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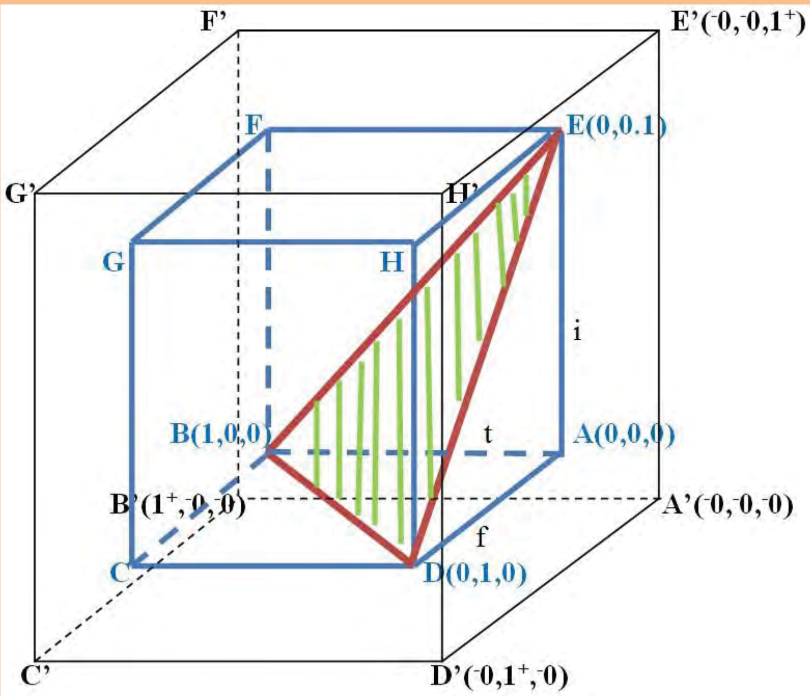

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*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Neutrosophic Soft Matrix and its application to Decision Making

Tuhin Bera[1], Nirmal Kumar Mahapatra [2]

[1] Department of Mathematics, Boror S. S. High School, Bagnan, Howrah-711312,WB, India. E-mail: tuhin78bera@gmail.com

[2]Department of Mathematics, Panskura Banamali College, Panskura RS-721152,WB, India. E-mail: nirmal_hridoy@yahoo.co.in (corresponding author)

Abstract: The motivation of this paper is to extend the concept of Neutrosophic soft matrix (NSM) theory. Some basic definitions of classical matrix theory in the parlance of neutrosophic soft set theory have been presented with proper examples. Then, a theoretical studies of some traditional operations of NSM have been developed. Finally, a decision making theory has been proposed by developing an appropriate solution algorithm, namely, score function algorithm and it has been illustrated by suitable examples.

Keywords: Intuitionistic fuzzy soft matrix, Neutrosophic soft set, Neutrosophic soft matrix and different operators, Application in decision making.

1 Introduction

Researchers in economics, sociology, medical science, engineering, environment science and many other several fields deal daily with the vague, imprecise and occasionally insufficient information of modeling uncertain data. Such uncertainties are usually handled with the help of the topics like probability, fuzzy sets [1], intuitionistic fuzzy sets [2], interval mathematics, rough sets etc. But, Molodtsov [3] has shown that each of the above topics suffers from inherent difficulties possibly due to inadequacy of their parametrization tool and there after, he initiated a novel concept 'soft set theory' for modeling vagueness and uncertainties. It is completely free from the parametrization inadequacy syndrome of different theories dealing with uncertainty. This makes the theory very convenient, efficient and easily applicable in practice. Molodtsov [3] successfully applied several directions for the applications of soft set theory, such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration and probability etc. In 2010, Cagman and Enginoglu [4] introduced a new soft set based decision making method which selects a set of optimum elements from the alternatives. Maji et al. [5, 6] have done further research on soft set theory.

Presence of vagueness demanded 'fuzzy soft set' [7] to come into picture. But satisfactory evaluation of membership values is not always possible because of the insufficiency in the available information situation. For that, Maji et al. [8, 9] have introduced the notion 'intuitionistic fuzzy soft set' in 2001. Matrices play an important role in the broad area of science and engineering. Classical matrix theory sometimes fails to solve the problems involving uncertainties. Hence, several authors proposed the matrix representation of soft set, fuzzy soft set, intuitionistic fuzzy soft set and applied these in certain decision making problems, for instance Cagman and Enginoglu [10], Yong and Chenli [11], Borah et al. [12], Neog and Sut [13], Broumi et al. [14], Mondal and Roy [15], Chetia and Das [16], Basu et al. [17], Rajarajeswari and Dhanalakshmi [18].

Evaluation of non-membership values is also not always possible for the same reason as in case of membership values and so, there exist an indeterministic part upon which hesitation survives. As a result, Smarandache [19, 20] has introduced the concept of **Neutrosophic Set (NS)** which is a generalisation of classical sets, fuzzy set, intuitionistic fuzzy set etc. Later, Maji [21] has introduced a combined concept **Neutrosophic soft set (NSS)**. Using this concept, several mathematicians have produced their research works in different mathematical structures, for instance Deli [22, 24], Broumi and Smarandache [25]. Later, this concept has been modified by Deli and Broumi [26]. Accordingly, Bera and Mahapatra [23, 27-31] introduce some view on algebraic structure on neutrosophic soft set. The development of decision making algorithms over neutrosophic soft set theory are seen in the literatures [32-37].

The present study aims to extend the NSM theory by developing the basic definitions of classical matrix theory and by establishing some results in NSS theory context. The organisation of the paper is as following :

Section 2 deals some preliminary necessary definitions which will be used in rest of this paper. In Section 3, the concept of NSM has been discussed broadly with suitable examples. Then, some traditional operators of NSM are proposed along with some properties in Section 4. In Section 5, a decision making algorithm has been developed and applied in two different situations. Firstly, it has been adopted in a class room to select the best student in an academic year and then in national security system to emphasize the security management in five mega cities. This algorithm is much more brief and simple rather than others. Moreover, a decision can be made with respect to a lot of parameters concerning the fact easily by that. That is why, this algorithm is more generous, we think. Finally, the conclusion of the present work has been stated in Section 6.

2 Preliminaries

In this section, we recall some necessary definitions related to fuzzy set, intuitionistic fuzzy soft matrix, neutrosophic set, neutrosophic soft set, NSM for the sake of completeness.

2.1 Definition [28]

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t - norm if $*$ satisfies the following conditions:
(i) $*$ is commutative and associative.
(ii) $*$ is continuous.
(iii) $a * 1 = 1 * a = a,\ \forall a \in [0,1]$.
(iv) $a * b \leq c * d$ if $a \leq c,\ b \leq d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous t-norm are $a * b = ab, a * b = \min\{a, b\}, a * b = \max\{a + b - 1, 0\}$.

2.2 Definition [28]

A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t - conorm (s - norm) if $\diamond$ satisfies the following conditions :
(i) $\diamond$ is commutative and associative.
(ii) $\diamond$ is continuous.
(iii) $a \diamond 0 = 0 \diamond a = a,\ \forall a \in [0,1]$.
(iv) $a \diamond b \leq c \diamond d$ if $a \leq c,\ b \leq d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous s-norm are $a \diamond b = a + b - ab, a \diamond b = \max\{a, b\}, a \diamond b = \min\{a + b, 1\}$.

2.3 Definition [16]

Let U be an initial universe, E be the set of parameters and $A \subseteq E$. Let, (f_A, E) be an intuitionistic fuzzy soft set over U. Then a subset of $U \times E$ is uniquely defined by $R_A = \{(u, e) : e \in A, u \in f_A(e)\}$ which is called a relation form of (f_A, E). The membership function and non-membership functions are written by $\mu_{R_A} : U \times E \to [0,1]$ and $\nu_{R_A} : U \times E \to [0,1]$ where $\mu_{R_A}(u, e) \in [0,1]$ and $\nu_{R_A}(u, e) \in [0,1]$ are the membership value and non-membership value, respectively of $u \in U$ for each $e \in E$. If $(\mu_{ij}, \nu_{ij}) = (\mu_{R_A}(u_i, e_j), \nu_{R_A}(u_i, e_j))$, we can define a matrix $[(\mu_{ij}, \nu_{ij})]_{m \times n} =$

$$\begin{pmatrix} (\mu_{11}, \nu_{11}) & (\mu_{12}, \nu_{12}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & \cdots & (\mu_{2n}, \nu_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}, \nu_{m1}) & (\mu_{m2}, \nu_{m2}) & \cdots & (\mu_{mn}, \nu_{mn}) \end{pmatrix}$$

which is called an $m \times n$ IFSM of the IFSS (f_A, E) over U. Therefore, we can say that a IFSS (f_A, E) is uniquely characterised by the matrix $[(\mu_{ij}, \nu_{ij})]_{m \times n}$ and both concepts are interchangeable. The set of all $m \times n$ IFS matrices over U will be denoted by IFSM$_{m \times n}$.

2.4 Definition [20]

Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$. That is $T_A, I_A, F_A : X \to]^-0, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x),\ F_A(x)$ and so, $^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

2.5 Definition [3]

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U. Then for $A \subseteq E$, a pair (F, A) is called a soft set over U, where $F : A \to P(U)$ is a mapping.

2.6 Definition [21]

Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then for $A \subseteq E$, a pair (F, A) is called an NSS over U, where $F : A \to NS(U)$ is a mapping.

This concept has been modified by Deli and Broumi [26] as given below.

2.7 Definition [26]

Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then, a neutrosophic soft set N over U is a set defined by a set valued function f_N representing a mapping $f_N : E \to NS(U)$ where f_N is called approximate function of the neutrosophic soft set N. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $NS(U)$ and therefore it can be written as a set of ordered pairs,

$$N = \{(e, \{< x, T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) >: x \in U\}) : e \in E\}$$

where $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \in [0,1]$ are respectively called truth-membership, indeterminacy-membership, falsity-membership function of $f_N(e)$. Since supremum of each T, I, F is 1 so the inequality $0 \leq T_{f_N(e)}(x) + I_{f_N(e)}(x) + F_{f_N(e)}(x) \leq 3$ is obvious.

2.7.1 *Example*

Let $U = \{h_1, h_2, h_3\}$ be a set of houses and $E = \{e_1(\text{beautiful}), e_2(\text{good location}), e_3, (\text{green surrounding})\}$ be a set of parameters describing the nature of houses. Let,

$$f_N(e_1) = \{< h_1,(0.5,0.6,0.3) >, < h_2,(0.4,0.7,0.6) >, < h_3,(0.6,0.2,0.3) >\}$$
$$f_N(e_2) = \{< h_1,(0.6,0.3,0.5) >, < h_2,(0.7,0.4,0.3) >, < h_3,(0.8,0.1,0.2) >\}$$
$$f_N(e_3) = \{< h_1,(0.7,0.4,0.3) >, < h_2,(0.6,0.7,0.2) >, < h_3,(0.7,0.2,0.5) >\}$$

Then $N = \{[e_1, f_N(e_1)], [e_2, f_N(e_2)], [e_3, f_N(e_3)]\}$ is an NSS over (U, E). The tabular representation of the NSS N is given in Table 1.

	$f_N(e_1)$	$f_N(e_2)$	$f_N(e_3)$
h_1	(0.5,0.6,0.3)	(0.6,0.3,0.5)	(0.7,0.4,0.3)
h_2	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.6,0.7,0.2)
h_3	(0.6,0.2,0.3)	(0.8,0.1,0.2)	(0.7,0.2,0.5)

Table 1 : Tabular form of NSS N.

2.8 Definition [26]

1. The complement of a neutrosophic soft set N is denoted by N^o and is defined by :

$$N^o = \{(e, \{< x, F_{f_N(e)}(x), 1 - I_{f_N(e)}(x), T_{f_N(e)}(x) >: x \in U\}) : e \in E\}$$

2. Let N_1 and N_2 be two NSSs over the common universe (U, E). Then N_1 is said to be the neutrosophic soft subset of N_2 if $\forall e \in E$ and $x \in U$

$$T_{f_{N_1}(e)}(x) \le T_{f_{N_2}(e)}(x), \quad I_{f_{N_1}(e)}(x) \ge I_{f_{N_2}(e)}(x), \quad F_{f_{N_1}(e)}(x) \ge F_{f_{N_2}(e)}(x).$$

We write $N_1 \subseteq N_2$ and then N_2 is the neutrosophic soft superset of N_1.

3. Let N_1 and N_2 be two NSSs over the common universe (U, E). Then their union is denoted by $N_1 \cup N_2 = N_3$ and is defined by

$$N_3 = \{(e, \{< x, T_{f_{N_3}(e)}(x), I_{f_{N_3}(e)}(x), F_{f_{N_3}(e)}(x) >: x \in U\}) : e \in E\}$$

where $T_{f_{N_3}(e)}(x) = T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x), I_{f_{N_3}(e)}(x) = I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x), F_{f_{N_3}(e)}(x) = F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x)$.

4. Let N_1 and N_2 be two NSSs over the common universe (U, E). Then their intersection is denoted by $N_1 \cap N_2 = N_4$ and is defined by :

$$N_4 = \{(e, \{< x, T_{f_{N_4}(e)}(x), I_{f_{N_4}(e)}(x), F_{f_{N_4}(e)}(x) >: x \in U\}) : e \in E\}$$

where $T_{f_{N_4}(e)}(x) = T_{f_{N_1}(e)}(x) * T_{f_{N_2}(e)}(x), I_{f_{N_4}(e)}(x) = I_{f_{N_1}(e)}(x) \diamond I_{f_{N_2}(e)}(x), F_{f_{N_4}(e)}(x) = F_{f_{N_1}(e)}(x) \diamond F_{f_{N_2}(e)}(x)$.

2.9 Definition [26]

1. Let N be a neutrosophic soft set over $N(U)$. Then a subset of $N(U) \times E$ is uniquely defined by : $R_N = \{(f_N(x), x) : x \in E, f_N(x) \in N(U)\}$ which is called a relation form of (N, E). The characteristic function of R_N is written as :

$$\Theta_{R_N} : N(U) \times E \to [0,1] \times [0,1] \times [0,1] \quad \text{by}$$
$$\Theta_{R_N}(u, x) = (T_{f_N(x)}(u), I_{f_N(x)}(u), F_{f_N(x)}(u))$$

where $T_{f_N(x)}(u), I_{f_N(x)}(u), F_{f_N(x)}(u)$ are truth-membership, indeterminacy-membership and falsity-membership of $u \in U$, respectively.

2. Let $U = \{u_1, u_2, \cdots, u_m\}, E = \{x_1, x_2, \cdots, x_n\}$ and N be a neutrosophic soft set over $N(U)$. Then,

R_N	$f_N(x_1)$	$f_N(x_2)$	$\cdots$	$f_N(x_n)$
u_1	$\Theta_{R_N}(u_1, x_1)$	$\Theta_{R_N}(u_1, x_2)$	$\cdots$	$\Theta_{R_N}(u_1, x_n)$
u_2	$\Theta_{R_N}(u_2, x_1)$	$\Theta_{R_N}(u_2, x_2)$	$\cdots$	$\Theta_{R_N}(u_2, x_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$\Theta_{R_N}(u_m, x_1)$	$\Theta_{R_N}(u_m, x_2)$	$\cdots$	$\Theta_{R_N}(u_m, x_n)$

If $a_{ij} = \Theta_{R_N}(u_i, x_j)$, we can define a matrix

$$[a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

such that $a_{ij} = (T_{f_N(x_j)}(u_i), I_{f_N(x_j)}(u_i), F_{f_N(x_j)}(u_i)) = (T^a_{ij}, I^a_{ij}, F^a_{ij})$, which is called an $m \times n$ neutrosophic soft matrix (NS-matrix) of the neutrosophic soft set N over $N(U)$.

According to this definition, a neutrosophic soft set N is uniquely characterised by a matrix $[a_{ij}]_{m\times n}$. Therefore, we shall identify any neutrosophic soft set with it's soft NS-matrix and use these two concepts as interchangeable. The set of all $m \times n$ NS-matrix over $N(U)$ will be denoted by $\tilde{N}_{m\times n}$. From now on we shall delete the subscripts $m \times n$ of $[a_{ij}]_{m\times n}$, we use $[a_{ij}]$ instead of $[a_{ij}]_{m\times n}$, since $[a_{ij}] \in \tilde{N}_{m\times n}$ means that $[a_{ij}]$ is an $m \times n$ NS-matrix for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$.

2.10 Definition [26]

Let $[a_{ij}], [b_{ij}] \in \tilde{N}_{m\times n}$. Then,

1. $[a_{ij}]$ is a zero NS-matrix, denoted by $[\tilde{0}]$, if $a_{ij} = (0, 1, 1), \ \forall i, j$.

2. $[a_{ij}]$ is a universal NS-matrix, denoted by $[\tilde{1}]$, if $a_{ij} = (1, 0, 0), \ \forall i, j$.

3. $[a_{ij}]$ is an NS-submatrix of $[b_{ij}]$, denoted by $[a_{ij}]\tilde{\subseteq}[b_{ij}]$, if $T^a_{ij} \le T^b_{ij}, \ I^a_{ij} \ge I^b_{ij}, \ F^a_{ij} \ge F^b_{ij}, \ \forall i, j$.

4. $[a_{ij}]$ and $[b_{ij}]$ are equal NS- matrices, denoted by $[a_{ij}] = [b_{ij}]$, if $a_{ij} = b_{ij}, \ \forall i, j$.

5. Complement of $[a_{ij}]$ is denoted by $[a_{ij}]^o$ and is defined as another NS-matrix $[c_{ij}]$ such that $c_{ij} = (F^a_{ij}, 1 - I^a_{ij}, T^a_{ij}), \ \forall i, j$.

3 Neutrosophic soft matrix

In this section, we have introduced some definitions and have included some new operations related to NSM.

3.1 Definition

Let $U = \{u_1, u_2, \cdots, u_m\}$ and $E = \{e_1, e_2, \cdots, e_n\}$ be the universal set of objects and the parametric set, respectively. Suppose, N be a neutrosophic soft set over (U, E) given by $N = \{<(e, f_N(e)) >: e \in E\}$ where

$$f_N(e) = \{< u, (T_{f_N(e)}(u), I_{f_N(e)}(u), F_{f_N(e)}(u)) >: u \in U\}.$$

Thus, $f_N(e)$ corresponds a relation on $\{e\} \times U$ i.e., $f_N(e) = \{(e, u_i) : 1 \leq i \leq m\}$ for each $e \in E$. It is obviously a symmetric relation. Now, consider a relation R_E on $U \times E$ given by $R_E = \{(u, e) : e \in E, u \in f_N(e)\}$. It is called a relation form of the NSS N over (U, E). The characteristic function of R_E is $\chi_{R_E} : U \times E \to [0, 1] \times [0, 1] \times [0, 1]$ and is defined as : $\chi_{R_E}(u, e) = (T_{f_N(e)}(u), I_{f_N(e)}(u), F_{f_N(e)}(u))$. The tabular representation of R_E is given in Table 2.

	e_1	e_2	$\cdots$	e_n
u_1	$\chi_{R_E}(u_1, e_1)$	$\chi_{R_E}(u_1, e_2)$	$\cdots$	$\chi_{R_E}(u_1, e_n)$
u_2	$\chi_{R_E}(u_2, e_1)$	$\chi_{R_E}(u_2, e_2)$	$\cdots$	$\chi_{R_E}(u_2, e_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$\chi_{R_E}(u_m, e_1)$	$\chi_{R_E}(u_m, e_2)$	$\cdots$	$\chi_{R_E}(u_m, e_n)$

Table 2 : Tabular form of R_E

If $a_{ij} = \chi_{R_E}(u_i, e_j)$, then we can define a matrix

$$[a_{ij}]_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where $a_{ij} = (T_{f_N(e_j)}(u_i), I_{f_N(e_j)}(u_i), F_{f_N(e_j)}(u_i)) = (T^a_{ij}, I^a_{ij}, F^a_{ij})$.

Thus, we shall identify any neutrosophic soft set with it's NSM and use these two concepts as interchangeable. Since we consider the full parametric set E, so each NSS N over (U, E) corresponds a unique NSM $[a_{ij}]_{m\times n}$ where cardinality of U and E are m and n, respectively. To get another NSM of the same order over (U, E), we need to define another NSS over (U, E). The set of all NSMs of order $m \times n$ is denoted by $NSM_{m\times n}$. Whenever U and E are fixed, we get all NSMs of unique order i.e., to obtain an NSM of distinct order, at least any of U and E will have to be changed.

3.1.1 *Example*

Consider the Example [2.7.1]. The relation form of the NSS N over the said (U, E) is

	e_1	e_2	e_3
h_1	(0.5,0.6,0.3)	(0.6,0.3,0.5)	(0.7,0.4,0.3)
h_2	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.6,0.7,0.2)
h_3	(0.6,0.2,0.3)	(0.8,0.1,0.2)	(0.7,0.2,0.5)

Hence, the NSM corresponding to this NSS N over (U, E) is :

$$[a_{ij}]_{3\times 3} = \begin{pmatrix} (0.5, 0.6, 0.3) & (0.6, 0.3, 0.5) & (0.7, 0.4, 0.3) \\ (0.4, 0.7, 0.6) & (0.7, 0.4, 0.3) & (0.6, 0.7, 0.2) \\ (0.6, 0.2, 0.3) & (0.8, 0.1, 0.2) & (0.7, 0.2, 0.5) \end{pmatrix}$$

Next, let $E_1 = \{e_1(\text{cheap}), e_2(\text{moderate}), e_3(\text{high}), e_4(\text{very high})\}$ be another set of parameters describing the cost of houses in U. The relation form of an NSS M over (U, E_1) is written as :

	e_1	e_2	e_3	e_4
h_1	(.4, .5, .5)	(.5, .7, .6)	(.2, .5, .8)	(.5, .6, .4)
h_2	(.6, .4, .7)	(.6, .3, .4)	(.7, .6, .5)	(.8, .4, .3)
h_3	(.7, .3, .4)	(.5, .2, .5)	(.8, .4, .4)	(.1, .6, .6)

Here, the NSM corresponding to the NSS M over (U, E_1) is $[b_{ij}]_{3\times 4} =$

$$\begin{pmatrix} (.4, .5, .5) & (.5, .7, .6) & (.2, .5, .8) & (.5, .6, .4) \\ (.6, .4, .7) & (.6, .3, .4) & (.7, .6, .5) & (.8, .4, .3) \\ (.7, .3, .4) & (.5, .2, .5) & (.8, .4, .4) & (.1, .6, .6) \end{pmatrix}$$

3.2 Definition

Let $A = [a_{ij}] \in NSM_{m\times n}$ where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then,

1. A is called a square NSM if $m = n$ i.e., if the number of rows and the number of columns are equal. The NSS corresponding to this NSM has the same number of objects and parameters.

2. A square NSM $A = [a_{ij}]_{n\times n}$ is called upper triangular NSM if $a_{ij} = (0, 1, 1),\ \forall i > j$ and is called lower triangular NSM if $a_{ij} = (0, 1, 1),\ \forall i < j$.

A is called triangular NSM if it is either neutrosophic soft upper triangular or neutrosophic soft lower triangular matrix.

3. The transpose of a square NSM $A = [a_{ij}]_{n\times n}$ is another square NSM of same order obtained from $[a_{ij}]$ by interchanging it's rows and columns. It is denoted by A^t. Thus $A^t = [a_{ij}]^t = [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^t = [(T^a_{ji}, I^a_{ji}, F^a_{ji})]$. The NSS corresponding to A^t becomes a new NSS over the same universe and the same parametric set.

4. A square NSM $A = [a_{ij}]_{n\times n}$ is said to be a symmetric NSM if $A^t = A$ i.e., if $a_{ij} = a_{ji}$ or $(T^a_{ij}, I^a_{ij}, F^a_{ij}) = (T^a_{ji}, I^a_{ji}, F^a_{ji}),\ \forall i, j$.

3.3 Definition

Let $A = [a_{ij}] \in NSM_{m\times n}$, where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then, the scalar multiple of NSM A by a scalar k is defined by $kA = [ka_{ij}]_{m\times n}$ where $0 \leq k \leq 1$.

3.3.1 *Example*

Let $A = [a_{ij}]_{2\times 3} =$

$$\begin{pmatrix} (0.4,0.5,0.5) & (0.5,0.7,0.6) & (0.5,0.6,0.4) \\ (0.6,0.4,0.7) & (0.7,0.3,0.4) & (0.8,0.4,0.3) \end{pmatrix}$$

be an NSM. Then the scalar multiple of this matrix by $k = 0.5$ is $kA = [ka_{ij}]_{2\times 3} =$

$$\begin{pmatrix} (0.20,0.25,0.25) & (0.25,0.35,0.30) & (0.25,0.30,0.20) \\ (0.30,0.20,0.35) & (0.35,0.15,0.20) & (0.40,0.20,0.15) \end{pmatrix}$$

3.4 Proposition

Let $A = [a_{ij}], B = [b_{ij}] \in NSM_{m\times n}$ where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. For two scalars $s, k \in [0,1]$,
(i) $s(kA) = (sk)A$. (ii) $s \le k \Rightarrow sA \le kA$. (iii) $A \subseteq B \Rightarrow sA \subseteq sB$.

Proof.

$$\begin{aligned} \text{(i)}\ \ s(kA) &= s[ka_{ij}] = s[(kT^a_{ij}, kI^a_{ij}, kF^a_{ij})] \\ &= [(skT^a_{ij}, skI^a_{ij}, skF^a_{ij})] = sk[(T^a_{ij}, I^a_{ij}, F^a_{ij})] \\ &= sk[a_{ij}] = (sk)A. \end{aligned}$$

(ii) Since $T^a_{ij}, I^a_{ij}, F^a_{ij} \in [0,1],\ \forall i,j$ so, $sT^a_{ij} \le kT^a_{ij}$, $sI^a_{ij} \le kI^a_{ij}$, $sF^a_{ij} \le kF^a_{ij}$.
Now, $sA = [(sT^a_{ij}, sI^a_{ij}, sF^a_{ij})] \le [(kT^a_{ij}, kI^a_{ij}, kF^a_{ij})] = kA$.

$$\begin{aligned} \text{(iii)}\ \ A \subseteq B &\Rightarrow [a_{ij}] \subseteq [b_{ij}] \\ &\Rightarrow T^a_{ij} \le T^b_{ij}, I^a_{ij} \ge I^b_{ij}, F^a_{ij} \ge F^b_{ij}, \forall i,j \\ &\Rightarrow sT^a_{ij} \le sT^b_{ij}, sI^a_{ij} \ge sI^b_{ij}, sF^a_{ij} \ge sF^b_{ij}, \forall i,j \\ &\Rightarrow s[a_{ij}] \subseteq s[b_{ij}] \\ &\Rightarrow sA \subseteq sB \end{aligned}$$

3.5 Theorem

Let $A = [a_{ij}]_{m\times n}$ be an NSM where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then,
(i) $(kA)^t = k\,A^t$ for $k \in [0,1]$ being a scalar.
(ii) $(A^t)^t = A$.
(iii) If $A = [a_{ij}]_{n\times n}$ is an upper triangular (lower triangular) NSM, then A^t is lower triangular (upper triangular) NSM.

Proof.(i) Here $(kA)^t,\ k\,A^t \in NSM_{n\times m}$. Now,

$$\begin{aligned} (kA)^t &= [(kT^a_{ij}, kI^a_{ij}, kF^a_{ij})]^t = [(kT^a_{ji}, kI^a_{ji}, kF^a_{ji})] \\ &= k[(T^a_{ji}, I^a_{ji}, F^a_{ji})] = k[(T^a_{ij}, I^a_{ij}, F^a_{ij})]^t = k\,A^t. \end{aligned}$$

(ii) Here $A^t \in NSM_{n\times m}$ and so $(A^t)^t \in NSM_{m\times n}$. Now,

$$\begin{aligned} (A^t)^t &= ([(T^a_{ij}, I^a_{ij}, F^a_{ij})]^t)^t = [(T^a_{ji}, I^a_{ji}, F^a_{ji})]^t \\ &= [(T^a_{ij}, I^a_{ij}, F^a_{ij})] = A. \end{aligned}$$

(iii) Straight forward.

3.6 Definition

Let $A = [a_{ij}] \in NSM_{m\times n}$, where $m = n$ and $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then, the trace of NSM A is denoted by $tr(A)$ and is defined as $tr(A) = \sum_{i=1}^m [T^a_{ii} - (I^a_{ii} + F^a_{ii})]$.

3.6.1 *Example*

Let $A = [a_{ij}]_{3\times 3} =$

$$\begin{pmatrix} (0.5,0.6,0.3) & (0.6,0.3,0.5) & (0.7,0.4,0.3) \\ (0.4,0.7,0.6) & (0.7,0.4,0.3) & (0.6,0.7,0.2) \\ (0.6,0.2,0.3) & (0.8,0.1,0.2) & (0.7,0.2,0.5) \end{pmatrix}$$

be an NSM. Then $tr(A) = (0.5-0.6-0.3)+(0.7-0.4-0.3)+(0.7-0.2-0.5) = -0.4$

3.7 Proposition

Let $A = [a_{ij}] \in NSM_{n\times n}$, where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij})$. If $k \in [0,1]$ is a scalar, then $tr(kA) = k\,tr(A)$.

Proof. $tr(kA) = \sum_{i=1}^n [kT^a_{ii} - (kI^a_{ii} + kF^a_{ii})] = k\sum_{i=1}^n [T^a_{ii} - (I^a_{ii} + F^a_{ii})] = k\,tr(A)$.

3.8 Max-Min Product of NSMs

Two NSMs A and B are said to be conformable for the product $A \otimes B$ if the number of columns of the NSM A be equal to the number of rows of the NSM B and this product becomes also an NSM. If $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$, then $A \otimes B = [c_{ik}]_{m\times p}$ where $a_{ij} = (T^a_{ij}, I^a_{ij}, F^a_{ij}), b_{jk} = (T^b_{jk}, I^b_{jk}, F^b_{jk})$ and $c_{ik} = (\max_j \min(T^a_{ij}, T^b_{jk}), \min_j \max(I^a_{ij}, I^b_{jk}), \min_j \max(F^a_{ij}, F^b_{jk}))$. Clearly, $B \otimes A$ can not be defined here.

3.8.1 *Example*

Let $A = [a_{ij}]_{3\times 2} = \begin{pmatrix} (0.5,0.6,0.3) & (0.6,0.3,0.5) \\ (0.4,0.7,0.6) & (0.7,0.4,0.3) \\ (0.6,0.2,0.3) & (0.8,0.1,0.2) \end{pmatrix}$

and $B = [b_{jk}]_{2\times 3} =$

$$\begin{pmatrix} (0.4,0.5,0.5) & (0.5,0.7,0.6) & (0.5,0.6,0.4) \\ (0.6,0.4,0.7) & (0.7,0.3,0.4) & (0.8,0.4,0.3) \end{pmatrix}$$

be two NSMs. Then, $A \otimes B = [c_{ik}]_{3\times 3} =$

$$\begin{pmatrix} (0.6,0.4,0.5) & (0.6,0.3,0.5) & (0.6,0.4,0.4) \\ (0.6,0.4,0.6) & (0.7,0.4,0.4) & (0.7,0.4,0.3) \\ (0.6,0.4,0.5) & (0.7,0.3,0.4) & (0.8,0.4,0.3) \end{pmatrix}$$

One calculation is provided herewith for convenience of $A \otimes B$.

$$\begin{aligned} T^c_{21} &= \max_j\{\min(T^a_{21}, T^b_{11}), \min(T^a_{22}, T^b_{21})\} \\ &= \max\{\min(0.4, 0.4), \min(0.7, 0.6)\} = 0.6 \\ I^c_{21} &= \min_j\{\max(I^a_{21}, I^b_{11}), \max(I^a_{22}, I^b_{21})\} \\ &= \min\{\max(0.7, 0.5), \max(0.4, 0.4)\} = 0.4 \\ F^c_{21} &= \min_j\{\max(F^a_{21}, F^b_{11}), \max(F^a_{22}, F^b_{21})\} \\ &= \min\{\max(0.6, 0.5), \max(0.3, 0.7)\} = 0.6 \end{aligned}$$

Thus, $c_{21} = (0.6, 0.4, 0.6)$ and so on.

3.9 Theorem

Let $A = [a_{ij}]_{m\times n}, B = [b_{jk}]_{n\times p}$ be two NSMs where a_{ij}=$(T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then, $(A \otimes B)^t = B^t \otimes A^t$

Proof. Let $A \otimes B = [c_{ik}]_{m\times p}$. Then $(A \otimes B)^t = [c_{ki}]_{p\times m}, A^t$ = $[a_{ji}]_{n\times m}, B^t = [b_{kj}]_{p\times n}$ and so the order of $(B^t \otimes A^t)$ is $(p\times m)$. Now,

$$\begin{aligned} & (A \otimes B)^t \\ =\; & [(T^c_{ki}, I^c_{ki}, F^c_{ki})]_{p\times m} \\ =\; & [(\max_j \min(T^b_{kj}, T^a_{ji}),\ \min_j \max(I^b_{kj}, I^a_{ji}), \\ & \min_j \max(F^b_{kj}, F^a_{ji}))]_{p\times m} \\ =\; & [(T^b_{kj}, I^b_{kj}, F^b_{kj})]_{p\times n} \otimes [(T^a_{ji}, I^a_{ji}, F^a_{ji})]_{n\times m} = B^t \otimes A^t. \end{aligned}$$

4 Operators of NSMs

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})],\ B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})] \in NSM_{m\times n}$. Then,

(i) **Union** $A \cup B = C$ where $T^c_{ij} = T^a_{ij} \diamond T^b_{ij},\ I^c_{ij} = I^a_{ij} * I^b_{ij},\ F^c_{ij} = F^a_{ij} * F^b_{ij},\ \forall i, j.$

(ii) **Intersection** $A \cap B = C$ where $T^c_{ij} = T^a_{ij} * T^b_{ij},\ I^c_{ij} = I^a_{ij} \diamond I^b_{ij},\ F^c_{ij} = F^a_{ij} \diamond F^b_{ij},\ \forall i, j.$

(iii)**Arithmetic mean** $A \circledast B = C$ where $T^c_{ij} = \frac{T^a_{ij}+T^b_{ij}}{2},\ I^c_{ij} = \frac{I^a_{ij}+I^b_{ij}}{2},\ F^c_{ij} = \frac{F^a_{ij}+F^b_{ij}}{2},\ \forall i, j.$

(iv) **Weighted arithmetic mean** $A \circledast^w B = C$ where $T^c_{ij} = \frac{w_1 T^a_{ij}+w_2 T^b_{ij}}{w_1+w_2}, I^c_{ij} = \frac{w_1 I^a_{ij}+w_2 I^b_{ij}}{w_1+w_2},\ F^c_{ij} = \frac{w_1 F^a_{ij}+w_2 F^b_{ij}}{w_1+w_2},\ \forall i, j$ and $w_1, w_2 > 0.$

(v) **Geometric mean** $A \circledcirc B = C$ where $T^c_{ij} = \sqrt{T^a_{ij} \cdot T^b_{ij}},\ I^c_{ij} = \sqrt{I^a_{ij} \cdot I^b_{ij}},\ F^c_{ij} = \sqrt{F^a_{ij} \cdot F^b_{ij}},\ \forall i, j.$

(vi) **Weighted geometric mean** $A \circledcirc^w B = C$ where

$$T^c_{ij} = \sqrt[(w_1+w_2)]{(T^a_{ij})^{w_1} \cdot (T^b_{ij})^{w_2}},$$
$$I^c_{ij} = \sqrt[(w_1+w_2)]{(I^a_{ij})^{w_1} \cdot (I^b_{ij})^{w_2}},$$
$$F^c_{ij} = \sqrt[(w_1+w_2)]{(F^a_{ij})^{w_1} \cdot (F^b_{ij})^{w_2}},\ \forall i, j \text{ and } w_1, w_2 > 0.$$

(vii) **Harmonic mean** $A \boxdot B = C$ where $T^c_{ij} = \frac{2T^a_{ij}T^b_{ij}}{T^a_{ij}+T^b_{ij}}, I^c_{ij} = \frac{2I^a_{ij}I^b_{ij}}{I^a_{ij}+I^b_{ij}}, F^c_{ij} = \frac{2F^a_{ij}F^b_{ij}}{F^a_{ij}+F^b_{ij}},\ \forall i, j.$

(viii) **Weighted harmonic mean** $A \boxdot^w B = C$ where $T^c_{ij} = \frac{w_1+w_2}{\frac{w_1}{T^a_{ij}}+\frac{w_2}{T^b_{ij}}},\ I^c_{ij} = \frac{w_1+w_2}{\frac{w_1}{I^a_{ij}}+\frac{w_2}{I^b_{ij}}}, F^c_{ij} = \frac{w_1+w_2}{\frac{w_1}{F^a_{ij}}+\frac{w_2}{F^b_{ij}}},\ \forall i, j$ and $w_1, w_2 > 0.$

4.1 Proposition

Let $A = [a_{ij}], B = [b_{ij}] \in NSM_{m\times n}$, where a_{ij} = $(T^a_{ij}, I^a_{ij}, F^a_{ij})$. Then,
(i) $(A \cup B)^t = A^t \cup B^t,\quad (A \cap B)^t = A^t \cap B^t.$
(ii) $(A \circledast B)^t = A^t \circledast B^t,\quad (A \circledast^w B)^t = A^t \circledast^w B^t.$
(iii) $(A \circledcirc B)^t = A^t \circledcirc B^t,\quad (A \circledcirc^w B)^t = A^t \circledcirc^w B^t.$
(iv) $(A \boxdot B)^t = A^t \boxdot B^t,\quad (A \boxdot^w B)^t = A^t \boxdot^w B^t.$

Proof. (i) Here $A \cup B, (A \cup B)^t, A^t, B^t, A^t \cup B^t \in NSM_{m\times n}$. Now,

$$\begin{aligned} (A \cup B)^t &= [(T^a_{ij} \diamond T^b_{ij}, I^a_{ij} * I^b_{ij}, F^a_{ij} * F^b_{ij})]^t \\ &= [(T^a_{ji} \diamond T^b_{ji}, I^a_{ji} * I^b_{ji}, F^a_{ji} * F^b_{ji})] \\ &= [(T^a_{ji}, I^a_{ji}, F^a_{ji})] \cup [(T^b_{ji}, I^b_{ji}, F^b_{ji})] \\ &= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^t \cup [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^t \\ &= A^t \cup B^t. \end{aligned}$$

Next $A \cap B, (A \cap B)^t, A^t \cap B^t \in NSM_{m\times n}$. Now,

$$\begin{aligned} (A \cap B)^t &= [(T^a_{ij} * F^b_{ij}, I^a_{ij} \diamond (1 - I^b_{ij}), F^a_{ij} \diamond T^b_{ij})]^t \\ &= [(T^a_{ji} * F^b_{ji}, I^a_{ji} \diamond (1 - I^b_{ji}), F^a_{ji} \diamond T^b_{ji})] \\ &= [(T^a_{ji}, I^a_{ji}, F^a_{ji})] \cap [(T^b_{ji}, I^b_{ji}, F^b_{ji})] \\ &= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^t \cap [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^t \\ &= A^t \cap B^t. \end{aligned}$$

Remaining others can be proved in the similar manner.

4.2 Proposition

Let $A = [a_{ij}], B = [b_{ij}]$ are upper triangular (lower triangular) NSMs of same order. Then (i) $A \cup B,\ A \cap B$ (ii) $A \circledast B,\ A \circledast^w B$ (iii) $A \circledcirc B,\ A \circledcirc^w B$ all are upper triangular (lower triangular) NSMs.

Proof. Straight forward.

4.3 Theorem

Let $A = [a_{ij}], B = [b_{ij}]$ be two symmetric NSMs of same order. Then,

(i) $A \cup A^t, A \cup B, A \cap B, A \circledast B, A \circledast^w B, A \circledcirc B, A \circledcirc^w B, A \boxdot B, A \boxdot^w B$ are so.

(ii) $A \otimes B$ is symmetric iff $A \otimes B = B \otimes A$.
(iii) $A \otimes A^t$, $A^t \otimes A$ both are symmetric.

Proof. Here $A^t = A$ and $B^t = B$ as both are symmetric NSMs. Clearly $A \cup A^t, A \cup B, A \cap B, A \circledast B, A \circledast^w B, A \circledcirc B, A \circledcirc^w B, A \boxdot B, A \boxdot^w B, A \otimes B, B \otimes A, A \otimes A^t, A^t \otimes A$ all are well defined as both the NSMs are same order and square. Now,

(i) These are left to the reader.
(ii) $(A \otimes B)^t = B^t \otimes A^t = B \otimes A = A \otimes B$.
(iii) $(A \otimes A^t)^t = (A^t)^t \otimes A^t = A \otimes A^t$ and $(A^t \otimes A)^t = A^t \otimes (A^t)^t = A^t \otimes A$.

4.4 Proposition

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})], B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})] \in NSM_{m\times n}$. Then,

(i) $(A \cup B)^o = A^o \cap B^o$, $(A \cap B)^o = A^o \cup B^o$.
(ii) $(A \circledast B)^o = A^o \circledast B^o$, $(A \circledast^w B)^o = A^o \circledast^w B^o$.

Proof. (i) Here $(A \cup B)^o, A^o \cap B^o \in NSM_{m\times n}$. Now,

$$\begin{aligned}
(A \cup B)^o &= [(T^a_{ij} \diamond T^b_{ij}, I^a_{ij} * I^b_{ij}, F^a_{ij} * F^b_{ij})]^o \\
&= [(F^a_{ij} * F^b_{ij}, 1 - (I^a_{ij} * I^b_{ij}), T^a_{ij} \diamond T^b_{ij})] \\
&= [(F^a_{ij} * F^b_{ij}, (1 - I^a_{ij}) \diamond (1 - I^b_{ij}), T^a_{ij} \diamond T^b_{ij})] \\
&= [(F^a_{ij}, 1 - I^a_{ij}, T^a_{ij})] \cap [(F^b_{ij}, 1 - I^b_{ij}, T^b_{ij})] \\
&= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^o \cap [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^o \\
&= A^o \cap B^o.
\end{aligned}$$

Next, $(A \cap B)^o, A^o \cup B^o \in NSM_{m\times n}$. Now,

$$\begin{aligned}
(A \cap B)^o &= [(T^a_{ij} * T^b_{ij}, I^a_{ij} \diamond I^b_{ij}, F^a_{ij} \diamond F^b_{ij})]^o \\
&= [(F^a_{ij} \diamond F^b_{ij}, 1 - (I^a_{ij} \diamond I^b_{ij}), T^a_{ij} * T^b_{ij})] \\
&= [(F^a_{ij} \diamond F^b_{ij}, (1 - I^a_{ij}) * (1 - I^b_{ij}), T^a_{ij} * T^b_{ij})] \\
&= [(F^a_{ij}, 1 - I^a_{ij}, T^a_{ij})] \cup [(F^b_{ij}, 1 - I^b_{ij}, T^b_{ij})] \\
&= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^o \cup [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^o \\
&= A^o \cup B^o.
\end{aligned}$$

Note : Here, $(1 - I^a_{ij}) \diamond (1 - I^b_{ij}) = 1 - (I^a_{ij} * I^b_{ij})$ and $(1 - I^a_{ij}) * (1 - I^b_{ij}) = 1 - (I^a_{ij} \diamond I^b_{ij})$ hold for dual pairs of non-parameterized t-norms and s-norms e.g., $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, $a * b = \max\{a + b - 1, 0\}$ and $a \diamond b = \min\{a + b, 1\}$ etc.

(ii) Here $(A \circledast B)^o, A^o \circledast B^o \in NSM_{m\times n}$.

$$\begin{aligned}
(A \circledast B)^o &= [(\frac{T^a_{ij} + T^b_{ij}}{2}, \frac{I^a_{ij} + I^b_{ij}}{2}, \frac{F^a_{ij} + F^b_{ij}}{2})]^o \\
&= [(\frac{F^a_{ij} + F^b_{ij}}{2}, 1 - \frac{I^a_{ij} + I^b_{ij}}{2}, \frac{T^a_{ij} + T^b_{ij}}{2})] \\
&= [(\frac{F^a_{ij} + F^b_{ij}}{2}, \frac{(1 - I^a_{ij}) + (1 - I^b_{ij})}{2}, \frac{T^a_{ij} + T^b_{ij}}{2})] \\
&= [(F^a_{ij}, 1 - I^a_{ij}, T^a_{ij})] \circledast [(F^b_{ij}, 1 - I^b_{ij}, T^b_{ij})] \\
&= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^o \circledast [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^o \\
&= A^o \circledast B^o.
\end{aligned}$$

Next, for $w_1, w_2 > 0$, we have,

$$\begin{aligned}
&(A \circledast^w B)^o \\
&= [(\frac{w_1 T^a_{ij} + w_2 T^b_{ij}}{w_1 + w_2}, \frac{w_1 I^a_{ij} + w_2 I^b_{ij}}{w_1 + w_2}, \frac{w_1 F^a_{ij} + w_2 F^b_{ij}}{w_1 + w_2})]^o \\
&= [(\frac{w_1 F^a_{ij} + w_2 F^b_{ij}}{w_1 + w_2}, 1 - \frac{w_1 I^a_{ij} + w_2 I^b_{ij}}{w_1 + w_2}, \frac{w_1 T^a_{ij} + w_2 T^b_{ij}}{w_1 + w_2})] \\
&= [(\frac{w_1 F^a_{ij} + w_2 F^b_{ij}}{w_1 + w_2}, \frac{w_1(1 - I^a_{ij}) + w_2(1 - I^b_{ij})}{w_1 + w_2}, \\
&\quad \frac{w_1 T^a_{ij} + w_2 T^b_{ij}}{w_1 + w_2})] \\
&= [(F^a_{ij}, 1 - I^a_{ij}, T^a_{ij})] \circledast^w [(F^b_{ij}, 1 - I^b_{ij}, T^b_{ij})] \\
&= [(T^a_{ij}, I^a_{ij}, F^a_{ij})]^o \circledast^w [(T^b_{ij}, I^b_{ij}, F^b_{ij})]^o = A^o \circledast^w B^o.
\end{aligned}$$

4.5 Proposition (Commutative law)

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})], B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})] \in NSM_{m\times n}$. Then,

(i) $A \cup B = B \cup A$, $A \cap B = B \cap A$ (ii) $A \circledast B = B \circledast A$, $A \circledast^w B = B \circledast^w A$ (iii) $A \circledcirc B = B \circledcirc A$, $A \circledcirc^w B = B \circledcirc^w A$ (iv) $A \boxdot B = B \boxdot A$, $A \boxdot^w B = B \boxdot^w A$.

Proof. Obvious

4.6 Proposition (Associative law)

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})], B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})], C = [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \in NSM_{m\times n}$. Then,

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$ (iii) $(A \circledast B) \circledast C \neq A \circledast (B \circledast C)$ (iv) $(A \circledcirc B) \circledcirc C \neq A \circledcirc (B \circledcirc C)$ (v) $(A \boxdot B) \boxdot C \neq A \boxdot (B \boxdot C)$.

Proof. (i) Clearly $(A \cup B) \cup C, A \cup (B \cup C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
&(A \cup B) \cup C \\
&= [(T^a_{ij} \diamond T^b_{ij}, I^a_{ij} * I^b_{ij}, F^a_{ij} * F^b_{ij})] \cup [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \\
&= [((T^a_{ij} \diamond T^b_{ij}) \diamond T^c_{ij}, (I^a_{ij} * I^b_{ij}) * I^c_{ij}, (F^a_{ij} * F^b_{ij}) * F^c_{ij})] \\
&= [(T^a_{ij} \diamond (T^b_{ij} \diamond T^c_{ij}), I^a_{ij} * (I^b_{ij} * I^c_{ij}), F^a_{ij} * (F^b_{ij} * F^c_{ij}))] \\
&= A \cup (B \cup C)
\end{aligned}$$

Similarly, the other results can be verified.

4.7 Proposition (Distributive law)

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})], B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})], C = [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \in NSM_{m\times n}$. Then,

(i) $A \cap (B \circledast C) = (A \cap B) \circledast (A \cap C)$, $(A \circledast B) \cap C =$

$(A \cap C) \circledast (B \cap C)$.
(ii) $A \cup (B \circledast C) = (A \cup B) \circledast (A \cup C)$, $(A \circledast B) \cup C = (A \cup C) \circledast (B \cup C)$.

Proof. (i) Here $A \cap (B \circledast C)$, $(A \cap B) \circledast (A \cap C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
& A \cap (B \circledast C) \\
= \;& [(T^a_{ij}, I^a_{ij}, F^a_{ij})] \cap [(\frac{T^b_{ij}+T^c_{ij}}{2}, \frac{I^b_{ij}+I^c_{ij}}{2}, \frac{F^b_{ij}+F^c_{ij}}{2})] \\
= \;& [(T^a_{ij} * \frac{T^b_{ij}+T^c_{ij}}{2}, I^a_{ij} \diamond \frac{I^b_{ij}+I^c_{ij}}{2}, F^a_{ij} \diamond \frac{F^b_{ij}+F^c_{ij}}{2})] \\
= \;& [(\frac{T^a_{ij} * T^b_{ij} + T^a_{ij} * T^c_{ij}}{2}, \frac{I^a_{ij} \diamond I^b_{ij} + I^a_{ij} \diamond I^c_{ij}}{2}, \\
& \frac{F^a_{ij} \diamond F^b_{ij} + F^a_{ij} \diamond F^c_{ij}}{2})] \\
= \;& [(T^a_{ij} * T^b_{ij}, I^a_{ij} \diamond I^b_{ij}, F^a_{ij} \diamond F^b_{ij})] \\
& \circledast [(T^a_{ij} * T^c_{ij}, I^a_{ij} \diamond I^c_{ij}, F^a_{ij} \diamond F^c_{ij})] \\
= \;& (A \cap B) \circledast (A \cap C)
\end{aligned}$$

Next $(A \circledast B) \cap C$, $(A \cap C) \circledast (B \cap C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
& (A \circledast B) \cap C \\
= \;& [(\frac{T^a_{ij}+T^b_{ij}}{2}, \frac{I^a_{ij}+I^b_{ij}}{2}, \frac{F^a_{ij}+F^b_{ij}}{2})] \cap [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \\
= \;& [(\frac{T^a_{ij}+T^b_{ij}}{2} * T^c_{ij}, \frac{I^a_{ij}+I^b_{ij}}{2} \diamond I^c_{ij}, \frac{F^a_{ij}+F^b_{ij}}{2} \diamond F^c_{ij})] \\
= \;& [(\frac{T^a_{ij} * T^c_{ij} + T^b_{ij} * T^c_{ij}}{2}, \frac{I^a_{ij} \diamond I^c_{ij} + I^b_{ij} \diamond I^c_{ij}}{2}, \\
& \frac{F^a_{ij} \diamond F^c_{ij} + F^b_{ij} \diamond F^c_{ij}}{2})] \\
= \;& [(T^a_{ij} * T^c_{ij}, I^a_{ij} \diamond I^c_{ij}, F^a_{ij} \diamond F^c_{ij})] \\
& \circledast [(T^b_{ij} * T^c_{ij}, I^b_{ij} \diamond I^c_{ij}, F^b_{ij} \diamond F^c_{ij})] \\
= \;& (A \cap C) \circledast (B \cap C)
\end{aligned}$$

In a similar way, the remaining can be established.

4.8 Proposition (Distributive law)

Let $A = [(T^a_{ij}, I^a_{ij}, F^a_{ij})], B = [(T^b_{ij}, I^b_{ij}, F^b_{ij})], C = [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \in NSM_{m\times n}$.
If $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, then

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

(iii) $A \circledast (B \cup C) = (A \circledast B) \cup (A \circledast C)$, $(A \cup B) \circledast C = (A \circledast C) \cup (B \circledast C)$.
$A \circledast (B \cap C) = (A \circledast B) \cap (A \circledast C)$, $(A \cap B) \circledast C = (A \circledast C) \cap (B \circledast C)$.

(iv) $A \circledcirc (B \cup C) = (A \circledcirc B) \cup (A \circledcirc C)$, $(A \cup B) \circledcirc C = (A \circledcirc C) \cup (B \circledcirc C)$.
$A \circledcirc (B \cap C) = (A \circledcirc B) \cap (A \circledcirc C)$, $(A \cap B) \circledcirc C = (A \circledcirc C) \cap (B \circledcirc C)$.

(v) $A \boxdot (B \cup C) = (A \boxdot B) \cup (A \boxdot C)$, $(A \cup B) \boxdot C = (A \boxdot C) \cup (B \boxdot C)$.
$A \boxdot (B \cap C) = (A \boxdot B) \cap (A \boxdot C)$, $(A \cap B) \boxdot C = (A \boxdot C) \cap (B \boxdot C)$.

Proof. We shall here prove (i), (iv) and (v) only. The others can be proved in the similar fashion.

(i) Here $A \cap (B \cup C)$, $(A \cap B) \cup (A \cap C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
& A \cap (B \cup C) \\
= \;& [(T^a_{ij}, I^a_{ij}, F^a_{ij})] \cap [(\max\{T^b_{ij}, T^c_{ij}\}, \min\{I^b_{ij}, I^c_{ij}\}, \\
& \min\{F^b_{ij}, F^c_{ij}\})] \\
= \;& [(\min\{T^a_{ij}, \max\{T^b_{ij}, T^c_{ij}\}\}, \max\{I^a_{ij}, \min\{I^b_{ij}, I^c_{ij}\}\}, \\
& \max\{F^a_{ij}, \min\{F^b_{ij}, F^c_{ij}\}\})] \\
= \;& [(\max\{\min\{T^a_{ij}, T^b_{ij}\}, \min\{T^a_{ij}, T^c_{ij}\}\}, \min\{\max\{I^a_{ij}, I^b_{ij}\}, \\
& \max\{I^a_{ij}, I^c_{ij}\}\}, \min\{\max\{F^a_{ij}, F^b_{ij}\}, \max\{F^a_{ij}, F^c_{ij}\}\})] \\
= \;& [(\min\{T^a_{ij}, T^b_{ij}\}, \max\{I^a_{ij}, I^b_{ij}\}, \max\{F^a_{ij}, F^b_{ij}\})] \\
& \cup [(\min\{T^a_{ij}, T^c_{ij}\}, \max\{I^a_{ij}, I^c_{ij}\}, \max\{F^a_{ij}, F^c_{ij}\})] \\
= \;& (A \cap B) \cup (A \cap C)
\end{aligned}$$

Next $(A \cup B) \cap C, (A \cap C) \cup (B \cap C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
& (A \cup B) \cap C \\
= \;& [(\max\{T^a_{ij}, T^b_{ij}\}, \min\{I^a_{ij}, I^b_{ij}\}, \min\{F^a_{ij}, F^b_{ij}\})] \\
& \cap [(T^c_{ij}, I^c_{ij}, F^c_{ij})] \\
= \;& [(\min\{\max\{T^a_{ij}, T^b_{ij}\}, T^c_{ij}\}, \max\{\min\{I^a_{ij}, I^b_{ij}\}, I^c_{ij}\}, \\
& \max\{\min\{F^a_{ij}, F^b_{ij}\}, F^c_{ij}\})] \\
= \;& [(\max\{\min\{T^a_{ij}, T^c_{ij}\}, \min\{T^b_{ij}, T^c_{ij}\}\}, \min\{\max\{I^a_{ij}, I^c_{ij}\}, \\
& \max\{I^b_{ij}, I^c_{ij}\}\}, \min\{\max\{F^a_{ij}, F^c_{ij}\}, \max\{F^b_{ij}, F^c_{ij}\}\})] \\
= \;& [(\min\{T^a_{ij}, T^c_{ij}\}, \max\{I^a_{ij}, I^c_{ij}\}, \max\{F^a_{ij}, F^c_{ij}\})] \\
& \cup [(\min\{T^b_{ij}, T^c_{ij}\}, \max\{I^b_{ij}, I^c_{ij}\}, \max\{F^b_{ij}, F^c_{ij}\})] \\
= \;& (A \cap C) \cup (B \cap C)
\end{aligned}$$

(iv) Here $A \circledcirc (B \cup C), (A \circledcirc B) \cup (A \circledcirc C) \in NSM_{m\times n}$. Now,

$$\begin{aligned}
& A \circledcirc (B \cup C) \\
= \;& [(T^a_{ij}, I^a_{ij}, F^a_{ij})] \circledcirc [(\max\{T^b_{ij}, T^c_{ij}\}, \min\{I^b_{ij}, I^c_{ij}\}, \\
& \min\{F^b_{ij}, F^c_{ij}\})] \\
= \;& [(\sqrt{T^a_{ij} \cdot \max\{T^b_{ij}, T^c_{ij}\}}, \sqrt{I^a_{ij} \cdot min\{I^b_{ij}, I^c_{ij}\}}, \\
& \sqrt{F^a_{ij} \cdot min\{F^b_{ij}, F^c_{ij}\}})] \\
= \;& [(\max\{\sqrt{T^a_{ij} \cdot T^b_{ij}}, \sqrt{T^a_{ij} \cdot T^c_{ij}}\}, \min\{\sqrt{I^a_{ij} \cdot I^b_{ij}}, \\
& \sqrt{I^a_{ij} \cdot I^c_{ij}}\}, \min\{\sqrt{F^a_{ij} \cdot F^b_{ij}}, \sqrt{F^a_{ij} \cdot F^c_{ij}}\})]
\end{aligned}$$

$$
\begin{aligned}
&= [(\sqrt{T_{ij}^a \cdot T_{ij}^b}, \sqrt{I_{ij}^a \cdot I_{ij}^b}, \sqrt{F_{ij}^a \cdot F_{ij}^b})] \\
&\quad \cup[(\sqrt{T_{ij}^a \cdot T_{ij}^c}, \sqrt{I_{ij}^a \cdot I_{ij}^c}, \sqrt{F_{ij}^a \cdot F_{ij}^c})] \\
&= (A \circledcirc B) \cup (A \circledcirc C)
\end{aligned}
$$

Next $(A \cup B) \circledcirc C, (A \circledcirc C) \cup (B \circledcirc C) \in NSM_{m\times n}$. Now,

$$
\begin{aligned}
&(A \cup B) \circledcirc C \\
&= [(\max\{T_{ij}^a, T_{ij}^b\}, \min\{I_{ij}^a, I_{ij}^b\}, \min\{F_{ij}^a, F_{ij}^b\})] \\
&\quad \circledcirc[(T_{ij}^c, I_{ij}^c, F_{ij}^c)] \\
&= [(\sqrt{\max\{T_{ij}^a, T_{ij}^b\} \cdot T_{ij}^c}, \sqrt{\min\{I_{ij}^a, I_{ij}^b\} \cdot I_{ij}^c}, \\
&\quad \sqrt{\min\{F_{ij}^a, F_{ij}^b\} \cdot F_{ij}^c},)] \\
&= [(\max\{\sqrt{T_{ij}^a \cdot T_{ij}^c}, \sqrt{T_{ij}^b \cdot T_{ij}^c}\}, \min\{\sqrt{I_{ij}^a \cdot I_{ij}^c}, \\
&\quad \sqrt{I_{ij}^b \cdot I_{ij}^c}\}, \min\{\sqrt{F_{ij}^a \cdot F_{ij}^c}, \sqrt{F_{ij}^b \cdot F_{ij}^c}\})] \\
&= [(\sqrt{T_{ij}^a \cdot T_{ij}^c}, \sqrt{I_{ij}^a \cdot I_{ij}^c}, \sqrt{F_{ij}^a \cdot F_{ij}^c})] \\
&\quad \cup[(\sqrt{T_{ij}^b \cdot T_{ij}^c}, \sqrt{I_{ij}^b \cdot I_{ij}^c}, \sqrt{F_{ij}^b \cdot F_{ij}^c})] \\
&= (A \circledcirc C) \cup (B \circledcirc C)
\end{aligned}
$$

(v) Here $A \boxdot (B \cup C), (A \boxdot B) \cup (A \boxdot C) \in NSM_{m\times n}$. Now,

$$
\begin{aligned}
&A \boxdot (B \cup C) \\
&= [(T_{ij}^a, I_{ij}^a, F_{ij}^a)] \boxdot \\
&\quad [(\max\{T_{ij}^b, T_{ij}^c\}, \min\{I_{ij}^b, I_{ij}^c\}, \min\{F_{ij}^b, F_{ij}^c\})] \\
&= [(\frac{2 \cdot T_{ij}^a \cdot \max\{T_{ij}^b, T_{ij}^c\}}{T_{ij}^a + \max\{T_{ij}^b, T_{ij}^c\}}, \frac{2 \cdot I_{ij}^a \cdot \min\{I_{ij}^b, I_{ij}^c\}}{I_{ij}^a + \min\{I_{ij}^b, I_{ij}^c\}}, \\
&\quad \frac{2 \cdot F_{ij}^a \cdot \min\{F_{ij}^b, F_{ij}^c\}}{F_{ij}^a + \min\{F_{ij}^b, F_{ij}^c\}})] \\
&= [(\max\{\frac{2T_{ij}^a T_{ij}^b}{T_{ij}^a + T_{ij}^b}, \frac{2T_{ij}^a T_{ij}^c}{T_{ij}^a + T_{ij}^c}\}, \min\{\frac{2I_{ij}^a I_{ij}^b}{I_{ij}^a + I_{ij}^b}, \\
&\quad \frac{2I_{ij}^a I_{ij}^c}{I_{ij}^a + I_{ij}^c}\}, \min\{\frac{2F_{ij}^a F_{ij}^b}{F_{ij}^a + F_{ij}^b}, \frac{2F_{ij}^a F_{ij}^c}{F_{ij}^a + F_{ij}^c}\})] \\
&= [(\frac{2T_{ij}^a T_{ij}^b}{T_{ij}^a + T_{ij}^b}, \frac{2I_{ij}^a I_{ij}^b}{I_{ij}^a + I_{ij}^b}, \frac{2F_{ij}^a F_{ij}^b}{F_{ij}^a + F_{ij}^b})] \\
&\quad \cup[(\frac{2T_{ij}^a T_{ij}^c}{T_{ij}^a + T_{ij}^c}, \frac{2I_{ij}^a I_{ij}^c}{I_{ij}^a + I_{ij}^c}, \frac{2F_{ij}^a F_{ij}^c}{F_{ij}^a + F_{ij}^c})] \\
&= (A \boxdot B) \cup (A \boxdot C)
\end{aligned}
$$

Next $(A \cup B) \boxdot C, (A \boxdot C) \cup (B \boxdot C) \in NSM_{m\times n}$. Now,

$$
\begin{aligned}
&(A \cup B) \boxdot C \\
&= [(\max\{T_{ij}^a, T_{ij}^b\}, \min\{I_{ij}^a, I_{ij}^b\}, \min\{F_{ij}^a, F_{ij}^b\})] \\
&\quad \boxdot[(T_{ij}^c, I_{ij}^c, F_{ij}^c)] \\
&= [(\frac{2 \cdot \max\{T_{ij}^a, T_{ij}^b\} \cdot T_{ij}^c}{\max\{T_{ij}^b, T_{ij}^c\} + T_{ij}^c}, \frac{2 \cdot \min\{I_{ij}^a, I_{ij}^b\} \cdot I_{ij}^c}{\min\{I_{ij}^b, I_{ij}^c\} + I_{ij}^c}, \\
&\quad \frac{2 \cdot \min\{F_{ij}^a, F_{ij}^b\} \cdot F_{ij}^c}{\min\{F_{ij}^b, F_{ij}^c\} + F_{ij}^c})] \\
&= [(\max\{\frac{2T_{ij}^a T_{ij}^c}{T_{ij}^a + T_{ij}^c}, \frac{2T_{ij}^b T_{ij}^c}{T_{ij}^b + T_{ij}^c}\}, \min\{\frac{2I_{ij}^a I_{ij}^c}{I_{ij}^a + I_{ij}^c}, \\
&\quad \frac{2I_{ij}^b I_{ij}^c}{I_{ij}^b + I_{ij}^c}\}, \min\{\frac{2F_{ij}^a F_{ij}^c}{F_{ij}^a + F_{ij}^c}, \frac{2F_{ij}^b F_{ij}^c}{F_{ij}^b + F_{ij}^c}\})] \\
&= [(\frac{2T_{ij}^a T_{ij}^c}{T_{ij}^a + T_{ij}^c}, \frac{2I_{ij}^a I_{ij}^c}{I_{ij}^a + I_{ij}^c}, \frac{2F_{ij}^a F_{ij}^c}{F_{ij}^a + F_{ij}^c})] \\
&\quad \cup[(\frac{2T_{ij}^b T_{ij}^c}{T_{ij}^b + T_{ij}^c}, \frac{2I_{ij}^b I_{ij}^c}{I_{ij}^b + I_{ij}^c}, \frac{2F_{ij}^b F_{ij}^c}{F_{ij}^b + F_{ij}^c})] \\
&= (A \boxdot C) \cup (B \boxdot C)
\end{aligned}
$$

4.9 Proposition (Idempotent law)

Let $A = [(T_{ij}^a, I_{ij}^a, F_{ij}^a)] \in NSM_{m\times n}$. Then,
(i) $A \circledast^w A = A$ (ii) $A \circledcirc^w A = A$ (iii) $A \boxdot^w A = A$.

Proof. For all i, j and $w_1, w_2 > 0$ we have,

(i) $A \circledast^w A = [(\frac{w_1 T_{ij}^a + w_2 T_{ij}^a}{w_1 + w_2}, \frac{w_1 I_{ij}^a + w_2 I_{ij}^a}{w_1 + w_2}, \frac{w_1 F_{ij}^a + w_2 F_{ij}^a}{w_1 + w_2},)] = [(T_{ij}^a, I_{ij}^a, F_{ij}^a)] = A$.

(ii) $A \circledcirc^w A = [(\sqrt[(w_1+w_2)]{(T_{ij}^a)^{w_1} \cdot (T_{ij}^a)^{w_2}}, \sqrt[(w_1+w_2)]{(I_{ij}^a)^{w_1} \cdot (I_{ij}^a)^{w_2}}, \sqrt[(w_1+w_2)]{(F_{ij}^a)^{w_1} \cdot (F_{ij}^a)^{w_2}})]$
$= [(\sqrt[(w_1+w_2)]{(T_{ij}^a)^{w_1+w_2}}, \sqrt[(w_1+w_2)]{(I_{ij}^a)^{w_1+w_2}}, \sqrt[(w_1+w_2)]{(F_{ij}^a)^{w_1+w_2}})] = [(T_{ij}^a, I_{ij}^a, F_{ij}^a)] = A$.

(iii) $A \boxdot^w A = [(\frac{w_1+w_2}{\frac{w_1}{T_{ij}^a} + \frac{w_2}{T_{ij}^a}}, \frac{w_1+w_2}{\frac{w_1}{I_{ij}^a} + \frac{w_2}{I_{ij}^a}}, \frac{w_1+w_2}{\frac{w_1}{F_{ij}^a} + \frac{w_2}{F_{ij}^a}})] = [(T_{ij}^a, I_{ij}^a, F_{ij}^a)] = A$.

5 Neutrosophic soft matrix theory in decision making (score function algorithm)

5.1 Definition

1. Let $A = [a_{ij}]_{m\times n}$ be an NSM where $a_{ij} = (T_{ij}^a, I_{ij}^a, F_{ij}^a)$. Then the value of the matrix A is denoted by $V(A)$ and is defined as : $V(A) = [v_{ij}^a]_{m\times n}$ where $v_{ij}^a = T_{ij}^a - I_{ij}^a - F_{ij}^a, \ \forall i, j$.

2. The score of two NSMs A and B is defined as $S(A, B) = [s_{ij}]_{m\times n}$ where $s_{ij} = v_{ij}^a + v_{ij}^b$. So, $S(A, B) = V(A) + V(B)$.

3. The total score for each object in U is $\Sigma_{j=1}^n s_{ij}$.

5.2 Properties of Score Function

Value matrices are classical real matrices which follow all properties of classical real matrices. The score function is basically a real matrix in classical sense derived from two or more value matrices. So score functions obey all properties of real matrices.

5.3 Methodology

Suppose, N number of decision makers wish to select an object jointly from m number of objects i.e., universal set U with respect to n number of features i.e., parametric set E. Each decision maker forms an NSS over (U, E) and corresponding to each NSS, each get an NSM of order $m \times n$. It needs to compute the value matrix corresponding to each matrix. Then the score matrix and finally, the total score of each object will be calculated.

5.3.1 Algorithm

Step 1 : Construct the NSMs from the given NSSs.
Step 2 : Calculate the value matrices of corresponding NSMs.
Step 3 : Compute the score matrix from value matrices and the total score for each object in U.
Step 4 : Find the object of maximum score and it is the optimal solution.
Step 5 : If score is maximum for more than one object, then find $\Sigma_{j=1}^{n}(s_{ij})^k$, $k \geq 2$ successively. Choose the object of maximum score and hereby the optimal solution.

5.3.2 Case study 1 (application in class room)

Three students $\{s_1, s_2, s_3\}$ from class - x in a school have been shortened to win the best student award in an academic session. A team of three teachers $\{T_1, T_2, T_3\}$ has been formed by the Head Master of that school for this purpose. Final selection is based on the set of parameters $\{e_1, e_2, e_3, e_4, e_5\}$ indicating the quality of student, participation in school cultural programme, class room interactions, maintenance of discipline in class room, daily attendance, respectively. Teachers have given their valuable opinions by the following NSSs separately i.e., first NSS given by first teacher and so on.

$M = \{f_M(e_1), f_M(e_2), f_M(e_3), f_M(e_4), f_M(e_5)\}$ where

$$\begin{aligned}
f_M(e_1) &= \{< s_1, (0.7, 0.2, 0.6) >, < s_2, (0.6, 0.3, 0.5) >, \\
&\quad < s_3, (0.8, 0.3, 0.5) >\} \\
f_M(e_2) &= \{< s_1, (0.4, 0.6, 0.7) >, < s_2, (0.7, 0.6, 0.3) >, \\
&\quad < s_3, (0.5, 0.5, 0.4) >\} \\
f_M(e_3) &= \{< s_1, (0.5, 0.5, 0.3) >, < s_2, (0.7, 0.4, 0.4) >, \\
&\quad < s_3, (0.6, 0.4, 0.6) >\} \\
f_M(e_4) &= \{< s_1, (0.6, 0.6, 0.5) >, < s_2, (0.5, 0.8, 0.6) >, \\
&\quad < s_3, (0.4, 0.7, 0.4) >\}
\end{aligned}$$

$N = \{f_N(e_1), f_N(e_2), f_N(e_3), f_N(e_4), f_N(e_5)\}$ where

$$\begin{aligned}
f_N(e_1) &= \{< s_1, (0.6, 0.4, 0.5) >, < s_2, (0.7, 0.4, 0.2) >, \\
&\quad < s_3, (0.9, 0.4, 0.2) >\} \\
f_N(e_2) &= \{< s_1, (0.5, 0.5, 0.6) >, < s_2, (0.8, 0.5, 0.1) >, \\
&\quad < s_3, (0.6, 0.7, 0.5) >\} \\
f_N(e_3) &= \{< s_1, (0.7, 0.3, 0.4) >, < s_2, (0.8, 0.5, 0.3) >, \\
&\quad < s_3, (0.5, 0.6, 0.7) >\} \\
f_N(e_4) &= \{< s_1, (0.7, 0.5, 0.3) >, < s_2, (0.6, 0.7, 0.5) >, \\
&\quad < s_3, (0.5, 0.5, 0.5) >\} \\
f_N(e_5) &= \{< s_1, (0.6, 0.4, 0.6) >, < s_2, (0.6, 0.3, 0.7) >, \\
&\quad < s_3, (0.8, 0.3, 0.3) >\}\}
\end{aligned}$$

$P = \{f_P(e_1), f_P(e_2), f_P(e_3), f_P(e_4), f_P(e_5)\}$ where

$$\begin{aligned}
f_P(e_1) &= \{< s_1, (0.8, 0.3, 0.3) >, < s_2, (0.8, 0.5, 0.3) >, \\
&\quad < s_3, (1.0, 0.4, 0.2) >\} \\
f_P(e_2) &= \{< s_1, (0.6, 0.4, 0.5) >, < s_2, (0.7, 0.6, 0.2) >, \\
&\quad < s_3, (0.8, 0.5, 0.4) >\} \\
f_P(e_3) &= \{< s_1, (0.8, 0.4, 0.1) >, < s_2, (0.7, 0.5, 0.5) >, \\
&\quad < s_3, (0.6, 0.7, 0.3) >\} \\
f_P(e_4) &= \{< s_1, (0.6, 0.6, 0.2) >, < s_2, (0.8, 0.6, 0.4) >, \\
&\quad < s_3, (0.7, 0.3, 0.6) >\} \\
f_P(e_5) &= \{< s_1, (0.8, 0.4, 0.2) >, < s_2, (0.6, 0.4, 0.3) >, \\
&\quad < s_3, (0.7, 0.5, 0.4) >\}\}
\end{aligned}$$

The above three NSSs are represented by the NSMs A, B and C, respectively, as following :

$$\begin{pmatrix}
(.7, .2, .6) & (.4, .6, .7) & (.5, .5, .3) & (.6, .6, .5) & (.8, .3, .4) \\
(.6, .3, .5) & (.7, .6, .3) & (.7, .4, .4) & (.5, .8, .6) & (.7, .2, .6) \\
(.8, .3, .5) & (.5, .5, .4) & (.6, .4, .6) & (.4, .7, .4) & (.9, .1, .2)
\end{pmatrix}$$

$$\begin{pmatrix}
(.6, .4, .5) & (.5, .5, .6) & (.7, .3, .4) & (.7, .5, .3) & (.6, .4, .6) \\
(.7, .4, .2) & (.8, .5, .1) & (.8, .5, .3) & (.6, .7, .5) & (.6, .3, .7) \\
(.9, .4, .2) & (.6, .7, .5) & (.5, .6, .7) & (.5, .5, .5) & (.8, .3, .3)
\end{pmatrix}$$

$$\begin{pmatrix}
(.8, .3, .3) & (.6, .4, .5) & (.8, .4, .1) & (.6, .6, .2) & (.8, .4, .2) \\
(.8, .5, .4) & (.7, .6, .2) & (.7, .5, .5) & (.8, .6, .4) & (.6, .4, .3) \\
(1, .4, .2) & (.8, .5, .4) & (.6, .7, .3) & (.7, .3, .6) & (.7, .5, .4)
\end{pmatrix}$$

Then the corresponding value matrices are :

$$V(A) = \begin{pmatrix}
-.1 & -.9 & -.3 & -.5 & 0.1 \\
-.2 & -.2 & -.1 & -.9 & -.1 \\
0.0 & -.4 & -.4 & -.7 & 0.6
\end{pmatrix}$$

$$V(B) = \begin{pmatrix}
-.3 & -.6 & 0.0 & -.1 & -.4 \\
0.1 & 0.2 & 0.0 & -.6 & -.4 \\
0.3 & -.6 & -.8 & -.5 & 0.2
\end{pmatrix}$$

$$V(C) = \begin{pmatrix}
0.2 & -.3 & 0.3 & -.2 & 0.2 \\
-.1 & -.1 & -.3 & -.2 & -.1 \\
0.4 & -.1 & -.4 & -.2 & -.2
\end{pmatrix}$$

The score matrix is :

$$S(A,B,C) = \begin{pmatrix} -0.2 & -1.8 & 00.0 & -0.8 & -0.1 \\ -0.2 & -0.1 & -0.4 & -1.7 & -0.6 \\ 00.7 & -1.1 & -1.6 & -1.4 & 00.6 \end{pmatrix}$$

and the total score $= \begin{pmatrix} -2.9 \\ -3.0 \\ -2.8 \end{pmatrix}$

Hence, the student s_3 will be selected for the best student award from class-x in that academic session.

5.3.3 Case study 2 (application in security management)

An important discussion on internal security management has been arranged by the order of Home Minister. Two officers have mate in that discussion to analyse and arrange the security management in five mega-cities e.g., Delhi(D), Mumbai(M), Kolkata(K), Chennai(C), Bengaluru(B). The priority of management is given to the cities based on the set of parameters $\{a, b, c\}$ indicating their geographical position(e.g., having international boarder line, having sea coast etc), population density, past history of terrorist attack, respectively. Following NSSs refer the opinions of two officers individually regarding that matter.

$N_1 = \{f_{N_1}(a), f_{N_1}(b), f_{N_1}(c)\}$ where

$$\begin{aligned} f_{N_1}(a) &= \{< D,(0.9,0.4,0.5) >, < M,(0.8,0.5,0.4) >, \\ &\quad < K,(0.7,0.6,0.6) >, < C,(0.6,0.4,0.7) >, \\ &\quad < B,(0.5,0.3,0.8) >\} \\ f_{N_1}(b) &= \{< D,(0.8,0.5,0.5) >, < M,(0.9,0.3,0.3) >, \\ &\quad < K,(0.7,0.6,0.5) >, < C,(0.6,0.7,0.8) >, \\ &\quad < B,(0.6,0.8,0.5) >\} \\ f_{N_1}(c) &= \{< D,(0.7,0.5,0.4) >, < M,(0.9,0.3,0.2) >, \\ &\quad < K,(0.5,0.6,0.7) >, < C,(0.7,0.4,0.6) >, \\ &\quad < B,(0.6,0.3,0.4) >\} \end{aligned}$$

$N_2 = \{f_{N_2}(a), f_{N_2}(b), f_{N_2}(c)\}$ where

$$\begin{aligned} f_{N_2}(a) &= \{< D,(1.0,0.5,0.4) >, < M,(0.9,0.4,0.5) >, \\ &\quad < K,(0.7,0.7,0.5) >, < C,(0.6,0.5,0.3) >, \\ &\quad < B,(0.6,0.7,0.4) >\} \\ f_{N_2}(b) &= \{< D,(0.9,0.4,0.5) >, < M,(0.9,0.2,0.3) >, \\ &\quad < K,(0.8,0.5,0.4) >, < C,(0.7,0.7,0.6) >, \\ &\quad < B,(0.6,0.8,0.7) >\} \\ f_{N_2}(c) &= \{< D,(0.8,0.3,0.2) >, < M,(0.9,0.2,0.1) >, \\ &\quad < K,(0.4,0.5,0.6) >, < C,(0.5,0.6,0.6) >, \\ &\quad < B,(0.7,0.4,0.3) >\} \end{aligned}$$

These two NSSs are represented by the NSMs A and B, respectively, as following :

$$A = \begin{pmatrix} (0.9,0.4,0.5) & (0.8,0.5,0.5) & (0.7,0.5,0.4) \\ (0.8,0.5,0.4) & (0.9,0.3,0.3) & (0.9,0.3,0.2) \\ (0.7,0.6,0.6) & (0.7,0.6,0.5) & (0.5,0.6,0.7) \\ (0.6,0.4,0.7) & (0.6,0.7,0.8) & (0.7,0.4,0.6) \\ (0.5,0.3,0.8) & (0.6,0.8,0.5) & (0.6,0.3,0.4) \end{pmatrix}$$

$$B = \begin{pmatrix} (1.0,0.5,0.4) & (0.9,0.4,0.5) & (0.8,0.3,0.2) \\ (0.9,0.4,0.5) & (0.9,0.2,0.3) & (0.9,0.2,0.1) \\ (0.7,0.7,0.5) & (0.8,0.5,0.4) & (0.4,0.5,0.6) \\ (0.6,0.5,0.3) & (0.7,0.7,0.6) & (0.5,0.6,0.6) \\ (0.6,0.7,0.4) & (0.6,0.8,0.7) & (0.7,0.4,0.3) \end{pmatrix}$$

Then the corresponding value matrices are :

$$V(A) = \begin{pmatrix} 0.0 & -.2 & -.2 \\ -.1 & 0.3 & 0.4 \\ -.5 & -.4 & -.8 \\ -.5 & -.9 & -.3 \\ -.6 & -.7 & -.1 \end{pmatrix}$$

$$V(B) = \begin{pmatrix} 0.1 & 0.0 & 0.3 \\ 0.0 & 0.4 & 0.6 \\ -.5 & -.1 & -.7 \\ -.2 & -.6 & -.7 \\ -.5 & -.9 & 0.0 \end{pmatrix}$$

The score matrix and the total score for selection are :

$$S(A,B) = \begin{pmatrix} 00.1 & -0.2 & 00.1 \\ -0.1 & 00.7 & 01.0 \\ -1.0 & -.5 & -1.5 \\ -0.7 & -1.5 & -1.0 \\ -1.1 & -1.6 & -0.1 \end{pmatrix}$$

$$\text{Total score} = \begin{pmatrix} 00.0 \\ 01.6 \\ -3.0 \\ -3.2 \\ -2.8 \end{pmatrix}$$

Hence, the priority of security management should be given in descending order to Mumbai, Delhi, Bangaluru, Kolkata and Chennai.

6 Conclusion

In this paper, some definitions regarding neutrosophic soft matrices have been brought and some new operators have been included, illustrated by suitable examples. Moreover, application of neutrosophic soft matrix theory in decision making problems have been made. We expect, this paper will promote the future study on different algorithms in several other decision making problems.

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NSS University of New Mexico

Interval neutrosophic sets applied to ideals in BCK/BCI-algebras

Seok-Zun Song[1], Madad Khan[2], Florentin Smarandache[3], Young Bae Jun[4,*]

[1]Department of Mathematics, Jeju National University, Jeju 63243, Korea. E-mail: szsong@jejunu.ac.kr

[2]Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, Pakistan. E-mail: madadmath@yahoo.com

[3]Mathematics & Science Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA. E-mail: fsmarandache@gmail.com

[4]Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea. E-mail: skywine@gmail.com

*Correspondence: skywine@gmail.com

Abstract: For $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, the notion of $(T(i, j), I(k, l), F(m, n))$-interval neutrosophic ideals in BCK/BCI-algebras is introduced, and their properties and relations are investigated.

Keywords: interval neutrosophic set; interval neutrosophic ideal.

1 Introduction

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [3], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets (= MV-algebras). Also, Iséki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra (see [4]). The neutrosophic set developed by Smarandache [7, 8, 9] is a formal framework which generalizes the concept of the classic set, fuzzy set [14], interval valued fuzzy set, intuitionistic fuzzy set [1], interval valued intuitionistic fuzzy set and paraconsistent set etc. Neutrosophic set theory is applied to various part, including algebra, topology, control theory, decision making problems, medicines and in many real life problems. Wang et al. [11, 12, 13] presented the concept of interval neutrosophic sets, which is more precise and more fl xible than the single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership (t, i, f) functions are independent, and their values belong to the unit interval $[0, 1]$. The interval neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information which exists in real world. Jun et al. [5] discussed interval neutrosophic sets in BCK/BCI-algebras, and introduced the notion of $(T(i, j), I(k, l), F(m, n))$-interval neutrosophic subalgebras in BCK/BCI-algebras for $i, j, k, l, m, n \in \{1, 2, 3, 4\}$. They also introduced the notion of interval neutrosophic length of an interval neutrosophic set, and investigated related properties.

In this article, we apply the notion of interval neutrosophic sets to ideal theory in BCK/BCI-algebras. We introduce the notion of $(T(i, j), I(k, l), F(m, n))$-interval neutrosophic ideals in BCK/BCI-algebras for $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, and investigate their properties and relations.

2 Preliminaries

By a *BCI-algebra* (see [2, 6]) we mean a system $X := (X, *, 0)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0,$

(II) $(x * (x * y)) * y = 0,$

(III) $x * x = 0,$

(IV) $x * y = y * x = 0 \Rightarrow x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfie $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra* (see [2, 6]).

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* (see [2, 6]) of X if $x * y \in S$ for all $x, y \in S$.

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [2] and [6] for further information regarding BCK/BCI-algebras.

By a *fuzzy structure* over a nonempty set X we mean an ordered pair (X, ρ) of X and a fuzzy set ρ on X.

Definition 2.1 ([10]). A fuzzy structure (X, μ) over $(X, *, 0) \in \mathcal{B}(X)$ is called a

- *fuzzy ideal* of $(X, *, 0)$ with type 1 (briefl , 1-*fuzzy ideal* of $(X, *, 0)$) if

$$(\forall x \in X)\, (\mu(0) \geq \mu(x))\,, \tag{2.1}$$
$$(\forall x, y \in X)\, (\mu(x) \geq \min\{\mu(x * y), \mu(y)\})\,, \tag{2.2}$$

- *fuzzy ideal* of $(X,*,0)$ with type 2 (briefl , 2-*fuzzy ideal* of $(X,*,0)$) if

$$(\forall x \in X)\,(\mu(0) \leq \mu(x))\,, \tag{2.3}$$
$$(\forall x, y \in X)\,(\mu(x) \leq \min\{\mu(x*y), \mu(y)\})\,, \tag{2.4}$$

- *fuzzy ideal* of $(X,*,0)$ with type 3 (briefl , 3-*fuzzy ideal* of $(X,*,0)$) if it satisfie (2.1) and

$$(\forall x, y \in X)\,(\mu(x) \geq \max\{\mu(x*y), \mu(y)\})\,, \tag{2.5}$$

- *fuzzy ideal* of $(X,*,0)$ with type 4 (briefl , 4-*fuzzy ideal* of $(X,*,0)$) if it satisfie (2.3) and

$$(\forall x, y \in X)\,(\mu(x) \leq \max\{\mu(x*y), \mu(y)\})\,. \tag{2.6}$$

Let X be a non-empty set. A neutrosophic set (NS) in X (see [8]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0,1]$ is a truth membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function.

An interval neutrosophic set (INS) A in X is characterized by truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0,1]$ (see [12, 13]).

In what follows, let $(X,*,0) \in \mathcal{B}(X)$ and $\mathcal{P}^*([0,1])$ be the family of all subintervals of $[0,1]$ unless otherwise specifie .

Definition 2.2 ([12, 13]). An *interval neutrosophic set* in a nonempty set X is a structure of the form:

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}$$

where

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1])$$

which is called *interval truth-membership function*,

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1])$$

which is called *interval indeterminacy-membership function*, and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1])$$

which is called *interval falsity-membership function*.

For the sake of simplicity, we will use the notation $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ for the interval neutrosophic set

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}.$$

Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we consider the following functions (see [5]):

$$\mathcal{I}[T]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[T](x)\}$$
$$\mathcal{I}[I]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[I](x)\}$$
$$\mathcal{I}[F]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[F](x)\}$$

and

$$\mathcal{I}[T]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[T](x)\}$$
$$\mathcal{I}[I]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[I](x)\}$$
$$\mathcal{I}[F]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[F](x)\}.$$

3 Interval neutrosophic ideals

Definition 3.1. For any $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is called a $(T(i, j), I(k, l), F(m, n))$-*interval neutrosophic ideal* of X if the following assertions are valid.

(1) $(X, \mathcal{I}[T]_{\inf})$ is an i-fuzzy ideal of $(X,*,0)$ and $(X, \mathcal{I}[T]_{\sup})$ is a j-fuzzy ideal of $(X,*,0)$,

(2) $(X, \mathcal{I}[I]_{\inf})$ is a k-fuzzy ideal of $(X,*,0)$ and $(X, \mathcal{I}[I]_{\sup})$ is an l-fuzzy ideal of $(X,*,0)$,

(3) $(X, \mathcal{I}[F]_{\inf})$ is an m-fuzzy ideal of $(X,*,0)$ and $(X, \mathcal{I}[F]_{\sup})$ is an n-fuzzy ideal of $(X,*,0)$.

Example 3.2. Consider a BCK-algebra $X = \{0,1,2,3\}$ with the binary operation $*$ which is given in Table 1 (see [6]).

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

(1) Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X,*,0)$ for which $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1])\ \ x \mapsto \begin{cases} [0.4, 0.6) & \text{if } x = 0, \\ (0.3, 0.6] & \text{if } x = 1, \\ [0.2, 0.7) & \text{if } x = 2, \\ [0.1, 0.8] & \text{if } x = 3, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1])\ \ x \mapsto \begin{cases} [0.5, 0.6) & \text{if } x = 0, \\ (0.4, 0.6) & \text{if } x = 1, \\ [0.2, 0.9] & \text{if } x = 2, \\ [0.5, 0.7) & \text{if } x = 3, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.4, 0.5) & \text{if } x = 0, \\ (0.3, 0.5) & \text{if } x = 1, \\ [0.1, 0.7] & \text{if } x = 2, \\ (0.2, 0.8] & \text{if } x = 3. \end{cases}$$

It is routine to verify that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic ideal of $(X, *, 0)$.

(2) Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ for which $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.1, 0.4) & \text{if } x = 0, \\ (0.2, 0.7) & \text{if } x = 1, \\ [0.3, 0.8] & \text{if } x = 2, \\ [0.4, 0.6) & \text{if } x = 3, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} (0.2, 0.5) & \text{if } x = 0, \\ [0.5, 0.6] & \text{if } x = 1, \\ (0.6, 0.7] & \text{if } x = 2, \\ [0.3, 0.8] & \text{if } x = 3, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.3, 0.4) & \text{if } x = 0, \\ (0.4, 0.7) & \text{if } x = 1, \\ (0.6, 0.8) & \text{if } x = 2, \\ [0.4, 0.6] & \text{if } x = 3. \end{cases}$$

By routine calculations, we know that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,4), I(4,4), F(4,4))$-interval neutrosophic ideal of $(X, *, 0)$.

Example 3.3. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 2 (see [6]).

Table 2: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ where $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.33, 0.91) & \text{if } x = 0, \\ (0.72, 0.91) & \text{if } x = a, \\ [0.72, 0.82) & \text{if } x = b, \\ (0.55, 0.82] & \text{if } x = c, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.22, 0.65) & \text{if } x = 0, \\ [0.52, 0.55] & \text{if } x = a, \\ (0.62, 0.65) & \text{if } x = b, \\ [0.62, 0.55) & \text{if } x = c, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} (0.25, 0.63) & \text{if } x = 0, \\ [0.45, 0.63] & \text{if } x = a, \\ (0.35, 0.53] & \text{if } x = b, \\ [0.45, 0.53) & \text{if } x = c. \end{cases}$$

Routine calculations show that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,1), I(4,1), F(4,1))$-interval neutrosophic ideal of $(X, *, 0)$. But it is not a $(T(2,1), I(2,1), F(2,1))$-interval neutrosophic ideal of $(X, *, 0)$ since

$$\mathcal{I}[T]_{\inf}(a) = 0.72 > 0.55 = \min\{\mathcal{I}[T]_{\inf}(a * b), \mathcal{I}[T]_{\inf}(b)\},$$

$$\mathcal{I}[I]_{\inf}(b) = 0.62 > 0.52 = \min\{\mathcal{I}[I]_{\inf}(b * c), \mathcal{I}[I]_{\inf}(c)\},$$

and/or

$$\mathcal{I}[F]_{\inf}(c) = 0.45 > 0.35 = \min\{\mathcal{I}[F]_{\inf}(c * a), \mathcal{I}[F]_{\inf}(c)\}.$$

Also, it is not a $(T(4,3), I(4,3), F(4,3))$-interval neutrosophic ideal of $(X, *, 0)$ since

$$\mathcal{I}[T]_{\sup}(c) = 0.82 < 0.91 = \max\{\mathcal{I}[T]_{\inf}(c * b), \mathcal{I}[T]_{\inf}(b)\}$$

and/or

$$\mathcal{I}[F]_{\sup}(b) = 0.35 < 0.62 = \max\{\mathcal{I}[F]_{\inf}(b * a), \mathcal{I}[F]_{\inf}(a)\}.$$

We also know that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is not a $(T(2,3), I(2,3), F(2,3))$-interval neutrosophic ideal of $(X, *, 0)$.

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X. We consider the following sets (see [5]):

$$U(\mathcal{I}[T]_\psi; \alpha_I) := \{x \in X \mid \mathcal{I}[T]_\psi(x) \geq \alpha_I\},$$
$$L(\mathcal{I}[T]_\psi; \alpha_S) := \{x \in X \mid \mathcal{I}[T]_\psi(x) \leq \alpha_S\},$$

$$U(\mathcal{I}[I]_\psi; \beta_I) := \{x \in X \mid \mathcal{I}[I]_\psi(x) \geq \beta_I\},$$
$$L(\mathcal{I}[I]_\psi; \beta_S) := \{x \in X \mid \mathcal{I}[I]_\psi(x) \leq \beta_S\},$$

and

$$U(\mathcal{I}[F]_\psi; \gamma_I) := \{x \in X \mid \mathcal{I}[F]_\psi(x) \geq \gamma_I\},$$
$$L(\mathcal{I}[F]_\psi; \gamma_S) := \{x \in X \mid \mathcal{I}[F]_\psi(x) \leq \gamma_S\},$$

where $\psi \in \{\inf, \sup\}$, and α_I, α_S, β_I, β_S, γ_I and γ_S are numbers in $[0, 1]$.

Theorem 3.4. *Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in $(X, *, 0)$, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,4), I(1,4), F(1,4))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $L(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $U(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $L(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,1), I(4,1), F(4,1))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $L(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $U(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $L(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $U(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,1), I(1,1), F(1,1))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $U(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $U(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $U(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(4) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,4), I(4,4), F(4,4))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $L(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $L(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $L(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $L(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

Proof. (1) Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic ideal of $(X,*,0)$. Then $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 1-fuzzy ideals of X; and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 4-fuzzy ideals of X. Let $\alpha_I, \alpha_S \in [0,1]$ be such that $U(\mathcal{I}[T]_{\inf}; \alpha_I)$ and $L(\mathcal{I}[T]_{\sup}; \alpha_S)$ are nonempty. Obviously, $0 \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$ and $0 \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$. Let $x, y \in X$ be such that $x * y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$ and $y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$. Then $\mathcal{I}[T]_{\inf}(x*y) \geq \alpha_I$ and $\mathcal{I}[T]_{\inf}(y) \geq \alpha_I$, and so

$$\mathcal{I}[T]_{\inf}(x) \geq \min\{\mathcal{I}[T]_{\inf}(x*y), \mathcal{I}[T]_{\inf}(y)\} \geq \alpha_I,$$

that is, $x \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$. If $x * y \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$ and $y \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$, then $\mathcal{I}[T]_{\sup}(x*y) \leq \alpha_S$ and $\mathcal{I}[T]_{\sup}(y) \leq \alpha_S$, which imply that

$$\mathcal{I}[T]_{\sup}(x) \leq \max\{\mathcal{I}[T]_{\sup}(x*y), \mathcal{I}[T]_{\sup}(y)\} \leq \alpha_S,$$

that is, $x \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$. Hence $U(\mathcal{I}[T]_{\inf}; \alpha_I)$ and $L(\mathcal{I}[T]_{\sup}; \alpha_S)$ are ideals of $(X,*,0)$ for all α_I, $\alpha_S \in [0,1]$. Similarly, we can prove that $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebras of $(X,*,0)$ for all β_I, β_S, γ_I, $\gamma_S \in [0,1]$. By the similarly way to the proof of (1), we can prove that (2), (3) and (4) are true. □

Corollary 3.5. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X,*,0)$*, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(3,4), I(3,4), F(3,4))$*-interval neutrosophic ideal of* $(X,*,0)$ *or a* $(T(i,2), I(i,2), F(i,2))$*-interval neutrosophic ideal of* $(X,*,0)$ *for* $i \in \{1,3\}$*, then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $L(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $U(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $L(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,3), I(4,3), F(4,3))$*-interval neutrosophic ideal of* $(X,*,0)$ *or a* $(T(2,j), I(2,j), F(2,j))$*-interval neutrosophic ideal of* $(X,*,0)$ *for* $j \in \{1,3\}$*, then* $L(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $U(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $L(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $U(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(3,1), I(3,1), F(3,1))$*-interval neutrosophic ideal of* $(X,*,0)$ *or a* $(T(i,3), I(i,3), F(i,3))$*-interval neutrosophic ideal of* $(X,*,0)$ *for* $i \in \{1,3\}$*, then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $U(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $U(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $U(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(4) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(2,4), I(2,4), F(2,4))$*-interval neutrosophic ideal of* $(X,*,0)$ *or a* $(T(i,2), I(i,2), F(i,2))$*-interval neutrosophic ideal of* $(X,*,0)$ *for* $i \in \{2,4\}$*, then* $L(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $L(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $L(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $L(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

Proof. Straightforward since every 3-fuzzy (resp., 2-fuzzy) ideal is a 1-fuzzy (resp., 4-fuzzy) ideal. □

Theorem 3.6. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X,*,0)$*, the following assertions are valid.*

(1) *If* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $L(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $U(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $L(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are nonempty ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*, then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,4), I(1,4), F(1,4))$*-interval neutrosophic ideal of* $(X,*,0)$*.*

(2) *If* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $U(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $U(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $U(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are nonempty ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*, then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,1), I(1,1), F(1,1))$*-interval neutrosophic ideal of* $(X,*,0)$*.*

(3) *If* $L(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $U(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $L(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $U(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are nonempty ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*, then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,1), I(4,1), F(4,1))$*-interval neutrosophic ideal of* $(X,*,0)$*.*

(4) *If* $L(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $L(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $L(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $L(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are*

nonempty ideals of $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$, *then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4, 4), I(4, 4), F(4, 4))$*-interval neutrosophic ideal of* $(X, *, 0)$.

Proof. (1) Suppose that $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are nonempty ideals of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$. If $(X, \mathcal{I}[T]_{\inf})$ is not a 1-fuzzy ideal of $(X, *, 0)$, then there exist $x, y \in X$ such that

$$\mathcal{I}[T]_{\inf}(x) < \min\{\mathcal{I}[T]_{\inf}(x * y), \mathcal{I}[T]_{\inf}(y)\}.$$

If we take $\alpha_I = \min\{\mathcal{I}[T]_{\inf}(x*y), \mathcal{I}[T]_{\inf}(y)\}$, then $x*y,\ y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$ but $x \notin U(\mathcal{I}[T]_{\inf}; \alpha_I)$. This is a contradiction, and so $(X, \mathcal{I}[T]_{\inf})$ is a 1-fuzzy ideal of $(X, *, 0)$. If $(X, \mathcal{I}[T]_{\sup})$ is not a 4-fuzzy ideal of $(X, *, 0)$, then

$$\mathcal{I}[T]_{\sup}(a) > \max\{\mathcal{I}[T]_{\sup}(a * b), \mathcal{I}[T]_{\sup}(b)\}$$

for some $a, b \in X$, and so $a * b,\ b \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$ and $a \notin L(\mathcal{I}[T]_{\sup}; \alpha_S)$ by taking

$$\alpha_S := \max\{\mathcal{I}[T]_{\sup}(a * b), \mathcal{I}[T]_{\sup}(b)\}.$$

This is a contradiction, and therefore $(X, \mathcal{I}[T]_{\sup})$ is a 4-fuzzy ideal of $(X, *, 0)$. Similarly, we can verify that $(X, \mathcal{I}[I]_{\inf})$ is a 1-fuzzy ideal of $(X, *, 0)$ and $(X, \mathcal{I}[I]_{\sup})$ is a 4-fuzzy ideal of $(X, *, 0)$, and $(X, \mathcal{I}[F]_{\inf})$ is a 1-fuzzy ideal of $(X, *, 0)$ and $(X, \mathcal{I}[F]_{\sup})$ is a 4-fuzzy ideal of $(X, *, 0)$. Consequently, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1, 4), I(1, 4), F(1, 4))$-interval neutrosophic ideal of $(X, *, 0)$. The assertions (2), (3) and (4) can be proved by the similar way to the proof of (1). □

Theorem 3.7. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$ *is a* $(T(2, 3), I(2, 3), F(2, 3))$*-interval neutrosophic ideal of* $(X, *, 0)$*, then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ *are either empty or ideals of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

Proof. Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be a $(T(2, 3), I(2, 3), F(2, 3))$-interval neutrosophic ideal of $(X, *, 0)$. Then

(1) $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 2-fuzzy ideals of $(X, *, 0)$,

(2) $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 3-fuzzy ideals of $(X, *, 0)$.

Let α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$ be such that $U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ are nonempty. Then there exist $x, y, z, a, b, d \in X$ such that $x \in U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $a \in L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $y \in U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $b \in L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $z \in U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $d \in L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$. Hence

$\mathcal{I}[T]_{\inf}(0) \leq \mathcal{I}[T]_{\inf}(x) < \alpha_I$ and $\mathcal{I}[T]_{\sup}(0) \geq \mathcal{I}[T]_{\sup}(a) > \alpha_S$,

$\mathcal{I}[I]_{\inf}(0) \leq \mathcal{I}[I]_{\inf}(y) < \beta_I$ and $\mathcal{I}[I]_{\sup}(0) \geq \mathcal{I}[I]_{\sup}(b) > \beta_S$,

$\mathcal{I}[F]_{\inf}(0) \leq \mathcal{I}[F]_{\inf}(z) < \gamma_I$ and $\mathcal{I}[F]_{\sup}(0) \geq \mathcal{I}[F]_{\sup}(d) > \gamma_S$,

and so $0 \in U(\mathcal{I}[T]_{\inf}; \alpha_I)^c \cap L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $0 \in U(\mathcal{I}[I]_{\inf}; \beta_I)^c \cap L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, and $0 \in U(\mathcal{I}[F]_{\inf}; \gamma_I)^c \cap L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$. Let $x, y \in X$ be such that $x * y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$ and $y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$. Then $\mathcal{I}[T]_{\inf}(x*y) < \alpha_I$ and $\mathcal{I}[T]_{\inf}(y) < \alpha_I$. Hence

$$\mathcal{I}[T]_{\inf}(x) \leq \min\{\mathcal{I}[T]_{\inf}(x * y), \mathcal{I}[T]_{\inf}(y)\} < \alpha_I,$$

and so $x \in U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$. Thus $U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$ is an ideal of $(X, *, 0)$. Similarly, we can verify that

- If $x * y \in L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$ and $y \in L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, then $x \in L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$,
- If $x * y \in U(\mathcal{I}[I]_{\inf}; \beta_I)^c$ and $y \in U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, then $x \in U(\mathcal{I}[I]_{\inf}; \beta_I)^c$,
- If $x * y \in L(\mathcal{I}[I]_{\sup}; \beta_S)^c$ and $y \in L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, then $x \in L(\mathcal{I}[I]_{\sup}; \beta_S)^c$,
- If $x * y \in U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $y \in U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$, then $x \in U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$,
- If $x * y \in L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ and $y \in L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$, then $x \in L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$.

Therefore $L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ are ideals of $(X, *, 0)$. □

The converse of Theorem 3.7 is not true in general as seen in the following example.

Example 3.8. Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the binary operation $*$ which is given in Table 3 (see [6]).

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ where $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

$$\mathcal{I}[T] : X \to \tilde{\mathcal{P}}([0, 1]),\ \ x \mapsto \begin{cases} [0.25, 0.85) & \text{if } x = 0, \\ (0.45, 0.83] & \text{if } x = 1, \\ [0.55, 0.73] & \text{if } x = a, \\ (0.65, 0.73] & \text{if } x = b, \\ [0.65, 0.75) & \text{if } x = c, \end{cases}$$

$$\mathcal{I}[I] : X \to \tilde{\mathcal{P}}([0,1]),\ \ x \mapsto \begin{cases} [0.3, 0.75) & \text{if } x = 0, \\ (0.3, 0.70] & \text{if } x = 1, \\ [0.6, 0.63] & \text{if } x = a, \\ (0.5, 0.63] & \text{if } x = b, \\ [0.6, 0.68) & \text{if } x = c, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \tilde{\mathcal{P}}([0,1]),\ \ x \mapsto \begin{cases} [0.44, 0.9) & \text{if } x = 0, \\ (0.55, 0.9] & \text{if } x = 1, \\ [0.55, 0.7] & \text{if } x = a, \\ (0.66, 0.8] & \text{if } x = b, \\ [0.66, 0.7) & \text{if } x = c. \end{cases}$$

Then

$$U(\mathcal{I}[T]_{\inf}; \alpha_I)^c = \begin{cases} \emptyset & \text{if } \alpha_I \in [0, 0.25], \\ \{0\} & \text{if } \alpha_I \in (0.25, 0.45], \\ \{0,1\} & \text{if } \alpha_I \in (0.45, 0.55], \\ \{0,1,a\} & \text{if } \alpha_I \in (0.55, 0.65], \\ X & \text{if } \alpha_I \in (0.65, 1.0], \end{cases}$$

$$L(\mathcal{I}[T]_{\sup}; \alpha_S)^c = \begin{cases} \emptyset & \text{if } \alpha_S \in [0.85, 1.0], \\ \{0\} & \text{if } \alpha_S \in [0.83, 0.85), \\ \{0,1\} & \text{if } \alpha_S \in [0.75, 0.83), \\ \{0,1,c\} & \text{if } \alpha_S \in [0.73, 0.75), \\ X & \text{if } \alpha_S \in [0, 0.73), \end{cases}$$

$$U(\mathcal{I}[I]_{\inf}; \beta_I)^c = \begin{cases} \emptyset & \text{if } \beta_I \in [0, 0.3], \\ \{0,1\} & \text{if } \beta_I \in (0.3, 0.5], \\ \{0,1,b\} & \text{if } \beta_I \in (0.5, 0.6], \\ X & \text{if } \beta_I \in (0.6, 1.0], \end{cases}$$

$$L(\mathcal{I}[I]_{\sup}; \beta_S)^c = \begin{cases} \emptyset & \text{if } \beta_S \in [0.75, 1.0], \\ \{0\} & \text{if } \beta_S \in [0.70, 0.75), \\ \{0,1\} & \text{if } \beta_S \in [0.68, 0.70), \\ \{0,1,c\} & \text{if } \beta_S \in [0.63, 0.68), \\ X & \text{if } \beta_S \in [0, 0.63), \end{cases}$$

$$U(\mathcal{I}[F]_{\inf}; \gamma_I)^c = \begin{cases} \emptyset & \text{if } \gamma_I \in [0, 0.44], \\ \{0\} & \text{if } \gamma_I \in (0.44, 0.55], \\ \{0,1,a\} & \text{if } \gamma_I \in (0.55, 0.66], \\ X & \text{if } \gamma_I \in (0.66, 1.0], \end{cases}$$

$$L(\mathcal{I}[F]_{\sup}; \gamma_S)^c = \begin{cases} \emptyset & \text{if } \gamma_S \in [0.9, 1.0], \\ \{0,1\} & \text{if } \gamma_S \in [0.8, 0.9), \\ \{0,1,b\} & \text{if } \gamma_S \in [0.7, 0.8), \\ X & \text{if } \gamma_S \in [0, 0.7). \end{cases}$$

Hence the nonempty sets $U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ are ideals of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$. But $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is not a $(T(2,3)$, $I(2,3)$, $F(2,3))$-interval neutrosophic ideal of $(X, *, 0)$ since

$$\mathcal{I}[T]_{\inf}(c) = 0.65 > 0.55 = \min\{\mathcal{I}[T]_{\inf}(c * a), \mathcal{I}[T]_{\inf}(a)\},$$

$$\mathcal{I}[T]_{\sup}(a) = 0.73 < 0.75 = \max\{\mathcal{I}[T]_{\sup}(a * c), \mathcal{I}[T]_{\sup}(c)\},$$

$$\mathcal{I}[I]_{\inf}(c) = 0.6 > 0.5 = \min\{\mathcal{I}[I]_{\inf}(c * a), \mathcal{I}[I]_{\inf}(a)\},$$

$$\mathcal{I}[I]_{\sup}(a) = 0.63 < 0.68 = \max\{\mathcal{I}[I]_{\sup}(a * c), \mathcal{I}[I]_{\sup}(c)\},$$

$$\mathcal{I}[F]_{\inf}(c) = 0.66 > 0.55 = \min\{\mathcal{I}[F]_{\inf}(c * a), \mathcal{I}[F]_{\inf}(a)\},$$

and/or

$$\mathcal{I}[F]_{\sup}(a) = 0.7 < 0.8 = \max\{\mathcal{I}[F]_{\sup}(a * c), \mathcal{I}[F]_{\sup}(c)\}.$$

Using the similar way to the proof of Theorem 3.7, we have the following theorems.

Theorem 3.9. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$*, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(2,2), I(2,2), F(2,2))$*-interval neutrosophic ideal of* $(X, *, 0)$*, then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $U(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ *are either empty or ideals of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(3,2), I(3,2), F(3,2))$*-interval neutrosophic ideal of* $(X, *, 0)$*, then* $L(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $L(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $U(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ *are either empty or ideals of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(3,3), I(3,3), F(3,3))$*-interval neutrosophic ideal of* $(X, *, 0)$*, then* $L(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $L(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ *are either empty or ideals of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

Using the similar way to the proofs of Theorems 3.4 and 3.7, we have the following theorem.

Theorem 3.10. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$*, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,2), I(1,2), F(1,2))$*-interval neutrosophic ideal of* $(X, *, 0)$*, then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $U(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ *are either empty or ideals of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,3), I(1,3), F(1,3))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $U(\mathcal{I}[T]_{\inf};\alpha_I)$, $L(\mathcal{I}[T]_{\sup};\alpha_S)^c$, $U(\mathcal{I}[I]_{\inf};\beta_I)$, $L(\mathcal{I}[I]_{\sup};\beta_S)^c$, $U(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup};\gamma_S)^c$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(2,1), I(2,1), F(2,1))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $U(\mathcal{I}[T]_{\inf};\alpha_I)^c$, $U(\mathcal{I}[T]_{\sup};\alpha_S)$, $U(\mathcal{I}[I]_{\inf};\beta_I)^c$, $U(\mathcal{I}[I]_{\sup};\beta_S)$, $U(\mathcal{I}[F]_{\inf};\gamma_I)^c$ *and* $U(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(4) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(3,1), I(3,1), F(3,1))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $L(\mathcal{I}[T]_{\inf};\alpha_I)^c$, $U(\mathcal{I}[T]_{\sup};\alpha_S)$, $L(\mathcal{I}[I]_{\inf};\beta_I)^c$, $U(\mathcal{I}[I]_{\sup};\beta_S)$, $L(\mathcal{I}[F]_{\inf};\gamma_I)^c$ *and* $U(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(5) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(2,4), I(2,4), F(2,4))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $U(\mathcal{I}[T]_{\inf};\alpha_I)^c$, $L(\mathcal{I}[T]_{\sup};\alpha_S)$, $U(\mathcal{I}[I]_{\inf};\beta_I)^c$, $L(\mathcal{I}[I]_{\sup};\beta_S)$, $U(\mathcal{I}[F]_{\inf};\gamma_I)^c$ *and* $L(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(6) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(3,4), I(3,4), F(3,4))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $L(\mathcal{I}[T]_{\inf};\alpha_I)^c$, $L(\mathcal{I}[T]_{\sup};\alpha_S)$, $L(\mathcal{I}[I]_{\inf};\beta_I)^c$, $L(\mathcal{I}[I]_{\sup};\beta_S)$, $L(\mathcal{I}[F]_{\inf};\gamma_I)^c$ *and* $L(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(7) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,2), I(4,2), F(4,2))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $L(\mathcal{I}[T]_{\inf};\alpha_I)$, $U(\mathcal{I}[T]_{\sup};\alpha_S)^c$, $L(\mathcal{I}[I]_{\inf};\beta_I)$, $U(\mathcal{I}[I]_{\sup};\beta_S)^c$, $L(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup};\gamma_S)^c$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(8) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,3), I(4,3), F(4,3))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $L(\mathcal{I}[T]_{\inf};\alpha_I)$, $L(\mathcal{I}[T]_{\sup};\alpha_S)^c$, $L(\mathcal{I}[I]_{\inf};\beta_I)$, $L(\mathcal{I}[I]_{\sup};\beta_S)^c$, $L(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup};\gamma_S)^c$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

Proposition 3.11. *Every* $(T(1,4), I(1,4), F(1,4))$*-interval neutrosophic ideal* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *of* $(X,*,0)$ *satisfies*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) \leq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) \geq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) \leq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) \geq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) \leq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.1}$$

for all $x, y \in X$ *with* $x \leq y$.

Proof. If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic ideal of $(X,*,0)$, then $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 1-fuzzy ideals of $(X,*,0)$, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 4-fuzzy ideals of $(X,*,0)$. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\begin{aligned} \mathcal{I}[T]_{\inf}(x) &\geq \min\{\mathcal{I}[T]_{\inf}(x*y), \mathcal{I}[T]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[T]_{\inf}(0), \mathcal{I}[T]_{\inf}(y)\} = \mathcal{I}[T]_{\inf}(y), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[T]_{\sup}(x) &\leq \max\{\mathcal{I}[T]_{\sup}(x*y), \mathcal{I}[T]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[T]_{\sup}(0), \mathcal{I}[T]_{\sup}(y)\} = \mathcal{I}[T]_{\sup}(y), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\inf}(x) &\geq \min\{\mathcal{I}[I]_{\inf}(x*y), \mathcal{I}[I]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[I]_{\inf}(0), \mathcal{I}[I]_{\inf}(y)\} = \mathcal{I}[I]_{\inf}(y), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\sup}(x) &\leq \max\{\mathcal{I}[I]_{\sup}(x*y), \mathcal{I}[I]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[I]_{\sup}(0), \mathcal{I}[I]_{\sup}(y)\} = \mathcal{I}[I]_{\sup}(y), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\inf}(x) &\geq \min\{\mathcal{I}[F]_{\inf}(x*y), \mathcal{I}[F]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[F]_{\inf}(0), \mathcal{I}[F]_{\inf}(y)\} = \mathcal{I}[F]_{\inf}(y), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\sup}(x) &\leq \max\{\mathcal{I}[F]_{\sup}(x*y), \mathcal{I}[F]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[F]_{\sup}(0), \mathcal{I}[F]_{\sup}(y)\} = \mathcal{I}[F]_{\sup}(y). \end{aligned}$$

This completes the proof. □

Using the similar way to the proof of Proposition 3.11, we have the following proposition.

Proposition 3.12. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X,*,0)$*, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,1), I(1,1), F(1,1))$*-*

interval neutrosophic ideal of $(X,*,0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) \geq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) \geq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) \geq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) \geq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) \geq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.2}$$

for all $x, y \in X$ *with* $x \leq y$*.*

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,1), I(4,1), F(4,1))$*-interval neutrosophic ideal of* $(X,*,0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) \geq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) \leq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) \geq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) \leq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) \geq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.3}$$

for all $x, y \in X$ *with* $x \leq y$*.*

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,4), I(4,4), F(4,4))$*-interval neutrosophic ideal of* $(X,*,0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) \leq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) \leq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) \leq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) \leq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) \leq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.4}$$

for all $x, y \in X$ *with* $x \leq y$*.*

Proposition 3.13. *For every* $(i,j) \in \{(2,2),(2,3),(3,2),(3,3)\}$*, Every* $(T(i,j), I(i,j), F(i,j))$*-interval neutrosophic ideal* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *of* $(X,*,0)$ *satisfies*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases} \tag{3.5}$$

for all $x, y \in X$ *with* $x \leq y$*.*

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(2,3), I(2,3), F(2,3))$-interval neutrosophic ideal of $(X,*,0)$. Then $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 2-fuzzy ideals of $(X,*,0)$, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 3-fuzzy ideals of $(X,*,0)$. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and thus

$$\begin{aligned} \mathcal{I}[T]_{\inf}(x) &\leq \min\{\mathcal{I}[T]_{\inf}(x*y), \mathcal{I}[T]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[T]_{\inf}(0), \mathcal{I}[T]_{\inf}(y)\} = \mathcal{I}[T]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[T]_{\sup}(x) &\geq \max\{\mathcal{I}[T]_{\sup}(x*y), \mathcal{I}[T]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[T]_{\sup}(0), \mathcal{I}[T]_{\sup}(y)\} = \mathcal{I}[T]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\inf}(x) &\leq \min\{\mathcal{I}[I]_{\inf}(x*y), \mathcal{I}[I]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[I]_{\inf}(0), \mathcal{I}[I]_{\inf}(y)\} = \mathcal{I}[I]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\sup}(x) &\geq \max\{\mathcal{I}[I]_{\sup}(x*y), \mathcal{I}[I]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[I]_{\sup}(0), \mathcal{I}[I]_{\sup}(y)\} = \mathcal{I}[I]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\inf}(x) &\leq \min\{\mathcal{I}[F]_{\inf}(x*y), \mathcal{I}[F]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[F]_{\inf}(0), \mathcal{I}[F]_{\inf}(y)\} = \mathcal{I}[F]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\sup}(x) &\geq \max\{\mathcal{I}[F]_{\sup}(x*y), \mathcal{I}[F]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[F]_{\sup}(0), \mathcal{I}[F]_{\sup}(y)\} = \mathcal{I}[F]_{\inf}(0). \end{aligned}$$

It follows that $\mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0)$, $\mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0)$, $\mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0)$, $\mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0)$, $\mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0)$ and $\mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0)$ for all $x, y \in X$ with $x \leq y$. Similarly, we can verify that (3.5) is true for $(i,j) \in \{(2,2),(3,2),(3,3)\}$. □

Using the similar way to the proof of Propositions 3.11 and 3.13, we have the following proposition.

Proposition 3.14. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X,*,0)$*, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,j), I(1,j), F(1,j))$*-interval neutrosophic ideal of* $(X,*,0)$ *for* $j \in \{2,3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) \geq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) \geq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases} \tag{3.6}$$

for all $x, y \in X$ *with* $x \leq y$*.*

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(i,1), I(i,1), F(i,1))$*-*

interval neutrosophic ideal of $(X, *, 0)$ *for* $i \in \{2, 3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) \geq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) \geq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) \geq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.7}$$

for all $x, y \in X$ *with* $x \leq y$.

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(i, 4), I(i, 4), F(i, 4))$*-interval neutrosophic ideal of* $(X, *, 0)$ *for* $i \in \{2, 3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) \leq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) \leq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) \leq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.8}$$

for all $x, y \in X$ *with* $x \leq y$.

(4) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4, j), I(4, j), F(4, j))$*-interval neutrosophic ideal of* $(X, *, 0)$ *for* $j \in \{2, 3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) \leq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) \leq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases} \tag{3.9}$$

for all $x, y \in X$ *with* $x \leq y$.

Proposition 3.15. *Every* $(T(1, 4), I(1, 4), F(1, 4))$*-interval neutrosophic ideal* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *of* $(X, *, 0)$ *satisfies*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) \leq \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) \geq \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) \leq \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) \geq \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) \leq \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases} \tag{3.10}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

Proof. Let $x, y, z \in X$ be such that $x*y \leq z$. Then $(x*y)*z = 0$, and so

$$\begin{aligned} \mathcal{I}[T]_{\inf}(x) &\geq \min\{\mathcal{I}[T]_{\inf}(x*y), \mathcal{I}[T]_{\inf}(y)\} \\ &\geq \min\{\min\{\mathcal{I}[T]_{\inf}((x*y)*z), \mathcal{I}[T]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[T]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[T]_{\inf}(0), \mathcal{I}[T]_{\inf}(z)\}, \mathcal{I}[T]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{I}[T]_{\sup}(x) &\leq \max\{\mathcal{I}[T]_{\sup}(x*y), \mathcal{I}[T]_{\sup}(y)\} \\ &\leq \max\{\max\{\mathcal{I}[T]_{\sup}((x*y)*z), \mathcal{I}[T]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[T]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[T]_{\sup}(0), \mathcal{I}[T]_{\sup}(z)\}, \mathcal{I}[T]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\inf}(x) &\geq \min\{\mathcal{I}[I]_{\inf}(x*y), \mathcal{I}[I]_{\inf}(y)\} \\ &\geq \min\{\min\{\mathcal{I}[I]_{\inf}((x*y)*z), \mathcal{I}[I]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[I]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[I]_{\inf}(0), \mathcal{I}[I]_{\inf}(z)\}, \mathcal{I}[I]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\sup}(x) &\leq \max\{\mathcal{I}[I]_{\sup}(x*y), \mathcal{I}[I]_{\sup}(y)\} \\ &\leq \max\{\max\{\mathcal{I}[I]_{\sup}((x*y)*z), \mathcal{I}[I]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[I]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[I]_{\sup}(0), \mathcal{I}[I]_{\sup}(z)\}, \mathcal{I}[I]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\inf}(x) &\geq \min\{\mathcal{I}[F]_{\inf}(x*y), \mathcal{I}[F]_{\inf}(y)\} \\ &\geq \min\{\min\{\mathcal{I}[F]_{\inf}((x*y)*z), \mathcal{I}[F]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[F]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[F]_{\inf}(0), \mathcal{I}[F]_{\inf}(z)\}, \mathcal{I}[F]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\sup}(x) &\leq \max\{\mathcal{I}[F]_{\sup}(x*y), \mathcal{I}[F]_{\sup}(y)\} \\ &\leq \max\{\max\{\mathcal{I}[F]_{\sup}((x*y)*z), \mathcal{I}[F]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[F]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[F]_{\sup}(0), \mathcal{I}[F]_{\sup}(z)\}, \mathcal{I}[F]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\}. \end{aligned}$$

This completes the proof. □

Using the similar way to the proof of Proposition 3.15, we have the following proposition.

Proposition 3.16. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$*, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1, 1), I(1, 1), F(1, 1))$*-*

interval neutrosophic ideal of $(X, *, 0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) \geq \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) \geq \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) \geq \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) \geq \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) \geq \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,1),\ I(4,1),\ F(4,1))$*-interval neutrosophic ideal of* $(X, *, 0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) \geq \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) \leq \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) \geq \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) \leq \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) \geq \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,4),\ I(4,4),\ F(4,4))$*-interval neutrosophic ideal of* $(X, *, 0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) \leq \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) \leq \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) \leq \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) \leq \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) \leq \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

Proposition 3.17. *For every* $(i, j) \in \{(2,2), (2,3), (3,2), (3,3)\}$*, Every* $(T(i,j),\ I(i,j),\ F(i,j))$*-interval neutrosophic ideal* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *of* $(X, *, 0)$ *satisfies*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases} \tag{3.11}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(2,3),\ I(2,3),\ F(2,3))$-interval neutrosophic ideal of $(X, *, 0)$. Then $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 2-fuzzy ideals of $(X, *, 0)$, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 3-fuzzy ideals of $(X, *, 0)$. Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and thus

$$\begin{aligned} \mathcal{I}[T]_{\inf}(x) &\leq \min\{\mathcal{I}[T]_{\inf}(x * y), \mathcal{I}[T]_{\inf}(y)\} \\ &\leq \min\{\min\{\mathcal{I}[T]_{\inf}((x * y) * z), \mathcal{I}[T]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[T]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[T]_{\inf}(0), \mathcal{I}[T]_{\inf}(z)\}, \mathcal{I}[T]_{\inf}(y)\} \\ &= \mathcal{I}[T]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[T]_{\sup}(x) &\geq \max\{\mathcal{I}[T]_{\sup}(x * y), \mathcal{I}[T]_{\sup}(y)\} \\ &\geq \max\{\max\{\mathcal{I}[T]_{\sup}((x * y) * z), \mathcal{I}[T]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[T]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[T]_{\sup}(0), \mathcal{I}[T]_{\sup}(z)\}, \mathcal{I}[T]_{\sup}(y)\} \\ &= \mathcal{I}[T]_{\sup}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\inf}(x) &\leq \min\{\mathcal{I}[I]_{\inf}(x * y), \mathcal{I}[I]_{\inf}(y)\} \\ &\leq \min\{\min\{\mathcal{I}[I]_{\inf}((x * y) * z), \mathcal{I}[I]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[I]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[I]_{\inf}(0), \mathcal{I}[I]_{\inf}(z)\}, \mathcal{I}[I]_{\inf}(y)\} \\ &= \mathcal{I}[I]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\sup}(x) &\geq \max\{\mathcal{I}[I]_{\sup}(x * y), \mathcal{I}[I]_{\sup}(y)\} \\ &\geq \max\{\max\{\mathcal{I}[I]_{\sup}((x * y) * z), \mathcal{I}[I]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[I]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[I]_{\sup}(0), \mathcal{I}[I]_{\sup}(z)\}, \mathcal{I}[I]_{\sup}(y)\} \\ &= \mathcal{I}[I]_{\sup}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\inf}(x) &\leq \min\{\mathcal{I}[F]_{\inf}(x * y), \mathcal{I}[F]_{\inf}(y)\} \\ &\leq \min\{\min\{\mathcal{I}[F]_{\inf}((x * y) * z), \mathcal{I}[F]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[F]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[F]_{\inf}(0), \mathcal{I}[F]_{\inf}(z)\}, \mathcal{I}[F]_{\inf}(y)\} \\ &= \mathcal{I}[F]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\sup}(x) &\geq \max\{\mathcal{I}[F]_{\sup}(x * y), \mathcal{I}[F]_{\sup}(y)\} \\ &\geq \max\{\max\{\mathcal{I}[F]_{\sup}((x * y) * z), \mathcal{I}[F]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[F]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[F]_{\sup}(0), \mathcal{I}[F]_{\sup}(z)\}, \mathcal{I}[F]_{\sup}(y)\} \\ &= \mathcal{I}[F]_{\sup}(0). \end{aligned}$$

Since $\mathcal{I}[T]_{\inf}(0) \leq \mathcal{I}[T]_{\inf}(x)$, $\mathcal{I}[T]_{\sup}(0) \geq \mathcal{I}[T]_{\sup}(x)$, $\mathcal{I}[I]_{\inf}(0) \leq \mathcal{I}[I]_{\inf}(x)$, $\mathcal{I}[I]_{\sup}(0) \geq \mathcal{I}[I]_{\sup}(x)$, $\mathcal{I}[F]_{\inf}(0) \leq \mathcal{I}[F]_{\inf}(x)$ and $\mathcal{I}[F]_{\sup}(0) \geq \mathcal{I}[F]_{\sup}(x)$, it follows that $\mathcal{I}[T]_{\inf}(0) = \mathcal{I}[T]_{\inf}(x)$, $\mathcal{I}[T]_{\sup}(0) = \mathcal{I}[T]_{\sup}(x)$, $\mathcal{I}[I]_{\inf}(0) = \mathcal{I}[I]_{\inf}(x)$, $\mathcal{I}[I]_{\sup}(0) = \mathcal{I}[I]_{\sup}(x)$, $\mathcal{I}[F]_{\inf}(0) =$

$\mathcal{I}[F]_{\inf}(x)$ and $\mathcal{I}[F]_{\sup}(0) = \mathcal{I}[F]_{\sup}(x)$. Similarly, we can verify that (3.11) is true for $(i, j) \in \{(2, 2), (3, 2), (3, 3)\}$. □

Using the similar way to the proof of Propositions 3.15 and 3.17, we have the following proposition.

Proposition 3.18. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$*, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1, j), I(1, j), F(1, j))$*-interval neutrosophic ideal of* $(X, *, 0)$ *for* $j \in \{2, 3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) \geq \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) \geq \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$*.*

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(i, 1), I(i, 1), F(i, 1))$*-interval neutrosophic ideal of* $(X, *, 0)$ *for* $i \in \{2, 3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) \geq \min\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) \geq \min\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) \geq \min\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$*.*

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(i, 4), I(i, 4), F(i, 4))$*-interval neutrosophic ideal of* $(X, *, 0)$ *for* $i \in \{2, 3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) \leq \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) \leq \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) \leq \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$*.*

(4) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4, j), I(4, j), F(4, j))$*-interval neutrosophic ideal of* $(X, *, 0)$ *for* $j \in \{2, 3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \max\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) \leq \max\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) \leq \max\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$*.*

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NSS University of New Mexico

On Separation Axioms in an Ordered Neutrosophic Bitopological Space

[1]R. Narmada Devi, [2] R. Dhavaseelan, [3]S. Jafari

[1]Department of Mathematics, Vel Tech Rangarajan Dr.Sagunthala R and D Institute of Science and Technology,Chennai,Tamil Nadu, India.

E-mail :narmadadevi23@gmail.com

[2] Department of Mathematics, Sona College of Technology, Salem-636005, Tamil Nadu, India. E-mail: dhavaseelan.r@gmail.com

[3] Department of Mathematics, College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark. E-mail: jafaripersia@gmail.com

Abstract: In this paper we introduce the concept of a new class of an ordered neutrosophic bitopological spaces. Besides giving some interesting properties of these spaces. We also prove analogues of Uryshon's lemma and Tietze extension theorem in an ordered neutrosophic bitopological spaces.

Keywords: Ordered neutrosophic bitopological space; lower(resp.upper) pairwise neutrosophic G_δ-α-locally T_1-ordered space; pairwise neutrosophic G_δ-α-locally T_1-ordered space; pairwise neutrosophic G_δ-α-locally T_2-ordered space; weakly pairwise neutrosophic G_δ-α-locally T_2-ordered space; almost pairwise neutrosophic G_δ-α-locally T_2-ordered space and strongly pairwise neutrosophic G_δ-α-locally normally ordered space.

1 Introduction and Preliminaries

The concept of fuzzy sets was introduced by Zadeh [17]. Fuzzy sets have applications in many fields such as information theory [15] and control theory [16]. The theory of fuzzy topological spaces was introduced and developed by Chang [7]. Atanassov [2] introduced and studied intuitionistic fuzzy sets. On the other hand, Coker [8] introduced the notions of an intuitionistic fuzzy topological space and some other related concepts. The concept of an intuitionistic fuzzy α-closed set was introduced by B. Krsteshka and E. Ekici [5]. G. Balasubramanian [3] was introduced the concept of fuzzy G_δ set. Ganster and Reilly used locally closed sets [10] to define LC-continuity and LC-irresoluteness. The concept of an ordered fuzzy topological space was introduced and developed by A. K. Katsaras [11]. Later G. Balasubmanian [4] introduced and studied the concepts of an ordered L-fuzzy bitopological spaces. F. Smarandache [[13], [14] introduced the concepts of neutrosophy and neutrosophic set. The concepts of neutrosophic crisp set and neutrosophic crisp topological space were introduced by A. A. Salama and S. A. Alblowi [12].

In this paper, we introduce the concepts of pairwise neutrosophic G_δ-α-locally T_1-ordered space, pairwise neutrosophic G_δ-α-locally T_2-ordered space, weakly pairwise neutrosophic G_δ-α-locally T_2-ordered space, almost pairwise neutrosophic G_δ-α-locally T_2-ordered space and strongly pairwise neutrosophic G_δ-α-locally normally ordered space. Some interesting propositions are discussed. Urysohn's lemma and Tietze extension theorem of an strongly pairwise neutrosophic G_δ-α-locally normally ordered space are studied and established.

Definition 1.1. [7] Let X be a nonempty set and $A \subset X$. The characteristic function of A is denoted and defined by $\chi_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}$

Definition 1.2. [13, 14] Let T,I,F be real standard or non standard subsets of $]0^-, 1^+[$, with $sup_T = t_{sup}, inf_T = t_{inf}$
$sup_I = i_{sup}, inf_I = i_{inf}$
$sup_F = f_{sup}, inf_F = f_{inf}$
$n - sup = t_{sup} + i_{sup} + f_{sup}$
$n - inf = t_{inf} + i_{inf} + f_{inf}$. T,I,F are neutrosophic components.

Definition 1.3. [13, 14] Let X be a nonempty fixed set. A neutrosophic set [briefly NS] A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set A.

Remark 1.1. [13, 14]

(1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A \rangle$ in $]0^-, 1^+[$ on X.

(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A \rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.4. [12] Let X be a nonempty set and the neutrosophic sets A and B in the form

$A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;

(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(c) $\bar{A} = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$; [Complement of A]

(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$;

(e) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$;

(f) $[\;]A = \{\langle x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x)\rangle : x \in X\}$;

(g) $\langle\rangle A = \{\langle x, 1 - \gamma_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.5. [12] Let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in X. Then

(a) $\bigcap A_i = \{\langle x, \wedge\mu_{A_i}(x), \wedge\sigma_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in X\}$;

(b) $\bigcup A_i = \{\langle x, \vee\mu_{A_i}(x), \vee\sigma_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in X\}$.

Since our main purpose is to construct the tools for developing neutrosophic topological spaces, we must introduce the neutrosophic sets 0_N and 1_N in X as follows:

Definition 1.6. [12] $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$.

Definition 1.7. [9] A neutrosophic topology (NT) on a nonempty set X is a family T of neutrosophic sets in X satisfying the following axioms:

(i) $0_N, 1_N \in T$,

(ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$,

(iii) $\cup G_i \in T$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq T$.

In this case the ordered pair (X, T) or simply X is called a neutrosophic topological space (NTS) and each neutrosophic set in T is called a neutrosophic open set (NOS). The complement $\overline{A}$ of a NOS A in X is called a neutrosophic closed set (NCS) in X.

Definition 1.8. [9] Let A be a neutrosophic set in a neutrosophic topological space X. Then

$Nint(A) = \bigcup\{G \mid G$ is a neutrosophic open set in X and $G \subseteq A\}$ is called the neutrosophic interior of A;

$Ncl(A) = \bigcap\{G \mid G$ is a neutrosophic closed set in X and $G \supseteq A\}$ is called the neutrosophic closure of A.

Corollary 1.1. [9] Let A,B,C be neutrosophic sets in X. Then the basic properties of inclusion and complementation:

(a) $A \subseteq B$ and $C \subseteq D \Rightarrow A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$,

(b) $A \subseteq B$ and $A \subseteq C \Rightarrow A \subseteq B \cap C$,

(c) $A \subseteq C$ and $B \subseteq C \Rightarrow A \cup B \subseteq C$,

(d) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$,

(e) $\overline{A \cup B} = \overline{A} \cap \overline{B}$,

(f) $\overline{A \cap B} = \overline{A} \cup \overline{B}$,

(g) $A \subseteq B \Rightarrow \overline{B} \subseteq \overline{A}$,

(h) $\overline{\overline{(A)}} = A$,

(i) $\overline{1_N} = 0_N$,

(j) $\overline{0_N} = 1_N$.

Now we shall define the image and preimage of neutrosophic sets. Let X and Y be two nonempty sets and $f : X \to Y$ be a function.

Definition 1.9. [9]

(a) If $B = \{\langle y, \mu_B(y), \sigma_B(y), \gamma_B(y)\rangle : y \in Y\}$ is a neutrosophic set in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is the neutrosophic set in X defined by
$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\sigma_B)(x), f^{-1}(\gamma_B)(x)\rangle : x \in X\}$.

(b) If $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ is a neutrosophic set in X,then the image of A under f, denoted by $f(A)$, is the neutrosophic set in Y defined by
$f(A) = \{\langle y, f(\mu_A)(y), f(\sigma_A)(y), (1 - f(1 - \gamma_A))(y)\rangle : y \in Y\}$. where

$$f(\mu_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$f(\sigma_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \sigma_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$(1 - f(1 - \gamma_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \gamma_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 1, & \text{otherwise,} \end{cases}$$

For the sake of simplicity, let us use the symbol $f_-(\gamma_A)$ for $1 - f(1 - \gamma_A)$.

Corollary 1.2. [9] Let A , $A_i (i \in J)$ be neutrosophic sets in X, B, $B_i (i \in K)$ be neutrosophic sets in Y and $f : X \to Y$ a function. Then

(a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$,

(b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

(c) $A \subseteq f^{-1}(f(A))$ { If f is injective,then $A = f^{-1}(f(A))$ } ,

(d) $f(f^{-1}(B)) \subseteq B$ { If f is surjective,then $f(f^{-1}(B)) = B$ },

(e) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$,

(f) $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$,

(g) $f(\bigcup A_i) = \bigcup f(A_i)$,

(h) $f(\bigcap A_i) \subseteq \bigcap f(A_i)$ { If f is injective,then $f(\bigcap A_i) = \bigcap f(A_i)$},

(i) $f^{-1}(1_N) = 1_N$,

(j) $f^{-1}(0_N) = 0_N$,

(k) $f(1_N) = 1_N$, if f is surjective,

(l) $f(0_N) = 0_N$,

(m) $\overline{f(A)} \subseteq f(\overline{A})$, if f is surjective,

(n) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Definition 1.10. [1] A neutrosophic set A in a neutrosophic topological space (X, T) is called a neutrosophic α-open set $(N\alpha OS)$ if $A \subseteq Nint(Ncl(Nint(A)))$.

2 Ordered neutrosophic G_δ-α-locally bitopological Spaces

In this section, the concepts of a neutrosophic G_δ set, neutrosophic α-closed set, neutrosophic G_δ-α-locally closed set, upper pairwise neutrosophic G_δ-α-locally T_1-ordered space, lower pairwise neutrosophic G_δ-α-locally T_1-ordered space, pairwise neutrosophic G_δ-α-locally T_1-ordered space, pairwise neutrosophic G_δ-α-locally T_2-ordered space, weakly pairwise neutrosophic G_δ-α-locally T_2-ordered space, almost pairwise neutrosophic G_δ-α-locally T_2-ordered space and strongly pairwise neutrosophic G_δ-α-locally normally ordered space are introduced. Some basic properties and characterizations are discussed. Urysohn's lemma and Tietze extension theorem of an strongly pairwise neutrosophic G_δ-α-locally normally ordered space are studied and established.

Definition 2.1. Let (X, T) be a neutrosophic topological space. Let $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ be a neutrosophic set of a neutrosophic topological space X. Then A is said to be a neutrosophic G_δ set (briefly NG_δS) if $A = \bigcap_{i=1}^{\infty} A_i$, where each $A_i \in T$ and $A_i = \langle x, \mu_{A_i}, \sigma_{A_i}, \gamma_{A_i} \rangle$.

The complement of neutrosophic G_δ set is said to be a neutrosophic F_σ set(briefly NF_σS).

Definition 2.2. Let (X, T) be a neutrosophic topological space. Let $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ be a neutrosophic set on a neutrosophic topological space (X, T). Then A is said be a neutrosophic G_δ-α-locally closed set (in short,NG_δ-α-lcs) if $A = B \cap C$, where B is a neutrosophic G_δ set and C is an neutrosophic α-closed set.

The complement of a neutrosophic G_δ-α-locally closed set is said to be a neutrosophic G_δ-α-locally open set (in short,NG_δ-α-los).

Definition 2.3. Let (X, T) be a neutrosophic topological space. Let $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ be a neutrosophic set in a neutrosophic topological space (X, T). The neutrosophic G_δ-α-locally closure of A is denoted and defined by

NG_δ-α-$lcl(A)$=$\bigcap\{B$:$B = \langle x, \mu_B, \sigma_B, \gamma_B \rangle$ is a neutrosophic G_δ-α-locally closed set in X and $A \subseteq B\}$.

Definition 2.4. Let (X, T) be a neutrosophic topological space. Let $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ be a neutrosophic set in a neutrosophic topological space (X, T). The neutrosophic G_δ-α-locally interior of A is denoted and defined by

NG_δ-α-$lint(A)$=$\bigcup\{B$:$B = \langle x, \mu_B, \sigma_B, \gamma_B \rangle$ is a neutrosophic G_δ-α-locally open set in X and $B \subseteq A\}$.

Definition 2.5. Let X be a nonempty set and $x \in X$ a fixed element in X. If $r, t \in I_0 = (0, 1]$ and $s \in I_1 = [0, 1)$ are fixed real numbers such that $0 < r + t + s < 3$, then $x_{r,t,s} = \langle x, r, t, s \rangle$ is called a neutrosophic point (briefly NP) in X, where r denotes the degree of membership of $x_{r,t,s}$, t denotes the degree of indeterminacy and s denotes the degree of nonmembership of $x_{r,t,s}$ and $x \in X$ the support of $x_{r,t,s}$.

The neutrosophic point $x_{r,t,s}$ is contained in the neutrosophic $A(x_{r,t,s} \in A)$ if and only if $r < \mu_A(x), t < \sigma_A(x), s > \gamma_A(x)$.

Definition 2.6. A neutrosophic set $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ in a neutrosophic topological space (X, T) is said to be a neutrosophic neighbourhood of a neotrosophic point $x_{r,t,s}, x \in X$, if there exists a neutrosophic open set $B = \langle x, \mu_B, \sigma_B, \gamma_B \rangle$ with $x_{r,t,s} \subseteq B \subseteq A$.

Definition 2.7. A neutrosophic set $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ in a neutrosophic topological space (X, T) is said to be a neutrosophic G_δ-α-locally neighbourhood of a neutrosophic point $x_{r,t,s}, x \in X$, if there exists a neutrosophic G_δ-α-locally open set $B = \langle x, \mu_B, \sigma_B, \gamma_B \rangle$ with $x_{r,t,s} \subseteq B \subseteq A$.

Notation 2.1. In what follows, we denote neutrosophic neighbourhood A of a in X by neutrosphic neighbourhood A of a neutrsophic point $a_{r,t,s}$ for $a \in X$.

Definition 2.8. A neutrosophic set $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ in a partially ordered set $(X, \leq)$ is said to be an

(i) increasing neutrosophic set if $x \leq y$ implies $A(x) \subseteq A(y)$.That is,
$\mu_A(x) \leq \mu_A(y)$, $\sigma_A(x) \leq \sigma_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$.

(ii) decreasing neutrosophic set if $x \leq y$ implies $A(x) \supseteq A(y)$.That is,
$\mu_A(x) \geq \mu_A(y)$,$\sigma_A(x) \geq \sigma_A(y)$and $\gamma_A(x) \leq \gamma_A(y)$.

Definition 2.9. An ordered neutrosophic bitopological space is a neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ (where τ_1 and τ_2 are neutrosophic topologies on X) equipped with a partial order $\leq$.

Definition 2.10. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be an upper pairwise neutrosophic T_1-ordered space if $a, b \in X$ such that $a \nleq b$, there exists a decreasing τ_1 neutrosophic neighbourhood (or) an decreasing τ_2 neutrosophic neighbourhood A of b such that $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ is not a neutrosophic neighbourhood of a.

Definition 2.11. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be a lower pairwise neutrosophic T_1-ordered space if $a, b \in X$ such that $a \nleq b$, there exists an increasing τ_1 neutrosophic neighbourhood (or) an increasing τ_2 neutrosophic neighbourhood A of a such that $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ is not a neutrosophic neighbourhood of b.

Example 2.1. Let $X = \{1, 2\}$ with a partial order relation $\leq$. Let $\tau_1 = \{0_N, 1_N, A\}$ and $\tau_2 = \{0_N, 1_N, B\}$ where $A = \langle (0.3, 0.3, 0.5), (0.7, 0.7, 0.4) \rangle$ and $B = \langle (0.5, 0.5, 0.5), (0.5, 0.5, 0.5) \rangle$ be any two topologies on X. Then $(X, \tau_1, \tau_2, \leq)$ is an ordered neutrosophic bitopological space. Let $1_{(0.25,0.3,0.5)}$ and $2_{(0.25,0.25,0.35)}$ be any two neutrosophic points on X. For $1_{(0.25,0.3,0.5)} \nleq 2_{(0.25,0.25,0.35)}$, there exists an increasing τ_1 neutrosphic neighbourhood A of $1_{(0.25,0.3,0.5)}$ such that A is not neutrosophic neighbourhood of $2_{(0.25,0.25,0.35)}$. Therefore $(X, \tau_1, \tau_2, \leq)$ is a lower pairwise neutrosophic T_1-ordered space.

Definition 2.12. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be a pairwise neutrosophic T_1-ordered space if and only if it is both upper and lower pairwise neutrosophic T_1-ordered space.

Definition 2.13. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be an upper pairwise neutrosophic G_δ-α-locally T_1-ordered space if $a, b \in X$ such that $a \nleq b$, there exists a decreasing τ_1 neutrosophic G_δ-α-locally neighbourhood (or) a decreasing τ_2 neutrosophic G_δ-α-locally neighbourhood $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ of b such that A is not a neutrosophic G_δ-α-locally neighbourhood of a.

Definition 2.14. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be a lower pairwise neutrosophic G_δ-α-locally T_1-ordered space if $a, b \in X$ such that $a \nleq b$, there exists an increasing τ_1 neutrosophic G_δ-α-locally neighbourhood (or) an increasing τ_2 neutrosophic G_δ-α-locally neighbourhood $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ of a such that A is not a neutrosophic G_δ-α-locally neighbourhood of b.

Definition 2.15. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be a pairwise neutrosophic G_δ-α-locally T_1-ordered space if and only if it is both upper and lower pairwise neutrosophic G_δ-α-locally T_1-ordered space.

Proposition 2.1. For an ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ the following are equivalent

(i) X is a lower (resp. upper) pairwise neutrosophic G_δ-α-locally T_1-ordered space.

(ii) For each $a, b \in X$ such that $a \nleq b$, there exists an increasing (resp. decreasing) τ_1 neutrosophic G_δ-α-locally open set(or) an increasing (resp.decreasing) τ_2 neutrosophic G_δ-α-locally open set $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ such that $A(a) > 0$ (resp. $A(b) > 0$) and A is not a neutrosophic G_δ-α-locally neighbourhood of b (resp.a).

Proof:
(i)⇒(ii) Let X be a lower pairwise neutrosophic G_δ-α-locally T_1-ordered space. Let $a, b \in X$ such that $a \nleq b$. There exists an increasing τ_1 neutrosophic G_δ-α-locally neighbourhood (or) an increasing τ_2 neutrosophic G_δ-α-locally neighbourhood A of a such that A is not a neutrosophic G_δ-α-locally neighbourhood of b. It follows that there exists a τ_i neutrosophic G_δ-α-locally open set ($i = 1(or)2$), $A_i = \langle x, \mu_{A_i}, \sigma_{A_i}, \gamma_{A_i} \rangle$ with $A_i \subseteq A$ and $A_i(a) = A(a) > 0$. As A is an increasing neutrosophic set, $A(a) > A(b)$ and since A is not a neutrosophic G_δ-α-locally neighbourhood of b, $A_i(b) < A(b)$ implies $A_i(a) = A(a) > A(b) \geq A_i(b)$. This shows that A_i is an increasing neutrosophic set and A_i is not a neutrosophic G_δ-α-locally neighbourhood of b, since A is not a neutrosophic G_δ-α-locally neighbourhood of b.
(ii)⇒(i) Since A_1 is an increasing τ_1 neutrosophic G_δ-α-locally open set (or) increasing τ_2 neutrosophic G_δ-α-locally open set. Now, A_1 is a neutrosophic G_δ-α-locally neighbourhood of a with $A_1(a) > 0$. By (ii), A_1 is not a neutrosophic G_δ-α-locally neighbourhood of b. This implies, X is a lower pairwise neutrosophic G_δ-α-locally T_1-ordered space.

Remark 2.1. Similar proof holds for upper pairwise neutrosophic G_δ-α-locally T_1-ordered space.

Proposition 2.2. If $(X, \tau_1, \tau_2, \leq)$ is a lower (resp. upper) pairwise neutrosophic G_δ-α-locally T_1-ordered space and $\tau_1 \subseteq \tau_1^*, \tau_2 \subseteq \tau_2^*$, then $(X, {\tau_1}^*, {\tau_2}^*, \leq)$ is a lower (resp. upper) pairwise neutrosophic G_δ-α-locally T_1-ordered space.
Proof:
Let $(X, \tau_1, \tau_2, \leq)$ be a lower pairwise neutrosophic G_δ-α-locally T_1-ordered space. Then if $a, b \in X$ such that $a \nleq b$, there exists an increasing τ_1 neutrosophic G_δ-α-locally neighbourhood (or) an increasing τ_2 neutrosophic G_δ-α-locally neighbourhood $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ of a such that A is not a neutrosophic G_δ-α-locally neighbourhood of b. Since $\tau_1 \subseteq \tau_1^*$ and $\tau_2 \subseteq \tau_2^*$. Therefore, if $a, b \in X$ such that $a \nleq b$, there exists an increasing ${\tau_1}^*$ neutrosophic G_δ-α-locally neighbourhood (or) an increasing ${\tau_2}^*$ neutrosophic G_δ-α-locally neighbourhood $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ of a such that A is not a neutrosophic G_δ-α-locally neighbourhood of b. Thus $(X, {\tau_1}^*, {\tau_2}^*, \leq)$ is a lower pairwise neutrosophic G_δ-α-locally T_1-ordered space.

Remark 2.2. Similar proof holds for upper pairwise neutrosophic G_δ-α-locally T_1-ordered space.

Definition 2.16. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be a pairwise neutrosophic T_2-ordered space if for $a, b \in X$ with $a \nleq b$, there exist a neutrosophic open sets $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ and $B = \langle x, \mu_B, \sigma_B, \gamma_B \rangle$ such that

A is an increasing τ_i neutrosophic neighbourhood of a, B is a decreasing τ_j neutrosophic neighbourhood of b ($i, j = 1, 2$ and $i \neq j$) and $A \cap B = 0_N$.

Definition 2.17. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be a pairwise neutrosophic G_δ-α-locally T_2-ordered space if for $a, b \in X$ with $a \nleq b$, there exist a neutrosophic G_δ-α-locally open sets $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ and $B = \langle x, \mu_B, \sigma_B, \gamma_B \rangle$ such that A is an increasing τ_i neutrosophic G_δ-α-locally neighbourhood of a, B is a decreasing τ_j neutrosophic G_δ-α-locally neighbourhood of b ($i, j = 1, 2$ and $i \neq j$) and $A \cap B = 0_N$.

Definition 2.18. Let $(X, \leq)$ be a partially ordered set. Let $G = \{(x, y) \in X \times X \mid$
$x \leq y, y = f(x)\}$. Then G is called a graph of the partially ordered $\leq$.

Definition 2.19. Let X be any nonempty set. Let $A \subseteq X$. Then we define a neutrosophic set χ^*_A is of the form $\langle x, \chi_A(x), \chi_A(x), 1 - \chi_A(x) \rangle$.

Definition 2.20. Let $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ be a neutrosophic set in an ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$. Then for $i = 1(or)2$, we define

I_{τ_i}-G_δ-α-$li(A)$ = increasing τ_i neutrosophic G_δ-α-locally interior of A
= the greatest increasing τ_i neutrosophic G_δ-α-locally open
set contained in A

D_{τ_i}-G_δ-α-$li(A)$ = decreasing τ_i neutrosophic G_δ-α-locally interior of A
= the greatest decreasing τ_i neutrosophic G_δ-α-locally open
set contained in A

I_{τ_i}-G_δ-α-$lc(A)$ = increasing τ_i neutrosophic G_δ-α-locally closure of A
= the smallest increasing τ_i neutrosophic G_δ-α-locally closed
set containing in A

D_{τ_i}-G_δ-α-$lc(A)$ = decreasing τ_i neutrosophic G_δ-α-locally closure of A
= the smallest decreasing τ_i neutrosophic G_δ-α-locally closed
set containing in A.

Notation 2.2. (i) The complement of a neutrosophic set $\chi_G{}^*$, where G is the graph of the partial order of X is denoted by $\chi^*_{\overline{G}}$.

(ii) I_{τ_i}-G_δ-α-$lc(A)$ is denoted by $I_i(A)$ and D_{τ_j}-G_δ-α-$lc(A)$ is denoted by $D_j(A)$, where $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ is a neutrosophic set in an ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$, for $i, j = 1, 2$ and $i \neq j$.

(iii) I_{τ_i}-G_δ-α-$li(A)$ is denoted by $I_i{}^\circ(A)$ and D_{τ_j}-G_δ-α-$li(A)$ is denoted by $D_j{}^\circ(A)$, where $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ is a neutrosophic set in an ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$, for $i, j = 1, 2$ and $i \neq j$.

Definition 2.21. Let A and B be any two neutrosophic sets of a nonempty set X. Then a neutrosophic set $A \times B$ on $X \times X$ is of the form $A \times B = \langle (x, y), \mu_{A\times B}, \sigma_{A\times B}, \gamma_{A\times B} \rangle$ where $\mu_{A\times B}((x, y)) = \mu_A(x) \wedge \mu_B(y), \sigma_{A\times B}((x, y)) = \sigma_A(x) \wedge \sigma_B(y)$ and $\gamma_{A\times B}((x, y)) = \gamma_A(x) \vee \gamma_B(y)$, for every $(x, y) \in X \times X$

Proposition 2.3. For an ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ the following are equivalent

(i) X is a pairwise neutrosophic G_δ-α-locally T_2-ordered space.

(ii) For each pair $a, b \in X$ such that $a \nleq b$, there exist a τ_i neutrosophic G_δ-α-locally open set $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ and τ_j neutrosophic G_δ-α-locally open set $B = \langle x, \mu_B, \sigma_B, \gamma_B \rangle$ such that $A(a) > 0, B(b) > 0$ and $A(x) > 0, B(y) > 0$ together imply that $x \nleq y$.

(iii) The neutrosophic set $\chi^*_{\overline{G}}$, where G is the graph of the partial order of X is a τ^*-neutrosophic G_δ-α-locally closed set, where τ^* is either $\tau_1 \times \tau_2$ or $\tau_2 \times \tau_1$ in $X \times X$.

Proof:
(i)⇒(ii) Let X be a pairwise neutrosophic G_δ-α-locally T_2-ordered space.
Assume that suppose $A(x) > 0$, $B(y) > 0$ and $x \leq y$. Since A is an increasing τ_i neutrosophic G_δ-α-locally open set and B is a decreasing τ_j neutrosophic G_δ-α-locally open set, $A(x) \leq A(y)$ and $B(y) \leq B(x)$. Therefore $0 < A(x) \cap B(y) \leq A(y) \cap B(x)$, which is a contradiction to the fact that $A \cap B = 0_N$. Therefore $x \nleq y$.
(ii)⇒(i) Let $a, b \in X$ with $a \nleq b$, there exists a neutrosophic sets A and B satisfying the properties in (ii). Since $I_i{}^\circ(A)$ is an increasing τ_i neutrosophic G_δ-α-locally open set and $D_j{}^\circ(B)$is decreasing τ_j neutrosophic G_δ-α-locally open set, we have $I_i{}^\circ(A) \cap D_j{}^\circ(B)$=$0_N$. Suppose $z \in X$ is such that $I_i{}^\circ(A)(z)$ $\cap$ $D_j{}^\circ(B)(z)$ >0. Then $I_i{}^\circ(A) > 0$ and $D_j{}^\circ(B)(z) > 0$. If $x \leq z \leq y$, then $x \leq z$ implies that $D_j{}^\circ(B)(x) \geq D_j{}^\circ(B)(z)$ >0 and $z \leq y$ implies that $I_i{}^\circ(A)(y) \geq I_i{}^\circ(A)(z)$ >0 then $D_j{}^\circ(B)(x)$ >0 and $I_i{}^\circ(A)(y)$ >0. Hence by (ii), $x \nleq y$ but then $x \leq y$. This is a contradiction. This implies that X is pairwise neutrosophic G_δ-α-locally T_2-ordered space.
(i)⇒ (iii) We want to show that $\chi^*_{\overline{G}}$ is a τ^* neutrosophic G_δ-α-locally closed set. That is to show that $\chi^*_{\overline{G}}$ is τ^* neutrosophic G_δ-α-locally open set. It is sufficient to prove that $\chi^*_{\overline{G}}$ is a neutrosophic G_δ-α-locally neighbourhood of a point $(x, y) \in X \times X$ such that $\chi^*_{\overline{G}}(x, y) > 0$. Suppose $(x, y) \in X \times X$ is such that $\chi^*_{\overline{G}}(x, y) > 0$. We have $\chi^*_G(x, y) < 1$. This means $\chi^*_G(x, y) = 0$. Thus $(x, y) \notin G$ and hence $x \nleq y$. Therefore by assumption (i), there exist neutrosophic G_δ-α-locally open sets A and B such that A is an increasing τ_i neutrosophic G_δ-α-locally neighbourhood of a , B is an decreasing τ_j neutrosophic G_δ-α-locally neighbourhood of b ($i, j = 1, 2$ and $i \neq j$) and $A \cap B = 0_N$. Clearly $A \times B$ is an $IF\tau^*$ G_δ-α-locally neighbourhood of (x, y). It is easy to verify that $A \times B \subseteq \chi_{\overline{G}}$. Thus we find that $\chi_{\overline{G}}$ is an τ^* NG_δ-α-locally open set. Hence (iii) is

established.

(iii)⇒(i) Suppose $x \nleq y$. Then $(x, y) \notin G$,where G is a graph of the partial order. Given that $\chi^*_{\overline{G}}$ is τ^* neutrosophic G_δ-α-locally closed set. That is $\chi^*_{\overline{G}}$ is an τ^* neutrosophic G_δ-α-locally open set. Now $(x, y) \notin G$ implies that $\chi^*_{\overline{G}}(x, y) > 0$. Therefore $\chi^*_{\overline{G}}$ is an τ^* neutrosophic G_δ-α-locally neighbourhood of $(x, y) \in X \times X$. Hence we can find that τ^* neutrosophic G_δ-α-locally open set $A \times B$ such that $A \times B \subseteq \chi^*_{\overline{G}}$ and A is τ_i neutrosophic G_δ-α-locally open set such that $A(x) > 0$ and B is an τ_j neutrosophic G_δ-α-locally open set such that $B(y) > 0$. We now claim that $I_i{}^\circ(A) \cap D_j{}^\circ(B)$=$0_N$. For if $z \in X$ is such that $(I_i{}^\circ(A) \cap D_j{}^\circ(B))$(z)> 0, then $I_i{}^\circ(A)(z) \cap D_j{}^\circ(B)(z) > 0$. This means $I_i{}^\circ(A)(z)$> 0 and $D_j{}^\circ(B)(z)$> 0. And if $a \leq z \leq b$, then $z \leq b$ implies that $I_i{}^\circ(A)(b)$≥ $I_i{}^\circ(A)(z)$> 0 and $a \leq z$ implies that $D_j{}^\circ(B)(a)$≥ $D_j{}^\circ(B)(z)$> 0. Then $D_j{}^\circ(B)(a)$> 0 and $I_i{}^\circ(A)(b)$> 0 implies that $a \nleq b$ but then $a \leq b$. This is a contradiction. Hence (i) is established.

Definition 2.22. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be a weakly pairwise neutrosophic T_2-ordered space if given $b < a$ (that is $b \leq a$ and $b \neq a$), there exist an τ_i neutrosophic open set $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ such that $A(a) > 0$ and τ_j neutrosophic open set $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ such that $B(b) > 0$ $(i, j = 1, 2$ and $i \neq j)$ such that if $x, y \in X$, $A(x) > 0$, $B(y) > 0$ together imply that $y < x$.

Definition 2.23. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be a weakly pairwise neutrosophic G_δ-α-locally T_2-ordered space if given $b < a$ (that is $b \leq a$ and $b \neq a$), there exist an τ_i neutrosophic G_δ-α-locally open set $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ such that $A(a) > 0$ and τ_j neutrosophic G_δ-α-locally open set $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ such that $B(b) > 0$ $(i, j = 1, 2$ and $i \neq j)$ such that if $x, y \in X$, $A(x) > 0$, $B(y) > 0$ together imply that $y < x$.

Definition 2.24. The symbol $x \parallel y$ means that $x \leq y$ and $y \leq x$.

Definition 2.25. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be an almost pairwise neutrosophic T_2-ordered space if given $a \parallel b$, there exist a τ_i neutrosophic open set $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ such that $A(a) > 0$ and τ_j neutrosophic open set $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ such that $B(b) > 0$ (i,j=1,2 and $i \neq j$) such that if $x, y \in X$, $A(x) > 0$ and $B(y) > 0$ together imply that $x \parallel y$.

Definition 2.26. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be an almost pairwise neutrosophic G_δ-α-locally T_2-ordered space if given $a \parallel b$, there exist a τ_i neutrosophic G_δ-α-locally open set $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ such that $A(a) > 0$ and τ_j neutrosophic G_δ-α-locally open set $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ such that $B(b) > 0$ $(i, j = 1, 2$ and $i \neq j)$ such that if $x, y \in X$, $A(x) > 0$ and $B(y) > 0$ together imply that $x \parallel y$.

Proposition 2.4. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is a pairwise neutrosophic G_δ-α-locally T_2-ordered space if and only if it is a weakly pairwise neutrosophic G_δ-α-locally T_2-ordered and almost pairwise neutrosophic G_δ-α-locally T_2-ordered space.

Proof:

Let $(X, \tau_1, \tau_2, \leq)$ be a pairwise neutrosophic G_δ-α-locally T_2-ordered space. Then by Proposition 3.3 and Definition 3.20, it is a weakly pairwise neutrosophic G_δ-α-locally T_2-ordered space. Let $a \parallel b$. Then $a \nleq b$ and $b \nleq a$.Since $a \nleq b$ and X is a pairwise neutrosophic G_δ-α-locally T_2-ordered space. We have τ_i neutrosophic G_δ-α-locally open set $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ and τ_j neutrosophic G_δ-α-locally open set $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ such that $A(a) > 0$, $B(b) > 0$ and $A(x) > 0$, $B(y) > 0$ together imply that $x \nleq y$. Also since $b \nleq a$,there exist τ_i neutrosophic G_δ-α-locally open set A^*=$\langle x, \mu_{A^*}, \gamma_{A^*}\rangle$ and τ_j neutrosophic G_δ-α-locally open set B^*=$\langle x, \mu_{B^*}, \gamma_{B^*}\rangle$ such that $A^*(a) > 0$, $B^*(b) > 0$ and $A^*(x) > 0$, $B^*(y) > 0$ together imply that $y \nleq x$. Thus $I_i{}^\circ(A \cap A^*)$ is an τ_i neutrosophic G_δ-α-locally open set such that $I_i{}^\circ(A \cap A^*)(a) > 0$ and $I_j{}^\circ(B \cap B^*)$ is a τ_j neutrosophic G_δ-α-locally open set such that $I_j{}^\circ(B \cap B^*)(b) > 0$.Also $I_i{}^\circ(A \cap A^*)(x) > 0$ and $I_j{}^\circ(B \cap B^*)(y) > 0$ togetherimply that $x \parallel y$. Hence X is an almost pairwise neutrosophic G_δ-α-locally T_2-ordered space.

Conservely, let X be a weakly pairwise neutrosophic G_δ-α-locally T_2-ordered and almost pairwise neutrosophic G_δ-α-locally T_2-ordered space. We want to show that X is a pairwise neutrosophic G_δ-α-locally T_2-ordered space. Let $a \nleq b$. Then either $b < a$ (or) $b \nleq a$. If $b < a$ then X being weakly pairwise neutrosophic G_δ-α-locally T_2-ordered space, there exist τ_i neutrosophic G_δ-α-locally open set A and τ_j neutrosophic G_δ-α-locally open set B such that $A(a) > 0$,$B(b) > 0$ and such that $A(x) > 0$,$B(y) > 0$ together imply that $y < x$. Thus $x \nleq y$. If $b \nleq a$, then $a \parallel b$ and the result follows easily since X is an almost pairwise neutrosophic G_δ-α-locally T_2-ordered space. Hence X is a pairwise neutrosophic G_δ-α-locally T_2-ordered space.

Definition 2.27. Let $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ and $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ be neutrosophic sets in an ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$. Then A is said to be a τ_i neutrosophic neighbourhood of B if $B \subseteq A$ and there exists τ_i neutrosophic open set C=$\langle x, \mu_C, \sigma_C, \gamma_C\rangle$ such that $B \subseteq C \subseteq A$,$(i = 1(or)2)$.

Definition 2.28. Let $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ and $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ be neutrosophic sets in an ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$. Then A is said to be a τ_i neutrosophic G_δ-α-locally neighbourhood of B if $B \subseteq A$ and there exists τ_i neutrosophic G_δ-α-locally open set $C = \langle x, \mu_C, \sigma_C, \gamma_C\rangle$ such that $B \subseteq C \subseteq A$,
$(i = 1(or)2)$.

Definition 2.29. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is said to be a strongly pairwise neutrosophic G_δ-α-locally normally ordered space if for every pair $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ is a decreasing τ_i neutrosophic G_δ-α-locally closed set and $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ is an decreasing τ_j neutrosophic G_δ-α-locally open set such that $A \subseteq B$ then there exist decreasing τ_j neutrosophic G_δ-α-locally open set A_1=$\langle x, \mu_{A_1}, \gamma_{A_1}\rangle$ such that $A \subseteq A_1 \subseteq D_i(A_1) \subseteq B$,$(i, j = 1, 2$ and $i \neq j)$.

Proposition 2.5. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ the following are equivalent

(i) $(X, \tau_1, \tau_2, \leq)$ is a strongly pairwise neutrosophic G_δ-α-locally normally ordered space.

(ii) For each increasing τ_i neutrosophic G_δ-α-locally open set A=$\langle x, \mu_A, \sigma_A, \gamma_A\rangle$ and decreasing τ_j neutrosophic G_δ-α-locally open set B=$\langle x, \mu_B, \sigma_B, \gamma_B\rangle$ with $A \subseteq B$ there exists an decreasing τ_j neutrosophic G_δ-α-locally open set A_1 such that $A \subseteq A_1 \subseteq NG_\delta$-$\alpha$-$lcl_{\tau_i}(A_1) \subseteq B$,$(i, j = 1, 2$ and $i \neq j)$.

Proof: The Proof is simple.

Notation 2.3. (i) The collection of all neutrosophic set in nonempty set X is denoted by ζ^X.

(ii) Let X be any nonempty set and $A \in \zeta^X$. Then for $x \in X$, $\langle \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle$ is denoted by $A^\sim$.

Definition 2.30. A neutrosophic real line $\mathbb{R}_{\mathbb{I}}(I)$ is the set of all monotone decreasing neutrosophic $A \in \zeta^{\mathbb{R}}$ satisfying $\cup\{A(t) : t \in \mathbb{R}\} = 1^\sim$ and $\cap\{A(t) : t \in \mathbb{R}\} = 0^\sim$ after the identification of neutrosophic sets $A, B \in \mathbb{R}_{\mathbb{I}}(I)$ if and only if $A(t-) = B(t-)$ and $A(t+) = B(t+)$ for all $t \in \mathbb{R}$ where $A(t-) = \cap\{A(s) : s < t\}$ and $A(t+) = \cup\{A(s) : s > t\}$.

The neutrosophic unit interval $\mathbb{I}_{\mathbb{I}}(I)$ is a subset of $\mathbb{R}_{\mathbb{I}}(I)$ such that $[A] \in \mathbb{I}_{\mathbb{I}}(I)$ if the membership , indeterminancy and non-membership of A are defined by

$$\mu_A(t) = \begin{cases} 1, & \text{t<0;} \\ 0, & \text{t>1.} \end{cases} \quad \sigma_A(t) = \begin{cases} 1, & \text{t<0;} \\ 0, & \text{t>1.} \end{cases} \quad \text{and } \gamma_A(t) = \begin{cases} 0, & \text{t<0;} \\ 1, & \text{t>1.} \end{cases}$$

respectively. The natural neutrosophic topology on $\mathbb{R}_{\mathbb{I}}(I)$ is generated from the subbasis $\{L^{\mathbb{I}}{}_t, R^{\mathbb{I}}{}_t : t \in \mathbb{R}\}$ where $L^{\mathbb{I}}_t, R^{\mathbb{I}}_t : \mathbb{R}_{\mathbb{I}}(I) \to \mathbb{I}_{\mathbb{I}}(I)$ are given by $L^{\mathbb{I}}_t[A] = \overline{A(t-)}$ and $R^{\mathbb{I}}_t[A] = A(t+)$, respectively.

Definition 2.31. Let $(X, \tau_1, \tau_2, \leq)$ be an ordered neutrosophic bitopological space. A function $f : X \to \mathbb{R}_{\mathbb{I}}(I)$ is said to be a τ_i $lower^*(resp.upper^*)$ neutrosophic G_δ-α-locally continuous function if $f^{-1}(R^{\mathbb{I}}{}_t)$ $(resp.f^{-1}(\overline{L^{\mathbb{I}}{}_t}))$ is an increasing (or)an decreasing τ_i(resp.τ_j) neutrosophic G_δ-α-locally open set, for each $t \in \mathbb{R}$ $(i, j = 1, 2$ and $i \neq j)$.

Proposition 2.6. Let $(X, \tau_1, \tau_2, \leq)$ be an ordered neutrosophic bitopological space. Let $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ be a neutrosophic set in X and let $f : X \to \mathbb{R}_{\mathbb{I}}(I)$ be such that

$$f(x)(t) = \begin{cases} 1^\sim & if \quad t < 0 \\ A^\sim & if \quad 0 \leq t \leq 1 \\ 0^\sim & if \quad t > 1 \end{cases}$$

for all $x \in X$ and $t \in \mathbb{R}$. Then f is a τ_i $lower^*(resp.\tau_j upper^*)$ neutrosophic G_δ-α-locally continuous function if and only if A is an increasing (or) a decreasing τ_i (resp. τ_j) neutrosophic G_δ-α-locally open (resp. closed) set $(i, j = 1, 2$ and $i \neq j)$.

Proof:

$$f^{-1}(R^{\mathbb{I}}{}_t) = \begin{cases} 1^\sim & if \quad t < 0 \\ A^\sim & if \quad 0 \leq t \leq 1 \\ 0^\sim & if \quad t > 1 \end{cases}$$

implies that f is τ_i $lower^*$ neutrosophic G_δ-α-locally continuous function if and only if A is an increasing (or) a decreasing τ_i neutrosophic G_δ-α-locally open set in X.

$$f^{-1}(L^{\mathbb{I}}{}_t) = \begin{cases} 1^\sim & if \quad t < 0 \\ A^\sim & if \quad 0 \leq t \leq 1 \\ 0^\sim & if \quad t > 1 \end{cases}$$

implies that f is τ_j $upper^*$ neutrosophic G_δ-α-locally continuous function if and only if A is an increasing (or) a decreasing τ_j neutrosophic G_δ-α-locally closed set in X $(i, j = 1, 2$ and $i \neq j)$.

Uryshon's lemma

Proposition 2.7. An ordered neutrosophic bitopological space $(X, \tau_1, \tau_2, \leq)$ is a strongly pairwise neutrosophic G_δ-α-locally normally ordered space if and only if for every $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ is decreasing τ_i neutrosophic closed set and $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ is an increasing τ_j neutrosophic closed set with $A \subseteq \overline{B}$, there exists increasing neutrosophic function $f : X \to \mathbb{I}_{\mathbb{I}}(I)$ such that $A \subseteq f^{-1}(\overline{L_1}) \subseteq f^{-1}(R_0) \subseteq B$ and f is a τ_i $upper^*$ neutrosophic G_δ-α-locally continuous function and τ_j $lower^*$ neutrosophic G_δ-α-locally continuous function $(i, j = 1, 2$ and $i \neq j)$.

Proof:

Suppose that there exists a function f satisfying the given conditions. Let $C = \langle x, \mu_C, \sigma_C, \gamma_C\rangle = f^{-1}(\overline{L^{\mathbb{I}}{}_t})$ and $D = \langle x, \mu_D, \sigma_D, \gamma_D\rangle$=$f^{-1}(R^{\mathbb{I}}{}_t)$ for some $0 \leq t \leq 1$. Then $\overline{C} \in \tau_i$ and $D \in \tau_j$ and such that $A \subseteq C \subseteq D \subseteq \overline{B}$. It is easy to verify that D is a decreasing τ_j neutrosophic G_δ-α-locally open set and C is an increasing τ_i neutrosophic G_δ-α-locally closed set. Then there exists decreasing τ_j neutrosophic G_δ-α-locally open set C_1 such that $C \subseteq C_1 \subseteq D_i(C_1) \subseteq D$, $(i, j = 1, 2$ and $i \neq j)$. This proves that X is a strongly pairwise neutrosophic G_δ-α-locally normally ordered space.

Conversely, let X be a strongly pairwise neutrosophic G_δ-α-locally normally ordered space. Let A be a decreasing τ_i neutrosophic G_δ-α-locally closed set and B be an increasing τ_j neutrosophic G_δ-α-locally closed set. By the Proposition 3.6, we can construct a collection $\{C_t \mid t \in \mathbb{I}\} \subseteq \tau_j$, where $C = \langle x, \mu_{C_t}, \gamma_{C_t}\rangle$, $t \in \mathbb{I}$ such that $A \subseteq C_t \subseteq B$, NG_δ-α-$lcl_{\tau_i}(C_s) \subseteq C_t$ whenever $s < t$, $A \subseteq C_0$ $C_1 = B$ and $C_t = 0_N$ for $t < 0$,$C_t = 1_N$ for $t > 1$. We define a function $f : X \to \mathbb{I}_{\mathbb{I}}(I)$ by $f(x)(t) = C_{1-t}(x)$. Clearly f is well defined. Since $A \subseteq C_{1-t} \subseteq B$, for $t \in \mathbb{I}$. We have $A \subseteq f^{-1}(\overline{L^{\mathbb{I}}{}_1}) \subseteq f^{-1}(R^{\mathbb{I}}{}_0) \subseteq B$. Furthermore $f^{-1}(R^{\mathbb{I}}{}_t) = \bigcup_{s<1-t} C_s$ is a τ_j neutrosophic G_δ-α-locally open set and $f^{-1}(\overline{L^{\mathbb{I}}{}_t}) = \bigcap_{s>1-t} C_s = \bigcap_{s>1-t} NG_\delta$-$\alpha$-$lcl_{\tau_i}(C_s)$ is an τ_i neutrosophic G_δ-α-locally closed set. Thus f is a τ_j $lower^*$ neutrosophic G_δ-α-locally continuous function and τ_i $upper^*$ neutrosophic G_δ-α-locally continuous function and is an increasing neutrosophic function.

Tietze extension theorem

Proposition 2.8. Let $(X, \tau_1, \tau_2, \leq)$ be an ordered neutrosophic bitopological space the following statements are equivalent.

(i) $(X, \tau_1, \tau_2, \leq)$ is a strongly pairwise neutrosophic G_δ-α-locally normally ordered space.

(ii) If $g, h : X \to \mathbb{R}_{\mathbb{I}}(I)$,$g$ is an τ_i $upper^*$ neutrosophic G_δ-α-locally continuous function, h is a τ_j $lower^*$ neutrosophic G_δ-α-locally continuous function and $g \subseteq h$, then there exists $f : X \to \mathbb{I}_{\mathbb{I}}(I)$ such that $g \subseteq f \subseteq h$ and f is a τ_i $upper^*$ neutrosophic G_δ-α-locally continuous function and τ_j $lower^*$ neutrosophic G_δ-α-locally continuous function $(i, j = 1, 2$ and $i \neq j)$.

Proof:
(ii)⇒(i) Let A=$\langle x, \mu_A, \sigma_A, \gamma_A\rangle$ and B=$\langle x, \mu_B, \sigma_B, \gamma_B\rangle$ be a neutrosophic G_δ-α-locally open sets such that $A \subseteq B$. Define $g, h : X \to \mathbb{R}_{\mathbb{I}}(I)$ by

$$g(x)(t) = \begin{cases} 1^{\sim} & if \quad t < 0 \\ A^{\sim} & if \quad 0 \leq t \leq 1 \\ 0^{\sim} & if \quad t > 1 \end{cases} \quad \text{and} \quad h(x)(t) = \begin{cases} 1^{\sim} & if \quad t < 0 \\ B^{\sim} & if \quad 0 \leq t \leq 1 \\ 0^{\sim} & if \quad t > 1 \end{cases}$$

for each $x \in X$. By Proposition 3.6, g is an τ_i $upper^*$ neutrosophic G_δ-α-locally continuous function and h is an τ_j $lower^*$ neutrosophic G_δ-α-locally continuous function. Clearly, $g \subseteq h$ holds,so that there exists $f : X \to \mathbb{R}_{\mathbb{I}}(I)$ such that $g \subseteq f \subseteq h$. Suppose $t \in (0, 1)$. Then $A = g^{-1}(R^{\mathbb{I}}{}_t) \subseteq f^{-1}(R^{\mathbb{I}}{}_t) \subseteq f^{-1}(\overline{L^{\mathbb{I}}}_t) \subseteq h^{-1}(\overline{L^{\mathbb{I}}}_t) = B$. By Proposition 3.7, X is a strongly pairwise neutrosophic G_δ-α-locally normal ordered space.
(i)⇒(ii) Define two mappings $A, B : Q \to I$ by $A(r) = A_r = h^{-1}(\overline{R^{\mathbb{I}}}_r)$ and $B(r) = B_r = g^{-1}(L^{\mathbb{I}}{}_r)$, for all $r \in Q$ (Q is the set of all rationals). Clearly, A and B are monotone increasing families of a decreasing τ_i neutrosophic G_δ-α-locally closed sets and decreasing τ_j neutrosophic G_δ-α-locally open sets of X. Moreover $A_r \subset B_{r'}$ if $r < r'$. By Proposition 3.5, there exists an decreasing τ_j neutrosophic G_δ-α-locally open set $C = \langle x, \mu_C, \sigma_C, \gamma_C\rangle$ such that $A_r \subseteq NG_\delta$-α-$lint_{\tau_i}(C_r)$, NG_δ-α-$lcl_{\tau_i}(C_r) \subseteq NG_\delta$-$\alpha$-$lint_{\tau_i}(C_{r'})$, NG_δ-α-$lcl_{\tau_i}(C_r) \subseteq B_{r'}$ whenever $r < r'$ $(r, r' \in Q)$. Letting $V_t = \bigcap_{r<t} \overline{C_r}$ for $t \in R$, we define a monotone decreasing family $\{V_t \mid t \in R\} \subseteq I$. Moreover we have NG_δ-α-$lcl_{\tau_i}(V_t) \subseteq NG_\delta$-$\alpha$-$lint_{\tau_i}(V_s)$ whenever $s < t$. We have,

$$\begin{aligned} \bigcup_{t \in R} V_t &= \bigcup_{t \in R} \bigcap_{r<t} \overline{C_r} \\ &\supseteq \bigcup_{t \in R} \bigcap_{r<t} \overline{B_r} \\ &= \bigcup_{t \in R} \bigcap_{r<t} g^{-1}(\overline{L^{\mathbb{I}}}_r) \\ &= \bigcup_{t \in R} g^{-1}(\overline{L^{\mathbb{I}}}_r) \\ &= g^{-1}(\bigcup_{t \in R} \overline{L^{\mathbb{I}}}_t) \\ &= 1_N \end{aligned}$$

Similarly, $\bigcap_{t \in R} V_t = 0_N$. Now define a function $f : (X, \tau_1, \tau_2, \leq) \to \mathbb{R}_{\mathbb{I}}(I)$ satisfying the required conditions. Let $f(x)(t) = V_t(x)$, for all $x \in X$ and $t \in R$. By the above discussion, it follows that f is well defined. To prove f is a τ_i $upper^*$ neutrosophic G_δ-α-locally continuous function and τ_j $lower^*$ neutrosophic G_δ-α-locally continuous function (i,j=1,2 and $i \neq j$). Observe that $\bigcup_{s>t} V_s = \bigcup_{s>t} NG_\delta$-$\alpha$-$lint_{\tau_i}(V_s)$ and $\bigcap_{s>t} V_s = \bigcap_{s>t} NG_\delta$-$\alpha$-$lcl_{\tau_i}(V_s)$. Then $f^{-1}(R_t) = \bigcup_{s>t} V_s = \bigcup_{s>t} NG_\delta$-$\alpha$-$lint_{\tau_i}(V_s)$ is an increasing τ_i neutrosophic G_δ-α-locally open set. Now $f^{-1}(\overline{L_t}) = \bigcap_{s>t} V_s = \bigcap_{s>t} NG_\delta$-$\alpha$-$lcl_{\tau_i}(V_s)$ is a decreasing τ_j neutrosophic G_δ-α-locally closed set. So that f is a τ_i $upper^*$ neutrosophic G_δ-α-locally continuous function and τ_j $lower^*$ neutrosophic G_δ-α-locally continuous function. To conclude the proof it remains to show that $g \subseteq f \subseteq h$. That is $g^{-1}(\overline{L^{\mathbb{I}}}_t) \subseteq f^{-1}(\overline{L^{\mathbb{I}}}_t) \subseteq h^{-1}(\overline{L^{\mathbb{I}}}_t)$ and $g^{-1}(R^{\mathbb{I}}{}_t) \subseteq f^{-1}(R^{\mathbb{I}}{}_t) \subseteq h^{-1}(R^{\mathbb{I}}{}_t)$ for each $t \in R$. We have,

$$\begin{aligned} g^{-1}(\overline{L^{\mathbb{I}}}_t) &= \bigcap_{s<t} g^{-1}(\overline{L^{\mathbb{I}}}_s) \\ &= \bigcap_{s<t} \bigcap_{r<s} g^{-1}(\overline{L^{\mathbb{I}}}_r) \\ &= \bigcap_{s<t} \bigcap_{r<s} \overline{B_r} \\ &\subseteq \bigcap_{s<t} \bigcap_{r<s} \overline{C_r} \\ &= \bigcap_{s<t} V_s \\ &= f^{-1}(\overline{L^{\mathbb{I}}}_t) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\overline{L^{\mathbb{I}}_t}) &= \bigcap_{s<t} V_s \\ &= \bigcap_{s<t}\bigcap_{r<s} \overline{C_r} \\ &\subseteq \bigcap_{s<t}\bigcap_{r<s} \overline{A_r} \\ &= \bigcap_{s<t}\bigcap_{r<s} h^{-1}(\overline{R^{\mathbb{I}}_r}) \\ &= \bigcap_{s<t} h^{-1}(\overline{L^{\mathbb{I}}_s}) \\ &= h^{-1}(\overline{L_t}) \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} g^{-1}(R^{\mathbb{I}}_t) &= \bigcup_{s>t} g^{-1}(R^{\mathbb{I}}_s) \\ &= \bigcup_{s>t}\bigcup_{r>s} g^{-1}(\overline{L^{\mathbb{I}}_r}) \\ &= \bigcup_{s>t}\bigcup_{r>s} \overline{B_r} \\ &\subseteq \bigcup_{s>t}\bigcup_{r>s} \overline{C_r} \\ &= \bigcup_{s>t} V_s \\ &= f^{-1}(\overline{R^{\mathbb{I}}_t}) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\overline{R^{\mathbb{I}}_t}) &= \bigcup_{s>t} V_s \\ &= \bigcup_{s>t}\bigcup_{r>s} \overline{C_r} \\ &\subseteq \bigcup_{s>t}\bigcup_{r>s} \overline{A_r} \\ &= \bigcup_{s>t}\bigcup_{r>s} h^{-1}(\overline{R^{\mathbb{I}}_r}) \\ &= \bigcup_{s>t} h^{-1}(\overline{R^{\mathbb{I}}_s}) \\ &= h^{-1}(\overline{R^{\mathbb{I}}_t}) \end{aligned}$$

Hence the proof.

Proposition 2.9. Let $(X, \tau_1, \tau_2, \leq)$ be a strongly pairwise neutrosophic G_δ-α-locally normally ordered space. Let $\overline{A} \in \tau_1$ and $\overline{A} \in \tau_2$ be crisp and let $f : (A, \tau_1/A, \tau_2/A) \to \mathbb{I}_{\mathbb{I}}(I)$ be a τ_i $upper^*$ neutrosophic G_δ-α-locally continuous function and τ_j $lower^*$ neutrosophic G_δ-α-locally continuous function (i,j=1,2 and $i \neq j$). Then f has a neutrosophic extension over $(X, \tau_1, \tau_2, \leq)$ (that is, $F : (X, \tau_1, \tau_2, \leq) \to \mathbb{I}_{\mathbb{I}}(I)$.
Proof:

Define $g : X \to \mathbb{I}_{\mathbb{I}}(I)$ by

$$\begin{aligned} g(x) &= f(x) \quad if \quad x \in A \\ &= [A_0] \quad if \quad x \notin A \end{aligned}$$

and also define $h : X \to \mathbb{I}_{\mathbb{I}}(I)$ by

$$\begin{aligned} h(x) &= f(x) \quad if \quad x \in A \\ &= [A_1] \quad if \quad x \notin A \end{aligned}$$

where $[A_0]$ is the equivalence class determined by $A_0 : \mathbb{R}_{\mathbb{I}}(I) \to \mathbb{I}_{\mathbb{I}}(I)$ such that

$$\begin{aligned} A_0(t) &= 1^{\sim} \quad if \quad t < 0 \\ &= 0^{\sim} \quad if \quad t > 0 \end{aligned}$$

and $[A_1]$ is the equivalence class determined by $A_1 : \mathbb{R}_{\mathbb{I}}(I) \to \mathbb{I}_{\mathbb{I}}(I)$ such that

$$\begin{aligned} A_1(t) &= 1^{\sim} \quad if \quad t < 1 \\ &= 0^{\sim} \quad if \quad t > 1 \end{aligned}$$

g is a τ_i $upper^*$ neutrosophic G_δ-α-locally continuous function and h is a τ_j $lower^*$ neutrosophic G_δ-α-locally continuous function and $g \subseteq h$. Hence by Proposition 3.8, there exists a function $F : X \to \mathbb{I}_{\mathbb{I}}(I)$ such that F is a τ_i $upper^*$ neutrosophic G_δ-α-locally continuous function and τ_j $lower^*$ neutrosophic G_δ-α-locally continuous function and $g(x) \subseteq F(x) \subseteq h(x)$ for all $x \in X$. Hence for all $x \in A$, $f(x) \subseteq F(x) \subseteq f(x)$. So that F is a required extension of f over X.

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NSS
University of New Mexico

On Neutrosophic Semi Alpha Open Sets

Qays Hatem Imran[1], F. Smarandache[2], Riad K. Al-Hamido[3] and R. Dhavaseelan[4]

[1]Department of Mathematics, College of Education for Pure Science, Al-Muthanna University, Samawah, Iraq.
E-mail: qays.imran@mu.edu.iq
[2]Department of Mathematics, University of New Mexico 705 Gurley Ave. Gallup, NM 87301, USA.
E-mail: smarand@unm.edu
[3]Department of Mathematics, College of Science, Al-Baath University, Homs, Syria.
E-mail: riad-hamido1983@hotmail.com
[4]Department of Mathematics, Sona College of Technology, Salem-636005,Tamil Nadu, India.
E-mail: dhavaseelan.r@gmail.com

Abstract. In this paper, we presented another concept of neutrosophic open sets called neutrosophic semi-α-open sets and studied their fundamental properties in neutrosophic topological spaces. We also present neutrosophic semi-α-interior and neutrosophic semi-α-closure and study some of their fundamental properties.
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1. Introduction

In 2000, G.B. Navalagi [4] presented the idea of semi-α-open sets in topological spaces. The concept of "neutrosophic set" was first given by F. Smarandache [2,3]. A.A. Salama and S.A. Alblowi [1] presented the concept of neutrosophic topological space (briefly NTS). The objective of this paper is to present the concept of neutrosophic semi-α-open sets and study their fundamental properties in neutrosophic topological spaces. We also present neutrosophic semi-α-interior and neutrosophic semi-α-closure and obtain some of its properties.

2. Preliminaries

Throughout this paper, $(\mathcal{U},T)$ (or simply $\mathcal{U}$) always mean a neutrosophic topological space. The complement of a neutrosophic open set (briefly N-OS) is called a neutrosophic closed set (briefly N-CS) in $(\mathcal{U},T)$. For a neutrosophic set $\mathcal{A}$ in a neutrosophic topological space $(\mathcal{U},T)$, $Ncl(\mathcal{A})$, $Nint(\mathcal{A})$ and $\mathcal{A}^c$ denote the neutrosophic closure of $\mathcal{A}$, the neutrosophic interior of $\mathcal{A}$ and the neutrosophic complement of $\mathcal{A}$ respectively.

Definition 2.1:
A neutrosophic subset $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U},T)$ is said to be:
(i) A neutrosophic pre-open set (briefly NP-OS) [7] if $\mathcal{A} \subseteq Nint(Ncl(\mathcal{A}))$. The complement of a NP-OS is called a neutrosophic pre-closed set (briefly NP-CS) in $(\mathcal{U},T)$. The family of all NP-OS (resp. NP-CS) of $\mathcal{U}$ is denoted by NPO$(\mathcal{U})$ (resp. NPC$(\mathcal{U})$).
(ii) A neutrosophic semi-open set (briefly NS-OS) [6] if $\mathcal{A} \subseteq Ncl(Nint(\mathcal{A}))$. The complement of a NS-OS is called a neutrosophic semi-closed set (briefly NS-CS) in $(\mathcal{U},T)$. The family of all NS-OS (resp. NS-CS) of $\mathcal{U}$ is denoted by NSO$(\mathcal{U})$ (resp. NSC$(\mathcal{U})$).
(iii) A neutrosophic α-open set (briefly Nα-OS) [5] if $\mathcal{A} \subseteq Nint(Ncl(Nint(\mathcal{A})))$. The complement of a Nα-OS is called a neutrosophic α-closed set (briefly Nα-CS) in $(\mathcal{U},T)$. The family of all Nα-OS (resp. Nα-CS) of $\mathcal{U}$ is denoted by NαO$(\mathcal{U})$ (resp. NαC$(\mathcal{U})$).

Definition 2.2:
(i) The neutrosophic pre-interior of a neutrosophic set $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U},T)$ is the union of all NP-OS contained in $\mathcal{A}$ and is denoted by $PNint(\mathcal{A})$[7].
(ii) The neutrosophic semi-interior of a neutrosophic set $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U},T)$ is the union of all NS-OS contained in $\mathcal{A}$ and is denoted by $SNint(\mathcal{A})$[6].
(iii) The neutrosophic α-interior of a neutrosophic set $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U},T)$ is the union of all Nα-OS contained in $\mathcal{A}$ and is denoted by $\alpha Nint(\mathcal{A})$[5].

Definition 2.3:
(i) The neutrosophic pre-closure of a neutrosophic set $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U},T)$ is the intersection of all NP-CS that contain $\mathcal{A}$ and is denoted by $PNcl(\mathcal{A})$[7].
(ii) The neutrosophic semi-closure of a neutrosophic set $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U},T)$ is the

intersection of all NS-CS that contain $\mathcal{A}$ and is denoted by $SNcl(\mathcal{A})$[6].
(iii) The neutrosophic α-closure of a neutrosophic set $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U}, T)$ is the intersection of all Nα-CS that contain $\mathcal{A}$ and is denoted by $\alpha Ncl(\mathcal{A})$[5].

Proposition 2.4 [5]:
In a neutrosophic topological space $(\mathcal{U}, T)$, then the following statements hold, and the equality of each statement are not true:
(i) Every N-OS (resp. N-CS) is a Nα-OS (resp. Nα-CS).
(ii) Every Nα-OS (resp. Nα-CS) is a NS-OS (resp. NS-CS).
(iii) Every Nα-OS (resp. Nα-CS) is a NP-OS (resp. NP-CS).

Proposition 2.5 [5]:
A neutrosophic subset $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U}, T)$ is a Nα-OS iff $\mathcal{A}$ is a NS-OS and NP-OS.

Lemma 2.6:
(i) If $\mathcal{K}$ is a N-OS, then $SNcl(\mathcal{K}) = Nint(Ncl(\mathcal{K}))$.
(ii) If $\mathcal{A}$ is a neutrosophic subset of a neutrosophic topological space $(\mathcal{U}, T)$, then $SNint(Ncl(\mathcal{A})) = Ncl(Nint(Ncl(\mathcal{A})))$.
Proof: This follows directly from the definition)2.1) and proposition (2.4).

3. Neutrosophic Semi-α-Open Sets

In this section, we present and study the neutrosophic semi-α-open sets and some of its properties.

Definition 3.1:
A neutrosophic subset $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U}, T)$ is called neutrosophic semi-α-open set (briefly NSα-OS) if there exists a Nα-OS $\mathcal{H}$ in $\mathcal{U}$ such that $\mathcal{H} \subseteq \mathcal{A} \subseteq Ncl(\mathcal{H})$ or
equivalently if $\mathcal{A} \subseteq Ncl(\alpha Nint(\mathcal{A}))$. The family of all NSα-OS of $\mathcal{U}$ is denoted by NSαO($\mathcal{U}$).

Definition 3.2:
The complement of NSα-OS is called a neutrosophic semi-α-closed set (briefly NSα-CS). The family of all NSα-CS of $\mathcal{U}$ is denoted by NSαC($\mathcal{U}$).

Proposition 3.3:
It is evident by definitions that in a neutrosophic topological space $(\mathcal{U}, T)$, the following hold:
(i) Every N-OS (resp. N-CS) is a NSα-OS (resp. NSα-CS).
(ii) Every Nα-OS (resp. Nα-CS) is a NSα-OS (resp. NSα-CS).

The converse of the above proposition need not be true as seen from the following example.

Example 3.4:
Let $\mathcal{U} = \{u\}$, $\mathcal{A} = \{\langle u, 0.5, 0.5, 0.4\rangle : u \in \mathcal{U}\}$, $\mathcal{B} = \{\langle u, 0.4, 0.5, 0.8\rangle : u \in \mathcal{U}\}, \mathcal{C} = \{\langle u, 0.5, 0.6, 0.4\rangle : u \in \mathcal{U}\}, \mathcal{D} = \{\langle u, 0.4, 0.6, 0.8\rangle : u \in \mathcal{U}\}$.
Then $T = \{0_N, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, 1_N\}$ is a neutrosophic topology on $\mathcal{U}$.
(i) Let $\mathcal{H} = \{\langle u, 0.5, 0.1, 0.3\rangle : u \in \mathcal{U}\}$, $\mathcal{A} \subseteq \mathcal{H} \subseteq Ncl(\mathcal{A}) = \langle u, 0.6, 0.4, 0.2\rangle$, the neutrosophic set $\mathcal{H}$ is a NSα-OS but is not N-OS. It is clear that $\mathcal{H}^c = \{\langle u, 0.5, 0.9, 0.7\rangle : u \in \mathcal{U}\}$ is a NSα-CS but is not N-CS.
(ii) Let $\mathcal{K} = \{\langle u, 0.5, 0.1, 0.2\rangle : u \in \mathcal{U}\}$, $\mathcal{A} \subseteq \mathcal{K} \subseteq Ncl(\mathcal{A}) = \langle u, 0.6, 0.4, 0.2\rangle$, the neutrosophic set $\mathcal{K}$ is a NSα-OS, $\mathcal{K} \nsubseteq Nint(Ncl(Nint(\mathcal{K})) = Nint(Ncl(\langle u, 0.5, 0.5, 0.4\rangle)) = Nint(\langle u, 0.6, 0.4, 0.2\rangle) = \langle u, 0.5, 0.5, 0.4\rangle$, the neutrosophic set $\mathcal{K}$ is not Nα-OS. It is clear that $\mathcal{K}^c = \{\langle u, 0.5, 0.9, 0.8\rangle : u \in \mathcal{U}\}$ is a NSα-CS but is not Nα-CS.

Remark 3.5:
The concepts of NSα-OS and NP-OS are independent, as the following examples shows.

Example 3.6:
In example (3.4), then the neutrosophic set $\mathcal{H} = \{\langle u, 0.5, 0.1, 0.3\rangle : u \in \mathcal{U}\}$ is a NSα-OS but is not NP-OS, because $\mathcal{H} \nsubseteq Nint(Ncl(\mathcal{H})) = Nint(\langle u, 0.6, 0.4, 0.2\rangle) = \langle u, 0.5, 0.5, 0.4\rangle$.

Example 3.7:
Let $\mathcal{U} = \{a, b\}$, $\mathcal{A} = \{\langle 0.4, 0.8, 0.9\rangle, \langle 0.7, 0.5, 0.3\rangle\}$, $\mathcal{B} = \{\langle 0.5, 0.8, 0.6\rangle, \langle 0.8, 0.4, 0.3\rangle\}$, $\mathcal{C} = \{\langle 0.4, 0.7, 0.9\rangle, \langle 0.6, 0.4, 0.4\rangle\}$, $\mathcal{D} = \{\langle 0.5, 0.7, 0.5\rangle, \langle 0.8, 0.4, 0.6\rangle\}$.
Then $T = \{0_N, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, 1_N\}$ is a neutrosophic topology on $\mathcal{U}$.
Then the neutrosophic set $\mathcal{K} = \{\langle 1, 1, 0.3\rangle, \langle 0.7, 0.3, 0.6\rangle\}$ is a NP-OS but is not NSα-OS.

Remark 3.8:
(i) If every N-OS is a N-CS and every nowhere neutrosophic dense set is N-CS in any neutrosophic topological space $(\mathcal{U}, T)$, then every NSα-OS is a N-OS.
(ii) If every N-OS is a N-CS in any neutrosophic topological space $(\mathcal{U}, T)$, then every NSα-OS is a Nα-OS.

Remark 3.9:
(i) It is clear that every NS-OS and NP-OS of any neutrosophic topological space $(\mathcal{U}, T)$ is a NSα-OS (by proposition (2.5) and proposition (3.3) (ii)).
(ii) A NSα-OS in any neutrosophic topological space $(\mathcal{U}, T)$ is a NP-OS if every N-OS of $\mathcal{U}$ is a N-CS (from proposition (2.4) (iii) and remark (3.8) (ii)).

Theorem 3.10:
For any neutrosophic subset $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U}, T)$, $\mathcal{A} \in$ NαO($\mathcal{U}$) iff there exists a N-OS $\mathcal{H}$ such that $\mathcal{H} \subseteq \mathcal{A} \subseteq Nint(Ncl(\mathcal{H}))$.

Proof: Let $\mathcal{A}$ be a Nα-OS. Hence $\mathcal{A} \subseteq Nint(Ncl(Nint(\mathcal{A})))$, so let $\mathcal{H} = Nint(\mathcal{A})$, we get $Nint(\mathcal{A}) \subseteq \mathcal{A} \subseteq Nint(Ncl(Nint(\mathcal{A})))$. Then there exists a N-OS $Nint(\mathcal{A})$ such that $\mathcal{H} \subseteq \mathcal{A} \subseteq Nint(Ncl(\mathcal{H}))$, where $\mathcal{H} = Nint(\mathcal{A})$.
Conversely, suppose that there is a N-OS $\mathcal{H}$ such that $\mathcal{H} \subseteq \mathcal{A} \subseteq Nint(Ncl(\mathcal{H}))$.
To prove $\mathcal{A} \in$ NαO$(\mathcal{U})$.
$\mathcal{H} \subseteq Nint(\mathcal{A})$ (since $Nint(\mathcal{A})$ is the largest N-OS contained in $\mathcal{A}$).
Hence $Ncl(\mathcal{H}) \subseteq Nint(Ncl(\mathcal{A}))$, then $Nint(Ncl(\mathcal{H})) \subseteq Nint(Ncl(Nint(\mathcal{A})))$.
But $\mathcal{H} \subseteq \mathcal{A} \subseteq Nint(Ncl(\mathcal{H}))$ (by hypothesis). Then $\mathcal{A} \subseteq Nint(Ncl(Nint(\mathcal{A})))$.
Therefore, $\mathcal{A} \in$ NαO$(\mathcal{U})$.

Theorem 3.11:
For any neutrosophic subset $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U},T)$. The following properties are equivalent:
(i) $\mathcal{A} \in$ NSαO$(\mathcal{U})$.
(ii) There exists a N-OS say $\mathcal{H}$ such that $\mathcal{H} \subseteq \mathcal{A} \subseteq Ncl(Nint(Ncl(\mathcal{H})))$.
(iii) $\mathcal{A} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$.
Proof:
$(i) \Longrightarrow (ii)$ Let $\mathcal{A} \in$ NSαO$(\mathcal{U})$. Then there exists $\mathcal{K} \in$ NαO$(\mathcal{U})$, such that $\mathcal{K} \subseteq \mathcal{A} \subseteq Ncl(\mathcal{K})$. Hence there exists $\mathcal{H}$ N-OS such that $\mathcal{H} \subseteq \mathcal{K} \subseteq Nint(Ncl(\mathcal{H}))$ (by theorem (3.10)). Therefore, $Ncl(\mathcal{H}) \subseteq Ncl(\mathcal{K}) \subseteq Ncl(Nint(Ncl(\mathcal{H})))$, implies that $Ncl(\mathcal{K}) \subseteq Ncl(Nint(Ncl(\mathcal{H})))$. Then $\mathcal{H} \subseteq \mathcal{K} \subseteq \mathcal{A} \subseteq Ncl(\mathcal{K}) \subseteq Ncl(Nint(Ncl(\mathcal{H})))$. Therefore, $\mathcal{H} \subseteq \mathcal{A} \subseteq Ncl(Nint(Ncl(\mathcal{H})))$, for some $\mathcal{H}$ N-OS.
$(ii) \Longrightarrow (iii)$ Suppose that there exists a N-OS $\mathcal{H}$ such that $\mathcal{H} \subseteq \mathcal{A} \subseteq Ncl(Nint(Ncl(\mathcal{H})))$. We know that $Nint(\mathcal{A}) \subseteq \mathcal{A}$. On the other hand, $\mathcal{H} \subseteq Nint(\mathcal{A})$ (since $Nint(\mathcal{A})$ is the largest N-OS contained in $\mathcal{A}$). Hence $Ncl(\mathcal{H}) \subseteq Ncl(Nint(\mathcal{A}))$, then $Nint(Ncl(\mathcal{H})) \subseteq Nint(Ncl(Nint(\mathcal{A})))$, therefore $Ncl(Nint(Ncl(\mathcal{H}))) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$.
But $\mathcal{A} \subseteq Ncl(Nint(Ncl(\mathcal{H})))$ (by hypothesis). Hence $\mathcal{A} \subseteq Ncl(Nint(Ncl(\mathcal{H}))) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$, then $\mathcal{A} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$.
$(iii) \Longrightarrow (i)$ Let $\mathcal{A} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$.
To prove $\mathcal{A} \in$ NSαO$(\mathcal{U})$. Let $\mathcal{K} = Nint(\mathcal{A})$; we know that $Nint(\mathcal{A}) \subseteq \mathcal{A}$. To prove $\mathcal{A} \subseteq Ncl(Nint(\mathcal{A}))$.
Since $Nint(Ncl(Nint(\mathcal{A}))) \subseteq Ncl(Nint(\mathcal{A}))$. Hence, $Ncl(Nint(Ncl(Nint(\mathcal{A})))) \subseteq Ncl(Ncl(Nint(\mathcal{A}))) = Ncl(Nint(\mathcal{A}))$. But $\mathcal{A} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$ (by hypothesis). Hence, $\mathcal{A} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$ $\subseteq Ncl(Nint(\mathcal{A})) \Longrightarrow \mathcal{A} \subseteq Ncl(Nint(\mathcal{A}))$. Hence, there exists a N-OS say $\mathcal{K}$, such that $\mathcal{K} \subseteq \mathcal{A} \subseteq Ncl(\mathcal{A})$. On the other hand, $\mathcal{K}$ is a Nα-OS (since $\mathcal{K}$ is a N-OS). Hence $\mathcal{A} \in$ NSαO$(\mathcal{U})$.
Corollary 3.12:
For any neutrosophic subset $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U},T)$, the following properties are equivalent:
(i) $\mathcal{A} \in$ NSαC$(\mathcal{U})$.
(ii) There exists a N-CS $\mathcal{F}$ such that $Nint(Ncl(Nint(\mathcal{F})))$ $\subseteq \mathcal{A} \subseteq \mathcal{F}$.
(iii) $Nint(Ncl(Nint(Ncl(\mathcal{A})))) \subseteq \mathcal{A}$.
Proof:
$(i) \Longrightarrow (ii)$ Let $\mathcal{A} \in$ NSαC$(\mathcal{U})$, then $\mathcal{A}^c \in$ NSαO$(\mathcal{U})$. Hence there is $\mathcal{H}$ N-OS such that $\mathcal{H} \subseteq \mathcal{A}^c \subseteq Ncl(Nint(Ncl(\mathcal{H})))$ (by theorem (3.11)). Hence $(Ncl(Nint(Ncl(\mathcal{H}))))^c \subseteq \mathcal{A}^{c^c} \subseteq \mathcal{H}^c$, i.e., $Nint(Ncl(Nint(\mathcal{H}^c))) \subseteq \mathcal{A} \subseteq \mathcal{H}^c$. Let $\mathcal{H}^c = \mathcal{F}$, where $\mathcal{F}$ is a N-CS in $\mathcal{U}$. Then $Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{A} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-CS.
$(ii) \Longrightarrow (iii)$ Suppose that there exists $\mathcal{F}$ N-CS such that $Nint\left(Ncl\left(Nint(\mathcal{F})\right)\right) \subseteq \mathcal{A} \subseteq \mathcal{F}$, but $Ncl(\mathcal{A})$ is the smallest N-CS containing $\mathcal{A}$. Then $Ncl(\mathcal{A}) \subseteq \mathcal{F}$, and therefore: $Nint\left(Ncl(\mathcal{A})\right) \subseteq Nint(\mathcal{F}) \Longrightarrow$ $Ncl\left(Nint\left(Ncl(\mathcal{A})\right)\right) \subseteq Ncl\left(Nint(\mathcal{F})\right) \Longrightarrow$ $Nint(Ncl(Nint(Ncl(\mathcal{A})))) \subseteq Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{A} \Longrightarrow Nint(Ncl(Nint(Ncl(\mathcal{A})))) \subseteq \mathcal{A}$.
$(iii) \Longrightarrow (i)$ Let $Nint(Ncl(Nint(Ncl(\mathcal{A})))) \subseteq \mathcal{A}$.
To prove $\mathcal{A} \in$ NSαC$(\mathcal{U})$, i.e., to prove $\mathcal{A}^c \in$ NSαO$(\mathcal{U})$.
Then $\mathcal{A}^c \subseteq (Nint(Ncl(Nint(Ncl(\mathcal{A})))))^c = Ncl(Nint(Ncl(Nint(\mathcal{A}^c))))$, but $(Nint(Ncl(Nint(Ncl(\mathcal{A})))))^c = Ncl(Nint(Ncl(Nint(\mathcal{A}^c))))$.
Hence $\mathcal{A}^c \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}^c))))$, and therefore $\mathcal{A}^c \in$ NSαO$(\mathcal{U})$, i.e., $\mathcal{A} \in$ NSαC$(\mathcal{U})$.

Proposition 3.13:
The union of any family of Nα-OS is a Nα-OS.
Proof: Let $\{\mathcal{A}_i\}_{i\in\Lambda}$ be a family of Nα-OS of $\mathcal{U}$.
To prove $\bigcup_{i\in\Lambda}\mathcal{A}_i$ is a Nα-OS,
i.e., $\bigcup_{i\in\Lambda}\mathcal{A}_i \subseteq Nint(Ncl(Nint(\bigcup_{i\in\Lambda}\mathcal{A}_i)))$.
Then $\mathcal{A}_i \subseteq Nint(Ncl(Nint(\mathcal{A}_i)))$, $\forall i \in \Lambda$.
Since $\bigcup_{i\in\Lambda} Nint(\mathcal{A}_i) \subseteq Nint(\bigcup_{i\in\Lambda}\mathcal{A}_i)$ and $\bigcup_{i\in\Lambda} Ncl(\mathcal{A}_i) \subseteq Ncl(\bigcup_{i\in\Lambda}\mathcal{A}_i)$ hold for any neutrosophic topology.
We have
$$\begin{aligned}\bigcup_{i\in\Lambda}\mathcal{A}_i &\subseteq \bigcup_{i\in\Lambda} Nint(Ncl(Nint(\mathcal{A}_i)))\\ &\subseteq Nint(\bigcup_{i\in\Lambda} Ncl(Nint(\mathcal{A}_i)))\\ &\subseteq Nint(Ncl(\bigcup_{i\in\Lambda}(Nint(\mathcal{A}_i)))\\ &\subseteq Nint(Ncl(Nint(\bigcup_{i\in\Lambda}\mathcal{A}_i))).\end{aligned}$$
Hence $\bigcup_{i\in\Lambda}\mathcal{A}_i$ is a Nα-OS.

Theorem 3.14:
The union of any family of NSα-OS is a NSα-OS.
Proof: Let $\{\mathcal{A}_i\}_{i\in\Lambda}$ be a family of NSα-OS. To prove $\bigcup_{i\in\Lambda}\mathcal{A}_i$ is a NSα-OS. Since $\mathcal{A}_i \in$ NSαO$(\mathcal{U})$. Then there is a Nα-OS $\mathcal{B}_i$ such that $\mathcal{B}_i \subseteq \mathcal{A}_i \subseteq Ncl(\mathcal{B}_i)$, $\forall i \in \Lambda$. Hence $\bigcup_{i\in\Lambda}\mathcal{B}_i \subseteq \bigcup_{i\in\Lambda}\mathcal{A}_i \subseteq \bigcup_{i\in\Lambda} Ncl(\mathcal{B}_i) \subseteq Ncl(\bigcup_{i\in\Lambda}\mathcal{B}_i)$.
But $\bigcup_{i\in\Lambda}\mathcal{B}_i \in$ NαO$(\mathcal{U})$ (by proposition (3.13)).
Hence $\bigcup_{i\in\Lambda}\mathcal{A}_i \in$ NSαO$(\mathcal{U})$.

Corollary 3.15:

The intersection of any family of NSα-CS is a NSα-CS.
Proof: This follows directly from the theorem (3.14).

Remark 3.16:
The following diagram shows the relations among the different types of weakly neutrosophic open sets that were studied in this section:

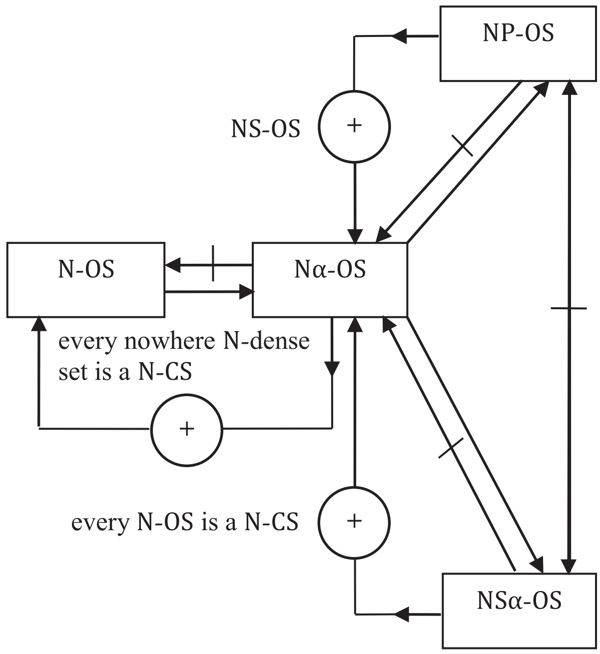

Diagram (3.1)

4. Neutrosophic Semi-α-Interior and Neutrosophic Semi-α-Closure

We present neutrosophic semi-α-interior and neutrosophic semi-α-closure and obtain some of its properties in this section.

Definition 4.1:
The union of all NSα-OS in a neutrosophic topological space $(\mathcal{U}, T)$ contained in $\mathcal{A}$ is called neutrosophic semi-α-interior of $\mathcal{A}$ and is denoted by $S\alpha Nint(\mathcal{A})$, $S\alpha Nint(\mathcal{A}) = \cup\{\mathcal{B}: \mathcal{B} \subseteq \mathcal{A}, \mathcal{B}$ is a NSα-OS$\}$.

Definition 4.2:
The intersection of all NSα-CS in a neutrosophic topological space $(\mathcal{U}, T)$ containing $\mathcal{A}$ is called neutrosophic semi-α-closure of $\mathcal{A}$ and is denoted by $S\alpha Ncl(\mathcal{A})$, $S\alpha Ncl(\mathcal{A}) = \cap\{\mathcal{B}: \mathcal{A} \subseteq \mathcal{B}, \mathcal{B}$ is a NSα-CS$\}$.

Proposition 4.3:
Let $\mathcal{A}$ be any neutrosophic set in a neutrosophic topological space $(\mathcal{U}, T)$, the following properties are true:
(i) $S\alpha Nint(\mathcal{A}) = \mathcal{A}$ iff $\mathcal{A}$ is a NSα-OS.
(ii) $S\alpha Ncl(\mathcal{A}) = \mathcal{A}$ iff $\mathcal{A}$ is a NSα-CS.
(iii) $S\alpha Nint(\mathcal{A})$ is the largest NSα-OS contained in $\mathcal{A}$.
(iv) $S\alpha Ncl(\mathcal{A})$ is the smallest NSα-CS containing $\mathcal{A}$.
Proof: (i), (ii), (iii) and (iv) are obvious.

Proposition 4.4:
Let $\mathcal{A}$ be any neutrosophic set in a neutrosophic topological space $(\mathcal{U}, T)$, the following properties are true:
(i) $S\alpha Nint(1_N - \mathcal{A}) = 1_N - (S\alpha Ncl(\mathcal{A}))$,
(ii) $S\alpha Ncl(1_N - \mathcal{A}) = 1_N - (S\alpha Nint(\mathcal{A}))$.
Proof: (i) By definition, $S\alpha Ncl(\mathcal{A}) = \cap\{\mathcal{B}: \mathcal{A} \subseteq \mathcal{B}, \mathcal{B}$ is a NSα-CS$\}$

$1_N - (S\alpha Ncl(\mathcal{A})) = 1_N - \cap\{\mathcal{B}: \mathcal{A} \subseteq \mathcal{B}, \mathcal{B}$ is a NSα-CS$\}$
$= \cup\{1_N - \mathcal{B}: \mathcal{A} \subseteq \mathcal{B}, \mathcal{B}$ is a NSα-CS$\}$
$= \cup\{\mathcal{H}: \mathcal{H} \subseteq 1_N - \mathcal{A}, \mathcal{H}$ is a NSα-OS$\}$
$= S\alpha Nint(1_N - \mathcal{A})$.
(ii) The proof is similar to (i).

Theorem 4.5:
Let $\mathcal{A}$ and $\mathcal{B}$ be two neutrosophic sets in a neutrosophic topological space $(\mathcal{U}, T)$. The following properties hold:
(i) $S\alpha Nint(0_N) = 0_N$, $S\alpha Nint(1_N) = 1_N$.
(ii) $S\alpha Nint(\mathcal{A}) \subseteq \mathcal{A}$.
(iii) $\mathcal{A} \subseteq \mathcal{B} \Longrightarrow S\alpha Nint(\mathcal{A}) \subseteq S\alpha Nint(\mathcal{B})$.
(iv) $S\alpha Nint(\mathcal{A} \cap \mathcal{B}) \subseteq S\alpha Nint(\mathcal{A}) \cap S\alpha Nint(\mathcal{B})$.
(v) $S\alpha Nint(\mathcal{A}) \cup S\alpha Nint(\mathcal{B}) \subseteq S\alpha Nint(\mathcal{A} \cup \mathcal{B})$.
(vi) $S\alpha Nint(S\alpha Nint(\mathcal{A})) = S\alpha Nint(\mathcal{A})$.
Proof: (i), (ii), (iii), (iv), (v) and (vi) are obvious.

Theorem 4.6:
Let $\mathcal{A}$ and $\mathcal{B}$ be two neutrosophic sets in a neutrosophic topological space $(\mathcal{U}, T)$. The following properties hold:
(i) $S\alpha Ncl(0_N) = 0_N$, $S\alpha Ncl(1_N) = 1_N$.
(ii) $\mathcal{A} \subseteq S\alpha Ncl(\mathcal{A})$.
(iii) $\mathcal{A} \subseteq \mathcal{B} \Longrightarrow S\alpha Ncl(\mathcal{A}) \subseteq S\alpha Ncl(\mathcal{B})$.
(iv) $S\alpha Ncl(\mathcal{A} \cap \mathcal{B}) \subseteq S\alpha Ncl(\mathcal{A}) \cap S\alpha Ncl(\mathcal{B})$.
(v) $S\alpha Ncl(\mathcal{A}) \cup S\alpha Ncl(\mathcal{B}) \subseteq S\alpha Ncl(\mathcal{A} \cup \mathcal{B})$.
(vi) $S\alpha Ncl(S\alpha Ncl(\mathcal{A})) = S\alpha Ncl(\mathcal{A})$.
Proof: (i) and (ii) are evident.
(iii) By part (ii), $\mathcal{B} \subseteq S\alpha Ncl(\mathcal{B})$. Since $\mathcal{A} \subseteq \mathcal{B}$, we have $\mathcal{A} \subseteq S\alpha Ncl(\mathcal{B})$. But $S\alpha Ncl(\mathcal{B})$ is a NSα-CS. Thus $S\alpha Ncl(\mathcal{B})$ is a NSα-CS containing $\mathcal{A}$. Since $S\alpha Ncl(\mathcal{A})$ is the smallest NSα-CS containing $\mathcal{A}$, we have $S\alpha Ncl(\mathcal{A}) \subseteq S\alpha Ncl(\mathcal{B})$. Hence, $\mathcal{A} \subseteq \mathcal{B} \Longrightarrow S\alpha Ncl(\mathcal{A}) \subseteq S\alpha Ncl(\mathcal{B})$.
(iv) We know that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B}$.
Therefore, by part (iii), $S\alpha Ncl(\mathcal{A} \cap \mathcal{B}) \subseteq S\alpha Ncl(\mathcal{A})$ and $S\alpha Ncl(\mathcal{A} \cap \mathcal{B}) \subseteq S\alpha Ncl(\mathcal{B})$.
Hence $S\alpha Ncl(\mathcal{A} \cap \mathcal{B}) \subseteq S\alpha Ncl(\mathcal{A}) \cap S\alpha Ncl(\mathcal{B})$.
(v) Since $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B}$, it follows from part (iii) that $S\alpha Ncl(\mathcal{A}) \subseteq S\alpha Ncl(\mathcal{A} \cup \mathcal{B})$ and $S\alpha Ncl(\mathcal{B}) \subseteq S\alpha Ncl(\mathcal{A} \cup \mathcal{B})$.
Hence $S\alpha Ncl(\mathcal{A}) \cup S\alpha Ncl(\mathcal{B}) \subseteq S\alpha Ncl(\mathcal{A} \cup \mathcal{B})$.
(vi) Since $S\alpha Ncl(\mathcal{A})$ is a NSα-CS, we have by proposition (4.3) part (ii), $S\alpha Ncl(S\alpha Ncl(\mathcal{A})) = S\alpha Ncl(\mathcal{A})$.

Proposition 4.7:
For any neutrosophic subset $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U}, T)$, then:

(i) $Nint(\mathcal{A}) \subseteq \alpha Nint(\mathcal{A}) \subseteq S\alpha Nint(\mathcal{A}) \subseteq S\alpha Ncl(\mathcal{A}) \subseteq \alpha Ncl(\mathcal{A}) \subseteq Ncl(\mathcal{A})$.
(ii) $Nint(S\alpha Nint(\mathcal{A})) = S\alpha Nint(Nint(\mathcal{A})) = Nint(\mathcal{A})$.
(iii) $\alpha Nint(S\alpha Nint(\mathcal{A})) = S\alpha Nint(\alpha Nint(\mathcal{A})) = \alpha Nint(\mathcal{A})$.
(iv) $Ncl(S\alpha Ncl(\mathcal{A})) = S\alpha Ncl(Ncl(\mathcal{A})) = Ncl(\mathcal{A})$.
(v) $\alpha Ncl(S\alpha Ncl(\mathcal{A})) = S\alpha Ncl(\alpha Ncl(\mathcal{A})) = \alpha Ncl(\mathcal{A})$.
(vi) $S\alpha Ncl(\mathcal{A}) = \mathcal{A} \cup Nint(Ncl(Nint(Ncl(\mathcal{A}))))$.
(vii) $S\alpha Nint(\mathcal{A}) = \mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A}))))$.
(viii) $Nint(Ncl(\mathcal{A})) \subseteq S\alpha Nint(S\alpha Ncl(\mathcal{A}))$.

Proof: We shall prove only (ii), (iii), (iv), (vii) and (viii).

(ii) To prove $Nint(S\alpha Nint(\mathcal{A})) = S\alpha Nint(Nint(\mathcal{A})) = Nint(\mathcal{A})$. Since $Nint(\mathcal{A})$ is a N-OS, then $Nint(\mathcal{A})$ is a NSα-OS. Hence $Nint(\mathcal{A}) = S\alpha Nint(Nint(\mathcal{A}))$ (by proposition (4.3)). Therefore:
$Nint(\mathcal{A}) = S\alpha Nint(Nint(\mathcal{A}))$......................(1)
Since $Nint(\mathcal{A}) \subseteq S\alpha Nint(\mathcal{A}) \Rightarrow Nint(Nint(\mathcal{A})) \subseteq Nint(S\alpha Nint(\mathcal{A})) \Rightarrow Nint(\mathcal{A}) \subseteq Nint(S\alpha Nint(\mathcal{A}))$.
Also, $S\alpha Nint(\mathcal{A}) \subseteq \mathcal{A} \Rightarrow Nint(S\alpha Nint(\mathcal{A})) \subseteq Nint(\mathcal{A})$. Hence:
$Nint(\mathcal{A}) = Nint(S\alpha Nint(\mathcal{A}))$......................(2)
Therefore by (1) and (2), we get $Nint(S\alpha Nint(\mathcal{A})) = S\alpha Nint(Nint(\mathcal{A})) = Nint(\mathcal{A})$.

(iii)To prove $\alpha Nint(S\alpha Nint(\mathcal{A})) = S\alpha Nint(\alpha Nint(\mathcal{A})) = \alpha Nint(\mathcal{A})$. Since $\alpha Nint(\mathcal{A})$ is Nα-OS, therefore $\alpha Nint(\mathcal{A})$ is NSα-OS. Therefore by proposition (4.3):
$\alpha Nint(\mathcal{A}) = S\alpha Nint(\alpha Nint(\mathcal{A}))$......................(1)
Now, to prove $\alpha Nint(\mathcal{A}) = \alpha Nint(S\alpha Nint(\mathcal{A}))$. Since $\alpha Nint(\mathcal{A}) \subseteq S\alpha Nint(\mathcal{A}) \Rightarrow \alpha Nint(\alpha Nint(\mathcal{A})) \subseteq \alpha Nint(S\alpha Nint(\mathcal{A})) \Rightarrow$
$\alpha Nint(\mathcal{A}) \subseteq \alpha Nint(S\alpha Nint(\mathcal{A}))$.
Also, $S\alpha Nint(\mathcal{A}) \subseteq \mathcal{A} \Rightarrow \alpha Nint(S\alpha Nint(\mathcal{A})) \subseteq \alpha Nint(\mathcal{A})$. Hence:
$\alpha Nint(\mathcal{A}) = \alpha Nint(S\alpha Nint(\mathcal{A}))$......................(2)
Therefore by (1) and (2), we get $\alpha Nint(S\alpha Nint(\mathcal{A})) = S\alpha Nint(\alpha Nint(\mathcal{A})) = \alpha Nint(\mathcal{A})$.

(iv) To prove $Ncl(S\alpha Ncl(\mathcal{A})) = S\alpha Ncl(Ncl(\mathcal{A})) = Ncl(\mathcal{A})$. We know that $Ncl(\mathcal{A})$ is a N-CS, so it is NSα-CS. Hence by proposition (4.3), we have:
$Ncl(\mathcal{A}) = S\alpha Ncl(Ncl(\mathcal{A}))$......................(1)
To prove $\boldsymbol{Ncl(\mathcal{A}) = Ncl(S\alpha Ncl(\mathcal{A}))}$.
Since $\boldsymbol{S\alpha Ncl(\mathcal{A}) \subseteq Ncl(\mathcal{A})}$ (by part (i)).
Then $\boldsymbol{Ncl(S\alpha Ncl(\mathcal{A})) \subseteq Ncl(Ncl(\mathcal{A})) = Ncl(\mathcal{A}) \Rightarrow Ncl(S\alpha Ncl(\mathcal{A})) \subseteq Ncl(\mathcal{A})}$. Since $\boldsymbol{\mathcal{A} \subseteq S\alpha Ncl(\mathcal{A}) \subseteq Ncl(S\alpha Ncl(\mathcal{A}))}$, then $\boldsymbol{\mathcal{A} \subseteq Ncl(S\alpha Ncl(\mathcal{A}))}$. Hence $\boldsymbol{Ncl(\mathcal{A}) \subseteq Ncl\left(Ncl(S\alpha Ncl(\mathcal{A}))\right) = Ncl(S\alpha Ncl(\mathcal{A}))}$
$\boldsymbol{\Rightarrow Ncl(\mathcal{A}) \subseteq Ncl(S\alpha Ncl(\mathcal{A}))}$ and therefore: $\boldsymbol{Ncl(\mathcal{A}) = Ncl(S\alpha Ncl(\mathcal{A}))}$...................................(2)
Now, by (1) and (2), we get that $Ncl(S\alpha Ncl(\mathcal{A})) = S\alpha Ncl(Ncl(\mathcal{A}))$.
Hence $Ncl(S\alpha Ncl(\mathcal{A})) = S\alpha Ncl(Ncl(\mathcal{A})) = Ncl(\mathcal{A})$.

(vii) To prove $S\alpha Nint(\mathcal{A}) = \mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A}))))$.
Since $S\alpha Nint(\mathcal{A}) \in \text{NS}\alpha\text{O}(\mathcal{U}) \Rightarrow S\alpha Nint(\mathcal{A}) \subseteq Ncl(Nint(Ncl(Nint(S\alpha Nint(\mathcal{A})))))$
$= Ncl(Nint(Ncl(Nint(\mathcal{A}))))$ (by part (ii)).
Hence $S\alpha Nint(\mathcal{A}) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$, also $S\alpha Nint(\mathcal{A}) \subseteq \mathcal{A}$. Then:
$S\alpha Nint(\mathcal{A}) \subseteq \mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A}))))$.............(1)
To prove $\mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A}))))$ is a NSα-OS contained in $\mathcal{A}$.
It is clear that $\mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A})))) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$ and also it is clear that $Nint(\mathcal{A}) \subseteq Ncl(Nint(\mathcal{A})) \Rightarrow Nint(Nint(\mathcal{A})) \subseteq Nint(Ncl(Nint(\mathcal{A}))) \Rightarrow Nint(\mathcal{A}) \subseteq Nint(Ncl(Nint(\mathcal{A}))) \Rightarrow Ncl(Nint(\mathcal{A})) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$ and $Nint(\mathcal{A}) \subseteq Ncl(Nint(\mathcal{A}))$ $\Rightarrow Nint(\mathcal{A}) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$ and $Nint(\mathcal{A}) \subseteq \mathcal{A} \Rightarrow Nint(\mathcal{A}) \subseteq \mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A}))))$.
We get $Nint(\mathcal{A}) \subseteq \mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A})))) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$.
Hence $\mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A}))))$ is a NSα-OS (by proposition (4.3)). Also, $\mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A}))))$ is contained in $\mathcal{A}$. Then $\mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A})))) \subseteq S\alpha Nint(\mathcal{A})$ (since $S\alpha Nint(\mathcal{A})$ is the largest NSα-OS contained in $\mathcal{A}$). Hence:
$\mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A})))) \subseteq S\alpha Nint(\mathcal{A})$.............(2)
By (1) and (2), $S\alpha Nint(\mathcal{A}) = \mathcal{A} \cap Ncl(Nint(Ncl(Nint(\mathcal{A}))))$.

(viii) To prove that $Nint(Ncl(\mathcal{A})) \subseteq S\alpha Nint(S\alpha Ncl(\mathcal{A}))$.
Since $S\alpha Ncl(\mathcal{A})$ is a NSα-CS, therefore $Nint(Ncl(Nint(Ncl(S\alpha Ncl(\mathcal{A}))))) \subseteq S\alpha Ncl(\mathcal{A})$ (by corollary (3.12)). Hence $Nint(Ncl(\mathcal{A})) \subseteq Nint(Ncl(Nint(Ncl(\mathcal{A})))) \subseteq S\alpha Ncl(\mathcal{A})$ (by part (iv)).
Therefore, $S\alpha Nint\left(Nint(Ncl(\mathcal{A}))\right) \subseteq S\alpha Nint(S\alpha Ncl(\mathcal{A})) \Rightarrow$
$Nint(Ncl(\mathcal{A})) \subseteq S\alpha Nint(S\alpha Ncl(\mathcal{A}))$ (by part (ii)).

Theorem 4.8:
For any neutrosophic subset $\mathcal{A}$ of a neutrosophic topological space $(\mathcal{U}, T)$. The following properties are equivalent:
(i) $\mathcal{A} \in \text{NS}\alpha\text{O}(\mathcal{U})$.
(ii) $\mathcal{H} \subseteq \mathcal{A} \subseteq Ncl(Nint(Ncl(\mathcal{H})))$, for some N-OS $\mathcal{H}$.
(iii) $\mathcal{H} \subseteq \mathcal{A} \subseteq SNint(Ncl(\mathcal{H}))$, for some N-OS $\mathcal{H}$.
(iv) $\mathcal{A} \subseteq SNint(Ncl(Nint(\mathcal{A})))$.
Proof:
$(i) \Rightarrow (ii)$ Let $\mathcal{A} \in \text{NS}\alpha\text{O}(\mathcal{U})$, then $\mathcal{A} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$ and $Nint(\mathcal{A}) \subseteq \mathcal{A}$. Hence $\mathcal{H} \subseteq \mathcal{A} \subseteq Ncl(Nint(Ncl(\mathcal{H})))$, where $\mathcal{H} = Nint(\mathcal{A})$.
$(ii) \Rightarrow (iii)$ Suppose $\mathcal{H} \subseteq \mathcal{A} \subseteq Ncl(Nint(Ncl(\mathcal{H})))$, for some N-OS $\mathcal{H}$.

But $SNint(Ncl(\mathcal{H})) = Ncl(Nint(Ncl(\mathcal{H})))$ (by lemma (2.6)).
Then $\mathcal{H} \subseteq \mathcal{A} \subseteq SNint(Ncl(\mathcal{H}))$, for some N-OS $\mathcal{H}$.
$(iii) \Longrightarrow (iv)$ Suppose that $\mathcal{H} \subseteq \mathcal{A} \subseteq SNint(Ncl(\mathcal{H}))$, for some N-OS $\mathcal{H}$. Since $\mathcal{H}$ is a N-OS contained in $\mathcal{A}$. Then $\mathcal{H} \subseteq Nint(\mathcal{A}) \Longrightarrow Ncl(\mathcal{H}) \subseteq Ncl(Nint(\mathcal{A}))$
$\Longrightarrow SNint(Ncl(\mathcal{H})) \subseteq SNint(Ncl(Nint(\mathcal{A})))$.
But $\mathcal{A} \subseteq SNint(Ncl(\mathcal{H}))$ (by hypothesis), then
$\mathcal{A} \subseteq SNint(Ncl(Nint(\mathcal{A})))$.
$(iv) \Longrightarrow (i)$ Let $\mathcal{A} \subseteq SNint(Ncl(Nint(\mathcal{A})))$. But
$SNint(Ncl(Nint(\mathcal{A}))) = Ncl(Nint(Ncl(Nint(\mathcal{A}))))$
(by lemma (2.6)). Hence $\mathcal{A} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{A}))))$
$\Longrightarrow \mathcal{A} \in \text{NS}\alpha\text{O}(\mathcal{U})$.

Corollary 4.9:
For any neutrosophic subset $\mathcal{B}$ of a neutrosophic topological space $(\mathcal{U}, T)$, the following properties are equivalent:
(i) $\mathcal{B} \in \text{NS}\alpha\text{C}(\mathcal{U})$.
(ii) $Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{B} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-CS.
(iii) $SNcl(Nint(\mathcal{F})) \subseteq \mathcal{B} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-CS.
(iv) $SNcl(Nint(Ncl(\mathcal{B}))) \subseteq \mathcal{B}$.
Proof:
$(i) \Longrightarrow (ii)$ Let $\mathcal{B} \in \text{NS}\alpha\text{C}(\mathcal{U}) \Longrightarrow$
$Nint(Ncl(Nint(Ncl(\mathcal{B})))) \subseteq \mathcal{B}$ (by corollary (3.12))
and $\mathcal{B} \subseteq Ncl(\mathcal{B})$. Hence we get
$Nint(Ncl(Nint(Ncl(\mathcal{B})))) \subseteq \mathcal{B} \subseteq Ncl(\mathcal{B})$.
Therefore $Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{B} \subseteq \mathcal{F}$, where $\mathcal{F} = Ncl(\mathcal{B})$.
$(ii) \Longrightarrow (iii)$ Let $Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{B} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-CS. But $Nint(Ncl(Nint(\mathcal{F}))) = SNcl(Nint(\mathcal{F}))$ (by lemma (2.6)). Hence $SNcl(Nint(\mathcal{F})) \subseteq \mathcal{B} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-CS.
$(iii) \Longrightarrow (iv)$ Let $SNcl(Nint(\mathcal{F})) \subseteq \mathcal{B} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-CS. Since $\mathcal{B} \subseteq \mathcal{F}$ (by hypothesis), hence $Ncl(\mathcal{B}) \subseteq \mathcal{F}$
$\Longrightarrow Nint(Ncl(\mathcal{B}) \subseteq Nint(\mathcal{F}) \Longrightarrow SNcl(Nint(Ncl(\mathcal{B})))$
$\subseteq SNcl(Nint(\mathcal{F})) \subseteq \mathcal{B} \Longrightarrow SNcl(Nint(Ncl(\mathcal{B}))) \subseteq \mathcal{B}$.
$(iv) \Longrightarrow (i)$ Let $SNcl(Nint(Ncl(\mathcal{B}))) \subseteq \mathcal{B}$.
But $SNcl(Nint(Ncl(\mathcal{B}))) = Nint(Ncl(Nint(Ncl(\mathcal{B}))))$ (by lemma (2.6)). Hence $Nint(Ncl(Nint(Ncl(\mathcal{B})))) \subseteq \mathcal{B} \Longrightarrow \mathcal{B} \in \text{NS}\alpha\text{C}(\mathcal{U})$.

5. Conclusion

In this work, we have defined new class of neutrosophic open sets called neutrosophic semi-α-open sets and studied their fundamental properties in neutrosophic topological spaces. The neutrosophic semi-α-open sets can be used to derive a new decomposition of neutrosophic continuity, neutrosophic compactness, and neutrosophic connectedness.

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University of New Mexico

IN-cross Entropy Based MAGDM Strategy under Interval Neutrosophic Set Environment

Shyamal Dalapati[1], Surapati Pramanik[2*] Shariful Alam[1], Florentin Smarandache [3] and Tapan Kumar Roy[1]

[1] Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, P.O.-Botanic Garden, Howrah-711103, West Bengal, India.

[2] Department of Mathematics, Nandalal Ghosh B.T. College, Panpur, P.O.-Narayanpur, District –North 24 Parganas, Pin code-743126, West Bengal, India. *E-mail: sura_pati@yahoo.co.in

[3]Department of Mathematics & Science, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu

*Correspondence: e-mail: sura_pati@yahoo.co.in ; Tel.: +91-9477035544

Abstract. Cross entropy measure is one of the best way to calculate the divergence of any variable from the priori one variable. We define a new cross entropy measure under interval neutrosophic set (INS) environment, which we call IN-cross entropy measure and prove its basic properties. We also develop weighted IN-cross entropy measure and investigats its basic properties. Based on the weighted IN-cross entropy measure, we develop a novel strategy for multi attribute group decision making (MAGDM) strategy under interval neutrosophic environment. The proposed multi attribute group decision making strategy is compared with the existing cross entropy measure based strategy in the literature under interval neutrosophic set environment. Finally, an illustratative example of multi attribute group decision making problem is solved to show the feasibility, validity and efficiency of the proposed MAGDM strategy.

Keywords: Interval neutrosophic set, IN-cross entropy measure, MAGDM strategy.

1. Introduction

In our daily life we frequently meet with the quantitative measure to take appropriate decision for solving many problems. Entropy measure provides us a quantitative measure of two variables. In 1968, Zadeh [1] introduced fuzzy entropy measure. According to Liu [2], under fuzzy environment, entropy should meet at least three basic following requirements: the entropy of a crisp number is zero; the entropy of an equipossible fuzzy variable is maximum and the entropy is applicable not only to finite and infinite cases but also to discrete and continuous cases. Shang and Jiang [3] proposed a cross entropy measure and symmetric discrimination measure between fuzzy sets. Atanassov [4] introduced intuitionistic fuzzy set (IFS) in 1989, which is the extension of fuzzy set. Some recent applications of IFS are found in [5-11] in the literature. Vlachos and Sergiadis [12] defined cross entropy measure in IFS environment and showed a mathematical connection between the notions of entropy for fuzzy sets and IFSs in terms of fuzziness and intuitionism. In 1998, Smarandache [13] introduced the concept of neutrosophic set (NS) by introducing truth membership, falsity membership and indeterminacy membership functions as independent components and their sum lies ($^-0$, 3^+). Thereafter, Wang et al. [14] introduced single valued neutrosophic set (SVNS) as a subclass of NS. Thereafter, many researchers paid attention to apply NS and SVNS in many field of research such as conflict resolution [15], clustering analysis [16, 17], decision making [18-47], educational problem [48, 49], image processing [50, 52], medical diagnosis [53], optimization [54-59], social problem [60, 61]. Ye [62] introduced cross entropy measure in SVNS and applied it to multi criteria decision- making (MCDM) problems. Ye [63] defined an improved cross entropy measure for SVNS to overcome drawbacks in [62]. In 2005, Wang et al. [64] introduced interval neutrosophic set (INS) considering truth membership, indeterminate membership and falsity membership as interval number in [0, 1]. Broumi and Smarandache [65] defined correlation coefficient of INS and proved its basic properties. Zhang et al. [66] defined correlation coefficient for interval neutrosophic number (INN) and applied it MAGDM problems. Zhang et al. [67] presented an outranking approach for INS and applied its MCDM problems. Recently, Yu et al. [68] use VIKOR method to solve MAGDM problem with INN. Ye [69] defined similarity measure in INS environment and applied to solve MCDM problem. Pramanik and Mondal [70] extended the single valued neutrosophic grey relational analysis strategy to interval neutrosophic environment and applied it to multi-attribute decision-making (MADM) problems. Zhao et al. [71] proposed a MADM strategy based on generalized weighted aggregation operator with INS. Zhang et al. [72] proposed a MCDM strategy based on two interval neutrosophic number aggregation operators. Sahin [73] defined two cross entropy measures with INS based on fuzzy cross entropy measure and single valued neutrosophic cross entropy measure and applied for solving MCDM problem. Tian et al. [74] proposed a cross entropy measure with INS and TOPSIS for solving MCDM problems.

Sahin [73], Tian et al. [74] proposed cross entropy measures under the interval-valued neutrosophic set environment, which is suitable for single decision maker only. So multiple decision maker cannot participate in their strategies in [73, 74].

The aforementioned applications of cross entropy [63, 73, 74] can be effective in dealing with neutrosophic MADM problems. However, they also bear some limitations, which are outlined below:

i. The strategies [63, 73, 74] are capable of solving neutrosophic MADM problems.
ii. In the strategies [73, 74], interval-valued neutrosophic set are transformed to SVNS by suitable transform operators.
iii. The strategies [63, 73, 74] have a single decision-making structure, and not enough attention is paid to improving robustness when processing the assessment information.

Research gap:

MAGDM strategy based on cross entropy measure.

This study answers the following research questions:

i. Is it possible to define a new cross entropy measure under interval-valued neutrosophic set environment that is free from asymmetrical phenomena?
ii. Is it possible to define a new weighted cross entropy measure under interval-valued neutrosophic set that is free from asymmetrical phenomena?
iii. Is it possible to develop a new MAGDM strategy based on the proposed cross entropy measure under interval-valued neutrosophic set environment?
iv. Is it possible to develop a new MAGDM strategy based on the proposed weighted cross entropy measure under interval-valued neutrosophic set environment?

Motivation:

The above-mentioned analysis describes the motivation behind proposing a novel IN-cross entropy-based strategy for tackling MAGDM under the interval-valued neutrosophic environment. This study develops a novel IN-cross entropy-based MAGDM strategy that can deal with multiple decision-makers and free from the drawbacks that exist in [63, 72, 73].

The objectives of the paper are:

1. i. To define a new cross entropy measure under interval-valued neutrosophic set environment without using any transformation operator and prove its basic properties,
2. ii. To define a new weighted cross measure and prove its basic properties.
3. iii. To develop a new MAGDM strategy based on weighted cross entropy measure under interval-valued neutrosophic set environment.

To fill the research gap, we propose IN-cross entropy-based MAGDM, which is capable of dealing with multiple decision-makers.

The main contributions of this paper are summarized below:

1. i. We define a new IN-cross entropy measure and prove its basic properties. It is straightforward symmetric.
2. ii. We define a new weighted IN-cross entropy measure in the single-valued neutrosophic set environment and prove its basic properties. It is straightforward symmetric
3. iii. In this paper, we develop a new MAGDM strategy based on weighted IN cross entropy to solve MAGDM problems.
4. iv. In this paper, we solve a MAGDM problem based on the proposed MAGDM strategy.
5.

The paper unfolds as follows: In section 2, we describe the basic definitions and operations of SVNS, INS. In section 3, we present the definition of proposed IN-cross entropy measure, weighted IN-cross entropy measure and their basic properties. In section 4, we develop a MAGDM strategy with the proposed weighted IN-cross entropy measure. In section 5, we solve a MAGDM problem to show the feasibility, validity and efficiency of the proposed strategy. In section 6, we present conclusion and future direction of this study.

2. Preliminaries

2.1 Definition: Single valued neutrosophic set (SVNS) [14]

Assume that U be a space of points (objects) with generic elements $u \in U$. A SVNS H in U is characterized by a truth-membership function $T_H(u)$, an indeterminacy-membership function $I_H(u)$, and a falsity-membership function $F_H(u)$, where $T_H(u)$, $I_H(u)$, $F_H(u) \in [0, 1]$ for each point u in U. Therefore, a SVNS A can be expressed as H = $\{ u, T_H(u), I_H(u), F_H(u) \mid u \in U\}$, whereas, the sums of $T_H(u)$, $I_H(u)$ and $F_H(u)$ satisfy the condition

$$0 \leq T_H(u) + I_H(u) + F_H(u) \leq 3.$$

2.2 Definition: Interval neutrosophic sets (INSs) [64]

Assume that U be a space of points (objects) with generic elements $u \in U$. An INSs J in U is characterized by a truth-membership measure $T_J(u)$, an indeterminacy-membership measure $I_J(u)$, and a falsity-membership measure $F_J(u)$, where,

$T_J(u)=[T_J^-(u),T_J^+(u)]$, $I_J(u)=[I_J^-(u),I_J^+(u)]$,

$F_J(u)=[F_J^-(u),F_J^+(u)]$ for each point u in U. Therefore, a INSs J can be expressed as J = $\{u, [T_J^-(u),T_J^+(u)]$, $[I_J^-(u),I_J^+(u)]$, $[F_J^-(u),F_J^+(u)] \mid u \in U\}$. Where, $T_J^-(u),T_J^+(u)$, $I_J^-(u),I_J^+(u)$, $F_J^-(u),F_J^+(u) \subseteq [0,1]$.

2.3 Definition: Inclusion of two INSs [64]

Let $J_1 = \{u, [T_{J_1}^-(u),T_{J_1}^+(u)], [I_{J_1}^-(u),I_{J_1}^+(u)], [F_{J_1}^-(u),F_{J_1}^+(u)] \mid u \in U\}$ and $J_2 = \{u, [T_{J_2}^-(u),T_{J_2}^+(u)], [I_{J_2}^-(u),I_{J_2}^+(u)], [F_{J_2}^-(u),F_{J_2}^+(u)] \mid u \in U\}$ be any two INSs in U, then $J_1 \subseteq J_2$ iff $T_{J_1}^-(u) \leq T_{J_2}^-(u)$, $T_{J_1}^+(u) \leq T_{J_2}^+(u)$, $I_{J_1}^-(u) \geq I_{J_2}^-(u)$, $I_{J_1}^+(u) \geq I_{J_2}^+(u)$, $F_{J_1}^-(u) \geq F_{J_2}^-(u)$, $F_{J_1}^+(u) \geq F_{J_2}^+(u)$ for all $u \in U$.

2.4 Definition: Complement of an INS [64]

The complement J^c of an INS J = $\{u, [T_J^-(u),T_J^+(u)], [I_J^-(u),I_J^+(u)], [F_J^-(u),F_J^+(u)] \mid u \in U\}$ is defined as follows:

$J^c = \{u, [1-T_J^+(u), 1-T_J^-(u)], [1-I_J^+(u), 1-I_J^-(u)], [1-F_J^+(u), 1-F_J^-(u)] \mid u \in U\}$.

2.5 Definition: Equality of two INSs [64]

Let $J_1 = \{u, [T_{J_1}^-(u),T_{J_1}^+(u)], [I_{J_1}^-(u),I_{J_1}^+(u)], [F_{J_1}^-(u),F_{J_1}^+(u)] \mid u \in U\}$ and $J_2 = \{u, [T_{J_2}^-(u),T_{J_2}^+(u)], [I_{J_2}^-(u),I_{J_2}^+(u)], [F_{J_2}^-(u),F_{J_2}^+(u)] \mid u \in U\}$ be any two INSs in U, then $J_1 = J_2$ iff $T_{J_1}^-(u) = T_{J_2}^-(u)$, $T_{J_1}^+(u) = T_{J_2}^+(u)$, $I_{J_1}^-(u) = I_{J_2}^-(u)$, $I_{J_1}^+(u) = I_{J_2}^+(u)$, $F_{J_1}^-(u) = F_{J_2}^-(u)$, $F_{J_1}^+(u) = F_{J_2}^+(u)$ for all $u \in U$.

3. Definition: IN-cross-entropy measure

Let J_1 and J_2 be any two INSs in U = $\{ u_1, u_2, u_3, \ldots, u_n \}$. Then, the interval neutrosophic cross-entropy measure of J_1 and J_2 is denoted by $CE_{IN}(J_1, J_2)$ and defined as follows:

$$
\begin{aligned}
CE_{IN}(J_1,J_2) = \frac{1}{4}\sum_{i=1}^{n}\Bigg\{\Bigg\langle & \left[\frac{2\left|T_{J_1}^-(u_i)-T_{J_2}^-(u_i)\right|}{\sqrt{1+\left|T_{J_1}^-(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^-(u_i)\right|^2}} + \frac{2\left|(1-T_{J_1}^-(u_i))-(1-T_{J_2}^-(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^-(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^-(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|T_{J_1}^+(u_i)-T_{J_2}^+(u_i)\right|}{\sqrt{1+\left|T_{J_1}^+(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^+(u_i)\right|^2}} + \frac{2\left|(1-T_{J_1}^+(u_i))-(1-T_{J_2}^+(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^+(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^+(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|I_{J_1}^-(u_i)-I_{J_2}^-(u_i)\right|}{\sqrt{1+\left|I_{J_1}^-(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^-(u_i)\right|^2}} + \frac{2\left|(1-I_{J_1}^-(u_i))-(1-I_{J_2}^-(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^-(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^-(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|I_{J_1}^+(u_i)-I_{J_2}^+(u_i)\right|}{\sqrt{1+\left|I_{J_1}^+(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^+(u_i)\right|^2}} + \frac{2\left|(1-I_{J_1}^+(u_i))-(1-I_{J_2}^+(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^+(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^+(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|F_{J_1}^-(u_i)-F_{J_2}^-(u_i)\right|}{\sqrt{1+\left|F_{J_1}^-(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^-(u_i)\right|^2}} + \frac{2\left|(1-F_{J_1}^-(u_i))-(1-F_{J_2}^-(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^-(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^-(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|F_{J_1}^+(u_i)-F_{J_2}^+(u_i)\right|}{\sqrt{1+\left|F_{J_1}^+(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^+(u_i)\right|^2}} + \frac{2\left|(1-F_{J_1}^+(u_i))-(1-F_{J_2}^+(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^+(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^+(u_i))\right|^2}}\right]\Bigg\rangle\Bigg\}
\end{aligned}
\tag{1}
$$

Theorem 1.

Interval-valued neutrosophic cross entropy $CE_{IN}(J_1, J_2)$ for any two INSs J_1 and J_2 of U, satisfies the following properties:

i) $CE_{IN}(J_1, J_2) \geq 0$.

ii) $CE_{IN}(J_1, J_2) = 0$ if and only if $T_{J_1}^-(u_i) = T_{J_2}^-(u_i)$, $T_{J_1}^+(u_i) = T_{J_2}^+(u_i)$, $I_{J_1}^-(u_i) = I_{J_2}^-(u_i)$, $I_{J_1}^+(u_i) = I_{J_2}^+(u_i)$, $F_{J_1}^-(u_i) = F_{J_2}^-(u_i)$, $F_{J_1}^+(u_i) = F_{J_2}^+(u_i)$ for all $\forall u_i \in U$.

iii) $CE_{IN}(J_1, J_2) = CE_{IN}(J_1^c, J_2^c)$

iv) $CE_{IN}(J_1, J_2) = CE_{IN}(J_2, J_1)$

Proof: i)

For all values of $u_i \in U$, $\left|T^-_{J_1}(u_i)\right| \ge 0$, $\left|T^-_{J_2}(u_i)\right| \ge 0$,

$\left|T^-_{J_1}(u_i) - T^-_{J_2}(u_i)\right| \ge 0$, $\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2} \ge 0$, $\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2} \ge 0$,

$\left|(1-T^-_{J_1}(u_i))\right| \ge 0$, $\left|(1-T^-_{J_2}(u_i))\right| \ge 0$,

$\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right| \ge 0$, $\sqrt{1+\left|(1-T^-_{J_1}(u_i))\right|^2} \ge 0$,

$\sqrt{1+\left|(1-T^-_{J_2}(u_i))\right|^2} \ge 0$

$\Rightarrow$

$$\left[\frac{2\left|T^-_{J_1}(u_i)-T^-_{J_2}(u_i)\right|}{\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{J_2}(u_i))\right|^2}}\right] \ge 0$$

and $\left|T^+_{J_1}(u_i)\right| \ge 0$, $\left|T^+_{J_2}(u_i)\right| \ge 0$, $\left|T^+_{J_1}(u_i) - T^+_{J_2}(u_i)\right| \ge 0$,

$\sqrt{1+\left|T^+_{J_1}(u_i)\right|^2} \ge 0$, $\sqrt{1+\left|T^+_{J_2}(u_i)\right|^2} \ge 0$,

$\left|(1-T^+_{J_1}(u_i))\right| \ge 0$, $\left|(1-T^+_{J_2}(u_i))\right| \ge 0$,

$\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right| \ge 0$, $\sqrt{1+\left|(1-T^+_{J_1}(u_i))\right|^2} \ge 0$,

$\sqrt{1+\left|(1-T^+_{J_2}(u_i))\right|^2} \ge 0$

$\Rightarrow$

$$\left[\frac{2\left|T^+_{J_1}(u_i)-T^+_{J_2}(u_i)\right|}{\sqrt{1+\left|T^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{J_2}(u_i))\right|^2}}\right] \ge 0$$

Similarly, we can show that

$$\left[\frac{2\left|I^-_{J_1}(u_i)-I^-_{J_2}(u_i)\right|}{\sqrt{1+\left|I^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^-_{J_1}(u_i))-(1-I^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{J_2}(u_i))\right|^2}}\right] \ge 0$$

,

$$\left[\frac{2\left|I^+_{J_1}(u_i)-I^+_{J_2}(u_i)\right|}{\sqrt{1+\left|I^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{J_1}(u_i))-(1-I^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{J_2}(u_i))\right|^2}}\right] \ge 0$$

,

$$\left[\frac{2\left|F^-_{J_1}(u_i)-F^-_{J_2}(u_i)\right|}{\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{J_1}(u_i))-(1-F^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{J_2}(u_i))\right|^2}}\right] \ge 0$$

and

$$\left[\frac{2\left|F^+_{J_1}(u_i)-F^+_{J_2}(u_i)\right|}{\sqrt{1+\left|F^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{J_1}(u_i))-(1-F^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{J_2}(u_i))\right|^2}}\right] \ge 0$$

.

Hence, we can conclude that $CE_{IN}(J_1, J_2) \ge 0$.

ii). For all values of $u_i \in U$,

$$\left[\frac{2\left|T^-_{J_1}(u_i)-T^-_{J_2}(u_i)\right|}{\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow T^-_{J_1}(u_i) = T^-_{J_2}(u_i)$

$$\left[\frac{2\left|T^+_{J_1}(u_i)-T^+_{J_2}(u_i)\right|}{\sqrt{1+\left|T^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow T^+_{J_1}(u_i) = T^+_{J_2}(u_i)$.

$$\left[\frac{2\left|I^-_{J_1}(u_i)-I^-_{J_2}(u_i)\right|}{\sqrt{1+\left|I^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J_2}(u_i)\right|^2}}+\frac{2\left\|(1-I^-_{J_1}(u_i))-(1-I^-_{J_2}(u_i))\right\|}{\sqrt{1+\left|(1-I^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow I^-_{J_1}(u_i) = I^-_{J_2}(u_i)$

$$\left[\frac{2\left|I^+_{J_1}(u_i)-I^+_{J_2}(u_i)\right|}{\sqrt{1+\left|I^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{J_1}(u_i))-(1-I^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow I^+_{J_1}(u_i) = I^+_{J_2}(u_i)$

$$\left[\frac{2\left|F^-_{J_1}(u_i)-F^-_{J_2}(u_i)\right|}{\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{J_1}(u_i))-(1-F^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow F^-_{J_1}(u_i) = F^-_{J_2}(u_i)$

$$\left[\frac{2\left|F^+_{J_1}(u_i)-F^+_{J_2}(u_i)\right|}{\sqrt{1+\left|F^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{J_1}(u_i))-(1-F^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow F^{+}_{J_1}(u_i)=F^{+}_{J_2}(u_i)$$

So, $CE_{IN}(J_1,J_2)=0$ if and only if

$T^{-}_{J_1}(u_i)=T^{-}_{J_2}(u_i)$, $T^{+}_{J_1}(u_i)=T^{+}_{J_2}(u_i)$, $I^{-}_{J_1}(u_i)=I^{-}_{J_2}(u_i)$,

$I^{+}_{J_1}(u_i)=I^{+}_{J_2}(u_i)$, $F^{-}_{J_1}(u_i)=F^{-}_{J_2}(u_i)$, $F^{+}_{J_1}(u_i)=F^{+}_{J_2}(u_i)$ $\forall u_i\in U$.

Hence complete the proof.

iii). Using definition (2.4), we obtain the following expression:

$$CE_{IN}(J_1^c,J_2^c)=\frac{1}{4}\left\{\sum_{i=1}^{n}\left\langle\left[\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}+\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}+\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}+\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}+\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}+\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}+\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}\right]\right\rangle\right\}$$

$$=\frac{1}{4}\left\{\sum_{i=1}^{n}\left\langle\left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right]\right\rangle\right\}=CE_{IN}(J_1,J_2).$$

Hence complete the proof.

iv).

$$CE_{IN}(J_1,J_2)=$$

$$\frac{1}{4}\left\{\sum_{i=1}^{n}\left\langle\left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right]\right\rangle\right\}$$

$$=\frac{1}{4}\left\{\sum_{i=1}^{n}\left\langle\left[\frac{2\left|T^{-}_{J_2}(u_i)-T^{-}_{J_1}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J_2}(u_i))-(1-T^{-}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^{+}_{J_2}(u_i)-T^{+}_{J_1}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_2}(u_i))-(1-T^{+}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{-}_{J_2}(u_i)-I^{-}_{J_1}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_2}(u_i))-(1-I^{-}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{+}_{J_2}(u_i)-I^{+}_{J_1}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_2}(u_i))-(1-I^{+}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{-}_{J_2}(u_i)-F^{-}_{J_1}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_2}(u_i))-(1-F^{-}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{+}_{J_2}(u_i)-F^{+}_{J_1}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_2}(u_i))-(1-F^{+}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}}\right]\Bigg\}\Bigg\rangle$$

$=CE_{IN}(J_2, J_1)$.

Hence complete the proof.

3.1 Definition: Weighted IN-cross-entropy measure

We consider the weight w_i (i = 1, 2, 3,…, n) of u_i (i = 1, 2, 3, …, n) with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$.

Then the weighted cross entropy measure between J_1 and J_2 can be defined as follows:

$$CE^{w}_{IN}(J_1, J_2)=\frac{1}{4}\left\langle \sum_{i=1}^{n} w_i\left\{\left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right]\right\}\right\rangle \quad (2)$$

Theorem 2.

Interval neutrosophic weighted cross-entropy measure $CE^{w}_{IN}(J_1, J_2)$ satisfies the following properties:

i). $CE^{w}_{IN}(J_1, J_2) \geq 0$.

ii). $CE^{w}_{IN}(J_1, J_2) = 0$, if and only if $T^{-}_{J_1}(u_i) = T^{-}_{J_2}(u_i)$, $T^{+}_{J_1}(u_i) = T^{+}_{J_2}(u_i)$, $I^{-}_{J_1}(u_i) = I^{-}_{J_2}(u_i)$, $I^{+}_{J_1}(u_i) = I^{+}_{J_2}(u_i)$, $F^{-}_{J_1}(u_i) = F^{-}_{J_2}(u_i)$, $F^{+}_{J_1}(u_i) = F^{+}_{J_2}(u_i)$ for all $\forall u_i \in U$.

iii). $CE^{w}_{IN}(J_1, J_2) = CE^{w}_{IN}(J^{c}_1, J^{c}_2)$

iv). $CE^{w}_{IN}(J_1, J_2) = CE^{w}_{IN}(J_2, J_1)$

Proof:

i). For all values of $u_i \in U$, $\left|T^{-}_{J_1}(u_i)\right| \geq 0$, $\left|T^{-}_{J_2}(u_i)\right| \geq 0$,

$\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right| \geq 0$, $\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2} \geq 0$,

$\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2} \geq 0$, $\left|(1-T^{-}_{J_1}(u_i))\right| \geq 0$, $\left|(1-T^{-}_{J_2}(u_i))\right| \geq 0$,

$\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right| \geq 0$,

$\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2} \geq 0$

$$\Rightarrow\left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

and $\left|T^{+}_{J_1}(u_i)\right| \geq 0$, $\left|T^{+}_{J_2}(u_i)\right| \geq 0$, $\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right| \geq 0$,

$\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2} \geq 0$, $\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2} \geq 0$,

$\left|(1-T^{+}_{J_1}(u_i))\right| \geq 0$, $\left|(1-T^{+}_{J_2}(u_i))\right| \geq 0$, $\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right| \geq 0$,

$\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2} \geq 0$

$$\Rightarrow\left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

Similarly, we can show that

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

,

$$\left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

,

$$\left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

and

$$\left[\frac{2\left|F^+_{J_1}(u_i)-F^+_{J_2}(u_i)\right|}{\sqrt{1+\left|F^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{J_1}(u_i))-(1-F^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{J_2}(u_i))\right|^2}}\right]\ge 0$$

Since $w_i\in[0,1], \sum_{i=1}^{n} w_i=1$, we have, $CE^w_{IN}(J_1,J_2)\ge 0$. Hence complete the proof.

ii).

$$\left[\frac{2\left|T^-_{J_1}(u_i)-T^-_{J_2}(u_i)\right|}{\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow T^-_{J_1}(u_i)=T^-_{J_2}(u_i)$

$$\left[\frac{2\left|T^+_{J_1}(u_i)-T^+_{J_2}(u_i)\right|}{\sqrt{1+\left|T^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow T^+_{J_1}(u_i)=T^+_{J_2}(u_i)$

$$\left[\frac{2\left|I^-_{J_1}(u_i)-I^-_{J_2}(u_i)\right|}{\sqrt{1+\left|I^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^-_{J_1}(u_i))-(1-I^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow I^-_{J_1}(u_i)=I^-_{J_2}(u_i)$

$$\left[\frac{2\left|I^+_{J_1}(u_i)-I^+_{J_2}(u_i)\right|}{\sqrt{1+\left|I^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{J_1}(u_i))-(1-I^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow I^+_{J_1}(u_i)=I^+_{J_2}(u_i)$

$$\left[\frac{2\left|F^-_{J_1}(u_i)-F^-_{J_2}(u_i)\right|}{\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{J_1}(u_i))-(1-F^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow F^-_{J_1}(u_i)=F^-_{J_2}(u_i)$

$$\left[\frac{2\left|F^+_{J_1}(u_i)-F^+_{J_2}(u_i)\right|}{\sqrt{1+\left|F^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{J_1}(u_i))-(1-F^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow F^+_{J_1}(u_i)=F^+_{J_2}(u_i)$, For all values of $u_i\in U$.

Since, $w_i\in[0,1], \sum_{i=1}^{n} w_i=1, w_i\ge 0$, we can show that

$CE^w_{IN}(J_1,J_2)=0$ iff $T^-_{J_1}(u_i)=T^-_{J_2}(u_i)$, $T^+_{J_1}(u_i)=T^+_{J_2}(u_i)$, $I^-_{J_1}(u_i)=I^-_{J_2}(u_i)$, $I^+_{J_1}(u_i)=I^+_{J_2}(u_i)$, $F^-_{J_1}(u_i)=F^-_{J_2}(u_i)$, $F^+_{J_1}(u_i)=F^+_{J_2}(u_i)$ and $T_{J_1}(u_i)=T_{J_2}(u_i), I_{J_1}(u_i)=I_{J_2}(u_i), F_{J_1}(u_i)=F_{J_2}(u_i)$ for all $u_i\in U$.

iii).

Using definition (2.4), we obtain the following expression:

$$CE^w_{IN}(J^c_1,J^c_2)=\frac{1}{4}\left\langle\sum_{i=1}^{n} w_i\left\{\left[\frac{2\left|T^-_{J^c_1}(u_i)-T^-_{J^c_2}(u_i)\right|}{\sqrt{1+\left|T^-_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{J^c_1}(u_i))-(1-T^-_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{J^c_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^+_{J^c_1}(u_i)-T^+_{J^c_2}(u_i)\right|}{\sqrt{1+\left|T^+_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{J^c_1}(u_i))-(1-T^+_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{J^c_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^-_{J^c_1}(u_i)-I^-_{J^c_2}(u_i)\right|}{\sqrt{1+\left|I^-_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-I^-_{J^c_1}(u_i))-(1-I^-_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{J^c_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^+_{J^c_1}(u_i)-I^+_{J^c_2}(u_i)\right|}{\sqrt{1+\left|I^+_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{J^c_1}(u_i))-(1-I^+_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{J^c_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^-_{J^c_1}(u_i)-F^-_{J^c_2}(u_i)\right|}{\sqrt{1+\left|F^-_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{J^c_1}(u_i))-(1-F^-_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{J^c_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F^+_{J^c_1}(u_i)-F^+_{J^c_2}(u_i)\right|}{\sqrt{1+\left|F^+_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{J^c_1}(u_i))-(1-F^+_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{J^c_2}(u_i))\right|^2}}\right]\right\}\right\rangle$$

$$=\frac{1}{4}\left\langle\sum_{i=1}^{n} w_i\left\{\left[\frac{2\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{J_2}(u_i))\right|^2}}+\frac{2\left|T^-_{J_1}(u_i)-T^-_{J_2}(u_i)\right|}{\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{J_2}(u_i))\right|^2}}+\frac{2\left|T^+_{J_1}(u_i)-T^+_{J_2}(u_i)\right|}{\sqrt{1+\left|T^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^-_{J_1}(u_i))-(1-I^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{J_2}(u_i))\right|^2}}+\frac{2\left|I^-_{J_1}(u_i)-I^-_{J_2}(u_i)\right|}{\sqrt{1+\left|I^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^+_{J_1}(u_i))-(1-I^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{J_2}(u_i))\right|^2}}+\frac{2\left|I^+_{J_1}(u_i)-I^+_{J_2}(u_i)\right|}{\sqrt{1+\left|I^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-F^-_{J_1}(u_i))-(1-F^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{J_2}(u_i))\right|^2}}+\frac{2\left|F^-_{J_1}(u_i)-F^-_{J_2}(u_i)\right|}{\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|(1-F^+_{Q_1}(u_i))-(1-F^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{Q_2}(u_i))\right|^2}}+\frac{2\left|F^+_{Q_1}(u_i)-F^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{Q_2}(u_i)\right|^2}}\right]\right\}\right\rangle$$

$$=\frac{1}{4}\left\langle\sum_{i=1}^{n}w_i\left\{\left[\frac{2\left|T^-_{J_1}(u_i)-T^-_{J_2}(u_i)\right|}{\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{J_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^+_{J_1}(u_i)-T^+_{J_2}(u_i)\right|}{\sqrt{1+\left|T^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^-_{J_1}(u_i)-I^-_{J_2}(u_i)\right|}{\sqrt{1+\left|I^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^-_{J_1}(u_i))-(1-I^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^+_{J_1}(u_i)-I^+_{J_2}(u_i)\right|}{\sqrt{1+\left|I^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{J_1}(u_i))-(1-I^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^-_{J_1}(u_i)-F^-_{J_2}(u_i)\right|}{\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{J_1}(u_i))-(1-F^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{J_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F^+_{J_1}(u_i)-F^+_{J_2}(u_i)\right|}{\sqrt{1+\left|F^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{J_1}(u_i))-(1-F^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{J_2}(u_i))\right|^2}}\right]\right\}\right\rangle$$

$$=CE^w_{IN}(J_1,J_2),\forall u_i\in U.$$

Hence complete the proof.

iv).

Since,

$\left|T^-_{J_1}(u_i)-T^-_{J_2}(u_i)\right|=\left|T^-_{J_2}(u_i)-T^-_{J_1}(u_i)\right|$,

$\left|I^-_{J_1}(u_i)-I^-_{J_2}(u_i)\right|=\left|I^-_{J_2}(u_i)-I^-_{J_1}(u_i)\right|$,

$\left|F^-_{J_1}(u_i)-F^-_{J_2}(u_i)\right|=\left|F^-_{J_2}(u_i)-F^-_{J_1}(u_i)\right|$,

$\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right|=\left|(1-T^-_{J_2}(u_i))-(1-T^-_{J_1}(u_i))\right|$,

$\left|(1-I^-_{J_1}(u_i))-(1-I^-_{J_2}(u_i))\right|=\left|(1-I^-_{J_2}(u_i))-(1-I^-_{J_1}(u_i))\right|$,

$\left|(1-F^-_{J_1}(u_i))-(1-F^-_{J_2}(u_i))\right|=\left|(1-F^-_{J_2}(u_i))-(1-F^-_{J_1}(u_i))\right|$.

Then, we obtain

$\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2}=\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|I^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J_2}(u_i)\right|^2}=\sqrt{1+\left|I^-_{J_2}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}=\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-T^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-T^-_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-T^-_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-T^-_{J_1}(u_i)\right|^2}$,

, $\sqrt{1+\left|(1-I^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-I^-_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-I^-_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-I^-_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-F^-_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-F^-_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-F^-_{J_1}(u_i)\right|^2}$,

$\forall u_i\in U$.

Similarly, $\left|T^+_{J_1}(u_i)-T^+_{J_2}(u_i)\right|=\left|T^+_{J_2}(u_i)-T^+_{J_1}(u_i)\right|$,

$\left|I^+_{J_1}(u_i)-I^+_{J_2}(u_i)\right|=\left|I^+_{J_2}(u_i)-I^+_{J_1}(u_i)\right|$,

$\left|F^+_{J_1}(u_i)-F^+_{J_2}(u_i)\right|=\left|F^+_{J_2}(u_i)-F^+_{J_1}(u_i)\right|$,

$\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right|=\left|(1-T^+_{J_2}(u_i))-(1-T^+_{J_1}(u_i))\right|$,

$\left|(1-I^+_{J_1}(u_i))-(1-I^+_{J_2}(u_i))\right|=\left|(1-I^+_{J_2}(u_i))-(1-I^+_{J_1}(u_i))\right|$,

$\left|(1-F^+_{J_1}(u_i))-(1-F^+_{J_2}(u_i))\right|=\left|(1-F^+_{J_2}(u_i))-(1-F^+_{J_1}(u_i))\right|$,

then

$\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|I^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J_2}(u_i)\right|^2}=\sqrt{1+\left|I^+_{J_2}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|F^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J_2}(u_i)\right|^2}=\sqrt{1+\left|F^+_{J_2}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-T^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-T^+_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-T^+_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-T^+_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-I^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-I^+_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-I^+_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-I^+_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-F^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-F^+_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-F^+_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-F^+_{J_1}(u_i)\right|^2}$,

$\forall u_i\in U$.

And $w_i\in[0,1],\ \sum_{i=1}^{n}w_i=1,\ w_i\geq 0$.

So, $CE^w_{IN}(J_1,J_2)=CE^w_{IN}(J_2,J_1)$. Hence complete the proof.

4. Multi attribute group decision making strategy using IN-cross entropy measure in interval neutrosophic set environment

In this section we develop a novel MAGDM strategy based on proposed IN- cross entropy measure.

The MAGDM problem can be consider as follows:
Let $A=\{A_1,A_2,A_3,...,A_m\}$ and $G=\{G_1,G_2,G_3,...,G_n\}$ be the discrete set of alternatives and attribute respectively. Let $W=\{w_1,w_2,w_3,...,w_n\}$ be the weight vector of attributes G_j (j = 1, 2, 3, …, n), where $w_j\geq 0$ and $\sum_{j=1}^{n}w_j=1$. Let $E=\{E_1,E_2,E_3,...,E_\rho\}$ be the set of decision makers who are employ to evaluate the alternative. The weight vector of the decision makers $E_k\ (k=1,2,3,...,\rho)$ is $\lambda=\{\lambda_1,\lambda_2,\lambda_3,...,\lambda_\rho\}$ (where, $\lambda\geq 0$ and $\sum_{k=1}^{\rho}\lambda_k=1$), which can be determined according to the decision makers expertise, judgment quality and decision making knowledge.

Now, we describe the steps of the proposed MAGDM strategy (See Figure 1.) using weighted IN-cross entropy measure.

MAGDM strategy using IN-cross entropy measure

Step: 1. Formulate the decision matrices

For MAGDM with INSs information, the rating values of the alternatives $A_i\ (i=1,2,3,...,m)$ on the basis of critera $G_j\ (j=1,2,3,...,n)$ by the k-th decision maker can be expressed in INN as $a_{ij}^k = <[{}^-T_{ij}^k, {}^+T_{ij}^k], [{}^-I_{ij}^k, {}^+I_{ij}^k], [{}^-F_{ij}^k, {}^+F_{ij}^k]>$ (i = 1, 2, 3, …, m; j = 1, 2, 3, …, n; k = 1, 2, 3, …, ρ) . We arrange these rating values of alternatives provided by the decision makers in matrix form as follows:

$$M^k = \begin{pmatrix} & G_1 & G_2 & \dots & G_n \\ A_1 & a_{11}^k & a_{12}^k & \cdots & a_{1n}^k \\ A_2 & a_{21}^k & a_{22}^k & & a_{2n}^k \\ . & . & & \dots & . \\ A_m & a_{m1}^k & a_{m2}^k & \cdots & a_{mn}^k \end{pmatrix} \dots\dots\dots(3)$$

Step: 2. Formulate the weighted aggregated decision matrix

For obtaining one group decision, we aggregate all individual decision matrices (M^k) to an aggregated decision matrix (M) using interval-valued neutrosophic weighted averaging (INNWA) operator ([72]) as follows:

$$a_{ij} = INNWA_\lambda(a_{ij}^1, a_{ij}^2, a_{ij}^3, \dots, a_{ij}^\rho) =$$
$$(\lambda_1 a_{ij}^1 \oplus \lambda_2 a_{ij}^2 \oplus \lambda_3 a_{ij}^3 \oplus \dots \oplus \lambda_\rho a_{ij}^\rho) =$$
$$<[1-\prod_{k=1}^{\rho}(1-{}^-T_{ij}^k)^{\lambda_k}, 1-\prod_{k=1}^{\rho}(1-{}^+T_{ij}^k)^{\lambda_k}], [\prod_{k=1}^{\rho}({}^-I_{ij}^k)^{\lambda_k}, \prod_{k=1}^{\rho}({}^+I_{ij}^k)^{\lambda_k}],$$
$$[\prod_{k=1}^{\rho}({}^-F_{ij}^k)^{\lambda_k}, \prod_{k=1}^{\rho}({}^+F_{ij}^k)^{\lambda_k}]> \qquad (4)$$

(i = 1, 2, 3, …, m; j = 1, 2, 3, …, n; k = 1, 2, 3, …, ρ).

Therefore, the aggregated decision matrix is defined as follows:

$$M = \begin{pmatrix} & G_1 & G_2 & \dots & G_n \\ A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ A_2 & a_{21} & a_{22} & & a_{2n} \\ . & . & & \dots & . \\ A_m & a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \dots\dots\dots(5)$$

Step: 3. Formulate priori/ ideal decision matrix

In the MAGDM processes, the priori decision matrix is used to select the best alternatives among the set of collected feasible alternatives. In this decision making processes we use the following decision matrix as priori decision matrix.

$$P = \begin{pmatrix} & G_1 & G_2 & \dots & G_n \\ A_1 & a_{11}^* & a_{12}^* & \cdots & a_{1n}^* \\ A_2 & a_{21}^* & a_{22}^* & & a_{2n}^* \\ . & . & & \dots & . \\ A_m & a_{m1}^* & a_{m2}^* & \cdots & a_{mn}^* \end{pmatrix} \dots\dots\dots\dots\dots(6)$$

Where, $a_{ij}^* = <[1,1],[0,0],[0,0]>$ for benefit type attributes and $a_{ij}^* = <[0,0],[1,1],[1,1]>$ for cost type attributes, (i = 1, 2, 3, …, m; j = 1, 2, 3, …, n).

Step: 4. Formulate the weighted IN-cross entropy matrix

Using equation (2), we calculate weighted cross entropy value between aggregate matrix and priori matrix. The cross entropy value can be present in matrix form as follows:

$$^{INS}M_{CE}^{w} = \begin{pmatrix} CE_{IN}^{w}(A_1) \\ CE_{IN}^{w}(A_2) \\ \dots\dots\dots \\ \dots\dots\dots \\ CE_{IN}^{w}(A_m) \end{pmatrix} \dots\dots\dots\dots (7)$$

Step: 5. Rank the priority

Smaller value of the cross entropy reflect that an alternative is closer to the ideal alternative. Therefore, the priority order of all the alternatives can be determined according to the increasing order of the cross entropy values $CE_{IN}^{w}(A_i)$ (i = 1, 2, 3, …, m). Smallest cross entropy value indicates the best alternative and greatest cross entropy value indicates the worst alternative.

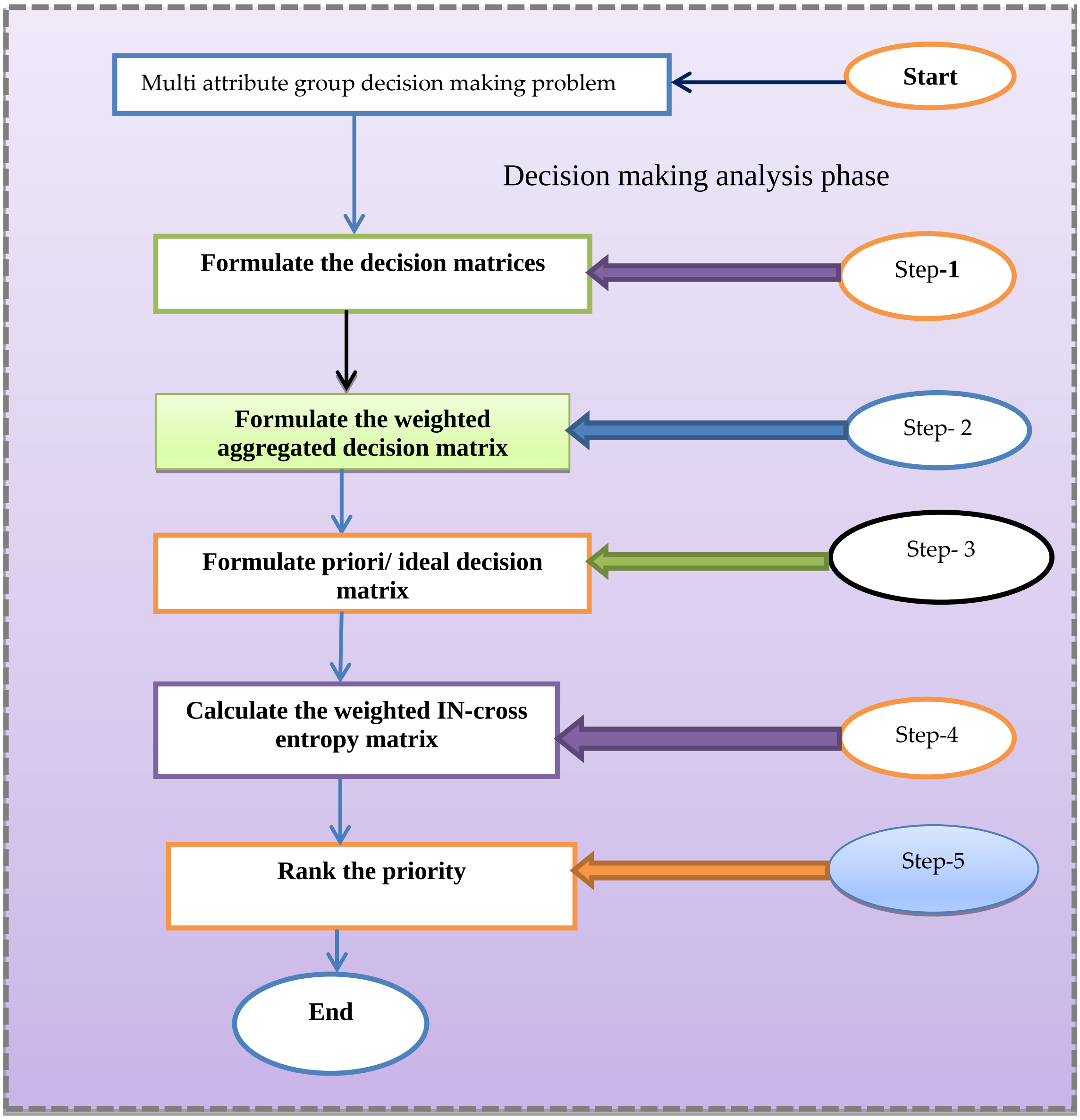

Figure.1 Decision making procedure of proposed MAGDM method

5. Illustrative example

In this section, we provide an illustrative example of MAGDM problems to reflect the validity and efficiency of our proposed strategy under INSs environment.

Now, we solve an illustrative example adapted from [9] for cultivation and analysis. A venture capital firm intends to make evaluation and selection to five enterprises with the investment potential:

1) Automobile company (A_1)
2) Military manufacturing enterprise (A_2)
3) TV media company (A_3)
4) Food enterprises (A_4)
5) Computer software company (A_5)

On the basis of four attributes namely:

1) Social and political factor (G_1)
2) The environmental factor (G_2)
3) Investment risk factor (G_3)
4) The enterprise growth factor (G_4).

The investment firm makes a panel of three decision makers $E = \{E_1, E_2, E_3\}$ having their weights vector

λ $\{0.42, 0.28, 0.30\}$ and weight vector of attributes is W = $\{0.24,\ 0.25,\ 0.23,\ 0.28\}$.

The steps of decision making strategy to rank alternatives are presented below:

Step: 1. Formulate the decision matrices

We represent the rating values of alternatives A_i (i = 1, 2, 3, 4, 5) with respects to the attributes G_j (j = 1, 2, 3, 4) provided by the decision-makers E_k (k = 1, 2, 3) in matrix form as follows:

Decision matrix for E_1 decision maker

$$M^1 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & \langle[.7,.9],[.3,.4],[.3,.4]\rangle & \langle[.6,.7],[.3,.4],[.4,.5]\rangle & \langle[.6,.7],[.2,.3],[.2,.4]\rangle & \langle[.4,.5],[.3,.4],[.7,.8]\rangle \\ A_2 & \langle[.6,.7],[.1,.2],[.2,.3]\rangle & \langle[.7,.8],[.2,.4],[.2,.3]\rangle & \langle[.7,.9],[.5,.6],[.4,.5]\rangle & \langle[.7,.9],[.1,.2],[.1,.3]\rangle \\ A_3 & \langle[.6,.8],[.2,.4],[.3,.4]\rangle & \langle[.5,.7],[.3,.4],[.1,.2]\rangle & \langle[.8,.9],[.5,.7],[.3,.6]\rangle & \langle[.6,.7],[.1,.3],[.2,.3]\rangle \\ A_4 & \langle[.4,.5],[.7,.8],[.6,.7]\rangle & \langle[.3,.6],[.2,.3],[.3,.4]\rangle & \langle[.6,.7],[.1,.2],[.4,.5]\rangle & \langle[.4,.5],[.3,.4],[.6,7]\rangle \\ A_5 & \langle[.7,.8],[.3,.4],[.2,.3]\rangle & \langle[.4,.5],[.2,.4],[.3,.5]\rangle & \langle[.5,.6],[.2,.4],[.3,.4]\rangle & \langle[.7,.9],[.6,.7],[.4,.5]\rangle \end{pmatrix} \ldots\ldots\ldots(8)$$

Decision matrix for E_2 decision maker

$$M^2 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & \langle[.6,.7],[.1,.2],[.2,.3]\rangle & \langle[.3,.5],[.2,.4],[.4,.5]\rangle & \langle[.7,.9],[.3,.4],[.3,.5]\rangle & \langle[.4,.6],[.4,.5],[.2,.3]\rangle \\ A_2 & \langle[.4,.7],[.2,.4],[.3,.4]\rangle & \langle[.6,.7],[.2,.3],[.3,.4]\rangle & \langle[.5,.7],[.1,.3],[.3,.4]\rangle & \langle[.4,.6],[.3,.4],[.2,.3]\rangle \\ A_3 & \langle[.3,.6],[.2,.4],[.3,.4]\rangle & \langle[.4,.5],[.2,.3],[.3,.5]\rangle & \langle[.8,.9],[.2,.5],[.3,.4]\rangle & \langle[.5,.6],[.3,.5],[.3,.6]\rangle \\ A_4 & \langle[.5,.7],[.3,.5],[.1,.3]\rangle & \langle[.5,.6],[.1,.3],[.4,.6]\rangle & \langle[.4,.7],[.1,.4],[.3,.4]\rangle & \langle[.6,.8],[.3,.5],[.3,4]\rangle \\ A_5 & \langle[.6,.9],[.3,.4],[.2,.3]\rangle & \langle[.3,.6],[.3,.4],[.2,.5]\rangle & \langle[.6,.8],[.3,.5],[.,4.6]\rangle & \langle[.3,.5],[.3,.4],[.4,.5]\rangle \end{pmatrix} \ldots\ldots\ldots(9)$$

Decision matrix for E_3 decision maker

$$M^3 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & \langle[.4,.7],[.1,.2],[.3,.5]\rangle & \langle[.3,.6],[.2,.4],[.3,.4]\rangle & \langle[.6,.7],[.2,.4],[.3,.5]\rangle & \langle[.8,.9],[.2,.4],[.1,.3]\rangle \\ A_2 & \langle[.3,.6],[.4,.5],[.4,.5]\rangle & \langle[.7,.9],[.1,.3],[.3,.4]\rangle & \langle[.5,.7],[.2,.4],[.2,.3]\rangle & \langle[.6,.8],[.2,.4],[.3,.5]\rangle \\ A_3 & \langle[.7,.8],[.1,.3],[.4,.5]\rangle & \langle[.8,.9],[.1,.3],[.3,.4]\rangle & \langle[.6,.8],[.2,.3],[.3,.4]\rangle & \langle[.6,.7],[.2,.3],[.3,.4]\rangle \\ A_4 & \langle[.6,.9],[.2,.3],[.2,.4]\rangle & \langle[.5,.6],[.1,.3],[.2,.4]\rangle & \langle[.3,.5],[.1,.2],[.2,.4]\rangle & \langle[.5,.7],[.2,.3],[.3,5]\rangle \\ A_5 & \langle[.7,.8],[.1,.3],[.2,.3]\rangle & \langle[.5,.6],[.2,.4],[.1,.3]\rangle & \langle[.4,.6],[.1,.3],[.2,.4]\rangle & \langle[.5,.7],[.2,.3],[.3,.5]\rangle \end{pmatrix} \ldots\ldots\ldots\ldots(10)$$

Step: 2. Formulate the weighted aggregated decision matrix

Using equation (4), the aggregated decision matrix is presented below:

Aggregated decision matrix

$$M = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & \langle[.6,.8],[.2,.3],[.3,.4]\rangle & \langle[.5,.6],[.2,.4],[.4,.4]\rangle & \langle[.6,.8],[.2,.3],[.2,.4]\rangle & \langle[.6,.7],[.3,.4],[.3,.4]\rangle \\ A_2 & \langle[.5,.7],[.2,.3],[.3,.4]\rangle & \langle[.7,.8],[.2,.3],[.2,.4]\rangle & \langle[.6,.8],[.2,.4],[.3,.4]\rangle & \langle[.6,.8],[.2,.3],[.2,.3]\rangle \\ A_3 & \langle[.6,.8],[.2,.4],[.3,.4]\rangle & \langle[.6,.8],[.2,.3],[.2,.3]\rangle & \langle[.8,.9],[.3,.5],[.3,.5]\rangle & \langle[.6,.7],[.2,.3],[.2,.4]\rangle \\ A_4 & \langle[.5,.7],[.4,.5],[.3,.5]\rangle & \langle[.4,.6],[.1,.3],[.3,.4]\rangle & \langle[.5,.6],[.1,.2],[.3,.4]\rangle & \langle[.5,.7],[.3,.4],[.4,.5]\rangle \\ A_5 & \langle[.7,.8],[.2,.4],[.2,.3]\rangle & \langle[.4,.6],[.2,.4],[.2,.4]\rangle & \langle[.5,.7],[.2,.4],[.3,.4]\rangle & \langle[.6,.8],[.4,.5],[.4,.5]\rangle \end{pmatrix} \ldots\ldots\ldots(11)$$

Step: 3. Formulate priori/ ideal decision matrix

Priori/ ideal decision matrix

$$M^1 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle \\ A_2 & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle \\ A_3 & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle \\ A_4 & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle \\ A_5 & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle \end{pmatrix} \ldots\ldots\ldots\ldots(12)$$

Step: 4. Calculate the weighted IN-cross entropy matrix

Using equation (2), we calculate the interval neutrosophic weighted cross entropy values between ideal matrixes (12) and weighted aggregated decision matrix (11).

$$^{IN}M^{w}_{CE} = \begin{pmatrix} 0.86 \\ 0.77 \\ 0.78 \\ 0.95 \\ 0.90 \end{pmatrix} \ldots\ldots\ldots\ldots(13)$$

Step: 5. Rank the priority

The position of cross entropy values of alternatives arranging in increasing order is

0.77 < 0.78 < 0.86 < 0.90 < 0.95. Since, smallest values of cross entropy indicate the alternative is closer to the ideal alternative. Thus the ranking priority of alternatives is $A_2 > A_3 > A_1 > A_5 > A_4$. Hence, military manufacturing enterprise (A_2) is the best alternative for investment.

In Figure 2, we draw a bar diagram to represent the cross entropy values of alternatives which shows that A_2 is the

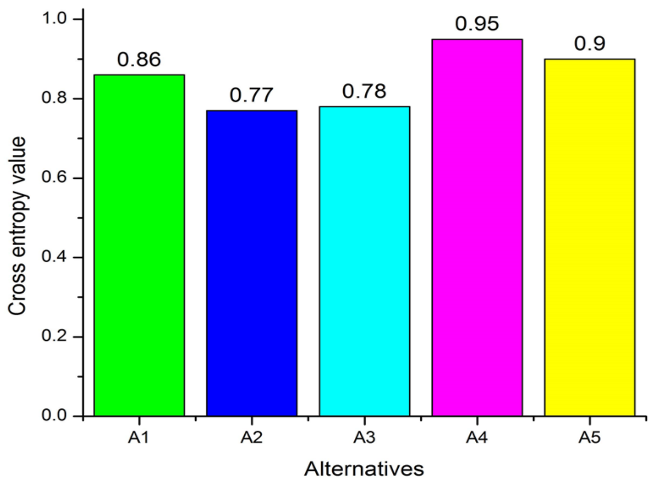

Figure.2. Bar diagram of alternatives versus cross entropy values of alternatives

2. Conclusion

In this paper we have defined IN-cross entropy measure in INS environment which is free from all the drawback of existence cross entropy measures under interval neutrosophic set environment. We have proved the basic properties of the cross entropy measures. We have also defined weighted IN- cross entropy measure and proved its basic properties. Based on the weighted IN-cross entropy measure, we have proposed a novel MAGDM strategy. Finally, we solve a MAGDM problem to show the feasibility and efficiency of the proposed MAGDM making strategy. The proposed IN-cross entropy based MAGDM strategy can be employed to solve a variety of problems such as logistics center selection, teacher selection, renewable energy selection, fault diagnosis, etc.

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University of New Mexico

Computing Operational Matrices in Neutrosophic Environments: A Matlab Toolbox

Said Broumi[1], Le Hoang Son[2], Assia Bakali[3],
Mohamed Talea[1], Florentin Smarandache[4], Ganeshsree Selvachandran[5]

[1]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco. E-mail: broumisaid78@gmail.com, taleamohamed@yahoo.fr
[2]VNU University of Science, Vietnam National University, Vietnam. E-mail:sonlh@vnu.edu.vn
[3]Ecole Royale Navale, Casablanca, Morocco. E-mail: assiabakali@yahoo.fr
[4]Department of Mathematics, University of New Mexico,705 Gurley Avenue, Gallup, NM 87301, USA .E-mail: fsmarandache@gmail.com
[5]Faculty of Business and Information Science,UCSIUniversity, Kuala Lumpur, Malaysia. E-mail: ganeshsree86@yahoo.com

Abstract. Neutrosophic set is a generalization of classical set, fuzzy set, and intuitionistic fuzzy set by employing a degree of truth (T), a degree of indeterminacy (I), and a degree of falsehood (F) associated with an element of the dataset. One of the most essential problems is studying set-theoretic operators in order to be applied to practical applications. Developing Matlab toolboxes for computing the operational matrices in neutrosophic environments is essential to gain more widely-used of neutrosophic algorithms. In this paper, we propose some computing procedures in Matlab for neutrosophic operational matrices, especially i) computing the single-valued neutrosophic matrix; ii) determining complement of a single-valued neutrosophic matrix; iii) computing max-min-min and min-max-max of two single-valued neutrosophic matrices; v) computing power of a single-valued neutrosophic matrix; vi) computing additional operation and subtraction of two single-valued neutrosophic matrices; and ix) computing transpose of a single-valued neutrosophic matrix. Numerical examples are given to illustrate their applicability.

Keywords: Matlab toolbox; Neutrosophic set; Single valued neutrosophic matrices; Set-theoretic operators

1 Introduction

There are many evidences in complex systems that an event or phenomenon cannot be modeled by a classical set [11,18]. For instance, the Schrödinger's Cat Theory says that the quantum state of a photoncan basically be in more than one place in the same time, which means that an element (quantum state) belongs and does not belong to a set (one place) inthe same time; or an element (quantum state) belongs to two different sets (two different places)in the same time [24]. Again, it is hard to judge the truth-value of a metaphor, or of an ambiguous statement, or of a social phenomenon which is positive from a standpoint and negative from another standpoint [24]. The classical mathematics does not practice any kind ofuncertainty in its tools, excluding possibly the case of probability, where it can handle a particular kind of uncertainty called randomness [11]. Therefore new techniques and modification of classical tools arerequired to model such uncertain phenomena [9]. Neutrosophic set (NS) [33] is a generalization of classical set, fuzzy set, and intuitionistic fuzzy set by employing a degree of truth (T), a degree of indeterminacy (I), and a degree of falsehood (F) associated with an element of the dataset proposed in 1998 by Smarandache. It has been successfully applied to many fields such as control theory[1], databases [4,5], medical diagnosis [7], decision making [23],topology [27]and graph theory [12-21].

NS has many advantages over other preceding sets. Specifically, triangular fuzzy numbers (TFNs) and neutrosophic numbers (NNs) are both generalizations of fuzzy numbers that are each characterized by three components [33]. TFNs and NNs have been widely used to represent uncertain and vague information in various areas such as engineering, medicine, communication science and decision science. However, NNs are far more accurate and convenient to be used to represent the uncertainty and hesitancy that exists in information, as compared to TFNs. NNs are characterized by three components, each of which clearly represents the degree of truth membership, indeterminacy membership and falsity membership of a NN with respect to an attribute. Therefore, we are able to tell the belongingness of the NN to the set of attributes that are being studied, by just looking at its structure. This provides a clear, concise and comprehensive method of representation of the different components of the membership of the number. This is in contrast to the structure of the TFN which only provides us with the maximum, minimum and initial values of the TFN, all of which can only tell us the path of the TFN, but does not tell us anything about the degree of non-belongingness of the TFN with respect to the set of attributes that are being studied. Furthermore, the structure of the TFN is not able to capture the hesitancy that naturally exists within the user in the process of assigning membership values. These reasons clearly show the advantages of NNs compared to TFNs.

One of the most essential problems in NS is studying set-theoretic operators (or operational matrices) in order to be applied to practical applications. Smarandache [33] and Wang et al.[41]proposed the concept of single-valued neutrosophic set and provided its set-theoretic operations and properties. Broumi and Smarandache [10] proposed some operations on interval neutrosophic sets (INSs) and studied their properties. Ye [43] defined the similarity measures between INSs on the basis of the hamming and Euclidean distances. Some set theoretic operations such as union, intersection and complement have also been proposed by Wang et al. [42].Broumi and Smarandache [8] also defined the correlation coefficient of interval neutrosophic set.Liu and Tang [26] presented some new operational laws for interval neutrosophic sets and studied their properties. More recent works on operational law and applications can be retrieved in [9, 24-26, 34, 44-45,47-50].

In practical point of view, developing Matlab toolboxes for computing the operational matrices in neutrosophic environments is essential to gain more widely-used of neutrosophic algorithms and methods. Zahariev [46] presented a new software package for fuzzy calculus in MATLAB environment whose main feature is solving fuzzy linear systems of equations and inequalities in fuzzy algebra. Peeva and Kyosev[30] developed a library for fuzzy relational calculus over the fuzzy algebra([0,1], max,min). The library includes various operations and compositions with fuzzy relation and intuitionistic fuzzy solving direct and inverse problem. Recently, Mumtaz et al. [3] implemented some functions in MATLAB for computing algebraic neutrosophic measures in medical diagnosis. Ashbacher [6] analyzed and developed some computing procedures for neutrosophic operations.Albeanu [2] described some neutrosophic computational models in order to identify a set of requirement for software implementation. Salama et al. [32] developed an Excel package for calculating neutrosophic data and analyzed them. Karunambigai and Kalaivani [22] developed a MATLAB program for computing power of an intuitionistic fuzzy matrix, strength of connectedness and index matrix of intuitionistic fuzzy graphs with suitable examples.

However, the existing Fuzzy Toolboxes in MATLAB does not propose options to evaluate the operations in neutrosophic environments. Thus, in this paper, we propose some computing procedures in Matlab for neutrosophic operational matrices, especially i) computing the single-valued neutrosophic matrix; ii) determining complement of a single-valued neutrosophic matrix; iii) computing max-min-min of two single-valued neutrosophic matrices; iv) computing min-max-max of two single-valued neutrosophic matrices; v) computing power of a single-valued neutrosophic matrix; vi) computing additional operation of two single-valued neutrosophic matrices; vii) computing subtraction of two single-valued neutrosophic matrices; and viii) computing transpose of a single-valued neutrosophic matrix. In order to illustrate their applicability, numerical examples are given and discussed.

The rest of this paper is organized as follows. Section 2 recalls some basic concepts of Neutrosophic Set. Section 3 presents the computing procedures in Matlab. Section 4 describes the numerical examples. Section 5 delineates conclusions and further studies of this research.

2 Fundamental and Basic Concepts

Definition 1[31]. ***Neutrosophic Set(NS)***

Let X be a space of points and let $x \in X$. A neutrosophic set $\overline{S}$ in X is characterized by a truth membership function $T_{\overline{S}}$, an indeterminacy membership function $I_{\overline{S}}$, and a falsehood membership function $F_{\overline{S}}$. $T_{\overline{S}}$, $I_{\overline{S}}$ and $F_{\overline{S}}$ are real standard or non-standard subsets of $]0^-,1^+[$. The neutrosophic set can be represented as

$$\overline{S} = \left\{ \left(x, T_{\overline{S}}(x), I_{\overline{S}}(x), F_{\overline{S}}(x) \right) : x \in X \right\}$$

The sum of $T_{\bar{S}}(x), I_{\bar{S}}(x)$ and $F_{\bar{S}}(x)$ is $0^{-} \leq T_{\bar{S}}(x)+I_{\bar{S}}(x)+F_{\bar{S}}(x) \leq 3^{+}$.

To use neutrosophic set in the real life applications such as engineering and scientific problems, it is necessary to consider the interval $[0,1]$ instead of $]0^{-},1^{+}[$ for technical applications.

Definition 2 [31]. Let $\tilde{A}_1=(T_1,I_1,F_1)$ and $\tilde{A}_2=(T_2,I_2,F_2)$ betwo single-valued neutrosophicnumbers. Then, the operations for NNs are defined as below:

(i) $\tilde{A}_1 \oplus \tilde{A}_2=(T_1+T_2-T_1T_2, I_1I_2, F_1F_2)$

(ii) $\tilde{A}_1 \otimes \tilde{A}_2=(T_1T_2, I_1+I_2-I_1I_2, F_1+F_2-F_1F_2)$

(iii) $\lambda\tilde{A} =(1-(1-T_1)^{\lambda}), I_1^{\lambda}, F_1^{\lambda})$

(iv) $\tilde{A}_1^{\lambda}=(T_1^{\lambda}, 1-(1-I_1)^{\lambda}, 1-(1-F_1)^{\lambda})$ where $\lambda>0$

Definition3[31]. Let $\tilde{A}_1=(T_1,I_1,F_1)$ be a single-valued neutrosophic number. Then, the score function $s(\tilde{A}_1)$, the accuracy function $a(\tilde{A}_1)$ and the certainty function $c(\tilde{A}_1)$ of SVNN are defined as follows:

(i) $s(\tilde{A}_1)=\dfrac{2+T_1-I_1-F_1}{3}$

(ii) $a(\tilde{A}_1)=T_1-F_1$

(iii) $a(\tilde{A}_1)=T_1$

Definition 4[31]. Let $\tilde{A}_1=(T_1,I_1,F_1)$ and $\tilde{A}_2=(T_2,I_2,F_2)$ betwo single-valued neutrosophic numbers then

(i) $\tilde{A}_1 \prec \tilde{A}_2$ if $s(\tilde{A}_1) \prec s(\tilde{A}_2)$

(ii) $\tilde{A}_1 \succ \tilde{A}_2$ if $s(\tilde{A}_1) \succ s(\tilde{A}_2)$

(iii) $\tilde{A}_1 = \tilde{A}_2$ if $s(\tilde{A}_1) = s(\tilde{A}_2)$

Definition 5 [31]. The unit 0_n is defined by one of the four types:

(0_1) Type 1. $0_n=\{<x,(0,0,1)>: x \in X\}$

(0_2) Type 2. $0_n=\{<x,(0,1,1)>: x \in X\}$

(0_3) Type 3. $0_n=\{<x,(0,1,0)>: x \in X\}$

(0_4) Type 4. $0_n=\{<x,(0,0,0)>: x \in X\}$

Definition 6 [31]. The unit 1_n is defined by one of the four types:

(1_1) Type 1. $1_n=\{<x,(1,0,0)>: x \in X\}$

(1_2) Type 2. $1_n=\{<x,(1,0,1)>: x \in X\}$

(1_3) Type 3. $1_n=\{<x,(1,1,0)>: x \in X\}$

(1_4) **Type 4.** $1_n=\{<x,(1,1,1)>: x \in X\}$

III. Computing procedures for set-theoretic operations

For the sake of brevity, we use the following notations to denote the following types of matrices:

- a.m: Membership matrix.
- a.i: Indeterminacy membership matrix.
- a.n: Non-membership matrix.

3.1.Computing the single-valued neutrosophic matrix

The procedure is described as follows.

```
Function nm_out=nm(varargin); %single valued neutrosophic matrix class con-structor.

%A = nm(Am,Ai,An) creates a single val-ued neutrosophic matrix

% with membership degrees from matrix Am

% indeterminate membership degrees from matrix Ai

%   and non-membership degrees from Ma-trix An.

% If the new matrix is not neutrosophic i.e. Am(i,j)+Ai(i,j+An(i,j)>3

% appears warning message, but the new object will be constructed.

If

length(varargin)==3

Am = varargin{1}; % Cell array indexing

Ai = varargin{2};

An = varargin{3};

end

nm_.m=Am;

nm_.i=Ai;

nm_.n=An;

nm_out=class(nm_,'im');

if ~checknm(nm_out)
```

```
disp('Warning! The created new object is NOT a Single valued neutrosophic matrix')

end
```

3.2. Determining complement of a single-valued neutrosophic matrix

In the literature, there are two definitions of complement of neutrosophic sets. They are implemented in this extended software package. To obtain the complement of a type 1 and type 2 of a single-valued neutrosophic matrix, simple call of the function named “complement1.m” or “complement2.m”.

```
Function At=complement1(A);

% complement of type1 single valued neutrosophic matrix A

% "A" have to be single valued neutrosophic  matrix - "nm" object:

a.m=A.n;

a.i=A.i;

a.n=A.m;

At=nm(a.m,a.i,a.n);
```

```
Function At=complement2(A);

% complement of type2 single valued neutrosophic matrix A

% "A" have to be single valued neutrosophic  matrix - "nm" object:

a.m=1-A.m;

a.i=1-A.i;

a.n=1-A.n;

At=nm(a.m,a.i,a.n);
```

3.3. Computing max-min-min of two single-valued neutrosophic matrices

To obtain the max-min min of two single-valued neutrosophic matrices, simple call of the following function named “maxminmin.m” is needed:

```
Function At=maxminmin(A,B);
```

```
% maxminmin of two single valued neutrosophic matrix A and B

% "A" have to be single valued neutrosophic  matrix - "nm" object:

%"B" have to be single valued neutrosophic  matrix - "nm" object:

a.m=max(A.m,B.m);

a.i=min(A.i,B.i);

a.n=min(A.n,B.n);

At=nm(a.m,a.i,a.n);
```

3.4. Computing min-max-max of two single-valued neutrosophic matrices

To obtain the min-max max of two single-valued neutrosophic matrices, simple call of the following function named “minmaxmax.m” is needed:

```
Function At=minmaxmax(A,B);

% minmaxmax of two single valued neutrosophic matrix A and B

% "A" have to be single valued neutrosophic  matrix - "nm" object:

%"B" have to be single valued neutrosophic  matrix - "nm" object:

a.m=min(A.m,B.m);

a.i=max(A.i,B.i);

a.n=max(A.n,B.n);

At=nm(a.m,a.i,a.n);
```

3.5. Computing power of a single-valued neutrosophic matrix

To obtain the power of single-valued neutrosophic matrix, simple call of the following function named “power.m” is needed:

```
Function At=power(A,k);
```

```
%power of single valued neutrosophic
matrix A

% "A" have to be single valued neutro-
sophic  matrix - "nm" object:

for i =2 :k

a.m=(A.m).^k;

a.i=(A.i).^k;

a.m=(A.m).^k;

At=nm(a.m,a.i,a.m);

end
```

3.6. Computing additional operation of two single-valued neutrosophic matrices

To obtain the additional operation of two single-valued neutrosophic soft matrices, simple call of the following function named "softadd.m" is needed:

```
Function At=softadd(A,B);

% addition operations of two single
valued neutrosophic soft  matrix A and
B

% "A" have to be single valued neutro-
sophic  matrix - "nm" object:

a.m=max(A.m,B.m);

a.i=(A.i+B.i)/2;

a.n=min(A.n,B.n);

At=nm(a.m,a.i,a.n);
```

3.7. Computing subtraction of two single-valued neutrosophic matrices

To obtain the subtraction operation of two single-valued neutrosophic soft matrices, simple call of the following function named "softsub.m" is needed:

```
Function At=softsub(A,B);

% function st=disp_intui(A);
```

```
% substraction operations of two single
valued neutrosophic soft  matrix A and
B
```

```
% "A" have to be single valued neutro-
sophic  matrix - "nm" object:

a.m=min(A.m,B.n);

a.i=(A.i+B.i)/2;

a.n=max(A.n,B.m);

At=nm(a.m,a.i,a.n);
```

3.8. Computing transpose of a single-valued neutrosophic matrix

To obtain the power of single-valued neutrosophic matrix, simple call of the following function named "transpose.m" is needed:

```
Function At=transpose(A);

% transpose Single valued neutrosophic
matrix A

% "A" have to be single valued neutro-
sophic matrix - "nm" object:

a.m=(A.m)';

a.i=(A.i)';

a.n=(A.n)';

At=nm(a.m,a.i,a.n);
```

VI. Numerical Examples

In this section, we give several examples to illustrate solving some operations of the single-valued neutrosophic matrices.

Example 1. Input a neutrosophic matrix by a given structure in the toolbox.

```
%Enter the degree of membership of A in the variable a.m

>>a.m = [0 .5 .5 ;.3 0 .1 ;.3 .1 0 ; .1 .2 .1];
```

```
%Enter the degree of indterminate-membership of A in the variable a.i

>>a.i = [1 .3 .2;.3 1 .4 ; .1 .5 1;.1 .5 .7 ];

%Enter the degree of non-membership of A in the variable a.n
```

>>a.n = [0 .2 .3 ;.4 0 .5 ;.6 .1 0 ; .3 .5 .5];

%Enter the degree of membership of Bin the variable b.m

>>b.m = [0 .4 .2 ;.4 0 .1 ;.3 .2 0 ; .3 .3 .1];

%Enter the degree of indterminate-membership of Bin the variable b.i

>>b.i = [0 .5 .4 ;.3 0 .5 ;.8 .1 0 ; .3 .2 .4];

%Enter the degree of non-membership of Bin the variable b.n

>>b.n = [0 .5 .4 ;.3 0 .5 ;.8 .1 0 ; .3 .2 .4];

>>**A=nm(a.m,a.i,a.n)**

%This command returns a matrix A with degree of membership a.m,degree of indeterminate-membership a.i and degree of non-membership a.n%

A =

<0.00, 1.00, 0.00><0.50, 0.30, 0.20><0.50, 0.20, 0.30>

<0.30, 0.30, 0.40><0.00, 1.00, 0.00><0.10, 0.40, 0.50>

<0.30, 0.10, 0.60><0.10, 0.50, 0.10><0.00, 1.00, 0.00>

<0.10, 0.10, 0.30><0.20, 0.50, 0.50><0.10, 0.70, 0.50>

>>**B=nm(b.m,b.i,b.n)**

%This command returns a N matrix B with degree of membership b.m, degree of indeterminate-membership b.i and degree of non- membership b.n %

B =

<0.00, 0.00, 0.10><0.40, 0.50, 0.40><0.20, 0.40, 0.30>

<0.40, 0.30, 0.30><0.00, 0.00, 1.00><0.10, 0.50, 0.40>

<0.30, 0.80, 0.10><0.20, 0.10, 0.60><0.00, 0.00, 1.00>

<0.30, 0.30, 0.10><0.30, 0.20, 0.30><0.10, 0.40, 0.60>

Example 2. Evaluate the complement type 1 of the following matrix:

A=

$$\begin{pmatrix} <0.00, 1.00, 0.00> & <0.20, 0.30, 0.50> & <0.30, 0.20, 0.50> \\ <0.40, 0.30, 0.30> & <0.00, 1.00, 0.00> & <0.50, 0.40, 0.10> \\ <0.60, 0.10, 0.30> & <0.10, 0.50, 0.10> & <0.00, 1.00, 0.00> \\ <0.30, 0.10, 0.10> & <0.50, 0.50, 0.20> & <0.50, 0.70, 0.10> \end{pmatrix}$$

>>**complement1(A)**

% This command returns the complement1of N matrices A .

ans =

<0.00, 1.00, 0.00><0.20, 0.30, 0.50><0.30, 0.20, 0.50>

<0.40, 0.30, 0.30><0.00, 1.00, 0.00><0.50, 0.40, 0.10>

<0.60, 0.10, 0.30><0.10, 0.50, 0.10><0.00, 1.00, 0.00>

<0.30, 0.10, 0.10><0.50, 0.50, 0.20><0.50, 0.70, 0.10>

Example 3. Evaluate the complement type 2 of matrix above

>>**complement2(A)**

% This command returns the complement2

ans =

<1.00, 0.00, 1.00><0.50, 0.70, 0.80><0.50, 0.80, 0.70>

<0.70, 0.70, 0.60><1.00, 0.00, 1.00><0.90, 0.60, 0.50>

<0.70, 0.90, 0.40><0.90, 0.50, 0.90><1.00, 0.00, 1.00>

<0.90, 0.90, 0.70><0.80, 0.50, 0.50><0.90, 0.30, 0.50>

Example 4. Evaluate the min-max-max and max-min-min of these matrices:

A=

$$\begin{pmatrix} <0.00, 1.00, 0.00> & <0.20, 0.30, 0.50> & <0.30, 0.20, 0.50> \\ <0.40, 0.30, 0.30> & <0.00, 1.00, 0.00> & <0.50, 0.40, 0.10> \\ <0.60, 0.10, 0.30> & <0.10, 0.50, 0.10> & <0.00, 1.00, 0.00> \\ <0.30, 0.10, 0.10> & <0.50, 0.50, 0.20> & <0.50, 0.70, 0.10> \end{pmatrix}$$

B=

$$\begin{pmatrix} <0.00, 0.00, 0.10> & <0.40, 0.50, 0.40> & <0.20, 0.40, 0.30> \\ <0.40, 0.30, 0.30> & <0.00, 0.00, 1.00> & <0.10, 0.50, 0.40> \\ <0.30, 0.80, 0.10> & <0.20, 0.10, 0.60> & <0.00, 0.00, 1.00> \\ <0.30, 0.30, 0.10> & <0.30, 0.20, 0.30> & <0.10, 0.40, 0.60> \end{pmatrix}$$

>>**minmaxmax(A,B)**

% This command returns the min-max-max

ans =

<0.00, 1.00, 0.10><0.40, 0.50, 0.40><0.20, 0.40, 0.30>

<0.30, 0.30, 0.40><0.00, 1.00, 1.00><0.10, 0.50, 0.50>

<0.30, 0.80, 0.60><0.10, 0.50, 0.60><0.00, 1.00, 1.00>

<0.10, 0.30, 0.30><0.20, 0.50, 0.50><0.10, 0.70, 0.60>

>>**maxminmin(A,B)**

% This command returns the max-min-min

ans =

<0.00, 0.00, 0.00><0.50, 0.30, 0.20><0.50, 0.20, 0.30>

<0.40, 0.30, 0.30><0.00, 0.00, 0.00><0.10, 0.40, 0.40>

<0.30, 0.10, 0.10><0.20, 0.10, 0.10><0.00, 0.00, 0.00>

<0.30, 0.10, 0.10><0.30, 0.20, 0.30><0.10, 0.40, 0.50>

Example 5. Evaluate the additional and subtraction operations of the matrices in Example

>>**softadd(A,B)**

% This command returns the addition of two neutrosophic matrices A and B

ans =

<0.00, 0.50, 0.00><0.50, 0.40, 0.20><0.50, 0.30, 0.30>

<0.40, 0.30, 0.30><0.00, 0.50, 0.00><0.10, 0.45, 0.40>

<0.30, 0.45, 0.10><0.20, 0.30, 0.10><0.00, 0.50, 0.00>

<0.30, 0.20, 0.10><0.30, 0.35, 0.30><0.10, 0.55, 0.50>

>>**softsub(A,B)**

% This command returns the substraction of two neutrosophic matrices A and B

ans =

<0.00, 0.50, 0.00><0.40, 0.40, 0.40><0.30, 0.30, 0.30>

<0.30, 0.30, 0.40><0.00, 0.50, 0.00><0.10, 0.45, 0.50>

<0.10, 0.45, 0.60><0.10, 0.30, 0.20><0.00, 0.50, 0.00>

<0.10, 0.20, 0.30><0.20, 0.35, 0.50><0.10, 0.55, 0.50>

Example 6. Return the transpose of the matrix below:

A=

$$\begin{pmatrix} <0.00,1.00,0.00> & <0.20,0.30,0.50> & <0.30,0.20,0.50> \\ <0.40,0.30,0.30> & <0.00,1.00,0.00> & <0.50,0.40,0.10> \\ <0.60,0.10,0.30> & <0.10,0.50,0.10> & <0.00,1.00,0.00> \\ <0.30,0.10,0.10> & <0.50,0.50,0.20> & <0.50,0.70,0.10> \end{pmatrix}$$

>>**transpose(A)**

% This command returns the power of matrix A .

ans =

<0.00, 1.00, 0.00><0.30, 0.30, 0.40><0.30, 0.10, 0.60><0.10, 0.10, 0.30>

<0.50, 0.30, 0.20><0.00, 1.00, 0.00><0.10, 0.50, 0.10><0.20, 0.50, 0.50>

<0.50, 0.20, 0.30><0.10, 0.40, 0.50><0.00, 1.00, 0.00><0.10, 0.70, 0.50

Note: The functions described above enables us to compute the operations on fuzzy matrices and intuitionistic fuzzy matrices

Fuzzy matrix:

$$A_{FS}=\begin{pmatrix} <0.5,0,0> & <0.2,0,0> & <0.4,0,0> \\ <0.3,0,0> & <0.3,0,0> & <0.8,0,0> \\ <0.4,0,0> & <0.6,0,0> & <1,0,0> \\ <0.6,0,0> & <0.5,0,0> & <0.2,0,0> \end{pmatrix}$$

Intuitionisticfuzzy matrix:

$$A_{IFS}=$$

$$\begin{pmatrix} <0.5,0,0.2> & <0.2,0,0.1> & <0.4,0,0.4> \\ <0.3,0,0.2> & <0.3,0,0.4> & <0.8,0,0.3> \\ <0.4,0,0.3> & <0.6,0,0.8> & <0.3,0,0.5> \\ <0.6,0,0.5> & <0.5,0,0.9> & <0.2,0,0.2> \end{pmatrix}$$

CONCLUSION

This paper aimed to propose some new computing procedures in Matlab forset-theoretic operations in the neutrosophic set. The toolbox consists of 8 operations including forming the single-valued neutrosophic matrix, computing complement, power and transpose of a single-valued neutrosophic matrix, calculating the max-min-min, min-max-max, additional and subtraction operations of two single-valued neutrosophic matrices.The neutrosophic software package gives the ability for easy calculation of operations in associated problems.The high level of userfriendliness of the programs and functions also makes it very convenient to be used, and gives it a higher level of computational efficiency compared to the existing software packages for fuzzy calculus. We hope that they will support researches who are working in the field of neutrosophic decision making and neutrosophic graph theory.

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NSS University of New Mexico

Selection of Transportation Companies and Their Mode of Transportation for Interval Valued Data

Dalbinder Kour[1] and Kajla Basu[2]

[1] Department of Mathematics, Michael Madhusudan Memorial College, Kabi Guru Sarani, City Centre,Durgapur-713216, West Bengal,India. E-mail: dalbinder2011@gmail.com

[2] National Institute of Technology, Mahatma Gandhi Avenue,Durgapur-713209, West Bengal,India. E-mail: kajla.basu@gmail.com

Abstract. The paper presents selection of transportation companies and their mode of transportation for interval valued neutrosophic data .The paper focuses on the application of distance measures to select mode of transportation for transportation companies. The paper also presents the application of multi-criteria decision making method using weighted correlation coefficient and extended TOPSIS for transportation companies. The multi-criteria decision making problem (MCDM) is taken in which there are different criteria and different modes. The selection is done among different modes and then it is done among four transportation companies in which data is taken as Interval Valued Neutrosophic Set (IVNS). The first method is concerned with a multi-criteria fuzzy decision making method based on weighted correlation coefficients under interval valued neutrosophic fuzzy environment. The second method utilizes the extended TOPSIS method to solve the problem with data as IVNS and given attribute weights. The ranking is done and the most appropriate transportation company with the most appropriate mode is selected. The methods are illustrated with numerical examples.

Keywords: multi-criteria decision making problem ; Interval Valued Neutrosophic Set (IVNS); weighted correlation coefficients; TOPSIS; positive ideal solution (PIS).

1 Introduction

Multi criteria decision making (MCDM) problems are focussed at selecting the best alternative among different available alternatives with different criteria. There are different classical methods for different MCDM problems. In real life due to uncertainties and lack of time and knowledge decision makers' preferences are provided as fuzzy data. Fuzzy set theory was introduced by Zadeh [27]. Intuitionistic fuzzy set (IFS) was introduced as a generalization of fuzzy set (FS). IFS was introduced by Atanassov [23] including two membership functions - membership (or called truth-membership) ($T(x)$)and non-membership (or called falsity-membership) ($F(x)$), and satisfying the conditions $T(x), F(x)\epsilon[0,1]$ and $0\leq T(x) + F(x) \leq 1$.

Atanassov & Gargov [24] introduced the concept of interval-valued intuitionistic fuzzy sets (IVIFSs) as a further generalization of IFS. Atanassov [25] also defined some operational laws of IVIFSs. De et al. [39] applied the max-min-max composition to medical diagnosis via IFSs. By following their reasoning, Szmidt & Kacprzyk [6] applied the distance measures to IFSs in the medical diagnosis.

The concept of neutrosophic set was introduced as a generalization of crisp set, fuzzy set [27], IFS [23] by Smarandache ([7],[9]) .The Indeterminacy function (I) was added to the two available parameters: Truth (T) and Falsity (F) membership functions. In neutrosophic set, the indeterminacy is quantified explicitly and truth-membership, indeterminacy membership and false-membership are completely independent. In intuitionistic fuzzy sets, and the indeterminacy is 1-T (x)-F (x) i.e. hesitancy or unknown degree by default. In neutrosophy, the indeterminacy membership ($I_A(x)$)is introduced as a new subcomponent so as to include the degree to which the decision maker is not sure. This type of treatment of the problem was out of scope of IFSs. The single valued neutrosophic set (SVNS) was introduced for the first time by Wang et al. [15] in 1998. Wang et al. [15] introduced the concept of interval valued neutrosophic set (IVNS) and provided the set-theoretic operators and various properties of SVNS and IVNS. SVNS and IVNS present uncertainty,

imprecise, inconsistent and incomplete information existing in real world.
Bustince & Burillo[13] proposed the concept of correlation and correlation coefficient of IVIFSs along with their properties. They also introduced two decomposition theorems – one in terms of the correlation of interval valued fuzzy sets and entropy of IFS and the other theorem is in terms of correlation of IFSs. Luo et al.[44] proposed a multi-criteria fuzzy decision-making method based on weighted correlation coefficients under interval-valued intuitionistic fuzzy environment with known criterion weight information. Wang et al. [47] proposed an approach to MADM with incomplete attribute weight information where individual assessments are provided as IVIFSs. Elhassouny, and Smarandache[1] used simplified TOPSIS for neutrosophic MCDM problems. Bausys et al. [35] and Bausys et al. [36]) used COPRAS and VIKOR respectively to solve neutrosophic MCDM problems. Ye [20] proposed MADM method with completely unknown weight information. Based on the correlation coefficient studied by Gerstenkorn & Manko [42], Ye [18],[19]) of IVIFSs, Park et al. ([3],[17]) investigated the group decision making problems in which the information about attribute weights is partially known. Ye [20] developed the MCDM method using the correlation coefficient under single-valued neutrosophic environment. Ye [22] also developed an extended TOPSIS method for MADM based on single valued neutrosophic linguistic numbers. Entropy based grey relational analysis method was used for MADM under single valued neutrosophic assessments by Biswas et al. [30]. An MCDM method based on single-valued trapezoidal neutrosophic preference relations with complete weight information was applied by Liang, et al. [37]. Neutrosophic MADM problems with unknown weight information was solved by Biswas et al. [31]. Mondal and Pramanik [26] Pramanik et al. [41] investigated neutrosophic tangent similarity measure and hybrid vector similarity measures respectively and their application to MADM. Sahin [38] also observed cross-entropy measure on interval neutrosophic sets and its applications in MCDM. Xu et al. [5] extended TODIM method for single-valued neutrosophic MADM. Z. *Zhang* and C. *Wu* [51] also developed a novel method for single-valued *neutrosophic* MCDM with incomplete weight information.

The technique for order of preference by similarity to ideal solution (TOPSIS) is a well-known method for solving decision making problems proposed by Hwang & Yoon [2]. Lai et al. [46] applied the concept of TOPSIS on multiple objective decision making (MODM) problems. Abo- Sinha & Amer [28] extended TOPSIS method for solving multi-objective large-scale nonlinear programming problems. Opricovic & Tzeng [40] conducted a comparative analysis of TOPSIS and VIKOR. Many researchers (Chi & Liu [33], Jahanshaloo et al. [10], [11], Kour et al. [4] ; Wang & Lee[47], Opricovic & Tzeng [40] extended TOPSIS approach to fuzzy environment as a natural generalization of TOPSIS models. Chen & Tsao [43] extended the concept of TOPSIS to develop a method for solving MADM problems with interval-valued fuzzy data. Xu [49] developed some geometric aggregation operators, such as the interval-valued intuitionistic fuzzy geometric (IIFG) operator and interval-valued intuitionistic fuzzy weighted geometric (IIFWG) operator and applied them to multiple attribute group decision making (MAGDM) with interval-valued intuitionistic fuzzy information. Xu & Chen [50] and Wei & Wang[12] respectively developed some geometric aggregation operators, such as the interval-valued intuitionistic fuzzy ordered weighted geometric (IIFOWG) operator and interval-valued intuitionistic fuzzy hybrid geometric (IIFHG) operator and applied them to MAGDM with interval-valued intuitionistic fuzzy information. However, they used the IIFWG, IIFWOG and IIFHG operators in the situation where the information about attribute weights is completely known. Chi & Liu [33] extended TOPSIS to IVNS environment in which the attribute weights are unknown and the attribute values are presented in terms of IVNS.
Kulak & Kahraman [29] studied a transportation company selection problem using axiomatic design and analytic hierarchy process (AHP) with partially known weight information in fuzzy environment. Kour et al. [4] applied the two methods on multi-criteria fuzzy decision making problems with IVIFS - the first one using correlation coefficient with unknown weights and the second one using TOPSIS method with known weights for the selection of transportation companies. TOPSIS method for MADM under single-valued neutrosophic environment was applied by Biswas et al. [32].
The present paper introduces the relation between the different criteria and different modes of transportation to select mode using distance measures for transportation companies for interval valued neutrosophic data. The present paper also extended the application of multi-criteria fuzzy decision making method with IVNSs to selection of transportation companies with given weights. A transportation company selection problem is taken with four different transportation companies and the data for the different criteria ad modes are taken as IVNSs.
The application of distance measures is done to select the best mode of transportation for transportation companies for interval valued neutrosophic data after calculating the minimum distance between the transportation companies and the modes. Then the selection is done for the best

transportation company. The first method involves determining correlation coefficient between an alternative and the ideal alternative. The ranking is then done using this coefficient and the best alternative is selected. The second method focuses the extended TOPSIS method. The weighted collective interval valued neutrosophic decision matrix is constructed. Then the interval valued neutrosophic PIS and NIS are determined using a defined score function. The distance measures are used to calculate the relative closeness of each alternative to the interval valued neutrosophic PIS. The alternatives are ranked and the best one is selected.

No other authors till date have considered the concept of correlation coefficient for IVNSs. Further to find the PIS and NIS for TOPSIS, a new score function has been introduced. And both the methods have been applied to solve a new type of transportation company selection problem in which mode selection is also introduced which has not been done by any other author before.

2 Basic Concept

2.1 Neutrosophic Set

Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$ as by Smarandache [7].

$$A = \{x, T_A(x), I_A(x), F_A(x) |, x \in X\}$$

The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of]0^-, 1^+ [. That is $T_A(x)$: X ϵ] 0^-, 1^+ [, $I_A(x)$: X ϵ] 0^-, 1^+ [, and $F_A(x)$: X ϵ] 0^-, 1^+ [.

There is no restriction on the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$, so 0^- ≤sup $T_A(x)$ +sup $I_A(x)$ + sup $F_A(x)$ ≤3^+.

2.2 Complement of Neutrosophic set

The complement of a neutrosophic set A is denoted by c A) and is defined by Smarandache[7] as $T^C(x)$ = {1^+} – $T_A(x)$, $I^C(x)$ = {1^+} - $I_A(x)$, and $F^C(x)$= {1^+} – $F_A(x)$ for every x in X.

2.3 Subset of Neutrosophic set

A neutrosophic set A is contained in the other neutrosophic set B, A ⊆B if and only if inf $T_A(x)$ ≤ inf $T_B(x)$, sup $T_A(x)$ ≤ sup $T_B(x)$, inf $I_A(x)$ ≥ inf $I_B(x)$, sup $I_A(x)$ ≥ sup $I_B(x)$, inf $F_A(x)$ ≥ inf $F_B(x)$, and sup $F_A(x)$ ≥ sup $F_B(x)$ for every x in X (Smarandache[7]).

2.4 Single Valued Neutrosophic Set (SVNS)

A SVNS [15] A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$ for each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x)$ ϵ [0, 1].

When X is continuous, an SVNS A can be written as

$$A = \int_X \frac{\langle T_A(x), I_A(x), F_A(x)\rangle}{x}, x \in X$$

When X is discrete, an SVNS A can be written as

$$A = \sum_{i=1}^{n} \frac{\langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle}{x_i}, x_i \in X$$

2.5 Interval Valued Neutrosophic Set (IVNS)

Let X be a universe of discourse, with a generic element in X denoted by x. An interval neutrosophic set A in X is defined by Wang et al.[14]. as $A = \{x, T_A(x), I_A(x), F_A(x) |, x \in X\}$ where, $T_A(x), I_A(x), F_A(x)$ are the truth-membership function, indeterminacy-membership function, and the falsity membership function, respectively. For each point x in X, we have $T_A(x), I_A(x), F_A(x) \subseteq [0,1]$ and

$$0 \le \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \le 3$$

For convenience, we take an interval-valued neutrosophic set (IVNS), $\widetilde{A} = ([a,b],[c,d][e,f])$ where $[a,b],[c,d],[e,f] \subset [0,1], 0 \le b + d + f \le 3$

2.6 Algebraic Rules of IVNS (Wang et al.[14])

Let

$$\widetilde{A} = ([a_1, b_1],[c_1, d_1],[e_1, f_1])$$

and

$$\widetilde{B} = ([a_2, b_2],[c_2, d_2],[e_2, f_2])$$

be two IVNS,then

The complement of
$\tilde{A} = ([a_1, b_1], [c_1, d_1], [e_1, f_1])$
is given by

$$\tilde{A}^C = ([e_1, f_1], [1 - c_1, 1 - d_1], [a_1, b_1])$$

1. $\tilde{A} \oplus \tilde{B} = ([a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2], [c_1 c_2, d_1 d_2], [e_1 e_2, f_1 f_2])$
2. $\tilde{A} \otimes \tilde{B} = ([a_1 a_2, b_1 b_2], [c_1 + c_2 - c_1 c_2, d_1 + d_2 - d_1 d_2], [e_1 + e_2 - e_1 e_2, f_1 + f_2 - f_1 f_2])$
3. $n\tilde{A} = ([1 - (1 - a_1)^n, 1 - (1 - b_1)^n], [c_1^n, d_1^n], [e_1^n, f_1^n]), n > 0$

5. $\tilde{A}^n = ([a_1^n, b_1^n], [1 - (1 - c_1)^n, 1 - (1 - d_1)^n] [1 - (1 - e_1)^n, 1 - (1 - f_1)^n],), n > 0$

2.7 Score of IVNS

Let $R = (\tilde{r}_{ij})_{mn}$, where $\tilde{r}_{ij} = [a_{ij}, b_{ij}], [c_{ij}, d_{ij}], [e_{ij}, f_{ij}]$ the collective interval - valued neutrosophic decision matrix be. Then $S = (s_{ij})_{mn}$ is defined as the score matrix of $R = (\tilde{r}_{ij})_{mn}$, where

$$s_{ij} = s(\tilde{r}_{ij}) = \frac{1}{3}\left(2 + a_{ij} - c_{ij} - e_{ij} + b_{ij} - d_{ij} - f_{ij}\right), i = 1,2,....,n \tag{1}$$

And $s(\tilde{r}_{ij})$ is called the score of $\tilde{r}_{ij}$

Example2.7.1 Let

$\tilde{A} = ([0.3,0.4], [0.1,0.2], [0.5,0.7])$
$\tilde{B} = ([0.4,0.5], [0.2,0.3], [0.5,0.6])$ be two INVSs.

Then by Definition 2.7,

$$s(\tilde{A}_{ij}) = \frac{1}{3}\left(2 + 0.3 - 0.1 - 0.5 + 0.4 - 0.2 - 0.7)\right) = 0.4$$

$$s(\tilde{B}_{ij}) = \frac{1}{3}\left(2 + 0.4 - 0.2 - 0.5 + 0.5 - 0.3 - 0.6\right)$$
$$= 0.433$$

Hence, $s(\tilde{A}_{ij}) < s(\tilde{B}_{ij})$

Properties2.7.2 Let $\tilde{r}_{ij} = [a_{ij}, b_{ij}], [c_{ij}, d_{ij}], [e_{ij}, f_{ij}]$ be an INVS. Then the score of $\tilde{r}_{ij}$ has some properties as follows:

(i) $s(\tilde{r}_{ij}) = 0$ if and only if $a_{ij} + b_{ij} = c_{ij} + d_{ij} + e_{ij} + f_{ij} - 2$.

(ii) $s(\tilde{r}_{ij}) = 1$ if and only if $a_{ij} + b_{ij} = c_{ij} + d_{ij} + e_{ij} + f_{ij} + 1$.

(iii) $s(\tilde{r}_{ij}) = -1$ if and only if $a_{ij} + b_{ij} = c_{ij} + d_{ij} + e_{ij} + f_{ij} - 1$.

2.8 Distance between two IVNS

Let $X = ([a_{i1}, b_{i1}], [c_{i1}, d_{i1}], [e_{i1} f_{i1}])$ and $Y = ([a_{i2}, b_{i2}], [c_{i2}, d_{i2}], [e_{i2} f_{i2}])$ be two IVNSs. The normalized Hamming distance between X and Y is defined by Chi & Liu [33] as

$$d_H(X,Y) = \frac{1}{6n}\sum_{i=1}^{n}\left(| a_{i1} - a_{i2} | + | b_{i1} - b_{i2} | + | c_{i1} - c_{i2} | + | d_{i1} - d_{i2} | + [e_{i1} - e_{i2} | + | f_{i1} - f_{i2} |\right] \tag{2}$$

3. Problem description and methodology

3.1Problem Description

The present paper deals with the selection of transportation company and their mode of transportation in interval valued neutrosophic environment. At first the neutrosophic relation Q from a set of different transportation companies T to a set of different criteria C like transportation cost, defective rate, tardiness rate, flexibility, etc. is considered. Then it follows the second relation R from the set of different criteria C to a set of different mode M of transportation like roadways, railways, waterways and airways. The composition of the two neutrosophic relation Q and R is the relation S from the set of transportation companies to the set of different modes which gives the best mode of transportation for each of the transportation companies. Finally, the best transportation company is to be selected among the given different companies. The problem can be solved by different methods available in this context taking into account the different criteria. The present paper focuses on two methods. The first one involves weighted correlation coefficient method. The second one involves extended TOPSIS method. The different weights are given for different criteria.

3.2 Methodology

A. Application of normalized hamming distance for interval valued neutrosophic set

Let there be a neutrosophic relation X: A_i ->B_j and Y: B_j ->C_k. Using the distance between two IVNSs in Definition 2.8 the normalized Hamming distance for all the elements of the A_i from the C_k is equal to

$$d_H(A_i,C_k)=\frac{1}{6n}\sum_{j=1}^{n}\begin{pmatrix}|\mu_{jL}(A_i)-\mu_{jL}(C_k)|+|\mu_{jU}(A_i)-\mu_{jU}(C_k)|+\\ |\upsilon_{jL}(A_i)-\upsilon_{jL}(C_k)|+|\upsilon_{jU}(A_i)-\upsilon_{jU}(C_k)|+\\ |r_{Lj}(A_i)-r_{jL}(C_k)|+|r_{Uj}(A_i)-r_{jU}(C_k)|\end{pmatrix} \quad (3)$$

B. Multi-criteria decision making method based on weighted correlation coefficients in interval valued neutrosophic environment

Let $A=\{A_1,A_2,A_3,....,A_m\}$ be a set of alternatives and let $C=\{C_1,C_2,C_3,........,C_n\}$ be a set of criteria. An alternative A_i is represented by the following IVNS: $A_i=\{(C_j,[\mu_{A_iL}(C_j),\mu_{A_iU}(C_j)],[\upsilon_{A_iL}(C_j),\upsilon_{A_iU}(C_j)]$ $[r_{A_iL}(C_j),r_{A_iU}(C_j)]:C_j\in C\}$ where $0\le\mu_{A_iU}(C_j)+\upsilon_{A_iU}(C_j)\le 1$ $\mu_{A_iL}(C_j)\ge 0$ $\upsilon_{A_iL}(C_j)\ge 0$ j = 1; 2;... ; n, and i= 1,2,...,m.

The IVNS that consists of Intervals $\mu_A(C_j)=[a_{ij},b_{ij}]$ $\upsilon_{A_i}(C_j)=[c_{ij},d_{ij}]$ $r_{A_i}(C_j)=[e_{ij},f_{ij}]$ for $C_j\in C$ is denoted by $\alpha_{ij}=([a_{ij},b_{ij}],[c_{ij},d_{ij}],[e_{ij},f_{ij}])$ for convenience.

We can express an interval-valued neutrosophic decision matrix $D=(\alpha_{ij})_{mn}$.

Ye ([18],[19]) established a model for weighted correlation coefficient between each alternative and the ideal alternative for single valued neutrosophic sets (SVNSs) using known weights of the criterion. Though the ideal alternative does not exist in real world, it does provide a useful theoretical construct against which to evaluate alternatives. Ye ([18],[19]) defined the ideal alternative for SVNSs as $\alpha^*=(a^*_{ij},b^*_{ij},c^*_{ij})=(1,0,0)$.

If the information about weight w_j of the criterion Cj (j= 1,2,...,n) is completely known, for determining the criterion weight from the decision matrix D we can establish an exact model for the weighted correlation coefficient between an alternative Ai and the ideal alternative A* represented by the IVNS as in Equation (4).We define the ideal alternative A^* as the IVNS

$$\alpha^*=([a^*_{ij},b^*_{ij}],[c^*_{ij},d^*_{ij}],[e^*_{ij},f^*_{ij}])=([1,1],[0,0],[0,0])$$

.

$$W_i(A_i,A^*)=\frac{\sum_{j=1}^{n}w_j[a_{ij}.a^*_j+b_{ij}.b^*_j+c_{ij}.c^*_j+d_{ij}.d^*_j+e_{ij}.e^*_j+f_{ij}.f^*_j]}{\sqrt{\sum_{j=1}^{n}w_j[a_{ij}{}^2+b_{ij}{}^2+c_{ij}{}^2+d_{ij}{}^2+e_{ij}{}^2+f_{ij}{}^2]}\sqrt{\sum_{j=1}^{n}w_j[.a^{*2}_j+b^{*2}_j+.c^{*2}_j+d^{*2}_j+e^{*2}_j+f^{*2}_j]}}$$

(4)

Then the bigger the value of the weighted correlation coefficient W_i is, the better the alternative A_i is. Therefore all the alternatives can be ranked according to the value of the weighted correlation coefficients so that the best alternative can be selected.

C. TOPSIS method to solve the multi-attribute decision making problem with the given information about attribute weights in interval valued neutrosophic environment

In the situations where the information about weights is completely known, that is, the weights w_i = (w_1, w_2, ..., w_m)^T of the c_j (j = 1,2,...,n) can be completely determined in advance, then we can construct the weighted collective interval-valued neutrosophic decision matrix $R^*=(\tilde{r}^*_{ij})_{mn}$ where

$$\tilde{r}^*_{ij}=w_i\tilde{r}_{ij}=\{[1-(1-a_{ij})^{w_i},1-(1-b_{ij})^{w_i}],[c^{w_i}_{ij},d^{w_i}_{ij}],[e^{w_i}_{ij},f^{w_i}_{ij}]\} \quad (5)$$

is the weighted IVNS, i = 1,2,...,m; j = 1,2,...,n, and w_i is weight of the attribute u_i such that $w_i\ge 0$ and $\sum_{i=1}^{m}w_i=1$.

Now, we denote by

$$\tilde{r}^*_{ij}=([a^*_{ij},b^*_{ij}],[c^*_{ij},d^*_{ij}],[e^*_{ij},f^*_{ij}] \text{ where i= 1; 2;... ;m; j= 1; 2;... ; n} \quad (6)$$

Let J_1 be a collection of benefit attributes (i.e., the larger u_i, the greater preference) and J_2 be a collection of cost attributes (i.e., the smaller u_i , the greater preference). The interval-valued neutrosophic PIS, denoted by A^*, and the interval-valued neutrosophic NIS, denoted by A^-, are defined as follows:

$$A^* = \{\{c_j, (\max_i \tilde{r}_{ij}^* : j \in J_1), (\min_i \tilde{r}_{ij}^* : j \in J_2)\} : j = 1,2,...., n\}^T = (\tilde{r}_1^+, \tilde{r}_2^+, ..., \tilde{r}_n^+)^T \quad (7)$$

$$A^- = \{\{c_j, (\min_i \tilde{r}_{ij}^* : i \in J_1), (\max_i \tilde{r}_{ij}^* : i \in J_2)\} : j = 1,2,...., n\}^T = (\tilde{r}_1^-, \tilde{r}_2^-, ..., \tilde{r}_n^-)^T \quad (8)$$

where $\tilde{r}_i^+ = ([a_i^+, b_i^+], [c_i^+, d_i^+][e_i^+, f_i^+]$ and $\tilde{r}_i^- = ([a_i^-, b_i^-], [c_i^-, d_i^-][e_i^+, f_i^+]$,i=1,2,..,m

Burillo & Bustince [13] method has been extended to find the separation measures for interval valued intuitionistic fuzzy numbers in Park et al. [17] and in Kour et al, [4]. The extension of this in IVNS has been used here to find separation measures based on the Hamming distance.

$$S_{i^+}^{d_1} = \frac{1}{6}\sum_{j=1}^{n}\begin{bmatrix} |a_{ij}^* - a_i^+| + |b_{ij}^* - b_i^+| + |c_{ij}^* - c_i^+| + \\ |d_{ij}^* - d_i^+| + |e_{ij}^* - e_i^+| + |f_{ij}^* - f_i^+| \end{bmatrix} \quad (9)$$

$$S_{i^-}^{d_1} = \frac{1}{6}\sum_{j=1}^{n}\begin{bmatrix} |a_{ij}^* - a_i^-| + |b_{ij}^* - b_i^-| + |c_{ij}^* - c_i^-| \\ + |d_{ij}^* - d_i^-| + |e_{ij}^* - e_i^-| + |f_{ij}^* - f_i^-| \end{bmatrix} \quad (10)$$

The relative closeness of an alternative A_i with respective to interval-valued neutrosophic PIS A^* is defined as the following:

$$C_i^+ = \frac{S_i^-}{S_i^+ + S_i^-} \text{ where i =1, 2 ,….,m} \quad (11)$$

The bigger the closeness coefficient C_i^+, the better the alternative A_i will be, as the alternative A_i is closer to the interval-valued neutrosophic PIS A^*,. Therefore, the alternatives *Ai* (i = 1, 2 ,..., m) can be ranked according to the closeness coefficients so that the best alternative can be selected.

3.3 Solution Procedure:

A. Algorithm for the method based on normalized hamming distance

Let $T = \{T_1, T_2, T_3, ..., T_m\}$ be a set of transportation companies, $C = \{C_1, C_2, C_3, ..., C_n\}$ be a set of criteria and $M = \{M_1, M_2, M_3, ..., M_p\}$ be a set of modes of transportation where each of the C_j of T_i and M_k is represented by IVNS.

$$C(T_i) = ([\mu_{jL}(T_i), \mu_{jU}(T_i)], [\upsilon_{jL}(T_i), \upsilon_{jU}(T_i)], [r_{Lj}(T_i), r_{Uj}(T_i)])$$

$$M_k = ([\mu_{jL}(M_k), \mu_{jU}(M_k)], [\upsilon_{jL}(M_k), \upsilon_{jU}(M_k)], [r_{Lj}(M_k), r_{Uj}(M_k)]) \quad (7)$$

Using the distance between two IVNSs in Definition 2.8 the Normalized Hamming distance for all the criteria of the i-th transportation company from the k-th modes is equal to (8)

$$d_H(C(T_i), M_k) = \frac{1}{30}\sum_{j=1}^{5}\begin{pmatrix} |\mu_{jL}(T_i) - \mu_{jL}(M_k)| + |\mu_{jU}(T_i) - \mu_{jU}(M_k)| + |\upsilon_{jL}(T_i) - \upsilon_{jL}(M_k)| + \\ |\upsilon_{jU}(T_i) - \upsilon_{jU}(M_k)| + |r_{Lj}(T_i) - r_{jL}(M_k)| + |r_{Uj}(T_i) - r_{jU}(M_k)| \end{pmatrix} \quad (12)$$

The minimum distance determines the appropriate mode of each transportation company.

B. Algorithm for the method based on weighted correlation coefficients using given weights

Step 1: Calculate the weighted correlation coefficient $W_i(A^*, A_i)$ (i = 1,2,...,m) by using Eq. (4).

Step 2: Rank the alternatives according to the obtained correlation coefficients, and then obtain the best choice.

C. Algorithm for TOPSIS method with the given information about attribute weights

Step1. Calculate the weighted collective interval-valued neutrosophic decision matrix $R^* = (\tilde{r}_{ij}^*)_{mn}$

Step 2: Calculate the score matrix $S = (s_{ij})_{mxn}$ of the collective interval-valued neutrosophic decision matrix R using Equation(1) from Definition 2.7.

Step3. Determine the interval-valued neutrosophic PIS A^*, and interval-valued neutrosophic NIS A^- using Equations(7) , (8) and score matrix S obtained above in Step 2 .

Step 4.Calculate the separation measures S_i^+ and S_i^- of each alternative A_i (i = 1,2,...,m) from interval-valued neutrosophic PIS A^* and interval-valued neutrosophic NIS A^-, respectively using Equations (9) and (10).

Step 5: Calculate the relative closeness C_i^+ of each alternative A_i (i = 1,2,...m) to the interval-valued neutrosophic PIS A^* using Equation(11).

Table 1. Data of transportation companies and their criteria in form of interval valued neutrosophic fuzzy numbers

Step 6. Rank the alternatives A_i (i = 1,2,...,m), according to the relative closeness to the interval-valued neutrosophic PIS A^* and then select the most desirable one (s).

4. Numerical Illustration:

4.1 Example

An international company needs a freight transportation company to carry its goods. The company determined four possible transportation companies. The criteria considered in the selection process are transportation costs, defective rate, tardiness rate, flexibility and documentation ability. Transportation cost is the cost to carry one ton along one kilometre. Tardiness rate is computed as ''the number of days delayed/the number of days expected for delivery. In Kulak & Kahraman [29], Transportation costs, defective rate and tardiness rate are taken to be crisp variables and the other criteria ''flexibility'' and ''documentation ability'' are taken as linguistic variables just to find only the best transportation company. In Kour et al. [4], the problem is taken in Interval valued Intuitionistic fuzzy environment in which each element of the decision matrix is taken as interval valued intuitionistic fuzzy numbers and the best appropriate transportation company is selected.
In the present paper, the problem is modified as the best transportation company and also their mode of transportation is selected under interval valued neutrosophic environment.

Alternative Transportation Companies	Criteria				
	Transportation Cost	Defective Rate	Tardiness Rate	Flexibility	Documentation Ability
Trans. Comp .1	([0.7,0.8],[0.01,0.02],[0.2,0.4])	([0.8,0.85],[0.02,0.03] ,[0.3,0.5])	([0.3,0.4],[0.2,0.4] ,[0.1,0.2])	([0.6,0.8],[0.01,0.02],[0.2,0.3])	([0.4,0.5],[0.1,0.3] ,[0.1,0.2])
Trans. Comp .2	([0.8,0.85],[0.01,0.03],[0.2,0.3])	([0.01,0.03],[0.8,0.9],[0.3,0.5])	([0.8,0.92],[0.01,0.04] ,[0.2,0.3])	([0.01,0.02],[0.4,0.6],[0.2,0.3])	([0.85,0.9],[0.01,0.02] ,[0.2,0.4])
Trans. Comp .3	([0.85,0.89],[0.02,0.05],[0.3,0.5])	([0.4,0.6],[0.1,0.3],[0.2,0.4])	([0.9,0.95],[0.01,0.02] ,[0.3,0.4])	([0.9,0.92],[0.01,0.03] , [0.3,0.5])	([0.7,0.8],[0.02,0.04],[0.2,0.4])
Trans. Comp .4	([0.8,0.9],[0.01,0.02],[0.2,0.5])	([0.2,0.4],[0.6,0.7],[0.3,0.4])	([0.2,0.3],[0.3,0.6],[0.3,0.4])	([0.5,0.6],[0.1,0.2],[0.2,0.3])	([0.7,0.8],[0.3,0.4],[0.02,0.1])

Let the set of transportation companies be T = {TC1, TC2, TC3, TC4}. Let the set of different criteria of the transportation companies be denoted by C = {Transportation cost (TC), Defective rate (DR), Tardiness rate (TR), Flexibility (F), Documentation ability (DA)}. The data of degree of satisfaction, indeterminacy and rejection of each criterion by each transportation company is represented by an IVNS in Table 1. The IVNS is denoted by a set of Intervals $T_i = (C_j, [\mu_{T_iL}, \mu_{T_iU}], [\upsilon_{T_iL}, \upsilon_{T_iU}][r_{T_iL}, r_{T_iU}]:$

$C_j \in C) = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}], [e_{ij}, f_{ij}])$

Table 2. Data of criteria of transportation companies and their mode of transportation in form of interval valued neutrosophic fuzzy numbers

The IVNS is usually elicited from the evaluated score to which the alternative TC_i satisfies the criterion Cj by means of a score law and data processing or from appropriate membership functions in practice. Therefore,

Alternative Criteria	Mode of transportation			
	Roadways	Railways	Waterways	Airways
Transportation Cost	([0.7,0.85], [0.02,0.03], [0.1,0.15])	([0.8,0.9], [0.02,0.03], [0.01,0.04])	([0.5,0.6], [0.1,0.2] , [0.3,0.35])	([0.3,0.4], [0.2,0.3] , [0.4,0.5])
Defective Rate	([0.3,0.4], [0.1,0.2], [0.5,0.6])	([0.6,0.7], [0.03,0.04], [0.2,0.25])	([0.65,0.75], [0.02,0.05], [0.1,0.2])	([0.8,0.9], [0.01,0.02], [0.01,0.1])
Tardiness Rate	([0.3,0.5], [0.02,0.04] , [0.4,0.45])	([0.5,0.65], [0.01,0.02], [0.2,0.25])	([0.4,0.5], [0.01,0.05], [0.2,0.3])	([0.75,0.85], [0.02,0.03], [0.1,0.15])
Flexibility	([0.8,0.9], [0.2,0.3], [0.01,0.08])	([0.6,0.7], [0.1,0.2] , [0.2,0.25])	([0.5,0.6], [0.01,0.02], [0.15,0.2])	([0.4,0.5], [0.02,0.04], [0.2,0.3])
Documentation Ability	([0.6,0.7], [0.01,0.02], [0.2,0.25])	([0.65,0.8], [0.03,0.05], [0.15,0.2])	([0.7,0.8], [0.2,0.4], [0.1,0.15])	([0.75,0.85], [0.03,0.04], [0.05,0.1])

we can express an interval-valued neutrosophic decision matrix D = $(\alpha_{ij})_{mxn}$.

Similarly let the set of different transportation modes is denoted by M = {Roadways, Railways, Waterways, Airways}. The data of degree of satisfaction, indeterminacy and rejection of each criterion for each mode is represented by an IVNS in Table 2.

$C_j = (M_k, [\mu_{C_jL}, \mu_{CjU}], [\upsilon_{C_jL}, \upsilon_{C_jU}][r_{C_jL}, r_{C_jU}]:$

$M_k \in M) = ([a_{jk}, b_{jk}], [c_{jk}, d_{jk}], [e_{jk}, f_{jk}])$

And it can be denoted by an interval-valued neutrosophic decision matrix D' = $(\beta_{jk})_{nxp}$.

The weights are taken as w_1=0.38, w_2=0.17, w_3=0.21, w_4=0.24, w_5=0.00

4.2 Solution

The given problem is a multi criteria decision making problem in interval valued neutrosophic environment and is solved in two sections. The first section follows up with selecting the best mode of transportation for each transportation company using distance measures. The second section includes the selection of the most appropriate transportation company by the two above mentioned methods. The results are obtained as follows:

A. Solution with method based on Application of Normalized Hamming Distance for Interval valued neutrosophic set

The Equation (3) is used to find the distance for all the criteria of the i-th transportation company from the k-th modes using the normalised Hamming distance as in Table 3. In the definition 2.8, the normalized hamming distance between X and Y (defined by Chi & Liu [33]) is given in Equation (2) which means the distance between any two IVNS. This definition is utilized to calculate the minimum distance between two IVNS in two different but related tables with IVNS as in Equation (3). Then the Equation (3) is utilized to find the Normalized Hamming distance for all the criterion of the i-th transportation company from the k-th modes as in Equation (12) taking data from the related tables Table 1 and Table 2. The minimum distance determines the appropriate mode of each transportation company. For Example - The minimum distance for all the criteria of the **transportation company TC2** is **0.2337** from the **Railways** mode. That means the appropriate mode for transportation companyTC2 is Railways. Similarly, the appropriate mode for each transportation company is given in Table 4.

Table 3. Data of distances for each transportation company from the considered set of their possible modes of transportation

Alternative Transportation Companies	Mode of transportation			
	Roadways	Railways	Waterways	Airways
Trans.Comp.1	0.1737	0.1333	**0.1283**	0.1847
Trans.Comp.2	0.2393	**0.2337**	0.361	0.292
Trans.Comp.3	0.172	**0.1303**	0.1727	0.2087
Trans.Comp.4	0.194	0.1923	**0.1887**	0.2743

Table 4. Appropriate Mode for each transportation company

Transportation companies	**Minimum Distance**	**Appropriate Mode**
Trans.Comp.1	0.1283	Waterways
Trans.Comp.2	0.2337	Railways
Trans.Comp.3	0.1303	Railways
Trans.Comp.4	0.1887	Waterways

B. Solution with method based on weighted correlation coefficients

The attribute weights are taken as w_1=0.38, w_2=0.17, w_3=0.21, w_4=0.24, w_5=0.00

Step 1: The weighted correlation coefficient between an alternative Ai and the ideal alternative A* represented by the IVNS

Is given by Equation (4).

Then taking weight attributes as w_1=0.38, w_2=0.17, w_3=0.21, w_4=0.24, w_5=0.00, the weighted correlation coefficient can be calculated for the data mentioned in Table 1 by applying Equation (4).

By applying Equation (4), we can compute $W_i(A^*, A_i)$ (i = 1, 2, 3, 4) as

$W_1(A^*, A_1) = 0.6737$; $W_2(A^*, A_2) = 0.4811$; $W_3(A^*, A_3) = 0.8942$; $W_4(A^*, A_4) = 0.7076$

Step 2: From the weighted correlation coefficients between the alternatives and the ideal alternative, the ranking order is $A_3 \prec A_4 \prec A_1 \prec A_2$

which is given in Table 5.

Table 5 Ranking based on Weighted Correlation Coefficient

Alternatives	Value of $W_i(A^*, A_i)$	Rank
Trans.Comp.1	0.6737	3
Trans.Comp.2	0.4811	4
Trans.Comp.3	0.8942	1
Trans.Comp.4	0.7076	2

Therefore, we can see that the alternative $TC3$ is the best choice, which is the same result as Kulak & Kahraman [29] and by method of weighted correlation coefficient in Kour et al.[4].

C. Solution with TOPSIS method with the given information about attribute weights

The attribute weights are taken as w_1=0.38, w_2=0.17, w_3=0.21, w_4=0.24, w_5=0.00
Step 1: The weighted collective interval-valued neutrosophic decision matrix $R^* = (\tilde{r}_{ij}^*)_{mn}$ is calculated (Table 6) applying Equation (5).

Step 2: The score matrix $S = (s_{ij})_{mxn}$ of the collective interval-valued neutrosophic decision matrix R is calculated using Equation (1) from Definition2.7 as in Table 7.

Step 3: Using Equations. (7), (8) and score matrix obtained above , the interval-valued neutrosophic PIS A^* and interval-valued neutrosophic NIS A^- is determined as in Table 8.

Step 4: The separation measures S_i^+ and S_i^- of each alternative A_i (i = 1, 2, 3, 4) are calculated from interval-valued neutrosophic PIS A^* and interval-valued neutrosophic NIS A^-, respectively, based on the Hamming distance using Equations. (9) - (10) (Table 9).

Step 5: The relative closeness C_i^+ of each alternative A_i (i = 1, 2, 3, 4) to the interval-valued neutrosophic PIS A^* is calculated with the different separation measures, based on the Hamming distance, using Eq. (11) (Table 10).

Step6. Rank the preference order of alternatives A_i (i = 1, 2, 3, 4) (Table 6), according to the relative closeness to the interval-valued neutrosophic PIS $A*$ and the ranking order is $A_4 \prec A_3 \prec A_1 \prec A_2$.

Therefore, we can see that the alternative $TC4$ is the best choice and then the most desirable alternative is Transportation company $TC4$ as by TOPSIS in Kour et al. [4].

Table 6 Weighted collective interval valued neutrosophic fuzzy decision matrix

Alternative	Criteria				
Transportation Companies	Transportation Cost	Defective Rate	Tardiness Rate	Flexibilty	Documentation Ability
Trans.Comp.1	([0.37,0.46], [0.17,0.22], [0.54,0.71])	([0.24,0.28], [0.51,0.55], [0.81,0.89])	([0.07,0.10], [0.7,0.83], [0.62,0.71])	([0.2,0.32], [0.33,0.39], [0.68,0.75])	([0,0], [1,1], [1,1])
Trans.Comp.2	([0.46,0.51], [0.17,0.26], [0.54,0.63])	([0.0017,0.005], [0.963,0.982], [0.815,0.888])	([0.29,0.41], [0.38,0.51], [0.71,0.78])	([0.002,0.005], [0.8,0.88], [0.68,0.75])	([0,0], [1,1], [1,1])
Trans.Comp.3	([0.51,0.57], [0.23,0.32], [0.63,0.77])	([0.08,0.14], [0.68,0.81], [0.76,0.86])	([0.38,0.47], [0.38,0.44], [0.78,0.83])	([0.42,0.45], [0.33,0.43], [0.75,0.85])	([0,0], [1,1], [1,1])
Trans.Comp.4	([0.46,0.58], [0.17,0.23], [0.54,0.77])	([0.04,0.08], [0.92,0.94], [0.81,0.86])	([0.05,0.07], [0.78,0.9], [0.78,0.83])	([0.15,0.2], [0.58,0.68], [0.68,0.75])	([0,0], [1,1], [1,1])

Table 7 Score matrix of the Weighted collective interval valued neutrosophic fuzzy decision matrix

Alternative	Criteria	
Transporta-	Minimize	Maximize

tion companies	Transportation Cost	Defective Rate	Tardiness Rate	Flexibilty	Documentation Ability
Trans.Comp. 1	0.3967	***-0.08***	-0.2333	0.1233	-0.6667
Trans.Comp. 2	***0.45667***	**-0.5473**	0.1067	***-0.3677***	-0.6667
Trans.Comp. 3	**0.3767**	-0.2967	***0.14***	**0.17**	-0.6667
Trans.Comp. 4	0.4433	-0.47	**-0.39**	-0.1133	-0.6667

Table 8 Interval valued PIS and NIS

	Minimize			Maximize	
	Transportation Cost	Defective Rate	Tardiness Rate	Flexibilty	Documentation Ability
PIS	([0.51,0.57], [0.23,0.32], [0.63,77])	[0.0017,0.005], [0.963,0.982],[0.815,0.888])	[0.05,0.07], [0.78,0.9], [0.78,0.83])	([0.42,0.45], [0.33,0.43], [0.75,0.85])	([0,0], [1,1], [1,1])
NIS	([0.46,0.51], [0.17,0.26], [,0.54,0.63])	([0.24,0.28], [0.51,0.55], [0.81,0.89])	([0.38,0.47] ,[0.38,0.44] ,[0.78,0.83])	([0.002,0.005],[0.8,0.88], [0.68,0.75])	([0,0], [1,1], [1,1])

Table9 Separation measures based on Hamming distance

Alternatives	S_i^+	S_i^-
Trans.Comp.1	0.4997	0.5688
Trans.Comp.2	0.6505	0.29073
Trans.Comp.3	0.39033	0.5372
Trans.Comp.4	0.287	0.6372

Table 10 Relative closeness C_i^+ based on Hamming Distance

Alternatives	Value of C_i^+	Rank
Trans.Comp.1	0.53234	3
Trans.Comp.2	0.30888	4
Trans.Comp.3	0.57917	2
Trans.Comp.4	0.68946	1

5. Results and comparison

In this paper, the distance measures on interval valued neutrosophic set using the normalized hamming distance help to find the best modes of transportation for each transportation company as in Table 4. The paper helps to find the appropriate transportation company. It follows with two methods. The first method which is based on weighted correlation coefficient gives the best transportation company as TC3. The result is same as in the Kour et al. [4] for the method to find the best transportation company based on weighted correlation coefficient under interval valued intuitionistic fuzzy environment. The second method which is the extended TOPSIS gives the best transportation company as TC4. The result is same as in the Kour et al. [4] for the extended TOPSIS method to find the best transportation company under interval valued intuitionistic fuzzy environment. In addition, this paper also helps to find the best mode of transportation for the selected transportation companies. In the first result, the selected transportation company TC3 opt for Railways whereas in the second result, the selected transportation company TC4 chooses Waterways as their mode of transportation. The present paper also deals with degree of indeterminacy along with the degree of acceptance and rejection of the different attributes as in Kour et al. [4]. The results can be compared with the help of the below mentioned tables (Table 11, Table 12, Table 13 and Table 14).

Table11 Solution as in [4] under interval valued intuitionistic fuzzy environment

Alternatives	**Rank with Weighted Correlation Coefficient Method(unknown**	**Rank with Extended TOPSIS(known weights)**

	weights)	
Trans.Comp.1	3	3
Trans.Comp.2	4	4
Trans.Comp.3	1	2
Trans.Comp.4	2	1

Table12 Appropriate Transportation Company in [4] under interval valued intuitionistic fuzzy environment

Weighted Correlation Coefficient Method(unknown weights)	ExtendedTOPSIS(known weights)
Trans Comp 3	Trans Comp 4

Table13 Solution as in the present paper under interval valued neutrosophic environment

Alternatives	Rank with Weighted Correlation Coefficient Method(known weights)	Rank with Extended TOPSIS(known weights)
Trans.Comp.1	3	3
Trans.Comp.2	4	4
Trans.Comp.3	1	2
Trans.Comp.4	2	1

Table14 Appropriate Transportation Company and their mode in the present paper under interval valued neutrosophic environment

Methods	Weighted Correlation Coefficient Method (unknown weights)	Extended TOPSIS (known weights)
Best Transportation Company	Trans Comp 3	Trans Comp 4
Best Transportation Mode	Railways	Waterways

6. Conclusion

- A new type of transportation company selection problem is constructed in which the mode of transportation is also selected along with the best transportation company which gives a greater scope of its application in real life circumstances to achieve better requirements of the transportation companies.
- The method for the application of normalized hamming distance on interval valued neutrosophic set helps the users to relate the given two different relational tables consisting of transportation companies, their criteria and their mode of transportation and thus to find the appropriate mode of each transportation companies for the first time.
- The weighted correlation coefficient method helps the users to solve the multi-criteria decision making problems with given weight information which has been done for the first time in Interval valued neutrosophic environment
- The extended TOPSIS method provides us an effective and practical way to solve the same type of problems, where the data is characterized by IVNSs and the information about weights is completely known. A score function has been defined for interval valued neutrosophic sets for the first time and is used to find the interval valued neutrosophic PIS and NIS.
- The interval valued neutrosophic set data can be seen as real life uncertainties and so represents more practical solutions of the problem where the degree of acceptance, indeterminacy and rejection of the different attributes are taken into account.

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University of New Mexico

An Application of Single-Valued Neutrosophic Sets in Medical Diagnosis

Gulfam Shahzadi[1], Muhammad Akram[1] and Arsham Borumand Saeid[2]

1. Department of Mathematics, University of the Punjab, Lahore, Pakistan. E-mail: gulfamshahzadi22@gmail.com, m.akram@pucit.edu.pk

2. Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran. E-mail: arsham@mail.uk.ac.ir

Abstract:
In this paper, we present the use of single-valued neutrosophic sets in medical diagnosis by using distance measures and similarity measures. Using interconnection between single-valued neutrosophic sets and symptoms of patient, we determine the type of disease. We define new distance formulas for single valued neutrosophic sets. We develop two new medical diagnosis algorithms under neutrosophic environment. We also solve a numerical example to illustrate the proposed algorithms and finally, we compare the obtained results.

Keywords: Single-valued neutrosophic sets, distance, similarity measures, medical diagnosis.

1 Introduction

The notion of fuzzy set was introduced by Zadeh [1] to deal with ambiguity, vagueness and imprecision. Atanassov [2] popularized the concept of intuitionistic fuzzy set, as a generalization of fuzzy set. Adlassnig [3] employed fuzzy set theory to formalize medical relationships and fuzzy logic to model the diagnostic process and developed a computerized diagnosis system. Important developments and applications of some medical expert systems based on fuzzy set theory were reported in the literature [4-8]. De et al. [9] first proposed an application of intuitionistic fuzzy sets in medical diagnosis. Davvaz and Sadrabadi [10] discussed an application of intuitionistic fuzzy sets in medicine. Several authors [10-15] employed intuitionistic fuzzy sets in medical diagnosis and cited De et al. [9]. However, Hung and Tuan [16] pointed out that the approach studied in [9] contains questionable results that may lead to false diagnosis of patients' symptoms.

It is widely recognized that the information available to the medical practitioners about his/her patient and about medical relationships in general is inherently uncertain. Even information is incomplete as it continually becomes enlarged and gets changed. Heisenberg's Uncertainty Principle [17] reflects that nature possibly is fundamentally indeterministic. It is widely accepted that knowledge may differ according to culture, education, religion, social status, etc., and therefore information derived from different sources may be inconsistent. We may recall Godel's Theorem [18] which clearly reflects that contradictions within a system cannot be eliminated by the system itself. So uncertainty, incomplete and inconsistency should be addressed in medical diagnosis problem which can be dealt with neutrosophic set [19] introduced by Florentin Smarandache. Neutrosophic set [19] consists of three independent objects called truth-membership (μ), indeterminacy-membership (σ) and falsity-membership (ν) whose values are real standard or non-standard subset of unit interval $]0^-, 1^-[$. In 1998, the idea of single-valued neutrosophic set was given by Smarandache [19] and the term "single valued neutrosophic set" was coined in 2010 by Wang et al. [20].

Yang et al. [21] presented the theory of single-valued neutrosophic relation based on single-valued neutrosophic set. In almost every scientific field, the idea of similarity is essentially important. To measure the degree of similarity between fuzzy sets, many methods have been introduced [22-25]. These methods are not suitable to deal with the similarity measures of neutrosophic sets (NSs). Majumdar and Samanta [26] presented several similarity measures of single valued neutrosophic sets based on distances, a matching function, membership grades, and then proposed an entropy measure. Several studies dealt with similarity measures for neutrosophic sets and single-valued neutrosophic sets [27-31]. Salama et. al. [32] defined the neutrosophic correlation coefficients which are another types of similarity measurement. Ye [33] discussed similarity measures on interval neutrosophic set [34] based on Hamming distance and Euclidean distance and showed how these measures can be used in decision making problems. Furthermore, on the domain of neutrosophic sets, Pramanik et al. [35] studied hybrid vector similarity measures for single valued neutrosophic sets as well as interval neutrosophic sets. In medical diagnosis, Ye [36] presented the improved cosine similarity measures of single valued neutrosophic sets as well as interval neutrosophic sets and employed them to medical diagnosis problems. Mondal and Pramanik [37] propose tangent similarity measure and weighted tangent similarity measure for single valued neutrosophic sets and employed them to medical diagnosis.

In medical diagnosis problem, symptoms and inspecting data of some disease may be changed in different time intervals. It leads to the question that whether only by using a single

period inspection one can conclude for a particular patient with a particular decease or not. Sometimes symptoms of different diseases may appear for a person under treatment. Then, natural question arises, how can we decide a proper diagnosis for the particular patient by using one inspection? To answer this question Ye [38] proposed multi-period medical diagnosis (i.e. dynamic medical diagnosis) strategy based on neutrosophic tangent function. Several medical strategies [39-52] have been reported in the literature in neutrosophic environment including neutrosophic hybrid set environment. Nguyen et al. [53] made a survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. The aforementioned strategies [36, 37, 38] employed cosine similarity measure and tangent similarity measure under neutrosophic environment.
The use of single-valued neutrosophic sets in medical diagnosis by using distance measures and similarity measure which have not been addressed in the literature. In this paper, we present two algorithms for medical diagnosis by using distance measures and similarity measures under neutrosophic environment. This study answers the following research questions:

1. Is it possible to formulate a new algorithm for medical diagnosis by using normalized Hamming distance and similarity measure?
2. Is it possible to formulate a new algorithm for medical diagnosis by using normalized Euclidean distance and similarity measure?
3. Is it possible to develop a new algorithm for medical diagnosis by using new distance formula and similarity measure?

The above-mentioned analysis describes the motivation behind proposing two new medical diagnosis algorithms under single valued neutrosophic environment using new distance formulas and similarity measures. This study develops two novel medical diagnosis algorithms under single valued neutrosophic environment. The Objectives of the paper are stated as follows:

1. To define two new neutrosophic distance formulas.
2. To develop two new medical diagnosis algorithms under single valued neutrosophic environment.
3. To show numerical example of medical diagnosis using the proposed algorithms.
4. To compare the obtained results derived from the proposed two algorithms with the algorithms based on normalized Hamming and normalized Euclidean distance.
5. To fill the research gap, we propose two algorithms for medical diagnosis by using distance measures and new similarity measures under neutrosophic environment.

The proposed algorithms can be effective in dealing with medical diagnosis under single valued neutrosophic set environment. It can be extended to interval neutrosophic environment and neutrosophic hybrid environment. The main contributions of this paper are summarized below:

i. We define two new distance formulas for neutrosophic sets.
ii. We develop two new algorithms for medical diagnosis based on new distance formulas and similarity measure.
iii. We present the comparison between the proposed algorithms with the algorithms based on normalized Hamming and normalized Euclidean distance.

The rest of the paper unfolds as follows: In section 2, we describe some basic definitions and operations of single valued neutrosophic sets (SVNSs). In section 3, we present the definition of proposed distance formulas and develop two new algorithms for medical diagnosis and present comparison with numerical example. In section 4, we present conclusion and future scope of the study.

2 Preliminaries

In this section, we review some basic concepts related to neutrosophic sets.

Definition 1. *[19] Let Z be a space of points (objects). A neutrosophic set M in Z is characterized by a truth-membership function $(\mu_M(z))$, an indeterminacy-membership function $(\sigma_M(z))$ and a falsity-membership function $(\nu_M(z))$. The functions $(\mu_M(z))$, $(\sigma_M(z))$, and $(\nu_M(z))$ are real standard or non-standard subsets of $]0^-, 1^+[$, that is, $\mu_M(z) : Z \to\ ]0^-, 1^+[, \sigma_M(z) : Z \to\ ]0^-, 1^+[$ and $\nu_M(z) : Z \to\ ]0^-, 1^+[$ and $0^- \leq \mu_M(z) + \sigma_M(z) + \nu_M(z) \leq 3^+$.*
From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]0^-, 1^+[$. In real life applications in scientific and engineering problems, it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]0^-, 1^+[$, , where $0^- = 0 - \epsilon$, $1^+ = 1 + \epsilon$, ϵ is an infinitesimal number > 0. To apply neutrosophic set in real-life problems more conveniently, Smarandache and Wang et al. [20] defined single-valued neutrosophic sets which takes the value from the subset of [0, 1]. Thus, a single-valued neutrosophic set is a special case of neutrosophic set. It has been proposed as a generalization of crisp sets, fuzzy sets, and intuitionistic fuzzy sets in order to deal with incomplete information.

Definition 2. *Let $Z = \{z_1, z_2, ..., z_n\}$ be a discrete confined set. Consider M, N, O be three neutrosophic sets in Z. For all $z_i \in Z$ we have:*

$$d_H(M,N)=H(M,N)=\max\{|\mu_M(z_i) - \mu_N(z_i)|, |\sigma_M(z_i) - \sigma_N(z_i)|, |\nu_M(z_i) - \nu_N(z_i)|\}.$$

where $d_H(M,N)=H(M,N)$ denotes the extended Hausdroff distance between between two neutrosophic sets M and N.
The above defined distance $d_H(M,N)$ between neutrosophic sets

M and N satisfies the following properties:
(D1) $d_H(M,N) \geq 0$,
(D2) $d_H(M,N) = 0$ if and only if $M = N$; for all $M, N \in NS$,
(D3) $d_H(M,N) = d_H(N,M)$,
(D4) If $M \subseteq N \subseteq O$ for all $M, N, O \in NS$, then $d_H(M,O) \geq d_H(M,N)$ and $d_H(M,O) \geq d_H(N,O)$.
then d is called the distance measure between two neutrosophic sets.

Definition 3. *A mapping $S : NS(Z) \times NS(Z) \longrightarrow [0,1]$, $NS(Z)$ denotes the set of all NS in $Z = \{z_1, z_2, ..., z_n\}$, $S(M,N)$ is said to be the degree of similarity between $M \in NS$ and $N \in NS$, if $S(M,N)$ satisfies the properties of conditions (S1-S4):*
(S1) $S(M,N)$=$S(N,M)$,
(S2) $S(M,N)$=(1,0,0). If $M = N$ for all $M, N \in NS$,
(S3) $S_\mu(M,N) \geq 0$, $S_\sigma(M,N) \geq 0$, $S_\nu(M,N) \geq 0$,
(S4) If $M \subseteq N \subseteq O$ for all $M, N, O \in NS$, then $S(M,N) \geq S(M,O)$and $S(N,O) \geq S(M,O)$.

Definition 4. *The normalized Hamming distance between two neutrosophic sets M and N is defined by*

$$d_3(M,N) = \frac{1}{2n}\sum_{j=1}^{n}(|\mu_M(z_j) - \mu_N(z_j)| + |\sigma_M(z_j) - \sigma_N(z_j)| + |\nu_M(z_j) - \nu_N(z_j)|).$$

Definition 5. *The normalized Euclidean distance between two neutrosophic sets M and N is defined by*

$$d_4(M,N) = \left\{\frac{1}{2n}\sum_{j=1}^{n}((\mu_M(z_j) - \mu_N(z_j))^2 + (\sigma_M(z_j) - \sigma_N(z_j))^2 + (\nu_M(z_j) - \nu_N(z_j))^2)\right\}^{\frac{1}{2}}.$$

3 Neutrosophic Sets in Medical Diagnosis

We first correct the formulas for the Definitions 4 and 5, where in both of them the we should put "$\frac{1}{3n}$" instead of "$\frac{1}{2n}$" in order for the Hamming distance and respectively Euclidean distance to be "normalized". These formulas are extended from intuitionistic fuzzy sets, where indeed one uses "$\frac{1}{2n}$ " since there are only two intuitionistic fuzzy sets memberships (membership and non-membership). But, we have three components in neutrosophic sets.
For example, if we compute the Hamming distance between the neutrosophic numbers: $(1,1,1)$ and $(0,0,0)$, we get $\frac{1}{2}\{|1-0| + |1-0| + |1-0|\} = \frac{3}{2} = 1.5 > 1$. Therefore, it is not normalized since the result is not in $[0,1]$. Similarly for the Euclidean formula, where we get for the same neutrosophic numbers: $\sqrt{\frac{1}{2}\{|1-0| + |1-0| + |1-0|\}} = \sqrt{\frac{3}{2}} > 1$.

We write normalized formulae for two neutrosophic sets as follows.

Definition 6. *The normalized Hamming distance between two neutrosophic sets M and N is defined by*

$$d_3(M,N) = \frac{1}{3n}\sum_{j=1}^{n}(|\mu_M(z_j) - \mu_N(z_j)| + |\sigma_M(z_j) - \sigma_N(z_j)| + |\nu_M(z_j) - \nu_N(z_j)|).$$

Definition 7. *The normalized Euclidean distance between two neutrosophic sets M and N is defined by*

$$d_4(M,N) = \left\{\frac{1}{3n}\sum_{j=1}^{n}((\mu_M(z_j) - \mu_N(z_j))^2 + (\sigma_M(z_j) - \sigma_N(z_j))^2 + (\nu_M(z_j) - \nu_N(z_j))^2)\right\}^{\frac{1}{2}}.$$

In this section, we give new concepts for medical diagnosis via distances between neutrosophic sets. In fact our purpose is to find an accurate diagnosis for each patient $p_i, i = 1,2,3$. The relation between neutrosophic sets for all the symptoms of the i-th patient from the k-th diagnosis is as follows:

$$d_1(p_i, d_k) = \frac{1}{n}\sum_{j=1}^{n}\Big[\frac{1}{6}\big[|\mu_{p_i}(z_j) - \mu_{d_k}(z_j)| + |\sigma_{p_i}(z_j) - \sigma_{d_k}(z_j)| + |\nu_{p_i}(z_j) - \nu_{d_k}(z_j)|\big] + \frac{1}{3}\big[\max(|\mu_{p_i}(z_j) - \mu_{d_k}(z_j)|, |\sigma_{p_i}(z_j) - \sigma_{d_k}(z_j)|, |\nu_{p_i}(z_j) - \nu_{d_k}(z_j)|)\big]\Big]. \quad (1)$$

We take $n = 5$.
We consider there are three patients: Ali, Hamza, Imran and symptoms of patient are Temperature, Insulin, Blood pressure, Blood plates, Cough and finally we get diagnosis as Diabates, Dengue, Tuberculosis.
In Table 1, the data are explained by three parameters: membership function (μ), non-membership function (ν) and indeterminacy function (σ). In Table 2, the symptoms are described by (μ, σ, ν). For example, Diabates temperature is low ($\mu = 0.2, \sigma = 0.0, \nu = 0.8$), while Dengue temperature is high ($\mu = 0.9, \sigma = 0.0, \nu = 0.1$).

Table 1. Membership function μ, Indeterminacy function σ and non-membership function ν.

I_1	Ali	Hamza	Imran
Temperature	(0.8,0.1,0.1)	(0.6,0.2,0.2)	(0.4,0.2,0.4)
Insulin	(0.2,0.2,0.6)	(0.9,0.0,0.1)	(0.2,0.1,0.7)
Blood pressure	(0.4,0.2,0.4)	(0.1,0.1,0.8)	(0.1,0.2,0.7)
Blood plates	(0.8,0.1,0.1)	(0.2,0.1,0.7)	(0.3,0.1,0.6)
Cough	(0.3,0.3,0.4)	(0.5,0.1,0.4)	(0.8,0.0,0.2)

Table 2. Symptoms

I_2	Temperature	Insulin	Blood pressure	Blood plates	Cough
Diabates	(0.2,0.0,0.8)	(0.9,0.0,0.1)	(0.1,0.1,0.8)	(0.1,0.1,0.8)	(0.1,0.1,0.8)
Dengue	(0.9,0.0,0.1)	(0.0,0.2,0.8)	(0.8,0.1,0.1)	(0.9,0.0,0.1)	(0.1,0.1,0.8)
Tuberculosis	(0.6,0.2,0.2)	(0.0,0.1,0.9)	(0.4,0.2,0.4)	(0.0,0.2,0.8)	(0.9,0.0,0.1)

By using formula (1), for $n = 5$, we obtain Table 3.

Table 3. Using formula (1), for $n = 5$.

I	Ali	Hamza	Imran
Diabates	0.38	0.14	0.27
Dengue	0.15	0.40	0.34
Tuberculosis	0.25	0.25	0.14

The best medical diagnosis in each column is identified by the lowest difference. Therefore, in the first column, Ali suffers from Dengue, in the second column, Hamza suffers from Diabates, in the third column, Imran suffers from Tuberculosis. Now we define another relation for the best medical diagnosis:

$$d_2(p_i, d_k) = \frac{1}{3}\sqrt[r]{n}\left\{\sum_{j=1}^{n}(|\mu_{p_i}(z_j) - \mu_{d_k}(z_j)| + |\sigma_{p_i}(z_j) - \sigma_{d_k}(z_j)| + |\nu_{p_i}(z_j) - \nu_{d_k}(z_j)|)^r\right\}^{\frac{1}{r}}. \quad (2)$$

and r is a positive number. We take $n = 5$. We examine the above relation for $r = 1, 2, ..., 10$. First, for $r = 1$ we calculate Table 4.

Table 4. Using formula (2), for $r = 1$.

I	Ali	Hamza	Imran
Diabates	0.39	0.15	0.26
Dengue	0.16	0.4	0.36
Tuberculosis	0.25	0.25	0.15

Now, for $r = 2$ we get Table 5.

Table 5. Using formula (2), for $r = 2$.

I	Ali	Hamza	Imran
Diabates	0.4	0.22	0.32
Dengue	0.19	0.43	0.38
Tuberculosis	0.32	0.32	0.15

The result for $r = 3$ is given in Table 6.

Table 6. Using formula (2), for $r = 3$.

I	Ali	Hamza	Imran
Diabates	0.41	0.25	0.35
Dengue	0.2	0.45	0.39
Tuberculosis	0.35	0.37	0.16

For $r = 4$, we obtain Table 7.

Table 7. Using formula (2), for $r = 4$.

I	Ali	Hamza	Imran
Diabates	0.42	0.28	0.37
Dengue	0.21	0.47	0.41
Tuberculosis	0.39	0.41	0.17

For $r = 5$, we get Table 8.

Table 8. Using formula (2), for $r = 5$.

I	Ali	Hamza	Imran
Diabates	0.43	0.3	0.39
Dengue	0.22	0.48	0.41
Tuberculosis	0.41	0.44	0.17

By calculation for $r = 6$, we find Table 9.

Table 9. Using formula (2), for $r = 6$.

I	Ali	Hamza	Imran
Diabates	0.43	0.31	0.4
Dengue	0.23	0.49	0.41
Tuberculosis	0.42	0.46	0.17

For $r = 7$, we find Table 10.

Table 10. Using formula (2), for $r = 7$.

I	Ali	Hamza	Imran
Diabates	0.43	0.32	0.41
Dengue	0.23	0.5	0.42
Tuberculosis	0.43	0.48	0.18

For $r = 8$, we get Table 11.

Table 11. Using formula (2), for $r = 8$.

I	Ali	Hamza	Imran
Diabates	0.44	0.33	0.41
Dengue	0.24	0.51	0.43
Tuberculosis	0.44	0.49	0.18

For $r = 9$, we get Table 12.

Table 12. Using formula (2), for $r = 9$.

I	Ali	Hamza	Imran
Diabates	0.44	0.33	0.42
Dengue	0.24	0.51	0.43
Tuberculosis	0.45	0.5	0.18

For $r = 10$, we obtain Table 13.

Table 13. Using formula (2), for $r = 10$.

I	Ali	Hamza	Imran
Diabates	0.45	0.34	0.43
Dengue	0.24	0.52	0.43
Tuberculosis	0.45	0.51	0.18

As r becoming larger, the difference between the data in tables become inferior, that is, the data approaches to the real amount. In Tables 4-13, the results are same. In fact in all tables, in the first column, the lowest difference is related to Ali and Dengue, so Ali suffers from Dengue, also in the second column Hamza suffers from Diabates, in the third column Imran suffers from Tuberculosis.
The normalized Hamming distance for all the symptoms of the i-th patient from the k-th diagnosis [**?**] is

$$d_3(p_i,d_k) = \frac{1}{3n}\sum_{j=1}^{n}(|\mu_{p_i}(z_j)-\mu_{d_k}(z_j)| + |\sigma_{p_i}(z_j)-\sigma_{d_k}(z_j)| + |\nu_{p_i}(z_j)-\nu_{d_k}(z_j)|). \quad (3)$$

and the normalized Euclidean distance [**?**] is

$$d_4(p_i,d_k) = \left\{\frac{1}{3n}\sum_{j=1}^{n}((\mu_{p_i}(z_j)-\mu_{d_k}(z_j))^2 + (\sigma_{p_i}(z_j)-\sigma_{d_k}(z_j))^2 + (\nu_{p_i}(z_j)-\nu_{d_k}(z_j))^2)\right\}^{\frac{1}{2}}. \quad (4)$$

We set $n = 5$.
By formulas (3), (4) respectively, the results are given in Tables 14 and 15.

Table 14. Using formula (3).

I	Ali	Hamza	Imran
Diabates	0.39	0.15	0.26
Dengue	0.16	0.4	0.36
Tuberculosis	0.25	0.25	0.15

Table 15. Using formula (4).

I	Ali	Hamza	Imran
Diabates	0.46	0.24	0.37
Dengue	0.20	0.49	0.43
Tuberculosis	0.35	0.37	0.18

Thus, we studied results that have been obtained from formulas (3), (4) are same with relations (1), (2). Another idea for medical diagnosis is

$$d(M,N) = \max(|\mu_M(z_i)-\mu_N(z_i)|, |\sigma_M(z_i)-\sigma_N(z_i)|, |\nu_M(z_i)-\nu_N(z_i)|). \quad (5)$$

Table 16. Medical diagnosis.

I	Ali	Hamza	Imran
Diabates	0.7	0.6	0.7
Dengue	0.4	0.9	0.7
Tuberculosis	0.8	0.9	0.3

The similarity measures between two neutrosophic sets M and N is defined as follows :

$$S_1(M,N) = \frac{1}{n}\sum_{i=1}^{n}\Big[\big[\min(\mu_M(z_i),\mu_N(z_i)) + \min(\sigma_M(z_i),\sigma_N(z_i)) + \min(\nu_M(z_i),\nu_N(z_i))\big] \div \big[\max(\mu_M(z_i),\mu_N(z_i)) + \max(\sigma_M(z_i),\sigma_N(z_i)) + \max(\nu_M(z_i),\nu_N(z_i))\big]\Big]. \quad (6)$$

We set $n = 5$ (Table 17).

Table 17. Using formula (6), for $n = 5$.

I	Ali	Hamza	Imran
Diabates	0.28	0.70	0.45
Dengue	0.63	0.27	0.32
Tuberculosis	0.51	0.52	0.65

$$S_2(M,N) = \frac{1}{n}\Big[\sum_{i=1}^{n}(1-\frac{1}{3}(|\mu_M(z_i)-\mu_N(z_i)| + |\sigma_M(z_i)-\sigma_N(z_i) + |\nu_M(z_i)-\nu_N(z_i))\Big]. \quad (7)$$

We set $n = 5$ (Table 18).

Table 18. Using formula (7), for $n = 5$.

I	Ali	Hamza	Imran
Diabates	0.69	0.45	0.72
Dengue	0.84	0.4	0.66
Tuberculosis	0.55	0.55	0.85

$$S_3(M,N) = \sum_{i=1}^{n}\big[\min(\mu_M(z_i),\mu_N(z_i)) + \min(\sigma_M(z_i),\sigma_N(z_i)) + \min(\nu_M(z_i),\nu_N(z_i))\big] \div \sum_{i=1}^{n}\big[\max(\mu_M(z_i),\mu_N(z_i)) + \max(\sigma_M(z_i),\sigma_N(z_i)) + \max(\nu_M(z_i),\nu_N(z_i))\big]. \quad (8)$$

We set $n = 5$ (Table 19).

Table 19. Using formula (8), for $n = 5$.

I	Ali	Hamza	Imran
Diabates	0.27	0.64	0.41
Dengue	0.61	0.25	0.31
Tuberculosis	0.45	0.45	0.64

$$S_4(M,N) = 1-\frac{1}{3}(\max_i(|\mu_M(z_i)-\mu_N(z_i)|) + \max_i(|\sigma_M(z_i)-\sigma_N(z_i)|) + \max_i(|\nu_M(z_i)-\nu_N(z_i)|)). \quad (9)$$

We set $n = 5$ (Table 20).

Table 20. Using formula (9), for $n = 5$.

I	Ali	Hamza	Imran
Diabates	0.47	0.6	0.5
Dengue	0.5	0.1	0.25
Tuberculosis	0.67	0.4	0.73

$$S_5(M,N) = 1 - \Bigg[\sum_{i=1}^{n} \big[|\mu_M(z_i) - \mu_N(z_i)| + |\sigma_M(z_i) - \sigma_N(z_i)| + |\nu_M(z_i) - \nu_N(z_i)|\big] \div \sum_{i=1}^{n} \big[|\mu_M(z_i) + \mu_N(z_i)| + |\sigma_M(z_i) + \sigma_N(z_i)| + |\nu_M(z_i) + \nu_N(z_i)|\big] \Bigg]. \tag{10}$$

We set $n = 5$ (Table 21).

Table 21. Using formula (10), for $n = 5$.

I	Ali	Hamza	Imran
Diabates	0.42	0.78	0.58
Dengue	0.76	0.4	0.46
Tuberculosis	0.62	0.62	0.78

We can see that the results obtained by using the relations S_1, S_2, S_3, S_4, S_5 are different from relations $1-5$. Therefore, these similarity measures are not applicable.
The new similarity measures between neutrosophic sets M and N are defined as follows. The first one is

$$S_{new1} = \frac{1}{1-\exp(-n)} \Bigg[1 - \exp(-\frac{1}{3} \sum_{i=1}^{n} (|\mu_M(z_i) - \mu_N(z_i)| + |\sigma_M(z_i) - \sigma_N(z_i)| + |\nu_M(z_i) - \nu_N(z_i)|)) \Bigg]. \tag{11}$$

We set $n = 5$ (Table 22).

Table 22. Using formula (11), for $n = 5$.

I	Ali	Hamza	Imran
Diabates	0.86	0.52	0.75
Dengue	0.55	0.88	0.84
Tuberculosis	0.73	0.73	0.52

The second one is

$$S_{new2} = \frac{1}{1-\exp(-n)} \Bigg[1 - \exp(-\frac{1}{3} \sum_{i=1}^{n} (|\sqrt{\mu_M(z_i)} - \sqrt{\mu_N(z_i)}| + |\sqrt{\sigma_M(z_i)} - \sqrt{\sigma_N(z_i)}| + |\sqrt{\nu_M(z_i)} - \sqrt{\nu_N(z_i)}|)) \Bigg]. \tag{12}$$

We set $n = 5$ (Table 23).

Table 23. Using formula (12), for $n = 5$.

I	Ali	Hamza	Imran
Diabates	0.83	0.50	0.75
Dengue	0.60	0.86	0.84
Tuberculosis	0.74	0.75	0.55

The obtained relations from $S_{new1}(M,N), S_{new2}(M,N)$ are closely same with relations $1-5$. Consequently, the obtained results from the relations between neutrosophic sets (1), (2), (5), (11), (12) are equivalent to the results of formula (3), (4). By using the distance and similarity measures formulas between neutrosophic sets, we establish the most applicable medical diagnosis that in all tables are related to the lowest difference in each column. Finally, we conclude that the methods which have the results equivalent to normalized hamming and normalized Euclidean formulas are best to determine the diseases of a patient.
Now we present our first method in the following algorithm 1.

Algorithm 1:

Step 1. Input the truth membership, indeterminacy and non-membership values of patients and diagnosis.

Step 2. Compute the diseases by different distance measures given in steps $3-7$.

Step 3.

$$d_1(p_i,d_k) = \frac{1}{n} \sum_{j=1}^{n} \Bigg[\frac{1}{6} \big[|\mu_{p_i}(z_j) - \mu_{d_k}(z_j)| + |\sigma_{p_i}(z_j) - \sigma_{d_k}(z_j)| + |\nu_{p_i}(z_j) - \nu_{d_k}(z_j)|\big] + \frac{1}{3} \big[\max(|\mu_{p_i}(z_j) - \mu_{d_k}(z_j)|, |\sigma_{p_i}(z_j) - \sigma_{d_k}(z_j)|, |\nu_{p_i}(z_j) - \nu_{d_k}(z_j)|) \big] \Bigg].$$

Step 4.

$$d_2(p_i,d_k) = \frac{1}{3} \sqrt[r]{n} \Bigg\{ \sum_{j=1}^{n} (|\mu_{p_i}(z_j) - \mu_{d_k}(z_j)| + |\sigma_{p_i}(z_j) - \sigma_{d_k}(z_j)| + |\nu_{p_i}(z_j) - \nu_{d_k}(z_j)|)^r \Bigg\}^{\frac{1}{r}}.$$

Step 5.

$$d_3(p_i,d_k) = \frac{1}{3n} \sum_{j=1}^{n} (|\mu_{p_i}(z_j) - \mu_{d_k}(z_j)| + |\sigma_{p_i}(z_j) - \sigma_{d_k}(z_j)| + |\nu_{p_i}(z_j) - \nu_{d_k}(z_j)|).$$

Step 6.

$$d_4(p_i,d_k) = \Bigg\{ \frac{1}{3n} \sum_{j=1}^{n} ((\mu_{p_i}(z_j) - \mu_{d_k}(z_j))^2 + (\sigma_{p_i}(z_j) - \sigma_{d_k}(z_j))^2 + (\nu_{p_i}(z_j) - \nu_{d_k}(z_j))^2) \Bigg\}^{\frac{1}{2}}.$$

Step 7.

$$d(M,N) = \max(|\mu_M(z_i) - \mu_N(z_i)|, |\sigma_M(z_i) - \sigma_N(z_i)|, |\nu_M(z_i) - \nu_N(z_i)|).$$

W present our second method in the following algorithm 2.

Algorithm 2:

Step 1. Input the truth membership, indeterminacy and non-membership values of patients and diagnosis.

Step 2. Also compute the diseases by similarity measures given in steps $3-9$.

Step 3.

$$S_1(M,N) = \frac{1}{n}\sum_{i=1}^{n}\Big[\big[\min(\mu_M(z_i),\mu_N(z_i)) + \min(\sigma_M(z_i),\sigma_N(z_i)) + \min(\nu_M(z_i),\nu_N(z_i))\big] \div \big[\max(\mu_M(z_i),\mu_N(z_i)) + \max(\sigma_M(z_i),\sigma_N(z_i)) + \max(\nu_M(z_i),\nu_N(z_i))\big]\Big].$$

Step 4.

$$S_2(M,N) = \frac{1}{n}\Big[\sum_{i=1}^{n}(1 - \frac{1}{3}(|\mu_M(z_i) - \mu_N(z_i)| + |\sigma_M(z_i) - \sigma_N(z_i) + |\nu_M(z_i) - \nu_N(z_i))\Big].$$

Step 5.

$$S_3(M,N) = \sum_{i=1}^{n}\big[\min(\mu_M(z_i),\mu_N(z_i)) + \min(\sigma_M(z_i),\sigma_N(z_i)) + \min(\nu_M(z_i),\nu_N(z_i))\big] \div \sum_{i=1}^{n}\big[\max(\mu_M(z_i),\mu_N(z_i)) + \max(\sigma_M(z_i),\sigma_N(z_i)) + \max(\nu_M(z_i),\nu_N(z_i))\big].$$

Step 6.

$$S_4(M,N) = 1 - \frac{1}{3}(\max_i(|\mu_M(z_i) - \mu_N(z_i)|) + \max_i(|\sigma_M(z_i) - \sigma_N(z_i)|) + \max_i(|\nu_M(z_i) - \nu_N(z_i)|)).$$

Step 7.

$$S_5(M,N) = 1 - \Big[\sum_{i=1}^{n}\big[|\mu_M(z_i) - \mu_N(z_i)| + |\sigma_M(z_i) - \sigma_N(z_i)| + |\nu_M(z_i) - \nu_N(z_i)|\big] \div \sum_{i=1}^{n}\big[|\mu_M(z_i) + \mu_N(z_i)| + |\sigma_M(z_i) + \sigma_N(z_i)| + |\nu_M(z_i) + \nu_N(z_i)|\big]\Big].$$

Step 8.

$$S_{new1} = \frac{1}{1-\exp(-n)}\Big[1 - \exp(-\frac{1}{3}\sum_{i=1}^{n}(|\mu_M(z_i) - \mu_N(z_i)| + |\sigma_M(z_i) - \sigma_N(z_i)| + |\nu_M(z_i) - \nu_N(z_i)|))\Big].$$

Step 9.

$$S_{new2} = \frac{1}{1-\exp(-n)}\Big[1 - \exp(-\frac{1}{3}\sum_{i=1}^{n}(|\sqrt{\mu_M(z_i)} - \sqrt{\mu_N(z_i)}| + |\sqrt{\sigma_M(z_i)} - \sqrt{\sigma_N(z_i)}| + |\sqrt{\nu_M(z_i)} - \sqrt{\nu_N(z_i)}|))\Big].$$

Finally, We compare these methods to normalized hamming and normalized Euclidean formulas and conclude that the methods which have results equivalent to normalized hamming and normalized Euclidean formulas are the best methods to determine the disease of a patient.

4 Conclusion

In this we have developed two new algorithms for medical diagnosis using the proposed distance formula and similarity measures. We have solved a numerical example and compared the obtained results derived from the proposed two algorithms with the algorithms based on normalized Hamming and normalized Euclidean distance. The proposed algorithms can be extended to interval neutrosophic set environment and other neutrosophic hybrid environment for medical diagnosis.

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Editors-in-Chief:

Prof. Florentin Smarandache
Department of Mathematics and Science
University of New Mexico
705 Gurley Avenue
Gallup, NM 87301, USA
E -mail: smarand@unm.edu

Dr. Surapati Pramanik, Assistant Professor, Department of Mathematics,
Nandalal Ghosh B.T. College, Panpur,
Narayanpur, Dist-North 24 Parganas,
West Bengal, India-743126
E -mail: sura_pati@yaho.co.in

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