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Neutrosophic Sets and Systems

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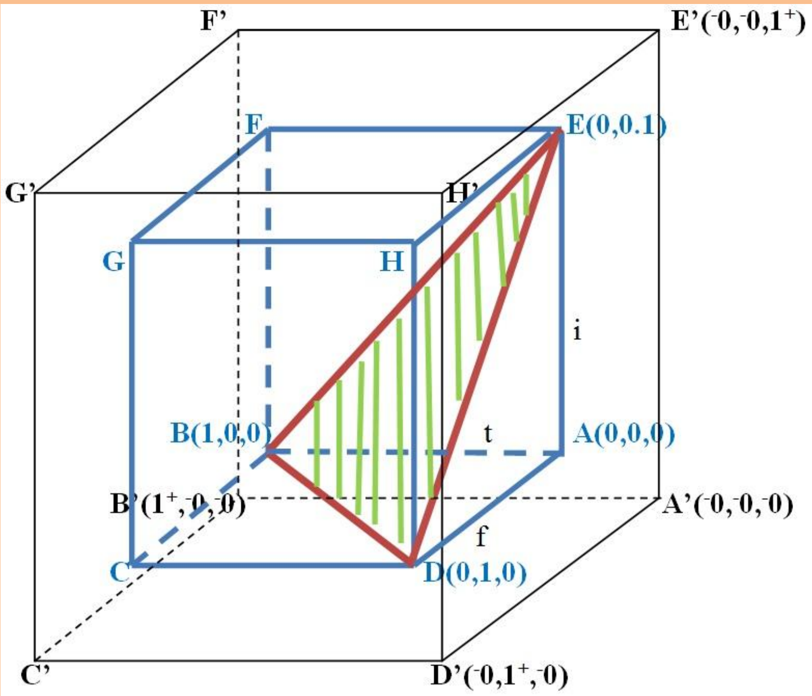

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Contents

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Information for Authors and Subscribers

Neutrosophic Sets and Systems has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Mappings on Neutrosophic Soft Classes

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Abstract. In 1995 Smarandache introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. In 2013 Maji introduced the concept of neutrosophic soft set theory as a general mathematical tool for dealing with uncertainty. In this paper we define the notion of a mapping on classes where the neutrosophic soft classes are collections of neutrosophic soft set. We also define and study the properties of neutrosophic soft images and neutrosophic soft inverse images of neutrosophic soft sets.

Keywords: soft set, fuzzy soft set, neutrosophic soft set, neutrosophic soft classes, mapping on neutrosophic soft classes, neutrosophic soft images, neutrosophic soft inverse images.

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1 Introduction

Most of the problems in engineering, medical science, economics, environments etc. have various uncertainties. in 1995, Smarandache talked for the first time about neutrosophy and in 1999 and 2005 [15, 16] he initiated the theory of neutrosophic set as a new mathematical tool for handling problems involving imprecise, indeterminacy, and inconsistent data. Molodtsov [8] initiated the concept of soft set theory as a mathematical tool for dealing with uncertainties. Chen et al. [7] and Maji et al. [11, 9] studied some different operations and application of soft sets. Furthermore Maji et al. [10] presented the definition of fuzzy soft set and Roy et al. [12] presented the applications of this notion to decision making problems. Alkhazaleh et al. [4] generalized the concept of fuzzy soft set to neutrosophic soft set and they gave some applications of this concept in decision making and medical diagnosis. They also introduced the concept of fuzzy parameterized interval-valued fuzzy soft set [3], where the mapping is defined from the fuzzy set parameters to the interval-valued fuzzy subsets of the universal set, and gave an application of this concept in decision making. Alkhazaleh and Salleh [2] introduced the concept of soft expert sets where the user can know the opinion of all experts in one model and gave an application of this concept in decision making problem. As a generalization of Molodtsov's soft set, Alkhazaleh et al. [5] presented the definition of a soft multiset and its basic operations such as complement, union and intersection. In 2012 Alkhazaleh and Salleh [6] introduced the concept of fuzzy soft multiset as a combination of soft multiset and fuzzy set and studied its properties and operations. They presented the applications of this concept to decision making problems. In 2012 Salleh et al. [1] introduced the notion of multiparameterized soft set and studied its properties. In 2010 Kharal and Ahmad [14] introduced the notion of mapping on soft classes where the soft classes are collections of soft sets. They also defined and studied the properties of soft images and soft inverse images of soft sets and gave the application of this mapping in medical diagnosis. They defined the notion of a mapping on classes of fuzzy soft sets. They also defined and studied the properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets (see [13]). In 2009 Bhowmik and Pal [18] studied the concept of intuitionistic neutrosophic set, and Maji [17] introduced neutrosophic soft set, established its application in decision making, and thus opened a new direction, new path of thinking to engineers, mathematicians, computer scientists and many others in various tests. In 2013 Said and Smarandache [19] defined the concept of intuitionistic neutrosophic soft set and introduced some operations on intuitionistic neutrosophic soft set and some properties of this concept have been established. In this paper we define the notion of a mapping on classes where the neutrosophic soft classes are collections of neutrosophic soft set. We also define and study the properties of neutrosophic soft images and neutrosophic soft inverse images of neutrosophic soft sets.

2 Preliminaries

In this section, we recall some basic notions in neutrosophic set theory, soft set theory and neutrosophic soft set theory . Smarandache defined neutrosophic set in the following way.

Definition 2.1 *[15] A neutrosophic set A on the universe of discourse X is defined as* $A=\{<x,T_A(x),I_A(x),F_A(x)>,x\in X\}$ *where* T, I, $F:X\to]^-0,1^+[$ *and* $^-0\le T_A(x)+I_A(x)+F_A(x)\le 3^+$.

Smarandache explained his concept as follows: "for example, neutrosophic logic is a generalization of the fuzzy logic. In neutrosophic logic a proposition is $T\equiv true$, $I\equiv indeterminate$, and $F\equiv false$. For example, let's analyze the following proposition: "Pakistan will win against India in the next soccer game". This proposition can be $(0.6,0.3,0.1)$ which means that there is possibility of $60\%\equiv$ that Pakistan wins, $30\%\equiv$ that Pakistan has a tie game, and $10\%\equiv$ that Pakistan looses in the next game vs. India."

Molodtsov defined soft set in the following way. Let U be a universe and E be a set of parameters. Let $P(U)$ denote the power set of U and $A\subseteq E$.

Definition 2.2 *[8] A pair (F,A) is called a soft set over U, where F is a mapping*

$$F:A\to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon\in A, F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F,A).

Definition 2.3 *[17] Let U be an initial universe set and E be a set of parameters. Consider $A\subset E$. Let $P(U)$ denotes the set of all neutrosophic sets of U. The collection (F,A) is termed to be the neutrosophic soft set (NSS in short) over U, where F is a mapping given by $F:A\to P(U)$.*

Example 2.1 *Suppose that $U=\{c_1,c_2,c_3\}$ is the set of color cloths under consideration, $A=\{e_1,e_2,e_3\}$ is the set of parameters, where e_1 stands for the parameter 'color' which consists of red, green and blue, e_2 stands for the parameter 'ingredient' which is made from wool, cotton and acrylic, and e_3 stands for the parameter 'price' which can be various: high, medium and low. We define neutrosophic soft set as follows:*

$$F(e_1)=\{<c_1,0.4,0.2,0.3>,<c_2,0.7,0.3,0.4>,<c_3,0.5,0.2,0.2>\},$$
$$F(e_2)=\{<c_1,0.6,0.2,0.6>,<c_2,0.9,0.4,0.1>,<c_3,0.4,0.3,0.3>\},$$
$$F(e_3)=\{<c_1,0.3,0.3,0.7>,<c_2,0.7,0.2,0.4>,<c_3,0.5,0.6,0.4>\}.$$

Definition 2.4 *[17] Let (F,A) and (G,B) be two neutrosophic soft sets over the common universe U. (F,A) is said to be neutrosophic soft subset of (G,B) if $A\subset B$, and* $T_F(e)(x)\le T_G(e)(x)$, $I_F(e)(x)\le I_G(e)(x)$, $F_F(e)(x)\ge F_G(e)(x)$, $\forall e\in A, x\in U$. *We denote it by $(F,A)\subseteq(G,B)$. (F,A) is said to be neutrosophic soft super set of (G,B) if (G,B) is a neutrosophic soft subset of (F,A). We denote it by $(F,A)\supseteq(G,B)$.*

Definition 2.5 *[17] Let (H,A) and (G,B) be two NSSs over the common universe U. Then the union of (H,A) and (G,B) is denoted by `$(H,A)\cup(G,B)$` and is defined by $(H,A)\cup(G,B)=(K,C)$, where $C=A\cup B$ and the truth-membership, indeterminacy-membership and falsity-membership of (K,C) are defined as follows:*

$$T_K(e)(m)=T_H(e)(m),\quad \text{if } e\in A-B,$$
$$=T_G(e)(m),\quad \text{if } e\in B-A,$$
$$=max(T_H(e)(m),T_G(e)(m)),\quad \text{if } e\in A\cap B.$$
$$I_K(e)(m)=I_H(e)(m),\quad \text{if } e\in A-B,$$
$$=I_G(e)(m),\quad \text{if } e\in B-A,$$
$$=\frac{I_H(e)(m)+I_G(e)(m))}{2},\quad \text{if } e\in A\cap B.$$
$$F_K(e)(m)=F_H(e)(m),\quad \text{if } e\in A-B,$$
$$=F_G(e)(m),\quad \text{if } e\in B-A,$$
$$=min(F_H(e)(m),F_G(e)(m)),\quad \text{if } e\in A\cap B.$$

Definition 2.6 *[17] Let (H,A) and (G,B) be two NSSs over the common universe U. Then the intersection of (H,A) and (G,B) is denoted by `$(H,A)\cap(G,B)$` and is defined by $(H,A)\cap(G,B)=(K,C)$, where $C=A\cup B$ and the truth-membership, indeterminacy-membership and falsity-membership of (K,C) are as follows:*

$$T_K(e)(m)=min(T_H(e)(m),T_G(e)(m)),$$
$$I_K(e)(m)=\frac{I_H(e)(m)+I_G(e)(m))}{2},$$
$$F_K(e)(m)=max(F_H(e)(m),F_G(e)(m)),\quad \forall e\in C.$$

Definition 2.7 *[17] Let (H,A) and (G,B) be two NSSs over the common universe U. Then the 'AND' operation on them is denoted by `$(H,A)\bigwedge(G,B)$` and is defined by $(H,A)\bigwedge(G,B)=(K,A\times B)$, where the truth-membership, indeterminacy-membership and falsity-membership of $(K,A\times B)$ are as follows:*

$$T_K(\alpha,\beta)(m)=min(T_H(\alpha)(m),T_G(\beta)(m)),$$
$$I_K(\alpha,\beta)(m)=\frac{I_H(\alpha)(m)+I_G(\beta)(m))}{2}$$ and

$$F_K(\alpha,\beta)(m) = max(F_H(\alpha)(m), F_G(\beta)(m)), \quad \forall\alpha \in A, \forall\beta \in B.$$

Definition 2.8 *[17] Let* (H,A) *and* (G,B) *be two NSSs over the common universe* U. *Then the 'OR' operation on them is denoted by* `$(H,A)\bigvee(G,B)$` *and is defined by* $(H,A)\bigvee(G,B) = (O, A\times B)$ *, where the truth-membership, indeterminacy-membership and falsity-membership of* $(O, A\times B)$ *are as follows:*

$$T_O(\alpha,\beta)(m) = max(T_H(\alpha)(m), T_G(\beta)(m)),$$

$$I_O(\alpha,\beta)(m) = \frac{I_H(\alpha)(m) + I_G(\beta)(m))}{2}$$

and

$$F_O(\alpha,\beta)(m) = min(F_H(\alpha)(m), F_G(\beta)(m)), \quad \forall\alpha \in A, \forall\beta \in B.$$

3 Mapping on Neutrosophic Soft Classes

In this section, we introduce the notion of mapping on neutrosophic soft classes. Neutrosophic soft classes are collections of neutrosophic soft sets. We also define and study the properties of neutrosophic soft images and neutrosophic soft inverse images of neutrosophic soft sets, and support them with example and theorems.

Definition 3.1 *Let* X *be a universe and* E *be a set of parameters. Then the collection of all neutrosophic soft sets over* X *with parameters from* E *is called a neutrosophic soft class and is denoted as* $(\widetilde{X,E})$.

Definition 3.2 *Let* $(\widetilde{X,E})$ *and* $(\widetilde{Y,E'})$ *be neutrosophic soft classes. Let* $r: X\to Y$ *and* $s: E\to E'$ *be mappings. Then a mapping* $f:(\widetilde{X,E})\to(\widetilde{Y,E'})$ *is defined as follows:*

For a neutrosophic soft set (F,A) in $(\widetilde{X,E})$, $f(F,A)$ is a neutrosophic soft set in $(\widetilde{Y,E'})$ obtained as follows:

$$f(F,A)(\beta)(y) = \begin{cases} \bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha} F(\alpha)\right), \\ if\ r^{-1}(y)\neq\varnothing \ and \quad s^{-1}(\beta)\cap A\neq\varnothing, \\ (0,0,0) \quad otherwise. \end{cases}$$

For $\beta\in s(E)\subseteq E'$, $y\in Y$ and $\forall\alpha\in s^{-1}(\beta)\cap A$.

$f(F,A)$ is called a *neutrosophic soft image* of the neutrosophic soft set (F,A).

Definition 3.3 *Let* $(\widetilde{X,E})$ *and* $(\widetilde{Y,E'})$ *be neutrosophic soft classes. Let* $r: X\to Y$ *and* $s: E\to E'$ *be mappings. Then a mapping* $f^{-1}:(\widetilde{Y,E'})\to(\widetilde{X,E})$ *is defined as follows:*

For a neutrosophic soft set (G,B) in $(\widetilde{Y,E'})$, $f^{-1}(G,B)$ is a neutrosophic soft set in $(\widetilde{X,E})$ obtained as follows:

$$f^{-1}(G,B)(\alpha)(x) = \begin{cases} G(s(\alpha))(r(x),) & if\ s(\alpha)\in B, \\ (0,0,0) & otherwise. \end{cases}$$

For $\alpha\in s^{-1}(B)\subseteq E$ and $x\in X$. $f^{-1}(G,B)$ is called a *neutrosophic soft inverse image* of the neutrosophic soft set (G,B).

Example 3.1 *Let* $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2,y_3\}$ *and let* $E=\{e_1,e_2,e_3\}$ *and* $E'=\{e_1',e_2',e_3'\}$. *Suppose that* $(\widetilde{X,E})$ *and* $(\widetilde{Y,E'})$ *are neutrosophic soft classes. Define* $r: X\to Y$ *and* $s: E\to E'$ *as follows:*

$$r(x_1)=y_1,\ r(x_2)=y_3,\ r(x_3)=y_3,$$
$$s(e_1)=e_1',\ s(e_2)=e_3',\ s(e_3)=e_2'.$$

Let (F,A) and (G,B) be two neutrosophic soft sets over X and Y respectively such that

$$\begin{aligned}(F,A)=\{&(e_1,\{<x_1,0.4,0.2,0.3>,<x_2,0.7,0.3,0.4>,<x_3,0.5,0.2,0.2>\}),\\ &(e_2,\{<x_1,0.2,0.2,0.7>,<x_2,0.3,0.1,0.8>,<x_3,0.2,0.3,0.6>\}),\\ &(e_3,\{<x_1,0.8,0.2,0.1>,<x_2,0.9,0.1,0.1>,<x_3,0.1,0.4,0.5>\})\},\end{aligned}$$

$$\begin{aligned}(G,B)=\{&(e_1',\{<y_1,0.2,0.4,0.5>,<y_2,0.1,0.2,0.6>,<y_3,0.2,0.5,0.3>\}),\\ &(e_2',\{<y_1,0.8,0.1,0.1>,<y_2,0.5,0.5,0.5>,<y_3,0.3,0.4,0.4>\}),\\ &(e_3',\{<y_1,0.7,0.3,0.3>,<y_2,0.9,0.2,0.1>,<y_3,0.8,0.2,0.1>\})\},\end{aligned}$$

Then we define a mapping $f:(\widetilde{X,E})\to(\widetilde{Y,E'})$ as follows:

For a neutrosophic soft set (F,A) in $(\widetilde{X,E})$, $f(F,A)$ is a neutrosophic soft set in $(\widetilde{Y,E'})$ and is obtained as follows:

$$f(F,A)(e_1')(y_1) = \left(\bigvee_{x\in r^{-1}(y_1)}\left(\bigvee_{\alpha}F(\alpha)\right)\right)$$

$$=\left(\bigvee_{x\in\{x_1\}}\left(\bigvee_{\alpha\in\{e_1\}}F(\alpha)\right)\right)$$

$$=\left(\bigvee_{x\in\{x_1\}}\left(\{<x_1,0.4,0.2,0.3>,<x_2,0.7,0.3,0.4>,<x_3,0.5,0.2,0.2>\}\right)\right)$$

$=(0.4,0.2,0.3)$.

$$f(F,A)(e_1')(y_2)=\left(\bigvee_{x\in r^{-1}(y_2)}\left(\bigvee_{\alpha}F(\alpha)\right)\right)$$

$=(0,0,0)$ as $r^{-1}(y_2)=\varnothing$.

$$f(F,A)(e_1')(y_3)=\left(\bigvee_{x\in r^{-1}(y_3)}\left(\bigvee_{\alpha}F(\alpha)\right)\right)$$

$$=\left(\bigvee_{x\in\{x_2,x_3\}}\left(\bigvee_{\alpha\in\{e_1\}}F(\alpha)\right)\right)$$

$$=\left(\bigvee_{x\in\{x_2,x_3\}}\left(\{<x_1,0.4,0.2,0.3>,<x_2,0.7,0.3,0.4>,<x_3,0.5,0.2,0.2>\}\right)\right)$$

$$=\left(\max(0.7,0.5),\frac{0.3+0.2}{2},\min(0.4,0.2)\right)=(0.7,0.25,0.2).$$

By similar calculations, consequently, we get

$$\begin{aligned}(f(F,A),B)=\{&(e_1',\{<y_1,0.4,0.2,0.3>,<y_2,0,0,0>,<y_3,0.7,0.25,0.2>\}),\\&(e_2',\{<y_1,0.8,0.2,0.1>,<y_2,0,0,0>,<y_3,0.9,0.25,0.1>\}),\\&(e_3',\{<y_1,0.2,0.2,0.7>,<y_2,0,0,0>,<y_3,0.3,0.2,0.6>\})\}.\end{aligned}$$

Next for the neutrosophic soft inverse images, the mapping $f^{-1}:(\widetilde{Y,E'})\to(\widetilde{X,E})$ is defined as follows:

For a neutrosophic soft set (G,B) in $(\widetilde{Y,E'})$, $(f^{-1}(G,B),A)$ is a neutrosophic soft set in $(\widetilde{X,E})$ obtained as follows:

$$f^{-1}(G,B)(e_1)(x_1)=\left(G(s(e_1))(r(x_1))\right)$$

$=\left(G(e_1')(y_1)\right)$

$=(0.2,0.4,0.5)$,

$$f^{-1}(G,B)(e_1)(x_2)=\left(G(s(e_1))(r(x_2))\right)$$

$=\left(G(e_1')(y_3)\right)$

$=(0.2,0.5,0.3)$,

$$f^{-1}(G,B)(e_1)(x_3)=\left(G(s(e_1))(r(x_3))\right)$$

$=\left(G(e_1')(y_3)\right)$

$=(0.2,0.5,0.3)$.

By similar calculations, consequently, we get

$$\begin{aligned}(f^{-1}(G,B),A)=\{&(e_1,\{<x_1,0.2,0.4,0.5>,<x_2,0.2,0.5,0.3>,<x_3,0.2,0.5,0.3>\}),\\&(e_2,\{<x_1,0.7,0.3,0.3>,<x_2,0.8,0.2,0.1>,<x_3,0.8,0.2,0.1>\}),\\&(e_3,\{<x_1,0.8,0.1,0.1>,<x_2,0.3,0.4,0.4>,<x_3,0.3,0.4,0.4>\})\}.\end{aligned}$$

Definition 3.4 *Let $f:(\widetilde{X,E})\to(\widetilde{Y,E'})$ be a mapping and (F,A) and (G,B) neutrosophic soft sets in $(\widetilde{X,E})$. Then for $\beta\in E'$, $y\in Y$, the neutrosophic soft union and intersection of neutrosophic soft images (F,A) and (G,B) are defined as follows:*

$$\left(f(F,A)\tilde{\bigvee}f(G,B)\right)(\beta)(y)=f(F,A)(\beta)(y)\bigvee f(G,B)(\beta)(y).$$

$$\left(f(F,A)\tilde{\wedge}f(G,B)\right)(\beta)(y)=f(F,A)(\beta)(y)\wedge f(G,B)(\beta)(y).$$

Definition 3.5 *Let $f:(\widetilde{X,E})\to(\widetilde{Y,E'})$ be a mapping and (F,A) and (G,B) neutrosophic soft sets in (X,E'). Then for $\alpha\in E$, $x\in X$, the neutrosophic soft union and intersection of neutrosophic soft inverse images (F,A) and (G,B) are defined as follows:*

$$\left(f^{-1}(F,A)\tilde{\bigvee}f^{-1}(G,B)\right)(\alpha)(x)=f^{-1}(F,A)(\alpha)(x)\bigvee f^{-1}(G,B)(\alpha)(x).$$

$$\left(f^{-1}(F,A)\tilde{\bigwedge}f^{-1}(G,B)\right)(\alpha)(x)=f^{-1}(F,A)(\alpha)(x)\bigwedge f^{-1}(G,B)(\alpha)(x).$$

Theorem 3.1 *Let $f:(\widetilde{X,E})\to(\widetilde{Y,E'})$ be a mapping. Then for neutrosophic soft sets (F,A) and (G,B) in the neutrosophic soft class $(\widetilde{X,E})$, [a.]*

a. $f(\varnothing)=\varnothing$.

b. $f(X)\subseteq Y$.

c. $f\left((F,A)\tilde{\bigvee}(G,B)\right)=f(F,A)\tilde{\bigvee}f(G,B)$.

d. $f\left((F,A)\tilde{\bigwedge}(G,B)\right)\subseteq f(F,A)\tilde{\bigwedge}f(G,B)$.

e. If $(F,A)\subseteq(G,B)$, then $f(F,A)\subseteq f(G,B)$.

Proof. For (a), (b) and (e) the proof is trivial, so we just give the proof of (c) and (d).

c. For $\beta\in E'$ and $y\in Y$, we want to prove that

$$f\left((F,A)\tilde{\bigvee}(G,B)\right)(\beta)(y)=f(F,A)(\beta)(y)\tilde{\bigvee}f(G,B)(\beta)(y)$$

For left hand side, consider

$f\left((F,A)\tilde{\bigvee}(G,B)\right)(\beta)(y)=f(H,A\cup B)(\beta)(y)$. Then

$$f(H,A\cup B)(\beta)(y)=\begin{cases}\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha}H(\alpha)\right),\\ \text{if } r^{-1}(y)\neq\varnothing \text{ and } s^{-1}(\beta)\cap(A\cup B)\neq\varnothing,\\ (0,0,0) \quad \text{otherwise.}\end{cases} \quad (1)$$

where $H(\alpha)=\cup(F(\alpha),G(\alpha))$.

Considering only the non-trivial case, then Equation 1 becomes:

$$f(H,A\cup B)(\beta)(y)=\bigvee_{x\in r^{-1}(y)}\left(\bigvee\cup(F(\alpha),G(\alpha))\right) \quad (2)$$

For right hand side and by using Definition 3.4, we have

$$\left(f(F,A)\tilde{\bigvee}f(G,B)\right)(\beta)(y)=f(F,A)(\beta)(y)\bigvee f(G,B)(\beta)(y)$$

$$=\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap A}F(\alpha)\right)\right)\bigvee\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap B}G(\alpha)\right)\right)$$

$$=\left(\bigvee_{x\in r^{-1}(y)}\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cup B)}\left(F(\alpha)\bigvee G(\alpha)\right)\right)$$

$$=\bigvee_{x\in r^{-1}(y)}\left(\bigvee\cup(F(\alpha),G(\alpha))\right) \quad (3)$$

From Equations 2 and 3, we get (c).

d. For $\beta\in E'$ and $y\in Y$, and using Definition 3.4, we have

$$f\left((F,A)\tilde{\bigwedge}(G,B)\right)(\beta)(y)=f(H,A\cup B)(\beta)(y)$$

$$=\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cup B)}H(\alpha)\right)$$

$$=\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cup B)}F(\alpha)\bigwedge G(\alpha)\right)$$

$$=\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cup B)}F(\alpha)\bigwedge G(\alpha)\right)$$

$$\subseteq\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap A}F(\alpha)\right)\right)\bigwedge\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap B}G(\alpha)\right)\right)$$

$$=f((F,A))(\beta)(y)\bigwedge((G,B))(\beta)(y)$$

$$=\left(f(F,A)\tilde{\bigwedge}f(G,B)\right)(\beta)(y).$$

This gives (d).

Theorem 3.2 *Let* $f:(\tilde{X,E})\to(\tilde{Y,E'})$ *be mapping. Then for neutrosophic soft sets* (F,A), (G,B) *in the neutrosophic soft class* (X,E'), *we have:*

1. $f^{-1}(\varnothing)=\varnothing$.
2. $f^{-1}(Y)=X$.
3. $f^{-1}\left((F,A)\tilde{\bigvee}(G,B)\right)=f^{-1}(F,A)\tilde{\bigvee}f^{-1}(G,B)$.
4. $f^{-1}\left((F,A)\tilde{\bigwedge}(G,B)\right)=f^{-1}(F,A)\tilde{\bigwedge}f^{-1}(G,B)$.
5. If $(F,A)\subseteq(G,B)$, then $f^{-1}(F,A)\subseteq f^{-1}(G,B)$.

Proof. We use the same method as in the previous proof.

4 Conclusion

In this paper we have defined the notion of a mapping on classes where the neutrosophic soft classes are collections of neutrosophic soft set. The properties of neutrosophic soft images and neutrosophic soft inverse images of neutrosophic soft sets have been defined and studied.

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On Neutrosophic Implications

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Abstract: In this paper, we firstly review the neutrosophic set, and then construct two new concepts called neutrosophic implication of type 1 and of type 2 for neutrosophic sets. Furthermore, some of their basic properties and some results associated with the two neutrosophic implications are proven.

Keywords: *Neutrosophic Implication, Neutrosophic Set, N-norm, N-conorm*.

1 Introduction

Neutrosophic set (NS) was introduced by Florentin Smarandache in 1995 [1], as a generalization of the fuzzy set proposed by Zadeh [2], interval-valued fuzzy set [3], intuitionistic fuzzy set [4], interval-valued intuitionistic fuzzy set [5], and so on. This concept represents uncertain, imprecise, incomplete and inconsistent information existing in the real world. A NS is a set where each element of the universe has a degree of truth, indeterminacy and falsity respectively and with lies in] 0^- , 1^+ [, the non-standard unit interval.

NS has been studied and applied in different fields including decision making problems [6, 7, 8], Databases [10], Medical diagnosis problem [11], topology [12], control theory [13], image processing [14, 15, 16] and so on.

In this paper, motivated by fuzzy implication [17] and intutionistic fuzzy implication [18], we will introduce the definitions of two new concepts called neutrosophic implication for neutrosophic set.

This paper is organized as follow: In section 2 some basic definitions of neutrosophic sets are presented. In section 3, we propose some sets operations on neutrosophic sets. Then, two kind of neutrosophic implication are proposed. Finally, we conclude the paper.

2 Preliminaries

This section gives a brief overview of concepts of neutrosophic sets, single valued neutrosophic sets, neutrosophic norm and neutrosophic conorm which will be utilized in the rest of the paper.

Definition 1 **(Neutrosophic set) [1]**

Let X be a universe of discourse then, the neutrosophic set A is an object having the form:

$A = \{< x: T_A\ x\ ,\ I_A\ x\ ,\ F_A\ x >, x \in X\}$, where the functions T, I, F : X→ $]^-0, 1^+[$ define respectively the degree of membership (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $x \in X$ to the set A with the condition.

$$^-0 \leq T_A\ x\ + I_A\ x + F_A\ x \leq 3^+. \qquad (1)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. So instead of $]^-0, 1^+[$, we need to take the interval [0, 1] for technical applications, because $]^-0, 1^+[$will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2 **(Single-valued Neutrosophic sets) [20]**

Let X be an universe of discourse with generic elements in X denoted by x. An SVNS A in X is characterized by a truth-membership function $T_A\ x$, an indeterminacy-membership function $I_A\ x$, and a falsity-membership function $F_A\ x$, for each point x in X, $T_A\ x$, $I_A\ x$, $F_A\ x$, $\in$ [0, 1].

When X is continuous, an SVNS A can be written as

$$A = {}_X \frac{<T_A\ x\ , I_A\ x\ , F_A\ x\ ,>}{x}, x \in X. \qquad (2)$$

When X is discrete, an SVNS A can be written as

$$A = {}_1^n \frac{<T_A\ x_i\ , I_A\ x_i\ , F_A\ x_i\ ,>}{x_i}, x_i \in X \qquad (3)$$

Definition 3 **(Neutrosophic norm, n-norm) [19]**

Mapping N_n: $(]-0,1+[\times]-0,1+[\times]-0,1+[)^2 \rightarrow]-0,1+[\times]-0,1+[\times]-0,1+[$

N_n (x(T_1, I_1, F_1), y(T_2, I_2, F_2)) = (N_nT(x,y), N_nI(x,y), N_nF(x,y), where

N_nT(.,.),N_nI(.,.),N_nF(.,.)

are the truth/membership, indeterminacy, and respectively falsehood/ nonmembership components.

N_n have to satisfy, for any x, y, z in the neutrosophic logic/set M of the universe of discourse X, the following axioms

a) Boundary Conditions: N_n (x, 0) = 0, N_n (x, 1) = x.

b) Commutativity: N_n (x, y) = N_n (y, x).

c) Monotonicity: If x ≤y, then N_n (x, z) ≤ N_n (y, z).

d) Associativity: N_n (N_n (x, y), z) = N_n (x, N_n (y, z)).

N_n represents the intersection operator in neutrosophic set theory.

Let J ∈ {T, I, F} be a component.

Most known N-norms, as in fuzzy logic and set the T-norms, are:

• The Algebraic Product N-norm: $N_{n-algebraic}$J(x, y) = x · y

• The Bounded N-Norm: $N_{n-bounded}$J(x, y) = max{0, x + y −1}

• The Default (min) N-norm: N_{n-min} (x, y) = min{x, y}.

A general example of N-norm would be this.

Let x(T_1, I_1, F_1) and y (T_2, I_2, F_2) be in the neutrosophic set M. Then:

N_n (x, y) = ($T_1 \wedge T_2$, $I_1 \vee I_2$, $F_1 \vee F_2$) (4)

where the "∧" operator is a N-norm (verifying the above N-norms axioms); while the "∨" operator, is a N-conorm.

For example, ∧ can be the Algebraic Product T-norm/N-norm, so $T_1 \wedge T_2 = T_1 \cdot T_2$ and ∨ can be the Algebraic Product T-conorm/N-conorm, so $T_1 \vee T_2 = T_1 + T_2 - T_1 \cdot T_2$

Or ∧ can be any T-norm/N-norm, and ∨ any T-conorm/N-conorm from the above.

Definition 4 **(Neutrosophic conorm, N-conorm) [19]**

Mapping N_c: ( ]-0,1+[×]-0,1+[×]-0,1+[ )2→]-0,1+[×]-0,1+[×]-0,1+[

N_c (x(T_1, I_1, F_1), y(T_2, I_2, F_2)) = (N_cT(x,y), N_cI(x,y), N_cF(x,y)),

where N_cT(.,.),N_cI(.,.),N_cF(.,.) are the truth/membership, indeterminacy, and respectively falsehood/non membership components.

N_c have to satisfy, for any x, y, z in the neutrosophic logic/set M of universe of discourse X, the following axioms:

a) Boundary Conditions: N_c (x, 1) = 1, N_c (x, 0) = x.

b) Commutativity: N_c (x, y) = N_c (y, x).

c) Monotonicity: if x ≤y, then N_c (x, z) ≤ N_c (y, z).

d) Associativity: N_c (N_c (x, y), z) = N_c (x, N_c (y, z))

N_c represents respectively the union operator in neutrosophic set theory.

Let J ∈ {T, I, F} be a component. Most known N-conorms, as in fuzzy logic and set the T-conorms, are:

• The Algebraic Product N-conorm: $N_{c-algebraic}$ J(x, y) = x + y −x · y

• The Bounded N-conorm: $N_{c-bounded}$ J(x, y) = min{1, x + y}

• The Default (max) N-conorm: N_{c-max} J(x, y) = max{x, y}.

A general example of N-conorm would be this.

Let x(T_1, I_1, F_1) and y(T_2, I_2, F_2) be in the neutrosophic set/logic M. Then:

N_c (x, y) = (T1∨T2, I1∧I2, F1∧F2) (5)

where the "∧" operator is a N-norm (verifying the above N-conorms axioms); while the "∨" operator, is a N-norm.

For example, ∧ can be the Algebraic Product T-norm/N-norm, so T1∧T2= T1·T2 and ∨ can be the Algebraic Product T-conorm/N-conorm, so T1∨T2= T1+T2-T1·T2.

Or ∧ can be any T-norm/N-norm, and ∨ any T-conorm/N-conorm from the above.

In 2013, A. Salama [21] introduced beside the intersection and union operations between two neutrosophic set A and B, another operations defined as follows:

Definition 5

Let A, B two neutrosophic sets

A $\cap_1$ B = min (T_A , T_B) ,max (I_A , I_B) , max(F_A , F_B)

A $\cup_1$ B = (max (T_A , T_B) , max (I_A , I_B) ,min(F_A , F_B))

A $\cap_2$ B={ min (T_A , T_B), min (I_A , I_B), max (F_A , F_B)}

A $\cup_2$ B = (max (T_A , T_B) , min (I_A , I_B) ,min(F_A , F_B))

A^C= (F_A , I_A , T_A).

Remark

For the sake of simplicity we have denoted:

$\cap_2$ = min min max, $\cup_2$ = max min min

$\cap_1$ = min max max, $\cup_1$ = max max min.

Where $\cap_1$, $\cup_2$ represent the intersection set and the union set proposed by Florentin Smarandache and $\cap_2$, $\cup_1$ represent the intersection set and the union set proposed by A.Salama.

3 Neutrosophic Implications

In this subsection, we introduce the set operations on neutrosophic set, which we will work with. Then, two neutrosophic implication are constructed on the basis of single valued neutrosophic set .The two neutrosophic implications are denoted by $_{NS1}$ and $_{NS2}$. Also, important properties of $_{NS1}$ and $_{NS2}$ are demonstrated and proved.

Definition 6 (Set Operations on Neutrosophic sets)

Let A and B two neutrosophic sets , we propose the following operations on NSs as follows:

$A \ @ \ B = (\frac{T_A + T_B}{2}, \frac{I_A + I_B}{2}, \frac{F_A + F_B}{2})$ where $< T_A, I_A, F_A > \in A$,$< T_B, I_B, F_B > \in B$

$A \ \$ \ B = (\overline{T_A T_B}, \ \overline{I_A \ I_B}, \ \overline{F_A F_B})$,where $< T_A, I_A, F_A > \in A$,$< T_B, I_B, F_B > \in B$

$A \ \# \ B = (\frac{2 T_A T_B}{T_A + T_B}, \frac{2 I_A I_B}{I_A + I_B}, \frac{2 F_A F_B}{F_A + F_B})$, where $< T_A, I_A, F_A > \in A$,$< T_B, I_B, F_B > \in B$

$A \oplus$ B=(T_A +T_B-T_A T_B , I_A I_B, F_A F_B) ,where $< T_A, I_A, F_A > \in A$,$< T_B, I_B, F_B > \in B$

$A \otimes$ B= (T_B T_A , I_A+I_B-I_A I_B, F_A +F_B-F_A F_B), where $< T_A, I_A, F_A > \in A$,$< T_B, I_B, F_B > \in B$

Obviously, for every two A and B, (A @ B), (A $ B), (A# B) , A⊕ B and A⊗ B are also NSs.
Based on definition of standard implication denoted by "A → B", which is equivalent to "non A or B". We extended it for neutrosophic set as follows:

Definition 7
Let A(x) ={<x**,** $T_A(x)$, $I_A(x)$, $F_A(x)$> | **x** ∈ **X**} and B(x) ={<x, $T_B(x)$, $I_B(x)$, $F_B(x)$> | x ∈ X} , A, B ∈ NS(X). So, depending on how we handle the indeterminacy, we can defined two types of neutrosophic implication, then $_{NS1}$ is the neutrosophic type1 defined as

A $_{NS1}$ B ={< x, $F_A(x) \vee T_B(x)$, $I_A(x) \wedge I_B(x)$, $T_A(x) \wedge F_B(x)$ > | x ∈ X} (6)

And
$_{NS2}$ is the neutrosophic type 2 defined as

A $_{NS2}$ B =={< x, $F_A(x) \vee T_B(x)$, $I_A(x) \vee I_B(x)$, $T_A(x) \wedge F_B(x)$ > | x ∈ X} (7)

by ∨ and ∧ we denote a neutrosophic norm (N-norm) and neutrosophic conorm (N-conorm).

Note: The neutrosophic implications are not unique, as this depends on the type of functions used in N-norm and N-conorm**.**
Throughout this paper, we used the function (dual) min/ max.

Theorem 1
For A, B and C ∈ NS(X),

i. A $\cup_1$ B $_{NS1}$ C =(A $_{NS1}$ C) $\cap_1$ (B $_{NS1}$ C)
ii. A $_{NS1}$ B $\cap_1$ C =(A $_{NS1}$ B) $\cap_1$ (A $_{NS1}$ C)
iii. A $\cap_1$ B $_{NS1}$ C = (A $_{NS1}$ C) $\cup_1$ (B $_{NS1}$ C)
iv. A $_{NS1}$ B $\cup_1$ C =(A $_{NS1}$ B)$\cup_1$ (A $_{NS1}$ C)

Proof
(i) From definition in (5) ,we have
A $\cup_1$ B $_{NS1}$ C ={<x ,Max(min(F_A , F_B), T_C) , Min(max (I_A, I_B), I_C) , Min(max (T_A , T_B), F_C) >| x ∈ X} (8)
and
(A $_{NS1}$ C) $\cap_1$ (B $_{NS1}$ C)= {<x, Min(max(F_A , T_C), max(F_B, T_C)) , Max (min (I_A , I_C), min (I_B , I_C)), Max(min (T_A , F_C), min (T_B , F_C)) >| x ∈ X} (9)
Comparing the result of (8) and (9), we get
Max(min(F_A , F_B), T_C)= Min(max(F_A , T_C), max(F_B , T_C))
Min(max (I_A , I_B), I_C)= Max (min (I_A , I_C), min (I_B , I_C))
Min(max (T_A , T_B), F_C)= Max**(**min (T_A , F_C), min (T_B , F_C))

Hence, A $\cup_1$ B $_{NS1}$ C = (A $_{NS1}$ C) $\cap_1$ (B $_{NS1}$ C)

(ii) From definition in (5), we have
A $_{NS1}$ B $\cap_1$ C={Max(F_A, min(T_B , T_C)) , Min(I_A ,max (I_B, I_C)) , Min(T_A, max (F_B , F_C) >| x ∈ X} (10)
and (A $_{NS1}$ B) $\cap_1$ (A $_{NS1}$ C) = {<x, Min (max (F_A , T_B), max(F_A , T_C)) , Max (min (I_A , I_B), min (I_A , I_C)), Max(min (T_A , F_B), min (T_A , F_C) >| x ∈ X}
(11)
Comparing the result of (10) and (11), we get
Max(F_A, min(T_B , T_C))= Min(max(F_A , T_B), max(F_A , T_C))
Min(I_A ,max (I_B , I_C))= Max (min (I_A , I_B), min (I_A , I_C))
Min(T_A, max (F_B , F_C)= Max(min (T_A , F_B), min (T_A , F_C))
Hence, A $\cap_1$ B $_{NS1}$ C = (A $_{NS1}$ C) $\cup_1$ (B $_{NS1}$ C)

(iii) From definition in (5), we have
A $\cap_1$ B $_{NS1}$ C ={< x , Max(max(F_A , F_B), T_C) , Min(min(I_A , I_B), I_C) , Min(min (T_A , T_B), F_C) >| x ∈ X} (12)
and
(A $_{NS1}$ C)$\cup_1$ (B $_{NS1}$ C) = {<x, Max(max(F_A , T_C), max(F_B , T_C)) , Max (min (I_A , I_C), min (I_B , I_C)), Min(min (T_A , F_C), min (T_B , F_C)) >| x ∈ X}
(13)
Comparing the result of (12) and (13), we get
Max(max(F_A , F_B), T_C)= Max(max(F_A , T_C), max(F_B , T_C))
Min(min(I_A , I_B), I_C)= Max (min (I_A , I_C), min (I_B, I_C))
Min(min (T_A , T_B), F_C)= Min(min (T_A , F_C), min (T_B , F_C)),
Hence, A $\cap_1$ B $_{NS1}$ C = (A $_{NS1}$ C) $\cup_1$ (B $_{NS1}$ C)
(iv) From definition in (5), we have
A $_{NS1}$ B $\cup_1$ C ={<x, Max (F_A ,Max (T_B , T_C)), Min (I_A , Max (I_B , I_C)), Min(T_A, Min(F_B , F_C)) >| x ∈ X} (14)
and
(A $_{NS1}$ B) $\cup_1$ (A $_{NS1}$ C) = {<x, Max(max (F_A, T_B), max(F_A , T_C)) , Max (min (I_A , I_B), min (I_A , I_C)), Min(min (T_A , F_B), min (T_A , F_C)) >| x ∈ X} (15)
Comparing the result of (14) and (15), we get
Max (F_A , Max (T_B , T_C)) = Max(max(F_A , T_B), max(F_A , T_C))
Min (I_A , Max (I_B , I_C)) = Max (min (I_A , I_B), min (I_A , I_C))
Min (T_A, Min(F_B , F_C)) = Min(min (T_A , F_B), min (T_A , F_C))
hence, A $_{NS1}$ B $\cup_1$ C = (A $_{NS1}$ B)$\cup_1$ (A $_{NS1}$ C)

In the following theorem, we use the operators: $\cap_2$ = min min max , $\cup_2$ = max min min.

Theorem 2 For A, B and C ∈ NS(X),

i. A $\cup_2$ B $_{NS1}$ C =(A $_{NS1}$ C) $\cap_2$ (B $_{NS1}$ C)

ii. A $_{NS1}$ B $\cap_2$ C =(A $_{NS1}$ B) $\cap_2$ (A $_{NS1}$ C)
iii. A $\cap_2$ B $_{NS1}$ C = (A $_{NS1}$ C) $\cup_2$ (B $_{NS1}$ C)
iv. A $_{NS1}$ B $\cup_2$ C =(A $_{NS1}$ B) $\cup_2$ (A $_{NS1}$ C)

Proof

The proof is straightforward.

In view of A $_{NS2}$ B ={< x, $F_A \vee T_B$, $I_A \vee I_B$, $T_A \wedge F_B$ >| x ∈ X} , we have the following theorem:

Theorem 3

For A, B and C ∈ NS(X),

i. A $\cup_1$ B $_{NS2}$ C =(A $_{NS2}$ C) $\cap_1$ (B $_{NS2}$ C)
ii. A $_{NS2}$ B $\cap_1$ C =(A $_{NS2}$ B) $\cap_1$ (A $_{NS2}$ C)
iii. A ∩ B $_{NS2}$ C = (A $_{NS2}$ C) $\cup_1$ (B $_{NS2}$ C)
iv. A $_{NS2}$ B $\cup_1$ C =(A $_{NS2}$ B)$\cup_1$ (A $_{NS2}$ C)

Proof

(i) From definition in (5), we have

A $\cup_1$ B $_{NS2}$ C ={<x, Max(min(F_A , F_B), T_C), Max(max (I_A, I_B), I_C) , Min(max (T_A , T_B), F_C) >| x ∈ X} (16)

and

(A $_{NS2}$ C)$\cap_1$ (B $_{NS2}$ C)= {<x, Min(max(F_A , T_C), max(F_B , T_C)) , Max (max (I_A , I_C), max (I_B , I_C)), Max(min (T_A , F_C), min (T_B , F_C)) >| x ∈ X} (17)

Comparing the result of (16) and (17), we get

Max(min(F_A , F_B), T_C)= Min(max(F_A , T_C), max(F_B , T_C))
Max(max (I_A , I_B), I_C)= Max (max(I_A , I_C), max (I_B , I_C))
Min(max (T_A , T_B), F_C)= Max(min (T_A , F_C), min (T_B , F_C))

hence, A $\cup_1$ B $_{NS2}$ C = (A $_{NS2}$ C) $\cap_1$ (B $_{NS2}$ C)

(ii) From definition in (5) ,we have

A $_{NS2}$ B $\cap_1$ C={<x ,Max(F_A, min(T_B , T_C)) , Max(I_A , max (I_B , I_C)) , Min(T_A, max (F_B , F_C) >| x ∈ X} (18)

and

(A $_{NS2}$ B) ∩ (A $_{NS2}$ C) = {<x,Min(max(F_A , T_B), max(F_A , T_C)) , Max (max (I_A , I_B),max (I_A , I_C)), Max(min (T_A , F_B), min (T_A , F_C)) >| x ∈ X} (19)

Comparing the result of (18) and (19), we get

Max(F_A, min(T_B , T_C))= Min(max(F_A , T_B), max(F_A , T_C))
Max(I_A ,max (I_B , I_C))= Max (max(I_A , I_B), max (I_A , I_C))
Min(T_A, max (F_B , F_C)= Max(min (T_A , F_B), min (T_A , F_C))

Hence , A $_{NS2}$ B$\cap_1$ C =(A $_{NS2}$ B) $\cap_1$ (A $_{NS2}$ C)

(iii) From definition in (5), we have

A $\cap_1$ B $_{NS2}$ C ={<x, Max(max(F_A , F_B), T_C) , Max(max (I_A, I_B), I_C) , Min(min (T_A , T_B), F_C) >| x ∈ X} (20)

and

(A $_{NS2}$ C) $\cup_1$ (B $_{NS2}$ C) = {Max(max(F_A , T_C), max(F_B, T_C)) , Max (max (I_A , I_C), max (I_B , I_C)), Min(min (T_A , F_C), min (T_B , F_C)) } (21)

Comparing the result of (20) and (21), we get

Max(max(F_A , F_B), T_C)= Max(max(F_A , T_C), max(F_B , T_C))
Max(max(I_A , I_B), I_C)= Max (max (I_A , I_C), max (I_B , I_C))
Min(min (T_A , T_B), F_C)= Min(min (T_A , F_C), min (T_B , F_C)),

hence, A $\cap_1$ B $_{NS2}$ C = (A $_{NS2}$ C) $\cup_1$ (B $_{NS2}$ C)

(iv) From definition in (5) ,we have

A $_{NS2}$ B $\cup_1$ C ={<x, Max (F_A , Max (T_B , T_C)), Max (I_A , Max (I_B , I_C)) , Min (T_A, Min(F_B , F_C))> | x ∈ X} (22)

and

(A $_{NS2}$ B) $\cup_1$ (A $_{NS2}$ C)= Max(max(F_A , T_B), max(F_A , T_C)) , Max (max (I_A , I_B), max (I_A , I_C)), Min(min (T_A , F_B), min (T_A , F_C)) (23).

Comparing the result of (22) and (23), we get

Max (F_A , Max (T_B , T_C)) = Max(max(F_A , T_B), max(F_A , T_C))
Max (I_A , Max (I_B , I_C)) = Max (max (I_A , I_B), max (I_A , I_C))
Min (T_A, Min(F_B , F_C)) = Min(min (T_A , F_B), min (T_A , F_C))

hence , A $_{NS2}$ B $\cup_1$ C =(A $_{NS2}$ B)$\cup_1$ (A $_{NS2}$ C)

Using the two operators $\cap_2$ = min min max , $\cup_2$ = max min min, we have

Theorem 4

For A, B and C ∈ NS(X),

i. A $\cup_2$ B $_{NS2}$ C =(A $_{NS2}$ C) $\cap_2$ (B $_{NS2}$ C)
ii. A $_{NS2}$ B $\cap_2$ C =(A $_{NS2}$ B) $\cap_2$ (A $_{NS2}$ C)
iii. A $\cap_2$ B $_{NS2}$ C = (A $_{NS2}$ C) $\cup_2$ (B $_{NS2}$ C)
iv. A $_{NS2}$ B $\cup_2$ C =(A $_{NS2}$ B)$\cup_2$ (A $_{NS2}$ C)

Proof

The proof is straightforward.

Theorem 5

For A, B ∈ NS(X),

i. A $_{NS2}$ B^C = A^C $\cup_1$ B^C
ii. (A $_{NS2}$ B^C)^c =(A^C $\cup_1$ B^C)^c = A $\cap_1$ B
iii. (A $_{NS1}$ B^C)^c = A $\cap_2$ B
iv. A^C $_{NS1}$ B = A $\cup_2$ B
v. A $_{NS1}$ B^C = (A $\cap_2$ B)^c

Proof

(i) From definition in (5) ,we have

A $_{NS2}$ B^C ={<x, max (F_A , F_B) ,min (I_A , I_B) , min (T_A , T_B) | x ∈ X} (24)

and

A^C $\cup_1$ B^C ={ max (F_A , F_B) ,min (I_A , I_B) , min (T_A, T_B)} (25)

From (24) and (25), we get A $_{NS2}$ $B^C = A^C \ \cup_1 B^C$

(ii) From definition in (5), we have

$A^C \ \cup_1 B^C$ ={<x, max (F_A , F_B) ,min (I_A , I_B) , min (T_A, T_B) > | x ∈ X} (26)

and

($A^C \ \cup_1 \ B^C$)^c= {<x, min (T_A , T_B),min (I_A , I_B) ,max (F_A , F_B) > | x ∈ X}

(27)

From (26) and (27), we get (A $_{NS2}$ $B^C)^c$ =($A^C \ \cup_1 \ B^C$)^c= A $\cap_1$ B

(iii) From definition in (5) ,we have

(A $_{NS1}$ $B^C)^c$ ={ <x, min (T_A , T_B), min (I_A , I_B), max (F_A, F_B) > | x ∈ X} (28)

and

A $\cap_2$ B={ min (T_A , T_B), min (I_A , I_B), max (F_A , F_B)} (29)

From (28) and (29), we get (A $_{NS1}$ $B^C)^c$ = A $\cap_2$ B

(iv)

A^C $_{NS1}$ B = $A \ \cup_2 \ B$ ={ <x, max (T_A , T_B) , min (I_A , I_B), min (F_A , F_B) > | x ∈ X}

(v)

A $_{NS1}$ B^C ={<x, max (F_A , F_B),min (I_A , I_B) , max (T_A , T_B) > | x ∈ X} (30)

and

($A \ \cap_2 \ B$)^c ={<x, max (F_A , F_B),min (I_A , I_B) , max (T_A, T_B) > | x ∈ X} (31)

From (30) and (31), we get A $_{NS1}$ B^C = ($A \ \cap_2 \ B$)^c

Theorem 6

For A, B ∈ NS(X),

i. (A B) ^{c} $_{NS1}$ (A @ B) = (A@B)^c $_{NS1}$ (A B) =(A⊕B)

ii. (A⊗B)^c $_{NS1}$ (A @ B) = (A@B)^c $_{NS1}$ (A ⊗ B) = (A@ B)

iii. (A⊗B)^c $_{NS1}$ (A # B) = (A# B)^c $_{NS1}$ (A ⊗ B) = (A# B)

iv. (A⊕B) ^{c} $_{NS1}$ (A \$ B) = (A\$ B)^c $_{NS1}$ (A ⊕ B) = (A ⊕B)

v. (A⊗B) ^{c} $_{NS1}$ (A \$ B) = (A\$ B)^c $_{NS1}$ (A ⊗ B) = (A \$ B)

vi. (A⊗B) ^{c} $_{NS1}$ (A ⊕ B) = (A⊕ B)^c $_{NS1}$ (A ⊗ B) = (A ⊕ B)

Proof

Let us recall following simple fact for any two real numbers a and b.

Max(a, b) +Min(a, b) = a +b.

Max(a, b) x Min(a, b) = a x b.

(i) From definition in (6), we have

(A⊕ B) ^{c} $_{NS1}$ (A @ B) = {<x,Max(T_A +T_B-T_A T_B, $\frac{T_A+T_B}{2}$) ,Min(I_A I_B, $\frac{I_A+I_B}{2}$) ,Min(F_A F_B, $\frac{F_A+F_B}{2}$) > | x ∈ X} = (T_A +T_B-T_A T_B, I_A I_B, F_A F_B)

= A⊕B (32)

and

(A@B)^c $_{NS1}$ (A ⊕B) = ($\frac{F_A+F_B}{2}$, $\frac{I_A+I_B}{2}$, $\frac{T_A+T_B}{2}$) $_{NS1}$ (T_A +T_B-T_A T_B , I_A I_B, F_A F_B)

= {<x, Max($\frac{T_A+T_B}{2}$, T_A +T_B-T_A T_B) ,Min($\frac{I_A+I_B}{2}$,I_A I_B) ,Min($\frac{F_A+F_B}{2}$, F_A F_B) > | x ∈ X} (33)

=(T_A +T_B-T_A T_B, I_A I_B, F_A F_B)

= A⊕B

From (32) and (33), we get the result (i)

(ii) From definition in (6), we have

$(A⊗B)^c = (T_B\, T_A\, , I_A + I_B - I_A\, I_B, F_A\, + F_B - F_A\, F_B)^c$= $(F_A\, + F_B - F_A\, F_B, I_A + I_B - I_A\, I_B, T_B\, T_A)$

(A⊗B)^c $_{NS1}$ (A @ B) =

= $(F_A\, + F_B - F_A\, F_B, I_A + I_B - I_A\, I_B, T_B\, T_A)$ $_{NS1}$ ($\frac{T_A+T_B}{2}$, $\frac{I_A+I_B}{2}$, $\frac{F_A+F_B}{2}$)

={<x, Max (T_B T_A, $\frac{T_A+T_B}{2}$) ,Min($I_B + I_B - I_A\, I_B$, $\frac{I_A+I_B}{2}$) ,Min($F_A\, + F_B - F_A\, F_B$, $\frac{F_A+F_B}{2}$) > | x ∈ X}

=($\frac{T_A+T_B}{2}$, $\frac{I_A+I_B}{2}$, $\frac{F_A+F_B}{2}$) = (A @ B) (34)

and

(A@B)^c $_{NS1}$ (A ⊗ B) =

=($\frac{F_A+F_B}{2}$, $\frac{I_A+I_B}{2}$, $\frac{T_A+T_B}{2}$) $_{NS1}$(T_A T_B, I_A +I_B-I_A I_B, F_A +F_B-F_A F_B)

={< x, Max($\frac{T_A+T_B}{2}$, T_A T_B), Min ($\frac{I_A+I_B}{2}$, I_A +I_B-I_A I_B), Min ($\frac{F_A+F_B}{2}$, F_A +F_B-F_A F_B) | x ∈ X}

=($\frac{T_A+T_B}{2}$, $\frac{I_A+I_B}{2}$, $\frac{F_A+F_B}{2}$) = (A @ B) (35)

From (34) and (35), we get the result (ii)

(iii) From definition in (6) ,we have

(A⊗B)^c $_{NS1}$ (A # B) =$(F_A\, + F_B - F_A\, F_B, I_A + I_B - I_A\, I_B, T_B\, T_A)$ $_{NS1}$ ($\frac{2\,T_A T_B}{T_A+T_B}$, $\frac{2\,I_A I_B}{I_A+I_B}$, $\frac{2\,F_A F_B}{F_A+F_B}$)

= {<x , Max (T_B T_A , $\frac{2\,T_A T_B}{T_A+T_B}$) ,Min($I_A + I_B - I_A\, I_B$, $\frac{2\,I_A I_B}{I_A+I_B}$), Min($F_A\, + F_B - F_A\, F_B$, $\frac{2\,F_A F_B}{F_A+F_B}$) > | x ∈ X}

=($\frac{2\,T_A T_B}{T_A+T_B}$, $\frac{2\,I_A I_B}{I_A+I_B}$, $\frac{2\,F_A F_B}{F_A+F_B}$)

=(A# B) (36)

and

(A# B)^c $_{NS1}$ (A ⊗ B)=($\frac{2\,F_A F_B}{F_A+F_B}$, $\frac{2\,I_A I_B}{I_A+I_B}$, $\frac{2\,T_A T_B}{T_A+T_B}$) $_{NS1}$ (T_B T_A , I_A+I_B-I_A I_B, F_A +F_B-F_A F_B)

={<x, Max ($\frac{2\,T_A T_B}{T_A+T_B}$, T_B T_A) ,Min($\frac{2\,I_A I_B}{I_A+I_B}$, I_A+I_B-I_A I_B), Min($\frac{2\,F_A F_B}{F_A+F_B}$, F_A +F_B-F_A F_B) > | x ∈ X}

$= \left(\frac{2T_A T_B}{T_A + T_B}, \frac{2I_A I_B}{I_A + I_B}, \frac{2F_A F_B}{F_A + F_B}\right) = (A\#\ B)$ (37)

From (36) and (37), we get the result (iii).

(iv) From definition in (6), we have

$(A\oplus B)^{c}\ _{NS1}\ (A\ \$\ B) = (F_A F_B,\ , I_A I_B, T_A + T_B - T_A T_B)\ _{NS1}(\ \overline{T_A T_B},\ \ \overline{I_A\ I_B},\ \ \overline{F_A F_B})$

$= \{<x, \text{Max}\ (T_A + T_B - T_A T_B,\ \ \overline{T_A T_B}), \text{Min}(I_A I_B,\ \overline{I_A\ I_B},), \text{Min}(F_A F_B,\ \ \overline{F_A F_B}) > \mid x \in X\}$

$= (\ \overline{T_A T_B},\ \ \overline{I_A\ I_B},\ \ \overline{F_A F_B})$

$= (A\ \$\ B)$ (38)

and

$(A\$ B)^{c}\ _{NS1}\ (A \oplus B) = (\ \overline{F_A F_B},\ \ \overline{I_A\ I_B},\ \ \overline{T_A T_B})\ _{NS1}(T_A + T_B - T_A T_B, I_A I_B, F_A F_B)$

$= \{<x, \text{Max}\ (\ \overline{T_A T_B}, T_A + T_B - T_A T_B), \text{Min}(\ \overline{I_A\ I_B}, I_A I_B), \text{Min}(\ \overline{F_A F_B}, F_A F_B) > \mid x \in X\}$

$= (\ \overline{T_A T_B},\ \ \overline{I_A\ I_B},\ \ \overline{F_A F_B})$

$= (A\ \$\ B)$ (39)

From (38) and (39), we get the result (iv).

(v) From definition in (6), we have

$(A\otimes B)^{c}\ _{NS1}\ (A\ \$\ B) = (F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A)\ _{NS1}\ (\ \overline{T_A T_B},\ \ \overline{I_A\ I_B},\ \ \overline{F_A F_B})$

$= \{<x, \text{Max}\ (T_B T_A,\ \ \overline{T_A T_B}), \text{Min}(I_A + I_B - I_A I_B,\ \overline{I_A\ I_B}), \text{Min}(F_A + F_B - F_A F_B,\ \ \overline{F_A F_B}) > \mid x \in X\}$

$= (\ \overline{T_A T_B},\ \ \overline{I_A\ I_B},\ \ \overline{F_A F_B})$

$= (A\ \$\ B)$ (40)

and

$(A\$ B)^{c}\ _{NS1}\ (A \otimes B) = (\ \overline{F_A F_B},\ \ \overline{I_A\ I_B},\ \ \overline{T_A T_B})\ _{NS1}(\ T_B T_A, I_A + I_B - I_A I_B, F_A + F_B - F_A F_B\ )$

$= \{<x, \text{Max}\ (\ \overline{T_A T_B}, T_B T_A\ ), \text{Min}(\ \overline{I_A\ I_B}, I_A + I_B - I_A I_B), \text{Min}(\ \overline{F_A F_B}, F_A + F_B - F_A F_B) > \mid x \in X\}$

$= (\ \overline{T_A T_B},\ \ \overline{I_A\ I_B},\ \ \overline{F_A F_B})$

$= (A\ \$\ B)$ (41)

From (40) and (41), we get the result (v).

(vi) From definition in (6), we have

$(A\otimes B)^{c}\ _{NS1}\ (A \oplus B) = (A\oplus B)^{c}\ _{NS1}\ (A \otimes B) = (A \oplus B)$

$(A\otimes B)^{c}\ _{NS1}\ (A \oplus B) = (F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A)\ _{NS1}(\ T_A + T_B - T_A T_B, I_A I_B, F_A F_B)$

$= \{<x, \text{Max}\ (T_B T_A, T_A + T_B - T_A T_B\ ), \text{Min}(I_A + I_B - I_A I_B, I_A I_B\ ), \text{Min}(F_A + F_B - F_A F_B, F_A F_B) > \mid x \in X\}$

$= (T_A + T_B - T_A T_B, I_A I_B, F_A F_B)$

$= (A \oplus B)$ (42)

and

$(A\oplus B)^{c}\ _{NS1}\ (A \otimes B) = (F_A F_B, I_A I_B, T_A + T_B - T_A T_B)\ _{NS1}\ (\ T_B T_A, I_A + I_B - I_A I_B, F_A + F_B - F_A F_B\ )$

$= \{<x, \text{Max}\ (T_A + T_B - T_A T_B, T_B T_A\ ), \text{Min}\ (I_B I_A, I_A + I_B - I_A I_B), \text{Min}(F_A F_B, F_A + F_B - F_A F_B) > \mid x \in X\}$

$= (T_A + T_B - T_A T_B, I_A I_B, F_A F_B)$

$= (A \oplus B)$ (43)

From (42) and (43), we get the result (vi).

The following theorem is not valid.

Theorem 7

For A, B $\in$ NS(X),

i. $(A\ \ \ B)\ _{NS1}\ (A @ B)^{C\ c} = (A@B)\ _{NS1}\ (A\ \ \ B)^{C\ c} = (A@B)$

ii. $(A\otimes B)\ _{NS1}\ (A @ B)^{C\ c} = (A@B)\ _{NS1}\ (A \otimes B)^{C\ c} = (A\otimes B)$

iii. $(A\oplus B)\ _{NS1}\ (A \#\ B)^{C\ c} = (A\# B)\ _{NS1}\ (A \oplus B)^{C\ c} = (A\#\ B)$

iv. $(A\otimes B)\ _{NS1}\ (A \#\ B)^{C\ c} = (A\# B)\ _{NS1}\ (A \otimes B)^{C\ c} = (A\otimes B)$

v. $(A\oplus B)\ _{NS1}\ (A\ \$\ B)^{C\ c} = (A\$ B)\ _{NS1}\ (A \oplus B)^{C\ c} = (A\ \$\ B)$

vi. $(A\otimes B)\ _{NS1}\ (A\ \$\ B)^{C\ c} = (A\$ B)\ _{NS1}\ (A \otimes B)^{C\ c} = (A \otimes B)$

Proof

The proof is straightforward.

Theorem 8

For A, B $\in$ NS(X),

i. $(A\ \ \ B)\ _{NS2}\ (A @ B)^{C\ c} = (A@B)\ _{NS2}\ (A\ \ \ B)^{C\ c} = (A@B)$

ii. $(A\otimes B)\ _{NS2}\ (A @ B)^{C\ c} = (A@B)\ _{NS2}\ (A \otimes B)^{C\ c} = (A\otimes B)$

iii. $(A\oplus B)\ _{NS2}\ (A \#\ B)^{C\ c} = (A\# B)\ _{NS2}\ (A \oplus B)^{C\ c} = (A\#\ B)$

iv. $(A\otimes B)\ _{NS2}\ (A \#\ B)^{C\ c} = (A\#B)\ _{NS2}\ (A \otimes B)^{C\ c} = (A\otimes B)$

v. $(A\oplus B)\ _{NS2}\ (A\ \$\ B)^{C\ c} = (A\$ B)\ _{NS2}\ (A \oplus B)^{C\ c} = (A\ \$\ B)$

vi. $(A\otimes B)\ _{NS2}\ (A\ \$\ B)^{C\ c} = (A\$ B)\ _{NS2}\ (A \otimes B)^{C\ c} = (A \otimes B)$

Proof

(i) From definition in (6), we have

$(A \quad B)_{NS2}$ $(A @ B)^{C\ c} = (T_A + T_B - T_A T_B\ , I_A I_B\ , F_A F_B)_{NS2}$ $\left(\frac{F_A+F_B}{2}, \frac{I_A+I_B}{2}, \frac{T_A+T_B}{2}\right)$

={<x , $\begin{matrix} \text{Max} \quad F_A F_B, \frac{F_A+F_B}{2} \ , \text{Max} \quad I_A I_B, \frac{I_A+I_B}{2} \ , \\ \text{Min} \quad T_A + T_B - T_A T_B, \frac{T_A+T_B}{2} \end{matrix}^{c}$ >| x ∈ X}

$= \frac{F_A+F_B}{2}, \frac{I_A+I_B}{2}, \frac{T_A+T_B}{2}\ ^{c}$

$=(\frac{T_A+T_B}{2}, \frac{I_A+I_B}{2}, \frac{F_A+F_B}{2})$

$=(A@B)$ (44)

and

$(A@B)_{NS2}$ $(A \oplus B)^{C\ c}$

$= (\frac{T_A+T_B}{2}, \frac{I_A+I_B}{2}, \frac{F_A+F_B}{2})_{NS2} (F_A F_B, I_A I_B, T_A + T_B - T_A T_B)^{c}$

=

$\text{Max} \ \frac{F_A+F_B}{2}, F_A F_B \ , Max \ \frac{I_A+I_B}{2}, I_A I_B \ , Min(\frac{T_A+T_B}{2}, T_A + T_B - T_A T_B)^{c}$

$= \frac{F_A+F_B}{2}, \frac{I_A+I_B}{2}, \frac{T_A+T_B}{2}\ ^{c}$

$=(\frac{T_A+T_B}{2}, \frac{I_A+I_B}{2}, \frac{F_A+F_B}{2})$

$=(A@B)$ (45)

From (44) and (45), we get the result (i).

(ii) From definition in (6) ,we have

$(A \otimes B)_{NS2}$ $(A @ B)^{C\ c}$ = $\text{Max} \quad F_A + F_B - F_A F_B, \frac{F_A+F_B}{2}$, $\text{Max} \quad I_A + I_B - I_A I_B, \frac{I_A+I_B}{2}$, $\text{Min} \quad T_B T_A, \frac{T_A+T_B}{2}\ ^{c}$

$= F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A\ ^{c}$

$=(T_B T_A\ , I_A + I_B - I_A I_B\ , F_A + F_B - F_A F_B)$

$= (A \otimes B)$ (46)

and

$(A@B)_{NS2}$ $(A \otimes B)^{C\ c}$ ={<x,$(\frac{T_A+T_B}{2}, \frac{I_A+I_B}{2}, \frac{F_A+F_B}{2})$ $(F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A)$ >| x ∈ X}

=

$\begin{matrix} \text{Max} \quad \frac{F_A+F_B}{2}, F_A + F_B - F_A F_B \ , \\ \text{Max} \quad \frac{I_A+I_B}{2}, I_A + I_B - I_A I_B \ , \text{Min} \quad \frac{T_A+T_B}{2}, T_B T_A \end{matrix}^{c}$

$= F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A\ ^{c}$

$=(T_B T_A, I_A + I_B - I_A I_B\ , F_A + F_B - F_A F_B)$

$=(A \otimes B)$ (47)

From (46) and (47), we get the result (ii).

(iii)From definition in (6), we have

$(A \oplus B)_{NS2}$ $(A \# B)^{C\ c}$ =

=

$\begin{matrix} \text{Max} \quad F_A F_B, \frac{2F_A F_B}{F_A+F_B} \ , \text{Max} \quad I_A I_B, \frac{2I_A I_B}{I_A+I_B} \ , \text{Min} \quad T_A + \\ T_B - T_A T_B\ , \frac{2T_A T_B}{T_A+T_B} \end{matrix}^{c}$

$= \frac{2F_A F_B}{F_A+F_B}, \frac{2I_A I_B}{I_A+I_B}, \frac{2T_A T_B}{T_A+T_B}\ ^{c}$

$= \frac{2T_A T_B}{T_A+T_B}, \frac{2I_A I_B}{I_A+I_B}, \frac{2F_A F_B}{F_A+F_B}$

$= (A\# B)$ (48)

and

$(A\# B)_{NS2}$ $(A \oplus B)^{C\ c} = \frac{2T_A T_B}{T_A+T_B}, \frac{2I_A I_B}{I_A+I_B}, \frac{2F_A F_B}{F_A+F_B}$ $_{NS2}$ $(F_A F_B, \ , I_A I_B, T_A + T_B - T_A T_B)$

=

$\begin{matrix} \text{Max} \quad \frac{2F_A F_B}{F_A+F_B}, F_A F_B \ , \text{Max} \quad \frac{2I_A I_B}{I_A+I_B}, I_A I_B \ , \\ \text{Min} \quad \frac{2T_A T_B}{T_A+T_B}, T_A + T_B - T_A T_B \end{matrix}^{c}$

$= \frac{2F_A F_B}{F_A+F_B}, \frac{2I_A I_B}{I_A+I_B}, \frac{2T_A T_B}{T_A+T_B}\ ^{c}$

$= \frac{2T_A T_B}{T_A+T_B}, \frac{2I_A I_B}{I_A+I_B}, \frac{2F_A F_B}{F_A+F_B}$

$= (A\# B)$ (49)

From (48) and (49), we get the result (iii).

(iv) From definition in (6), we have

$(A \otimes B)_{NS2}$ $(A \# B)^{C\ c}$ =

=

$\begin{matrix} \text{Max} \quad F_A + F_B - F_A F_B, \frac{2F_A F_B}{F_A+F_B} \ , \\ \text{Max} \quad I_A + I_B - I_A I_B, \frac{2I_A I_B}{I_A+I_B} \ , \text{Min} \quad T_B T_A, \frac{2T_A T_B}{T_A+T_B} \end{matrix}^{c}$

$= F_A + F_B - F_A F_B, \quad I_A + I_B - I_A I_B, \ T_B T_A\ ^{c}$

$= T_B T_A, \quad I_A + I_B - I_A I_B, \ F_A + F_B - F_A F_B$

$=(A \otimes B)$ (50)

and

$(A\# B)_{NS2}$ $(A \otimes B)^{C\ c}$ =

$= \begin{matrix} \text{Max} \quad \frac{2F_A F_B}{F_A+F_B}, \ F_A + F_B - \\ F_A F_B \ , \text{Max} \quad \frac{2I_A I_B}{I_A+I_B}, I_A + I_B - \\ I_A I_B \ , \ , \text{Min} \quad \frac{2T_A T_B}{T_A+T_B}, T_B T_A \end{matrix}^{c}$

$= F_A + F_B - F_A F_B, \quad I_A + I_B - I_A I_B, \ T_B T_A\ ^{c}$

$= T_B T_A, \quad I_A + I_B - I_A I_B, \ F_A + F_B - F_A F_B$

$=(A \otimes B)$ (51)

From (50) and (51), we get the result (iv).

(v) From definition in (6), we have

$(A \oplus B)_{NS2}$ $(A \$ B)^{C\ c}$ =

$= \begin{matrix} \text{Max} \quad F_A F_B, \quad \overline{F_A F_B} \ , \text{Max} \ I_A I_B\ , \quad \overline{I_A I_B} \ , \\ \text{Min} \ T_A + T_B - T_A T_B, \quad \overline{T_A T_B} \end{matrix}^{c}$

$= \overline{F_A F_B}, \ \overline{I_A I_B}, \ \overline{T_A T_B}\ ^{c}$

$=(\overline{T_A T_B}\ , \ \overline{I_A I_B}\ , \ \overline{F_A F_B})$

$=(A \$ B)$ (52)

and

$(A\$B)\ _{NS2}\ (A\oplus B)^{C\ c}=$

$=\ \text{Max}\ \ \overline{F_A F_B},\ F_A F_B\ ,\text{Max}\ \ \overline{I_A\ I_B},\ I_A I_B\ ,\ ^{c}$
$\text{Min}\ \ \overline{T_A T_B}\ ,T_A\ +T_B - T_A T_B$

$=\ \ \overline{F_A F_B},\ \ \overline{I_A\ I_B},\ \ \overline{T_A T_B}\ ^{c}$

$=(\ \overline{T_A T_B}\ ,\ \ \overline{I_A\ I_B}\ ,\ \ \overline{F_A F_B})$

$=(A\ \$\ B)$ (53)

From (52) and (53), we get the result (v).

(vi) From definition in (2), we have

$(A\otimes B)\ _{NS2}\ (A\ \$\ B)^{C\ c}$

$=$

$Max\ \ F_A\ +F_B - F_A F_B,\ \ \overline{F_A F_B}\ ,\ ^{c}$
$Max\ \ I_A + I_B - I_A I_B\ ,\ \ \overline{I_A\ I_B}\ \ ,Min\ \ T_B T_A,\ \ \overline{T_A T_B}$

$=\ F_A\ +F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A\ ^{c}$

$=\ T_B T_A, I_A + I_B - I_A I_B, F_A\ +F_B - F_A F_B$

$=(A\otimes B)$ (54)

and

$(A\$B)\ _{NS2}\ (A\otimes B)^{C\ c}=$

$=\ \text{Max}\ \ \overline{F_A F_B}\ ,\ F_A\ +F_B - F_A F_B\ \ ,\ ^{c}$
$\text{Max}\ \ \overline{I_A\ I_B}\ ,\ I_A + I_B - I_A I_B\ ,\text{Min}\ \ \overline{T_A T_B},\ T_B T_A$

$=\ F_A\ +F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A\ ^{c}$

$=\ T_B T_A, I_A + I_B - I_A I_B, F_A\ +F_B - F_A F_B$

$=(A\otimes B)$ (55)

From (54) and (55), we get the result (v).

The following are not valid.

$< T_A\ , F_A>$	$< T_B\ , F_B>$	$A\ _{NS1} B$	$A\ _{NS1} B$	$V(A\to B)$
< 0 ,1>	**< 0 ,1>**	**< 1 ,0>**	**< 1 ,0>**	**< 1 ,0>**
< 0 ,1>	**< 1 ,0>**	**< 1 ,0>**	**< 1 ,0>**	**< 1 ,0>**
< 1 ,0>	**< 0 ,1>**	**< 0 ,1>**	**< 0 ,1>**	**< 0 ,1>**
< 1 ,0>	**< 1 ,0>**	**< 1 ,0>**	**< 1 ,0>**	**< 1 ,0>**

Theorem 9

1- $(A\quad B)^{c}\ _{NS2}\ (A\ @\ B)\ =\ (A@B)^{c}\ _{NS2}\ (A\quad B)$ $=(A\oplus B)$

2- $(A\otimes B)^{c}\ _{NS2}\ (A\ @\ B)\ =$

$(A@B)^{c}\ _{NS2}\ (A\otimes B)\ =(A@\ B)$

3- $(A\oplus B)\ _{NS2}\ (A\ \#\ B)^{C\ c}\ =$

$(A\#B)\ _{NS2}\ (A\oplus B)^{C\ c}=(A\#\ B)$

4- $(A\otimes B)^{c}\ _{NS2}\ (A\ \#\ B)\ =$

$(A\#\ B)^{c}\ _{NS2}\ (A\otimes B)\ =(A\#\ B)$

5- $(A\oplus B)^{c}\ _{NS2}\ (A\ \$\ B)\ =\ (A\$\ B)^{c}$
$_{NS2}\ (A\oplus B)\ =(A\oplus B)$

6- $(A\otimes B)^{c}\ _{NS2}\ (A\ \$\ B)\ =\ (A\$\ B)^{c}$
$_{NS2}\ (A\otimes B)\ =(A\ \$\ B)$

8- $(A\otimes B)^{c}\ _{NS2}\ (A\ \$\ B)\ =\ (A\$\ B)^{c}$
$_{NS2}\ (A\otimes B)\ =(A\ \$\ B)$

9- $(A\otimes B)^{c}\ _{NS2}\ (A\oplus B)\ =\ (A\oplus B)^{c}$
$_{NS2}\ (A\otimes B)\ =(A\oplus B)$

Example

We prove only the (i)

1- $(A\quad B)^{c}\ _{NS2}\ (A\ @\ B)\ =$
$F_A F_B, I_A I_B, T_A\ +T_B - T_A T_B\ _{NS2}\ (\frac{T_A+T_B}{2}, \frac{I_A+I_B}{2}, \frac{F_A+F_B}{2})$

$=\{<x,\ \max\ (T_A\ +T_B - T_A T_B\ , \frac{T_A+T_B}{2})\ ,\max(I_A I_B, \frac{I_A+I_B}{2})\ ,\min\ (F_A F_B\ , \frac{F_A+F_B}{2}) > |\ \mathbf{x}\in \boldsymbol{X}\}$

$=\{<x,\ T_A\ +T_B - T_A T_B\ \ , \frac{I_A+I_B}{2}\ , \frac{F_A+F_B}{2} > |\ \mathbf{x}\in \boldsymbol{X}\}\neq (A\oplus B)$

The same thing, for $(A@B)^{c}\ _{NS2}\ (A\quad B)$

Then,

$(A\quad B)^{c}\ _{NS2}\ (A\ @\ B)\ =\ (A@B)^{c}$
$_{NS2}\ (A\quad B)\ \neq (A\oplus B).$

Remark

We remark that if the indeterminacy values are restricted to 0, and the membership /non-membership are restricted to 0 and 1. The results of the two neutrosophic implications $_{NS1}$ and $_{NS2}$ collapse to the fuzzy /intuitionistic fuzzy implications defined ($V(A\to B)$) in [17]

Table

Comparison of three kind of implications

From the table, we conclude that fuzzy /intuitionistic fuzzy implications are special case of neutrosophic implication.

Conclusion

In this paper, the neutrosophic implication is studied. The basic knowledge of the neutrosophic set is firstly reviewed, a two kind of neutrosophic implications are constructed, and its properties.

These implications may be the subject of further research, both in terms of their properties or comparison with other neutrosophic implication, and possible applications.

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NSS

Neutrosophic Hierarchical Clustering Algoritms

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Abstract. Interval neutrosophic set (INS) is a generalization of interval valued intuitionistic fuzzy set (IVIFS), whose the membership and non-membership values of elements consist of fuzzy range, while single valued neutrosophic set (SVNS) is regarded as extension of intuitionistic fuzzy set (IFS). In this paper, we extend the hierarchical clustering techniques proposed for IFSs and IVIFSs to SVNSs and INSs respectively. Based on the traditional hierarchical clustering procedure, the single valued neutrosophic aggregation operator, and the basic distance measures between SVNSs, we define a single valued neutrosophic hierarchical clustering algorithm for clustering SVNSs. Then we extend the algorithm to classify an interval neutrosophic data. Finally, we present some numerical examples in order to show the effectiveness and availability of the developed clustering algorithms.

Keywords: Neutrosophic set, interval neutrosophic set, single valued neutrosophic set, hierarchical clustering, neutrosophic aggregation operator, distance measure.

1 Introduction

Clustering is an important process in data mining, pattern recognition, machine learning and microbiology analysis [1, 2, 14, 15, 21]. Therefore, there are various types of techniques for clustering data information such as numerical information, interval-valued information, linguistic information, and so on. Several of them are clustering algorithms such as partitional, hierarchical, density-based, graph-based, model-based. To handle uncertainty, imprecise, incomplete, and inconsistent information which exist in real world, Smarandache [3, 4] proposed the concept of neutrosophic set (NS) from philosophical point of view. A neutrosophic set [3] is a generalization of the classic set, fuzzy set [13], intuitionistic fuzzy set [11] and interval valued intuitionistic fuzzy set [12]. It has three basic components independently of one another, which are truht-membership, indeterminacy-membership, and falsity-membership. However, the neutrosophic sets is be diffucult to use in real scientiffic or engineering applications. So Wang et al. [5, 6] defined the concepts of single valued neutrosophic set (SVNS) and interval neutrosophic set (INS) which is an instance of a neutrosophic set. At present, studies on the SVNSs and INSs is progressing rapidly in many different aspects [7, 8, 9, 10, 16, 18]. Yet, until now there has been little study on clustering the data represented by neutrosophic information [9]. Therefore, the existing clustering algorithms cannot cluster the neutrosophic data, so we need to develop some new techniques for clustering SVNSs and INSs.

2 Preliminaries

In this section we recall some definitions, operations and properties regarding NSs, SVNSs and INSs, which will be used in the rest of the paper.

2.1 Neutrosophic sets

Definition 1. [3] Let X be a space of points (objects) and $x \in X$. A neutrosophic set N in X is characterized by a truth-membership function T_N, an indeterminacy-membership function I_N and a falsity-membership function F_N, where $T_N(x)$, $I_N(x)$ and $F_N(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is, $T_N : U \to]0^-, 1^+[$, $I_N : U \to]0^-, 1^+[$ and $F_N : U \to]0^-, 1^+[$.

There is no restriction on the sum of $T_N(x)$, $I_N(x)$ and $F_N(x)$, so

$$0^- \leq \sup T_N(x) + \sup I_N(x) + \sup F_N(x) \leq 3^+.$$

Neutrosophic sets is difficult to apply in real scientific and engineering applications [5]. So Wang et al. [5] proposed the concept of SVNS, which is an instance of neutrosophic set.

2.2 Single valued neutrosophic sets

A single valued neutrosophic set has been defined in [5] as follows:

Definition 2. Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form:

$$A = \{\langle x, u_A(x), w_A(x), v_A(x)\rangle : x \in X\},$$

where $u_A: X \to [0,1]$, $w_A: X \to [0,1]$ and $v_A: X \to [0,1]$ with the condition

$$0 \leq u_A(x) + w_A(x) + v_A(x) \leq 3, \ \ \forall x \in X.$$

The numbers $u_A(x)$, $w_A(x)$ and $v_A(x)$ denote the degree of truth-membership, indeterminacy membership and falsity-membership of x to X, respectively.

Definition 3. Let A and B be two single valued neutrosophic sets,

$$A = \{\langle x, u_A(x), w_A(x), v_A(x)\rangle : x \in X\}$$
$$B = \{\langle x, u_B(x), w_B(x), v_B(x)\rangle : x \in X\}$$

Then we can give two basic operations of A and B as follows:

1. $A + B = \{< x, u_A(x) + u_B(x) - u_A(x).u_B(x),$ $w_A(x).w_B(x), v_A(x).v_B(x) >: x \in X\};$
2. $\lambda A = \{< x, 1 - (1 - u_A(x))^\lambda, (w_A(x))^\lambda, (v_A(x))^\lambda >:$ $x \in X$ and $\lambda > 0\}$

Definition 4. Let $X = \{x_1, x_2, ..., x_n\}$ be a universe of discourse. Consider that the elements x_i (i = 1,2,...,n) in the universe X may have different importance, let $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of x_i (i = 1,2,...,n), with $\omega_i \geq 0$, i = 1,2,...,n, $\sum_{i=1}^n \omega i = 1$. Assume that

$$A = \{\langle x, u_A(x), w_A(x), v_A(x)\rangle : x \in X\} \text{ and}$$
$$B = \{\langle x, u_B(x), w_B(x), v_B(x)\rangle : x \in X\}$$

be two SVNSs. Then we give the following distance measures:

The weighted Hamming distance and normalized Hamming distance [9]

$$e_1^\omega(A, B) = (\tfrac{1}{3}\textstyle\sum_{i=1}^n \omega_i(|u_A(x_i) - u_B(x_i)| + |w_A(x_i) - w_B(x_i)| + |v_A(x_i) - v_B(x_i)|)). \quad (1)$$

Assume that $\omega = (1/n, 1/n, \dots, 1/n)^T$, then Eq. (1) is reduced to the normalized Hamming distance

$$e_2^n(A, B) = (\tfrac{1}{3n}\textstyle\sum_{i=1}^n(|u_A(x_i) - u_B(x_i)| + |w_A(x_i) - w_B(x_i)| + |v_A(x_i) - v_B(x_i)|)) \quad (2)$$

The weighted Euclidean distance and normalized Euclidean distance [7]

$$e_3^\omega(A, B) = \left(\tfrac{1}{3}\textstyle\sum_{i=1}^n \omega_i(|u_A(x_i) - u_B(x_i)|)^2 + (|w_A(x_i) - w_B(x_i)|)^2 + (|v_A(x_i) - v_B(x_i)|)^2\right)^{\frac{1}{2}} \quad (3)$$

Assume that $\omega = (1/n, 1/n, ..., 1/n)^T$, then Eq. (3) is reduced to the normalized Euclidean distance

$$e_4^n(A, B) = \left(\tfrac{1}{3n}\textstyle\sum_{i=1}^n(|u_A(x_i) - u_B(x_i)|)^2 + (|w_A(x_i) - w_B(x_i)|)^2 + (|v_A(x_i) - v_B(x_i)|)^2\right)^{\frac{1}{2}} \quad (4)$$

2.3 Interval neutrosophic sets

Definition 5. [3] Let X be a set and $\mathrm{Int}[0,1]$ be the set of all closed subsets of [0,1]. An INS $\tilde{A}$ in X is defined with the form

$$\tilde{A} = \{\langle x, u_{\tilde{A}}(x), w_{\tilde{A}}(x), v_{\tilde{A}}(x)\rangle : x \in X\}$$

where $u_{\tilde{A}}: X \to \mathrm{Int}[0,1]$, $w_{\tilde{A}}: X \to \mathrm{Int}[0,1]$ and $v_{\tilde{A}}: X \to \mathrm{Int}[0,1]$ with the condition

$$0 \leq \sup u_{\tilde{A}}(x) + \sup w_{\tilde{A}}(x) + \sup v_{\tilde{A}}(x) \leq 3,$$

for all $x \in X$.

The intervals $u_{\tilde{A}}(x)$, $w_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ denote the truth-membership degree, the indeterminacy membership degree and the falsity-membership degree of x to $\tilde{A}$, respectively.

For convenience, if let

$$u_{\tilde{A}}(x) = [u_{\tilde{A}}^+(x), u_{\tilde{A}}^-(x)]$$
$$w_{\tilde{A}}(x) = [w_{\tilde{A}}^+(x), w_{\tilde{A}}^-(x)]$$
$$v_{\tilde{A}}(x) = [v_{\tilde{A}}^+(x), v_{\tilde{A}}^-(x)]$$

then

$$\tilde{A} = \{\langle x, [u_{\tilde{A}}^-(x), u_{\tilde{A}}^+(x)], [w_{\tilde{A}}^-(x), w_{\tilde{A}}^+(x)], [v_{\tilde{A}}^-(x), v_{\tilde{A}}^+(x)]\rangle\}: x \in X\}$$

with the condition

$$0 \leq \sup u_{\tilde{A}}^+(x) + \sup w_{\tilde{A}}^+(x) + \sup v_{\tilde{A}}^+(x) \leq 3,$$

for all $x \in X$. If $w_{\tilde{A}}(x) = [0,0]$ and $\sup u_{\tilde{A}}^+(x) + \sup v_{\tilde{A}}^+ \leq 1$ then $\tilde{A}$ reduces to an interval valued intuitionistic fuzzy set.

Definition 6. [20] Let $\tilde{A}$ and $\tilde{B}$ be two interval neutrosophic sets,

$$\tilde{A} = \{\langle x, [u_{\tilde{A}}^-(x), u_{\tilde{A}}^+(x)], [w_{\tilde{A}}^-(x), w_{\tilde{A}}^+(x)], [v_{\tilde{A}}^-(x), v_{\tilde{A}}^+(x)]\rangle : x \in X\},$$
$$\tilde{B} = \{\langle x, [u_{\tilde{B}}^-(x), u_{\tilde{B}}^+(x)], [w_{\tilde{B}}^-(x), w_{\tilde{B}}^+(x)], [v_{\tilde{B}}^-(x), v_{\tilde{B}}^+(x)]\rangle : x \in X\}.$$

Then two basic operations of $\tilde{A}$ and $\tilde{B}$ are given as follows:

1. $\tilde{A} + \tilde{B} = \{< x, [u_{\tilde{A}}^-(x) + u_{\tilde{B}}^-(x) -, u_{\tilde{A}}^+(x) \cdot u_{\tilde{B}}^-(x), u_{\tilde{A}}^+(x) +$ $u_{\tilde{B}}^+(x) - u_{\tilde{A}}^+(x) \cdot u_{\tilde{B}}^+(x)], [u_{\tilde{A}}^-(x) \cdot w_{\tilde{B}}^-(x), w_{\tilde{A}}^+(x) \cdot w_{\tilde{B}}^+(x)],$ $[v_{\tilde{A}}^-(x) \cdot v_{\tilde{B}}^-(x), v_{\tilde{A}}^+(x) \cdot v_{\tilde{B}}^+(x)]: x \in X\}$
2. $\lambda\tilde{A} = \{< x, \left[1 - \left(1 - u_{\tilde{A}}^-(x)\right)^\lambda, 1 - \left(1 - u_{\tilde{A}}^+(x)\right)^\lambda\right],$ $\left[\left(w_{\tilde{A}}^-(x)\right)^\lambda, \left(w_{\tilde{A}}^+(x)\right)^\lambda\right], \left[\left(v_{\tilde{A}}^-(x)\right)^\lambda, \left(v_{\tilde{A}}^+(x)\right)^\lambda\right] >: x \in X \text{ and } \lambda > 0\}.$

Definition 7. Let $X = \{x_1, x_2, \dots, x_n\}$ be a universe of discourse. Consider that the elements x_i (i = 1,2,...,n) in the universe X may have different importance, let $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of x_i (i = 1,2,...,n), with $\omega_i \geq 0$, i = 1,2,...,n, $\sum_{i=1}^n \omega_i = 1$. Suppose that $\tilde{A}$ and $\tilde{B}$ are two interval neutrosophic sets. Ye [6] has defined the distance measures for INSs as follows:

The weighted Hamming distance and normalized Hamming distance:

$$d_1^\omega(\tilde{A}, \tilde{B}) = (\tfrac{1}{6}\textstyle\sum_{i=1}^n \omega_i(|u_{\tilde{A}}^-(x) - u_{\tilde{B}}^-(x)| + |u_{\tilde{A}}^+(x) - u_{\tilde{B}}^+(x)| + |w_{\tilde{A}}^+(x) - w_{\tilde{B}}^+(x)| + |v_{\tilde{A}}^-(x) - u_{\tilde{B}}^-(x)| + |v_{\tilde{A}}^+(x) - u_{\tilde{B}}^+(x)|) \quad (5)$$

Assume that $\omega = (1/n, 1/n, \dots, 1/n)^T$, then Eq. (5) is reduced to the normalized Hamming distance

$$d_2^\omega(\tilde{A}, \tilde{B}) = (\tfrac{1}{6n}\textstyle\sum_{i=1}^n(|u_{\tilde{A}}^-(x) - u_{\tilde{B}}^-(x)| + |u_{\tilde{A}}^+(x) - u_{\tilde{B}}^+(x)| + |w_{\tilde{A}}^+(x) - w_{\tilde{B}}^+(x)| + |v_{\tilde{A}}^-(x) - u_{\tilde{B}}^-(x)| +$$

$|v_{\tilde{A}}^{+}(x) - u_{\tilde{B}}^{+}(x)|)$ (6)

The weighted Euclidean distance and normalized Hamming distance

$$d_3^{\omega}(\tilde{A},\tilde{B}) = \Big(\tfrac{1}{6}\textstyle\sum_{i=1}^{n}\omega_i(|u_{\tilde{A}}^{-}(x) - u_{\tilde{B}}^{-}(x)|^2 + |u_{\tilde{A}}^{+}(x) - u_{\tilde{B}}^{+}(x)|^2 + |w_{\tilde{A}}^{-}(x) - w_{\tilde{B}}^{-}(x)|^2 + |w_{\tilde{A}}^{+}(x) - w_{\tilde{B}}^{+}(x)|^2 + |v_{\tilde{A}}^{-}(x) - u_{\tilde{B}}^{-}(x)|^2 + |v_{\tilde{A}}^{+}(x) - u_{\tilde{B}}^{+}(x)|^2)\Big)^{\frac{1}{2}} \quad (7)$$

Assume that $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then Eq. (7) is reduced to the normalized Hamming distance

$$d_4^{n}(\tilde{A},\tilde{B}) = \Big(\tfrac{1}{6n}\textstyle\sum_{i=1}^{n}(|u_{\tilde{A}}^{-}(x) - u_{\tilde{B}}^{-}(x)|^2 + |u_{\tilde{A}}^{+}(x) - u_{\tilde{B}}^{+}(x)|^2 + |w_{\tilde{A}}^{-}(x) - w_{\tilde{B}}^{-}(x)|^2 + |w_{\tilde{A}}^{+}(x) - w_{\tilde{B}}^{+}(x)|^2 + |v_{\tilde{A}}^{-}(x) - u_{\tilde{B}}^{-}(x)|^2 + |v_{\tilde{A}}^{+}(x) - u_{\tilde{B}}^{+}(x)|^2)\Big)^{\frac{1}{2}} \quad (8)$$

Definition 8. [20] Let

$$\tilde{A}_k = \langle [u_{\tilde{A}}^{-}(x), u_{\tilde{A}}^{+}(x)], [w_{\tilde{A}}^{-}(x), w_{\tilde{A}}^{+}(x)], [v_{\tilde{A}}^{-}(x), v_{\tilde{A}}^{+}(x)] \rangle$$

$k = 1,2, \ldots, .n)$ be a collection of interval neutrosophic sets. A mapping $\tilde{F}\omega : INS^n \rightarrow INS$ is called an interval neutrosophic weighted averaging operator of dimension n if it is satisfies

$$\tilde{F}\omega(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = \textstyle\sum_{k=1}^{n}\omega_k\tilde{A}_k$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\tilde{A}_k$ ($k = 1,2,\ldots,n$), $\omega_k \in [0,1]$ and $\sum_{k=1}^{n}\omega_k = 1$.

Theorem 1. [20] Suppose that

$$\tilde{A}_k = \langle [u_{\tilde{A}}^{-}(x), u_{\tilde{A}}^{+}(x)], [w_{\tilde{A}}^{-}(x), w_{\tilde{A}}^{+}(x)], [v_{\tilde{A}}^{-}(x), v_{\tilde{A}}^{+}(x)] \rangle$$

$k = 1,2, \ldots, .n)$ are interval neutrosophic sets. Then the aggregation result through using the interval neutrosophic weighted averaging operator F_ω is an interval neutrosophic set and

$$\begin{aligned}\tilde{F}\omega(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) &= \tilde{A}_k \\ &=< \Big[1 - \textstyle\prod_{k=1}^{n}\big(1 - u_{\tilde{A}_k}^{-}(x)\big)^{\omega_k}, 1 - \prod_{k=1}^{n}\big(1 - u_{\tilde{A}_k}^{+}(x)\big)^{\omega_k}\Big], \\ &\quad \Big[\textstyle\prod_{k=1}^{n}\big(w_{\tilde{A}_k}^{-}(x)\big)^{\omega_k}, \prod_{k=1}^{n}\big(w_{\tilde{A}_k}^{+}(x)\big)^{\omega_k}\Big], \\ &\quad \Big[\textstyle\prod_{k=1}^{n}\big(v_{\tilde{A}_k}^{-}(x)\big)^{\omega_k}, \prod_{k=1}^{n}\big(v_{\tilde{A}_k}^{+}(x)\big)^{\omega_k}\Big] > \end{aligned} \quad (9)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\tilde{A}_k$ ($k = 1,2,\ldots,n$), $\omega_k \in [0,1]$ and $\sum_{k=1}^{n}\omega_k = 1$.

Suppose that $\omega = (1/n, 1/n, \ldots, 1/n)^T$ then the $\tilde{F}\omega$ is called an arithmetic average operator for INSs.

Since INS is a generalization of SVNS, according to Definition 8 and Theorem 1, the single valued neutrosophic weighted averaging operator can be easily obtained as follows.

Definition 9. *Let*

$$A_k = \langle u_{A_k}, w_{A_k}, v_{A_k} \rangle$$

$(k = 1,2,\ldots,n)$ *be* a collection single valued neutrosophic sets. A mapping $F\omega : SVNS^n \rightarrow SVNS$ *is* called a single valued neutrosophic weighted averaging operator of dimension n if it is satisfies

$$F\omega\,(A_1, A_2, \ldots, A_n) = \textstyle\sum_{k=1}^{n}\omega_k A_k$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $A_k (k = 1,2,\ldots,n)$, $\omega_k \in [0,1]$ and $\sum_{k=1}^{n}\omega_k = 1$.

Theorem 2. Suppose that

$$A_k = \langle u_{A_k}, w_{A_k}, v_{A_k} \rangle$$

$(k = 1,2,\ldots,n)$ are single valued neutrosophic sets. Then the aggregation result through using the single valued neutrosophic weighted averaging operator $F\omega$ is single neutrosophic set and

$$\begin{aligned}F\omega\,(A_1, A_2, \ldots, A_n) &= A_k \\ &=< 1 - \textstyle\prod_{k=1}^{n}\big(1 - u_{A_k}(x)\big)^{\omega_k}, \\ &\quad \textstyle\prod_{k=1}^{n}\big(w_{A_k}(x)\big)^{\omega_k}, \prod_{k=1}^{n}\big(v_{A_k}(x)\big)^{\omega_k} > \end{aligned} \quad (10)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $A_k (k = 1,2,\ldots,n)$, $\omega_k \in [0,1]$ and $\sum_{k=1}^{n}\omega_k = 1$.

Suppose that $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the $F\omega$ is called an arithmetic average operator for SVNSs.

3 Neutrosophic hierarchical algorithms

The traditional hierarchical clustering algorithm [17, 19] is generally used for clustering numerical information. By extending the traditional hierarchical clustering algorithm, Xu [22] introduced an intuitionistic fuzzy hierarchical clustering algorithm for clustering IFSs and extended it to IVIFSs. However, they fail to deal with the data information expressed in neutrosophic environment. Based on extending the intuitionistic fuzzy hierarchical clustering algorithm and its extended form, we propose the neutrosophic hierarchical algorithms which are called the single valued neutrosophic hierarchical clustering algorithm and interval neutrosophic hierarchical clustering algorithm.

Algorithm 1. Let us consider a collection of *n* SVNSs $A_k (k = 1,2,\ldots,n)$. In the first stage, the algorithm starts by assigning each of the n SVNSs to a single cluster. Based on the weighted Hamming distance (1) or the weighted Euclidean distance (3), the SVNSs $A_k (k = 1,2,\ldots,n)$ are then compared among themselves and are merged them into a single cluster according to the closest (with smaller distance) pair of clusters. The process are continued until all the SVNSs A_k are merged into one cluster i.e., clustered into a single cluster of size *n*. In each stage, only two clusters can be merged and they cannot be separated after they are merged, and the center of each cluster is recalculated by using the arithmetic average (from Eq. (10)) of the SVNSs proposed to the cluster. The distance between the centers of each cluster is considered as the distance between two clusters.

However, the clustering algorithm given above cannot cluster the interval neutrosophic data. Therefore, we need another clustering algorithm to deal with the data represented by INSs.

Algorithm 2. Let us consider a collection of n INSs $\tilde{A}_k(k = 1,2,\dots,n)$. In the first stage, the algorithm starts by assigning each of the n INSs to a single cluster. Based on the weighted Hamming distance (5) or the weighted Euclidean distance (7), the INSs $\tilde{A}_k(k = 1,2,\dots,n)$ are then compared among themselves and are merged them into a single cluster according to the closest (with smaller distance) pair of clusters. The process are continued until all the INSs $\tilde{A}_k$ are merged into one cluster i.e., clustered into a single cluster of size n. In each stage, only two clusters can be merged and they cannot be separated after they are merged, and the center of each cluster is recalculated by using the arithmetic average (from Eq. (9)) of the INSs proposed to the cluster. The distance between the centers of each cluster is considered as the distance between two clusters.

3.1 Numerical examples.

Let us consider the clustering problem adapted from [21].

Example 1. Assume that five building materials: sealant, floor varnish, wall paint, carpet, and polyvinyl chloride flooring, which are represented by the SVNSs $A_k(k = 1,2,\dots,5)$ in the feature space $X = \{x_1, x_2, \dots, x_8\}$. $\omega = (0.15,0.10,0.12,0.15,0.10,0.13,0.14,0.11)$ is the weight vector of $x_i(i = 1,2,\dots,8)$, and the given data are listed as follows:

$$A_1 = \{(x_1, 0.20,0.05,0.50), (x_2, 0.10,0.15,0.80), (x_3, 0.50,0.05,0.30), (x_4, 0.90,0.55,0.00), (x_5, 0.40,0.40,0.35), (x_6, 0.10,0.40,0.90), (x_7, 0.30,0.15,0.50), (x_8, 1.00,0.60,0.00),\}$$

$$A_2 = \{(x_1, 0.50,0.60,0.40), (x_2, 0.60,0.30,0.15)\}, (x_3, 1.00,0.60,0.00), (x_4, 0.15,0.05,0.65), (x_5, 0.00,0.25,0.80), (x_6, 0.70,0.65,0.15), (x_7, 0.50,0.50,0.30), (x_8, 0.65,0.05,0.20)\}$$

$$A_3 = \{(x_1, 0.45,0.05,0.35), (x_2, 0.60,0.50,0.30)\}, (x_3, 0.90,0.05,0.00), (x_4, 0.10,0.60,0.80), (x_5, 0.20,0.35,0.70), (x_6, 0.60,0.40,0.20), (x_7, 0.15,0.05,0.80), (x_8, 0.20,0.60,0.65)\}$$

$$A_4 = \{(x_1, 1.00,0.65,0.00), (x_2, 1.00,0.25,0.00)\}, (x_3, 0.85,0.65,0.10), (x_4, 0.20,0.05,0.80), (x_5, 0.15,0.30,0.85), (x_6, 0.10,0.60,0.70), (x_7, 0.30,0.60,0.70), (x_8, 0.50,0.35,0.70)\}$$

$$A_5 = \{(x_1, 0.90,0.20,0.00), (x_2, 0.90,0.40,0.10), (x_3, 0.80,0.05,0.10), (x_4, 0.70,0.45,0.20), (x_5, 0.50,0.25,0.15), (x_6, 0.30,0.30,0.65), (x_7, 0.15,0.10,0.75), (x_8, 0.65,0.50,0.80)\}$$

Now we utilize Algorithm 1 to classify the building materials $A_k(k = 1,2,\dots,5)$:

Step1 In the first stage, each of the SVNSs $A_k(k = 1,2,\dots,5)$ is considered as a unique cluster $\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}$.

Step2 Compare each SVNS A_k with all the other four SVNSs by using Eq. (1):

$$\begin{aligned}
e_1^\omega(A_1, A_2) &= d_1\,(A_2, A_1) = 0.6403\\
e_1^\omega(A_1, A_3) &= d_1\,(A_3, A_1) = 0.5191\\
e_1^\omega(A_1, A_4) &= d_1\,(A_4, A_1) = 0.7120\\
e_1^\omega(A_1, A_5) &= d_1\,(A_5, A_1) = 0.5435\\
e_1^\omega(A_2, A_3) &= d_1\,(A_3, A_2) = 0.5488\\
e_1^\omega(A_2, A_4) &= d_1\,(A_4, A_2) = 0.4546\\
e_1^\omega(A_2, A_5) &= d_1\,(A_5, A_2) = 0.6775\\
e_1^\omega(A_3, A_4) &= d_1\,(A_4, A_3) = 0.3558\\
e_1^\omega(A_3, A_5) &= d_1\,(A_5, A_3) = 0.2830\\
e_1^\omega(A_4, A_5) &= d_1\,(A_5, A_4) = 0.3117
\end{aligned}$$

and hence

$$e_1^\omega(A_1, A_3) = \min\{e_1^\omega(A_1, A_2), e_1^\omega(A_1, A_3), e_1^\omega(A_1, A_4), e_1^\omega(A_1, A_5)\} = 0.5191,$$

$$e_1^\omega(A_2, A_4) = \min\{e_1^\omega(A_2, A_1), e_1^\omega(A_2, A_3), e_1^\omega(A_2, A_4), e_1^\omega(A_2, A_5)\} = 0.4546,$$

$$e_1^\omega(A_3, A_5) = \min\{e_1^\omega(A_3, A_1), e_1^\omega(A_3, A_2), e_1^\omega(A_3, A_4), e_1^\omega(A_3, A_5)\} = 0.2830.$$

Then since only two clusters can be merged in each stage, the SVNSs $A_k(k = 1,2,\dots,5)$ can be clustered into the following three clusters at the second stage $\{A_1\}, \{A_2, A_4\}, \{A_3, A_5\}$.

Step3 Calculate the center of each cluster by using Eq. (10)

$$c\{A_1\} = A_1$$

$$c\{A_2, A_4\} = F_\omega(A_2, A_4) = \{(x_1, 1.00,0.62,0.00), (x_2, 1.00,0.27,0.00), (x_3, 1.00,0.62,0.00), (x_4, 0.17,0.05,0.72), (x_5, 0.07,0.27,0.82), (x_6, 0.48,0.62,0.32), (x_7, 0.40,0.54,0.45), (x_8, 0.58,0.13,0.37)\}$$

$$c\{A_3, A_5\} = F_\omega(A_3, A_5) = \{(x_1, 0.76,0.10,0.00), (x_2, 0.80,0.44,0.17), (x_3, 0.85,0.05,0.00), (x_4, 0.48,0.51,0.40), (x_5, 0.36,0.29,0.32), (x_6, 0.47,0.34,0.36), (x_7, 0.15,0.07,0.77), (x_8, 0.47,0.54,0.72)\}.$$

and then compare each cluster with the other two clusters by using Eq. (1):

$$e_1^{\omega}(c\{A_1\}, c\{A_2, A_4\}) = e_1^{\omega}(c\{A_2, A_4\}, c\{A_1\}) = 0.7101,$$
$$e_1^{\omega}(c\{A_1\}, c\{A_3, A_5\}) = e_1^{\omega}(c\{A_3, A_5\}, c\{A_1\}) = 5266,$$
$$e_1^{\omega}(c\{A_2, A_4\}, c\{A_3, A_5\}) = e_1^{\omega}(c\{A_3, A_5\}, c\{A_2, A_4\}) = 0.4879.$$

Subsequently, the SVNSs $A_k(k = 1,2,\ldots,5)$ can be clustered into the following two clusters at the third stage $\{A_1\}, \{A_2, A_3, A_4, A_5\}$.

Finally, the above two clusters can be further clustered into a unique cluster $\{A_1, A_2, A_3, A_4, A_5\}$.

All the above processes can be presented as in Fig. 1.

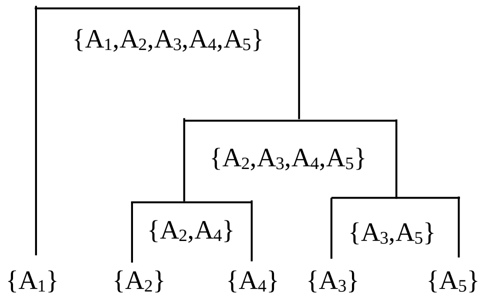

FIGURE 1: Classification of the building materials $A_k(k = 1,2,\ldots,5)$

Example 2. Consider four enterprises, represented by the INSs $\tilde{A}_k(k = 1,2,3,4)$ in the attribute set $X = \{x_1, x_2, \ldots, x_6\}$, where (1) x_1 –the ability of sale; (2) x_2 –the ability of management; (3) x_3 –the ability of production; (4) x_4 –the ability of technology; (5) x_5 –the ability of financing; (6) x_6 –the ability of risk bearing (the weight vector of $x_i(i = 1,2,\ldots,6)$ *is* $\omega = (0.25,0.20,15,0.10,0.15,0.15)$. The given data are listed as follows.

$$\tilde{A}_1 = \{(x_1, [0.70,0.75], [0.25,0.45], [0.10,0.15]), (x_2, [0.00,0.10], [0.15,0.15], [0.80,0.90]), (x_3, [0.15,0.20], [0.05,0.35], [0.60,0.65]), (x_4, [0.50,0.55], [0.45,0.55], [0.30,0.35]), (x_5, [0.10,0.15], [0.40,0.60], [0.50,0.60]), (x_6, [0.70,0.75], [0.20,0.25], [0.10,0.15])\}$$

$$\tilde{A}_2 = (x_1, [0.40,0.45], [0.00,0.15], [0.30,0.35]), (x_2, [0.60,0.65], [0.10,0.25], [0.20,0.30]), (x_3, [0.80,1.00], [0.05,0.75], [0.00,0.00]), (x_4, [0.70,0.90], [0.35,0.65], [0.00,1.00]), (x_5, [0.70,0.75], [0.15,0.55], [0.10,0.20]), (x_6, [0.90,1.00], [0.30,0.35], [0.00,0.00])\}.$$

$$\tilde{A}_3 = (x_1, [0.20,0.30], [0.85,0.60], [0.40,0.45), (x_2, [0.80,0.90], [0.10,0.25], [0.00,0.10]), (x_3, [0.10,0.20], [0.00,0.05], [0.70,0.80]), (x_4, [0.15,0.20], [0.25,0.45], [0.70,0.75]), (x_5, [0.00,0.10], [0.25,0.35], [0.80,0.90]), (x_6, [0.60,0.70], [0.15,0.25], [0.20,0.30])\}.$$

$$\tilde{A}_4 = (x_1, [0.60,0.65], [0.05,0.10], [0.30,0.35]), (x_2, [0.45,0.50], [0.45,0.55], [0.30,0.40]), (x_3, [0.20,0.25], [0.05,0.25], [0.65,0.70]), (x_4, [0.20,0.30], [0.35,0.45], [0.50,0.60]), (x_5, [0.00,0.10], [0.35,0.75], [0.75,0.80]), (x_6, [0.50,0.60], [0.00,0.05], [0.20,0.25])\}.$$

Here Algorithm 2 can be used to classify the enterprises$\tilde{A}_k(k = 1,2,3,4)$:

Step 1 In the first stage, each of the INSs $\tilde{A}_k(k = 1,2,3,4)$ is considered as a unique cluster $\{\tilde{A}_1\}, \{\tilde{A}_2\}, \{\tilde{A}_3\}, \{\tilde{A}_4\}$

Step 2 Compare each INS $\tilde{A}_k$ with all the other three INSs by using Eq. (5)

$$d_1^{\omega}(\tilde{A}_1, \tilde{A}_2) = d_1^{\omega}(\tilde{A}_2, \tilde{A}_1) = 0.3337,$$
$$d_1^{\omega}(\tilde{A}_1, \tilde{A}_3) = d_1^{\omega}(\tilde{A}_3, \tilde{A}_1) = 0.2937,$$
$$d_1^{\omega}(\tilde{A}_1, \tilde{A}_4) = d_1^{\omega}(\tilde{A}_4, \tilde{A}_1) = 0.2041,$$
$$d_1^{\omega}(\tilde{A}_2, \tilde{A}_3) = d_1^{\omega}(\tilde{A}_3, \tilde{A}_2) = 0.3508,$$
$$d_1^{\omega}(\tilde{A}_2, \tilde{A}_4) = d_1^{\omega}(\tilde{A}_4, \tilde{A}_2) = 0.2970,$$
$$d_1^{\omega}(\tilde{A}_3, \tilde{A}_4) = d_1^{\omega}(\tilde{A}_4, \tilde{A}_3) = 0.2487,$$

then the INSs $\tilde{A}_k(k = 1,2,3,4)$ can be clustered into the following three clusters at the second stage $\{\tilde{A}_1, \tilde{A}_4\}, \{\tilde{A}_2\}, \{\tilde{A}_3\}$.

Step 3 Calculate the center of each cluster by using Eq. (9)

$$c\{\tilde{A}_2\} = \{\tilde{A}_2\}, c\{\tilde{A}_3\} = \{\tilde{A}_3\},$$
$$c\{\tilde{A}_1, \tilde{A}_4\} = F_{\omega}(\tilde{A}_1, \tilde{A}_4) = (x_1, [0.60,0.70], [0.11,0.21], [0.17,0.22]), (x_2, [0.25,0.32], [0.25,0.28], [0.48,0.60]), (x_3, [0.17,0.22], [0.05,0.29], [0.62,0.67]), (x_4, [0.36,0.43], [0.39,0.49], [0.38,0.45]), (x_5, [0.05,0.12], [0.37,0.67], [0.61,0.69]), (x_6, [0.61,0.68], [0.00,0.011], [0.14,0.19])\}.$$

and then compare each cluster with the other two clusters by using Eq. (5)

$d_1^{\omega}(c\{\tilde{A}_2\}, c\{\tilde{A}_3\}) = d_1^{\omega}(c\{\tilde{A}_3\}, c\{\tilde{A}_2\}) = 0.3508$
$d_1^{\omega}(c\{\tilde{A}_2\}, c\{\tilde{A}_1, \tilde{A}_4\}) = d_1^{\omega}(c\{\tilde{A}_4, \tilde{A}_1\}, c\{\tilde{A}_2\}) = 0.3003$
$d_1^{\omega}(c\{\tilde{A}_3\}, c\{\tilde{A}_1, \tilde{A}_4\}) = d_1^{\omega}(c\{\tilde{A}_4, \tilde{A}_1\}, c\{\tilde{A}_3\}) = 0.2487.$

then the INSs $\tilde{A}_k(k = 1,2,3,4)$ can be clustered into the following two clusters in the third stage $\{\tilde{A}_2\}, \{\tilde{A}_1, \tilde{A}_3, \tilde{A}_4\}$.

In the final stage, the above two clusters can be further clustered into a unique cluster $\{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4\}$.

Note that the clustering results obtained in Example 1 and 2 are different from ones in [21].

All the above processes can be presented as in Fig. 2.

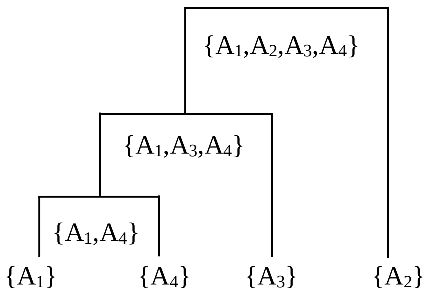

FIGURE 2: Classification of the enterprises $\tilde{A}_k(k = 1,2,3,4)$

Interval neutrosophic information is a generalization of interval valued intuitionistic fuzzy information while the single valued neutrosophic information extends the intuitionistic fuzzy information. In other words, The components of IFS and IVIFS are defined with respect to T and F, i.e., membership and nonmembership only, so they can only handle incomplete information but not the indetermine information. Hence INS and SVNS, whose components are the truth membership, indeterminacy-membership and falsity membership functions, are more general than others that do not include the indeterminacy-membership. Therefore, it is a natural outcome that the neutrosophic hierarchical clustering algorithms developed here is the extension of both the intuitionistic hierarchical clustering algorithm and its extend form. The above expression clearly indicates that clustering analysis under neutrossophic environment is more general and more practical than existing hierarchical clustering algorithms.

4 Conclusion

To cluster the data represented by neutrosophic information, we have discussed on the clustering problems of SVNSs and INSs. Firstly, we have proposed a single valued neutrosophic hierarchical algorithm for clustering SVNSs, which is based on the traditional hierarchical clustering procedure, the single valued neutrosophic aggregation operator, and the basic distance measures between SVNSs. Then, we have extented the algorithm to INSs for clustering interval neutrosophic data. Finally, an illustrative example is presented to demonstrate the application and effectiveness of the developed clustering algorithms. Since the NSs are a more general platform to deal with uncertainties, the proposed neutrosophic hierarchical algorithms are more priority than the other ones. In the future we will focus our attention on the another clustering methods of neutrosophic information.

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Neutrosophic Crisp Sets & Neutrosophic Crisp Topological Spaces

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Abstract. In this paper, we generalize the crisp topological spaces to the notion of neutrosophic crisp topological space, and we construct the basic concepts of the neutrosophic crisp topology. In addition to these, we introduce the definitions of neutrosophic crisp continuous function and neutrosophic crisp compact spaces. Finally, some characterizations concerning neutrosophic crisp compact spaces are presented and one obtains several properties. Possible application to GIS topology rules are touched upon.

Keywords: *Neutrosophic Crisp Set; Neutrosophic Topology; Neutrosophic Crisp Topology.*

1 Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their crisp and fuzzy counterparts, the most used one being the neutrosophic set theory [6, 7, 8]. After the introduction of the neutrosophic set concepts in [1, 2, 3, 4, 5, 9, 10, 11, 12] and after haven given the fundamental definitions of neutrosophic set operations, we generalize the crisp topological space to the notion of neutrosophic crisp set. Finally, we introduce the definitions of neutrosophic crisp continuous function and neutrosophic crisp compact space, and we obtain several properties and some characterizations concerning the neutrosophic crisp compact space.

2 Terminology

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [6, 7, 8, 12], and Salama et al. [1, 2, 3, 4, 5, 9, 10, 11, 12]. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $]^-0, 1^+[$ is non-standard unit interval.

Hanafy and Salama et al. [10, 12] considered some possible definitions for basic concepts of the neutrosophic crisp set and its operations. We now improve some results by the following.

3 Neutrosophic Crisp Sets

3.1 Definition

Let X be a non-empty fixed set. A neutrosophic crisp set (NCS for short) A is an object having the form $A = \langle A_1, A_2, A_3 \rangle$ where A_1, A_2 and A_3 are subsets of X satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$.

3.1 Remark

A neutrosophic crisp set $A = \langle A_1, A_2, A_3 \rangle$ can be identified as an ordered triple $\langle A_1, A_2, A_3 \rangle$, where A_1, A_2, A_3 are subsets on X, and one can define several relations and operations between NCSs.

Since our purpose is to construct the tools for developing neutrosophic crisp sets, we must introduce the types of NCSs ϕ_N, X_N in X as follows:

1) ϕ_N may be defined in many ways as a NCS, as follows:

i) $\phi_N = \langle \phi, \phi, X \rangle$, or

ii) $\phi_N = \langle \phi, X, X \rangle$, or

iii) $\phi_N = \langle \phi, X, \phi \rangle$, or

iv) $\phi_N = \langle \phi, \phi, \phi \rangle$

2) X_N may also be defined in many ways as a NCS:

i) $X_N = \langle X, \phi, \phi \rangle$,

ii) $X_N = \langle X, X, \phi \rangle$,

iii) $X_N = \langle X, X, \phi \rangle$,

Every crisp set A formed by three disjoint subsets of a non-empty set is obviously a NCS having the form $A = \langle A_1, A_2, A_3 \rangle$.

3.2 Definition

Let $A = \langle A_1, A_2, A_3 \rangle$ a NCS on , then the complement of the set A, (A^c for short may be defined in three different ways:

C_1 $A^c = \langle A_1^{\ c}, A_2^{\ c}, A_3^{\ c} \rangle$,

C_2 $A^c = \langle A_3, A_2, A_1 \rangle$

C_3 $A^c = \langle A_3, A_2^{\ c}, A_1 \rangle$

One can define several relations and operations between NCSs as follows:

3.3 Definition

Let X be a non-empty set, and the NCSs A and B in the form $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$, then we may consider two possible definitions for subsets $A \subseteq B$

$A \subseteq B$ may be defined in two ways:

1) $A \subseteq B \Leftrightarrow A_1 \subseteq B_1, A_2 \subseteq B_2$ and $A_3 \supseteq B_3$
or

2) $A \subseteq B \Leftrightarrow A_1 \subseteq B_1, A_2 \supseteq B_2$ and $A_3 \supseteq B_3$

3.1 Proposition

For any neutrosophic crisp set A the following hold:

i) $\phi_N \subseteq A$, $\phi_N \subseteq \phi_N$.

ii) $A \subseteq X_N$, $X_N \subseteq X_N$.

3.4 Definition

Let X is a non-empty set, and the NCSs A and B in the form $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$. Then:

1) $A \cap B$ may be defined in two ways:

i) $A \cap B = \langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3 \rangle$ or

ii) $A \cap B = \langle A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3 \rangle$

2) $A \cup B$ may also be defined in two ways:

i) $A \cup B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3 \rangle$ or

ii) $A \cup B = \langle A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3 \rangle$

3) $[\ ]A = \langle A_1, A_2, A_1^{\ c} \rangle$.

4) $<> A = \langle A_3^{\ c}, A_2, A_3 \rangle$.

3.2 Proposition

For all two neutrosophic crisp sets A and B on X, then the followings are true:

1) $(A \cap B)^c = A^c \cup B^c$.

2) $(A \cup B)^c = A^c \cap B^c$.

We can easily generalize the operations of intersection and union in definition 3.2 to arbitrary family of neutrosophic crisp subsets as follows:

3.3 Proposition

Let $\{A_j : j \in J\}$ be arbitrary family of neutrosophic crisp subsets in X, then

1) $\cap A_j$ may be defined as the following types :

i) $\cap A_j = \langle \cap Aj_1, \cap A_{j_2}, \cup A_{j_3} \rangle$, or

ii) $\cap A_j = \langle \cap Aj_1, \cup A_{j_2}, \cup A_{j_3} \rangle$.

2) $\cup A_j$ may be defined as the following types :

i) $\cup A_j = \langle \cup Aj_1, \cap A_{j_2}, \cap A_{j_3} \rangle$ or

ii) $\cup A_j = \langle \cup Aj_1, \cup A_{j_2}, \cap A_{j_3} \rangle$.

3.5 Definition

The product of two neutrosophic crisp sets A and B is a neutrosophic crisp set $A \times B$ given by
$A \times B = \langle A_1 \times B_1, A_2 \times B_2, A_3 \times B_3 \rangle$.

4 Neutrosophic Crisp Topological Spaces

Here we extend the concepts of topological space and intuitionistic topological space to the case of neutrosophic crisp sets.

4.1 Definition

A neutrosophic crisp topology (NCT for short) on a non-empty set is a family of neutrosophic crisp subsets in satisfying the following axioms

i) $\phi_N, X_N \in \Gamma$.

ii) $A_1 \cap A_2 \in \Gamma$ for any A_1 and $A_2 \in \Gamma$.

iii) $\cup A_j \in \Gamma \quad \forall \{A_j : j \in J\} \subseteq \Gamma$.

In this case the pair (X, Γ) is called a neutrosophic crisp topological space ($NCTS$ for short) in X. The elements in Γ are called neutrosophic crisp open sets (NCOSs for short) in X. A neutrosophic crisp set F is closed if and only if its complement F^C is an open neutrosophic crisp set.

4.1 Remark

Neutrosophic crisp topological spaces are very natural generalizations of topological spaces and intuitionistic topological spaces, and they allow more general functions to be members of topology.

$$TS \longrightarrow ITS \longrightarrow NCTS$$

4.1 Example

Let $X = \{a, b, c, d\}$, ϕ_N, X_N be any types of the universal and empty subsets, and A, B two neutrosophic crisp subsets on X defined by $A = \langle\{a\}, \{b, d\}, \{c\}\rangle$, $B = \langle\{a\}, \{b\}, \{c\}\rangle$, then the family $\Gamma = \{\phi_N, X_N, A, B\}$ is a neutrosophic crisp topology on X.

4.2 Example

Let $(X, \tau_\circ)$ be a topological space such that $\tau_\circ$ is not indiscrete. Suppose $\{G_i : i \in J\}$ be a family and $\tau_\circ = \{X, \phi\} \cup \{G_i : i \in J\}$. Then we can construct the following topologies as follows

i) Two intuitionistic topologies

a) $\tau_1 = \{\phi_I, X_I\} \cup \{\langle G_i, \phi\rangle, i \in J\}$.

b) $\tau_2 = \{\phi_I, X_I\} \cup \left\{\left\langle \phi, G_i^c \right\rangle, i \in J\right\}$

ii) Four neutrosophic crisp topologies

a) $\Gamma_1 = \{\phi_N, X_N\} \cup \left\{\left\langle \phi, \phi, G_i^c \right\rangle, i \in J\right\}$

b) $\Gamma_2 = \{\phi_N, X_N\} \cup \{\langle G_i, \phi, \phi\rangle, i \in J\}$

c) $\Gamma_3 = \{\phi_N, X_N\} \cup \left\{\left\langle G_i, \phi, G_i^c \right\rangle, i \in J\right\}$,

d) $\Gamma_4 = \{\phi_N, X_N\} \cup \left\{\left\langle G_i^c \ , \phi, \phi \right\rangle, i \in J\right\}$

4.2 Definition

Let $(X, \Gamma_1), (X, \Gamma_2)$ be two neutrosophic crisp topological spaces on X. Then Γ_1 is said be contained in Γ_2 (in symbols $\Gamma_1 \subseteq \Gamma_2$) if $G \in \Gamma_2$ for each $G \in \Gamma_1$. In this case, we also say that Γ_1 is coarser than Γ_2.

4.1 Proposition

Let $\{\Gamma_j : j \in J\}$ be a family of NCTs on X. Then $\cap\Gamma_j$ is a neutrosophic crisp topology on X. Furthermore, $\cap\Gamma_j$ is the coarsest NCT on X containing all topologies.

Proof

Obvious. Now, we define the neutrosophic crisp closure and neutrosophic crisp interior operations on neutrosophic crisp topological spaces:

4.3 Definition

Let (X, Γ) be NCTS and $A = \langle A_1, A_2, A_3\rangle$ be a NCS in X. Then the *neutrosophic crisp closure* of A ($NCCl(A)$ for short) and *neutrosophic interior crisp* ($NCInt\ (A)$ for short) of A are defined by

$$NCCl(A) = \cap\{K : K \text{ is an NCS in X and } A \subseteq K\}$$
$$NCInt\ (A) = \cup\{G : G \text{ is an NCOS in X and } G \subseteq A\},$$

where NCS is a neutrosophic crisp set, and NCOS is a neutrosophic crisp open set.

It can be also shown that $NCCl\ (A)$ is a NCCS (neutrosophic crisp closed set) and $NCInt\,(A)$ is a CNOS in X

a) A is in X if and only if $NCCl(A) \supseteq A$.

b) A is a NCCS in X if and only if $NCInt(A) = A$.

4.2 Proposition

For any neutrosophic crisp set A in (X, Γ) we have

(a) $NCCl(A^c) = (NCInt(A))^c$,

(b) $NCInt(A^c) = (NCCl(A))^c$.

Proof

a) Let $A = \langle A_1, A_2, A_3\rangle$ and suppose that the family of neutrosophic crisp subsets contained in A are indexed by the family if NCSs contained in A are indexed by the family $A = \{< A_{j1}, A_{j2}, A_{j3} >: i \in J\}$. Then we see that we have two types of $NCInt(A) = \{< \cup A_{j1}, \cup A_{j2}, \cap A_{j3} >\}$ or $NCInt(A) = \{< \cup A_{j1}, \cap A_{j2}, \cap A_{j3} >\}$ hence $(NCInt(A))^c = \{< \cap A_{j1}, \cap A_{j2}, \cup A_{j3} >\}$ or $(NCInt(A))^c = \{< \cap A_{j1}, \cup A_{j2}, \cup A_{j3} >\}$.

Hence $NCCl(A^c) = (NCInt(A))^c$, which is analogous to (a).

4.3 Proposition

Let (X,Γ) be a NCTS and A,B be two neutrosophic crisp sets in X. Then the following properties hold:

(a) $NCInt(A)\subseteq A,$
(b) $A\subseteq NCCl(A),$
(c) $A\subseteq B \Rightarrow NCInt(A)\subseteq NCInt(B),$
(d) $A\subseteq B \Rightarrow NCCl(A)\subseteq NCCl(B),$
(e) $NCInt(A\cap B)=NCInt(A)\cap NCInt(B),$
(f) $NCCl(A\cup B)=NCCl(A)\cup NCCl(B),$
(g) $NCInt(X_N)=X_N,$
(h) $NCCl(\phi_N)=\phi_N$

Proof. (a), (b) and (e) are obvious; (c) follows from (a) and from definitions.

5 Neutrosophic Crisp Continuity

Here come the basic definitions first

5.1 Definition

(a) If $B=\langle B_1,B_2,B_3\rangle$ is a NCS in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is a NCS in X defined by $f^{-1}(B)=\left\langle f^{-1}(B_1), f^{-1}(B_2), f^{-1}(B_3)\right\rangle$.

(b) If $A=\langle A_1,A_2,A_3\rangle$ is a NCS in X, then the image of A under f, denoted by $f(A)$, is the a NCS in Y defined by $f(A)=\left\langle f(A_1), f(A_2), f(A_3)^c\right\rangle$.

Here we introduce the properties of images and preimages some of which we shall frequently use in the following sections .

5.1 Corollary

Let A, $\{A_i : i\in J\}$, be NCSs in X, and B, $\{B_j : j\in K\}$ NCS in Y, and $f:X\to Y$ a function. Then

(a) $A_1\subseteq A_2 \Leftrightarrow f(A_1)\subseteq f(A_2),$
$B_1\subseteq B_2 \Leftrightarrow f^{-1}(B_1)\subseteq f^{-1}(B_2),$

(b) $A\subseteq f^{-1}(f(A))$ and if f is injective, then $A=f^{-1}(f(A))$.

(c) $f^{-1}(f(B))\subseteq B$ and if f is surjective, then $f^{-1}(f(B))=B,$.

(d) $f^{-1}(\cup B_i)=\cup f^{-1}(B_i),\ f^{-1}(\cap B_i)=\cap f^{-1}(B_i),$

(e) $f(\cup A_i)=\cup f(A_i);\ f(\cap A_i)\subseteq \cap f(A_i);$ and if f is injective, then $f(\cap A_i)=\cap f(A_i);$

(f) $f^{-1}(Y_N)=X_N,\ f^{-1}(\phi_N)=\phi_N$.

(g) $f(\phi_N)=\phi_N,\ f(X_N)=Y_N,$ if f is subjective.

Proof

Obvious.

5.2 Definition

Let (X,Γ_1) and (Y,Γ_2) be two NCTSs, and let $f:X\to Y$ be a function. Then f is said to be continuous iff the preimage of each NCS in Γ_2 is a NCS in Γ_1.

5.3 Definition

Let (X,Γ_1) and (Y,Γ_2) be two NCTSs and let $f:X\to Y$ be a function. Then f is said to be open iff the image of each NCS in Γ_1 is a NCS in Γ_2.

5.1 Example

Let (X,Γ_o) and (Y,ψ_o) be two NCTSs

(a) If $f:X\to Y$ is continuous in the usual sense, then in this case, f is continuous in the sense of Definition 5.1 too. Here we consider the NCTs on X and Y, respectively, as follows :
$\Gamma_1=\{\langle G,\phi,G^c\rangle : G\in\Gamma_o\}$ and
$\Gamma_2=\{\langle H,\phi,H^c\rangle : H\in\Psi_o\},$
In this case we have, for each $\langle H,\phi,H^c\rangle\in\Gamma_2,$ $H\in\Psi_o,$

$$f^{-1}\langle H,\phi,H^c\rangle=\left\langle f^{-1}(H), f^{-1}(\phi), f^{-1}(H^c)\right\rangle$$
$$=\left\langle f^{-1}H, f(\phi), (f(H))^c\right\rangle\in\Gamma_1.$$

(b) If $f:X\to Y$ is open in the usual sense, then in this case, f is open in the sense of Definition 3.2. Now we obtain some characterizations of continuity:

5.1 Proposition

Let $f:(X,\Gamma_1)\to(Y,\Gamma_2)$.

f is continuous if the preimage of each CNCS (crisp neutrosophic closed set) in Γ_2 is a CNCS in Γ_2.

5.2 Proposition

The following are equivalent to each other:

(a) $f:(X,\Gamma_1)\to(Y,\Gamma_2)$ is continuous.
(b) $f^{-1}(CNInt(B)\subseteq CNInt(f^{-1}(B))$ for each CNS B in Y.
(c) $CNCl(f^{-1}(B))\subseteq f^{-1}(CNCl(B))$ for each CNC B in Y.

5.2 Example

Let (Y, Γ_2) be a NCTS and $f : X \to Y$ be a function. In this case $\Gamma_1 = \{f^{-1}(H) : H \in \Gamma_2\}$ is a NCT on X. Indeed, it is the coarsest NCT on X which makes the function $f : X \to Y$ continuous. One may call it the initial neutrosophic crisp topology with respect to f.

6 Neutrosophic Crisp Compact Space (NCCS)

First we present the basic concepts:

6.1 Definition

Let (X, Γ) be an NCTS.

(a) If a family $\{\langle G_{i_1}, G_{i_2}, G_{i_3}\rangle : i \in J\}$ of NCOSs in X satisfies the condition $\cup\{\langle X, G_{i_1}, G_{i_2}, G_{i_3}\rangle : i \in J\} = X_N$, then it is called an neutrosophic open cover of X.

(b) A finite subfamily of an open cover $\{\langle G_{i_1}, G_{i_2}, G_{i_3}\rangle : i \in J\}$ on X, which is also a neutrosophic open cover of X , is called a neutrosophic finite subcover $\{\langle G_{i_1}, G_{i_2}, G_{i_3}\rangle : i \in J\}$.

(c) A family $\{\langle K_{i_1}, K_{i_2}, K_{i_3}\rangle : i \in J\}$ of NCCSs in X satisfies the finite intersection property (FIP for short) iff every finite subfamily $\{\langle K_{i_1}, K_{i_2}, K_{i_3}\rangle : i = 1,2,\ldots,n\}$ of the family satisfies the condition $\cap\{\langle K_{i_1}, K_{i_2}, K_{i_3}\rangle : i \in J\} \neq \phi_N$.

6.2 Definition

A NCTS (X, Γ) is called neutrosophic crisp compact iff each crisp neutrosophic open cover of X has a finite subcover.

6.1 Example

a) Let $X = N$ and let's consider the NCSs (neutrosophic crisp sets) given below:

$A_1 = \langle\{2,3,4,\ldots\}, \phi, \phi\rangle$, $A_2 = \langle\{3,4,\ldots\}, \phi, \{1\}\rangle$,

$A_3 = \langle\{4,5,6,\ldots\}, \phi, \{1,2\}\rangle$, ...

$$A_n = \langle\{n+1, n+2, n+3, \ldots\}, \phi, \{1,2,3,..n-1\}\rangle.$$

Then $\Gamma = \{\phi_N, X_N\} \cup \{A_{n:} = 3,4,5,\ldots\}$ is a NCT on X and (X, Γ) is **a** neutrosophic crisp compact.

b) Let $X = (0,1)$ and let's take the NCSs

$$A_n = \left\langle X, \left(\frac{1}{n}, \frac{n-1}{n}\right), \phi, \left(0, \frac{1}{n}\right)\right\rangle, n = 3,4,5,\ldots \text{in X}.$$

In this case $\Gamma = \{\phi_N, X_N\} \cup \{A_{n:} = 3,4,5,\ldots\}$ is an NCT on X, which is not a neutrosophic crisp compact.

6.1 Corollary

A NCTS (X, Γ) is a neutrosophic crisp compact iff every family $\{\langle X, G_{i_1}, G_{i_2}, G_{i_3}\rangle : i \in J\}$ of NCCSs in X having the FIP has nonempty intersection.

6.2 Corollary

Let (X, Γ_1), (Y, Γ_2) be NCTSs and $f : X \to Y$ be a continuous surjection. If (X, Γ_1) is a neutrosophic crisp compact, then so is (Y, Γ_2)

6.3 Definition

(a) If a family $\{\langle G_{i_1}, G_{i_2}, G_{i_3}\rangle : i \in J\}$ of NCCSs in X satisfies the condition $A \subseteq \cup\{\langle G_{i_1}, G_{i_2}, G_{i_3}\rangle : i \in J\}$, then it is called a neutrosophic crisp open cover of A.

(b) Let's consider a finite subfamily of a neutrosophic crisp open subcover of $\{\langle G_{i_1}, G_{i_2}, G_{i_3}\rangle : i \in J\}$.

A neutrosophic crisp set $A = \langle A_1, A_2, A_3\rangle$ in a NCTS (X, Γ) is called neutrosophic crisp compact iff every neutrosophic crisp open cover of A has a finite neutrosophic crisp open subcover.

6.3 Corollary

Let (X, Γ_1), (Y, Γ_2) be NCTSs and $f : X \to Y$ be a continuous surjection. If A is a neutrosophic crisp compact in (X, Γ_1), then so is $f(A)$ in (Y, Γ_2).

7 Conclusion

In this paper we introduce both the neutrosophic crisp topology and the neutrosophic crisp compact space, and we present properties related to them.

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Neutrosophic Logic for Mental Model Elicitation and Analysis

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Abstract. Mental models are personal, internal representations of external reality that people use to interact with the world around them. They are useful in multiple situations such as muticriteria decision making, knowledge management, complex system learning and analysis. In this paper a framework for mental models elicitation and analysis based on neutrosophic Logic is presented. An illustrative example is provided to show the applicability of the proposal. The paper ends with conclusion future research directions.

Keywords: *mental model, neutrosophic Logic, neutrosophic cognitive maps, static analysis.*

1 Introduction

Mental models are useful in multiple situations such as muticriteria decision making [1], knowledge management, complex system learning and analysis [2]. In this paper, we propose the use of an innovative technique for processing uncertainty and indeterminacy in mental models.

The outline of this paper is as follows: Section 2 is dedicated to mental models and neutrosophic logic and neutrosophic cognitive maps. The proposed framework is presented in Section 3. An illustrative example is discussed in Section 4. The paper closes with concluding remarks, and discussion of future work in Section 5.

2 Mental Models and neutrosophic Logic

Mental models are personal, internal representations of external reality that people use to interact with the world around them [3]. The development of more effective end-user mental modelling tools is an active area of research [4].

A cognitive map is form of structured knowledge representation introduced by Axelrod [5]. Mental models have been studied using cognitive mapping [6].

Another approach is based in fuzzy cognitive maps [7]. FCM utilizes fuzzy logic in the creation of a directed cognitive map. FCM are a further extension of Axelrod's definition of cognitive maps [7] .

Neutrosophic logic is a generalization of fuzzy logic based on neutrosophy [8]. If indeterminacy is introduced in cognitive mapping it is called Neutrosophic Cognitive Map (NCM) [9].

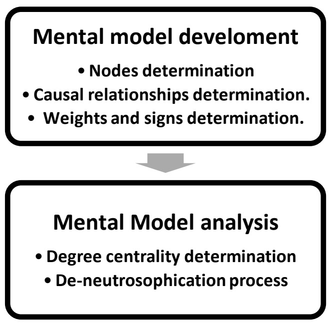
NCM are based on neutrosophic logic to represent uncertainty and indeterminacy in cognitive maps [8]. A NCM is a directed graph in which at least one edge is an indeterminacy denoted by dotted lines [6].

3 Proposed Framework

The following steps will be used to establish a framework for mental model elicitation and analysis with NCM (Fig. 1).

Figure 1: Mental model.

- **Mental model development.**



This Activity begins with determination of nodes. Finally causal relationships, its weights and signs are elicited [10].

- **Mental model analysis**

Static analysis is develop to define the importance of each node based on the degree centrality measure [11]. A de-neutrosophication process gives an interval number for centrality. Finally the nodes are ordered.

4 Illustrative example

In this section, we present an illustrative example in order to show the applicability of the proposed model. We selected a group of concepts related to people factor in agile software develoment projects sucess (Table 1) [12].

Table I. FCM nodes

Node	Description
A	Competence and expertise of team members
B	Motivation of tem members
C	Managers knowledge of agile development
D	Team training
E	Customer relationship
F	Customer commitment and presence

The FCM is developed integrating knowledge from one expert. The FCM with weighs is represented in Fig. 4.

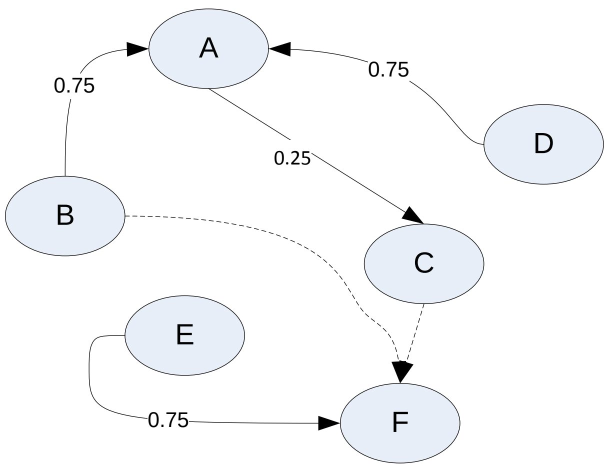


Figure 2: Mental model.

The neutrosophic score of each node based on the centrality measure is as follows:

A	1.75
B	0.75+I
C	0.25+I
D	0.75
E	0.75
F	0.75+2I

The next step is the de-neutrosophication process as proposes by Salmeron and Smarandache [13]. $I \in [0,1]$ is repalaced by both maximum and minimum values.

A	1.75
B	[0.75,1.75]
C	[0.25,1.25]
D	0.75
E	0.75
F	[0.75,2.75]

Finally we work with extreme values [14] for giving a total order:

$$A \sim F \succ B \succ C \sim D \sim E$$

Competence and expertise of team members, Customer commitment and presence are the more important factors in his mental model.

5 Conclusions

In this paper, we propose a new framework for processing uncertainty and indeterminacy in mental models. Future research will focus on conducting further real life experiments and the development of a tool to automate the process. The use of the computing with words (CWW) is another area of research.

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On Neutrosophic Canonical Hypergroups and Neutrosophic Hyperrings

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Abstract. A neutrosophic hyperstructure is an algebraic structure generated by a given hyperstructure *H* and an indeterminacy factor *I* under the hyperoperation(s) of *H*. The objective of this paper is to study canonical hypergroups and hyperrings in which addition and multiplication are hyperoperations in a neutrosophic environment. Some basic properties of neutrosophic canonical hypergroups and neutrosophic hyperrings are presented. Quotient neutrosophic canonical hypergroups and neutrosophic hyperrings are presented.

Keywords: neutrosophic canonical hypergroup, neutrosophic subcanonical hypergroup, neutrosophic hyperring, neutrosophic subhyperring, neutrosophic hyperideal.

1 Introduction

Given any hyperstructure *H*, a new hyperstructure *H(I)* may be generated by *H* and *I* under the hyperoperation(s) of *H*. Such new hyperstructures *H(I)* are called neutrosophic hyperstructures where *I* is an indeterminate or a neutrosophic element. Generally speaking, *H(I)* is an extension of *H* but some properties of *H* may not hold in *H(I)*. However, *H(I)* may share some properties with *H* and at times may possess certain algebraic properties not present in *H*.

Neutrosophic theory was introduced by F. Smarandache in 1995 and some known algebraic structures in the literature include neutrosophic groups, neutrosophic semigroups, neutrosophic loops, neutrosophic rings, neutrosophic fields, neutrosophic vector spaces, neutrosophic modules etc. Further introduction to neutrosophy and neutrosophic algebraic structures can be found in [1,2,3,4,16,26,27].

In 1934, Marty [18] introduced the theory of hyperstructures at the 8th Congress of Scandinavian Mathematicians. In 1972, Mittas [21] introduced the theory of canonical hypergroups. A class of hyperrings (R, +, .), where + and . are hyperoperations are introduced by De Salvo [15]. This class of hyperrings has been further studied by Asokkumar and Velrajan [5,22] and Davvaz and Leoranu-Fotea [14]. Further contributions to the theory of hyperstructures can be found in [7,8,9,10,14,22].

Agboola and Davaaz introduced and studied neutrosophic hypergroups in [4]. The present paper is concerned with the study of canonical hypergroups and hyperrings in a neutrosophic environment. Basic properties of neutrosophic canonical hypergroups and neutrosophic hyperrings are presented. Quotient neutrosophic canonical hypergroups and neutrosophic hyperrings are also presented.

2 A Review of Well Known Definitions

In this section, we provide basic definitions, notations and results that will be used in the sequel.

Definition 2.1. Let (G, $*$) be any group and let $G(I)=\langle G\cup I\rangle$. The couple $(G(I),*)$ is called a *neutrosophic group* generated by G and I under the binary operation $*$. The indeterminacy factor I is such that $I*I=I$. If $*$ is ordinary multiplication, then $I*I*I...*I=I^n=I$ and $*$ if is ordinary addition, then $I*I*I...*I=nI$ for $n\in N$.

G(I) is said to be *commutative* if $a*b=b*a$ for all $a,b\in G(I)$.

Theorem 2.2. [26] Let G(I) be a neutrosophic group.
(1) G(I) in general is not a group;
(2) G(I) always contain a group.

Definition 2.3. Let G(I) be a neutrosophic group.
(1) A proper subset A(I) of G(I) is said to be a neutrosophic subgroup of G(I) if A(I) is a neutrosophic group, that is, A(I) contains a proper subset which is a group;
(2) A(I) is said to be a pseudo neutrosophic group if it does not contain a proper subset which is a group.

Definition 2.4. Let A(I) be a neutrosophic subgroup (pseudo neutrosophic subgroup) of a neutrosophic group G(I).
(1) A(I) is said to be *normal* in G(I) if there exist $x,y\in G(I)$ such that $xA(I)y=A(I)$.
(2) G(I) is said to be *simple* if it has no non-trivial

neutrosophic normal subgroup.

Example 1. [3] Let G(I)={e, a, b, c, I, aI, bI, cI} be a set, where $a^2=b^2=c^2=e$, bc=cb=a, ac=ca=b, ab=ba=c. Then (G(I),.) is a commutative neutrosophic group, and H(I)={e,a,I,aI}, K(I)={e,b,I,bI} and PI={e,c,I,cI} are neutrosophic subgroups of G(I).

Theorem 2.5. [3] Let H(I) be a non-empty proper subset of a neutrosophic group (G(I),$*$). Then, H(I) is a neutrosophic subgroup of G(I) if and only if the following conditions hold:

(1) $a,b \in H(I)$ implies that $a*b \in H(I)$;
(2) There exists a proper subset A of H(I) such that (A, $*$) is a group.

Definition 2.6. Let (G_1(I),$*$) and (G_1(I),$*$') be two neutrosophic groups and let $\phi: G_1(I) \to G_2(I)$ be a mapping of G_1(I) into G_2(I). Then, ϕ is said to be a *homomorphism* if the following conditions hold:

(1) ϕ is a group of homomorphism;
(2) $\phi(I) = I$.

In addition, if ϕ is a bijection, then ϕ is called a neutrosophic group isomorphism and we write $G_1(I) \cong G_2(I)$.

Definition 2.7. Let (R,+,.) be any ring. A *neutrosophic ring* is a triple (R(I),+,.) generated by R and I, that is, $R(I) = \langle R \cup I \rangle$.

Indeed, $R(I) = \{x = a + bI : a,b \in R\}$, where if x=a+bI and y=c+dI are elements of R, then

$x \oplus y = (a+bI) \oplus (c+dI) = (a+c) + (b+d)I$,
$x \odot y = (a+bI) \odot (c+dI) =$
$(ac) + (ad + bc + bd)I$.

Example 2. Let $\mathbb{Z}_n$ be a ring of integers modulo n. Then, $\mathbb{Z}_n(I) = \{x = a + bI : a,b \in \mathbb{Z}_n\}$ is a neutrosophic ring of integers modulo n.

Theorem 2.8. [27] Let (R(I),+,.) be a *neutrosophic ring*. Then, (R(I),+,.) is a ring.

Definition 2.9. Let (R(I),+,.) be a neutrosophic ring. A non-empty subset S(I) of R(I) is said to be a *neutrosophic subring* if (S(I),+,.) is a neutrosophic ring. It is essential that S(I) must contain a proper subset which is a ring. Otherwise, S(I) is called a *pseudo neutrosophic subring* of R(I).

Example 3. Let $(\mathbb{Z}_{12}(I),+,\cdot)$ be a neutrosophic ring of integers modulo 12 and let S(I) and T(I) be subsets of $\mathbb{Z}_{12}(I)$ given by S(I)={0, 6, I, 2I, 3I, ..., 11I, 6+I, 6+2I, 6+3I, ..., 6+11I} and T(I)={0, 2I, 4I, 6I, 10I}. Then, (S(I),+,.) is a neutrosophic subring of $\mathbb{Z}_{12}(I)$ while (T(I),+,.) is a pseudo neutrosophic ring of $\mathbb{Z}_{12}(I)$.

Definition 2.10. Let (R(I),+,.) be a neutrosophic ring and let S(I) be a neutrosophic subring (pseudo neutrosophic subring) of R(I). Then, S(I) is called a *neutrosophic ideal (pseudo neutrosophic ideal)* of R(I) if for all $r \in R(I)$ and $s \in S(I)$, $r \cdot s, s \cdot r \in S(I)$.

Definition 2.11. Let $(R_1(I),+,\cdot)$ and $(R_2(I),+,\cdot)$ be two neutrosophic rings and let $\phi: R_1(I) \to R_2(I)$ be a mapping of $R_1(I)$ into $R_2(I)$. Then, ϕ is said to be a *homomorphism* if the following conditions hold:

(1) ϕ is a group of homomorphism;
(2) $\phi(I) = I$.

Moreover, if ϕ is a bijection, then ϕ is called a neutrosophic ring *isomorphism* and we write $R_1(I) \cong R_2(I)$. The kernel of ϕ denoted by $Ker\phi$ is the set $\{x \in R_1(I) : \phi(x) = 0\}$ and the image of ϕ denoted by $\operatorname{Im}\phi$ is the set $\{\phi(x) : x \in R_1(I)\}$.

It should be noted that $\operatorname{Im}\phi$ is a neutrosophic subring of $R_2(I)$ and $Ker\phi$ is always a subring of R_1 and never a neutrosophic subring (ideal) of R_1(I).

Definition 2.12. A map $\cdot : S \times S \to P^*(S)$ is called *hyperoperation* on the set S, where S is non-empty set and $P^*(S)$ denotes the set of all non-empty subsets of S.

A *hyperstructure* or *hypergroupoid* is the pair $(S,\cdot)$, where $\cdot$ is a hyperoperation on the set S.

Definition 2.13. A hyperstructure $(S,\cdot)$ is called a semihypergroup if for all $x,y,z \in S, (x \cdot y) \cdot z = x \cdot (y \cdot z)$, which means that

$$\bigcup_{u \in x \cdot y} u \cdot z = \bigcup_{v \in y \cdot z} x \cdot v.$$

Definition 2.14. A non-empty subset A of a semihypergroup $(S,\cdot)$ is called a *subsemihypergroup*. In other words, a non-empty subset A of a semihypergroup $(S,\cdot)$ is a subsemihypergroup if $A \cdot A \subseteq A$.

If $x \in S$ and A,B are non-empty subsets of S, then

$$A \cdot B = \bigcup_{\substack{a \in A \\ b \in B}} a \cdot b, A \cdot x = A \cdot \{x\}, \text{ and}$$

$$x \cdot B = \{x\} \cdot B.$$

Definition 2.15. A hypergroupoid $(H,\cdot)$ is called a *quasihypergroup* if for all a of H we have $a \cdot H = H \cdot a = H$. This condition is also called the *reproduction axiom*.

Definition 2.16. A hypergroupoid $(H,\cdot)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup.*

Definition 2.17. Let H be a non-empty set and let + be a hyperoperation on H. The couple $(H,+)$ is called *canonical hypergroup* if the following conditions hold:

(1) x+y=y+x, for all $x, y \in H$;
(2) x+(y+z)=(x+y)+z, for all $x, y, z \in H$;
(3) there exists a neutral element $0 \in H$ such that x+0={x}=0+x, for all $x \in H$;
(4) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x+(-x) \cap (-x)+x$;
(5) $z \in x+y$ implies $y \in -x+z$ and $x \in z-y$, for all $x, y, z \in H$.

A non-empty subset A of H is called a *subcanonical hypergroup* if A is canonical hypergroup under the same hyperaddition as that of H that is, for every $a, b \in A$, $a-b \in A$. In addition, if $a+A-a \subseteq A$ for all $a \in H$, A is said to be *normal*.

Definition 2.18. A *hyperring* is a triple $(R,+,\cdot)$ satisfying the following axioms:
(1) $(R,+)$ is a canonical hypergroup;
(2) $(R,\cdot)$ is a semihypergroup such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$, that is,) is a bilateral absorbing element;
(3) $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(x+y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$.

Definition 2.19. Let $(R,+,\cdot)$ be a hyperring and A be a non-empty subset of R. Then, A is said to be *subhyperring* of R if $(A,+,\cdot)$ is itself a hyperring.

Definition 2.20. Let A be a subhyperring of a hyperring R.
(1) A is called a *left hyperideal* of R if $r \cdot a \subseteq A$ for all $r \in R, a \in A$.
(2) A is called a *right hyperideal* of R if $a \cdot r \subseteq A$ for all $r \in R, a \in A$.
(3) A is called a *hyperideal* of R if A is both left and right hyperideal of R.
(4) A hyperideal A is said to be normal if $r+a-r \subseteq A$ for all $r \in R$.

Definition 2.21. Let $(H_1,+)$ and $(H_2,+)$ be two canonical hypergroups. A mapping $\phi: H_1 \rightarrow H_2$, is called
(1) a *homomorphism* if (i) for all $x, y \in H_1$, $\phi(x+y) \subseteq \phi(x)+\phi(y)$ and (ii) $\phi(0)=0$.
(2) a *good or strong homomorphism* if (i) for all $x, y \in H_1$, $\phi(x+y) = \phi(x)+\phi(y)$ and (ii) $\phi(0)=0$.
(3) an *isomorphism (strong isomorphism*) if ϕ is a bijective homomorphism (strong homomorphism).

Definition 2.22. [4] Let $(H,\circ)$ be any hypergroup and let $\langle H \cup I \rangle = \{(a,bI): a,b \in H\}$. The couple $H(I) = (\langle H \cup I \rangle, \circ)$ is called a neutrosophic hypergroup generated by H and I under the hyperoperation $\circ$, where for all $(a,bI),(c,dI) \in H(I)$, the composition element of H(I) is defined by

$$(a,bI) \circ (c,dI) = \{(x,yI): x \in a \circ c, y \in a \circ d \cup b \circ c \cup b \circ d\}.$$

3 Development of Neutrosophic Canonical Hypergroups and Neutrosophic Hyperrings

In this section, we develop the concepts of neutrosophic canonical hypergroups and neutrosophic hyperrings. Necessary definitions are given and examples are provided.

Definition 3.1. Let $(H,+)$ be any canonical hypergroup and let I be an indeterminate. Let $H(I) = \langle H \cup I \rangle = \{(a,bI): a,b \in H\}$ be a set generated by H and I. The hyperstructure $(H(I),+)$ is called a *neutrosophic canonical hypergroup*, where for all $(a,bI),(c,dI) \in H(I)$ with $b \neq 0$ or $d \neq 0$, we define

$$(a,bI)+(c,dI) \in \{(x,yI): x \in a+c, y \in a+d \cup b+c \cup b+d\}$$

and

$$(x,0)+(y,0) = \{(u,0): u \in x+y\}.$$

The element I is represented by (0,I) in H(I) and any element $x \in H$ is represented by (x,0) in H(I). For any non-empty subset A[I] of H(I) , we define $-A[I] = \{-(a,bI) = (-a,-bI): a,b \in H\}$.

Lemma 3.2. *Let* $H \neq \{0\}$ *be a canonical hypergroup and let H(I) be the corresponding neutrosophic canonical hypergroup. Then, (0,0) the neutral element of H is not a neutral element of H(I).*

Proof. Suppose that (0,0) is the neutral element of H(I) and suppose that $(a,bI) \in H(I)$ such that b is non-zero and $a \neq b$.

Then

$$\begin{aligned}(a,bI)+(0,0) &= \{(u,vI): u \in a+0, v \in a+0 \cup b+0 \cup b+0\} \\ &= \{(u,vI): u \in \{a\}, v \in \{a,b\}\} \\ &\neq (a,bI),\end{aligned}$$

a contradiction. Hence, (0,0) is not a neutral element of H(I).

Definition 3.3. Let (H(I),+) be a neutrosophic canonical hypergroup.
(1) A non-empty subset A[I] of H(I) is called a *neutrosophic subcanonical hypergroup* of H(I) if (A[I],+) is itself a neutrosophic canonical hypergroup. It is essential that A[I] must contain a proper subset which is a subcanonical hypergroup of H. If A[I] does not contain a proper subset which is a subcanonical hypergroup of H, then it is called a pseudo *neutrosophic subcanonical hypergroup* of H(I).
(2) If A[I] is a neutrosophic subcanonical hypergroup (pseudo neutrosophic subcanonical hypergroup), then A[I] is said to be normal in H(I) if for all $(a,bI) \in H(I)$,

$$(a,bI)+A[I]-(a,bI) \subseteq A[I].$$

Lemma 3.4. *Let (H(I),+) be a neutrosophic canonical hypergroup and let A[I] be a non-empty proper subset of*

H(I). Then, A[I] is a neutrosophic subcanonical hypergroup if and only if the following conditions hold:

(1) for all $(a,bI),(c,dI)\in A[I],(a,bI)-(c,dI)\subseteq A[I]$,

(2) *A[I] contains a proper subset which is a canonical hypergroup of H.*

Lemma 3.5. *Let (H(I),+) be a neutrosophic canonical hypergroup and let A[I] be a non-empty proper subset of H(I). Then, A[I] is a pseudo neutrosophic subcanonical hypergroup if and only if the following conditions hold:*

(1) for all $(a,bI),(c,dI)\in A[I],(a,bI)-(c,dI)\subseteq A[I]$,

(2) *A[I] does not contain a proper subset which is a canonical hypergroup of H.*

Definition 3.6. Let A[I] and B[I] be any two neutrosophic subcanonical hypergroups of a neutrosophic canonical hypergroup H(I). The sum of A[I] and B[I] denoted by A[I]+B[I] is defined as the set:

$$A[I]+B[I]=\bigcup_{\substack{(a,bI)\in A[I]\\(c,dI)\in B[I]}}(a,bI)+(c,dI).$$

Definition 3.7. Let H(I) be a neutrosophic canonical hypergroup and let A[I] be a neutrosophic subcanonical hypergroup of H(I). If K is a subcanonical hypergroup of H, we define the set

$$A[I]+K=\bigcup_{\substack{(a,bI)\in A[I]\\(k,0)\in K}}(a,bI)+(k,0).$$

Definition 3.8. Let $(R,+,\cdot)$ be any hyperring and let I be an indeterminate. The hyperstructure $(R(I),\oplus,\odot)$ generated by R and I, that is, R(I)=$\langle R\cup I\rangle$, is called a *neutrosophic hyperring*, where for all

$(a,bI),(c,dI)\in R(I)$,

$(a,bI)\oplus(c,dI)=\{(x,yI):x\in a+c,y\in b+d\}$;

for all $(a,bI),(c,dI)\in R(I)$ with $b\neq 0$ or $d\neq 0$,

$(a,bI)\odot(c,dI)=\{(x,yI):x\in a.c,$
$y\in a.d\cup b.c\cup b.d\}$

and

$(x,0)\odot(y,0)=\{(u,0):u\in x\cdot y\}$.

We usually use + and $\cdot$ instead of $\oplus$ and $\odot$.

Lemma 3.9. *Let R(I) be a neutrosophic hyperring. Then, $(0,0)\in R(I)$ is bilaterally absorbing element.*

Proof. Suppose that $(a,bI)\in R(I)$. Then, $(a,bI).(0,0)=$
$\{(u,vI):u\in a\cdot 0,v\in a\cdot 0\cup b\cdot 0\cup b\cdot 0\}=$
$\{(u,vI):u\in\{0\},v\in\{0\}\}=\{(0,0)\}$.

Hence, $(0,0)\in R(I)$ is a bilaterally absorbing element.

Definition 3.10. Let $(R(I),+,\cdot)$ be a neutrosophic hyperring and let A[I] be a non-empty subset of R(I). Then, A[I] is called a *neutrosophic subhyperring* of R(I) if $(A[I],+,\cdot)$ is itself a neutrosophic hyperring. It is essential that A[I] must contain a proper subset which is a hyperring. Otherwise, A[I] is called a *pseudo neutrosophic subhyperring* of R(I).

Definition 3.11. Let $(R(I),+,\cdot)$ be a neutrosophic hyperring and let A[I] be a neutrosophic subhyperring of R(I).

(1) A[I] is called a *left neutrosophic hyperideal* if for all $(r,sI)\in R(I),(a,bI)\in A[I]$, $(r,sI)\cdot(a,bI)\subseteq A[I]$.

(2) A[I] is called a *right neutrosophic hyperideal* if for all $(r,sI)\in R(I),(a,bI)\in A[I]$, $(a,bI)\cdot(r,sI)\subseteq A[I]$.

(3) A[I] is called a *neutrosophic hyperideal* if A[I] is both a left and right neutrosophic hyperideal.

A neutrosophic hyperideal A[I] of R[I] is said to be *normal* in R(I) if for all $(r,sI)\in R(I)$,
$(r,sI)+A[I]-(r,sI)\subseteq A[I]$.

Lemma 3.12. *Let $(R(I),+,\cdot)$ be a neutrosophic hyperring and let A[I] be a non-empty subset of R(I). Then, A[I] is a neutrosophic hyperideal if and only if the following conditions hold:*

(1) For all $(a,bI),(c,dI)\in A[I],(a,bI)-(c,dI)\subseteq A[I]$;

(2) For all $(r,sI)\in R(I),(a,bI)\in A[I]$, $(a,bI)\cdot(r,sI)\subseteq A[I]$ and $(r,sI)\cdot(a,bI)\subseteq A[I]$;

(3) A[I] contains a proper subset which is a hyperring.

Lemma 3.13. *Let $(R(I),+,\cdot)$ be a neutrosophic hyperring and let A[I] be a non-empty subset of R(I). Then, A[I] is a pseudo neutrosophic hyperideal if and only if the following conditions hold:*

(1) For all $(a,bI),(c,dI)\in A[I],(a,bI)-(c,dI)\subseteq A[I]$;

(2) For all $(r,sI)\in R(I),(a,bI)\in A[I]$, $(a,bI)\cdot(r,sI)\subseteq A[I]$ and $(r,sI)\cdot(a,bI)\subseteq A[I]$;

(3) A[I] does not contain a proper subset which is a hyperring.

Definition 3.14. Let A[I] and B[I] be any two neutrosophic hyperideals of a neutrosophic hyperring R(I). The sum of A[I] and B[I] denoted by A[I]+B[I] is defined as the set $\{(x,yI):(x,yI)\in(a,bI)+(c,dI)$, where $(a,bI)\in A[I],(c,dI)\in B[I]\}$.

Definition 3.15. Let R(I) be a neutrosophic hyperring and let A[I] be a neutrosophic hyperideal of R(I). If K is a hyperideal (pseudo hyperideal) of R, the sum of A[I] and K denoted by A[I]+K is defined as the set $\{(x,yI):(x,yI)\in(a,bI)+(k,0),$ where $(a,bI)\in A[I],(k,0)\in K\}.$

Example 4. Let R(I)={(0,0),(x,0),(0,xI),(x,xI)} be a set and let + and $\cdot$ be hyperoperations on R(I) defined in the tables below.

+	(0,0)	(x,0)	(0,xI)	(x,xI)
(0,0)	{(0,0)}	{(x,0)}	{(0,0), (0,xI)}	{(x,xI)}
(x,0)	{(x,0)}	{(0,0), (x,0)}	{(x,0), (x,xI)}	R(I)
(0,xI)	{(0,0), (0,xI)}	{(x,0), (x,xI)}	{(0,0), (0,xI)}	{(x,0), (x,xI)}
(x,xI)	{(x,xI)}	R(I)	{(x,0), (x,xI)}	R(I)

Table 1.

$\cdot$	(0,0)	(x,0)	(0,xI)	(x,xI)
(0,0)	{(0,0)}	{(0,0)}	{(0,0)}	{(0,0)}
(x,0)	{(0,0)}	{(0,0), (x,0)}	{(0,0), (0,xI)}	R(I)
(0,xI)	{(0,0)}	{(0,0), (0,xI)}	{(0,0), (0,xI)}	{(0,0), (0,xI)}
(x,xI)	{(0,0)}	R(I)	{(0,0), (0,xI)}	R(I)

Table 2.

It is clear from the tables that (R(I),+) is a neutrosophic canonical hypergroup and $(R(I),+,\cdot)$ is a neutrosophic hyperring.

4 Properties of Neutrosophic Canonical Hypergroups

In this section, we present some basic properties of neutrosophic canonical hypergroups.

Proposition 4.1. *Let (H(I),+) be a neutrosophic canonical hypergroup. Then,*

(1) (H(I),+) in general is not a canonical hypergroup.
(2) (H(I),+) always contain a canonical hypergroup.

Lemma 4.2. *Let (H(I),+) be a neutrosophic canonical hypergroup.*

(1) –(0,0)=(0,0).
(2) $-(-(a,bI))=(a,bI)$ for all $(a,bI)\in H(I).$
(3) $-((a,bI)+(c,dI)=-(a,bI)-(c,dI)$ for all $(a,bI),(c,dI)\in H(I).$

Proposition 4.3. *Let (H(I),+) and (G(I),+') be any two neutrosophic canonical hypergroups. Then,* $H(I)\times G(I)$ *is a neutrosophic canonical hypergroup, where*

$$((a,bI),(c,dI)+''((e,fI),(g,hI)) =\{((p,qI),(x,yI)):(p,qI)\in(a,bI)+(e,fI), (x,yI)\in(c,dI)+'(g,hI)\},$$

for all $((a,bI),(c,dI)),((e,fI),(g,hI))\in H(I)\times G(I).$

Proposition 4.4. *Let (H(I),+) be a neutrosophic canonical hypergroup and let (K,+') be a canonical hypergroup. Then, H(I)xK is a neutrosophic canonical hypergroup, where*

$$((a,bI),(m,0)+''((c,dI),(n,0)) =\{((x,yI),(k,0)):(x,yI)\in(a,bI)+(c,dI), (k,0)\in(m,0)+'(n,0)\},$$

for all $((a,bI),(m,0)),((c,dI),(n,0))\in H(I)\times K.$

Proposition 4.5. *Let A[I] and B[I] be any two neutrosophic subcanonical hypergroups of a neutrosophic canonical hypergroup H(I). Then,*

(1) A[I]+B[I] is a neutrosophic subcanonical hypergroup of H(I).
(2) A[I]+A[I]=A[I].
(3) $A[I]\cap B[I]$ *is a neutrosophic subcanonical hypergroup of H(I).*

Proposition 4.6. *Let H(I) be a neutrosophic canonical hypergroup and let A[I] and B[I] be any neutrosophic subcanonical hypergroup and pseudo neutrosophic subcanonical hypergroup of H(I), respectively. Then,*

(1) A[I]+B[I] is a neutrosophic subcanonical hypergroup of H(I).
(2) $A[I]\cap B[I]$ *is a pseudo neutrosophic subcanonical hypergroup of H(I).*

Proposition 4.7. *Let H(I) be a neutrosophic canonical hypergroup and let A[I] and B[I] be any neutrosophic subcanonical hypergroup and pseudo neutrosophic subcanonical hypergroup respectively. If K is any subcanonical hypergroup of H, then*

(1) A[I]+K is a neutrosophic subcanonical hypergroup of H(I).
(2) B[I]+K is a neutrosophic subcanonical hypergroup of H(I).

Proposition 4.8. *Let (H(I),+) be a neutrosophic canonical hypergroup and let A be a subcanonical hypergroup of H. If A is normal in H, A[I] is not necessarily normal to H(I).*

Proposition 4.9. Let (H(I),+) be a neutrosophic canonical hypergroup and let A be a normal neutrosophic subcanonical hypergroup of H. Then, $(a,aI)+A-(a,aI)\subseteq A[I]$ for all $(a,aI)\in H(I).$

Proof. Suppose that A is normal in H. Let (h,0) be an arbitrary element of A. Then, for all $(a,aI)\in H(I)$ with $a\neq 0$, we have

$$\begin{aligned}&(a,aI)+(h,0)-(a,aI)\\&=(a,aI)+\{(x,yI):x\in h-a,y\in h-a\cup 0-a\}\\&=\{(u,v1):u\in a+x,v\in a+y\cup a+x,\\&\quad x\in h-a,y\in h-a\cup -a\}\end{aligned}$$

$=\{(u,vI):u\in a+h-a,$
$v\in a+h-a\cup a-a\cup a+h-a\},$

from which we obtain $u\in A$ and $v\in A$. Therefore, $(u,vI)\in A[I]$. Since $(h,0)\in A$ is arbitrary, the required results follow.

Definition 4.10. Let (H(I),+) be a neutrosophic canonical hypergroup and let A[I] be a neutrosophic subcanonical hypergroup of H(I). We consider the quotient

$(H(I):A[I])=\{(a,bI)+A[I]:(a,bI)\in H[I]\}$

and we put $(a,bI)+A[I]=[(a,bI)]$. For all $[(a,bI)],[(c,dI)]\in(H(I):A[I])$, we define the hyperoperation $\oplus$ on $(H(I):A[I])$ as

$[(a,bI)]\oplus[(c,dI)]=\{[(e,fI)]:(e,fI)\in (a,bI)+(c,dI)\}$.

Then the couple $((H(I):A[I]),\oplus)$ is called the *quotient neutrosophic canonical hypergroup*. If A[I] is a pseudo neutrosophic subcanonical hypergroup, then we call $((H(I):A[I]),\oplus)$ a *pseudo quotient neutrosophic canonical hypergroup*.

Proposition 4.11. *Let H(I) be a neutrosophic canonical hypergroup and A[I] be a neutrosophic subcanonical hypergroup (pseudo neutrosophic subcanonical hypegroup) of H(I). Then,* $((H(I):A[I]),\oplus)$ *is generally not a canonical hypergroup.*

Definition 4.12. Let $(H_1(I),+)$ and $(H_2(I),+)$ be two neutrosophic canonical hypergroups and let $\phi:H_1(I)\rightarrow H_2(I)$ be a mapping from $H_1(I)$ into $H_2(I)$.

(1) ϕ is called a *homomorphism* if
  a. ϕ is a canonical hypergroup homomorphism;
  b. $\phi((0,I))=(0,I)$.

(2) ϕ is called a *good* or *strong homomorphism* if
  a. ϕ is a good or strong canonical hypergroup homomorphism;
  b. $\phi((0,I))=(0,I)$.

(3) ϕ is called a is*omorphism* (*strong isomorphism*) if ϕ is a bijective homomorphism (strong homomorphism).

Definition 4.13. Let $\phi:H_1(I)\rightarrow H_2(I)$ be a homomorphism from a neutrosophic canonical hypergroup $H_1(I)$ into a neutrosophic canonical hypergroup $H_2(I)$.

(1) The kernel of ϕ denoted by $Ker\phi$ is the set $\{(a,bI)\in H_1(I):\phi((a,bI))=(0,0)\}$.

(2) The kernel of ϕ denoted by $\operatorname{Im}\phi$ is the set $\{\phi((a,bI)):(a,bI)\in H_1(I)\}$.

5 Properties of Neutrosophic Hyperrings

In this section, we present some basic properties of neutrosophic hyperrings.

Proposition 5.1. *Let* $(R(I),+,\cdot)$ *be a neutrosophic hyperring. Then,*

(1) $(R(I),+,\cdot)$ *in general is not a hyperring.*

(2) $(R(I),+,\cdot)$ *always contain a hyperring.*

Proof. (1) It has been presented in part (1) of Proposition 4.1. that $(R(I),+,\cdot)$ is not a canonical hypergroup. Also, distributive laws are not valid in $(R(I),+,\cdot)$. Hence, $(R(I),+,\cdot)$ is not a hyperring. (2) Follows from the definition.

Proposition 5.2. *Let* $(R(I),+,\cdot)$ *and* $(S(I),+',\cdot')$ *be any two neutrosophic hyperrings. Then,* $R(I)\times S(I)$ *is a neutrosophic hyperring, where*

$((a,bI),(c,dI)+''((e,fI),(g,hI))$
$=\{((p,qI),(x,yI)):(p,qI)\in(a,bI)+(e,fI),$
$(x,yI)\in(c,dI)+'(g,hI)\},$

and

$((a,bI),(c,dI)\cdot''((e,fI),(g,hI))$
$=\{((p,qI),(x,yI)):(p,qI)\in(a,bI).(e,fI),$
$(x,yI)\in(c,dI).'(g,hI)\},$

for all

$((a,bI),(c,dI)),((e,fI),(g,hI))\in R(I)\times S(I).$

Proposition 5.3. *Let* $(R(I),+,\cdot)$ *be a neutrosophic hyperring and let* $(K,+',\cdot')$ *be a hyperrings. Then,* $R(I)\times K$ *is a neutrosophic hyperring, where*

$((a,bI),(m,0)+''((c,dI),(n,0))$
$=\{((x,yI),(k,0)):(x,yI)\in(a,bI)+(c,dI),$
$(k,0)\in(m,0)+'(n,0)\},$

and

$((a,bI),(m,0)\cdot''((c,dI),(n,0))$
$=\{((x,yI),(k,0)):(x,yI)\in(a,bI).(c,dI),$
$(k,0)\in(m,0).'(n,0)\},$

for all

$((a,bI),(m,0)),((c,dI),(n,0))\in R(I)\times K.$

Lemma 5.4. Let A[I] be any neutrosophic hyperideal of a neutrosophic hyperring R(I). Then,

(1) A[I]+A[I]=A[I].

(2) (a,bI)+A[I]=A[I] for all $(a,bI)\in A[I]$.

Proposition 5.5. *Let* $(R(I),+,\cdot)$ *be a neutrosophic hyperring and let A[I] and B[I] be left (right) neutrosophic ideals of R(I). Then,*

(1) $A[I]\cap B[I]$ *is a left (right) neutrosophic hyperideal of R(I).*

(2) $A[I]+B[I]$ *is a left (right) neutrosophic hyperideal of R(I).*

Proposition 5.6. *Let R(I) be a neutrosophic hyperring and let A[I] and B[I] be any neutrosophic hyperideal and pseudo neutrosophic hyperideal of R(I) respectively. Then,*

(1) $A[I]+B[I]$ *is a neutrosophic hyperideal of R(I).*

(2) $A[I]\cap B[I]$ *is a pseudo neutrosophic hyperideal of R(I).*

Proposition 5.7. *Let R(I) be a neutrosophic hyperring* *and let A[I] and B[I] be any neutrosophic hyperideal and pseudo neutrosophic hyperideal, respectively. If K is any subhyperring of R, then*

(1) $A[I]+K$ is a neutrosophic hyperideal of R(I).

(2) $B[I]+K$ is a neutrosophic hyperideal of R(I).

Proposition 5.8. *Let $(R(I),+,\cdot)$ be a neutrosophic hyperring and let A be a hyperideal of R. If A is normal in R, A[I] is not necessarily normal in R(I).*

Proposition 5.9. *Let $(R(I),+,\cdot)$ be a neutrosophic hyperring and let A be a normal hyperideal of R. Then,*

$$(a,aI)+A-(a,aI)\subseteq A[I]$$

for all $(a,aI)\in R(I)$.

Definition 5.10. Let $(R(I),+,\cdot)$ be a neutrosophic hyperring and let $(A(I),+,\cdot)$ be a neutrosophic hyperideal of R(I). We consider the quotient $(R(I):A[I])=$ $\{(a,bI)+A[I]:(a,bI)\in R(I)\}$ and we put (a,bI)+A[I]=[(a,bI)].

For all $[(a,bI)],[(c,dI)]\in((R(I):A[I])$, we consider the hyperoperation $\oplus$ as defined in the Definition 4.10 and we define the hyperoperation $\square$ on $((R(I):A[I])$ as $[(a,bI)]\square[(c,dI)]=\{[(e,fI)]: (e,fI)\in(a,bI).(c,dI)\}$.

Then, the triple $((R(I):A[I]),\oplus,\square)$ is called the *quotient neutrosophic hyperring*. If A[I] is a pseudo neutrosophic hyperideal, then we call $((R(I):A[I]),\oplus,\square)$ a *pseudo neutrosophic hyperring*.

Proposition 5.11. *Let R(I) be a neutrosophic hyperring and let A[I] be a neutrosophic hyperideal (pseudo neutrosophic hyperideal) of R(I). Then, $((R(I):A[I]),\oplus,\square)$ is generally not a hyperring*.

Definition 5.12. Let $(R_1(I),+)$ and $(R_2(I),+)$ be two neutrosophic hyperrings and let $\phi:R_1(I)\to R_2(I)$ be a mapping from $R_1(I)$ into $R_2(I)$.

(1) ϕ is called a *homomorphism* if
   a. ϕ is a hyperring homomorphism;
   b. $\phi((0,I))=(0,I)$.

(2) ϕ is called a *good* or *strong homomorphism* if
   a. ϕ is a good or strong hyperring homomorphism;
   b. $\phi((0,I))=(0,I)$.

(3) ϕ is called a is*omorphism* (*strong isomorphism*) if ϕ is a bijective homomorphism (strong homomorphism).

Definition 5.13. Let $\phi:R_1(I)\to R_2(I)$ be a homomorphism from a neutrosophic hyperring $R_1(I)$ into a neutrosophic hyperring $R_2(I)$.

(1) The kernel of ϕ denoted by $Ker\phi$ is the set $\{(a,bI)\in R_1(I):\phi((a,bI))=(0,0)\}$.

(2) The kernel of ϕ denoted by $\operatorname{Im}\phi$ is the set $\{\phi((a,bI)):(a,bI)\in R_1(I)\}$.

Proposition 5.14. *Let $\phi:R_1(I)\to R_2(I)$ be a homomorphism from a neutrosophic hyperring $R_1(I)$ into a neutrosophic hyperring $R_2(I)$. Then,*

(1) $Ker\phi$ is a subhyperring of R_1 and never be a neutrosophic hyperring (neutrosophic hyperideal) of $R_1(I)$.

(2) $\operatorname{Im}\phi$ is a neutrosophic subhyperring of $R_2(I)$.

Question 1: Does there exist:

(1) A neutrosophic canonical hypergroup with normal neutrosophic subcanonical hypergroups?

(2) A neutrosophic hyperring with normal neutrosophic hyperideals?

(3) A simple neutrosophic canonical hypergroup?

(4) A simple neutrosophic hyperring?

6 Conclusion

In this paper, we have introduced and studied *neutrosophic* canonical hypergroups and *neutrosophic* hyperrings. We presented elementary properties of *neutrosophic* canonical hypergroups and *neutrosophic* hyperrings. Also, we studied quotient *neutrosophic* canonical hypergroups and quotient *neutrosophic* hyperrings.

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Neutrosophic Lattices

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Abstract**.** In this paper authors for the first time define a new notion called neutrosophic lattices. We define few properties related with them. Three types of neutrosophic lattices are defined and the special properties about these new class of lattices are discussed and developed. This paper is organised into three sections. First section introduces the concept of partially ordered neutrosophic set and neutrosophic lattices. Section two introduces different types of neutrosophic lattices and the final section studies neutrosophic Boolean algebras. Conclusions and results are provided in section three.

Keywords: Neutrosophic set, neutrosophic lattices and neutrosophic partially ordered set.

1 Introduction to partially ordered neutrosophic set

Here we define the notion of a partial order on a neutrosophic set and the greatest element and the least element of it. Let N(P) denote a neutrosophic set which must contain I, 0, 1 and 1 + I; that is 0, 1, I and $1+ I \in$ N(P). We call 0 to be the least element so $0 < 1$ and $0 < I$ is assumed for the working. Further by this N(P) becomes a partially ordered set. We define 0 of N(P) to be the least element and $I \cup 1 = 1 + I$ to be the greatest element of N(P).

Suppose N(P) = $\{0, 1, I, 1 + I, a_1, a_2, a_3, a_1I, a_2I, a_3I\}$ then N(P) with $0 < a_i$, $0 < a_iI$, $1 \le i \le 3$. Further $1 > a_i$; $I > a_iI$; $1 \le i \le 3$ $a_i \not< a_j$ if $i \ne j$ for $1 \le i, j \le 3$ and $Ia_i \not< Ia_j$; $i \ne j$ for $1 \le i, j \le 3$.

We will define the notion of Neutrosophic lattice.

DEFINITION 1.1: *Let N(P) be a partially ordered set with 0, 1, I, $1+I = 1 \cup I \in N(P)$.*

Define min and max on N(P) that is max {x, y} and min {x, y} $\in N(P)$. 0 is the least element and $1 \cup I = 1 + I$ is the greatest element of N(P). {N(P), min, max} is defined as the neutrosophic lattice.

We will illustrate this by some examples.

Example 1.1: Let N(P) = $\{0, 1, I, I \cup 1 = 1 + I, a, aI\}$ be a partially ordered set; N(P) is a neutrosophic lattice.

We know in case of usual lattices [1-4]. Hasse defined the notion of representing finite lattices by diagrams known as Hasse diagrams [1-4]. We in case of Neutrosophic lattices represent them by the diagram which will be known as the neutrosophic Hasse diagram.

The neutrosophic lattice given in example 1.1 will have the following Hasse neutrosophic diagram.

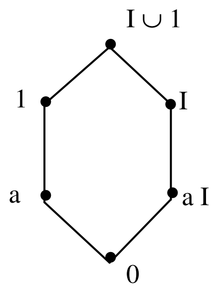

Figure 1.1

Example 1.2: Let N(P) = $\{0, 1, I, 1 \cup I, a_1, a_2, a_1I, a_2I\}$ be a neutrosophic lattice associated with the following Hasse neutrosophic diagram.

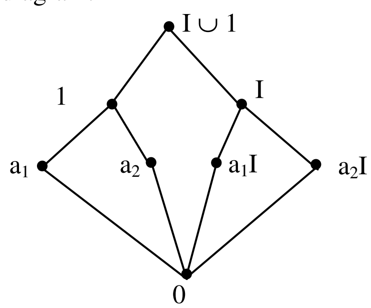

Figure 1.2

Example 1.3: Let N(P) = $\{0\ 1, I, 1 \cup I\}$ be a neutrosophic lattice given by the following neutrosophic Hasse diagram.

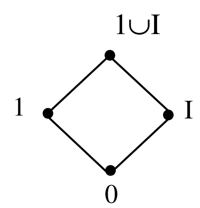

Figure 1.3

It is pertinent to observe that if N(P) is a neutrosophic lattice then 0, 1, I, $1 \cup I \in$ N(P) and so that N(P) given in example 1.3 is the smallest neutrosophic lattice.

Example 1.4: Let N(P) = $\{0, 1, I, 1 \cup I = 1 + I, a_1, a_2, a_1I, a_2I, a_1 < a_2\}$ be the neutrosophic lattice. The Hasse diagram of the neutrosophic lattice N(P) is as follows:

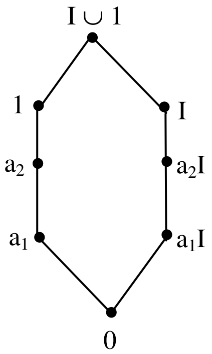

Figure 1.4

We can have neutrosophic lattices which are different.

Example 1.5: Let N(P) = $\{0, 1, I, a_1, a_2, a_3, a_4, a_1I, a_2I, a_3I, a_4I, 1 + I = I \cup 1\}$ be the neutrosophic lattice of finite order. (a_i is not comparable with a_j if $i \neq j$, $1 \leq i, j \leq 4$).

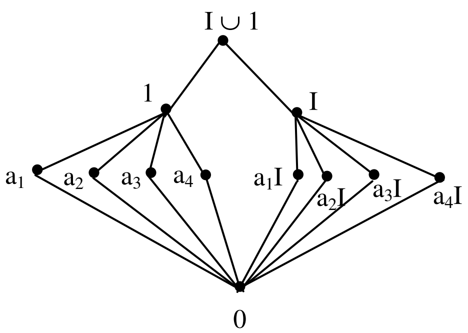

Figure 1.5

We see N(P) is a neutrosophic lattice with the above neutrosophic Hasse diagram.

In the following section we proceed onto discuss various types of neutrosophic lattices.

2. Types of Neutrosophic Lattices

The concept of modular lattice, distributive lattice, super modular lattice and chain lattices can be had from [1-4]. We just give examples of them and derive a few properties associated with them. In the first place we say a neutrosophic lattice to be a pure neutrosophic lattice if it has only neutrosophic coordinates or equivalently all the co ordinates (vertices) are neutrosophic barring 0.

In the example 1.5 we see the pure neutrosophic part of the neutrosophic lattice figure 2.1;

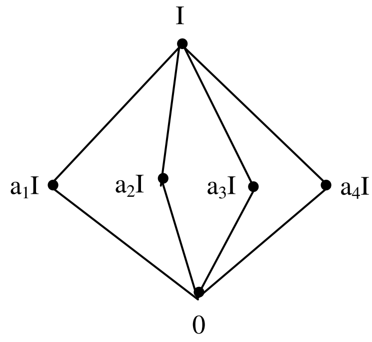

Figure 2.1

whose Hasse diagram is given is the pure neutrosophic sublattice lattice from figure 1.5. Likewise we can have the Hasse diagram of the usual lattice from example 1.5.

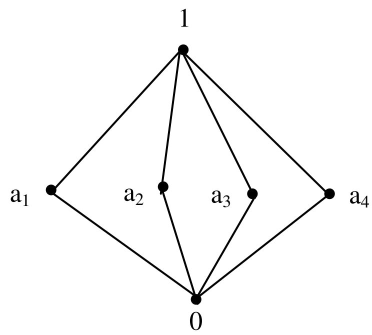

Figure 2.2

We see the diagrams are identical as diagrams one is pure neutrosophic where as the other is a usual lattice. As we have no method to compare a neutrosophic number and a non neutrosophic number, we get two sublattices identical in diagram of a neutrosophic lattice. For the modular identity, distributive identity and the super modular identity and their related properties refer [1-4].

The neutrosophic lattice given in example 1.5 has a sublattice which is a modular pure neutrosophic lattice and sublattice which is a usual modular lattice.

The neutrosophic lattice given in example 1.3 is a distributive lattice with four elements. However the neutrosophic lattice given in example 1.5 is not distributive as it contains sublattices whose homomorphic image is isomorphic to the neutrosophic modular lattice $N(M_4)$; where $N(M_4)$ is a lattice of the form

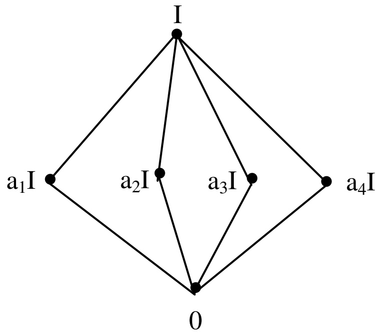

Figure 2.3

Likewise by $N(M_n)$ we have a pure neutrosophic lattice of the form given below in figure 2.4.

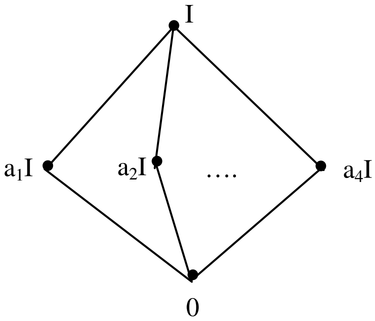

Figure 2.4

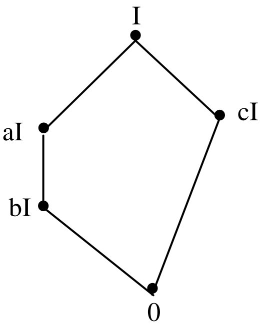

Figure 2.5

The neutrosophic pentagon lattice is given in figure 2.5 which is neither distributive nor modular.

The lattice $N(M_4)$ is not neutrosophic super modular we see the neutrosophic lattice in example 1.5 is not modular for it has sublattices whose homomorphic image is isomorphic to the pentagon lattice.

So we define a neutrosophic lattice N(L) to be a quasi modular lattice if it has atleast one sublattice (usual) which is modular and one sublattice which is a pure neutrosophic modular lattice.

Thus we need to modify the set S and the neutrosophic set N(S) of S. For if $S = \{a_1, \ldots, a_n\}$ we define $N(S) = \{a_1I, a_2I, \ldots, a_nI\}$ and take with $S \cup N(S)$ and the elements 0, 1, I, and $1 \cup I = 1 + I$. Thus to work in this way is not interesting and in general does not yield modular neutrosophic lattices.

We define the strong neutrosophic set of a set S as follows

Let $A = \{a_1, a_2, \ldots, a_n\}$, the strong neutropshic set of A;

$SN(A) = \{a_i, a_jI, a_i \cup a_jI = a_i + a_jI; 0, 1, I, 1 + I, 1 \leq i, j \leq n\}$.

S(L) the strong neutrosophic lattice is defined as follows:

$S(L) = \{0, 1, I, 1 + I, a_i, a_jI,$

$I \cup a_i = I + a_i \;\; a_1I \cup 1 = a_iI + 1, a_i + a_jI = a_i \cup a_jI \;\; 0 \underset{\neq}{\leq} a_i \underset{\neq}{\leq} 1; 0 < a_jI < I, 1 \leq i, j \leq n\}$.

S(L) with max, min is defined as the strong neutrosophic lattice.

We will illustrate this situation by some examples.

Example 2.1: Let $S(L) = \{0, 1, I, 1 + I, a, aI, a + aI, 1 + aI, I + a\}$

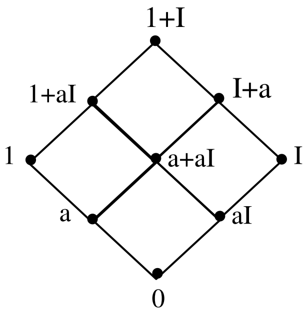

Figure 2.6

be a strong neutrosophic lattice.

We have several sublattices both strong neutrosophic sublattice as well as usual lattice.

For

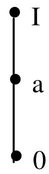

Figure 2.7

is the usual lattice.

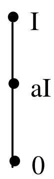

Figure 2.8

is the pure neutrosophic lattice.

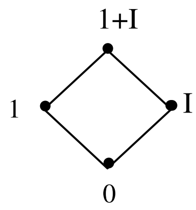

Figure 2.9

is the strong neutrosophic lattice.

These lattices have the edges to be real. Only vertices are indeterminates or neutrosophic numbers. However we can have lattices where all its vertices are real but some of the lines (or edges) are indeterminates.

Example 2.2: For consider

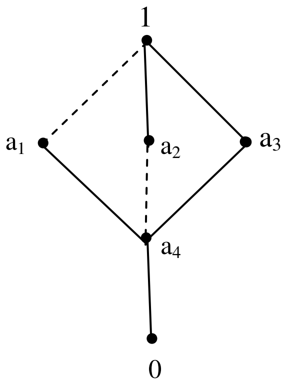

Figure 2.10

Such type of lattices will be known as edge neutrosophic lattices.

In case of edge neutrosophic lattices, we can have edge neutrosophic distributive lattices, edge neutrosophic modular lattices and edge neutrosophic super modular lattices and so on.

We will only illustrate these by some examples.

Example 2.3: Consider the following Hasse diagram.

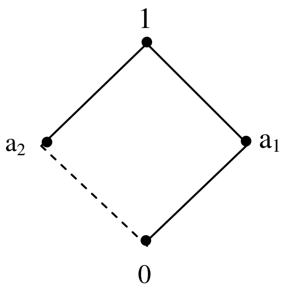

Figure 2.11

This is a edge neutrosophic lattice as the edge connecting 0 to a_2 is an indeterminate.

Example 2.4: Let us consider the following Hasse diagram of a lattice L.

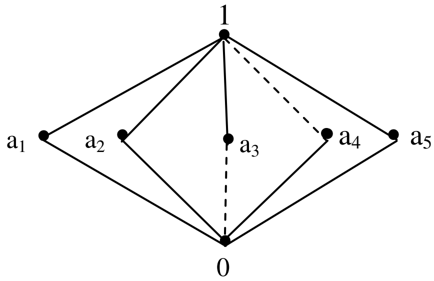

Figure 2.12

L is a edge neutrosophic modular lattice.
The edges connecting 0 to a_3 and 1 to a_4 are neutrosophic edges and the rest of the edges are reals. However all the vertices are real and it is a partially ordered set. We take some of the edges to be an indeterminate.

Example 2.5: Let L be the edge neutrosophic lattice whose Hassee diagram is as follows:

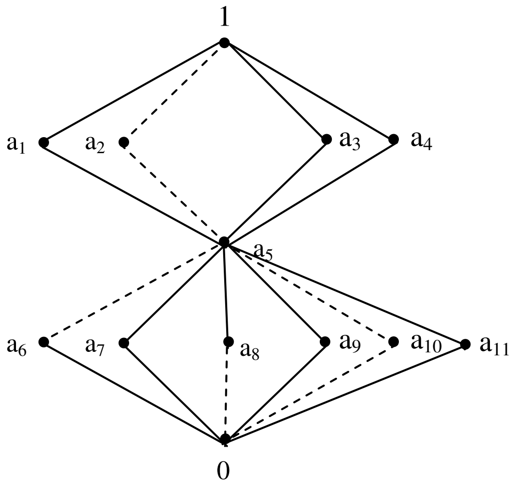

Figure 2.13

Clearly L is not a distributive edge neutrosophic lattice. However L has modular edge neutrosophic sublattices as well as modular lattices which are not neutrosophic.

Inview of this we have the following theorem.

THEOREM 2.1: *Let L be a edge neutrosophic lattice. Then L in general have sublattices which are not edge neutrosophic.*

Proof follows from the simple fact that every vertex is a sublattice and all vertices of the edge neutrosophic lattice which are not neutrosophic; but real is an instance of a not an edge neutropshic lattice.

We can have pure neutrosophic lattice which have the edges as well the vertices to be neutrosophic.

The following lattices with the Hasse diagram are pure neutrosophic lattices.

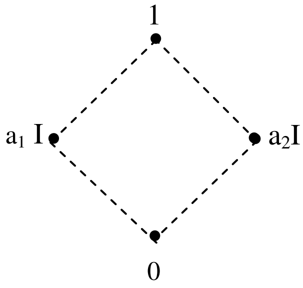

Figure 2.14

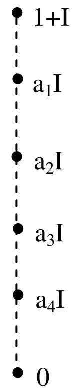

Figure 2.15

These two pure neutrosophic lattices cannot have edge neutrosophic sublattice or vertex neutrosophic sublattice.

3. Neutrosophic Boolean Algebras

Let us consider the power set of a neutrosophic set S = {a + bI | a = 0 or b = 0 can occur with 0 as the least element and 1 + I as the largest element}. P(S) = {Collection of all subsets of the set S} {P(S), $\cup$, $\cap$, ϕ, S} is a lattice defined as the neutrosophic Boolean algebra of order $2^{|P(S)|}$.

We will give examples of them.

Example 3.1: Let S = {0, 1, 1 +I, I}. P(S) = {ϕ, {0}, {1}, {1+I}, {I}, {0,1}, {0, I}, {0, 1+I}, {1, I}, {1, 1+I}, {I, 1+I}, {0, 1, I}, {0, 1, 1+I}, {0, I, 1+I}, {1, I, 1+I}, S} be the collection of all subsets of S including the empty set ϕ and the set S. |P(S)| = 16. P(S) is a neutrosophic Boolean algebra under '$\cup$' and '$\cap$' as the operations on P(S) and the containment relation of subsets as the partial order relation on P(S).

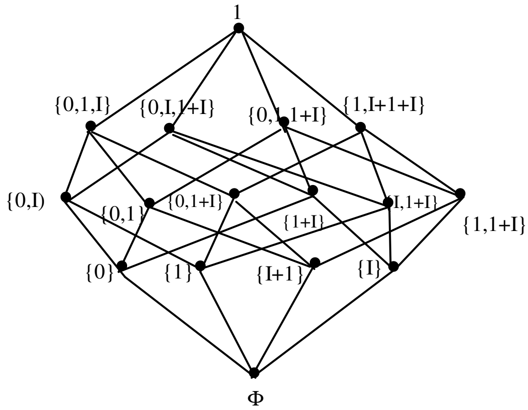

Figure 3.1

Example 3.2: Let S = {0, 1, I, 1+I, a, aI, a+I, aI+1, aI+a} be the neutrosophic set; 0 < a < 1. P(S) be the power set of S. |P(S)| = 2^9. P(S) is a neutrosophic Boolean algebra of order 2^9.

Example 3.3: Let S = {0, 1, I, 1+I, a_1, a_2, a_1I, a_2I, a_1+I, a_2+I, 1+a_1I, 1+a_2I, 1+a_1I+a_2, a_1+a_2, 1 + a_1I + a_2I, …} be the neutrosophic set with $a_1 \not< a_2$ or $a_2 \not< a_1$, $0 < a_1 < 1$, $0 < a_2 < 1$. P(S) is a neutrosophic Boolean algebra.

Now these neutrosophic Boolean algebras cannot be edge neutrosophic lattices. We make it possible to define edge neutrosophic lattice. Let L be a lattice given by the following Hasse-diagram.

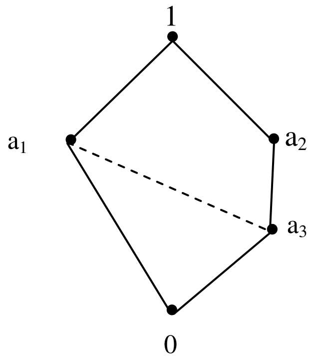

Figure 3.2

a_1 and a_3 are not comparable but we can have a neutrosophic edge given by the above diagram.

So we see the lattice has become a edge neutrosophic lattice.

Let L be a lattice given by the following diagram.

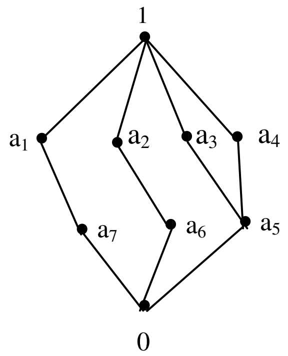

Figure 3.3

Clearly a_1 and a_6 are not comparable, a_2 and a_5 are not comparable a_4 and a_7 are not comparable.

We can have the following Hasse diagram which has neutrosophic edges.

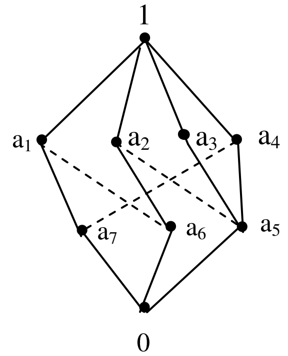

Figure 3.4

Clearly L is a edge neutrosophic lattice where we have some neutrosophic edges which are not comparable in the original lattice.

So we can on usual lattices L remake it into a edge neutrosophic lattice this is done if one doubts that a pair of elements $\{a_1, a_2\}$ of L with $a_1 \neq a_2$, min $\{a_1, a_2\} \neq a_1$ or a_2 or max $\{a_1, a_2\} \neq a_1$ or a_2.

If some experts needs to connect a_1 with a_2 by edge then the resultant lattice becomes a edge neutrosophic lattice.

Conclusion: Here for the first time we introduce the concept of neutrosophic lattices. Certainly these lattices will find applications in all places where lattices find their applications together with some indeterminancy. When one doubts a connection between two vertices one can have a neutrosophic edge.

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Single Valued Neutrosophic Similarity Measures for Multiple Attribute Decision-Making

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Abstract. Similarity measures play an important role in data mining, pattern recognition, decision making, machine learning, image process etc. Then, single valued neutrosophic sets (SVNSs) can describe and handle the indeterminate and inconsistent information, which fuzzy sets and intuitionistic fuzzy sets cannot describe and deal with. Therefore, the paper proposes new similarity meas-ures between SVNSs based on the minimum and maxi-mum operators. Then a multiple attribute decision-making method based on the weighted similarity measure of SVNSs is established in which attribute values for al-ternatives are represented by the form of single valued neutrosophic values (SVNVs) and the attribute weights and the weights of the three independent elements (i.e., truth-membership degree, indeterminacy-membership degree, and falsity-membership degree) in a SVNV are considered in the decision-making method. In the decision making, we utilize the single-valued neutrosophic weighted similarity measure between the ideal alternative and an alternative to rank the alternatives corresponding to the measure values and to select the most desirable one(s). Finally, two practical examples are provided to demonstrate the applications and effectiveness of the single valued neutrosophic multiple attribute decision-making method.

Keywords: *Neutrosophic set, single valued neutrosophic set, similarity measure, decision making.*

1 Introduction

Since fuzzy sets [1], intuitionistic fuzzy sets (IFSs) [2], interval-valued intuitionistic fuzzy sets (IVIFSs) [3] were introduced, they have been widely applied in data mining, pattern recognition, information retrieval, decision making, machine learning, image process and so on. Although they are very successful in their respective domains, fuzzy sets, IFSs, and IVIFSs cannot describe and deal with the indeterminate and inconsistent information that exists in real world. To handle uncertainty, imprecise, incomplete, and inconsistent information, Smarandache [4] proposed the concept of a neutrosophic set. The neutrosophic set is a powerful general formal framework which generalizes the concepts of the classic set, fuzzy set, IFS, IVIFS etc. [4]. In the neutrosophic set, truth-membership, indeterminacy-membership, and falsity-membership are represented independently. However, the neutrosophic set generalizes the above mentioned sets from philosophical point of view and its functions TA(x), IA(x) and FA(x) are real standard or nonstandard subsets of $]^-0, 1^+[$, i.e., TA(x): X → $]^-0, 1^+[$, IA(x): X → $]^-0, 1^+[$, and FA(x): X → $]^-0, 1^+[$. Thus, it is difficult to apply in real scientific and engineering areas. Therefore, Wang et al. [5, 6] introduced a single valued neutrosophic set (SVNS) and an interval neutrosophic set (INS), which are the subclass of a neutrosophic set. They can describe and handle indeterminate information and inconsistent information, which fuzzy sets, IFSs, and IVIFSs cannot describe and deal with. Recently, Ye [7-9] proposed the correlation coefficients of SVNSs and the cross-entropy measure of SVNSs and applied them to single valued neutrosophic decision-making problems. Then, Ye [10] introduced similarity measures based on the distances between INSs and applied them to multicriteria decision-making problems with interval neutrosophic information. Chi and Liu [11] proposed an extended TOPSIS method for the multiple attribute decision making problems with interval neutrosophic information. Furthermore, Ye [12] introduced the concept of simplified neutrosophic sets and presented simplified neutrosophic weighted aggregation operators, and then he applied them to multicriteria decision-making problems with simplified neutrosophic information. Majumdar and Samanta [13] introduced several similarity measures between SVNSs based on distances, a matching function, membership grades, and then proposed an entropy measure for a SVNS. Broumi and Smarandache [14] defined the distance between neutrosophic sets on the basis of the Hausdorff distance and some similarity measures based on the distances, set theoretic approach, and matching function to calculate the similarity degree between neutrosophic sets.

Because the concept of similarity is fundamentally important in almost every scientific field and SVNSs can describe and handle the indeterminate and inconsistent information, this paper proposes new similarity measures between SVNSs based on the minimum and maximum operators and establishes a multiple attribute decision-making method based on the weighted similarity measure of SVNSs under single valued neutrosophic environment. To do so, the rest of the article is organized as follows. Section 2 introduces some basic concepts of SVNSs. Section 3 proposes new similarity measures between SVNSs based on the minimum and maximum operators and investigates their properties. In Section 4, a single valued neutrosophic decision-making approach is proposed based on the weighted similarity measure of SVNSs. In Section 5, two practical examples are given to demonstrate the applications and the effectiveness of the proposed decision-making approach. Conclusions and further research are contained in Section 6.

2 Some basic concepts of SVNSs

Smarandache [4] originally introduced the concept of a neutrosophic set from philosophical point of view, which generalizes that of fuzzy set, IFS, and IVIFS etc..

Definition 1 [4]. Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]^-0, 1^+[$. That is $T_A(x)$: $X \to]^-0, 1^+[$, $I_A(x)$: $X \to]^-0, 1^+[$, and $F_A(x)$: $X \to]^-0, 1^+[$. Thus, there is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $^-0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.

Obviously, it is difficult to apply in real scientific and engineering areas. Hence, Wang et al. [6] introduced the definition of a SVNS.

Definition 2 [6]. Let X be a universal set. A SVNS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then, a SVNS A can be denoted by

$$A = \left\{ \left\langle x, T_A(x), I_A(x), F_A(x) \right\rangle \mid x \in X \right\},$$

where $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$ for each point x in X. Therefore, the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ satisfies the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 3 [6]. The complement of a SVNS A is denoted by A^c and is defined as $T_A^c(x) = F_A(x)$, $I_A^c(x) = 1 - I_A(x)$, $F_A^c(x) = T_A(x)$ for any x in X. Then, it can be denoted by

$$A^c = \left\{ \left\langle x, F_A(x), 1 - I_A(x), T_A(x) \right\rangle \mid x \in X \right\}.$$

Definition 4 [6]. A SVNS A is contained in the other SVNS B, $A \subseteq B$, if and only if $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$ for any x in X.

Definition 5 [6]. Two SVNSs A and B are equal, i.e., $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

3 Similarity measures of SVNSs

This section proposes several similarity measures of SVNSs based on the minimum and maximum operators and investigates their properties.

In general, a similarity measure between two SVNSs A and B is a function defined as S: $N(X)^2 \to [0, 1]$ which satisfies the following properties:

(SP1) $0 \le S(A, B) \le 1$;

(SP2) $S(A, B) = 1$ if $A = B$;

(SP3) $S(A, B) = S(B, A)$;

(SP4) $S(A, C) \le S(A, B)$ and $S(A, C) \le S(B, C)$ if $A \subseteq B \subseteq C$ for a SVNS C.

Let two SVNSs A and B in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ be $A = \left\{ \left\langle x_i, T_A(x_i), I_A(x_i), F_A(x_i) \right\rangle \mid x_i \in X \right\}$ and $B = \left\{ \left\langle x_i, T_B(x_i), I_B(x_i), F_B(x_i) \right\rangle \mid x_i \in X \right\}$, where $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i) \in [0, 1]$ for every $x_i \in X$. Based on the minimum and maximum operators, we present the following similarity measure between A and B:

$$S_1(A,B) = \frac{1}{3n} \sum_{i=1}^{n} \left(\frac{\min\left(T_A(x_i), T_B(x_i)\right)}{\max\left(T_A(x_i), T_B(x_i)\right)} + \frac{\min\left(I_A(x_i), I_B(x_i)\right)}{\max\left(I_A(x_i), I_B(x_i)\right)} + \frac{\min\left(F_A(x_i), F_B(x_i)\right)}{\max\left(F_A(x_i), F_B(x_i)\right)} \right). \quad (1)$$

The similarity measure has the following proposition.

Proposition 1. Let A and B be two SVNSs in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. The single valued neutrosophic similarity measure $S_1(A, B)$ should satisfy the following properties:

(SP1) $0 \le S_1(A, B) \le 1$;

(SP2) $S_1(A, B) = 1$ if $A = B$;

(SP3) $S_1(A, B) = S_1(B, A)$;

(SP4) $S_1(A, C) \le S_1(A, B)$ and $S_1(A, C) \le S_1(B, C)$ if $A \subseteq B \subseteq C$ for a SVNS C.

Proof. It is easy to remark that $S_1(A, B)$ satisfies the properties (SP1)-(SP3). Thus, we must prove the property (SP4).

Let $A \subseteq B \subseteq C$, then, $T_A(x_i) \le T_B(x_i) \le T_C(x_i)$, $I_A(x_i) \ge I_B(x_i) \ge I_C(x_i)$, and $F_A(x_i) \ge F_B(x_i) \ge F_C(x_i)$ for every $x_i \in X$. According to these inequalities, we have the following similarity measures:

$$S_1(A,B) = \frac{1}{3n}\sum_{i=1}^{n}\left(\frac{T_A(x_i)}{T_B(x_i)} + \frac{I_B(x_i)}{I_A(x_i)} + \frac{F_B(x_i)}{F_A(x_i)}\right),$$

$$S_1(A,C) = \frac{1}{3n}\sum_{i=1}^{n}\left(\frac{T_A(x_i)}{T_C(x_i)} + \frac{I_C(x_i)}{I_A(x_i)} + \frac{F_C(x_i)}{F_A(x_i)}\right)$$

$$S_1(B,C) = \frac{1}{3n}\sum_{i=1}^{n}\left(\frac{T_B(x_i)}{T_C(x_i)} + \frac{I_C(x_i)}{I_B(x_i)} + \frac{F_C(x_i)}{F_B(x_i)}\right).$$

Since there are $\frac{T_A(x_i)}{T_B(x_i)} \ge \frac{T_A(x_i)}{T_c(x_i)}$, $\frac{I_B(x_i)}{I_A(x_i)} \ge \frac{I_C(x_i)}{I_A(x_i)}$, and $\frac{F_B(x_i)}{F_A(x_i)} \ge \frac{F_C(x_i)}{F_A(x_i)}$, we can obtain that $S_1(A, C) \le S_1(A, B)$.

Similarly, there are $\frac{T_B(x_i)}{T_C(x_i)} \ge \frac{T_A(x_i)}{T_C(x_i)}$, $\frac{I_C(x_i)}{I_B(x_i)} \ge \frac{I_C(x_i)}{I_A(x_i)}$, and $\frac{F_C(x_i)}{F_B(x_i)} \ge \frac{F_C(x_i)}{F_A(x_i)}$. Then, we can obtain that $S_1(A, C) \le S_1(B, C)$.

Thus $S_1(A, B)$ satisfies the property (SP4).

Therefore, we finish the proof. □

If the important differences are considered in the three independent elements (i.e., truth-membership, indeterminacy-membership, and falsity-membership) in a SVNS, we need to take the weights of the three independent terms in Eq.(1) into account. Therefore, we develop another similarity measure between SVNSs:

$$S_2(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left(\alpha\frac{\min(T_A(x_i),T_B(x_i))}{\max(T_A(x_i),T_B(x_i))} + \beta\frac{\min(I_A(x_i),I_B(x_i))}{\max(I_A(x_i),I_B(x_i))} + \gamma\frac{\min(F_A(x_i),F_B(x_i))}{\max(F_A(x_i),F_B(x_i))}\right), \quad (2)$$

where α, β, γ are the weights of the three independent elements (i.e., truth-membership, indeterminacy-membership, and falsity-membership) in a SVNS and $\alpha + \beta + \gamma = 1$. Especially, when $\alpha = \beta = \gamma = 1/3$, Eq. (2) reduces to Eq. (1).

Then, the similarity measure of $S_2(A, B)$ also has the following proposition:

Proposition 2. Let A and B be two SVNSs in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. The single valued neutrosophic similarity measure $S_2(A, B)$ should satisfy the following properties:

(SP1) $0 \le S_2(A, B) \le 1$;

(SP2) $S_2(A, B) = 1$ if $A = B$;

(SP3) $S_2(A, B) = S_2(B, A)$;

(SP4) $S_2(A, C) \le S_2(A, B)$ and $S_2(A, C) \le S_2(B, C)$ if $A \subseteq B \subseteq C$ for a SVNS C.

By the similar proof method in Proposition 1, we can prove that the similarity measure of $S_2(A, B)$ also satisfies the properties (SP1)-(SP4) (omitted).

Furthermore, if the important differences are considered in the elements in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, we need to take the weight of each element x_i ($i = 1, 2,\ldots, n$) into account. Therefore, we develop a weighted similarity measure between SVNSs.

Let w_i be the weight for each element x_i ($i = 1, 2,\ldots, n$), $w_i \in [0, 1]$, and $\sum_{i=1}^{n} w_i = 1$, and then we have the following weighted similarity measure:

$$S_3(A,B) = \sum_{i=1}^{n} w_i\left(\alpha\frac{\min(T_A(x_i),T_B(x_i))}{\max(T_A(x_i),T_B(x_i))} + \beta\frac{\min(I_A(x_i),I_B(x_i))}{\max(I_A(x_i),I_B(x_i))} + \gamma\frac{\min(F_A(x_i),F_B(x_i))}{\max(F_A(x_i),F_B(x_i))}\right). \quad (3)$$

Similarly, the weighted similarity measure of $S_3(A, B)$ also has the following proposition:

Proposition 3. Let A and B be two SVNSs in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. Then, the single valued neutrosophic similarity measure $S_3(A, B)$ should satisfy the following properties:

(SP1) $0 \le S_3(A, B) \le 1$;

(SP2) $S_3(A, B) = 1$ if $A = B$;

(SP3) $S_3(A, B) = S_3(B, A)$;

(SP4) $S_3(A, C) \le S_3(A, B)$ and $S_3(A, C) \le S_3(B, C)$ if $A \subseteq B \subseteq C$ for a SVNS C.

Similar to the proof method in Proposition 1, we can prove that the weighted similarity measure of $S_3(A, B)$ also satisfies the properties (SP1)–(SP4) (omitted).
If $w = (1/n, 1/n,\ldots, 1/n)^T$, then Eq. (3) reduces to Eq. (2).

For Example, Assume that we have the following three SVNSs in a universe of discourse $X = \{x_1, x_2\}$:

$A = \{<x_1, 0.3, 0.6, 0.7>, <x_2, 0.4, 0.4, 0.6>\}$,

$B = \{<x_1, 0.5, 0.4, 0.5>, <x_2, 0.5, 0.3, 0.4>\}$,

$C = \{<x_1, 0.7, 0.2, 0.3>, <x_2, 0.8, 0.2, 0.2>\}$.

Then, there are $A \subseteq B \subseteq C$, with $T_A(x_i) \le T_B(x_i) \le T_C(x_i)$, $I_A(x_i) \ge I_B(x_i) \ge I_C(x_i)$, and $F_A(x_i) \ge F_B(x_i) \ge F_C(x_i)$ for each x_i in $X = \{x_1, x_2\}$.

By using Eq. (1), the similarity measures between the SVNSs are as follows:

$S_1(A, B) = 0.6996$, $S_1(B, C) = 0.601$, and $S_1(A, C) = 0.4206$.

Thus, there are $S_1(A, C) \le S_1(A, B)$ and $S_1(A, C) \le S_1(B, C)$.

If the weight values of the three independent elements (i.e., truth-membership degree, indeterminacy-membership degree, and falsity-membership degree) in a SVNS are $\alpha = 0.25$, $\beta = 0.35$, and $\gamma = 0.4$, by applying Eq. (2) the similarity measures between the SVNSs are as follows:

$S_2(A, B) = 0.6991$, $S_2(B, C) = 0.5916$, and $S_2(A, C) = 0.4143$.

Then, there are $S_2(A, C) \le S_2(A, B)$ and $S_2(A, C) \le S_2(B, C)$.

Assume that the weight vector of the two attributes is $w = (0.4, 0.6)^T$ and the weight values of the three independent elements (i.e., truth-membership degree, indeterminacy-membership degree, and falsity-membership degree) in a SVNS are $\alpha = 0.25$, $\beta = 0.35$, and $\gamma = 0.4$. By applying Eq. (3), the weighted similarity measures between the SVNSs are as follows:

$S_3(A, B) = 0.7051$, $S_3(B, C) = 0.4181$, and $S_3(A, C) = 0.5912$.

Hence, there are $S_3(A, C) \le S_3(A, B)$ and $S_3(A, C) \le S_3(B, C)$.

4 Decisions making method using the weighted similarity measure of SVNSs

In this section, we propose a multiple attribute decision-making method based on the weighted similarity measures between SVNSs under single valued neutrosophic environment.

Let $A = \{A_1, A_2,\ldots, A_m\}$ be a set of alternatives and $C = \{C_1, C_2,\ldots, C_n\}$ be a set of attributes. Assume that the weight of the attribute C_j ($j = 1, 2,\ldots, n$) is w_j with $w_j \in [0, 1]$, $\sum_{j=1}^{n} x_j = 1$ and the weights of the three independent elements (i.e., truth-membership, indeterminacy-membership, and falsity-membership) in a SVNS are α, β, and γ and $\alpha + \beta + \gamma = 1$, which are entered by the decision-maker. In this case, the characteristic of the alternative A_i ($i = 1, 2,\ldots, m$) on an attribute C_j ($j = 1, 2,\ldots, n$) is represented by a SVNS form:

$$A_i = \{\langle C_j, T_{A_i}(C_j), I_{A_i}(C_j), F_{A_i}(C_j)\rangle \mid C_j \in C\},$$

where $F_{A_i}(C_j)$, $I_{A_i}(C_j)$, $F_{A_i}(C_j) \in [0, 1]$ and $0 \le T_{A_i}(C_j) + I_{A_i}(C_j) + F_{A_i}(C_j) \le 3$ for $C_j \in C$, $j = 1, 2, \ldots, n$, and $i = 1, 2, \ldots, m$.

For convenience, the three elements $T_{A_i}(C_j)$, $I_{A_i}(C_j)$, $F_{A_i}(C_j)$ in the SVNS are denoted by a single valued neutrosophic value (SVNV) $a_{ij} = \langle t_{ij}, i_{ij}, f_{ij}\rangle$ ($i = 1, 2, \ldots, m$; $j = 1, 2,\ldots, n$), which is usually derived from the evaluation of an alternative A_i with respect to an attribute C_j by the expert or decision maker. Hence, we can establish a single valued neutrosophic decision matrix $D = (a_{ij})_{m\times n}$:

$$D = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

In multiple attribute decision making environments, the concept of ideal point has been used to help identify the best alternative in the decision set [7, 8]. Generally, the evaluation attributes can be categorized into two kinds: benefit attributes and cost attributes. Let H be a collection of benefit attributes and L be a collection of cost attributes. In the presented decision-making method, an ideal alternative can be identified by using a maximum operator for the benefit attributes and a minimum operator for the cost attributes to determine the best value of each attribute among all alternatives. Therefore, we define an ideal SVNV for a benefit attribute in the ideal alternative A^* as

$$a_j^* = \left\langle t_j^*, i_j^*, f_j^* \right\rangle = \left\langle \max_i(t_{ij}), \min_i(i_{ij}), \min_i(f_{ij}) \right\rangle \text{ for } j \in H;$$

while for a cost attribute, we define an ideal SVNV in the ideal alternative A^* as

$$a_j^* = \left\langle t_j^*, i_j^*, f_j^* \right\rangle = \left\langle \min_i(t_{ij}), \max_i(i_{ij}), \max_i(f_{ij}) \right\rangle \text{ for } j \in L.$$

Thus, by applying Eq. (3), the weighted similarity measure between an alternative A_i and the ideal alternative A^* are written as

$$S_4(A_i, A^*) = \sum_{j=1}^{n} w_j \left(\alpha \frac{\min(t_{ij}, t_j^*)}{\max(t_{ij}, t_j^*)} + \beta \frac{\min(i_{ij}, i_j^*)}{\max(i_{ij}, i_j^*)} + \gamma \frac{\min(f_{ij}, f_j^*)}{\max(f_{ij}, f_j^*)} \right) \quad (4)$$

which provides the global evaluation for each alternative regarding all attributes. According to the weighted similarity measure between each alternative and the ideal alternative, the bigger the measure value $S_4(A_i, A^*)$ (i = 1, 2, 3, 4), the better the alternative A_i. Hence, the ranking order of all alternatives can be determined and the best one can be easily selected as well.

5 Practical examples

This section provides two practical examples for multiple attribute decision-making problems with single valued neutrosophic information to demonstrate the applications and effectiveness of the proposed decision-making method.

Example 1. Let us consider the decision-making problem adapted from [7, 8]. There is an investment company, which wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investment company must take a decision according to the three attributes: (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is the environmental impact, where C_1 and C_2 are benefit attributes and C_3 is a cost attribute. The weight vector of the three attributes is given by w = $(0.35, 0.25, 0.4)^T$. The four possible alternatives are to be evaluated under the above three attributes by the form of SVNVs.

For the evaluation of an alternative A_i (i =1, 2, 3, 4) with respect to an attribute C_j (j =1, 2, 3), it is obtained from the questionnaire of a domain expert. For example, when we ask the opinion of an expert about an alternative A_1 with respect to an attribute C_1, he/she may say that the possibility in which the statement is good is 0.4 and the statement is poor is 0.3 and the degree in which he/she is not sure is 0.2. For the neutrosophic notation, it can be expressed as $a_{11} = \langle 0.4, 0.2, 0.3 \rangle$. Thus, when the four possible alternatives with respect to the above three attributes are evaluated by the expert, we can obtain the following single valued neutrosophic decision matrix D:

$$D = \begin{bmatrix} \langle 0.4,0.2,0.3 \rangle & \langle 0.4,0.2,0.3 \rangle & \langle 0.8,0.2,0.5 \rangle \\ \langle 0.6,0.1,0.2 \rangle & \langle 0.6,0.1,0.2 \rangle & \langle 0.5,0.2,0.8 \rangle \\ \langle 0.3,0.2,0.3 \rangle & \langle 0.5,0.2,0.3 \rangle & \langle 0.5,0.3,0.8 \rangle \\ \langle 0.7,0.0,0.1 \rangle & \langle 0.6,0.1,0.2 \rangle & \langle 0.6,0.3,0.8 \rangle \end{bmatrix}.$$

Without loss of generality, let the weight values of the three independent elements (i.e., truth-membership degree, indeterminacy-membership degree, and falsity-membership degree) in a SVNV be $\alpha = \beta = \gamma = 1/3$. Then we utilize the developed approach to obtain the most desirable alternative(s).

Firstly, from the single valued neutrosophic decision matrix we can yield the following ideal alternative:

$$A^* = \{\langle C_1, 0.7, 0.0, 0.1 \rangle, \langle C_2, 0.6, 0.1, 0.2 \rangle, \langle C_3, 0.5, 0.3, 0.8 \rangle\}.$$

Then, by using Eq. (4) we can obtain the values of the weighted similarity measure $S_4(A_i, A^*)$ (i =1, 2, 3, 4):

$S_4(A_1, A^*) = 0.6595$, $S_4(A_2, A^*) = 0.9805$,

$S_4(A_3, A^*) = 0.7944$, and $S_4(A_4, A^*) = 0.9828$.

Thus, the ranking order of the four alternatives is $A_4 \succ A_2 \succ A_3 \succ A_1$. Therefore, the alternative A_4 is the best choice among the four alternatives.

From the above results we can see that the ranking order of the alternatives and best choice are in agreement with the results (i.e., the ranking order is $A_4 \succ A_2 \succ A_3 \succ A_1$ and the best choice is A_4.) in Ye's method [8], but not in agreement with the results (i.e., the ranking order is $A_2 \succ A_4 \succ A_3 \succ A_1$ and the best choice is A_2.) in Ye's method [7]. The reason is that different measure methods may yield different ranking orders of the alternatives in the decision-making process.

Example 2. A multi-criteria decision making problem adopted from Ye [9] is concerned with a manufacturing company which wants to select the best global supplier according to the core competencies of suppliers. Now suppose that there are a set of four suppliers $A = \{A_1, A_2, A_3, A_4\}$ whose core competencies are evaluated by means of the four attributes (C_1, C_2, C_3, C_4): (1) the level of technology innovation (C_1), (2) the control ability of flow (C_2), (3) the ability of management (C_3), (4) the level of service (C_4), which are all benefit attributes. Then, the weight vector for the four attributes is $w = (0.3, 0.25, 0.25, 0.2)^T$. The four possible alternatives are to be evaluated under the above four attributes by the form of SVNVs.

For the evaluation of an alternative A_i (i =1, 2, 3, 4) with respect to an attribute C_j (j =1, 2, 3, 4), by the similar evaluation method in Example 1 it is obtained from the questionnaire of a domain expert. For example, when we ask the opinion of an expert about an alternative A_1 with respect to an attribute C_1, he/she may say that the possibility in which the statement is good is 0.5 and the statement is poor is 0.3 and the degree in which he/she is not sure is 0.1. For the neutrosophic notation, it can be expressed as $a_{11} = \langle 0.5, 0.1, 0.3\rangle$. Thus, when the four possible alternatives with respect to the above four attributes are evaluated by the similar method from the expert, we can establish the following single valued neutrosophic decision matrix D:

$$D = \begin{pmatrix} \langle 0.5,0.1,0.3\rangle & \langle 0.5,0.1,0.4\rangle & \langle 0.7,0.1,0.2\rangle & \langle 0.3,0.2,0.1\rangle \\ \langle 0.4,0.2,0.3\rangle & \langle 0.3,0.2,0.4\rangle & \langle 0.9,0.0,0.1\rangle & \langle 0.5,0.3,0.2\rangle \\ \langle 0.4,0.3,0.1\rangle & \langle 0.5,0.1,0.3\rangle & \langle 0.5,0.0,0.4\rangle & \langle 0.6,0.2,0.2\rangle \\ \langle 0.6,0.1,0.2\rangle & \langle 0.2,0.2,0.5\rangle & \langle 0.4,0.3,0.2\rangle & \langle 0.7,0.2,0.1\rangle \end{pmatrix}.$$

Without loss of generality, let the weight values of the three independent elements (i.e., truth-membership degree, indeterminacy-membership degree, and falsity-membership degree) in a SVNV be $\alpha = \beta = \gamma = 1/3$. Then the proposed decision-making method is applied to solve this problem for selecting suppliers.

From the single valued neutrosophic decision matrix, we can yield the following ideal alternative:

$$A^* = \{\langle C_1, 0.6, 0.1, 0.1\rangle, \langle C_2, 0.5, 0.1, 0.3\rangle, \langle C_3, 0.9, 0.0, 0.1\rangle, \langle C_4, 0.7, 0.2, 0.1\rangle\}.$$

By applying Eq. (4), the weighted similarity measure values between an alternative A_i (i = 1, 2, 3, 4) and the ideal alternative A^* are as follows:

$S_4(A_1, A^*) = 0.7491$, $S_4(A_2, A^*) = 0.7433$,
$S_4(A_3, A^*) = 0.7605$, and $S_4(A_4, A^*) = 0.6871$.

According to the measure values, the ranking order of the four suppliers is $A_3 \succ A_1 \succ A_2 \succ A_4$. Hence, the best supplier is A_3. From the results we can see that the ranking order of the alternatives and best choice are in agreement with the results in Ye's method [9].

From the above two examples, we can see that the proposed single valued neutrosophic multiple attribute decision-making method is more suitable for real scientific and engineering applications because it can handle not only incomplete information but also the indeterminate information and inconsistent information which exist commonly in real situations. Especially, in the proposed decision-making method we consider the important differences in the three independent elements (i.e., truth-membership degree, indeterminacy-membership degree, and falsity-membership degree) in a SVNV and can adjust the weight values of the three independent elements. Thus, the proposed single valued neutrosophic decision-making method is more flexible and practical than the existing decision-making methods [7-9]. The technique proposed in this paper extends the existing decision-making methods and provides a new way for decision-makers.

6 Conclusion

This paper has developed three similarity measures between SVNSs based on the minimum and maximum operators and investigated their properties. Then the proposed weighted similarity measure of SVNSs has been applied to multiple attribute decision-making problems under single valued neutrosophic environment. The proposed method differs from previous approaches for single valued neutrosophic multiple attribute decision making not only due to the fact that the proposed method use the weighted similarity measure of SVNSs, but also due to considering the weights of the truth-membership, indeterminacy-membership, and falsity-membership in SVNSs, which makes it have more flexible and practical than existing decision making methods [7-9] in real decision making problems. Through the weighted similarity measure between each alternative and the ideal alternative, we can obtain the ranking order of all alternatives and the best alternative. Finally, two practical examples demonstrated the applications and effectiveness of the decision-making approach under single valued neutrosophic environments. The proposed decision-making method can effectively deal with decision-making problems with the incomplete, indeterminate, and inconsistent information which exist commonly in real situations. Furthermore, by the similar method we can easily extend the proposed weighted similarity measure of SVNSs and its decision-making method to that of INSs. In the future, we shall investigate similarity measures between SVNSs and between INSs in the applications of other domains, such as pattern recognition, clustering analysis, image process, and medical diagnosis.

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Soft Neutrosophic Bigroup and Soft Neutrosophic N-Group

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Abstract. Soft neutrosophic group and soft neutrosophic subgroup are generalized to soft neutrosophic bigroup and soft neutrosophic N-group respectively in this paper. Different kinds of soft neutrosophic bigroup and soft neutrosophic N-group are given. The structural properties and theorems have been discussed with a lot of examples to disclose many aspects of this beautiful man made structure.

Keywords: Neutrosophic bigroup, Neutrosophic N-group, soft set, soft group, soft subgroup, soft neutrosophic bigroup, soft neutrosophic subbigroup, soft neutrosophic N-group, soft neutrosophic sub N-group.

1 Introduction

Neutrosophy is a new branch of philosophy which is in fact the birth stage of neutrosophic logic first found by Florentin Smarandache in 1995. Each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic. In fact neutrosophic set is the generalization of classical sets, conventional fuzzy set $[1]$, intuitionistic fuzzy set $[2]$ and interval valued fuzzy set $[3]$. This mathematical tool is handling problems like imprecise, indeterminacy and inconsistent data etc. By utilizing neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache dig out neutrosophic algebraic structures in $[11]$. Some of them are neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, and neutrosophic bigroupoids and so on.

Molodtsov in $[11]$ laid down the stone foundation of a richer structure called soft set theory which is free from the parameterization inadequacy, syndrome of fuzzy se theory, rough set theory, probability theory and so on. In many areas it has been successfully applied such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, and probability. Recently soft set theory has attained much attention since its appearance and the work based on several operations of soft sets introduced in $[2,9,10]$. Some more exciting properties and algebra may be found in $[1]$. Feng et al. introduced the soft semirings $[5]$. By means of level soft sets an adjustable approach to fuzzy soft sets based decision making can be seen in $[6]$. Some other new concept combined with fuzzy sets and rough sets was presented in $[7,8]$. AygÄunoglu et al. introduced the Fuzzy soft groups $[4]$. This paper is a mixture of neutrosophic bigroup,neutrosophic N -group and soft set theory which is infact a generalization of soft neutrosophic group. This combination gave birth to a new and fantastic approach called "Soft Neutrosophic Bigroup and Soft Neutrosophic N -group".

2.1 Neutrosophic Bigroup and N-Group

Definition 1 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a non empty subset with two binary operations on $B_N(G)$ satisfying the following conditions:

1) $B_N(G)=\{B(G_1)\cup B(G_2)\}$ where $B(G_1)$ and $B(G_2)$ are proper subsets of $B_N(G)$.

2) $(B(G_1),*_1)$ is a neutrosophic group.

3) $(B(G_2),*_2)$ is a group $\cdot$

Then we define $(B_N(G),*_1,*_2)$ to be a neutrosophic bigroup. If both $B(G_1)$ and $B(G_2)$ are neutrosophic groups. We say $B_N(G)$ is a strong neutrosophic bigroup. If both the groups are not neutrosophic group, we say $B_N(G)$ is just a bigroup.

Example 1 Let $B_N(G)=\{B(G_1)\cup B(G_2)\}$ where $B(G_1)=\{g/g^9=1\}$ be a cyclic group of order 9 and $B(G_2)=\{1,2,I,2I\}$ neutrosophic group under multiplication modulo 3. We call $B_N(G)$ a neutrosophic bigroup.

Example 2 Let $B_N(G)=\{B(G_1)\cup B(G_2)\}$ Where $B(G_1)=\{1,2,3,4,I,2I,3I,4I\}$ a neutrosophic group under multiplication modulo 5.

$B(G_2)=\{0,1,2,I,2I,1+I,2+I,1+2I,2+2I\}$ is a neutrosophic group under multiplication modulo 3. Clearly $B_N(G)$ is a strong neutrosophic bigroup.

Definition 2 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup. A proper subset $P=\{P_1\cup P_2,*_1,*_2\}$ is a neutrosophic subbigroup of $B_N(G)$ if the following conditions are satisfied $P=\{P_1\cup P_2,*_1,*_2\}$ is a neutrosophic bigroup under the operations $*_1,*_2$ i.e. $(P_1,*_1)$ is a neutrosophic subgroup of $(B_1,*_1)$ and $(P_2,*_2)$ is a subgroup of $(B_2,*_2)$. $P_1=P\cap B_1$ and $P_2=P\cap B_2$ are subgroups of B_1 and B_2 respectively. If both of P_1 and P_2 are not neutrosophic then we call $P=P_1\cup P_2$ to be just a bigroup.

Definition 3 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup. If both $B(G_1)$ and $B(G_2)$ are commutative groups, then we call $B_N(G)$ to be a commutative bigroup.

Definition 4 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup. If both $B(G_1)$ and $B(G_2)$ are cyclic, we call $B_N(G)$ a cyclic bigroup.

Definition 5 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup. $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic bigroup. $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ is said to be a neutrosophic normal subbigroup of $B_N(G)$ if $P(G)$ is a neutrosophic subbigroup and both $P(G_1)$ and $P(G_2)$ are normal subgroups of $B(G_1)$ and $B(G_2)$ respectively.

Definition 6 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup of finite order. Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup of $B_N(G)$. If $o(P(G))/o(B_N(G))$ then we call $P(G)$ a Lagrange neutrosophic subbigroup, if every neutrosophic subbigroup P is such that $o(P)/o(B_N(G))$ then we call $B_N(G)$ to be a Lagrange neutrosophic bigroup.

Definition 7 If $B_N(G)$ has atleast one Lagrange neutrosophic subbigroup then we call $B_N(G)$ to be a weak Lagrange neutrosophic bigroup.

Definition 8 If $B_N(G)$ has no Lagrange neutrosophic subbigroup then $B_N(G)$ is called Lagrange free neutrosophic bigroup.

Definition 9 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup. Suppose $P=\{P(G_1)\cup P(G_2),*_1,*_2\}$ and

$K=\{K(G_1)\cup K(G_2),*_1,*_2\}$ be any two neutrosophic subbigroups. we say P and K are conjugate if each $P(G_i)$ is conjugate with $K(G_i), i=1,2$, then we say P and K are neutrosophic conjugate subbigroups of $B_N(G)$.

Definition 10 A set $(\langle G\cup I\rangle,+,\circ)$ with two binary operations ` $+$ ' and ` $\circ$ ' is called a strong neutrosophic bigroup if

1) $\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle$,
2) $(\langle G_1\cup I\rangle,+)$ is a neutrosophic group and
3) $(\langle G_2\cup I\rangle,\circ)$ is a neutrosophic group.

Example 3 Let $\{\langle G\cup I\rangle,*_1,*_2\}$ be a strong neutrosophic bigroup where $\langle G\cup I\rangle=\langle Z\cup I\rangle\cup\{0,1,2,3,4,I,2I,3I,4I\}$. $\langle Z\cup I\rangle$ under ` $+$ ' is a neutrosophic group and $\{0,1,2,3,4,I,2I,3I,4I\}$ under multiplication modulo 5 is a neutrosophic group.

Definition 11 A subset $H\neq\phi$ of a strong neutrosophic bigroup $(\langle G\cup I\rangle,*,\circ)$ is called a strong neutrosophic subbigroup if H itself is a strong neutrosophic bigroup under ` $*$ ' and ` $\circ$ ' operations defined on $\langle G\cup I\rangle$.

Definition 12 Let $(\langle G\cup I\rangle,*,\circ)$ be a strong neutrosophic bigroup of finite order. Let $H\neq\phi$ be a strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*,\circ)$. If $o(H)/o(\langle G\cup I\rangle)$ then we call H, a Lagrange strong neutrosophic subbigroup of $\langle G\cup I\rangle$. If every strong neutrosophic subbigroup of $\langle G\cup I\rangle$ is a Lagrange strong neutrosophic subbigroup then we call $\langle G\cup I\rangle$ a Lagrange strong neutrosophic bigroup.

Definition 13 If the strong neutrosophic bigroup has at least one Lagrange strong neutrosophic subbigroup then we call $\langle G\cup I\rangle$ a weakly Lagrange strong neutrosophic bigroup.

Definition 14 If $\langle G\cup I\rangle$ has no Lagrange strong neutrosophic subbigroup then we call $\langle G\cup I\rangle$ a Lagrange free strong neutrosophic bigroup.

Definition 15 Let $(\langle G\cup I\rangle,+,\circ)$ be a strong neutrosophic bigroup with $\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle$. Let $(H,+,\circ)$ be a neutrosophic subbigroup where $H=H_1\cup H_2$. We say H is a neutrosophic normal subbigroup of G if both H_1 and H_2 are neutrosophic normal subgroups of $\langle G_1\cup I\rangle$ and $\langle G_2\cup I\rangle$ respectively.

Definition 16 Let $G=\langle G_1\cup G_2,*,\otimes\rangle$, be a neutrosophic bigroup. We say two neutrosophic strong subbigroups $H=H_1\cup H_2$ and $K=K_1\cup K_2$ are conjugate neutrosophic subbigroups of $\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle$ if H_1 is conjugate to K_1 and H_2 is conjugate to K_2 as neutrosophic subgroups of $\langle G_1\cup I\rangle$ and $\langle G_1\cup I\rangle$ respectively.

Definition 17 Let $(\langle G\cup I\rangle,*_1,...,*_N)$ be a nonempty set with N -binary operations defined on it. We say $\langle G\cup I\rangle$ is a strong neutrosophic N -group if the following conditions are true.

1) $\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle\cup...\cup\langle G_N\cup I\rangle$ where $\langle G_i\cup I\rangle$ are proper subsets of $\langle G\cup I\rangle$.
2) $(\langle G_i\cup I\rangle,*_i)$ is a neutrosophic group, $i=1,2,...,N$.
3) If in the above definition we have
   a. $\langle G\cup I\rangle=G_1\cup\langle G_2\cup I\rangle\cup...\cup\langle G_k\cup I\rangle\cup\langle G_{k+1}\cup I\rangle\cup...\cup G_N$
   b. $(G_i,*_i)$ is a group for some i or
4) $(\langle G_j\cup I\rangle,*_j)$ is a neutrosophic group for some j. Then we call $\langle G\cup I\rangle$ to be a neutrosophic N -group.

Example 4 Let
$\langle G\cup I\rangle=(\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle\cup\langle G_3\cup I\rangle\cup\langle G_4\cup I\rangle,*_1,*_2,*_3,*_4)$
be a neutrosophic 4 -group where
$\langle G_1\cup I\rangle=\{1,2,3,4,I,2I,3I,4I\}$
neutrosophic group under multiplication modulo 5.
$\langle G_2\cup I\rangle=\{0,1,2,I,2I,1+I,2+I,1+2I,2+2I\}$
a neutrosophic group under multiplication modulo 3,

$\langle G_3 \cup I \rangle = \langle Z \cup I \rangle$, a neutrosophic group under addition and $\langle G_4 \cup I \rangle = \{(a,b) : a,b \in \{1, I, 4, 4I\}\}$, component-wise multiplication modulo $5\}$.

Hence $\langle G \cup I \rangle$ is a strong neutrosophic 4 -group.

Example 5 Let

$\left(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle \cup G_3 \cup G_4, *_1, *_2, *_3, *_4\right)$

be a neutrosophic 4 -group, where

$\langle G_1 \cup I \rangle = \{1, 2, 3, 4, I, 2I, 3I, 4I\}$ a neutrosophic group under multiplication modulo 5.

$\langle G_2 \cup I \rangle = \{0, 1, I, 1+I\}$,a neutrosophic group under multiplication modulo 2. $G_3 = S_3$ and $G_4 = A_5$, the alternating group. $\langle G \cup I \rangle$ is a neutrosophic 4 -group.

Definition 18 Let

$\left(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle \cup ... \cup \langle G_N \cup I \rangle, *_1, ..., *_N\right)$

be a neutrosophic N -group. A proper subset $(P, *_1, ..., *_N)$ is said to be a neutrosophic sub N -group of $\langle G \cup I \rangle$ if $P = (P_1 \cup ... \cup P_N)$ and each $(P_i, *_i)$ is a neutrosophic subgroup (subgroup) of $(G_i, *_i), 1 \le i \le N$.

It is important to note $(P, *_i)$ for no i is a neutrosophic group.

Thus we see a strong neutrosophic N -group can have 3 types of subgroups viz.

1) Strong neutrosophic sub N -groups.
2) Neutrosophic sub N -groups.
3) Sub N -groups.

Also a neutrosophic N -group can have two types of sub N -groups.

1) Neutrosophic sub N -groups.
2) Sub N -groups.

Definition 19 If $\langle G \cup I \rangle$ is a neutrosophic N -group and if $\langle G \cup I \rangle$ has a proper subset T such that T is a neutrosophic sub N -group and not a strong neutrosophic sub N -group and $o(T) / o(\langle G \cup I \rangle)$ then we call T a Lagrange sub N -group. If every sub N -group of $\langle G \cup I \rangle$ is a Lagrange sub N -group then we call $\langle G \cup I \rangle$ a Lagrange N -group.

Definition 20 If $\langle G \cup I \rangle$ has atleast one Lagrange sub N -group then we call $\langle G \cup I \rangle$ a weakly Lagrange neutrosophic N-group.

Definition 21 If $\langle G \cup I \rangle$ has no Lagrange sub N -group then we call $\langle G \cup I \rangle$ to be a Lagrange free N -group.

Definition 22 Let

$\left(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle \cup ... \cup \langle G_N \cup I \rangle, *_1, ..., *_N\right)$

be a neutrosophic N -group. Suppose $H = \{H_1 \cup H_2 \cup ... \cup H_N, *_1, ..., *_N\}$ and $K = \{K_1 \cup K_2 \cup ... \cup K_N, *_1, ..., *_N\}$ are two sub N -groups of $\langle G \cup I \rangle$, we say K is a conjugate to H or H is conjugate to K if each H_i is conjugate to K_i $(i = 1, 2, ..., N)$ as subgroups of G_i .

2.2 Soft Sets

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A \subset E$. Molodtsov defined the soft set in the following manner:

Definition 23 A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $x \in A$, $F(x)$ may be considered as the set of x -elements of the soft set (F, A) , or as the set of e-approximate elements of the soft set.

Example 6 Suppose that U is the set of shops. E is the set of parameters and each parameter is a word or sentence. Let

$$E = \left\{ \begin{array}{l} \text{high rent,normal rent,} \\ \text{in good condition,in bad condition} \end{array} \right\}.$$

Let us consider a soft set (F, A) which describes the attractiveness of shops that Mr. Z is taking on rent. Suppose that there are five houses in the universe $U = \{s_1, s_2, s_3, s_4, s_5\}$ under consideration, and that $A = \{x_1, x_2, x_3\}$ be the set of parameters where

x_1 stands for the parameter 'high rent,

x_2 stands for the parameter 'normal rent,

x_3 stands for the parameter 'in good condition.
Suppose that

$$F(x_1) = \{s_1, s_4\},$$
$$F(x_2) = \{s_2, s_5\},$$
$$F(x_3) = \{s_3, s_4, s_5\}.$$

The soft set (F,A) is an approximated family $\{F(e_i), i = 1,2,3\}$ of subsets of the set U which gives us a collection of approximate description of an object. Then (F,A) is a soft set as a collection of approximations over U, where

$$F(x_1) = high\ \ rent = \{s_1, s_2\},$$
$$F(x_2) = normal\ \ rent = \{s_2, s_5\},$$
$$F(x_3) = in\ \ good\ \ condition = \{s_3, s_4, s_5\}.$$

Definition 24 For two soft sets (F,A) and (H,B) over U, (F,A) is called a soft subset of (H,B) if

1. $A \subseteq B$ and
2. $F(x) \subseteq H(x)$, for all $x \in A$.

This relationship is denoted by $(F,A) \subset (H,B)$. Similarly (F,A) is called a soft superset of (H,B) if (H,B) is a soft subset of (F,A) which is denoted by $(F,A) \supset (H,B)$.

Definition 25 Two soft sets (F,A) and (H,B) over U are called soft equal if (F,A) is a soft subset of (H,B) and (H,B) is a soft subset of (F,A).

Definition 26 Let (F,A) and (K,B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by $(F,A) \cap_R (K,B) = (H,C)$ where (H,C) is defined as $H(c) = F(c) \cap \mathrm{K}(c)$ for all $c \in C = A \cap B$.

Definition 27 The extended intersection of two soft sets (F,A) and (K,B) over a common universe U is the soft set (H,C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & \text{if c} \in A - B, \\ G(c) & \text{if c} \in B - A, \\ F(c) \cap G(c) & \text{if c} \in A \cap B. \end{cases}$$

We write $(F,A) \cap_\varepsilon (K,B) = (H,C)$.

Definition 28 The restricted union of two soft sets (F,A) and (K,B) over a common universe U is the soft set (H,C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as $H(c) = F(c) \cup G(c)$ for all $c \in C$. We write it as $(F,A) \cup_R (K,B) = (H,C).$

Definition 29 The extended union of two soft sets (F,A) and (K,B) over a common universe U is the soft set (H,C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & \text{if c} \in A - B, \\ G(c) & \text{if c} \in B - A, \\ F(c) \cup G(c) & \text{if c} \in A \cap B. \end{cases}$$

We write $(F,A) \cup_\varepsilon (K,B) = (H,C)$.

2.3 Soft Groups

Definition 30 Let (F,A) be a soft set over G. Then (F,A) is said to be a soft group over G if and only if $F(x) \prec G$ for all $x \in A$.

Example 7 Suppose that

$$G = A = S_3 = \{e,(12),(13),(23),(123),(132)\}.$$

Then (F,A) is a soft group over S_3 where

$$F(e) = \{e\},$$
$$F(12) = \{e,(12)\},$$
$$F(13) = \{e,(13)\},$$
$$F(23) = \{e,(23)\},$$
$$F(123) = F(132) = \{e,(123),(132)\}.$$

Definition 31 Let (F,A) be a soft group over G. Then

1. (F,A) is said to be an identity soft group over G if $F(x) = \{e\}$ for all $x \in A$, where e is the identity element of G and
2. (F,A) is said to be an absolute soft group if $F(x) = G$ for all $x \in A$.

3 Soft Neutrosophic Bigroup

Definition 32 Let

$$B_N(G) = \{B(G_1) \cup B(G_2), *_1, *_2\}$$

be a neutrosophic bigroup and let (F,A) be a soft set over $B_N(G)$. Then (F,A) is said to be soft neutrosophic bigroup over $B_N(G)$ if and only if $F(x)$ is a subbigroup of $B_N(G)$ for all $x \in A$.

Example 8 Let

$$B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$$

be a neutrosophic bigroup, where

$$B(G_1)=\{0,1,2,3,4,I,2I,3I,4I\}$$

is a neutrosophic group under multiplication modulo 5. $B(G_2)=\{g:g^{12}=1\}$ is a cyclic group of order 12.

Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup where $P(G_1)=\{1,4,I,4I\}$ and $P(G_2)=\{1,g^2,g^4,g^6,g^8,g^{10}\}$.

Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another neutrosophic subbigroup where $Q(G_1)=\{1,I\}$ and $Q(G_2)=\{1,g^3,g^6,g^9\}$.

Then (F,A) is a soft neutrosophic bigroup over $B_N(G)$, where

$$F(e_1)=\{1,4,I,4I,1,g^2,g^4,g^6,g^8,g^{10}\}$$

$$F(e_2)=\{1,I,1,g^3,g^6,g^9\}.$$

Theorem 1 Let (F,A) and (H,A) be two soft neutrosophic bigroup over $B_N(G)$. Then their intersection $(F,A)\cap(H,A)$ is again a soft neutrosophic bigroup over $B_N(G)$.

Proof Straight forward.

Theorem 2 Let (F,A) and (H,B) be two soft neutrosophic bigroups over $B_N(G)$ such that $A\cap B=\phi$, then their union is soft neutrosophic bigroup over $B_N(G)$.

Proof Straight forward.

Proposition 1 The extended union of two soft neutrosophic bigroups (F,A) and (K,D) over $B_N(G)$ is not a soft neutrosophic bigroup over $B_N(G)$.

To prove it, see the following example.

Example 9 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$, where $B(G_1)=\{1,2,3,4I,2I,3I,4I\}$ and $B(G_2)=S_3$.

Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup where $P(G_1)=\{1,4,I,4I\}$ and $P(G_2)=\{e,(12)\}$.

Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another neutrosophic subbigroup where $Q(G_1)=\{1,I\}$ and $Q(G_2)=\{e,(123),(132)\}$.

Then (F,A) is a soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1)=\{1,4,I,4I,e,(12)\}$$

$$F(x_2)=\{1,I,e,(123),(132)\}.$$

Again let $R(G)=\{R(G_1)\cup R(G_2),*_1,*_2\}$ be another neutrosophic subbigroup where $R(G_1)=\{1,4,I,4I\}$ and $R(G_2)=\{e,(13)\}$.

Also $T(G)=\{T(G_1)\cup T(G_2),*_1,*_2\}$ be a neutrosophic subbigroup where $T(G_1)=\{1,I\}$ and $T(G_2)=\{e,(23)\}$.

Then (K,D) is a soft neutrosophic bigroup over $B_N(G)$, where

$$K(x_2)=\{1,4,I,4I,e,(13)\},$$

$$K(x_3)=\{1,I,e,(23)\}.$$

The extended union $(F,A)\cup_{\varepsilon}(K,D)=(H,C)$ such that $C=A\cup D$ and for $x_2\in C$, we have

$$H(x_2)=F(x_2)\cup K(x_2)=\{1,4,I,4I,e,(13)(123),(132)\}$$

is not a subbigroup of $B_N(G)$.

Proposition 2 The extended intersection of two soft neutrosophic bigroups (F,A) and (K,D) over $B_N(G)$ is again a soft neutrosophic bigroup over $B_N(G)$.

Proposition 3 The restricted union of two soft neutrosophic bigroups (F,A) and (K,D) over $B_N(G)$ is not a soft neutrosophic bigroup over $B_N(G)$.

Proposition 4 The restricted intersection of two soft neutrosophic bigroups (F,A) and (K,D) over $B_N(G)$ is a soft neutrosophic bigroup over $B_N(G)$.

Proposition 5 The AND operation of two soft neutrosophic bigroups over $B_N(G)$ is again soft neutrosophic bigroup over $B_N(G)$.

Proposition 6 The OR operation of two soft neutrosophic bigroups over $B_N(G)$ may not be a soft nuetrosophic bigroup.

Definition 33 Let (F,A) be a soft neutrosophic bigroup over $B_N(G)$. Then

1) (F,A) is called identity soft neutrosophic bigroup if $F(x)=\{e_1,e_2\}$ for all $x\in A$, where e_1 and e_2 are the identities of $B(G_1)$ and $B(G_2)$ respectively.
2) (F,A) is called Full-soft neutrosophic bigroup if $F(x)=B_N(G)$ for all $x\in A$.

Theorem 3 Let $B_N(G)$ be a neutrosophic bigroup of prime order P, then (F,A) over $B_N(G)$ is either identity soft neutrosophic bigroup or Full-soft neutrosophic bigroup.

Definition 34 Let (F,A) and (H,K) be two soft neutrosophic bigroups over $B_N(G)$. Then (H,K) is soft neutrosophi subbigroup of (F,A) written as $(H,K)\prec(F,A)$, if

1) $K\subset A$,
2) $K(x)\prec F(x)$ for all $x\in A$.

Example 10 Let

$B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ where

$$B(G_1)=\left\{\begin{array}{l}0,1,2,3,4,I,2I,3I,4I,1+I,2+I,3+I,4+I,\\ 1+2I,2+2I,3+2I,4+2I,1+3I,2+3I,\\ 3+3I,4+3I,1+4I,2+4I,3+4I,4+4I\end{array}\right\}$$

be a neutrosophic group under multiplication modulo 5 and $B(G_2)=\{g:g^{16}=1\}$ a cyclic group of order 16. Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup where

$P(G_1)=\{0,1,2,3,4,I,2I,3I,4I\}$

and

be another neutrosophic subbigroup where

$P(G_2)=\{g^2,g^4,g^6,g^8,g^{10},g^{12},g^{14},1\}$.

Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$

$Q(G_1)=\{0,1,4,I,4I\}$

and

$Q(G_2)=\{g^4,g^8,g^{12},1\}$.

Again let $R(G)=\{R(G_1)\cup R(G_2),*_1,*_2\}$ be a neutrosophic subbigroup where

$R(G_1)=\{0,1,I\}$ and $R(G_2)=\{1,g^8\}$.

Let (F,A) be a soft neutrsophic bigroup over $B_N(G)$ where

$F(x_1)=\{0,1,2,3,4,I,2I,3I,4I,g^2,g^4,g^6,g^8,g^{10},g^{12},g^{14},1\}$,

$F(x_2)=\{0,1,4,I,4I,g^4,g^8,g^{12},1\}$,

$F(x_3)=\{0,1,I,g^8,1\}$.

Let (H,K) be another soft neutrosophic bigroup over $B_N(G)$, where

$H(x_1)=\{0,1,2,3,4,g^4,g^8,g^{12},1\}$,

$H(x_2)=\{0,1,I,g^8,1\}$.

Clearly $(H,K)\prec(F,A)$.

Definition 35 Let $B_N(G)$ be a neutrosophic bigroup. Then (F,A) over $B_N(G)$ is called commutative soft neutrosophic bigroup if and only if $F(x)$ is a commutative subbigroup of $B_N(G)$ for all $x\in A$.

Example 11 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup where $B(G_1)=\{g:g^{10}=1\}$ be a cyclic group of order 10 and $B(G_2)=\{1,2,3,4,I,2\mathrm{I},3I,4I\}$ be a neutrosophic group under mltiplication modulo 5.
Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a commutative neutrosophic subbigroup where $P(G_1)=\{1,g^5\}$ and $P(G_2)=\{1,4,I,4I\}$. Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another commutative neutrosophic subbigroup where $Q(G_1)=\{1,g^2,g^4,g^6,g^8\}$ and $Q(G_2)=\{1,I\}$.
Then (F,A) is commutative soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1)=\{1,g^5,1,4,I,4I\},$$
$$F(x_2)=\{1,g^2,g^4,g^6,g^8,1,I\}.$$

Theorem 4 Every commutative *soft neutrosophic bigroup* (F,A) over $B_N(G)$ is a soft *neutrosophic bigroup* but the converse is not true.

Theorem 5 If $B_N(G)$ is commutative neutrosophic bigroup. Then (F,A) over $B_N(G)$ is commutative soft neutrosophic bigroup but the converse is not true.

Theorem 6 If $B_N(G)$ is cyclic neutrosophic bigroup. Then (F,A) over $B_N(G)$ is commutative soft neutrosophic bigroup.

Proposition 7 Let (F,A) and (K,D) be two commutative soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $B_N(G)$ is not commutative soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ over $B_N(G)$ is commutative soft neutrosophic bigroup over $B_N(G)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $B_N(G)$ is not commutative soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $B_N(G)$ is commutative soft neutrosophic bigroup over $B_N(G)$.

Proposition 8 Let (F,A) and (K,D) be two commutative soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is commutative soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not commutative soft neutrosophic bigroup over $B_N(G)$.

Definition 36 Let $B_N(G)$ be a neutrosophic bigroup. Then (F,A) over $B_N(G)$ is called cyclic soft neutrosophic bigroup if and only if $F(x)$ is a cyclic subbigroup of $B_N(G)$ for all $x\in A$.

Example 12 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup where $B(G_1)=\{g:g^{10}=1\}$ be a cyclic group of order 10 and $B(G_2)=\{0,1,2,I,2I,1+I,2+I,1+2I,2+2I\}$ be a neutrosophic group under multiplication modulo 3.
Le $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a cyclic neutrosophic subbigroup where $P(G_1)=\{1,g^5\}$ and $\{1,1+I\}$.
Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another cyclic neutrosophic subbigroup where $Q(G_1)=\{1,g^2,g^4,g^6,g^8\}$ and $Q(G_2)=\{1,2+2I\}$.
Then (F,A) is cyclic soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1)=\{1,g^5,1,1+I\},$$
$$F(x_2)=\{1,g^2,g^4,g^6,g^8,1,2+2I\}.$$

Theorem 7 If $B_N(G)$ is a cyclic neutrosophic soft bigroup, then (F,A) over $B_N(G)$ is also cyclic soft neutrosophic bigroup.

Theorem 8 Every cyclic soft neutrosophic bigroup (F,A) over $B_N(G)$ is a soft neutrosophic bigroup but the converse is not true.

Proposition 9 Let (F,A) and (K,D) be two cyclic soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $B_N(G)$ is not cyclic soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F,A)\bigcap_\varepsilon(K,D)$ over $B_N(G)$ is cyclic soft neutrosophic bigroup over $B_N(G)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $B_N(G)$ is not cyclic soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $B_N(G)$ is cyclic soft neutrosophic bigroup over $B_N(G)$.

Proposition 10 Let (F,A) and (K,D) be two cyclic soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is cyclic soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not cyclic soft neutrosophic bigroup over $B_N(G)$.

Definition 37 Let $B_N(G)$ be a neutrosophic bigroup. Then (F,A) over $B_N(G)$ is called normal soft neutrosophic bigroup if and only if $F(x)$ is normal subbigroup of $B_N(G)$ for all $x\in A$.

Example 13 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup, where

$$B(G_1)=\begin{Bmatrix} e,y,x,x^2,xy,x^2y,I, \\ Iy,Ix,Ix^2,Ixy,Ix^2y \end{Bmatrix}$$

is a neutrosophic group under multiplaction and $B(G_2)=\{g:g^6=1\}$ is a cyclic group of order 6.

Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a normal neutrosophic subbigroup where $P(G_1)=\{e,y\}$ and

$$P(G_2)=\{1,g^2,g^4\}.$$

Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another normal neutrosophic subbigroup where

$Q(G_1)=\{e,x,x^2\}$ and $Q(G_2)=\{1,g^3\}$.

Then (F,A) is a normal soft neutrosophic bigroup over $B_N(G)$ where

$$F(x_1)=\{e,y,1,g^2,g^4\},$$
$$F(x_2)=\{e,x,x^2,1,g^3\}.$$

Theorem 9 Every normal soft neutrosophic bigroup (F,A) over $B_N(G)$ is a soft neutrosophic bigroup but the converse is not true.

Theorem 10 If $B_N(G)$ is a normal neutrosophic bigroup. Then (F,A) over $B_N(G)$ is also normal soft neutrosophic bigroup.

Theorem 11 If $B_N(G)$ is a commutative neutrosophic bigroup. Then (F,A) over $B_N(G)$ is normal soft neutrosophic bigroup.

Theorem 12 If $B_N(G)$ is a cyclic neutrosophic bigroup. Then (F,A) over $B_N(G)$ is normal soft neutrosophic bigroup.

Proposition 11 Let (F,A) and (K,D) be two normal soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $B_N(G)$ is not normal soft neutrosophic bigroup over

$B_N(G)$.

2) Their extended intersection $(F,A)\cap_\varepsilon (K,D)$ over $B_N(G)$ is normal soft neutrosophic bigroup over $B_N(G)$.

3) Their restricted union $(F,A)\cup_R (K,D)$ over $B_N(G)$ is not normal soft neutrosophic bigroup over $B_N(G)$.

4) Their restricted intersection $(F,A)\cap_R (K,D)$ over $B_N(G)$ is normal soft neutrosophic bigroup over $B_N(G)$.

Proposition 12 Let (F,A) and (K,D) be two normal soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is normal soft neutrosophic bigroup over $B_N(G)$.

2) Their OR operation $(F,A)\vee(K,D)$ is not normal soft neutrosophic bigroup over $B_N(G)$.

Definition 38 Let (F,A) be a soft neutrosophic bigroup over $B_N(G)$. If for all $x\in A$ each $F(x)$ is a Lagrange subbigroup of $B_N(G)$, then (F,A) is called Lagrange soft neutosophic bigroup over $B_N(G)$.

Example 14 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup, where

$$B(G_1)=\begin{Bmatrix} e,y,x,x^2,xy,x^2y,I, \\ Iy,Ix,Ix^2,Ixy,Ix^2y \end{Bmatrix}$$

is a neutrosophic symmetric group of and $B(G_2)=\{0,1,I,1+I\}$ be a neutrosophic group under addition modulo 2. Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup where $P(G_1)=\{e,y\}$ and $P(G_2)=\{0,1\}$.

Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another neutrosophic subbigroup where $Q(G_1)=\{e,Iy\}$ and $Q(G_2)=\{0,1+I\}$.

Then (F,A) is Lagrange soft neutrosophic bigroup over $B_N(G)$, where

$F(x_2)=\{e,y,0,1\}$,

$F(x_2)=\{e,yI,0,1+I\}$.

Theorem 13 If $B_N(G)$ is a Lagrange neutrosophic bigroup, then (F,A) over $B_N(G)$ is Lagrange soft neutrosophic bigroup.

Theorem 14 Every Lagrange soft *neutrosophic bigroup* (F,A) over $B_N(G)$ is a soft *neutrosophic bigroup* but the converse is not true.

Proposition 13 Let (F,A) and (K,D) be two Lagrange soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_\varepsilon (K,D)$ over $B_N(G)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.

2) Their extended intersection $(F,A)\cap_\varepsilon (K,D)$ over $B_N(G)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.

3) Their restricted union $(F,A)\cup_R (K,D)$ over $B_N(G)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.

4) Their restricted intersection $(F,A)\cap_R (K,D)$ over $B_N(G)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.

Proposition 14 Let (F,A) and (K,D) be two Lagrange soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.

2) Their OR operation $(F,A)\vee(K,D)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.

Definition 39 Let (F,A) be a soft neutrosophic bigroup over $B_N(G)$. Then (F,A) is called weakly Lagrange soft neutosophic bigroup over $B_N(G)$ if atleast one $F(x)$ is a Lagrange subbigroup of $B_N(G)$, for some $x \in A$.

Example 15 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup, where

$$B(G_1)=\left\{\begin{array}{l}0,1,2,3,4,I,2I,3I,4I,1+I,2+I,3+I,4+I,\\ 1+2I,2+2I,3+2I,4+2I,1+3I,2+3I,\\ 3+3I,4+3I,1+4I,2+4I,3+4I,4+4I\end{array}\right\}$$

is a neutrosophic group under multiplication modulo 5 and $B(G_2)=\{g:g^{10}=1\}$ is a cyclic group of order 10. Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup where $P(G_1)=\{0,1,4,I,4I\}$ and $P(G_2)=\{g^2,g^4,g^6,g^8,1\}$. Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another neutrosophic subbigroup where $Q(G_1)=\{0,1,4,I,4I\}$ and $Q(G_2)=\{g^5,1\}$. Then (F,A) is a weakly Lagrange soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1)=\{0,1,4,I,4I,g^2,g^4,g^6,g^8,1\},$$
$$F(x_2)=\{0,1,4,I,4I,g^5,1\}.$$

Theorem 15 Every weakly Lagrange *soft neutrosophic bigroup* (F,A) over $B_N(G)$ is a soft *neutrosophic bigroup* but the converse is not true.

Proposition 15 Let (F,A) and (K,D) be two weakly Lagrange soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_{\varepsilon}(K,D)$ over $B_N(G)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,D)$ over $B_N(G)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $B_N(G)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $B_N(G)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.

Proposition 16 Let (F,A) and (K,D) be two weakly Lagrange soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.

Definition 40 Let (F,A) be a soft neutrosophic bigroup over $B_N(G)$. Then (F,A) is called Lagrange free soft neutrosophic bigroup if each $F(x)$ is not Lagrange subbigroup of $B_N(G)$, for all $x \in A$.

Example 16 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup, where $B(G_1)=\{0,1,I,1+I\}$ is a neutrosophic group under addition modulo 2 of order 4 and $B(G_2)=\{g:g^{12}=1\}$ is a cyclic group of order 12. Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup where $P(G_1)=\{0,I\}$ and $P(G_2)=\{g^4,g^8,1\}$. Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another neutrosophic subbigroup where $Q(G_1)=\{0,1+I\}$ and $Q(G_2)=\{1,g^3,g^6,g^9\}$. Then (F,A) is Lagrange free soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1)=\{0,I,1,g^4,g^8\},$$
$$F(x_2)=\{0,1+I,1,g^3,g^6,g^9\}.$$

Theorem 16 If $B_N(G)$ is Lagrange free neutrosophic bigroup, and then (F,A) over $B_N(G)$ is Lagrange free soft neutrosophic bigroup.

Theorem 17 Every Lagrange free *soft neutrosophic bigroup* (F,A) over $B_N(G)$ is a soft *neutrosophic bigroup* but the converse is not true.

Proposition 17 Let (F,A) and (K,D) be two Lagrange free soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_{\varepsilon}(K,D)$ over $B_N(G)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,D)$ over $B_N(G)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.
3) Their restricted union $(F,A)\cup_{R}(K,D)$ over $B_N(G)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_{R}(K,D)$ over $B_N(G)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.

Proposition 18 Let (F,A) and (K,D) be two Lagrange free soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.

Definition 41 Let $B_N(G)$ be a neutrosophic bigroup. Then (F,A) is called conjugate soft neutrosophic bigroup over $B_N(G)$ if and only if $F(x)$ is neutrosophic conjugate subbigroup of $B_N(G)$ for all $x\in A$.

Example 17 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a soft neutrosophic bigroup, where

$$B(G_1)=\{e,y,x,x^2,xy,x^2y\}$$

is Klien 4 -group and

$$B(G_2)=\begin{Bmatrix}0,1,2,3,4,5,I,2I,3I,4I,5I,\\1+I,2+I,3+I,...,5+5I\end{Bmatrix}$$

be a neutrosophic group under addition modulo 6.
Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup of $B_N(G)$, where $P(G_1)=\{e,y\}$ and $P(G_2)=\{0,3,3I,3+3I\}$. Again let $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another neutrosophic subbigroup of $B_N(G)$, where $Q(G_1)=\{e,x,x^2\}$ and $Q(G_2)=\{0,2,4,2+2I,4+4I,2I,4I\}$. Then (F,A) is conjugate soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1)=\{e,y,0,3,3I,3+3I\},$$
$$F(x_2)=\{e,x,x^2,0,2,4,2+2I,4+4I,2I,4I\}.$$

Theorem 18 If $B_N(G)$ is conjugate neutrosophic bigroup, then (F,A) over $B_N(G)$ is conjugate soft neutrosophic bigroup.

Theorem 19 Every conjugate soft neutrosophic bigroup (F,A) over $B_N(G)$ is a soft neutrosophic bigroup but the converse is not true.

Proposition 19 Let (F,A) and (K,D) be two conjugate soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_{\varepsilon}(K,D)$ over $B_N(G)$ is not conjugate soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,D)$ over $B_N(G)$ is conjugate soft neutrosophic bigroup over $B_N(G)$.

3) Their restricted union $(F,A)\cup_R(K,D)$ over $B_N(G)$ is not conjugate soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $B_N(G)$ is conjgate soft neutrosophic bigroup over $B_N(G)$.

Proposition 20 Let (F,A) and (K,D) be two conjugate soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is conjugate soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not conjugate soft neutrosophic bigroup over $B_N(G)$.

3.3 Soft Strong Neutrosophic Bigroup

Definition 42 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup. Then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is called soft strong neutrosophic bigroup if and only if $F(x)$ is a strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$ for all $x\in A$.

Example 18 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup, where $\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle$ with $\langle G_1\cup I\rangle=\langle Z\cup I\rangle$, the neutrosophic group under addition and $\langle G_2\cup I\rangle=\{0,1,2,3,4,I,2I,3I,4I\}$ a neutrosophic group under multiplication modulo 5. Let $H=H_1\cup H_2$ be a strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$, where $H_1=\{\langle 2Z\cup I\rangle,+\}$ is a neutrosophic subgroup and $H_2=\{0,1,4,I,4I\}$ is a neutrosophic subgroup. Again let $K=K_1\cup K_2$ be another strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$, where $K_1=\{\langle 3Z\cup I\rangle,+\}$ is a neutrosophic subgroup and $K_2=\{0,1,I,2I,3I,4I\}$ is a neutrosophic subgroup. Then clearly (F,A) is a soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$, where

$$F(x_1)=\{0,\pm2,\pm4,\ldots,1,4,I,4I\},$$
$$F(x_2)=\{0,\pm3,\pm6,\ldots,1,I,2I,3I,4I\}.$$

Theorem 20 Every soft strong neutrosophic bigroup (F,A) is a soft neutrosophic bigroup but the converse is not true.

Theorem 21 If $(\langle G\cup I\rangle,*_1,*_2)$ is a strong neutrosophic bigroup, then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is soft strong neutrosophic bigroup.

Proposition 21 Let (F,A) and (K,D) be two soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Proposition 22 Let (F,A) and (K,D) be two soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Definition 43 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup. Then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is called Lagrange soft strong neutrosophic bigroup if and only if $F(x)$ is Lagrange subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$ for all $x\in A$.

Example 19 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup of order 15, where $\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle$ with $\langle G_1\cup I\rangle=\{0,1,2,1+I,I,2I,2+I,2+2I,1+2I\}$, the neutrosophic group under mltiplication modulo 3 and $\langle G_2\cup I\rangle=\langle A_3\cup I\rangle=\{e,x,x^2,I,xI,x^2I\}$. Let $H=H_1\cup H_2$ be a strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$, where $H_1=\{1,2+2I\}$ is a neutrosophic subgroup and $H_2=\{e,x,x^2\}$ is a neutrosophic subgroup. Again let $K=K_1\cup K_2$ be another strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$, where $K_1=\{1,1+I\}$ is a neutrosophic subgroup and $K_2=\{I,xI,x^2I\}$ is a neutrosophic subgroup. Then clearly (F,A) is Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$, where

$$F(x_1)=\{1,2+2I,e,x,x^2\},$$
$$F(x_2)=\{1,1+I,I,xI,x^2I\}.$$

Theorem 22 Every Lagrange soft strong neutrosophic bigroup (F,A) is a soft neutrosophic bigroup but the converse is not true.

Theorem 23 Every Lagrange soft strong neutrosophic bigroup (F,A) is a soft strong neutrosophic bigroup but the converse is not true.

Theorem 24 If $(\langle G\cup I\rangle,*_1,*_2)$ is a Lagrange strong neutrosophic bigroup, then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is a Lagrange soft strong neutrosophic soft bigroup.

Proposition 23 Let (F,A) and (K,D) be two Lagrange soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Proposition 24 Let (F,A) and (K,D) be two Lagrange soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Definition 44 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup. Then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is called weakly Lagrange soft strong neutrosophic bigroup if atleast one $F(x)$ is a Lagrange subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$ for some $x\in A$.

Example 20 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup of order 15, where $\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle$ with $\langle G_1\cup I\rangle=\{0,1,2,1+I,I,2I,2+I,2+2I,1+2I\}$, the neutrosophic under mltiplication modulo 3 and

$\langle G_2 \cup I \rangle = \{e, x, x^2, I, xI, x^2I\}$. Let $H = H_1 \cup H_2$ be a strong neutrosophic subbigroup of $(\langle G \cup I \rangle, *_1, *_2)$, where $H_1 = \{1, 2, I, 2I\}$ is a neutrosophic subgroup and $H_2 = \{e, x, x^2\}$ is a neutrosophic subgroup. Again let $K = K_1 \cup K_2$ be another strong neutrosophic subbigroup of $(\langle G \cup I \rangle, *_1, *_2)$, where $K_1 = \{1, 1+I\}$ is a neutrosophic subgroup and $K_2 = \{e, I, xI, x^2I\}$ is a neutrosophic subgroup.

Then clearly (F, A) is weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$, where

$$F(x_1) = \{1, 2, I, 2I, e, x, x^2\},$$

$$F(x_2) = \{1, 1+I, e, I, xI, x^2I\}.$$

Theorem 25 Every weakly Lagrange soft strong neutrosophic bigroup (F, A) is a soft neutrosophic bigroup but the converse is not true.

Theorem 26 Every weakly Lagrange soft strong neutrosophic bigroup (F, A) is a soft strong neutrosophic bigroup but the converse is not true.

Proposition 25 Let (F, A) and (K, D) be two weakly Lagrange soft strong neutrosophic bigroups over $(\langle G \cup I \rangle, *_1, *_2)$. Then

1) Their extended union $(F, A) \cup_\varepsilon (K, D)$ over $(\langle G \cup I \rangle, *_1, *_2)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$.
.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, D)$ over $(\langle G \cup I \rangle, *_1, *_2)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$.
3) Their restricted union $(F, A) \cup_R (K, D)$ over $(\langle G \cup I \rangle, *_1, *_2)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$. .
4) Their restricted intersection $(F, A) \cap_R (K, D)$ over $(\langle G \cup I \rangle, *_1, *_2)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$.

Proposition 26 Let (F, A) and (K, D) be two weakly Lagrange soft strong neutrosophic bigroups over $(\langle G \cup I \rangle, *_1, *_2)$. Then

1) Their AND operation $(F, A) \wedge (K, D)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$. .
2) Their OR operation $(F, A) \vee (K, D)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$.

Definition 45 Let $(\langle G \cup I \rangle, *_1, *_2)$ be a strong neutrosophic bigroup. Then (F, A) over $(\langle G \cup I \rangle, *_1, *_2)$ is called Lagrange free soft strong neutrosophic bigroup if and only if $F(x)$ is not Lagrange subbigroup of $(\langle G \cup I \rangle, *_1, *_2)$ for all $x \in A$.

Example 21 Let $(\langle G \cup I \rangle, *_1, *_2)$ be a strong neutrosophic bigroup of order 15, where $\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle$ with $\langle G_1 \cup I \rangle = \{0, 1, 2, 3, 4, I, 2I, 3I, 4I\}$, the neutrosophic under mltiplication modulo 5 and $\langle G_2 \cup I \rangle = \{e, x, x^2, I, xI, x^2I\}$, a neutrosophic symmetric group. Let $H = H_1 \cup H_2$ be a strong neutrosophic subbigroup of $(\langle G \cup I \rangle, *_1, *_2)$, where $H_1 = \{1, 4, I, 4I\}$ is a neutrosophic subgroup and $H_2 = \{e, x, x^2\}$ is a neutrosophic subgroup. Again let $K = K_1 \cup K_2$ be another strong neutrosophic subbigroup of $(\langle G \cup I \rangle, *_1, *_2)$, where $K_1 = \{1, I, 2I, 3I, 4I\}$ is a neutrosophic subgroup and $K_2 = \{e, x, x^2\}$ is a neutrosophic subgroup.

Then clearly (F, A) is Lagrange free soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$, where

$$F(x_1)=\{1,4,I,4I,e,x,x^2\},$$
$$F(x_2)=\{1,I,2I,3I,4I,e,x,x^2\}.$$

Theorem 27 Every Lagrange free soft strong neutrosophic bigroup (F,A) is a soft neutrosophic bigroup but the converse is not true.

Theorem 28 Every Lagrange free soft strong neutrosophic bigroup (F,A) is a soft strong neutrosophic bigroup but the converse is not true.

Theorem 29 If $(\langle G\cup I\rangle,*_1,*_2)$ is a Lagrange free strong neutrosophic bigroup, then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is also Lagrange free soft strong neutrosophic bigroup.

Proposition 27 Let (F,A) and (K,D) be weakly Lagrange free soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Proposition 28 Let (F,A) and (K,D) be two Lagrange free soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Definition 46 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup. Then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is called soft normal strong neutrosophic bigroup if and only if $F(x)$ is normal strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$ for all $x\in A$.

Theorem 30 Every soft normal *strong neutrosophic bigroup* (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is a soft *neutrosophic bigroup* but the converse is not true.

Theorem 31 Every soft normal *strong neutrosophic bigroup* (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is a soft strong *neutrosophic bigroup* but the converse is not true.

Proposition 29 Let (F,A) and (K,D) be two soft normal strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not soft normal strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is soft normal strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not soft normal strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is soft normal strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Proposition 30 Let (F,A) and (K,D) be two soft normal strong neutrosophic bigroups over $\left(\langle G\cup I\rangle,*_1,*_2\right)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft normal strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft normal strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.

Definition 47 Let $\left(\langle G\cup I\rangle,*_1,*_2\right)$ be a strong neutrosophic bigroup. Then (F,A) over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is called soft conjugate strong neutrosophic bigroup if and only if $F(x)$ is conjugate neutrosophic subbigroup of $\left(\langle G\cup I\rangle,*_1,*_2\right)$ for all $x\in A$.

Theorem 32 Every soft conjugate *strong neutrosophic bigroup* (F,A) over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is a soft *neutrosophic bigroup* but the converse is not true.

Theorem 33 Every soft conjugate *strong neutrosophic bigroup* (F,A) over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is a soft strong neutrosophic bigroup *but the converse is not true.*

Proposition 31 Let (F,A) and (K,D) be two soft conjugate strong neutrosophic bigroups over $\left(\langle G\cup I\rangle,*_1,*_2\right)$. Then

1) Their extended union $(F,A)\cup_{\varepsilon}(K,D)$ over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is not soft conjugate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,D)$ over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is soft conjugate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is not soft conjugate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is soft conjugate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.

Proposition 32 Let (F,A) and (K,D) be two soft conjugate strong neutrosophic bigroups over $\left(\langle G\cup I\rangle,*_1,*_2\right)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft conjugate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft conjgate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.

4.1 Soft Neutrosophic N-Group

Definition 48 Let $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$ be a neutrosophic N-group. Then (F,A) over $\left(\langle G\cup I\rangle,*_1,\ldots,*_2\right)$ is called soft neutrosophic N-group if and only if $F(x)$ is a sub N-group of $\left(\langle G\cup I\rangle,*_1,\ldots,*_2\right)$ for all $x\in A$.

Example 22 Let $\left(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle\cup\langle G_3\cup I\rangle,*_1,*_2,*_3\right)$ be a neutrosophic 3-group, where $\langle G_1\cup I\rangle=\langle Q\cup I\rangle$ a neutrosophic group under multiplication. $\langle G_2\cup I\rangle=\{0,1,2,3,4,I,2I,3I,4I\}$ neutrosophic group under multiplication modulo 5 and $\langle G_3\cup I\rangle=\{0,1,2,1+I,2+I,I,2I,1+2I,2+2I\}$ a neutrosophic group under multiplication modulo 3. Let

$$P=\left\{\left\{\left\langle\frac{1}{2^n},2^n,\frac{1}{(2I)^n},(2I)^n,I,1\right\rangle\right\},(1,4,I,4I),(1,2,I,2I)\right\},$$

$T=\{Q\setminus\{0\},\{1,2,3,4\},\{1,2\}\}$ and

$X=\{Q\setminus\{0\},\{1,2,I,2I\},\{1,4,I,4I\}\}$ are sub 3-groups.

Then (F,A) is clearly soft neutrosophic 3-group over $\left(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle\cup\langle G_3\cup I\rangle,*_1,*_2,*_3\right)$, where

$$F(x_1)=\left\{\left\{\left\langle \frac{1}{2^n},2^n,\frac{1}{(2I)^n},(2I)^n,I,1\right\rangle\right\},(1,4,I,4I),(1,2,I,2I)\right\},$$

$$F(x_2)=\{Q\setminus\{0\},\{1,2,3,4\},\{1,2\}\},$$

$$F(x_3)=\{Q\setminus\{0\},\{1,2,I,2I\},\{1,4,I,4I\}\}.$$

Theorem 34 Let (F,A) and (H,A) be two soft neutrosophic N-groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then their intersection $(F,A)\cap(H,A)$ is again a soft neutrosophic N-group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Proof The proof is straight forward.

Theorem 35 Let (F,A) and (H,B) be two soft neutrosophic N-groups over $(\langle G\cup I\rangle,*_1,...,*_N)$ such that $A\cap B-=\phi,$ then their union is soft neutrosophic N-group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Proof The proof can be established easily.

Proposition 33 Let (F,A) and (K,D) be two soft neutrosophic N-groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ is not soft neutrosophic N-group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ is soft neutrosophic N-group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ is not soft neutrosophic N-group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ is soft neutrosophic N-group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Proposition 34 Let (F,A) and (K,D) be two soft neutrosophic N-groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft neutrosophic N-group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft neutrosophic N-group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Definition 49 Let (F,A) be a soft neutrosophic N-group over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) (F,A) is called identity soft neutrosophic N-group if $F(x)=\{e_1,...,e_N\}$ for all $x\in A,$ where $e_1,...,e_N$ are the identities of $\langle G_1\cup I\rangle,...,\langle G_N\cup I\rangle$ respectively.
2) (F,A) is called Full soft neutrosophic N-group if $F(x)=(\langle G\cup I\rangle,*_1,...,*_N)$ for all $x\in A.$

Definition 50 Let (F,A) and (K,D) be two soft neutrosophic N-groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then (K,D) is soft neutrosophic sub N-group of (F,A) written as $(K,D)\prec(F,A)$, if

1) $D\subset A,$
2) $K(x)\prec F(x)$ for all $x\in A.$

Example 23 Let (F,A) be as in example 22. Let (K,D) be another soft neutrosophic soft N-group over $(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle\cup\langle G_3\cup I\rangle,*_1,*_2,*_3)$, where

$$K(x_1)=\left\{\left\{\left\langle\frac{1}{2^n},2^n\right\rangle\right\},\{1,4,I,4I\},\{1,2,I,2I\}\right\},$$

$$K(x_2)=\{Q\setminus\{0\},\{1,4\},\{1,2\}\}.$$

Clearly $(K,D)\prec(F,A)$.

Thus a soft neutrosophic N-group can have two types of soft neutrosophic sub N-groups, which are following

Definition 51 A soft neutrosophic sub N-group (K,D) of a soft neutrosophic N-group (F,A) is called soft strong neutrosophic sub N-group if

1) $D\subset A,$
2) $K(x)$ is neutrosophic sub N-group of $F(x)$ for

all $x \in A$.

Definition 52 A soft neutrosophic sub N -group (K,D) of a soft neutrosophic N -group (F,A) is called soft sub N -group if

1) $D \subset A$,
2) $K(x)$ is only sub N -group of $F(x)$ for all $x \in A$.

Definition 53 Let $(\langle G \cup I \rangle, *_1, ..., *_N)$ be a neutrosophic N -group. Then (F,A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is called soft Lagrange neutrosophic N -group if and only if $F(x)$ is Lagrange sub N -group of $(\langle G \cup I \rangle, *_1, ..., *_N)$ for all $x \in A$.

Example 24 Let $(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup G_2 \cup G_3, *_1, *_2, *_3)$ be neutrosophic N -group, where $\langle G_1 \cup I \rangle = \{\langle Z_6 \cup I \rangle\}$ is a group under addition modulo 6 , $G_2 = A_4$ and $G_3 = \langle g : g^{12} = 1 \rangle$, a cyclic group of order 12, $o(\langle G \cup I \rangle) = 60$.

Take $P = (\langle P_1 \cup I \rangle \cup P_2 \cup P_3, *_1, *_2, *_3)$, a neutrosophic sub 3-group where

$\langle T_1 \cup I \rangle = \{0, 3, 3I, 3+3I\}$,

$$P_2 = \left\{ \begin{pmatrix} 1234 \\ 1234 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2143 \end{pmatrix}, \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}, \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} \right\},$$

$P_3 = \{1, g^6\}$. Since P is a Lagrange neutrosophic sub 3-group where order of $P = 10$.

Let us Take $T = (\langle T_1 \cup I \rangle \cup T_2 \cup T_3, *_1, *_2, *_3)$, where $\langle T_1 \cup I \rangle = \{0, 3, 3I, 3+3I\}, T_2 = P_2$ and $T_3 = \{g^3, g^6, g^9, 1\}$ is another Lagrange sub 3-group where $o(T) = 12$.

Let (F,A) is soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup G_2 \cup G_3, *_1, *_2, *_3)$, where

$$F(x_1) = \left\{ 0, 3, 3I, 3+3I, 1, g^6, \begin{pmatrix} 1234 \\ 1234 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2143 \end{pmatrix}, \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}, \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} \right\},$$

$$F(x_2) = \left\{ 0, 3, 3I, 3+3I, 1, g^3, g^6, g^9, \begin{pmatrix} 1234 \\ 1234 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2143 \end{pmatrix}, \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}, \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} \right\}.$$

Theorem 36 Every soft Lagrange neutrosophic N -group (F,A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is a soft neutrosophic N -group but the converse is not true.

Theorem 37 If $(\langle G \cup I \rangle, *_1, ..., *_N)$ is a Lagrange neutrosophic N -group, then (F,A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is also soft Lagrange neutrosophic N -group.

Proposition 35 Let (F,A) and (K,D) be two soft Lagrange neutrosophic N -groups over $(\langle G \cup I \rangle, *_1, ..., *_N)$. Then

1) Their extended union $(F,A) \cup_{\varepsilon} (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
2) Their extended intersection $(F,A) \cap_{\varepsilon} (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
3) Their restricted union $(F,A) \cup_R (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
4) Their restricted intersection $(F,A) \cap_R (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

Proposition 36 Let (F,A) and (K,D) be two soft Lagrange neutrosophic N -groups over $(\langle G \cup I \rangle, *_1, ..., *_N)$. Then

1) Their AND operation $(F,A) \wedge (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
2) Their OR operation $(F,A) \vee (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

Definition 54 Let $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ be a neutrosophic N -group. Then (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is called soft weakly Lagrange neutrosophic N -group if atleast one $F(x)$ is Lagrange sub N -group of $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ for some $x\in A$.

Examp 25 Let $\left(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup G_2\cup G_3,*_1,*_2,*_3\right)$ be neutrosophic N -group, where $\langle G_1\cup I\rangle=\left\{\langle Z_6\cup I\rangle\right\}$ is a group under addition modulo 6 , $G_2=A_4$ and $G_3=\langle g:g^{12}=1\rangle$, a cyclic group of order 12, $o\left(\langle G\cup I\rangle\right)=60.$

Take $P=\left(\langle P_1\cup I\rangle\cup P_2\cup P_3,*_1,*_2,*_3\right)$, a neutrosophic sub 3-group where

$$\langle T_1\cup I\rangle=\{0,3,3I,3+3I\},$$

$$P_2=\left\{\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\},$$

$P_3=\{1,g^6\}$. Since P is a Lagrange neutrosophic sub 3-group where order of $P=10$.

Let us Take $T=\left(\langle T_1\cup I\rangle\cup T_2\cup T_3,*_1,*_2,*_3\right)$, where $\langle T_1\cup I\rangle=\{0,3,3I,3+3I\},T_2=P_2$ and $T_3=\{g^4,g^8,1\}$ is another Lagrange sub 3-group.

Then (F,A) is soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup G_2\cup G_3,*_1,*_2,*_3\right)$, where

$$F(x_1)=\left\{0,3,3I,3+3I,1,g^6,\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\},$$

$$F(x_2)=\left\{0,3,3I,3+3I,1,g^4,g^8,\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\}.$$

Theorem 38 Every soft weakly Lagrange neutrosophic N -group (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is a soft neutrosophic N -group but the converse is not tue.

Theorem 39 If $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is a weakly Lagrange neutrosophi N -group, then (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is also soft weakly Lagrange neutrosophic N -group.

Proposition 37 Let (F,A) and (K,D) be two soft weakly Lagrange neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1. Their extended union $(F,A)\cup_\varepsilon(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2. Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
3. Their restricted union $(F,A)\cup_R(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
4. Their restricted intersection $(F,A)\cap_R(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Proposition 38 Let (F,A) and (K,D) be two soft weakly Lagrange neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Definition 55 Let $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ be a neutrosophic N -group. Then (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is called soft Lagrange free neutrosophic N -group if $F(x)$ is not Lagrange sub N -group of $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$ for all $x\in A$.

Example 26 Let $\left(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup G_2\cup G_3,*_1,*_2,*_3\right)$ be neutrosophic 3-group, where $\langle G_1\cup I\rangle=\left\{\langle Z_6\cup I\rangle\right\}$ is a group under addition modulo 6 , $G_2=A_4$ and $G_3=\left\langle g:g^{12}=1\right\rangle$, a cyclic group of order 12, $o\left(\langle G\cup I\rangle\right)=60.$

Take $P=\left(\langle P_1\cup I\rangle\cup P_2\cup P_3,*_1,*_2,*_3\right)$, a neutrosophic sub 3-group where

$P_1=\{0,2,4\}$,

$$P_2=\left\{\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\},$$

$P_3=\left\{1,g^6\right\}$. Since P is a Lagrange neutrosophic sub 3-group where order of $P=10.$

Let us Take $T=\left(\langle T_1\cup I\rangle\cup T_2\cup T_3,*_1,*_2,*_3\right)$, where $\langle T_1\cup I\rangle=\{0,3,3I,3+3I\},T_2=P_2$ and $T_3=\left\{g^4,g^8,1\right\}$ is another Lagrange sub 3-group.

Then (F,A) is soft Lagrange free neutrosophic 3-group over $\left(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup G_2\cup G_3,*_1,*_2,*_3\right)$, where

$$F(x_1)=\left\{0,2,4,1,g^6,\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\},$$

$$F(x_2)=\left\{0,3,3I,3+3I,1,g^4,g^8,\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\}$$

Theorem 40 Every soft Lagrange free neutrosophic N -group (F,A) over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$ is a soft neutrosophic N -group but the converse is not true.

Theorem 41 If $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$ is a Lagrange free neutrosophic N -group, then (F,A) over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$ is also soft Lagrange free neutrosophic N -group.

Proposition 39 Let (F,A) and (K,D) be two soft Lagrange free neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ is not soft Lagrange free neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ is not soft Lagrange free neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ is not soft Lagrange free neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ is not soft Lagrange free neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$.

Proposition 40 Let (F,A) and (K,D) be two soft Lagrange free neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not soft Lagrange free neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft Lagrange free neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$.

Definition 56 Let $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$ be a neutrosophic N -group. Then (F,A) over $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$ is called soft normal neutrosophic N -group if $F(x)$ is normal sub N -group of $\left(\langle G\cup I\rangle,*_1,\ldots,*_N\right)$ for all $x\in A$.

Example 27 Let $\left(\langle G_1\cup I\rangle=\langle G_1\cup I\rangle\cup G_2\cup\langle G_3\cup I\rangle,*_1,*_2,*_3\right)$ be a soft neutrosophic N -group, where $\langle G_1\cup I\rangle=\left\{e,y,x,x^2,xy,x^2y,I,yI,xI,x^2I,xyI,x^2yI\right\}$

is a neutrosophic group under multiplaction, $G_2 = \{g : g^6 = 1\}$, a cyclic group of order 6 and $\langle G_3 \cup I \rangle = \langle Q_8 \cup I \rangle = \{\pm 1, \pm i, \pm j, \pm k, \pm I, \pm iI, \pm jI, \pm kI\}$ is a group under multiplication. Let $P = (\langle P_1 \cup I \rangle \cup P_2 \cup \langle P_3 \cup I \rangle, *_1, *_2, *_3)$, a normal sub 3-group where $P_1 = \{e, y, I, yI\}$, $P_2 = \{1, g^2, g^4\}$ and $P_3 = \{1, -1\}$. Also $T = (\langle T_1 \cup I \rangle \cup T_2 \cup \langle T_3 \cup I \rangle, *_1, *_2, *_3)$ be another normal sub 3-group where $\langle T_1 \cup I \rangle = \{e, I, xI, x^2 I\}, T_2 = \{1, g^3\}$ and $\langle T_3 \cup I \rangle = \{\pm 1, \pm i\}$. Then (F, A) is a soft normal neutrosophic N-group over $(\langle G_1 \cup I \rangle = \langle G_1 \cup I \rangle \cup G_2 \cup \langle G_3 \cup I \rangle, *_1, *_2, *_3)$, where

$$F(x_1) = \{e, y, I, yI, 1, g^2, g^4, \pm 1\},$$
$$F(x_2) = \{e, I, xI, x^2 I, 1, g^3, \pm 1, \pm i\}.$$

Theorem 42 Every soft normal neutrosophic N-group (F, A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is a soft neutrosophic N-group but the converse is not true.

Proposition 41 Let (F, A) and (K, D) be two soft normal neutrosophic N-groups over $(\langle G \cup I \rangle, *_1, ..., *_N)$. Then

1) Their extended union $(F, A) \cup_\varepsilon (K, D)$ is not soft normal neutrosophic soft N-group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, D)$ is soft normal neutrosophic N-group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
3) Their restricted union $(F, A) \cup_R (K, D)$ is not soft normal neutrosophic N-group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
4) Their restricted intersection $(F, A) \cap_R (K, D)$ is soft normal neutrosophic N-group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

Proposition 42 Let (F, A) and (K, D) be two soft normal neutrosophic N-groups over $(\langle G \cup I \rangle, *_1, ..., *_N)$. Then

1) Their AND operation $(F, A) \wedge (K, D)$ is soft normal neutrosophic N-group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
2) Their OR operation $(F, A) \vee (K, D)$ is not soft normal neutrosophic N-group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

Definition 56 Let $(\langle G \cup I \rangle, *_1, ..., *_N)$ be a neutrosophic N-group. Then (F, A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is called soft conjugate neutrosophic N-group if $F(x)$ is conjugate sub N-group of $(\langle G \cup I \rangle, *_1, ..., *_N)$ for all $x \in A$.

Theorem 43 Every soft conjugate neutrosophic N-group (F, A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is a soft neutrosophic N-group but the converse is not true.

Proposition 43 Let (F, A) and (K, D) be two soft conjugate neutrosophic N-groups over $(\langle G \cup I \rangle, *_1, ..., *_N)$. Then

1) Their extended union $(F, A) \cup_\varepsilon (K, D)$ is not soft conjugate neutrosophic N-group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, D)$ is soft conjugate neutrosophic N-group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
3) Their restricted union $(F, A) \cup_R (K, D)$ is not soft conjugate neutrosophic N-group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
4) Their restricted intersection $(F, A) \cap_R (K, D)$ is soft conjugate neutrosophic N-group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

Proposition 44 Let (F, A) and (K, D) be two soft conjugate neutrosophic N-groups over

$(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft conjugate neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft conjugate neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

4.2 Soft Strong Neutrosophic N-Group

Definition 57 Let $(\langle G\cup I\rangle,*_1,...,*_N)$ be a neutrosophic N -group. Then (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is called soft strong neutrosophic N -group if and only if $F(x)$ is a strong neutrosophic sub N -group for all $x\in A$.

Example 28 Let $(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle\cup\langle G_3\cup I\rangle,*_1,*_2,*_3)$ be a neutrosophic 3-group, where $\langle G_1\cup I\rangle=\langle Z_2\cup I\rangle=\{0,1,I,1+I\}$, a neutrosophic group under multiplication modulo 2. $\langle G_2\cup I\rangle=\{O,1,2,3,4,I,2I,3I,4I\}$, neutrosophic group under multiplication modulo 5 and $\langle G_3\cup I\rangle=\{0,1,2,I,2I\}$,a neutrosophic group under multiplication modulo 3. Let

$$P=\left\{\left\{\left\langle\frac{1}{2^n},2^n,\frac{1}{(2I)^n},(2I)^n,I,1\right\rangle\right\},\{1,4,I,4I\},\{1,2,I,2I\}\right\},$$

and $X=\{Q\setminus\{0\},\{1,2,I,2I\},\{1,I\}\}$ are neutrosophic sub 3-groups.

Then (F,A) is clearly soft strong neutrosophic 3-group over

$$(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle\cup\langle G_3\cup I\rangle,*_1,*_2,*_3),$$

where

$$F(x_1)=\left\{\left\{\left\langle\frac{1}{2^n},2^n,\frac{1}{(2I)^n},(2I)^n,I,1\right\rangle\right\},\{1,4,I,4I\},\{1,I\}\right\},$$

$$F(x_2)=\{Q\setminus\{0\},\{1,2,I,2I\},\{1,I\}\}.$$

Theorem 44 Every soft strong neutrosophic soft N -group (F,A) is a soft neutrosophic N -group but the converse is not true.

Theorem 89 (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is soft strong neutrosophic N -group if $(\langle G\cup I\rangle,*_1,...,*_N)$ is a strong neutrosophic N -group.

Proposition 45 Let (F,A) and (K,D) be two soft strong neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ is not soft strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ is not soft strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ is not soft strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ is not soft strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Proposition 46 Let (F,A) and (K,D) be two soft strong neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not soft strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Definition 58

Let (F,A) and (H,K) be two soft strong neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then (H,K) is called soft strong neutrosophic sub $x\in A$ -group of (F,A) written as $(H,K)\prec(F,A)$, if

1) $K \subset A$,
2) $K(x)$ is soft neutrosophic soft sub N -group of $F(x)$ for all $x \in A$.

Theorem 45 If $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ is a strong neutrosophic N -group. Then every soft neutrosophic sub N -group of (F, A) is soft strong neutosophic sub N -group.

Definition 59 Let $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ be a strong neutrosophic N -group. Then (F, A) over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ is called soft Lagrange strong neutrosophic N -group if $F(x)$ is a Lagrange neutrosophic sub N -group of $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ for all $x \in A$.

Theorem 46 Every soft Lagrange strong neutrosophic N -group (F, A) over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ is a soft neutrosophic soft N -group but the converse is not true.

Theorem 47 Every soft Lagrange strong neutrosophic N -group (F, A) over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ is a soft srtong neutrosophic N -group but the converse is not tue.

Theorem 48 If $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ is a Lagrange strong neutrosophic N -group, then (F, A) over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ is also soft Lagrange strong neutrosophic N -group.

Proposition 47 Let (F, A) and (K, D) be two soft Lagrange strong neutrosophic N -groups over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$. Then

1) Their extended union $(F, A) \cup_{\varepsilon} (K, D)$ is not soft Lagrange strong neutrosophic N -group over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$.
2) Their extended intersection $(F, A) \cap_{\varepsilon} (K, D)$ is not soft Lagrange strong neutrosophic N -group over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$.
3) Their restricted union $(F, A) \cup_R (K, D)$ is not soft Lagrange strong neutrosophic N -group over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$.
4) Their restricted intersection $(F, A) \cap_R (K, D)$ is not soft Lagrange strong neutrosophic N -group over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$.

Proposition 48 Let (F, A) and (K, D) be two soft Lagrange strong neutrosophic N -groups over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$. Then

1) Their AND operation $(F, A) \wedge (K, D)$ is not soft Lagrange strong neutrosophic N -group over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$.
2) Their OR operation $(F, A) \vee (K, D)$ is not soft Lagrange strong neutrosophic N -group over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$.

Definition 60 Let $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ be a strong neutrosophic N -group. Then (F, A) over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ is called soft weakly Lagrange strong neutrosophic soft N -group if atleast one $F(x)$ is a Lagrange neutrosophic sub N -group of $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ for some $x \in A$.

Theorem 49 Every soft weakly Lagrange strong neutrosophic N -group (F, A) over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ is a soft neutrosophic soft N -group but the converse is not true.

Theorem 50 Every soft weakly Lagrange strong neutrosophic N -group (F, A) over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$ is a soft strong neutrosophic N -group but the converse is not true.

Proposition 49 Let (F, A) and (K, D) be two soft weakly Lagrange strong neutrosophic N -groups over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$. Then

1) Their extended union $(F, A) \cup_{\varepsilon} (K, D)$ is not soft weakly Lagrange strong neutrosophic N -group over $\left(\langle G \cup I \rangle, *_1, ..., *_N\right)$.
2) Their extended intersection $(F, A) \cap_{\varepsilon} (K, D)$ is not soft weakly Lagrange strong neutrosophic N -group over

$\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.

3) Their restricted union $(F, A) \cup_R (K, D)$ is not soft weakly Lagrange strong neutrosophic N -group over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.
4) Their restricted intersection $(F, A) \cap_R (K, D)$ is not soft weakly Lagrange strong neutrosophic N -group over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.

Proposition 50 Let (F, A) and (K, D) be two soft weakly Lagrange strong neutrosophic N -groups over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$. Then

1) Their AND operation $(F, A) \wedge (K, D)$ is not soft weakly Lagrange strong neutrosophic N -group over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.
2) Their OR operation $(F, A) \vee (K, D)$ is not soft weakly Lagrange strong neutrosophic N -group over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.

Definition 61 Let $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ be a strong neutrosophic N -group. Then (F, A) over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ is called soft Lagrange free strong neutrosophic N -group if $F(x)$ is not Lagrange neutrosophic sub N -group of $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ for all N .

Theorem 51 Every soft Lagrange free strong neutrosophic N -group (F, A) over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ is a soft neutrosophic N -group but the converse is not true.

Theorem 52 Every soft Lagrange free strong neutrosophic N -group (F, A) over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ is a soft strong neutrosophic N -group but the converse is not true.

Theorem 53 If $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ is a Lagrange free strong neutrosophic N -group, then (F, A) over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ is also soft Lagrange free strong neutrosophic N -group.

Proposition 51 Let (F, A) and (K, D) be two soft Lagrange free strong neutrosophic N -groups over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$. Then

1) Their extended union $(F, A) \cup_\varepsilon (K, D)$ is not soft Lagrange free strong neutrosophic N -group over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, D)$ is not soft Lagrange free strong neutrosophic N -group over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.
3) Their restricted union $(F, A) \cup_R (K, D)$ is not soft Lagrange free strong neutrosophic N -group over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.
4) Their restricted intersection $(F, A) \cap_R (K, D)$ is not soft Lagrange free strong neutrosophic N -group over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.

Proposition 52 Let (F, A) and (K, D) be two soft Lagrange free strong neutrosophic N -groups over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$. Then

1) Their AND operation $(F, A) \wedge (K, D)$ is not soft Lagrange free strong neutrosophic N -group over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.
2) Their OR operation $(F, A) \vee (K, D)$ is not soft Lagrange free strong neutrosophic N -group over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$.

Definition 62 Let N be a strong neutrosophic N -group. Then (F, A) over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ is called sofyt normal strong neutrosophic N -group if $F(x)$ is normal neutrosophic sub N -group of $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ for all $x \in A$.

Theorem 54 Every soft normal strong neutrosophic N -group (F, A) over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ is a soft neutrosophic N -group but the converse is not true.

Theorem 55 Every soft normal strong neutrosophic N -group (F, A) over $\left(\langle G \cup I\rangle, *_1, ..., *_N\right)$ is a soft strong neutrosophic N -group but the converse is not true.

Proposition 53 Let (F, A) and (K, D) be two soft normal strong neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1) Their extended union $\left(F,A\right)\cup_{\varepsilon}\left(K,D\right)$ is not soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2) Their extended intersection $\left(F,A\right)\cap_{\varepsilon}\left(K,D\right)$ is soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
3) Their restricted union $\left(F,A\right)\cup_{R}\left(K,D\right)$ is not soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
4) Their restricted intersection $\left(F,A\right)\cap_{R}\left(K,D\right)$ is soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Proposition 54 Let $\left(F,A\right)$ and $\left(K,D\right)$ be two soft normal strong neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1) Their AND operation $\left(F,A\right)\wedge\left(K,D\right)$ is soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2) Their OR operation $\left(F,A\right)\vee\left(K,D\right)$ is not soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Definition 63 Let $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ be a strong neutrosophic N -group. Then $\left(F,A\right)$ over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is called soft conjugate strong neutrosophic N -group if $F\left(x\right)$ is conjugate neutrosophic sub N -group of $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ for all $x\in A$.

Theorem 56 Every soft conjugate strong neutrosophic N -group $\left(F,A\right)$ over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is a soft neutrosophic N -group but the converse is not true.

Theorem 57 Every soft conjugate strong neutrosophic N -group $\left(F,A\right)$ over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is a soft strong neutrosophic N -group but the converse is not true.

Proposition 55 Let $\left(F,A\right)$ and $\left(K,D\right)$ be two soft conjugate strong neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1) Their extended union $\left(F,A\right)\cup_{\varepsilon}\left(K,D\right)$ is not soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2) Their extended intersection $\left(F,A\right)\cap_{\varepsilon}\left(K,D\right)$ is soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
3) Their restricted union $\left(F,A\right)\cup_{R}\left(K,D\right)$ is not soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
4) Their restricted intersection $\left(F,A\right)\cap_{R}\left(K,D\right)$ is soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Proposition 56 Let $\left(F,A\right)$ and $\left(K,D\right)$ be two soft conjugate strong neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1) Their AND operation $\left(F,A\right)\wedge\left(K,D\right)$ is soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2) Their OR operation $\left(F,A\right)\vee\left(K,D\right)$ is not soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Conclusion

This paper is about the generalization of soft neutrosophic groups. We have extended the concept of soft neutrosophic group and soft neutrosophic subgroup to soft neutrosophic bigroup and soft neutrosophic N-group. The notions of soft normal neutrosophic bigroup, soft normal neutrosophic N-group, soft conjugate neutrosophic bigroup and soft conjugate neutrosophic N-group are defined. We have given various examples and important theorems to illustrate the aspect of soft neutrosophic bigroup and soft neutrosophic N-group.

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Neutrosophic Game Theoretic Approach to Indo-Pak Conflict over Jammu-Kashmir

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Abstract**.** The study deals with the enduring conflict between India and Pakistan over Jammu and Kashmir since 1947. The ongoing conflict is analyzed as an enduring rivalry; characterized by three major wars (1947-48), 1965, 1971, low intensity military conflict (Siachen), mini war at Kargil (1999), internal insurgency, cross border terrorism. We examine the progress and the status of the dispute, as well as the dynamics of the India Pakistan relationship by considering the influence of USA and China in crisis dynamics. We discuss the possible solutions offered by the various study groups and persons. Most of the studies were done in crisp environment. Pramanik and Roy (S. Pramanik and T.K. Roy, Game theoretic model to the Jammu-Kashmir conflict between India and Pakistan. International Journal of Mathematical Archive (IJMA), 4(8) (2013), 162-170.) studied game theoretic model toJammu and Kashmir conflict in crisp environment. In the present study we have extended the concept of the game theoric model of the Jammu and Kashmir conflict in neutrosophic envirorment. We have explored the possibilities and developed arguments for an application of principle of neutrosophic game theory to understand properly of the Jammu and Kashmir conflict in terms of goals and strategy of either side. Standard 2×2 zero-sum game theoretic model used to identify an optimal solution.

Keywords: Conflict resolution, game theory, Jammu and Kashmir conflict, neutrosophic membership function, optimal solution saddle point, zero-sum game.

1 Introduction

The purpose of this study is to develop neutrosophic game theoretic model to India-Pakistan (Indo-Pak) crisis dynamics and contribute to the neutrosophic analysis of conflicts and their neutrosophic resolution. M. Intriligator [1] reviewed mathematical approaches to the study of conflict resolutions in crisp environment. He prepared a list of primary methodological thrusts as differential equations, decision and control theory, game and bargaining theory, uncertainty analysis, stability theory, action-reaction models and organization theory. Anandalingam and Apprey [2] proposed multilevel mathematical programming model in order to develop a conflict resolution theory based on the integration of non-cooperative game within a mathematical paradigm. They postulated conflict problem as a Stackelberg [3] optimization with leaders and followers. However, the model is suitable only for the normal version of information distribution [4] when the strategy of all lower-level players is completely known to the leader. Yakir Plessner [5] employed the game theoretic model to resolve the conflict between Israel and the Palestinians. Pramanik and Roy [6] studied game theoretic model to the J&K conflict between India and Pakistan in crisp environment. But the situation and relation between India and Pakistan are not static but dynamic and neutrosophic in nature. So new approach is required to deal with the conflict.

Our contribution to the literature is to discuss briefly the genesis of the conflict and apply neutrosophic game theory for conflict resolution.

Rest of the paper is organized in the following way. Section 2 presents some basics of neutrosophy and neutrosophic sets and their operations. Section 3 describes a brief history and the genesis of Jammu and Kashmir conflict. Section 4 is devoted to formulation neutrosophic game theoretic model to Jammu and Kashmir conflict between India and Pakistan. Section 5 presents concluding remarks.

2. Basics of neutrosophy and neutrosophic sets

In this section, we present some basic definitions of neutrosophy, and neutrosophic sets and their operations due to Smrandache [7] and Wang et al.[8].

Definition 1. Neutrosophy: A new branch of philosophy, introduced by Florentin Smarandache that presents the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

Neutrosophy is the basis of neutrosophic set, neutrosophic probability, and neutrosophic statistics.

Definition 2. Infinitesimal number: ε is said to be infinitesimal number if and only if for all positive integers n, $|\varepsilon| < 1/n$

Definition 3. *Hyper-real number:* Let $\varepsilon > 0$ be an infinitesimal number. The *hyper-real number set* is an extension of the real number set, which includes classes of infinite numbers and classes of infinitesimal numbers.

Definition 4. Non-standard finite number: $1^+ = 1+\varepsilon$, where "1" is its standard part and "ε" its $^-$ non-standard part

Definition 5. Non-standard finite number: $^-0 = 0-\varepsilon$, and "0" is standard part and "ε" is its non-standard part.

Definition 6. A non-standard unit interval: A non-standard unit interval can be defined as $\|{-}^-0, 1^+{-}\|$. Here $^-0$ is non-standard number infinitely small but less than 0 and 1^+ is non-standard number infinitely small but greater than 1.

Main Principle: Between an idea $<\psi>$ and its opposite $<$Anti-$\psi>$, there is a continuum-power spectrum of neutralities $<$Neut-$\psi>$.

Fundamental Thesis: Any idea <X> is T% true, I% indeterminate, and F% false, where T, I, F belong to subset of nonstandard unit interval $\|^-0, 1^+\|$ and their sum is not restricted to 100%.

Definition 7. Let X be a space of points (objects) with generic element in X denoted by x. Then a neutrosophic set A in X is characterized by a truth membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A. The functions T_A, I_A and F_A are real standard or non-standard subsets of$]\,0^-, 1^+[$ i.e.
$T_A : X \to\,]0^-, 1^+[$; $I_A : X \to\,]0^-, 1^+[$; $F_A : X \to\,]0^-, 1^+[$

It should be noted that there is no restriction on the sum of $T_A(x)$, $I_A(x)$, $F_A(x)$ i.e. $0^- \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

Definition 8. The complement of a neutrosophic set *A* is denoted by A^c and is defined by

$T_{A^c}(x) = \{1^+\} - T_A(x)$; $I_{A^c}(x) = \{1^+\} - I_A(x)$;

$F_{A^c}(x) = \{1^+\} - F_A(x)$

Definition 9. A neutrosophic set *A* is contained in the other neutrosophic set *B*, $A \subseteq B$ if and only if the following result holds.

$$\inf T_A(x) \le \inf T_B(x),\ \sup T_A(x) \le \sup T_B(x) \quad (1)$$
$$\inf I_A(x) \ge \inf I_B(x),\ \sup I_A(x) \ge \sup I_B(x) \quad (2)$$
$$\inf F_A(x) \ge \inf F_B(x),\ \sup F_A(x) \ge \sup F_B(x) \quad (3)$$

for all x in *X*.

Definition 10. Single-valued neutrosophic set (SVNS): Let *X* be a universal space of points (objects) with a generic element of *X* denoted by x. A single-valued neutrosophic set $\tilde{\mathcal{N}} \subset X$ is characterized by a true membership function $T_{\tilde{\mathcal{N}}}(x)$, a falsity membership function $F_{\tilde{\mathcal{N}}}(x)$ and an indeterminacy function $I_{\tilde{\mathcal{N}}}(x)$ with $T_{\tilde{\mathcal{N}}}(x)$, $I_{\tilde{\mathcal{N}}}(x)$, $F_{\tilde{\mathcal{N}}}(x) \in$ [0, 1] for all x in *X*.

When X is continuous a SVNSs, $\tilde{\mathcal{N}}$ can be written as

$$\tilde{\mathcal{N}} = \int_x \left\langle T_{\tilde{\mathcal{N}}}(x), I_{\tilde{\mathcal{N}}}(x), F_{\tilde{\mathcal{N}}}(x) \right\rangle \Big/ x, \quad \forall x \in X.$$

and when X is discrete a SVNSs $\tilde{\mathcal{N}}$ can be written as

$$\tilde{\mathcal{N}} = \sum_{i=1}^{m} \left\langle T_{\tilde{\mathcal{N}}}(x), I_{\tilde{\mathcal{N}}}(x), F_{\tilde{\mathcal{N}}}(x) \right\rangle / x, \quad \forall x \in X.$$

SVNS is an instance of neutrosophic set that can be used in real life situations like decision making, scientific and engineering applications. In case of SVNS, the degree of the truth membership $T_{\tilde{\mathcal{N}}}(x)$, the indeterminacy membership $I_{\tilde{\mathcal{N}}}(x)$ and the falsity membership $F_{\tilde{\mathcal{N}}}(x)$ values belong to [0, 1].

It should be noted that for a SVNS $\tilde{\mathcal{N}}$,

$$0 \le \sup T_{\tilde{\mathcal{N}}}(x) + \sup I_{\tilde{\mathcal{N}}}(x) + \sup F_{\tilde{\mathcal{N}}}(x) \le 3,\ \forall x \in X. \quad (4)$$

and for a neutrosophic set, the following relation holds

$$0^- \le \sup T_{\tilde{\mathcal{N}}}(x) + \sup I_{\tilde{\mathcal{N}}}(x) + \sup F_{\tilde{\mathcal{N}}}(x) \le 3^+,\ \forall x \in X. \quad (5)$$

Definition 11. The complement of a neutrosophic set $\tilde{\mathcal{N}}$ is denoted by $\tilde{\mathcal{N}}^c$ and is defined by

$T_{\tilde{\mathcal{N}}^c}(x) = F_{\tilde{\mathcal{N}}}(x)$; $I_{\tilde{\mathcal{N}}^c}(x) = 1 - I_{\tilde{\mathcal{N}}}(x)$; $F_{\tilde{\mathcal{N}}^c}(x) = T_{\tilde{\mathcal{N}}}(x)$

Definition 12. A SVNS $\tilde{\mathcal{N}}_A$ is contained in the other SVNS $\tilde{\mathcal{N}}_B$, denoted as $\tilde{\mathcal{N}}_A \subseteq \tilde{\mathcal{N}}_B$, if and only if

$T_{\tilde{\mathcal{N}}_A}(x) \le T_{\tilde{\mathcal{N}}_B}(x)$; $I_{\tilde{\mathcal{N}}_A}(x) \ge I_{\tilde{\mathcal{N}}_B}(x)$; $F_{\tilde{\mathcal{N}}_A}(x) \ge F_{\tilde{\mathcal{N}}_B}(x)$ $\forall x \in X$.

Definition 13. Two SVNSs $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ are equal, i.e. $\tilde{\mathcal{N}}_A = \tilde{\mathcal{N}}_B$, if and only if $\tilde{\mathcal{N}}_A \subseteq \tilde{\mathcal{N}}_B$ and $\tilde{\mathcal{N}}_A \supseteq \tilde{\mathcal{N}}_B$.

Definition 14. Union: The union of two SVNSs $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ is a SVNS $\tilde{\mathcal{N}}_C$, written as $\tilde{\mathcal{N}}_C = \tilde{\mathcal{N}}_A \cup \tilde{\mathcal{N}}_B$. Its truth membership, indeterminacy-membership and falsity membership functions are related as follows:

$T_{\tilde{\mathcal{N}}_C}(x) = \max(T_{\tilde{\mathcal{N}}_A}(x), T_{\tilde{\mathcal{N}}_B}(x))$;

$I_{\tilde{\mathcal{N}}_C}(x) = \max(I_{\tilde{\mathcal{N}}_A}(x), I_{\tilde{\mathcal{N}}_B}(x))$;

$F_{\tilde{\mathcal{N}}_C}(x) = \min(F_{\tilde{\mathcal{N}}_A}(x), F_{\tilde{\mathcal{N}}_B}(x))$ for all x in X.

Definition 15. Intersection: The intersection of two SVNSs $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ is a SVNS $\tilde{\mathcal{N}}_C$, written as $\tilde{\mathcal{N}}_C = \tilde{\mathcal{N}}_A \cap \tilde{\mathcal{N}}_B$, whose truth membership, indeterminacy-membership and falsity membership functions are related as follows:

$T_{\tilde{\mathcal{N}}_C}(x) = \min(T_{\tilde{\mathcal{N}}_A}(x), T_{\tilde{\mathcal{N}}_B}(x))$;

$I_{\tilde{\mathcal{N}}_C}(x) = \min(I_{\tilde{\mathcal{N}}_A}(x), I_{\tilde{\mathcal{N}}_B}(x))$;

$F_{\tilde{\mathcal{N}}_C}(x) = \max(F_{\tilde{\mathcal{N}}_A}(x), F_{\tilde{\mathcal{N}}_B}(x))$ for all x in X.

3. Brief history and the genesis of Jammu and Kashmir conflict

It is said that Kashmir is more beautiful than the heaven, and the benefactor of the supreme blessing and happiness. The account of Kashmir is found in the oldest extant book- "Nilamat Purana". Kalhan, Kashmir's greatest historian scholarly depicted the history of Kashmir starting just before the great Mahabharata War. According to Kalhan, the Mauryan emperor Ashoka annexed Kashmir in 250 B.C. He embraced Buddhism after the Kalinga war. He made it a state religion. He built many Bihars, temples specially Shiva temple. According to Chinese traveler, Huen Tsang over five thousand Buddhist Monks settled down in Kashmir during the reign of Ashoka. After the fall of Maurya dynasty, it is believed that Kashmir for over two hundred years was ruled by Indo-Greek Kings before the start of "Turushka" (Kushan) rule in the state. Thus, the people of Kashmir came in contact with the Greeks. The reflection of which is found on the beautiful architectural and sculptural style of old Kashmiri temples, and the coinage of the later Kashmiri kings.

The zenith of Buddhist power in Kashmir was reached in the reign of king Kanishka. Influenced by Indian culture, Kanishka adopted Buddhism and made it the state religion. During his reign, it is believed that the forth Buddhist Council was held at Kundalavana in Kashmir. It was enthusiastically attended by a large number of scholars, theoreticians, and commentators. . During his reign, Buddhism propagated in Tibet, China and Central Asia. However, Buddhism was followed by a revival of Hinduism and Hindu rulers ruled Kashmir up to 1320.

Rinchan (1320-1323) ascended the throne on 6th October 1320. He was the first converted Islam ruler in the history of Kashmir. Shah Mir ascended the throne with the title of Sultan Shamsuddin (1339-1342) in 1339 A.D. and Shah-Mir dynasty (1339-1561) ruled the state for 222 years. Shah Mir dynasty is one of the most important in the annals of Kashmir, in as much as Islam was firmly established here. During Chak rule (1561-1586) Sunnni Muslims and Hindus alike were persecuted.

Akbar, the Mughal Emperor annexed Kashmir in 1586. It is important to note that under the Mughal reign (1586-1752), people got slight relief and lived honorably. However, the Mughal used forced labor in their visits to Kashmir in terms of a huge retinue of unpaid laborers to carry their goods and other supplies for the journey.

Afghan rule (1752-1819) succeeded in maintaining their suzerainty over Kashmir for a span of sixty-seven years. The Afghans were highly unscrupulous in the employment of forced labor. The common Kashmirian people were tired of their ferocity, barbarity and persecution. It is true history of Kashmir that all sections of people suffered during Afghan rule but the principal victims of these cruel were the peasants. During this era all cruel and inhuman measures of Afghan rulers could not put an end the basic tradition of Kashmiri.

The reign of Sikh Power (1819-1846) in Kashmir lasted for only 27 years. It is to be noted that the Sikhs continued with the practice of forced labor in order to transport of goods and materials. According to Lawrence [9], "to all classes in Kashmir to see the downfall of the evil rule of Pathan, and to none was the relief greater than to the peasants who had been cruelly fleeced by the rapacious sardars of Kabul. I do not mean to suggest that the Sikh rule was benign or good, but it was at any rate better that that of the Pathans."

3.1 Dogra rule (1846-1947)

Dogra dynesty played an in important role in developing Jammu and Kahmir State.

3.1.1 Gulab Singh (1846-1857)

The State of Jammu was conferred on Gulab Singh with the title of Raja by Maharaja Ranjit Singh of Punjab in 1820. He annexed Ladakh in September 1842. Some parts of Gilgit and Baltistan were invaded before 1846. The State of Jammu and Kashmir (J&K) is founded through Amritsar treaty in 1846 between the East India Company and Raja Gulab Singh who buys Kashmir Valley from the East India Company for Rs. 7.5 million and annexes it to Jammu and Ladakh already under his rule. Thus the Dogra dynesty establishes in 1846. Gulab Singh conquered Muzaffarabad in 1854.

3.1.2 Ranbir Singh (1857-1885)

Ranbir Singh (1857-1885) ascended the throne after his father death in 1857 A.D., who ruled from 1857 to 1885 A.D. Lord Northbook's Government recommended for a political officer to reside permanently at the Maharaja's Court in September 26, 1873 A.D. A British Resident remained permanently at the court of Maharaja relating to the external relations of British India from 1873.

3.1.3 Maharaja Pratap Singh (1885-1925)

Maharaja Pratap Singh (1885-1925) ascended the throne after his father death in 1885. During his rule, British power was deeply interested in Kashmir and through British Resident Maharaja Pratap Singh was kept under pressure. In September 1885 during the initial stage of Pratap Singh's rule, the British Government changed the status of the British officer Special Duty in Kashmir to that of a political Resident. Pratap Singh's Address in Durbar October 19, 1885 revealed that the position of political officer in Kashmir has been placed on the same footing with that of Residents in other Indian States in subordinate alliance with the Government. British Government of India disposed Maharaja in 1889. Maharaja was offered an allowance, which was ungenerously described as sufficient for dignity but not for extravagance, would be made to him. No specific period was set for this arrangement to come to an end. Colonel Nisbet, Resident of Kashmir became the virtual ruler because although the Council of minister would have full powers of administration, they would be expected to exercise those powers under the guidance of the British Resident. Without consulting with him, Council would not take any important decision and the Council would follow Resident's advice whenever it was offered.

In 1889, the British Government instituted Gilgit Agency under the direct rule of British political agent. Colonel Algeron Durand [10], the first British Agent in Gilgit records the Russian influence for creation of Gilgit Agency in his Book, "The Making of a Frontier". He remarked in a statement "Why it has been asked should it be worth our while to interfere there whatever happened? The answer is of course Russia…Expensive as the Gilgit game might have been, it was worth the Candle." Viceroy Lord Curzon reinstated Maharaja Pratap Singh in power in 1905 A.D. The State Council is abolished in May 1906 A.D.

3.1.4 Hari Singh (1925-1947)

Hari Singh (1925-1947) ascended the throne after his grandfather, Pratap Singh's death in 1925. During his rule the agitation against the Dogra rule started mainly against the misrule, corrupt administration, autocratic rule, repression on the subjects at the slightest excuse and lack of administrative efficiency. Maharaja Hari ruthlessly crushed a mass uprising in 1931. Hari Singh constituted Grievances Enquiry Commission headed by B .J. Glancy on 12 November 1931 for a probe into the complaints of the people of Kashmir. In April 1932, the commission recommended its suggestions. Among these recommendations, the important one was the step to be taken for propagating education for Kashmiri Muslims. The Commission recommended to give payment [11] Kashmiri people for Government work. In the order dated 31 May 1932, Maharaja Hari Singh accepted the recommendation of the President of the Kashmir Constitutional Reforms Committee, B. J. Glancy and appointed a Franchise Committee to put forward tentative suggestion regarding the important question of the franchise and the composition of the assembly. In this background All Jammu and Kashmir Muslim Conference (AJKMC) was formed under the leadership of Sheikh Abdullah in 1932 in October in Srinagar. The conference held from 15 to 17 October 1932.

In 1934, the Muslim Conference demonstrated its secular view when it forwarded memorandum drafted by Ghulam Abbus to the Maharaja demanding early implementation of the report of Glancy Commission and specifically urged for a system of joint electorates in the State. In 1934, Maharaja introduced a Legislative Assembly. However 35 of its 75 members were to be nominated. 8 per cent of the population was allowed to cast vote. To become a voter literacy and property qualifications were specified. The Assembly enjoyed only consultative powers. Maharaja further reformed making the provision for Council of Ministers and a judicial and legislative branch of public administration in 1939. However, Maharaja enjoyed most of the decision powers under the new reforms.

On 26 March 1938 Sheik Abdullah iterated two important points: i) to put an end communalism by ceasing to think in terms of Muslims-non-Muslims when discussing political problems. ii) Universal suffrage on the basis of joint electorate. It is to be noted that the national demand issued in August 1938 was signed among others by Pandit Jia Lal Kilam, Pandit Lal Saraf, Pandit Kasyap Bandhu. Under the leadership of Sheik Abdullah AJKMC felt the necessity of common platform to struggle against the rule of Maharaja. After series of discussions and debates, the working committee of AJKMC took the historic decision of re-christening to Jammu and Kashmir National Conference (or simply National Conference) on 24 June 1938. On 27 April 1939, National Conference came into being. Its secular credentials set a new pace for the politics of Jammu Kashmir. National Conference [12] consisted of many leaders of minority communities like Hindu, Sikh etc during 1940s.

In the history of India subcontinent, the Pakistan resolution demanding the creation of an independent state comprised of all regions in which Muslims are the majority is passed at Iqbal Park, Lahore on March 23, 1940 by Muslim League.

The secularization of Kashmir politics and redefinition of the goal helped immensely National Conference to come in close contact with the Indian National Congress. In 1942 'New Kashmir' manifesto was formulated under the leadership of Dr. N. N. Raina by a brilliant group of young communist operating within the National Conference who were mostly responsible for introducing the nationalist movement to the concept of socialist pattern of society based on equality, democracy and free from exploitation. It consists of two parts: a) the constitution of the state; b) the National Economic Plan. Under the sound leadership of Abdullah, National Conference led a powerful mass movement in order to find a new political and economic order in Kashmir and other parts of Jammu region. The National Conference started agitation against the Dogra rule in 1945. In the grave political situation, offering him all charges, Ram Chandra Kak was appointed as Prime Minister in order to bring the agitation in control. In May 1946 National Conference launched "Quit Kashmir" movement following the "Quit India Movement" in 1942 led by the Indian National Congress. Mohamod Ali Jinnah was not interested in the 'Quit Kashmir Movement'[13] rather blamed the movement as act of Gundas. In March 1946 Crisps Mission came to visit India. Sheikh Abdullah sent a telegram by demanding freedom of people of Kashmir on withdrawal of British power from India.

Prime Minister of J&K Ram Chandra Kak declared emergency to crackdown the movement. Abdullah was arrested on 20 May 1946. The State Government employed a wave of arrests and a policy of repression throughout the State. The people protested strongly and several agitated Kashmiri people were killed and injured due to clash with armed forces of Maharaja. The Indian National Congress and the All India States peoples' Conference supported National Conference strongly. Sheik Abdullah was imprisoned for three years for antinational activities. National Conference was banned. In January 1947, National Conference boycotted elections because of repression. Muslim Conference grabbed the opportunity and won 16 out of 21 Muslim seats.

3.2 Muslim Conference

Muslim Conference did not support the 'Quit Kashmir' agitation. Muslim Conference discouraged the people of Kashmir from joining the agitation in the same tune of Muslim League. On 30 May 1946, Chaudhury Gulam Abbas the President of Muslim Conference stated that the agitation had started at Congress leaders' behest in order to "restore the lost prestige of the Nationalist." The Muslim Conference adopts the Azad Kashmir Resolution on 26 July 1946 calling for the end of autocratic Dogra rule in the region and claiming the right to elect their own constituent assembly. He said that the primary task [14]was to restore the unity of the Muslim nation and there be "no other place for an honest and self-respecting Muslim but in his own organization." On 25 October 1946, State Government arrested and detained four top leaders of Muslim Conferences.

3.3 British Cabinet Mission

In March 1946, The British Cabinet Mission held conference about a week at Simla with four representatives, two each of the Congress and the Muslim Leagues and the conference broke down on the issue of Pakistan and parity in the proposed interim government. On 16 May 1946, the Cabinet Mission announced their own proposals, the essence of which was the creation of a Constituent Assembly to frame the Constitution of India, which was to be based on the principle that the Center would control only three subjects, viz., Defense, Foreign Affairs and Communications and the creation of three group of provinces-two of the areas claimed by Muslim League for Pakistan in the east and the west and the third of the rest of the subcontinent [15].

3.4 Interim Government announced

On 25 June 1946, the Congress Working Committee announced their rejection of the plan of Interim Government. On June 26, 1946, Lord Wavell announced that he would set up a temporary 'caretaker' Government of officials to carry on in the interim period.

In July 1946, the Muslim League withdrew its acceptance of the Cabinet Mission's plan and resolved that "now the time has come for the Muslim nation to resort to direct action to achieve Pakistan, to assert their just rights, to vindicate their honor and to get rid of the present British slavery and the contemplated future 'caste- Hindu domination" at a meeting in Bombay.

Accepting the invitation from the Viceroy to constitute an interim Government, on 6 August 1946, Jawaharlal Nehru formed it, which consisted of six Hindus, including one Depressed Class member, three Muslims of whom two belonged neither to the Congress nor to the League, one Sikh, one Christian-and one Parsee. It started functioning on 2nd September 1946. The League joined the Interim Government in the last week of October 1946 but was not prepared to join the Constituent Assembly, which led every day a more and more difficult and delicate on account of the differences between the cabinet ministers of Congress and the Muslim League. 0n 26 November 1946, Mr. Atlee invited Lord Wavell and representatives of the Congress and the Muslim League to meet in London to attempt to resolve the deadlock The discussions were held from 3 to 6 December 1946 but did not yield any agreed settlement. The first meeting of the Constituent Assembly of India was held in on 11 December 1946. The Muslim League boycotted it and it developed a stake in sabotaging the Assembly's work.

On 20 February 1947, Prime Minister Atlee declared that Britain would transfer power by June 1948, by which time the Congress and the Muslim League were supposed to resolve their differences. On 24 March Mountbatten was sworn in as Viceroy and Governor General of India in place of Wavell. After negotiations with the leaders of different political parties, Viceroy, Lord Mountbatten announced that long before June 1948, the Dominions of India and Pakistan would be created and that the question of Indian states would be dealt with in the light of the Cabinet Mission's memorandum [16] of 12 May 1946.To approve the Mountbatten plan accepted by British Cabinet, a conference between Mountbatten and representatives of the Congress and the Muslim League was held on 2 June 1947. On 3 June 1947 a White paper was issued which stated the detail procedure of the partition of India. Regarding the Princely States it declared that British policy towards Indian States contained in the Cabinet Mission's memorandum [17] of 12th may, 1946 remained unchanged.

3.5 Partition Plan accepted by Congress

On 14 June 1947, in a historic session of All India Congress Committee (AICC) in New Delhi, Pandit Ballabh Pant moved the resolution dealing with the Mountbatten plan for partition British India. Mahatma Gandhi intervened in the debate in the second day and expressed that he was always against the partition but situation had changed and appealed to support the resolution. On 15 June 1947, the resolution was passed with 29 votes in favor and 15 against.

Mr. Jinnah clearly expressed Muslim League view [18] on the question of Princely States on 17 June 1947 by saying "Constitutionally and legally the Indian states will be independent sovereign states on the termination of paramountcy and they will be free to decide for themselves and adopt any course they like; it is open to them to join the Hindustan Constitutional Assembly or decide to remain independent. In case they opt for independence they would enter into such agreements or relationships with Hindustan or Pakistan as they may choose".

3.6 Partition and riots

Calcutta, capital of Bengal witnessed a beginning of holocaust on an unprecedented scale on 16 August 1946, which was declared a public holiday by the Muslim League Government of Bengal. It was estimated that Jinnah's direct action [18] caused death of more than 5000 lives, and over 15000 people were injured, besides 100000 being rendered homeless. After a fortnight 560 people were killed in Bombay. After Calcutta, on October 1946, serious anti- Hindu riots erupted in Noakhali in East Bengal followed by massacred of Muslims in Bihar The chain reaction of riots started in the Punjab causing large scale killings of Hindus, Sikhs, and Muslims shortly afterwards.

3.7 Development in Jammu and Kashmir

Based on two-nation theory, India was partitioned into Pakistan and India in August 14, 1947. The princely states were offered the right under the 'Indian Independent Act 1947' and 'Government of Indian Act 1935' [19] to accede either to India or Pakistan or remain independent. It seemed that Hari Singh, the then Maharaja of Jammu and Kashmir hoped to create independent Kingdom or autonomy from India and Pakistan. He did not accede to either of two successor dominions at the time of accession. All Jammu and Kashmir Rajya Hindu Sabha passed a resolution [20] expressing its faith in Maharaja Hari Singh and extended its "support to whatever he was doing or might do on the issue of accession" in 1947. On 15 June 1947, an important resolution [21] regarding the princely states saying the lapse of paramountcy does not lead to the independence of the princely states was adopted by AICC unanimously. Contrary to this, Mr. Jinnah clearly expressed the view [18] of Muslim League on the question of Princely States on 17 June 1947 by saying "Constitutionally and legally the Indian states will be independent sovereign states on the termination of paramountcy and they will be free to decide for themselves and adopt any course they like; it is open to them to join the Hindustan Constitutional Assembly or the Pakistan Constituent Assembly, or decide to remain independent. In the last case, they enter into such agreements or relationship with Hindusthan or Pakistan as they may choose ".

On 19 July 1947, the working committee of Muslim Conference passed a modified resolution [22] in favor of independence, which respectfully and fervently appealed to the Maharaja to declare internal autonomy of the state and accede to Pakistan regarding to defense, communication and external affairs. Khurshid Ahmad, Jinnah's personal Secretary during his stay in Kashmir on the crucial days for the question of accession gave Maharaja assurance [23] that "Pakistan would not touch a hair of his head or take away a iota of his power". Before partition British Government restored the Gilgit area, an important strategic region, hitherto administered by a British agency, to J&K without taking the verdict of the local people.

3.8 Standstill Agreement

Pakistan became independent on 14 august 1947. India and few princely states, which did not join either of India or Pakistan, became independent on 15 August 1947. In this way J&K attained the status of independent on 15 August 1947. On 15 August post offices in J&K hoisted the Pakistani flags. Maharaja Hai Singh signed a standstill agreement with Pakistan on 16 August 1947 with regard to State's postal services, railways, and communications and hoped to sign similar agreement with India with regard to external affairs, control of state forces, defense etc. India [23] did not show any interest in the acceptance of the offer of standstill agreement. . On 18 August 1947 a controversy came into light when Sir Cyril Radcliffe awarded a portion of Muslim majority Gurudaspur District to India causing fundamental differences in J&K's geopolitical situation. The subcontinent experienced communal riots during these days. By this time, Muslim majority Poonch estate within the Jammu region experienced serious troubles with regard to some local demands like the rehabilitation of 60,000 demobilized soldiers of the British army belonging to the area. The agitation finally transformed into communal form having mixed with other issues. The state army refused to fire on the demonstrators with whom they had religious and ethnic ties. The agitation turned to the form of armed revolt because of mass desertion from army. The supply of arms and ammunition and other assistance from outside the border magnified the revolt. The Kashmir Socialist party passed a resolution on 18 September 1947 to join Pakistan and not India. The party impressed on Maharaja that without any further unnecessary delay he should make an announcement accordingly. It is to be noted here that a convention of Muslim Conference workers formally asked for accession to Pakistan on 22 September 1947.

Maharaja Hari Singh released Sheikh Abdullah from prison along with some other National Conference workers on 29 September 1947 but he did not release the workers of Muslim Conference due to grave situation of the state. Pakistan termed Abdullah's release as a conspiracy because workers of Muslim Conference were not simultaneously released. By October, communal riots spread all over J&K. The mass infiltration baked by Pakistani army jeopardized the environment of the state. Pakistan violated the standstill agreement by stopping regular supply of food, salt, petrol and essential commodities from Pakistan. The communication system controlled by Pakistani Government did not render proper service.

On 21 October 1947, Pakistan decided to settle the future of Kashmir with the power of gun suspecting that Maharaja was likely to accede to India. Jinnah, the Governor Gen-

eral of Pakistan personally authorized a plan [25] to launch "a clandestine invasion by a force comprised of Pathan (Afghan) tribesmen, ex-servicemen and soldiers on leave".
It was witnessed that charges and counter charges were being made by both the government of J&K and Pakistan during the month of October and finally On 22 October 1947, 2000 tribesmen from Northwest Frontier Province (NWFP) of Pakistan and other Pakistani nationals fully armed with modern arms, under the command of trained generals, started invasion to capturing the state's territory. The Muslims in the Western part of Kashmir established their own independent (Azad) Kashmir Government on 24 October 1947. The State forces were wiped out in fighting. The tribesmen resorted to "indiscriminate slaughter of both Hindus and Muslims"[26]. They reached within 15 miles from capital Srinagar. Under this great emergency of the situation, Maharaja sought Indian military assistance in his letter dated 26 October 1947 along with the 'Instrument of Accession' [27] to Mountbatten, the Governor General of India. Thereafter the Maharaja signed the instrument of accession, which the Governor General Mountbatten accepted on 27 October 1947 by adding that the question of accession [28] should be settled by a referendum. Indian forces [29] airlifted at Srinagar almost at the crucial moment, for, "a few minutes later the airfield might well have been in enemy hands". Members of the National Conference provided logistical support for the Indian forces. Infuriated by Indian intervention, on 27 October 1947, Pakistani Governor General, Mohammed Ali Jinnah ordered Lt. General Sir Douglas Gracey, Chief of the Pakistani Army, to send Pakistani regular troops to Kashmir, but Field Marshall Auchinleck, the Supreme Commander of the transition period succeeded in persuading him to withdraw his orders. A message [30] was sent to the Governor General and the Prime Minister of India to go to Lahore for discussion regarding Kashmir.

3.9 Indo-Pak talks

On 1 November 1947, at a meeting of Governors General of India and Pakistan at Lahore, Mountbatten offered to resolve the J&K issue by holding referendum. Rejecting the Mountbatten formula, M.A. Jinnah remarked that a plebiscite was "redundant and undesirable". H.V. Hodson [31] has recorded in his book, The Great Divide, that M.A. Jinnah "objected that with Indian troops present and Sheikh Abdullah in power the people would be frightened to vote for Pakistan". Jinnah proposed a simultaneous withdrawal of all forces- the Indian troops and the invading forces. Here it is interesting to note that when he was asked how anyone could guarantee that the latter would also be withdrawn, Jinnah [30] replied "If you do this I will call the whole thing off". In connection with the steps to ascertain the wishes of the people of J&K, Mountbatten was in favor of a plebiscite under the auspices of United Nations while M. L. Jinnah proposed that he and Mountbatten should have plenary power to control and supervise the plebiscite. Ultimately, the first direct bilateral talks broke down.
On 1 January 1948, based on the advice of Mountbatten, India lodged a complaint with the Security Council invoking articles 35 of Chapter VI of the UN Charter to "recommend appropriate procedures or methods of adjustment" for the pacific settlement of disputes and not for "action" with respect to acts of aggression as provided for in Chapter VII of the Charter [32]. India reiterated her pledge of her conditional commitment to a plebiscite under international auspices once the aggressor was evicted. Pakistan contradicted the validity of the Maharaja's accession to India [33], and urged the Security Council to appoint a commission for securing a cease-fire and ensuring withdrawal of outside forces, and conducting a plebiscite in order to determine the future of J&K.

3.10 Role of the United Nation Security Council (UNSC)

Both India and Pakistan denied implementing the UN resolutions [34-36] for a free and impartial plebiscite in order to put an end to the situation for the accession of J&K. Having taken note of the developments in J&K, the United Nations Commission for India and Pakistan UNCIP submitted a draft resolution [36] consisting of three parts to the council on 13 August 1948.
Part I of the resolution comprised of instruction for a ceasefire.
Part II of the resolution dealt with the principle of a truce agreement which called for Pakistan to withdraw tribesmen, Pakistani nationals not normally resident therein who had entered the State of J&K for the purpose of fighting, to evacuate the territory occupied by Pakistan and after the notice of the implementation of the above stipulation by the UNCIP India was to withdraw the bulk of her forces in stages from J&K leaving minimum strength with the approval of the commission in order to ensure law, order and peace in the State.
Part III of the resolution appeared to be important as it clearly expressed that both the Government of India and the Government of Pakistan reaffirm their wish that the future status of the State of J&K shall be determined in accordance with the will of the people.
The second resolution [37] specified the basic principle of plebiscite was formally adopted on 5 January 1949 after acceptance of India and Pakistan on 23 and 25 December 1948 respectively.
An important development occurred when both India and Pakistan agreed to the cease-fire line in 1949. This enabled the UN to finally send a Military observer Group to supervise the line [38]. The ceasefire came into effect on 1 January 1949. The most important long- term outcome of the first Indo-Pak war was the creation of ceasefire line. Thus UNCIP succeeded in implementing the important provision of Part I of the resolution. In order to monitor to the

ceasefire line (CFL), the UNCIP sent a Monitoring Group for India and Pakistan (UNMGIP) to J&K on 24 January 1949 relying on its resolution of 13 august 1948. In Karachi on 27 July 1949, the military representatives of India and Pakistan, duly authorized, approved CFL and thus approved the presence of UNMGIP [39].

In March 1949, the conflicting attitudes came into light as India and Pakistan expressed their viewpoints before the truce subcommittee of the UNCIP. On 15 April 1949, UNCIP transmitted to the governments of India and Pakistan its own proposals [40], which were:

i) to create a cease-fire line, eliminating all no man's lands and based on the factual position of the troops in January 1949.
ii) to draw a phased program of withdrawal of Pakistani troops to be completed in seven weeks, and the withdrawal of all Pakistani nationals.
iii) to ask Indian forces also to withdraw in accordance with a phased program after the withdrawn of tribesmen and Pakistani national and after the declaration of UNCIP's satisfaction regarding the troops withdrawal of Pakistan.
iv) to release all prisoners of war within one month.
v) to repeal all emergency laws.
vi) to release all political prisoners.

Both India and Pakistan [41] could not accept the proposals because of their own interest.
The UNCIP proposed arbitration on the issues regarding the part II of the resolution in a letter to the two Governments on 26 August 1949 and named Fleet Admiral Chester Nimitz as the Arbitrator. Pakistan accepted the proposal on 7 September 1949 but India rejected this proposal of arbitration. The Czechoslovak representative of the UNCIP, Dr. Oldrich Chyle (Chyle took the post after resignation of Korbel) criticized the UNCIP's work [42]. According to him, the arbitration move was a pre-planned attempt on the part of the USA and UK to intervene in the dispute.

On 17 December 1949, the UNSC named its president General A. G. L. McNaughton of Canada as the Informal Mediator [43], instead of commission to negotiate a demilitarization plan in consultation with India and Pakistan. He submitted his proposal on 22 December 1949. Pakistan accepted the proposals, suggesting minor amendments while India suggested major amendments: one calling for the disbanding and disarming of Azad forces, and the other dealing with the return of the Northern Areas to India for purposes of defense and administration of J&K. Pakistan was unable to accept Indian amendments [44] as a clear rejection of the proposals. Pakistan agreed to simultaneous demilitarization but Indian rejected it on the grounds of the legal and moral aspects of the plan.
The UNSC adopted another resolution introduced by C. Blanco of Cuba on behalf of four powers Cuba, Norway, UK and USA on 14 March 1950, which called upon the two nations, without prejudice to their rights or claims to prepare and execute within the stipulated period of five months for the demilitarization of J&K based on proposals of McNaughton and for self determination [45] through an impartial plebiscite. The resolution terminated the UNCIP and transferred their powers and responsibilities to a UN representative.

3.10.1 Dixon mediation

Sir Owen Dixon, UN Representative submitted his recommendations to the UN on 15 September 1950. He suggesting a unique proposal [46] limiting the plebiscite only to the Kashmir Valley claimed by both by Pakistan due to its Muslim majority and the waters of Jhelum. India and Pakistan rejected the plan.
UN representatives worked to negotiate for free and impartial plebiscite in J&K until 1953 but their efforts brought no fruit. The UN continued its efforts for a plebiscite [47], but all attempts of UN failed due to the conflicting and divergent attitude of the Governments of India and Pakistan towards the dispute and the cold war [18]. The fifth report of Dr. Frank P. Graham [48] suggested direct negotiations between India and Pakistan. Thus the UN attempts at solving the problem of J&K came to end which reflected the limitations of the UN.
The armies of India and Pakistan waged an inconclusive war (1947-48) for over a year in J&K. The Indian army occupied almost two third of J&K remaining 1/3 portion was under the control of Pakistan which is called Azad Kashmir or Pakistan occupied Kashmir (POK).

3.11 Indo-Pak negotiation (1962-1963)

India experienced a huge defeat in 1962 war against China. The J&K dispute became the subject of Indo-Pak negotiation in late 1962 but no agreement could be signed for resolution of J& K question despite six round talks between an Indian delegation headed by Swaran Singh and a Pakistani delegation headed by Z.A. Bhutto from 27 December 1962 and 16 May 1963.

3.12 Sino-Pak border agreement 1963

On 2 March 1963, the Sino-Pakistan Border Agreement was signed in Peking and they had agreed that after the settlement of the Kashmir dispute between Pakistan and India, the sovereign authority concerned would reopen negotiations with China on the boundary as described in Article. By this agreement Pakistan [49] succeeded in stabilizing Pakistan's position regarding Kashmir in the eyes of Chinese Government and compelling her "to reject unequivocally the contention that Kashmir belonged to India".

3.13 The Kutch conflict- a low intensity war

In 19 April 1965, Pakistani permanent representative [50] in UN made claims about 8960 square kilometers area of Rann. Pakistani claim to the Rann of Kutch was based on the fact that the Rann was a lake and according to international law [51], the boundary line between India and Pakistan must be drawn through the middle of the Rann. On other hand India argued that the Rann of Kutch was a “marsh” land rather than a lake. India asked Pakistan to restore the status quo ante. Tikka Khan, in command of the 18 Infantry Division, did painstakingly prepare for the operations and succeeded in advancing inside Indian territories in strength, causing to the fall of the Indian forward post hastily positioned there. India and Pakistan fought a low intensity war. The important aspect of the conflict lies in the historic fact that both India and Pakistan accepted a ceasefire and arbitration on British intervention. On the other hand, India captured some Pakistani Posts in the Kargil area of Ladakh. The Kutch dispute [52] was referred to a tribunal comprising of three members, one nominated by India, another by Pakistan and a Chairman chosen by the UN Secretary General. After a long deliberation the tribunal awarded Pakistan 317 square miles out of 3500 square miles claimed by her. India left the occupied posts of Pakistan in Kargil.

3.14 Indo-Pak war in 1965

The Pakistani Government was greatly emboldened by presumably military success in the Rann of Kutch in 1965. In August 1965 infiltration had started in Jammu and Kashmir to wage what Zulfikar Ali Bhutto called a “war of liberation”. On 10 August 1965, Z. A. Bhutto [53] publicly declared his country’s full support to the people of Kashmir but denied his country’s involvement in the Kashmir trouble. On 1 September, 1965 Indian forces crossed the international border and sealed the borders of Kashmir. On 4 September, Malaysia moved a resolution co-sponsored by Bolivia, the Ivory Cost, Jordan, the Netherlands, and Uruguay proposing an immediate ceasefire in Kashmir without calling Pakistan as an aggressor in the UNSC [54]. But it did not succeed in stopping the fighting. Ayub Khan backed the infiltration with a full-fledged attack in the Chhamb sector by crossing the international border, leading to effective progress to reach Jaurian. On 5 September 1965, Indian forces launched three-pronged thrust in of West Pakistan in Lahore Sector and in Sialkot sector a day later. Following this development, Malaysian representative submitted another resolution [55] supported by Bolivia, the Ivory Cost, Jordan, the Netherlands, and Uruguay calling upon both the countries to cease hostilities and withdraw their troops to the positions held by 5 August 1965, which was passed unanimously on 5 September 1965. The goodwill mission to India and Pakistan by the U.N. Secretary General, U Thant did not succeed. Both countries were requested by U Thant to stop fighting without imposing any condition on each other [56]. India accepted unconditional ceasefire but President Ayub Khan [57] imposed certain pre-conditions: (i) Withdrawal of all forces of both India and Pakistan (ii) Induction of foreign forces, preferably Afro-Asian under UN auspicious, (iii) Holding a Plebiscite in Kashmir within three months of the cease fire.

Armies of both the countries engaged in large - scale combat in a series of sharp and intense actions along the ceasefire line in J&K and the international border in Punjab, Rajasthan, and Gujarat by employing import weaponry system but outmoded war stretegies. They reached to the point of exhaustion, battle fatigue. The representative of the Netherlands moved the draft resolution [58], which was accepted, by both India and Pakistan in the UNSC on 20 September 1965. It was adopted by ten votes to nil, with Jordan abstaining. On 20 September 1965, the super power USA concurred with USSR in the Security Council on calling ceasefire within 48 hours. Pakistan and India accepted the call [59] on 21 and 22 September 1965 respectively. The ceasefire, the UN enforced became effective at 03:30 hours of 23 September 1965. Both India and Pakistan lost nearly 3000 people each in the war. Economy of both the countries suffered a setback.

Although fighting ended inconclusive both India and Pakistan claimed victories,. China identified India as an aggressor and supported the Kashmiri’s right of self-determination.

3.15 Tashkent agreement 1966

The Tashkent Declaration was signed between Indian Prime Minister L. B. Shastri and Pakistani President after six days of hard bargaining on 10 January 1966. They agreed that all armed personnel of the two countries should be drawn not later than 25 February 1966 to the position they held prior to 5 August 1965, and both sides should observe the ceasefire terms on the ceasefire line. They affirmed to employ peaceful means to solve their conflicts. Neither side was allowed to enjoy the gains of war. Pakistan was not even mentioned as the aggressor nor did it admit having engineered the infiltration in J&K.

3.16 Indo-Pak war in 1971

In the general election held in Pakistan in 7 December 1970, the Awami League led by Mujibur Rehman secured majority in the national assembly by winning 158 seats out of 300 seats. He demanded complete autonomy for East Pakistan. The East Pakistanis formed Mukti Bahini (Liberation Force) and civil war erupted in East Pakistan. India supported the Movement. Pakistan used armed forces to curve the movement. The fighting forced 10 million East Pakistanis to flee in Indian territories. India accused the Government of Pakistan of committing brutal genocide in the East Pakistan. India asked Pakistan to negotiate with Rehaman for a political settlement. On 3 December 1971, Pakistan launched attack on Indian airfields along the frontier of Punjab, Rajasthan, and J&K [60]. On the other hand, Pakistan alleged that Indian forces attacked on 21 November 1971 in the south- eastern sector of East Pakistan. India is the first country who recognized formally the birth of Bangladesh [61] on 6 December 1971 . The Indian Army along with the Mukti Bahini (Liberation Army) fought the Pakistani armed forces. The news of sending a naval task force from the US Seven Fleet [62] to the Bay of Bengal from the Indo-China theatre caught much attention. But the USSR [63] confirmed India that the Soviet powerful naval fleet would follow the Seven Fleet. On 15 December 1971 the Indian army reached the outskirts of Dacca. On 16 December 1971, 9000 Pakistani forces along with their commander General Niazi surrendered to the Joint Command of India and Bangladesh. India declared a unilateral ceasefire [64] effective from 20:00 hours on 17 December 1971 and Yahya Khan accepted it. Yahya Khan had to resign because of huge defeat in East Pakistan. He handed power to Z.A. Bhutto. Although India and Pakistan fought a third war over East Pakistan, J&K dispute was only a peripheral issue but vital one in the case of J&K. At time of ceasefire, India occupied 204. 7 sq kms of territory of Pakistan administered Kashmir, 957.31 sq km of Punjab and 12198.84 sq kms of Kutch while Pakistan occupied 134.58 sq kms of territory of Indian administered J&K in the Chhamb sector, 175.87 sq kms in Punjab and 1.48 sq kms in Rajasthan [65].

3.17 Role of UN

The UN intervened to arrange cease-fires during the war 1971. USSR exercised her veto power several times in favor of India. The Secretary General [66] was authorized to appointment, if necessary, a special representative to help in the solution of humanitarian problem. The issue of Indo-Pak conflict came to an end on 25 December 1971 with the appointment by U Thant, the Secretary General, of V. W. Guicciardi, as Secretary General’s special representative for humanitarian problems in India and Pakistan.

3.18 Simla agreement 1972

The Prime Minister of India and President of Pakistan had talks in Simla from 28 June 1972 to 2 July 1972 and signed the Simla Agreement [67] on 2 July, 1972. By signing the agreement, both India and Pakistan committed themselves to settling their differences through bilateral negotiations or by any other peaceful means mutually agreed upon between them. Hopefully, they also agreed that in “Jammu and Kashmir, the line of control (LOC) resulting from the cease-fire of December 17, 1971, shall be respected by both sides without prejudice to the recognized position of either side”.

The Simla Agreement was ratified by both countries [68] and it came into force on 5 August 1972. To delineate the line of control General Bhagat and General Hamid Khan had to hold ten meetings between 10 August to 7 December 1972. On 11 December 1972, they [69] met at Suchetgarh and jointly signed 19 maps delineating the line of control from Chhamb to Turtuk, covering about 800 kilometers. Both the Governments approved the delineation [70] almost next day. On completion of adjustment in the line of control, India and Pakistan withdrew troops from the occupied territories in order to restore the status quo ante on the international border on 20 December 1972. Pakistan [71] has recognized Bangladesh in February 1974. The issue regarding prisoner of wars [72] closed with the repatriation of the last group along with Gen. Niazi at Wagah on 29 April 1974. East Pakistan crisis reflected that the two-nation theory failed miserably in the subcontinent.

3.19 The conflict at Siachen (1984 onwards)

The conflict between India & Pakistan over Siachen originated due to the non-demarcations on the western side of the map beyond a grid point known as NJ 9842. The CFL, which was established because of the first Indo-Pak war of 1947-48 and the intervention of the UN, runs along the international Indo-Pak border and then north and northeast until map grid-point NJ 9842, located near the Shyok River at the base of the Saltoro mountain range. Unfortunately, it was not delineated beyond the grid point known as NJ 9842 as far as the Chinese border but both countries agreed vaguely that the CFL extends to the terminal point NJ 9842, and "thence north to the Glaciers". After second Indo-Pak war in 1965, obeying the Tashkent Agreement both countries withdrew forces along the 1949 CFL. After third Indo-Pak war 1971, the Simla Agreement of 1972 created a new LOC based on December 1971 cease-fire. However, the Siachen Glacier region was left un-delineated where no hostilities occurred. The authorities of both countries showed no interest to clarify the position of the LOC beyond NJ 9842. Due to lack of strategic viewpoint and seriousness the LOC was poorly described as running from Nerlin (inclusive to India), Brilman (inclusive to Pakistan), up to Chorbat La in the Turtok sector. In April 1984, Indian army occupied key mountain Passes and established permanent posts at the Siachen heights. Indian troops brought control over two out of three passes on the Siachen, Sia La and Bilfond La, while the third pass, Gyong La remained under Pakistan's control. The Indian forces are permanently deployed all along the 110-km long Actual Ground Position Line (AGPL). Armies of both India and Pakistan fight in lethal 10-22000 foot altitude in Siachen Glacier. Pakistan retaliates in the world’s highest war zone.

3.20 Kashmir insurgency in Indian administered Kashmir

The assembly elections in J&K on 23 March 1987 were partly manipulated and rigged which the National Conference-Congress coalition won a landslide victory. The opposition party Muslim United Front (MUF) called the victory as blatantly fraudulent and rigged. A large number young people of Kashmir were alienated by this perception. State Government of J&K witnessed various demonstration and agitation between mid-1987 and mid-1989 based manifestation of an accumulated anger comprised of many components such as administrative (the curtail number of Offices that move to the winter capital Jammu), the regional autonomy, economic policy (increase of power tariffs), religious sentiments, civil liberties (custodian death), and anti-India demonstration of 14 and 15 August, 26 October (accession day) and 26 January. On 8 December 1989, the militants kidnapped Rubaiye Sayeed, daughter of Indian Home Minster Mufti Mohammed Sayeed. The prestige of Farooq Abdullah led State Government suffered serious setbacks for repression of any form of protest Farooq Abdullah's resignation with the appointment of Jagmohan as Governor for the second time on 19 January 1990, brought Central Government into direct confrontation with the various rebel groups. At 5 a. m. on 20 January 1990, Indian paramilitary forces cracked down on a part of Srinagar city and began the most intense house-to-house search and rounded up over three hundred people. Most of them, however, were later released and arrested persons complained to be beaten up or dragged out of their houses. People got frightened first, but discovering the courage of desperation, the people started pouring out into the street defying the curfew, to protest against the alleged excessive use of force in search operation in next day. The administration got completely unnerved and gave orders to fire at when most of the groups of demonstrators converged at Gau Kadal. The number of deaths [73, 74] is disputed; however, the press reported 35 dead. Then the implicit support for the separatists for independence transformed into explicit due to mainly the high-handed searches ordered by Jagmohan, the Governor of J&K. On 19 February 1990 Governor dissolved the State assembly and Governor rule was imposed. The Jagmohan regime [13] witnessed sadly the exodus of almost the entire small Kashmiri Pandit community from the valley and 20000 thousand Muslim had been forced to migrate. The State assembly election of 1990 resulted in Abdullah downfall following the outbreak of a Muslim uprising. During the 1990s, several new militant groups were formed, having radical Islamic views. The large numbers of Islamic Jihadis, who had fought in Afghanistan against the Soviet Union in the 1980s, joined the movement

Many umbrella groups were responsible for the uprising in J&K. Among them, the first umbrella group is tied to the Jammu and Kashmir Liberation Front (JKLF). They demaned independent Kashmir. The second group comprised of Muslim fundamentalists and has links with the fundamentalist Pakistan party, Jammait-I- Islam. No doubt the group has a pro-Pakistan Orientation. The third group is Jammu and Kashmir People`s League that has a pro-Pakistan orientation. The groups demanded plebiscite so that people of J&K could exercise their right of self determination. India adopted a multiple prolonged approach to deal with the insurgency in J&K. In 1990, the then Governor Jagmohan announced the implementation of Armed Forces Special Powers Act of 1958 (AFSPA) for J&K and J&K Disturbed Areas Act to put down the militancy. Indian security forces allegedly committed a series of human right abuses [75] in J&K. It is observed that the encounter between Indian security forces and the militants caused more than 50, 000 deaths [75] including many hundreds of innocent civilians. Kashmiri militants have been also accused of killing moderate Muslim leaders, Hindus, bombing passenger busses and railway bridges and public establishments. In September 1996, National Conference had won a landslide victory in J&K Assembly election, although the 30-disparate party coalition, known as All-Party Hurriyat Conference (APHC) did boycott the election. Indian authorities formed several Muslim counterinsurgency groups to combat the insurgency along with Indian security forces. Due to the acute failure of Indian authorities to address the socio-economic problems and ambition of autonomy to some extent of the people of J&K and Pakistan's active role in fostering cross border terrorism, the situation in J&K becomes more complex and volatile and neutrosophic in nature.

3.21 Nuclear rivalry between India and Pakistan

India had conducted her first nuclear device in 1974. In May 1998, India conducted several nuclear tests in the desert of Rajasthan. Pakistan got the opportunity to conduct nuclear test, and hopefully grabbed the opportunity and conducted six tests in Baluchistan in order to balancing nuclear power with India. The arm race between Indo-Pak caught international attention. The UNSC condemned both the countries for conducting nuclear tests and urged them to stop all nuclear weapons program. On 23 September 1998, new development occurred following at UN General Assembly session. Both India and Pakistan agreed to try to resolve the Kashmir question peacefully and to focus on trade and "people to people contact". Pakistan sent her cricket team in India as goodwill gesture on November 1998 after a decade absence. On the other hand, India agreed to buy sugar and powder from Pakistan. In February 1999, bus service between New Delhi to Lahore started. Accepting an invitation from Sharif, Vajpayee visited Lahore by bus. His visit to Pakistan is known as bus diplomacy. It drew much attention and at end of the summit they issued Lahore Declaration that was backed by Memorandum of Understanding (MOU) [76].

3.22 Kargil conflict in 1999

In May 1999, Pakistan –backed militants together with Pakistani regular forces crossed the LOC and infiltrated into India-held Kargil region of North Kashmir. The militants occupied covertly more than thirty well-fortified positions the most inhospitable frigidly cold ridges at an altitude of 16000 -18000 feet, in the Great Himalayan range facing Dras, Kargil, Batalik and the Mushko Valley sectors. India retaliated by launching air attacks known as 'Operation Vijay (victory) on 26 May 1999. India identified Pakistan as an aggressor that violated the LOC. As the battle turned more intense, the Clinton administration intervened to help defuse the conflict. It was witnessed that on 15 June Clinton made separate telephonic conversations with both the Prime Ministers of India and Pakistan. He asked Sharif to withdraw infiltrators from across the LOC. He cordially appreciated Vajpayee for his display of restraint in the conflict. Pakistan was isolated from world community regarding the Kargil-issue, only Saudi Arabia and United Emirates supported Pakistan. On 4 July 1999, Sharif and Clinton held a three-hour meeting and issued a joint communiqué in which Sharif agreed to withdraw the intruders. On 11 July 1999, in accordance with the agreement the infiltrators started retreating from Kargil as India set 16 July, 1999 as the dead line for the total withdrawal. On 12 July 1999, Sharif defended his 4 July agreement with Mr. Clinton and defended his Kargil policy that designed to draw the international attention of the international community to Kashmir issue. In the war [77], India lost more than 400 forces. 670 intruders and 30 Pakistani regular forces were also killed excluding the injured.

3.23 Agra summit (14-16 July 2001)

Agra Summit was held between the Indian Prime Minister Atal Behari Vajpayee and Pakistan's President Pervez Musharraf in Agra, from July 14 to 16, 2001. The summit began amid high hopes of resolving various disputes between the two countries including complex J&K issue. Both sides remained inflexible on the core issue of J&K and ultimately the bilateral summit failed to produce any formal agreement.

3.24 The threat of war between India and Pakistan and the role of Bush administration

On 13 December 2001, five militants attacked Indian national parliament house causing the deaths of 13 people including five terrorists. India held Pakistan responsible for the incidence. India immediately reacted by deploying a sizeable force along the LOC in J&K. Pakistan followed suit, until both nations had aligned a vast array of troops and weapons against one another. Armies of both countries frequently exchanged of artillery fires. The mobilization of troops sparked worldwide fears of a deadly military conflict between India and Pakistan.

In order to defuse the growing tensions Bush Administration took initiatives and succeeded in getting both sides to make conciliatory move. On 12 January 2002, in his address to his nation, Musharraf committed Pakistan's "political, diplomatic and moral" support to the struggle of people of Kashmir. He went on to criticize the Pakistani militant Islamic groups for i) creating violent activities, ii) aggravating internal instability, iii) harboring sectarianism in Pakistan politics iv) creating war like situation against India. He banned two militant groups, the Lasker-e-Toiba and Jais-e- Mohammed. In the following weeks, 2,000 activists of the banned militant groups were arrested in Pakistan. Musharraf regime closed down some of militants groups' offices and recruiting centers. India welcomed these measures cautiously and the tension was somewhat defused. On May 14, 2002, three militants disguised in Indian Army uniforms shot passengers indiscriminately on a public bus and then killed 40 people (mostly wives and children of army personnel) including eight bus passengers at Kaluchak of Jammu before militants were gunned down. India reacted by threatening to strike at the terrorist camps situated in POK and took tough stand declaring some measures: i) expelled the Ambassador of Pakistan to India, ii) withdrew her diplomatic personnel from Pakistan, iii) imposed ban on Pakistani commercial air flights from Indian air space, iv) mobilized 100000 Indian forces close to LOC.

On 22 May 2002, on his visit to the frontlines in J&K, Vajpayee called for a "decisive battle". Pakistani authority declared that it would defend Pakistani administered Kashmir by any cost. Musharraf mobilized 50,000 troops to the borders. On 27 May 2002, Musharraf [78] warned India by declaring, "if war is thrust upon us, we will respond with full might".

Even Pakistan threatened to use nuclear weapons against India. This threat drew pointed attention to the USA and UK. The British Foreign Secretary, Jack Straw and US Deputy of Secretary, Richard Armitage and Defense Secretary, Donald Ramsfeld rushed to both India and Pakistan in May-June 2002 in order to defuse tension. They were successful in their mission to defuse tension and succeeded in getting promise from Musharraf to stop cross-border infiltration into J&K. After witnessing a slowdown in infiltration, India tried to improve her relation with Pakistan by reestablishing diplomatic ties, recalling her naval ships to their Bombay base, and opening her airspace to Pakistani commercial flights.

3.25 Musharraf's proposals for J&K resolution

On 25 October 2004, while addressing an 'Iftar party', President Musharraf announced an important declaration regarding settlement of the J&K acceptable to Pakistan, India and people of J&K. He remarked that the solution would have to be met thee steps:

i) Both sides should identify the regions on both sides of LOC,
ii) Demilitarize these regions,
iii) Determine their status through independence or joint control or UN mandate.

He opinioned that Pakistanis demand for a plebiscite was impractical while India's offer for making LOC a permanent border was unacceptable. The Musharraf's announcements drew much attention but Indian Prime Minister M. Singh refused to comment describing them as "of the cut remarks".

President Musharraf [79] proposed four point solutions regarding the resolution of J&K disputes as follows:

i) troops will be withdrawn from the region in a staggered manner
ii) the border will remain unchanged, however people will be allowed to move freely in the region
iii) self governance or autonomy but not independence
iv) a joint management mechanism will be created with India, Pakistan and Kashmiri Representatives

On 5 December 2006, during an interview with NDTV President Musharraf opinioned that Pakistan is prepared to give up her claim on Kashmir, also ready to give up her old demand for a plebiscite and forget all UN resolution if India accepts the four-point resolution of Kashmir dispute offered by him. He remarked that Pakistan is absolutely against the independence of Kashmir so is India. He stated that for compromise solution both countries would have to give up their positions and step back.

On 31 December 2006, G. N. Azad, the Chief Minister of J&K stated that 'joint mechanism" is possible in the field of trade, water, tourism and culture, and this could lead the way for a resolution to the longstanding Kashmir problem.

On 8 January 2007, he further stated that the latest four point-solution offered by President Musharraf should not be put aside without discussing positively.

On 19 January 2007, following the meeting with Indian External Affairs Minister, Chief Minister, G. N. Azad of J&K said "The date for the composite dialogue has been fixed for 13-14 March 2007 and I am sure all outstanding issues and proposals floated from time to time by President Musharraf will be discussed." On the same day APHC leader Mirwaiz Umar Farooq told the BBC that the next three months would be crucial.

On 2 February 2007, President Musharraf said: "We cannot take people on board who believe in confrontation and who think that only militancy solves the problem". On 3 February 2007, Indian Prime Minister M. Singh welcomed President Musharraf's statement that militancy or violence cannot resolve the Kashmir issue. On 4 February 2007, Indian External Affairs Minister, Pranab Mukherjee commented on Musharraf's proposals: "It is good. Everybody should advise terrorists to give up violence and join the process of dialogue." The idea of four point resolution was purely personal that did not have the mandate. However, Musharraf had to resign from his post for internal problem. His endeavor to resolve the J&K problem failed due to no response from India.

3.26 Terror attack at Mumbai

On 26 November 2008, Mumbai, the capital of Maharastra and financial capital of India witnessed deadly terror attacks. India adopted a tougher-than –usual stand against Pakistan in the wake of the Mumbai terror attack and demanded to hand over 20 terrorists including Pakistan-based underworld Dawood Ibrahim and Jaish-e-Mohammad chief Maulana Masood Azhar staying in Pakistan. . To defuse the tension between India and Pakistan, Secretary of State of the USA, Condoleeza Rice visited Indian subcontinent. Ultimately USA succeeded in defusing the tension.

3.27 Is China a third party in J&K conflict?

Indian stand on the question is contradictory, ambiguous and unclear and neutrosophic in nature. India strongly objected the border agreement between Pakistan and China signed in 1963 by which China gained 2700 square miles of the Pakistan occupied Kashmir. Based on the Simla Agreement 1972 between India and Pakistan, India strongly is of the opinion that J&K problem is a bilateral dispute. Third party involvement is not welcomed by India to resolve the issue. However, Pakistan wants that China would play a definite role to resolve the J&K conflict. China adopts a neutral role as seen in the Kargil conflict in 1999. So, depending upon Simla Agreement, Indian point of view and present status, J&K conflict is considered as a bilateral problem between India and Pakistan.

3.28 Internal development in Indian administered J&K

Special status was conferred on J&K under Article 370 of Indian Constitution [80]. Constituent Assembly was elected by J&K on 31 October, 1951. The accession of J&K to India was ratified by the State's Constituent Assembly in 1954. The Constituent Assembly also ratified the Maharaja's accession of 1947 in 17 November 1956. In 1956, the category of part B was abolished and J&K was included as one of the States of India under article 1. On January 26, 1957 Constituent Assembly dissolves itself. On 30 March 1965, article 249 of Indian Constitution extended to J&K whereby the Central Government at New Delhi could legislate on any matter enumerated in state list. Designation like Prime Minister and Sarder-i-Riyasat are replaced by Chief Minster and Governor respectively. In 1964, decision to extend Articles 356 and 357 of the Union Constitution of India to J&K announced. On 12 February 1975, Chief Minister Abdullah signed Kashmir Accord that affirmed its status as a constituent unit of the India and the State J&K will be governed by Article 370 of the Constitution of India.

3.29 Development of Pakistan administered Jammu and Kashmir

Azad Kashmir was established on 24 October 1947. The UNCIP resolution depicts the status of Azad Kashmir as neither a sovereign state nor a province of Pakistan, but rather a "local authority" with responsibility over the area assigned to it under the ceasefire agreement. The "local authority" or the provisional government of Azad Kashmir had handed over matters related to defense, foreign affairs, negotiations with the UNCIP and coordination of all affairs relating to Gilgit and Baltistan to Pakistan under the Karachi Agreement [81] of April 28, 1949.

Zulfiqar Ali Bhutto’s government virtually annexed the POK by promulgating the Azad Jammu and Kashmir Act of 1974. Azad Kashmir adopts Islam as the state religion vide Article 3. The constitution prohibits activities prejudicial or detrimental to the ideology of the State's accession to Pakistan (Article 7). It disqualifies non-Muslims from election to the Presidency and prescribed in the oath of office the pledge 'to remain loyal to the country and the cause of accession of the State of Jammu and Kashmir to Pakistan'. It provides two executive forums, namely the Azad Kashmir government in Muzaffarabad and the Azad Kashmir council in Islamabad. The Pakistan government can dismiss any elected government in Azad Kasmir irrespective of the support it may enjoy in the Azad Kasmir Legislative Assembly [82] by applying the Section 56 of the constitution,. The Northern Areas have the status of a Federally Administered Area.

3.30 The Indo-Pak conflict over Jammu and Kashmir

The conflict was based on the neutrosophic explanation and understanding of the neutrosophic situation in India and Pakistan. From Pakistani point of view, she hoped J&K was going to accede to Pakistan as the majority of the population being Muslims. If Junagadh, despite its Muslim rulers’ accession to Pakistan, belonged to India because of its Hindu majority, then Kashmir surely belonged to Pakistan. Princely state Hydrabad became Independent on 15 August 1947 like J&K. But India invaded it because of Hindu majority. Pakistan regarded the Accession of J&K as a coerced attempt to force the hand of Maharaja. Popular outbursts took place but J&K had acceded to India, because the ruler was a Hindu. From Indian point of view, J&K had acceded to her. Armed Pakistani raiders having Pakistani complicity and support invaded some portion of J&K. Both the countries failed to implement the UN resolutions. The plebiscite has never been held. India viewed that that the time has changed. India strongly argues that legislative measures subsequently legitimized the question of accession. After Simla Agreement, J&K problem became bilateral issue and its solution should be based on Simla Agreement 1972.

4 Neutrosophic game theoretic model formulation of the Indo-Pak conflict over Jammu and Kashmir

Following the above discussion and based on Simla Agreement 1972 , game theoretic model is formulated.
The problem is modeled as a standard two person (2×2) zero-sum game.

Table 1: Payoff matrix

		Pakistan I	Pakistan II
India	I	1	-1
	II	0	-1
	III	0	-1
	IV	1	0

Pakistan strategy vector:
i) Full compliance with Simla Agreement 1972,
ii) Partial or non-compliance with Simla agreement 1972
India’s strategy vector:
i) Make territorial concessions,
ii) Accept the third party mediation (USA),
iii) Apply the strategy of all-out military campaign,
iv) Continue fencing along the LOC (see Fig. 1).

Fig.1 Photograph of fence along LOC

The above payoff matrix has been constructed with reference to the row player i.e. India. In the process of formulating the payoff matrix, it is assumed that the combination (I, I) will hopefully resolve the conflict while the combination (IV, II) will basically imply a status quo with continuing conflict. If Pakistan can get India to either make territorial concessions (Muslims dominated Kashmir valley or other important stragic areas of J&K such Kargil) or accept the third party mediation like USA without fully complying with the Simla Agreement i.e. strategy combinations (I, II) and (II, II), then it reflects that Pakistan will be benefited but India will be loser. If India accepts the third party mediation and Pakistan agrees to comply fully with the Simla Agreement, then though it potentially ends the conflict, there should be a political jeopardy in India as a result of lack of strategic and political concensus among the political parties and so the strategy combination (II, I) is not a favorable payoff for India. If India employs an all-out military campaign, an devastating war seems to occur as both the countries are capable of using nuclear powers i.e. strategy combination (III, I) would not produce a positive payoff for either side. If there occurs a unexpected and sudden change of mind set up for J&K within the Pakistani leaderships (from inside or outside pressure) and Pakistan chooses to fully abide by the Simla Agreement 1972 considering LOC as the permanent international border i.e. strategy combination (IV, I) will bring a potential end to the conflict as both countries may think that they will be loser and winner at the same time (neutrosophically true) in the sense that they will not get the whole J&K but Pakistan can cansole her saying that she has gained one third of J&K while India may think she gained two third of J&K.

In the payoff matrix (see Table 1), all the elements of the first, second and third rows are less than or equal to the corresponding elements of the forth row, therefore from the game theory [83] point of view, forth row dominates the first three rows. On the other hand, every element of the first column is greater than the corresponding elements of the second column, therefore, first column is dominated by second column. It shows that the above game has a saddle point having the strategy combination (IV, II), which reflects that in their very attempt to out-bargain each other both countries actually end up continuing the conflict indefinitely! Thus the model model offers an equilibrium solution.

In the subcontinent political arena, Pakistani leadership's best interest was to transform the conflict more complex and keep the conflict more alive with full strenght to gain political support from inside and outside and ultimately compelled India to make territorial or other concessions. However, for international pressure mainly from USA, Pakistan had to state some overt declaration that negotiated settlement over J&K based on Simla Agreement 1972 is possible. However, Pakistan covertly continues her sincere help to separatist groups by means of monetary, logical, psychological and military equipments. By doing so Pakistan is now in deep troubal with various militant groups and Jihadi groups. She has to deal internal security probles caused by Pakistani Taliban groups. Under such volatile circumstances, it would be quite impossible for Pakistan to chalk out a distinct governing strategy to meet with counter strategy.

Both the countries, in general, played their games under international pressure. Although Pakistan signed Lahore declaration with India by the then Prime Minister Nawaj Sarif, Pakistani military boss Mr. Musharraf occupied some heights of Kargil in 1999 to derail the peace process and draw international attention to the J&K conflict. An important lesson of the Kargil conflict seems to be that no military expedition could be a success if it is pursued without paying to serious attention to the totality of the scenarios having domestic, political, economic, geographical, international opinions and sensitivities. Another important of Kargil conflict seems that national community does not want to military solution relating to J&K problem. However, one positive aspect of the Lahore declaration reflects that both the countries are capable of transforming the game scenario an open one in the sense that the conflicting countries are capable of dynamically constructing and formulating objectives and strategies in the course of their peaceful, mutual interaction within a formally defined socio- political set up.

During the Agra summit in 2001, when probably President Musharraf was thinking to make the conflict very alive while offering the impression to the other side (India) that they were wholeheartedly seeking strategies to put an end the conflict. President Musharraf played very clever and diplomatic role by using the media very cautiously and cleverly to make the international community and his country understand that he tries his level best to reach a meaningful, desicive and effective agreement but fails due to Indian rigid position regarding J&K issue. He left Agra and thereby tried to obtain his acceptance to his own nation and international community. This immediately shows why such a negotiation would break down at early stage.

The government of India and Pakistan are dealing with militancy and terrorism in own land. But main issue remains the J&K conflict.

4.1 Case for applying neutrosophic game theory

It is experienced that none of the strategy vectors available to either side will remain temporarily stationary because crucial events come into light on the global political arena, in general, and the south Asian subcontinent in particular. Moreover, there is a broad variety of ambiguities about the motives behind Pakistan authority's primary goals about the driving force behind Pakistan authority's primary goal and the strategies it adopts to achieve that goal. Pakistan's principal ally USA is also a great factor. The influence of USA has a great impact on forming strategies. The terrorist activities by Pakistan baked terrorist groups are sometimes monitored by the wishes of USA. Although Pakistan has not hundred percent control over foreign mercenaries coming from different part of world namely, Saudi Arabia, Afghanistan, Chechnia, Sudan etc. Pakistan is constantly trying to bring India under pressure by harboring terrorist attacks on Indian ruled Kashmir and destabilizing the normalcy in the state in order to understand the international community that international intervention is requierd to resolve the J&K conflict. It is also difficult to state apart a true bargaining strategy from one just meant to be a political decay. In the horizon of continuous conflict, we believe and advocate an application of the conceptual framework of the neutrosophic game theory as a generalization of the dynamic fuzzy game paradigm.

It generalized terms, a well-specified dynamic game at a particular time t is a particular interaction between decision makers with well-defined rules and regulations and roles for the decision makers, which remain in place at time t but are free to change over time. However, it is likely that the decision makers may suffer from the role of ambiguity i.e. a typical situation where none of the decision makers are exactly sure what to expect from others or what the other decision makers expect from them. In the context of indo-Pak continuing conflict, for example, Pakistan leadership would probably not have been sure of its role when Mr. Musharraf met with Indian prime minister Atal Behari Vajpayee at the Agra summit to chalk out a peace agreement. Mr. Musharuff went to that summit under the international pressure or to prove himself to be an efficient leader of Pakistan or against his free will and would have liked to avoid Agra if he could because he did not want to sign any final agreement on J&K which can be against common feeling of people of Pakistan. Mushauff demanded for declaration of J&K to be a disputed territory at least. Having no such capitulation forthcoming from India, Musharuff left Agra without signing any joint statement.

In this political context, Pakistani leadership's best interest was to keep the conflict alive with full strength ultimately compelled India to make territorial or other concession. However, for international pressure mainly from USA, Pakistan had to offer some overt declaration that negotiated settlement over Jammu and Kashmir based on Simla Agreement 1972 is possible. Pakistan covertly continues her sincere help separatist groups by means of monetary, logical, psychological and military equipments. Under such volatile circumstances, it would be quite impossible to chalk out a distinct governing strategy to meet with counter strategy.

However, both the countries, in general, played their games under international pressure. Although Pakistan signed Lahore declaration with India by the then Prime Minister Nawaj Sarif, Pakistani military boss Mr. Musharraf occupied some heights of Kargil in 1999 to derail the peace process. An important lesson of the Kargil conflict seems to be that no military expedition could be a success if it is pursued without paying to serious attention to the totality of the scenario having domestic, political, economic, geographical, international opinions and sensitivities. Another important of Kargil conflict seems that national community does not want to military solution relating to Kashmir problem. However, one positive aspect about Lahore declaration or Indo-Pak joint declaration indicates they transform the game scenario an open one in the sense that the conflicting parties are capable of dynamically constructing and formulating objectives and strategies in the course of their peaceful, mutual interaction within a formally defined socio-political set up.

Thus, the negotiation space may be represented as:

$N_{Pakistan} \cap N_{India}$.

According to the opinion of Burns and Rowzkowska [84] each players personal views constitute a deal. The fuzzy semantic space comprises of such deals i.e. personal views. Personal value judgments, acquired experiences and expectations about the possible best or worst outcomes from a negotiation are crucial to constitute such views. The fuzzy semantic space is a dynamical system and is free to modify according to the need and desire of the players and practical situations in the light of new information.

The semantic space however remains fuzzy owing to vagueness about the exact objectives and lack of precise understanding of the exact stand, which the opponent parties have from their viewpoints. That is why none of the conflicting parties can effectively read and precisely understand each other's nature of expectations.

This was reflected in Agra summit when probably Musharraf (Pakistan) was thinking in terms of keeping the conflict alive while offering the impression to the other side (India) that they were wholeheartedly seeking strategies to put an end the conflict. Mr. Musharraf played very diplomatic role by using the media very cautiously and cleverly to make the international community and his country understand he tried his level best to reach a fruitful agreement but failed and left at midnight and thereby tried to obtain his acceptance to his own nation and international community. This immediately comes to light why such a negotiation would break down at early stage.

If the Indo-Pak conflict over Jammu and Kashmir is constituted as fuzzy dynamic fuzzy bargaining game, the players' fuzzy set judgment functions over expected outcomes can be formulated as follows according to the rules of well-developed fuzzy set theory due to Zadeh [85]

For Pakistan, the fuzzy evaluative judgment function at time t, $\mu_{J(P,t)}$ (P, t) will be represented by fuzzy set membership function as follows:

$$\mu_{J(P,t)} = \left\langle \begin{array}{ll} \in (0.5,1), & \text{for } O_{Worst} < x < O_{Best} \\ 0.5, & \text{for } x = O_{Worst} \\ 0, & \text{for } x < O_{Worst} \end{array} \right\rangle$$

Here, the symbol O_{Best} indicates the best possible outcome that Pakistan would expect; which would probably the annexation of Jammu-Kashmir to Pakistan according to the two-nation theory of Muslim League. Similarly, O_{Worst} indicates probably the conversion of LOC as the permanent international borderline.

On the other hand, for India the fuzzy evaluative judgment function at time t, J (I, t) will be represented by the fuzzy set membership function $\mu_{J(I,t)}$ as follows:

$$\mu_{J(I,t)} = \left\langle \begin{array}{ll} 1, & \text{for } y \geq \Theta_{Best} \\ \in (0.5,1), & \text{for } \Theta_{Worst} < y < \Theta_{Best} \\ 0.5, & \text{for } y = \Theta_{Worst} \\ 0, & \text{for } y \leq \Theta_{Worst} \end{array} \right\rangle$$

Here Θ_{Worst} indicates the worst possible negotiation outcome India could expect, which would be coincidence with the best-expected outcome for Pakistan.

It is to be noted that semantic space $N_{Pakistan} \cap N_{India}$ is more generally framed as a neutrosophic semantic space, which considers a three-way generalization of the fuzzy semantic space. Since neutrosophic semantic space comprises of neutral possibility, it cannot be defuzzified into two crisp zero-one states owing to the incorporation of an intervening state of "indeterminacy". Such indeterminacy would be practically encountered due to the fact any mediated, two-way negotiation process is likely to become over catalyst by the subjective utility preferences of the mediator.

Let T, I, F represent real subsets of the real standard unit interval [0,1]. Statically, T, I, F are subsets while dynamically, in the context of a dynamic game, they may be considered as set-valued vector functions. If a logical proposition is said to be t % true in T, i % indeterminate in I and f % false in F, then T, I, F are considered as the neutrosophic components. According to Smarandache [7] neutrosophic probability is useful to events that are shrouded in a veil of indeterminacy like the actual implied volatile of long-term options. Bhattacharya et al.y applied the concept of neutrosophic probability in order to formulate neutrosophic game theretic model [86] to Israel –Palestine conflict. It is worthy of mention that the proposed approach uses a subset-approximation for truth-values, indeterminacy and falsity-values. It is capable of providing a better approximation than classical probability to uncertain events.

Therefore, the neutrosophic evaluative function for Pakistan at time t, $J_N(P, t)$ will be represented by the neutrosophic set membership function $\mu_{J_N(P,t)}(x)$ as follows:

$$\mu_{J_N(P,t)}(x) = \left\langle \begin{array}{l} (.5,1), \text{for } O_{Worst} < x < O_{Best}, x \in T \\ .5, \text{for } x = O_{Worst}, x \in T \\ 0, \text{ for } x < O_{Worst}, x \in T \end{array} \right\rangle$$

On the other hand, the neutrosophic evaluative judgment function for India at time t, $J_N(I, t)$ will be represented by the neutrosophic set membership function $\mu_{J_N(I,t)}(y)$ as follows:

$$\mu_{J_N(I,t)}(y) = \left\langle \begin{array}{ll} 1, & \text{for } y \geq \Theta_{Best}, y \in F \\ \in (0.5,1), & \text{for } \Theta_{Worst} < y < \Theta_{Best}, y \in F \\ 0.5, & \text{for } y = \Theta_{Worst}, y \in F \\ 0, & \text{for } y \leq \Theta_{Worst}, y \in F \end{array} \right\rangle$$

Relying on three-way classification of neutrosophic semantic space, it is t %true in sub-space that bilateral negotiation will produce a favourable outcome within the evaluative judgment space of the Pakistan while it is f % false in the sub-space F that the outcome will be favorable within the evaluative judgment space of the Pakistan. However, there is an i % indeterminacy in sub-space I whereby the nature of the outcome may be neither favorable nor unfavorable within the evaluative judgment space of either competitor.

5 Conclusion

We have discussed the crisis dynamics of the continuing Indo-Pak conflict over J&K. we have briefly examined the efforts made by various study groups and persons and India and Pakistan in resolving the conflict and the reasons for their failure. We have looked the Indo-Pak relations and recent developments and their views on J&K situations. Alternative possible solutions are also considered. Most of the solutions are either impractical or unacceptable to India, Pakistan, and or the various militant groups. Pro Pakistani militant groups and Pakistan would opt for free and impartial plebiscite. Even some militant groups would oppose the plebiscite because the UNSC resolutions do not offer them the very option of independence. In the process of formulating the payoff matrix for game theory model, the combination (I, I) will hopefully resolve the conflict i.e. maintaining the status quo along the LOC with some border adjustments favorable to Pakistan. Otherwise, the the proposed model offers the solution wich state that both the countries will continue the conflict indefinitely. The application of game theoretic method to the ongoing Indo-Pak conflict over J&K is based on identifying and evaluating the best options that each side has and is trying to achieve chosen options. The Simla Agreement 1972 is used as an instance with Pakistan being left to choose between two mutually exclusive options.

The solution reflects the real facts that Pakistan does not want to ever agree to have full compliance with the Simla Agreement 1972, as she will see always herself worse off that way. Realizing that Pakistan will never actually comply with Simla Agreement 1972, India will find her best interest to continue the status quo with an ongoing conflict, as she will see herself ending up on the worse end of the bargaining if she chooses to apply any other strategy. It is experienced that none of the strategy vectors available to either side will remain temporally stationary due to action and reaction of militant groups and Indain securty forces in J&K. Moreover, there exists a broad variety of ambiguities concerning primary goals of the two countries and the strategies they adopt to achieve those goals.

Due to the impact of globalization, people have to interact with people from other countries. In the days of cross-border strategic alliances and emphasis on various groups, it would really be tragic if other nations remain standstill or ignore the conflict. The government of India fails to solve its own problems of northeast states such as Nagaland, Assam, Mizoram, and Arunachal Pradesh. On the other hand, Pakistan is constantly facing with the myriad problems of democracy and absence of it. Pakistan fails to curve insurgencies in Balochistan, the largest province of Pakistan having resource of natural gas and mineral and sparsely populated. We have discussed the development of the Jammu and Kashmir conflict. We have examined the various efforts by Indo-Pak and other countries to resolve the grave conflict. The effect of nuclear power acquisition by both the countries,9/11 terror attack on USA and USA and its allies invasion in Iraq. The most beneficiary party of Jammu and Kashmir conflict is the republic of China. It invaded about 38000 square kilometers territory from Indian ruled Jammu and Kashmir in 1962 war and later Pakistan ceded 5180 square kilometers areas to Beijing under a 1963 pact. The ongoing conflict between and Pakistan reflects the fact that both the countries are incapable of solving J&K conflict and other core issues. At the international level, third party arrangements can be established with the participation of an intervening state, group of states, or international organizations in order to play a crucial role in helping to overcome a strategic impasse. The question remains whether India and Pakistan would accept third party mediation. Limits on international direct intervention in the J&K grave conflict have historically had to do with India's insistence that the 1972 Simla Agreement between India and Pakistan is sufficient to deal with the issue bilaterally. It is to noted that when well conceived and pragmatic, India has come to appreciate mediation by international third party specially USA mediation in Kargil crisis in 1999 and 2001-2002 mobilization crisis.

$\mu_{J_N(P,t)}(x)$ or $\mu_{J_N(I,t)}(y)$ would be interpreted as Pakistan's (or India's) degree of satisfaction with the negotiation based settlement. It is likely that Pakistan authority's ultimate objective is to annex J&K and if that is the case then of course there will always be an unbridgeable incongruence between $\mu_{J_N(P,t)}(x)$ and $\mu_{J_N(I,t)}(y)$ due to mutually inconsistent evaluative judgment spaces between India and Pakistan to the conflict. Therefore, for any form of negotiation in order to produce a positive result, the first and foremost requirement should be to make the evaluative judgment spaces consistent. On the other hand, if the evaluative judgment spaces are inconsistent, the negotiation space will generate into a null set at the very onset of the bargaining process, thereby pre-empting any equilibrium solution different from the status quo. Since these spaces are not crisp, according to Reiter [87] they are malleable to some extent. Therefore, even while retaining their core forms, subtle changes could be induced for making these spaces workably consistent. Then the goal of the mediator should be such that it will allow India and Pakistan to redefine their primary objectives without necessarily feeling that such redefinition itself reflects a concession. When this type of redefinition of primary objectives has been achieved, the evaluative judgment spaces generate a negotiation space that will not become null ab initio. However, it should be mentioned that there exists also an indeterminate aspect to any process of mediated bilateral dialogues between the two countries becasue of the catalyzation effect brought about the subjective utility preferences of the mediator.

We build on an earlier attempted justification of a game theoretic explanation of the Indo-Pak conflict over J&K by Pramanik and Roy [6] and go on to argue in favor of a neutrosophic adaptation of the standard 2x2 zero-sum game theoretic model in order to identify an optimal outcome. We hope that the concept of the interval neutrosophic set [88] can be used to formulate interval neutrosophic game theoretic model of the Indo-Pak conflict in more general way. From ancient period J&K was a place of religious center and different people with their different faiths live together peacefully. But, poeple of J&K are at present neutrosophically divided in the question of independent of J&K. The disintegration of India based on two nation theory does not provide any good for the people of Indian subcontinent. So people of Indian subcontinent can neutrosophically hope that India and Pakisatn will rethink their decision of partition based on vaguely defined two nation theory. Rather India and Pakistancan form a union of indepent states by considering their origin, culteral heristage, common interest, blood relation. If East Germany and West Germany are able to get united, why not the subcontinent? According to neutrosophy nothing is impossible. So according to neutrosophy, the resolution of J&K is neutrosophically possible. The present paper provides the conceptual framework of neutrosophic game theroetic model of the complex J&K conflict hoping that neotrosophic linear programming will be able to solve the problem in near future.

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Entropy Based Grey Relational Analysis Method for Multi-Attribute Decision Making under Single Valued Neutrosophic Assessments

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Abstract. In this paper we investigate multi-attribute decision making problem with single-valued neutrosophic attribute values. Crisp values are inadequate to model real life situation due to imprecise information frequently used in decision making process. Neutrosophic set is one such tool that can handle these situations. The rating of all alternatives is expressed with single-valued neutrosophic set which is characterised by truth-membership degree, indeterminacy-membership degree, and falsity-membership degree. Weight of each attribute is completely unknown to decision maker. We extend the grey relational analysis method to neutrosophic environment and apply it to multi-attribute decision making problem. Information entropy method is used to determine the unknown attribute weights. Neutrosophic grey relational coefficient is determined by using Hamming distance between each alternative to ideal neutrosophic estimates reliability solution and the ideal neutrosophic estimates un-reliability solution. Then neutrosophic relational degree is defined to determine the ranking order of all alternatives. Finally, an example is provided to illustrate the application of the proposed method.

Keywords: Neutrosophic set; Single-valued neutrosophic set; Grey relational analysis; Information Entropy; Multi-attribute decision making.

1 Introduction

Multiple attribute decision making (MADM) problems in the area of operation research, management science, economics, systemic optimization, urban planning and many other fields have achieved very much attention to the researchers during the last several decades. It is often used to solve various decision making and/or selection problems. These problems generally consist of choosing the most desirable alternative that has the highest degree of satisfaction from a set of alternatives with respect to their attributes. In this approach, the decision makers have to provide qualitative and/ or quantitative assessments for determining the performance of each alternative with respect to each attribute, and the relative importance of evaluation attribute.

In classical MADM methods, such as TOPSIS (Hwang & Yoon [1]), PROMETHEE (Brans et al. [2]), VIKOR (Opricovic [3-4]), ELECTRE (Roy [5]) the weight of each attributes and rating of each alternative are naturally considered with crisp numbers. However, in real complex situation, decision maker may prefer to evaluate the attributes by using linguistic variables rather than exact values due to his time pressure, lack of knowledge and lack of information processing capabilities about the problem domain. In such situations, the preference information of alternatives provided by the decision maker may be vague, imprecise or incomplete. Fuzzy set (Zadeh [6]) is one of such tool that utilizes this impreciseness in a mathematical form. MADM with imprecise information can be modelled quite well by using fuzzy set theory into the field of decision making.

Bellman and Zadeh [7] first investigated decision making problem in fuzzy environment. Chen [8] extended one of known classical MADM method, technique for order preference by similarity to ideal solution (TOPSIS). He developed a methodology for solving multi-criteria decision making problems in fuzzy environment. Zeng [9] solved fuzzy MADM problem with known attribute weight by using expected value operator of fuzzy variables. However, fuzzy set can only focus on the membership grade of vague parameters or events. It fails to handle non-membership degree and indeterminacy degree of imprecise parameters.

Atanassov [10] introduced intuitionistic fuzzy set (IFS). It is characterized by the membership degree, non-membership degree simultaneously. Impreciseness of the objectives can be well expressed by using IFS than fuzzy sets (Atanassov [11]). Therefore it has gained more and more attention to the researchers. Boran et.al [12] extended the TOPSIS method for multi-criteria intuitionistic decision making problem. Z.S. Xu[13] studied fuzzy multiple attribute decision making problems, in which all attribute values are given as intuitionistic fuzzy numbers and the preference information on alternatives can be provided by the decision maker. Z. Xu [14] proposed a solving method for MADM problem with interval-valued intuitionistic fuzzy decision making by using distance measure.

In IFSs, sum of membership degree and non-membership degree of a vague parameter is less than unity. Therefore, a certain amount of incomplete information or indeterminacy arises in an intuitionistic fuzzy set. It cannot handle all types of uncertainties successfully in different real physical problems such as problems involving incomplete information. Hence further generalizations of fuzzy as well as intuitionistic fuzzy sets are required.

Florentin Smarandache [15] introduced neutrosophic set (NS) and neutrosophic logic. It is actually generalization of different type of FSs and IFSs. The term "neutrosophy" means "knowledge of neutral thought". This "neutral" concept makes the differences between NSs and other sets like FSs, IFSs. Wang et al. [16] proposed single-valued neutrosophic set (SVNS) which is a sub-class of NSs. SVNS is characterized by truth membership degree (T), indeterminacy membership degree (I) and falsity membership degree (F) that are independent to each other. This is the key characteristic of NSs other than IFSs or fuzzy sets.

Such formulation is helpful for modelling MADM with neutrosophic set information for the most general ambiguity cases, including paradox. The assessment of attribute values by the decision maker takes the form of single-valued neutrosophic set. Ye [17] studied multi-criteria decision making problem under SVNS environment. He proposed a method for ranking of alternatives by using weighted correlation coefficient. Ye [18] also discussed single-valued neutrosophic cross entropy for multi-criteria decision making problems. He used similarity measure for interval valued neutrosophic set for solving multi-criteria decision making problems. Grey relational analysis (GRA) is widely used for MADM problems. Deng [19-20] developed the GRA method that is applied in various areas, such as economics, marketing, personal selection and agriculture. Zhang et al. [21] discussed GRA method for multi attribute decision making with interval numbers. An improved GRA method proposed by Rao & Singh [22] is applied for making a decision in manufacturing situations. Wei [23] studied the GRA method for intuitionistic fuzzy multi-criteria decision making. Therefore, it is necessary to pay attention to this issue for neutrosophic environment.

The aim of this paper is to extend the concept of GRA to develop a methodology for solving MADM problems with single valued neutrosophic set information. The information taken from expert's opinion about attribute values takes the form of single valued neutrosophic set. It is assumed that the information about attribute weights is completely unknown to decision maker. Entropy method is used for determining the unknown attribute weights. In this modified GRA method, the ideal neutrosophic estimates reliability solution and the ideal neutrosophic estimate un-reliability solution has been developed. Neutrosophic grey relational coefficient of each alternative is determined to rank the alternatives.

In order to do so, the remaining of this paper is organized as follows: Section 2 briefly introduce some preliminaries relating to neutrosophic set and the basics of single-valued neutrosophic set. In Section 3, Hamming distance between two single-valued neutrosophic sets is defined. Section 4 represents the model of MADM with SVNSs and discussion about modified GRA method to solve MADM problems. In section 5, an illustrative example is provided to show the effectiveness of the proposed model. Finally, section 6 presents the concluding remarks.

2 Preliminaries of Neutrosophic sets and Single valued neutrosophic set

Neutrosophic set is a part of neutrosophy, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra (Smarandache [15]), and is a powerful general formal framework, which generalizes the above mentioned sets from philosophical point of view. Smarandache [15] gave the following definition of a neutrosophic set.

2.1 Definition of neutrosophic set

Definition 1 Let X be a space of points (objects) with generic element in X denoted by x. Then a neutrosophic set A in X is characterized by a truth membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A. The functions T_A, I_A and F_A are real standard or non-standard subsets of $]0^-, 1^+[$ that is

$T_A : X \to]0^-, 1^+[$; $I_A : X \to]0^-, 1^+[$; $F_A : X \to]0^-, 1^+[$

It should be noted that there is no restriction on the sum of $T_A(x)$, $I_A(x)$, $F_A(x)$ i.e. $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$

Definition 2 The complement of a neutrosophic set A is denoted by A^c and is defined by

$$T_{A^c}(x) = \{1^+\} - T_A(x)\,;\; I_{A^c}(x) = \{1^+\} - I_A(x)\,;$$

$$F_{A^c}(x) = \{1^+\} - F_A(x)$$

Definition 3 (Containment) A neutrosophic set A is contained in the other neutrosophic set B, $A \subseteq B$ if and only if the following result holds.

$$\inf T_A(x) \leq \inf T_B(x)\,,\; \sup T_A(x) \leq \sup T_B(x) \qquad (1)$$

$$\inf I_A(x) \geq \inf I_B(x)\,,\; \sup I_A(x) \geq \sup I_B(x) \qquad (2)$$

$$\inf F_A(x) \geq \inf F_B(x)\,,\; \sup F_A(x) \geq \sup F_B(x) \qquad (3)$$

for all x in *X*.

2.2 Some basics of single valued neutrosophic sets (SVNSs)

In this section we provide some definitions, operations and properties about single valued neutrosophic sets due to Wang et al. [16]. It will be required to develop the rest of the paper.

Definition 4 (Single-valued neutrosophic set). Let *X* be a universal space of points (objects), with a generic element of X denoted by x. A single-valued neutrosophic set $\tilde{\mathcal{N}} \subset X$ is characterized by a true membership function $T_{\tilde{\mathcal{N}}}(x)$, a falsity membership function $F_{\tilde{\mathcal{N}}}(x)$ and an indeterminacy function $I_{\tilde{\mathcal{N}}}(x)$ with $T_{\tilde{\mathcal{N}}}(x)$, $I_{\tilde{\mathcal{N}}}(x)$, $F_{\tilde{\mathcal{N}}}(x) \in [0, 1]$ for all x in *X*.

When X is continuous a SVNSs $\tilde{\mathcal{N}}$ can be written as

$$\tilde{\mathcal{N}} = \int_x \left\langle T_{\tilde{\mathcal{N}}}(x), I_{\tilde{\mathcal{N}}}(x), F_{\tilde{\mathcal{N}}}(x)\right\rangle \Big/ x, \quad \forall x \in X.$$

and when X is discrete a SVNSs $\tilde{\mathcal{N}}$ can be written as

$$\tilde{\mathcal{N}} = \sum_{i=1}^{m} \left\langle T_{\tilde{\mathcal{N}}}(x), I_{\tilde{\mathcal{N}}}(x), F_{\tilde{\mathcal{N}}}(x)\right\rangle / x, \quad \forall x \in X.$$

Actually, SVNS is an instance of neutrosophic set which can be used in real life situations like decision making, scientific and engineering applications. In case of SVNS, the degree of the truth membership $T_{\tilde{\mathcal{N}}}(x)$, the indeterminacy membership $I_{\tilde{\mathcal{N}}}(x)$ and the falsity membership $F_{\tilde{\mathcal{N}}}(x)$ values belong to [0, 1] instead of non standard unit interval] 0^-, 1^+ [as in the case of ordinary neutrosophic sets.

It should be noted that for a SVNS $\tilde{\mathcal{N}}$,

$$0 \le \sup T_{\tilde{\mathcal{N}}}(x) + \sup I_{\tilde{\mathcal{N}}}(x) + \sup F_{\tilde{\mathcal{N}}}(x) \le 3, \forall x \in X. \quad (4)$$

and for a neutrosophic set, the following relation holds

$$0^- \le \sup T_{\tilde{\mathcal{N}}}(x) + \sup I_{\tilde{\mathcal{N}}}(x) + \sup F_{\tilde{\mathcal{N}}}(x) \le 3^+, \forall x \in X. \quad (5)$$

For example, suppose ten members of a political party will critically review their specific agenda. Five of them agree with this agenda, three of them disagree and rest of two members remain undecided. Then by neutrosophic notation it can be expressed as $x\langle 0.5, 0.2, 0.3\rangle$.

Definition 5 The complement of a neutrosophic set $\tilde{\mathcal{N}}$ is denoted by $\tilde{\mathcal{N}}^c$ and is defined by

$$T_{\tilde{\mathcal{N}}^c}(x) = F_{\tilde{\mathcal{N}}}(x); I_{\tilde{\mathcal{N}}^c}(x) = 1 - I_{\tilde{\mathcal{N}}}(x); F_{\tilde{\mathcal{N}}^c}(x) = T_{\tilde{\mathcal{N}}}(x)$$

Definition 6 A SVNS $\tilde{\mathcal{N}}_A$ is contained in the other SVNS $\tilde{\mathcal{N}}_B$, denoted as $\tilde{\mathcal{N}}_A \subseteq \tilde{\mathcal{N}}_B$, if and only if

$$T_{\tilde{\mathcal{N}}_A}(x) \le T_{\tilde{\mathcal{N}}_B}(x); I_{\tilde{\mathcal{N}}_A}(x) \ge I_{\tilde{\mathcal{N}}_B}(x); F_{\tilde{\mathcal{N}}_A}(x) \ge F_{\tilde{\mathcal{N}}_B}(x)$$

$\forall x \in X$.

Definition 7 Two single valued neutrosophic sets $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ are equal, i.e. $\tilde{\mathcal{N}}_A = \tilde{\mathcal{N}}_B$, if and only if $\tilde{\mathcal{N}}_A \subseteq \tilde{\mathcal{N}}_B$ and $\tilde{\mathcal{N}}_A \supseteq \tilde{\mathcal{N}}_B$.

Definition 8 (Union) The union of two SVNSs $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ is a SVNS $\tilde{\mathcal{N}}_C$, written as $\tilde{\mathcal{N}}_C = \tilde{\mathcal{N}}_A \cup \tilde{\mathcal{N}}_B$. Its truth membership, indeterminacy-membership and falsity membership functions are related to those of $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ by

$$T_{\tilde{\mathcal{N}}_C}(x) = \max(T_{\tilde{\mathcal{N}}_A}(x), T_{\tilde{\mathcal{N}}_B}(x));$$

$$I_{\tilde{\mathcal{N}}_C}(x) = \max(I_{\tilde{\mathcal{N}}_A}(x), I_{\tilde{\mathcal{N}}_B}(x));$$

$$F_{\tilde{\mathcal{N}}_C}(x) = \min(F_{\tilde{\mathcal{N}}_A}(x), F_{\tilde{\mathcal{N}}_B}(x)) \text{ for all x in X.}$$

Definition 9 (Intersection) The intersection of two SVNSs $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ is a SVNS $\tilde{\mathcal{N}}_C$, written as $\tilde{\mathcal{N}}_C = \tilde{\mathcal{N}}_A \cap \tilde{\mathcal{N}}_B$, whose truth membership, indeterminacy-membership and falsity membership functions are related to those of $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ by $T_{\tilde{\mathcal{N}}_C}(x) = \min(T_{\tilde{\mathcal{N}}_A}(x), T_{\tilde{\mathcal{N}}_B}(x))$;

$$I_{\tilde{\mathcal{N}}_C}(x) = \min(I_{\tilde{\mathcal{N}}_A}(x), I_{\tilde{\mathcal{N}}_B}(x));$$

$$F_{\tilde{\mathcal{N}}_C}(x) = \max(F_{\tilde{\mathcal{N}}_A}(x), F_{\tilde{\mathcal{N}}_B}(x)) \text{ for all x in X.}$$

3 Distance between two neutrosophic sets.

Similar to fuzzy or intuitionistic fuzzy set, the general SVNS having the following pattern

$\tilde{\mathcal{N}} = \{(x/(T_{\tilde{\mathcal{N}}}(x), I_{\tilde{\mathcal{N}}}(x), F_{\tilde{\mathcal{N}}}(x)) : x \in X\}$. For finite SVNSs can be represented by the ordered tetrads:

$$\begin{aligned}\tilde{\mathcal{N}} = \{&(x_1/(T_{\tilde{\mathcal{N}}}(x_1), I_{\tilde{\mathcal{N}}}(x_1), F_{\tilde{\mathcal{N}}}(x_1)),\\ &\ldots, x_m/(T_{\tilde{\mathcal{N}}}(x_m), I_{\tilde{\mathcal{N}}}(x_m), F_{\tilde{\mathcal{N}}}(x_m))\}\end{aligned}, \quad \forall x \in X$$

Definition 10 Let

$$\begin{aligned}\tilde{\mathcal{N}}_A = \{&(x_1/(T_{\tilde{\mathcal{N}}_A}(x_1), I_{\tilde{\mathcal{N}}_A}(x_1), F_{\tilde{\mathcal{N}}_A}(x_1)),\\ &\ldots, x_n/(T_{\tilde{\mathcal{N}}_A}(x_n), I_{\tilde{\mathcal{N}}_A}(x_n), F_{\tilde{\mathcal{N}}_A}(x_n))\}\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}_B = \{&(x_1/(T_{\tilde{\mathcal{N}}_B}(x_1), I_{\tilde{\mathcal{N}}_B}(x_1), F_{\tilde{\mathcal{N}}_B}(x_1)),\\ &\ldots, x_n/(T_{\tilde{\mathcal{N}}_B}(x_n), I_{\tilde{\mathcal{N}}_B}(x_n), F_{\tilde{\mathcal{N}}_B}(x_n))\}\end{aligned} \quad (6)$$

be two single-valued neutrosophic sets (SVNSs) in $X = \{x_1, x_2, \ldots, x_n\}$.

Then the Hamming distance between two SVNSs $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ is defined as follows:

$$d_{\tilde{\mathcal{N}}}\left(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B\right) = \sum_{i=1}^{n} \left\{ \begin{aligned} &\left|T_{\tilde{\mathcal{N}}_A}(x_i) - T_{\tilde{\mathcal{N}}_B}(x_i)\right| + \left|I_{\tilde{\mathcal{N}}_A}(x_i) - I_{\tilde{\mathcal{N}}_B}(x_i)\right| \\ &+ \left|F_{\tilde{\mathcal{N}}_A}(x_i) - F_{\tilde{\mathcal{N}}_B}(x_i)\right| \end{aligned} \right\} \quad (7)$$

and normalized Hamming distance between two SVNSs $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ is defined as follows:

$$^{N}d_{\tilde{\mathcal{N}}}\left(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B\right) =$$

$$\frac{1}{3n}\sum_{i=1}^{n}\left\{\left|T_{\tilde{\mathcal{N}}_A}(x_1) - T_{\tilde{\mathcal{N}}_B}(x_1)\right| + \left|I_{\tilde{\mathcal{N}}_A}(x_1) - I_{\tilde{\mathcal{N}}_B}(x_1)\right| + \left|F_{\tilde{\mathcal{N}}_A}(x_1) - F_{\tilde{\mathcal{N}}_B}(x_1)\right|\right\} \quad (8)$$

with the following two properties

1. $0 \le d_{\tilde{\mathcal{N}}}\left(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B\right) \le 3n$ (9)
2. $0 \le {}^{N}d_{\tilde{\mathcal{N}}}\left(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B\right) \le 1$ (10)

Proof: The proofs are obvious from the basic definition of SVNS.

4 GRA method for multiple attribute decision making problem with single valued neutrosophic information

Consider a multi-attribute decision making problem with *m* alternatives and *n* attributes. Let A_1, A_2, ..., A_m and $\mathbf{C_1}$, C_2, ..., C_n denote the alternatives and attributes respectively. The rating describes the performance of alternative A_i against attribute $\mathbf{C_j}$. For MADM weight vector W = {w_1, w_2,...,w_n} is assigned to the attributes. The weight $w_j > 0$ (j = 1, 2, ..., n) reflects the relative importance of attributes C_j (j = 1, 2, ..., m) to the decision making process. The weights of the attributes are usually determined on subjective basis. They represent the opinion of a single decision maker or synthesize the opinions of a group of experts using a group decision technique, as well. The values associated with the alternatives for MADM problems presented in the decision table.

Table 1 Decision table of attribute values

$$D = \langle d_{ij} \rangle_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ . \\ . \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ d_{m1} & d_{m2} & \ldots & d_{mn} \end{bmatrix} \end{array} \quad (11)$$

GRA is one of the derived evaluation methods for MADM based on the concept of grey relational space. The main procedure of GRA method is firstly translating the performance of all alternatives into a comparability sequence. This step is called data pre-processing. According to these sequences, a reference sequence (ideal target sequence) is defined. Then, the grey relational coefficient between all comparability sequences and the reference sequence for different values of distinguishing coefficient are calculated. Finally, based on these grey relational coefficients, the grey relational degree between the reference sequence and every comparability sequences is calculated. If an alternative gets the highest grey relational grade with the reference sequence, it means that the comparability sequence is most similar to the reference sequence and that alternative would be the best choice (Fung [24]). The steps of improved GRA under SVNS are described below:

Step 1 Determine the most important criteria.

Generally, there are many criteria or attributes in decision making problems where some of them are important and others may not be so important. So it is crucial, to select the proper criteria or attribute for decision making situations. The most important criteria may be chosen with help of experts' opinions or by some others method that are technically sound.

Step 2 Data pre-processing

Assuming for a multiple attribute decision making problem having *m* alternatives and *n* attributes, the general form of decision matrix can be presented as shown in Table-1. It may be mentioned here that the original GRA method can effectively deal mainly with quantitative attributes. However, there exists some difficulty in the case of qualitative attributes. In the case of a qualitative attribute (i.e. quantitative value is not available); an assessment value is taken as SVNSs.

Step 3 Construct the decision matrix with SVNSs

For multi-attribute decision making problem, the rating of alternative A_i (i = 1, 2,…m) with respect to attribute C_j (j = 1, 2,…n) is assumed as SVNS. It can be represented with the following looks

$$A_i = \left\{ \frac{C_1}{\langle T_{i1}, I_{i1}, F_{i1}\rangle}, \frac{C_2}{\langle T_{i2}, I_{i2}, F_{i2}\rangle}, \ldots, \frac{C_n}{\langle T_{in}, I_{in}, F_{in}\rangle} : C_j \in C \right\}.$$

$$= \left\langle \frac{C_j}{\langle T_{ij}, I_{ij}, F_{ij}\rangle} : C_j \in C \right\rangle \text{ for } j = 1, 2, \ldots, n.$$

T_{ij}, I_{ij}, F_{ij} are the degrees of truth membership, degree of indeterminacy and degree of falsity membership of the alternative A_i satisfying the attribute C_j, respectively where $0 \le T_{ij} \le 1$, $0 \le I_{ij} \le 1$, $0 \le F_{ij} \le 1$ and $0 \le T_{ij} + I_{ij} + F_{ij} \le 3$. The decision matrix can be taken in the form:

Table 2 Decision table with SVNSs

$$D_{\tilde{\mathcal{N}}} = \langle T_{ij}, I_{ij}, F_{ij} \rangle_{m\times n}$$

$$= \begin{array}{c} \\ A_1 \\ A_2 \\ . \\ . \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & & C_n \end{array} \\ \begin{bmatrix} \langle T_{11}, I_{11}, F_{11}\rangle & \langle T_{12}, I_{12}, F_{12}\rangle & \ldots & \langle T_{1n}, I_{1n}, F_{1n}\rangle \\ \langle T_{21}, I_{21}, F_{21}\rangle & \langle T_{22}, I_{22}, F_{22}\rangle & \ldots & \langle T_{2n}, I_{2n}, F_{2n}\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \langle T_{m1}, I_{m1}, F_{m1}\rangle & \langle T_{m2}, I_{m2}, F_{m2}\rangle & \ldots & \langle T_{mn}, I_{mn}, F_{mn}\rangle \end{bmatrix} \end{array} \quad (12)$$

Step 4: Determine the weights of criteria.

In the decision-making process, decision makers may often face with unknown attribute weights. It may happens that the importance of the decision makers are not equal. Therefore, we need to determine reasonable attribute weight for making a proper decision. Many methods are available to determine the unknown attribute weight in the literature such as maximizing deviation method (Wu and Chen [25]), entropy method (Wei and Tang [26]; Xu and Hui [27]), optimization method (Wang and Zhang [28-29]) etc. In this paper, we propsoe information entropy method.

4.1 Entropy method:

Entropy has an important contribution for measuring uncertain information (Shannon [30-31]). Zadeh [32] introduced the fuzzy entropy for the first time. Similarly Bustince and Burrillo [33] introduced the intuitionistic fuzzy entropy. Szmidt and Kacprzyk [34] extended the axioms of De Luca and Termini's [35] non-probabilistic entropy in the setting of fuzzy set theory into intuitionistic fuzzy information entropy. Vlachos and Sergiadis [36] also studied intuitionistic fuzzy information entropy. Majumder and Samanta [37] developed some similarity and entropy measures for SVNSs. The entropy measure can be used to determine the attributes weights when it is unequal and completely unknown to decision maker. Hwan and Yoon (1981) developed a method to determine the attribute weights based on information entropy.

In this paper we propose an entropy method for determining attribute weight. According to Majumder and Samanta [37], the entropy measure of a SVNS $\tilde{\mathcal{N}}_A = \langle T_{\tilde{\mathcal{N}}_A}(x_1), I_{\tilde{\mathcal{N}}_A}(x_1), F_{\tilde{\mathcal{N}}_A}(x_1)\rangle$ is

$$E_i(\tilde{\mathcal{N}}_A) = 1 - \frac{1}{n}\sum_{i=1}^{m}\left(T_{\tilde{\mathcal{N}}_A}(x_i) + F_{\tilde{\mathcal{N}}_A}(x_i)\right)\left| I_{\tilde{\mathcal{N}}_A}(x_i) - I^C_{\tilde{\mathcal{N}}_A}(x_i)\right| \quad (13)$$

which has the following properties:

1. $E_i(\tilde{\mathcal{N}}_A) = 0$ if $\tilde{\mathcal{N}}_A$ is a crisp set and $I_{\tilde{\mathcal{N}}_A}(x_i) = 0$ $\forall x \in X$.
2. $E_i(\tilde{\mathcal{N}}_A) = 1$ if $\langle T_{\tilde{\mathcal{N}}_A}(x_1), I_{\tilde{\mathcal{N}}_A}(x_1), F_{\tilde{\mathcal{N}}_A}(x_1)\rangle = \langle 0.5, 0.5, 0.5\rangle$ $\forall x \in X$.
3. $E_i(\tilde{\mathcal{N}}_A) \ge E_i(\tilde{\mathcal{N}}_B)$ if $\tilde{\mathcal{N}}_A$ is more uncertain than $\tilde{\mathcal{N}}_B$ i.e. $T_{\tilde{\mathcal{N}}_A}(x_1) + F_{\tilde{\mathcal{N}}_A}(x_1) \le T_{\tilde{\mathcal{N}}_B}(x_1) + F_{\tilde{\mathcal{N}}_B}(x_1)$ and $\left| I_{\tilde{\mathcal{N}}_A}(x_i) - I_{\tilde{\mathcal{N}}_{A^c}}(x_i)\right| \le \left| I_{\tilde{\mathcal{N}}_B}(x_i) - I_{\tilde{\mathcal{N}}_{B^c}}(x_i)\right|$
4. $E_i(\tilde{\mathcal{N}}_A) = E_i(\tilde{\mathcal{N}}_{A^c})$ $\forall x \in X$.

In order to obtain the entropy value E_j of the j-th attribute C_j (j = 1, 2,…, n), equation (13) can be written as :

$$E_j = 1 - \frac{1}{n}\sum_{i=1}^{m}\left(T_{ij}(x_i) + F_{ij}(x_i)\right)\left| I_{ij}(x_i) - I^C_{ij}(x_i)\right|$$

for i = 1, 2,..,m; j = 1, 2,…,n. (14)

It is also noticed that $E_j \in [0,1]$. Due to Hwang and Yoon [1], and Wang and Zhang [29] the entropy weight of the j-th attibute C_j is presented by

$$w_j = \frac{1 - E_j}{\sum_{j=1}^{n}(1 - E_j)} \quad (15)$$

We get weight vector W= (w_1, w_2,…,w_n)^T of attributes C_j (j = 1,2,…, n) with $w_j \ge 0$ and $\sum_{j=1}^{n} w_j = 1$

Step 5. Determine the ideal neutrosophic estimates reliability solution (INERS) and the ideal neutrosophic estimates un-reliability solution (INEURS) for neutrosophic decision matrix.

For a neutrosophic decision making matrix $D_{\tilde{\mathcal{N}}} = [q_{\tilde{\mathcal{N}}ij}]_{m\times n}$ $= \langle T_{ij}, I_{ij}, F_{ij}\rangle_{m\times n}$, T_{ij}, I_{ij}, F_{ij} are the degrees of membership, degree of indeterminacy and degree of non membership of the alternative A_i of A satisfying the attribute C_j of C. The neutrosophic estimate reliability estimation can be easily determined from the concept of SVNS cube proposed by Dezert [38].

Definition 11 From the neutrosophic cube, the membership grade represents the estimates reliability. The ideal neutrosophic estimates reliability solution (INERS) $Q^+_{\tilde{\mathcal{N}}} = [q^+_{\tilde{\mathcal{N}}1}, q^+_{\tilde{\mathcal{N}}2}, ..., q^+_{\tilde{\mathcal{N}}n}]$ is a solution in which every component $q^+_{\tilde{\mathcal{N}}j} = \langle T^+_j, I^+_j, F^+_j\rangle$, where $T^+_j = \max_i\{T_{ij}\}$, $I^+_j = \min_i\{I_{ij}\}$ and $F^+_j = \min_i\{F_{ij}\}$ in the neutrosophic decision matrix $D_{\tilde{\mathcal{N}}} = \langle T_{ij}, I_{ij}, F_{ij}\rangle_{m\times n}$ for i = 1, 2, .., m; j = 1, 2, …, n.

Definition 12 Similarly, in the neutrosophic cube maximum un-reliability happens when the indeterminacy membership grade and the degree of falsity membership reaches maximum simultaneously. Therefore, the ideal neutrosophic estimates un-reliability solution (INEURS) $Q_{\tilde{\mathcal{N}}}^{-} = [q_{\tilde{\mathcal{N}}1}^{-}, q_{\tilde{\mathcal{N}}2}^{-}, ..., q_{\tilde{\mathcal{N}}n}^{-}]$ can be taken as a solution in the form $q_{\tilde{\mathcal{N}}j}^{-} = \langle T_j^{-}, I_j^{-}, F_j^{-} \rangle$, where $T_j^{-} = \min_i \{T_{ij}\}$,

$I_j = \max_i \{I_{ij}\}$ and $F_j^{-} = \max_i \{F_{ij}\}$ in the neutrosophic decision matrix $D_{\tilde{\mathcal{N}}} = \langle T_{ij}, I_{ij}, F_{ij} \rangle_{m\times n}$ for i = 1, 2,..,m; j = 1, 2,…,n.

Step 6 Calculate neutrosophic grey relational coefficient of each alternative from INERS and INEURS.

Grey relational coefficient of each alternative from INERS is:

$$\chi_{ij}^{+} = \frac{\min_i \min_j \Delta^{+}_{ij} + \rho \max_i \max_j \Delta^{+}_{ij}}{\Delta^{+}_{ij} + \rho \max_i \max_j \Delta^{+}_{ij}} \text{, where}$$

$$\Delta^{+}_{ij} = d\left(q_{\tilde{\mathcal{N}}j}^{+}, q_{\tilde{\mathcal{N}}ij}\right) \text{, for i= 1, 2,…,m. and j=1, 2,…,n.} \quad (16)$$

Grey relational coefficient of each alternative from INEURS is:

$$\chi_{ij}^{-} = \frac{\min_i \min_j \Delta^{-}_{ij} + \rho \max_i \max_j \Delta^{-}_{ij}}{\Delta^{-}_{ij} + \rho \max_i \max_j \Delta^{-}_{ij}} \text{, where, } \Delta^{-}_{ij} =$$

$$d\left(q_{\tilde{\mathcal{N}}ij}, q_{\tilde{\mathcal{N}}j}^{-}\right) \text{, for i= 1, 2,…,m. and j=1, 2,…,n.} \quad (17)$$

$\rho \in [0, 1]$ is the distinguishable coefficient or the identification coefficient used to adjust the range of the comparison environment, and to control level of differences of the relation coefficients. When $\rho = 1$, the comparison environment is unaltered; when $\rho = 0$, the comparison environment disappears. Smaller value of distinguishing coefficient will yield in large range of grey relational coefficient. Generally, $\rho = 0.5$ is considered for decision- making situation.

Step 7. Calculate of neutrosophic grey relational coefficient.

Calculate the degree of neutrosophic grey relational coefficient of each alternative from INERS and INEURS using the following equation respectively:

$$\chi_i^{+} = \sum_{j=1}^{n} w_j \chi_{ij}^{+} \quad (18)$$

and $\chi_i^{-} = \sum_{j=1}^{n} w_j \chi_{ij}^{-}$ for i = 1, 2,…, m. (19)

Step 8. Calculate the neutrosophic relative relational degree.

We calculate the neutrosophic relative relational degree of each alternative from ITFPIS with the help of following equations:

$$R_i = \frac{\chi_i^{+}}{\chi_i^{+} + \chi_i^{-}} \text{, for i = 1, 2,…, m.} \quad (20)$$

Step 9. Rank the alternatives.

According to the relative relational degree, the ranking order of all alternatives can be determined. The highest value of R_i yields the most important alternative.

5 . Illustrative Examples

In this section, a multi-attribute decision-making problem is considered to demonstrate the application as well as the effectiveness of the proposed method. We consider the decision-making problem adapted from Ye [39]. Suppose there is an investment company, which wants to invest a sum of money to the best one from these four possible alternatives (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; and (4) A_4 is an arms company. The investment company must take a decision according to the following three criteria: (1) C_1 is the risk analysis; (2) C_2 is the growth analysis; and (3) C_3 is the environmental impact analysis. Thus, when the four possible alternatives with respect to the above three criteria are evaluated by the expert, we can obtain the following single-valued neutrosophic decision matrix:

$$D_{\tilde{\mathcal{N}}} = \langle T_{ij}, I_{ij}, F_{ij} \rangle_{4\times 3} =$$

$$\begin{array}{c c} & \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} \langle 0.4, 0.2, 0.3 \rangle & \langle 0.4, 0.2, 0.3 \rangle & \langle 0.2, 0.2, 0.5 \rangle \\ \langle 0.6, 0.1, 0.2 \rangle & \langle 0.6, 0.1, 0.2 \rangle & \langle 0.5, 0.2, 0.2 \rangle \\ \langle 0.3, 0.2, 0.3 \rangle & \langle 0.5, 0.2, 0.3 \rangle & \langle 0.5, 0.3, 0.2 \rangle \\ \langle 0.7, 0.0, 0.1 \rangle & \langle 0.6, 0.1, 0.2 \rangle & \langle 0.4, 0.3, 0.2 \rangle \end{bmatrix} \end{array} \quad (21)$$

Step1: Determine the weights of attribute

Entropy value E_j of the j-th (j = 1, 2, 3) attributes can be determined from SVN decision matrix $D_{\tilde{\mathcal{N}}}$ (21) and equation (14) as: $E_1 = 0.50$; $E_2 = 0.2733$ and $E_3 = 0.5467$.

Then the corresponding entropy weights w_1, w_2, w_3 of all attributes according to equation (15) are obtained by $w_1 = 0.2958$; $w_2 = 0.4325$ and $w_3 = 0.2697$ such that $\sum_{j=1}^{3} w_j = 1$.

Step1: Determine the ideal neutrosophic estimates reliability solution (INERS):

$$Q^{+}_{\tilde{N}} = [q^{+}_{\tilde{N}_1}, q^{+}_{\tilde{N}_2}, q^{+}_{\tilde{N}_3}] =$$

$$\left[\begin{array}{l} \left\langle \max_i\{T_{i1}\}, \min_i\{I_{i1}\}, \min_i\{F_{i1}\}\right\rangle, \left\langle \max_i\{T_{i2}\}, \min_i\{I_{i2}\}, \min_i\{F_{i2}\}\right\rangle, \\ \qquad\qquad\qquad\qquad \left\langle \max_i\{T_{i3}\}, \min_i\{I_{i3}\}, \min_i\{F_{i3}\}\right\rangle \end{array}\right]$$

$$= \left[\langle 0.7, 0.0, 0.1\rangle, \langle 0.6, 0.1, 0.2\rangle, \langle 0.5, 0.2, 0.2\rangle\right]$$

Step 2: Determine the ideal neutrosophic estimates un-reliability solution (INEURS):

$$Q^{+}_{\tilde{N}} = [q^{-}_{\tilde{N}_1}, q^{-}_{\tilde{N}_2}, q^{-}_{\tilde{N}_3}] =$$

$$\left[\begin{array}{l} \left\langle \min_i\{T_{i1}\}, \max_i\{I_{i1}\}, \max_i\{F_{i1}\}\right\rangle, \left\langle \min_i\{T_{i2}\}, \max_i\{I_{i2}\}, \max_i\{F_{i2}\}\right\rangle, \\ \qquad\qquad\qquad\qquad \left\langle \min_i\{T_{i3}\}, \max_i\{I_{i3}\}, \max_i\{F_{i3}\}\right\rangle \end{array}\right]$$

$$= \left[\langle 0.4, 0.2, 0.3\rangle, \langle 0.4, 0.2, 0.3\rangle, \langle 0.2, 0.3, 0.5\rangle\right]$$

Step 3: Calculation of the neutrosophic grey relational co-efficient of each alternative from INERS and INEURS.

By using Equation (16) the neutrosophic grey relational coefficient of each alternative from INERS can be obtained

as: $$\left[\chi^{+}_{ij}\right]_{4\times 3} = \begin{bmatrix} 0.3636 & 0.5000 & 0.4000 \\ 0.5714 & 1.0000 & 1.0000 \\ 0.3333 & 0.5714 & 0.8000 \\ 1.0000 & 1.0000 & 0.6666 \end{bmatrix} \tag{22}$$

Similarly, from Equation (17) the neutrosophic grey relational coefficient of each alternative from INEURS is

$$\left[\chi^{-}_{ij}\right]_{4\times 3} = \begin{bmatrix} 1.0000 & 1.0000 & 0.7778 \\ 0.4667 & 0.4667 & 0.3333 \\ 0.7778 & 0.7778 & 0.3684 \\ 0.3333 & 0.4667 & 0.4111 \end{bmatrix} \tag{23}$$

Step 4: Determine the degree of neutrosophic grey relational co-efficient of each alternative from INERS and INEURS. The required neutrosophic grey relational co-efficient corresponding to INERS is obtained by using equations (18) as:

$$\chi^{+}_1 = 0.43243\,;\ \chi^{+}_2 = 0.87245\,;\ \chi^{+}_3 = 0.56222\,;$$
$$\chi^{+}_4 = 0.91004 \tag{24}$$

and corresponding to INEURS is obtained with the help of equation (19) as:

$$\chi^{-}_1 = 0.9111\,;\ \chi^{-}_2 = 0.4133\,;\ \chi^{-}_3 = 0.6140\,;$$

$$\chi^{-}_4 = 0.3978 \tag{25}$$

Step 5: Thus neutrosophic relative degree of each alternative from INERS can be obtained with the help of equation (20) as: R_1= 0.31507; R_2= 0.66949; R_3= 0.54275 and R_4= 0.68835.

Step 6: The ranking order of all alternatives can be determined according the value of neutrosophic relational degree i.e. $R_4 > R_2 > R_3 > R_1$. It is seen that the highest value of neutrosophic relational degree is R_4 therefore A_4 i.e. Arms Company is the best alternative for investment purpose.

6 Conclusion

In practical applications for MADM process, the assessments of all attributes are convenient to use the linguistic variables rather than numerical values. In most ambiguity cases, SVNS plays an important role to model MADM problem. In this paper, we study about SVNS based MADM in which all the attribute weight information is unknown. Entropy based modified GRA analysis method is proposed to solve this MADM problem. Neutrosophic grey relation coefficient is proposed for solving multiple attribute decision-making problems. Finally, an illustrative example is provided to show the feasibility of the developed approach. This proposed method can also be applied in the application of the multiple attribute decision-making with interval valued neutrosophic set and to other domains, such as decision making, pattern recognition, medical diagnosis and clustering analysis.

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