$; then B **is a NS but is not an IFS; we can call it intuitionistic set** (from intuitionistic logic, which deals with incomplete information).

Remarkably, in a NS one can have elements which have paraconsistent information (sum of components >1), or incomplete information (sum of components <1), or consistent information (in the case when the sum of components $=1$).

2.4 Definition: Complement of a Neutrosophic Set [18]

The complement of a neutrosophic set S is denoted by $c(S)$ and is defined by $T_{c(S)}(x)=F_S(x), I_{c(S)}(x)=1-I_S(x), F_{c(S)}(x)=T_S(x) \forall x \in X$

2.5 Definition: Neutrosophic Soft Set [12]

Let U be an initial universe set and E be a set of parameters. Consider $A \subseteq E$. Let $P(U)$ denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the **neutrosophic soft set** over U, where F is a mapping given by $F: A \to P(U)$.

2.6 Traditional Assignment Problems [15]

Sometimes we are faced with a type of problem which consists in assigning men to offices, jobs to machines, classes in a school to rooms, drivers to trucks, delivery trucks to different routs or problems to different research teams etc in which the assignees posses varying degree of efficiency, called cost or effectiveness. The basic assumption of this type of problem is that one person can perform one job at a time with respect to one criterion. An assignment plan is optimal if it optimizes the total effectiveness of performing all the jobs.

Example 2.1

Let us consider the assignment problem represented by the following cost matrix (Table-1) in which the elements represent the cost in lacs required by a machine to perform the corresponding job. The problem is to allocate the jobs to the machines so as to minimize the total cost.

Table-1: Cost Matrix

JOBS \ MACHINES	M_1	M_2	M_3	M_4
A	7	25	16	10
B	12	27	3	25
C	37	18	17	14
D	18	25	23	9

3 Neutrosophic Multi-Criteria Assignment Problems With Multiple Decision Makers

Normally in traditional assignment problems one person is assigned for one job with respect to a single criterion but in real life there are different problems in which one person can be assigned for one job with respect to more than one criteria. Such type of problems is known as **Multi-Criteria Assignment Problem(MCAP)**. Moreover in such MCAP if atleast one criterion be neutrosophic in nature then the problems will be called **Neutrosophic Multi-Criteria Assignment Problem(NMCAP)**. Now there may be such type of NMCAP in which the criteria matrices are determined by more than one decision makers according to their own opinions. In such type of problems there may be more than one matrices associated with a single criterion as the criteria are determined by multiple decision makers. Now we will discuss these new type of NMCAP with more than one decision makers and develop an algorithm to solve such type of problems.

3.1 General Formulation of a Neutrosophic Multi-Criteria Assignment Problem With Multiple Decision Makers

Let m jobs have to be performed by m number of machines depending on p number of criteria (each criterion is neutrosophic in nature) according to q number of decision makers. Now suppose that to perform j-th job by i-th machine it will take the degree of efficiency ξ_q^k for the k-th criterion according to the q-th decision maker. Then the k-th ($k=1,2,...., p$) criteria matrix according to q-th decision maker will be as given in Table-2.

Table-2: criteria matrix of k -th criterion for q -th decision maker

MACHINES

JOBS

	M_1	M_2	M_3	$\cdots$	M_m
J_1	ξ_{q11}^k	ξ_{q12}^k	ξ_{q13}^k	$\cdots$	ξ_{q1m}^k
J_3	ξ_{q31}^k	ξ_{q32}^k	ξ_{q33}^k	$\cdots$	ξ_{q3m}^k
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
J_m	ξ_{qm1}^k	ξ_{qm2}^k	ξ_{qm3}^k	$\cdots$	ξ_{qmm}^k

If the number of jobs and machines be equal in a criteria matrix then it is called a balanced criteria matrix otherwise it is known as unbalanced criteria matrix. Now the problem is to assign each machine with a unique job in such a way that the total degree of efficiency for an allocation will be optimized for all criteria which is illustrated in the following example.

Example 3.1

Let us consider a NMCAP represented by the following cost matrices and time matrix in which the criteria are neutrosophic in nature and the elements of the matrices are representing the degree of cost and time required by a machine to perform the corresponding job according to two decision makers Mr. X and Mr.Y.

Table-3: Cost Matrix by Mr.X

MACHINES

JOBS

	M_1	M_2	M_3
	(0.8,0.2,0.6)	(0.2,0.5,0.9)	(0.6,0.4,0.4)
	(0.2,0.6,0.8)	(0.7,0.2,0.5)	(0.6,0.3,0.5)
	(0.6,0.3,0.5)	(0.6,0.2,0.7)	(0.6,0.1,0.5)

Table-4: Cost Matrix by Mr.Y

MACHINES

JOBS

	M_1	M_2	M_3
J_1	(0.7,0.4,0.3)	(0.2,0.5,0.9)	(0.5,0.4,0.6)
J_2	(0.3,0.6,0.8)	(0.7,0.2,0.4)	(0.6,0.4,0.3)
J_3	(0.5,0.3,0.6)	(0.6,0.3,0.5)	(0.5,0.2,0.7)

Table-5:Time Matrix by Mr.X and Mr.Y

MACHINES

JOBS

	M_1	M_2	M_3
	(0.3,0.5,0.8)	(0.7,0.2,0.4)	(0.5,0.2,0.6)
J_2	(0.8,0.3,0.3)	(0.2,0.5,0.9)	(0.5,0.3,0.7)
J_3	(0.5,0.3,0.6)	(0.5,0.4,0.5)	(0.4,0.3,0.7)

The problem is to allocate the jobs J_1, J_2, J_3 to the machines M_1, M_2, M_3 so as to minimize the total cost and time collectively and simultaneously.

4 The Concept of Multi-Dimensional Neutrosophic Soft Set

4.1 Definition: Multi-Dimensional Neutrosophic Soft Set

Let $U_1, U_2, ..., U_n$ be n non-null finite sets of n different type of objects such that, $U_1 = \{O_1, O_2, ..., O_{m1}\}, U_2 = \{O_1^{'}, O_2^{'}, ..., O_{m2}^{'}\},.$ $.........., U_n = \{O_1^{(n-1)'}, O_2^{(n-1)'}, ..., O_{mn}^{(n-1)'}\};$ where $m1, m2, ..., mn$ respectively be the cardinalities of $U_1, U_2, ..., U_n$ and let $U = U_1 \times U_2 \times .... \times U_n$. Now let E be the set of parameters clarifying all types of objects $O_{i1}, O_{i2}^{'}, ..., O_{in}^{(n-1)'};$ $i1 = 1,2,..., m1; i2 = 1,2,..., m2; ....; in = 1,2,..., mn$ and each parameter is a neutrosophic word or neutrosophic sentence involving neutrosophic words and $A \subseteq E$. Suppose that N^U denotes all neutrosophic sets of U . Now a mapping F is defined from the parameter set A to N^U , i.e., $F : A \to \mathrm{N}^U$, then the algebraic structure (F, A) is said to be a n **-Dimensional neutrosophic soft set** over U .

Now n may be finite or, infinite. If $n = 1$ then (F, A) will be the conventional neutrosophic soft set, if $n = 2$ then (F, A) is said to be a two-dimensional neutrosophic soft set, if $n = 3$ then

(F, A) is said to be a three-dimensional neutrosophic soft set and so on.

4.2 The Features of Multi-Dimensional Neutrosophic Soft Set Compared to Neutrosophic Soft Set

Neutrosophic soft set is just a special type of multi-dimensional neutrosophic soft set where the dimension i.e., the number of the set of objects is one.

A neutrosophic soft set indicates that how a single set of objects is involved with a single set of parameters (or, criteria) where as a n-dimensional neutrosophic soft set(n may be any positive integer) reveals the involvement of n number of sets of different types of objects with a single set of parameters(or, criteria). **So from the perspective of application, multi-dimensional neutrosophic soft set has more vast scope than the conventional neutrosophic soft set.** Now we will discuss the example, operations and properties of two-dimensional neutrosophic soft set and for the higher dimensional neutrosophic soft set they can also be established in the identical manner.

Example 4.1: Let U_1 be the set of three jobs, say, $U_1 = \{J_1, J_2, J_3\}$ and let U_2 be the set of four machines, say, $U_2 = \{M_1, M_2, M_3, M_4\}$. Now let

$$E = \{ \text{cost requirement}, \text{time requirement}, \text{troublesome due to transportation} \} = \{e_1, e_2, e_3\}(\text{say}).$$

Let $A = \{e_1, e_2\}$

Now let $U = U_1 \times U_2$ and $F : A \to \mathrm{N}^U$, s.t.,

$$F(\text{cost requirement}) = \{(J_1, M_1)/(.8,0.3,0.4), (J_1, M_2)/(.3,.2,.8), (J_1, M_3)/(.5,.4,.6), (J_1, M_4)/(.7,.2,.3), (J_2, M_1)/(.2,.3,.9), (J_2, M_2)/(.7,.3,.4)(J_2, M_3)/(.5,.5,.6), (J_2, M_4)/(.3,.2,.8), (J_3, M_1)/(.6,.4,.6), (J_3, M_2)/(.4,.2,.6), (J_3, M_3)/(.3,.4,.8), (J_3, M_4)/(.7,.2,.5)\}$$

and

$$F(\text{time requirement}) = \{(J_1, M_1)/(.2,.3,.9), (J_1, M_2)/(.6,.3,.5), (J_1, M_3)/(.5,.3,.7), (J_1, M_4)/(.4,.5,.8), (J_2, M_1)/(.7,.2,.5), (J_2, M_2)/(.2,.3,.9), (J_2, M_3)/(.6,.3,.5), (J_2, M_4)/(.6,.2,.7), (J_3, M_1)/(.4,.3,.7), (J_3, M_2)/(.5,.6,.7), (J_3, M_3)/(.6,.3,.5), (J_3, M_4)/(.3,.4,.8)\}$$

Now the two-dimensional neutrosophic soft set (F, A) describing the requirements for the objects is given by,

$$(F, A) = \{\text{cost requirement} = \{(J_1, M_1)/(.8,0.3,0.4), (J_1, M_2)/(.3,.2,.8), (J_1, M_3)/(.5,.4,.6), (J_1, M_4)/(.7,.2,.3), (J_2, M_1)/(.2,.3,.9), (J_2, M_2)/(.7,.3,.4) (J_2, M_3)/(.5,.5,.6), (J_2, M_4)/(.3,.2,.8), (J_3, M_1)/(.6,.4,.6), (J_3, M_2)/(.4,.2,.6), (J_3, M_3)/(.3,.4,.8), (J_3, M_4)/(.7,.2,.5)\},$$

$$\text{time requirement} = \{(J_1, M_1)/(.2,.3,.9), (J_1, M_2)/(.6,.3,.5), (J_1, M_3)/(.5,.3,.7), (J_1, M_4)/(.4,.5,.8), (J_2, M_1)/(.7,.2,.5), (J_2, M_2)/(.2,.3,.9), (J_2, M_3)/(.6,.3,.5), (J_2, M_4)/(.6,.2,.7), (J_3, M_1)/(.4,.3,.7), (J_3, M_2)/(.5,.6,.7), (J_3, M_3)/(.6,.3,.5), (J_3, M_4)/(.3,.4,.8)\}$$

The Tabular Representation of the two-dimensional neutrosophic soft set (F, A) is as follows:

Table-6

Tabular Representation of (F, A)

	e_1	e_2
(J_1, M_1)	(.8,0.3,0.4)	(.2,.3,.9)
(J_1, M_2)	(.3,.2,.8)	(.6,.3,.5)
(J_1, M_3)	(.5,.4,.6)	(.5,.3,.7)
(J_1, M_4)	(.7,.2,.3)	(.4,.5,.8)
(J_2, M_1)	(.2,.3,.9)	(.7,.2,.5)
(J_2, M_2)	(.7,.3,.4)	(.2,.3,.9)
(J_2, M_3)	(.5,.5,.6)	(.6,.3,.5)
(J_2, M_4)	(.3,.2,.8)	(.6,.2,.7)
(J_3, M_1)	(.6,.4,.6)	(.4,.3,.7)
(J_3, M_2)	(.4,.2,.6)	(.5,.6,.7)
(J_3, M_3)	(.3,.4,.8)	(.6,.3,.5)
(J_3, M_4)	(.7,.2,.5)	(.3,.4,.8)

4.3 Definition: Choice Value:

According to a decision making problem the parameters of a decision maker's choice or requirement which forms a subset of the whole

parameter set of that problem are known as **choice parameters**.
Choice value of an object is the sum of the true-membership values of that object corresponding to all the choice parameters associated with a decision making problem.

4.4 Definition: Rejection Value:
Rejection value of an object is the sum of the falsity-membership values of that object corresponding to all the choice parameters associated with a decision making problem.

4.5 Definition: Confusion Value:
Confusion value of an object is the sum of the indeterminacy-membership values of that object corresponding to all the choice parameters associated with a decision making problem.

4.6 Definition: Null Two-dimensional Neutrosophic Soft Set:
Let $U_1 \times U_2$ be the initial universe set, E be the universe set of parameters and $A \subset E$. Then a two-dimensional neutrosophic soft set (F,A) is said to be a **null two-dimensional neutrosophic soft set (** ϕ_A) with respect to the parameter set A if for each $e \in A$

$$F(e) = \{(O_i, O_j^{'})/0.0\} \forall (O_i, O_j^{'}) \in U_1 \times U_2\}$$

4.7 Definition: Universal Two-dimensional Neutrosophic Soft Set:
Let $U_1 \times U_2$ be the initial universe set, E be the universe set of parameters and $A \subset E$. Then a two-dimensional neutrosophic soft set (F,A) is said to be a **universal two-dimensional neutrosophic soft set (**U_A) with respect to the parameter set A if for each $e \in A$

$$F(e) = \{(O_i, O_j^{'})/1.0\} \forall (O_i, O_j^{'}) \in U_1 \times U_2\}$$

4.8 Definition: Complement of a Two-dimensional Neutrosophic Soft Set
The **complement** of a two-dimensional neutrosophic soft set (F,A) over the universe U where $U = U_1 \times U_2; U_1 = \{O_1, O_2, ..., O_i\}$, $U_2 = \{O_1^{'}, O_2^{'}, ..., O_j^{'}\}; i, j \in N$ over the parameter set E (where each parameter is a neutrosophic word or neutrosophic sentence involving neutrosophic words)is denoted by $(F,A)^C$ and is defined by $(F,A)^C = (F^C, |A)$ where $F^C : |A \to \mathrm{N}^U$ where $|A$ is the NOT set of the parameter set A.

4.9 Definition: Union
The **union** of two two-dimensional neutrosophic soft sets (F,A) and (G,B) over the same universe U (where $U = U_1 \times U_2; U_1 = \{O_1, O_2, ..., O_i\}$, $U_2 = \{O_1^{'}, O_2^{'}, ..., O_j^{'}\}; i, j \in N$) and over the parameter set E (where $A, B \subseteq E$ and each parameter is a neutrosophic word or neutrosophic sentence involving neutrosophic words)is denoted by $(F,A) \tilde{\cup} (G,B)$ and is defined by $(F,A) \tilde{\cup} (G,B) = (H,C)$ where

$$H(e) = \begin{cases} F(e), & if\ e \in (A-B) \\ G(e), & if\ e \in (B-A) \\ \{(O_i, O_j^{'})/ max\{\mu_{F(e)}(O_i, O_j^{'}), \mu_{G(e)}(O_i, O_j^{'})\} \forall (O_i, O_j^{'}) \in U_1 \times U_2\}, & if\ e \in A \cap B \end{cases}$$

where $\mu_{F(e)}(O_i, O_j^{'})$ and $\mu_{G(e)}(O_i, O_j^{'})$ denote the membership values of $(O_i, O_j^{'})$ w.r.t the functions F and G respectively associated with the parameter e.

4.10 Definition: Intersection
The **intersection** of two two-dimensional neutrosophic soft sets (F,A) and (G,B) over the same universe U where $U = U_1 \times U_2; U_1 = \{O_1, O_2, ..., O_i\}$, $U_2 = \{O_1^{'}, O_2^{'}, ..., O_j^{'}\}; i, j \in N$ and over the parameter set E (where $A, B \subseteq E$ and each parameter is a neutrosophic word or neutrosophic sentence involving neutrosophic words)is denoted by $(F,A) \tilde{\cap} (G,B)$ and is defined by $(F,A) \tilde{\cap} (G,B) = (H,C)$ where

$$H(e) = \begin{cases} F(e), & if\ e \in (A-B) \\ G(e), & if\ e \in (B-A) \\ \{(O_i, O_j^{'})/ min\{\mu_{F(e)}(O_i, O_j^{'}), \mu_{G(e)}(O_i, O_j^{'})\} \forall (O_i, O_j^{'}) \in U_1 \times U_2\}, & if\ e \in A \cap B \end{cases}$$

where $\mu_{F(e)}(O_i, O_j^{'})$ and $\mu_{G(e)}(O_i, O_j^{'})$ denote the membership values of $(O_i, O_j^{'})$ w.r.t the functions F and G respectively associated with the parameter e.

4.11 Properties:

Let $(F,A),(G,B)$ and (H,C) be three two-dimensional neutrosophic soft sets over the same universe U and parameter set E. Then we have,

$(i)(F,A)\tilde{\cup}((G,B)\tilde{\cup}(H,C)) = ((F,A)\tilde{\cup}(G,B))\tilde{\cup}(H,C)$

$(ii)(F,A)\tilde{\cup}(G,B) = (G,B)\tilde{\cup}(F,A)$

$(iii)((F,A)^C)^C = (F,A)$

$(iv)(F,A)\tilde{\cup}(F,A) = (F,A)$

$(v)(F,A)\tilde{\cap}(F,A) = (F,A)$

$(vi)(F,A)\tilde{\cup}\phi_A = (F,A)$, where ϕ_A is the null two-dimensional neutrosophic soft set with respect to the parameter set A.

$(vii)(F,A)\tilde{\cap}\phi_A = \phi_A$

$(viii)(F,A)\tilde{\cup}U_A = U_A$, where U_A is the universal two-dimensional neutrosophic soft set with respect to the parameter set A.

$(ix)(F,A)\tilde{\cap}U_A = (F,A)$

4.12 De Morgan's laws in two-dimensional neutrosophic soft set theory:

The well known De Morgan's type of results hold in two-dimensional neutrosophic soft set theory for the newly defined operations: complement, union and intersection.

Theorem 4.1

Let (F,A) and (G,B) be two two-dimensional neutrosophic soft sets over a common universe U and parameter set E. Then

$i)((F,A)\tilde{\cup}(G,B))^C = (F,A)^C\tilde{\cap}(G,B)^C$

$ii)((F,A)\tilde{\cap}(G,B))^C = (F,A)^C\tilde{\cup}(G,B)^C$

5 The Methodology Based On Two-Dimensional Neutrosophic Soft Set For Solving Neutrosophic Multi-Criteria Assignment Problems With Multiple Decision Makers

In many real life problems we have to assign each object of a set of objects to another object in a different set of objects such as assigning men to offices, jobs to machines, classes in a school to rooms, drivers to trucks, delivery trucks to different routs or problems to different research teams etc. in which the assignees posses varying degree of efficiency, depending on neutrosophic multiple criteria such as cost, time etc. The basic assumption of this type of problem is that one person can perform one job at a time. To solve such type of problems our aim is to make such assignment that optimize the criteria i.e., minimize the degree of cost and time or maximizes the degree of profit. **Since in such type of problems the degrees of each criterion (or, parameter) of a set of criteria (or, parameter set) are evaluated with respect to two different types of objects, to solve such problems we can apply two-dimensional neutrosophic soft set and their various operations.**

The stepwise procedure to solve such type of problems is given below.

<u>$2-DNS$ **Algorithm**</u>**:**

Step 1: Convert each unbalanced criteria matrix to balanced by adding a fictitious job or machine with zero cost of efficiency.

Step 2: From these balanced criteria matrices construct a two-dimensional neutrosophic soft set (F_i,E_i) according to each decision maker $d_i; i=1,2,...,q;$ q be the number of decision makers.

Step 3: Combining the opinions of all the decision makers about the criteria, take the union of all these two-dimensional neutrosophic soft sets $(F_i,E_i); i=1,2,...,q$ as follows

$$(F,E) = \tilde{\cup}_{i=1}^{q}(F_i,E_i)$$

Step 4: Then compute the complement $(F,E)^C$ of the two-dimensional neutrosophic soft set (F,E) if our aim be to minimize the criteria (such as cost, time etc.).

Step 5: Construct the tabular representation of (F,E) or, $(F,E)^C$ according to maximization or minimization problem with row wise sum of parametric values which is known as choice value ($C_{(J_i,M_j)}$).

Step 6: Now for i-th job, consider the choice values $C_{(J_i,M_j)}, \forall j$ and point out the maximum choice value $C^{max}_{(J_i,M_j)}$ with a $*$.

Step 7: If $C^{max}_{(J_i,M_j)}$ holds for all distinct j's then assign M_j machine for J_i job and put a tick mark(√) beside the choice values corresponding to M_j to indicate that already M_j machine has been assigned.

Step 8: If for more than one i, $C^{max}_{(J_i,M_j)}$ hold for the same j, ie., if there is a tie for the assignment of M_j machine in more than one job then we have to consider the difference value ($V_{d_{(J_i,M_j)}}$) between the maximum and the next to maximum choice values(corresponding to those machines which are not yet assigned). If $V_{d_{(J_{i_1},M_j)}} < V_{d_{(J_{i_2},M_j)}}$ then M_j machine will be assigned for the job J_{i_2}. Now if the difference values also be same, i.e., $V_{d_{(J_{i_1},M_j)}} = V_{d_{(J_{i_2},M_j)}}$ then go to the next step.

Step 9: Now for i -th job, consider the rejection values $R_{(J_i,M_j)}, \forall j$ and point out the minimum rejection value $R^{min}_{(J_i,M_j)}$ with a $*$.

Step 10: If for more than one i, $R^{min}_{(J_i,M_j)}$ hold for the same j, consider the difference value ($V_{dR_{(J_i,M_j)}}$) between the minimum and the next to minimum rejection values(corresponding to those machines which are not yet assigned). If $V_{dR_{(J_{i_1},M_j)}} < V_{dR_{(J_{i_2},M_j)}}$ then M_j machine will be assigned for the job J_{i_2}. Now if the difference values also be same then go to the final step.

Step 11: Now for i -th job, consider the confusion values $\zeta_{(J_i,M_j)}, \forall j$ and point out the minimum confusion value $\zeta^{min}_{(J_i,M_j)}$ with a $*$.

Step 12: If for more than one i, $\zeta^{min}_{(J_i,M_j)}$ hold for the same j, consider the difference value ($V_{d\zeta_{(J_i,M_j)}}$) between the minimum and the next to minimum confusion values(corresponding to those machines which are not yet assigned). If $V_{d\zeta_{(J_{i_1},M_j)}} < V_{d\zeta_{(J_{i_2},M_j)}}$ then M_j machine will be assigned for the job J_{i_2}. Now if the difference values also be same i.e., $V_{d\zeta_{(J_{i_1},M_j)}} = V_{d\zeta_{(J_{i_2},M_j)}}$ then M_j machine may be assigned to any one of the jobs J_{i_1} or J_{i_2}.

6 Application of $2-DNS$ Algorithm For Solving Neutrosophic Multi-Criteria Assignment Problems in Medical Science

In medical science there also exist neutrosophic multi-criteria assignment problems and we may apply the $2-DNS$ Algorithm for solving those problems. Now we will discuss a such type of problem with its solution.

Problem 1: In medical science[19] there are different types of diseases and various modalities of treatments in respect to them. On the basis of different aspects of the treatment procedure (such as degree of pain relief, cost and time requirements for treatment etc.) we may measure the degree of effectiveness of the treatment for the disease. Here we consider three common diseases of oral cavity such as dental caries, gum disease and oral ulcer. Now medicinal treatment, extraction and scaling that are commonly executed, have more or less impacts on the treatment of these three diseases. According to the statistics, (true-membership value, indeterminacy-membership value, falsity-membership value) of pain relief in case of medicinal treatment on the basis of pain score for dental caries, gum disease, oral ulcer are $(0.7,0.7,0.5),(0.6,0.8,0.5)$ and $(0.9,0.5,0.2)$ respectively; by extraction the degrees of pain relief for dental caries, gum disease and oral ulcer are $(0.8,0.5,0.3)$, $(0.8,0.7,0.4)$ and $(0.5,0.7,0.6)$ respectively and by scaling the degrees of pain relief for dental caries, gum disease and oral ulcer are $(0.3,0.8,0.8)$, $(0.9,0.4,0.2)$ and $(0.6,0.7,0.5)$ respectively. Now the degree of cost to avail the medicinal treatment, extraction and scaling for both the diseases dental caries, gum disease are $(0.4,0.3,0.8),(0.3,0.2,0.7)$ and $(0.5,0.4,0.6)$ respectively and that for oral ulcer are $(0.3,0.2,0.8),(0.2,0.3,0.9)$ and $(0.4,0.4,0.7)$ respectively. Moreover the degree of time taken to the medicinal treatment, extraction and scaling for gum disease are $(0.6,0.3,0.5),(0.4,0.2,0.8),(0.5,0.5,0.6)$ and for oral ulcer are $(0.6,0.4,0.7),(0.4,0.3,0.8),(0.5,0.5,0.5)$ respectively and that of for dental caries are $(0.6,0.2,0.3),(0.5,0.4,0.7)$ and $(0.3,0.2,0.9)$ respectively. **Now the problem is to assign a treatment for each disease so that to maximize the pain relief and minimize the cost and time simultaneously as much as possible.**

<u>Solution By $2-DNS$ Algorithm</u>

The set of universe $U = U_1 \times U_2$ where

$U_1 = \{dental\,caries, gum\,disease, oral\,ulcer\}$

$= \{d_1, d_2, d_3\}$,

$U_2 = \{medicinal\,treatment, extraction, scaling\}$

$= \{t_1, t_2, t_3\}$

and the set of parameters

$E = \{$ *pain score, cost requirement, time requirement*

$= \{e_1, e_2, e_3\}$(say)

Now from the given data we have the following criteria matrices:

Table-7(Pain Score Matrix)

TREATMENTS

DISEASES	t_1	t_2	t_3
d_1	(0.5,0.2,0.6)	(0.4,0.3,0.8)	(0.2,0.6,0.9)
d_2	(0.2,0.5,0.9)	(0.6,0.3,0.5)	(0.5,0.3,0.6)
d_3	(0.2,0.5,0.9)	(0.6,0.3,0.5)	(0.5,0.3,0.6)

Table-8(Cost Matrix)

TREATMENTS

DISEASES	t_1	t_2	t_3
d_1	(0.4,0.3,0.8)	(0.3,0.2,0.7)	(0.5,0.4,0.6)
d_2	(0.4,0.2,0.7)	(0.3,0.3,0.8)	(0.5,0.4,0.6)
d_3	(0.3,0.2,0.8)	(0.2,0.3,0.9)	(0.4,0.4,0.7)

Table-9(Time Matrix)

TREATMENTS

	t_1	t_2	t_3
d_1	(0.6,0.2,0.3)	(0.5,0.4,0.7)	(0.3,0.2,0.9)
d_2	(0.6,0.3,0.5)	(0.4,0.2,0.8)	(0.5,0.5,0.6)
d_3	(0.6,0.4,0.7)	(0.4,0.3,0.8)	(0.5,0.5,0.5)

To solve this problem by $2-DNS$ algorithm at first we have to form the two-dimensional neutrosophic soft set (F,E) describing the impact of the treatments for the diseases from the given criteria matrices as:

$(F,E) = \{$*degree of pain score* $= \{(d_1,t_1)/(0.5,0.3,0.7),$
$(d_1,t_2)/(0.3,0.2,0.7),(d_1,t_3)/(0.8,0.2,0.3),$
$(d_2,t_1)/(0.5,0.2,0.6),(d_2,t_2)/(0.4,0.3,0.8),$
$(d_2,t_3)/(0.2,0.6,0.9),(d_3,t_1)/(0.3,0.2,0.8),$
$(d_3,t_2)/(0.6,0.3,0.5),(d_3,t_3)/(0.5,0.3,0.6)\},$
degree of cost requirement $= \{(d_1,t_1)/(0.4,0.3,0.8),$
$(d_1,t_2)/(0.3,0.2,0.7),(d_1,t_3)/(0.5,0.4,0.6),$
$(d_2,t_1)/(0.4,0.2,0.7),(d_2,t_2)/(0.3,0.3,0.8),$
$(d_2,t_3)/(0.5,0.4,0.6),(d_3,t_1)/(0.3,0.2,0.8),$
$(d_3,t_2)/(0.2,0.3,0.9),(d_3,t_3)/(0.4,0.4,0.7)\},$
degree of time requirement $= \{(d_1,t_1)/(0.6,0.2,0.3),$
$(d_1,t_2)/(0.5,0.4,0.7),(d_1,t_3)/(0.3,0.2,0.9),$
$(d_2,t_1)/(0.6,0.3,0.5),(d_2,t_2)/(0.4,0.2,0.8),$
$(d_2,t_3)/(0.5,0.5,0.6),(d_3,t_1)/(0.6,0.4,0.7),$
$(d_3,t_2)/(0.4,0.3,0.8),(d_3,t_3)/(0.5,0.5,0.5)\}\}$

Here,

$\rceil E = \{$ *pain relief, not requirement of cost, not requirement of time* $\} = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\},$

then

$(F,E)^c = \{$*degree of pain relief* $= \{(d_1,t_1)/(0.7,0.7,0.5),$
$(d_1,t_2)/(0.7,0.8,0.3),(d_1,t_3)/(0.3,0.8,0.8),$
$(d_2,t_1)/(0.6,0.8,0.5),(d_2,t_2)/(0.8,0.2,0.4),$
$(d_2,t_3)/(0.9,0.4,0.2),(d_3,t_1)/(0.8,0.8,0.3),$
$(d_3,t_2)/(0.5,0.7,0.6),(d_3,t_3)/(0.6,0.7,0.5)\},$
degree of not requirement of cost $= \{(d_1,t_1)/(0.8,0.7,0.4),$
$(d_1,t_2)/(0.7,0.8,0.3),(d_1,t_3)/(0.6,0.6,0.5),$
$(d_2,t_1)/(0.7,0.8,0.4),(d_2,t_2)/(0.8,0.7,0.3),$
$(d_2,t_3)/(0.6,0.6,0.5),(d_3,t_1)/(0.8,0.8,0.3),$
$(d_3,t_2)/(0.9,0.7,0.2),(d_3,t_3)/(0.7,0.6,0.4)\},$
degree of not requirement of time $= \{(d_1,t_1)/(0.3,0.8,0.6),$
$(d_1,t_2)/(0.7,0.6,0.5),(d_1,t_3)/(0.9,0.8,0.3),$
$(d_2,t_1)/(0.5,0.7,0.6),(d_2,t_2)/(0.8,0.8,0.4),$
$(d_2,t_3)/(0.6,0.5,0.5),(d_3,t_1)/(0.7,0.6,0.6),$
$(d_3,t_2)/(0.8,0.7,0.4),(d_3,t_3)/(0.5,0.5,0.5)\}\}$

Therefore the tabular representation of $(F,E)^c$ is as follows:

Table-10

Tabular Representation of $(F,E)^c$ with choice rejection and confusion values $(C_{(d_i,t_j)}, R_{(d_i,t_j)}, \zeta_{(d_i,t_j)})$

	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$C_{(d_i,t_j)}$	$R_{(d_i,t_j)}$	$\zeta_{(d_i,t_j)}$
(d_1,t_1)	(0.7,0.7,0.5),	(0.8,0.7,0.4)	(0.3,0.8,0.6)	1.8	1.5	2.2
(d_1,t_2)	(0.7,0.8,0.3)	(0.7,0.8,0.3)	(0.7,0.6,0.5)	*2.1	1.1	2.2
(d_1,t_3)	(0.3,0.8,0.8)	(0.6,0.6,0.5)	(0.9,0.8,0.3)	1.8	1.6	2.2 √
(d_2,t_1)	(0.6,0.8,0.5)	(0.7,0.8,0.4)	(0.5,0.7,0.6)	1.8	1.5	2.3
(d_2,t_2)	(0.8,0.2,0.4)	(0.8,0.7,0.3)	(0.8,0.8,0.4)	*2.4	1.1	1.7 √
(d_2,t_3)	(0.9,0.4,0.2)	(0.6,0.6,0.5)	(0.6,0.5,0.5)	2.1	1.2	1.5
(d_3,t_1)	(0.8,0.8,0.3)	(0.8,0.8,0.3)	(0.7,0.6,0.6)	*2.3 √	1.2	2.2
(d_3,t_2)	(0.5,0.7,0.6)	(0.9,0.7,0.2)	(0.8,0.7,0.4)	2.2	1.2	2.1
(d_3,t_3)	(0.6,0.7,0.5)	(0.7,0.6,0.4)	(0.5,0.5,0.5)	1.8	1.4	1.8

Now among the choice values $C_{(d_3,t_j)}; j=1,2,3$, $C_{(d_3,t_1)}$ is maximum(2.3), which implies that t_1 **treatment has to be assigned for the disease** d_3.

But for both the diseases d_1 and d_2, $C_{(d_i,t_j)}; j=1,2,3$ take the maximum value at $j=2$, i.e., for the assignment of t_2 treatment there is a tie between the diseases d_1 and d_2. We have to consider the difference value $V_{d_{(d_i,t_j)}}; i=1,2; j=2,3$ between the maximum and the next to maximum choice values(corresponding to those treatments which are not yet assigned).

Now since $V_{d_{(d_1,t_j)}} = 0.3 = V_{d_{(d_2,t_j)}}$ for $j=2,3$; we have to consider the rejection values. But for both the diseases d_1 and d_2, $R_{(d_i,t_j)}; j=1,2,3$ take the minimum value at $j=2$, therefore we have to consider their confusion values. Now since $\zeta_{(d_2,t_j)}; j=2,3$ take the minimum value (1.7) at $j=2$, t_2 **treatment has to be assigned for the disease** d_2 **and the rest treatment** t_3 **is assigned for the disease** d_1**.**

Block Diagram of 2DNS-Algorithm to Assign a Treatment for a Disease

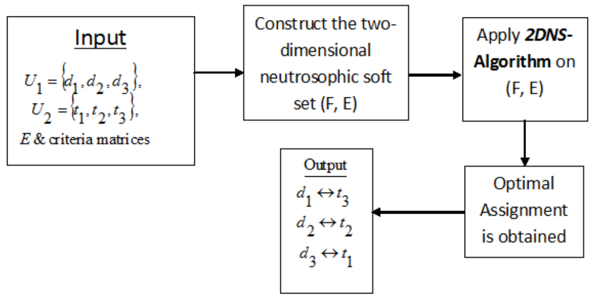

Figure 1: Block Diagram of $2DNS$ -Algorithm to Assign a Treatment for a Disease

7 Conclusion:

In this paper, we have introduced a new concept of multi-dimensional neutrosophic soft set. Using this new idea, an algorithm named $2-DNS$ has been proposed to solve neutrosophic multi-criteria assignment problems with multiple decision makers. Finally, our newly proposed $2-DNS$ algorithm has been applied to solve an assignment problem in medical science.

References

[1] L.A. Zadeh. Fuzzy Sets. Infor.And Control, 8(1965),338-353.
[2] K Atanassov. Intuitionistic fuzzy sets. Fuzzy Set.Syst., 20(1986),87-96.
[3] W. L. Gau and D. J. Buehrer. Vague Sets. IEEE Trans. System Man Cybernet., 23(2)(1993),610-614.
[4] M.B. Gorzalczany. A method of inference in approximate reasoning based on interval valued fuzzy sets. Fuzzy Set. Syst., 21 (1987),1-17.
[5] Z Pawlak. Rough Sets. Int.J.Inform.Comput.Sci., 11 (1982),341-356.
[6] D Molodtsov. Soft Set Theory-First Results. Comput.Math.Appl., 37 (1999),19-31.
[7] P. K. Maji, R. Biswas and A. R. Roy. An application of soft sets in a decision making problem. Comput. Math. Appl., 44 (2002),1077-1083.
[8] P. K. Maji, R. Biswas and A. R. Roy. Soft Set Theory, An application of soft sets in a decision making problem. Comput.Math.Appl., 45 (2003),555-562.
[9] P. K. Maji, R. Biswas and A. R. Roy. Fuzzy Soft Sets. J.Fuzzy Math., 9(3) (2001),589-602.
[10] P. K. Maji, A. R. Roy. A Fuzzy Soft Set Theoretic Approach to Decision Making Problems. J.Comput.Appl.Math., 203 (2007),412-418.
[11] P. K. Maji. Neutrosophic soft set. Annals of Fuzzy Mathematics and Informatics, 5(1)(2013), 157-168.

[12] P. K. Maji. A neutrosophic soft set approach to a decision making problem. Annals of Fuzzy Mathematics and Informatics, 3(2)(2013),313-319.
[13] F Smarandache. Neutrosophic set - a generalisation of the intuitionistic fuzzy sets. International Journal of Pure and Applied Mathematics, 24 (2005), 287-297.
[14] T.M. Basu, N.K. Mahapatra and S.K. Mondal. A Balanced Solution of a Fuzzy Soft Set Based Decision Making Problem in Medical Science. Applied Soft Computing, 12(2012),3260-3275.
[15] H.W. Kuhn. The Hungarian Method for the assignment problem. Naval Research Logistic Quarterly, 2(1955)83-97.
[16] P. K. De and Y.Bharti. An Algorithm to Solve Multi-Objective Assignment Problem Using Interactive Fuzzy Goal Programming Approach. Int.J. Contemp.Math.Sci., 6(34) (2011), 1651-1662.
[17] S.Kar, K. Basu and S. Mukherjee. Solution of Multi-Criteria Assignment Problem using Neutrosophic Set Theory. Neutrosophic Sets and Systems, 10 (2015),31-38.
[18] H Wang, F Smarandache, QY Zhang, R Sunderraman. Single valued neutrosophic sets. Multispace Multistruct., 4(2010),410-413.
[19] Carranza. Carranza's Clinical Periodontology 10th edition. Elsevier(A Divison of Reed Elsevier Pvt. Ltd.)2006.

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GRA for Multi Attribute Decision Making in Neutrosophic Cubic Set Environment

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Abstract. In this paper, multi attribute decision making problem based on grey relational analysis in neutrosophic cubic set environment is investigated. In the decision making situation, the attribute weights are considered as single valued neutrosophic sets. The neutrosophic weights are converted into crisp weights. Both positve and negative GRA coefficients, and weighted GRA coefficients are determined. Hamming distances for weighted GRA coefficients and standard (ideal) GRA coefficients are determined. The relative closeness coefficients are derived in order to rank the alternatives. The relative closeness coefficients are designed in ascending order. Finally, a numerical example is solved to demonstrate the applicability of the proposed approach.

Keywords: Grey relational coefficient, interval valued neutrosophic set, multi attribute decision making, neutrosophic set, neutrosophic cubic set, relative closeness coefficient

1 Introduction

In management section, banking sector, factory, plant multi attribute decision making (MADM) problems are to be extensively encountered. In a MADM situation, the most appropriate alternative is selecting from the set of alternatives based on highest degree of acceptance. In a decision making situation, decision maker (DM) considers the efficiency of each alternative with respect to each attribute. In crisp MADM, there are several approaches [1, 2, 3, 4, 5] in the literature. The weight of each attribute and the elements of decision matrix are presented by crisp numbers. But in real situation, DMs may prefer to use linguistic variables like 'good', 'bad', 'hot', 'cold', 'tall', etc. So, there is an uncertainty in decision making situation which can be mathematically explained by fuzzy set [6]. Zadeh [6] explained uncertainty mathematically by defining fuzzy set (FS). Bellman and Zadeh [7] studied decision making in fuzzy environment. Atanassov [8, 9] narrated uncertainty by introducing non-membership as independent component and defined intuitionistic fuzzy set (IFS). Degree of indeterminacy (hesitency) is not independent .

Later on DMs have recognized that indeterminacy plays an important role in decision making. Smarandache [10] incorporated indeterminacy as independent component and developed neutrosophic set (NS) and together with Wang et a. [11] defined single valued neutrosophic set (SVNS) which is an instance of neutrosophic set. Ye [12] proposed a weighted correlation coefficients for ranking the alternatives for multicriteria decision making (MCDM). Ye [13] established single valued neutrosophic cross entropy for MCDM problem. Sodenkamp [14] studied multiple-criteria decision analysis in neutrosophic environment. Mondal and Pramanik [15] defined neutrosophic tangent similarity measure and presented its application to MADM. Biswas et al. [16] studied cosine similarity measure based MADM with trapezoidal fuzzy neutrosophic numbers. Mondal and Pramanik [17] presented multi-criteria group decision making (MCGDM) approach for teacher recruitment in higher education. Mondal and Pramanik [18] studied neutrosophic decision making model of school choice. Liu and Wang [19] presented MADM method based on single-valued neutrosophic normalized weighted Bonferroni mean. Biswas et al. [20] presented TOPSIS method for MADM under single-valued neutrosophic environment. Chi and Liu [21] presented extended TOPSIS method for MADM on interval neutrosophic set. Broumi et al. [22] presented extended TOPSIS method for MADM based on interval neutrosophic uncertain linguistic variables. Nabdaban and Dzitac [23] presented a very short review of TOPSIS in neutrosophic environment. Pramanik et al. [24] studied hybrid vector similarity measures and their applications to MADM under neutrosophic environment. Biswas et al. [25] presented triangular fuzzy neutrosophic set information and its application to MADM. Sahin and Liu [26] studied maximizing deviation method for neutrosophic MADM with incomplete weight information. Ye [27] studied bidirectional projection method for MADM with neutrosophic numbers of the form *a* + *bI*, where *I* is characterized by indeterminacy. Biswas et al. [28] presented value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to MADM. Dey et al. [29] studied extended projection-based models for solving MADM problems with interval-valued neutrosophic information.

Deng [30, 31] studied grey relational analysis (GRA). Pramanik and Mukhopadhyaya [32] developed GRA based intuitionistic fuzzy multi criteria decision making (MCDM) approach for teacher selection in higher education. Dey et al. [33] established MCDM in intuitionistic fuzzy environment based on GRA for weaver selection in Khadi institution. Rao, and Singh [34] established modified GRA method for decision making in manufacturing situation. Wei [35] presented GRA method for intuitionistic fuzzy MCDM. Biswas et al. [36] studied GRA method for MADM under single valued neutrosophic assessment based on entropy. Dey et al. [37] presented extended GRA based neutrosophic MADM in interval uncertain linguistic setting. Pramanik and K. Mondal [38] employed GRA for interval neutrosophic MADM and presented numerical examples.

Several neutrosophic hybrid sets have been recently proposed in the literature, such as neutrosophic soft set proposed by Maji [39], single valued soft expert set proposed by Broumi and Smarandache [40], rough neutrosophic set proposed by Broumi, et al. [41], neutrosophic bipolar set proposed by Deli et al. [42], rough bipolar neutrosophic set proposed by Pramanik and Mondal [43], neutrosophic cubic set proposed by Jun et al. [44] and Ali et al. [45]. Jun et al. [44] presented the concept of neut-rosophic cubic set by extending the concept of cubic set proposed by Jun et al. [46] and introduced the notions of truth-internal (indeterminacy-internal, falsity-internal) neut-rosophic cubic sets and truth-external (indeterminacy-external, falsity-external) and investigated related properties. Ali et al. [45] presented concept of neutrosophic cubic set by extending the concept of cubic set [46] and defined internal neutrosophic cubic set (INCS) and external neutrosophic cubic set (ENCS). In their study, Ali et al.[45] also introduced an adjustable approach to neutrosophic cubic set based decision making.

GRA based MADM/ MCDM problems have been proposed for various neutrosophic hybrid environments [47, 48, 49, 50]. MADM with neutrosophic cubic set is yet to appear in the literature. It is an open area of research in neutrosophic cubic set environment.

The present paper is devoted to develop GRA method for MADM in neutrosophic cubic set environment. The attribute weights are described by single valued neutrosophic sets. Positive and negative grey relational coefficients are determined. We define ideal grey relational coefficients and relative closeness coefficients in neutrosophic cubic set environment. The ranking of alternatives is made in descending order.

The rest of the paper is designed as follows: In Section 2, some relevant definitions and properties are recalled. Section 3 presents MADM in neutrosophic cubic set environment based on GRA. In Section 4, a numerical example is solved to illustrate the proposed approach. Section 5 presents conclusions and future scope of research.

2 Preliminaries

In this section, we recall some established definitions and properties which are connected in the present article.

2.1 Definition (Fuzzy set) [6]

Let *W* be a universal set. Then a fuzzy set F over *W* can be defined by $F=\{<w, \mu_F(w)>: w \in W\}$ where $\mu_F(w): W \to [0, 1]$ is called membership function of F and $\mu_F(w)$ is the degree of membership to which $w \in F$.

2.2 Definition (Interval valued fuzzy set) [52]

Let *W* be a universal set. Then, an interval valued fuzzy set F over *W* is defined by $F = \{[F^-(w), F^+(w)]/ w : w \in W\}$, where $F^-(w)$ and $F^+(w)$ are referred to as the lower and upper degrees of membership $w \in W$ where

$0 \le F^-(w) + F^+(w) \le 1$, respectively.

2.3 Definition (Cubic set) [46]

Let W be a non-empty set. A cubic set C in W is of the form $c = \{w, F(w), \lambda(w))/ w \in W\}$ where F is an interval valued fuzzy set in *W* and λ is a fuzzy set in *W*.

2.4 Definition (Neutrosophic set (NS)) [10]

Let *W* be a space of points (objects) with generic element w in *W*. A neutrosophic set N in *W* is denoted by $N= \{< w: T_N(w), I_N(w), F_N(w)>: w \in W\}$ where T_N, I_N, F_N represent membership, indeterminacy and non-membership function respectively. T_N, I_N, F_N can be defined as follows:

$$T_N : W \to]^-0, 1^+[$$

$$I_N : W \to]^-0, 1^+[$$

$$F_A : W \to]^-0, 1^+[$$

Here, $T_N(w)$, $I_N(w)$, $F_N(w)$ are the real standard and non-standard subset of $]^-0, 1^+[$ and

$$^-0 \le T_N(w)+I_N(w)+F_N(w) \le 3^+.$$

2.5 Definition (Complement of neutrosophic set) [10]

The complement of a neutrosophic set N is denoted by N' and defined as

N′= {<w: $T_{N'}(w)$, $I_{N'}(w)$, $F_{N'}(w)$>, w ∈ W }

$T_{N'}(w) = \{1^+\} - T_N(w)$

$I_{N'}(w) = \{1^+\} - I_N(w)$

$F_{N'}(w) = \{1^+\} - F_N(w)$

2.6 Definition (Containment) [10, 20]

A neutrosophic set P is contained in the other neutrosophic set Q, P ⊆ Q, if and only if

$\inf (T_P) \le \inf (T_Q), \sup (T_P) \le \sup (T_Q)$,

$\inf (I_P) \ge \inf (I_Q), \sup (I_P) \ge \sup (I_Q)$.

$\inf (F_P) \ge \inf (F_Q), \sup (F_P) \ge \sup (F_Q)$.

2.7 Definition (Union) [10]

The union of two neutrosophic sets P and Q is a neutrosophic set R, written as R = P ∪ Q, whose truth-membership, indeterminacy-membership and falsity membership functions are related to those of P and Q by

$T_R(w) = T_P(w) + T_Q(w) - T_P(w) \times T_Q(w)$,

$I_R(w) = I_P(w) + I_Q(w) - I_P(w) \times I_Q(w)$,

$F_R(w) = F_P(w) + F_Q(w) - F_P(w) \times F_Q(w)$, for all $w \in W$.

2.8 Definition (Intersection) [10]

The intersection of two neutrosophic sets P and Q is a neutrosophic set C, written as R =P∪Q, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of P and Q by

$T_R(w) = T_P(w) \times T_Q(w)$,

$I_R(w) = I_P(w) \times I_Q(w)$,

$F_R(w) = F_P(w) \times F_Q(w)$, for all $w \in W$.

2.9 Definition (Hamming distance) [20, 53]

Let $P = \{\langle w_i : T_P(w_i), I_P(w_i), F_P(w_i)\rangle, i = 1, 2, \ldots, n\}$ and $Q = \{\langle w_i : T_P(w_i), I_Q(w_i), F_Q(w_i)\rangle, i = 1, 2, \ldots, n\}$ be any two neutrosophgic sets. Then the Hamming distance between P and Q can be defined as follows:

$$d(P,Q) = \sum_{i=1}^{n} \left(|T_P(w_i) - T_Q(w_i)| + |I_P(w_i) - I_Q(w_i)| + |F_P(w_i) - F_Q(w_i)| \right) \quad (1)$$

2.10 Definition (Normalized Hamming distance)

The normalized Hamming distance between two SVNSs, A and B can be defined as follows:

$${}_N d(P,Q) = \frac{1}{3n} \sum_{i=1}^{n} \left(|T_P(w_i) - T_Q(w_i)| + |I_P(w_i) - I_Q(w_i)| + |F_P(w_i) - F_Q(w_i)| \right) \quad (2)$$

2. 11 Definition (Interval neutrosophic set) [51]

Let *W* be a non-empty set. An interval neutrosophic set (INS) P in *W* is characterized by the truth-membership function P_T, the indeterminacy-membership function P_I and the falsity-membership function P_F. For each point w ∈ *W*, $P_T(w)$, $P_I(w)$, $P_F(w)) \subseteq [0,1]$. Here P can be presented as follows:

$$P = \{< w, [P_T^L(w), P_T^U(w)], [P_I^L(w), P_I^U(w)], [P_F^L(w), P_F^U(w)] > : w \in W\}.$$

2.12 Definition (Neutrosophic cubic set) [44, 45]

Let W be a set. A neutrosophic cubic set (NCS) in *W* is a pair (P, Λ) where $P = \{\langle w, P_T(w), P_I(w), P_F(w)\rangle / w \in W\}$ is an interval neutrosophic set in *W* and $\Lambda = \{\langle w, \lambda_T(w), \lambda_I(w), \lambda_F(w)\rangle / w \in W\}$ is a neutrosophic set in W.

3 GRA for MADM in neutrosophic cubic set environment

We consider a MADM problem with r alternatives {A_1, A_2, …, A_r} and s attributes {C_1, C_2, …, C_s}. Every attribute is not equally important to decision maker. Decision maker provides the neutrosophic weights for each attribute. Let $W = \{w_1, w_2, \ldots, w_s\}^T$ be the neutrosophic weights of the attributes.

Step 1 Construction of decision matrix

Step1.The decision matrix (see Table 1) is constructed as follows:

Table 1: Decision matrix

$$A = (a_{ij})_{r \times s} = \begin{pmatrix} & C_1 & C_2 & \ldots & C_s \\ A_1 & (A_{11}, \Lambda_{11}) & (A_{12}, \Lambda_{12}) & \ldots & (A_{1s}, \Lambda_{1s}) \\ A_2 & (A_{21}, \Lambda_{21}) & (A_{22}, \Lambda_{22}) & \ldots & (A_{2s}, \Lambda_{2s}) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ A_r & (A_{r1}, \Lambda_{r1}) & (A_{r2}, \Lambda_{r2}) & \ldots & (A_{rs}, \Lambda_{rs}) \end{pmatrix}_{r \times s}$$

Here $a_{ij} = (A_{ij}, \Lambda_{ij})$, $A_{ij} = ([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U])$, $\Lambda_{ij} = (T_{ij}, I_{ij}, F_{ij})$, a_{ij} means the rating of alternative A_i with respect to the attribute C_j. Each weight component w_j of attribute C_j has been taken as neutrosophic set and

$$w_j = (T_j, I_j, F_j), \; A_{ij} = ([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U])$$

are interval neutrosophic set and $\Lambda_{ij} = (T_{ij}, I_{ij}, F_{ij})$ is a neutrosophic set.

Step 2 Crispification of neutrosophic weight set

Let $w_j = (T_j, I_j, F_j)$ be the j – th neutrosophic weight for the attribute C_j. The equivalent crisp weight of C_j is defined as follows:

$$w_j^c = \frac{\sqrt{T_j^2 + I_j^2 + F_j^2}}{\sum_{j=1}^{n}\sqrt{T_j^2 + I_j^2 + F_j^2}} \text{ and } \sum_{j=1}^{s} w_j^c = 1 .$$

Step 3 Conversion of interval neutrosophic set into neutrosophic set decision matrix

In the decision matrix (1), each $A_{ij} = ([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U])$ is an INS. Taking mid value of each interval the decision matrix reduces to single valued neutrosophic decision matrix (See Table 2).

Table 2: Neutrosophic decision matrix

$$M = (m_{ij})_{r\times s} = \begin{pmatrix} & C_1 & C_2 & \dots & C_s \\ A_1 & (M_{11}, \Lambda_{11}) & (M_{12}, \Lambda_{12}) & \dots & (M_{1s}, \Lambda_{1s}) \\ A_2 & (M_{21}, \Lambda_{21}) & (M_{22}, \Lambda_{22}) & \dots & (M_{2s}, \Lambda_{2s}) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ A_r & (M_{r1}, \Lambda_{r1}) & (M_{r2}, \Lambda_{r2}) & \dots & (M_{rs}, \Lambda_{rs}) \end{pmatrix}_{r\times s}$$

where each $m_{ij} = (M_{ij}, \Lambda_{ij})$ and

$$M_{ij} = \left(\frac{T_{ij}^L + T_{ij}^U}{2}, \frac{I_{ij}^L + I_{ij}^U}{2}, \frac{F_{ij}^L + F_{ij}^U}{2}\right) = (T_{ij}^m, I_{ij}^m, F_{ij}^m).$$

Step 4 Some definitions of GRA method for MADM with NCS

The GRA method for MADM with NCS can be presented in the following steps:

Step 4.1 Definition:

The ideal neutrosophic estimates reliability solution (INERS) can be denoted as

$(M^+, \Lambda^+) = [(M_1^+, \Lambda_1^+), (M_2^+, \Lambda_2^+), \dots, (M_q^+, \Lambda_q^+)]$

and defined as $M_j^+ = (T_j^+, I_j^+, F_j^+)$, where $T_j^+ = \max_i T_{ij}^m$, $I_j^{m+} = \min_i I_{ij}^m$, $F_j^{m+} = \min_i F_{ij}^m$ and $\Lambda_j^+ = (T_j^+, I_j^+, F_j^+)$ where $T_j^+ = \max_i T_{ij}, I_j^+ = \min_i I_{ij}, F_j^+ = \min_i F_{ij}$ in the neutrosophic cubic decision matrix $M = (m_{ij})_{p\times q}$, i = 1,2,...,r and j = 1, 2, ..., s.

Step 4.2 Definition:

The ideal neutrosophic estimates unreliability solution (INEURS) can be denoted as

$(M^-, \Lambda^-) = [(M_1^-, \Lambda_1^-), (M_2^-, \Lambda_2^-), \dots, (M_s^-, \Lambda_s^-)]$

and defined as $M_j^- = (T_j^{m-}, I_j^{m-}, F_j^{m-})$ where $T_j^{m-} = \min_i T_{ij}^m$, $I_j^{m-} = \max_i I_{ij}^m$, $F_j^{m-} = \max_i F_{ij}^m$ and $\Lambda_j^- = (T_j^-, I_j^-, F_j^-)$ where $T_j^- = \min_i T_{ij}, I_j^- = \max_i I_{ij}, F_j^- = \max_i F_{ij}$ in the neutrosophic cubic decision matrix $M = (m_{ij})_{r\times s}$, i = 1,2,...,r and j = 1, 2, ...,s.

Step 4.3 Definition:

The grey relational coefficients of each alternative from INERS can be defined as:

$$(\eta_{ij}^+, \xi_{ij}^+) = \left(\frac{\min_i \min_j \delta_{ij}^+ + \lambda \max_i \max_j \delta_{ij}^+}{\delta_{ij}^+ + \lambda \max_i \max_j \delta_{ij}^+}, \frac{\min_i \min_j \Omega_{ij}^+ + \lambda \max_i \max_j \Omega_{ij}^+}{\Omega_{ij}^+ + \lambda \max_i \max_j \Omega_{ij}^+}\right)$$

Here,

$$\delta_{ij}^+ = d(M_j^+, M_{ij}) = \sum_{i=1}^{r}\left(\left|T_j^{m+} - T_{ij}^m\right| + \left|I_j^{m+} - I_{ij}^m\right| + \left|F_j^{m+} - F_{ij}^m\right|\right)$$

and $\Omega_{ij}^+ = d(\Lambda_j^+, \Lambda_{ij}) = \sum_{i=1}^{r}\left(\left|T_j^+ - T_{ij}\right| + \left|I_j^+ - I_{ij}\right| + \left|F_j^+ - F_{ij}\right|\right)$,

i = 1, 2 ,..., r and j = 1, 2, ..., s, $\lambda \in [0,1]$.

We call $(\eta_{ij}^+, \xi_{ij}^+)$ as positive grey relational coefficient.

Step 4.4 Definition:

The grey relational coefficient of each alternative from INEURS can be defined as:

$$(\eta_{ij}^-, \xi_{ij}^-) = \left(\frac{\min_i \min_j \delta_{ij}^- + \lambda \max_i \max_j \delta_{ij}^-}{\delta_{ij}^- + \lambda \max_i \max_j \delta_{ij}^-}, \frac{\min_i \min_j \Omega_{ij}^- + \lambda \max_i \max_j \Omega_{ij}^-}{\Omega_{ij}^- + \lambda \max_i \max_j \Omega_{ij}^-}\right)$$

Here,

$$\delta_{ij}^- = d(M_j^-, M_{ij}) = \sum_{i=1}^{r}\left(\left|T_j^{m-} - T_{ij}^m\right| + \left|I_j^{m-} - I_{ij}^m\right| + \left|F_j^{m-} - F_{ij}^m\right|\right)$$

and:

$\Omega_{ij}^- = d(\Lambda_j^-, \Lambda_{ij}) = \sum_{i=1}^{r}\left(\left|T_j^- - T_{ij}\right| + \left|I_j^- - I_{ij}\right| + \left|F_j^- - F_{ij}\right|\right)$, i = 1, 2,..., r and j = 1, 2, ..., s, $\lambda \in [0,1]$.

We call $(\eta_{ij}^-, \xi_{ij}^-)$ as negative grey relational coefficient.

λ is called distinguishable coefficient or identification coefficient and it is used to reflect the range of comparison environment that controls the level of differences of the grey relational coefficient. $\lambda = 0$ indicates comparison environment disappears and $\lambda = 1$ indicates comparison environment is unaltered. Generally, $\lambda = 0.5$ is assumed for decision making.

Step 4.5 Calculation of weighted grey relational coefficients for MADM with NCS

We can construct two $r\times s$ order matrices namely $M_{GR}^{+}=\left(\eta_{ij}^{+},\xi_{ij}^{+}\right)_{r\times s}$ and $M_{GR}^{-}=\left(\eta_{ij}^{-},\xi_{ij}^{-}\right)_{r\times s}$. The crisp weight is to be multiplied with the corresponding elements of M_{GR}^{+} and M_{GR}^{-} to obtain weighted matrices ${}_{w}M_{GR}^{+}$ and ${}_{w}M_{GR}^{-}$ and defined as:

$${}_{w}M_{GR}^{+}=\left(w_j^c\eta_{ij}^{+},w_j^c\xi_{ij}^{+}\right)_{r\times s}=\left(\tilde{\eta}_{ij}^{+},\tilde{\xi}_{ij}^{+}\right)_{r\times s}$$

and $${}_{w}M_{GR}^{-}=\left(w_j^c\eta_{ij}^{-},w_j^c\xi_{ij}^{-}\right)_{r\times s}=\left(\tilde{\eta}_{ij}^{-},\tilde{\xi}_{ij}^{-}\right)_{r\times s}$$

Step 4.6

From the definition of grey relational coefficient, it is clear that grey relational coefficients of both types must be less than equal to one. This claim is going to be proved in the following theorems.

Theorem 1

The positive grey relational coefficient is less than unity i.e. $\eta_{ij}^{+}\le 1$, and $\xi_{ij}^{+}\le 1$.

Proof:

From the definition

$$\eta_{ij}^{+}=\frac{\min\limits_i\min\limits_j\delta_{ij}^{+}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{+}}{\delta_{ij}^{+}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{+}}$$

Now, $\min\limits_i\min\limits_j\delta_{ij}^{+}\le\delta_{ij}^{+}$

$$\Rightarrow\min\limits_i\min\limits_j\delta_{ij}^{+}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{+}\le\delta_{ij}^{+}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{+}$$

$$\Rightarrow\frac{\min\limits_i\min\limits_j\delta_{ij}^{+}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{+}}{\delta_{ij}^{+}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{+}}\le 1$$

$$\Rightarrow\eta_{ij}^{+}\le 1$$

Again, from the definition, we can write:

$$\xi_{ij}^{+}=\frac{\min\limits_i\min\limits_j\Omega_{ij}^{+}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{+}}{\Omega_{ij}^{+}+\lambda^{+}\max\limits_i\max\limits_j\Omega_{ij}^{+}}$$

Now, $\min\limits_i\min\limits_j\Omega_{ij}^{+}\le\Omega_{ij}^{+}$

$$\Rightarrow\min\limits_i\min\limits_j\Omega_{ij}^{+}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{+}\le\Omega_{ij}^{+}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{+}$$

$$\Rightarrow\xi_{ij}^{+}=\frac{\min\limits_i\min\limits_j\Omega_{ij}^{+}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{+}}{\Omega_{ij}^{+}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{+}}$$

$$\Rightarrow\xi_{ij}^{+}\le 1.$$

Theorem 2

The negative grey relational coefficient is less than unity i.e. $\eta_{ij}^{-}\le 1$, $\xi_{ij}^{-}\le 1$.

Proof:

From the definition, we can write

$$\eta_{ij}^{-}=\frac{\min\limits_i\min\limits_j\delta_{ij}^{-}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{-}}{\delta_{ij}^{-}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{-}}$$

Now, $\min\limits_i\min\limits_j\delta_{ij}^{-}\le\delta_{ij}^{-}$

$$\Rightarrow\min\limits_i\min\limits_j\delta_{ij}^{-}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{-}\le\delta_{ij}^{-}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{-}$$

$$\eta_{ij}^{-}=\frac{\min\limits_i\min\limits_j\delta_{ij}^{-}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{-}}{\delta_{ij}^{-}+\lambda\max\limits_i\max\limits_j\delta_{ij}^{-}}$$

$$\Rightarrow\eta_{ij}^{-}\le 1$$

Again, from the definition

$$\xi_{ij}^{-}=\frac{\min\limits_i\min\limits_j\Omega_{ij}^{-}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{-}}{\Omega_{ij}^{-}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{-}}$$

Now, $\min\limits_i\min\limits_j\Omega_{ij}^{-}\le\Omega_{ij}^{-}$

$$\Rightarrow\min\limits_i\min\limits_j\Omega_{ij}^{-}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{-}\le\Omega_{ij}^{-}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{-}$$

$$\Rightarrow\xi_{ij}^{-}=\frac{\min\limits_i\min\limits_j\Omega_{ij}^{-}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{-}}{\Omega_{ij}^{-}+\lambda\max\limits_i\max\limits_j\Omega_{ij}^{-}}$$

$$\Rightarrow\xi_{ij}^{-}\le 1.$$

Note 1:

i. Since $\eta_{ij}^{+}\le 1$, $w_j^c\le 1$ then $\eta_{ij}^{+}w_j^c\le 1\Rightarrow\tilde{\eta}_{ij}^{+}\le 1$

ii. Since $\eta_{ij}^{-}\le 1$, $w_j^c\le 1$ then $\eta_{ij}^{-}w_j^c\le 1\Rightarrow\tilde{\eta}_{ij}^{-}\le 1$

iii. Since $\xi_{ij}^{+}\le 1$, $w_j^c\le 1$ then $\xi_{ij}^{+}w_j^c\le 1\Rightarrow\tilde{\xi}_{ij}^{+}\le 1$

iv. Since $\xi_{ij}^{-}\le 1$, $w_j^c\le 1$ then $\xi_{ij}^{-}w_j^c\le 1\Rightarrow\tilde{\xi}_{ij}^{-}\le 1$

Step 4.7

We define the ideal or standard grey relational coefficient as (1, 1). Then we construct ideal grey relational coefficient matrix of order $r\times s$ (see Table 3).

Table 3: Ideal grey relational coefficient matrix of order $r\times s$

$$I=\begin{pmatrix}(1,1)\,(1,1)\ldots(1,1)\\(1,1)\,(1,1)\ldots(1,1)\\\ldots\ldots\ldots\ldots\\(1,1)\,(1,1)\ldots(1,1)\end{pmatrix}_{r\times s}$$

Step 5 Determination of Hamming distances

We find the distance d_i^{+} between the corresponding elements of i-th row of I and ${}_{w}M_{GR}^{+}$ by employing Hamming distance. Similarly, d_i^- can be determined between I and ${}_wM_{GR}^-$ by employing Hamming distance as follows:

$$d_i^+ = \frac{1}{2s}[\sum_{j=1}^{s}\{|1-\tilde{\eta}_{ij}^+| + |1-\tilde{\xi}_{ij}^+|\}],\ i = 1, 2, \ldots, r.$$

$$d_i^- = \frac{1}{2s}[\sum_{j=1}^{s}\{|1-\tilde{\eta}_{ij}^-| + |1-\tilde{\xi}_{ij}^-|\}],\ i = 1, 2, \ldots, r.$$

Step 6 Determination of relative closeness coefficient

The relative closeness coefficient can be calculated as:

$$\Delta_i = \frac{d_i^+}{d_i^+ + d_i^-} \quad i = 1, 2, ..., r.$$

Step 7 Ranking the alternatives

According to the relative closeness coefficient, the ranking order of all alternatives is determined. The ranking order is made according to descending order of relative closeness coefficients.

4 Numerical example

Consider a hypothetical MADM problem. The problem consists of single decision maker, three alternatives with three alternatives {A_1, A_2, A_3} and four attributes {C_1, C_2, C_3, C_4}. The solution of the problem is presented using the following steps:

Step 1. Construction of neutrosophic cubic decision matrix

The decison maker forms the decision matrix which is displayed in the Table 4, at the end of article.

Step 2. Crispification of neutrosophic weight set

The neutrosophic weights of the attributes are taken as:

$$W = \{(0.5, 0.2, 0.1), (0.6, 0.1, 0.1), (0.9, 0.2, 0.1), (0.6, 0.3, 0.4)\}^T$$

The equivalent crisp weights are

$$W^c = \{(0.1907), (0.2146), (0.3228), (0.2719)\}^T$$

Step 3 Conversion of interval neutrosophic set into neutrosophic set in decision matrix

Taking the mid value of INS in the Table 4, the new decision matrix is presented in the following Table 5, at the end of article.

Step 4 Some Definitions of GRA method for MADM with NCS

The ideal neutrosophic estimates reliability solution (INERS) (M^+, Λ^+) and the ideal neutrosophic estimates unreliability solution (INEURS) (M^-, Λ^-) are presented in the Table 6, at the end of article.

$\delta^+ = (\delta_{ij}^+) = (d(M_j^+, M_{ij}))\ \forall i, j$ is presented as below:

$$\delta^+ = \begin{pmatrix} 0.85 & 0.95 & 0.05 & 0.15 \\ 0.65 & 0 & 0.7 & 0.25 \\ 0.05 & 0.15 & 0.25 & 0.45 \end{pmatrix}$$

The $\Omega^+ = (\Omega_{ij}^+) = (d(\Lambda_j^+, \Lambda_{ij}))\ \forall i, j$ is presented as below:

$$\Omega^+ = \begin{pmatrix} 0.45 & 1.2 & 0.4 & 0.15 \\ 0.05 & 0.5 & 0.2 & 0.2 \\ 0.25 & 0.3 & 0.2 & 0.5 \end{pmatrix}$$

$\delta^- = (\delta_{ij}^-) = (d(M_j^-, M_{ij}))\ \forall i, j$ is presented as below:

$$\delta^- = \begin{pmatrix} 0.25 & 0.3 & 0.7 & 0.55 \\ 0.45 & 1.2 & 0 & 0.45 \\ 1.05 & 0.65 & 0.6 & 0.25 \end{pmatrix}$$

The $\Omega^- = (\Omega_{ij}^-) = (d(\Lambda_j^-, \Lambda_{ij}))\ \forall i, j$ is presented as:

The positive grey relational coefficient $M_{GR}^+ = (\eta_{ij}^+, \xi_{ij}^+)_{3\times4}$ is presented in the Table 7, at the end of article.

The negative grey relational coefficient $M_{GR}^- = (\eta_{ij}^-, \xi_{ij}^-)_{3\times4}$ is presented in the Table 8, at the end of article.

Now, we multiply the crisp weight with the corresponding elements of M_{GR}^+ and M_{GR}^- to get weighted matrices ${}_wM_{GR}^+$ and ${}_wM_{GR}^-$ and which are described in the Table 9 and 10 respectively, at the end of article.

Step 5 Determination of Hamming distances

Hamming distances are calculated as follows:

$d_1^+ = 0.84496$, $d_1^- = 0.83845625$,

$d_2^+ = 0.82444375$, $d_2^- = 0.85328875$,

$d_3^+ = 0.82368675$, $d_3^- = 0.85277$.

Step 6 Determination of relative closeness coefficient

The relative closeness coefficients are calculated as:

$$\Delta_1 = \frac{d_1^+}{d_1^+ + d_1^-} = 0.501932$$

$$\Delta_2 = \frac{d_2^+}{d_2^+ + d_2^-} = 0.491403576$$

$$\Delta_3 = \frac{d_3^+}{d_3^+ + d_3^-} = 0.49132$$

Step 7 Ranking the alternatives

The ranking of alternatives is made according to descending order of relative closeness coefficients. The ranking order is shown in the Table 11 below.

Alternatives	Ranking order
A_3	1
A_2	2
A_1	3

Conclusion

This paper develops GRA based MADM in neutrosophic cubic set environment. This is the first approach of GRA in MADM in neutrosophic cubic set environment. The proposed approach can be applied to other decision making problems such as pattern recognition, personnel selection, etc.

The proposed approach can be applied for decision making problem described by internal NCSs and external NCSs. We hope that the proposed approach will open up a new avenue of research in newly developed neutrosophic cubic set environment.

References

[1] C. L. Hwang, and K. Yoon. Multiple attribute decision making: methods and applications: a state-of-the-art survey, Springer, London, (1981).

[2] J.P. Brans, P. Vinvke, and B. Mareschal. How to select and how to rank projects: The PROMETHEE method. European Journal of Operation Research, 24(1986), 228–238.

[3] S. Opricovic. Multicriteria optimization of civil engineering systems. Faculty of Civil Engineering, Belgrade (1998).

[4] S. Opricovic, and G. H. Tzeng. Compromise solution by MCDM methods: a comparative analysis of VIKOR and TOPSIS. European Journal of Operation Research, 156 (2004), 445–455.

[5] B. Roy. The outranking approach and the foundations of ELECTRE methods. Theory Decision, 31(1991), 49–73.

[6] L. A. Zadeh. Fuzzy Sets. Information and Control, 8 (3) (1965), 338-353.

[7] R. Bellman, and L. A. Zadeh. Decision making in a fuzzy environment, Management Science, 17B (4) (1970), 141-164.

[8] K. T. Atanassov. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20 (1986), 87–96.

[9] K. T. Atanassov. On Intuitionistic fuzzy set theory, studies in fuzziness and soft computing, Springer- Verlag, Berlin (2012).

[10] F. Smarandache. A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic. Rehoboth: American Research Press (1998).

[11] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman. Single valued neutrosophic sets, Multispace and Multistructure, 4(2010), 410–413.

[12] J. Ye. Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. International Journal of General Systems, 42(4) (2013), 386-394.

[13] J. Ye. Single valued neutrosophic cross entropy for multicriteria decision-making problems. Applied Mathematical Modeling, (2013), doi:10.1016/j.apm.2013.07.020.

[14] M. Sodenkamp, Models, methods and applications of group multiple-criteria decision analysis in complex and uncertain systems. Dissertation, University of Paderborn, Germany, 2013.

[15] K. Mondal, and S. Pramanik. Neutrosophic tangent similarity measure and its application to multiple attribute decision-making. Neutrosophic Sets and Systems, 9 (2015), 85-92.

[16] P. Biswas, S. Pramanik, and B. C. Giri. Cosine similarity measure based multi-attribute decision-making with trapezoidal fuzzy neutrosophic numbers. Neutrosophic Sets and Systems, 8 (2015), 47-57.

[17] K. Mondal, and S. Pramanik. Multi-criteria group decision-making approach for teacher recruitment in higher education under simplified Neutrosophic environment. Neutrosophic Sets and Systems, 6 (2014), 28-34.

[18] K. Mondal, and S. Pramanik. Neutrosophic decision-making model of school choice. Neutrosophic Sets and Systems, 7 (2015), 62-68.

[19] P. Liu, and Y. Wang. Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean. Neural Computing and Applications, 25(7) (2014), 2001–2010.

[20] P. Biswas, S. Pramanik, and B. C. Giri. TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment. Neural Computing and Applications. doi: 10.1007/s00521-015-1891-2, 2015.

[21] P. Chi, and P. Liu. An extended TOPSIS method for the multi-attribute decision making problems on interval neutrosophic set. Neutrosophic Sets and Systems, 1 (2013), 63–70.

[22] S. Nabdaban, and S. Dzitac. Neutrosophic TOPSIS: a general view. 6th International Conference on Computers Communications and Control, 2016, 250-253.

[23] S. Broumi, J. Ye, and F. Smnarandache. An extended TOPSIS method for multiple attribute decision making based on interval neutrosophic uncertain linguistic variables. Neutrosophic Sets and Systems, 8 (2015), 22-31.

[24] S. Pramanik, P. Biswas, and B. C. Giri. Hybrid vector similarity measures and their applications to multi-attribute decision making under neutrosophic environment. Neural Computing and Applications, (2015), doi: 10.1007/s00521-015-2125-3.

[25] P. Biswas, S. Pramanik, and B. C. Giri. Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making, NSS, 12 (2016), 20-40.

[26] R. Sahin,and P. Liu. Maximizing deviation method for neutrosophic multiple attribute decision making with incomplete weight information. Neural Computing and Applications, (2015), doi: 10.1007/s00521-015-1995-8.

[27] J. Ye. Bidirectional projection method for multiple attribute group decision making with neutrosophic numbers. Neural Computing and Applications, (2015), doi: 10.1007/s00521-015-2123-5.

[28] P. Biswas, S. Pramanik, and B. C. Giri. Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multi-attribute decision making. Neutrosophic Sets and Systems, 12 (2016), 127-138.

[29] P. P. Dey, S. Pramanik, and B.C. Giri. Extended projection-based models for solving multiple attribute decision making problems with interval –valued neutrosophic information. In: New Trends in Neutrosophic Theory and Applications, eds. F. Smarandache and S. Pramanik, Pons Editions, Brussels, 2016, 127- 140.

[30] J. L. Deng. Introduction to grey system theory. The Journal of Grey System, 1(1) (1989), 1–24.

[31] J. L. Deng. The primary methods of grey system theory, Huazhong University of Science and Technology Press, Wuhan, (2005).

[32] S. Pramanik, and D. Mukhopadhyaya. Grey relational analysis based intuitionistic fuzzy multi criteria group decision making approach for teacher selection in higher education. International Journal of Computer Applications, 34 (10) (2011), 21 – 29.

[33] P.P. Dey, S. Pramanik, and B.C. Giri, Multi-criteria group decision making in intuitionistic fuzzy environment based on grey relational analysis for weaver selection in Khadi institution. Journal of Applied and Quantitative Methods, 10(4) (2015), 1-14.

[34] R. V. Rao, and D. Singh. An improved grey relational analysis as a decision making method for manufacturing situations. International Journal of Decision Science, Risk and Management, 2(2010), 1–23.

[35] G. W. Wei. Grey relational analysis method for intuitionistic fuzzy multiple attribute decision making. Expert Systems with Applications, 38(2011), 11671-11677.

[36] P. Biswas, S. Pramanik, and B.C. Giri. Entropy based grey relational analysis method for multi-attribute decision – making under single valued neutrosophic assessments. Neutrosophic Sets and Systems, 2 (2014), 102 – 110.

[37] P.P. Dey, S. Pramanik, and B.C. Giri. An extended grey relational analysis based multiple attribute decision making in interval neutrosophic uncertain linguistic setting. Neutrosophic Sets and Systems, 11 (2016), 21-30.

[38] S. Pramanik, and K. Mondal. Interval neutrosophic multi-attribute decision-making based on grey relational analysis. Neutrosophic Sets and Systems, 9(2015), 13-22.

[39] P. K. Maji. Neutrosophic soft set. Annals of fuzzy Mathematics and Informatics, 5 (1) (2013), 157 – 168.

[40] S. Broumi, and F. Smarandache. Single valued neutrosophic soft expert sets and their application in decision making. Journal of New Theory, 3 (2015), 67 – 88.

[41] S. Broumi, F. Smarandache, and M. Dhar. Rough neutrosophic sets. Neutrosophic Sets and Systems, 3 (2014), 62-67.

[42] I. Deli, M. Ali, and F. Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. Proceedings of the 2015 International Conference on Advanced Mechatronic Systems, Beijing, China, August (2015), 22-24.

[43] S. Pramanik, and K. Mondal. Rough bipolar neutrosophic set. Global Journal of Engineering Science and Research Management, 3 (6) (2015), 71- 81.

[44] Y. B. Jun, F. Smarandache, and C. S. Kim. Neutrosophic cubic sets. New mathematics and natural computation, 9 (2015), 1 – 15.

[45] M. Ali, I. Deli, and F. Smarandache, The theory of neutrosophic cubic sets and their applications in pattern recognition, Journal of intelligent and fuzzy systems, 30 (2016), 1957-1963.

[46] Y. B. Jun, C. S. Kim, K. O. Yang. Cubic sets. Annals Fuzzy Mathematics Information, 4 (1) (2012), 83 – 98.

[47] S. Pramanik, and S. Dalapati. GRA based multi criteria decision making in generalized neutrosophic soft set environment. Global Journal of Engineering Science and Research Management, 3 (5) (2016), 153 - 169.

[48] P.P. Dey, S. Pramanik, and B.C. Giri. Neutrosophic soft multi attribute group decision making based on grey relational analysis method. Journal of New Results in Science, 10 (2016), 25 – 37.

[49] K. Mondal, and S. Pramanik. Rough neutrosophic multi-Attribute decision-making based on grey relational analysis. Neutrosophic Sets and Systems, 7 (2014), 8-17.

[50] S. Pramanik, and K. Mondal. Interval neutrosophic multi-Attribute decision-making based on grey relational analysis. Neutrosophic Sets and Systems, 9 (2015), 13-22.

[51] H. Wang, F. Smarandache, Y.-Q. Zhang, and R. Sunderraman, Interval Neutrosophic Sets and Logic: Theory and Applications in Computing, Hexis, Phoenix, AZ, (2005).

[52] I.B. Turksen. Interval-valued fuzzy sets based on normal forms. Fuzzy Sets and Systems, 20 (1986), 191–210.

[53] R. W. Hamming. Error detecting and error correcting codes. Bell System Technical Journal, 29 (2) (1950), 147–160.

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Table 4: Construction of neutrosophic cubic decision matrix

$A=(a_{ij})_{3\times 4}=$

	C_1	C_2	C_3	C_4
A_1	(([0.2, 0.3],[0.3, 0.5],[0.2, 0.5]), (0.3, 0.2, 0.3))	([0.1, 0.3],[0.2, 0.4],[0.3, 0.6]), (0.2, 0.5, 0.4))	(([0.6, 0.9],[0.1, 0.2],[0, 0.2]), (0.4, 0.5, 0.1))	(([0.4, 0.7],[0.1, 0.3],[0.2, 0.3]), (0.7, 0.3, 0.2))
A_2	([0.6, 0.8],[0.4, 0.6],[0.3, 0.7]), (0.5, 0.2, 0.1))	([0.7, 0.9],[0.2, 0.3],[0.1, 0.3]), (0.7, 0.3, 0.3))	(([0.5, 0.7],[0.4, 0.6],[0.3, 0.5]), (0.4, 0.1, 0.2))	(([0.4, 0.5],[0.1, 0.3],[0.2, 0.3]), (0.6, 0.2, 0.1))
A_3	(([0.4, 0.9],[0.1, 0.4],[0, 0.2]), (0.25, 0.15, 0.1))	(([0.8, 0.9],[0.4, 0.7],[0.4, 0.6]), (0.8, 0.1, 0.2))	(([0.6, 0.9],[0.1, 0.3],[0, 0.3]), (0.5, 0.4, 0.3))	(([0.6, 0.8],[0.5, 0.7],[0.2, 0.4]), (0.5, 0.1, 0.4))

Table 5: Construction of neutrosophic decision matrix

$M=(m_{ij})_{3\times 4}=$

	C_1	C_2	C_3	C_4
A_1	((0.25, 0.4, 0.35), (0.3, 0.2, 0.3))	((0.2, 0.3, 0.45), (0.2, 0.5, 0.4))	((0.75, 0.15, 0.1), (0.4, 0.5, 0.1))	((0.55, 0.2, 0.25), (0.7, 0.3, 0.2))
A_2	((0.7,0.5, 0.5), (0.5, 0.2, 0.1))	((0.8, 0.25, 0.2), (0.7, 0.3, 0.3))	((0.6, 0.5, 0.4), (0.4, 0.1, 0.2))	((0.45, 0.2, 0.25), (0.6, 0.2, 0.1))
A_3	((0.65, 0.25, 0.1), (0.25, 0.15, 0.1))	((0.85, 0.55, 0.5), (0.8, 0.1, 0.2))	((0.75, 0.2, 0.15), (0.5, 0.4, 0.3))	((0.7, 0.6, 0.3), (0.5, 0.1, 0.4))

Table 6: The ideal neutrosophic estimates reliability solution (INERS) (M^+, Λ^+) and the ideal neutrosophic estimates unreliability solution (INEURS) (M^-, Λ^-)

(M^+, Λ^+)	$\begin{pmatrix}(0.7,0.25,0.1),\\(0.5,0.15,0.1)\end{pmatrix}$	$\begin{pmatrix}(0.85,0.25,0.2),\\(0.8,0.1,0.2)\end{pmatrix}$	$\begin{pmatrix}(0.75,0.15,0.1),\\(0.5,0.1,0.1)\end{pmatrix}$	$\begin{pmatrix}(0.7,0.2,0.25),\\(0.7,0.1,0.1)\end{pmatrix}$
(M^-, Λ^-)	$\begin{pmatrix}(0.25,0.5,0.5),\\(0.25,0.2,0.3)\end{pmatrix}$	$\begin{pmatrix}(0.2,0.55,0.5),\\(0.2,0.5,0.4)\end{pmatrix}$	$\begin{pmatrix}(0.6,0.5,0.4),\\(0.4,0.5,0.3)\end{pmatrix}$	$\begin{pmatrix}(0.45,0.6,0.3),\\(0.5,0.3,0.4)\end{pmatrix}$

Table 7: The positive grey relational coefficient $M^+_{GR} = (\eta^+_{ij}, \xi^+_{ij})_{3\times 4}$

$$M^+_{GR} = \begin{pmatrix} (0.3585, 0.6190) & (0.333, 0.3611) & (0.9048, 0.65) & (0.76, 0.7222) \\ (0.4222, \quad 1 \quad) & (1, 0.5909) & (0.4042, 0.8125) & (0.6552, 0.8125) \\ (0.9048, 0.7647) & (0.76, 0.7222) & (0.6552, 0.8125) & (0.5135, 0.5909) \end{pmatrix}$$

Table 8: The negative grey relational coefficient $M_{GR}^{-} = \left(\eta_{ij}^{-}, \xi_{ij}^{-}\right)_{3\times 4}$

$$M_{GR}^{-} = \begin{pmatrix} (0.7059, 0.5454) & (0.6667, 1) & (0.4615, 0.75) & (0.5217, 0.6) \\ (0.5714, 0.5714) & (0.3333, 0.4286) & (1, 0.5454) & (0.5714, 0.5454) \\ (0.3636, 0.7059) & (0.48, 0.3333) & (0.5, 0.75) & (0.7059, 0.75) \end{pmatrix}$$

Table 9: Weighted matrix ${}_{W}M_{GR}^{+}$ ${}_{W}M_{GR}^{+} =$

$$\begin{pmatrix} (0.06836, 0.11804) & (0.07153, 0.07749) & (0.29207, 0.20982) & (0.20664, 0.19637) \\ (0.08051, 0.1907) & (0.2146, 0.12681) & (0.13048, 0.26228) & (0.17815, 0.22092) \\ (0.17252, 0.14583) & (0.163096, 0.15498) & (0.21150, 0.26228) & (0.13962, 0.16066) \end{pmatrix}$$

Table 10: Weighted matrix ${}_{W}M_{GR}^{-}$

$${}_{W}M_{GR}^{-} = \begin{pmatrix} (0.13461, 0.10401) & (0.14307, 0.2146) & (0.14897, 0.2421) & (0.14185, 0.16314) \\ (0.10896, 0.10896) & (0.07153, 0.08173) & (0.3228, 0.17606) & (0.15536, 0.14829) \\ (0.06934, 0.13461) & (0.10301, 0.07153) & (0.1614, 0.2421) & (0.19193, 0.20392) \end{pmatrix}$$

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Bipolar Neutrosophic Projection Based Models for Solving Multi-attribute Decision Making Problems

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Abstract. Bipolar neutrosophic sets are the extension of neutrosophic sets and are based on the idea of positive and negative preferences of information. Projection measure is a useful apparatus for modelling real life decision making problems. In the paper, we define projection, bidirectional projection and hybrid projection measures between bipolar neutrosophic sets. Three new methods based on the proposed projection measures are developed for solving multi-attribute decision making problems. In the solution process, the ratings of performance values of the alternatives with respect to the attributes are expressed in terms of bipolar neutrosophic values. We calculate projection, bidirectional projection, and hybrid projection measures between each alternative and ideal alternative with bipolar neutrosophic information. All the alternatives are ranked to identify the best alternative. Finally, a numerical example is provided to demonstrate the applicability and effectiveness of the developed methods. Comparison analysis with the existing methods in the literature in bipolar neutrosophic environment is also performed.

Keywords: Bipolar neutrosophic sets; projection measure; bidirectional projection measure; hybrid projection measure; multi-attribute decision making.

1 Introduction

For describing and managing indeterminate and inconsistent information, Smarandache [1] introduced neutrosophic set which has three independent components namely truth membership degree (*T*), indeterminacy membership degree (*I*) and falsity membership degree (*F*) where *T*, *I*, and *F* lie in]$^-$0, 1$^+$[. Later, Wang et al. [2] proposed single valued neutrosophic set (SVNS) to deal real decision making problems where *T*, *I*, and *F* lie in [0, 1].

Zhang [3] grounded the notion of bipolar fuzzy sets by extending the concept of fuzzy sets [4]. The value of membership degree of an element of bipolar fuzzy set belongs to [-1, 1]. With reference to a bipolar fuzzy set, the membership degree zero of an element reflects that the element is irrelevant to the corresponding property, the membership degree belongs to (0, 1] of an element reflects that the element somewhat satisfies the property, and the membership degree belongs to [−1,0) of an element reflects that the element somewhat satisfies the implicit counter-property.

Deli et al. [5] extended the concept of bipolar fuzzy set to bipolar neutrosophic set (BNS). With reference to a bipolar neutrosophic set *Q*, the positive membership degrees $T_Q^+(x)$, $I_Q^+(x)$, and $F_Q^+(x)$ represent respectively the truth membership, indeterminate membership and falsity membership of an element $x \in X$ corresponding to the bipolar neutrosophic set *Q* and the negative membership degrees $T_Q^-(x)$, $I_Q^-(x)$, and $F_Q^-(x)$ denote respectively the truth membership, indeterminate membership and false membership degree of an element $x \in X$ to some implicit counter-property corresponding to the bipolar neutrosophic set *Q*.

Projection measure is a useful decision making device as it takes into account the distance as well as the included angle for measuring the closeness degree between two objects [6, 7]. Yue [6] and Zhang et al. [7] studied projection based multi-attribute decision making (MADM) in crisp environment i.e. projections are defined by ordinary numbers or crisp numbers. Yue [8] further investigated a new multi-attribute group decision making (MAGDM) method based on determining the weights of the decision makers by employing projection technique with interval data. Yue and Jia [9] established a methodology for MAGDM based on a new normalized projection measure, in which the attribute values are provided by decision makers in hybrid form with crisp values and interval data.

Xu and Da [10] and Xu [11] studied projection method for decision making in uncertain environment with preference information. Wei [12] discussed a MADM method based on the projection technique, in which the attribute values are presented in terms of intuitionistic fuzzy numbers. Zhang et al. [13] proposed a grey relational projection method for MADM based on intuitionistic trapezoidal fuzzy number. Zeng et al. [14] investigated projections on interval valued intuitionistic fuzzy numbers and developed algorithm to the MAGDM problems with interval-valued intuitionistic fuzzy information. Xu and Hu [15] developed two projection based models for MADM in intuitionistic fuzzy environment and interval valued intuitionistic fuzzy environment. Sun [16] presented a group decision making method based on projection method and score function under interval valued intuitionistic fuzzy environment. Tsao and Chen [17] developed a novel projection based compromising method for multi-criteria decision making (MCDM) method in interval valued intuitionistic fuzzy environment.

In neutrosophic environment, Chen and Ye [18] developed projection based model of neutrosophic numbers and presented MADM method to select clay-bricks in construction field. Bidirectional projection measure [19, 20] considers the distance and included angle between two vectors *x, y*. Ye [19] defined bidirectional projection measure as an improvement of the general projection measure of SVNSs to overcome the drawback of the general projection measure. In the same study, Ye [19] developed MADM method for selecting problems of mechanical design schemes under a single-valued neutrosophic environment. Ye [20] also presented bidirectional projection method for MAGDM with neutrosophic numbers.

Ye [21] defined credibility – induced interval neutrosophic weighted arithmetic averaging operator and credibility – induced interval neutrosophic weighted geometric averaging operator and developed the projection measure based ranking method for MADM problems with interval neutrosophic information and credibility information. Dey et al. [22] proposed a new approach to neutrosophic soft MADM using grey relational projection method. Dey et al. [23] defined weighted projection measure with interval neutrosophic assessments and applied the proposed concept to solve MADM problems with interval valued neutrosophic information. Pramanik et al. [24] defined projection and bidirectional projection measures between rough neutrosophic sets and proposed two new multi-criteria decision making (MCDM) methods based on projection and bidirectional projection measures in rough neutrosophic set environment.

In the field of bipolar neutrosophic environment, Deli et al. [5] defined score, accuracy, and certainty functions in order to compare BNSs and developed bipolar neutrosophic weighted average (BNWA) and bipolar neutrosophic weighted geometric (BNWG) operators to obtain collective bipolar neutrosophic information. In the same study, Deli et al. [5] also proposed a MCDM approach on the basis of score, accuracy, and certainty functions and BNWA, BNWG operators. Deli and Subas [25] presented a single valued bipolar neutrosophic MCDM through correlation coefficient similarity measure. Şahin et al. [26] provided a MCDM method based on Jaccard similarity measure of BNS. Uluçay et al. [27] defined Dice similarity, weighted Dice similarity, hybrid vector similarity, weighted hybrid vector similarity measures under BNSs and developed MCDM methods based on the proposed similarity measures. Dey et al. [28] defined Hamming and Euclidean distance measures to compute the distance between BNSs and investigated a TOPSIS approach to derive the most desirable alternative.

In this study, we define projection, bidirectional projection and hybrid projection measures under bipolar neutrosophic information. Then, we develop three methods for solving MADM problems with bipolar neutrosophic assessments. We organize the rest of the paper in the following way. In Section 2, we recall several useful definitions concerning SVNSs and BNSs. Section 3 defines projection, bidirectional projection and hybrid projection measures between BNSs. Section 4 is devoted to present three models for solving MADM under bipolar neutrosophic environment. In Section 5, we solve a decision making problem with bipolar neutrosophic information on the basis of the proposed measures. Comparison analysis is provided to demonstrate the feasibility and flexibility of the proposed methods in Section 6. Finally, Section 7 provides conclusions and future scope of research.

2 Basic Concepts Regarding SVNSs and BNSs

In this Section, we provide some basic definitions regarding SVNSs, BNSs which are useful for the construction of the paper.

2.1 Single valued neutrosophic sets [2]

Let *X* be a universal space of points with a generic element of *X* denoted by *x*, then a SVNS *P* is characterized by a truth membership function $T_P(x)$, an indeterminate membership function $I_P(x)$ and a falsity membership function $F_P(x)$. A SVNS *P* is expressed in the following way.

$$P = \{x, \langle T_P(x), I_P(x), F_P(x)\rangle \mid x \in X\}$$

where, $T_P(x)$, $I_P(x)$, $F_P(x) : X \rightarrow [0, 1]$ and $0 \leq T_P(x) + I_P(x) + F_P(x) \leq 3$ for each point $x \in X$.

2.2 Bipolar neutrosophic set [5]

Consider *X* be a universal space of objects, then a BNS *Q* in *X* is presented as follows:

$$Q = \{x, \langle T_Q^+(x), I_Q^+(x), F_Q^+(x), T_Q^-(x), I_Q^-(x), F_Q^-(x)\rangle \mid x \in X\},$$

where $T_Q^+(x)$, $I_Q^+(x)$, $F_Q^+(x): X \to [0, 1]$ and $T_Q^-(x)$, $I_Q^-(x)$, $F_Q^-(x): X \to [-1, 0]$.The positive membership degrees $T_Q^+(x)$, $I_Q^+(x)$, $F_Q^+(x)$ denote the truth membership, indeterminate membership, and falsity membership functions of an element $x \in X$ corresponding to a BNS Q and the negative membership degrees $T_Q^-(x)$, $I_Q^-(x)$, $F_Q^-(x)$ denote the truth membership, indeterminate membership, and falsity membership of an element $x \in X$ to several implicit counter property associated with a BNS Q. For convenience, a bipolar neutrosophic value (BNV) is presented as $\tilde{q} = < T_Q^+, I_Q^+, F_Q^+, T_Q^-, I_Q^-, F_Q^- >$.

Definition 1 [5]
Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be any two BNSs. Then $Q_1 \subseteq Q_2$ if and only if $T_{Q_1}^+(x) \le T_{Q_2}^+(x)$, $I_{Q_1}^+(x) \le I_{Q_2}^+(x)$, $F_{Q_1}^+(x) \ge F_{Q_2}^+(x)$; $T_{Q_1}^-(x) \ge T_{Q_2}^-(x)$, $I_{Q_1}^-(x) \ge I_{Q_2}^-(x)$, $F_{Q_1}^-(x) \le F_{Q_2}^-(x)$ for all $x \in X$.

Definition 2 [5]
Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be any two BNSs. Then $Q_1 = Q_2$ if and only if $T_{Q_1}^+(x) = T_{Q_2}^+(x)$, $I_{Q_1}^+(x) = I_{Q_2}^+(x)$, $F_{Q_1}^+(x) = F_{Q_2}^+(x)$; $T_{Q_1}^-(x) = T_{Q_2}^-(x)$, $I_{Q_1}^-(x) = I_{Q_2}^-(x)$, $F_{Q_1}^-(x) = F_{Q_2}^-(x)$ for all $x \in X$.

Definition 3 [5]
Let, $Q = \{x, \langle T_Q^+(x), I_Q^+(x), F_Q^+(x), T_Q^-(x), I_Q^-(x), F_Q^-(x) \rangle \mid x \in X\}$ be a BNS. The complement of Q is represented by Q^c and is defined as follows:

$T_{Q^c}^+(x) = \{1^+\} - T_Q^+(x)$, $I_{Q^c}^+(x) = \{1^+\} - I_Q^+(x)$, $F_{Q^c}^+(x) = \{1^+\} - F_Q^+(x)$;

$T_{Q^c}^-(x) = \{1^-\} - T_Q^-(x)$, $I_{Q^c}^-(x) = \{1^-\} - I_Q^-(x)$, $F_{Q^c}^-(x) = \{1^-\} - F_Q^-(x)$.

Definition 4
Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be any two BNSs. Their union $Q_1 \cup Q_2$ is defined as follows:

$Q_1 \cup Q_2$ = {Max ($T_{Q_1}^+(x)$, $T_{Q_2}^+(x)$), Min ($I_{Q_1}^+(x)$, $I_{Q_2}^+(x)$), Min ($F_{Q_1}^+(x)$, $F_{Q_2}^+(x)$), Min ($T_{Q_1}^-(x)$, $T_{Q_2}^-(x)$), Max ($I_{Q_1}^-(x)$, $I_{Q_2}^-(x)$), Max ($F_{Q_1}^-(x)$, $F_{Q_2}^-(x)$)}, $\forall$ $x \in X$.

Their intersection $Q_1 \cap Q_2$ is defined as follows:

$Q_1 \cap Q_2$ = {Min ($T_{Q_1}^+(x)$, $T_{Q_2}^+(x)$), Max ($I_{Q_1}^+(x)$, $I_{Q_2}^+(x)$), Max ($F_{Q_1}^+(x)$, $F_{Q_2}^+(x)$), Max ($T_{Q_1}^-(x)$, $T_{Q_2}^-(x)$), Min ($I_{Q_1}^-(x)$, $I_{Q_2}^-(x)$), Min ($F_{Q_1}^-(x)$, $F_{Q_2}^-(x)$)}, $\forall$ $x \in X$.

Definition 5 [5]
Let $\tilde{q}_1 = < T_{Q_1}^+, I_{Q_1}^+, F_{Q_1}^+, T_{Q_1}^-, I_{Q_1}^-, F_{Q_1}^- >$ and $\tilde{q}_2 = < T_{Q_2}^+, I_{Q_2}^+, F_{Q_2}^+, T_{Q_2}^-, I_{Q_2}^-, F_{Q_2}^- >$ be any two BNVs, then

i. $\beta . \tilde{q}_1 = < 1 - (1 - T_{Q_1}^+)^\beta, (I_{Q_1}^+)^\beta, (F_{Q_1}^+)^\beta, -(-T_{Q_1}^-)^\beta, -(-I_{Q_1}^-)^\beta, -(1 - (1 - (-F_{Q_1}^-))^\beta) >$;

ii. $(\tilde{q}_1)^\beta = < (T_{Q_1}^+)^\beta, 1 - (1 - I_{Q_1}^+)^\beta, 1 - (1 - F_{Q_1}^+)^\beta, -(1 - (1 - (-T_{Q_1}^-))^\beta), -(-I_{Q_1}^-)^\beta, (-F_{Q_1}^-)^\beta >$;

iii. $\tilde{q}_1 + \tilde{q}_2 = < T_{Q_1}^+ + T_{Q_2}^+ - T_{Q_1}^+ . T_{Q_2}^+, I_{Q_1}^+ . I_{Q_2}^+, F_{Q_1}^+ . F_{Q_2}^+, - T_{Q_1}^- . T_{Q_2}^-, -(-I_{Q_1}^- - I_{Q_2}^- - I_{Q_1}^- . I_{Q_2}^-), -(-F_{Q_1}^- - F_{Q_2}^- - F_{Q_1}^- . F_{Q_2}^-) >$;

iv. $\tilde{q}_1 . \tilde{q}_2 = < T_{Q_1}^+ . T_{Q_2}^+, I_{Q_1}^+ + I_{Q_2}^+ - I_{Q_1}^+ . I_{Q_2}^+, F_{Q_1}^+ + F_{Q_2}^+ - F_{Q_1}^+ . F_{Q_2}^+, -(-T_{Q_1}^- - T_{Q_2}^- - T_{Q_1}^- . T_{Q_2}^-), - I_{Q_1}^- . I_{Q_2}^-, - F_{Q_1}^- . F_{Q_2}^- >$ where $\beta > 0$.

3 Projection, bidirectional projection and hybrid projection measures of BNSs

This Section proposes a general projection, a bidirectional projection and a hybrid projection measures for BNSs.

Definition 6
Assume that $X = (x_1, x_2, \ldots, x_m)$ be a finite universe of discourse and Q be a BNS in X, then modulus of Q is defined as follows:

$$\| Q \| = \sqrt{\sum_{j=1}^{m} \alpha_j^2} = \sqrt{\sum_{j=1}^{m} [(T_{Q_j}^+)^2 + (I_{Q_j}^+)^2 + (F_{Q_j}^+)^2 + (T_{Q_j}^-)^2 + (I_{Q_j}^-)^2 + (F_{Q_j}^-)^2]} \quad (1)$$

where $\alpha_j = \langle T_{Q_j}^+(x), I_{Q_j}^+(x), F_{Q_j}^+(x), T_{Q_j}^-(x), I_{Q_j}^-(x), F_{Q_j}^-(x) \rangle$, j = 1, 2, ..., m.

Definition 7 [10, 29]

Assume that $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$ be two vectors, then the projection of vector u onto vector v can be defined as follows:

$$Proj\ (u)_v = \| u \| \operatorname{Cos}(u, v) = \sqrt{\sum_{j=1}^{m} u_j^2} \times \frac{\sum_{j=1}^{m}(u_j v_j)}{\sqrt{\sum_{j=1}^{m} u_j^2} \times \sqrt{\sum_{j=1}^{m} v_j^2}} = \frac{\sum_{j=1}^{m}(u_j v_j)}{\sqrt{\sum_{j=1}^{m} v_j^2}} \quad (2)$$

where, $Proj\ (u)_v$ represents that the closeness of u and v in magnitude.

Definition 8

Assume that $X = (x_1, x_2, \ldots, x_m)$ be a finite universe of discourse and R, S be any two BNSs in X, then

$$Proj\ (R)_S = \| R \| \operatorname{Cos}(R, S) = \frac{1}{\| S \|}(R.S) \quad (3)$$

is called the projection of R on S, where

$\|R\| = \sqrt{\sum_{i=1}^{m}[(T_R^+)^2(x_i) + (I_R^+)^2(x_i) + (F_R^+)^2(x_i) + (T_R^-)^2(x_i) + (I_R^-)^2(x_i) + (F_R^-)^2(x_i)]}$,

$\|S\| = \sqrt{\sum_{i=1}^{m}[(T_S^+)^2(x_i) + (I_S^+)^2(x_i) + (F_S^+)^2(x_i) + (T_S^-)^2(x_i) + (I_S^-)^2(x_i) + (F_S^-)^2(x_i)]}$,

and $R.S = \sum_{i=1}^{m}[T_R^+(x_i)T_S^+(x) + I_R^+(x_i)I_S^+(x_i) + F_R^+(x_i)F_S^+(x_i) + T_R^-(x_i)T_S^-(x_i) + I_R^-(x_i)I_S^-(x_i) + F_R^-(x_i)F_S^-(x_i)]$.

Example 1. Suppose that R = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, S = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be the two BNSs in X, then the projection of R on S is obtained as follows:

$$Proj\ (R)_S = \frac{1}{\| S \|}(R.S) = \frac{(0.5)(0.7) + (0.3)(0.3) + (0.2)(0.1) + (-0.2)(-0.4) + (-0.1)(-0.2) + (-0.05)(-0.3)}{\sqrt{(0.7)^2 + (0.3)^2 + (0.1)^2 + (-0.4)^2 + (-0.2)^2 + (-0.3)^2}}$$

= 0.612952

The bigger value of $Proj\ (R)_S$ reflects that R and S are closer to each other.

However, in single valued neutrosophic environment, Ye [20] observed that the general projection measure cannot describe accurately the degree of α close to β. We also notice that the general projection incorporated by Xu [11] is not reasonable in several cases under bipolar neutrosophic setting, for example let, $\alpha = \beta$ = < $a, a, a, -a, -a, -a$ > and γ = < $2a, 2a, 2a, -2a, -2a, -2a$ >, then $Proj\ (\alpha)_\beta = 2.44949\ \|a\|$ and $Proj\ (\gamma)_\beta = 4.898979\ \|a\|$. This shows that β is much closer to γ than α which is not true because $\alpha = \beta$. Ye [20] opined that α is equal to β whenever $Proj\ (\alpha)_\beta$ and $Proj\ (\beta)_\alpha$ should be equal to 1. Therefore, Ye [20] proposed an alternative method called bidirectional projection measure to overcome the limitation of general projection measure as given below.

Definition 9 [20]

Consider x and y be any two vectors, then the bidirectional projection between x and y is defined as follows:

$$B\text{-}proj\ (x, y) = \frac{1}{1 + \left| \frac{x.y}{\| x \|} - \frac{x.y}{\| y \|} \right|} = \frac{\| x \| \| y \|}{\| x \| \| y \| + |\, \| x \| - \| y \| \,|\, x.y} \quad (4)$$

where $\|x\|$, $\|y\|$ denote the moduli of x and y respectively, and $x.\ y$ is the inner product between x and y.

Here, $B\text{-}Proj\ (x, y) = 1$ if and only if $x = y$ and $0 \le B\text{-}Proj\ (x, y) \le 1$, i.e. bidirectional projection is a normalized measure.

Definition 10

Consider $R = \left\langle T_R^+(x_i), I_R^+(x_i), F_R^+(x_i), T_R^-(x_i), I_R^-(x_i), F_R^-(x_i) \right\rangle$ and $S = \left\langle T_S^+(x_i), I_S^+(x_i), F_S^+(x_i), T_S^-(x_i), I_S^-(x_i), F_S^-(x_i) \right\rangle$ be any two BNSs in $X = (x_1, x_2, \ldots, x_m)$, then the bidirectional projection measure between R and S is defined as follows:

$$B\text{-}Proj\ (R, S) = \frac{1}{1 + \left| \frac{R.S}{\| R \|} - \frac{R.S}{\| S \|} \right|} = \frac{\| R \| \| S \|}{\| R \| \| S \| + |\, \| R \| - \| S \| \,|\, R.S} \quad (5)$$

where

$\|R\| = \sqrt{\sum_{i=1}^{m}[(T_R^+)^2(x_i) + (I_R^+)^2(x_i) + (F_R^+)^2(x_i) + (T_R^-)^2(x_i) + (I_R^-)^2(x_i) + (F_R^-)^2(x_i)]}$,

$\|S\| = \sqrt{\sum_{i=1}^{m}[(T_S^+)^2(x_i) + (I_S^+)^2(x_i) + (F_S^+)^2(x_i) + (T_S^-)^2(x_i) + (I_S^-)^2(x_i) + (F_S^-)^2(x_i)]}$

and $R.S = \sum_{i=1}^{m}[T_R^+(x_i)T_S^+(x) + I_R^+(x_i)I_S^+(x_i) + F_R^+(x_i)F_S^+(x_i) + T_R^-(x_i)T_S^-(x_i) + I_R^-(x_i)I_S^-(x_i) + F_R^-(x_i)F_S^-(x_i)]$.

Proposition 1. Let $B\text{-}Proj\ (R)_S$ be a bidirectional projection measure between any two BNSs R and S, then

1. $0 \le B\text{-}Proj\ (R, S) \le 1$;
2. $B\text{-}Proj\ (R, S) = B\text{-}Proj\ (S, R)$;
3. $B\text{-}Proj\ (R, S) = 1$ for $R = S$.

Proof.

1. For any two non-zero vectors R and S,

$$\frac{1}{1 + \left| \frac{R.S}{\| R \|} - \frac{R.S}{\| S \|} \right|} > 0, \quad \because \frac{1}{1+x} > 0, \text{ when } x > 0$$

$\therefore$ *B*-Proj (*R*, *S*) > 0, for any two non-zero vectors *R* and *S*.
B-Proj (*R*, *S*) = 0 if and only if either || *R* || = 0 or || *S* || = 0 i.e. when either *R* = (0, 0, 0, 0, 0, 0) or *S* = (0, 0, 0, 0, 0, 0) which is trivial case.

$\therefore$ *B*-Proj (*R*, *S*) ≥ 0.
For two non-zero vectors *R* and *S*,
$|| R || \, || S || + | \, || R || - || S || \, | \, R.S \geq || R || \, || S ||$
$\therefore || R || \, || S || \leq || R || \, || S || + | \, || R || - || S || \, | \, R.S$

$$\therefore \frac{|| R || \, || S ||}{|| R || \, || S || + | \, || R || - || S || \, | \, R.S} \leq 1$$

$\therefore$ *B-Proj* (*R*, *S*) ≤ 1.
$\therefore 0 \leq$ *B-Proj* (*R*, *S*) ≤ 1;

2. From definition, *R.S* = *S.R*, therefore,

$$B\text{-}Proj\ (R,\ S) = \frac{||R|| ||S||}{||R|| ||S|| + | ||R|| - ||S|| | R.S} = \frac{||S|| ||R||}{||S|| \, ||R|| + | \, ||S|| - ||R|| \, | \, S.R} = B\text{-}Proj\ (S,\ R).$$

Obviously, *B-Proj* (*R*, *S*) = 1, only when || *R* || = || *S* || i. e. when $T_R^+(x_i) = T_S^+(x_i)$, $I_R^+(x_i) = I_S^+(x_i)$, $F_R^+(x_i) = F_S^+(x_i)$, $T_R^-(x_i) = T_S^-(x_i)$, $I_R^-(x_i) = I_S^-(x_i)$, $F_R^-(x_i) = F_S^-.(x_i)$
This completes the proof.

Example 2. Assume that *R* = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, *S* = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be the BNSs in *X*, then the bidirectional projection measure between *R* on *S* is computed as given below.
B-Proj (*R*, *S*) =

$$\frac{(0.6576473).(0.9380832)}{(0.6576473).(0.9380832) + | \, 0.9380832 - 06576473 \, | \, (0.575)}$$

= 0.7927845

Definition 11
Let *R* = $\left\langle T_R^+(x_i), I_R^+(x_i), F_R^+(x_i), T_R^-(x_i), I_R^-(x_i), F_R^-(x_i) \right\rangle$ and *S* = $\left\langle T_S^+(x_i), I_S^+(x_i), F_S^+(x_i), T_S^-(x_i), I_S^-(x_i), F_S^-(x_i) \right\rangle$ be any two BNSs in *X* = (x_1, x_2, …, x_m), then hybrid projection measure is defined as the combination of projection measure and bidirectional projection measure. The hybrid projection measure between *R* and *S* is represented as follows:
Hyb-Proj (*R*, *S*) = ρ *Proj* $(R)_S$ + (1 - ρ) *B-Proj* (*R*, *S*)

$$= \rho \frac{R.S}{|| S ||} + (1 - \rho) \frac{|| R || \, || S ||}{|| R || \, || S || + | \, || R || - || S || \, | \, R.S} \qquad (6)$$

where

$||R|| = \sqrt{\sum_{i=1}^{m}[(T_R^+)^2(x_i) + (I_R^+)^2(x_i) + (F_R^+)^2(x_i) + (T_R^-)^2(x_i) + (I_R^-)^2(x_i) + (F_R^-)^2(x_i)]}$,

$||S|| = \sqrt{\sum_{i=1}^{m}[(T_S^+)^2(x_i) + (I_S^+)^2(x_i) + (F_S^+)^2(x_i) + (T_S^-)^2(x_i) + (I_S^-)^2(x_i) + (F_S^-)^2(x_i)]}$,

and

$R.S = \sum_{i=1}^{m}[T_R^+(x_i)T_S^+(x) + I_R^+(x_i)I_S^+(x_i) + F_R^+(x_i)F_S^+(x_i) + T_R^-(x_i)T_S^-(x_i) + I_R^-(x_i)I_S^-(x_i) + F_R^-(x_i)\,F_S^-(x_i)]$

where $0 \leq \rho \leq 1$.

Proposition 2
Let *Hyb-Proj* (*R*, *S*) be a hybrid projection measure between any two BNSs *R* and *S*, then
1. $0 \leq$ *Hyb-Proj* (*R*, *S*) ≤ 1;
2. *Hyb-Proj* (*R*, *S*) = *B-Proj* (*S*, *R*);
3. *Hyb-Proj* (*R*, *S*) = 1 for *R* = *S*.

Proof. The proofs of the properties under Proposition 2 are similar as Proposition 1.
Example 3. Assume that *R* = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, *S* = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be the two BNSs, then the hybrid projection measure between *R* on *S* with ρ = 0.7 is calculated as given below.
Hyb-Proj (*R*, *S*) = (0.7). (0.612952) + (1 - 0.7). (0.7927845) = 0.6669018.

4 Projection, bidirectional projection and hybrid projection based decision making methods for MADM problems with bipolar neutrosophic information

In this section, we develop projection based decision making models to MADM problems with bipolar neutrosophic assessments. Consider *E* = {E_1, E_2, …, E_m}, ($m \geq 2$) be a discrete set of *m* feasible alternatives, *F* = {F_1, F_2, …, F_n}, ($n \geq 2$) be a set of attributes under consideration and $w = (w_1, w_2, …, w_n)^T$ be the weight vector of the attributes such that $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$. Now, we present three algorithms for MADM problems involving bipolar neutrosophic information.

4.1. Method 1

Step 1. The rating of evaluation value of alternative E_i (i = 1, 2, …, *m*) for the predefined attribute F_j (j = 1, 2, …, *n*) is presented by the decision maker in terms of bipolar neutrosophic values and the bipolar neutrosophic decision matrix is constructed as given below.

$$\left\langle q_{ij} \right\rangle_{m\times n} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ q_{m1} & q_{m2} & \cdots & q_{mn} \end{bmatrix}$$

where q_{ij} = < ($T_{ij}^{+}, I_{ij}^{+}, F_{ij}^{+}, T_{ij}^{-}, I_{ij}^{-}, F_{ij}^{-}$) > with $T_{ij}^{+}, I_{ij}^{+}, F_{ij}^{+}$, $-T_{ij}^{-}$, $-I_{ij}^{-}$, $-F_{ij}^{-}$ $\in$ [0, 1] and $0 \le T_{ij}^{+} + I_{ij}^{+} + F_{ij}^{+} - T_{ij}^{-} - I_{ij}^{-} - F_{ij}^{-} \le 6$ for i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. We formulate the bipolar weighted decision matrix by multiplying weights w_j of the attributes as follows:

$$w_j \otimes \langle q_{ij} \rangle_{m\times n} = \langle z_{ij} \rangle_{m\times n} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix}$$

where $z_{ij} = w_j . q_{ij}$ = < $1 - (1 - T_{ij}^{+})^{w_j}$, $(I_{ij}^{+})^{w_j}$, $(F_{ij}^{+})^{w_j}$, $-(-T_{ij}^{-})^{w_j}$, $-(-I_{ij}^{-})^{w_j}$, $-(1 - (1 - (-F_{ij}^{-}))^{w_j})$ > = < $\mu_{ij}^{+}, \nu_{ij}^{+}, \omega_{ij}^{+}, \mu_{ij}^{-}, \nu_{ij}^{-}, \omega_{ij}^{-}$ > with $\mu_{ij}^{+}, \nu_{ij}^{+}, \omega_{ij}^{+}, -\mu_{ij}^{-}, -\nu_{ij}^{-}, -\omega_{ij}^{-} \in$ [0, 1] and $0 \le \mu_{ij}^{+} + \nu_{ij}^{+} + \omega_{ij}^{+} - \mu_{ij}^{-} - \nu_{ij}^{-} - \omega_{ij}^{-} \le 6$ for i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. We identify the bipolar neutrosophic positive ideal solution (BNPIS) [27, 28] as follows:

$z^{PIS} = \langle e_j^{+}, f_j^{+}, g_j^{+}, e_j^{-}, f_j^{-}, g_j^{-} \rangle$ = < [{ $\underset{i}{\text{Max}}(\mu_{ij}^{+})$ |j $\in \sigma$ }; { $\underset{i}{\text{Min}}(\mu_{ij}^{+})$ |j $\in \varsigma$ }], [{ $\underset{i}{\text{Min}}(\nu_{ij}^{+})$ | j $\in \sigma$ }; { $\underset{i}{\text{Max}}(\nu_{ij}^{+})$ |j $\in \varsigma$ }], [{ $\underset{i}{\text{Min}}(\omega_{ij}^{+})$ |j $\in \sigma$ }; { $\underset{i}{\text{Max}}(\omega_{ij}^{-})$ |j $\in \varsigma$ }], [{ $\underset{i}{\text{Min}}(\mu_{ij}^{-})$ |j $\in \sigma$ }; { $\underset{i}{\text{Max}}(\mu_{ij}^{-})$ |j $\in \varsigma$ }], [{ $\underset{i}{\text{Max}}(\nu_{ij}^{-})$ |j $\in \sigma$ }; { $\underset{i}{\text{Min}}(\nu_{ij}^{-})$ |j $\in \varsigma$ }], [{ $\underset{i}{\text{Max}}(\omega_{ij}^{-})$ |j $\in \sigma$ }; { $\underset{i}{\text{Min}}(\omega_{ij}^{-})$ |j $\in \varsigma$ }] >, j = 1, 2, …, *n*, where σ and ς are benefit and cost type attributes respectively.

Step 4. Determine the projection measure between z^{PIS} and $Z^i = \langle z_{ij} \rangle_{m\times n}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* by using the following Eq.

$$Proj\,(Z^i)_{z^{PIS}} = \frac{\sum_{j=1}^{n}[\mu_{ij}^{+}e_j^{+} + \nu_{ij}^{+}f_j^{+} + \omega_{ij}^{+}g_j^{+} + \mu_{ij}^{-}e_j^{-} + \nu_{ij}^{-}f_j^{-} + \omega_{ij}^{-}g_j^{-}]}{\sqrt{\sum_{j=1}^{n}[(e_j^{+})^2 + (f_j^{+})^2 + (g_j^{+})^2 + (e_j^{-})^2 + (f_j^{-})^2 + (g_j^{-})^2]}} \quad (7)$$

Step 5. Rank the alternatives in a descending order based on the projection measure $Proj\,(Z^i)_{z^{PIS}}$ for i = 1, 2, …, *m* and bigger value of $Proj\,(Z^i)_{z^{PIS}}$ determines the best alternative.

4.2. Method 2

Step 1. Give the bipolar neutrosophic decision matrix $\langle q_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. Construct weighted bipolar neutrosophic decision matrix $\langle z_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. Determine $z^{PIS} = \langle e_j^{+}, f_j^{+}, g_j^{+}, e_j^{-}, f_j^{-}, g_j^{-} \rangle$; j = 1, 2, …, *n*.

Step 4. Compute the bidirectional projection measure between z^{PIS} and $Z^i = \langle z_{ij} \rangle_{m\times n}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* using the Eq. as given below.

$$B\text{-}Proj\,(Z^i, z^{PIS}) = \frac{\| Z^i \| \| z^{PIS} \|}{\| Z^i \| \| z^{PIS} \| + | \| Z^i \| - \| z^{PIS} \| | \, Z^i . z^{PIS}} \quad (8)$$

where $\| Z^i \| = \sqrt{\sum_{j=1}^{n}[(\mu_{ij}^{+})^2 + (\nu_{ij}^{+})^2 + (\omega_{ij}^{+})^2 + (\mu_{ij}^{-})^2 + (\nu_{ij}^{-})^2 + (\omega_{ij}^{-})^2]}$, i = 1, 2, ..., *m*.

$\| z^{PIS} \| = \sqrt{\sum_{j=1}^{n}[(e_j^{+})^2 + (f_j^{+})^2 + (g_j^{+})^2 + (e_j^{-})^2 + (f_j^{-})^2 + (g_j^{-})^2]}$ and

$Z^i . z^{PIS} = \sum_{j=1}^{n}[\mu_{ij}^{+}e_j^{+} + \nu_{ij}^{+}f_j^{+} + \omega_{ij}^{+}g_j^{+} + \mu_{ij}^{-}e_j^{-} + \nu_{ij}^{-}f_j^{-} + \omega_{ij}^{-}g_j^{-}]$, i = 1, 2, ..., *m*.

Step 5. According to the bidirectional projection measure *B-Proj* (Z^i, z^{PIS}) for i = 1, 2, …, *m* the alternatives are ranked and highest value of *B-Proj* (Z^i, z^{PIS}) reflects the best option.

4.3. Method 3

Step 1. Construct the bipolar neutrosophic decision matrix $\langle q_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. Formulate the weighted bipolar neutrosophic decision matrix $\langle z_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. Identify $z^{PIS} = \langle e_j^{+}, f_j^{+}, g_j^{+}, e_j^{-}, f_j^{-}, g_j^{-} \rangle$, j = 1, 2, …, *n*.

Step 4. By combining projection measure $Proj\,(Z^i)_{z^{PIS}}$ and bidirectional projection measure *B-Proj* (Z^i, z^{PIS}), we calculate the hybrid projection measure between z^{PIS} and $Z^i = \langle z_{ij} \rangle_{m\times n}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* as follows.

$Hyb\text{-}Proj\,(Z^{i}, z^{PIS}) = \rho\ Proj\ (Z^{i})_{z^{PIS}} + (1-\rho)\ B\text{-}Proj\,(Z^{i}, z^{PIS}) =$

$$\rho\ \frac{Z^{i}.z^{PIS}}{\|z^{PIS}\|} + (1-\rho)\ \frac{\|Z^{i}\|\|z^{PIS}\|}{\|Z^{i}\|\|z^{PIS}\| + |\,\|Z^{i}\| - \|z^{PIS}\|\,|\,Z^{i}.z^{PIS}} \quad (9)$$

where $\|Z^{i}\| = \sqrt{\sum_{j=1}^{n}[(\mu_{ij}^{+})^2 + (\nu_{ij}^{+})^2 + (\omega_{ij}^{+})^2 + (\mu_{ij}^{-})^2 + (\nu_{ij}^{-})^2 + (\omega_{ij}^{-})^2]}$, i = 1, 2, …, *m*,

$\|z^{PIS}\| = \sqrt{\sum_{j=1}^{n}[(e_j^{+})^2 + (f_j^{+})^2 + (g_j^{+})^2 + (e_j^{-})^2 + (f_j^{-})^2 + (g_j^{-})^2]}$,

$Z^{i}.z^{PIS} = \sum_{j=1}^{n}[\mu_{ij}^{+}e_j^{+} + \nu_{ij}^{+}f_j^{+} + \omega_{ij}^{+}g_j^{+} + \mu_{ij}^{-}e_j^{-} + \nu_{ij}^{-}f_j^{-} + \omega_{ij}^{-}g_j^{-}]$, i = 1, 2, …, *m*, with $0 \le \rho \le 1$.

Step 5. We rank all the alternatives in accordance with the hybrid projection measure *Hyb-Proj* (Z^{i}, z^{PIS}) and greater value of *Hyb-Proj* (Z^{i}, z^{PIS}) indicates the better alternative.

5 A numerical example

We solve the MADM studied in [5, 28] where a customer desires to purchase a car. Suppose four types of car (alternatives) *E*i, (i = 1, 2, 3, 4) are taken into consideration in the decision making situation. Four attributes namely Fuel economy (F_1), Aerod (F_2), Comfort (F_3) and Safety (F_4) are considered to evaluate the alternatives. Assume the weight vector [5] of the attribute is given by $w = (w_1, w_2, w_3, w_4)$ = (0.5, 0.25, 0.125, 0.125).

Method 1: The proposed projection measure based decision making with bipolar neutrosophic information for car selection is presented in the following steps:

Step 1: Construct the bipolar neutrosophic decision matrix

The bipolar neutrosophic decision matrix $\langle q_{ij} \rangle_{m\times n}$ presented by the decision maker as given below (see Table 1)

Table 1. The bipolar neutrosophic decision matrix

	F_1	F_2	F_3	F_4
E_1	<0.5, 0.7, 0.2, -0.7, -0.3, -0.6>	<0.4, 0.5, 0.4, -0.7, -0.8, -0.4>	<0.7, 0.7, 0.5, -0.8, -0.7, -0.6>	<0.1, 0.5, 0.7, -0.5, -0.2, -0.8>
E_2	<0.9, 0.7, 0.5, -0.7, -0.7, -0.1>	<0.7, 0.6, 0.8, -0.7, -0.5, -0.1>	<0.9, 0.4, 0.6, -0.1, -0.7, -0.5>	<0.5, 0.2, 0.7, -0.5, -0.1, -0.9>
E_3	<0.3, 0.4, 0.2, -0.6, -0.3, -0.7>	<0.2, 0.2, 0.2, -0.4, -0.7, -0.4>	<0.9, 0.5, 0.5, -0.6, -0.5, -0.2>	<0.7, 0.5, 0.3, -0.4, -0.2, -0.2>
E_4	<0.9, 0.7, 0.2, -0.8, -0.6, -0.1>	<0.3, 0.5, 0.2, -0.5, -0.5, -0.2>	<0.5, 0.4, 0.5, -0.1, -0.7, -0.2>	<0.2, 0.4, 0.8, -0.5, -0.5, -0.6>

Step 2. Construction of weighted bipolar neutrosophic decision matrix

The weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ is obtained by multiplying weights of the attributes to the bipolar neutrosophic decision matrix as follows (see Table 2).

Table 2. The weighted bipolar neutrosophic decision matrix

	F_1	F_2	F_3	F_4
E_1	<0.293, 0.837, 0.447,-0.837, -0.818, -0.182 >	<0.120, 0.795, 0.841, 0.915, -0.946, -0.120>	<0.140, 0.956, 0.917, 0.972, -0.956, -0.108>	<0.013, 0.917, 0.956, -0.917, -0.818, -0.182>
E_2	<0.684, 0.837, 0.707, -0.837, -0.837, -0.051>	<0.260, 0.880, 0.946, -0.915, -0.841, -0.026>	<0.250, 0.892, 0.938, -0.750, -0.956, -0.083>	<.083, 0.818, 0.956, 0.917, -0.750, -0.250>
E_3	<0.163, 0.632, 0.447, -0.774, -0.548, -0.452>	<0.054, 0.669, 0.669, - 0.795, -0.915, -0.120>	<0.250, 0.917, 0.917, -0.938, -0.917, -0.028>	<.140, 0.917, 0.860, -0.892, -0.818, -0.028>
E_4	<0.648, 0.837, 0.447, , -0.894,- -0.774, -0.051>	<0.085, 0.841, 0.669, -0.841, -0.841, -0.054>	<0.083, 0.892, 0.917, -0.750, -0.956, -0.028>	<0.062, 0.818, 0.972, -0.917, -0.917, -0.108>

Step 3. Selection of BNPIS

The BNRPIS (z^{PIS}) = $\langle e_j^{+}, f_j^{+}, g_j^{+}, e_j^{-}, f_j^{-}, g_j^{-} \rangle$, (j = 1, 2, 3, 4) is computed from the weighted decision matrix as follows:

$\langle e_1^{+}, f_1^{+}, g_1^{+}, e_1^{-}, f_1^{-}, g_1^{-} \rangle$ = < 0.684, 0.632, 0.447, -0.894, -0.548, -0.051 >;

$\langle e_2^{+}, f_2^{+}, g_2^{+}, e_2^{-}, f_2^{-}, g_2^{-} \rangle$ = < 0.26, 0.669, 0.669, -0.915, -0.841, -0.026 >;

$\langle e_3^{+}, f_3^{+}, g_3^{+}, e_3^{-}, f_3^{-}, g_3^{-} \rangle$ = < 0.25, 0.892, 0.917, -0.972, -0.917, -0.028 >;

$\langle e_4^{+}, f_4^{+}, g_4^{+}, e_4^{-}, f_4^{-}, g_4^{-} \rangle$ = < 0.14, 0.818, 0.86, -0.917, -0.75, -0.028 >.

Step 4. Determination of weighted projection measure

The projection measure between positive ideal bipolar neutrosophic solution z^{PIS} and each weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ can be obtained as follows:

$Proj\ (Z^1)_{z^{PIS}}$ = 3.4214, $Proj\ (Z^2)_{z^{PIS}}$ = 3.4972, $Proj\ (Z^3)_{z^{PIS}}$ = 3.1821, $Proj\ (Z^4)_{z^{PIS}}$ = 3.3904.

Step 5. Rank the alternatives

We observe that $Proj\ (Z^2)_{z^{PIS}}$ > $Proj\ (Z^1)_{z^{PIS}}$ > $Proj\ (Z^4)_{z^{PIS}}$ > $Proj\ (Z^3)_{z^{PIS}}$. Therefore, the ranking order of the cars is $E_2 \succ E_1 \succ E_4 \succ E_3$. Hence, E_2 is the best alternative for the customer.

Method 2: The proposed bidirectional projection measure based decision making for car selection is presented as follows:

Step 1. Same as Method 1

Step 2. Same as Method 1

Step 3. Same as Method 1

Step 4. Calculation of bidirectional projection measure

The bidirectional projection measure between positive ideal bipolar neutrosophic solution z^{PIS} and each weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ can be determined as given below.

B-Proj (Z^1, z^{PIS}) = 0.8556, *B-Proj* (Z^2, z^{PIS}) = 0.8101, *B-Proj* (Z^3, z^{PIS}) = 0.9503, *B-Proj* (Z^4, z^{PIS}) = 0.8969.

Step 5. Ranking the alternatives

Here, we notice that *B-Proj* (Z^3, z^{PIS}) > *B-Proj* (Z^4, z^{PIS}) > *B-Proj* (Z^1, z^{PIS}) > *B-Proj* (Z^2, z^{PIS}) and therefore, the ranking order of the alternatives is obtained as $E_3 \succ E_4 \succ E_1 \succ E_2$. Hence, E_3 is the best choice among the alternatives.

Method 3: The proposed hybrid projection measure based MADM with bipolar neutrosophic information is provided as follows:

Step 1. Same as Method 1

Step 2. Same as Method 1

Step 3. Same as Method 1

Step 4. Computation of hybrid projection measure

The hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are shown in the Table 3.

Table 3. Results of hybrid projection measure for different valus of ρ

Similarity measure	ρ	Measure values	Ranking order
Hyb-Proj (Z^i, Z^{PIS})	0.25	*Hyb-Proj* (Z^1, Z^{PIS}) = 1.4573 *Hyb-Proj* (Z^2, Z^{PIS}) = 1.4551 *Hyb-Proj* (Z^3, Z^{PIS}) = 1.5297 *Hyb-Proj* (Z^4, Z^{PIS}) = 1.5622	$E_4 > E_3 > E_1 > E_2$
Hyb-Proj (Z^i, Z^{PIS})	0.50	*Hyb-Proj* (Z^1, Z^{PIS}) = 2.1034 *Hyb-Proj* (Z^2, Z^{PIS}) = 2.0991 *Hyb-Proj* (Z^3, Z^{PIS}) = 2.0740 *Hyb-Proj* (Z^4, Z^{PIS}) = 2.1270	$E_4 > E_1 > E_2 > E_3$
Hyb-Proj (Z^i, Z^{PIS})	0.75	*Hyb-Proj* (Z^1, Z^{PIS}) = 2.4940 *Hyb-Proj* (Z^2, Z^{PIS}) = 2.7432 *Hyb-Proj* (Z^3, Z^{PIS}) = 2.6182 *Hyb-Proj* (Z^4, Z^{PIS}) = 2.6919	$E_2 > E_4 > E_3 > E_1$
Hyb-Proj (Z^i, Z^{PIS})	0.90	*Hyb-Proj* (Z^1, Z^{PIS}) = 3.1370 *Hyb-Proj* (Z^2, Z^{PIS}) = 3.1296 *Hyb-Proj* (Z^3, Z^{PIS}) = 2.9448 *Hyb-Proj* (Z^4, Z^{PIS}) = 3.0308	$E_1 > E_2 > E_4 > E_3$

6 Comparative analysis

In the Section, we compare the results obtained from the proposed methods with the results derived from other existing methods under bipolar neutrosophic environment to show the effectiveness of the developed methods.

Dey et al. [28] assume that the weights of the attributes are not identical and weights are fully unknown to the decision maker. Dey et al. [28] formulated maximizing deviation model under bipolar neutrosophic assessment to compute unknown weights of the attributes as w = (0.2585, 0.2552, 0.2278, 0.2585). By considering w = (0.2585, 0.2552, 0.2278, 0.2585), the proposed projection measures are shown as follows:

$Proj\ (Z^1)_{z^{PIS}}$ = 3.3954, $Proj\ (Z^2)_{z^{PIS}}$ = 3.3872, $Proj\ (Z^3)_{z^{PIS}}$ = 3.1625, $Proj\ (Z^4)_{z^{PIS}}$ = 3.2567.

Since, $Proj\ (Z^1)_{z^{PIS}}$ > $Proj\ (Z^2)_{z^{PIS}}$ > $Proj\ (Z^4)_{z^{PIS}}$ > $Proj\ (Z^3)_{z^{PIS}}$, therefore the ranking order of the four alternatives is given by $E_1 \succ E_2 \succ E_4 \succ E_3$. Thus, E_1 is the best choice for the customer.

Now, by taking w = (0.2585, 0.2552, 0.2278, 0.2585), the bidirectional projection measures are calculated as given below.

B-Proj (Z^1, z^{PIS}) = 0.8113, *B-Proj* (Z^2, z^{PIS}) = 0.8111, *B-Proj* (Z^3, z^{PIS}) = 0.9854, *B-Proj* (Z^4, z^{PIS}) = 0.9974.

Since, *B-Proj* (Z^4, z^{PIS}) > *B-Proj* (Z^3, z^{PIS}) > *B-Proj* (Z^1, z^{PIS}) > *B-Proj* (Z^2, z^{PIS}), consequently the ranking

order of the four alternatives is given by $E_4 \succ E_3 \succ E_1 \succ E_2$. Hence, E_4 is the best option for the customer.
Also, by taking w = (0.2585, 0.2552, 0.2278, 0.2585), the proposed hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are revealed in the Table 4.

Deli et al. [5] assume the weight vector of the attributes as w = (0.5, 0.25, 0.125, 0.125) and the ranking order based on score values is presented as follows:

$$E_3 \succ E_4 \succ E_2 \succ E_1$$

Thus, E_3 was the most desirable alternative.

Dey et al. [28] employed maximizing deviation method to find unknown attribute weights as w = (0.2585, 0.2552, 0.2278, 0.2585). The ranking order of the alternatives is presented based on the relative closeness coefficient as given below.

$$E_3 \succ E_2 \succ E_4 \succ E_1.$$

Obviously, E_3 is the most suitable option for the customer.

Dey et al. [28] also consider the weight vector of the attributes as w = (0.5, 0.25, 0.125, 0.125), then using TOPSIS method, the ranking order of the cars is represented as follows:

$$E_4 \succ E_2 \succ E_3 \succ E_1.$$

So, E_4 is the most preferable alternative for the buyer. We observe that different projection measure provides different ranking order and the projection measure is weight sensitive. Therefore, decision maker should choose the projection measure and weights of the attributes in the decision making context according to his/her needs, desires and practical situation.

Conclusion

In this paper, we have defined projection, bidirectional projection measures between bipolar neutrosophic sets. Further, we have defined a hybrid projection measure by combining projection and bidirectional projection measures. Through these projection measures we have developed three methods for multi-attribute decision making models under bipolar neutrosophic environment. Finally, a car selection problem has been solved to show the flexibility and applicability of the proposed methods. Furthermore, comparison analysis of the proposed methods with the other existing methods has also been demonstrated.

The proposed methods can be extended to interval bipolar neutrosophic set environment. In future, we shall apply projection, bidirectional projection, and hybrid projection measures of interval bipolar neutrosophic sets for group decision making, medical diagnosis, weaver selection, pattern recognition problems, etc.

Table 4. Results of hybrid projection measure for different values of ρ

Similarity measure	ρ	Measure values	Ranking order
Hyb-Proj (Z^i, Z^{PIS})	0.25	*Hyb-Proj* (Z^1, Z^{PIS}) = 1.4970 *Hyb-Proj* (Z^2, Z^{PIS}) = 1.4819 *Hyb-Proj* (Z^3, Z^{PIS}) = 1.5082 *Hyb-Proj* (Z^4, Z^{PIS}) = 1.5203	$E_4 > E_3 > E_1 > E_2$
Hyb-Proj (Z^i, Z^{PIS})	0.50	*Hyb-Proj* (Z^1, Z^{PIS}) = 2.1385 *Hyb-Proj* (Z^2, Z^{PIS}) = 2.1536 *Hyb-Proj* (Z^3, Z^{PIS}) = 2.0662 *Hyb-Proj* (Z^4, Z^{PIS}) = 2.1436	$E_4 > E_1 > E_2 > E_3$
Hyb-Proj (Z^i, Z^{PIS})	0.75	*Hyb-Proj* (Z^1, Z^{PIS}) = 2.7800 *Hyb-Proj* (Z^2, Z^{PIS}) = 2.8254 *Hyb-Proj* (Z^3, Z^{PIS}) = 2.6241 *Hyb-Proj* (Z^4, Z^{PIS}) = 2.7670	$E_2 > E_4 > E_3 > E_1$
Hyb-Proj (Z^i, Z^{PIS})	0.90	*Hyb-Proj* (Z^1, Z^{PIS}) = 3.1648 *Hyb-Proj* (Z^2, Z^{PIS}) = 3.2285 *Hyb-Proj* (Z^3, Z^{PIS}) = 2.9589 *Hyb-Proj* (Z^4, Z^{PIS}) = 3.1410	$E_2 > E_1 > E_4 > E_3$

References

[1] F. Smarandache. A unifying field of logics. Neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, 1998.
[2] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman. Single valued neutrosophic sets. Multi-space and Multi-Structure, 4 (2010), 410-413.
[3] W. R. Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. In: Proc. of IEEE Conf., 1994, 305-309. DOI: 10.1109/IJCF.1994.375115.
[4] L. A. Zadeh. Fuzzy sets. Information and Control 8 (3) (1965), 338-353.
[5] I. Deli, M. Ali, and F. Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. In: International conference on advanced mechatronic systems (ICAMechS), IEEE, 2015, 249-254.
[6] Z. L. Yue. Approach to group decision making based on determining the weights of experts by using projection method. Applied Mathematical Modelling, 36(7) (2012), 2900-2910.
[7] G. Zhang, Y. Jing, H. Huang, and Y. Gao. Application of improved grey relational projection method to evaluate sustainable building envelope performance. Applied Energy, 87(2) (2010), 710-720.
[8] Z. Yue. Application of the projection method to determine weights of decision makers for group decision making. Scientia Iranica, 19(3) (2012), 872-878.
[9] Z. Yue, and Y. Jia. A direct projection-based group decision-making methodology with crisp values and interval data. Soft Computing (2015). DOI 10.1007/s00500-015-1953-5.

[10] Z. S. Xu, and, Q. Da. Projection method for uncertain multi-attribute decision making with preference information on alternatives. International Journal of Information & Decision Making, 3 (2004), 429-434.
[11] Z. Xu. On method for uncertain multiple attribute decision making problems with uncertain multiplicative preference information on alternatives. Fuzzy optimization and Decision Making, 4 (2005), 131-139.
[12] G. W. Wei. Decision-making based on projection for intuitionistic fuzzy multiple attributes. Chinese Journal of Management, 6(9) (2009), 1154-1156.
[13] X. Zhang, F. Jin, and P. Liu. A grey relational projection method for multi-attribute decision making based on intuitionistic trapezoidal fuzzy number. Applied Mathematical Modelling, 37(5) (2013), 3467-3477.
[14] S. Zeng, T. Baležentis, J. Chen, and G. Luo. A projection method for multiple attribute group decision making with intuitionistic fuzzy information. Informatica, 24(3) (2013), 485-503.
[15] Z. Xu, and H. Hu. Projection models for intuitionistic fuzzy multiple attribute decision making. International Journal of Technology & Decision Making, 9(2) (2010), 267–280.
[16] G. Sun. A group decision making method based on projection method and score function under IVIFS environment. British Journal of Mathematics & Computer Science, 9(1) (2015), 62-72.
DOI: 10.9734/BJMCS/2015/9549.
[17] C. Y. Tsao, and T. Y. Chen. A projection-based compromising method for multiple criteria decision analysis with interval-valued intuitionistic fuzzy information. Applied Soft Computing, 45 (2016), 207-223.
[18] J. Chen, and J. Ye. A projection model of neutrosophic numbers for multiple attribute decision making of clay-brick selection. Neutrosophic Sets and Systems, 12 (2016), 139-142.
[19] J. Ye. Projection and bidirectional projection measures of single valued neutrosophic sets and their decision – making method for mechanical design scheme. Journal of Experimental & Theoretical Artificial Intelligence, (2016). DOI:10.1080/0952813X.2016.1259263.
[20] J. Ye. Bidirectional projection method for multiple attribute group decision making with neutrosophic numbers. Neural Computing and Applications, (2015). DOI: 10.1007/s00521-015-2123-5.
[21] J. Ye. Interval neutrosophic multiple attribute decision-making method with credibility information. International Journal of Fuzzy Systems, (2016). DOI: 10.1007/s40815-015-0122-4.
[22] P. P. Dey, S. Pramanik, and B. C. Giri. Neutrosophic soft multi-attribute decision making based on grey relational projection method. Neutrosophic Sets and Systems, 11 (2016), 98-106.
[23] P. P. Dey, S. Pramanik, and B. C. Giri. Extended projection based models for solving multiple attribute decision making problems with interval valued neutrosophic information. In: New Trends in Neutrosophic Theory and Applications, eds. F. Smarandache and S. Pramanik, Pons Editions, Brussels, 2016, 127-140.
[24] S. Pramanik, R. Roy, and T. K. Roy, Multi criteria decision making based on projection and bidirectional projection measures of rough neutrosophic sets, New Trends in Neutrosophic Theory and Applications-Vol-II. Pons Editions, Brussels, 2017. In Press.
[25] I. Deli, and Y. A. Subas. Multiple criteria decision making method on single valued bipolar neutrosophic set based on correlation coefficient similarity measure. In: International Conference on mathematics and mathematics education (ICMME-2016), Frat University, Elazg, Turkey, 2016.
[26] M. Şahin, I. Deli, and V. Uluçay. Jaccard vector similarity measure of bipolar neutrosophic set based on multi-criteria decision making. In: International Conference on Natural Science and Engineering (ICNASE'16), Killis, 2016.
[27] V. Uluçay, I. Deli, and M. Şahin. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. Neural Computing and Applications, (2016). DOI: 10.1007/s00521-016-2479-1.
[28] P. P. Dey, S. Pramanik, and B. C. Giri. TOPSIS for solving multi-attribute decision making problems under bi-polar neutrosophic environment. In: New Trends in Neutrosophic Theory and Applications, eds. F. Smarandache and S. Pramanik, Pons Editions, Brussels, 2016, 65-77.
[29] Z. S. Xu. Theory method and applications for multiple attribute decision-making with uncertainty, Tsinghua University Press, Beijing, 2004.

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Integrated Framework of Optimization Technique and Information Theory Measures for Modeling Neutrosophic Variables

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Abstract

Uncertainty and indeterminacy are two major problems in data analysis these days. Neutrosophy is a generalization of the fuzzy theory. Neutrosophic system is based on indeterminism and falsity of concepts in addition to truth degrees. Any neutrosophy variable or concept is defined by membership, indeterminacy and non-membership functions. Finding efficient and accurate definition for neutrosophic variables is a challenging process. This paper presents a framework of Ant Colony Optimization and entropy theory to define a neutrosophic variable from concrete data. Ant Colony Optimization is an efficient search algorithm presented to define parameters of membership, indeterminacy and non-membership functions. The integrated framework of information theory measures and Ant Colony Optimization is proposed. Experimental results contain graphical representation of the membership, indeterminacy and non-membership functions for the temperature variable of the forest fires data set. The graphs demonstrate the effectiveness of the proposed framework.

Keywords

Neutrosophic set, Ant Colony Optimization, Information Theory Measures, Entropy function.

1. Introduction

These days, Indeterminacy is the key idea of the information in reality issues. This term alludes to the obscure some portion of the information representation. The fuzzy logic[1][2][3]] serves the piece of information participation degree. Thus, the indeterminacy and non-participation ideas of the information ought to be fittingly characterized and served. The neutrosophic [4][16] theory characterizes the informational index in mix with their membership, indeterminacy and non-membership degrees. Thus, the decisions could be practically figured out from these well defined information.

Smarandache in [5][13][14], and Salama et al. in[4], [9],[10][11][12][12][16] present the mathematical base of neutrosophic system and principles of neutrosophic data. Neutrosophy creates the main basics for a new mathematics field through adding indeterminacy concept to traditional and fuzzy theories[1][2][3][15].

Handling neutrosophic system is a new, moving and appealing field for scientists. In literature, neutrosophic toolbox implementation using object oriented programming operations and formulation is introduced in[18]. Moreover, a data warehouse utilizing neutrosophic methodologies and sets is applied in [17]. Also, the problem of optimizing membership functions using Particle Swarm Optimization was introduced in [24]. This same mechanism could be generalized to model neutrosophic variable.

The neutrosophic framework depends actually on the factors or variables as basics. The neutrosophic variable definition is without a doubt the base in building a precise and productive framework. The neutroshophic variable is made out of a tuple of value, membership, indeterminacy and non-membership. Pronouncing the elements of participation, indeterminacy and non-enrollment and map those to the variable values would be an attainable arrangement or solution for neutroshophic variable formulation.

Finding the subsets boundary points of membership and non-membership functions within a variable data would be an interesting optimization problem. Ant Colony Optimization (ACO)[19][20] is a meta-heuristic optimization and search procedure[22] inspired by ants lifestyle in searching for food. ACO initializes a population of ants in the search space traversing for their food according to some probabilistic transition rule. Ants follow each other basing on rode pheromone level and ant desirability to go through a specific path. The main issue is finding suitable heuristic desirability which should be based on the information conveyed from the variable itself. Information theory measures [6][20][21], [23] collect information from concrete data. The entropy definition is the measure of information conveyed in a variable. Whereas, the mutual information is the measure of data inside a crossing point between two nearby subsets of a variable. These definitions may help in finding limits of a membership function of neutrosophic variable subsets depending on the probability distribution of the data as the heuristic desirability of ants.

In a similar philosophy, the non- membership of a neutrosophic variable might be characterized utilizing the entropy and mutual information basing on the data probability distribution complement. Taking the upsides of the neutrosophic set definition; the indeterminacy capacity could be characterized from the membership and non-membership capacities.

This paper exhibits an incorporated hybrid search model amongst ACO and information theory measures to demonstrate a neutrosophic variable. The rest of this paper is organized as follows. Section 2 shows the hypotheses and algorithms. Section 3 announces the proposed integrated framework. Section 4 talks about the exploratory outcomes of applying the framework on a general variable and demonstrating the membership, indeterminacy and non-membership capacities. Conclusion and future work is displayed in section 5.

2. Theory overview

2.1 Parameters of a neutrosophic variable

In the neutrosophy theory[5][13][14], every concept is determined by rates of truth $\mu_A(x)$, indeterminacy $\sigma_A(x)$, and negation $\nu_A(x)$ in various partitions. Neutrosophy is a generalization of the fuzzy hypothesis[1][2][3]] and an extension of the regular set. Neutrosophic is connected to concepts identified with indeterminacy. Neutrosophic data is defined by three main concepts to manage uncertainty. These concepts are joined together in the triple:

$$A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle \quad (1)$$

Where

$\mu_A(x)$ is the membership degree,

$\sigma_A(x)$ is the indeterminacy degree,

$\nu_A(x)$ is the falsity degree.

These three terms form the fundamental concepts and they are independent and explicitly quantified. In neutrosophic set [7], each value $x \in X$ in set A defined by Eq. 1 is constrained by the following conditions:

$$0^- \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1^+ \quad (2)$$

$$0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+ \quad (3)$$

Whereas, Neutrosophic intuitionistic set of type 1 [8] is subjected to the following:

$$0^- \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1^+ \quad (4)$$

$$\mu_A(x) \wedge \sigma_A(x) \wedge \nu_A(x) \leq 0.5 \quad (5)$$

$$0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+ \quad (6)$$

Neutrosophic intuitionistic set of type 2 [5] is obliged by to the following conditions:

$$0.5 \leq \mu_A(x), \sigma_A(x), \nu_A(x) \quad (7)$$

$$\mu_A(x) \wedge \sigma_A(x) \leq 0.5, \ \mu_A(x) \wedge \nu_A(x) \leq 0.5, \ \sigma_A(x) \wedge \nu_A(x) \leq 0.5 \quad (8)$$

$$0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 2^+ \quad (9)$$

2.2 Ant Colony Optimization (ACO)

The ACO [19][20]is an efficient search algorithm used to find feasible solutions for complex and high dimension problems. The intelligence of the ACO is based on a population of ants traversing the search workspace for their food. Each ant follows a specific path depending on information left previously from other ants. This information is characterized by the probabilistic transition rule Eq. 10.

$$p_j^m(t) = \frac{[\eta_j] \times [\tau_{ij}(t)]}{\sum_{i \in I_m} [\eta_i] \times [\tau_{ij}(t)]} \quad (10)$$

Where

η_j is the heuristic desirability of choosing node j and

τ_{ij} is the amount of virtual pheromone on edge (i, j)

The pheromone level guides the ant through its journey. This guide is a hint of the significance level of a node (exhibited by the ants went to the nodes some time recently). The pheromone level is updated by the algorithm using the fitness function.

$$\tau_{ij}(t+1) = (1-\rho). \tau_{ij}(t) + \Delta\tau_{ij}(t) \quad (11)$$

Where $0< \rho <1$ is a decay constant used to estimate the evaporation of the pheromone from the edges. $\Delta\tau_{ij}(t)$ is the amount of pheromone deposited by the ant.

The heuristic desirability η_j describes the association between a node j and the problem solution or the fitness function of the search. If a node has a heuristic value for a certain path then the ACO will use this node in the solution of the problem. The algorithm of ACO is illustrated in figure 1.

$$\eta_j = objective\ function \quad (12)$$

ACO Algorithm

Input :pd, N

%%%% pd number of decision variables in ant, N iterations, Present position (ant) in the search universe X_{id} , ρ evaporation rate, %%%%%%

Output: Best_Solution

Initianlize_Node_Graph();
Initialize_Phermoni_Node();
While (num_of_Iterations>0) do
foreach Ant
η_j ← objective function of the search space
TRANSITION_RULE[j]= $p_j^m(t) = \frac{[\eta_j] \times [\tau_{ij}(t)]}{\sum_{i \in I_m} [\eta_i] \times [\tau_{ij}(t)]}$
Select node with the highest $p_j^m(t)$
Update Pheromone level $\tau_{ij}(t+1) = (1-\rho). \tau_{ij}(t) + \Delta\tau_{ij}(t)$
num_of_Iterations--;
end While
Best_sol← solution with best η_j
output(Best_sol)

Figure 1: Pseudo code of ant colony optimization Algorithm

2.3 Entropy and Mutual Information

Information theory measures [6][20][23] collect information from raw data. The entropy of a random variable is a function which characterizes the unexpected events of a random variable. Consider a random variable X expressing the number on a roulette wheel or the number on a fair 6-sided die.

$$H(X) = \sum_{x \in X} -P(x) \log P(x) \qquad (13)$$

Joint entropy is the entropy of a joint probability distribution, or a multi-valued random variable. For example, consider the joint entropy of a distribution of mankind (X) defined by a characteristic (Y) like age or race or health status of a disease.

$$I(X;\ Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \qquad (14)$$

3. The proposed frame work

An Integrated hybrid model of ACO and information theory measures (entropy and mutual information) as the objective function is presented. The ACO[19][20] is a heuristic searching algorithm used to locate the ideal segments of the membership and non-membership functions of a neutrosophic variable. The indeterminacy function is calculated by the membership and non-membership functions basing on the definitions of neutrosophic set illustrated in section 2. The objective function is the amount of information conveyed from various partitions in the workspace. Therefore, the total entropy [21] is used as the objective function on the variables workspace. Total entropy calculates amount of information of various partitions and intersections between these partitions. Best points in declaring the membership function are the boundaries of the partitions. The ants are designed to form the membership and non-membership partitions as illustrated in figure 2. A typical triangle membership function would take the shape of figure 2.

The triangle function of a variable partition is represented by parameters (L, (L+U)/2, U). Finding best values of L and U for all partitions would optimize the membership (non-membership) function definition. Figure 3 give a view of the ant with n partitions for each fuzzy variable.

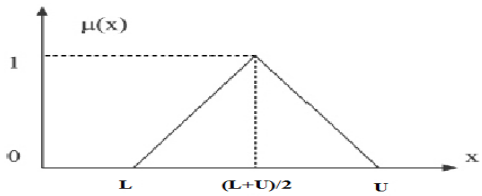

Figure 2 : corresponding to triangle fuzzy membership and its boundary parameters

Individual	L_1	U_1	L_2	U_2	……	L_n	U_n

Figure 3: Individual in ACO for Triangle function

One of the main difficulties in designing optimization problem using ACO is finding the heuristic desirability which formulates the transition rule. The amount of information deposited by neutrosophic variable inspires the ACO to calculate the transition rule and find parameters of membership, indeterminism and non-membership declarations. The membership function subsets are declared by ant parameters in figure 2. The histogram of a variable shows the data distribution of the different values. Therefore, the set of parameters are mapped to the histogram of a given variable data (Fig. 4).

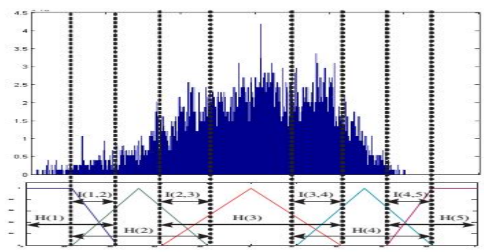

Figure 4: Fuzzy discretizing of the histogram into n joint subsets and m-1 intersections

The objective function is set as the total entropy of partitions[23]. By enhancing partition's parameters to optimize the total entropy of the histogram subsets, the optimal membership design of the variable is found.

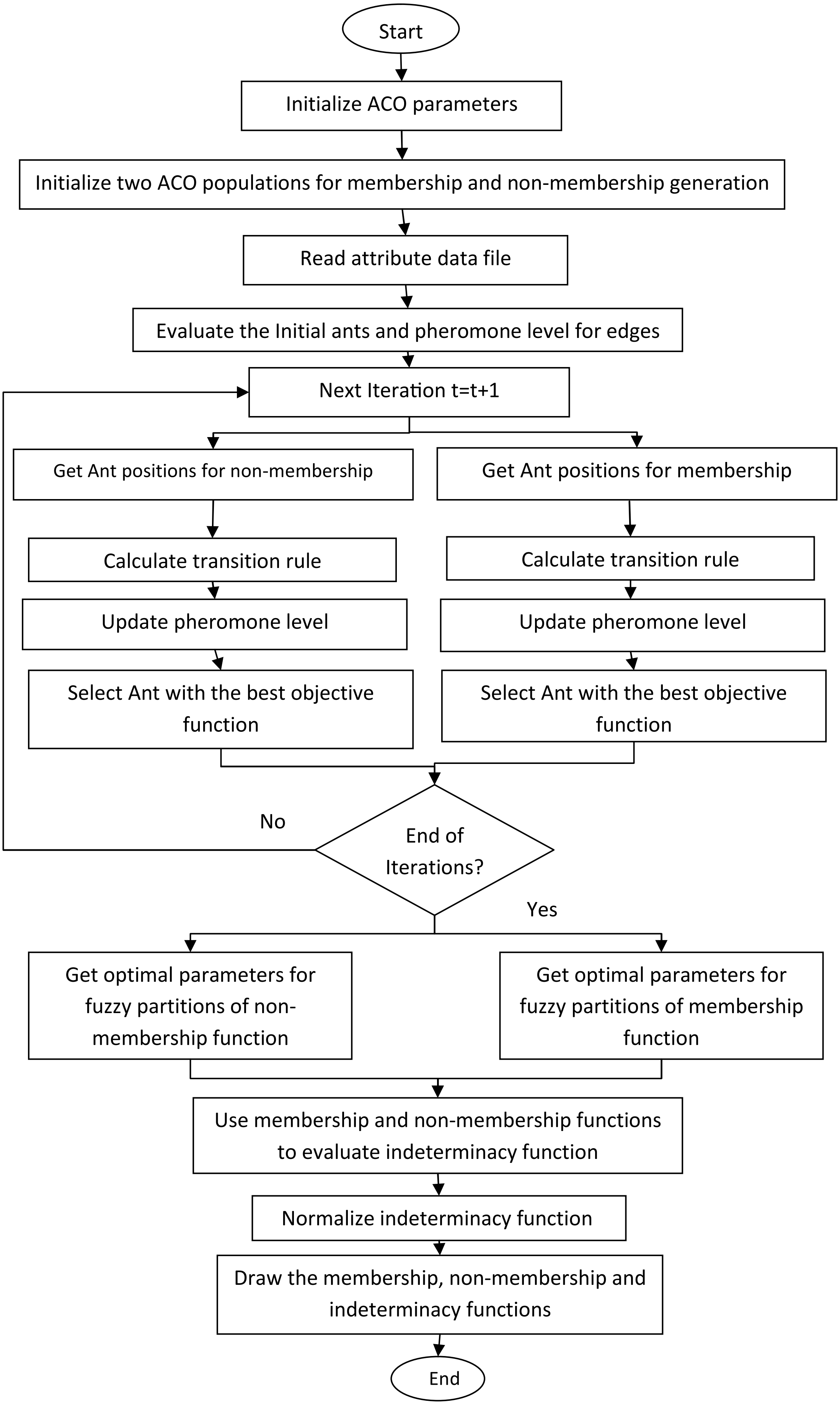

Figure 4: Flow chart for the modelling neuotrosophic variable using ACO

Input :pd, N, variable_datafile
%%%% pd number of decision variables in particle, N ieteration, Present position in the search universe X_{id} , ρ is the decay rate of phermone. %%%%%%
Output: membership, non-membership and indeterminacy function, conversion rate.
1: X←Initianlize_Ants(); % Each ant is composed of **pd** decision variables for fuzzy partitions
2:Att←Read_data(variable_datafile)
3:Objective_mem_ ← Evaluate _ Objective_of_Particles (X, P(Att)); % According to entropy and Mutual information
4: Objective_non_mem ← Evaluate _ Objective_of_Particles (X, 1-P(Att)); % According to entropy and Mutual information

5: **While** (num_of_Iterations<Max_iter)
% membership generation
6: **foreach** Ant
7: $\eta_j \leftarrow H = \sum_{i=1}^{n} H(i) - \sum_{j=1}^{n-1} I(j, j+1)$
8: $p_j^m(t) \leftarrow \frac{[\eta_j]\times[\tau_{ij}(t)]}{\sum_{i\in I_m}[\eta_i]\times[\tau_{ij}(t)]}$
9: $\tau_{ij}(t+1) = (1-\rho).\tau_{ij}(t) + \Delta\tau_{ij}(t)$
10:**end foreach**
11: Best_sol_mem← **max**(η_j) % Best found value until iteration t
% non-membership generation
12: **foreach** Ant
13 $\eta_j \leftarrow H = \sum_{i=1}^{n} H(i) - \sum_{j=1}^{n-1} I(j, j+1)$
14: $p_j^m(t) \leftarrow \frac{[\eta_j]\times[\tau_{ij}(t)]}{\sum_{i\in I_m}[\eta_i]\times[\tau_{ij}(t)]}$
15: $\tau_{ij}(t+1) = (1-\rho).\tau_{ij}(t) + \Delta\tau_{ij}(t)$
16:**end foreach**
17: Best_sol_non-mem← **max**(η_j) % Best found value until iteration t
18: **End While**
18: Best _mem ← Best_sol_mem
19: Best _non-mem ← Best_sol_non-mem
20: indeterminacy← calculate-ind(Best _mem, Best _non-mem);
21: Draw(Best _mem, Best _non-mem, indetrminancy)
22: Draw_conversions_rate()
23: **Output** membership, non-membership and indeterminacy function, conversion rate.

Function calculate-ind($\mu_A(x)$, $\nu_A(x)$)
1: Input:($\mu_A(x)$, $\nu_A(x)$)
2: Output: indeterminacy
3: $0^- - [\mu_A(x) + \nu_A(x)] \leq \sigma_A(x) \leq 3^+ - [\mu_A(x) + \nu_A(x)]$
4: indeterminacy ←**Normalize**($\sigma_A(x)$)**;**
5: Return indeterminacy
5: End Fun

Figure 5: Algorithm for the modelling neuotrosophic variable using ACO

To model (n) membership functions, variable histogram is partitioned into n overlapped subsets that produce n-1 intersections. Every joint partition corresponds to joint entropy and each overlap is modelled by mutual information. Eq.15 shows the total entropy which is assigned to the heuristic desirability of ants.

$$\eta_j = H = \sum_{i=1}^{n} H(i) - \sum_{j=1}^{n-1} I(j, j+1) \quad (15)$$

Where n is the number of partitions or subsets in the fuzzy variable,

H is the total entropy,

H(i) is the entropy of subset i,

I is the mutual information between to intersecting partitions(i,j).

In membership function modelling, the total entropy function Eq. 13, 14 and 15 are calculated by the probability distribution P(x) of the variable data frequency in various partitions and the intersecting between them. The complement of probability distribution $1 - P(x)$ is utilized to measure the non-membership of variable data in different partitions. Therefore, the non-membership objective function will compute Eq. 13, 14 and 15 with the variable data frequency complement in different partitions and overlapping.

According to Eq.3 & 6, the summation of the membership, non membership and indeterminacy values for the same instance is in the interval $[\,0^-, 3^+\,]$. Hence the indeterminacy function is declared by Eq. 16.

$$0^- - [\mu_A(x) + \nu_A(x)] \leq \sigma_A(x) \leq 3^+ - [\mu_A(x) + \nu_A(x)] \quad (16)$$

Where Eq. 9 states that the summation of the membership, non membership and indeterminacy values for the same instance is in the interval $[\,0^-, 2^+\,]$. Hence, the indeterminacy function is defined as Eq. 17.

$$0^- - [\mu_A(x) + \nu_A(x)] \leq \sigma_A(x) \leq 2^+ - [\mu_A(x) + \nu_A(x)] \quad (17)$$

By finding the membership and non-membership definition of x, the indeterminacy function $\sigma_A(x)$ could be driven easily from Eq. 15 or 16. The value of the indeterminacy function should be in the interval $[0^- \; 1^+]$, hence the $\sigma_A(x)$ function is normalized according to Eq. 18.

$$Normalized\_\sigma_A(x_i) = \frac{\sigma_A(x_i) - min(\sigma_A(x))}{max(\sigma_A(x)) - min(\sigma_A(x))} \quad (18)$$

Where $\sigma_A(x_i)$ is the indeterminacy function for the value x_i. The flow chart and algorithm of the integrated framework is illustrated in figure 5 and 6 respectively.

4. Experimental Results

The present reality issues are brimming with vulnerability and indeterminism. The neutrosophic field is worried by picking up information with degrees of enrollment, indeterminacy and non-participation. Neutrosophic framework depends on various neurtosophic factors or variables. Unfortunately, the vast majority of the informational indexes accessible are normal numeric qualities or unmitigated characteristics. Henceforth, creating approaches for characterizing a neutrosophic set from the current informational indexes is required.

The membership capacity function of a neutrosophy variable, similar to the fuzzy variable, can take a few sorts. Triangle membership is very popular due to its simplicity and accuracy. Triangle function is characterized by various overlapping partitions. These subsets are characterized by support, limit and core parameters. The most applicable parameter to a specific subset is the support which is the space of characterizing the membership degree. Finding the start and closure of a support over the universe of a variable could be an intriguing search issue suitable for optimization. Meta-heuristic search methodologies [22] give a intelligent procedure for finding ideal arrangement of solutions is any universe. ACO is a well defined search procedure that mimics ants in discovering their sustenance. Figure 3 presents the ant as an individual in a population for upgrading a triangle membership function through the ACO procedure. The ACO utilizes the initial ant population and emphasizes to achieve ideal arrangement.

Table 1:Parameters of ACO

Maximum Number of Iterations	50
Population Size (number of ants)	10
Decaying rate	0.1

The total entropy given by Eq. 15 characterizes the heuristic desirability which affects the probabilistic transition rule of ants in the ACO algorithm. The probability distribution $P(x)$ presented in Eq. 13, 14 and 15 is used to calculate the total entropy function. The ACO parameters like Maximum Number of Iterations, Population Size, and pheromone decaying rate are presented in table 1.

The non-membership function means the falsity degree in the variables values. Hence, the complement of a data probability distribution $1-P(x)$ is utilized to create the heuristic desirability of the ants in designing the non-membership function Eq. 13, 14 and 15.

The indeterminacy capacity of variable data is created by both membership and non-membership capacities of the same data using neutrosophic set declaration in section 2 and Eq. 16 or 17. Afterwards, Eq. 18 is used to normalize the indeterminacy capacity of the data. Through simulation, the ACO is applied by MATLAB , PC with Intel(R) Core (TM) CPU and 4 GB RAM. The simulation are implemented on the temperature variable from the Forest Fires data set created by: Paulo Cortez and Anbal Morais (Univ. Minho) [25]. The histogram of a random collection of the temperature data is shown in figure 7.

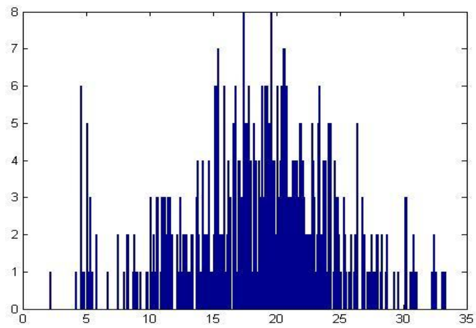

Figure 6: Temparature Variable Histogram

Figures 8: a, b and c presents the resulting membership, non-membership and indeterminacy capacities produced by applying the ACO on a random collection of the temperature data.

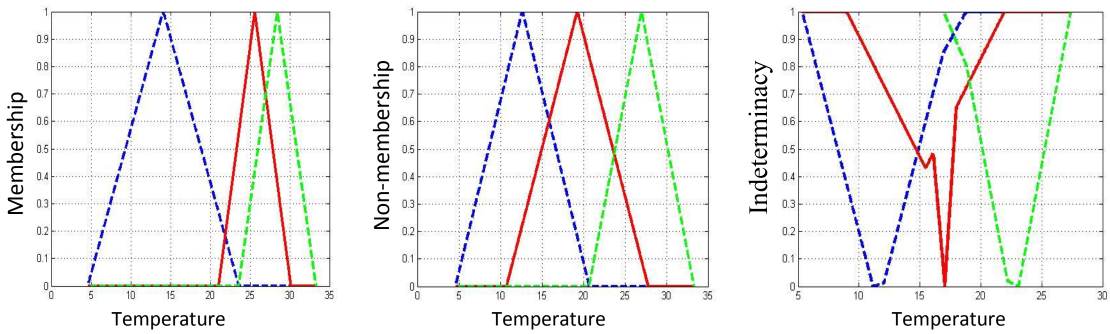

Figure 7: a. Membership Function b. Non-membership Function c. Inderminacy Function

5. Conclusion

A proposed framework utilizing the ant colony optimization and the total entropy measure for mechanizing the design of neutrosophic variable is exhibited. The membership, non-membership and indeterminacy capacities are utilized to represent the neutrosophy idea. The enrollment or truth of subset could be conjured from total entropy measure. The fundamental system aggregates the total entropy to the participation or truth subsets of a neutrosophic concept. The ant colony optimization is a meta-heuristic procedure which seeks the universe related to variable X to discover ideal segments or partitions parameters. The heuristic desirability of ants, for membership generation, is the total entropy based on the probability density function of random variable X. Thusly, the probability density complement is utilized to design non-membership capacity. The indeterminacy capacity is identified, as indicated by neutrosophic definition, by the membership and non-membership capacities. The results in light of ACO proposed system are satisfying. Therefore, the technique can be utilized as a part of data preprocessing stage within knowledge discovery system. Having sufficient data gathering, general neutrosophic variable outline for general data can be formulated.

References

[1] Atanassov, K. T.. Intuitionistic fuzzy sets, in V.Sgurev, ed.,Vii ITKRS Session, Sofia(June 1983 central Sci. and Techn. Library, Bulg. Academy of Sciences, (1984).

[2] Atanassov, K. T.. Intuitionistic fuzzy sets. *Fuzzy sets and Systems*, *20*(1), (1986), 87-96.

[3] Atanassov, K. (1988). Review and new results on intuitionistic fuzzy sets. *preprint IM-MFAIS-1-88, Sofia*, *5*, l.

[4] Alblowi, S. A., Salama, A. A., & Eisa, M. (2013). New Concepts of Neutrosophic Sets. *International Journal of Mathematics and Computer Applications Research (IJMCAR)*, *3*(4), 2013.

[5] Bhowmik, M., & Pal, M.. Intuitionistic neutrosophic set relations and some of its properties. *Journal of Information and Computing Science*, *5*(3), (2010), 183-192.

[6] Cover, T. M., & Thomas, J. A.. Entropy, relative entropy and mutual information. Elements of Information Theory, 2, (1991), 1-55.

[7] Salama, A. A., & Alblowi, S. A.. Generalized neutrosophic set and generalized neutrosophic topological spaces. *Journal computer Sci. Engineering*, *2*(7), (2012), 29-32.

[8] Salama, A. A., & Alblowi, S. A.. Neutrosophic set and neutrosophic topological spaces. *IOSR Journal of Mathematics (IOSR-JM)*, *3*(4), (2012), 31-35.

[9] Salama, A. A., & Smarandache, F.. Filters via Neutrosophic Crisp Sets. *Neutrosophic Sets and Systems*, *1*(1), (2013) ,34-38.

[10] Salama, A. A., & Alblowi, S. A.. Intuitionistic Fuzzy Ideals Topological Spaces. *Advances in Fuzzy Mathematics*, *7*(1), (2012), 51-60.

[11] Salama, A. A., Smarandache, F., & Kroumov, V.. Neutrosophic Crisp Sets & Neutrosophic Crisp Topological Spaces. *Neutrosophic Sets and Systems*, *2*, (2014), 25-30.

[12] Salama, A. A., Smarandache, F., & ALblowi, S. A.. New Neutrosophic Crisp Topological Concepts. *Neutrosophic Sets and Systems*, *4*, (2014), 50-54.

[13] Smarandache, F.. Neutrosophy and Neutrosophic Logic. In First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico, Gallup, NM (Vol. 87301), (2001).

[14] Smarandache, F.. A Unifying Field in Logics: Neutrosophic Logic. Philosophy, American Research Press, Rehoboth, NM, (1999), 1-141.

[15] Smarandache, F.. Neutrosophic set, a generialization of the intuituionistics fuzzy sets, Inter. J. Pure Appl. Math., (2005), 287 – 297.

[16] Hanafy, I. M., Salama, A. A., & Mahfouz, K. M. Neutrosophic Classical Events and Its Probability, International Journal of Mathematics and Computer Applications

Research(IJMCAR), 3(1) , (2013), 171 - 178.

[17] Salama, A. A, Ibrahim El-Henawy and Bondok, M.S.. New Structure of Data Warehouse via Neutrosophic Techniques, Neutrosophic Sets and Systems, 13. (2016) .

[18] Salama, A. A, Mohamed Abdelfattah, El-Ghareeb, H. A., Manie, A. M., Design and Implementation of Neutrosophic Data Operations Using Object Oriented Programming, International Journal of Computer Application, 5(4), (2014), 163-175.

[19] Dorigo, M., Birattari, M., & Stutzle, T. Ant colony optimization. IEEE computational intelligence magazine, 1(4), (2006), 28-39.

[20] Dorigo, M., Birattari, M., Blum, C., Clerc, M., Stützle, T., & Winfield, A. (Eds.). Ant Colony Optimization and Swarm Intelligence: 6th International Conference, ANTS 2008, Brussels, Belgium, (2008), Proceedings (Vol. 5217). Springer.

[21] Makrehchi, M., Basir, O., & Kamel, M. Generation of fuzzy membership function using information theory measures and genetic algorithm. In International Fuzzy Systems Association World Congress , (2003) (pp. 603-610). Springer Berlin Heidelberg.

[22] Osman, I. H., & Kelly, J. P. (Eds.). Meta-heuristics: theory and applications. Springer Science & Business Media, (2012).

[23] Paninski, L. Estimation of entropy and mutual information. Neural computation, 15(6), (2003), 1191-1253.

[24] Permana, K. E., & Hashim, S. Z. M. (2010). Fuzzy membership function generation using particle swarm optimization. Int. J. Open Problems Compt. Math, 3(1), 27-41.

[25] P. Cortez and A. Morais. A Data Mining Approach to Predict Forest Fires using Meteorological Data. In J. Neves, M. F. Santos and J. Machado Eds., New Trends in Artificial Intelligence, Proceedings of the 13th EPIA 2007 - Portuguese Conference on Artificial Intelligence, December, Guimaraes, Portugal, pp. 512-523, 2007. APPIA, ISBN-13 978-989-95618-0-9. Available at: http://www.dsi.uminho.pt/~pcortez/fires.pdf

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Neutrosophic Modal Logic

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Abstract: We introduce now for the first time the neutrosophic modal logic. The Neutrosophic Modal Logic includes the neutrosophic operators that express the modalities. It is an extension of neutrosophic predicate logic and of neutrosophic propositional logic.

Applications of neutrosophic modal logic are to neutrosophic modal metaphysics. Similarly to classical modal logic, there is a plethora of neutrosophic modal logics. Neutrosophic modal logics is governed by a set of neutrosophic axioms and neutrosophic rules.

Keywords: neutrosophic operators, neutrosophic predicate logic, neutrosophic propositional logic, neutrosophic epistemology, neutrosophic mereology.

1 Introduction.

The paper extends the fuzzy modal logic [1, 2, and 4], fuzzy environment [3] and neutrosophic sets, numbers and operators [5 – 12], together with the last developments of the neutrosophic environment {including (t, i, f)-neutrosophic algebraic structures, neutrosophic triplet structures, and neutrosophic overset / underset / offset} [13 - 15] passing through the symbolic neutrosophic logic [16], ultimately to neutrosophic modal logic.

All definitions, sections, and notions introduced in this paper were never done before, neither in my previous work nor in other researchers'.

Therefore, we introduce now the Neutrosophic Modal Logic and the Refined Neutrosophic Modal Logic.

Then we can extend them to Symbolic Neutrosophic Modal Logic and Refined Symbolic Neutrosophic Modal Logic, using labels instead of numerical values.

There is a large variety of neutrosophic modal logics, as actually happens in classical modal logic too. Similarly, the neutrosophic accessibility relation and possible neutrosophic worlds have many interpretations, depending on each particular application. Several neutrosophic modal applications are also listed.

Due to numerous applications of neutrosophic modal logic (see the examples throughout the paper), the introduction of the neutrosophic modal logic was needed.

Neutrosophic Modal Logic is a logic where some neutrosophic modalities have been included.

Let $\mathcal{P}$ be a neutrosophic proposition. We have the following types of **neutrosophic modalities**:

A) Neutrosophic Alethic Modalities (related to *truth*) has three neutrosophic operators:

i. ***Neutrosophic Possibility***: It is neutrosophically possible that $\mathcal{P}$.

ii. ***Neutrosophic Necessity***: It is neutrosophically necessary that $\mathcal{P}$.

iii. ***Neutrosophic Impossibility***: It is neutrosophically impossible that $\mathcal{P}$.

B) Neutrosophic Temporal Modalities (related to *time*)

It was the neutrosophic case that $\mathcal{P}$.
It will neutrosophically be that $\mathcal{P}$.
And similarly:
It has always neutrosophically been that $\mathcal{P}$.
It will always neutrosophically be that $\mathcal{P}$.

C) Neutrosophic Epistemic Modalities (related to *knowledge*):

It is neutrosophically known that $\mathcal{P}$.

D) Neutrosophic Doxastic Modalities (related to *belief*):

It is neutrosophically believed that $\mathcal{P}$.

E) Neutrosophic Deontic Modalities:

It is neutrosophically obligatory that $\mathcal{P}$.
It is neutrosophically permissible that $\mathcal{P}$.

2 Neutrosophic Alethic Modal Operators

The modalities used in classical (alethic) modal logic can be neutrosophicated by inserting the indeterminacy. We insert the **degrees of possibility** and **degrees of necessity**, as refinement of classical modal operators.

3 Neutrosophic Possibility Operator

The classical Possibility Modal Operator « $\Diamond P$ » meaning «It is possible that P» is extended to **Neutrosophic Possibility Operator**: $\Diamond_N \mathcal{P}$ meaning

«It is (t, i, f)-possible that $\mathcal{P}$ », using Neutrosophic Probability, where «(t, i, f)-possible» means t % possible (chance that $\mathcal{P}$ occurs), i % indeterminate (indeterminate-chance that $\mathcal{P}$ occurs), and f % impossible (chance that $\mathcal{P}$ does not occur).

If $\mathcal{P}(t_p, i_p, f_p)$ is a neutrosophic proposition, with t_p, i_p, f_p subsets of [0, 1], then the neutrosophic truth-value of the neutrosophic possibility operator is:

$$\lozenge_N \mathcal{P} = \left(\sup(t_p), \inf(i_p), \inf(f_p)\right),$$

which means that if a proposition P is t_p true, i_p indeterminate, and f_p false, then the value of the neutrosophic possibility operator $\lozenge_N \mathcal{P}$ is: $\sup(t_p)$ possibility, $\inf(i_p)$ indeterminate-possibility, and $\inf(f_p)$ impossibility.

For ***example***.

Let P = «It will be snowing tomorrow».

According to the meteorological center, the neutrosophic truth-value of $\mathcal{P}$ is:

$$\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}),$$

i.e. $[0.5, 0.6]$ true, $(0.2, 0.4)$ indeterminate, and $\{0.3, 0.5\}$ false.

Then the neutrosophic possibility operator is:

$$\lozenge_N \mathcal{P} = (\sup[0.5, 0.6], \inf(0.2, 0.4), \inf\{0.3, 0.5\}) = (0.6, 0.2, 0.3),$$

i.e. 0.6 possible, 0.2 indeterminate-possibility, and 0.3 impossible.

4 Neutrosophic Necessity Operator

The classical Necessity Modal Operator « $\square P$ » meaning «It is necessary that P» is extended to **Neutrosophic Necessity Operator**: $\square_N \mathcal{P}$ meaning «It is (t, i, f)-necessary that $\mathcal{P}$ », using again the Neutrosophic Probability, where similarly «(t, i, f)-necessity» means t % necessary (surety that $\mathcal{P}$ occurs), i % indeterminate (indeterminate-surety that $\mathcal{P}$ occurs), and f % unnecessary (unsurely that $\mathcal{P}$ occurs).

If $\mathcal{P}(t_p, i_p, f_p)$ is a neutrosophic proposition, with t_p, i_p, f_p subsets of [0, 1], then the neutrosophic truth value of the neutrosophic necessity operator is:

$$\square_N \mathcal{P} = \left(\inf(t_p), \sup(i_p), \sup(f_p)\right),$$

which means that if a proposition $\mathcal{P}$ is t_p true, i_p indeterminate, and f_p false, then the value of the neutrosophic necessity operator $\square_N \mathcal{P}$ is: $\inf(t_p)$ necessary, $\sup(i_p)$ indeterminate-necessity, and $\sup(f_p)$ unnecessary.

Taking the previous *example*:

$\mathcal{P}$ = «It will be snowing tomorrow», with $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$, then the neutrosophic necessity operator is:

$$\square_N \mathcal{P} = (\inf[0.5, 0.6], \sup(0.2, 0.4), \sup\{0.3, 0.5\}) = (0.5, 0.4, 0.5),$$

i.e. 0.5 necessary, 0.4 indeterminate-necessity, and 0.5 unnecessary.

5 Connection between *Neutrosophic Possibility Operator* and *Neutrosophic Necessity Operator*.

In classical modal logic, a modal operator is equivalent to the negation of the other:

$$\lozenge P \leftrightarrow \neg\square\neg P,$$

$$\square P \leftrightarrow \neg \lozenge \neg P.$$

In neutrosophic logic one has a class of neutrosophic negation operators. The most used one is:

$$\underset{N}{\neg} P(t, i, f) = \bar{P}(f, 1-i, t),$$

where t, i, f are real subsets of the interval [0, 1].

Let's check what's happening in the neutrosophic modal logic, using the previous *example*.

One had:

$$\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}),$$

then

$$\underset{N}{\neg} \mathcal{P} = \bar{\mathcal{P}}(\{0.3, 0.5\}, 1-(0.2, 0.4), [0.5, 0.6]) = \bar{\mathcal{P}}(\{0.3, 0.5\}, 1-(0.2, 0.4), [0.5, 0.6]) = \bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6]).$$

Therefore, denoting by $\underset{N}{\leftrightarrow}$ the neutrosophic equivalence, one has:

$$\underset{N}{\neg}\underset{N}{\square}\underset{N}{\neg}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}) \underset{N}{\leftrightarrow}$$

$\underset{N}{\leftrightarrow}$ It is not neutrosophically necessary that «It will not be snowing tomorrow»

$\underset{N}{\leftrightarrow}$ It is not neutrosophically necessary that $\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\underset{N}{\leftrightarrow}$ It is neutrosophically possible that $\underset{N}{\neg}\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\underset{N}{\leftrightarrow}$ It is neutrosophically possible that $\mathcal{P}([0.5, 0.6], 1-(0.6, 0.8), \{0.3, 0.5\})$

$\underset{N}{\leftrightarrow}$ It is neutrosophically possible that $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$

$$\underset{N}{\leftrightarrow}\underset{N}{\lozenge}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}) = (0.6, 0.2, 0.3).$$

Let's check the second neutrosophic equivalence.

$\underset{N}{\neg}\underset{N}{\lozenge}\underset{N}{\neg}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})\underset{N}{\leftrightarrow}$

$\underset{N}{\leftrightarrow}$ It is not neutrosophically possible that «It will not be snowing tomorrow»

$\underset{N}{\leftrightarrow}$ It is not neutrosophically possible that $\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\underset{N}{\leftrightarrow}$ It is neutrosophically necessary that $\underset{N}{\neg}\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\underset{N}{\leftrightarrow}$ It is neutrosophically necessary that $\mathcal{P}([0.5, 0.6], 1 - (0.6, 0.8), \{0.3, 0.5\})$

$\underset{N}{\leftrightarrow}$ It is neutrosophically necessary that $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$

$\underset{N}{\leftrightarrow}\underset{N}{\square}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}) = (0.6, 0.2, 0.3)$.

6 Neutrosophic Modal Equivalences

Neutrosophic Modal Equivalences hold within a certain accuracy, depending on the definitions of neutrosophic possibility operator and neutrosophic necessity operator, as well as on the definition of the neutrosophic negation – employed by the experts depending on each application. Under these conditions, one may have the following neutrosophic modal equivalences:

$$\lozenge_N\, \mathcal{P}(t_p, i_p, f_p)\underset{N}{\leftrightarrow}\underset{N}{\neg}\underset{N}{\square}\underset{N}{\neg}\mathcal{P}(t_p, i_p, f_p)$$

$$\square_N\mathcal{P}(t_p, i_p, f_p)\underset{N}{\leftrightarrow}\underset{N}{\neg}\underset{N}{\lozenge}\underset{N}{\neg}\mathcal{P}(t_p, i_p, f_p)$$

For example, other definitions for the **neutrosophic modal operators** may be:

$\lozenge_N\, \mathcal{P}(t_p, i_p, f_p) = \left(\sup(t_p), \sup(i_p), \inf(f_p)\right)$, or

$\lozenge_N\, \mathcal{P}(t_p, i_p, f_p) = \left(\sup(t_p), \frac{i_p}{2}, \inf(f_p)\right)$ etc.,

while

$\square_N\mathcal{P}(t_p, i_p, f_p) = \left(\inf(t_p), \inf(i_p), \sup(f_p)\right)$, or

$\square_N\mathcal{P}(t_p, i_p, f_p) = \left(\inf(t_p), 2i_p \cap [0,1], \sup(f_p)\right)$

etc.

7 Neutrosophic Truth Threshold

In neutrosophic logic, first we have to introduce a **neutrosophic truth threshold**, $TH = \langle T_{th}, I_{th}, F_{th}\rangle$, where T_{th}, I_{th}, F_{th} are subsets of [0, 1]. We use upper-case letters (T, I, F) in order to distinguish the neutrosophic components of the threshold from those of a proposition in general.

We can say that the proposition $\mathcal{P}(t_p, i_p, f_p)$ is **neutrosophically true** if:

$\inf(t_p) \geq \inf(T_{th})$ and $\sup(t_p) \geq \sup(T_{th})$;

$\inf(i_p) \leq \inf(I_{th})$ and $\sup(t_p) \leq \sup(I_{th})$;

$\inf(f_p) \leq \inf(F_{th})$ and $\sup(f_p) \leq \sup(F_{th})$.

For the particular case when all T_{th}, I_{th}, F_{th} and t_p, i_p, f_p are single-valued numbers from the interval [0, 1], then one has:

The proposition $\mathcal{P}(t_p, i_p, f_p)$ is **neutrosophically true** if:

$t_p \geq T_{th}$;

$i_p \leq I_{th}$;

$f_p \leq F_{th}$.

The neutrosophic truth threshold is established by experts in accordance to each applications.

8 Neutrosophic Semantics

Neutrosophic Semantics of the Neutrosophic Modal Logic is formed by a **neutrosophic frame** G_N, which is a non-empty neutrosophic set, whose elements are called **possible neutrosophic worlds**, and a **neutrosophic binary relation** $\mathcal{R}_N$, called **neutrosophic accesibility relation**, between the possible neutrosophic worlds. By notation, one has:

$$\langle G_N, \mathcal{R}_N\rangle.$$

A neutrosophic world w'_N that is neutrosophically accessible from the neutrosophic world w_N is symbolized as:

$$w_N\mathcal{R}_N w'_N.$$

In a **neutrosophic model** each neutrosophic proposition $\mathcal{P}$ has a **neutrosophic truth-value** $(t_{w_N}, i_{w_N}, f_{w_N})$ respectively to each neutrosophic world $w_N \in G_N$, where $t_{w_N}, i_{w_N}, f_{w_N}$ are subsets of [0, 1].

A **neutrosophic actual world** can be similarly noted as in classical modal logic as w_N*.

Formalization.

Let S_N be a set of neutrosophic propositional variables.

9 Neutrosophic Formulas

1) Every neutrosophic propositional variable $\mathcal{P} \in S_N$ is a neutrosophic formula.

2) If A, B are neutrosophic formulas, then $\underset{N}{\neg}A$, $A\underset{N}{\wedge}B$, $A\underset{N}{\vee}B$, $A\underset{N}{\rightarrow}B$, $A\underset{N}{\leftrightarrow}B$, and $\underset{N}{\lozenge}A$, $\underset{N}{\square}A$, are also neutrosophic formulas, where $\underset{N}{\neg}$, $\underset{N}{\wedge}$, $\underset{N}{\vee}$, $\underset{N}{\rightarrow}$, $\underset{N}{\leftrightarrow}$, and $\underset{N}{\lozenge}$,

$\underset{N}{\square}$ represent the neutrosophic negation, neutrosophic intersection, neutrosophic union, neutrosophic implication, neutrosophic equivalence, and neutrosophic possibility operator, neutrosophic necessity operator respectively.

10 Accesibility Relation in a Neutrosophic Theory

Let G_N be a set of neutrosophic worlds w_N such that each w_N chracterizes the propositions (formulas) of a given neutrosophic theory τ.

We say that the neutrosophic world w'_N is accesible from the neutrosophic world w_N, and we write: $w_N \mathcal{R}_N w'_N$ or $\mathcal{R}_N(w_N, w'_N)$, if for any proposition (formula) $\mathcal{P} \in w_N$, meaning the neutrosophic truth-value of $\mathcal{P}$ with respect to w_N is

$$\mathcal{P}\left(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}\right),$$

one has the neutrophic truth-value of $\mathcal{P}$ with respect to w'_N

$$\mathcal{P}\left(t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}\right),$$

where

$\inf\left(t_p^{w'_N}\right) \geq \inf\left(t_p^{w_N}\right)$ and $\sup\left(t_p^{w'_N}\right) \geq \sup\left(t_p^{w_N}\right)$;

$\inf\left(i_p^{w'_N}\right) \leq \inf\left(i_p^{w_N}\right)$ and $\sup\left(i_p^{w'_N}\right) \leq \sup\left(i_p^{w_N}\right)$;

$\inf\left(f_p^{w'_N}\right) \leq \inf\left(f_p^{w_N}\right)$ and $\sup\left(f_p^{w'_N}\right) \leq \sup\left(f_p^{w_N}\right)$

(in the general case when $t_p^{w_N}, i_p^{w_N}, f_p^{w_N}$ and $t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}$ are subsets of the interval [0, 1]).

But in the instant of $t_p^{w_N}, i_p^{w_N}, f_p^{w_N}$ and $t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}$ as single-values in [0, 1], the above inequalities become:

$$t_p^{w'_N} \geq t_p^{w_N},$$

$$i_p^{w'_N} \leq i_p^{w_N},$$

$$f_p^{w'_N} \leq f_p^{w_N}.$$

11 Applications

If the neutrosophic theory τ is the Neutrosophic Mereology, or Neutrosophic Gnosisology, or Neutrosophic Epistemology etc., the neutrosophic accesibility relation is defined as above.

12 Neutrosophic n-ary Accesibility Relation

We can also extend the classical **binary** accesibility relation $\mathcal{R}$ to a **neutrosophic n-ary accesibility relation**

$\mathcal{R}_N^{(n)}$, for n integer ≥ 2.

Instead of the classical $R(w, w')$, which means that the world w' is accesible from the world w, we generalize it to:

$$\mathcal{R}_N^{(n)}\left(w_{1_N}, w_{2_N}, \dots, w_{n_N}; w'_N\right),$$

which means that the neutrosophic world w'_N is accesible from the neutrosophic worlds $w_{1_N}, w_{2_N}, \dots, w_{n_N}$ all together.

13 Neutrosophic Kripke Frame

$k_N = \langle G_N, R_N \rangle$ is a neutrosophic Kripke frame, since:

$i.$ G_N is an arbitrary non-empty neutrosophic set of **neutrosophic worlds**, or **neutrosophic states**, or **neutrosophic situations**.

$ii.$ $R_N \subseteq G_N \times G_N$ is a **neutrosophic accesibility relation** of the neutrosophic Kripke frame. Actually, one has a degree of accesibility, degree of indeterminacy, and a degree of non-accesibility.

14 Neutrosophic (t, i, f)-Assignement

The Neutrosophic (t, i, f)-Assignement is a neutrosophic mapping

$$v_N: S_N \times G_N \to [0,1] \times [0,1] \times [0,1]$$

where, for any neutrosophic proposition $\mathcal{P} \in S_N$ and for any neutrosophic world w_N, one defines:

$$v_N(P, w_N) = \left(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}\right) \in [0,1] \times [0,1] \times [0,1]$$

which is the neutrosophical logical truth value of the neutrosophic proposition $\mathcal{P}$ in the neutrosophic world w_N.

15 Neutrosophic Deducibility

We say that the neutrosophic formula $\mathcal{P}$ is neutrosophically deducible from the neutrosophic Kripke frame k_N, the neutrosophic (t, i, f) – assignment v_N, and the neutrosophic world w_N, and we write as:

$$k_N, v_N, w_N \underset{N}{\vDash} \mathcal{P}.$$

Let's make the notation:

$$\alpha_N(\mathcal{P};\ k_N, v_N, w_N)$$

that denotes the neutrosophic logical value that the formula $\mathcal{P}$ takes with respect to the neutrosophic Kripke frame k_N, the neutrosophic (t, i, f)-assignement v_N, and the neutrosphic world w_N.

We define α_N by neutrosophic induction:

1. $\alpha_N(\mathcal{P};\ k_N, v_N, w_N) \overset{def}{=} v_N(\mathcal{P}, w_N)$ if $\mathcal{P} \in S_N$ and $w_N \in G_N$.
2. $\alpha_N\left(\underset{N}{\neg}\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \underset{N}{\neg}[\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]$.
3. $\alpha_N\left(\mathcal{P} \underset{N}{\wedge} Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)] \underset{N}{\wedge} [\alpha_N(Q;\ k_N, v_N, w_N)]$

4. $\alpha_N\left(\mathcal{P}\underset{N}{\vee}Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]\underset{N}{\vee}[\alpha_N(Q;\ k_N, v_N, w_N)]$

5. $\alpha_N\left(\mathcal{P}\underset{N}{\rightarrow}Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]\underset{N}{\rightarrow}[\alpha_N(Q;\ k_N, v_N, w_N)]$

6. $\alpha_N\left(\underset{N}{\lozenge}\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \langle\sup, \inf, \inf\rangle\{\alpha_N(\mathcal{P};\ k_N, v_N, w'_N), w' \in G_N \text{ and } w_N R_N w'_N\}.$

7. $\alpha_N\left(\underset{N}{\square}\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \langle\inf, \sup, \sup\rangle\{\alpha_N(\mathcal{P};\ k_N, v_N, w'_N), w'_N \in G_N \text{ and } w_N R_N w'_N\}.$

8. $\underset{N}{\vDash}\mathcal{P}$ if and only if $w_N * \vDash \mathcal{P}$ (a formula $\mathcal{P}$ is neutrosophically deducible if and only if $\mathcal{P}$ is neutrosophically deducible in the actual neutrosophic world).

We should remark that α_N has a degree of truth (t_{α_N}), a degree of indeterminacy (i_{α_N}), and a degree of falsehood (f_{α_N}), which are in the general case subsets of the interval [0, 1].

Applying $\langle\sup, \inf, \inf\rangle$ to α_N is equivalent to calculating:

$\langle\sup(t_{\alpha_N}), \inf(i_{\alpha_N}), \inf(f_{\alpha_N})\rangle$,

and similarly

$\langle\inf, \sup, \sup\rangle\alpha_N = \langle\inf(t_{\alpha_N}), \sup(i_{\alpha_N}), \sup(f_{\alpha_N})\rangle$.

16 Refined Neutrosophic Modal Single-Valued Logic

Using neutrosophic (*t*, *i*, *f*) - thresholds, we refine for the first time the neutrosophic modal logic as:

a) Refined Neutrosophic Possibility Operator.

$\underset{N}{\lozenge_1}\mathcal{P}_{(t,i,f)}$ = «It is very little possible (degree of possibility t_1) that $\mathcal{P}$», corresponding to the threshold (t_1, i_1, f_1), i.e. $0 \le t \le t_1$, $i \ge i_1$, $f \ge f_1$, for t_1 a very little number in [0, 1];

$\underset{N}{\lozenge_2}\mathcal{P}_{(t,i,f)}$ = «It is little possible (degree of possibility t_2) that $\mathcal{P}$», corresponding to the threshold (t_2, i_2, f_2), i.e. $t_1 < t \le t_2$, $i \ge i_2 > i_1$, $f \ge f_2 > f_1$;

… … …

and so on;

$\underset{N}{\lozenge_m}\mathcal{P}_{(t,i,f)}$ = «It is possible (with a degree of possibility t_m) that $\mathcal{P}$», corresponding to the threshold (t_m, i_m, f_m), i.e. $t_{m-1} < t \le t_m$, $i \ge i_m > i_{m-1}$, $f \ge f_m > f_{m-1}$.

b) Refined Neutrosophic Necessity Operator.

$\underset{N}{\square_1}\mathcal{P}_{(t,i,f)}$ = «It is a small necessity (degree of necessity t_{m+1}) that $\mathcal{P}$», i.e. $t_m < t \le t_{m+1}$, $i \ge i_{m+1} \ge i_m$, $f \ge f_{m+1} > f_m$;

$\underset{N}{\square_2}\mathcal{P}_{(t,i,f)}$ = «It is a little bigger necessity (degree of necessity t_{m+2}) that $\mathcal{P}$», i.e. $t_{m+1} < t \le t_{m+2}$, $i \ge i_{m+2} > i_{m+1}$, $f \ge f_{m+2} > f_{m+1}$;

… … …

and so on;

$\underset{N}{\square_k}\mathcal{P}_{(t,i,f)}$ = «It is a very high necessity (degree of necessity t_{m+k}) that $\mathcal{P}$», i.e. $t_{m+k-1} < t \le t_{m+k} = 1$, $i \ge i_{m+k} > i_{m+k-1}$, $f \ge f_{m+k} > f_{m+k-1}$.

17 Application of the Neutrosophic Threshold

We have introduced the term of (*t*, *i*, *f*)-physical law, meaning that a physical law has a degree of truth (*t*), a degree of indeterminacy (*i*), and a degree of falsehood (*f*). A physical law is 100% true, 0% indeterminate, and 0% false in perfect (ideal) conditions only, maybe in laboratory.

But our actual world ($w_N *$) is not perfect and not steady, but continously changing, varying, fluctuating.

For example, there are physicists that have proved a universal constant (c) is not quite universal (i.e. there are special conditions where it does not apply, or its value varies between $(c-\varepsilon, c+\varepsilon)$, for $\varepsilon > 0$ that can be a tiny or even a bigger number).

Thus, we can say that a proposition $\mathcal{P}$ is **neutrosophically nomological necessary**, if $\mathcal{P}$ is neutrosophically true at all possible neutrosophic worlds that obey the (*t*, *i*, *f*)-physical laws of the actual neutrosophic world $w_N *$.

In other words, at each possible neutrosophic world w_N, neutrosophically accesible from $w_N *$, one has:

$\mathcal{P}(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}) \ge TH(T_{th}, I_{th}, F_{th})$,

i.e. $t_p^{w_N} \ge T_{th}$, $i_p^{w_N} \le I_{th}$, and $f_p^{w_N} \ge F_{th}$.

18 Neutrosophic Mereology

Neutrosophic Mereology means the theory of the neutrosophic relations among the parts of a whole, and the neutrosophic relations between the parts and the whole.

A neutrosophic relation between two parts, and similarly a neutrosophic relation between a part and the whole, has a degree of connectibility (*t*), a degree of indeterminacy (*i*), and a degree of disconnectibility (*f*).

19 Neutrosophic Mereological Threshold

Neutrosophic Mereological Threshold is defined as:

$$TH_M = (\min(t_M), \max(i_M), \max(f_M))$$

where t_M is the set of all degrees of connectibility between the parts, and between the parts and the whole;

i_M is the set of all degrees of indeterminacy between the parts, and between the parts and the whole;

f_M is the set of all degrees of disconnectibility between the parts, and between the parts and the whole.

We have considered all degrees as single-valued numbers.

20 Neutrosophic Gnosisology

Neutrosophic Gnosisology is the theory of (*t, i, f*)-knowledge, because in many cases we are not able to completely (100%) find whole knowledge, but only a part of it (t %), another part remaining unknown (f %), and a third part indeterminate (unclear, vague, contradictory) (i %), where *t, i, f* are subsets of the interval [0, 1].

21 Neutrosophic Gnosisological Threshold

Neutrosophic Gnosisological Threshold is defined, similarly, as:

$$TH_G = (\min(t_G), \max(i_G), \max(f_G)),$$

where t_G is the set of all degrees of knowledge of all theories, ideas, propositions etc.,
i_G is the set of all degrees of indeterminate-knowledge of all theories, ideas, propositions etc.,
f_G is the set of all degrees of non-knowledge of all theories, ideas, propositions etc.
We have considered all degrees as single-valued numbers.

22 Neutrosophic Epistemology

And **Neutrosophic Epistemology**, as part of the Neutrosophic Gnosisology, is the theory of (*t, i, f*)-scientific knowledge.

Science is infinite. We know only a small part of it (t %), another big part is yet to be discovered (f %), and a third part indeterminate (unclear, vague, contradictort) (i %).

Of course, *t, i, f* are subsets of [0, 1].

23 Neutrosophic Epistemological Threshold

It is defined as:

$$TH_E = (\min(t_E), \max(i_E), \max(f_E))$$

where t_E is the set of all degrees of scientific knowledge of all scientific theories, ideas, propositions etc.,
i_E is the set of all degrees of indeterminate scientific knowledge of all scientific theories, ideas, propositions etc.,
f_E is the set of all degrees of non-scientific knowledge of all scientific theories, ideas, propositions etc.
We have considered all degrees as single-valued numbers.

24 Conclusions

We have introduced for the first time the Neutrosophic Modal Logic and the Refined Neutrosophic Modal Logic.

Symbolic Neutrosophic Logic can be connected to the neutrosophic modal logic too, where instead of numbers we may use labels, or instead of quantitative neutrosophic logic we may have a quantitative neutrosophic logic. As an extension, we may introduce **Symbolic Neutrosophic Modal Logic** and **Refined Symbolic Neutrosophic Modal Logic**, where the symbolic neutrosophic modal operators (and the symbolic neutrosophic accessibility relation) have qualitative values (labels) instead on numerical values (subsets of the interval [0, 1]).

Applications of neutrosophic modal logic are to neutrosophic modal metaphysics. Similarly to classical modal logic, there is a plethora of neutrosophic modal logics. Neutrosophic modal logics is governed by a set of neutrosophic axioms and neutrosophic rules. The neutrosophic accessibility relation has various interpretations, depending on the applications. Similarly, the notion of possible neutrosophic worlds has many interpretations, as part of possible neutrosophic semantics.

References

[1] Rod Girle, Modal Logics and Philosophy, 2nd ed., McGill-Queen's University Press, 2010.

[2] P. Hájek, D. Harmancová, A comparative fuzzy modal logic, in Fuzzy Logic in Artificial Intelligence, Lecture Notes in AI 695, 1993, 27-34.

[3] K. Hur, H. W. Kang and K. C. Lee, Fuzzy equivalence relations and fuzzy partitions, Honam Math. J. 28 (2006) 291-315.

[4] C. J. Liau, B. I-Pen Lin, Quantitative Modal Logic and Possibilistic Reasoning, 10th European Conference on Artificial Intelligence, 1992, 43-47.

[5] P.D. Liu, Y.C. Chu, Y.W. Li, Y.B. Chen, Some generalized neutrosophic number Hamacher aggregation operators and their application to Group Decision Making, Int. J. Fuzzy Syst., 2014,16(2): 242-255.

[6] P.D. Liu, L.L. Shi, The Generalized Hybrid Weighted Average Operator Based on Interval Neutrosophic Hesitant Set and Its Application to Multiple Attribute Decision Making, Neural Computing and Applications, 2015,26(2): 457-471.

[7] P.D. Liu, G.L. Tang, Some power generalized aggregation operators based on the interval neutrosophic numbers and their application to decision making, Journal of Intelligent & Fuzzy Systems 30 (2016): 2517–2528.

[8] P.D. Liu, Y.M. Wang, Interval neutrosophic prioritized OWA operator and its application to multiple attribute decision making, JOURNAL OF SYSTEMS SCIENCE & COMPLEXITY, 2016, 29(3): 681-697.

[9] P.D. Liu, H.G. Li, Multiple attribute decision making method based on some normal neutrosophic Bonferroni mean operators, Neural Computing and Applications, 2017, 28(1), 179-194.

[10] Peide Liu, Guolin Tang, Multi-criteria group decision-making based on interval neutrosophic uncertain linguistic variables and Choquet integral, Cognitive Computation, 8(6) (2016) 1036-1056.

[11] Peide Liu, Lili Zhang, Xi Liu, Peng Wang, Multi-valued Neutrosophic Number Bonferroni mean Operators and Their Application in Multiple Attribute Group Decision Making, INTERNATIONAL JOURNAL OF INFORMATION TECHNOLOGY & DECISION MAKING 15(5) (2016) 1181–1210.

[12] Peide Liu, The aggregation operators based on Archimedean t-conorm and t-norm for the single valued neutrosophic numbers and their application to Decision Making, Int. J. Fuzzy Syst., 2016,18(5):849–863.

[13] F. Smarandache, (t, i, f)-Physical Laws and (t, i, f)-Physical Constants, 47th Annual Meeting of the APS Division of Atomic, Molecular and Optical Physics, Providence, Rhode Island, Volume 61, Number 8, Monday-Friday, May 23-27, 2016; http://meetings.aps.org/Meeting/DAMOP16/Session/Q1.197

[14] F. Smarandache, M. Ali, Neutrosophic Triplet as extension of Matter Plasma, Unmatter Plasma, and Antimatter Plasma, 69th Annual Gaseous Electronics Conference, Bochum, Germany, Volume 61, Number 9, Monday-Friday, October 10-14, 2016; http://meetings.aps.org/Meeting/GEC16/Session/HT6.111

[15] Florentin Smarandache, Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics, 168 p., Pons Editions, Brussels, 2016; https://hal.archives-ouvertes.fr/hal-01340830; https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf

[16] Florentin Smarandache, Symbolic Neutrosophic Theory, EuropaNova, Brussels, 2015; https://arxiv.org/ftp/arxiv/papers/1512/1512.00047.pdf

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