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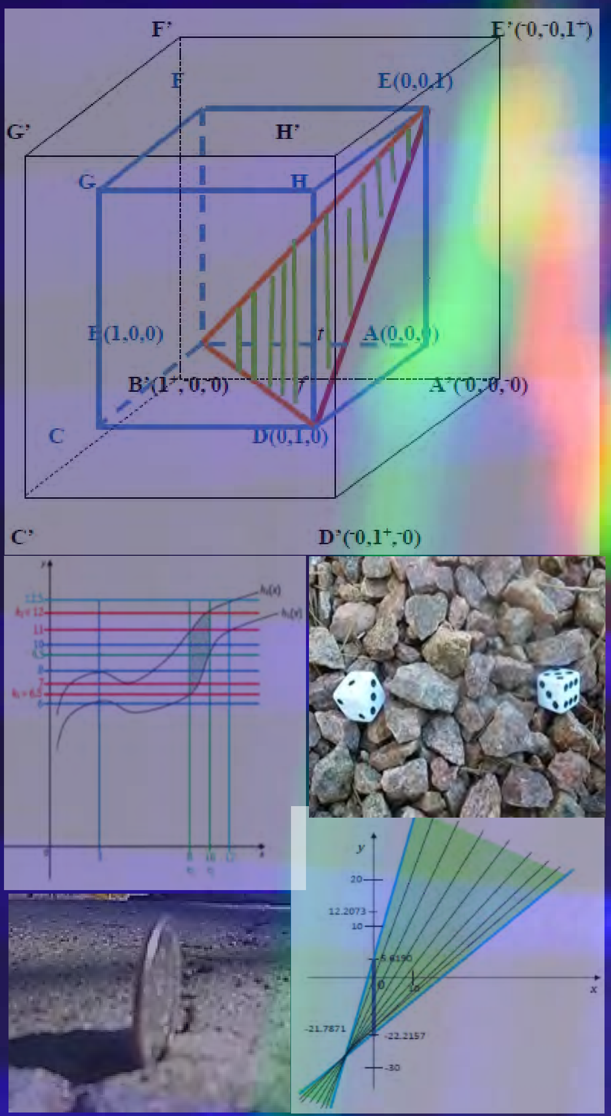

<A> <neutA> <antiA>

(updated)

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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Information for Authors and Subscribers

"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Introduction to Plithogenic Logic as generalization of MultiVariate Logic

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Abstract: A Plithogenic Logical proposition *P* is a proposition that is characterized by many degrees of truth-values with respect to many corresponding attribute-values (or random variables) that characterize *P*. Each degree of truth-value may be classical, fuzzy, intuitionistic fuzzy, neutrosophic, or other fuzzy extension type logic. At the end, a cumulative truth of *P* is computed.

Keywords: neutrosophic logic, plithogenic logic, plithogenic multi-variate analysis, cumulative truth, plithogenic logic application

1. Introduction

We recall the Plithogenic Logic and explain it in detail by showing a practical application.

A Plithogenic Logical proposition *P* is a proposition that is characterized by many degrees of truth-values with respect to many corresponding attribute-values (or random variables) that characterize *P*.

It is a pluri-logic.

We denote it by $P(V_1, V_2, \ldots, V_n)$, for $n \geq 1$, where $V_1, V_2, \ldots, V_n$ are the random variables that determine, each of them in some degree, the truth-value of *P*.

The variables may be independent one by one, or may have some degree of dependence among some of them. The degrees of independence and dependence of variables determine the plithogenic logic conjunctive operator to be used in the computing of the cumulative truth of *P*.

The random variables may be: classical, fuzzy, intuitionistic fuzzy, indeterminate, neutrosophic, and other types of fuzzy extensions.

$P(V_1) = t_1$ or the truth-value of the proposition *P* with respect to the random variable V_1.

$P(V_2) = t_2$ or the truth-value of the proposition *P* with respect to the random variable V_2.

…

And so on, $P(V_n) = t_n$ or the truth-value of the proposition *P* with respect to the random variable V_n.

The variables $V_1, V_2, \ldots, V_n$ are described by various types of probability distributions, $P(V_1)$, $P(V_2)$, …, $P(V_n)$. The whole proposition *P* is, therefore, characterized by *n* probability distributions, or n sub-truth-values. By combining all of them, we get a *cumulative truth-value* of the logical proposition *P*.

Plithogenic Logic / Set / Probability and Statistics were introduced by Smarandache [1] in 2017 and he further on (2018 -2020) developed them [2-6].

They were applied in many fields by various authors [7-27].

The Plithogenic MultiVariate Analysis used in the Set theory, Probability, and Statistics is now used in Logic, giving birth to the Plithogenic Logic.

Plithogenic MultiVariate Analysis is a generalization of the classical MultiVariate Analysis.

2. Classification of the Plithogenic Logics

Depending on the real-values of $t_1, t_2, \ldots, t_n$, we have:

2.1. Plithogenic Boolean (or Classical) Logic

It occurs when the degrees of truths $t_1, t_2, \ldots, t_n \in \{0,1\}$, where 0 = false, and 1 = true.

2.2. Plithogenic Fuzzy Logic

When the degrees of truths $t_1, t_2, \ldots, t_n$ are included in [0,1], and at least one of them is included in (0, 1), in order to distinguish it from the previous Plithogenic Boolean Logic.

Herein, we have:

2.2.1. Single-Valued Plithogenic Fuzzy Logic, if the degrees of truths $t_1, t_2, \ldots, t_n$ are single (crisp) numbers in [0, 1].

2.2.2. Subset-Valued (such as Interval-Valued, Hesitant-Valued, etc.) Plithogenic Fuzzy Logic, when the degrees of truths $t_1, t_2, \ldots, t_n$ are subsets (intervals, hesitant subsets, etc.) of [0, 1].

2.3. Plithogenic Intuitionistic Fuzzy Logic

When $P(V_j) = (t_j, f_j)$, when t_j, f_j are included in [0,1], $1 \le j \le n$, where t_j is the degree of truth, and f_j is the degree of falsehood of the proposition P, with respect to the variable V_j.

In the same way, we have:

2.3.1. Single-Valued Plithogenic Intuitionistic Fuzzy Logic, when all degrees of truths and falsehoods are single-valued (crisp) numbers in [0, 1].

2.3.2. Subset-Valued Plithogenic Intuitionistic Fuzzy Logic, when all degrees of truths and falsehoods are subset-values included in [0, 1].

2.4. Plithogenic Indeterminate Logic

When the probability distributions of the random variables $V_1, V_2, \ldots, V_n$ are indeterminate (neutrosophic) functions, i.e. functions with vague or unclear arguments and/or values.

2.5. Plithogenic Neutrosophic Logic

When $P(V_j) = (t_j, i_j, f_j)$, with t_j, i_j, f_j included in [0,1], $1 \le j \le n$, where t_j, i_j, f_j are the degrees of truth, indeterminacy, and falsehood respectively of the proposition P with respect to the random variable V_j.

Similarly, we have:

2.5.1. Single-Valued Plithogenic Neutrosophic Logic, when all degrees of truths, indeterminacies, and falsehoods are single-valued (crisp) numbers in [0, 1].

2.5.2. Subset-Valued Plithogenic Neutrosophic Logic, when all degrees of truths, indeterminacies, and falsehoods are subset-values included in [0, 1].

2.6. Plithogenic (other fuzzy extensions) Logic

Where other fuzzy extensions are, as of today: Pythagorean Fuzzy, Picture Fuzzy, Fermatean Fuzzy, Spherical Fuzzy, q-Rung Orthopair Fuzzy, Refined Neutrosophic Logic, and refined any other fuzzy-extension logic, etc.

2.7. Plithogenic Hybrid Logic

When $P(V_1), P(V_2), \ldots, P(V_n)$ are mixed types of the above probability distributions.

3. Applications

3.1. Pluri-Truth Variables

In our everyday life, we rarely have a "simple truth", we mostly deal with "complex truths".

For example:

- You like somebody for something, but dislike him for another thing;
- Or, you like somebody for something in a degree, and for another thing in a different degree;
- Similarly, for hating somebody for something in a degree, and for another thing in a different degree.

An egalitarian society (or system) does not exist in fact in our real world. It is too rigid. The individuals are different, and act differently.

Therefore, in our world, we deal with a "*plitho-logic*" (*plitho* means, in Greek, *many, pluri-*) or "complex logic". And this is best characterized by the *Plithogenic Logic*.

3.2. Types of Random Truth-Variables

The truth depends on many parameters (random variables), not only on a single one, and at the end we need to compute the *cumulative truth* (truth of all truths).

The random variables may be classical (with crisp/exact values), but often in our world they are vague, unclear, only partially known, with indeterminate data.

3.3. Weights of the Truth-Variables

Some truth may weight more than another truth.

For example, you may like somebody for something more than you dislike him/her for another thing.

Or the opposite, you may dislike somebody for something more than you like him/her for another thing.

3.4. Degrees of Subjectivity of the Truth Variables

In the soft sciences, such as: sociology, political science, psychology, linguistics, etc., or in the culture, literature, art, theatre, dance, there exists a significant *degree of subjectivity* in measuring the truth. *It is not beautiful what is beautiful, but it is beautiful what I like myself*, says a Romanian proverb.

4. Generalizations

Plithogenic Logic is a generalization of all previous logics: Boolean, Fuzzy, Intuitionistic Fuzzy, Neutrosophic Logic, and all other fuzzy-extension logics.

It is a MultiVariate Logic, whose truth variables may be in any type of the above logics.

5. Example

Let us consider an ordinary proposition P, defined as below:

$$P = \text{John loves his city}$$

and let's calculate its complex truth-value.

Of course, lots of attributes (truth-variables) may characterize a city (some of them unknown, other partially known, or approximately known). A complete spectrum of attributes to study is unreachable.

For the sake of simplicity, we consider the below five propositions as 100% independent two by two.

In this example we only chose a few variables V_j, for $1 \leq j \leq 5$:

V_1 : low / high percentage of COVID-19 virus infected inhabitants;

V_2 : nonviolent / violent;

V_3 : crowded / uncrowded;

V_4 : clean / dirty;

V_5 : quiet / noisy,

A more accurate representation of the proposition P is $P(V_1, V_2, V_3, V_4, V_5)$.

With respect to each variable V_j, the $P(V_j)$ included in [0, 1] has, in general, different truth-values, for $1 \leq j \leq 5$.

Suppose John prefer his city to have (or to be): low percentage of COVID-19 infected inhabitants, non-violent, uncrowded, clean, and quiet.

$P(V_j)$ is the degree in which John loves the city with respect to the way the variable V_j characterizes it.

5.1. Plithogenic Boolean (Classical) Logic

$P(V_1) = 1$
$P(V_2) = 0$
$P(V_3) = 1$
$P(V_4) = 0$
$P(V_5) = 1$

Therefore $P(V_1, V_2, V_3, V_4, V_5) = (1, 0, 1, 0, 1)$, or John loves his city in the following ways:

- in a degree of 100% with respect to variable V_1;
- in a degree of 0% with respect to variable V_2;
- in a degree of 100% with respect to variable V_3;
- in a degree of 0% with respect to variable V_4;
- in a degree of 100% with respect to variable V_5.

The cumulative truth-value will be, in the classical way, the classical conjunction ($\wedge_c$), where c stands for classical:

$$1 \wedge_c 0 \wedge_c 1 \wedge_c 0 \wedge_c 1 = 0,$$

or John likes his city in a cumulative classical degree of 0%!

The classical logic is rough, therefore more refined logics give a better accuracy, as follows.

5.2. Plithogenic Fuzzy Logic

The 100% or 0% truth-variables may not exactly fit John's preferences, but they may be close. For example:

$$P(V_1, V_2, V_3, V_4, V_5) = (0.95, 0.15, 0.80, 0.25, 0.85),$$

which means that John loves his city:

- in a degree of 95% with respect to variable V_1;
- in a degree of 15% with respect to variable V_2;
- in a degree of 80% with respect to variable V_3;
- in a degree of 25% with respect to variable V_4;
- in a degree of 85% with respect to variable V_5.

Using the fuzzy conjunction ($\wedge_F$) min operator, we get:

$$0.95 \wedge_F 0.15 \wedge_F 0.80 \wedge_F 0.25 \wedge_F 0.85 = \min\{0.95, 0.15, 0.80, 0.25, 0.85\} = 0.15$$

or John likes his city in a cumulative fuzzy degree of 15%.

5.3. Plithogenic Intuitionistic Fuzzy Logic

$$P(V_1, V_2, V_3, V_4, V_5) = (\ (0.80, 0.20), (0.15, 0.70), (0.92, 0.05), (0.10, 0.75), (0.83, 0.07)\ ),$$

which means that John loves his city in a degree of 80%, and dislikes it a degree of 20%, and so on with respect to the other variables.

Using the intuitionistic fuzzy conjunction ($\wedge_{IF}$) min/max operator in order to get the cummulative truth-value, one has:

$$(0.80, 0.20) \wedge_{IF} (0.15, 0.70) \wedge_{IF} (0.92, 0.05) \wedge_{IF} (0.10, 0.75) \wedge_{IF} (0.83, 0.07) =$$
$$= (\min\{0.80, 0.15, 0.92, 0.10, 0.83\}, \max\{0.20, 0.70, 0.05, 0.75, 0.07\}) = (0.10, 0.75),$$

or John likes and dislikes his city in a cumulative intuitionistic fuzzy degree of 10%, and respectively 75%.

5.4. Plithogenic Indeterminate Logic

$P(V_1, V_2, V_3, V_4, V_5)$ = (0.80 or 0.90, [0.10, 0.15], [0.60, unknown], > 0.13, 0.79),

which means that John loves his city:

- in a degree of 80% or 90% (he is not sure about) with respect to variable V_1;
- in a degree between 10% or 15% with respect to variable V_2;
- in a degree of 60% or greater with respect to variable V_3;
- in a degree greater than 13% (i.e. in the interval (0.13, 1]) with respect to variable V_4;
- in a degree of 79% with respect to variable V_5.

Therefore, the variables provide indeterminate (unclear, vague) values.

Applying the indeterminate conjunction ($\wedge_I$) min operator, we get:

min{(0.80 or 0.90), [0.10, 0.15], [0.60, unknown), (0.13, 1], 0.79}= 0.10.

5.5. Plithogenic Neutrosophic Logic

$P(V_1, V_2, V_3, V_4, V_5)$ = ((0.86, 0.12, 0.54), (0.18, 0.44, 0.72), (0.90, 0.05, 0.05), (0.09, 0.14, 0.82), (0.82, 0.09, 0.14)),

which means that John loves the city in a degree of 86%, the degree of indeterminate love is 12%, and the degree of dislike is 54% with respect to the variable V_1, and similarly with respect to the other variables.

Again, using the neutrosophic conjunction ($\wedge_N$) min/max/max operator in order to obtain the cumulative truth-value, one gets:

$$(0.86, 0.12, 0.54) \wedge_N (0.18, 0.44, 0.72) \wedge_N (0.90, 0.05, 0.05) \wedge_N (0.09, 0.14, 0.82) \wedge_N (0.82, 0.09, 0.14) = (\min\{0.86, 0.18, 0.90, 0.09, 0.82\}, \max\{0.12, 0.44, 0.05, 0.14, 0.09\}, \max\{0.54, 0.72, 0.05, 0.82, 0.14\}) = (0.09, 0.44, 0.82),$$

or John loves, is not sure (indeterminate), and dislikes his city with a cumulative neutrosophic degree of 9%, 44%, and 82% respectively.

5. Future Research

To construct the *plithogenic aggregation operators* (such as: intersection, union, negation, implication, etc.) of the variables $V_1, V_2, \ldots, V_n$ all together (cumulative aggregation), in the cases when variables V_i and V_j have some degree of dependence d_{ij} and degree of independence $1- d_{ij}$, with $d_{ij} \in [0, 1]$, for all $i, j \in \{1, 2, \ldots, n\}$, and $n \geq 2$.

6. Conclusions

We showed in this paper that the Plithogenic Logic is the largest possible logic of today. Since we live in a world full of indeterminacy and conflicting data, we have to deal, instead of a *simple truth* with a *complex truth*, where the last one is a cumulative truth resulted from the plithogenic aggregation of many truth-value random variables that characterize an item (or event).

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Conflicts of Interest: The author declares no conflict of interest.

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2-Additive Choquet Similarity Measures For Multi-Period Medical Diagnosis in Single-Valued Neutrosophic Set Setting

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Abstract. Medical diagnosis is a disease identification process that matches symptoms with diseases based on the symptoms of target patient. In this process, it is necessary to establish a similarity relation between symptoms and diseases so as to determine the correct diagnosis. Similarity measure theory is a beneficial way that is used to model this relationship mathematically under vary environment. In the literature, various similarity measures have been constructed in single-valued neutrosophic set setting. However, these similarity measures ignores the interaction between symptoms. To overcome this deficiency, we propose four new similarity measures by using the Choquet integral under single-valued neutrosophic environment that take into account both period and the interaction between symptoms. Moreover, we take advantage of the concept of 2-additivity to reduce the computational effort to obtain multi-period medical diagnosis results. We implement them to a multi-period medical diagnosis example existing in the literature. We also compare our results with some previous ones and we analyze the consistency of the results via some statistical methods.

Keywords: Single-valued neutrosophic set; similarity measure; Choquet integral; medical diagnosis

1. Introduction

Neutrosophic set theory was proposed by Smarandache [18] from a philosophical perspective as a generalisation of the concept of fuzzy set (FS) and intuitionistic fuzzy set (IFS). A neutrosophic set (NS) is characterized by a truth membership function T, an indeterminacy membership function I and a falsity membership function F and each membership degree is a real standard or non-standard subset of the non-standard unit interval $]^{-}0, 1^{+}[$. Besides, there is no restraint on the sum of the membership functions. The concept has various generalizations such as single-valued neutrosophic set (SVNS) [24], interval neutrosophic set (INS) [25], neutrosophic cubic set [9] and single-valued neutrosophic linguistic set [30].

In this paper, we focus on two multi-period medical diagnosis (MPMD) applications in SVNS setting. MPMD is a process of decision making on a disease which evaluates the effect of symptoms on the target patients according to several different periods. The most important factor that discriminates this process from other medical diagnosis processes is the presentation of the solution algorithm by paying attention to the period variable. The symptoms of the target patients or the effects of the symptoms on the target patients may change, as period progresses. A medical diagnosis includes a lot of incoherent and incomplete data because of the patient's imprecise data and the indeterminate information of the symptoms of the diseases. To solve the medical diagnosis problems in case of uncertainty, some solution methods have been proposed in the literature [1,4,11,15,17,26]. One of these methods is the determination of the disease of the target patient with the help of the similarity measures.

A similarity measure plays an important role to specify the degree of similarity between two sets such as FSs, IFSs and NSs. Similarity measures are frequently used to figure out medical diagnosis problems under neutrosophic environment [3,28,29]. The target patients and possible diseases are represented by SVNSs according to symptoms and the most accurate diagnosis is obtained by establishing a similarity between the target patients and the symptoms of the possible diseases. In a MPMD problem, period variable is also added to the problem. For example, Ye and Fu [27] proposed tangent similarity measures for SVNSs and apply them to a MPMD problem. Later, Chou et al. [7] introduced new similarity measures for SVNSs and applied them to the same MPMD problem. However, these similarity measures ignore the interaction between symptoms. A symptom may occur as a result of another symptom. In such a case, an interaction is valid between these symptoms. To overcome this deficiency, we propose new similarity measures for SVNSs based on the Choquet integral that considers the interaction between symptoms.

The concept of Choquet integral [6] was presented by Gustave Choquet in 1953 as a nonlinear continuous aggregation operator. A Choquet integral is characterized with a fuzzy measure [19] which is able to model interaction between elements of a set or between criteria in real life problems. Actually, the concept is an enlargement of the Lebesgue integral and a non-additive extension of the weighted arithmetic mean. Although, a fuzzy integral has more complicated structure due to the lack of additivity in contrast to the additive integrals such as Lebesgue integral, use of a fuzzy measure and a fuzzy integral is more effective in the aggregation. In [16], it is shown that the Choquet integral performs substantially more orders than the weighted arithmetic mean and that the difference gets larger when the number of the elements of the set gets larger. Moreover, it has been proved in [12] that when the number of the element of the finite set increases, the probability of getting more optimal ranking in the Choquet integral increases compared to the weighted arithmetic mean. Actually, fuzzy measures and fuzzy integrals let us to take the preferences into account that are not contained in the weights in the weighted arithmetic mean [23]. The notion of fuzzy measure is defined on the power set. As a result, the process of fuzzy measure identification is complicated for a set with a large number of elements due to the exponential increase in the number of the subsets. To facilitate this situation, researchers have proposed various methods and many authors studied various fuzzy measure identification methods (see, e.g., [10, 20, 21]). One of these methods is the concept of k-additive fuzzy measure proposed by Grabisch [8]. Whenever a fuzzy measure is k-additive, the notion of Choquet integral is expressed with the help of Möbius transform of the fuzzy measure. Moreover, the Möbius transform of a fuzzy measure corresponds to the correlation coefficients that indicate the direction and strength of the linear relationship between two or more random variables in probability theory and statistics. Thus, thanks to the k-additivity, the effort of fuzzy measure calculation can be reduced.

In this paper, we focus on constructing four new similarity measures based on the Choquet integral with respect to a 2-additive fuzzy measure under single-valued neutrosophic environment. Then, we give a MPMD method. We apply this method to a MPMD problem to demonstrate the effectiveness of the proposed method. The remainder of this paper is set out as follows. In Section 2, we recall the concept of SVNS. Then, we recall the MPMD methods of Ye and Fu [27] and Chou et al. [7]. In Section 3, the concepts of fuzzy measure, Möbius transform of a fuzzy measure and the concept of 2-additive fuzzy measure, the concept of Choquet integral with respect to a 2- additive fuzzy measure are recalled. In Section 4, we propose four similarity measures based on Choquet integral with respect to 2- additive fuzzy measure for SVNSs. Then, we propose the promised MPMD method. In Section 5, to indicate the effectiveness of the proposed method, we apply it to a MPMD problem from the literature. Then, the results of the problem are compared with some previous ones. Moreover, we give a consistency analysis of the results with Spearman's rank correlation coefficients. In Section 6, we give a conclusion.

2. **The Concept of SVNS and Some Existing MPMD Methods**

The concept of NS is a helpful mathematical tool that models uncertainty and inconsistent data. However, the set theoretical operators such as intersection, union and inclusion cannot be defined on the non-standard unit interval. Therefore, it is not easy to perform the applications of NSs. To come through this hassle, Wang et al. [24] presented the notion of SVNS.

Definition 2.1. [24] Let X be a universal set. A SVNS $\tilde{A}_1$ of X is given with

$$\tilde{A}_1 = \left\{ \left\langle \xi, T_{\tilde{A}_1}(\xi), I_{\tilde{A}_1}(\xi), F_{\tilde{A}_1}(\xi) \right\rangle : \xi \in X \right\} \tag{1}$$

where $T_{\tilde{A}_1}$, $I_{\tilde{A}_1}$ and $F_{\tilde{A}_1}$ are functions from X to closed interval $[0,1]$. The values $T_{\tilde{A}_1}(\xi), I_{\tilde{A}_1}(\xi)$ and $F_{\tilde{A}_1}(\xi)$ indicate the truth , the indeterminacy and the falsity membership degrees of the element ξ to the set $\tilde{A}_1$, respectively. Clearly, the sum of the three values satisfies the condition $0 \leq T_{\tilde{A}_1}(\xi) + I_{\tilde{A}_1}(\xi) + F_{\tilde{A}_1}(\xi) \leq 3$. Moreover, the triplet $\left\langle T_{\tilde{A}_1}(\xi), I_{\tilde{A}_1}(\xi), F_{\tilde{A}_1}(\xi) \right\rangle$ is called a single-valued neutrosophic value (SVNV).

Ye and Fu [27] proposed similarity measures T_1 and T_2 between SVNSs based on arithmetic mean and applied the similarity measure T_2 to a MPMD problem. Let $X = \{\xi_1, ..., \xi_m\}$ be a set of symptoms, let $T = \{t_1, ..., t_q\}$ be a set of periods and let $D = \{D_1, ..., D_n\}$ be a set of diseases. For a patient P_s with assorted symptoms, $C_j(t_k)$ denotes the SVNV between a patient and jth symptom ξ_j for $j = 1, ..., m$ in the kth period t_k for $k = 1, ..., q$ (see, Table 2 in [27]). It is represented as $C_j(t_k) = \langle T_j(t_k), I_j(t_k), F_j(t_k) \rangle$ in the form of a SVNV. Apparently, if $q = 1$, the MPMD problem is generally a single period medical diagnosis problem. Moreover, C_{ij} denotes the SVNV between the jth symptom ξ_j for $j = 1, ..., m$ and the ith noted disease D_i for $i = 1, ..., n$ (see, Table 3 in [27]). It is represented as $C_{ij} = \langle T_{ij}, I_{ij}, F_{ij} \rangle$ in the form of a SVNV.

Let weights of the symptoms be $0 \leq w_1, ..., w_m \leq 1$ with $\sum_{j=1}^{m} w_i = 1$ and the weights of the periods be $0 \leq \omega(t_1), ..., \omega(t_q) \leq 1$ with $\sum_{k=1}^{q} \omega(t_k) = 1$. The MPMD method is constructed as follows: Firstly, the similarity measure between a patient P_s and the noted disease D_i for $i = 1, ..., n$ in each period t_k for $k = 1, ..., q$ is calculated with the help of the weighted version of similarity T_2 with the following:

$$T_{w_i}(P_s, t_k) := 1 - \sum_{j=1}^{m} w_j \tan \left[\frac{\pi}{12} \left(|T_j(t_k) - T_{ij}| + |I_j(t_k) - I_{ij}| + |F_j(t_k) - F_{ij}| \right) \right]. \tag{2}$$

Then, the weighted aggregation value $M(P_s, D_i)$ for $i = 1, ..., n$ is obtained with the following:

$$M(P_s, D_i) := \sum_{k=1}^{q} \omega(t_k) T_{w_i}(P_s, t_k). \tag{3}$$

Finally, the weighted values with respect to D_i for $i = 1, ..., n$ are put in order and the highest value is determined as the most appropriate choice.

Chou et al. [7] constructed a MPMD method for SVNSs by motivating from Ye and Fu's working [27]. They proposed two weighted similarity measures M_{w1} and M_{w2}. Then, this two similarity measures are used in the same MPMD with the help of the same algorithm of Ye and Fu [27].

In this study, our aims are to express new similarity measures, motivating by [7] and [27] with the help of the 2-additive Choquet integral that pays attention to the interaction between the symptoms and to propose a MPMD method.

3. **Some Basic Concepts of Fuzzy Measure Theory and Choquet Integral**

The basis of Choquet integral is inherently fuzzy measure. Therefore, we recall the concept of fuzzy measure.

Definition 3.1. Let X be a non-emty set and let $P(X)$ be the power set of X. If
i) $\sigma(\varnothing) = 0$,
ii) $\sigma(X) = 1$,
iii) $A_1 \subseteq A_2$ implies $\sigma(A_1) \le \sigma(A_2)$ (monotonicity),
then the set function $\sigma : P(X) \to [0, 1]$ is called a fuzzy measure on X [6].

There exist $2^n = \sum_{k=0}^{n} \binom{n}{k}$ coefficients to be determined on the power set of a set with n elements. For this reason, the process of determining a fuzzy measure over a set with excess number of elements is quite difficult. Thus, Grabisch introduced a crucial kind of fuzzy measure which is named k-additive fuzzy measure to facilitate the process of determining a fuzzy measure on set with large elements [8]. For instance, if $k = 2$, it is enough to determine the fuzzy measure of $n(n-1)/2$ subsets so as to specify the whole fuzzy measure (see, [8]).

Definition 3.2. The Möbius transform of a set function σ on X is a set function $m : P(X) \to \mathbb{R}$ defined by

$$m(A_1) := \sum_{A_2 \subset A_1} (-1)^{|A_1 \setminus A_2|} \sigma(A_2). \quad (4)$$

A fuzzy measure σ is expressed as:

$$\sigma(A_1) = \sum_{A_2 \subset A_1} m(A_2) \quad (5)$$

for all $A_1 \in P(X)$ [5] whenever its Möbius transform m is given. As a result, the Möbius transform over singletons is equal to the fuzzy measure itself.

Definition 3.3. Let X be a finite set and let σ be a fuzzy measure on X. σ is said to be 2-additive if its Möbius transform m satisfies $m(A_1) = 0$ for all $A_1 \subset X$ such that $|A_1| > 2$ and there exist at least one subset $A_1 \subset X$ with $|A_1| = 2$ such that $m(A_1) \neq 0$ [8] .

The following important theorem gives some properties of the Möbius transform corresponding to a fuzzy measure and they can be used in the fuzzy measure identification process.

Theorem 3.4. *[5] Let X be a finite set and let $\sigma : P(X) \to \mathbb{R}$ be a function. σ is a fuzzy measure on X if and only if its Möbius transform m satisfies*

i) $m(\varnothing) = 0$,

ii) $\sum_{A_1 \subset X} m(A_1) = 1$,

iii) $\sum_{\xi \in A_2 \subset A_1} m(A_2) \geq 0$, *for all $A_1 \subset X$ and for all $\xi \in A_1$.*

Another crucial notion that is related to the fuzzy measure theory is the concept of interaction index (see, e.g., [8]). The following theorem gives the relationship between the interaction index and the Möbius transform of a fuzzy measure.

Theorem 3.5. *[8] Let X be a finite set and let m be the Möbius transform of a fuzzy measure on X. The interaction index I satisfies*

$$I(A_1) = \sum_{k=0}^{|X \setminus A_1|} \frac{1}{k+1} \sum_{\substack{A_2 \subset X \setminus A_1 \\ |A_2| = k}} m(A_1 \cup A_2) \tag{6}$$

for any $A_1 \subset X$.

From Theorem 3.5, we have

$$I(A_1) = \begin{cases} m(A_1), & |A_1| = 2 \\ 0, & |A_1| > 2 \end{cases} \tag{7}$$

whenever the fuzzy measure is 2-additive [8]. Interaction between at most two criteria can exist whenever the fuzzy measure is 2-additive. That is, there is no interaction between more than two criteria when σ is 2-additive.

Let $X = \{\xi_1, ..., \xi_n\}$ be a finite set and let $I_{ij} := I(\{\xi_i, \xi_j\})$.

(1) If $I_{ij} > 0$, then there is a positive interaction between the criteria ξ_i and ξ_j, and when they come together, their severity increases.

(2) If $I_{ij} < 0$, then there is negative interaction between the criteria ξ_i and ξ_j, and one of the criteria is more redundant. When these two criteria come together, their severity decreases.

(3) If $I_{ij} = 0$, then there is no interaction between the criteria ξ_i and ξ_j and they are independent from each other.

Definition 3.6. Let $X = \{\xi_1, ..., \xi_n\}$ be a finite set and let σ be a fuzzy measure on X. The Choquet integral [6] of a function $f : X \to [0, 1]$ with respect to σ is defined by

$$(C)\int_X f\,d\sigma := \sum_{k=1}^{n}\left(f(\xi_{(k)})-f(\xi_{(k-1)})\right)\sigma(E_{(k)}), \tag{8}$$

where the sequence $\left\{\xi_{(k)}\right\}_{k=0}^{n}$ is the permutation of the sequence $\{\xi_k\}_{k=0}^{n}$ such that $0 := f(\xi_{(0)}) \le f(\xi_{(1)}) \le f(\xi_{(2)}) \le ... \le f(\xi_{(n)})$ and $E_{(k)} := \left\{\xi_{(k)}, \xi_{(k+1)}, ..., \xi_{(n)}\right\}$.

If the fuzzy measure is 2-additive Definition 3.6 is equivalent to following expressions:

$$(C_{2-add.})\int_X f\ d\sigma =: \sum_{\xi_i\in X} m_i\, f(\xi_i) + \sum_{\{\xi_i,\xi_j\}\subseteq X} m_{ij}\min(f(\xi_i), f(\xi_j)) \tag{9}$$

where m is the Möbius transform of a 2-additive fuzzy measure σ on X and $m_i := m(\{\xi_i\})$, $m_{ij} := m(\{\xi_i,\xi_j\})$ [13,14].

From (7) and (9), we see that interaction indices are enough to calculate the Choquet integral with respect to a 2-additive fuzzy measure. Therefore, in Section 5 we use interaction indices.

4. **2-ADDITIVE CHOQUET SIMILARITY MEASURES FOR SVNSs**

In this section, we propose four new similarity measures for SVNSs by using a 2-additive Choquet integral and we give some propositions associated with these similarity measures. Moreover, the proposed similarity measures are integrated into a MPMD method with the help of the Choquet integral. Motivating by [7] and [27], now we define the following similarity measures.

Definition 4.1. Let $X = \{\xi_1, ..., \xi_n\}$ be a finite set, let $\tilde{A}_1$ and $\tilde{A}_2$ be two SVNSs of X and let σ be a 2-additive fuzzy measure on X. Two 2-additive Choquet similarity measures are given with

$$W_{T_1}^{(C_{2-add.},\sigma)}(\tilde{A}_1,\tilde{A}_2) := 1-(C_{2-add.})\int_X f^{(1)}_{\tilde{A}_1,\tilde{A}_2}\,d\sigma \tag{10}$$

$$W_{T_2}^{(C_{2-add.},\sigma)}(\tilde{A}_1,\tilde{A}_2) := 1-(C_{2-add.})\int_X f^{(2)}_{\tilde{A}_1,\tilde{A}_2}\,d\sigma \tag{11}$$

where

$$f^{(1)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j) := \max\left(\left|T_{\tilde{A}_1}(\xi_j)-T_{\tilde{A}_2}(\xi_j)\right|, \left|I_{\tilde{A}_1}(\xi_j)-I_{\tilde{A}_2}(\xi_j)\right|, \left|F_{\tilde{A}_1}(\xi_j)-F_{\tilde{A}_2}(\xi_j)\right|\right), \tag{12}$$

$$f^{(2)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j) := \frac{\left|T_{\tilde{A}_1}(\xi_j)-T_{\tilde{A}_2}(\xi_j)\right| + \left|I_{\tilde{A}_1}(\xi_j)-I_{\tilde{A}_2}(\xi_j)\right| + \left|F_{\tilde{A}_1}(\xi_j)-F_{\tilde{A}_2}(\xi_j)\right|}{3}, \tag{13}$$

for $j = 1, ..., n$.

Definition 4.2. Let $X = \{\xi_1, ..., \xi_n\}$ be a finite set, let $\tilde{A}_1$ and $\tilde{A}_2$ be two SVNSs of X and let σ be a 2-additive fuzzy measure on X. Two 2-additive Choquet similarity measures are given with

$$W_{T_3}^{(C_{2-add.},\sigma)}(\tilde{A}_1, \tilde{A}_2) := 1 - (C_{2-add.}) \int_X f^{(3)}_{\tilde{A}_1,\tilde{A}_2} \, d\sigma \tag{14}$$

$$W_{T_4}^{(C_{2-add.},\sigma)}(\tilde{A}_1, \tilde{A}_2) := 1 - (C_{2-add.}) \int_X f^{(4)}_{\tilde{A}_1,\tilde{A}_2} \, d\sigma \tag{15}$$

where

$$f^{(3)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j) := \tan\left[\frac{\pi}{4} \max\left(\left|T_{\tilde{A}_1}(\xi_j) - T_{\tilde{A}_2}(\xi_j)\right|, \left|I_{\tilde{A}_1}(\xi_j) - I_{\tilde{A}_2}(\xi_j)\right|, \left|F_{\tilde{A}_1}(\xi_j) - F_{\tilde{A}_2}(\xi_j)\right|\right)\right], \tag{16}$$

and

$$f^{(4)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j) := \tan\left[\frac{\pi}{12} \left(\left|T_{\tilde{A}_1}(\xi_j) - T_{\tilde{A}_2}(\xi_j)\right| + \left|I_{\tilde{A}_1}(\xi_j) - I_{\tilde{A}_2}(\xi_j)\right| + \left|F_{\tilde{A}_1}(\xi_j) - F_{\tilde{A}_2}(\xi_j)\right|\right)\right], \tag{17}$$

for $j = 1, ..., n$.

Note here that, if we consider additive measures, then we obtain the similarity measures of [7] and [27].

Proposition 4.3. *Let X be a finite set and let $\tilde{A}_1$ and $\tilde{A}_2$ be two SVNSs in X. The 2-additive Choquet similarity measure $W_{T_i}^{(C_{2-add.},\sigma)}$ for $i = 1, 2, 3, 4$ satisfies the following properties:*

($\mathbf{P_1}$) $0 \leq W_{T_i}^{(C_{2-add.},\sigma)}(\tilde{A}_1, \tilde{A}_2) \leq 1$;

($\mathbf{P_2}$) $W_{T_i}^{(C_{2-add.},\sigma)}(\tilde{A}_1, \tilde{A}_2) = W_{T_i}^{(C_{2-add.},\sigma)}(\tilde{A}_2, \tilde{A}_1)$;

($\mathbf{P_3}$) $\tilde{A}_1 = \tilde{A}_2$ *if and only if* $W_{T_i}^{(C_{2-add.},\sigma)}(\tilde{A}_1, \tilde{A}_2) = 1$,

($\mathbf{P_4}$) *If $\tilde{A}_3$ is a SVNS on X and $\tilde{A}_1 \subseteq \tilde{A}_2 \subseteq \tilde{A}_3$, then*

$$W_{T_i}^{(C_{2-add.},\sigma)}(\tilde{A}_1, \tilde{A}_3) \leq W_{T_i}^{(C_{2-add.},\sigma)}(\tilde{A}_1, \tilde{A}_2)$$

and

$$W_{T_i}^{(C_{2-add.},\sigma)}(\tilde{A}_1, \tilde{A}_3) \leq W_{T_i}^{(C_{2-add.},\sigma)}(\tilde{A}_2, \tilde{A}_3).$$

Proof. $(\mathbf{P_1})$ Since $T, I, F : X \to [0,1]$, we have $\left|T_{\tilde{A}_1}(\xi_j) - T_{\tilde{A}_2}(\xi_j)\right|$, $\left|I_{\tilde{A}_1}(\xi_j) - I_{\tilde{A}_2}(\xi_j)\right|, \left|F_{\tilde{A}_1}(\xi_j) - F_{\tilde{A}_2}(\xi_j)\right| \in [0,1]$. So, we obtain $f^{(1)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j)$, $f^{(2)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j) \in [0,1]$, for $j = 1,...,n$. Moreover, since the value of the tangent function is within $[0,1]$ when $\xi \in [0, \pi/4]$, we obtain $f^{(3)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j), f^{(4)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j) \in [0,1]$ for $j = 1,...,n$. As the Choquet integral is monotone, we have $0 \leq W^{(C_{2-add.},\sigma)}_{T_i}(\tilde{A}_1, \tilde{A}_2) \leq 1$, for $i = 1,2,3,4$.
$(\mathbf{P_2})$ Since $f^{(k)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j) = f^{(k)}_{\tilde{A}_2,\tilde{A}_1}(\xi_j)$ for any $j = 1,...,n$ and $k = 1,2,3,4$, the proof is trivial.
$(\mathbf{P_3})$ If $\tilde{A}_1 = \tilde{A}_2$, then $T_{\tilde{A}_1}(\xi_j) = T_{\tilde{A}_2}(\xi_j)$, $I_{\tilde{A}_1}(\xi_j) = I_{\tilde{A}_2}(\xi_j)$ and $F_{\tilde{A}_1}(\xi_j) = F_{\tilde{A}_2}(\xi_j)$ for $j = 1,...,n$. Then, we have $f^{(k)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j) = 0$ for $k = 1,2,3,4$. Therefore, we obtain that $W^{(C_{2-add.},\sigma)}_{T_i}(\tilde{A}_1, \tilde{A}_2) = 1$ for $i = 1,2,3,4$. Conversely, assume that $W^{(C_{2-add.},\sigma)}_{T_i}(\tilde{A}_1, \tilde{A}_2) = 1$, for $i = 1,2,3,4$. This implies $f^{(k)}_{A,B}(\xi_j) = 0$, for $k = 1,2,3,4$. Thus, we obtain $\left|T_{\tilde{A}_1}(\xi_i) - T_{\tilde{A}_2}(\xi_i)\right| = 0$, $\left|I_{\tilde{A}_1}(\xi_i) - I_{\tilde{A}_2}(\xi_i)\right| = 0$ and $\left|F_{\tilde{A}_1}(\xi_i) - F_{\tilde{A}_2}(\xi_i)\right| = 0$, and $\tan 0 = 0$. Therefore, we have $T_{\tilde{A}_1}(\xi_j) = T_{\tilde{A}_2}(\xi_j)$, $I_{\tilde{A}_1}(\xi_j) = I_{\tilde{A}_2}(\xi_j)$ and $F_{\tilde{A}_1}(\xi_j) = F_{\tilde{A}_2}(\xi_j)$, for $j = 1,...,n$. Hence, $\tilde{A}_1 = \tilde{A}_2$.
$(\mathbf{P_4})$ If $\tilde{A}_1 \subseteq \tilde{A}_2 \subseteq \tilde{A}_3$ then $T_{\tilde{A}_1}(\xi_j) \leq T_{\tilde{A}_2}(\xi_j) \leq T_{\tilde{A}_3}(\xi_j)$, $I_{\tilde{A}_1}(\xi_j) \geq I_{\tilde{A}_2}(\xi_j) \geq I_{\tilde{A}_3}(\xi_j)$ and $F_{\tilde{A}_1}(\xi_j) \geq F_{\tilde{A}_2}(\xi_j) \geq F_{\tilde{A}_3}(\xi_i)$, for all $j = 1,...,n$. Thus, we have

$$\left|T_{\tilde{A}_1}(\xi_j) - T_{\tilde{A}_2}(\xi_j)\right| \leq \left|T_{\tilde{A}_1}(\xi_j) - T_{\tilde{A}_3}(\xi_j)\right|, \left|T_{\tilde{A}_2}(\xi_j) - T_{\tilde{A}_3}(\xi_j)\right| \leq \left|T_{\tilde{A}_1}(\xi_j) - T_{\tilde{A}_3}(\xi_j)\right|,$$
$$\left|I_{\tilde{A}_1}(\xi_j) - I_{\tilde{A}_2}(\xi_j)\right| \leq \left|I_{\tilde{A}_1}(\xi_j) - I_{\tilde{A}_3}(\xi_j)\right|, \left|I_{\tilde{A}_2}(\xi_j) - I_{\tilde{A}_3}(\xi_j)\right| \leq \left|I_{\tilde{A}_1}(\xi_j) - I_{\tilde{A}_3}(\xi_j)\right|,$$
$$\left|F_{\tilde{A}_1}(\xi_j) - F_{\tilde{A}_2}(\xi_j)\right| \leq \left|F_{\tilde{A}_1}(\xi_j) - F_{\tilde{A}_3}(\xi_j)\right|, \left|F_{\tilde{A}_2}(\xi_j) - F_{\tilde{A}_3}(\xi_j)\right| \leq \left|F_{\tilde{A}_1}(\xi_j) - F_{\tilde{A}_3}(\xi_j)\right|.$$

So, we obtain $f^{(k)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j) \leq f^{(k)}_{\tilde{A}_1,\tilde{A}_3}(\xi_j)$ and $f^{(k)}_{\tilde{A}_2,\tilde{A}_3}(\xi_j) \leq f^{(k)}_{\tilde{A}_1,\tilde{A}_3}(\xi_j)$, for $k = 1,2$. Moreover, since the tangent function is increasing within the interval $[0, \pi/4]$, we obtain $f^{(k)}_{\tilde{A}_1,\tilde{A}_2}(\xi_j) \leq f^{(k)}_{\tilde{A}_1,\tilde{A}_3}(\xi_j)$ and $f^{(k)}_{\tilde{A}_1,\tilde{A}_2}(\xi_i) \leq f^{(k)}_{\tilde{A}_1,\tilde{A}_3}(\xi_i)$, for $k = 3,4$. Therefore, from monotonicity of the Choquet integral and definition of proposed similarity measures, we have $W^{(C_{2-add.},\sigma)}_{T_i}(\tilde{A}_1, \tilde{A}_3) \leq W^{(C_{2-add.},\sigma)}_{T_i}(\tilde{A}_1, \tilde{A}_2)$ and $W^{(C_{2-add.},\sigma)}_{T_i}(\tilde{A}_1, \tilde{A}_3) \leq W^{(C_{2-add.},\sigma)}_{T_i}(\tilde{A}_2, \tilde{A}_3)$, for $i = 1,2,3,4$. Hence, the proof is completed. □

Remark 4.4. In the proof of Proposition 4.3 we assume that $0 < \sigma(A) < 1$ where $A \neq \varnothing$, X, which is consistent with the nature of the decision making.

Note that, the proposed similarity measures take into account the interaction between symptoms thanks to Choquet integral. If we consider an additive measure instead of a fuzzy measure, then the similarity measures proposed in Definition 4.1 and 4.2 reduced to the weighted similarity measures in [7, 27].

Now, we construct a MPMD method by using proposed 2-additive Choquet similarity measures.

Step1: Let $X = \{\xi_1, ..., \xi_m\}$ be a set of symptoms. In this step, we will use the Möbius transform to construct a 2-additive fuzzy measure. We assume that the weights of the criteria are given and each weight is considered as the fuzzy measure of the corresponding singleton (criteria). The fuzzy measure σ will be constructed with the help of interaction indices (see Subsection 5.1).

Step2: Let $T = \{t_1, ..., t_q\}$ be a set of periods and let $D = \{D_1, ..., D_n\}$ be the set of diseases. For a patient P_s with various symptoms, SVNV between a patient and jth symptom ξ_j for $j = 1, ..., m$ in the kth period t_k for $k = 1, ..., q$ is indicated with $\left\langle T_s^{(t_k)}(\xi_j), I_s^{(t_k)}(\xi_j), F_s^{(t_k)}(\xi_j)\right\rangle$. Moreover, the SVNV between the jth symptom ξ_j for $j = 1, ..., m$ and the ith noted disease D_i for $i = 1, ..., n$ is indicated $\langle T_{D_i}(\xi_j), I_{D_i}(\xi_j), F_{D_i}(\xi_j)\rangle$. The similarity measures between a patient P_s and the noted disease D_i for $i = 1, ..., n$ in each period t_k for $k = 1, ..., q$ are calculated by using following formulas:

$$W_{T_l}^{(C_{2-add.},\sigma)}(t_k) := 1 - (C_{2-add.}) \int_X f_{P_s,D_i}^{(l)}(\xi_j)\, d\sigma \tag{18}$$

for $l = 1, 2, 3, 4$ where

$$f_{P_s,D_i}^{(1)}(\xi_j) := \max\left(\left|T_s^{(t_k)}(\xi_j) - T_{D_i}(\xi_j)\right|, \left|I_s^{(t_k)}(\xi_j) - I_{D_i}(\xi_j)\right|, \left|F_s^{(t_k)}(\xi_j) - F_{D_i}(\xi_j)\right|\right), \tag{19}$$

$$f_{P_s,D_i}^{(2)}(\xi_j) := \frac{\left|T_s^{(t_k)}(\xi_j) - T_{D_i}(\xi_j)\right| + \left|I_s^{(t_k)}(\xi_j) - I_{D_i}(\xi_j)\right| + \left|F_s^{(t_k)}(\xi_j) - F_{D_i}(\xi_j)\right|}{3}, \tag{20}$$

$$f_{P_s,D_i}^{(3)}(\xi_j) := \tan\left[\frac{\pi}{4}\max\left(\left|T_s^{(t_k)}(\xi_j) - T_{D_i}(\xi_j)\right|, \left|I_s^{(t_k)}(\xi_j) - I_{D_i}(\xi_j)\right|, \left|F_s^{(t_k)}(\xi_j) - F_{D_i}(\xi_j)\right|\right)\right], \tag{21}$$

$$f^{(4)}(\xi_j) := \tan\left[\frac{\pi}{12}\left(\left|T_s^{(t_k)}(\xi_j) - T_{D_i}(\xi_j)\right| + \left|I_s^{(t_k)}(\xi_j) - I_{D_i}(\xi_j)\right| + \left|F_s^{(t_k)}(\xi_j) - F_{D_i}(\xi_j)\right|\right)\right]. \tag{22}$$

Step3: We assume that a fuzzy measure η is given on the set of periods. We aggregate similarities obtained in Step 2 with respect to periods by using Choquet integral. We obtain the aggregated value $M_{f_l}^{(C,\eta)}(P_s, D_i)$ for each $l = 1, 2, 3, 4$ by the following formula:

$$M_{f_l}^{(C,\eta)}(P_s, D_i) := (C)\int_X W_{T_l}^{(C_{2-add.},\sigma)}\, d\eta := \sum_{k=1}^{q}\left(W_{T_l}^{(C_{2-add.},\sigma)}(t_{(k)}) - W_{T_l}^{(C_{2-add.},\sigma)}(t_{(k-1)})\right)\eta(E_{(k)}), \tag{23}$$

where the sequence $\{t_{(k)}\}_{k=0}^{q}$ is a new permutation of the sequence $\{t_k\}_{k=0}^{q}$ such that $0 := W_{T_l}^{(C_{2-add.},\sigma)}(t_{(0)}) \leq W_{T_l}^{(C_{2-add.},\sigma)}(t_{(1)}) \leq ... \leq W_{T_l}^{(C_{2-add.},\sigma)}(t_{(q)})$ and $E_{(k)} := \{t_{(k)}, t_{(k+1)}, ..., t_{(q)}\}$.

Step4: We rank all the weighted measures of $M_{f_l}^{(C,\eta)}(P_s, D_i)$ for $i = 1, ..., n$ in a descending order and give a proper diagnosis relative to the maximum weighted measure value.

5. MULTI-PERIOD MEDICAL DIAGNOSIS PROBLEM

In this section, we implement the proposed method to a MPMD problem from the literature. Then we compare our results with those obtained by Chou et al. [7] and Ye and Fu [27].

5.1. *Illustrative Example*

Example 5.1. Let us consider the set of symptoms and diagnoses as follows, respectively:

$$S = \left\{ \xi_1(\text{Temperature}), \xi_2(\text{Headache}), \xi_3(\text{Stomach pain}), \xi_4(\text{Cough}), \xi_5(\text{Chest pain}) \right\}$$

$$D = \left\{ D_1(\text{Viral fever}), D_2(\text{Malaria}), D_3(\text{Typhoid}), D_4(\text{Gastritis}), D_5(\text{Stenocardia}) \right\}.$$

Each diagnosis D_i, $i = 1, 2, 3, 4, 5$, is given as a SVNSs (see, Table 4 of [27]) and the patients P_1, P_2, P_3 and P_4 that have all the symptoms are represented with respect to t_1, t_2 and t_3 periods as SVNV (see, Table 5 of [27]).

So as to use the proposed Choquet integral based method, let us construct a 2-additive fuzzy measure σ. For this purpose, the weight of all criteria is taken equally. We use the Möbius transform to determine the fuzzy measures of the remaining two element subsets. We also know that whenever measure is 2-additive, the Möbius transform of subsets of two elements is equal to the interaction index (see Equation 7). As the sum of the Möbius transforms (fuzzy measures) of singletons is equal to 1 we have from (ii) of Theorem 3.4 that the sum of Möbius transforms of subsets of two elements should be equal to zero.

Now considering interaction of the symptoms we assign Möbius transforms (interaction indices) to the sets of two elements (see, Table 1).

Table 1. Möbius Representation of σ

$m(\{\xi_1, \xi_2\}) = -0.06$	$m(\{\xi_1, \xi_3\}) = 0$	$m(\{\xi_1, \xi_4\}) = -0.12$
$m(\{\xi_1, \xi_5\}) = 0$	$m(\{\xi_2, \xi_3\}) = 0$	$m(\{\xi_2, \xi_4\}) = 0$
$m(\{\xi_2, \xi_5\}) = 0.08$	$m(\{\xi_3, \xi_4\}) = 0$	$m(\{\xi_3, \xi_5\}) = 0.09$
$m(\{\xi_4, \xi_5\}) = 0.01$		

For example, since there is a redundancy between the symptoms ξ_1, ξ_2 we assign a negative value for $I_{1,2} = m_{1,2}$.

Now, we calculate the similarity between patients and diseases with respect to given symptoms:

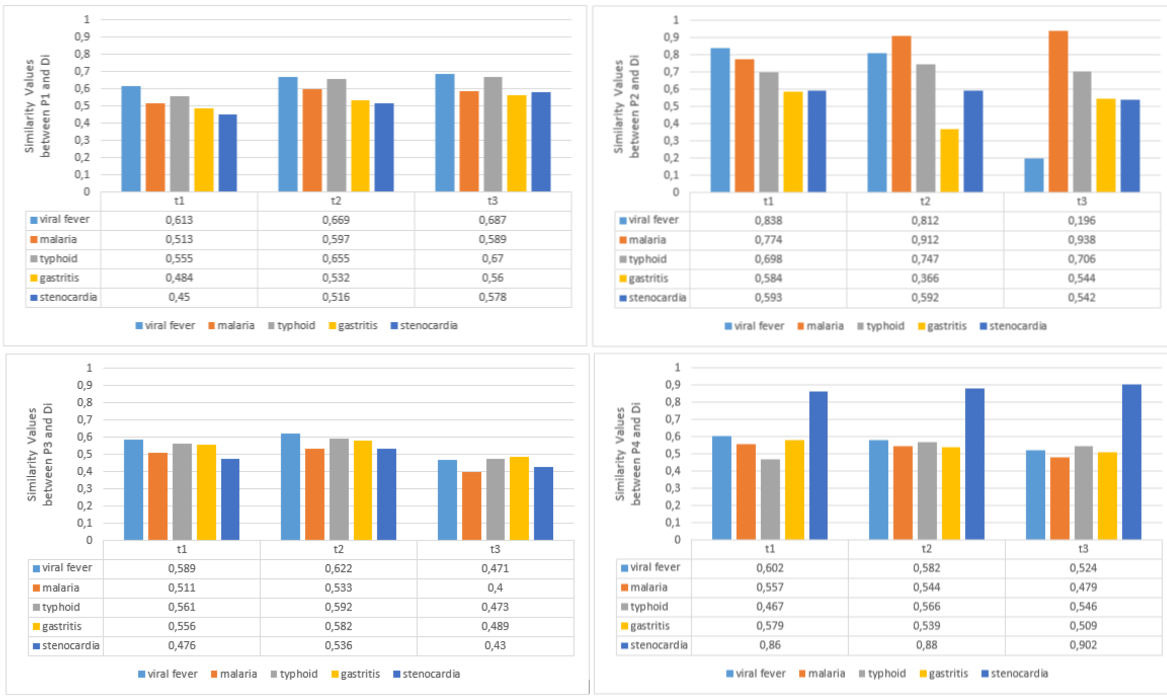

FIGURE 1. $W_{T_1}^{(C_{2-add.},\sigma)}$ for given patients with respect to periods

The results in Figure 1 show that for the P_1, similarity increase with viral fever, typhoid, gastritis and stenocardia. For the P_2 patient, the similarity of the symptoms with viral fever decreases, while the similarity with malaria increases. For the P_3, all diseases fluctuate over period. For the P_4, similarity decrease with viral fever, malaria, gastritis while increases with stenocardia.

The results in Figure 2 show that for the P_1, similarity increases with typhoid. Other diseases fluctuate over period For the P_2 patient, the similarity of the symptoms with viral fever and gastritis decreases, while the similarity with malaria increases. For the P_3, similarity decrease with malaria, typhoid. Other diseases fluctuate over period. For the P_4, similarity decrease with viral fever while fluctuating other disease over period.

The results in Figure 3 show that for the P_1, similarity increases with typhoid. Other diseases fluctuate over period For the P_2 patient, the similarity of the symptoms with viral fever and gastritis decreases, while the similarity with malaria increases. For the P_3, similarity

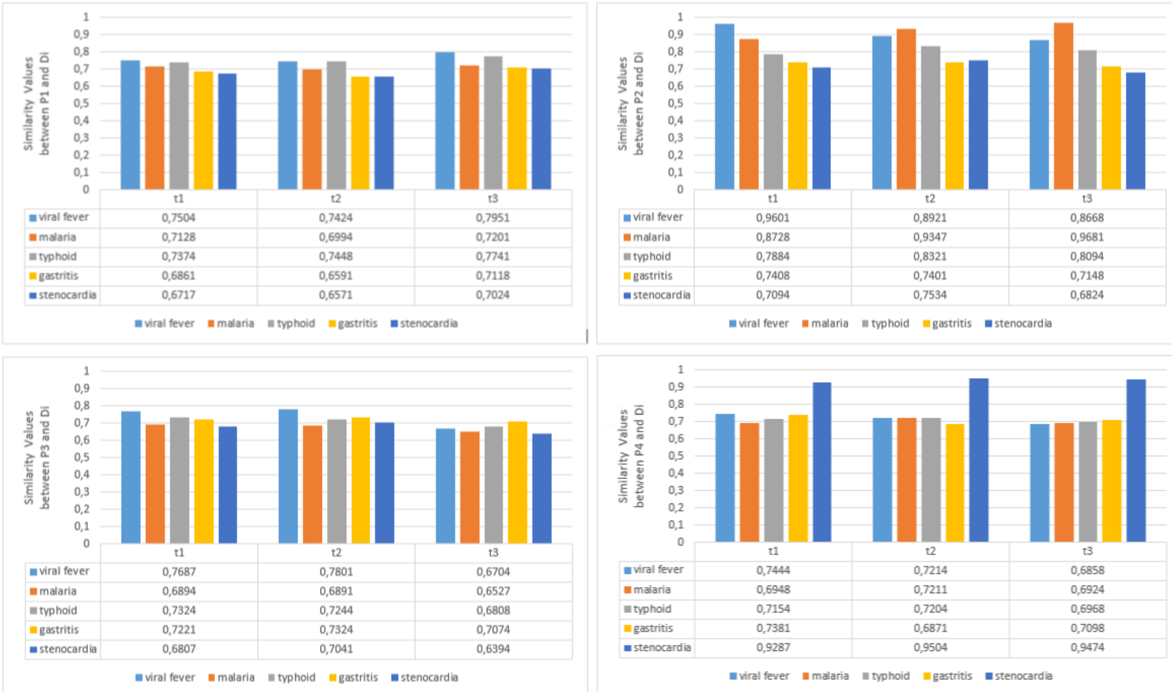

FIGURE 2. $W_{T_2}^{(C_{2-add.},\sigma)}$ for given patients with respect to periods

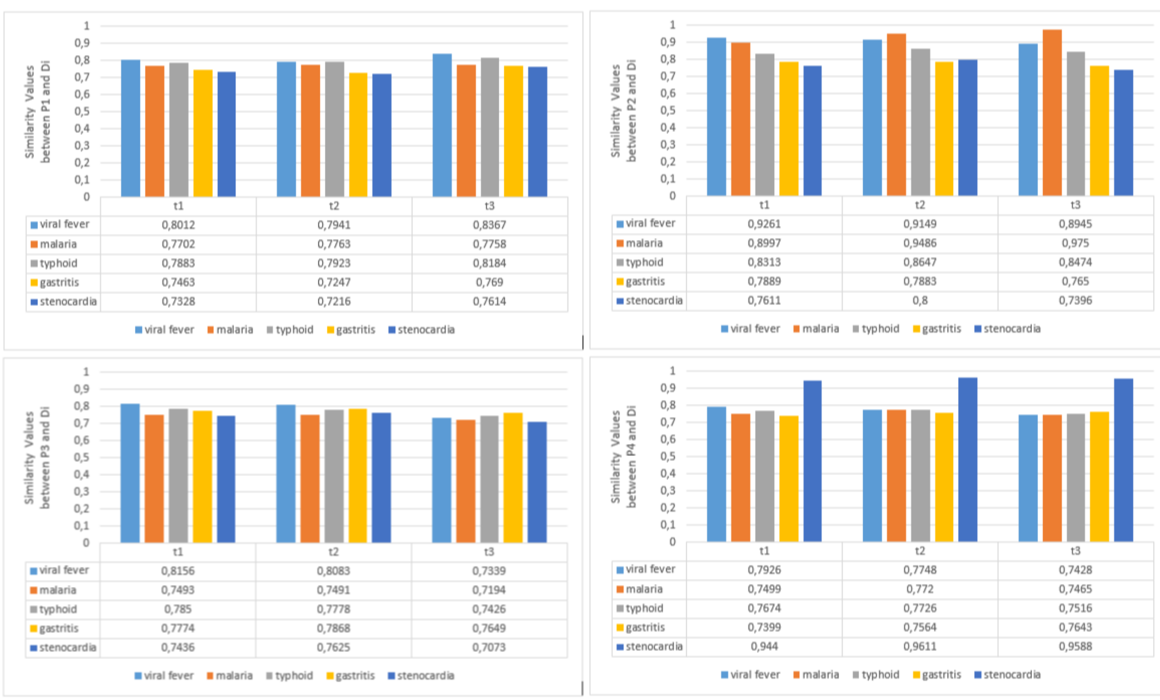

FIGURE 3. $W_{T_4}^{(C_{2-add.},\sigma)}$ for given patients with respect to periods

decrease with viral fever and malaria, typhoid. Other diseases fluctuate over period. For P_4, similarity decreases with viral fever while increases with gastritis.

For the sake of completeness, we show the calculation of the similarity between P_1 and D_1 with respect to $W_{T_4}^{(C_{2-add.},\sigma)}$ and t_1:

$$f^{(4)}(\xi_1) = \tan\left[\frac{\pi}{12}\left(|0.8-0.4|+|0.6-0.6|+|0.5-0|\right)\right] = 0.2400,$$
$$f^{(4)}(\xi_2) = \tan\left[\frac{\pi}{12}\left(|0.5-0.3|+|0.4-0.2|+|0.3-0.5|\right)\right] = 0.1583,$$
$$f^{(4)}(\xi_3) = \tan\left[\frac{\pi}{12}\left(|0.2-0.1|+|0.1-0.3|+|0.3-0.7|\right)\right] = 0.1853,$$
$$f^{(4)}(\xi_4) = \tan\left[\frac{\pi}{12}\left(|0.7-0.4|+|0.6-0.3|+|0.3-0.3|\right)\right] = 0.1583,$$
$$f^{(4)}(\xi_5) = \tan\left[\frac{\pi}{12}\left(|0.4-0.1|+|0.3-0.2|+|0.2-0.7|\right)\right] = 0.2400.$$

and

$$W_{T_4}^{(C_{2-add.},\sigma)}(t_1) = 1 - \left[\begin{array}{l} m(\{\xi_1\}) \times f^{(4)}(\xi_1) + m(\{\xi_2\}) \times f^{(4)}(\xi_2) + m(\{\xi_3\}) \times f^{(4)}(\xi_3) \\ +m(\{\xi_4\}) \times f^{(4)}(\xi_4) + m(\{\xi_5\}) \times f^{(4)}(\xi_5) + m(\{\xi_1,\xi_2\}) \times \min(f^{(4)}(\xi_1), f^{(4)}(\xi_2)) \\ +m(\{\xi_1,\xi_4\}) \times \min(f^{(4)}(\xi_1), f^{(4)}(\xi_4)) + m(\{\xi_2,\xi_5\}) \times \min(f^{(4)}(\xi_2), f^{(4)}(\xi_5)) \\ +m(\{\xi_3,\xi_5\}) \times \min(f^{(4)}(\xi_3), f^{(4)}(\xi_5)) + m(\{\xi_4,\xi_5\}) \times \min(f^{(4)}(\xi_4), f^{(4)}(\xi_5)) \end{array}\right]$$

$$\begin{aligned} &= 1 - (0.2(0.2400 + 0.1583 + 0.1853 + 0.1583 + 0.2400) - 0.06 \times \min(0.2400, 0.1583) \\ &-0.12 \times \min(0.2400, 0.1583) + 0.08 \times \min(0.1583, 0.2400) + 0.09 \times \min(0.1853, 0.2400) \\ &+0.01 \times \min(0.1583, 0.2400)) = 0.8012. \end{aligned}$$

We also show the aggregation of the similarities for $W_{T_4}^{(C_{2-add.},\sigma)}$ with respect to periods for P_1 and D_1. Consider the following fuzzy measure η on $T = \{t_1, t_2, t_3\}$ given as follows: $\eta(\{t_1\}) = 0.25$, $\eta(\{t_2\}) = 0.35$, $\eta(\{t_3\}) = 0.40$, $\eta(\{t_1, t_2\}) = 0.45$, $\eta(\{t_1, t_3\}) = 0.95$, $\eta(\{t_2, t_3\}) = 0.45$, $\eta(\{t_1, t_2, t_3\}) = 1$. The fuzzy measure of singletons is taken as the weights of the singletons proposed in [7] and [27]. It is also thought that the synergy is greater between the initial period and the end period .

For D_1 disease, $W_{T_4}^{(C_{2-add.},\sigma)}(t_2) \leq W_{T_4}^{(C_{2-add.},\sigma)}(t_1) \leq W_{T_4}^{(C_{2-add.},\sigma)}(t_3)$ and so

$$\begin{aligned} M_{f_4}^{(C,\eta)}(P_1, D_1) =& (C)\int_X W_{T_4}^{(C_{2-add.},\sigma)}\, d\eta = \sum_{k=1}^{3}\left(W_{T_4}^{(C_{2-add.},\sigma)}(t_{(k)}) - W_{T_4}^{(C_{2-add.},\sigma)}(t_{(k-1)})\right)\eta(E_{(k)}) \\ =& W_{T_4}^{(C_{2-add.},\sigma)}(t_2) + (W_{T_4}^{(C_{2-add.},\sigma)}(t_1) - W_{T_4}^{(C_{2-add.},\sigma)}(t_2))\eta(\{t_1, t_3\}) \\ &+(W_{T_4}^{(C_{2-add.},\sigma)}(t_3) - W_{T_4}^{(C_{2-add.},\sigma)}(t_1))\eta(\{t_3\}) \\ =& 0.7941 + (0.8012 - 0.7941) \times 0.95 + (0.8367 - 0.8012) \times 0.40 = 0.8150. \end{aligned}$$

We can see from Table 2 that other results except for $M_{f_4}^{(C,\eta)}(P_3, D_i)$ are consistent with previous studies. This difference is due to the consideration of the interaction between symptoms in the proposed MPMD method.

Table 2. Evaluation Scores for SVNSs

	Aggregation Values	D_1	D_2	D_3	D_4	D_5	Diagnosis
The results of [7]	$M_{w_1}(P_1, D_i)$	**0.6730**	0.5990	0.6560	0.5260	0.5270	viral fever
	$M_{w_1}(P_2, D_i)$	0.7970	**0.8730**	0.6780	0.5510	0.5330	malaria
	$M_{w_1}(P_3, D_i)$	**0.5860**	0.5040	0.5540	0.5300	0.4520	viral fever
	$M_{w_1}(P_4, D_i)$	0.5210	0.4750	0.5640	0.6020	**0.8910**	stenocardia
	$M_{w_2}(P_1, D_i)$	**0.7770**	0.7323	0.7483	0.6810	0.6510	viral fever
	$M_{w_2}(P_2, D_i)$	0.8683	**0.9250**	0.7897	0.6883	0.6670	malaria
	$M_{w_2}(P_3, D_i)$	**0.7573**	0.6983	0.6960	0.7133	0.6573	viral fever
	$M_{w_2}(P_4, D_i)$	0.6917	0.6617	0.7170	0.7577	**0.9443**	stenocardia
The results of [27]	$M(P_1, D_i)$	**0.8183**	0.7852	0.7966	0.7427	0.7167	viral fever
	$M(P_2, D_i)$	0.8985	**0.9409**	0.8315	0.7451	0.7220	malaria
	$M(P_3, D_i)$	**0.8058**	0.7554	0.7738	0.7701	0.7230	viral fever
	$M(P_4, D_i)$	0.7491	0.7214	0.7692	0.8036	**0.9562**	stenocardia
The results of proposed Choquet integral methods	$M_{f_1}^{(C,\eta)}(P_1, D_i)$	**0.6454**	0.5500	0.6060	0.5168	0.5045	viral fewer
	$M_{f_1}^{(C,\eta)}(P_2, D_i)$	0.4797	**0.8465**	0.7159	0.5451	0.5647	malaria
	$M_{f_1}^{(C,\eta)}(P_3, D_i)$	**0.5356**	0.4576	0.5234	0.5282	0.4717	viral fever
	$M_{f_1}^{(C,\eta)}(P_4, D_i)$	0.5551	0.5115	0.5095	0.5325	**0.8778**	stenocardia
	$M_{f_2}^{(C,\eta)}(P_1, D_i)$	**0.7678**	0.7150	0.7524	0.6950	0.6832	viral fever
	$M_{f_2}^{(C,\eta)}(P_2, D_i)$	0.8951	**0.9140**	0.8057	0.7263	0.7099	malaria
	$M_{f_2}^{(C,\eta)}(P_3, D_i)$	**0.7186**	0.6691	0.7024	0.7176	0.6661	viral fever
	$M_{f_2}^{(C,\eta)}(P_4, D_i)$	0.7075	0.7026	0.7069	0.7157	**0.9381**	stenocardia
	$M_{f_4}^{(C,\eta)}(P_1, D_i)$	**0.8150**	0.7728	0.8005	0.7543	0.7436	viral fever
	$M_{f_4}^{(C,\eta)}(P_2, D_i)$	0.9064	**0.9322**	0.8446	0.7756	0.7628	malaria
	$M_{f_4}^{(C,\eta)}(P_3, D_i)$	0.7692	0.7328	0.7602	**0.7738**	0.7302	gastritis
	$M_{f_4}^{(C,\eta)}(P_4, D_i)$	0.7616	0.7557	0.7605	0.7504	**0.9514**	stenocardia

5.2. *Ranking Analysis with Spearman's Rank Correlation Coefficient*

In this subsection, we use the Spearman's correlation coefficients to analyze the ranking differences between the obtained results. The Spearman's rank correlation coefficient, denoted by ρ, is shown below and the results of the test are presented in Table 3 and 4:

$$\rho =: 1 - \frac{6}{n(n^2-1)} \sum_{i=1}^{n} d_i^2 \tag{24}$$

where n is the number of results and d_i is difference between rankings of results obtained.

TABLE 3. Spearman's Rank Correlations between M and $M_{f_i}^{(C,\eta)}$ for $i = 1, 2, 4$

Patients	Similarity Measures	Correlation Value	Consistency Ranking
P_1	$M_{f_1}^{(C,\eta)}$	1.0	1
	$M_{f_2}^{(C,\eta)}$	1.0	1
	$M_{f_4}^{(C,\eta)}$	1.0	1
P_2	$M_{f_2}^{(C,\eta)}$	1.0	1
	$M_{f_4}^{(C,\eta)}$	1.0	1
	$M_{f_1}^{(C,\eta)}$	0.3	2
P_3	$M_{f_2}^{(C,\eta)}$	0.9	1
	$M_{f_1}^{(C,\eta)}$	0.8	2
	$M_{f_4}^{(C,\eta)}$	0.7	3
P_4	$M_{f_2}^{(C,\eta)}$	0.9	1
	$M_{f_1}^{(C,\eta)}$	0.5	2
	$M_{f_4}^{(C,\eta)}$	0.3	3

TABLE 4. Spearman's Rank Correlations between $M_{f_i}^{(C,\eta)}$ and M_{w_1} and M_{w_2}, for $i = 1, 2, 4$

Patients	Similarity Measures	Correlation Value	Consistency Ranking	Similarity Measures	Correlation Value	Consistency Ranking
P_1	$M_{f_1}^{(C,\eta)}$	0.9	1	$M_{f_1}^{(C,\eta)}$	1.0	1
	$M_{f_2}^{(C,\eta)}$	0.9	1	$M_{f_2}^{(C,\eta)}$	1.0	1
	$M_{f_4}^{(C,\eta)}$	0.9	1	$M_{f_4}^{(C,\eta)}$	1.0	1
P_2	$M_{f_2}^{(C,\eta)}$	1.0	1	$M_{f_2}^{(C,\eta)}$	1.0	1
	$M_{f_4}^{(C,\eta)}$	1.0	1	$M_{f_4}^{(C,\eta)}$	1.0	1
	$M_{f_1}^{(C,\eta)}$	0.3	2	$M_{f_1}^{(C,\eta)}$	0.3	2
P_3	$M_{f_2}^{(C,\eta)}$	0.9	1	$M_{f_2}^{(C,\eta)}$	0.9	1
	$M_{f_1}^{(C,\eta)}$	0.8	2	$M_{f_4}^{(C,\eta)}$	0.8	2
	$M_{f_4}^{(C,\eta)}$	0.7	3	$M_{f_1}^{(C,\eta)}$	0.7	3
P_4	$M_{f_2}^{(C,\eta)}$	0.9	1	$M_{f_2}^{(C,\eta)}$	0.9	1
	$M_{f_1}^{(C,\eta)}$	0.5	2	$M_{f_1}^{(C,\eta)}$	0.5	2
	$M_{f_4}^{(C,\eta)}$	0.3	3	$M_{f_4}^{(C,\eta)}$	0.3	3

6. Conclusion

In this paper, we focus on increasing the sensitivity of some existing fuzzy measures by taking into account the interaction between symptoms with the help of the Choquet integral. For this purpose, we propose four new similarity measures based on the Choquet integral for SVNSs, both by providing the opportunity to evaluate more symptoms with the help of 2-additivity, and taking into account the interaction between symptoms. We adapted the proposed similarity measures to a MPMD problem that exists in the literature and we compare the results with some existing results. The most of the obtained results are consistent with past results. The consistency between these results is showed with the Spearman's correlation coefficients. Inconsistent result may occur because of the novelty of the proposed relatively sensitive method.

Conflicts of Interest: "The authors declare no conflict of interest."

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Neutrosophic special dominating set in neutrosophic graphs

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Abstract. The neutrosophic graph is a new version of graph theory that has recently been proposed as an extension of fuzzy graph and intuitionistic fuzzy graph that provides more precision compatibility and flexibility than a fuzzy graph and an intuitionistic fuzzy graph in structuring and modelling many real-life problems. The aim of this paper is to offer for the first time the new concepts of neutrosophic highly strong arc, neutrosophic special dominating set, neutrosophic special domination numbers, neutrosophic special cobondage set and neutrosophic special cobondage numbers in the neutrosophic graph, as well as expressing some of relation and results of them and reduce the effect of adding a neutrosophic highly strong arc on neutrosophic special domination number parameter in a neutrosophic graph. Finally, an application related to decision making based on agents affecting the performance of the organization is provided.

Keywords: Neutrosophic graph, neutrosophic special dominating set, neutrosophic special domination number, neutrosophic special cobondage set, neutrosophic special cobondage number.)

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1. **Introduction**

The concept of neutrosophic sets (NSs) was offered by Smarandache [22] as a of the fuzzy sets [27], intuitionistic fuzzy sets [3], interval valued fuzzy set [26] and interval-valued intuitionistic fuzzy sets [4] theories. The neutrosophic set is a powerful mathematical tool for dealing with incomplete, indeterminate and inconsistent information in real world. The neutrosophic sets are characterized by a truth-membership function T, an indeterminacy-membership function I and a falsitymembership function F independently, which are within the real standard or nonstandard unit interval $]^{-}0, 1^{+}[$. Graph theory has now become a major branch of applied mathematics and it is generally regarded as a branch of combinatorics. Graph is a widely used tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, optimization and computer science. If one has uncertainty regarding either the set of nodes or arcs, or both, the model becomes a fuzzy graph. But, if the relations betwixt nodes (or nodes) in problems are indeterminate, the fuzzy graphs and their extensions fail. For this purpose, Smarandache [23, 24] given two main categories of neutrosophic graphs. In another study, Satham Hussain, Jahir Hussain and Smarandache [21] proposed the notion of domination in neutrosophic soft graphs. By considering the above-mentioned studies, the present paper seek to offer the concepts of neutrosophic special dominating set, neutrosophic special domination numbers, neutrosophic special cobondage set and neutrosophic special cobondage numbers in neutrosophic graphs.

2. Preliminaries

A *fuzzy graph* $G = (\phi, \psi)$ on simple graph $G^* = (\mathbb{V}, \mathbb{E})$ is a pair of functions $\phi : \mathbb{V} \to [0, 1]$ and $\psi : \mathbb{E} \to [0, 1]$ where, for each $zw \in \mathbb{E}$, $\psi(zw) \preceq \min\{\phi(z), \phi(w)\}$.

Definition 2.1. [22] If $\mathbb{V}$ is a space of points (objects) with general elements in $\mathbb{V}$ symbolized by z, then the neutrosophic set H is an object having the form

$$H = \{\langle z : T_H(z), I_H(z), F_H(z)\rangle, z \in \mathbb{V}\},$$

where the functions $T, I, F : \mathbb{V} \to]^-0, 1^+[$ describe respectively, the truth-membership function, the indeterminacy-membership function and the falsity-membership function of the element $z \in \mathbb{V}$ to the set H with the condition

$$^-0 \leq T_H(z) + I_H(z) + F_H(z) \leq 3^+,$$

the functions $T_H(z), I_H(z)$ and $F_H(z)$ are real standard or nonstandard subsets of $]^-0, 1^+[$.

Definition 2.2. [8] A neutrosophic graph on simple graph $G^* = (\mathbb{V}, \mathbb{E})$ is symbolized by $G = (K, L)$, where $K = (T_K, I_K, F_K)$ such that $T_K, I_K, F_K : \mathbb{V} \to [0, 1]$ with the condition

$$0 \leq T_K(z) + I_K(z) + F_K(z) \leq 3,$$

for all $z \in \mathbb{V}$ and $L = (T_L, I_L, F_L)$ where $T_L, I_L, F_L : \mathbb{E} \to [0, 1]$ with conditions

$$T_L(zw) \leq T_K(z) \wedge T_K(w),$$

$$I_L(zw) \geq I_K(z) \vee I_K(w),$$

$$F_L(zw) \geq F_K(z) \vee F_K(w),$$

and $0 \leq T_L(zw) + I_L(zw) + F_L(zw) \leq 3$ for all $zw \in \mathbb{E}$.

Definition 2.3. [10] Put $G = (K, L)$ be a neutrosophic graph on simple graph $G^* = (\mathbb{V}, \mathbb{E})$ and $u, v \in \mathbb{V}$. Then,

(i) *T-strength* of connectedness betwixt u and v is

$$T_L^\infty(uv) = \sup\{T_L^n(uv)\big|n = 1, 2, \cdots, m\},$$

and

$$T_L^n(uv) = \min\left\{T_L(uz_1), T_L(z_1z_2), \cdots, T_L(z_{n-1}v)\big|u, z_1, \cdots, z_{n-1}, v \in \mathbb{V}, n = 1, 2, \cdots, m\right\}.$$

(ii) *I-strength* of connectedness betwixt u and v is

$$I_L^\infty(uv) = \inf\{I_L^n(uv)\big|n = 1, 2, \cdots, m\},$$

and

$$I_L^k(uv) = \max\left\{I_L(uz_1), I_L(z_1z_2), \cdots, I_L(z_{n-1}v)\big|u, z_1, \cdots, z_{n-1}, v \in \mathbb{V}, n = 1, 2, \cdots, m\right\}.$$

(iii) *F-strength* of connectedness betwixt u and v is

$$F_L^\infty(uv) = \inf\{F_L^n(uv)\big|n = 1, 2, \cdots, m\},$$

and

$$F_L^n(uv) = \max\{F_L(uz_1), F_L(z_1z_2), \cdots, F_L(z_{n-1}v)\big|u, z_1, \cdots, z_{n-1}, v \in \mathbb{V}, n = 1, 2, \cdots, m\}.$$

Definition 2.4. [10] Put $G = (K, L)$ be a neutrosophic graph on simple graph $G^* = (\mathbb{V}, \mathbb{E})$. An arc $zw \in \mathbb{E}$ said to be a neutrosophic strong arc if

$$T_L(zw) \geq T_L^\infty(zw)\ ,\ I_L(zw) \leq I_L^\infty(zw) \text{ and } I_L(zw) \leq I_L^\infty(zw).$$

Notation 1. *From now on, in this paper we put* $G = (K, L)$ *be a neutrosophic graph on simple graph* $G^* = (\mathbb{V}, \mathbb{E})$ *and symbolized by NG.*

3. Study of neutrosophic special dominating set by addition of neutrosophic highly strong arcs

In this part, we describe the notions of neutrosophic highly strong arc, neutrosophic slightly isolated node, neutrosophic special dominating set, neutrosophic special domination numbers, neutrosophic slightly independent set and neutrosophic slightly independent numbers on neutrosophic graphs and we investigate some related results. Also we discuss about neutrosophic special domination of neutrosophic graph by adding a neutrosophic highly strong arc to this neutrosophic graph.

Definition 3.1. Put $G = (K, L)$ be a NG. Then,

(i) the neutrosophic order of G is given by,

$$|\mathbb{V}| = \sum_{v_i \in \mathbb{V}} \left(\frac{3 + T_K(v_i) - (I_K(v_i) + F_K(v_i))}{2} \right),$$

(ii) the neutrosophic size of G is given by,

$$|\mathbb{E}| = \sum_{v_i v_j \in \mathbb{E}} \left(\frac{3 + T_L(v_i v_j) - (I_L(v_i v_j) + F_L(v_i v_j))}{2} \right),$$

(iii) the neutrosophic cardinality of G is given by,

$$|G| = |\mathbb{V}| + |\mathbb{E}|,$$

(iv) for each $U \subset \mathbb{V}$, the neutrosophic node cardinality of U is symbolized by $O(U)$ and given by,

$$O(U) = \sum_{v_i \in U} \left(\frac{3 + T_K(v_i) - (I_K(v_i) + F_K(v_i))}{2} \right),$$

(v) for each $F \subset \mathbb{E}$, the neutrosophic arc cardinality of F is symbolized by $S(F)$ and given by,

$$S(F) = \sum_{v_i v_j \in F} \left(\frac{3 + T_L(v_i v_j) - (I_L(v_i v_j) + F_L(v_i v_j))}{2} \right).$$

Definition 3.2. An arc $e = zw$ in G is called a neutrosophic highly strong arc (NHStA), if

$$T_L(zw) > T_L^{\infty}(zw)\ ,\ I_L(zw) < I_L^{\infty}(zw)\ ,\ F_L(zw) < F_L^{\infty}(zw).$$

Definition 3.3. The neutrosophic highly strong neighborhood of $z \in \mathbb{V}$ is symbolized by $N_{hs}(z)$ and given as follows:

$$N_{hs}(z) = \{w \in \mathbb{V} \mid zw \text{ is a highly strong arc in } G\}.$$

Example 3.4. Investigate a NG G as Figure 1. Then, u_1u_3 and u_3u_4 are NHStAs and it is clear that $N_{hs}(u_3) = \{u_1, u_4\}$ and $N_{hs}(u_1) = N_{hs}(u_4) = \{u_3\}$.

Definition 3.5. Put G be a NG on simple graph $G^* = (\mathbb{V}, \mathbb{E})$ and $z, w \in \mathbb{V}$. Then:
(i) we say that z specially dominate w in G, if there is a NHStA betwixt z and w.
(ii) $S \subset \mathbb{V}$ said to be a neutrosophic special dominating set (NSpDS) in G, if for each $w \in \mathbb{V} \setminus S$, there is $z \in S$ where z specially dominates w.
(iii) A NSpDS S in G said to be a minimal neutrosophic special dominating set if no proper subset of S is a neutrosophic special dominating set.
(iv) Minimum neutrosophic node cardinality amongst all minimal NSpDSs of G said to be

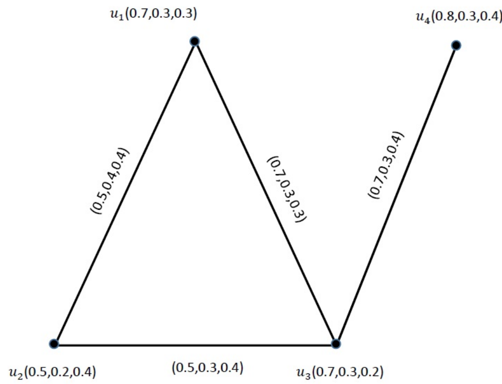

FIGURE 1. NG G.

lower neutrosophic special domination number of G and is symbolized by $(nsdn)_{\mathbb{V}}(G)$.
(v) Maximum neutrosophic node cardinality amongst all minimal NSpDSs of G said to be upper neutrosophic special domination number of G and is symbolized by $(NSDN)_{\mathbb{V}}(G)$.
(vi) The neutrosophic special domination number of G is symbolized by $NS\Delta N(G)$ and given by

$$NS\Delta N(G) = \frac{(nsdn)_{\mathbb{V}}(G) + (NSDN)_{\mathbb{V}}(G)}{2}.$$

Theorem 3.6. *A NSpDS D of a neutrosophic graph G is a minimal NSpDS iff for each node $z \in D$, one of the following conditions hold.*
(i) $N_{hs}(z) \cap D = \emptyset$,
(ii) there is a node $w \in \mathbb{V} \setminus D$ where $N_{hs}(w) \cap D = \{z\}$.

Proof. Suppose that D is a minimal NSpDS of G. Then, for each node $z \in D$, $D \setminus \{z\}$ is not a NSpDS. Thus there is $w \in \mathbb{V} \setminus (D \setminus \{z\})$ that is not specially dominated by any node in $D \setminus \{z\}$. If $w = z$, then w is not a neutrosophic strong neighbor of any node in D. If $w \neq z$, then w is not specially dominated by $D \setminus \{z\}$, but is specially dominated by D.
Conversely, consider that D is a NSpDS and for each node $z \in D$, one of the two conditions hold. Suppose D is not a minimal NSpDS. Then there is a node $z \in D$ where $D \setminus \{z\}$ is a NSpDS. Then z is a neutrosophic highly strong neighbor to at least one node in $D \setminus \{z\}$, and so (i) does not true. Also, every node w in $\mathbb{V} \setminus D$ is a neutrosophic highly strong neighbor to at least one node in $D \setminus \{z\}$. Thus (ii) does not true, that is a contradiction. Therefore, D is a minimal NSpDS. □

Example 3.7. Investigate a NG G as Figure 2. Then, $D_1 = \{u_1, u_3, u_4\}$, $D_2 = \{u_2, u_4, u_5\}$ and $D_3 = \{u_3, u_4, u_5\}$ are minimal NSpDSs and clearly $(nsdn)_{\mathbb{V}}(G) = 1.55$ and $(NSDN)_{\mathbb{V}}(G) =$

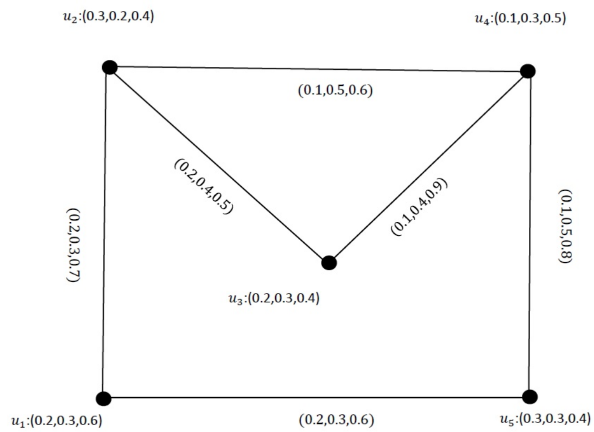

FIGURE 2. NG G.

2.8 and so,

$$NS\Delta N(G) = \frac{1.55 + 2.8}{2} = 2.175.$$

Definition 3.8. A node $z \in \mathbb{V}$ of a NG G is called a neutrosophic slightly isolated node (NSlIN) if does not specially dominate any other node of G and $N_{hs}(z) = \emptyset$.

Example 3.9. Investigate the NG G as Figure 1. Then, u_2 is a NSlIN in G.

If in graph $G^* = (\mathbb{V}, \mathbb{E})$ we add an arc e to $\mathbb{E}$, then we denote it by $\mathbb{E}_e = \mathbb{E} \cup \{e\}$ and $G^*_e = (\mathbb{V}, \mathbb{E}_e)$. Moreover, if neutrosophic graph $G = (K, L)$ on G^* extened on G^*_e, then we symbolized it by $G_e = (K_e, L_e)$.

Notation 2. *If arc e in NG G_e is a NHStA, then we denote $G_e^{hs} = (K_e^{hs}, L_e^{hs})$ insteade of $G_e = (K_e, L_e)$.*

Theorem 3.10. *Put $e = zw$ be an additional NHStA in G^*_e. Then*
(i) $NS\Delta N(G_e^{hs}) \leq NS\Delta N(G)$.
(ii) $0 \leq NS\Delta N(G) - NS\Delta N(G_e^{hs}) \leq \max\{O(\{z\}), O(\{w\})\}$.

Proof. (i) Suppose that D is a minimal NSpDS of G and $e = zw$ be an additional NHStA in G^*. If z or w is a NSlIN, then $D \setminus \{z\}$ or $D \setminus \{w\}$ is a minimal NSpDS in G_e^s. Otherwise, D is a minimal NSpDS in G_e^s. Hence, $(nsdn)_{\mathbb{V}}(G_e^{hs}) \leq (nsdn)_{\mathbb{V}}(G)$ and $(NSDN)_{\mathbb{V}}(G_e^{hs}) \leq (NSDN)_{\mathbb{V}}(G)$. Therefore, $NS\Delta N(G_e^{hs}) \leq NS\Delta N(G)$.
(ii) By the proof of (i), we have:

$$0 \leq (nsdn)_{\mathbb{V}}(G) - (nsdn)_{\mathbb{V}}(G_e^{hs}) \leq \max\{O(\{z\}), O(\{w\})\},$$

and

$$0 \leq (NSDN)_{\mathbb{V}}(G) - (NSDN)_{\mathbb{V}}(G_e^{hs}) \leq \max\{O(\{z\}), O(\{w\})\}.$$

Then,

$$0 \leq NS\Delta N(G) - NS\Delta N(G_e^{hs}) \leq \max\{O(\{z\}), O(\{w\})\}.$$

□

Theorem 3.11. *Put G be a NG and e be an additional arc in G_e^*. Then e is a NHStA in G_e iff there exist nodes z and w where $z - w$ neutrosophic path of G_e that includes e is the unique strongest neutrosophic path betwixt two nodes z and w.*

Proof. Put $e = xy$ be a NHStA in G_e. Then,

$$T_L(zw) > T_L^{\infty}(zw),\ I_L(zw) < I_L^{\infty}(zw),\ F_L(zw) < F_L^{\infty}(zw).$$

If we let $z = x$ and $w = y$, then the proof is clear.
Conversely, if there exist nodes z, w where $z - w$ neutrosophic path P_e of G_e that includes $e = xy$ is the unique strongest neutrosophic path betwixt two nodes z and w, then for each $x - y$ neutrosophic path P without arc $e = xy$ in G, we have:

$$T_L(zw) > T_P(zw),\ I_L(zw) < I_P(zw),\ F_L(zw) < F_P(zw).$$

Hence,

$$T_L(zw) > T_L^{\infty}(zw)\ ,\ I_L(zw) < I_L^{\infty}(zw)\ ,\ F_L(zw) < F_L^{\infty}(zw).$$

Therefore, $e = xy$ is a NHStA in G_e. □

Definition 3.12. An arc e in a NG G said to be a/an
(i) T-bridge if deleting e reduces the T-strength of connectedness betwixt some pair of nodes.
(ii) I-bridge if deleting e increases the I-strength of connectedness betwixt some pair of nodes.
(iii) F-bridge if deleting e increases the F-strength of connectedness betwixt some pair of nodes.
(iv) neutrosophic bridge if it is a T-bridge, I-bridge and F-bridge.

Theorem 3.13. *An arc $e = zw$ in G^* is a NHStA iff $e = zw$ is a neutrosophic bridge in G.*

Proof. Put $e = zw$ be a NHStA in G_e. Then,

$$T_L(zw) > T_L^{\infty}(zw),\ I_L(zw) < I_L^{\infty}(zw),\ F_L(zw) < F_L^{\infty}(zw).$$

It is clear that $e = zw$ is the unique strongest neutrosophic path betwixt z and w. Thus, deleting $e = zw$ reduces the T-strength and also increases I-strength and F-strength of connectedness betwixt z and w. Therefore $e = zw$ is a neutrosophic bridge in G.
Conversely, if we let $e = zw$ as a neutrosophic bridge in G, then the proof is clear. □

Example 3.14. Investigate the NG G as Figure 2. Then, $e_1 = u_1u_2$, $e_2 = u_1u_5$ and $e_3 = u_2u_3$ are NHStAs and so neutrosophic bridges in G.

Definition 3.15. Put G be a NG. Then:
(i) Two nodes $z,\ w \in \mathbb{V}$ are called neutrosophic slightly independent if there is not any NHStA betwixt them.
(ii) $S \subset \mathbb{V}$ is called a neutrosophic slightly independent set (NSlIS) in G if for each $z, w \in S$, $T_L(uv) \leq T_L^{\infty}(zw)$, $I_L(zw) \geq I_L^{\infty}(zw)$ and $F_L(zw) \geq F_L^{\infty}(zw)$.
(iii) A NSlIS S in G is called a maximal NSlIS if for each node $w \in \mathbb{V} \setminus S$, the set $S \cup \{w\}$ is not NSlIS.
(iv) Minimum neutrosophic node cardinality amongst all maximal NSlISs said to be lower neutrosophic slightly independent number of G and is symbolized by $(ni)_{\mathbb{V}}(G)$.
(v) Maximum neutrosophic node cardinality amongst all maximal NSlISs said to be upper neutrosophic slightly independent number of G and is symbolized by $(NI)_{\mathbb{V}}(G)$.
(vi) The neutrosophic slightly independent number of G is symbolized by $NI(G)$ and given as follows,

$$NI(G) = \frac{(ni)_{\mathbb{V}}(G) + (NI)_{\mathbb{V}}(G)}{2}.$$

Theorem 3.16. *Every maximal NSlIS in G is a minimal NSpDS.*

Proof. Assume that M is a maximal NSlIS in G. Then any node $v \in \mathbb{V} \setminus M$ is a NHSN to at least one node in M. Hence, M is a NSpDS in G. On the other hand, for each node $d \in M$, $N_{hs}(d) \cap D = \emptyset$. Therefore, by Theorem 3.6, M is a minimal neutrosophic special dominating set. □

Example 3.17. In Figure 2, $D_3 = \{u_3, u_4, u_5\}$ is a maximal NSlIS and so minimal NSpDS in G.

Theorem 3.18. *Put e be an additional NHStA in G_e^*. Then $NI(G_e^{hs}) \leq NI(G)$.*

Proof. Straightforward □

4. **Neutrosophic special cobondage numbers of a NG.**

In this part, we offer the concepts of neutrosophic special cobondage set and neutrosophic special cobondage numbers on NGs and investigated some related results.

Definition 4.1. (i) The neutrosophic special cobondage set (NSpCS) of a NG G is the set C of additional NHStAs to G, that reduces the neutrosophic special domination number, i.e,

$$NS\Delta N(G_C) < NS\Delta N(G).$$

(ii) A NSpCS C of G said to be a minimal NSpCS if no proper subset of C is a NSpCS.
(iii) Minimum neutrosophic arc cardinality amongst all minimal NSpCSs of G said to be lower neutrosophic special cobondage number of G and symbolized by $(nsbn)_{\mathbb{E}}(G)$.
(iv) Maximum neutrosophic arc cardinality amongst all minimal NSpCSs of G said to be upper neutrosophic special cobondage number of G and symbolized by $(NSBN)_{\mathbb{E}}(G)$.

Example 4.2. Investigate a NG G as Figure 3. Obviously $D_1^* = \{a, d\}$ and $D_2^* = \{b, c\}$ are

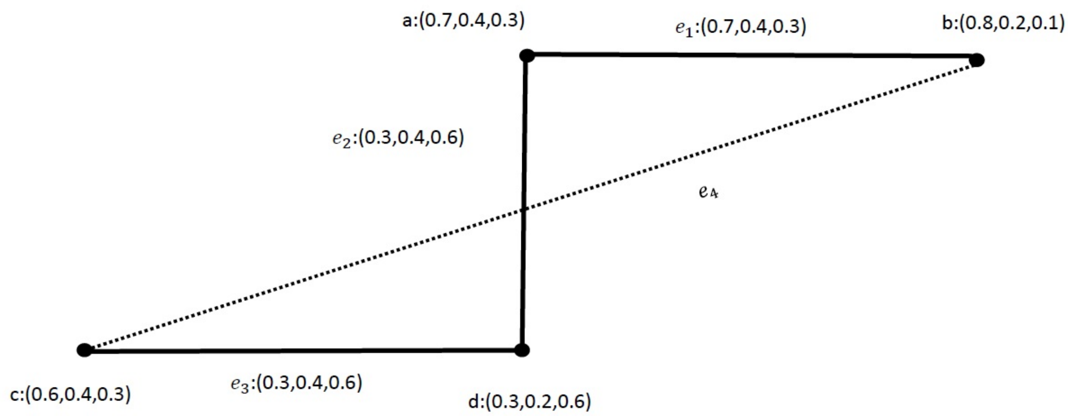

FIGURE 3. NG G.

the minimal NSpDSs of G ($(ndn)_{\mathbb{V}}(G) = 1.83$, $(NDN)_{\mathbb{V}}(G) = 2.67$ and $N\Delta N(G) = 2.25$). In this case, by adding $e_4 = (0.5, 0.4, 0.5)$, the set $D_1 = \{c, d\}$ is a minimal NSpDS with the neutrosophic node cardinality of 1.8. Then, by adding e_5 as $bd = (0.3, 0.2, 0.6)$, the set $D_2 = \{d\}$ is a minimal NSpDS with the neutrosophic node cardinality of 0.83, so $x_2 = \{e_5\}$ is a minimal NSpCS, and by adding e_6 as $ac = (0.6, 0.4, 0.3)$, the set $D_3 = \{a\}$ is a minimal NSpDS with the neutrosophic node cardinality of 1. Thus, $x_3 = \{e_6\}$ is a minimal NSpCS and so $(nbn)_{\mathbb{E}}(G)$ and $(NBN)_{\mathbb{E}}(G)$ are 0.83 and 0.97, respectively.

Theorem 4.3. *If a NG G has a NSlIN w, then*

$$(nsbn)_{\mathbb{E}}(G) \leq O(\{v\}).$$

Proof. Put w be a NSlIN of G. Then w belongs to every minimal NSpDS D of G. If $z \in D \setminus \{w\}$ and e is an NHStA betwixt w and z, then, $D \setminus \{w\}$ is a minimal NSpDS of G_e^{hs} and $(nsdn)_{\mathbb{V}}(G_e^{hs}) < (nsdn)_{\mathbb{V}}(G)$. Thus, $NS\Delta N(G_e^{hs}) < NS\Delta N(G)$. Also, we have $T_L(e) \leq T_K(w)$, $I_L(e) \geq I_K(w)$ and $F_L(e) \geq F_K(w)$. Hence,

$$S(e) \leq \left(\frac{3 + T_K(w) - (I_K(w) + F_K(w))}{2} \right),$$

and so

$$(nsbn)_{\mathbb{E}}(G) = S(e) \leq \left(\frac{3 + T_K(w) - (I_K(w) + F_K(w))}{2}\right) = O(\{w\}).$$

□

Theorem 4.4. *If G has not a NSlIN and* $e = zw$ *is a NHStA and* $N_{hs}(z) = w$, $N_{hs}(w) = z$, *then*

$$(nsbn)_{\mathbb{E}}(G) \leq O(\{z\}) + O(\{w\}).$$

Proof. If $e = zw$ is a NHStA in G where $N_{hs}(z) = w$, $N_{hs}(w) = z$, then one of z or w belongs to every minimal NSpDS D of G. Put $z \in D$ and $t \in D \setminus \{z\}$. By adding the NHStAs $e_1 = (zt)$ and $e_2 = (wt)$, the set $D \setminus \{z\}$ is a minimal NSpDS of G_C, where $C = \{e_1, e_2\}$. Thus $NS\Delta N(G_C) < NS\Delta N(G)$. Therefore,

$$\begin{aligned}(nsbn)_{\mathbb{E}}(G) &= S(C) \leq \left(\frac{3 + T_K(z) - (I_K(z) + F_K(z))}{2}\right) + \left(\frac{3 + T_K(w) - (I_K(w) + F_K(w))}{2}\right) \\ &= O(\{z\}) + O(\{w\}).\end{aligned}$$

□

5. Application

NG models have recently been used to model many real-life problems in several different fields of engineering and science. In this study, we present the idea of NSpDS in NG theory. The NSpDS in the neutrophic network can be used to solve many real problems.

5.1. *Decision making in gray conditions betwixt certainty and uncertainty*

NG models are one of the efficient models in various fields of modeling because they show more flexibility than various fuzzy graph models in dealing with real-life problems. Controlling and ensuring the compliance of decisions in various dimensions of the organization with the desired performance and predetermined performance standards despite the gray conditions betwixt certainty and uncertainty, is one of the main tasks of the leaders of an organization and plays an significant role in increasing the productivity and effectiveness of the organization. Therefore, proper management and modeling and optimization of an organization's success plan based on the agents affecting the performance of the organization in the gray conditions betwixt certainty and uncertainty is one of the significant issues considered by the leaders of an organization. The set of agents affecting the performance of an organization in gray conditions betwixt certainty and uncertainty can be considered as a NG. We describe the T-strength, I-strength and F-strength values in each node and arc (path) as follows. For each

$z, w \in V$ and $zw \in E$, we have:
$T_K(z)$: The weight of the direct effectiveness of agent z on the performance of the organization in gray conditions.
$I_K(z)$: The weight of the ineffectiveness of agent z on the performance of the organization in gray conditions.
$F_K(z)$: The weight of the indirect effectiveness of agent z on the performance of the organization in gray conditions.
$T_L(zw)$: The weight of direct impact zw on the performance of the organization in gray conditions.
$I_L(zw)$: The weight of the ineffectiveness zw on the performance of the organization in gray conditions.
$F_L(zw)$: The weight of indirect impact zw on the performance of the organization in gray conditions.
In this case, the following relations seem logical:

$$T_L(zw) \leq T_K(z) \wedge T_K(w),\ I_L(zw) \geq I_K(z) \vee I_K(w),\ F_L(zw) \geq F_K(z) \vee F_K(w).$$

The relationship betwixt z and w is effective when the xy is a NHStA. Thus, the NSpDS of this graph includes agents that other agents are specialiy dominated by at least one of the elements (agents) of this set. In fact, the NSpDS provides an opportunity for managers and leaders of the organization to focus on the agents of the NSpDS and align decisions with these agents instead of observing and controlling a large number of decision agents in gray conditions. This helps organizational leaders and managers make the best decisions with the utmost confidence in a short period of time. For example, Figure 4, displays the graph of agents affecting the performance of an organization, in which the set of $\{u_2, u_4, u_7\}$ is a minimal NSpDS (with minimum neutrosophic node cardinality 4.35). In other words, instead of controlling the 7 agents, only agents u_2, u_4, u_7 can be controlled and observed and be relatively sure about desirable performance in the decision-making process. It is worth noting that some factors such as common computational indices betwixt two agents, dependent calculation formula, and relationship betwixt the variables of calculating the indices of the agents play significant role in creating an effective relation betwixt the agents. For instance, in Figure 4, illustrates the optimal effective weight of the agent graph($S(F)$ where F is the set of all NHStAs of G) is 6.8 on desirable performance achievement.
Now, if possible, the optimal normal weight of the agent graph on desirable performance achievement can be increased by reinforcing the relation betwixt the agents, which leads to increased accuracy and confidence in the decision-making process and decreasing the neutrosophic node cardinality of the NSpDS. For instance, as shown in Figure 5, the NSpDS of agents decreases to the set $\{u_2, u_4\}$ when establishing an effective relationship is possible betwixt u_2

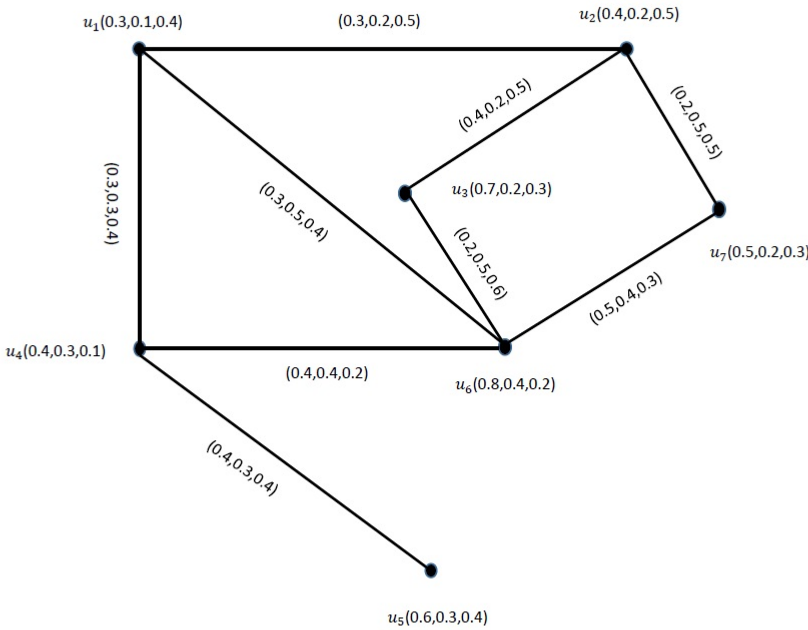

FIGURE 4. Neutrosophic graph G.

and u_7 agents with coordinates $(0.4, 0.2, 0.5)$, while the optimal effective weight of graph upgrades to 8.15.

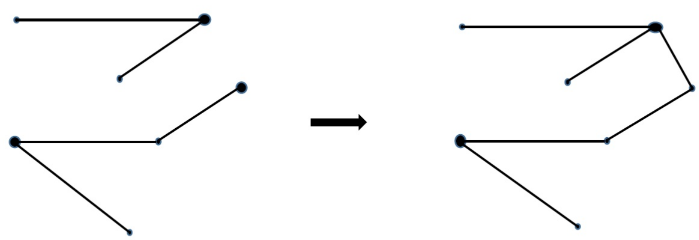

FIGURE 5. D and D'.

Neutrosophic special dominating set	$O(D)$	Optimal effective weight
$D = \{u_2, u_4, u_7\}$	4.35	6.8
$D' = \{u_2, u_4\}$	2.85	8.15

6. Conclusion

Many practical problems of interest can be illustrated with graphs. In general, graph theory has a wide range of applications in various fields. The notion of domination in graph is very important in both theoretical developments and applications. In this paper, for the first time the notions of neutrosophic special dominating set, neutrosophic special domination numbers, neutrosophic special cobondage set and neutrosophic special cobondage numbers in a NG are presented. Finally, by using the concept of neutrosophic special dominating set and the reduction effect of an additional neutrosophic highly strong arc on the neutrosophic special domination number parameter, a model for optimizing the neutrosophic special domination parameter was presented. In future works, we have a decision to study the concepts of neutrosophic special n-dominating set and inverse neutrosophic special dominating set in a NG.

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An Introduction To Refined Neutrosophic Number Theory

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Abstract: Number theory is concerned with properties of integers and Diophantine equations. The objective of this paper is dedicated to introduce the basic concepts in refined neutrosophic number theory such as division, divisors, congruencies, and Pell's equation in the refined neutrosophic ring of integers $Z(I_1, I_2)$. Also, algorithms to solve refined neutrosophic linear congruencies and refined neutrosophic Pell's equation will be presented and discussed.

Keywords: refined neutrosophic integer, refined Pell's equation, neutrosophic congruence , neutrosophic Diophantine equation.

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1. Introduction

Neutrosophy is a new kind of generalized logic proposed by F.Smarandache [12,36]. It becomes a useful tool in many areas of science such as number theory [16], solving equations [19], and medical studies [11,15,21]. Also, we find many applications of neutrosophic structures in statistics [14], optimization [8], and decision making [7].

On the other hand, the theory of neutrosophic rings began in [4], where Smarandache and Kandasamy defined concepts such neutrosophic ideals and homomorphisms. These notions were handled widely by Agboola, et.al in [5,6,10]. Where homomorphisms and AH-substructures were studied [3,13,17]. More and more application of neutrosophic sets and their generalizations can be found in [25-35].

Recently, there is an arising interesting by the number theoretical concepts in neutrosophic ring of integers, where Ceven et.al defined and studied division and primes in Z(I) [2], Sankari et.al solved the linear Diophantine equations in Z(I) and $Z(I_1, I_2)$ [16]. Also, in [1], we find algorithms to solve neutrosophic Pell's equation and neutrosophic linear congruencies. In addition, Euler's famous theorem was proved in Z(I).

In this work, we extend the study to the case of refined neutrosophic ring of integers, where we determine algorithms and conditions for division, congruencies, and Pell's equation. In addition, we prove that there are no primes in $Z(I_1, I_2)$.

2. Preliminaries

Definition 2.1: [4]

Let R be a ring, I be the indeterminacy with property $I^2 = I$, then the neutrosophic ring is defined as follows:

$R(I) = \{a + bI; a, b \in R\}$.

Definition 2.2: [4]

Let R(I) be a neutrosophic ring, it is called commutative if and only if $xy = yx \forall\ x, y \in R(I)$..

Definition 2.3: [5]

The element I can be split into two indeterminacies I_1 , I_2 with conditions:

$I_1{}^2 = I_1$, $I_2{}^2 = I_2$, $I_1 I_2 = I_2 I_1 = I_1$.

Definition 2.4: [5]

If X is a set then $X(I_1, I_2) = \{(a, bI_1, cI_2): a , b , c \in X \}$ is called the refined neutrosophic set generated by X , I_1, I_2.

Definition 2.5: [5]

Let (R,+,×) be a ring, (R(I_1, I_2) , + ,×) is called a refined neutrosophic ring generated by R ,I_1, I_2.

Example 2.6: [6]

The refined neutrosophic ring of integers is $Z(I_1, I_2)$={$(a, bI_1, cI_2); a, b, c \in Z$}.

Definition 2.7: [20]

Pell's equation is the Diophantine equation with form $X^2 - DY^2 = N; D, N \in Z$.

Theorem 2.8: [20]

If the equation $X^2 - DY^2 = 1$ has a solution, then $D > 0 \; and \; D$ is square free.

Theorem 2.9: [20]

$Z[\sqrt{d_1}]$ is an integral domain.

Theorem 2.10: [2]

Let Z(I)= $\{a + bI; a, b \in Z\}$ the neutrosophic ring of integers. Then primes in Z(I) have one of the following forms:

$x = \pm p + (\pm 1 \pm p)I \; or \; x = \pm 1 + (\pm p \pm 1)I; p \; is \; any \; prime \; in \; Z.$

Definition 2.11: [16]

Let $Z(I) = \{a + bI; \; a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation with two variables is defined as follows:

$AX + BY = C; A, B, C \in Z(I).$

Theorem 2.12: [16]

Let $Z(I) = \{a + bI; \; a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation $AX + BY = C$ with two variables $X = x_1 + x_2 I, Y = y_1 + y_2 I,$ where $A = a_1 + a_2 I, B = b_1 + b_2 I$ is equivalent to the following two classical Diophantine equations:

(1) $a_1 x_1 + b_1 y_1 = c_1.$

$$(2)(a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2.$$

Definition 2.13: [16]

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); \; a, b, c \in Z\}$ be the refined neutrosophic ring of integers. The refined neutrosophic linear Diophantine equation with two variables is defined as follows:

$AX + BY = C; A, B, C \in Z(I_1, I_2).$

Theorem 2.14: [16]

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); \; a, b, c \in Z\}$ be the refined neutrosophic ring of integers,

$AX + BY = C; A, B, C \in Z(I_1, I_2)$ be a refined neutrosophic linear Diophantine equation, where

$X = (x_0, x_1 I_1, x_2 I_2), Y = (y_0, y_1 I_1, y_2 I_2), A = (a_0, a_1 I_1, a_2 I_2),$

$B = (b_0, b_1 I_1, b_2 I_2), C = (c_0, c_1 I_1, c_2 I_2)$. Then $AX + BY = C$ is equivalent to the following three Diophantine equations:

(1) $a_0 x_0 + b_0 y_0 = c_0.$

$$(2)(a_0 + a_2)(x_0 + x_2) + (b_0 + b_2)(y_0 + y_2) = c_0 + c_2.$$

$(3)(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2.$

3. Refined neutrosophic number theory

Definition 3.1: (Division)

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in Z\}$ the refined neutrosophic ring of integers. For any $x, y \in Z(I_1, I_2)$, we say that $x|y$ if there is $r \in Z(I_1, I_2);\ r.x = y$.

Theorem 3.2: (Form of division in $Z(I_1, I_2)$)

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in Z\}$ the refined neutrosophic ring of integers, $x = (x_0, x_1I_1, x_2I_2), y = (y_0, y_1I_1, y_2I_2)$ be two arbitrary elements in $Z(I_1, I_2)$. Then $x|y$ if and only if

$$x_0|y_0, x_0 + x_2|y_0 + y_2, x_0 + x_1 + x_2|y_0 + y_1 + y_2.$$

Proof:

Suppose that $x|y$ in $Z(I_1, I_2)$, then there is $r = (r_0, r_1I_1, r_2I_2) \in Z(I_1, I_2)$ such that $r.x = y$(*).

By easy computing to equation (*) we get the following equivalent equations:

(a) $r_0x_0 = y_0$, i.e.$x_0|y_0$.

(b) $r_0x_2 + r_2x_2 + r_2x_0 = y_2$.

(c) $r_0x_1 + r_2x_1 + r_1x_0 + r_1x_1 + r_1x_2 = y_1$.

By adding equation (a) to (b) we get (**) $(r_0 + r_2)(x_0 + x_2) = y_0 + y_2, i.e. x_0 + x_2|y_0 + y_2$.

Now, we add equation (**) to (c) to get $(r_0 + r_1 + r_2)(x_0 + x_1 + x_2) = y_0 + y_1 + y_2, i.e.$

$x_0 + x_1 + x_2|y_0 + y_1 + y_2$.

For the converse, we assume that $x_0|y_0, x_0 + x_2|y_0 + y_2, x_0 + x_1 + x_2|y_0 + y_1 + y_2$.

There are

$a, b, c \in Z;\ ax_0 = y_0, b(x_0 + x_2) = y_0 + y_2, c(x_0 + x_1 + x_2) = y_0 + y_1 + y_2$.

We put

$r_0 = a, r_2 = b - a, r_1 = c - b$.

Now, we get $r = (r_0, r_1I_1, r_2I_2) \in Z(I_1, I_2), and r.x = y,$

hence $x|y$.

Definition 3.3: (Congruence)

Let $x=(x_0,x_1I_1,x_2I_2), y=(y_0,y_1I_1,y_2I_2), z=(z_0,z_1I_1,z_2I_2)$ be three elements in $Z(I_1,I_2)$. We say that $x\equiv y(modz)$ if and only if $z|x-y$.

We say that $z=\gcd(x,y)$ if and only if $z|x \ and \ z|y, and \ for \ every \ c|x \ and \ c|y, we \ have \ c|z.$

x,y are called relatively prime in Z(I) if and only if $\gcd(x,y)=(1,0,0)$.

Theorem 3.4: (Form of congruencies in $Z(I_1,I_2)$)

Let $x=(x_0,x_1I_1,x_2I_2), y=(y_0,y_1I_1,y_2I_2), z=(z_0,z_1I_1,z_2I_2)$ be three elements in $Z(I_1,I_2)$. Then $x\equiv y(modz)$ if and only if

$x_0\equiv y_0(modz_0), x_0+x_2\equiv y_0+y_2(modz_0+z_2), x_0+x_1+x_2\equiv y_0+y_1+y_2(modz_0+z_1+z_2)$.

Proof:

We assume that $x\equiv y(modz)$, then $z|x-y$. By Theorem 3.2, we find that $z_0|x_0-y_0, (z_0+z_2)|(x_0+x_2)-(y_0+y_2), (z_0+z_1+z_2)|(x_0+x_1+x_2)-(y_0+y_1+y_2)$, thus

$x_0\equiv y_0(modz_0), x_0+x_2\equiv y_0+y_2(modz_0+z_2), x_0+x_1+x_2\equiv y_0+y_1+y_2(modz_0+z_1+z_2)$.

The converse is trivial.

Example 3.5:

$(1,I_1,2I_2)\equiv(3,-I_1,0)(mod(2,-I_1,I_2))$, that is because

$1\equiv 3(mod2), 1+2=3\equiv(3+0)(mod\ 3), 1+1+2=4\equiv(3-1+0)(mod2\ )$.

Theorem 3.6: (Form of GCD)

Let $x=(x_0,x_1I_1,x_2I_2), y=(y_0,y_1I_1,y_2I_2)$be two elements in $Z(I_1,I_2)$. Then

$\mathrm{r}=\gcd(x,y)=(m,nI_1,tI_2); m=\gcd(x_0,y_0), m+n+t=\gcd(x_0+x_1+x_2, y_0+y_1+y_2), m+t=\gcd(x_0+x_2,y_0+y_2)$.

Proof:

It is clear that $r|x \ and \ r|y$. Let $z=(z_0,z_1I_1,z_2I_2)$ be a common divisor of $x \ and \ y$, then

(a) $z_0|x_0, z_0|y_0$, hence $z_0|m$.

(b) $z_0+z_2|x_0+x_2 and z_0+z_2|y_0+y_2$, hence $z_0+z_2|m+t$.

(c) $z_0+z_1+z_2|x_0+x_1+x_2 and z_0+z_1+z_2|y_0+y_1+y_2$, hence $z_0+z_1+z_2|m+n+t$.

According to the previous discussion, we get $z|r$. Thus $\mathrm{r} = \gcd(x, y) = (m, nI_1, tI_2)$.

Example 3.7:

Let $x = (2, -I_1, 3I_2), y = (1, 3I_1, I_2)$, then $\gcd(x, y) = (1,0,0)$.

Theorem 3.8: (Euclidian division theorem in $Z(I_1, I_2)$)

Let $x = (x_0, x_1I_1, x_2I_2), y = (y_0, y_1I_1, y_2I_2)$be two elements in $Z(I_1, I_2)$.

There are two corresponding elements $q = (q_0, q_1I_1, q_2I_2), r = (r_0, r_1I_1, r_2I_2) \in Z(I_1, I_2); x = qy + r$.

Proof:

By classical division in Z, we can find $s_0, p_0, s_1, p_1, s_2, p_2$such that

$x_0 = y_0s_0 + p_0, (x_0 + x_2) = s_2(y_0 + y_2) + p_2, (x_0 + x_1 + x_2) = s_1(y_0 + y_1 + y_2) + p_1$.

By putting $q_0 = s_0, q_1 = s_1 - s_2, q_2 = s_2 - s_0, r_0 = p_0, r_1 = p_1 - p_2, r_2 = p_2 - p_0$, we get

$x = qy + r$.

Example 3.9:

Consider $x = (2, I_1, -I_2), y = (1, 2I_1, 2I_2)$, then we have $q = (2, 0, -2I_2), r = (0, I_1, I_2)$, where

$x = qy + r$.

Remark 3.10: (Solvability of a linear congruence in $Z(I_1, I_2)$)

To solve a linear congruence $x \equiv y(mod z)$. We should take its corresponding equivalent linear congruencies according to Theorem 3.4. Then we can find its solution easily.

Example 3.11:

Consider the following refined neutrosophic linear congruence

$x \equiv (2, 3I_1, I_2)(mod(1, I_1, 4I_2))$.

The equivalent system of congruencies is

$x_0 \equiv 2(mod\ 1)(I), x_0 + x_2 \equiv 3(mod\ 5)(II), x_0 + x_1 + x_2 \equiv 6(mod\ 6)(III)$.

The congruence (I) has a solution $x_0 = 1$. (II) has a solution $x_0 + x_2 = 3, hence x_2 = 2$.

(III) has a solution $x_0 + x_1 + x_2 = 6, hence x_1 = 3$. Thus the solution of the refined neutrosophic linear congruence is $x = (1, 3I_1, 2I_2)$. It is easy to check that $(1, I_1, 4I_2)|[(1, 3I_1, 2I_2) - (2, 3I_1, I_2)]$.

Definition 3.12:

We define $p = (a, bI_1, cI_2)$ to be a refined neutrosophic prime integer if and only if p is not divided by any other neutrosophic integer different from (1,0,0) and p.

Remark 3.13:

Definition 3.12 is different from the definition of prime elements in a ring, where p is called prime element if it has the following property:

If $p = rq$, then $r\ or\ q$must be a unit.

Theorem 3.14:

$Z(I_1, I_2)$ has no refined neutrosophic primes.

Proof:

Let $p = (a, bI_1, cI_2)$ be any refined neutrosophic integer different from $(1,2I_1, -2I_2)$, we have:

$r = (1,2I_1, -2I_2)$is a divisor of p, that is because $1|a, 1-2|a+c, 1+2-2|a+b+c,$ which is different from (1,0,0) and p. Hence p can not be a refined neutrosophic prime.

If p=$(1,2I_1, -2I_2)$, we have $(1, -2I_1, 0)$ as a divisor different from p and (1,0,0), thus there are no refined neutrosophic primes.

The question about the structure of prime elements in the refined neutrosophic ring of integers is still open. It depends on the structure of the group of units in the refined neutrosophic ring of integers.

Definition 3.16. (Linear Combination in $Z(I\ 1\ , I\ 2\ )$)

Let u,v be non-zero refined neutrosophic integers. Then any refined neutrosophic integer that can be written in the form $ux + vy$ where $x, y \in Z(I\ 1\ , I\ 2\ )$ is called a linear combination of u and v.

Example 3.17:

Let (2,2I_1 ,8I_2),(8,3I_1 ,7I_2) ∈ Z(I 1 ,I 2), we can find refined neutrosophic integers in $Z(I\ 1\ , I\ 2\ )$ that can be written as a linear combination of (2,2I_1 ,8I_2), and (8,3I_1 ,7I_2).

To see this , Let $A(I\ 1\ , I\ 2\ )$ be the set of all linear combinations of (2,2I_1 ,8I_2), and (8,3I_1 ,7I_2).

Then

A(I 1 ,I 2) = (2,2I_1 ,8I_2)(x_0 ,x_1I_1 ,x_2I_2) + (8,3I_1 ,7I_2)(y_0 ,y_1I_1 ,y_2I_2)}

where (x_0 ,x_1I_1 ,x_2I_2),(y_0 ,y_1I_1 ,y_2I_2) ∈ $Z(I\ 1\ , I\ 2\ )$.

Now, let (m_0, m_1I_1, m_2I_2) = (2,2I_1 ,8I_2)(x_0 ,x_1I_1 ,x_2I_2) + (8,3I_1 ,7I_2)(y_0 ,y_1I_1 ,y_2I_2) for some (x_0 ,x_1I_1 ,x_2I_2) and (y_0 ,y_1I_1 ,y_2I_2).

Since

gcd((2,2I_1 ,8I_2),(8,3I_1 ,7I_2)) = (2, I_1,3 I_2).

Then

(m_0 ,m_1I_1 ,m_2I_2) = (2,2I 1 ,8I 2)(x_0 ,x_1I_1 ,x_2I_2) + (8,3I 1 ,7I 2)(y_0 ,y_1I_1 ,y_2I_2)

= (2, I_1,3 I_2)[(1,0I_1 ,I_2)(x_0 ,x_1I_1 ,x_2I_2) + (4,0I_1 ,$-I_2$)(y_0 ,y_1I_1 ,y_2I_2)].

We see that (2, I_1,3 I_2)|(m_0 ,m_1I_1 ,m_2I_2), whatever the values of (x_0 ,x_1I_1 ,x_2I_2) and (y_0 ,y_1I_1 ,y_2I_2).

Hence, (2, I_1,3 I_2)|(m_0 ,m_1I_1 ,m_2I_2) for all (m_0 ,m_1I_1 ,m_2I_2) ∈ $A(I\,1\,,I\,2\,)$. Thus, every member of $A(I\,1\,,I\,2\,)$ is a multiple of (2, I_1,3 I_2).

This observation is recorded in the following theorem.

Theorem 3.15:

Let u = (a_0 ,a_1I_1 ,a_2I_2),v = (b_0 ,b_1I_1 ,b_2I_2) and w = (g_0 ,g_1I_1 ,g_2I_2) be non-zero refined neutrosophic integers and let $w = gcd(u, v)$. Then every linear combination of u and v is a multiple of w. That is,

$w|up + vq,$

for all p = (p_0 ,p_1I_1 ,p_2I_2),q = (q_0 ,q_1I_1 ,q_2I_2) ∈ $Z(I\,1\,,I\,2\,)$.

Proof:

The proof is similar to the classical case.

4. Refined neutrosophic Pell's equation

Definition 4.1:

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in Z\}$ be the refined neutrosophic ring of integers. Refined Neutrosophic Pell's equation is defined as follows:

$X^2 - DY^2 = C; X = (x_0, x_1I_1, x_2I_2), Y = (y_0, y_1I_1, y_2I_2), D = (d_0, d_1I_1, d_2I_2), C = (c_0, c_1I_1, c_2I_2).$

Where $c_i, d_i, x_i, y_i \in Z.$

Theorem 4.2:

Let $Z(I_1,I_2)=\{(a,bI_1,cI_2);a,b,c\in Z\}$ be the refined neutrosophic ring of integers, $X^2-DY^2=C$ be a refined neutrosophic Pell's equation. Then it is equivalent to the following three classical Pell's equations:

(a) ${x_0}^2-d_0{y_0}^2=c_0$.

(b) $(x_0+x_2)^2-(d_0+d_2)(y_0+y_2)^2=c_0+c_2$.

(c) $(x_0+x_1+x_2)^2-(d_0+d_1+d_2)(y_0+y_1+y_2)^2=c_0+c_1+c_2$.

Proof:

We compute:

$X^2=({x_0}^2,[x_0x_1+x_1x_0+x_1x_1+x_1x_2+x_1x_2]I_1,[x_0x_2+x_2x_2+x_2x_0]I_2)$=

$({x_0}^2,[{x_1}^2+2x_0x_1+2x_1x_2]I_1,[{x_2}^2+2x_0x_2]I_2)$,

$DY^2=(d_0,d_1I_1,d_2I_2).({y_0}^2,[{y_1}^2+2y_0y_1+2y_1y_2]I_1,[{y_2}^2+2y_0y_2]I_2)$=

$(d_0{y_0}^2,[d_0{y_1}^2+2d_0y_0y_1+2d_0y_1y_2+d_1{y_0}^2+d_1{y_1}^2+2d_1y_0y_1+2d_1y_1y_2+d_1{y_2}^2+2d_1y_0y_2+$
$d_2{y_1}^2+2d_2y_0y_1+2d_2y_1y_2]I_1,[d_0{y_2}^2+2d_0y_0y_2+d_2{y_0}^2+d_2{y_2}^2+2d_2y_0y_2]I_2)$.

Now we have:

${x_0}^2-d_0{y_0}^2=c_0$. (Equation (a)).

(*)${x_2}^2+2x_0x_2-(d_0{y_2}^2+2d_0y_0y_2+d_2{y_0}^2+d_2{y_2}^2+2d_2y_0y_2)=c_2$.

(**)${x_1}^2+2x_0x_1+2x_1x_2-(\,d_0{y_1}^2+2d_0y_0y_1+2d_0y_1y_2+d_1{y_0}^2+d_1{y_1}^2+2d_1y_0y_1+2d_1y_1y_2+$
$d_1{y_2}^2+2d_1y_0y_2+d_2{y_1}^2+2d_2y_0y_1+2d_2y_1y_2)=c_1$.

By adding (a) to (*) we get:

$(x_0+x_2)^2-(d_0+d_2)(y_0+y_2)^2=c_0+c_2$. (Equation (b)).

By adding (b) to (**) we get:

$(x_0+x_1+x_2)^2-(d_0+d_1+d_2)(y_0+y_1+y_2)^2=c_0+c_1+c_2$.

The converse is clear.

Remark 4.3:

To solve a refined neutrosophic Pell's equation, follow these steps:

(1) Write the equivalent system of classical Pell's equations.

(2) Solve equation (a).

(3) Solve (b).

(4) Solve (c).

(5) Compute x_2, y_2, and then x_1, y_1.

Example 4.5:

Consider the following refined neutrosophic Pell's equation $X^2 - (2,0,I_2)Y^2 = (1,-6I_1,3I_2)$.

The equivalent system is:

(a) ${x_0}^2 - 2{y_0}^2 = 1$.

(b)$(x_0 + x_2)^2 - 3(y_0 + y_2)^2 = 4$.

(c) $(x_0 + x_1 + x_2)^2 - 3(y_0 + y_1 + y_2)^2 = -2$.

Equation (a) has a solution $x_0 = 3, y_0 = 2$. Equation (b) has a solution $y_0 + y_2 = 2, x_0 + x_2 = 4$.

Equation (c) has a solution $x_0 + x_1 + x_2 = 5, y_0 + y_1 + y_2 = 3$. Thus $y_2 = 0, x_2 = 1, y_1 = 1, x_1 = 1$, so $X = (3, I_1, I_2), Y = (2, I_1, 0)$.

5. Open questions

There are many open problems come to light according to this research. This section is devoted to present some important questions in the refined neutrosophic number theory.

Problem 1: Determine the form of prime elements in $Z(I_1, I_2)$.

Problem 2: Define Euler's function in $Z(I_1, I_2)$. Is Euler's Theorem still true in the case of refined neutrosophic integers.

Problem 3: Find an easy algorithm to solve a refined neutrosophic non linear congruence in a similar way to refined neutrosophic Pell's equation.

Problem 4: Find the form of the fundamental theorem in arithmetic in $Z(I_1, I_2)$.

4. Conclusions

In this article, we have established the basic theory of refined neutrosophic integers. Many important concepts and conditions about division, gcd, and congruencies in $Z(I_1, I_2)$. Also, refined neutrosophic Pell's equation was studied and we gave an algorithm to solve this kind of non linear Diophantine equations.

We have listed four open new problems concerning the refined neutrosophic number theory, their solution may lead to a big progression in neutrosophic number theory.

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Topology on Quadripartitioned Neutrosophic Sets

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Abstract: The focus of this paper is to introduce the notion of quadripartitioned neutrosophic topology (Q-NT) on quadripartitioned neutrosophic sets (Q-NS). In this paper, we define quadripartitioned neutrosophic closure, quadripartitioned neutrosophic interior operator of Q-NSs in quadripartitioned neutrosophic topological space (Q-NTS) and investigate several properties of them. Again, we introduce quadripartitioned neutrosophic semi-open (Q-NSO) set, quadripartitioned neutrosophic pre-open (Q-NPO) set, quadripartitioned neutrosophic *b*-open (Q-N-*b*-O) set, and quadripartitioned neutrosophic α-open (Q-Nα-O) set in Q-NTSs. Further, we furnish some suitable examples and prove some basic results on Q-NTS.

Keywords: *Quadripartitioned Neutrosophic Set*; *Q-NT*; *Q-NTS*; *Quadripartitioned-Neutrosophic Closure*; *Quadripartitioned-Neutrosophic Closure*; *Q-NPO*; *Q-N*α*-O*.

1. Introduction: In the year 2005, Smarandache [20] extended the concept of intuitionistic fuzzy set by introducing the notion of neutrosophic set (NS). Later on, many researchers use NS in their theoretical and practical research. In the year 2016, Chatterjee et. al. [4] grounded the idea of quadripartitioned neutrosophic set and defined several similarity measures between two quadripartitioned neutrosophic sets. The idea of neutrosophic topological space (NTS) was presented by Salama and Alblowi [18] in the year 2012. The neutrosophic semi-open mappings are studied by Arokiarani et. al. [2]. Afterwards, Iswaraya and Bageerathi [11] studied the concept of neutrosophic semi-open sets and neutrosophic semi-closed sets. Pushpalatha and Nandhini [15] grounded the idea of neutrosophic generalized closed sets in NTSs. The notion of neutrosophic *b*-open sets in NTSs was presented by Ebenanjar et al. [10]. Rao and Srinivasa [17] grounded the concept of pre open set and pre closed set via neutrosophic topological spaces. Thereafter, Maheswari et. al. [13] studied the neutrosophic generalized *b*-closed sets in NTSs. In the year 2019, Mohammed Ali Jaffer and Ramesh [14] studied the concept of neutrosophic generalized pre-regular closed sets. The generalized neutrosophic *b*-open sets in NTSs was introduced by Das and Pramanik [6]. Das and Pramanik [7] also defined the neutrosophic Φ-open sets and neutrosophic Φ-continuous mappings via NTSs. Recently, Ramesh [16] presented the notion of Ngpr homomorphism via neutrosophic topological spaces. In this study, we introduce the notion of Q-NT and present the concept of quadripartitioned neutrosophic closure and quadripartitioned neutrosophic interior operator in Q-NTSs. It is just the beginning of a new notion. In the future, based on these notions and various open sets on Q-NTSs, many new investigations (compectness, paracompectness, separation axioms) can be easily done. Also, the researchers can use the quadripartitioned neutrosophic topological operators in the area of multi criteria decision making problems.

Research Gap: No investigation on Q-NTSs has been reported in the recent literature.

Motivation: To reduce the gap of research, we present the notion of Q-NTSs.

The remaining part of this paper has been splited into following four sections:
In section-2, we recall some relevant definitions, properties, results of NSs, Q-NSs. Section-3 is on the notion of quadripartitioned neutrosophic topological spaces. In this section, we give some basic definitions, theorems, and propositions on Q-NTSs. In section-4, we conclude our work done in this paper and stating the future scope of research.

2. Some Relevant Definitions:

Definition 2.1. [4] Assume that Ω be a fixed set. Then, a quadripartitioned neutrosophic set (Q-NS) P over Ω is defined by:

$P = \{(q, T_P(q), C_P(q), G_P(q), F_P(q)): q\in\Omega\}$, where $T_P(q)$, $C_P(q)$, $G_P(q)$, and $F_P(q)$ ($\in [0,1]$) are the truth, contradiction, ignorance, and falsity membership values of $q\in\Omega$. So, $0\leq T_P(q)+C_P(q)+G_P(q)+F_P(q) \leq 4$.

Example 2.1. Suppose $\Omega=\{u, v\}$. Then, $P= \{(u,0.5,0.6,0.3,0.6), (v,0.9,0.3,0.4,0.2)\}$ is a Q-NS over Ω.

Definition 2.2. [4] The absolute Q-NS (1_{QN}) and the null Q-NS (0_{QN}) over Ω are defined as follows:

(i) $1_{QN} = \{(q, 1, 1, 0, 0): q\in\Omega\}$;

(ii) $0_{QN} = \{(q, 0, 0, 1, 1): q\in\Omega\}$.

Example 2.2. Suppose that $\Omega=\{u, v\}$. Then, $1_{QN} = \{(u,1, 1, 0, 0), (v, 1, 1, 0, 0)\}$ and $0_{QN} = \{(u,0, 0, 1, 1), (v, 0, 0, 1, 1)\}$.

Remark 2.1. Suppose that R be a Q-NS over Ω. Then, $0_{QN} \subseteq R \subseteq 1_{QN}$.

Definition 2.3.[4] Suppose that $X = \{(q, T_X(q), C_X(q), G_X(q), F_X(q)): q\in\Omega\}$ and $Y = \{(q, T_Y(q), C_Y(q), G_Y(q), F_Y(q)): q\in\Omega\}$ be two Q-NSs over Ω. Then, $X \subseteq Y \Leftrightarrow T_X(q)\leq T_Y(q), C_X(q)\leq C_Y(q), G_X(q)\geq G_Y(q), F_X(q)\geq F_Y(q)$, for all $q\in\Omega$.

Example 2.3. Assume that $\Omega=\{u, v\}$. Suppose that $X=\{(u,0.5,0.3,0.6,0.7), (v,0.2,0.4,0.8,0.8)\}$ and $Y=\{(u,0.3,0.3,0.8,0.8), (v,0.2,0.3,0.9,1.0)\}$ be two Q-NSs over Ω. Then, $Y\subseteq X$.

Definition 2.4.[4] Suppose that $X = \{(q, T_X(q), C_X(q), G_X(q), F_X(q)): q\in\Omega\}$ and $Y = \{(q, T_Y(q), C_Y(q), G_Y(q), F_Y(q)): q\in\Omega\}$ be two Q-NSs over Ω. Then, the union of X and Y is $X \cup Y = \{(q, \max \{T_X(q), T_Y(q)\}, \max \{C_X(q), C_Y(q)\}, \min \{G_X(q), G_X(q)\}, \min \{F_X(q), F_X(q)\}): q\in\Omega\}$.

Example 2.4. Assume that X and Y be two Q-NSs over $\Omega=\{u, v\}$ as shown in Example 2.3. Then, $X\cup Y= \{(u,0.5,0.3,0.6,0.7), (v,0.2,0.4,0.8,0.8)\}$.

Definition 2.5.[4] Suppose that X ={(q, $T_X(q)$, $C_X(q)$, $G_X(q)$, $F_X(q)$): $q\in\Omega$} and Y = {(q, $T_Y(q)$, $C_Y(q)$, $G_Y(q)$, $F_Y(q)$): $q\in\Omega$} be two Q-NSs over Ω. Then, the complement of X is X^c ={(q, $F_X(q)$, $G_X(q)$, $C_X(q)$, $T_X(q)$): $q\in\Omega$}.

Example 2.5. Suppose that Ω={u, v} and X={(u,0.8,0.8,0.5,1.0), (v,1.0,0.5,0.3,0.8)} be a Q-NS over Ω. Then, X^c={(u,1.0,0.5,0.8,0.8), (v,0.8,0.3,0.5,1.0)}.

Definition 2.6.[4] Suppose that X={(q, $T_X(q)$, $C_X(q)$, $G_X(q)$, $F_X(q)$): $q\in\Omega$} and Y = {(q, $T_Y(q)$, $C_Y(q)$, $G_Y(q)$, $F_Y(q)$): $q\in\Omega$} be two Q-NSs over Ω. Then, the intersection of X and Y is $X\cap Y$={(q, min {$T_X(q)$, $T_Y(q)$}, min {$C_X(q)$, $C_Y(q)$}, max {$G_X(q)$, $G_X(q)$}, max {$F_X(q)$, $F_X(q)$}): $q\in\Omega$}.

Example 2.6. Assume that X and Y be two Q-NSs over Ω={u, v} as shown in Example 2.3. Then, $X\cap Y$={(u,0.3,0.3, 0.8,0.8), (v,0.2,0.3,0.9, 1.0)}.

3. Quadripartitioned Neutrosophic Topology:

Definition 3.1. Let Ω be a fixed set. A collection $\Im$ of some Q-NSs over Ω is called a Q-NT on Ω, if the following conditions holds:

(i) 1_{QN}, $0_{QN}\in\Im$;

(ii) $M_1\cap M_2\in\Im$ whenever $M_1, M_2\in\Im$;

(iii) $\cup M_i\in\Im$, whenever $\{M_i: i\in\Delta\}\subseteq\Im$.

Then, $(\Omega,\Im)$ is called a Q-NTS. Every element of $\Im$ are called a quadripartitioned neutrosophic open set (Q-NOS). If $M\in\Im$, then M^c is called a quadripartitioned neutrosophic closed set (Q-NCS).

Remark 3.1. In every Q-NTS, 0_{QN} and 1_{QN} are both Q-NOS and Q-NCS.

Example 3.1. Let Ω = {u, v}. Assume that M = {(u,0.9,0.5,0.7,1.0), (v,0.3,0.1,0.5,0.7): u, v ∈ Ω} and N = {(u,0.9,0.7,0.1,0.9), (v,0.4,0.6,0.1,0.2): u, v ∈Ω} be two Q-NSs over Ω. Then, $\Im$ ={0_{QN}, 1_{QN}, M, N} forms a Q-NT on Ω.

The quadripartitioned-neutrosophic interior and quadripartitioned-neutrosophic closure of a Q-NS in a Q-NTS are defined as follows:

Definition 3.2. Let us consider a quadripartitioned neutrosophic subset X of a Q-NTS $(\Omega,\Im)$. Then, the quadripartitioned-neutrosophic closure (Q-N_{cl}) of X is the intersection of all Q-NCSs containing X and the quadripartitioned-neutrosophic interior (Q-N_{int}) of X is the union of all Q-NOSs contained in X, i.e.

$Q\text{-}N_{cl}(X) = \cap\{Z : X \subseteq Z$ and Z is a Q-NCS in $(\Omega,\Im)\}$;

$Q\text{-}N_{int}(X) = \cup\{Y : Y \subseteq X$ and Y is a Q-NOS in $(\Omega,\Im)\}$.

Remark 3.2. It is clear that $Q\text{-}N_{cl}(X)$ is the smallest Q-NCS in $(\Omega,\Im)$ that contains X and $Q\text{-}N_{int}(X)$ is the largest Q-NOS in $(\Omega,\Im)$ which is contained in X.

Theorem 3.1. If T and R be any two quadripartitioned neutrosophic subsets of a Q-NTS $(\Omega,\Im)$, then

(i) $Q\text{-}N_{int}(T)\subseteq T\subseteq Q\text{-}N_{cl}(T)$;

(ii)$T\subseteq R\Rightarrow Q\text{-}N_{cl}(T)\subseteq Q\text{-}N_{cl}(R)$;

(iii)$T\subseteq R\Rightarrow Q\text{-}N_{int}(T)\subseteq Q\text{-}N_{int}(R)$;

(iv)T is an N^*-OS iff $Q\text{-}N_{int}(T) = T$;

(v)T is an N^*-CS iff $Q\text{-}N_{cl}(T) =T$.

Proof. (i) From the previous definition, we have $Q\text{-}N_{int}(T) = \cup\{R$: R is a Q-NOS in $(\Omega,\Im)$ and $R\subseteq T\}$.

Since, each $R \subseteq T$, so $\cup\{R$: R is a Q-NOS in $(\Omega,\Im)$ and $R \subseteq T\} \subseteq T$, i.e. $Q\text{-}N_{int}(T) \subseteq T$.

Again, $Q\text{-}N_{cl}(T) = \cap\{Z$: Z is a Q-NCS in $(\Omega,\Im)$ and $T \subseteq Z\}$. Since, each $Z \supseteq T$, so $\cap\{Z$: Z is a Q-NCS in $(\Omega,\Im)$ and $T \subseteq Z\} \supseteq T$, i.e. $Q\text{-}N_{cl}(T) \supseteq T$.

Therefore, $Q\text{-}N_{int}(T) \subseteq T \subseteq Q\text{-}N_{cl}(T)$.

(ii) Assume that T and R be any two quadripartitioned neutrosophic subsets of a Q-NTS $(\Omega,\Im)$ such that $T \subseteq R$.

Now, $Q\text{-}N_{cl}(T) = \cap\{Z$: Z is a Q-NCS in $(\Omega,\Im)$ and $T \subseteq Z\}$

$\subseteq \cap\{Z$: Z is a Q-NCS in $(\Omega,\Im)$ and $R \subseteq Z\}$ [since $T \subseteq R$]

$= Q\text{-}N_{cl}(R)$

$\Rightarrow Q\text{-}N_{cl}(T) \subseteq Q\text{-}N_{cl}(R)$.

Therefore, $T \subseteq R \Rightarrow Q\text{-}N_{cl}(T) \subseteq Q\text{-}N_{cl}(R)$.

(iii) Assume that T and R be any two quadripartitioned neutrosophic subsets of a Q-NTS $(\Omega,\Im)$ such that $T \subseteq R$.

Now, $Q\text{-}N_{int}(T) = \cup\{Z$: Z is a Q-NOS in $(\Omega,\Im)$ and $Z \subseteq T\}$

$\subseteq \cup\{Z$: Z is a Q-NOS in $(\Omega,\Im)$ and $Z \subseteq R\}$ [since $T \subseteq R$]

$= Q\text{-}N_{int}(R)$

$\Rightarrow Q\text{-}N_{int}(T) \subseteq Q\text{-}N_{int}(R)$.

Therefore, $T \subseteq R \Rightarrow Q\text{-}N_{int}(T) \subseteq Q\text{-}N_{int}(R)$.

(iv) Assume that T be a Q-NOS in a Q-NTS $(\Omega,\Im)$. Now, $Q\text{-}N_{int}(T) = \cup\{Z$: Z is a Q-NOS in $(\Omega,\Im)$ and $Z \subseteq T\}$. Since, T is a Q-NOS in $(\Omega,\Im)$, so T is the largest Q-NOS, which is contained in T. Therefore, $\cup\{Z$: Z is a Q-NOS in $(\Omega,\Im)$ and $Z \subseteq T\} = T$. This implies, $Q\text{-}N_{int}(T) = T$.

(v) Assume that T be a Q-NCS in a Q-NTS $(\Omega,\Im)$. Now, $Q\text{-}N_{cl}(T) = \cap\{Z : Z$ is a Q-NCS in $(\Omega,\Im)$ and $T \subseteq Z\}$. Since, T is a Q-NCS in $(\Omega,\Im)$, so T is the smallest Q-NCS, which contains T. Therefore, $\cap\{Z$: Z is a Q-NCS in $(\Omega,\Im)$ and $T \subseteq Z\} = T$. This implies, $Q\text{-}N_{cl}(T) = T$.

Theorem 3.2. Let E be a quadripartitioned neutrosophic subset of a Q-NTS $(\Omega,\Im)$. Then,

(i) $(Q\text{-}N_{int}(E))^c = Q\text{-}N_{cl}(E^c)$;

(ii) $(Q\text{-}N_{cl}(E))^c = Q\text{-}N_{int}(E^c)$.

Proof. (i) Suppose that $(\Omega,\Im)$ be a Q-NTS and $E=\{(w, T_E(w), C_E(w), G_E(w), F_E(w))$: $w \in \Omega\}$ be a quadripartitioned neutrosophic subset of Ω. Now, $Q\text{-}N_{int}(E) = \cup\{Z_i$: $i \in \Delta$ and Z_i is a Q-NOS in $(\Omega,\Im)$ such that $Z_i \subseteq E\} = \{(w, \vee T_{Z_i}(w), \vee C_{Z_i}(w), \wedge G_{Z_i}(w), \wedge F_{Z_i}(w))$: $w \in \Omega\}$, where for all $i \in \Delta$ and Z_i is a Q-NOS in $(\Omega,\Im)$ such that $Z_i \subseteq E$. This implies, $(Q\text{-}N_{int}(E))^c = \{(w, \wedge T_{Z_i}(w), \wedge C_{Z_i}(w), \vee G_{Z_i}(w), \vee F_{Z_i}(w))$: $w \in \Omega\}$. Since, $\wedge T_{Z_i}(w) \leq T_E(w)$, $\wedge C_{Z_i}(w) \leq C_E(w)$, $\vee G_{Z_i}(w) \geq G_E(w)$, $\vee F_{Z_i}(w) \geq F_E(w)$, for each $i \in \Delta$ and $w \in \Omega$, so $Q\text{-}N_{cl}(E^c) = \{(w, \wedge T_{Z_i}(w), \wedge C_{Z_i}(w), \vee G_{Z_i}(w), \vee F_{Z_i}(w))$: $w \in \Omega\}\} = \cap\{Z_i$: $i \in \Delta$ and Z_i is a Q-NCS in $(\Omega,\Im)$ such that $E^c \subseteq Z_i\}$. Therefore, $(Q\text{-}N_{int}(E))^c = Q\text{-}N_{cl}(E^c)$.

(ii) Suppose that $(\Omega,\Im)$ be a Q-NTS and $E=\{(w, T_E(w), C_E(w), G_E(w), F_E(w))$: $w \in \Omega\}$ be a quadripartitioned neutrosophic subset of Ω. Now, $Q\text{-}N_{cl}(E) = \cap\{Z_i$: $i \in \Delta$ and Z_i is a Q-NCS in $(\Omega,\Im)$ such that $Z_i \supseteq E\} = \{(w, \wedge T_{Z_i}(w), \wedge C_{Z_i}(w), \vee G_{Z_i}(w), \vee F_{Z_i}(w))$: $w \in \Omega\}$, where for all $i \in \Delta$ and Z_i is a Q-NCS in $(\Omega,\Im)$ such that $Z_i \supseteq E$. This implies, $(Q\text{-}N_{cl}(E))^c = \{(w, \vee T_{Z_i}(w), \vee C_{Z_i}(w), \wedge G_{Z_i}(w), \wedge F_{Z_i}(w))$: $w \in \Omega\}$. Since, $\vee T_{Z_i}(w) \geq T_E(w)$, $\vee C_{Z_i}(w) \geq C_E(w)$, $\wedge G_{Z_i}(w) \leq G_E(w)$, $\wedge F_{Z_i}(w) \leq F_E(w)$, for each $i \in \Delta$ and $w \in \Omega$, so $Q\text{-}N_{int}(E^c) =$

$\{(w,\vee T_{Z_i}(w),\vee C_{Z_i}(w),\wedge G_{Z_i}(w),\wedge F_{Z_i}(w))$: $w\in\Omega\}=\cup\{Z_i$: $i\in\Delta$ and Z_i is a Q-NOS in $(\Omega,\Im)$ such that $Z_i\subseteq E^c\}$. Therefore, $(Q\text{-}N_{cl}(E))^c = Q\text{-}N_{int}(E^c)$.

Definition 3.3. Suppose that $(\Omega,\Im)$ be a Q-NTS. Then, a Q-NS W over Ω is called a

(i) quadripartitioned neutrosophic semi-open (Q-NSO) set iff $W \subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W))$;

(ii) quadripartitioned neutrosophic pre-open (Q-NPO) set iff $W \subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W))$.

The complement of Q-NSO sets and Q-NPO sets are called Q-NSC sets and Q-NPC sets respectively.

Theorem 3.3. Suppose that $(\Omega,\Im)$ be a Q-NTS. If W and M are two Q-NSO sets, then $W\cup M$ is also a Q-NSO set.

Proof. Suppose that $(\Omega,\Im)$ be a Q-NTS. Let W, M be two Q-NSO sets in $(\Omega,\Im)$. Therefore,

$W\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W))$ (1)

and $M\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(M))$ (2)

From (1) and (2), we have

$W\cup M\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W))\cup Q\text{-}N_{cl}(Q\text{-}N_{int}(M)) = Q\text{-}N_{cl}(Q\text{-}N_{int}(W) \cup Q\text{-}N_{int}(Y)) \subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W \cup M))$.

This implies, $W\cup M \subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W \cup M))$. Therefore, $W\cup M$ is a Q-NSO set in $(\Omega,\Im)$.

Theorem 3.4. Suppose that $(\Omega,\Im)$ be a Q-NTS. If W is a Q-NOS, then W is also a Q-NSO set.

Proof. Suppose that $(\Omega,\Im)$ be a Q-NTS and W be a Q-NOS. Therefore, $W=Q\text{-}N_{int}(W)$. It is known that $W\subseteq Q\text{-}N_{cl}(W)$. This implies, $W\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W))$. Therefore, W is a Q-NSO set.

Theorem 3.5. Suppose that $(\Omega,\Im)$ be a Q-NTS. If W and M are two Q-NPO sets, then $W\cup M$ is also a Q-NPO set.

Proof. Suppose that $(\Omega,\Im)$ be a Q-NTS. Let W, M be two Q-NPO sets in $(\Omega,\Im)$. Therefore,

$W\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W))$ (3)

and $M\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(M))$ (4)

From (3) and (4), we have,

$W\cup M \subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W))\cup Q\text{-}N_{int}(Q\text{-}N_{cl}(M)) \subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W)\cup Q\text{-}N_{cl}(M)) = Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cup M))$.

This implies, $W\cup M\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W \cup M))$. Therefore, $W\cup M$ is a Q-NPO set in $(\Omega,\Im)$.

Theorem 3.6. Assume that $(\Omega,\Im)$ be a Q-NTS. If W is a Q-NOS, then W is also a Q-NPO set.

Proof. Suppose that $(\Omega,\Im)$ be a Q-NTS and W be a Q-NOS in $(\Omega,\Im)$. So $W=Q\text{-}N_{int}(W)$. It is known that $W\subseteq Q\text{-}N_{cl}(W)$. This implies, $Q\text{-}N_{int}(W) \subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W))$ i.e. $W =Q\text{-}N_{int}(W) \subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W))$. Hence, $W\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W))$. Therefore, W is a Q-NPO set.

Definition 3.4. Let us assume that $(\Omega,\Im)$ be a Q-NTS. Then W, a Q-NS over Ω is called a quadripartitioned neutrosophic α-open (Q-Nα-O) set if and only if $W\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(Q\text{-}N_{int}(W)))$.

Remark 3.3. (i) The complement of a Q-Nα-O set is called a quadripartitioned neutrosophic α-closed (Q-Nα-C) set.

(ii) In a Q-NTS $(\Omega,\Im)$, every Q-NOS is a Q-Nα-O set.

(iii) In a Q-NTS $(\Omega,\Im)$, every Q-NCS is a Q-Nα-C set.

Theorem 3.7. In a Q-NTS $(\Omega,\Im)$,

(i) every Q-Nα-O set is a Q-NPO set;

(ii) every Q-Nα-O set is a Q-NSO set.

Proof. (i) Suppose that W be a Q-Nα-O set in a Q-NTS $(\Omega,\Im)$. Therefore, $W\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(Q\text{-}N_{int}(W)))$. It is known that, $Q\text{-}N_{int}(W)\subseteq W$. This implies, $Q\text{-}N_{cl}(Q\text{-}N_{int}(W))\subseteq Q\text{-}N_{cl}(W)$. Therefore,

$Q\text{-}N_{int}(Q\text{-}N_{cl}(Q\text{-}N_{int}(W)))\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W)$. This implies, $W\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W)$. Therefore, W is a Q-NPO set. Hence, every Q-Nα-O set is a Q-NPO set.

(ii) Suppose that W be a Q-Nα-O set in a Q-NTS $(\Omega,\Im)$. So, $W\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(Q\text{-}N_{int}(W)))$. It is known that, $Q\text{-}N_{int}(Q\text{-}N_{cl}(Q\text{-}N_{int}(W)))\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W))$. This implies, $W\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W))$. Therefore, W is a Q-NSO set. Hence, every Q-Nα-O set is a Q-NSO set.

Definition 3.5. Suppose that $(\Omega,\Im)$ be a Q-NTS. Then W, a Q-NS over Ω is called a quadripartitioned neutrosophic b-open (Q-N-b-O) set if and only if $W\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W))\cup Q\text{-}N_{cl}(Q\text{-}N_{int}(W))$. A quadripartitioned neutrosophic set W is called a quadripartitioned neutrosophic b-closed (Q-N-b-C) set if and only if W^c is a Q-N-b-O set.

Remark 3.4. A Q-NS W over Ω is called a Q-N-b-C set if and only if $Q\text{-}N_{int}(Q\text{-}N_{cl}(W))\cap Q\text{-}N_{cl}(Q\text{-}N_{int}(W))\subseteq W$.

Theorem 3.8. Suppose that $(\Omega,\Im)$ be a Q-NTS. If W, M be two Q-N-b-O sets in $(\Omega,\Im)$, then $W\cup M$ is also a Q-N-b-O set.

Proof. Suppose that $(\Omega,\Im)$ be a Q-NTS. Let W, M be two Q-N-b-O sets in $(\Omega,\Im)$. So

$$W\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W)) \cup Q\text{-}N_{cl}(Q\text{-}N_{int}(W)), \quad (5)$$

and $M\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(M)) \cup Q\text{-}N_{cl}(Q\text{-}N_{int}(M))$. (6)

We know that, $W\subseteq W\cup M$ and $M\subseteq W\cup M$. This implies,

$$Q\text{-}N_{cl}(Q\text{-}N_{int}(W))\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W\cup M)), \quad (7)$$

$$Q\text{-}N_{int}(Q\text{-}N_{cl}(W))\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cup M)), \quad (8)$$

$$Q\text{-}N_{cl}(Q\text{-}N_{int}(M))\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W\cup M)), \quad (9)$$

$$Q\text{-}N_{int}(Q\text{-}N_{cl}(M))\subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cup M)). \quad (10)$$

From Eq. (5) and Eq. (6), we have,

$$W\cup M\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W)) \cup Q\text{-}N_{int}(Q\text{-}N_{cl}(W)) \cup Q\text{-}N_{cl}(Q\text{-}N_{int}(M)) \cup Q\text{-}N_{int}(Q\text{-}N_{cl}(M))$$

$$\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W\cup M)) \cup Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cup M)) \cup Q\text{-}N_{cl}(Q\text{-}N_{int}(W\cup M)) \cup Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cup M))$$

[by eqs (7), (8), (9), (10)]

$$= Q\text{-}N_{cl}(Q\text{-}N_{int}(W\cup M)) \cup Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cup M)).$$

This implies, $W\cup M\subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W\cup M)) \cup Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cup M))$. Hence, $W\cup M$ is a Q-N-b-O set.

Theorem 3.9. Suppose that $(\Omega,\Im)$ be a Q-NTS. If W, M be two Q-N-b-C sets in $(\Omega,\Im)$, then $W\cap M$ is also a Q-N-b-C set.

Proof. Suppose that $(\Omega,\Im)$ be a Q-NTS. Let W, M be two Q-N-b-C sets in $(\Omega,\Im)$. So,

$$Q\text{-}N_{int}(Q\text{-}N_{cl}(W)) \cap Q\text{-}N_{cl}(Q\text{-}N_{int}(W)) \subseteq W \quad (11)$$

and $Q\text{-}N_{int}(Q\text{-}N_{cl}(M)) \cap Q\text{-}N_{cl}(Q\text{-}N_{int}(M)) \subseteq M$ (12)

We know that, $W\cap M\subseteq W$ and $W\cap M\subseteq M$. This implies,

$$Q\text{-}N_{cl}(Q\text{-}N_{int}(W\cap M)) \subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W)) \quad (13)$$

$$Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cap M)) \subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W)) \quad (14)$$

$$Q\text{-}N_{cl}(Q\text{-}N_{int}(W\cap M)) \subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(M)) \quad (15)$$

$$Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cap M)) \subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(M)) \quad (16)$$

From Eq. (11) and Eq. (12), we have

$$W\cap M \supseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W)) \cap Q\text{-}N_{cl}(Q\text{-}N_{int}(W)) \cap Q\text{-}N_{int}(Q\text{-}N_{cl}(M)) \cap Q\text{-}N_{cl}(Q\text{-}N_{int}(M))$$

$$\supseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cap M)) \cap Q\text{-}N_{cl}(Q\text{-}N_{int}(W\cap M)) \cap Q\text{-}N_{int}(Q\text{-}N_{cl}(W\cap M)) \cap Q\text{-}N_{cl}(Q\text{-}N_{int}(W\cap M))$$

[by using eqs. (13), (14), (15) & (16)]

$= Q\text{-}N_{int}(Q\text{-}N_{cl}(W \cap M)) \cap Q\text{-}N_{cl}(Q\text{-}N_{int}(W \cap M))$.

This implies, $W \cap M \supseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W \cap M)) \cap Q\text{-}N_{int}(Q\text{-}N_{cl}(W \cap M))$. Hence, $W \cap M$ is a Q-N-*b*-C set in $(\Omega,\Im)$.

Theorem 3.10. In a Q-NTS $(\Omega,\Im)$, every Q-NPO set is a Q-N-*b*-O set.

Proof. Let W be a Q-NPO set in a Q-NTS $(\Omega,\Im)$. So, $W \subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W))$. This implies, $W \subseteq Q\text{-}N_{int}(Q\text{-}N_{cl}(W)) \cup Q\text{-}N_{cl}(Q\text{-}N_{int}(W))$. Therefore, W is a Q-N-*b*-O set. Hence, every Q-NPO set is a Q-N-*b*-O set.

Theorem 3.11. In a Q-NTS $(\Omega,\Im)$, every Q-NSO set is a Q-N-*b*-O set.

Proof. Let W be a Q-NSO set in a Q-NTS $(\Omega,\Im)$. Therefore, $W \subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W))$. This implies, $W \subseteq Q\text{-}N_{cl}(Q\text{-}N_{int}(W)) \cup Q\text{-}N_{int}(Q\text{-}N_{cl}(W))$. Therefore, W is a Q-N-*b*-O set. Hence, every Q-NSO set is a Q-N-*b*-O set.

4. Conclusion: In this article, we introduce topology on Q-NSs. We study different types of open sets like Q-NPO set, Q-NSO set, Q-Nα-O set, and Q-N-*b*-O set, etc. By defining Q-NPO set, Q-NSO set, Q-N-*b*-O set, and Q-Nα-O set, we formulate some theorems, remarks on quadripartitioned neutrosophic topological space. Further, few illustrative examples are given. In the future, based on these notions and various open sets on Q-NTSs, many new investigations (compectness, para-compectness, conectedness, separation axioms) can be easily done. Also, the quadripartitioned neutrosophic topological operators can be used in the area of multi criteria decision making problems.

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On The Characterization Of Maximal and Minimal Ideals In Several Neutrosophic Rings

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Abstract: If R(I) is a neutrosophic ring, then every subset of R(I) has the form $M = P + SI$, where P,S are subsets of the classical ring R. The objective of this paper is to determine the necessary and sufficient condition on classical subsets P and S which makes M an ideal in R(I). The main result is proving that every classical ideal in a neutrosophic ring must be an AH-ideal and determining the form of maximal and minimal ideals in R(I). Also, a similar discussion of the case of refined neutrosophic rings will be presented.

Keywords: Neutrosophic ring, refined neutrosophic ring, maximal ideal, minimal ideal, AH-ideal.

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1. Introduction

Neutrosophy is a generalized view on intuitionistic fuzzy logic, it is considered as a new generalization of fuzzy ideas. The concept of neutrosophic set was built over the idea of dividing logical degrees into truth, falsity, and indeterminacy. This concept has an interesting effect in the study of optimization [16], computer science [18,22], decision making [15], and medical studies [19,21]. More applications of neutrosophy in many areas can be found in [33,34,35,36,37].

In the field of pure mathematics, we find many applications such as neutrosophic spaces [9,11,30], modules [4], groups [26], and rings [3,28,29].

The concept of neutrosophic ring was proposed by Smarandache and Kandasamy in [24], where they defined neutrosophic ring, neutrosophic ideal and neutrosophic isomorphism. Recently, many interesting results about neutrosophic rings were discussed [1,,3,20,14].

A neutrosophic ideal is an ideal by classical meaning i.e. it is a subset N from R(I) with the following properties: (N,+) is a subgroup of $(R(I), +)$, and $r.x \in N \text{ } for \text{ } all \text{ } x \in N \text{ } and \text{ } r \in R(I)$.

AH-ideals are subsets $N = P + QI$, where P,Q are two classical ideals in the classical ring R [1].

In [1], we find that AH-ideals are not supposed to be neutrosophic ideals, the converse is still unknown. A general study of AH-ideals and their relationships with Kothe's conjecture can be found in [31].

Through the first section of this paper, we present a characterization theorem of classical neutrosophic ideals in a neutrosophic ring R(I). We prove that each neutrosophic ideal must be an AH-ideal. In addition, we determine the necessary and sufficient condition for any subset $M = P + SI$ to be a neutrosophic ideal only using classical properties of P and S.

On the other hand, Agboola et.al presented a generalization of neutrosophic sets by splitting the degree of indeterminacy I into two degrees of indeterminacy I_1, I_2 . This idea was used widely in algebra by studying refined neutrosophic rings [6,7], and n-refined neutrosophic rings and modules [12,13,25].

AH-ideals in refined neutrosophic rings were defined in [2], as subsets with form (P, QI_1, SI_2), where P, Q, S are classical ideals in the ring R. According to [2], refined neutrosophic AH-ideals are not supposed to be ideals by classical meaning. In the second section of this work, we prove a characterization theorem of refined neutrosophic subsets to be classical refined ideals by depending on classical properties of P,Q,S only. This theorem ensures that each refined neutrosophic classical ideal must be a refined neutrosophic AH-ideal.

The main results of this work is to describe the structure of all non trivial maximal or minimal ideals in neutrosophic and refined neutrosophic rings.

This work is an extension of efforts to classify maximal and minimal ideals in neutrosophic rings in [38].

All rings through this paper are considered with unity 1.

Motivation

Our motivation is to close an important research gap by determining all maximal and minimal ideals and their forms in neutrosophic rings, and refined neutrosophic rings.

2.Preliminaries

Definition 2.1: [24]

Let R be a ring, I be the indeterminacy with property $I^2 = I$, then the neutrosophic ring is defined as follows:

$R(I) = \{a + bI; a, b \in R\}$.

Neutrosophic ring can be considered as an extension of classical ring by adding an indeterminacy element to R.

Definition 2.2 : [24]

Let R(I) be a neutrosophic ring, it is called commutative if and only if $xy = yx \forall\ x, y \in R(I)$..

Definition 2.3: [24]

Let R(I) be a neutrosophic ring, a non-empty subset P of R(I) is called a neutrosophic ideal if

.)a) P is a neutrosophic subring of R(I(

(b) $rx \in P\ for\ exery\ x \in P\ and\ r \in R(I)$.

Definition 2.4: [1]

Let R(I) be a neutrosophic ring and $P = P_0 + P_1 I = \{a_0 + a_1 I\ ;\ a_0 \in P_0\ , a_1 \in P_1\}$.

(a)We say that P is an AH-ideal if P_0, P_1 are ideals in the ring R.

(b)We say that P is an AHS-ideal if $P_0 = P_1$.

(c) The AH-ideal P is called null if $P_0, P_1 \in \{R\ , O\ \}$.

Theorem 2.5: [1]

Let R(I) be a neutrosophic ring and $P = P_0 + P_1 I$ be an AH-ideal, then P is not a neutrosophic ideal in general by the classical meaning.

Definition 2.6: [6]

The element I can be split into two indeterminacies $I_1\ , I_2$ with conditions:

$\mathrm{I}_1{}^2 = \mathrm{I}_1\ , I_2{}^2 = I_2\ , I_1 I_2 = I_2 I_1 = I_1$.

Definition 2.7: [6]

If X is a set then $\mathrm{X}(I_1, I_2) = \{(a, bI_1, cI_2) : a\ , b\ , c \in X\ \}$ is called the refined neutrosophic set generated by X , I_1, I_2.

Definition 2.8: [2]

Let (R,+,×) be a ring, (R(I_1, I_2) , + ,×) is called a 2-refined neutrosophic ring generated by R ,I_1, I_2.

Example 2.9: [6]

The refined neutrosophic ring of integers is $Z(I_1, \mathrm{I}_2)$={$(a, bI_1, cI_2); a, b, c \in Z$}.

Definition 2.10: [2]

Let (R(I_1, I_2) , + , .) be a refined neutrosophic ring and P_0, P_1, P_2 be ideals in the ring R then the set $P = (P_0, P_1 I_1, P_2 I_2) = \{(a, bI_1, cI_2): a \in P_0, b \in P_1, c \in P_2\}$ is called a refined neutrosophic AH-ideal.

If $P_0 = P_1 = P_2$ then P is called a refined neutrosophic AHS-ideal.

3. Ideals in Neutrosophic rings

Remark 3.1:

Since every neutrosophic ring R(I) can be understood as $R(I) = R + RI = (a + bI; a, b \in R\}$,

Then each subset of R(I) has the form $M = P + SI$; P,S are two subsets of R. We call P the real part, S the neutrosophic part of M.

An important question arises here. This question is:

When M is a neutrosophic ideal of R(I)?. In other words, what conditions on the real part P and neutrosophic part S which make M be an ideal?.

The following theorem clarifies the necessary and sufficient condition to answer the previous question.

Theorem 3.2:

Let R(I) be a neutrosophic ring, $M = P + SI$ be any subset of R(I), then

M is a neutrosophic ideal if and only if the following conditions are true:

(a) P is an ideal on R.

(b) P is contained in S.

(c) S is an ideal of R.

Proof:

Firstly, we assume that (a),(b), and (c) are true, we have:

$(M, +)$ is a subgroup of $(R(I), +)$, that is because if $a + bI, c + dI \in M; a, c \in P, b, d \in S$, we find

$(a + bI) - (c + dI) = (a - c) + (b - d)I \in M; a - c \in P, b - d \in S$..

Now, suppose that $a + bI \in M \ and \ r = m + nI \in R(I)$, we have

$r.(a + bI) = m.a + I[m.b + n.b + n.a]$, by the assumption, we regard that $m.b + n.b \in S, and \ n.a \in P \leq S$, thus $r.(a + bI) = m.a + I[m.b + n.b + n.a] \in P + SI = M$, which means that M is a neutrosophic ideal of R(I).

Conversely, we suppose that $M = P + SI$ is a neutrosophic ideal of R(I). Let a, c be two arbitrary elements in P, and b, d be two arbitrary elements in S, we have $a + bI, c + dI \in M$, by using the assumption we have M as an ideal, hence $(a + bI) - (c + dI) = (a - c) + (b - d)I \in M = P + SI$, thus

$a - c \in P, and\ b - d \in S$, thus $(P, +), (S, +)$ are two subgroups of $(R, +)$.

For every $r \in R, we\ have\ r = r + 0I \in R(I), and\ r.(a + bI) = r.a + r.bI \in M = P + SI$, thus

$r.a \in P, r.b \in S$, this means that P, S are ideals in the classical ring R.

Now, we prove that P is contained in S. We have $(1 - I) \in R(I)$, that is because R(I) has a unity 1. On the other hand, we can write $(1 - I)(a + bI) = (a - aI) \in M = P + SI$, and that is because M is an ideal of R(I), hence $-a \in S$, thus $a \in S$, by regarding that a is an arbitrary element of P, we get that $P \leq S$.

The previous theorem ensures that each ideal is an AH-ideal, since P,S are supposed to be classical ideals of R.

Example 3.3:

Let $R = Z$ be the ring of integers, $R(I) = Z(I) = \{a + bI;\ a, b \in Z\}$ be the corresponding neutrosophic ring, we have:

(a) $P =< 2 >, Q =< 4 >, S =< 3 >$, are three ideals of R, with $Q \leq P$.

(b) $M = Q + PI = \{4m + 2nI; m, n \in Z\}$ is an ideal of R(I).

(c) $N = P + SI = \{2m + 3nI; m, n \in Z\}$ is not a neutrosophic ideal, that is because P is not contained in S.

Example 3.4:

Let $R = Z_8$ be the ring of integers modulo 8. $R(I) = \{a + bI; a, b \in Z_8\}$, be the corresponding neutrosophic ring. Consider the set $M = \{0,4,2I,4I,6I,4 + 2I,4 + 6I,4 + 4I\}$. We have M as an ideal of R(I), that is because $M =< 4 > +< 2 > I\ and < 4 >\leq< 2 >$.

Theorem 3.5:

The following theorem determines the form of maximal ideals in R(I).

Let R(I) be a neutrosophic ring, $M = P + SI$ be an ideal of R(I), then M is maximal if and only if P is maximal in R with $S = R$ or $M = R(I)$.

Proof:

Suppose that M is maximal of R(I), let $N = V + WI$ be any ideal of R(I) with the property $M \leq N$, then $P \leq V$ and $S \leq W$, by the assumption of the maximality of M, we find that $N = M\ or\ N = R(I)$, this implies that

$(V = P\ with\ W = R) or\ (V = W = R)$, which means that P is maximal in R or $P = R$. On the other hand $P \leq S$ and P is maximal, thus $S = P\ or\ S = R$. Since $P + SI \leq P + RI$, hence the only non trivial maximal ideal is $M = P + RI$, with P as a maximal ideal in R.

The converse is clear.

Theorem 3.6:

The following theorem describes minimal ideals in R(I).

Let R(I) be a neutrosophic ring, $M = P + SI$ be an ideal of R(I), then M is minimal if and only if S is minimal in R and $P = \{0\}$.

Proof:

Suppose that M is minimal of R(I), let $N = V + WI$ be any ideal of R(I) with the property $N \leq M$, then $V \leq P$ and $W \leq S$, by the assumption of the minimality of M, we find that $N = M\ or\ N = \{0\}$, this implies that

$(V = P\ with\ W = S) or\ (W = N = \{0\})$, which means that P, S are minimal in R. On the other hand $P \leq S$ and S is minimal, thus $S = P\ or\ P = \{0\}$. Since SI is a sub-ideal of P+SI, hence $P = \{0\}$.

The converse is clear.

Remark 3.7:

According to Theorem 5.1 and Theorem 6.1, we get a full description of the structure of maximal and minimal ideals in the neutrosophic ring R(I).

(a) Non trivial Maximal ideals in R(I) has the form {P+RI}, where P is maximal in R.

(b) Non trivial minimal ideals have the form {{0}+SI} where S is minimal in R.

Example 3.8:

Let Z(I) be the neutrosophic ring of integers, non trivial maximal ideals in Z(I) are

$\{< p > +ZI\}$, where p is any prime number.

4. Ideals in refined neutrosophic rings

Remark 4.1:

Since every refined neutrosophic ring R(I_1, I_2) can be understood as $R(I_1, I_2) = (R, RI_1, RI_2) = \{(a, bI_1, cI_2); a, b, c \in R\}$,

Then each subset of $R(I_1, I_2)$ has the form $M = (P, QI_1, SI_2)$; P, Q, S are two subsets of R.

An important question arises here. This question is:

When M is a refined neutrosophic ideal of $R(I_1, I_2)$?. In other words, what conditions on P, Q, S which make M an ideal?.

The following theorem clarifies the necessary and sufficient condition to answer the previous question.

Theorem 4.2:

Consider the following:

$R(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in R\}$ be a refined neutrosophic ring, $M = (P, QI_1, SI_2)$ be a subset of $R(I_1, I_2)$

. M is an ideal of $R(I_1, I_2)$ if and only if

(a) P, Q, S are ideals on R

(b)$P \leq S \leq Q$.

Proof:

Suppose that M is an ideal, then we have for every $a, m \in P and\ b, n \in Q\ and\ c, t \in S$,

$x = (a, bI_1, cI_2), y = (m, nI_1, tI_2)$, are two elements of $R(I_1, I_2)$.

$x - y = (a - m, [b - n]I_1, [c - t]I_2) \in M, thus\ a - m \in P, b - n \in Q, c - t \in S, hence$(P, +), (Q, +), (S, +) are subgroups of $(R, +)$.

For every $r \in R, we\ have\ (r, 0,0) \in R(I_1, I_2) and (r, 0,0).(a, bI_1, cI_2) = (r.a, r.bI_1, r.cI_2) \in M$, thus $r.a \in P, r.b \in Q, r.c \in S$, thus P, Q, S are ideals of R.

On the other hand, we have $(1,0, -I_2) \in R(I_1, I_2), thus\ (1,0, -I_2).(a, bI_1, cI_2) = (a, 0, -aI_2) \in M, hence - a \in S\ and\ P \leq S$, that is because a is an arbitrary element in P.

Also, $(1, -I_1, 0) \in R(I_1, I_2), thus\ (1, -I_1, 0).(0, bI_1, cI_2) = (0, -cI_1, cI_2) \in M, hence - c \in Q\ and\ S \leq Q$.

That is because c is an arbitrary element in S.

For the converse, we suppose that (a) and (b) are true, we have (M,+) as a subgroup of $R(I_1, I_2)$.

Let $r = (m, nI_1, tI_2) \in R(I_1, I_2) and x = (a, bI_1, cI_2) \in M$, we have

$r.x = (m.a, [m.b + n.a + n.b + n.c + t.b]I_1, [m.c + t.a + t.c]I_2)$, it is clear that

$m.c + t.c \in S, t.a \in P \leq S, thus\ m.a + t.a + t.c \in S$. Also,

$m.b + n.b + t.b \in Q, and\ n.a + n.c \in S \leq Q, thus\ m.b + n.a + n.b + n.c + t.b \in Q$. This implies that $r.x \in M$, hence M is an ideal.

Example 4.3:

Let $Z(I_1, I_2)$ be the refined neutrosophic ring of integers, we have

$(< 8 >, < 2 > I_1, < 4 > I_2) = \{(8a, 2bI_1, 4cI_2); a, b, c \in Z\}$ is an ideal in $Z(I_1, I_2)$. That is because $< 8 > \leq < 4 > \leq < 2 >$.

Example 4.4:

Let $Z_{20}(I_1, I_2)$ be the refined neutrosophic ring of integers modulo 20, we have

$(0, < 5 > I_1, < 10 > I_2) =$

$\{(0,0,0), (0,5I_1, 0), (0,5I_1, 10I_2), (0,10I_1, 0), (0,10I_1, 10I_2), (0,15I_1, 0), (0,15I_1, 10I_2), (0,0,10I_2\}$.

Theorem 4.5:

Consider the following:

$R(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in R\}$ be a refined neutrosophic ring, $M = (P, QI_1, SI_2)$ be any non trivial maximal ideal of $R(I_1, I_2)$

M has the following form:

(P, RI_1, RI_2). Where P is any maximal ideal of R.

Proof:

We assume that M is a maximal ideal, and $N = (X, YI_1, ZI_2)\ is\ an\ ideal\ of\ R(I_1, I_2)\ with\ M \leq N$, hence $M = N\ or\ N = R(I_1, I_2)$, we have $P = X, Q = Y, S = Z$, or $X = Y = Z = R$. This implies that P, S, Q should be maximal; but we have that

$P \leq S \leq Q$, hence $(R = S, Q = R; P\ is\ maximal\ in\ R)$.

The converse is clear.

Theorem 4.6:

Consider the following:

$R(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in R\}$ be a refined neutrosophic ring, $M = (P, QI_1, SI_2)$ be any non trivial minimal ideal of $R(I_1, I_2)$

M has the following form:

$(0, PI_1, 0)$. Where P is any minimal ideal of R.

Proof:

The proof is similar to Theorem 5.2.

Example 4.7:

(a) Consider $Z_8(I_1, I_2)$ the refined neutrosophic ring of integers modulo 8, we have $< 4 >= \{0,4\}$ is a minimal ideal of Z_8. Hence $(0, < 4 > I_1, 0) = \{(0,0,0), (0,4I_1, 0)\}$ is a minimal ideal of $Z_8(I_1, I_2)$.

(b) $< \mathbf{2} >= \{\mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{0}\}$ is maximal in $\boldsymbol{Z_8}$. Hence $(< \mathbf{2} >, \boldsymbol{Z_8 I_1}, \boldsymbol{Z_8 I_2}) = \{(\boldsymbol{a}, \boldsymbol{bI_1}, \boldsymbol{cI_2}); \boldsymbol{a} \in < \mathbf{2} >$ $\boldsymbol{and}\ \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{Z_8}\}$ is maximal in $\boldsymbol{Z_8(I_1, I_2)}$

4. Conclusions

In this article, we have studied algebraic ideals in neutrosophic rings and refined neutrosophic rings, where we proved that every ideal in a neutrosophic or refined neutrosophic ring with unity must be an AH-ideal. Also, we have determined the structure of all maximal and minimal ideals in any neutrosophic ring and any refined neutrosophic ring with unity. In addition, many examples were built to clarify the validity of this work.

As a future research direction, we aim to classify neutrosophic factors and refined neutrosophic factors.

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Entropy Measures for Interval Neutrosophic Vague Sets and Their Application in Decision Making

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Abstract: Entropy measure is an important tool in measuring uncertain information and plays a vital role in solving Multi Criteria Decision Making (MCDM). At present, various entropy measures in literature are developed to measure the degree of fuzziness. However, they could not be used to deal with interval neutrosophic vague set (INVS) environment. In this study, two kinds of entropy measures are proposed as the extension of the entropy measure of single valued neutrosophic set (SVNS). First, we construct the axiomatic definition of INVS and propose a new formula for the entropy measure of INVS. Based on this measure, we develop two multi criteria decision making methods. Subsequently, an illustrative example of investment problems under INVS is given to demonstrate the proposed entropy measures. Finally, a comparative analysis is presented to illustrate the rationality and effectiveness of the proposed entropy measures.

Keywords: decision making ; fuzzy set theory; neutrosophic set theory; interval neutrosophic vague set theory; entropy measure; uncertainty.

1. Introduction

Different types of uncertainties arise in real life problems and many complex systems such as information fusion systems, medical diagnosis, decision making, and image processing. The issue of uncertainties in decision making recently become increasingly important since the appearance of classical mathematics. Hence, entropy measure notation has been introduced for measuring fuzzy information. Fuzziness, a characteristic of incomplete information, arises from the lack of crisp distinction between the elements belonging and not belonging to a set. Shanon and Weaver [1], [2] first proposed an entropy measure known as Shanon entropy. In 1968, Zadeh [3] extended the axiom of Shanon entropy to fuzzy entropy based on the fuzzy subset with respect to the concerned probability distribution. Later, Luca and Termimi [4] presented a formal definition of fuzzy entropy and defined several axioms. In addition, Sander [5] introduced Shanon fuzzy entropy measure and proved sharpness, valuation and general additivity and all properties of the fuzzy entropy. To investigate a more comprehensive entropy, Xie and Bedrosian [6] focused on the total fuzzy entropy. To counter the disadvantages of the total fuzzy entropy, Pal and Pal [7] introduced the objective measure in hybrid entropy used to get proper defuzzification of a certain fuzzy set. Shi and Yuan [8] suggested interval entropy, interval similarity measure, interval distance measure and interval inclusion measure of fuzzy set. As for the intuitionistic fuzzy sets (IFS) and their generalization by Attanasov [9], Burillo and Bustince [10] developed an intuitionistic fuzzy entropy measure and defined an axiomatic definition. Szmidt and Kacprzyk [11] suggested a new entropy measure that is based on a geometric representation of the intuitionistic fuzzy sets (IFS). Wei et al. [12] proposed entropy measures for interval-valued intuitionistic fuzzy sets (IVIFSs) and applied them in the case study of pattern recognition. Garg [13] developed an entropy measure under IVIFSs and used the proposed measure in solving MCDM with unknown attribute weights. Meanwhile, Rashid et al. [14] constructed the concept of entropy measure distance based on IVIFSs. All the above related entropy researches were dealt with under uncertain and fuzzy information. However, fuzzy sets cannot be dealt with indeterministic and inconsistent information.

Considering this limitation, Smarandache [15] proposed a neutrosophic set (NS) which is the three components of truth, indeterminacy, and falsity degrees and that can be denoted as T, I, F respectively. NS is characterized independently and the ranges of functions T, I, F are in form of real standard and the nonstandard interval $\left]^{-}0, 1^{+}\right[$ which cannot be used in real applications. Therefore, Wang et al. [16] proposed single valued neutrosophic set (SVNS) where the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree in form of real standard interval. Later, Wang [17] introduced interval neutrosophic set (INS) as an extension of SVNS whose values are interval rather than real numbers. Alkhazaleh [18], introduced a neutrosophic vague set (NV) by incorporating the features of SVNS and vague set [19]. Besides that, he also defined several operators for NV and proved related properties. NV has played a significant role in the uncertain information system. In certain NV sets, the degree of truth, falsity, and indeterminacy of a given statement cannot be strictly described in real-world contexts, but it is instead denoted by several possible interval values. To overcome this problem, Hashim et al. [20] introduced an interval neutrosophic vague set (INVS) by upgrading the membership functions in several interval membership degrees. The advantage of INVS is that it can deal with more uncertain information than NV on similar decision situations. In light of its significance, many scholars have worked to improve the concept of neutrosophic in decision making [21]–[28].

The recent rapid developments of NS and its generalization have heightened the need for measuring the fuzziness degree under NS setting. Therefore, Majumdar and Samanta [29] proposed the entropy and distance measures under SVNS. They defined the formula for entropy measure and proved related properties. Later, Wu et al. [30] suggested an entropy that overcomes the limitations in Majumdar and Samanta's entropy. They suggested a better concept of complement of SVNS where $A^{C} = \left\{ F_A(x), 1 - I_A(x), T_A(x) \middle| x \in X \right\}$ instead of $\bar{A} = \left\{ (1 - T_A(x), 1 - I_A(x), 1 - F_A(x)) \middle| x \in X \right\}$. Later, Garg [31] suggested SVNS entropy of order α. For various parameter values, the suggested entropy is more stable and scalable. In addition, Abu Qamar and Hassan [32] proposed several entropy, distance, and similarity measures for Q-Neutrosophic soft sets and applied this measure in medical diagnosis and decision making problems. The ranking method used in this example is based on the smallest entropy value. In 2020, Thao and Smarandache [33], proposed a new entropy measure based similarity measure under SVNS. They claimed that the proposed entropy by Majumdar and Samanta did not satisfy the axiomatic definition where it contradicts with the value of entropy in $[0,1]$. Hence, they proposed two entropy measures based on the intuitionistic fuzzy set entropy. Ye and Du [34] established some distances, similarity, and entropy measures and studied its relationship. They also compared the proposed entropy measure with existing entropy measures. Aydogdu and Sahin [35] defined two entropy measures for SVNS and INS. Based on this measure, they proposed a decision making problems under SVNS and INS. Quek et al. [36] proposed a new formula for the entropy measure under plithogenic sets. Very recently, Ye [37] suggested entropy measures based on trigonometric functions under simplified neutrosophic sets. In addition, he proposed a new ranking method of entropy values by considering positive and negative arguments.

Entropy measure also is a well-known approach for generating the weight in MCDM. Biswas et al. [38] used entropy proposed by [29] to measure the weight of criteria and applied it in grey relational analysis (GRA). Also, Thao and Smarandache [39] proposed complex proportional assessment (COPRAS) with weight calculated based on the proposed entropy. Besides that, Barukab et al. [40] proposed entropy measure under spherical fuzzy information to calculate unknown weights information and applied it in the technique for order performance by similarity to ideal solution (TOPSIS) method. Ye [41] proposed entropy weights-based correlation coefficients of IVIFS and used it to solve decision making problems with unknown information on criteria weights. Until now, there is no entropy measures under the INVS environment. A summary of the previous researcher's contribution is presented in Table 1.

In this paper, we introduce the INVS entropy measure that is extended from the concept of SVNS entropy in [29], [33]. This measure will resolve the limitation of the entropy proposed by Majumdar and Samanta that has been claimed as invalid by Thao and Smarandache [33]. The illustration of the limitation in [29], [33] is discussed in Subsection 3.1 and 3.2. These are the motivations that driven us to investigate a more appropriate concept of entropy measure under the INVS environment. To do so, the rest of this paper is organized as follows: Section 2 presents the definition of INVS and its relation on INVS. In Section 3, we introduce two types of entropy of INVS, propose its formula and prove related properties. The illustrative example based on the proposed entropy of INVS and comparative analysis are presented in Section 4 and the conclusion is presented in Section 5.

Table 1: Summary of contribution under neutrosophic environment

Author	Year	Set	Contributions	Gaps
Majumdar and Samanta [29]	2014	SVNS	• Introduce similarity and entropy of a neutrosophic set	• Did not satisfy the axiomatic definition for the proposed entropy measure.
Garg [31]	2016	SVNS	• Single-valued neutrosophic entropy of order α • Consider the pair of their membership functions as well as hesitation degree between them.	• Ranking methods of entropy values are not always reasonable
Wu et al. [30]	2018	SVNS	• Introduce similarity measure and cross-entropy of single-valued neutrosophic sets	• Did not consider the standard definition of a complement of SVNS
Abu Qamar and Hassan[32]	2018	Q-neutrosophic soft set	• Introduce entropy, distance, and similarity measure	• Ranking methods of entropy values are not always reasonable

Thao and Smarandache [33]	2019	SVNS	• Introduce entropy-based similarity measures of single valued neutrosophic sets • A natural extension of the concept of entropy measure of fuzzy sets and IFS	• Ranking methods of entropy values are not always reasonable
Quek et al. [36]	2020	Plithogenic sets	• Introduce Entropy Measures for Plithogenic Sets	• This entropy is limited when applied to the Plithogenic set. Due to the complexity and novelty of Plitogenic sets.
Ye [37]	2021	Simplified neutrosophic sets	• Entropy measures based on trigonometric functions	• Only considers the positive and negative arguments regarding the entropy values for different alternatives.

2. Preliminaries

In this section, we review some basic concepts related to INVS, which will be used in the rest of the paper.

Definition 2.1:[20] Let X be a universe discourse and an INVS A is written as follows:

$$A = \left\{x, \left[\bar{T}_A^L(x), \bar{T}_A^U(x)\right], \left[\bar{I}_A^L(x), \bar{I}_A^U(x)\right], \left[\bar{F}_A^L(x), \bar{F}_A^U(x)\right] > \middle| x \in X\right\} \quad (1)$$

Whose truth membership, indeterminacy membership, and falsity membership functions are defined as:

$$\bar{T}_A^L(x) = \left[T^{L-}, T^{L+}\right], \bar{T}_A^U(x) = \left[T^{U-}, T^{U+}\right], \bar{I}_A^L(x) = \left[I^{L-}, I^{L+}\right], \bar{I}_A^U(x) = \left[I^{U-}, I^{U+}\right] \text{ and}$$

$$\bar{F}_A^L(x) = \left[F^{L-}, F^{L+}\right], \bar{F}_A^U(x) = \left[F^{U-}, F^{U+}\right] \quad (2)$$

The symbols L and U denote the lower and upper of the intervals in which

$$T^{L+} = 1 - F^{L-}, F^{L+} = 1 - T^{L-}$$

$$F^{U-} = 1 - T^{U+}, T^{U-} = 1 - F^{U+}$$

and satisfying
$$0 \le T^{L-} + T^{U-} + I^{L-} + I^{U-} + F^{L-} + F^{U-} \le 4 \quad (3)$$

$$0 \le T^{L+} + T^{U+} + I^{L+} + I^{U+} + F^{L+} + F^{U+} \le 4 \quad (4)$$

Definition 2.2 [20] Let $A=\left\{x,\left[\bar{T}_A^L(x),\bar{T}_A^U(x)\right],\left[\bar{I}_A^L(x),\bar{I}_A^U(x)\right],\left[\bar{F}_A^L(x),\bar{F}_A^U(x)\right]>\mid x\in X\right\}$ and $B=\left\{x,\left[\bar{T}_B^L(x),\bar{T}_B^U(x)\right],\left[\bar{I}_B^L(x),\bar{I}_B^U(x)\right],\left[\bar{F}_B^L(x),\bar{F}_B^U(x)\right]>\mid x\in X\right\}$ be two INVSs. Then the relation of INVS is defined as follows:

i. $A=B$ if and only if $A=B$ if $T_A^L(x)=T_B^L(x)$, $T_A^U(x)=T_B^U(x)$, $I_A^L(x)=I_B^L(x)$

$I_A^U(x)=I_B^U(x)$, $F_A^L(x)=F_B^L(x)$ and $F_A^U(x)=F_B^U(x)$.

ii. $A\subseteq B$ if and only if $T_A^L(x)\le T_B^L(x)$, $T_A^U(x)\le T_B^U(x)$, $I_A^L(x)\ge I_B^L(x)$, $I_A^U(x)\ge I_B^U(x)$,

$F_A^L(x)\ge F_B^L(x)$ and $F_A^U(x)\ge F_B^U(x)$

iii. $A\cap(\cup)B=C$

$$T_C^L(x)=\left[\cap(\cup)\left(T_A^{L-},T_B^{L-}\right),\cap(\cup)\left(T_A^{L+},T_B^{L+}\right)\right],\ T_C^U(x)=\left[\cap(\cup)\left(T_A^{U-},T_B^{U-}\right),\cap(\cup)\left(T_A^{U+},T_B^{U+}\right)\right],$$

$$I_C^L(x)=\left[\cup(\cap)\left(I_A^{L-},I_B^{L-}\right),\cup(\cap)\left(I_A^{L+},I_B^{L+}\right)\right],\ I_C^U(x)=\left[\cup(\cap)\left(I_A^{U-},I_B^{U-}\right),\cup(\cap)\left(I_A^{U+},I_B^{U+}\right)\right],$$

$$F_C^L(x)=\left[\cup(\cap)\left(F_A^{L-},F_B^{L-}\right),\cup(\cap)\left(F_A^{L+},F_B^{L+}\right)\right],F_C^U(x)=\left[\cup(\cap)\left(F_A^{U-},F_B^{U-}\right),\cup(\cap)\left(F_A^{U+},F_B^{U+}\right)\right]$$

iv. A^C

$$\left(T^L\right)^C(x)=\left[1-T_A^{L+},1-T_A^{L-}\right],\left(T^U\right)^C(x)=\left[1-T_A^{U+},1-T_A^{U-}\right],$$

$$\left(I^L\right)^C(x)=\left[1-I_A^{L+},1-I_A^{L-}\right],\left(I^U\right)^C(x)=\left[1-I_A^{U+},1-I_A^{U-}\right],$$

$$\left(F^L\right)^C(x)=\left[1-F_A^{L+},1-F_A^{L-}\right],\left(F^U\right)^C(x)=\left[1-F_A^{U+},1-F_A^{U-}\right]$$

3. The entropy of INVS

In this section, we introduce two entropy to measure the fuzziness degree of INVS information. The entropy of INVS is defined by two formulas which are based on interval approximation and INVS entropy generalized from the existing entropy of SVNS by Majumdar and Samantha [29]. We first give the axiomatic definition of INVS entropy.

The definition is derived to satisfy several conditions need in INVS entropy, as shown below:

(i) The entropy will be null when the set is a crisp set,

(ii) The entropy will be maximum if the set is completely INVS,

(iii) The INVS entropy and its complement is equal, and

(iv) If the degree of lower and upper approximation for truth membership, indeterminacy membership and falsity membership of each element decreases, the sum will do as well, and therefore this set becomes fuzzier and consequently the entropy should increase.

In light of the conditions stated above, the axiomatic definition of INVS entropy is defined as follows:

Definition 3.1: Let $\text{INVS}(X)$ be a set of all INVSs on (X) and $E:\text{INVS}(X)\rightarrow[0,1]$ satisfying all following properties:

(E0) (Nonenegativity) $0\le E(A)\le 1$

(E1) (Minimality) $E(A)=0$ if A is a crisp set i.e

$\left[T_A^{L-}(x_i),T_A^{L+}(x_i)\right]=[1,1]$, $\left[T_A^{U-}(x_i),T_A^{U+}(x_i)\right]=[1,1]$, $\left[I_A^{L-}(x_i),I_A^{L+}(x_i)\right]=[0,0]$,

$\left[I_A^{U-}(x_i),I_A^{U+}(x_i)\right]=[0,0]$, $\left[F_A^{L-}(x_i),F_A^{L+}(x_i)\right]=[0,0]$, $\left[F_A^{L-}(x_i),F_A^{L+}(x_i)\right]=[0,0]$ or

$\left[T_A^{L-}(x_i),T_A^{L+}(x_i)\right]=[0,0]$, $\left[T_A^{U-}(x_i),T_A^{U+}(x_i)\right]=[0,0]$, $\left[I_A^{L-}(x_i),I_A^{L+}(x_i)\right]=[0,0]$,

$\left[I_A^{U-}(x_i),I_A^{U+}(x_i)\right]=[0,0]$, $\left[F_A^{L-}(x_i),F_A^{L+}(x_i)\right]=[1,1]$, $\left[F_A^{L-}(x_i),F_A^{L+}(x_i)\right]=[1,1]$ for all $x_i\in X$

(E2) (Maximality) $E(A)=1$ if $\left[T_A^{L-}(x_i),T_A^{L+}(x_i)\right]=[0.5,0.5]$, $\left[T_A^{U-}(x_i),T_A^{U+}(x_i)\right]=[0.5,0.5]$,

$\left[I_A^{L-}(x_i),I_A^{L+}(x_i)\right]=[0.5,0.5]$, $\left[I_A^{U-}(x_i),I_A^{U+}(x_i)\right]=[0.5,0.5]$, $\left[F_A^{L-}(x_i),F_A^{L+}(x_i)\right]=[0.5,0.5]$,

$\left[F_A^{U-}(x_i),F_A^{U+}(x_i)\right]=[0.5,0.5]$ for all $x_i\in X$

(E3) (Symmetric) $E(A)=E(A^C)$ for all $A\in\text{INVS}(X)$,

(E4) (Resolution) $E(A)\ge E(B)$ if , i.e,

$T_A^{L-}(x_i)\ge T_B^{L-}(x_i)$, $T_A^{L+}(x_i)\ge T_B^{L+}(x_i)$, $T_A^{U-}(x_i)\ge T_B^{U-}(x_i)$, $T_A^{U+}(x_i)\ge T_B^{U+}(x_i)$;

$I_A^{L-}(x_i)\le I_B^{L-}(x_i)$, $I_A^{L+}(x_i)\le I_B^{L+}(x_i)$, $I_A^{U-}(x_i)\le I_B^{U-}(x_i)$, $I_A^{U+}(x_i)\le I_B^{U+}(x_i)$;

$F_A^{L-}(x_i)\le F_B^{L-}(x_i)$, $F_A^{L+}(x_i)\le F_B^{L+}(x_i)$, $I_A^{U-}(x_i)\le I_B^{U-}(x_i)$, $F_A^{U+}(x_i)\le F_B^{U+}(x_i)$.

Now, we define the INVS entropy based on interval approximation $\left(E_{\Delta\text{INVS}}\right)$ and entropy $\left(E_{\text{INVS}}\right)$ generalized from SVNS entropy in Subsection 3.1 and 3.2. The notations and descriptions are used in the proposed entropy measures are presented in Table 2. The proposed entropy should satisfy the axiomatic Definition 3.1.

Table 2: Some notations and descriptions

Notation	Description
X , x	Universal set, element of X
T, I, F	Truth, indeterminacy and false membership functions
$\left[\bar{T}_A^L(x), \bar{T}_A^U(x)\right]$	Truth interval valued with respect to upper bound and lower bound.
$\bar{T}_A^L(x)=\left[T^{L-}, T^{L+}\right]$	Truth lower interval valued functions with respect to the beginning of interval and end of the interval.
$\bar{T}_A^U(x)=\left[T^{U-}, T^{U+}\right]$	Truth upper interval valued functions with respect to the beginning of interval and end of the interval.
$\bar{I}_A^L(x)=\left[I^{L-}, I^{L+}\right]$	Indeterminacy lower interval valued functions with respect to the beginning of interval and end of the interval.
$\bar{I}_A^U(x)=\left[I^{U-}, I^{U+}\right]$	Indeterminacy upper interval valued functions with respect to the beginning of interval and end of the interval.
$\bar{F}_A^L(x)=\left[F^{L-}, F^{L+}\right]$	Falsity lower interval valued functions with respect to the beginning of interval and end of the interval.
$\bar{F}_A^U(x)=\left[F^{U-}, F^{U+}\right]$	Falsity lower interval valued functions with respect to the beginning of interval and end of the interval.
x_i	Element set of criteria in the universe X
$E_{\Delta INVS}$	INVS entropy based on interval approximation
E_{INVS}	INVS entropy generalized from SVNS entropy

3.1. INVS entropy based on interval approximation

In this subsection, a new concept of INVS entropy is generated from the entropy measure proposed by Thao and Smarandache [33]. This measure can deal only with SVNS set. Moreover, this entropy used the concept of natural extension of the concept of entropy measure of fuzzy sets and IFS. The SVNS entropy measure is defined as follows:

$$E_T(A) = 1 - \frac{1}{n}\sum_{i=1}^{n}\frac{|T_A(x_i)-0.5|+|F_A(x_i)-0.5|+|I_A(x_i)-0.5|+|I_{A^c}(x_i)-0.5|}{2} \quad (5)$$

By using this measure, we presented a new concept of the INVS entropy based on interval approximation denoted as $E_{\Delta INVS}$. The interval approximation represents the average possible membership degree of truth, indeterminate, and falsity of element x . Definition of INVS entropy measure is presented as follows:

Definition 3.1.1: The entropy of the interval neutrosophic vague sets denoted as $E_{\Delta INVS}(A)$ and defined by

$$E_{\Delta INVS}(A) = 1 - \frac{1}{2n}\sum_{i=1}^{n} E_{\Delta INVS}(A)(x_i)$$

where

$$\sum_{i=1}^{n} E_{\Delta INVS}(A)(x_i) = \left|\frac{T_A^{L-}(x_i)+T_A^{L+}(x_i)+T_A^{U-}(x_i)+T_A^{U+}(x_i)}{4}-0.5\right| + \left|\frac{F_A^{L-}(x_i)+F_A^{L+}(x_i)+F_A^{U-}(x_i)+F_A^{U+}(x_i)}{4}-0.5\right| + \left|\frac{I_A^{L-}(x_i)+I_A^{L+}(x_i)+I_A^{U-}(x_i)+I_A^{U+}(x_i)}{4}-0.5\right| + \left|\frac{I_{A^c}^{L-}(x_i)+I_{A^c}^{L+}(x_i)+I_{A^c}^{U-}(x_i)+I_{A^c}^{U+}(x_i)}{4}-0.5\right|$$

for all $i = 1, 2, \ldots, n$ (6)

Theorem 1. $E_{\Delta INVS}(A)$ as defined in Definition 3.1.1 is entropy for INVSs

Now, we show that $E_{\Delta INVS}(A)$ satisfies all properties given in Definition 3.1.

Proof:

(E1) if A is a crisp set then $\left[T_A^{L-}(x_i), T_A^{L+}(x_i)\right] = [1,1]$, $\left[T_A^{U-}(x_i), T_A^{U+}(x_i)\right] = [1,1]$, $\left[I_A^{L-}(x_i), I_A^{L+}(x_i)\right] = [0,0]$, $\left[I_A^{U-}(x_i), I_A^{U+}(x_i)\right] = [0,0]$, $\left[F_A^{L-}(x_i), F_A^{L+}(x_i)\right] = [0,0]$, $\left[F_A^{U-}(x_i), F_A^{U+}(x_i)\right] = [0,0]$ or $\left[T_A^{L-}(x_i), T_A^{L+}(x_i)\right] = [0,0]$, $\left[T_A^{U-}(x_i), T_A^{U+}(x_i)\right] = [0,0]$, $\left[I_A^{L-}(x_i), I_A^{L+}(x_i)\right] = [0,0]$, $\left[I_A^{U-}(x_i), I_A^{U+}(x_i)\right] = [0,0]$, $\left[F_A^{L-}(x_i), F_A^{L+}(x_i)\right] = [1,1]$, $\left[F_A^{U-}(x_i), F_A^{U+}(x_i)\right] = [1,1]$

for all $x_i \in X$ we have

$$E_{\Delta INVS}(A)(x_i) = \left|\frac{1+1+1+1}{4}-0.5\right| + \left|\frac{0+0+0+0}{4}-0.5\right| + \left|\frac{0+0+0+0}{4}-0.5\right| + \left|\frac{1+1+1+1}{4}-0.5\right| = 2$$

It implies that, $E_{\Delta INVS}(A) = 1 - \frac{1}{2(1)}(2) = 0$ or

$$E_{\Delta INVS}(A)(x_i) = \left|\frac{0+0+0+0}{4}-0.5\right| + \left|\frac{1+1+1+1}{4}-0.5\right| + \left|\frac{0+0+0+0}{4}-0.5\right| + \left|\frac{1+1+1+1}{4}-0.5\right| = 2$$

It implies that, $E_{\Delta INVS}(A) = 1 - \frac{1}{2(1)}(2) = 0$

Therefore, the INVS entropy will be null $(E_{INVS}(A) = 0)$ when the set is a crisp set.

(E2) $\left[T_A^{L-}(x_i), T_A^{L+}(x_i)\right] = [0.5,0.5]$, $\left[T_A^{U-}(x_i), T_A^{U+}(x_i)\right] = [0.5,0.5]$, $\left[I_A^{L-}(x_i), I_A^{L+}(x_i)\right] = [0.5,0.5]$, $\left[I_A^{U-}(x_i), I_A^{U+}(x_i)\right] = [0.5,0.5]$, $\left[F_A^{L-}(x_i), F_A^{L+}(x_i)\right] = [0.5,0.5]$ and $\left[F_A^{U-}(x_i), F_A^{U+}(x_i)\right] = [0.5,0.5]$

$$E_{\Delta INVS}(A)(x_i) = \left|\frac{0.5+0.5+0.5+0.5}{4}-0.5\right| + \left|\frac{0.5+0.5+0.5+0.5}{4}-0.5\right| + \left|\frac{0.5+0.5+0.5+0.5}{4}-0.5\right| + \left|\frac{0.5+0.5+0.5+0.5}{4}-0.5\right| = 0$$

It implies that

$$E_{\Delta INVS}(A) = 1 - \frac{1}{2(1)}(0) = 1$$

Therefore, the entropy will be maximum $(E_{INVS}(A) = 1)$ if the set is completely INVS.

(E3) $E_{\Delta INVS}(A)=1-\frac{1}{2n}\sum_{i=1}^{n}\left|\frac{\left(1-T_A^{L+}(x_i)\right)+\left(1-T_A^{L-}(x_i)\right)+\left(1-T_A^{U+}(x_i)\right)+\left(1-T_A^{U-}(x_i)\right)}{4}-0.5\right|+$

$$\left|\frac{\left(1-F_A^{L+}(x_i)\right)+\left(1-F_A^{L-}(x_i)\right)+\left(1-F_A^{U+}(x_i)\right)+\left(1-F_A^{U-}(x_i)\right)}{4}-0.5\right|+$$

$$\left|\frac{\left(1-I_A^{L+}(x_i)\right)+\left(1-I_A^{L-}(x_i)\right)+\left(1-I_A^{U+}(x_i)\right)+\left(1-I_A^{U-}(x_i)\right)}{4}-0.5\right|+$$

$$\left|\frac{\left(1-I_{A^c}^{L+}(x_i)\right)+\left(1-I_{A^c}^{L-}(x_i)\right)+\left(1-I_{A^c}^{U+}(x_i)\right)+\left(1-I_{A^c}^{U-}(x_i)\right)}{4}-0.5\right|$$

$$=1-\frac{1}{2n}\sum_{i=1}^{n}\left|\frac{4-\left(T_A^{L+}(x_i)+T_A^{L-}(x_i)+T_A^{U+}(x_i)+T_A^{U-}(x_i)\right)}{4}-0.5\right|+\left|\frac{4-\left(F_A^{L-}(x_i)+F_A^{L+}(x_i)+F_A^{U-}(x_i)+F_A^{U+}(x_i)\right)}{4}-0.5\right|+$$

$$\left|\frac{4-\left(I_A^{L-}(x_i)+I_A^{L+}(x_i)+I_A^{U-}(x_i)+I_A^{U+}(x_i)\right)}{4}-0.5\right|+\left|\frac{4-\left(I_{A^c}^{L-}(x_i)+I_{A^c}^{L+}(x_i)+I_{A^c}^{U-}(x_i)+I_{A^c}^{U+}(x_i)\right)}{4}-0.5\right|$$

$$=1-\frac{1}{2n}\sum_{i=1}^{n}\left|\frac{4-\left(T_A^{L+}(x_i)+T_A^{L-}(x_i)+T_A^{U+}(x_i)+T_A^{U-}(x_i)\right)-0.5(4)}{4}\right|+\left|\frac{4-\left(F_A^{L-}(x_i)+F_A^{L+}(x_i)+F_A^{U-}(x_i)+F_A^{U+}(x_i)\right)-0.5(4)}{4}\right|+$$

$$\left|\frac{4-\left(I_A^{L-}(x_i)+I_A^{L+}(x_i)+I_A^{U-}(x_i)+I_A^{U+}(x_i)\right)-0.5(4)}{4}\right|+\left|\frac{4-\left(I_{A^c}^{L-}(x_i)+I_{A^c}^{L+}(x_i)+I_{A^c}^{U-}(x_i)+I_{A^c}^{U+}(x_i)\right)-0.5(4)}{4}\right|$$

$$=1-\frac{1}{2n}\sum_{i=1}^{n}\left|\frac{2-\left(T_A^{L+}(x_i)+T_A^{L-}(x_i)+T_A^{U+}(x_i)+T_A^{U-}(x_i)\right)}{4}\right|+\left|\frac{2-\left(F_A^{L-}(x_i)+F_A^{L+}(x_i)+F_A^{U-}(x_i)+F_A^{U+}(x_i)\right)}{4}\right|+$$

$$\left|\frac{2-\left(I_A^{L-}(x_i)+I_A^{L+}(x_i)+I_A^{U-}(x_i)+I_A^{U+}(x_i)\right)}{4}\right|+\left|\frac{2-\left(I_{A^c}^{L-}(x_i)+I_{A^c}^{L+}(x_i)+I_{A^c}^{U-}(x_i)+I_{A^c}^{U+}(x_i)\right)}{4}\right|$$

$$=1-\frac{1}{2n}\sum_{i=1}^{n}\left|\frac{T_A^{L+}(x_i)+T_A^{L-}(x_i)+T_A^{U+}(x_i)+T_A^{U-}(x_i)}{4}-0.5\right|+\left|\frac{F_A^{L-}(x_i)+F_A^{L+}(x_i)+F_A^{U-}(x_i)+F_A^{U+}(x_i)}{4}-0.5\right|+$$

$$\left|\frac{I_A^{L-}(x_i)+I_A^{L+}(x_i)+I_A^{U-}(x_i)+I_A^{U+}(x_i)}{4}-0.5\right|+\left|\frac{I_{A^c}^{L-}(x_i)+I_{A^c}^{L+}(x_i)+I_{A^c}^{U-}(x_i)+I_{A^c}^{U+}(x_i)}{4}-0.5\right|=E_{\Delta INVS}(A^c)$$

Therefore, INVS entropy and its complement is equal; $E_{\Delta INVS}(A)=E_{\Delta INVS}(A^c)$ for all $A\in INVS(X)$

(E4) we have $E_{INVS}(A)\geq E_{INVS}(B)$ if $T_A^{L-}(x_i)\geq T_B^{L-}(x_i)$, $T_A^{L+}(x_i)\geq T_B^{L+}(x_i)$, $T_A^{U-}(x_i)\geq T_B^{U-}(x_i)$, $T_A^{U+}(x_i)\geq T_B^{U+}(x_i)$, $I_A^{L-}(x_i)\leq I_B^{L-}(x_i)$, $I_A^{L+}(x_i)\leq I_B^{L+}(x_i)$, $I_A^{U-}(x_i)\leq I_B^{U-}(x_i)$, $I_A^{U+}(x_i)\leq I_B^{U+}(x_i)$; $F_A^{L-}(x_i)\leq F_B^{L-}(x_i)$, $F_A^{L+}(x_i)\leq F_B^{L+}(x_i)$, $I_A^{U-}(x_i)\leq I_B^{U-}(x_i)$, $F_A^{U+}(x_i)\leq F_B^{U+}(x_i)$ for $x_i\in X$;

Then we obtain the following relation:

a) $$\left|\frac{T_A^{L-}(x_i)+T_A^{L+}(x_i)+T_A^{U-}(x_i)+T_A^{U+}(x_i)}{4}-0.5\right| \geq \left|\frac{T_B^{L-}(x_i)+T_B^{L+}(x_i)+T_B^{U-}(x_i)+T_B^{U+}(x_i)}{4}-0.5\right|$$

b) $$\left|\frac{I_A^{L-}(x_i)+I_A^{L+}(x_i)+I_A^{U-}(x_i)+I_A^{U+}(x_i)}{4}-0.5\right| \leq \left|\frac{I_B^{L-}(x_i)+I_B^{L+}(x_i)+I_B^{U-}(x_i)+I_B^{U+}(x_i)}{4}-0.5\right|$$

c) $$\left|\frac{F_A^{L-}(x_i)+F_A^{L+}(x_i)+F_A^{U-}(x_i)+F_A^{U+}(x_i)}{4}-0.5\right| \leq \left|\frac{F_B^{L-}(x_i)+F_B^{L+}(x_i)+F_B^{U-}(x_i)+F_B^{U+}(x_i)}{4}-0.5\right|$$

d) $$\left|\frac{I_{A^c}^{L-}(x_i)+I_{A^c}^{L+}(x_i)+I_{A^c}^{U-}(x_i)+I_{A^c}^{U+}(x_i)}{4}-0.5\right| \leq \left|\frac{I_{A^c}^{L-}(x_i)+I_{A^c}^{L+}(x_i)+I_{A^c}^{U-}(x_i)+I_{A^c}^{U+}(x_i)}{4}-0.5\right|$$

Combining a), b), c), and d) we obtain

$$E_{\Delta INVS}(A)=1-\frac{1}{2n}\sum_{i=1}^{n}\left|\frac{T_A^{L-}(x_i)+T_A^{L+}(x_i)+T_A^{U-}(x_i)+T_A^{U+}(x_i)}{4}-0.5\right|+\left|\frac{F_A^{L-}(x_i)+F_A^{L+}(x_i)+F_A^{U-}(x_i)+F_A^{U+}(x_i)}{4}-0.5\right|+$$
$$\left|\frac{I_A^{L-}(x_i)+I_A^{L+}(x_i)+I_A^{U-}(x_i)+I_A^{U+}(x_i)}{4}-0.5\right|+\left|\frac{I_{A^c}^{L-}(x_i)+I_{A^c}^{L+}(x_i)+I_{A^c}^{U-}(x_i)+I_{A^c}^{U+}(x_i)}{4}-0.5\right| \geq$$
$$1-\frac{1}{2n}\sum_{i=1}^{n}\left|\frac{T_B^{L-}(x_i)+T_B^{L+}(x_i)+T_B^{U-}(x_i)+T_B^{U+}(x_i)}{4}-0.5\right|+\left|\frac{F_B^{L-}(x_i)+F_B^{L+}(x_i)+F_B^{U-}(x_i)+F_B^{U+}(x_i)}{4}-0.5\right|+$$
$$\left|\frac{I_B^{L-}(x_i)+I_B^{L+}(x_i)+I_B^{U-}(x_i)+I_B^{U+}(x_i)}{4}-0.5\right|+\left|\frac{I_{B^c}^{L-}(x_i)+I_{B^c}^{L+}(x_i)+I_{B^c}^{U-}(x_i)+I_{B^c}^{U+}(x_i)}{4}-0.5\right|$$

That is $E_{\Delta INVS}(A) \geq E_{\Delta INVS}(B)$. Thus, the property (E4) is satisfied.

The proof is completed.

Apart from INVS based on interval approximation, we also have the INVS entropy based on SVNS entropy. This definition is given as follows.

3.2. *INVS entropy generalized from SVNS entropy*

In this subsection, we developed another approach to measure the degree of fuzziness of an INVS. It is generalized from SVNS entropy proposed by Majumdar and Samanta [29]. The SVNS entropy is defined as follows:

$$E_{MM}(A)=1-\frac{1}{n}\sum_{i=1}^{n}\left[T_A(x_i)+F_A(x_i)\right]\left|I_A(x_i)-I_{A^C}(x_i)\right| \tag{7}$$

The SVNS entropy meets the principle of entropy measure. But, Thao and Smarandache [29] claimed in some conditions such as when $A=\{(x,0.8,0.0.7)\}$ on $X=\{x\}$ then we substitute in the SVNS entropy $E_{MM}(A)=1-\frac{1}{1}(0.8+0.7)|0-1|=-0.5\notin[0,1]$.

Therefore, to overcome this limitation we improved to

$E_{MM}(A)=1-\frac{1}{2n}\sum_{i=1}^{n}\left[T_A(x_i)+F_A(x_i)\right]\left|I_A(x_i)-I_{A^c}(x_i)\right|$. So we have,

$E_{MM}(A)=1-\frac{1}{2}(0.8+0.7)|0-1|=0.75\in[0,1]$.

For convenient and suitability of the INVS entropy, Equation 6 is simplified using the complement definition $I_{A^c}=1-I(x)$ as follows:

$$\begin{aligned}E_{MM}(A)&=1-\frac{1}{2n}\sum_{i=1}^{n}\left[T_A(x_i)+F_A(x_i)\right]\left|\, I_A(x_i)-\left(1-I_A(x_i)\right)\right|\\&=1-\frac{1}{2n}\sum_{i=1}^{n}\left[T_A(x_i)+F_A(x_i)\right]\left|\,2\,I_A(x_i)-1\,\right|\end{aligned} \tag{8}$$

Based on the SVNS entropy we generalized the entropy formula under the INVS environment. The INVS entropy based on SVNS entropy denoted as $E_{INVS}(A)$ is defined as follows:

Definition 3.2.1: An entropy $E_{INVS}(A)$ on interval neutrosophic vague sets is a function $E:INVS(X)\to[0,1]$ satisfying given condition in Definition 3.1. Then

$$E_{INVS}(A)=1-\frac{1}{2n}\sum_{i=1}^{n}E_{INVS}(A)(x_i)$$

where

$$\begin{aligned}\sum_{i=1}^{n}E_{INVS}(A)(x_i)=\sum_{i=1}^{n}&\left\{\left(\left|\frac{T_A^{L-}(x_i)+T_A^{L+}(x_i)}{2}-\frac{F_A^{L-}(x_i)+F_A^{L+}(x_i)}{2}\right|\cdot\left|I_A^{L-}(x_i)+I_A^{L+}(x_i)-1\right|\right)\right.\\&\left.+\left(\left|\frac{T_A^{U-}(x_i)+T_A^{U+}(x_i)}{2}-\frac{F_A^{U-}(x_i)+F_A^{U+}(x_i)}{2}\right|\cdot\left|I_A^{U-}(x_i)+I_A^{U+}(x_i)-1\right|\right)\right\}\end{aligned}$$

for all $i=1,2,\ldots,n$ (9)

Theorem 2. $E_{INVS}(A)$ as specified in Definition 3.2.1 is entropy for INVS

To show that $E_{INVS}(A)$ is a valid measure, we must show that it satisfies the axioms mentioned in Definition 3.1.

Proof

(E1) if A is a crisp set then $\left[T_A^{L-}(x_i),T_A^{L+}(x_i)\right]=[1,1]$, $\left[T_A^{U-}(x_i),T_A^{U+}(x_i)\right]=[1,1]$, $\left[I_A^{L-}(x_i),I_A^{L+}(x_i)\right]=[0,0]$, $\left[I_A^{U-}(x_i),I_A^{U+}(x_i)\right]=[0,0]$, $\left[F_A^{L-}(x_i),F_A^{L+}(x_i)\right]=[0,0]$, $\left[F_A^{U-}(x_i),F_A^{U+}(x_i)\right]=[0,0]$ or $\left[T_A^{L-}(x_i),T_A^{L+}(x_i)\right]=[0,0]$, $\left[T_A^{U-}(x_i),T_A^{U+}(x_i)\right]=[0,0]$, $\left[I_A^{L-}(x_i),I_A^{L+}(x_i)\right]=[0,0]$, $\left[I_A^{U-}(x_i),I_A^{U+}(x_i)\right]=[0,0]$, $\left[F_A^{L-}(x_i),F_A^{L+}(x_i)\right]=[1,1]$, $\left[F_A^{U-}(x_i),F_A^{U+}(x_i)\right]=[1,1]$ for all $x_i\in X$ we have

$$E_{INVS}(A)(x_i)=\left(\left|\frac{1+1}{2}-\frac{0+0}{2}\right|\cdot\left|0+0-1\right|\right)+\left(\left|\frac{1+1}{2}-\frac{0+0}{2}\right|\cdot\left|0+0-1\right|\right)=2$$

It implies that, $E_{INVS}(A)=1-\frac{1}{2(1)}(2)=0$ or

$$E_{INVS}(A)(x_i)=\left(\left|\frac{0+0}{2}-\frac{1+1}{2}\right|\cdot\left|0+0-1\right|\right)+\left(\left|\frac{0+0}{2}-\frac{1+1}{2}\right|\cdot\left|0+0-1\right|\right)=2$$

It implies that, $E_{INVS}(A)=1-\frac{1}{2(1)}(2)=0$

Therefore, the INVS entropy will be null $(E_{INVS}(A)=0)$ when the set is a crisp set.

(E2) $\left[T_A^{L-}(x_i),T_A^{L+}(x_i)\right]=\left[0.5,0.5\right]$, $\left[T_A^{U-}(x_i),T_A^{U+}(x_i)\right]=\left[0.5,0.5\right]$, $\left[I_A^{L-}(x_i),I_A^{L+}(x_i)\right]=\left[0.5,0.5\right]$, $\left[I_A^{U-}(x_i),I_A^{U+}(x_i)\right]=\left[0.5,0.5\right]$, $\left[F_A^{L-}(x_i),F_A^{L+}(x_i)\right]=\left[0.5,0.5\right]$ and $\left[F_A^{U-}(x_i),F_A^{U+}(x_i)\right]=\left[0.5,0.5\right]$ for all $x_i\in X$

$$E_{INVS}(A)(x_i)=\left(\left|\frac{0.5+0.5}{2}-\frac{0.5+0.5}{2}\right|\cdot\left|0.5+0.5-1\right|\right)+\left(\left|\frac{0.5+0.5}{2}-\frac{0.5+0.5}{2}\right|\cdot\left|0.5+0.5-1\right|\right)=0$$

It implies that, $E_{INVS}(A)=1-\frac{1}{2(1)}(0)=1$

Therefore, the entropy will be maximum $(E_{INVS}(A)=1)$ if the set is completely INVS.

(E3)

$$E_{INVS}(A)=1-\frac{1}{2n}\left\{\left(\left|\frac{\left(1-T_A^{L+}(x_i)\right)+\left(1-T_A^{L-}(x_i)\right)}{2}-\frac{\left(1-F_A^{L+}(x_i)\right)+\left(1-F_A^{L-}(x_i)\right)}{2}\right|\cdot\left|\left(1-I_A^{L+}(x_i)\right)+\left(1-I_A^{L-}(x_i)\right)-1\right|\right)\right.$$

$$\left.+\left(\left|\frac{\left(1-T_A^{U+}(x_i)\right)+\left(1-T_A^{U-}(x_i)\right)}{2}-\frac{\left(1-F_A^{U+}(x_i)\right)+\left(1-F_A^{U-}(x_i)\right)}{2}\right|\cdot\left|\left(1-I_A^{U+}(x_i)\right)+\left(1-I_A^{U-}(x_i)\right)-1\right|\right)\right\}$$

$$=1-\frac{1}{2n}\left\{\left(\left|\frac{2-2-T_A^{L+}(x_i)-T_A^{L-}(x_i)+F_A^{L+}(x_i)+F_A^{L-}(x_i)}{2}\right|\cdot\left|2-I_A^{L+}(x_i)-I_A^{L-}(x_i)-1\right|\right)+\right.$$

$$\left.\left(\left|\frac{2-2-T_A^{U+}(x_i)-T_A^{U-}(x_i)+F_A^{U+}(x_i)+F_A^{U-}(x_i)}{2}\right|\cdot\left|2-I_A^{U+}(x_i)-I_A^{U-}(x_i)\right|-1\right)\right\}$$

$$=1-\frac{1}{2n}\left\{\left(\left|\frac{-T_A^{L+}(x_i)-T_A^{L-}(x_i)+F_A^{L+}(x_i)+F_A^{L-}(x_i)}{2}\right|\cdot\left|1-I_A^{L+}(x_i)-I_A^{L-}(x_i)\right|\right)+\right.$$

$$\left.\left(\left|\frac{-T_A^{U+}(x_i)-T_A^{U-}(x_i)+F_A^{U+}(x_i)+F_A^{U-}(x_i)}{2}\right|\cdot\left|1-I_A^{U+}(x_i)-I_A^{U-}(x_i)\right|\right)\right\}$$

$$=1-\frac{1}{2n}\left\{\left(\left|\frac{T_A^{L+}(x_i)+T_A^{L-}(x_i)-F_A^{L+}(x_i)-F_A^{L-}(x_i)}{2}\right|\cdot\left|I_A^{L+}(x_i)+I_A^{L-}(x_i)-1\right|\right)+\right.$$

$$\left.\left(\left|\frac{T_A^{U+}(x_i)+T_A^{U-}(x_i)-F_A^{U+}(x_i)-F_A^{U-}(x_i)}{2}\right|\cdot\left|I_A^{U+}(x_i)+I_A^{U-}(x_i)-1\right|\right)\right\}$$

$$=1-\frac{1}{2n}\left\{\left(\left|\frac{T_A^{L+}(x_i)+T_A^{L-}(x_i)-\left(F_A^{L+}(x_i)+F_A^{L-}(x_i)\right)}{2}\right|\cdot\left|I_A^{L+}(x_i)+I_A^{L-}(x_i)-1\right|\right)+\right.$$

$$\left.\left(\left|\frac{\left(T_A^{U+}(x_i)+T_A^{U-}(x_i)\right)-\left(F_A^{U+}(x_i)+F_A^{U-}(x_i)\right)}{2}\right|\cdot\left|I_A^{U+}(x_i)+I_A^{U-}(x_i)-1\right|\right)\right\}=E_{INVS}(A^c)$$

Therefore, INVS entropy and its complement is equal; $E_{INVS}(A)=E_{INVS}(A^c)$ for all $A\in INVS(X)$

(**E4**) we have $E_{INVS}(A)\geq E_{INVS}(B)$ if $T_A^{L-}(x_i)\geq T_B^{L-}(x_i)$, $T_A^{L+}(x_i)\geq T_B^{L+}(x_i)$, $T_A^{U-}(x_i)\geq T_B^{U-}(x_i)$, $T_A^{U+}(x_i)\geq T_B^{U+}(x_i)$, $I_A^{L-}(x_i)\leq I_B^{L-}(x_i)$, $I_A^{L+}(x_i)\leq I_B^{L+}(x_i)$, $I_A^{U-}(x_i)\leq I_B^{U-}(x_i)$, $I_A^{U+}(x_i)\leq I_B^{U+}(x_i)$; $F_A^{L-}(x_i)\leq F_B^{L-}(x_i)$, $F_A^{L+}(x_i)\leq F_B^{L+}(x_i)$, $I_A^{U-}(x_i)\leq I_B^{U-}(x_i)$, $F_A^{U+}(x_i)\leq F_B^{U+}(x_i)$ for $x_i\in X$;

Then we obtain the following relation:

a) $$\left|\frac{T_A^{L-}(x_i)+T_A^{L+}(x_i)}{2}-\frac{F_A^{L-}(x_i)+F_A^{L+}(x_i)}{2}\right|\geq\left|\frac{T_B^{L-}(x_i)+T_B^{L+}(x_i)}{2}-\frac{F_B^{L-}(x_i)+F_B^{L+}(x_i)}{2}\right|,$$

$$\left|\frac{T_A^{U-}(x_i)+T_A^{U+}(x_i)}{2}-\frac{F_A^{U-}(x_i)+F_A^{U+}(x_i)}{2}\right|\geq\left|\frac{T_B^{U-}(x_i)+T_B^{U+}(x_i)}{2}-\frac{F_B^{U-}(x_i)+F_B^{U+}(x_i)}{2}\right|$$

b) $$\left|I_A^{L-}(x_i)+I_A^{L+}(x_i)-1\right|\leq\left|I_B^{L-}(x_i)+I_B^{L+}(x_i)-1\right|$$

$$\left|I_A^{U-}(x_i)+I_A^{U+}(x_i)-1\right|\leq\left|I_B^{U-}(x_i)+I_B^{U+}(x_i)-1\right|$$

Combining a) and b)

$$1-\frac{1}{2n}\sum_{i=1}^{n}E_{INVS}\left\{\left(\left|\frac{T_A^{L-}(x_i)+T_A^{L+}(x_i)}{2}-\frac{F_A^{L-}(x_i)+F_A^{L+}(x_i)}{2}\right|\cdot\left|I_A^{L-}(x_i)+I_A^{L+}(x_i)-1\right|\right)\right.$$

$$\left.+\left(\left|\frac{T_A^{U-}(x_i)+T_A^{U+}(x_i)}{2}-\frac{F_A^{U-}(x_i)+F_A^{U+}(x_i)}{2}\right|\cdot\left|I_A^{U-}(x_i)+I_A^{U+}(x_i)-1\right|\right)\right\}\geq$$

$$1-\frac{1}{2n}\sum_{i=1}^{n}\left\{\left(\left|\frac{T_B^{L-}(x_i)+T_B^{L+}(x_i)}{2}-\frac{F_B^{L-}(x_i)+F_B^{L+}(x_i)}{2}\right|\cdot\left|I_A^{L-}(x_i)+I_A^{L+}(x_i)-1\right|\right)+\right.$$

$$\left.\left(\left|\frac{T_B^{U-}(x_i)+T_B^{U+}(x_i)}{2}-\frac{F_B^{U-}(x_i)+F_B^{U+}(x_i)}{2}\right|\cdot\left|I_B^{U-}(x_i)+I_B^{U+}(x_i)-1\right|\right)\right\}$$

That is $E_{INVS}(A) \geq E_{INVS}(B)$. Thus, the property (E4) is satisfied.

The proof is completed.
The proposed entropy is further embedded into a MCDM.

4. MCDM problem based on proposed entropy

In this section, the proposed entropy measures are applied in MCDM problems. Measuring uncertainty is important in decision making problems, the DMs will obtain significant preference and priority to avoid losing out in the selection process. Based on the proposed entropy measures, when the entropy value is smaller, then DMs can provide more valuable knowledge from this alternative. As a result, the alternative with the lowest entropy value should be considered as a priority.

Consider the set of different alternatives denoted as $A = \{A_1, A_2, \ldots, A_m\}$ and set of criteria is denoted by $C = \{C_1, C_2, \ldots, C_n\}$ in INVS environment and the algorithm to evaluate the best alternative is presented as follows:

Step 1: Construction of decision making matrix

Organize each A_i alternatives under criteria C_j according to the DM's preferences in the form of INVS environment as follows
$D = (x_{ij})_{m \times n}$ where $x_{ij} = \left\{\left[T_{ij}^{L-}, T_{ij}^{L+}\right], \left[T_{ij}^{U-}, T_{ij}^{U+}\right]\right\}, \left\{\left[I_{ij}^{L-}, I_{ij}^{L+}\right], \left[I_{ij}^{U-}, I_{ij}^{U+}\right]\right\}, \left\{\left[F_{ij}^{L-}, F_{ij}^{L+}\right], \left[F_{ij}^{U-}, F_{ij}^{U+}\right]\right\}$
$0 \leq T^{L-} + T^{U-} + I^{L-} + I^{U-} + F^{L-} + F^{U-} \leq 4$ and $0 \leq T^{L+} + T^{U+} + I^{L+} + I^{U+} + F^{L+} + F^{U+} \leq 4$.
$\left[T_{ij}^{L-}, T_{ij}^{L+}\right]$ represents the degree that alternative A_i relatively satisfies the criteria C_j, $\left[T_{ij}^{U-}, T_{ij}^{U+}\right]$ represents the degree that alternative A_i absolutely satisfies the criteria C_j, $\left[I_{ij}^{L-}, I_{ij}^{L+}\right]$ represents the degree relatively indeterminant the criteria C_j, $\left[I_{ij}^{U-}, I_{ij}^{U+}\right]$ represents the degree absolutely indeterminant the criteria C_j, $\left[F_{ij}^{L-}, F_{ij}^{L+}\right]$ represents the degree that alternative A_i relatively doesn't satisfies the criteria C_j and $\left[F_{ij}^{U-}, F_{ij}^{U+}\right]$ represents the degree that alternative A_i absolutely doesn't satisfies the criteria C_j. Therefore, a decision matrix D is arranged as follows:

$$D = (x_{ij})_{m \times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \end{array}$$

Where x_{ij} is $x_{ij} = \left\{\left[T_{ij}^{L-}, T_{ij}^{L+}\right], \left[T_{ij}^{U-}, T_{ij}^{U+}\right]\right\}, \left\{\left[I_{ij}^{L-}, I_{ij}^{L+}\right], \left[I_{ij}^{U-}, I_{ij}^{U+}\right]\right\}, \left\{\left[F_{ij}^{L-}, F_{ij}^{L+}\right], \left[F_{ij}^{U-}, F_{ij}^{U+}\right]\right\}$

Step 2: Transform the INVS decision matrix $D=\left(x_{ij}\right)_{m\times n}$ into the normalized INVS decision matrix denoted as $\tilde{D}=\left(\tilde{x}_{ij}\right)_{m\times n}$ where

$$\tilde{x}_{ij}=\begin{cases}x_{ij}^{c};\, j\in B\\ x_{ij};\, j\in C\end{cases} \tag{10}$$

$$x_{ij}^{c}=\left\{\left[1-T_{ij}^{L+},1-T_{ij}^{L-}\right],\left[1-T_{ij}^{U+},1-T_{ij}^{U-}\right]\right\},\left\{\left[1-I_{ij}^{L+},1-I_{ij}^{L-}\right],\left[1-I_{ij}^{U+},1-I_{ij}^{U-}\right]\right\},\left\{\left[1-F_{ij}^{L+},1-F_{ij}^{L-}\right],\left[1-F_{ij}^{U+},1-F_{ij}^{U-}\right]\right\}$$

is complement of $x_{ij}=\left\{\left[T_{ij}^{L-},T_{ij}^{L+}\right],\left[T_{ij}^{U-},T_{ij}^{U+}\right]\right\},\left\{\left[I_{ij}^{L-},I_{ij}^{L+}\right],\left[I_{ij}^{U-},I_{ij}^{U+}\right]\right\},\left\{\left[F_{ij}^{L-},F_{ij}^{L+}\right],\left[F_{ij}^{U-},F_{ij}^{U+}\right]\right\}$.

Step 3: Calculate the entropy

$$E_{\Delta INVS}(A)=1-\frac{1}{2n}\sum_{i=1}^{n}E_{\Delta INVS}(A)(x_i) \tag{11}$$

or

$$E_{INVS}(A)=1-\frac{1}{2n}\sum_{i=1}^{n}E_{INVS}(A)(x_i) \tag{12}$$

Step 4: Select the best option A_i for $i=1,2,\ldots,m$ based on smallest entropy.

4.1. Illustrative example

In this section, the proposed entropy measure is applied to the case study of investment decisions adapted from [31]. There is a DM with four potential investment options namely A_1 is a food company, A_2 is a transport company, A_3 is an electronic company, and A_4 is a tire company. The DM makes a choice based on three criteria, C_1 is growth analysis, C_2 is risk analysis, C_3 is an environment impact analysis. We begin by using the INVS entropy based on interval approximation $\left(E_{\Delta INVS}\right)$.

Method 1: Using INVS entropy based on interval approximation $\left(E_{\Delta INVS}\right)$.

Step 1: The linguistic evaluation consists of $A=\left\{A_1,A_2,A_3,A_4\right\}$ with respect to criteria $C=\left\{C_1,C_2,C_3\right\}$ are obtained from the expert evaluation. The INVS decision matrix denoted as $D=\left(x_{ij}\right)_{m\times n}$ is represented as follows:

	C_1	C_2	C_3
A_1	$[0.5,0.7],[0.6,0.8]$	$[0.5,0.6],[0.4,0.9]$	$[0.7,0.8],[0.5,0.8]$
	$[0.2,0.3],[0.3,0.4]$	$[0.1,0.2],[0.3,0.4]$	$[0.1,0.2],[0.3,0.4]$
	$[0.3,0.5],[0.2,0.4]$	$[0.4,0.5],[0.1,0.6]$	$[0.2,0.3],[0.2,0.5]$
A_2	$[0.4,0.7],[0.6,0.9]$	$[0.3,0.6],[0.1,0.5]$	$[0.6,0.7],[0.5,0.9]$
	$[0.2,0.3],[0.4,0.5]$	$[0.2,0.4],[0.4,0.5]$	$[0.3,0.4],[0.4,0.5]$
	$[0.3,0.6],[0.1,0.4]$	$[0.4,0.7],[0.5,0.9]$	$[0.3,0.4],[0.1,0.5]$
A_3	$[0.4,0.9],[0.7,0.9]$	$[0.5,0.7],[0.5,0.9]$	$[0.5,0.6],[0.4,0.7]$
	$[0.3,0.4],[0.4,0.5]$	$[0.1,0.2],[0.2,0.3]$	$[0.1,0.2],[0.2,0.3]$
	$[0.1,0.6],[0.1,0.3]$	$[0.3,0.5],[0.1,0.5]$	$[0.4,0.5],[0.3,0.6]$
A_4	$[0.6,0.8],[0.5,0.9]$	$[0.2,0.5],[0.1,0.4]$	$[0.4,0.8],[0.5,0.9]$
	$[0.1,0.2],[0.3,0.5]$	$[0.2,0.3],[0.3,0.4]$	$[0.3,0.4],[0.4,0.5]$
	$[0.2,0.4],[0.1,0.5]$	$[0.5,0.6],[0.6,0.9]$	$[0.2,0.6],[0.1,0.5]$

Step 2: since the criteria C_1 is the benefit criteria and C_2, C_3 are cost criteria, so the INVS decision matrix is transformed into the normalized INVS decision matrix using Equation 10.

	C_1	C_2	C_3
A_1	$[0.3,0.5],[0.2,0.4]$	$[0.5,0.6],[0.4,0.9]$	$[0.7,0.8],[0.5,0.8]$
	$[0.7,0.8],[0.6,0.7]$	$[0.1,0.2],[0.3,0.4]$	$[0.1,0.2],[0.3,0.4]$
	$[0.5,0.7],[0.6,0.8]$	$[0.4,0.5],[0.1,0.6]$	$[0.2,0.3],[0.2,0.5]$
A_2	$[0.3,0.6],[0.1,0.4]$	$[0.3,0.6],[0.1,0.5]$	$[0.6,0.7],[0.5,0.9]$
	$[0.7,0.8],[0.5,0.6]$	$[0.2,0.4],[0.4,0.5]$	$[0.3,0.4],[0.4,0.5]$
	$[0.4,0.7],[0.6,0.9]$	$[0.4,0.7],[0.5,0.9]$	$[0.3,0.4],[0.1,0.5]$
A_3	$[0.1,0.6],[0.1,0.3]$	$[0.5,0.7],[0.5,0.9]$	$[0.5,0.6],[0.4,0.7]$
	$[0.6,0.7],[0.5,0.6]$	$[0.1,0.2],[0.2,0.3]$	$[0.1,0.2],[0.2,0.3]$
	$[0.4,0.9],[0.7,0.9]$	$[0.3,0.5],[0.1,0.5]$	$[0.4,0.5],[0.3,0.6]$
A_4	$[0.2,0.4],[0.1,0.5]$	$[0.2,0.5],[0.1,0.4]$	$[0.4,0.8],[0.5,0.9]$
	$[0.8,0.9],[0.5,0.7]$	$[0.2,0.3],[0.3,0.4]$	$[0.3,0.4],[0.4,0.5]$
	$[0.6,0.8],[0.5,0.9]$	$[0.5,0.6],[0.6,0.9]$	$[0.2,0.6],[0.1,0.5]$

Step 3: Calculate the aggregated entropy measure for all the alternative $A=\{A_1,A_2,A_3,A_4\}$

By equation 11, we calculate for $i=1, j=1$, then

$$=\left|\frac{0.3+0.5+0.2+0.4}{4}-0.5\right|+\left|\frac{0.5+0.7+0.6+0.8}{4}-0.5\right|+\left|\frac{0.7+0.8+0.6+0.7}{4}-0.5\right|$$

$$+\left|\frac{0.2+0.3+0.3+0.4}{4}-0.5\right|=0.7$$

Therefore;

$A_1: \sum_{i=1}^{3} E_{\Delta INVS}(A) = 0.7 + 0.7 + 0.9 = 2.3$ and $E_{\Delta INVS}(A) = 1 - \frac{1}{6}(2.3) = 0.6167$ and the rest of entropy measure for A_2, A_3, A_4 are calculated similarly and presented as follows:

$A_2 = 0.7250, A_3 = 0.6250, A_4 = 0.65$

Step 4: Based on the smallest entropy values, we conclude that the ranking of given alternatives is as follows:

$A_1 \prec A_3 \prec A_4 \prec A_2$

Since A_1 is the less uncertainty information**.** Therefore, the best option to invest is in a food company.

Method 2: Using INVS entropy generalized based on SVNS entropy (E_{INVS}).

Step 1: Similar to Step 1 of Method 1.
Step 2: Similar to Step 2 of Method 1.

Step 3: Calculate the aggregated entropy measure for all the alternative $A = \{A_1, A_2, A_3, A_4\}$

By equation 12, we calculate for $i = 1, j = 1$, then

$= \left\{ \left(\left| \frac{0.3 + 0.5}{2} - \frac{0.5 + 0.7}{2} \right| \cdot |0.7 + 0.8 - 1| \right) + \left(\left| \frac{0.2 + 0.4}{2} - \frac{0.6 + 0.8}{2} \right| \cdot |0.6 + 0.7 - 1| \right) \right\} = 0.22$. Hence,

$A_1: \sum_{i=1}^{3} E_{INVS}(A) = 0.22 + 0.16 + 0.44 = 0.82$ and $E_{INVS}(A) = 1 - \frac{1}{6}(0.82) = 0.8633$ and the rest of entropy measure for A_2, A_3, A_4 are calculated similarly and presented as follows:

$A_2 = 0.9463, A_3 = 0.8983, A_4 = 0.8817$

Step 4: Based on the smallest entropy values, we conclude that the ranking of given alternatives is as follows:

Since A_1 is the less uncertainty information, we conclude that the ranking of given alternatives is as follows:

$A_1 \prec A_3 \prec A_4 \prec A_2$

Since the smallest entropy value is A_1. Therefore, the best option to invest is in a food company.

According to the findings, investing in a food company is the best option based on the proposed entropy. Both entropy measure produce different fuzziness degree but the ranking of alternatives is similar which could assist DMs to choose the best alternative. The ranking of four alternatives using proposed entropy measures is presented in Figure 1.

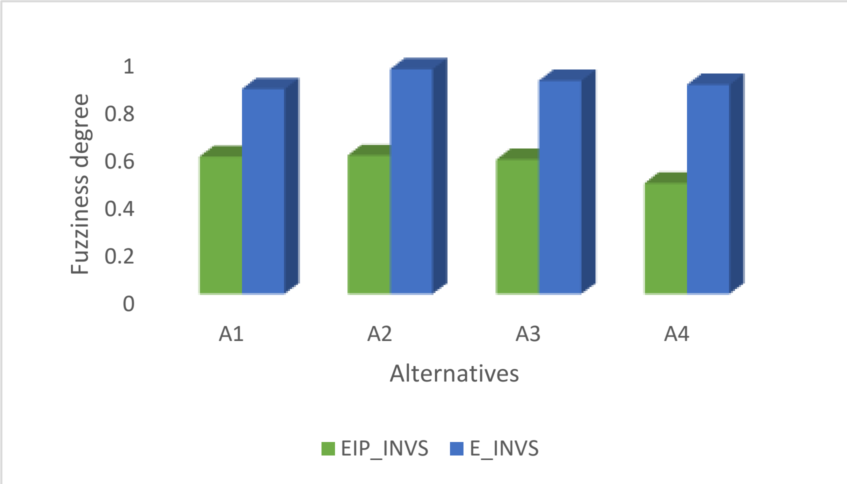

Figure 1: Ranking of the four alternatives using $(E_{\Delta INVS})$ and (E_{INVS})

4.2. Comparison analysis and discussion

Based on the illustrative example in [31] and the computational procedures in section 4, the proposed entropy measures are compared with two existing entropy measures under the INVS environment. The proposed entropy measures are denoted as $E_{\Delta INVS}$ and E_{INVS} are based on interval approximation and existing SVNS entropy, respectively. The existing entropy measures by Majumdar and Samanta E_{MM} [29], and entropy measures by Ali E_{δ} [35] are incorporated in this comparative analysis. Table 1 represents the ranking results based on entropy values.

Table 3: The comparison with other entropy measures

Entropy measure	Aggregated entropy measure $A_i\,(i=1,2,3,4)$	Ranking
Proposed entropy $E_{\Delta INVS}$	$A_1=0.6167$, $A_2=0.6250$ $A_3=0.7250$, $A_4=0.65$	$A_1 \prec A_3 \prec A_4 \prec A_2$
Proposed entropy E_{INVS}	$A_1=0.863$, $A_2=0.9463$ $A_3=0.8983$, $A_4=0.8817$	$A_1 \prec A_4 \prec A_3 \prec A_2$
Entropy E_{MM} [33]	$A_1=0.7267$, $A_2=0.8967$ $A_3=0.7967$, $A_4=0.7633$	$A_1 \prec A_4 \prec A_3 \prec A_2$
Entropy E_{δ} [35]	$A_1=0.4474$, $A_2=0.5690$ $A_3=0.4567$, $A_4=0.4860$	$A_1 \prec A_3 \prec A_4 \prec A_2$

The following conclusions are drawn from a comparison of different entropy measures:

- The ranking result of our proposed entropy is almost consistent with the existing entropy in the literature. The smallest entropy value is A_1 meanwhile the largest entropy value is A_2
- The entropy measures show a similar ranking of alternatives with different fuzziness degree.
- The proposed entropy measures are reliable in measuring the degree of fuzziness in terms of the INVS data set.
- The fuzziness degree in the proposed entropy measures may assist DMs to choose the most significant alternative based on the lowest fuzziness degree.
- The new entropy measures resolve the arguments claimed by Thao and Smarandache [33] towards the entropy measures proposed by Majumdar and Samanta.
- The suggested entropy measure can address the same decision making problem as existing entropy measures.
- The proposed entropy measures can take into account the incompleteness and vagueness environments and may assist to better understand the degree of fuzziness in terms of the INVS data set.

5. Conclusions

In this paper, we have presented two entropy measures of INVS and some desirable properties corresponding to these entropies including nonnegativity, minimality, maximality, symmetric and resolution have been proved. Based on the extension of SVNS entropy, we define the concept of INVS entropy by including some improvements. Specifically, the improved entropy measures resolve the arguments claimed in [33]. In addition, this entropy measures can measure the degree of fuzziness in terms of INVS environment. Then, the proposed entropies are applied in a MCDM problem, in which the alternatives on criteria are represented in the INVS environment. Subsequently, an illustrative example was presented to illustrate the application of the proposed MCDM. Finally, a comparative analysis with other entropy measures is presented.

The advantages of proposed entropy are in form of INVS where truth, indeterminacy, and falsity are defined by several membership degree and also complement the NV and INS in representing uncertain, indeterminate, and inconsistent information. The result shows that the proposed entropy measures are reliable in measuring the degree of fuzziness. Measuring uncertainty information is important in decision making, the least value of fuzziness degree will assist DMs to make effective decisions to prevent loss. The suggested entropy measures may be used to assess uncertainty information in other decision making problems such as selection of renewable energy, waste water treatment, and supplier selection. The limitation of the study is the idea to generalize entropy measure may be utilized for the interval concept only.

In the future, the proposed entropy measures can be extended further based on exponential entropy, symmetric entropy, and trigonometric functions. Under the INVS environment, we shall propose other information measures such as similarity and cross-entropy. Besides that, the proposed entropy measures may be used to measure the weight of criteria and DMs in MCDM.

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Multimodal Fusion: A Review, Taxonomy, Open Challenges, Research Roadmap and Future Directions

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Abstract: The present work collects a plethora of previous research work in the field of multimodal fusion which despite a lot of research could not handle the imperfections. These imperfections could be at any stage initiating from the imperfections in data and its sources to imperfections in fusion strategies. Further, the work explores various applications of Neutrosophy in the field of handling imperfections along with description of previous work in this regard. These applications include the one which addresses the notion of imperfection and uncertainty among multimodal data which is being collected for fusion. In this way, the present work tries to incorporate neutrosophic logic and its applications in the field of computer vision including multimodal data fusion and information systems. It is assumed that if the notion of uncertainty is included in multimodal research, the development of newer algorithms for solving the problems of imperfections in multimodal systems will provide impetus to the existing research in this field.

Keywords: Multimodal Data; Multimodal Fusion; Imperfections; Fuzzy Logic; Neutrosophic Logic; Machine Learning.

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1. Introduction

The present world is witnessing a change where the user is not only a consumer of information but a great producer of it. Earlier website owners were the main source of information production but in the current scenario, the social web has taken its position. This rapid development in the field of the web is termed Web 2.0. The repositories of multimedia content (Flicker, YouTube, Picasa and Twitter etc.) over the web is increasing at a faster pace than ever before. This plethora of content over the web as well as on personal computers has raised the issue of its effective storage, organization, indexing and retrieval. This multimedia content (image or video) has a multimodal (visual, textual) nature. These multimodalities are of utmost importance since the information conveyed by pixels covers only visual content which is totally different from tag information. These modalities must be combined in such a way that it gives more of the information needed on time. In order to combine the above-mentioned modalities it is important to consider the process of information fusion. This process at the initial level is carried forward in different ways. These may be data fusion (low level), feature fusion (intermediate level) and decision fusion (high level). When multiple sources of raw data are combined in such a way that the new source is more informative and synthetic than the previous two, it is called data fusion. Feature fusion combines features extracted from different sources into a single stand-alone feature vector. Decision fusion is clearly based on classifiers when aid in giving unbiased and accurate results. One of the main characteristics of the fusion process is imperfection as explained by Bloch [2001]. These imperfections are the main reason for the fusion process to be carried out more effectively [22]. These imperfections could mainly be imprecision, uncertainty and incompleteness. These imperfections occur at a different level of fusion. In this paper we have reviewed work on multimodal fusion, also we have reviewed work on neutrosophic technologies which could be employed in the field of multimodal fusion and systems. The following figure shows the workflow:

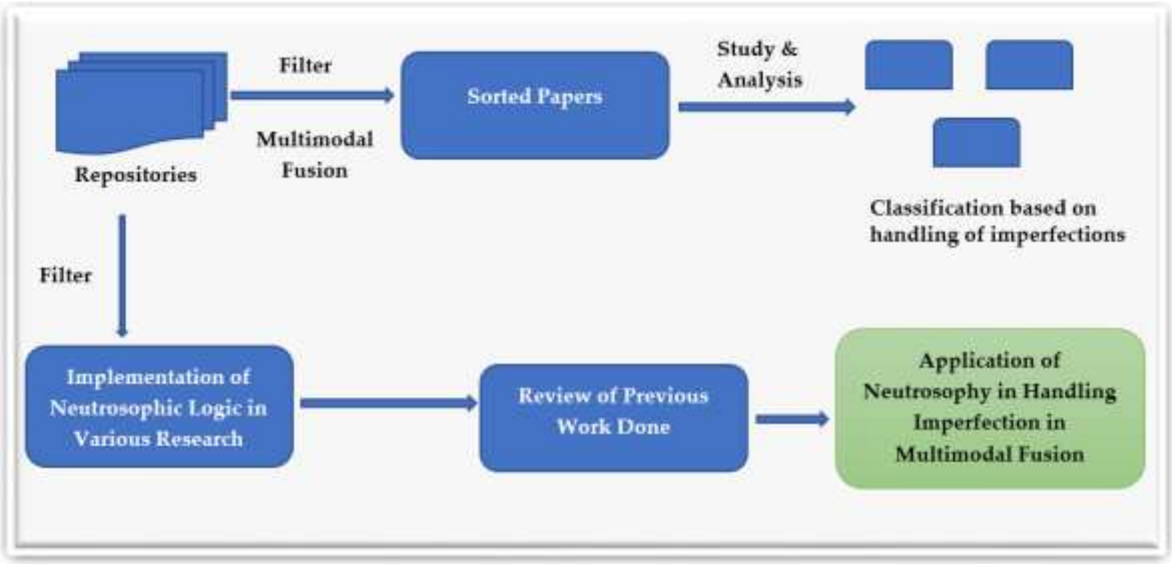

Figure 1 Block diagram for the process of research being adopted in the present manuscript

1.1 Background:

Multimodal fusion has been attaining exponential attention in multimodal information access and retrieval tasks and this has been well studied by Kludas and Marchand-Maillet [2011]; Souvannavong et al. [2005]; Marchand-Maillet et al. [2010] and Niaz and M´erialdo [2013] [23-26]. The facts related to this can be found in the study done by Atrey et al. [2010] [27]. The imperfections in textual modality are partially considered in the context of multimodal systems. Most of the state-of-the-art approaches which address the notion of textual imperfections always do so using relevancy e.g. imperfections in tags. The incompleteness issue has been well identified in the literature by Liu et al. [2009]; Tang et al. [2009] [28-29]; Wang et al. [2010] [40] but work on imprecision and uncertainty is left far behind e.g. noisy tags. Imperfection in different modalities has been studied by Bloch [2003] [41]. Though various research work has been done in the field of multimodal fusion but very little has focused on handling imperfections. Handling imperfections does not appear their primary goal while performing multimodal fusion tasks. Below Table 1 shows the work in above-mentioned field using different principles by various researchers around the globe.

Table 1 A Summary of the state-of-the-art approaches for Multimodal Fusion

S No.	Work	Principle	Handling Imperfections
1.	Romberg et al. (2012)	Probabilistic latent semantic analysis on tags co-occurrence matrix.	No
2.	Zhang et al.(2012)	Semantic BOW based on the Tag-to-Tag similarity	The incomplete data problem.
3.	Xioufis et al. (2011) .	Binary BOW representation with feature selection	No
4.	Kawanabe et al. (2011)	Binary BOW representing the presence/absence of tags with random walks over tags.	The incomplete data problem.
5.	Guillaumin et al. (2010)	Binary BOW representation representing the presence/absence of tags.	No
6.	Liu et al. (2013)	Histogram of Textual Concepts based on the Tag-to-Concept similarity	The incomplete data problem.
7.	Nagel et al. (2011)	BOW based on the tf-idf values of tags.	No
8.	Li et al. (2010)	Compare Tag and annotation concepts expansion vectors.	No
9.	Gao et al. (2010)	Probability based on the tag-concept co-occurrence.	No
10.	Wang et al. (2010)	Semantic Fields based on the tag-concept co-occurrence.	No
11.	S. Poria et al. (2015)	Aggregate semantic and affective information associated with data	No
12.	S. Poria et al. (2015)	Decision level data fusion	No
13.	S. Poria et al. (2016)	Deep neural network & multiple kernel learning classifier	No
14.	Minghai Chen et al. (2017)	Modality fusion at word level	No

15.	Kyung-Min Kim et al. (2018)	Residual learning fusion	No
16.	Feiran Huang et al. 2019	internal correlation among features (textual & visual)for joint sentiment classification	No

Above mentioned approaches whether related to early fusion, late fusion or transmedia fusion, do not tackle the problem of imperfections at the feature level. Now after defining the imperfections or uncertainties at various levels of fusion, let us understand from Table 2 what are the terms being used to describe these data/information imperfections by prominent researchers in their work.

Table 2 A summary of terms used to describe tag imperfections in Multimodal Fusion

S No.	Work	Imperfections terms used
1.	Jin et al. (2005)	Noisy
2.	Weinberger et al. (2008)	Ambiguous
3.	Xu et al. (2009)	Ambiguous
4.	Wang et al. (2010)	Incomplete, Ambiguous
5.	Liu et al. (2009)	Incomplete, Imprecise, Noisy
6.	Kennedy et al. (2009)	Unreliable, Noisy
7.	Tang et al. (2009)	Incorrect, Noisy, Incomplete
8.	Liu et al. (2010)	Incomplete, Biased, Incorrect
9.	Zhu et al. (2010)	Noisy
10.	Yang et al. (2011)	Ambiguous, Noisy
11.	Wu et al. (2012)	Inconsistent, Noisy, Incomplete, Unreliable
12.	Valentin Vielzeuf et al. (2017)	Noisy
13.	Natalia Neverova et al. (2014)	Uncertain, Noisy

14.	A. Tamrakar (2012)	Noisy
15.	Kyung-Min Kim et al. (2018)	Ambiguous
16.	Feiran Huan et al. (2019)	Inconsistent, Noisy, Incomplete,
17.	Yagya Raj Pandeya and Joonwhoan Lee (2019)	Lack of labelling

The motivation behind carrying out present work is the negligence of research community towards addressing the problem of imprecision in data, which is used in designing multimodal systems. This imprecision arises due to dependence of classifiers on incomplete and uncertain data that leads to an imprecise decision function. This also happens when scores produced by different classifiers are combined, fusion faces the problem of imperfection. These imperfections could lead to imperfections in machine learning algorithms at the decision level. To achieve our goal we have described various work done by researchers in multimodal fusion at each stage i.e. early fusion, late fusion and transmedia fusion. We have also explained the terms that are used for showing imperfections and imprecision in data by various researchers. Further, we have explained the neutrosophic theory which provides a way to deal with uncertainty, imprecision and imperfections, with a detailed description of work carried out using this theory to address the imperfections at various levels. Our aim is to acknowledge the problem of uncertainty and imprecision in multimodal fusion tasks and introduce new researchers working in the concerned field to the notion of Neutrosophy and its applications in handling imperfections in multimodal fusion.

The taxonomy of research challenges and opportunities in multimodal fusion together with potential research challenges are summarized and highlighted in this article. The main objectives of this work include:

- To identify the problem of imperfections in multimodal fusion. Interpreting existing research conducted in this domain.
- To interpret current studies conducted in this area of research.
- To identify a research gap in the field that needs to be further investigated by the researchers in the field.
- To identify and introduce new researchers with the concept of neutrosophy and its applications in multimodal fusion.
- To identify roadmap that requires investigation in future by concerned researchers in the field of multimodal information systems.

The rest of the paper is divided into five sections, Section 2 explains research conducted in the field of multimodal fusion, including early fusion, late fusion and transmedia fusion. It also explains the problem of imperfection encountered in multimodal fusion. Section 3 introduces new researchers working in the field of multimodal information access and retrieval with the concept called Neutrosophy. It also describes the current research conducted in the field of neutrosophy which could be employed in multimodal systems for handling imperfections. Section 4 summarizes and highlights the future research roadmap to multimodal fusion using Neutrosophy. Section 5 concludes the work.

2. Taxonomy of open issues and challenges in multimodal fusion approaches:

Multimodal fusion is one of the important steps in multimodal information access and retrieval. The accuracy of the framework depends on fusion strategies being adopted. Researchers dealing with multimodal fusion mainly use three strategies namely;

1. Early Fusion
2. Late Fusion
3. Transmedia Fusion

These strategies are well explained with their working principle in the work carried out by Mohd Anas Wajid and Aasim Zafar [2019] [83]. The above-mentioned fusion strategies could easily be understood by the following diagram.

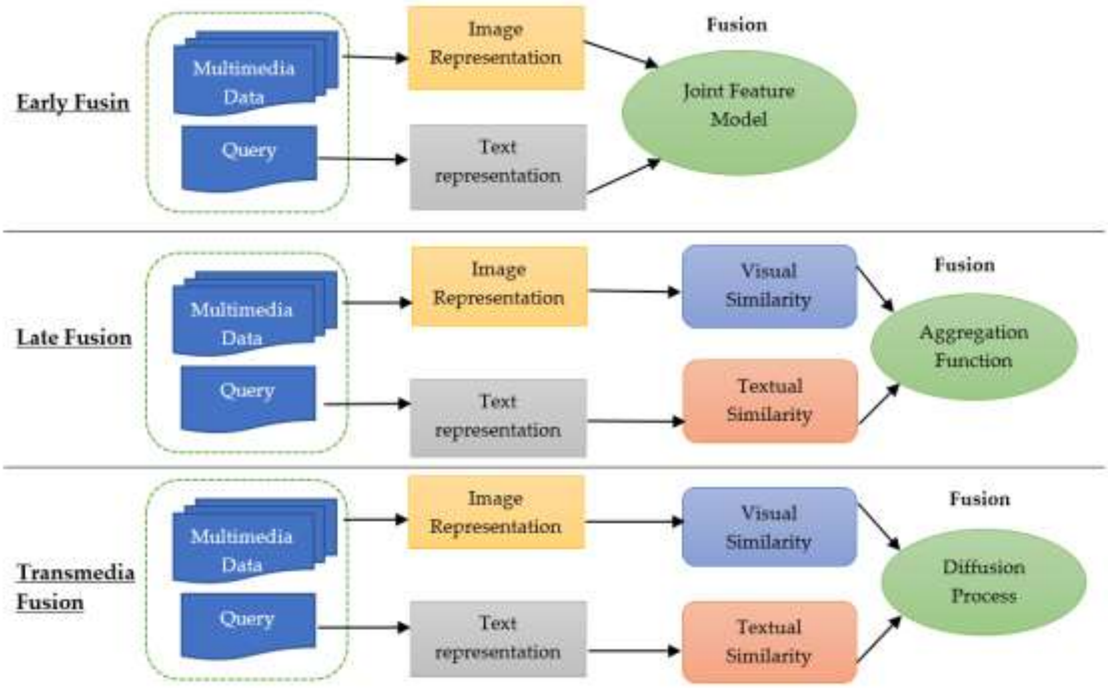

Figure 2 A summary of Fusion Approaches for Multimodal Fusion

2.1 Early Fusion:

The early fusion strategy has been adopted by a number of researchers around the globe. Though effective in many ways, it does not address the issue of imperfections while handling the data. Some of the prominent work using early fusion are explained below and later compared in the table on the basis of their fusion methodology being adopted.

Li et al. [2009] have used simple concatenation of visual aspects together with textual aspects of data for fusion [42]. Duygulu et al. [2002] have employed the correlation concept using an estimation maximization algorithm (EM). This is used for attaching words to the segmented image regions after the training phase is over. The model proposed is called the translation model [43].

Barnard et al. [2003] on one hand studied the joint distribution among the textual modality and the segmented image modality and on the other hand, used it to convert it in likelihood function between text and segmented image region [44].

According to Blei and Jordan [2003] Probabilistic Latent Semantic Analysis (pLSA) and Latent Dirichlet Allocation (LDA) can be used for correspondence between textual modality and the corresponding image modality. The model proposed by them is estimated using the EM algorithm [45].

Monay and Gatica-Perez [2003] in a similar fashion have used pLSA over the concatenated set of visual and textual modalities. The balance between the two modalities limits the size of visual representation [46]. A pLSA based model proposed by Lienhart et al. [2009] is again used to retrieve information in multimodal retrieval systems [47].

A. Tamrakar et al. [2012] have used BoW descriptors within Support Vector Machine (SVM). This is done for event detection and for this they have used many early and late fusion strategies [66]. Minghai Chen et al. [2017] have done multimodal fusion at the word level. They have emphasized Temporal Attention Layer for predicting sentiments in sentimental analysis. They have also described the noise that is present in data of different modality [39].

L. Morency et al. [2011] have proved the effectiveness of using different modalities for sentiment analysis. Though they have shown how the internet could be a source of information while using different modalities like audio, video and text but they have failed to address the imperfections present in data while carrying out the multimodal fusion [48].

How the error in sentiment classification is reduced while taking into consideration the combination of different modalities is studied by V. Pérez-Rosas [2013]. In their study authors have stressed that while using a single modality the error is 10% high as compared to using various modalities together [49]. A recent development in sentimental analysis and emotion recognition has been recorded by S. Poria et al. [2016]. Here authors have performed Emotion recognition and sentiment analysis using convolution MKL and SVM based approach [50].

The following table shows the work done by prominent researchers in the field of information retrieval using the early fusion strategy. Though all the approaches have their significance in fusing multimodal data yet they fail on the grounds of handling imperfections in data.

Table 3 A Summary of Approaches Based on Early Fusion

S No.	Work	Fusion Method	Handling Imperfections	Fusion Level
1.	Li et al. (2009)	concatenation of textual & visual modality	No	
2.	Duygulu et al. (2002)	Assigning words to segmented image regions using translation model.	No	
3.	Barnard et al. (2003)	joint distribution of text and segmented image	No	
4.	Blei and Jordan (2003)	LDA on visual and textual modality.	No	
5.	Monay and Gatica-Perez (2003)	pLSA for concatenating visual and textual features. .	No	
6.	Lienhart et al. (2009)	Multimodal pLSA multilayer model.	No	
7.	Chandrika and Jawahar (2010)	Multimodal pLSA	No	
8.	Nikolopoulos et al. (2013)	High Order pLSA.	No	**Early Fusion**
9.	Wang et al. (2009)	Visual tag dictionary using GMM.	No	
10.	A. Tamrakar (2012)	Event detection using BoW descriptors within SVM.	No	
11.	Minghai Chen et al. (2017)	Gated Multimodal Embedding LSTM with Temporal Attention	No	
12.	L. Morency et al. (2011)	Tri-model sentiment analysis using Gaussian mixtures and HMM	No	
13.	V. Pérez-Rosas et al. (2013)	BoW and OpenEAR an open source software with SVM	No	
14.	S. Poria (2016)	Emotion recognition using convolution MKL based approach	No	

***2.2 Late Fusion*:**

Now we describe multimodal fusion that is based on late fusion strategy. There exist a plethora of work using this strategy but they are all based on different methodologies which are discussed further.

Xioufis et al. [2011] in their approach worked in a different fashion. They introduced a multimodal fusion strategy based on late fusion. Their approach is totally based on predictions obtained by the classifiers from visual features. These predictions obtained from visual modality are averaged. Further, these are averaged with the predictions obtained from textual modality [82].

Wang et al. [2009a] worked in line with SVM where the scores from different classifiers are fed to SVM. The authors proposed to build two classifiers one for text modality and the other for visual modality. A third classifier is introduced to combine the confidence of the previous two and give final predictions [56].

Guillaumin et al. [2010] work with MKL framework which is considered to be a success for the feature fusion method. In the first step, their proposed semi-supervised method exploits both textual and visual features for learning a classifier. Later MKL framework is employed to predict text modality based on the visual content provided [33].

Kawanabe et al. [2011] have used a similar approach however it differs from the use of MKL. It deploys trained SVMs and uniform kernel weights and gives results approximately the same as MKL method [58]. Zhang et al. [2012a] have used the same method for combining kernels learned on textual and visual features [31].

Gao et al. [2010] have adopted a technique based on feature selection using Grouping Based Precision & Recall-Aided (GBPRA) in classifier combination which enriches the performance of classification [60]. Liu et al. [2011] have used Dempster's rule for combining classifiers predictions to achieve the best classification results [62]. Liu et al. (2013) worked on a fusion scheme termed Selective Weighted Late Fusion (SWLF). It works towards enhancing the mean average precision by selectively choosing the weights which in turn enhances the optimization [61].

Daeha Kim et al. (2017) have worked towards classifying human emotions using multimodal signals and neural networks. Though the data which they have used comprises of landmark, audio and image having various imperfections at fusion level but these are not been handled [63]. Moving towards a similar goal of emotion recognition, Valentin Vielzeuf et al. (2017) have explored several multimodal fusion strategies. They have used a supervised classifier to know emotion labels and later proposed 2D and 3D Convolution Neural Network approaches for better face descriptors [64].

Natalia Neverova et al. (2014) have worked towards gestures identification giving more stress on modality initialization and later on their fusion using late fusion strategy [65]. The work to use late fusion with dual attention mechanism has been mentioned by Kyung-Min Kim et al. (2018). This approach is utilized in proposing an architecture that could be utilized in designing effective Question-Answering (Q & A) systems [67].

Feiran Huan et al. (2019) have proposed a Deep Multimodal Attentive Fusion (DMAF), for sentimental analysis using data from social media platforms. Authors have used late fusion strategy for an effective fusion of modalities like image and text but when it comes to handling imperfections their approach seems to be lacking on this ground [68]. The work done by Escalante et al. [2008] is totally based on predictions obtained from classifiers. These are learned on textual and visual modalities and later combined in a linear way [54].

Getting inspired by the music-video combination, Yagya Raj Pandeya and Joonwhoan Lee (2019) have prepared a dataset that could be effectively utilized for sentiment analysis. In their approach, they have extracted features of music and video separately, later characterized using long short-term memory (LSTM) and for evaluating the emotions various machine learning algorithms are used [69].

Though all approaches have shown remarkable results in terms of the fusion of different modalities however they all lack on similar grounds i.e. handling imperfections. The Table 4 presents a summary of work using late fusion strategy compared on the basis of fusion method adopted.

Table 4 A Summary of Approaches Based on Late Fusion

S No.	Work	Fusion Method	Handling Imperfections	Fusion Level
1.	Escalante et al. (2008)	Prediction by different classifier is combined.	No	
2.	Xioufis et al. (2011)	Average rule used for late fusion.	No	
3.	Wang et al. (2009)	Predicted features are concatenated and SVM classifier is used.	No	
4.	Guillaumin et al. (2010)	Multiple Kernel Learning.	No	
5.	Kawanabe et al. (2011)	Multiple Kernel Learning.	No	
6.	Zhang et al. (2012)	Multiple Kernel Learning.	No	
7.	Gao et al. (2010)	Feature selection using Grouping Based Precision & Recall-Aided (GBPRA).	No	
8.	Liu et al. (2013)	Late fusion using selective weight	No	

9.	Liu et al. (2011)	Classifier predictions combined using Dempster's rule	Yes	
10.	Daeha Kim et al. (2017)	Semi supervised learning and neural network	No	**Late Fusion**
11.	Valentin Vielzeuf et al. (2017)	Temporal multimodal fusion	No	
12.	Natalia Neverova et al. (2014)	Multi-scale deep learning and localization	No	
13.	A. Tamrakar (2012)	BoW descriptors within SVM.	No	
14.	Kyung-Min Kim et al. (2018)	Residual learning fusion	No	
15.	Feiran Huan et al. (2019)	Internal correlation among features (textual & visual)for joint sentiment classification	No	
16.	Yagya Raj Pandeya and Joonwhoan Lee (2019)	Pre-trained neural networks	No	

2.3 Transmedia Fusion:

Transmedia fusion is also referred to as intermediate fusion or cross-media fusion. The basic notion of its functioning is to use visual features to accumulate image modality (Visually Nearest Neighbor) and later switch to the textual modality to collect features from the neighbors. All the approaches towards achieving transmedia fusion are listed in Table 5. It also mentions the fusion method being employed by the researchers. Though the results of the work are fully satisfying the goal of transmedia fusion; it does not handle imperfections present in different data modalities.

Table 5 A Summary of Approaches Based on Transmedia Fusion

S No.	**Work**	**Fusion Method**	**Handling Imperfections**	**Fusion Level**
1.	Makadia et al. (2008)	Nearest neighbors using Joint Equal Contribution.	No	
2.	Torralba et al. (2008)	Grasping texts from neighbors.	No	**Transmedia Fusion**

3.	Guillaumin et al. (2009)	Metric learning for text propagation.	No	
4.	Li et al. (2009)	Votes are accumulated for tag relevance	No	
5.	Feiran Huan et al. (2019)	internal correlation among features (textual & visual) for joint sentiment classification	No	
6.	Daeha Kim et al. (2017)	Neural networks based on multimodal signals	No	
7.	Valentin Vielzeuf et al. (2017)	Supervised classifier based on audio-visual signals	No	
8.	Natalia Neverova et al. (2014)	Gesture detection using multimodal and multiscale deep learning	No	
9.	A. Tamrakar (2012)	using BoW descriptors within an SVM approach for event detection	No	

3. Taxonomy of Research Work Handling Imperfections Using Neutrosophy:

Neutrosophic logic has gained alarming attention since its inception. At present is has left no areas of research untouched. Researchers all around the globe are employing its tools and techniques for the computation of uncertainty and imprecision which was a problem since time immemorial [57] [85-87]. But with the advent of neutrosophic sets and theory, the days are not far for computational intelligence to achieve its verge with the address of uncertainty and indeterminacy in machine learning algorithms and models. This theory was proposed by Florentin Smarandache [2005] which is extensively used since then for handling imperfections at various levels in mathematics and computer science. It is also referred to as Smarandache's logic [84]. It states that a proposition could have values in the range of [T, I, F] where T refers to membership degrees of truth, I refers to membership degrees indeterminacy and F refers to membership degrees falsity. Bouzina Salah (2016) have compared operational fuzzy logic to that of neutrosophic logic. The authors have shown how in fuzzy logic the membership of truth and falsity gets changed into truth, falsity and indeterminacy in neutrosophic logic. The authors argue that how a change in principle changes the whole system of working [10].

Now we describe some of the work performed by researchers using neutrosophic sets and systems. These works show that if we employ their strategy at an early stage in multimodal fusion then the problem of imperfections could easily be handled while carrying out this task. This would also enable us to remove imperfections in machine learning algorithms which in turn will not be transmitted to the modelling stage and our information access and retrieval will be more quick and accurate. Now let us understand how this work is carried out and what are strategies being followed by the researchers.

Ned Vito Quevedo Arnaiz et al. [2020] have proposed a method for dealing with unlabeled data. Their approach involves the usage of neutrosophic sets and systems. The treatment of unlabeled data is done by developing unsupervised Neutrosophic K-means algorithm. Their work is motivated due to the increasing amount of unlabeled data over the internet. The authors have taken data for experiments from a stored dataset of the City of Riobamba to show the effectiveness of their methodology [1].

Mouhammad Bakro et al. (2020) in their paper have adopted a neutrosophic approach to digital images. The elements of image modality are represented in the neutrosophic domain by dividing points of the image matrix into neutrosophic sets. The authors have also studied various methods and metrics for calculating similarity and dissimilarity between image modality. The authors have claimed that their approach would enable researchers in searching inside images and videos [2].

Abhijit Saha et al. (2020) have addressed the problem of incomplete data using neutrosophic soft sets taking in account various suitable examples. The authors have explained the inconsistent and consistent association among various parameters followed by definitions such as consistent association degree, consistent association number between the parameters, inconsistent association number between the parameters and inconsistent association degree to measure these associations. They have also proposed a data filling algorithm and proved its feasibility and validity [3].

Carmen Verónica Valenzuela-Chicaiza, et al. (2020) have done an analysis of emotional intelligence using Neutrosophic psychology. The experiment is carried out using 245 randomly selected students at the Autonomous University of Los Andes [4].

Ridvan Sahin (2014) have worked in achieving a Hierarchical clustering algorithm based on neutrosophy. This is achieved by extending algorithms proposed for Intuitionistic Fuzzy Set (IFS) and Interval Valued Intuitionistic Fuzzy Set (IVIFS) to Single Valued Neutrosophic Set (SVNS) and Interval Neutrosophic Set (INS). They have extended the algorithm for classifying neutrosophic data to show its effectiveness and applicability [5].

Yaman Akbulut et al. (2017) have worked towards enhancing the classification performance of k-Nearest Neighbour (k-NN) by the introduction of Neutrosophy. The authors have introduced Neutrosophic-k-NN. The authors have tested their approach on various datasets and have found good classification results as compared to k-NN [19]. Wen Ju et al. (2013) have introduced the Neutrosophic Support Vector Machine (N-SVM) [20].

A. A. Salama (2014) have done significant work in the domain of image processing by employing Neutrosophy in the field. They have proposed techniques to address imperfectly defined image modality. The authors have also worked towards similarity metrics for neutrosophic sets like Hamming distance and Euclidian distance. Possible applications to image processing are also touched upon [6]. The authors in the same year have worked extensively to introduce the researchers with neutrosophic linear regression and correlation [7].

Anjan Mukherjee et al. (2015) has studied Neutrosophy and its application in the field of pattern recognition. The authors have proposed a weighted similarity measure between two neutrosophic soft sets and verified its application in recognizing patterns in computer vision problems by taking some suitable examples [8].

A. A. Salama et al. (2016) have represented image modality features in the neutrosophic domain. For this purpose authors have stressed on textual modality. The authors have used these features extensively in training the model so that it could easily be used in image processing tasks [9].

Nguyen Xuan Thao et al. (2017) in their work mentioned various applications of Soft Computing. The authors have introduced a new concept of Support Neutrosophic Set (SNS) which is a combination of fuzzy set and neutrosophic set. They have also described the operations of these sets together with their properties [11].

Okpako Abugor Ejaita et al. (2017) have studied the uncertainties in medical diagnosis. Authors have stressed how negligence of uncertainty at the initial stage of diagnosis could lead to fatal problems in patients at a later stage. To overcome these authors have introduced a framework based on Neutrosophic Neural Network for diagnosis of confusable disease [12].

A. A. Salama et al. (2018) have worked towards enhancing the quality of image modality. For this reason authors have converted the image in the neutrosophic domain so that their contrast could be enhanced. This approach to the neutrosophic grayscale image domain would enable image processing to yield good results while performing information retrieval [13]. To achieve the same goal Ming Zhang et al. (2010) have proposed an image segmentation approach based on Neutrosophy [16]. Abdulkadir Sengur and Yanhui Guo (2011) have done colour, texture image segmentation based on neutrosophic set and wavelet transform [17].

D. Vitalio Ponce Ruiz et al. (2019) have introduced a new concept of linguistic modelling in Neutrosophy. This is done to remove the uncertainty which seems to be a big hurdle while modelling linguistic terms while performing information retrieval. The modelling is performed using LOWA operator. Their work seems to be a milestone achieved for modelling linguistic modality in multimodal systems [14].

Elyas Rashno et al. (2019) have worked towards recognizing noisy speech. Their approach of recognition employs Convolution Neural network (CNN) model based on Neutrosophy. They have proposed Neutrosophic Convolution Neural Network (NCNN) claiming that this would ease the task of classification [18].

G. Jayaparthasarathy et al. (2019) have discussed various applications of Neutrosophy in data mining. To illustrate their objective, authors have taken the medical domain as their field of research [15]. A survey of machine learning in neutrosophic environment is presented by Azeddine Elhassouny et al. (2019) [59].

Kritika Mishra et al. (2020) have performed sentiment analysis using neutrosophy. Their proposed framework works with audio files and calculates their Single-Valued Neutrosophic Sets (SVNS) and clusters them into positive-neutral-negative. Later, obtained results from the above tasks are combined with sentiment analysis results obtained from textual files of the same audio file. Their approach seems to yield good results [21]. Table 6 presents a summary of work using neutrosophy giving more stress on author's contribution in the field for handling imperfections.

Table 6 A Summary of Research Work for Handling Imperfections Using Neutrosophy

S No.	Author & Year	Primary Contribution	Handling Imperfections
1.	Ned Vito Quevedo Arnaiz et al. (2020)	Developing Neutrosophic K-means based method for treatment of unlabelled data.	Yes
2.	Mouhammad Bakro et al. (2020)	• Neutrosophic representation of digital image. • Points of digital picture matrix converted into neutrosophic sets.	Yes
3.	Abhijit Saha et al. (2020)	• Described neutrosophic soft sets having incomplete data. • Described consistent and inconsistent association between parameters.	Yes
4.	Carmen Verónica Valenzuela-Chicaiza, et al. (2020)	Classical statistical inference tools for emotional intelligence.	Yes

5.	Ridvan Sahin (2014)	Hierarchical clustering algorithm based on Neutrosophy.	Yes
6.	A. A. Salama et al. (2014)	Image modality processing using Neutrosophy.	Yes
7.	A. Salama et al. (2014)	Introduced neutrosophic simple regression and correlation.	Yes
8.	Anjan Mukherjee et al. (2015)	• Application of Neutrosophy in pattern recognition. • Proposed weighted similarity measure between two neutrosophic soft sets.	Yes
9.	A. A. Salama et al. (2016)	Representing features of image modality in neutrosophic domain.	Yes
10.	Bouzina, Salah (2016)	Compared fuzzy logic with neutrosophic logic.	Yes
11.	Nguyen Xuan Thao et al. (2017)	Introduces Support Neutrosophic Set (SNS).	Yes
12.	Okpako Abugor Ejaita et al. (2017)	• Addressed uncertainties in medical diagnosis using Neutrosophy. • Introduced a framework based on Neutrosophic Neural Network.	Yes
13.	A. A. Salama et al. (2018)	Introduced an approach to grayscale image in neutrosophic domain.	Yes

14.	D. Vitalio Ponce Ruiz et al. (2019)	• Treatment of uncertainty while retrieving information. • Linguistic modelling using Neutrosophy.	Yes
15.	G. Jayaparthasarathy et al. (2019)	Applications of Neutrosophy in data mining.	Yes
16.	Azeddine Elhassouny et al. (2019)	Presented a survey of machine learning in neutrosophic environment.	Yes
17.	Ming Zhang et al. (2010)	A neutrosophic approach to image segmentation.	Yes
18.	Abdulkadir Sengur &Yanhui Guo (2011)	Color, texture image segmentation based on neutrosophic set and wavelet transform.	Yes
19.	Elyas Rashno et al. (2019)	• Worked to recognize noisy speech. • A Convolution Neural Network model based on Neutrosophy.	Yes
20.	Yaman Akbulut et al. (2017)	• Enhanced classification performance of k-NN by the introduction of Neutrosophy. • Introduced Neutrosophic-k-NN.	Yes
21.	Wen Ju et al. (2013)	Introduced Neutrosophic Support Vector Machine.	Yes
22.	Kritika Mishra et al. (2020)	Sentiment analysis using Neutrosophy.	Yes

4. Future Research Trends and Directions for Handling Imperfections in Multimodal Fusion:

Based on our literature investigation and analysis of more than 80 articles, various research trends, research directions, and potential research topics are drawn for handling imperfections in multimodal fusion research and development. Though the direct handling of imperfection at the fusion stage will not yield fruitful results, we recommend handling imperfections at each stage starting from selecting data sources and collection of data in different modalities to fusing the features together. The procedure involved in this is summarized in the following Figure 3.

Figure 3 Potential Trends and Future Directions for Multimodal Fusion Research and Development Using Neutrosophy

5. Conclusion

The plethora of digital data over the internet has surged the need of on-time accurate information using various intelligent information systems. This need has enabled researchers to design multimodal information systems which mainly depend on multimodal fusion. As the data which is collected for modelling these systems is in no way free from imperfections so is the multimodal fusion which deals with such data. The motivation behind the current study is to introduce researchers working in this field of multimodal fusion with the notion of indeterminacy, uncertainty and imprecision (imperfections) present in existing approaches. This work also enables researchers to understand the field of Neutrosophic Sets and Systems by illustrating various work which are conducted using this theory to handle imperfections. The present works clearly mention how the imperfections could be handled using neutrosophy in multimodal systems. Though the work has explained well the applicability of neutrosophy in multimodal information access and retrieval systems for handling imperfections, it has not implemented the concepts in present work. The future work in this regard would include the use of neutrosophic logic, neutrosophic algorithms and converting modalities in the neutrosophic domain so that multimodal fusion is achieved addressing the notion of imperfection. If this work is performed as explained in present paper, it would enable the design of multimodal systems more effectively so that it could be used in other areas such as medical diagnosis, financial market information, robotics, security, information fusion system, expert system and bioinformatics.

Conflicts of Interest: "The authors declare no conflict of interest."

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Pentapartitioned Neutrosophic Topological Space

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Abstract:
The main focus of this study is to present the notions of pentapartitioned neutrosophic topological space. We introduce the notions of closure and interior operator of pentapartitioned neutrosophic sets in pentapartitioned neutrosophic topological space, and investigate some of their basic properties. Further, we define pentapartitioned neutrosophic pre-open (in short P-NPO) set, pentapartitioned neutrosophic semi-open (in short P-NSO) set, pentapartitioned neutrosophic *b*-open (in short P-N-*b*-O) set and pentapartitioned neutrosophic α-open (in short P-Nα-O) set via pentapartitioned neutrosophic topological spaces. By defining P-NPO set, P-NSO set, P-N-*b*-O set, P-Nα-O set, we furnish some suitable examples and formulate some basic results on pentapartitioned neutrosophic topological spaces.

Keywords: Neutrosophic Set; Pentapartitioned Neutrosophic Set; P-NPO; P-NSO; P-N-b-O; P-Nα-O.

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1. Introduction: In the year 1998, Smarandache [30] introduced the notions of Neutrosophic Set (in short N-S) by extending the notions of Fuzzy Set [33] and Intuitionistic Fuzzy Set [4]. Later on, the notions of Neutrosophic Topological Space (in short N-T-S) was grounded by Salama and Alblowi [29] in the year 2012. Thereafter, Arokiarani et al. [3] defined the notions of neutrosophic semi-open functions. In the year 2016, Iswaraya and Bageerathi [19] presented the concept of neutrosophic semi-open set and neutrosophic semi-closed set via N-T-Ss. Later on, Dhavaseelan and Jafari [17] introduced the idea of generalized neutrosophic closed sets. The notions of neutrosophic generalized closed sets via N-T-Ss was studied by Pushpalatha and Nandhini [27]. The idea of neutrosophic *b*-open sets in N-T-Ss was presented by Ebenanjar et al. [18]. Thereafter, Maheswari et al. [22] presented the concept of neutrosophic generalized *b*-closed sets via N-T-Ss. In the year 2019, the concept of generalized neutrosophic *b*-open set via N-T-Ss was studied by Das and Pramanik [10]. Das and Pramanik [11] also grounded the notion of neutrosophic Φ-open sets and neutrosophic Φ-continuous mappings via N-T-Ss. Afterwards, Das and Pramanik [12] presented the notions of neutrosophic simply soft open set via neutrosophic soft topological spaces. Das and Tripathy [16] introduced and studied the neutrosophic simply *b*-open set via N-T-S. Recently, Das et al. [7] applied the concept of topology on Quadripartitioned N-Ss [5] and introduced the notions of Quadripartitioned N-T-S.

In the year 2020, Mallick and Pramanik [23] grounded the notions of Pentapatitioned Neutrosophic Set (in short P-N-S) by extending the notions of N-S and Quadripartitioned N-S. The main focus of this article is to procure the notions of Pentapartitioned Neutrosophic Topological Space (in short Pentapartitioned N-T-S) and study several properties of them.

Research Gap: No investigation on pentapartitioned neutrosophic topological space has been reported in the recent literature.

Motivation: To reduce the research gap, we procure the notion of pentapartitioned neutrosophic topological space.

The remaining part of this article has been split into the following sections:

In section-2, we recall some relevant definitions and results on N-S, N-T-S, and P-N-S. In section-3, we present the notions of Pentapartitioned N-T-S and formulate some results on it. In section-4, we conclude the work done in this paper.

2. Preliminaries and Definitions:

In this section, we give some some basic definitions and results those are relevant to the main results of this article.

Definition 2.1. [23] Let W be a universe of discourse. Then P, a P-N-S over W is defined by:

$P = \{(q,T_P(q),C_P(q),G_P(q),U_P(q),F_P(q)): q\in W\}$, where $T_P(q)$, $C_P(q)$, $G_P(q)$, $U_P(q)$, $F_P(q)$ ($\in[0, 1]$) are the truth membership, contradiction membership, ignorance membership, unknown membership, and falsity membership values of $q\in W$. So, $0 \leq T_P(q) + C_P(q) + G_P(q) + U_P(q) + F_P(q) \leq 5$, for all $q\in W$.

Definition 2.2. [23] The absolute P-N-S (1_{PN}) and the null P-N-S (0_{PN}) over a fixed set W are defined as follows:

(*i*) $1_{PN} = \{(q,1,1,0,0,0): q\in W\}$;

(*ii*) $0_{PN} = \{(q,0,0,1,1,1): q\in W\}$.

The absolute P-N-S 1_{PN} and the null P-N-S 0_{PN} have other seven types of representations. They are given below:

$1_{PN} = \{(q,1,1,0,0,1): q\in W\}$;

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$1_{PN} = \{(q,1,1,0,1,1): q\in W\}$;

$1_{PN} = \{(q,1,1,1,0,1): q\in W\}$;

$1_{PN} = \{(q,1,1,1,1,0): q\in W\}$;

$1_{PN} = \{(q,1,1,1,1,1): q \in W\}$;

$0_{PN} = \{(q,0,0,1,1,0): q \in W\}$;

$0_{PN} = \{(q,0,0,1,0,1): q \in W\}$;

$0_{PN} = \{(q,0,0,0,1,1): q \in W\}$;

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$0_{PN} = \{(q,0,0,0,0,0): q \in W\}$.

Remark 2.1. Throughout this article we shall use $1_{PN} = \{(q,1,1,0,0,0): q \in W\}$ and $0_{PN} = \{(q,0,0,1,1,1): q \in W\}$, since the complement of 1_{PN} needs to be 0_{PN} and the complement of 0_{PN} needs to be 1_{PN}. But for any combination of 1_{PN} and 0_{PN} from the other seven types of combination, it does not hold.

Clearly, $0_{PN} \subseteq X \subseteq 1_{PN}$, for any P-N-S X over W.

Definition 2.3. [23] Let $X = \{(q,T_X(q),C_X(q),G_X(q),U_X(q),F_X(q)): q \in W\}$ and $Y = \{(q,T_Y(q),C_Y(q),G_Y(q),U_Y(q), F_Y(q)): q \in W\}$ be two P-N-Ss over a fixed set W. Then, $X \subseteq Y$ if and only if $T_X(q) \leq T_Y(q)$, $C_X(q) \leq C_Y(q)$, $G_X(q) \geq G_Y(q)$, $U_X(q) \geq U_Y(q)$, $F_X(q) \geq F_Y(q)$, for all $q \in W$.

Example 2.1. Let $W = \{m_1, m_2\}$. Consider two P-NSs $X = \{(m_1,0.4,0.3,0.7,0.7,0.8), (m_2,0.2,0.5,0.8,0.7,0.8)\}$ and $Y = \{(m_1,0.7,0.5,0.5,0.5,0.4), (m_2,0.8,0.7,0.5,0.5,0.5)\}$ over W. Then, $X \subseteq Y$.

Definition 2.4. [23] Let $X = \{(q,T_X(q),C_X(q),G_X(q),U_X(q),F_X(q)): q \in W\}$ and $Y = \{(q,T_Y(q),C_Y(q),G_Y(q),U_Y(q), F_Y(q)): q \in W\}$ be two P-N-Ss over a fixed set W. Then, the intersection of X and Y is defined by

$X \cap Y = \{(q, \min\{T_X(q),T_Y(q)\}, \min\{C_X(q),C_Y(q)\}, \max\{G_X(q),G_X(q)\}, \max\{U_X(q),U_X(q)\}, \max\{F_X(q),F_X(q)\}): q \in W\}$.

Example 2.2. Let $W = \{m_1, m_2\}$. Consider two P-N-Ss $X = \{(m_1,0.6,0.5,0.6,0.7,0.5), (m_2,0.8,0.5,0.6,0.7,0.8)\}$ and $Y = \{(m_1,0.7,0.6,0.5,0.5,0.2), (m_2,0.9,0.7,0.4,0.3,0.8)\}$ over W. Then, intersection of X and Y is $X \cap Y = \{(m_1,0.6,0.5,0.6,0.7,0.5), (m_2,0.8,0.5,0.6,0.7,0.8)\}$.

Definition 2.5. [23] Let $X = \{(q,T_X(q),C_X(q),G_X(q),U_X(q),F_X(q)): q \in W\}$ and $Y = \{(q,T_Y(q),C_Y(q),G_Y(q),U_Y(q), F_Y(q)): q \in W\}$ be two P-N-Ss over a fixed set W. Then, the union of X and Y is defined by

$X \cup Y = \{(q, \max\{T_X(q),T_Y(q)\}, \max\{C_X(q),C_Y(q)\}, \min\{G_X(q),G_X(q)\}, \min\{U_X(q),U_X(q)\}, \min\{F_X(q),F_X(q)\}): q \in W\}$.

Example 2.3. Let $W = \{m_1, m_2\}$. Consider two P-N-Ss $X = \{(m_1,0.5,0.5,0.4,0.7,0.6), (m_2,0.7,0.5,0.7,0.8,0.4)\}$ and $Y = \{(m_1,0.8,0.5,0.7,0.8,0.9), (m_2,1.0,0.8,0.7,0.6,0.5)\}$ over W. Then, $X \cup Y = \{(m_1,0.8,0.5,0.4,0.7,0.6), (m_2,1.0,0.8,0.7,0.6,0.4)\}$.

Definition 2.6. [23] Suppose that $X = \{(q,T_X(q),C_X(q),G_X(q),U_X(q),F_X(q)): q \in W\}$ be a P-N-S over W. Then, the complement of X is defined by $X^c = \{(q,F_X(q),U_X(q),1\text{-}G_X(q),C_X(q),T_X(q)): q \in W\}$.

Example 2.4. Let $W = \{m_1, m_2\}$. Consider a P-N-S $X = \{(m_1,0.7,0.8,0.6,0.8,1.0), (m_2,1.0,0.9,0.5,0.4,0.8)\}$ be a P-NS over W. Then, the complement of X is $X^c = \{(m_1,1.0,0.8,0.4,0.8,0.7), (m_2,0.8,0.4,0.5,0.9,1.0)\}$.

Now, we define the complement of a P-N-S in another way, which was given below:

Definition 2.7. Let $X = \{(q,T_X(q),C_X(q),G_X(q),U_X(q),F_X(q)): q\in W\}$ be a P-N-S over a fixed set W. Then, the complement of X i.e. X^c is defined by

$X^c = \{(q,1-T_X(q),1-C_X(q),1-G_X(q),1-U_X(q),1-F_X(q)): q\in W\}$.

Example 2.5. Let $W = \{m_1, m_2\}$. Let $X = \{(m_1,0.5,0.8,0.4,0.7,0.5), (m_2,0.5,0.4,0.5,0.8,0.7)\}$ be a P-N-S over W. Then, $X^c = \{(m_1,0.5,0.2,0.6,0.3,0.5), (m_2,0.5,0.6,0.5,0.2,0.3)\}$.

3. Pentapartitioned Neutrosophic Topology:

In this section, we procure the notions of pentapartitioned neutrosophic topology on P-N-Ss. Then, we introduce the interior and closure of a P-N-S from the point of view of pentapartitioned N-T-S, and prove some results on them.

Definition 3.1. Let W be a fixed set. Then, a set $\Im$ of P-N-Ss over W is called a Pentapartitioned Neutrosophic Topology (in short Pentapartitioned N-T) on W, if the following three conditions hold:

(i) $0_{PN}, 1_{PN} \in\Im$;

(ii) $Y_1, Y_2 \in\Im \Rightarrow Y_1\cap Y_2 \in\Im$;

(iii) $\{Y_i: i\in\Delta\} \subseteq\Im \Rightarrow \cup Y_i\in\Im$.

Then, the pair $(W,\Im)$ is called a Pentapartitioned Neutrosophic Topological Space (in short Pentapartitioned N-T-S). Each element of $\Im$ is called a pentapartitioned neutrosophic open sets (in short P-NOS). If $Y\in\Im$, then Y^c is called a pentapartitioned neutrosophic closed set (in short P-NCS).

Example 3.1. Let X, Y and Z be three P-N-Ss over a fixed set $W=\{p, q, r\}$ such that:

$X = \{(p,0.7,0.4,0.6,0.7,0.5), (q,0.5,0.6,0.4,0.5,0.1), (r,0.9,0.5,0.3,0.6,0.7): p, q, r\in W\}$;

$Y = \{(p,0.6,0.4,0.7,0.8,0.9), (q,0.5,0.4,0.6,0.8,0.3), (r,0.4,0.4,0.7,0.7,0.8): p, q, r\in W\}$;

$Z = \{(p,0.5,0.3,0.8,0.8,1.0), (q,0.4,0.3,0.8,0.9,0.4), (r,0.3,0.4,0.8,0.7,1.0): p, q, r\in W\}$.

Then, the collection $\Im=\{0_{PN}, 1_{PN}, X, Y, Z\}$ forms a Pentapartitioned N-T on W.

Remark 3.1. In a Pentapartitioned N-T-S $(W,\Im)$, the null P-N-S (0_{PN}) and the absolute P-N-S (1_{PN}) are both P-NOS and P-NCS in $(W,\Im)$.

The pentapartitioned neutrosophic interior and pentapartitioned neutrosophic closure of a P-N-S are defined as follows:

Definition 3.2. Let $(W,\Im)$ be a Pentapartitioned N-T-S. Let X be a P-N-S over W. Then, the pentapartitioned neutrosophic interior (in short $P\text{-}N_{int}$) of X is the union of all P-NOSs contained in X and the pentapartitioned neutrosophic closure (in short $P\text{-}N_{cl}$) of X is the intersection of all P-NCSs containing X, i.e.

$P\text{-}N_{int}(X) = \cup\{Y: Y\subseteq X$ and Y is a P-NOS in $(W,\Im)\}$,

and $P\text{-}N_{cl}(X) = \cap\{Z: X\subseteq Z$ and Z is a P-NCS in $(W,\Im)\}$.

Remark 3.2. It is clearly seen that $P\text{-}N_{int}(X)$ is the largest P-NOS in $(W,\Im)$, which is contained in X and $P\text{-}N_{cl}(X)$ is the smallest P-NCS in $(W,\Im)$ that contains X.

Theorem 3.1. Let $(W,\Im)$ be a Pentapartitioned N-T-S. Let Q and R be any two P-N-Ss over W. Then, the following holds:

(i) $P\text{-}N_{int}(Q) \subseteq Q \subseteq P\text{-}N_{cl}(Q)$;

(ii) $Q \subseteq R \Rightarrow P\text{-}N_{cl}(Q) \subseteq P\text{-}N_{cl}(R)$;

(iii) $Q \subseteq R \Rightarrow P\text{-}N_{int}(Q) \subseteq P\text{-}N_{int}(R)$;

(iv) $P\text{-}N_{cl}(Q\cup R) = P\text{-}N_{cl}(Q) \cup P\text{-}N_{cl}(R)$;

(v) $P\text{-}N_{cl}(Q\cap R) \subseteq P\text{-}N_{cl}(Q) \cap P\text{-}N_{cl}(R)$;

(vi) $P\text{-}N_{int}(Q\cup R) \supseteq P\text{-}N_{int}(Q) \cup P\text{-}N_{int}(R)$;

(vii) $P\text{-}N_{int}(Q\cap R) \subseteq P\text{-}N_{int}(Q) \cap P\text{-}N_{int}(R)$.

Proof. (i) From definition 3.2., we have $P\text{-}N_{int}(Q) = \cup\{R: R$ is a P-NOS in $(W,\Im)$ and $R\subseteq Q\}$. Since, each $R\subseteq Q$, so $\cup\{R: R$ is a P-NOS in $(W,\Im)$ and $R\subseteq Q\} \subseteq Q$, i.e. $P\text{-}N_{int}(Q)\subseteq Q$.

Again, $P\text{-}N_{cl}(Q) = \cap\{Z: Z$ is a P-NCS in $(W,\Im)$ and $Q\subseteq Z\}$. Since, each $Z\supseteq Q$, so $\cap\{Z: Z$ is a P-NCS in $(W,\Im)$ and $Q\subseteq Z\} \supseteq Q$, i.e. $P\text{-}N_{cl}(Q)\supseteq Q$.

Therefore, $P\text{-}N_{int}(Q)\subseteq Q\subseteq P\text{-}N_{cl}(Q)$.

(ii) Let $(W,\Im)$ be a Pentapartitioned N-T-S. Let Q and R be any two P-N-Ss over W such that $Q\subseteq R$.

Now, $P\text{-}N_{cl}(Q) = \cap\{Z: Z$ is a P-NCS in $(W,\Im)$ and $Q\subseteq Z\}$

$\subseteq \cap\{Z: Z$ is a P-NCS in $(W,\Im)$ and $R\subseteq Z\}$ [Since $Q\subseteq R$]

$= P\text{-}N_{cl}(R)$

$\Rightarrow P\text{-}N_{cl}(Q) \subseteq P\text{-}N_{cl}(R)$.

Therefore, $Q\subseteq R \Rightarrow P\text{-}N_{cl}(Q)\subseteq P\text{-}N_{cl}(R)$.

(iii) Let $(W,\Im)$ be a Pentapartitioned N-T-S. Let Q and R be any two P-N-Ss over W such that $Q\subseteq R$.

Now, $P\text{-}N_{int}(Q) = \cup\{Z: Z$ is a P-NOS in $(W,\Im)$ and $Z\subseteq Q\}$

$\subseteq \cup\{Z: Z$ is a P-NOS in $(W,\Im)$ and $Z\subseteq R\}$ [Since $Q\subseteq R$]

$= P\text{-}N_{int}(R)$

$\Rightarrow P\text{-}N_{int}(Q) \subseteq P\text{-}N_{int}(R)$.

Therefore, $Q\subseteq R \Rightarrow P\text{-}N_{int}(Q)\subseteq P\text{-}N_{int}(R)$.

(iv) Let Q and R be two pentapartitioned neutrosophic subsets of a Pentapartitioned N-T-S $(W,\Im)$. It is known that $Q\subseteq Q\cup R$ and $R\subseteq Q\cup R$.

Now, $Q \subseteq Q\cup R$

$\Rightarrow P\text{-}N_{cl}(Q) \subseteq P\text{-}N_{cl}(Q\cup R)$;

and $R \subseteq Q\cup R$

$\Rightarrow P\text{-}N_{cl}(R) \subseteq P\text{-}N_{cl}(Q\cup R)$.

Therefore, $P\text{-}N_{cl}(Q)\cup P\text{-}N_{cl}(R) \subseteq P\text{-}N_{cl}(Q\cup R)$ (1)

We have, $Q\subseteq P\text{-}N_{cl}(Q)$, $R\subseteq P\text{-}N_{cl}(R)$. Therefore, $Q\cup R \subseteq P\text{-}N_{cl}(Q)\cup P\text{-}N_{cl}(R)$. Further, it is known that $P\text{-}N_{cl}(Q)\cup P\text{-}N_{cl}(R)$ is a P-NCS in $(W,\Im)$. It is clear that, $P\text{-}N_{cl}(Q)\cup P\text{-}N_{cl}(R)$ is a P-NCS in $(W,\Im)$, which contains $Q\cup R$. But it is known that $P\text{-}N_{cl}(Q\cup R)$ is the smallest P-NCS in $(W,\Im)$, which contains $Q\cup R$.

Therefore, $P\text{-}N_{cl}(Q\cup R) \subseteq P\text{-}N_{cl}(Q)\cup P\text{-}N_{cl}(R)$ (2)

From eq. (1) and eq. (2), we have $P\text{-}N_{cl}(Q\cup R) = P\text{-}N_{cl}(Q)\cup P\text{-}N_{cl}(R)$.

(v) Let Q and R be two pentapartitioned neutrosophic subsets of a P-NTS (W,$\Im$). It is known that $Q\cap R\subseteq Q$, $Q\cap R\subseteq R$.

Now, $Q\cap R\subseteq Q$

$\Rightarrow P\text{-}N_{cl}(Q\cap R) \subseteq P\text{-}N_{cl}(Q)$;

and $Q\cap R\subseteq R$

$\Rightarrow P\text{-}N_{cl}(Q\cap R) \subseteq P\text{-}N_{cl}(R)$.

Therefore, $P\text{-}N_{cl}(Q\cap R) \subseteq P\text{-}N_{cl}(Q)\cap P\text{-}N_{cl}(R)$.

(vi) Let Q and R be two pentapartitioned neutrosophic subsets of a Pentapartitioned N-T-S (W,$\Im$). It is known that $Q\subseteq Q\cup R$ and $R\subseteq Q\cup R$.

Thus, we get

$Q\subseteq Q\cup R$

$\Rightarrow P\text{-}N_{int}(Q) \subseteq P\text{-}N_{int}(Q\cup R)$;

and $R\subseteq Q\cup R$

$\Rightarrow P\text{-}N_{int}(R) \subseteq P\text{-}N_{int}(Q\cup R)$.

Therefore, $P\text{-}N_{int}(Q)\cup P\text{-}N_{int}(R) \subseteq P\text{-}N_{int}(Q\cup R)$.

(vii) Let Q and R be two pentapartitioned neutrosophic subsets of a Pentapartitioned N-T-S (W,$\Im$). It is known that $Q\cap R\subseteq Q$, $Q\cap R\subseteq R$.

Now, $Q\cap R\subseteq Q$

$\Rightarrow P\text{-}N_{int}(Q\cap R) \subseteq P\text{-}N_{int}(Q)$;

and $Q\cap R\subseteq R$

$\Rightarrow P\text{-}N_{int}(Q\cap R) \subseteq P\text{-}N_{int}(R)$.

Therefore, $P\text{-}N_{int}(Q\cap R) \subseteq P\text{-}N_{int}(Q)\cap P\text{-}N_{int}(R)$.

Theorem 3.2. Let Q be a pentapartitioned neutrosophic subset of a Pentapartitioned N-T-S (W,$\Im$). Then, the following holds:

(i) $(P\text{-}N_{int}(Q))^c = P\text{-}N_{cl}(Q^c)$;

(ii) $(P\text{-}N_{cl}(Q))^c = P\text{-}N_{int}(Q^c)$.

Proof. (i) Let (W,$\Im$) be a Pentapartitioned N-T-S and $Q = \{(w, T_Q(w), C_Q(w), G_Q(w), U_Q(w), F_Q(w)): w\in W\}$ be a pentapartitioned neutrosophic subset of W.

We have,

$P\text{-}N_{int}(Q) = \cup\{Z_i : i\in\Delta$ and Z_i is a P-NOS in (W,$\Im$) such that $Z_i \subseteq Q\}$

$= \{(w, \vee T_{Z_i}(w), \vee C_{Z_i}(w), \wedge G_{Z_i}(w), \wedge U_{Z_i}(w), \wedge F_{Z_i}(w)) : w \in W\}$, where for all $i\in\Delta$ and Z_i is a P-NOS in (W,$\Im$) such that $Z_i\subseteq Q$.

This implies, $(P\text{-}N_{int}(Q))^c = \{(w, \wedge T_{Z_i}(w), \wedge C_{Z_i}(w), \vee G_{Z_i}(w), \vee U_{Z_i}(w), \vee F_{Z_i}(w)): w\in W\}$.

Since, $\wedge T_{Z_i}(w) \leq T_E(w)$, $\wedge C_{Z_i}(w) \leq C_E(w)$, $\vee G_{Z_i}(w) \geq G_E(w)$, $\vee U_{Z_i}(w) \geq U_E(w)$, $\vee F_{Z_i}(w) \geq F_E(w)$, for each $i\in\Delta$ and $w\in W$, so $P\text{-}N_{cl}(Q^c) = \{(w, \wedge T_{Z_i}(w), \wedge C_{Z_i}(w), \vee G_{Z_i}(w), \vee U_{Z_i}(w), \vee F_{Z_i}(w)): w\in W\}\} = \cap\{Z_i: i\in\Delta$ and Z_i is a P-NCS in (W,$\Im$) such that $Q^c\subseteq Z_i\}$. Therefore, $(P\text{-}N_{int}(Q))^c = P\text{-}N_{cl}(Q^c)$.

(*ii*) Let $(W,\Im)$ be a Pentapartitioned N-T-S and $Q = \{(w, T_Q(w), C_Q(w), G_Q(w), U_Q(w), F_Q(w)): w\in W\}$ be a pentapartitioned neutrosophic subset of W.

We have,

$P\text{-}N_{cl}(Q) = \cap\{Z_i: i\in\Delta$ and Z_i is a P-NCS in $(W,\Im)$ such that $Z_i \supseteq Q\}$

$= \{(w, \wedge T_{Z_i}(w), \wedge C_{Z_i}(w), \vee G_{Z_i}(w), \vee U_{Z_i}(w), \vee F_{Z_i}(w)): w\in W\}$, where Z_i is a P-NCS in $(W,\Im)$ such that $Z_i \supseteq Q$, for all $i\in\Delta$.

This implies, $(P\text{-}N_{cl}(Q))^c = \{(w, \vee T_{Z_i}(w), \vee C_{Z_i}(w), \wedge G_{Z_i}(w), \wedge U_{Z_i}(w), \wedge F_{Z_i}(w)): w\in W\}$.

Since $\vee T_{Z_i}(w) \geq T_E(w)$, $\vee C_{Z_i}(w) \geq C_E(w)$, $\wedge G_{Z_i}(w) \leq G_E(w)$, $\wedge U_{Z_i}(w) \leq U_E(w)$, $\wedge F_{Z_i}(w) \leq F_E(w)$, for each $i\in\Delta$ and $w\in W$, so $P\text{-}N_{int}(Q^c) = \{(w, \vee T_{Z_i}(w), \vee C_{Z_i}(w), \wedge G_{Z_i}(w), \wedge U_{Z_i}(w), \wedge F_{Z_i}(w)): w\in W\} = \cup\{Z_i: i\in\Delta$ and Z_i is a P-NOS in $(W,\Im)$ such that $Z_i \subseteq Q^c\}$. Therefore, $(P\text{-}N_{cl}(Q))^c = P\text{-}N_{int}(Q^c)$.

Theorem 3.3. Let X be a pentapartitioned neutrosophic subset of a Pentapartitioned N-T-S $(W,\Im)$. Then, the following holds:

(*i*) Q is a P-NOS if and only if $P\text{-}N_{int}(Q) = Q$;

(*ii*)Q is a P-NOS if and only if $P\text{-}N_{cl}(Q) =Q$.

Proof. (*i*) Let Q be a P-NOS in a Pentapartitioned N-T-S $(W,\Im)$. Now, $P\text{-}N_{int}(Q) = \cup\{Z$: Z is a P-NOS in $(W,\Im)$ and $Z \subseteq Q\}$. Since, Q is a P-NOS in $(W,\Im)$, so Q is the largest P-NOS, which is contained in Q. This implies, $\cup\{Z$: Z is a P-NOS in $(W,\Im)$ and $Z \subseteq Q\} = Q$. Therefore, $P\text{-}N_{int}(Q) = Q$.

(*ii*) Let Q be a P-NCS in a Pentapartitioned N-T-S $(W,\Im)$. Now, $P\text{-}N_{cl}(Q) = \cap\{Z$: Z is a P-NCS in $(W,\Im)$ and $Q\subseteq Z\}$. Since, Q is a P-NCS in $(W,\Im)$, so Q is the smallest P-NCS, which contains Q. This implies, $\cap\{Z$: Z is a P-NCS in $(W,\Im)$ and $Q\subseteq Z\} = Q$. Therefore, $P\text{-}N_{cl}(Q) = Q$.

Definition 3.3. Let $(W,\Im)$ be a Pentapartitioned N-T-S. Then X, a P-N-S over W is called a

(*i*) pentapartitioned neutrosophic semi-open (P-NSO) set if and only if $X \subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X))$;

(*ii*) pentapartitioned neutrosophic pre-open (P-NPO) set if and only if $X \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X)$.

Remark 3.3. The complement of P-NSO set and P-NPO set in a Pentapartitioned N-T-S $(W,\Im)$ are called pentapartitioned neutrosophic semi-closed (in short P-NSC) set and pentapartitioned neutrosophic pre-closed (in short P-NPC) set respectively.

Theorem 3.4. Let $(W,\Im)$ be a Pentapartitioned N-T-S. Then,

(*i*) every P-NOS is a P-NSO set.

(*ii*) every P-NOS is a P-NPO set.

Proof. (*i*) Let $(W,\Im)$ be a Pentapartitioned N-T-S. Let X be a P-NOS. Therefore, $X=P\text{-}N_{int}(X)$. It is known that $X\subseteq P\text{-}N_{cl}(X)$. This implies, $X\subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X))$. Therefore, X is a P-NSO set in $(W,\Im)$.

(*ii*) Let $(W,\Im)$ be a Pentapartitioned N-T-S. Let X be a P-NOS. Therefore, $X=P\text{-}N_{int}(X)$. It is known that, $X\subseteq P\text{-}N_{cl}(X)$. This implies, $P\text{-}N_{int}(X)\subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X))$ i.e. $X = P\text{-}N_{int}(X) \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X))$. Therefore, $X \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X))$. Hence, X is a P-NPO set in $(W,\Im)$.

Remark 3.4. The converse of the previous theorem may not be true in general. This follows from the following example.

Example 3.2. Let $(W,\Im)$ be a Pentapartitioned N-T-S, where $\Im=\{0_{PN}, 1_{PN}, \{(a,0.3,0.4,0.5,0.4,0.3), (b,0.4,0.3,0.7,0.3,0.4)\}, \{(a,0.4,0.6,0.4,0.4,0.1), (b,0.5,0.4,0.5,0.1,0.3)\}\}$. Then,

(*i*) $Q=\{(a,0.6,0.6,0.3,0.4,0.1), (b,0.9,0.8,0.4,0.1,0.2)\}$ is a P-NSO set but it is not a P-NOS in $(W,\Im)$.

(*ii*) $P=\{(a,0.3,0.7,0.2,0.9,0.2), (b,0.3,0.7,0.5,0.4,0.3)\}$ is a P-NPO set but it is not a P-NOS in $(W,\Im)$.

Theorem 3.5. In a Pentapartitioned N-T-S $(W,\Im)$, the union of two P-NSO sets is a P-NSO set.

Proof. Let X and Y be two P-NSO sets in a Pentapartitioned N-T-S $(W,\Im)$. Therefore,

$X\subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X))$ (3)

and $Y\subseteq P\text{-}N_{cl}(P\text{-}N_{int}(Y))$ (4)

From eq. (3) and eq. (4), we have

$X\cup Y \subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X))\cup P\text{-}N_{cl}(P\text{-}N_{int}(Y))$

$= P\text{-}N_{cl}(P\text{-}N_{int}(X)\cup P\text{-}N_{int}(Y))$

$\subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X\cup Y))$.

Therefore, $X\cup Y \subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X\cup Y))$. Hence, $X\cup Y$ is a P-NSO set in $(W,\Im)$.

Theorem 3.6. In a Pentapartitioned N-T-S $(W,\Im)$, the union of two P-NPO sets is also a P-NPO set.

Proof. Let X and Y be two P-NPO sets in a Pentapartitioned N-T-S $(W,\Im)$. Therefore,

$X\subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X))$ (5)

and $Y\subseteq P\text{-}N_{int}(P\text{-}N_{cl}(Y))$ (6)

From eq. (5) and eq. (6), we have,

$X\cup Y\subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X))\cup P\text{-}N_{int}(P\text{-}N_{cl}(Y))$

$\subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X)\cup P\text{-}N_{cl}(Y))$

$= P\text{-}N_{int}(P\text{-}N_{cl}(X\cup Y))$.

Therefore, $X\cup Y \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X\cup Y))$. Hence, $X\cup Y$ is a P-NPO set in $(W,\Im)$.

Definition 3.4. Let $(W,\Im)$ be a Pentapartitioned N-T-S. Then, a P-N-S X over W is called a pentapartitioned neutrosophic α-open (in short P-Nα-O) set if and only if $X\subseteq P\text{-}N_{int}(P\text{-}N_{cl}(P\text{-}N_{int}(X)))$. The complement of a P-Nα-O set is called a pentapartitioned neutrosophic α-closed (in short P-Nα-C) set.

Proposition 3.1. In a Pentapartitioned N-T-S $(W,\Im)$, every P-NOS is a P-Nα-O set.

Remark 3.5. The converse of the above proposition may not be true in general, which follows from the following example.

Example 3.3. Let us consider a Pentapartitioned N-T-S $(W,\Im)$ as shown in Example 3.2. Clearly, the pentapartitioned neutrosophic set $Q=\{(a,0.6,0.6,0.3,0.4,0.1), (b,0.9,0.8,0.4,0.1,0.2)\}$ is a P-Nα-O set but it is not a P-NOS in $(W,\Im)$.

Theorem 3.7. In a Pentapartitioned N-T-S $(W,\Im)$, every P-Nα-O set is a P-NSO set.

Proof. Let X be a P-Nα-O set in $(W,\Im)$. Therefore, $X\subseteq P\text{-}N_{int}(P\text{-}N_{cl}(P\text{-}N_{int}(X)))$. It is known that $P\text{-}N_{int}(P\text{-}N_{cl}(P\text{-}N_{int}(X))) \subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X))$. Thus we have, $X\subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X))$. Hence, X is a P-NSO set. Therefore, every P-Nα-O set is a P-NSO set.

Remark 3.6. The converse of the above example may not be true in general. This follows from the following example.

Example 3.4. Let $(W,\Im)$ be a Pentapartitioned N-T-S, where $\Im$={0_{PN}, 1_{PN}, {(*a*,0.5,0.6,0.5,0.7,0.8), (*b*,0.5,0.5,0.5,0.5,0.6)}, {(*a*,0.4,0.4,0.8,0.8,0.8), (*b*,0.5,0.5,0.8,0.8,0.8)}}. Then, it can be easily verified that *A*={(*a*,0.6,0.6,0.3,0.3,0.3), (*b*,0.5,0.5,0.4,0.4,0.4)} is a P-NSO set in $(W,\Im)$, but it is not a P-Nα-O set in $(W,\Im)$.

Theorem 3.8. In a Pentapartitioned N-T-S $(W,\Im)$, every P-Nα-O set is a P-NPO set.

Proof. Let $(W,\Im)$ be a Pentapartitioned N-T-S. Let X be a P-Nα-O set in $(W,\Im)$. Therefore, $X\subseteq P\text{-}N_{int}(P\text{-}N_{cl}(P\text{-}N_{int}(X)))$. It is known that $P\text{-}N_{int}(X)\subseteq X$. This implies, $P\text{-}N_{cl}(P\text{-}N_{int}(X))\subseteq P\text{-}N_{cl}(X)$. Which implies $P\text{-}N_{int}(P\text{-}N_{cl}(P\text{-}N_{int}(X))) \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X)$. Therefore, $X\subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X)$. Hence, X is a P-NPO set. Therefore, every P-Nα-O set is a P-NPO set in $(W,\Im)$.

Remark 3.7. The converse of the above example may not be true in general. This follows from the following example.

Example 3.5. Let us consider a Pentapartitioned N-T-S $(W,\Im)$ as shown in Example 3.2. Then, the pentapartitioned neutrosophic set *P*={(*a*,0.3,0.7,0.2,0.9,0.2), (*b*,0.3,0.7,0.5,0.4,0.3)} is a P-NPO set in $(W,\Im)$ but it is not a P-Nα-O set in $(W,\Im)$.

Definition 3.5. Let $(W,\Im)$ be a P-NTS. Then, a P-NS X over W is called a pentapartitioned neutrosophic *b*-open (in short P-N-*b*-O) set if and only if $X \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X)) \cup P\text{-}N_{cl}(P\text{-}N_{int}(X))$.

Remark 3.8. A pentapartitioned neutrosophic set X is called a pentapartitioned neutrosophic *b*-closed (in short P-N-*b*-C) set iff X^c is a P-N-*b*-O set i.e. if $P\text{-}N_{int}(P\text{-}N_{cl}(X))\cap P\text{-}N_{cl}(P\text{-}N_{int}(X)) \subseteq X$.

Theorem 3.9. In a P-NTS $(W,\Im)$, every P-NPO (P-NSO) set is a P-N-*b*-O set.

Proof. Suppose that X be a P-NPO set in a P-NTS $(W,\Im)$. Therefore, $X \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X))$. This implies, $X \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X)) \cup P\text{-}N_{cl}(P\text{-}N_{int}(X))$. Hence, X is a P-N-*b*-O set. Therefore, every P-NPO set is a P-N-*b*-O set.

Similarly, it can be shown that every P-NSO set is a P-N-*b*-O set.

Theorem 3.10. The union of two P-N-*b*-O sets in a P-NTS $(W,\Im)$ is a P-N-*b*-O set.

Proof. Let X and Y be two P-N-*b*-O sets in a P-NTS $(W,\Im)$.

Therefore, $X \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X)) \cup P\text{-}N_{cl}(P\text{-}N_{int}(X))$ (7)

and $Y \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(Y)) \cup P\text{-}N_{cl}(P\text{-}N_{int}(Y))$ (8)

It is known that, $X \subseteq X\cup Y$ and $Y \subseteq X\cup Y$.

Now, $X \subseteq X\cup Y$

$\Rightarrow P\text{-}N_{int}(X) \subseteq P\text{-}N_{int}(A\cup B)$

$\Rightarrow P\text{-}N_{cl}(P\text{-}N_{int}(X)) \subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X\cup Y))$ (9)

and $X \subseteq X\cup Y$

$\Rightarrow P\text{-}N_{cl}(X) \subseteq P\text{-}N_{cl}(A\cup B)$

$\Rightarrow P\text{-}N_{int}(P\text{-}N_{cl}(X)) \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X\cup Y))$ (10)

Similarly, it can be shown that

$P\text{-}N_{cl}(P\text{-}N_{int}(Y)) \subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X\cup Y))$ (11)

$P\text{-}N_{int}(P\text{-}N_{cl}(Y)) \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X\cup Y))$ (12)

From eq. (7) and eq. (8) we have,

$X\cup Y \subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X)) \cup P\text{-}N_{int}(P\text{-}N_{cl}(X)) \cup P\text{-}N_{cl}(P\text{-}N_{int}(Y)) \cup P\text{-}N_{int}(P\text{-}N_{cl}(Y))$

$\subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X\cup Y)) \cup P\text{-}N_{int}(P\text{-}N_{cl}(X\cup Y)) \cup P\text{-}N_{cl}(P\text{-}N_{int}(X\cup Y)) \cup P\text{-}N_{int}(P\text{-}N_{cl}(X\cup Y))$

[by eqs (9), (10), (11), & (12)]

$= P\text{-}N_{cl}(P\text{-}N_{int}(X\cup Y)) \cup P\text{-}N_{int}(P\text{-}N_{cl}(X\cup Y))$

$\Rightarrow X\cup Y \subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X\cup Y)) \cup P\text{-}N_{int}(P\text{-}N_{cl}(X\cup Y))$.

Therefore, $X\cup Y$ is a P-N-*b*-O set.

Hence, the union of two P-N-*b*-O sets is a P-N-*b*-O set.

Theorem 3.11. In a P-NTS $(W,\mathfrak{I})$, the intersection of two P-N-*b*-C sets is a P-N-*b*-C set.

Proof. Let $(W,\mathfrak{I})$ be a P-NTS. Let X and Y be two P-N-*b*-C sets in $(W,\mathfrak{I})$. Therefore,

$P\text{-}N_{int}(P\text{-}N_{cl}(X)) \cap P\text{-}N_{cl}(P\text{-}N_{int}(X)) \subseteq X$ (13)

and $P\text{-}N_{int}(P\text{-}N_{cl}(Y)) \cap P\text{-}N_{cl}(P\text{-}N_{int}(Y)) \subseteq Y$ (14)

Since, $X\cap Y \subseteq X$ and $X\cap Y \subseteq Y$, so we get

$P\text{-}N_{int}(X\cap Y) \subseteq P\text{-}N_{int}(X) \Rightarrow P\text{-}N_{cl}(P\text{-}N_{int}(X\cap Y)) \subseteq P\text{-}N_{cl}(P\text{-}N_{int}(X))$; (15)

$P\text{-}N_{cl}(X\cap Y) \subseteq P\text{-}N_{cl}(X) \Rightarrow P\text{-}N_{int}(P\text{-}N_{cl}(X\cap Y)) \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(X))$ (16)

$P\text{-}N_{int}(X\cap Y) \subseteq P\text{-}N_{int}(Y) \Rightarrow P\text{-}N_{cl}(P\text{-}N_{int}(X\cap Y)) \subseteq P\text{-}N_{cl}(P\text{-}N_{int}(Y))$ (17)

and $P\text{-}N_{cl}(X\cap Y) \subseteq P\text{-}N_{cl}(Y) \Rightarrow P\text{-}N_{int}(P\text{-}N_{cl}(X\cap Y)) \subseteq P\text{-}N_{int}(P\text{-}N_{cl}(Y))$ (18)

From eq. (13) and eq. (14) we get,

$X\cap Y \supseteq P\text{-}N_{int}(P\text{-}N_{cl}(X)) \cap P\text{-}N_{cl}(P\text{-}N_{int}(X)) \cap P\text{-}N_{int}(P\text{-}N_{cl}(Y)) \cap P\text{-}N_{cl}(P\text{-}N_{int}(Y))$

$\supseteq P\text{-}N_{int}(P\text{-}N_{cl}(X\cap Y)) \cap P\text{-}N_{cl}(P\text{-}N_{int}(X\cap Y)) \cap P\text{-}N_{int}(P\text{-}N_{cl}(X\cap Y)) \cap P\text{-}N_{cl}(P\text{-}N_{int}(X\cap Y))$

[by eqs (15), (16), (17) & (18)]

$= P\text{-}N_{int}(P\text{-}N_{cl}(X\cap Y)) \cap P\text{-}N_{cl}(P\text{-}N_{int}(X\cap Y))$

$\Rightarrow X\cap Y \supseteq P\text{-}N_{cl}(P\text{-}N_{int}(X\cap Y)) \cap P\text{-}N_{int}(P\text{-}N_{cl}(X\cap Y))$.

Hence, $X\cap Y$ is a P-N-*b*-C set in $(W,\mathfrak{I})$.

Therefore, the intersection of two P-N-*b*-C sets is again a P-N-*b*-C set.

4. Conclusion: In this study, we present the notions of pentapartitioned neutrosophic topological space and studied different types of open sets namely P-NPO set, P-NSO set, P-N-*b*-O set, and P-Nα-O set. By defining P-NPO set, P-NSO set, P-N-*b*-O set and P-Nα-O set, we formulate some results on Pentapartitioned N-T-Ss in the form of Theorems, Propositions, etc. We provide few illustrative counter examples where the results fail. We hope that, in the future, based on these notions and various open sets on Pentapartitioned N-T-S, many new investigation / research can be done. Further, the notion of pentapartitioned neutrosophic topological space can be used in area of decision making, data mining, etc.

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On Separation Axioms with New Constructions in Fuzzy Neutrosophic Topological Spaces

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Abstract: The purpose of this .research is, to define a fuzzy neutrosophic points in fuzzy neutrosophic. topological space namely [FNPs]. Also, we have study some new types of points in separation axioms T_i, where i= 0,1,2 with some new construction based of fuzzy neutrosophic topological spaces as extension of Fatimah et al. work [*1*]. Then, we investigate many theorems and examples to present and discuss.

Keywords: fuzzy neutrosophic point; fuzzy neutrosophic topology; separation axioms.

1. Introduction

The concept of neutrosophic set theory introduced by Smarandache [2] and get the introduction of the term neutrosophic. components,$(T,I,F,)$ which refers to the membership, non-membership and between them the indeterminacy values . Then, Salama et al..[3,4] study some .basic concepts and theire operations, of the neutrosophic.crisp set for building. new branches. of neutrosophic. mathematic. Then, many authors studied and presented the term of neutrosophic set theory and some of its applications in their works, (see[5,6,7,8,9,10]).

Recently, many concepts of neutrosophic topological spaces have been extended in fuzzy neutrosophic topological spaces by the authors (see [11-20]). In this work, we put some basic concepts of the neutrosophic. set, with their operations, and because, of their .useful and wide applications to solve many problems, we used these concepts of fuzzy neutrosophic sets as generalized of Ahmed et al. study [21] to define new types of neutrosophic points based of our space, also our interest is to study separation axioms T_0, T_1, T_2 with new construction as extension of Fatimah et al. work [1] in fuzzy neutrosophic topological spaces by definitions, propositions and counter examples so in this paper, several types of fuzzy neutrosophic points in separation axioms via fuzzy neutrosophic topological spaces are going to be studied. Finally, we used the new concepts and definitions to examine the relationship between them in details.

2. Preliminaries:

In this part of our research, we will refer to some definitions and operations which are useful in our study.

Definition 2.1 [9]: "Let X_N be a non-empty fixed set. The fuzzy neutrosophic set (FNS), S_N is an object having the form $S_N = \{< x, \mu_{S_N}(x), \sigma_{S_N}(x), v_{S_N}(x) >: x \in X_N\}$ where the functions $\mu_{S_N}, \sigma_{S_N}, v_{S_N}: X_N \to [0, 1]$ denote the degree of membership function (namely $\mu_{S_N}(x)$), the degree of indeterminacy function (namely $\sigma_{S_N}(x)$) and the degree of non-membership function (namely $v_{S_N}(x)$) respectively of each element $x \in X_N$ to the set S_N and $0 \leq \mu_{S_N}(x) + \sigma_{S_N}(x) + v_{S_N}(x) \leq 3$, for each $x \in X_N$."

Remark 2.2 [15]: "FNS $\mu_N = \{< u, \lambda_{\mu N}(u), \sigma_{\mu N}(u), V_{\mu N}(u) >: u \in U\}$ can be identified to an ordered triple $< u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ in [0,1] on U."

Definition 2.3 [9]: "Fuzzy neutrosophic topology (FNT) on a non-empty set X_N is a family τ of fuzzy neutrosophic subsets in X_N satisfying the following axioms.

i. $0_N, 1_N \in \tau$,

ii. $S_{N_1} \wedge S_{N_2} \in \tau$ for any $S_{N_1}, S_{N_2} \in \tau$,

iii. $\vee S_{N_j} \in \tau, \forall \{S_{N_j}: j \in J\} \subseteq \tau$.

In this case the pair (X_N, τ) is called fuzzy neutrosophic topological space (FNTS). The elements of τ are called fuzzy neutrosophic -open sets (FN-open set). The complement of FN-open set in the FNTS (X_N, τ) is called fuzzy neutrosophic-closed set (FN-closed set)."

Definition 2.4 [9]: "Let $S = \langle \mu_S(x), \sigma_S(x), \gamma_S(x) >$ be a NS on X_N, then the complement of the set S (S^c, for short) maybe defined as three kinds of complements:

(C_1) $S^c = \{< x, 1 - \mu_S(x), 1 - \gamma_S(x) >: x \in X_N\}$,

(C_2) $S^c = \{< x, \gamma_S(x), \sigma_S(x), \mu_S(x) >: x \in X_N\}$,

(C_3) $S^c = \{< x, \gamma_S(x), 1 - \sigma_S(x), \mu_S(x) >: x \in X_N\}$."

Definition 2.5 [10]**:** "Let X_N be a non-empty set and two NS*s* S with M in the form $S =< \mu_S(x), \sigma_S(x), \gamma_S(x) >$, $M = < \mu_M(x), \sigma_M(x), \gamma_M(x) >$, then we may consider two possible definitions for subsets $(S \subseteq M)$ may be defined as :

(1) $S \subseteq M \Leftrightarrow \mu_S(x) \leq \mu_M(x), \gamma_S(x) \geq \gamma_M(x)$ and, $\sigma_S(x) \leq \sigma_M(x) \ \forall x \in X_N$,

(2) $S \subseteq M \Leftrightarrow \mu_S(x) \leq \mu_M(x), \gamma_S(x) \geq \gamma_M(x)$ and $\sigma_S(x) \geq \sigma_M(x) \forall x \in X_N$."

Proposition 2.6 [10]**:** "For any neutrosophic set S the following are holds:

(1) $0_N \subseteq S$, $0_N \subseteq 0_N$,

(2) $S \subseteq 1_N$, $1_N \subseteq 1_N$."

Also, the intersection can be written as $S \wedge M$ and may be defined by:

(I_1) $S \wedge M = < x, \mu_S(x) \wedge \mu_M(x), \sigma_S(x) \wedge \sigma_M(x), \gamma_S(x) \vee \gamma_M(x) >$,

(I_2) $S \wedge M = < x, \mu_S(x) \wedge \mu_M(x), \sigma_S(x) \vee \sigma_M(x), \gamma_S(x) \vee \gamma_M(x) >$.

Finally, the union can be written as $S \vee M$ may be defined by:

(U_1) $S \vee M$ =<x,$\mu_S(x) \vee \mu_M(x), \sigma_S(x) \vee \sigma_M(x), \gamma_S(x) \wedge \gamma_M(x) >$,
(U_2) $S \vee M$ =<x,$\mu_S(x) \vee \mu_M(x), \sigma_S(x) \wedge \sigma_M(x), \gamma_S(x) \wedge \gamma_M(x) >$.

3. Some New Separation Axioms in Fuzzy Neutrosophic Topological Spaces

In this section, we present T_i- separation axioms where I = 0,1,2 based of fuzzy neutrosophic topological spaces and introduced it after giving some definitions of as follows:

Definition 3.1: The object having the from $S = \langle S_1 , S_2 , S_3 \rangle$ is called:

1. A fuzzy neutrosophic set of Type 1 [FNS/Type1] if satisfying
   $S_1 \wedge S_2 = 0, S_1 \wedge S_3 = 0$ and $S_2 \wedge S_3 = 0$,
2. A fuzzy neutrosophic set of Type 2 [FNS/Type2] if satisfying
   $S_1 \wedge S_2 = 0, S_1 \wedge S_3 = 0$ and $S_2 \wedge S_3 = 0 , S_1 \vee S_2 \vee S_3 = 1$,
3. A fuzzy neutrosophic set of Type 3 [FNS/Type3] if satisfying
   $S_1 \wedge S_2 \wedge S_3 = 0$, $S_1 \vee S_2 \vee S_3 = 1$.

Example 3.2: Let $X_N = \{a\}$, then:

1. $S =< 0.5, 0, 0 >$ is a FNS in X_N,

Type 1: $S_1 \wedge S_2 = 0.5 \wedge 0 = 0, S_1 \wedge S_3 = 0.5 \wedge 0 = 0, S_2 \wedge S_3 = 0$

Therefore FNS is FN_1.

2. $S =< 1,0,0 >$ is an (FNS) in X_N,

Type 2 : $S_1 \wedge S_2 = 1 \wedge 0 = 0$, $S_1 \wedge S_3 = 1 \wedge 0 = 0, S_2 \wedge S_3 = 0$ _

$S_1 \vee S_2 \vee S_3 = 1$

Therefore FNS is FN_2.

3. $S =< 0.8, 1, 0 >$ is an (FNS) in X_N.

Type 3 : $S_1 \wedge S_2 \wedge S_3 = 0$, $S_1 \vee S_2 \vee S_3 = 1$.

Therefore FNS is FN_3**.**

Remark 3.3: For the FNS we have:

1. Every FN_2 is FN_1,
2. Every FN_2 is FN_3.

The proof is directly.

The converse of Remark 3.3 is not true as it shown in the next example.

Example 3.4: Let $X_N = \{a\}$, then:

1. $S =< 0.5,0,0 >$ is an (FNS) in X_N,

Type 1: $S_1 \wedge S_2 = 0.5 \wedge 0 = 0, S_1 \wedge S_3 = 0.5 \wedge 0 = 0, S_2 \wedge S_3 = 0.$

Therefore FNS is FN_1 but is not FN_2.

2. $S =< 0.8, 1, 0 >$ is an (FNS) in X_N.

Type 3 : $S_1 \wedge S_2 \wedge S_3 = 0$, $S_1 \vee S_2 \vee S_3 = 1.$

Therefore FNS is FN_1 but is not FN_2.

Definition 3.5: Types of FNSs 0_N and 1_N in X_N can defined as follows :

1. 0_N may be defined in many ways as a FNS as four ways:
   1. Type 1 : $0_N =< 0,0,1 >$,
   2. Type 2 : $0_N =< 0,1,1 >$,
   3. Type 3 : $0_N =< 0,1,0 >$,
   4. Type 4 : $0_N =< 0,0,0 >$.
2. 1_N may be defined in many ways as a FNS as:
   1. Type 1 : $1_N =< 1,0,0 >$,
   2. Type 2 : $1_N =< 1,1,0 >$,
   3. Type 3 : $1_N =< 1,0,1 >$,
   4. Type 4 : $1_N =< 1,1,1 >$.

Definition 3.6: Let X_N be a non - empty set and the FNSs α and β in form $\alpha =< \alpha_1, \alpha_2, \alpha_3 >$, $\beta =< \beta_1, \beta_2, \beta_3 >$, then we may consider two possible definitions for subsets $\alpha \subseteq \beta$, may be defined in two ways :

1. $\alpha \subseteq \beta \Leftrightarrow \alpha_1 \subseteq \beta_1, \alpha_2 \subseteq \beta_2$ and $\alpha_3 \subseteq \beta_3$,
2. $\alpha \subseteq \beta \Leftrightarrow \alpha_1 \subseteq \beta_1, \beta_2 \subseteq \alpha_2$ and $\beta_3 \subseteq \alpha_3$.

Definition 3.7: For all x ,y, z belonging to a non – empty set X_N, the fuzzy neutrosophic points related to x ,y, z are defined as follows:

1. $x_{N_1} =< \{x\},0,0 >$, is called a fuzzy neutrosophic point (FNP_{N_1}) in X_N,
2. $y_{N_2} =< 0, \{y\},0 >$, is called a fuzzy neutrosophic point (FNP_{N_2}) in X_N,
3. $z_{N_3} =< 0,0, \{z\} >$, is called a fuzzy neutrosophic point (FNP_{N_3}) in X_N.

The set of all fuzzy neutrosophic points (FNP_{N_1}, FNP_{N_2}, FNP_{N_3}) is denoted by FNP_N.

Definition 3.8: Let X_N be anon - empty set and x, y, z $\in X_N$. Then the fuzzy neutrosophic point:

1. x_{N_1} is belonging to the fuzzy neutrosophic set $S = < S_1, S_2, S_3 >$, denoted by $x_{N_1} \in S$, if $x \in S_1$ where in x_{N_1} does not belong to the fuzzy neutrosophic set S denoted by $x_{N_1} \notin S$, if $x_{N_1} \notin S_1$,
2. y_{N_2} is belonging to the fuzzy neutrosophic set $S = < S_1, S_2, S_3 >$, denoted by $y_{N_2} \in S$, if $y \in S_2$ in contrast y_{N_2} does not belong to the fuzzy neutrosophic set S denoted by $y_{N_2} \notin S$, if $y_{N_2} \notin S_2$,
3. z_{N_3} is belonging to the fuzzy neutrosophic set $S = < S_1, S_2, S_3 >$, denoted by $z_{N_3} \in S$, if $z \in S_3$ in contrast z_{N_3} does not belong to the fuzzy neutrosophic set S denoted by $z_{N_3} \notin S$, if $z_{N_3} \notin S_3$.

Definition 3.9: Let (X_N, τ) be a FNTS, Then X_N is called:

1- $FN_1 T_0$ -space for every two different points from X_N are x_{N_1}, y_{N_1} there exists two fuzzy neutrosophic open set S ,M inX_N such that $x_{N_1} \in S$, $y_{N_1} \notin S$ and $x_{N_1} \notin M$, $y_{N_1} \in M$,

2- $FN_2 T_0$ -space for every two different points from X_N are x_{N_2}, y_{N_2} there exists two fuzzy neutrosophic open set S ,M inX_N such that $x_{N_2} \in S$, $y_{N_2} \notin S$ and $x_{N_2} \notin M$, $y_{N_2} \in M$,

3- $FN_3 T_0$ -space for every two different points from X_N are x_{N_3}, y_{N_3} there exists two fuzzy neutrosophic open set S, M inX_N such that $x_{N_3} \in S$, $y_{N_3} \notin S$ and $x_{N_3} \notin M$, $y_{N_3} \in M$.

Example 3.10: Let $X_N = \{a,b,c\}$ and $\tau = \{0_N, 1_N, S\}$,

1. If $S = \{< \frac{a}{0.8,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\}$.

So, $x_{N_1} = \{< \frac{a}{0.8,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \neq y_{N_1} = \{< \frac{a}{0.7,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \in X_N$.

There is a FNOS in (X_N, τ) say $x_{N_1} = \{< \frac{a}{0.8,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \in S$ but $y_{N_1} = \{< \frac{a}{0.7,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \notin S$.

Therefore, (X_N, τ) is $FN_1 T_0$ -space .

2. If $S = \{< \frac{a}{1,1,0} >, < \frac{b}{0,1,0} >, < \frac{c}{0,1,0} >\}$.So, $x_{N_2} = \{< \frac{a}{1,1,0} >, < \frac{b}{1,0,0} >, < \frac{c}{0,1,0} >\} \neq y_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,0} >, < \frac{c}{0,1,0} >\} \in X_N$. There is a FNOS in (X_N, τ) say $x_{N_2} = \{< \frac{a}{1,1,0} >, < \frac{b}{1,0,0} >, < \frac{c}{0,1,0} >\} \in S$ but, $y_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,0} >, < \frac{c}{0,1,0} >\} \notin S$.

Therefore, (X_N, τ) is $FN_2 T_0$ -space

3. If $S=\{<\frac{a}{1,0.9,0}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,1}>\}$.

So, $x_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0.9,1},>\}\neq y_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0.6,1,0}>\}\in X_N$.

There is a FNOS in (X_N,τ) say, $x_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0.9,1},>\}\in S$ but, $y_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0.6,1,0}>\}\notin S$. Therefore, (X_N,τ) is $FN_3\,T_0$ -space .

Definition 3.11: Suppose that (X_N,τ) is a FNTS, Then X_N is called:

1- $FN_1\,T_1$ -space for every two different points from X_N are x_{N_1}, y_{N_1} there exists two fuzzy neutrosophic open set S ,M inX_N such that $x_{N_1}\in S$, $y_{N_1}\notin S$ and $x_{N_1}\notin M$, $y_{N_1}\in M$,
2- $FN_2\,T_1$ -space for every two different points from X_N are x_{N_2}, y_{N_2} there exists two fuzzy neutrosophic open set S ,M inX_N such that $x_{N_2}\in S$, $y_{N_2}\notin S$ and $x_{N_2}\notin M$, $y_{N_2}\in M$,
3- $FN_3\,T_1$ -space for every two different points from X_N are x_{N_3}, y_{N_3} there exists two fuzzy neutrosophic open set S ,M inX_N such that $x_{N_3}\in S$, $y_{N_3}\notin S$ and $x_{N_3}\notin M$, $y_{N_3}\in M$.

Example 3.12: Let $X_N=\{a,b,c\}$, $\tau=\{0_N,1_N,S,M,S\wedge M,S\vee M\}$,Then.

1. If $S=\{<\frac{a}{0.5,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}$and

$M=\{<\frac{a}{0.3,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}$.

So, $x_{N_1}=\{<\frac{a}{0.5,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\neq y_{N_1}=\{<\frac{a}{0,0.3,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\in X_N$.

There is a FNOS in (X_N,τ), say $x_{N_1}=\{<\frac{a}{0.5,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\in S$, $x_{N_1}=\{<\frac{a}{0.5,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\notin M$ and $y_{N_1}=\{<\frac{a}{0,0.3,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\in M$, $y_{N_1}=\{<\frac{a}{0,0.3,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\notin S$. Therefore, (X_N,τ) is $FN_1\,T_1$ -space.

2. If $S=\{<\frac{a}{1,1,0}>,<\frac{b}{0,1,0}>,<\frac{c}{0,1,0}>\}$, and

$M=\{<\frac{a}{0,0,0}>,<\frac{b}{0,1,0}>,<\frac{c}{0,1,0}>\}$

So, $x_{N_2} = \{< \frac{a}{0,1,0} >,< \frac{b}{1,1,0} >,< \frac{c}{0,1,0} >\} \neq y_{N_2} = \{< \frac{a}{0,1,0} >,< \frac{b}{0,0,0} >,< \frac{c}{0,1,0} >\} \in X_N$

There is a FNOS in (X_N, τ), say $x_{N_2} = \{< \frac{a}{0,1,0} >,< \frac{b}{1,1,0} >,< \frac{c}{0,1,0} >\} \in S$, $x_{N_2} = \{< \frac{a}{0,1,0} >,< \frac{b}{1,1,0} >,< \frac{c}{0,1,0} >\} \notin M$ .and $y_{N_2} = \{< \frac{a}{0,1,0} >,< \frac{b}{0,0,0} >,< \frac{c}{0,1,0} >\}$

$\in M$,$y_{N_2} = \{< \frac{a}{0,1,0} >,< \frac{b}{0,0,0} >,< \frac{c}{0,1,0} >\} \notin S$. Therefore, (X_N, τ) is $FN_2\, T_1$ -space .

3. If $S = \{< \frac{a}{1,0.2,0} >,< \frac{b}{0,0,1} >,< \frac{c}{0,0,1} >\}$ and

   $M = \{< \frac{a}{0,0.7,1} >, < \frac{b}{0,0,1} >,< \frac{c}{0,0,1} >\}$

$S \wedge M = \{< \frac{a}{0,0.2,0} >,< \frac{b}{0,0,1} >,< \frac{c}{0,0,1} >\}$

$S \vee M = \{< \frac{a}{1,0.7,1} >,< \frac{b}{0,0,1} >,< \frac{c}{0,0,1} >\}$.

So, $x_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.2,1} >\} \neq y_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.7,1} >\} \in X_N$

There is a FNOS in (X_N, τ) say, $x_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.2,1} >\} \in S$,$x_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.2,1} >\} \notin M$ and $y_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.7,1} >\} \in M$, $y_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.7,1} >\} \notin S$. Therefore, (X_N, τ) is $FN_3\, T_1$ -space .

Definition 3.13: Suppose that (X_N, τ) is a FNTS. Then X_N is called:

1- $FN_1\, T_2$ -space for every two different points from X_N are x_{N_1}, y_{N_1} there exists two fuzzy neutrosophic open set S,M inX_N such that $x_{N_1} \in S$, $y_{N_1} \notin S$ and $x_{N_1} \notin M, y_{N_1} \in M$ with $S \wedge M =< 0,0,0 >$,
2- $FN_2\, T_2$ -space for every two different points from X_N are x_{N_2}, y_{N_2} there exists two fuzzy neutrosophic open set S ,M inX_N such that $x_{N_2} \in S$, $y_{N_2} \notin S$ and $x_{N_2} \notin M$, $y_{N_2} \in M$ with $S \wedge M =< 0,1,0 >$,
3- $FN_3\, T_2$ -space for every two different points from X_N are x_{N_3}, y_{N_3} there exists two fuzzy neutrosophic open set S ,M inX_N such that $x_{N_3} \in S$, $y_{N_3} \notin S$ and $x_{N_3} \notin M, y_{N_3} \in M$ with $S \wedge M = < 0,0,1 >$.

Example 3.14: Let $X_N = \{a, b, c\}$, $\tau = \{0_N, 1_N, S, M, S\vee M\}$,Then.

1. If $S = \{< \frac{a}{0.6,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\}$, and

$M = \{< \frac{a}{0,0.8,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\}$

$S\vee M = \{< \frac{a}{0,6,0.8,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\}$.

So, $S \wedge M =< 0,0,0 >$

Let $x_{N_1} = \{< \frac{a}{0.6,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \neq y_{N_1} = \{< \frac{a}{0,0.8,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \in X_N$

There is a FNOS in (X_N, τ), say, $x_{N_1} = \{< \frac{a}{0.6,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \in S$, $x_{N_1} = \{< \frac{a}{0.6,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \notin M$ and $y_{N_1} = \{< \frac{a}{0,0.8,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \in M$, $y_{N_1} = \{< \frac{a}{0,0.8,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \notin S$. Therefore, (X_N, τ) is $N_1\ T_2$ -space .

2. If $S = \{< \frac{a}{1,1,0} >, < \frac{b}{0,1,,0} >, < \frac{c}{0,1,,0} >\}$and

$M = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,,0} >, < \frac{c}{0,1,,0} >\}$

So, $S \wedge M =< 0,1,0 >$

Then, $x_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{1,1,,0} >, < \frac{c}{0,1,,0} >\} \neq y_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,,0} >, < \frac{c}{0,1,,0} >\} \in X_N$

There is a FNOS in(X_N, τ), say $x_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{1,1,,0} >, < \frac{c}{0,1,,0} >\} \in S$, $x_{N_2} = \{< \frac{a}{0}, \frac{b}{1}, \frac{c}{0} >, < \frac{a}{1}, \frac{b}{1}, \frac{c}{0} >, < \frac{a}{0}, \frac{b}{1}, \frac{c}{0} >\} \notin M$ and $y_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,,0} >, < \frac{c}{0,1,,0} >\} \in M$, $y_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,,0} >, < \frac{c}{0,1,,0} >\} \notin S$. Therefore, (X_N, τ) is $FN_2\ T_2$ -space .

3. If $S = \{< \frac{a}{1,0.9,0} >, < \frac{b}{0,0,1} >, < \frac{c}{0,0,1} >\}$, and

$M = \{< \frac{a}{0,0,1} >, < \frac{b}{0,0,1} >, < \frac{c}{0,0,1} >\}$

$S\vee M=\{<\frac{a}{1,0.9,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,1}>\}$

So, $S\wedge M=<0,0,1>$

Then, $x_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{1,0.9,1}>\}\neq y_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,1}>\}\in X_N$

There is a FNOS in (X_N,τ) say, $x_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{1,0.9,1}>\}\in S$, $x_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{1,0.9,1}>\}\notin M$ and $y_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,1}>\}\in M$, $y_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,1}>\}\notin S$. Therefore, (X_N,τ) is FN_3 T_2-space .

Note: Veereswari Y. [11] defined and construct several FNTSs as in the next definition so, we used it to study some new kinds of separation axioms with some relations and examples.

Definition 3.15 [11]: Let (X_N,τ) be a FNTS on X_N Then ,we can also construct several FNTSs on X_N in the following ways:

1- $\tau_{0.1}=\{[\,]S: S\in\tau\}$, where $[\,]S=<x,\mu_S(x),\sigma_S(x),1-\mu_S(x)>=$
 $FS=<S_1,S_2,S_1^c>$,
2- $\tau_{0.2}=\{<>S: S\in\tau\}$, where $<>S=<x,1-V_S(x),\sigma_S(x),V_S(x)>=$
 $SE=<S_3^c,S_2,S_3>$.

Now, we defined and construct two new FNTSs from the FNTS (X_N,τ) as the next definition.

Definition 3.16: Let (X_N,τ) be a FNTS such that τ is not indiscrete such that $\tau=\{0_N,1_N\}\vee\{S_i, i\in J\}$. Then we can construct two (FNTSs) on X_N as follows:

1- $\tau^1=\{0_N,1_N\}\vee\{S_1\}$,
2- $\tau^2=\{0_N,1_N\}\vee\{S_2\}$.

Example 3.17: Let $X_N=\{a,b,c\}$, $\tau=\{0_N,1_N,S,M,S\vee M\}$,Then.

1. If $S=\{<\frac{a}{0.6,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}$, and

 $M=\{<\frac{a}{0,0.8,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}$

 $S\vee M=\{<\frac{a}{0,6,0.8,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}$.

So, $S \wedge M = < 0,0,0 >$

$\tau^1 = \{0_N, 1_N\} \vee \{0.6,0,0\} \vee \{0,0.8,0\}$

$\tau^1 = \{0_N, 1_N, \{0.6,0,0\}, \{0,0.8,0\}\}$.

$\tau^2 = \{0_N, 1_N\} \vee \{S_2\} \vee \{M_2\}$

$\tau^2 = \{0_N, 1_N\} \vee \{0,0,0\} \vee \{0,0,0\}$

$\tau^2 = \{0_N, 1_N, \{0,0,0\}, \{0,0,0\}\}$.

Let $x_{N_1} = \{< \frac{a}{0.6,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \neq y_{N_1} = \{< \frac{a}{0,0.8,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \in X_N$

There is a FNOS in (X_N, τ) say, $x_{N_1} = \{< \frac{a}{0.6,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \in S$, $x_{N_1} = \{< \frac{a}{0.6,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \notin M$ and $y_{N_1} = \{< \frac{a}{0,0.8,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \in M$, $y_{N_1} = \{< \frac{a}{0,0.8,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \notin S$. Therefore, (X_N, τ) is $FN_1\, T_2$-space .

2. If $S = \{< \frac{a}{1,1,0} >, < \frac{b}{0,1,,0} >, < \frac{c}{0,1,,0} >\}$ and

$M = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,,0} >, < \frac{c}{0,1,,0} >\}$

So, $S \wedge M = < 0,1,0 >$

$\tau^1 = \{0_N, 1_N\} \vee \{S_1\} \vee \{M_1\}$

$\tau^1 = \{0_N, 1_N\} \vee \{1,1,0\} \vee \{0,1,0\}$

$\tau^1 = \{0_N, 1_N, \{1,0,0\}, \{0,1,0\}\}$.

$\tau^2 = \{0_N, 1_N\} \vee \{S_2\} \vee \{M_2\}$

$\tau^2 = \{0_N, 1_N\} \vee \{0,1,0\} \vee \{0,1,0\}$

$\tau^2 = \{0_N, 1_N, \{0,1,0\}, \{0,1,0\}\}$.

If $x_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{1,1,,0} >, < \frac{c}{0,1,,0} >\} \neq y_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,,0} >, < \frac{c}{0,1,,0} >\} \in X_N$

There is a FNOS in (X_N, τ) say, $x_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{1,1,,0} >, < \frac{c}{0,1,,0} >\} \in S$, $x_{N_2} = \{< \frac{a}{0}, \frac{b}{1}, \frac{c}{0} >, < \frac{a}{1}, \frac{b}{1}, \frac{c}{0} >, < \frac{a}{0}, \frac{b}{1}, \frac{c}{0} >\} \notin M$ and $y_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,,0} >, < \frac{c}{0,1,,0} >\} \in M$, $y_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,,0} >, < \frac{c}{0,1,,0} >\} \notin S$. Therefore, (X_N, τ) is $FN_2\, T_2$ -space .

3. If $S = \{< \frac{a}{1,0.9,0} >, < \frac{b}{0,0,1} >, < \frac{c}{0,0,1} >\}$, and

   $M = \{< \frac{a}{0,0,1} >, < \frac{b}{0,0,1} >, < \frac{c}{0,0,1} >\}$

$S \vee M = \{< \frac{a}{1,0.9,1} >, < \frac{b}{0,0,1} >, < \frac{c}{0,0,1} >\}$

So, $S \wedge M = < 0,0,1 >$ and ,$\tau^1 = \{0_N, 1_N\} \vee \{S_1\} \vee \{M_1\}$.

$\tau^1 = \{0_N, 1_N\} \vee \{1,0.9,1\} \vee \{0,0,1\}$

$\tau^1 = \{0_N, 1_N, \{1,0.9,1\}, \{0,0,0\}\}$,

$\tau^2 = \{0_N, 1_N\} \vee \{S_2\} \vee \{M_2\}$

$\tau^2 = \{0_N, 1_N\} \vee \{0,0,1\} \vee \{0,0,1\}$

$\tau^2 = \{0_N, 1_N, \{0,0,1\}, \{0,0,1\}\}$.

If we put, $x_{N_3} = \{< \frac{a}{0,0,1} >, < \frac{b}{0,0,1} >, < \frac{c}{1,0.9,1} >\} \neq y_{N_3} = \{< \frac{a}{0,0,1} >, < \frac{b}{0,0,1} >, < \frac{c}{0,0,1} >\} \in X_N$

There is a FNOS in (X_N, τ) say, $x_{N_3} = \{< \frac{a}{0,0,1} >, < \frac{b}{0,0,1} >, < \frac{c}{1,0.9,1} >\} \in S$,

$x_{N_3} = \{< \frac{a}{0,0,1} >, < \frac{b}{0,0,1} >, < \frac{c}{1,0.9,1} >\} \notin M$ and$y_{N_3} = \{< \frac{a}{0,0,1} >, < \frac{b}{0,0,1} >, < \frac{c}{0,0,1} >\} \in M$, $y_{N_3} = \{< \frac{a}{0,0,1} >, < \frac{b}{0,0,1} >, < \frac{c}{0,0,1} >\} \notin S$. Therefore, (X_N, τ) is $FN_3\, T_2$ -space .

Definition 3.18: A FNTS (X_N, τ) is called:

1. FNT_0-space if (X_N, τ) is FN_1T_0-space, FN_2T_0-space and FN_3T_0-space.
2. FNT_1-space if (X_N, τ) is FN_1T_1-space, FN_2T_1-space and FN_3T_1-space.
3. FNT_2-space if (X_N, τ) is FN_2T_0-space, FN_2T_2-space and FN_3T_2-space.

The next theorem gave the relations between the FNT_1-space and the new defined construction $\tau_{0.1}$ and τ^1.

Theorem 3.19: Let (X_N, τ) be a FNTS, then the following are equivalent:

(i)- (X_N, τ) is a FN T_1 -space,

(ii)- $(X_N, \tau_{0.1})$ is a FN T_1 -space,

(iii)- (X_N, τ^1) is a FN T_1 -space.

Proof. (i) $\Longrightarrow$ (ii) Let $x_N, y_N \in X_N$ such that $x_N \neq y_N$ then there exist.

$U_{x_N} = < S_1, S_2, S_3 >$ and $V_{y_N} = < M_1, M_2, M_3 >$, such that

$x_N \in U_{x_N}$, if $x_N \in S_1$ and $y_N \in V_{y_N}$, if $y_N \in M_1$

Since $FU_{x_N} = < S_1, S_2, S_1^c >$ and $FV_{y_N} = < M_1, M_2, M_1^c >$.

Then, $x_N \in S_1$ and $x_N \notin S_1^c$, So $x_N \in FU_{x_N}$, $x_N \notin FV_{y_N} \Longrightarrow x_N \notin M_1$ or $x_N \in M_1^c$.

Now if $x_N \notin M_1$, then $x_N \in M_1^c$.

Therefore, $x_N \notin FV_{y_N}$. If $x_N \in M_1^c$,then $x_N \notin M_1^c$ and since $M_1 \wedge M_1^c = 0_N$.

So, $x_N \in M_1^c$,Thus $x_N \notin FV_{y_N}$.

Similarly; $y_N \in V_{y_N}$ and $x_N \notin FV_{y_N}$. Therefore, $(X_N, \tau_{0.1})$ is a FN T_1 -space .

(ii) $\Longrightarrow$ (iii): Suppose that $x_N, y_N \in X_N$ such that $x_N \neq y_N$, then there exist.

$FU_{x_N} = < S_1, S_2, S_1^c >$ and $FV_{y_N} = < M_1, M_2, M_1^c >$in $\tau_{0.1}$.

Where: $U_{x_N} = < S_1, S_2, S_3 >$ and $V_{y_N} = < M_1, M_2, M_3 >$ in τ.

such that $x_N \in FU_{x_N}, y_N \in FV_{y_N}$, $x_N \notin FV_{y_N}$ and $y_N \notin FU_{x_N}$.

Thus , $x_N \in S_1$and not in M_1 and $y_N$in M_1not in S_1, there (X_N, τ^1) is a FN T_1 -space.

(iii) $\Longrightarrow$ (i) Let $x_N, y_N \in X_N$ such that $x_N \neq y_N$ then, there exist, $x_N \in S_1$ and $x_N \notin M_1$ with $y_N \in M_1$, $y_N \notin S_1$, where $S_1$and M_1 are in τ^1.

Put, $U_{x_N} = < S_1, S_2, S_3 >$ and $V_{y_N} = < M_1, M_2, M_3 >$.

So, $U_{x_N}$and $V_{y_N}$are in τ and satisfy T_1 . Therefore, (X_N, τ) is a FNT_1 -space.

Example 3.20: Let $X_N = \{a,b,c\}$, $\tau = \{0_N, 1_N, S, M, S\vee M, S\wedge M\}$,Then.

$S = \{<\frac{a}{1,0.4,0}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,1}>\}$,and

$M = \{<\frac{a}{0,0.5,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,1}>\}$

$S\vee M = \{<\frac{a}{1,0.5,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,1}>\}$

$S\wedge M = \{<\frac{a}{0,0.4,0}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,1}>\}$

If $x_{N_3} = \{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{1,0.4,0}>\} \neq y_{N_3} = \{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0.5,1}>\} \in X_N$.

For the set S, $\tau_{0.1} = \{<\frac{a}{1,0.4,0}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,6,1}>\}$.

For the set M, $\tau_{0.1} = \{<\frac{a}{0,0.5,1}>,<\frac{b}{0,0,1}>,<\frac{c}{1,0.5,0}>\}$.

Then, for τ we have $\tau_{0.1} = \{0_N, 1_N, <\frac{a}{1,0.4,0}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,6,1}>,<\frac{a}{0,0.5,1}>,<\frac{b}{0,0,1}>,<\frac{c}{1,0.5,0}>\}$and

$\tau^1 = \{0_N,1_N\} \vee \{S_1\} \vee \{M_1\} = \{0_N,1_N\} \vee \{1,0.4,0\} \vee \{0,0.5,1\}$

That is $\tau^1 = \{0_N,1_N, \{1,0.4,0\}, \{0,0.5,1\}\}$.

Then, There is a FNOS in X_N say, $x_{N_3} = \{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{1,0.4,0}>\} \in S$, $x_{N_3} = \{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{1,0.4,0}>\} \notin M$ and $y_{N_3} = \{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0.5,1}>\} \in M$, $y_{N_3} = \{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0.5,1}>\} \notin S$. Therefore, (X_N, τ) is $FN_3\,T_2$ -space .

The next theorem gave the relations between the FNT_1-space and the new defined construction $\tau_{0.2}$ and τ^2.

Theorem 3.21: Let (X_N, τ) be a FN T_1 -space, then:

(i)- $(X_N, \tau_{0.2})$ is a FN T_1–space, (ii)- (X_N, τ^2) is a FN T_1 –space.

Proof. (i) Let (X_N, τ) be a FNT_1 -space and Let x_N, y_N be any elements in X_N such that $x_N \neq y_N$ then there exists $U_{x_N} = < S_1, S_2, S_3 >$and $V_{y_N} = < M_1, M_2, M_3 >$, such that $x_N \in U_{x_N}, y_N \in V_{y_N}$, $x_N \notin V_{y_N}$ and $y_N \notin U_{x_N}$.

Thus $x_N \in U_{x_N}$if $x_N \in S_1$, $x_N \notin S_3$ and $y_N \in M_1$, $y_N \notin M_3$.

Also, $x_N \notin M_1$, $y_N \notin S_1$.

Since $SU_{x_N} = < S_3^c, S_2, S_3 >$ and $SV_{y_N} = < M_3^c, M_2, M_3 >$. Then $x_N \notin S_3$, so $x_N \in S_3^c$ and $y_N \in M_3^c$.

Thus $x_N \in SU_{x_N}$ and $y_N \in SV_{y_N}$.

Similarly, we can show $x_N \notin SV_{y_N}$ and $y_N \notin SU_{x_N}$.

Therefore, $(X_N, \tau_{0.2})$ is a FN T_1 -space.

(ii) Suppose that $x_N, y_N \in X_N$ such that $x_N \neq y_N$ then, there exists $SU_{x_N} = < S_3^c, S_2, S_3 >$ and $SV_{y_N} = < M_3^c, M_2, M_3 >$ in $\tau_{0.2}$. So, there exist $U_{x_N} = < S_1, S_2, S_3 >$and $V_{y_N} = < M_1, M_2, M_3 >$ in τ such that $x_N \in SU_{x_N}$, $y_N \in SV_{y_N}$, $x_N \notin SV_{y_N}$ and $y_N \notin SU_{x_N}$.

Thus $x_N \in S_3$ and not in M_3 and $y_N \in M_3$ not in S_3.

Therefore, (X_N, τ^2) is a FN T_1-space .

Remark 3.22: The converse of Theorem 3.21 is not true in general .The following examples show these cases .

Example 3.23: Let $X_N = \{a, b, c\}$,and Let $\tau = \{0_N, 1_N, S, M, S\wedge M, S\vee M\}$,where

$S = \{< (\frac{a}{1,0.7,0}),(\frac{b}{1,0.5,0}),(\frac{c}{1,0.8,0}) >\}$, and $M = \{< (\frac{a}{1,0.7,0}),(\frac{b}{1,0,0}),(\frac{c}{1,0.4,0}) >\}$,

$S\wedge M = \{< (\frac{a}{1,0.7,0}),(\frac{b}{1,0,0}),(\frac{c}{1,0.8,0}) >\}$, and $S\vee M = \{< (\frac{a}{1,0.7,0}),(\frac{b}{1,0.5,0}),(\frac{c}{1,0.4,0}) >\}$

1. For the set S, $\tau_{0.2} = \{<> S: S \in \tau\}$, where $<> S =< x, 1 - V_S(x), \sigma_S(x), V_S(x) >$

   $=< (\frac{a}{0,0.2,1}), (\frac{b}{1,0.5,0}), (\frac{c}{1,0.8,0}) >$, and

2. For the set M, $\tau_{0.2} = \{<> M: M \in \tau\}$, where $<> M =< x, 1 - V_M(x), \sigma_M(x), V_M(x) >$

   $=< (\frac{a}{0,0.6,1}),(\frac{b}{1,0,0}),(\frac{c}{1,0.4,0}) >$

Then, for τ we have $\tau_{0.2} = \{0_N, 1_N, (\frac{a}{0,0.2,1}), (\frac{b}{1,0.5,0}), (\frac{c}{1,0.8,0}), (\frac{a}{0,0.6,1}), (\frac{b}{1,0,0}), (\frac{c}{1,0.4,0})\}$.

Also, $\tau^2 = \{0_N, 1_N\} \vee \{S_2\} \vee \{M_2\}$

$= \{0_N, 1_N\} \vee \{(\frac{b}{1,0.5,0})\} \vee \{(\frac{b}{1,0,0})\}$.

Remark 3.24: For the FNTS (X_N, τ) we have:

1. Every FNT_2-space is FNT_1-space.
2. Every FNT_1-space is FNT_0-space.

The proof is directly from definitions 3.9, 3.11 and 3.13.

The converse of Remark 3.24 is not true as it shown in the next example.

Example 3.25: Let $X_N = \{a,b,c\}$, $\tau = \{0_N, 1_N, S\}$,

1. If $S = \{< \frac{a}{0.8,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\}$.

So, $x_{N_1} = \{< \frac{a}{0.8,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \neq y_{N_1} = \{< \frac{a}{0.7,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \in X_N$.

There is a FNOS in (X_N, τ) say $x_{N_1} = \{< \frac{a}{0.8,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \in S$ but $y_{N_1} = \{< \frac{a}{0.7,0,0} >, < \frac{b}{0,0,0} >, < \frac{c}{0,0,0} >\} \notin S$.

Therefore, (X_N, τ) is $FN_1 T_0$ –space but is not $FN_1 T_1$ –space and not $FN_1 T_2$ –space.

2. If $S = \{< \frac{a}{1,1,0} >, < \frac{b}{0,1,0} >, < \frac{c}{0,1,0} >\}$.

So, $x_{N_2} = \{< \frac{a}{1,1,0} >, < \frac{b}{1,0,0} >, < \frac{c}{0,1,0} >\} \neq y_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,0} >, < \frac{c}{0,1,0} >\} \in X_N$. There is a FNOS in X_N say $x_{N_2} = \{< \frac{a}{1,1,0} >, < \frac{b}{1,0,0} >, < \frac{c}{0,1,0} >\} \in S$ but, $y_{N_2} = \{< \frac{a}{0,1,0} >, < \frac{b}{0,1,0} >, < \frac{c}{0,1,0} >\} \notin S$.

Therefore, (X_N, τ) is $FN_2 T_0$ -space but, is not $FN_2 T_1$ –space and not $FN_2 T_2$ –space.

3. If $S=\{<\frac{a}{1,0.9,0}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0,1}>\}$.

So, $x_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0.9,1},>\}\neq y_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0.6,1,0}>\}\in X_N$.

There is a FNOTS in (X_N,τ) say, $x_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0,0.9,1},>\}\in S$ but, $y_{N_3}=\{<\frac{a}{0,0,1}>,<\frac{b}{0,0,1}>,<\frac{c}{0.6,1,0}>\}\notin S$.

Therefore, (X_N,τ) is $FN_3\,T_0$ -space but is not $FN_3\,T_1$ -space and not $FN_3\,T_2$ -space.

Example 3.26: Let $X_N=\{a,b,c\}$, $\tau=\{0_N,1_N,S,M,S\wedge M,S\vee M\}$,Then.

1. If $S=\{<\frac{a}{0.5,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}$and

   $M=\{<\frac{a}{0.3,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}$.

   So, $x_{N_1}=\{<\frac{a}{0.5,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\neq y_{N_1}=\{<\frac{a}{0,0.3,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\in X_N$.

There is a FNOS in (X_N,τ) say $x_{N_1}=\{<\frac{a}{0.5,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\in S$, $x_{N_1}=\{<\frac{a}{0.5,0,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\notin M$ and $y_{N_1}=\{<\frac{a}{0,0.3,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\in M$, $y_{N_1}=\{<\frac{a}{0,0.3,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,0,0}>\}\notin S$. Therefore, (X_N,τ) is $FN_1\,T_1$ –space but is not $FN_1\,T_2$ -space.

2. If $S=\{<\frac{a}{1,1,0}>,<\frac{b}{0,1,0}>,<\frac{c}{0,1,0}>\}$, and

   $M=\{<\frac{a}{0,0,0}>,<\frac{b}{0,1,0}>,<\frac{c}{0,1,0}>\}$

So, $x_{N_2}=\{<\frac{a}{0,1,0}>,<\frac{b}{1,1,0}>,<\frac{c}{0,1,0}>\}\neq y_{N_2}=\{<\frac{a}{0,1,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,1,0}>\}\in X_N$

There is a FNOS in (X_N,τ) say $x_{N_2}=\{<\frac{a}{0,1,0}>,<\frac{b}{1,1,0}>,<\frac{c}{0,1,0}>\}\in S$, $x_{N_2}=\{<\frac{a}{0,1,0}>,<\frac{b}{1,1,0}>,<\frac{c}{0,1,0}>\}\notin M$ .and $y_{N_2}=\{<\frac{a}{0,1,0}>,<\frac{b}{0,0,0}>,<\frac{c}{0,1,0}>\}$

$\in$ M,$y_{N_2} = \{< \frac{a}{0,1,0} >,< \frac{b}{0,0,0} >,< \frac{c}{0,1,0} >\} \notin$ S.

Therefore, (X_N, τ) is $FN_2\, T_1$ -space but is not $FN_2\, T_2$ -space.

3. If $S = \{< \frac{a}{1,0.2,0} >,< \frac{b}{0,0,1} >,< \frac{c}{0,0,1} >\}$ and

$M = \{< \frac{a}{0,0.7,1} >, < \frac{b}{0,0,1} >,< \frac{c}{0,0,1} >\}$

$S\wedge M = \{< \frac{a}{0,0.2,0} >,< \frac{b}{0,0,1} >,< \frac{c}{0,0,1} >\}$

$S\vee M = \{< \frac{a}{1,0.7,1} >,< \frac{b}{0,0,1} >,< \frac{c}{0,0,1} >\}$.

So, $x_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.2,1} >\} \neq y_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.7,1} >\} \in X_N$

There is a FNOS in (X_N, τ) say, $x_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.2,1} >\} \in S, x_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.2,1} >\} \notin M$ and $y_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.7,1} >\} \in M$, $y_{N_3} = \{< \frac{a}{0,0,1} >,<, \frac{b}{0,0,1} >,< \frac{c}{0,0.7,1} >\} \notin S$. Therefore, (X_N, τ) is $FN_3\, T_1$ -space but is not $FN_3\, T_2$.

4. Conclusions

In this research, the new type of fuzzy neutrosophic separation axioms has been defined in the fuzzy neutrosophic topological spaces by several new types of points and new constructions was studied. And many useful examples are presented to clear the newconcepts introduced. Also, proof some new theorems and characterizations relations among the new concepts and the other type are going to be found.

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A New Approach of Multi-Dimensional Single Valued Plithogenic Neutrosophic Set in Multi Criteria Decision Making

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Abstract: In the wider problem-solving process, decision-making requires knowledge to choose the possible and optimum solution in the real time. Decision making become further complicated if the available criteria are more. In this research work our intend is to study the behaviour of Multi-Dimensional Single valued Plithogenic Neutrosophic Sets(MSVPNS) used in multi criteria decision making with multi values of attributes. We also introduce a novel method to find the optimum solution of Single valued Plithogenic Neutrosophic Sets(SVPNS) with its operators. We apply this concept in the field of agriculture which deals with multi values of attribute and obtain a fruitful result for practising agriculture in a successful way.

Keywords: Decision making, Multi criteria decision making, Neutrosophic set, Plithogenic set, Plithogenic Neutrosophic set.

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1. Introduction

Decision-making process may be termed as the investigation, identification and choice of alternatives, the most appropriate option for the perseverance. It is generally called a cognitive analysis, since it involves conceptual and logical reasoning. There are some strategies in decision-making that are worth exploring, but there is little interest in the number of different alternatives, rather than in describing all possible solutions and select the one with the greatest likelihood of success, or the one that best matches the specific target or purpose.

Decision-making is a process that eliminates uncertainty to a significant degree. In most decisions, uncertainty is minimized rather than removed. Just in a few cases decisions are taken with absolute certainty. This means that most decisions require a certain amount of risk.

If there is no uncertainty, so there is no decision; only since you have to act and assume a determined conclusion. Decisions decide the progress of the project, and often there are tough times when they seem not to be as straightforward as we assume they are tougher.

Zadeh [14] brought a successful revolution by introducing a new theory of sets (i.e.) Fuzzy sets (FS) in the area of problem solving world and mathematics. Fuzzy sets accept the view that the knowledge available in the real world is not always definite or crisp, but keeps the hand of uncertainty and the analysis of this uncertainty will aid a great deal in the decision making process.

Atanassov [2] coined Intuitionistic fuzzy set (IFS) to manage vagueness which is an extension of the FS. IFS allocates both membership and non-membership degree for each component with the constraint that the addition of these two evaluations is less than or equal to unity. IFS plays a major role in resolving vagueness or uncertainty in decision making.

Smarandache [7] proposed Neutrosophic sets (NSs), a generalization of FS and IFS.NSs is highly supportive for dealing with insufficient, indefinite, and varying data that occurs in the the real world. NSs are characterized by functions of truth (T), indeterminacy (I) and falsity (F) membership functions. This concept is very essential in decision making process since indeterminacy is clearly enumerated and the truth, indeterminacy, and falsity membership functions are independent.

Smarandache [6] introduced the Plithogenic set (PS) as a generalization of neutrosophy in 2017. The components of PS are represented by one or many number of attributes and each of it have numerous values. Each values of attribute have its appurtenance degree for the component x (say) to the PS (say P) with reference to certain constraints. For the first time, Smarandache introduced the dissimilarity degree between each value of attribute and the predominant value of attribute which results in getting the enhanced accurateness for the plithogenic aggregation operators.

In this research work, we study how the single valued plithogenic neutrosophic set used in multi criteria decision making with multi values of attributes.

Section 1 gives the brief introduction with the organisation of the paper. Section 2 deals with the preliminary concepts. In this section we give the basic definitions, important results that is needed for our research work. Section 3 explains uni attribute value SVPNS with their operators. Section 4 is an extension of section 3 which is our proposed concept dealing with MSVPNS with their aggregation operators. Section 5 gives an algorithm for computing the optimum solution for numerical data. Section 6 explains the application of the constructed algorithm in the field of agriculture. Section 7 gives the results and discussions of the numerical problem and Section 8 concludes the present research work with the future work.

2. Preliminaries

Definition 2.1 [14] Let J be a universal set and the fuzzy set $F = \left\{\left\langle j, \gamma_f(j)\right\rangle \mid j \in J\right\}$ is termed by a belonging degree γ_f as $\gamma_f : J \to [0,1]$.

Definition 2.2 [2] Let H be a non-void set. The set $B = \left\{\left\langle h, \mu_B, \varphi_B \right\rangle h \in H\right\}$ is called an intuitionistic fuzzy set (in short, IFS) of H where the function $\mu_B : H \to [0,1]$, $\varphi_B : H \to [0,1]$ represents the belonging degree (say $\mu_B(h)$) and non- belonging degree (say $\varphi_B(h)$) of each component $h \in H$ to the set B and satisfies the constraint that $0 \le \mu_B(h) + \varphi_B(h) \le 1$.

Definition 2.3 [9] Let H be a non-void set. The set $B = \left\{\left\langle n, \lambda_B, \phi_B, \gamma_B \right\rangle h \in H\right\}$ is called a neutrosophic set (say NS) of H where the function $\lambda_B : H \to [0,1]$, $\phi_B : H \to [0,1]$ and $\gamma_B : H \to [0,1]$

represents the belonging degree (say $\lambda_B(h)$), neutral degree (say $\phi_B(h)$), and non- belonging degree (say $\gamma_B(h)$)of each component $h \in H$ to the set B and satisfies the limitation that $0 \le \lambda_B(h) + \phi_B(h) + \gamma_B(h) \le 3$.

Definition 2.4 [6] Plithogenic set (PS) is a generalization of a crisp set, a fuzzy set (FS), an intuitionistic fuzzy set (IFS) and a neutrosophic set (NS), while these four categories are represented by a particular values of attribute (appurtenance): single value (belonging)-for a crisp set and a FS, two values (belonging, non-belonging)-for an IFS, or triple values (belonging, non-belonging and indeterminacy) for NS.

In general, PS is a set whose members are determined by a set of elements with four or more values of attributes.

Definition 2.5 [6] Let Z be the universal set. A non-void set $B = \{\beta_1, \beta_2, ...\beta_s\}, s \ge 1$ of uni-dimensional parameters and $\beta \in B$ attributes is known as the values of attribute continuum of the PS. A given value whose range of all probable values is the non-void set U, is any finite discrete set $U = \{u_1, u_2, ..., u_s\}, 1 \le s < \infty$, or infinitely countable set $U = \{u_1, u_2, ..., u_\infty\}$, or infinitely uncountable set $U =]x, y[, x < y, where\ ]...[$ where U can be any open, quasi-open or closed interval from the set of real numbers of another universal set.

Definition 2.6 [10] Let R be a non-void subset of U, where R is the collection of the values of all attributes that the researchers need for their application. Every component $y \in$ P is described by the values of all attributes in $R = \{r_1, r_2, ...r_m\}, m \ge 1$.

Definition 2.7 [11] Generally there is a predominant values of attribute (DAV) within the value set R of the attribute, which is defined by the researchers upon their application. Predominant value is the most significant value of the attribute in which the researchers are involved. There are situations where such DAV may not be taken into consideration or does not exist, or several predominant (essential) values of attributes may exist when various methods would be applied.

Definition 2.8 [10] Each values of attribute $r \in R$ has its respective appurtenance degree $d(y, r)$ of the element y to the set P, with reference to some given criteria. The appurtenance degree can be: a fuzzy or intuitionistic fuzzy or neutrosophic to the plithogenic set. Therefore the values of attribute appurtenance degree function is $\forall x \in X, d : X \times W \to X([0,1]^T)$, so $d(y, r)$ is a subset of $[0,1]^T$, where $X([0,1]^T)$ is the power set of the $[0,1]^T$, where T=1 for FS, T=2 for IFS or T = 3 for NS.

Definition 2.9 [6] Let the cardinal $|R| \ge 1$. Let $C : R * R \to [0,1]$ be the values of attribute dissimilarity degree function between any two values of attributes r_1 and r_2 represented by $C(r_1, r_2)$ which satisfies the following conditions

(i) $C(r_1, r_2) = 0$, the dissimilarity degree among the same values of attribute is zero;

(ii) $C(r_1, r_2) = C(r_2, r_1)$ commutativity.

Remarks

1. The degree of dissimilarity is often determined between uni-dimensional values of attributes. We divide multi-dimensional value of attribute into its equivalent uni-dimensional values of attribute.
2. The dissimilarity function of the values of attribute allows the plithogenic operators and the relationship of plithogenic partial order to achieve a precise result.
3. In every domain where the PS is used in connection with the application, the values of attribute dissimilarity degree function is designed to solve. If the aggregation is overlooked, it still works, but the result will lose exactness.

Definition 2.10 [6] Plithogenic aggregation operators

The degree of dissimilarity for the values of attribute is calculated between each values of attribute with reference to the DAV represented by r_d. Most of the plithogenic aggregation operators (Intersection, Union, Partial orders) are linear combination of the fuzzy ℓ_{norm} (symbolized by $\wedge_f$) and fuzzy ℓ_{conorm} (symbolized by $\vee_f$).

If one imposes the ℓ_{norm} on DAV represented by r_d, and the dissimilarity between r_d and r_2 is $C(r_d, r_2)$, then onto values of attribute r_2 one imposes

$$[1 - C(r_d, r_2)] * \ell_{norm}(r_d, r_2) + C(r_d, r_2) * \ell_{conorm}(r_d, r_2)$$

or by using notations

$$[1 - C(r_d, r_2)] * \wedge_f (r_d, r_2) + C(r_d, r_2) * \vee_f (r_d, r_2).$$

Likewise if one imposes the ℓ_{conorm} on DAV represented by r_d, and the dissimilarity between r_d and r_2 is $C(r_d, r_2)$, then onto values of attribute r_2 one imposes

$$[1 - C(r_d, r_2)] * \ell_{conorm}(r_d, r_2) + C(r_d, r_2) * \ell_{norm}(r_d, r_2)$$

or by using notations

$$[1 - C(r_d, r_2)] * \vee_f (r_d, r_2) + C(r_d, r_2) * \wedge_f (r_d, r_2).$$

3. One Attribute Single valued Plithogenic Neutrosophic set (OASVPNS)

The attribute is Φ = "appurtenance"

The set of values of attributes R= {belonging, indeterminacy, non-belonging}, whose cardinal |R| = 3;

The DAV = belonging;

The values of attribute appurtenance degree function:

$d : P * R \to [0,1],\ d(y, belonging) \in [0,1], d(y, indeterminacy) \in [0,1], d(y, non\ belonging) \in [0,1]$

with $0 \le d(y, belonging) + d(y, indeterminacy) + d(y, non\,belonging) \le 3;$

and the values of attribute dissimilarity degree function:

$C : R * R \to [0,1]$,

$C(belonging\ , belonging\ ) = C(indeterminacy, indeterminacy) = C(non\,belonging\ \ , nonbelonging = 0,$

$C(belonging\ \ , nonbelonging\ ) = 0,$

$C(belonging\ \ , indeterminacy\ ) = c(non\,belonging\ \ , indeterminacy\ ) = \frac{1}{2},$

which means that for the SVPNS aggregation operators (Intersection, Union, Complement etc.), if one imposes the ℓ_{norm} on belonging function, then one has to impose the ℓ_{conorm} on non-belonging (and mutually), while on indeterminacy one imposes the average of ℓ_{norm} and ℓ_{conorm} .

3.1 OASVPNS operators

Let us consider the single valued plithogenic neutrosophic degree of appurtenance of values of attribute r of x to the set P with reference to some given criteria:

$$d^N \kappa(r) = (\kappa_1, \kappa_2, \kappa_3) \in [0,1]^3 \ and\ d^N{}_\varepsilon(r) = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in [0,1]^3$$

3.1.1 OASVPNS Intersection

$$(\kappa_1, \kappa_2, \kappa_3) \wedge_P (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\kappa_1 \wedge_P \varepsilon_1, \frac{1}{2}(\kappa_2 \wedge_f \varepsilon_2 + \kappa_2 \vee_f \varepsilon_2), \kappa_3 \vee_P \varepsilon_3)$$

3.1.2 OASVPNS Union

$$(\kappa_1, \kappa_2, \kappa_3) \vee_P (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\kappa_1 \vee_P \varepsilon_1, \frac{1}{2}(\kappa_2 \wedge_f \varepsilon_2 + \kappa_2 \vee_f \varepsilon_2), \kappa_3 \wedge_P \varepsilon_3)$$

3.1.3 OASVPNS Negation

$$\neg_P (\kappa_1, \kappa_2, \kappa_3) = (\kappa_3, \kappa_2, \kappa_1)$$
$$\neg_P (\kappa_1, \kappa_2, \kappa_3) = (\kappa_3, 1-\kappa_2, \kappa_1)$$
$$\neg_P (\kappa_1, \kappa_2, \kappa_3) = (1-\kappa_1, \kappa_2, 1-\kappa_3) \quad etc.,$$

3.1.4 OASVPNS Inclusions (Partial orders)

(i) Simple Neutrosophic Inclusion

$$(\kappa_1, \kappa_2, \kappa_3) \le_N (\varepsilon_1, \varepsilon_2, \varepsilon_3)\ if\ \kappa_1 \le \varepsilon_1, \kappa_2 \ge \varepsilon_2, \kappa_3 \ge \varepsilon_3$$

(ii) Complete Neutrosophic Inclusion

$$(\kappa_1, \kappa_2, \kappa_3) \le_P (\varepsilon_1, \varepsilon_2, \varepsilon_3)\ if\ \kappa_1 \le \varepsilon_1, \kappa_2 \ge 0.5 * \varepsilon_2, \kappa_3 \ge \varepsilon_3$$

3.1.5 OASVPNS Equality

(i) Simple Neutrosophic equality

$$(\kappa_1, \kappa_2, \kappa_3) =_N (\varepsilon_1, \varepsilon_2, \varepsilon_3)\ if\ (\kappa_1, \kappa_2, \kappa_3) \le_N (\varepsilon_1, \varepsilon_2, \varepsilon_3)\ and (\varepsilon_1, \varepsilon_2, \varepsilon_3) \le_N (\kappa_1, \kappa_2, \kappa_3)$$

(ii) Complete Neutrosophic equality

$$(\kappa_1, \kappa_2, \kappa_3) =_P (\varepsilon_1, \varepsilon_2, \varepsilon_3)\ if\ (\kappa_1, \kappa_2, \kappa_3) \le_P (\varepsilon_1, \varepsilon_2, \varepsilon_3)\ and (\varepsilon_1, \varepsilon_2, \varepsilon_3) \le_P (\kappa_1, \kappa_2, \kappa_3)$$

4. Proposed Multi-Dimensional Single valued Plithogenic Neutrosophic set (MSVPNS)

Consider a universal set E and $A, B \subset E$ be two single valued plithogenic neutrosophic sets.

Let $\beta_{[n]} = \beta_1 * \beta_2 * \ldots \beta_n$ be an n-dimensional attribute for $n \geq 1$, and every attribute β_i, $1 \leq i \leq n,$ has $v_i \geq 1$ values:

$$R_i = \{r_{i1}, r_{i2}, \ldots r_{iv_i}\}$$

An element $x \in P$ is characterized by $v_1 * v_2 * \ldots v_n = v\, values$:

$$R = \sum_{i=1}^{m} \{r_{i1}, r_{i2}, \ldots r_{iv_i}\} = \{r_{11}, r_{12}, \ldots r_{1v_1}\} * \{r_{21}, r_{22}, \ldots r_{2r_{v2}}\} * \ldots \{r_{m1}, r_{m2}, \ldots r_{mr_m}\}$$
$$= \{r_{1j_1}, r_{2j_2}, \ldots r_{nj_n}\}, 1 \leq j_1 \leq v_1, 1 \leq j_2 \leq v_2, \ldots 1 \leq j_n \leq v_n\}.$$

Let $C(r_{iD}, r_{ik}) = C_{ik} \subseteq [0,1]^3$ be the neutrosophic degree of dissimilarity between the attribute β_i predominant value (represented by r_{iD}) and other attribute ψ_i value (represented by r_{ik}) for $1 \leq i \leq n,$ and $1 \leq k \leq v_i$. And C_{ik} as a part of the unit interval [0, 1], may be a subset, or an interval, or a hesitant set, or a single number etc.

We break up the n dimensional attribute into n uni dimensional attribute. And when applying the plithogenic aggregation operators onto an n-uple $(r_{1j_1}, r_{2j_2}, \ldots r_{nj_n})$, we independently apply the ℓ_{norm}, ℓ_{conorm} or a linear combination of its n- components: $r_{1j_1}, r_{2j_2}, \ldots r_{nj_n}$

Let $d_A : P * R_i \rightarrow \mathrm{P}([0,1])^3$ for each $1 \leq i \leq n,$ be the appurtenance neutrosophic degree function, whereas $\mathrm{P}([0,1])$ is the power set of the unit interval [0, 1], i.e. all subsets of [0, 1].

Upon the values of attribute degree function, the ℓ_{norm}, ℓ_{conorm} and their linear combinations are adjusted to the neutrosophic sets.

Consequently $d_B : P * R_i \rightarrow \mathrm{P}([0,1])^3$

4.1 Multi-Dimensional Single valued Plithogenic Neutrosophic set operators (MSVPNS)

Let us consider the notations for two n-uple PSVNS denoted by

$$x_A = \{d_A(x, w_1), \ldots d_A(x, w_i), \ldots d_A(x, w_n)\} \text{ and}$$
$$x_B = \{d_B(x, w_1), \ldots d_B(x, w_i), \ldots d_B(x, w_n)\}$$

4.1.1 MSVPNS Intersection and Union

Let w_{id} be the attribute β_i predominant value and w_i be any of the attribute β_i value, $i \in \{1,2,\ldots,n\}$

$$x_A \wedge_p x_B = \{(1 - C(w_{id}, w_i)) * [d_A(x, w_{id}) \wedge_f d_B(x, w_i)] + C(w_{id}, w_i) * [d_A(x, w_{id}) \vee_f d_B(x, w_i)] 1 \leq i \leq n$$

$$x_A \vee_p x_B = \{(1 - C(w_{id}, w_i)) * [d_A(x, w_{id}) \vee_f d_B(x, w_i)] + C(w_{id}, w_i) * [d_A(x, w_{id}) \wedge_f d_B(x, w_i)] 1 \leq i \leq n$$

4.1.2 MSVPNS Negation

Without loss of generality, we assume the values of attribute dissimilarity degrees are

$$C(w_{1d}, w_1),...C(w_{id}, w_i),..., C(w_{nd}, w_n).$$

The plithogenic neutrosophic element values of attributes are $\{w_1,..., w_i,..., w_n\}$.The values of attributes appurtenance degree: $\{d_A(x, w_1),...d_A(x, w_i),...d_A(x, w_n)\}$. Then the plithogenic neutrosophic complement (negation) is

$$1 - C(w_{1d}, w_1),...1 - C(w_{id}, w_i),...,1 - C(w_{nd}, w_n), anti(w_1),..., anti(w_i),..., anti(w_n)$$

Or

$$\neg_p x_A = \{ d_A(x, anti(w_1) = d_A(x, w_1),..., d_A(x, anti(w_i) = d_A(x, w_i),..., d_A(x, anti(w_n) = d_A(x, w_n)$$

where $anti(w_i), 1 \le i \le n,$ is the attribute β_i contradictory value of w_i or $C(w_{id}, anti(w_i)) = [1 - C(w_{id}, w_i)]$

4.1.3 MSVPNS Partial order

Consider a partial order relation $x_A \le_p x_B$ on $P([0,1])^3$

if and only if

$$d_A(x, w_i) \le (1 - C(w_{id,} w_i)) * d_B(x, w_i), \; for\, 0 \le C(w_{id,} w_i) < 0.5 \text{ and}$$
$$d_A(x, w_i) \ge (1 - C(w_{id,} w_i)) * d_B(x, w_i), \; for\, C(w_{id,} w_i) \in [0.5,1] \; for\, all \; 1 \le i \le n$$

4.1.4 MSVPNS Equality

Consider a relation of total order has been represented on $P([0,1])^3$ then

$$x_A =_p x_B \; iff \; x_A \le_p x_B \; and \; x_B \le_p x_A.$$

5. Proposed Method to find the optimal solutions of MSVPNS

Step 1: Classify the problem with the attributes and its corresponding values of attribute.

Step 2: The cardinal number can be found as per the multi attribute dimension (say 'm') and denote it by R_m and find $|r_m|$

Step 3: Split the multi-dimensional attribute into its equivalent uni-dimensional attribute and compute the dissimilarity degree. Also Dissimilarity degree between two different attributes are zero.

Step 4: Choose the predominant values of attribute for each corresponding uni-dimensional attribute.

Step 5: Calculate the SVPNS intersection for n attribute which is given by

(i) for interior degrees of dissimilarity

$$((\lambda_{i1}, \lambda_{i2}, \lambda_{i3}), 1 \le i \le m) \wedge_p (\mu_{i1}, \mu_{i2}, \mu_{i3}, 1 \le i \le m) = [\lambda_{i1} \wedge_p \mu_{i1}, \frac{1}{2}(\lambda_{i2} \wedge_f \mu_{i2} + \lambda_{i2} \vee_f \mu_{i2}), \lambda_{i3} \vee_p \mu_{i3}], 1 \le i \le m)$$

(ii)

$$x_A \wedge_p x_B = \{(1 - C(w_{id}, w_i)) * [d_A(x, w_{id}) \wedge_f d_B(x, w_i)] + C(w_{id}, w_i) * [d_A(x, w_{id}) \vee_f d_B(x, w_i)] 1 \le i \le m$$

Select the optimal representation of x from the intersection of x_A *and* x_B

Note. Here we have used the intersection operator. But the option is free for the reader to collaborate with other operators (union, complement, partial order and equality) of their choice.

6. Application

In this section, we give a numerical example to find the optimum solution of Multi Single valued Plithogenic Neutrosophic Set which has 40 values of attribute.

Let P be a plithogenic neutrosophic set representing the factors needed for agriculture.

According to the experts A and B, $x \in P$ be the type of agriculture characterized by 3 attributes (Soil, Water, Crops) that has to be evaluated

Soil - whose values of attributes are {sandy, clay, loamy, Red, Black} = $\{s_1, s_2, s_3, s_4, s_5\}$

Water- whose values of attributes are {Rain-fed farming, Irrigation} = $\{w_1, w_2\}$

Crops- whose values of attributes are {Food, cash, plantation, Horticulture} = $\{t_1, t_2, t_3, t_4\}$.

The multi attribute of dimension 3 is,

$$R_3 = \{(s_i, w_j, t_k), \text{ for all } 1 \le i \le 5, 1 \le j \le 2, 1 \le k \le 4\}$$

The cardinal of R_3 *is* $|R_3| = 5 * 2 * 4 = 40$.

The predominant values of attributes are s_1, w_1, t_1 respectively for every uni-dimensional attribute correspondingly.

The uni- dimensional attribute dissimilarity degrees are:

$$c(s_1, s_2) = \frac{1}{4}, c(s_1, s_3) = \frac{2}{4}, c(s_1, s_4) = \frac{3}{4}, c(s_1, s_5) = 1$$

$$c(w_1, w_2) = 1$$

$$c(t_1, t_2) = \frac{1}{3}, c(t_1, t_3) = \frac{2}{3}, c(t_1, t_4) = 1.$$

Let us use $fuzzy\ \ell_{norm} = a \wedge_f b = ab$ *and* $fuzzy\ \ell_{conorm} = a \vee_f b = a + b - ab$

Dissimilarity Degree	**0**	**1/4**	**2/4**	**3/4**	**1**	**0**	**1**	**0**	**1/3**	**2/3**	**1**
Values of Attribute	Sandy	clay	loamy	Red	Black	Rain-fed farming	Irrigation	Food	cash	Plantation	Horticulture

Expert A Plithogenic Neutrosophic Degree	(0.1,0.5,0.3)	(0.2,0.3,0.1)	(0.6,0.1,0.2)	(0.2,0.6,0.5)	(0.4,0.2,0.1)	(0.8,0.2,0.3)	(0.5,0.2,0.3)	(0.1,0.6,0.3)	(0.4,0.6,0.5)	(0.2,0.5,0.7)	(0.3,0.2,0.9)
Expert B Plithogenic Neutrosophic Degree	(0.2,0.3,0.4)	(0.7,0.1,0.4)	(0.5,0.7,0.3)	(0.9,0.1,0.4)	(0.1,0.6,0.3)	(0.5,0.2,0.7)	(0.6,0.1,0.5)	(0.3,0.8,0.1)	(0.4,0.3,0.1)	(0.8,0.1,0.2)	(0.1,0.3,0.2)

Tri-dimensional SVPNS Intersection

Let $x_A = \{d_A(x, s_i, w_j, t_k)\ for all\ 1 \le i \le 5, 1 \le j \le 2, 1 \le k \le 4\}$

and $x_B = \{d_B(x, s_i, w_j, t_k)\ for all\ 1 \le i \le 5, 1 \le j \le 2, 1 \le k \le 4\}$

Then

$$x_A(s_i, w_j, t_k) \wedge_p x_B(s_i, w_j, t_k) =$$
$$\{(1 - c(s_{iD}, s_i)) * [d_A(x, s_{iD}) \wedge_f d_B(x, s_i)] + c(s_{iD}, s_i) * [d_A(x, s_{iD}) \vee_f d_B(x, s_i)] 1 \le i \le 5;$$
$$(1 - c(w_{jD}, w_j)) * [d_A(x, w_{jD}) \wedge_f d_B(x, w_j)] + c(w_{jD}, w_j) * [d_A(x, w_{jD}) \vee_f d_B(x, w_j)] 1 \le j \le 2;$$
$$(1 - c(t_{kD}, t_k)) * [d_A(x, t_{kD}) \wedge_f d_B(x, t_k)] + c(t_{kD}, t_k) * [d_A(x, t_{kD}) \vee_f d_B(x, t_k)] 1 \le k \le 4\}$$

Let us have

$$x_A[d_A(s_1) = (0.1,0.5,0.3), d_A(w_2) = (0.5,0.2,0.3), d_A(t_3) = (0.2,0.5,0.7)]\ and$$
$$x_B[d_B(s_1) = (0.2,0.3,0.4), d_B(w_2) = (0.6,0.1,0.5), d_B(t_3) = (0.8,0.1,0.2)\ ]$$

We take only 3-values of attribute: (s_1, w_2, t_3) for the other 39 3-values of attributes follow the same procedure.

$$x_A \wedge_P x_B = (0.1,0.5,0.3) \wedge_p (0.2,0.3,0.4); (0.5,0.2,0.3) \wedge_p (0.6,0.1,0.5); (0.2,0.5,0.7) \wedge_p (0.8,0.1,0.2)$$
$$where$$
$$d^N A(x, s_1) \wedge_p d^N B(x, s_1) = (0.1,0.5,0.3) \wedge_p (0.2,0.3,0.4)$$

First use the interior 'n' degree of dissimilarity among the 'n' components T, I and F (i.e.) 0, ½, 1.

$$(0.1,0.5,0.3) \wedge_p (0.2,0.3,0.4) = (0.1 \wedge_p 0.2, \frac{1}{2}[(0.5 \wedge_f 0.3) + (0.5 \vee_f 0.3)], 0.3 \vee_p 0.4)$$
$$= [(1-0)(0.1 \wedge_f 0.2) + 0.(0.1 \vee_f 0.2), \frac{1}{2}(0.5 \wedge_p 0.3 + 0.5 \vee_p 0.3), (1-0)(0.3 \vee_f 0.4) + 0 * (0.3 \wedge_f 0.4)]$$
$$= (0.2,0.4,0.12)$$

Similarly $d^N{}_A(x, w_2) \wedge_p d^N{}_B(x, w_2) = (0.5,0.2,0.3) \wedge_p (0.6,0.1,0.5) = (0.8,0.15,0.15)$ and also

$$d^N{}_A(x,t_3)\wedge_p d^N{}_B(x,t_3)=(0.2,0.5,0.7)\wedge_p(0.8,0.1,0.2)=(0.61,0.03,0.34)$$

Hence $x_A \wedge_p x_B(s_1,w_2,t_3)\approx((0.2,0.4,0.12);(0.8,0.15,0.15);(0.61,0.03,0.34))$.

We need to intersect the MSVPNS of the experts A and B to obtain the optimal representation of x.

7. Results and Discussions

Based on the Expert's (A and B) data the optimal condition for the given scenario is obtained at $s_5, w_2\ and\ t_4$ with the values

$$x_A \wedge_p x_B(s_4,w_2,t_4)\approx((0.8,0.2,0.5);(0.8,0.4,0.5);(0.8,0.5,0.7))$$

Therefore, black soil with irrigation water to cultivate horticulture is the best method for the factors needed for agriculture.

The above procedure is more generalized as it uses MSVPNS which deals with more attributes simultaneously. The beauty of this method is its ease as the researcher need not to manage with complex lengthy computation based operators. Also this method has a practical approach of using broad spectrum that can engage modifications according to the necessity of the provided environment. We can generalize the model of this method in plithogenic neutrosophic environments that can manage difficulties of the physical world.

8. Conclusion and Future Work

In this research work, we studied the application of multi-dimensional single valued plithogenic neutrosophic set in MCDM problems specifically in the field of agriculture. We apply the concept in the research areas which dealing with multi values of attributes. Thus the plithogenic aggregation operators gives the optimal solutions for the plithogenic neutrosophic environment. In future we can extend the concept to interval valued plithogenic neutrosophic sets which may help abundantly in the areas related with decision making.

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A New Approach to Group Decision Making Problem in Medical Diagnosis using Interval Neutrosophic Soft Matrix

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Abstract: The main objective of this article is to introduce the notion of interval neutrosophic soft matrix (IVNS-Matrix), which is an extension of the neutrosophic soft matrix and reveals various types of IVNS-Matrix along with different algebraic operations on them. A new method has been proposed to solve interval neutrosophic soft set based real-life group decision making problem effectively by introducing IVNSM-Algorithm. Finally, this algorithm has been applied in medical science for disease diagnosis.

Keywords: Interval neutrosophic matrix; IVNSM-Algorithm; Choice Matrix; Decision making.

1. Introduction

Our life is surrounded by an aura of uncertainties or ambiguities or vagueness. So, when we are going to solve various real-life problems, which contain uncertainties, then we realize that such types of problems cannot be handled by traditional mathematical tools. Because in such cases we do not analyze data appropriately, as we do in case of precise and deterministic data. So, there is a problem in real decision making. The introduction of fuzzy set theory [1] by Zadeh (1965) handled the vague concept to some extent by introducing the membership function. The membership function determines the degree of belongingness of each element in a set and the membership value lies in the interval [0, 1]. Fuzzy set theory has been used extensively in different fields. In the fuzzy set theory, there is no scope of considering non-membership value as we find that the concept of non-membership value is equally as important as membership value. Practically also we used to find the dual character of an object. So, to make a balance in the characteristic of an object, an intuitionistic fuzzy set [2] was introduced by Atanassov (1986) where, for every membership function there corresponds a non-membership function and both belong to the interval [0, 1] and their sum cannot exceed one. Membership or non-membership value only assign a single real value. But sometimes the concept of uncertainty cannot be defined by a single real value. For that purpose interval-valued fuzzy set [3] was introduced(1987). Due to more complexity in the environment of uncertainty and for the dire need of an hour, many other theoretical concepts and the properties, whose base is the fuzzy set, have been introduced. Some of them are given in [4-6]. According to the nature of the problem domain, we will decide which tool is suitable for us to handle a particular problem.

Zadeh's fuzzy set theory is the most appropriate theory to deal with uncertainty with the help of the membership function. But in 1999, Molodtsov [7] observed some limitations of the fuzzy set theory. In fuzzy set theory, a concept is handled by a membership function. But we should not impose only one way to define a membership function.The nature of a membership function is extremely individual. For example, to define the attractiveness of a house it is difficult to define a membership function. If one considers the membership degree as 0.6 then everyone may understand this in his or her own manner. For instance, Mr., X may understand that the house is highly attractive. Again Mr., Y may understand that it is very highly attractive. So there is a possibility of information loss in each particular case. Molodtsov said that the reason behind these limitations is the inadequacy of the parametrization of the theory. Then, to overcome this drawback he initiated the idea of soft set theory in the parametric form which is free from the above difficulties. In soft set theory, to define an object, no need to introduce a membership function. It is the more general form to represent the concept of vagueness parametrically. As we know that, because of the amalgamation of two or more concept together, gives a better shape and it provides more flexibility to handle various uncertain problems which we face in our day to day life, that's why by embedding the idea of the soft set and other sets, some major contributions are developed in [8-13].

The concept of indeterminacy or neutrality is common in real life. For example, when there is a fight between two players there are other people surrounding them who do not support neither of the two players. In the real decision-making problem, the concept of indeterminacy is very important. There are various types of indeterminacy involved in real-world problems. To eradicate such difficulty, Smarandache(2005) introduced the notion of Neutrosophic set in[14]. With an aid of a neutrosophic set, we deal with uncertainty, incompleteness ,and indeterminacy involving a pragmatic problem. It has been used successfully in various fields such as Sociology, Economics, Logic, Artificial intelligence, Machine learning, Optimization problem, Game theory ,etc. Some extensions of the neutrosophic set have been discussed in [15-21].

Matrices play a significant role in representing, manipulating ,and modeling such a large number of data because of their compact form. In the field of computer science, mainly in Data Science, there is an abundant use of Matrix. The classical matrix theory cannot solve the problems based on uncertainty. Matrix representation of the fuzzy soft set is initiated by Yang et al. in [22] and it is successfully used in decision-making problems in [23]. Some other extensions are intuitionistic fuzzy soft matrix[24-25], interval-valued fuzzy soft matrix[26], interval-valued intuitionistic fuzzy soft matrix and its application in medical diagnosis[27], interval-valued neutrosophic matrix[28] ,etc. Cagman et al. (2010), in [29], defined soft matrices and their operations and construct a decision-making method that can be used successfully to the problems that contain uncertainty. Also, in [30-31], Wang et al. discussed single-valued neutrosophic sets and interval neutrosophic sets respectively, Deli, in [32], used interval-valued neutrosophic soft sets in decision making, Smarandache[33] proposed the extension of the soft set by introducing hypersoft set, pilthogenic hypersoft set.

Group decision-making leads to improve outcomes and it involves two or more people. It occurs when individuals collectively make a selection from the alternatives. Group decisions are subject to factors such as social influence, peer pressure ,and group dynamics. It interacts and collaborated to reach a collective decision. In the proposed study we have used the concept of the neutrosophic set because it helps to study the higher degree of uncertainty, present in various real-life problems, in an effective manner. As compared to other mathematical tools namely, fuzzy set, intuitionistic fuzzy set, interval-valued fuzzy set ,etc., the neutrosophic set is a more flexible and more general form to handle uncertain concepts. For more clarity, we know that the neutrosophic set has three components namely $T, I, F \in [0,1]$ and each component has its own importance and they are independent of each other, which makes it superior to the other existing mathematical tools of vagueness. For example, if we consider the component $I = 0$ in the neutrosophic set, then also it has the potential to tackle complete or incomplete or paraconsistent or conflicting information. We also added intervals with neutrosophic sets to address the issues where the components used in the neutrosophic sets cannot be expressed by a single real value. In interval neutrosophic set $T, I, F \in Int([0,1])$, where $Int([0,1])$ denotes the set of all subsets of $[0,1]$. Moreover, with an aid of a matrix with an interval neutrosophic set, we made our calculation speedy and it helps to store big data in a computer easily. Some recent works based on neutrosophic set discussed in [34-37].

In [38], the concept of the neutrosophic soft matrix has been used in a group decision making problem. In our work, we have extended the earlier concept by introducing the notion of interval neutrosophic soft matrix and its types with concrete examples. We also introduce the choice matrix associated with interval neutrosophic soft set. An IVNSM-Algorithm has been proposed to solve real-life group decision making problem. Finally, with the help of an application, the IVNSM-algorithm has been successfully executed.

2. Preliminaries

Here we recall some basic definitions with examples that are appropriate as far as the article is concerned.

2.1 Definition[7]

Let X be an initial universe and E be a set of parameters. Let $P(X)$ denotes power set of X and $A \subseteq E$.

Then the pair (F_A, E) is called a soft set over X, where F_A is a mapping given by $F_A : E \to P(X)$.

A soft set is a parameterized family of subsets of the universe X.

2.2 Definition [14]

Let X be a universe of discourse,with a generic element in X denoted by x, the neutrosophic(NS) set A is defined as

$$A = \left\{ \left\langle x : \mu_A(x), \nu_A(x), \omega_A(x) \right\rangle, x \in X \right\},$$

Where the functions $\mu, \nu, \omega : X \to \left]^{-}0, 1^{+}\right[$ define respectively the degree of membership (or Truth), the degree of indeterminacy (neutrality), and the degree of non-membership (Falsehood) of the element $x \in X$ to the set A with the condition

$$^{-}0 \le \mu_A(x) + \nu_A(x) + \omega_A(x) \le 3^{+}$$

If t (0.4,0.1,0.5) belongs to, then it means that 40% A belongs to A, 50% does not belong to A and 10% is undecidable(not known exactly). If there is no indeterminacy involve in set A, then it reduces to an intuitionistic fuzzy set. Therefore, a neutrosophic set can be an intuitionistic fuzzy set.

2.3 Definition [30]

Let X be a universe of discourse with a generic element x. A single-valued neutrosophic set A is characterized by a truth-membership function $T_A(x)$, falsity-membership function $F_A(x)$ and the indeterminacy-membership function $I_A(x)$ and it is denoted by A = {x,< $T_A(x)$, $I_A(x)$, $F_A(x)$ >: $x \in X$ }, where $T_A(x)$, $I_A(x)$, $F_A(x)$ $\in$ [0,1] subject to the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Throughout our discussion we use the concept of single-valued neutrosophic set as it has a definite range.

2.4 Definition [31]

Let U be a space of points (objects), with a generic element u. An interval-valued neutrosophic set (IVN-set) A in U is characterized by a truth-membership function T_A , an indeterminacy-membership function I_A ,and a falsity-membership function F_A. For each point $u \in U$; T_A, I_A and $F_A \subseteq [0,1]$.

Thus, an IVN-sets over U can be represented by the set of

$$A = \left\{ \left\langle T_A(u), I_A(u), F_A(u) \right\rangle / u : u \in U \right\}$$

Here, $\left(T_A(u), I_A(u), F_A(u)\right)$ is called interval-valued neutrosophic number for all $u \in U$ and all interval-valued neutrosophic numbers over U will be denoted by $IVN(U)$.

2.4.1 Example

Let $U = \{u_1, u_2\}$ be the universe of discourse and A be an interval-valued neutrosophic set in U . Then A can be expressed as follows:

$$A = \left\{ \left\langle [0.5, 0.7], [0.5, 0.6], [0.5, 0.7] \right\rangle / u_1, \left\langle [0.4, 0.6], [0.7, 0.8], [0.3, 0.6] \right\rangle / u_2 \right\}$$

2.5 Definition [32]

Let $U = \{c_1, c_2, c_3, ..., c_m\}$ be a Universal set and E be the set of parameters given by $E = \{e_1, e_2, e_3, ..., e_n\}$. Let $A \subseteq E$ and (F, A) be a interval-valued neutrosophic soft set over U ,where F is a mapping given by $F{:}A \to I^U$,where I^U denotes the the collection of all interval-valued neutrosophic subsets of U. Then the interval-valued neutrosophic soft set can be expressed in a matrix form as follows:

$$\hat{A}_{m\times n} = \left[a_{ij} \right]_{m\times n} \quad \text{or} \quad \hat{A} = \left[a_{ij} \right] \; i = 1, 2, ..., m; \; j = 1, 2, ..., n$$

$$a_{ij} = \begin{cases} \left(\left\langle \left[T^l{}_j(c_i), T^u{}_j(c_i) \right], \left[I^l{}_j(c_i), I^u{}_j(c_i) \right], \left[F^l{}_j(c_i), F^u{}_j(c_i) \right] \right\rangle \right) & if \quad e_j \in A \\ \left(\left\langle [0,0], [1,1], [1,1] \right\rangle \right) & if \quad e_j \notin A \end{cases}$$

$\left[T^l{}_j(c_i), T^u{}_j(c_i)\right]$ represents the truth-membership value of c_i in the interval-valued neutrosophic set $F(e_j)$ $\left[I^l{}_j(c_i), I^u{}_j(c_i)\right]$ represents the indeterminacy-membership value of c_i in the interval-valued neutrosophic set $F(e_j)$ and $\left[F^l{}_j(c_i), F^u{}_j(c_i)\right]$ represents the falsity-membership value of c_i in the interval-valued neutrosophic set $F(e_j)$ with the condition $T^u{}_j(c_i) + I^u{}_j(c_i) + F^u{}_j(c_i) \leq 3$.

2.5.1 Example

Let $U = \{c_1, c_2\}$ be a set of cars under consideration and E is a set of parameters which is a neutrosophic word. Let $E = \{e_1 = \text{value},\ e_2 = \text{mileage},\ e_3 = \text{safety},\ e_4 = \text{performance},\ e_5 = \text{looks},\ e_6 = \text{sit capacity}\}$

We give an interval valued neutrosophic soft sets(ivn-soft sets) over U as follows:

$$\Upsilon_k = \left\{ \begin{array}{l} \left(e_1, \left\{\left\langle [0.2,0.9],[0.3,0.6],[0.4,0.7]\right\rangle / u_1, \left\langle [0.1,0.7],[0.2,0.8],[0.4,0.5]\right\rangle / u_2\right\}\right), \\ \left(e_2, \left\{\left\langle [0.4,0.8],[0.3,0.7],[0.5,0.8]\right\rangle / u_1, \left\langle [0.3,0.9],[0.1,0.8],[0.4,0.7]\right\rangle / u_2\right\}\right), \\ \left(e_3, \left\{\left\langle [0.0,0.6],[0.5,0.6],[0.3,0.7]\right\rangle / u_1, \left\langle [0.5,0.7],[0.8,0.9],[0.4,0.7]\right\rangle / u_2\right\}\right), \\ \left(e_4, \left\{\left\langle [0.3,0.6],[0.5,0.8],[0.6,0.9]\right\rangle / u_1, \left\langle [0.5,0.8],[0.6,0.8],[0.1,0.8]\right\rangle / u_2\right\}\right), \\ \left(e_5, \left\{\left\langle [0.2,0.9],[0.1,0.5],[0.4,0.9]\right\rangle / u_1, \left\langle [0.6,0.8],[0.1,0.8],[0.2,0.5]\right\rangle / u_2\right\}\right) \end{array} \right\}$$

The tabular representation of ivn-soft set Υ_k is as follows:

U	c_1	c_2
e_1	$\langle [0.2,0.9],[0.3,0.6],[0.4,0.7] \rangle$	$\langle [0.1,0.7],[0.2,0.8],[0.4,0.5] \rangle$
e_2	$\langle [0.1,0.7],[0.2,0.8],[0.4,0.5] \rangle$	$\langle [0.3,0.9],[0.1,0.8],[0.4,0.7] \rangle$
e_3	$\langle [0.0,0.6],[0.5,0.6],[0.3,0.7] \rangle$	$\langle [0.5,0.7],[0.8,0.9],[0.4,0.7] \rangle$
e_4	$\langle [0.3,0.6],[0.5,0.8],[0.6,0.9] \rangle$	$\langle [0.5,0.8],[0.6,0.8],[0.1,0.8] \rangle$
e_5	$\langle [0.2,0.9],[0.1,0.5],[0.4,0.9] \rangle$	$\langle [0.6,0.8],[0.1,0.8],[0.2,0.5] \rangle$

Table 1: Tabular representation of the ivn-soft set Υ_k

2.6 Definition [29]

Let (F_A, E) be a soft set over *U*. Then a subset of $U \times E$ is uniquely defined by

$R_A = \{(u, e) : e \in A, u \in F_A(e)\}$, which is called a relation form of (F_A, E). Now the characteristic function of R_A is written by,

$$\chi_{R_A} : U \times E \to \{0,1\}, \chi_{R_A} = \begin{cases} 1, (u,e) \in R_A \\ 0, (u,e) \notin R_A \end{cases}$$

Let $U = \{u_1, u_2, ..., u_m\}$, $E = \{e_1, e_2, ..., e_n\}$, then R_A can be presented by a table as in the following form

	e_1	e_2	…………..	e_n
u_1	$\chi_{R_A}(u_1,e_1)$	$\chi_{R_A}(u_1,e_2)$	…………….	$\chi_{R_A}(u_1,e_n)$
u_2	$\chi_{R_A}(u_2,e_1)$	$\chi_{R_A}(u_2,e_2)$	……………	$\chi_{R_A}(u_2,e_n)$
………	……………	……………	……………	…………
u_m	$\chi_{R_A}(u_m,e_1)$	$\chi_{R_A}(u_m,e_2)$	…………….	$\chi_{R_A}(u_m,e_n)$

Table 2 Tabular representation of R_A of the soft set (F_A, E)

If $a_{ij} = \chi_{R_A}(u_i,e_j)$, we can define a matrix

$$[a_{ij}]_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Which is called a soft matrix of the order $m \times n$ corresponding to the soft set (F_A, E) over U. A soft set (F_A, E) is uniquely characterized by the matrix $[a_{ij}]_{m\times n}$. Therefore we shall identify any soft set with its soft matrix and use these two concepts as interchangeable.

3. Some Notions of Interval Neutrosophic Soft Matrix Theory

3.1 Definition

Let $(F_A^{\neg}, E)$ be an interval neutrosophic soft set (here soft set in the sense of Cagman and Enginglu) over U, where $F_A^{\neg} : E \to IVN^U$, here IVN^U denotes the set of all interval neutrosophic sets over U, then a subset of $U \times E$ is uniquely defined by

$$R_A^{\neg} = \{(u,e) : e \in A, u \in F_A^{\neg}\}$$

The relation $R_A^{\neg}$ is described by the truth-membership function $T_A : U \times E \to \subseteq [0,1]$, indeterminacy-membership function $I_A : U \times E \to \subseteq [0,1],$ and the falsity-membership function $F_A : U \times E \to \subseteq [0,1]$.

Suppose $U = \{u_1, u_2, ..., u_m\}$ and $E = \{e_1, e_2, ..., e_m\}$, then the relation set $R_A^{\neg}$ can be expressed in the following matrix form

	e_1	e_2	....	e_m
u_1	$\left\langle [T_{11}^l, T_{11}^u], [I_{11}^l, I_{11}^u], [F_{11}^l, F_{11}^u] \right\rangle$	$\left\langle [T_{12}^l, T_{12}^u], [I_{12}^l, I_{12}^u], [F_{12}^l, F_{12}^u] \right\rangle$	....	$\left\langle [T_{1m}^l, T_{1m}^u], [I_{1m}^l, I_{1m}^u], [F_{1m}^l, F_{1m}^u] \right\rangle$
u_2	$\left\langle [T_{21}^l, T_{21}^u], [I_{21}^l, I_{21}^u], [F_{21}^l, F_{21}^u] \right\rangle$	$\left\langle [T_{22}^l, T_{22}^u], [I_{22}^l, I_{22}^u], [F_{22}^l, F_{22}^u] \right\rangle$	....	$\left\langle [T_{2m}^l, T_{2m}^u], [I_{2m}^l, I_{2m}^u], [F_{2m}^l, F_{2m}^u] \right\rangle$
…… ..	....	....	....	....
u_m	$\left\langle [T_{m1}^l, T_{m1}^u], [I_{m1}^l, I_{m1}^u], [F_{m1}^l, F_{m1}^u] \right\rangle$	$\left\langle [T_{m2}^l, T_{m2}^u], [I_{m2}^l, I_{m2}^u], [F_{m2}^l, F_{m2}^u] \right\rangle$	....	$\left\langle [T_{mm}^l, T_{mm}^u], [I_{mm}^l, I_{mm}^u], [F_{mm}^l, F_{mm}^u] \right\rangle$

Table3 Tabular representation of $R_A^{\neg}$ of IVN^U

The above matrix representation is useful for computer storage of such a big expression in concise form so that we can retrieve it very easily and it is handy for doing different algebraic operations under certain condition.

3.2 Definition

An interval neutrosophic soft matrix is said to be a null or void interval neutrosophic soft matrix if all the entries of $R_A^{\neg}$ are $\left\langle [0,0],[0,0],[1,1] \right\rangle$ and it is denoted by $\Phi^{\neg}$.

3.3 Definition

An interval neutrosophic soft matrix is said to be a complete interval neutrosophic soft matrix if all the entries of $R_A^{\neg}$ are $\left\langle [1,1],[0,0],[0,0] \right\rangle$ and it is denoted by□ ¬ .

3.4 Definition

Let $X^{\neg}$ be an interval neutrosophic soft matrix. Then the transpose of $X^{\neg}$ is obtained by interchanging its rows and columns and it is denoted by $\left(X^{\neg}\right)^t$.

3.5 Definition

Let $\mathrm{A}^{\neg}$ and $\mathrm{B}^{\neg}$ be two interval neutrosophic soft matrices of the same order. Then $\mathrm{A}^{\neg}$ is said to be a interval neutrosophic soft sub matrix of $\mathrm{B}^{\neg}$ if for every elements of $\mathrm{A}^{\neg}$ their corresponds another element in $\mathrm{B}^{\neg}$ such that $T_A^l \le T_B^l$, $T_A^u \le T_B^u$; $I_A^l \ge I_B^l$, $I_A^u \ge I_B^u$ and $F_A^l \ge F_B^l$, $F_A^u \ge F_B^u$ and it is denoted by

$\mathrm{A}^{\neg} \subseteq \mathrm{B}^{\neg}$.

3.6 Definition

Let $\mathrm{A}^{\neg}$ and $\mathrm{B}^{\neg}$ be two interval neutrosophic soft matrices of the same order. Then their sum is denoted by $\mathrm{A}^{\neg} \oplus \mathrm{B}^{\neg}$ and it is defined as

$$\mathrm{A}^{\neg} \oplus \mathrm{B}^{\neg} = \left\langle \left[\max\left(T_A^l, T_B^l\right), \max\left(T_A^u, T_B^u\right)\right], \left[\frac{I_A^l + I_B^l}{2}, \frac{I_A^u + I_B^u}{2}\right], \left[\min\left(F_A^l, F_B^l\right), \min\left(F_A^u, F_B^u\right)\right] \right\rangle$$

We can extend it for more than two matrices.

3.7 Definition

Let $\mathrm{A}^{\neg}$ and $\mathrm{B}^{\neg}$ be two interval neutrosophic soft matrices of the same order. Then their sum is denoted by $\mathrm{A}^{\neg} \ominus \mathrm{B}^{\neg}$ and it is defined as

$$\mathrm{A}^{\neg} \ominus \mathrm{B}^{\neg} = \left\langle \left[\min\left(T_A^l, T_B^l\right), \min\left(T_A^u, T_B^u\right)\right], \left[\frac{I_A^l - I_B^l}{2}, \frac{I_A^u - I_B^u}{2}\right], \left[\max\left(F_A^l, F_B^l\right), \max\left(F_A^u, F_B^u\right)\right] \right\rangle$$

3.8 Definition

Let $\mathrm{A}^{\neg}$ and $\mathrm{B}^{\neg}$ be two interval neutrosophic soft matrices of the same order. Then the product of $\mathrm{A}^{\neg}$ and $\mathrm{B}^{\neg}$ exist if the number of column in $\mathrm{A}^{\neg}$ is equal to the number of rows in $\mathrm{B}^{\neg}$ and it is denoted by $\mathrm{A}^{\neg} \otimes \mathrm{B}^{\neg}$ and it is defined as

$$\mathrm{A}^{\neg} \otimes \mathrm{B}^{\neg}$$

$$= \left\langle \left[\max\left(\min\left(T_{A_j}^l, T_{B_j}^l\right)\right), \max\left(\min\left(T_{A_j}^u, T_{B_j}^u\right)\right)\right], \left[\max\left(\min\left(I_{A_j}^l, I_{B_j}^l\right)\right), \min\left(\max\left(I_{A_j}^u, I_{B_j}^u\right)\right)\right], \left[\begin{matrix}\min\left(\max\left(F_{A_j}^l, F_{B_j}^l\right)\right), \\ \min\left(\max\left(F_{A_j}^u, F_{B_j}^u\right)\right)\end{matrix}\right] \right\rangle$$

3.9 Definition

Let $\mathrm{X}^{\neg}$ be an interval neutrosophic soft matrix. Then the complement of $\mathrm{X}^{\neg}$ is obtained by interchanging the truth-membership and falsity-membership intervals, without altering the indeterminacy-membership interval of all the elements of $\mathrm{X}^{\neg}$ and it is denoted by $\left(\mathrm{X}^{\neg}\right)^c$.

3.10 Definition

A square interval neutrosophic soft matrix $\mathrm{X}^{\neg}$ is said to be a diagonal interval neutrosophic soft matrix if all of its non-diagonal elements are $\langle [0,0],[0,0],[1,1] \rangle$.

3.11 Definition

Let $X^{\neg}$ be a square interval neutrosophic soft matrix of order $m \times n$, where m=n. Then the Trace of $X^{\neg}$ is denoted by $tr(X^{\neg})$ and it is defined as

$$tr(X^{\neg}) = \left(\left[\max(T_j^{\,l}(c_i)), \max(T_j^{\,u}(c_i))\right]\left[\max(I_j^{\,l}(c_i)), \max(I_j^{\,u}(c_i))\right]\left[\min(F_j^{\,l}(c_i)), \min(F_j^{\,u}(c_i))\right]\right).$$

3.12 Definition

Let X^* be a choice matrix corresponding to a square interval neutrosophic soft matrix $X^{\neg}$. The elements x_{ij}^* of X^* are defined as follows

$$x_{ij}^* = \begin{cases} \langle [1,1][0,0][0,0] \rangle, \text{ where both the entries are the entries of the choice parameters of the decision-makers} \\ \langle [0,0][0,0][1,1] \rangle, \text{ where atleast one of the parameters be not under choice} \end{cases}$$

Choice matrices depend upon the number of decision makers. To get a clear idea about choice matrices, we consider the following algorithm and apply this algorithm in example 4.1

4. **Construction of IVNSM-algorithm for decision making and its application**

For solving a real decision making problem we consider the following steps:

Step 1: Input the interval neutrosophic soft set over the set of attributes and construct interval neutrosophic soft matrix.

Step 2: Compute the product interval neutrosophic soft matrices by multiplying the given interval neutrosophic soft matrix with the combined choice matrices as per the rule of multiplication of interval neutrosophic soft matrices.

Step 3: Calculate the sum of all the product interval neutrosophic soft matrices as per the rule of matrix addition of interval neutrosophic soft matrices.

Step 4: Construct the lower-value matrix and the upper-value matrix corresponding to the resultant matrix.

Step 5: Compute the value matrices corresponding to the lower-value matrix and the upper-value matrix.

Step 6: Find the row sum of the value matrices.

Step 7: By adding the corresponding elements of the row sum of the value matrices, we obtain the weight of each object. Among these, the highest weight becomes the optimal choice object. If more than one object having the highest weight then any one of them may be chosen as the optimal choice object.

To understand IVNSM-algorithm properly, we have the following example.

4.1 Example

Suppose Mr., Debnath wants to buy a house and for that purpose, he appointed three brokers to inspect and report the house. According to their report, he will choose the house which fulfills the optimality criteria i.e the best option he affords. But the problem is that each broker has its own point of view and the owner has come to one decision. Keeping it in mind we consider the following problem:

Let $U=\{h_1,h_2,h_3,h_4\}$ be the set of four houses under consideration and E= {good location, cheap, green surrounding, costly}= $\{e_1,e_2,e_3,e_4\}$ be the set of parameters. The set of brokers is denoted by B= {Mrs., Rama, Mr., Advik, Mrs., Shewly}.

Now, let us construct the interval neutrosophic soft set $(F_A^{\neg},E)$ which describes the attractiveness of the houses and it is given by

$$\left(F_A^{\neg},E\right)=\left\{\begin{array}{l} \textit{good location of houses}=\{h_1/\langle[0.4,0.7][0.5,0.8][0.5,0.7]\rangle,h_2/\langle[0.5,0.6][0.3,0.5][0.4,0.6]\rangle,h_3/\langle[0.2,0.5][0.4,0.6][0.6,0.7]\rangle,h_4/\langle[0.3,0.5][0.4,0.6][0.5,0.8]\rangle\} \\ \textit{cheap house}=\{h_1/\langle[0.3,0.6][0.7,0.9][0.6,0.8]\rangle,h_2/\langle[0.1,0.3][0.3,0.6][0.5,0.7]\rangle,h_3/\langle[0.3,0.4][0.5,0.7][0.8,0.9]\rangle,h_4/\langle[0.4,0.5][0.2,0.4][0.3,0.6]\rangle\} \\ \textit{houses at green surroundings}=\{h_1/\langle[0.6,0.9][0.1,0.2][0.6,0.7]\rangle,h_2/\langle[0.3,0.4][0.2,0.4][0.5,0.6]\rangle,h_3/\langle[0.3,0.4][0.6,0.7][0.3,0.5]\rangle,h_4/\langle[0.4,0.5][0.3,0.5][0.7,0.8]\rangle\} \\ \textit{costly house}=\{h_1/\langle[0.2,0.5][0.5,0.7][0.4,0.6]\rangle,h_2/\langle[0.3,0.6][0.4,0.6][0.5,0.7]\rangle,h_3/\langle[0.1,0.3][0.3,0.5][0.7,0.8]\rangle,h_4/\langle[0.4,0.5][0.4,0.6][0.6,0.8]\rangle\} \end{array}\right\}$$

The matrix representation of the set $(F_A^{\neg},E)$ is given by

$$M=\begin{bmatrix} \langle[0.4,0.7][0.5,0.8][0.5,0.7]\rangle & \langle[0.3,0.6][0.7,0.9][0.6,0.8]\rangle & \langle[0.6,0.9][0.1,0.2][0.6,0.7]\rangle & \langle[0.2,0.5][0.5,0.7][0.4,0.6]\rangle \\ \langle[0.5,0.6][0.3,0.5][0.4,0.6]\rangle & \langle[0.1,0.3][0.3,0.6][0.5,0.7]\rangle & \langle[0.3,0.4][0.2,0.4][0.5,0.6]\rangle & \langle[0.3,0.6][0.4,0.6][0.5,0.7]\rangle \\ \langle[0.2,0.5][0.4,0.6][0.6,0.7]\rangle & \langle[0.3,0.4][0.5,0.7][0.8,0.9]\rangle & \langle[0.3,0.4][0.6,0.7][0.3,0.5]\rangle & \langle[0.1,0.3][0.3,0.5][0.7,0.8]\rangle \\ \langle[0.3,0.5][0.4,0.6][0.5,0.8]\rangle & \langle[0.4,0.5][0.2,0.4][0.3,0.6]\rangle & \langle[0.4,0.5][0.3,0.5][0.7,0.8]\rangle & \langle[0.4,0.5][0.4,0.6][0.6,0.8]\rangle \end{bmatrix}$$

Suppose the choice parameter sets of Mrs., Rama, Mr., Advik ,and Mrs., Shewly are respectively $X=\{e_2,e_3,e_4\}$, $Y=\{e_1,e_3,e_4\}$ and $Z=\{e_1,e_2,e_4\}$ and all these are the subsets of E.

By the definition of the choice matrix, we consider the separate choice matrices of Mrs., Rama, Mr., Advik and Mrs. Shewly are given by,

$$\left(x^{*}_{ij}\right)_X=\begin{pmatrix} \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \end{pmatrix}$$

$$\left(x^*_{ij}\right)_Y = \begin{pmatrix} \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \end{pmatrix}$$

and $$\left(x^*_{ij}\right)_Z = \begin{pmatrix} \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle \end{pmatrix}$$

Now combined choice matrix of Mrs. Rama and Mr. Advik , i.e the matrix in which they come to one decision

$$\left(x^*_{ij}\right)_{X\wedge Y} = \begin{pmatrix} \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \end{pmatrix}$$

Similarly, we can find another two combined choice matrices given by

$$\left(x^*_{ij}\right)_{Y\wedge Z} = \begin{pmatrix} \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle \end{pmatrix}$$

and $$\left(x^*_{ij}\right)_{Z\wedge X} = \begin{pmatrix} \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle \end{pmatrix}$$

Next, we consider another three choice matrices which predict that when any of the two brokers are agree with their decision then the third broker also agrees. They can be presented in the following form

$$\left(x^*_{ij}\right)_{(X\wedge Y,Z)} = \begin{pmatrix} \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle \end{pmatrix}$$

$$\left(x_{ij}^{*}\right)_{(Y\wedge Z,\,X)}=\begin{pmatrix}\langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle\\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle\\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle\\ \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle\end{pmatrix}$$

and

$$\left(x_{ij}^{*}\right)_{(Z\wedge X,\,Y)}=\begin{pmatrix}\langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle\\ \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle\\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle\\ \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle\end{pmatrix}$$

By the definition 3.8, we find the following products :

$$M\otimes\left(x_{ij}^{*}\right)_{(X\wedge Y,\,Z)}=$$

$$\begin{bmatrix}\langle[0.4,0.7][0.5,0.8][0.5,0.7]\rangle & \langle[0.3,0.6][0.7,0.9][0.6,0.8]\rangle & \langle[0.6,0.9][0.1,0.2][0.6,0.7]\rangle & \langle[0.2,0.5][0.5,0.7][0.4,0.6]\rangle\\ \langle[0.5,0.6][0.3,0.5][0.4,0.6]\rangle & \langle[0.1,0.3][0.3,0.6][0.5,0.7]\rangle & \langle[0.3,0.4][0.2,0.4][0.5,0.6]\rangle & \langle[0.3,0.6][0.4,0.6][0.5,0.7]\rangle\\ \langle[0.2,0.5][0.4,0.6][0.6,0.7]\rangle & \langle[0.3,0.4][0.5,0.7][0.8,0.9]\rangle & \langle[0.3,0.4][0.6,0.7][0.3,0.5]\rangle & \langle[0.1,0.3][0.3,0.5][0.7,0.8]\rangle\\ \langle[0.3,0.5][0.4,0.6][0.5,0.8]\rangle & \langle[0.4,0.5][0.2,0.4][0.3,0.6]\rangle & \langle[0.4,0.5][0.3,0.5][0.7,0.8]\rangle & \langle[0.4,0.5][0.4,0.6][0.6,0.8]\rangle\end{bmatrix}$$

$$\otimes$$

$$\begin{pmatrix}\langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle\\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle\\ \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle\\ \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle\end{pmatrix}$$

$$=\begin{bmatrix}\langle[0.6,0.9][0.0,0.2][0.4,0.6]\rangle & \langle[0.6,0.9][0.0,0.2][0.4,0.6]\rangle & \langle[0.0,0.0][0.0,0.2][1,1]\rangle & \langle[0.6,0.9][0.0,0.2][0.4,0.6]\rangle\\ \langle[0.3,0.6][0.0,0.4][0.5,0.6]\rangle & \langle[0.3,0.6][0.0,0.4][0.5,0.6]\rangle & \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.3,0.6][0.0,0.4][0.5,0.6]\rangle\\ \langle[0.3,0.4][0.0,0.5][0.3,0.5]\rangle & \langle[0.3,0.4][0.0,0.5][0.3,0.5]\rangle & \langle[0.0,0.0][0.0,0.5][1,1]\rangle & \langle[0.3,0.4][0.0,0.5][0.8,0.9]\rangle\\ \langle[0.4,0.5][0.0,0.4][0.6,0.8]\rangle & \langle[0.4,0.5][0.0,0.4][0.6,0.8]\rangle & \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.4,0.5][0.0,0.4][0.6,0.8]\rangle\end{bmatrix}$$

$$M \otimes \left(x_{ij}^{*}\right)_{(Y \wedge Z,\, X)} =$$

$$\begin{bmatrix} \langle[0.4,0.7][0.5,0.8][0.5,0.7]\rangle & \langle[0.3,0.6][0.7,0.9][0.6,0.8]\rangle & \langle[0.6,0.9][0.1,0.2][0.6,0.7]\rangle & \langle[0.2,0.5][0.5,0.7][0.4,0.6]\rangle \\ \langle[0.5,0.6][0.3,0.5][0.4,0.6]\rangle & \langle[0.1,0.3][0.3,0.6][0.5,0.7]\rangle & \langle[0.3,0.4][0.2,0.4][0.5,0.6]\rangle & \langle[0.3,0.6][0.4,0.6][0.5,0.7]\rangle \\ \langle[0.2,0.5][0.4,0.6][0.6,0.7]\rangle & \langle[0.3,0.4][0.5,0.7][0.8,0.9]\rangle & \langle[0.3,0.4][0.6,0.7][0.3,0.5]\rangle & \langle[0.1,0.3][0.3,0.5][0.7,0.8]\rangle \\ \langle[0.3,0.5][0.4,0.6][0.5,0.8]\rangle & \langle[0.4,0.5][0.2,0.4][0.3,0.6]\rangle & \langle[0.4,0.5][0.3,0.5][0.7,0.8]\rangle & \langle[0.4,0.5][0.4,0.6][0.6,0.8]\rangle \end{bmatrix}$$

$$\otimes$$

$$\begin{pmatrix} \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \end{pmatrix}$$

$$= \begin{bmatrix} \langle[0.0,0.0][0.0,0.2][1,1]\rangle & \langle[0.4,0.7][0.0,0.2][0.4,0.6]\rangle & \langle[0.4,0.7][0.0,0.2][0.4,0.6]\rangle & \langle[0.4,0.7][0.0,0.2][0.4,0.6]\rangle \\ \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.5,0.6][0.0,0.4][0.4,0.6]\rangle & \langle[0.5,0.6][0.0,0.4][0.4,0.6]\rangle & \langle[0.5,0.6][0.0,0.4][0.4,0.6]\rangle \\ \langle[0.0,0.0][0.0,0.5][1,1]\rangle & \langle[0.2,0.5][0.0,0.5][0.6,0.7]\rangle & \langle[0.2,0.5][0.0,0.5][0.6,0.7]\rangle & \langle[0.2,0.5][0.0,0.5][0.6,0.7]\rangle \\ \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.4,0.5][0.0,0.4][0.5,0.8]\rangle & \langle[0.4,0.5][0.0,0.4][0.5,0.8]\rangle & \langle[0.4,0.5][0.0,0.4][0.5,0.8]\rangle \end{bmatrix}$$

$$M \otimes \left(x_{ij}^{*}\right)_{(Z\wedge X, Y)} =$$

$$\begin{bmatrix} \langle[0.4,0.7][0.5,0.8][0.5,0.7]\rangle & \langle[0.3,0.6][0.7,0.9][0.6,0.8]\rangle & \langle[0.6,0.9][0.1,0.2][0.6,0.7]\rangle & \langle[0.2,0.5][0.5,0.7][0.4,0.6]\rangle \\ \langle[0.5,0.6][0.3,0.5][0.4,0.6]\rangle & \langle[0.1,0.3][0.3,0.6][0.5,0.7]\rangle & \langle[0.3,0.4][0.2,0.4][0.5,0.6]\rangle & \langle[0.3,0.6][0.4,0.6][0.5,0.7]\rangle \\ \langle[0.2,0.5][0.4,0.6][0.6,0.7]\rangle & \langle[0.3,0.4][0.5,0.7][0.8,0.9]\rangle & \langle[0.3,0.4][0.6,0.7][0.3,0.5]\rangle & \langle[0.1,0.3][0.3,0.5][0.7,0.8]\rangle \\ \langle[0.3,0.5][0.4,0.6][0.5,0.8]\rangle & \langle[0.4,0.5][0.2,0.4][0.3,0.6]\rangle & \langle[0.4,0.5][0.3,0.5][0.7,0.8]\rangle & \langle[0.4,0.5][0.4,0.6][0.6,0.8]\rangle \end{bmatrix}$$

$$\otimes$$

$$\begin{pmatrix} \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \\ \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[0,0][0,0][1,1]\rangle \\ \langle[1,1][0,0][0,0]\rangle & \langle[0,0][0,0][1,1]\rangle & \langle[1,1][0,0][0,0]\rangle & \langle[1,1][0,0][0,0]\rangle \end{pmatrix}$$

$$= \begin{bmatrix} \langle[0.3,0.6][0.0,0.2][0.4,0.6]\rangle & \langle[0.0,0.0][0.0,0.2][1,1]\rangle & \langle[0.3,0.6][0.0,0.2][0.4,0.6]\rangle & \langle[0.3,0.6][0.0,0.2][0.4,0.6]\rangle \\ \langle[0.3,0.6][0.0,0.4][0.5,0.7]\rangle & \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.3,0.6][0.0,0.4][0.5,0.7]\rangle & \langle[0.3,0.6][0.0,0.4][0.5,0.7]\rangle \\ \langle[0.3,0.4][0.0,0.5][0.7,0.8]\rangle & \langle[0.0,0.0][0.0,0.5][1,1]\rangle & \langle[0.3,0.4][0.0,0.5][0.7,0.8]\rangle & \langle[0.3,0.4][0.0,0.5][0.7,0.8]\rangle \\ \langle[0.4,0.5][0.0,0.4][0.3,0.6]\rangle & \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.4,0.5][0.0,0.4][0.3,0.6]\rangle & \langle[0.4,0.5][0.0,0.4][0.3,0.6]\rangle \end{bmatrix}$$

By the definition of 3.6, we take the sum of all the product matrices and obtain

$$\left(M \otimes \left(x_{ij}^{*}\right)_{(X\wedge Y, Z)}\right) \oplus \left(M \otimes \left(x_{ij}^{*}\right)_{(Y\wedge Z, X)}\right) \oplus \left(M \otimes \left(x_{ij}^{*}\right)_{(Z\wedge X, Y)}\right)$$

=

$$\begin{bmatrix} \langle[0.6,0.9][0.0,0.2][0.4,0.6]\rangle & \langle[0.6,0.9][0.0,0.2][0.4,0.6]\rangle & \langle[0.0,0.0][0.0,0.2][1,1]\rangle & \langle[0.6,0.9][0.0,0.2][0.4,0.6]\rangle \\ \langle[0.3,0.6][0.0,0.4][0.5,0.6]\rangle & \langle[0.3,0.6][0.0,0.4][0.5,0.6]\rangle & \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.3,0.6][0.0,0.4][0.5,0.6]\rangle \\ \langle[0.3,0.4][0.0,0.5][0.3,0.5]\rangle & \langle[0.3,0.4][0.0,0.5][0.3,0.5]\rangle & \langle[0.0,0.0][0.0,0.5][1,1]\rangle & \langle[0.3,0.4][0.0,0.5][0.8,0.9]\rangle \\ \langle[0.4,0.5][0.0,0.4][0.6,0.8]\rangle & \langle[0.4,0.5][0.0,0.4][0.6,0.8]\rangle & \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.4,0.5][0.0,0.4][0.6,0.8]\rangle \end{bmatrix}$$

$$\oplus$$

$$\begin{bmatrix} \langle[0.0,0.0][0.0,0.2][1,1]\rangle & \langle[0.4,0.7][0.0,0.2][0.4,0.6]\rangle & \langle[0.4,0.7][0.0,0.2][0.4,0.6]\rangle & \langle[0.4,0.7][0.0,0.2][0.4,0.6]\rangle \\ \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.5,0.6][0.0,0.4][0.4,0.6]\rangle & \langle[0.5,0.6][0.0,0.4][0.4,0.6]\rangle & \langle[0.5,0.6][0.0,0.4][0.4,0.6]\rangle \\ \langle[0.0,0.0][0.0,0.5][1,1]\rangle & \langle[0.2,0.5][0.0,0.5][0.6,0.7]\rangle & \langle[0.2,0.5][0.0,0.5][0.6,0.7]\rangle & \langle[0.2,0.5][0.0,0.5][0.6,0.7]\rangle \\ \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.4,0.5][0.0,0.4][0.5,0.8]\rangle & \langle[0.4,0.5][0.0,0.4][0.5,0.8]\rangle & \langle[0.4,0.5][0.0,0.4][0.5,0.8]\rangle \end{bmatrix}$$

$$\oplus$$

$$\begin{bmatrix} \langle[0.3,0.6][0.0,0.2][0.4,0.6]\rangle & \langle[0.0,0.0][0.0,0.2][1,1]\rangle & \langle[0.3,0.6][0.0,0.2][0.4,0.6]\rangle & \langle[0.3,0.6][0.0,0.2][0.4,0.6]\rangle \\ \langle[0.3,0.6][0.0,0.4][0.5,0.7]\rangle & \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.3,0.6][0.0,0.4][0.5,0.7]\rangle & \langle[0.3,0.6][0.0,0.4][0.5,0.7]\rangle \\ \langle[0.3,0.4][0.0,0.5][0.7,0.8]\rangle & \langle[0.0,0.0][0.0,0.5][1,1]\rangle & \langle[0.3,0.4][0.0,0.5][0.7,0.8]\rangle & \langle[0.3,0.4][0.0,0.5][0.7,0.8]\rangle \\ \langle[0.4,0.5][0.0,0.4][0.3,0.6]\rangle & \langle[0.0,0.0][0.0,0.4][1,1]\rangle & \langle[0.4,0.5][0.0,0.4][0.3,0.6]\rangle & \langle[0.4,0.5][0.0,0.4][0.3,0.6]\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle[0.6,0.9][0.0,0.3][0.4,0.6]\rangle & [0.6,0.9][0.0,0.3][0.4,0.6] & [0.4,0.7][0.0,0.3][0.4,0.6] & [0.6,0.9][0.0,0.3][0.4,0.6] \\ [0.3,0.6][0.0,0.6][0.5,0.6] & [0.5,0.6][0.0,0.6][0.4,0.6] & [0.5,0.6][0.0,0.6][0.4,0.6] & [0.5,0.6][0.0,0.6][0.4,0.6] \\ [0.3,0.4][0.0,0.75][0.3,0.5] & [0.3,0.5][0.0,0.75][0.3,0.5] & [0.3,0.5][0.0,0.75][0.6,0.7] & [0.3,0.5][0.0,0.75][0.6,0.7] \\ [0.4,0.5][0.0,0.6][0.3,0.6] & [0.4,0.5][0.0,0.6][0.5,0.8] & [0.4,0.5][0.0,0.6][0.3,0.6] & [0.4,0.5][0.0,0.6][0.3,0.6] \end{bmatrix}$$

Let

$$S= \begin{bmatrix} \langle[0.6,0.9][0.0,0.3][0.4,0.6]\rangle & [0.6,0.9][0.0,0.3][0.4,0.6] & [0.4,0.7][0.0,0.3][0.4,0.6] & [0.6,0.9][0.0,0.3][0.4,0.6] \\ [0.3,0.6][0.0,0.6][0.5,0.6] & [0.5,0.6][0.0,0.6][0.4,0.6] & [0.5,0.6][0.0,0.6][0.4,0.6] & [0.5,0.6][0.0,0.6][0.4,0.6] \\ [0.3,0.4][0.0,0.75][0.3,0.5] & [0.3,0.5][0.0,0.75][0.3,0.5] & [0.3,0.5][0.0,0.75][0.6,0.7] & [0.3,0.5][0.0,0.75][0.6,0.7] \\ [0.4,0.5][0.0,0.6][0.3,0.6] & [0.4,0.5][0.0,0.6][0.5,0.8] & [0.4,0.5][0.0,0.6][0.3,0.6] & [0.4,0.5][0.0,0.6][0.3,0.6] \end{bmatrix}$$

Taking all the lower limits and all the upper limits separately of each entry of S we construct another two matrices, given by

$$S^l = \begin{bmatrix} \langle 0.6,0.0,0.4\rangle & \langle 0.6,0.0,0.4\rangle & \langle 0.4,0.0,0.4\rangle & \langle 0.6,0.0,0.4\rangle \\ \langle 0.3,0.0,0.5\rangle & \langle 0.5,0.0,0.4\rangle & \langle 0.5,0.0,0.4\rangle & \langle 0.5,0.0,0.4\rangle \\ \langle 0.3,0.0,0.3\rangle & \langle 0.3,0.0,0.3\rangle & \langle 0.3,0.0,0.6\rangle & \langle 0.3,0.0,0.6\rangle \\ \langle 0.4,0.0,0.3\rangle & \langle 0.4,0.0,0.5\rangle & \langle 0.4,0.0,0.3\rangle & \langle 0.4,0.0,0.3\rangle \end{bmatrix}$$

and $$S^u = \begin{bmatrix} \langle 0.9,0.3,0.6\rangle & \langle 0.9,0.3,0.6\rangle & \langle 0.7,0.3,0.6\rangle & \langle 0.9,0.3,0.6\rangle \\ \langle 0.6,0.6,0.6\rangle & \langle 0.6,0.6,0.6\rangle & \langle 0.6,0.6,0.6\rangle & \langle 0.6,0.6,0.6\rangle \\ \langle 0.4,0.75,0.5\rangle & \langle 0.5,0.75,0.5\rangle & \langle 0.5,0.75,0.7\rangle & \langle 0.5,0.75,0.7\rangle \\ \langle 0.5,0.6,0.6\rangle & \langle 0.5,0.6,0.8\rangle & \langle 0.5,0.6,0.6\rangle & \langle 0.5,0.6,0.6\rangle \end{bmatrix}$$

Compute the value matrices corresponding to S^l and S^u are

$$V(S^l) = \begin{bmatrix} 0.2 & 0.2 & 0.0 & 0.2 \\ -0.2 & 0.1 & 0.1 & 0.1 \\ 0.0 & 0.0 & -0.3 & -0.3 \\ 0.1 & -0.1 & 0.1 & 0.1 \end{bmatrix}$$

$$V(S^u) = \begin{bmatrix} 0.6 & 0.6 & 0.4 & 0.6 \\ 0.6 & 0.6 & 0.6 & 0.6 \\ 0.65 & 0.75 & 0.55 & 0.55 \\ 0.5 & 0.3 & 0.5 & 0.5 \end{bmatrix}$$

$$\sum_{Row} V(S^l) = \begin{bmatrix} 0.6 \\ 0.1 \\ -0.6 \\ 0.2 \end{bmatrix} \text{ and } \sum_{Row} V(S^u) = \begin{bmatrix} 2.2 \\ 2.4 \\ 2.5 \\ 1.8 \end{bmatrix}$$

$$W(h_1) = 0.6 + 2.2 = 2.8$$

$$W(h_2) = 0.1 + 2.4 = 2.5$$

$$W(h_3) = -0.6 + 2.5 = 1.9$$

$$W(h_4) = 0.2 + 1.8 = 2.0$$

Among all the values above, h_1 has the highest weight. So, Mr., Debnath will prefer to buy the house h_1 .

5. Application in medical science

In medical science, one symptom is related to various diseases. For instance, the symptom fever is related to different diseases including typhoid, peptic ulcer, food poisoning ,etc. So, proper disease diagnosis is a very difficult task. For medical diagnosis problems, we consider the following example.

Suppose Mr. X is suffering from a fever having the symptoms of body pain , breathing difficulty, headache ,and cough and the possible diseases, as per experts advice, relating to the proposed symptoms are viral fever,dengue,food poisoning ,and diptheria. But all experts should come to a common decision so that it helps to diagnose the patient. For this, we consider the set of four diseases $U =${viral fever, dengue, food poisoning, diphtheria}, as a universal set and some related symptoms of these four diseases $E =$ {body pain, breathing difficulty, headache, cough, }, as a set of parameters and a set of doctors D= { d_1, d_2, d_3 },called a set of decision makers or experts or doctors. In reference to example 4.2, if we compare the set of houses as a set of diseases , set of parameters as a set of symptoms , set of decision makers as a set of doctors and then by using this information we construct interval neutrosophic soft set frame work, by considering the same data set proposed in section 4.1. Then, with the help of IVNSM-algoritm, proposed in section 4, we come to the conclusion that Mr. X is suffering from viral fever. The main advantage of using IVNSM-algoritm is that it helps the doctors to diagnose the correct disese and so it prevents the wrong treatment of the patient and it saves time as well. In case no such conclusion can be drawn with the given information then we need to reassess all the symptoms with the help of expert and then repeat all the steps proposed in IVNSM-algorithm.

6. Conclusions

In this paper, we have studied the various concept of interval neutrosophic soft set, which is an extension of neutrosophic soft set. We also introduce the choice matrices, which are associated with the interval neutrosophic soft sets. An algorithm, called IVNSM-algorithm, has been introduced for real decision making problem with the help of a group of decision makers. Finally, it has been discussed, how we use the IVNSM-algorithm in medical diagnosis problem.

7. Future Scope

In our work , we do not consider the information 'time duration' of the symptom or attribute though it has its own significance together with the belongingness or non-belongingness or indeterminacy level of a symptom for proper diagnosis of a patient. Which may be cover up by introducing complex interval neutrosophic soft set. Also, instead of introducing a soft set, we introduce a hypersoft set, pilthogenic hypersoft set, introduced by Smarandache(2018) in [33], to extend the concept used in this paper.

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Some Results on Single Valued Neutrosophic Bi-ideals in Ordered Semigroups

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Abstract. The importance of the theory of neutrosophy is due to its connections with several fields of sciences, engineering, and technology. So, the aim of this paper is to relate neutrosophy with some algebraic structures mainly the ordered semigroups. In this regard, we define single valued neutrosophic sets in ordered semigroups. More precisely, we study single valued neutrosophic ideals of ordered semigroups and single valued neutrosophic bi-ideals of ordered semigroups.

Keywords and phrases: semigroup, ordered semigroup, single valued neutrosophic set, neutrosophic ideal, neutrosophic ordered ideal, neutrosophic ordered bi-ideal.
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1. **Introduction**

The neutrosophic set is a generalization of intuitionistic fuzzy set which in return is a generalization of fuzzy set. Fuzzy set was introduced by Zadeh [18] in 1965 where he considered that every element in a certain set has a degree of membership (truth value) in the unit interval [0,1]. Then in 1986, Atanassov [5] introduced intuitionistic fuzzy set as a generalization of the fuzzy set where he considered that every element in the set has a degree of membership (truth value) and degree of non-membership (falsityhood value). Later in 1998, Smarandache [15] generalized these two concepts and introduced the neutrosophic sets where he considered that each element has a truth value (T), falsity value (F) and indeterminacy value (I).

Neutrosophy is considered as a connection of mathematics with philosophy where it studies the idea in a philosophical way. It is applicable in psychology, sociology, decision making, engineering, information technology, probability and statistics, etc. It proposes any idea and studies its origin, nature, scope and interactions with ideational spectra and reveals that the world is full of indeterminacy. It considers the idea studied as $< A >$, its opposite as $< Anti-A >$, its negation as $< Non-A >$ and its spectrum of neutralities as $< Neut-A >$. For details about neutrosophy, we refer to [15–17]. The new topic of neutrosophy had grabbed the attention of many algebraists and as a result, neutrosophic algebraic structures was introduced. For details about neutrosophic algebraic structures, we refer to [2,3,12–14]. Recently, a relation between neutrosophy and ordered algebraic structures was defined [1, 4].

In 2020, Al-Tahan el al. [1] defined single valued neutrosophic sets in ordered groupoids and investigated the various properties of single valued neutrosophic ideals in it. In this paper, we study the relation between neutrosophy and ordered semigroups and it is organized as follows: After a brief introduction about neutrosophy, Section 2 presents some preliminaries about some algebraic structures such as semigroups and ordered semigroups and give some concepts about neutrosophy. Section 3 presents some definitions, properties and examples about single valued neutrosophic ideals in ordered semigroups. Finally, Section 5 presents single valued neutrosophic bi-ideals in ordered semigroups and provides some related important theorems and examples.

2. Preliminaries

In this section, we present some definitions, concepts and examples related to (ordered) semigroups, neutrosophy, and single valued neutrosophic sets that are used throughout the paper.

2.1. *Semigroups and ordered semigroups*

Definition 2.1. [7] Let S be a non-empty set of elements and $\star$ be a binary operation defined on S. Then S is said to be semigroup if it is binary closed and the associative property holds. In other words, for every x, y and z in the set S, $(x \star y) \star z = x \star (y \star z)$.

Example 2.2. Let $\mathbb{Z}$ be the set of integers, then $(\mathbb{Z}, \cdot)$, where "$\cdot$" is the usual multiplication, is a semigroup.

Remark 2.3. (1) A semigroup is an associative groupoid.
(2) Every semigroup is a groupoid but not every groupoid is a semigroup.
(3) A semigroup with identity is called a monoid.

Definition 2.4. [11] Let $(S, \star)$ be a semigroup. Then a subset A of S is called a subsemigroup if $(A, \star)$ is a semigroup.

Remark 2.5. To prove that a non-empty subset A of a semigroup $(S, \star)$ is a subsemigroup, it suffices to show that $A \star A \subseteq A$.

Example 2.6. Let $(\mathbb{Z}, +)$ be the semigroup of integers under standard addition. Then $(\mathbb{N}, +)$, the set of non-negative integers under standard addition, is a subsemigroup of $(\mathbb{Z}, +)$.

Definition 2.7. [11] Let $(S, \star)$ be a semigroup and $A \subseteq S$ a subsemigroup of S. Then A is called a:

(1) Right ideal if $A \star S \subseteq A$,
(2) Left ideal if $S \star A \subseteq A$,
(3) Ideal if it is both right and left ideal of S,
(4) Bi-ideal if $A \star S \star A \subseteq S$.

Example 2.8. Let $(\mathbb{Z}, \cdot)$ be the semigroup of integers under standard multiplication and let $I = n\mathbb{Z} = \{nq | q \in \mathbb{Z}\}$. Then I is both right and left ideal of $(\mathbb{Z}, \cdot)$. Hence, it is an ideal of $(\mathbb{Z}, \cdot)$.

Definition 2.9. [6] Let G be a non-empty set of elements. A partial order is a binary relation "$\leq$" over a set G such that $\leq$ is reflexive, antisymmetric, and transitive.
In other words, for all $a, b, c \in G, \leq$ satisfies:

(1) $a \leq a$,
(2) If $a \leq b$ and $b \leq a$ then $a = b$,
(3) If $a \leq b$ and $b \leq c$ then $a \leq c$.

Definition 2.10. [6] A total order is a binary relation "$\leq$" over a set G such that $\leq$ is a partial order and every two elements in G are comparable.
In other words, for all $x, y \in G, x \leq y$ or $y \leq x$.

Definition 2.11. [8] Let $(S, \cdot)$ be a semigroup and "$\leq$" be a partial order on S. Then $(S, \cdot, \leq)$ is said to be an ordered semigroup if for all $z \in S$ the following condition holds

$$\text{if } x \leq y \text{ then } z \cdot x \leq z \cdot y \text{ and } x \cdot z \leq y \cdot z \text{ for all } x, y \in S.$$

Remark 2.12. Every ordered semigroup is an ordered groupoid. But the converse is not true.

Example 2.13. The set defined in Example 2.2 is an ordered semigroup under the partial order "$\leq$" which is for every $x, y \in \mathbb{Z}$, $x \leq y$ if and only if $x = y$.
This order is called **trivial order**.

Definition 2.14. [8] Let $(S, \cdot, \leq)$ be an ordered semigroup and $A \subseteq S$. Then

$$(A] = \{x \in S | x \leq a \text{ for some } a \in A\}.$$

Remark 2.15. If $(S, \cdot, \leq)$ is an ordered semigroup and $A \subseteq S$ then $A \subseteq (A]$.

Definition 2.16. [8] Let $(S, \cdot, \leq)$ be an ordered semigroup and $A \subseteq S$, then A is called:

(1) Left ideal of S if $S \cdot A \subseteq A$ and $(A] \subseteq A$,
(2) Right ideal of S if $A \cdot S \subseteq A$ and $(A] \subseteq A$,
(3) Ideal if it is both right and left ideal,
(4) Bi-ideal if $A \cdot S \cdot A \subseteq A$ and $(A] \subseteq A$.

Example 2.17. Let $S = \{a, b, c, d\}$ and let $(S, \star)$ be the semigroup defined by the following table:

$\star$	a	b	c	d
a	d	d	d	d
b	d	d	d	d
c	d	d	d	d
d	d	d	d	a

And let $\leq$ be defined as:

$$\leq = \{(a, a), (b, b), (b, c), (c, c), (d, d)\}.$$

Then $(S, \star, \leq)$ is an ordered semigroup. Now, let $I = \{a, b, d\}$, then I is a right and left ideal of the ordered semigroup $(S, \star, \leq)$. Hence I is an ideal of $(S, \star)$.

2.2. *Neutrosophy and single valued neutrosophic sets*

Definition 2.18. [15] Neutrosophy is a new branch of philosophy which studies the origin, nature, scope and interactions of neutralities with ideational spectra.
It considers:

- Any idea, proposition, theory or event by $< A >$,
- Its opposite by $< Anti - A >$,
- Its negation by $< Non - A >$,
- Its of spectrum of neutralities in between them by $< Neut - A >$.

Remark 2.19. In the theory of neutrosophy, every idea $< A >$ has a truth membership value (T), false membership value (F) and indeterminacy membership value (I).

Definition 2.20. [15] Let X be a non-empty space of elements (objects). A single valued neutrosophic set(SVNS) A on X is characterized by its truth-membership function (T_A), its indeterminacy-membership function (I_A), and its falsity-membership function (F_A) where for each element $x \in X$, $0 \leq T_A(x), I_A(x), F_A(x) \leq 1$.

Remark 2.21. Let X be a non-empty space of elements (objects). A single valued neutrosophic set(SVNS) A on X is defined by $N_A(x) = (T_A(x), I_A(x), F_A(x))$ for all $x \in X$.

Definition 2.22. [1] Let X be a non empty set of elements and A and B be two single valued neutrosophic sets over X defined as follows:

$$A = \{\frac{x}{(T_A(x), I_A(x), F_A(x))} | x \in X\} \text{ and } B = \{\frac{x}{(T_B(x), I_B(x), F_B(x))} | x \in X\}$$

Then

- $A \cap B$, which is the intersection of A and B, is a single valued neutrosophic set over X defined as follows:
$$A \cap B = \{\frac{x}{(T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x))} | x \in X\}$$
where $T_{A\cap B}(x) = T_A(x) \wedge T_B(x)$, $I_{A\cap B}(x) = I_A(x) \wedge I_B(x)$ and $F_{A\cap B}(x) = F_A(x) \vee F_B(x)$ for all $x \in X$.
- $A \cup B$, which is the union of A and B, is a single valued neutrosophic set over X defined as follows:
$$A \cup B = \{\frac{x}{(T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x))} | x \in X\}$$
where $T_{A\cup B}(x) = T_A(x) \vee T_B(x)$, $I_{A\cup B}(x) = I_A(x) \vee I_B(x)$ and $F_{A\cup B}(x) = F_A(x) \wedge F_B(x)$ for all $x \in X$.

3. Single valued neutrosophic ideals in ordered semigroups

Inspired by the work in [9] done by Khan et al. related to fuzzy ideals in ordered semigroups and by the definition of single valued neutrosophic sets in ordered groupoids [1], we consider single valued neutrosophic sets in ordered semigroups. More precisely, we define single valued neutrosophic left ideals, single valued neutrosophic right ideals, study their properties, and provide some examples.

Definition 3.1. Let $(S, \cdot, \leq)$ be an ordered semigroup and A be a single valued neutrosophic set over S. Then A is a single valued neutrosophic subsemigroup of S if for all $x, y \in S$, the following conditions hold:

(1) $T_A(x \cdot y) \geq T_A(x) \wedge T_A(y)$,
(2) $I_A(x \cdot y) \geq I_A(x) \wedge I_A(y)$,
(3) $F_A(x \cdot y) \leq F_A(x) \vee F_A(y)$,
(4) If $x \leq y$ then $T_A(x) \geq T_A(y)$, $I_A(x) \geq I_A(y)$ and $F_A(x) \leq F_A(y)$.

Definition 3.2. Let $(S, \cdot, \leq)$ be an ordered semigroup and A be a single valued neutrosophic set over S. Then A is a single valued neutrosophic left ideal of S if for all $x, y \in S$, the following conditions hold:

(1) $T_A(x \cdot y) \geq T_A(y)$,
(2) $I_A(x \cdot y) \geq I_A(y)$,
(3) $F_A(x \cdot y) \leq F_A(y)$,
(4) If $x \leq y$ then $T_A(x) \geq T_A(y)$, $I_A(x) \geq I_A(y)$ and $F_A(x) \leq F_A(y)$.

Definition 3.3. Let $(S, \cdot, \leq)$ be an ordered semigroup and A be a single valued neutrosophic set over S. Then A is a single valued neutrosophic right ideal of S if for all $x, y \in S$, the following conditions hold:

(1) $T_A(x \cdot y) \geq T_A(x)$,
(2) $I_A(x \cdot y) \geq I_A(x)$,
(3) $F_A(x \cdot y) \leq F_A(x)$,
(4) If $x \leq y$ then $T_A(x) \geq T_A(y)$, $I_A(x) \geq I_A(y)$ and $F_A(x) \leq F_A(y)$.

Definition 3.4. Let $(S, \cdot, \leq)$ be an ordered semigroup and A be a single valued neutrosophic set over S. Then A is said to be a single valued neutrosophic ideal of S if it is both single valued neutrosophic right and left ideal of S.

Remark 3.5. Let $(S, \cdot, \leq)$ be a commutative semigroup and A be a single valued neutrosophic right (left) ideal of S. Then A is a single valued neutrosophic ideal of S.

Remark 3.6. Let $(S, \cdot, \leq)$ be a commutative semigroup and α, β, $\gamma \in [0, 1]$. Then

$$A = \{\tfrac{x}{(\alpha,\beta,\gamma)} | x \in S\}$$

is a single valued neutrosophic ideal of S and it is called the **trivial single valued neutrosophic ideal** of S.

Example 3.7. Let $S = \{1, 2, 3, 4\}$ and $(S, \cdot)$ be defined by the following table:

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1

Let " $\leq$ " be the partial order on S defined as follows:

$$\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$$

Then $(S,.,\leq)$ is an ordered semigroup since "$\cdot$" is binary closed and associative, and "$\leq$" satisfies the monotone property (i.e. for all $x,y\in S, x\leq y$ and for all $z\in S$, $x\cdot z=z\cdot x=1\leq 1=y\cdot z=z\cdot y$).

Let A be an SVNS on S defined by N_A as follows

$$N_A(1)=(0.9,0.8,0.1),\ N_A(2)=(0.7,0.6,0.2),\ N_A(3)=(0.6,0.6,0.2) \text{ and } N_A(4)=(0.5,0.4,0.5).$$

Then A is a single valued neutrosophic ideal of S since for all $x,y\in S$, we have

$T_A(x\cdot y)=T_A(1)=0.9\geq T_A(x)\vee T_A(y)$;

$I_A(x\cdot y)=I_A(1)=0.8\geq I_A(x)\vee I_A(y)$;

$F_A(x\cdot y)=F_A(1)=0.1\leq F_A(x)\wedge F_A(y)$;

Moreover, $1\leq 2\leq 3\leq 4$ implies that $T_A(1)\geq T_A(2)\geq T_A(3)\geq T_A(4)$, $I_A(1)\geq I_A(2)\geq I_A(3)\geq I_A(4)$, and $F_A(1)\leq F_A(2)\leq F_A(3)\leq F_A(4)$.

Proposition 3.8. *Let $(S,\cdot,\leq)$ be an ordered semigroup with identity "e" and A be a single valued neutrosophic set over S. Then A is a single valued neutrosophic left(right) ideal of S if and only if A is the trivial single valued neutrosophic ideal of S.*

Proof. The proof is similar to the case in ordered groupoids [1]. □

Example 3.9. The only single valued neutrosophic right(left) ideal of the semigroup of non-negative integers under addition is the trivial single valued neutrosophic set.

Lemma 3.10. *Let $(S,\cdot,\leq)$ be an ordered semigroup and A_α a single valued neutrosophic left ideal, right ideal or subsemigroup of S. Then $\bigcap_\alpha A_\alpha$ which is the intersection of A_α for all α is a single valued neutrosophic left ideal, right ideal or subsemigroup of S.*

Proof. Let A_α be a single valued neutrosophic left ideal of S. Then for all $x,y\in S$, $T_{A_\alpha}(x\cdot y)\geq T_{A_\alpha}(y)$, $I_{A_\alpha}(x\cdot y)\geq I_{A_\alpha}(y)$, $F_{A_\alpha}(x\cdot y)\leq F_{A_\alpha}(y)$ for all α. This latter implies that

$T_{\bigcap_\alpha A_\alpha}(x\cdot y)=\inf_\alpha T_{A_\alpha}(x\cdot y)\geq \inf_\alpha T_{A_\alpha}(y)=T_{\bigcap_\alpha A_\alpha}(y)$;

$I_{\bigcap_\alpha A_\alpha}(x\cdot y)=\inf_\alpha I_{A_\alpha}(x\cdot y)\geq \inf_\alpha I_{A_\alpha}(y)=I_{\bigcap_\alpha A_\alpha}(y)$;

$F_{\bigcap_\alpha A_\alpha}(x\cdot y)=\sup_\alpha F_{A_\alpha}(x\cdot y)\leq \sup_\alpha F_{A_\alpha}(y)=F_{\bigcap_\alpha A_\alpha}(y)$.

Let $x\leq y$. Then $T_{A_\alpha}(x)\geq T_{A_\alpha}(y)$, $I_{A_\alpha}(x)\geq I_{A_\alpha}(y)$, $F_{A_\alpha}(x)\leq F_{A_\alpha}(y)$. So,

$T_{\bigcap_\alpha A_\alpha}(x)=\inf_\alpha T_{A_\alpha}(x)\geq \inf_\alpha T_{A_\alpha}(y)=T_{\bigcap_\alpha A_\alpha}(y)$;

$I_{\bigcap_\alpha A_\alpha}(x)=\inf_\alpha I_{A_\alpha}(x)\geq \inf_\alpha I_{A_\alpha}(y)=I_{\bigcap_\alpha A_\alpha}(y)$;

and $F_{\bigcap_\alpha A_\alpha}(x)=\sup_\alpha F_{A_\alpha}(x)\leq \sup_\alpha F_{A_\alpha}(y)=F_{\bigcap_\alpha A_\alpha}(y)$.

Therefore, $\bigcap_\alpha A_\alpha$ is a single valued neutrosophic left ideal of S. Similarly, we can prove that the intersection of single valued neutrosophic right ideals or subsemigroups of S is a single valued neutrosophic right ideal of S. □

Remark 3.11. Let $(S, \cdot, \leq)$ be an ordered semigroup and A_α a single valued neutrosophic subsemigroup of S. Then $\bigcup_\alpha A_\alpha$ is not necessarily a single valued neutrosophic subsemigroup of S.

Example 3.12. Let $(\mathbb{N}, +, \leq)$ be the ordered semigroup of natural numbers under standard addition and trivial order and let A and B be single valued neutrosophic sets on $\mathbb{N}$ defined as follows:

$$N_A(x) = \begin{cases} (0.9, 0.7, 0.2) & \text{if x is divisible by 5 ;} \\ (0.4, 0.3, 0.4) & \text{otherwise.} \end{cases}$$

$$N_B(x) = \begin{cases} (0.9, 0.7, 0.2) & \text{if x is divisible by 7 ;} \\ (0.4, 0.3, 0.4) & \text{otherwise.} \end{cases}$$

Then A and B are single valued neutrosophic subsemigroups of $\mathbb{N}$. But $A \cup B$ is not a single valued neutrosophic subsemigroup of $\mathbb{N}$ since $N_{A\cup B}(5+7) = N_{A\cup B}(12) = (0.4, 0.3, 0.4)$ so we will have that $T_{A\cup B}(5+7) = N_{A\cup B}(12) = 0.4 \not\geq N_{A\cup B}(5) \wedge N_{A\cup B}(7) = 0.9$.

Example 3.13. Let $(S, \cdot, \leq)$ be the ordered semigroup defined in Example 3.7. Let A and B be the single valued neutrosophic sets on S defined by N_A and N_B respectively as follows:

$N_A(1) = (0.9, 0.8, 0.1)$, $N_A(2) = (0.7, 0.6, 0.2)$, $N_A(3) = (0.6, 0.6, 0.2)$, $N_A(4) = (0.5, 0.4, 0.5)$;
$N_B(1) = (0.9, 0.8, 0.1)$, $N_B(2) = (0.8, 0.7, 0.2)$, $N_B(3) = (0.7, 0.6, 0.3)$, $N_B(4) = (0.6, 0.4, 0.6)$;

It is clear that A and B are single valued neutrosophic subsemigroups of S. Also $A \cup B$ and $A \cap B$, defined by $N_{A\cup B}$ and $N_{A\cap B}$ respectively as follows.

$$N_{A\cup B}(1) = (0.9, 0.8, 0.1),\ N_{A\cup B}(2) = (0.8, 0.7, 0.2),\ N_{A\cup B}(3) = (0.7, 0.6, 0.2),$$
$$N_{A\cup B}(4) = (0.6, 0.4, 0.5);$$
$$N_{A\cap B}(1) = (0.9, 0.8, 0.1),\ N_{A\cap B}(2) = (0.7, 0.6, 0.2),\ N_{A\cap B}(3) = (0.6, 0.6, 0.3),$$
$$N_{A\cap B}(4) = (0.5, 0.4, 0.6);$$

are also single valued neutrosophic subsemigoups of S.

Lemma 3.14. *Let $(S, \cdot, \leq)$ be an ordered semigroup and A_α a single valued neutrosophic ideal of S. Then $\bigcap_\alpha A_\alpha$ is a single valued neutrosophic ideal of S.*

Proof. Let A_α be a single valued neutrosophic ideal of S. Then A_α is both, a single valued neutrosophic right and left ideal of S. So, by Lemma 3.10, $\bigcap_\alpha A_\alpha$ is both, a single valued neutrosophic right and left ideal of S. Therefore $\bigcap_\alpha A_\alpha$ is a single valued neutrsophic ideal of S. □

Lemma 3.15. *Let $(S, \cdot, \leq)$ be an ordered semigroup and A_α a single valued neutrosophic ideal of S. Then $\bigcup_\alpha A_\alpha$ is a single valued neutrosophic ideal of S.*

Proof. The proof is similar to that of ordered groupoids [1]. □

Example 3.16. Let $(S, \star, \leq)$ be an ordered semigroup where $(S, \star)$ is defined by the following table:

$\star$	1	2	3
1	1	1	1
2	1	1	3
3	1	3	1

and $\leq = \{(1,1),(1,2),(1,3),(2,2),(3,3)\}$. And let A and B be single valued neutrosophic sets over S defined by N_A and N_B respectively as follows:

$$N_A(1) = (0.9, 0.8, 0.1),\ N_A(2) = (0.6, 0.6, 0.3),\ N_A(3) = (0.4, 0.5, 0.3);$$
$$N_B(1) = (0.9, 0.8, 0.1),\ N_B(2) = (0.8, 0.5, 0.5),\ N_B(3) = (0.6, 0.5, 0.3).$$

It is clear that A and B are single valued neutrosophic ideals of S. Then $A \cap B$ and $A \cup B$ defined by $N_{A\cap B}$ and $N_{A\cup B}$ respectively as follows:

$$N_{A\cap B}(1) = (0.9, 0.8, 0.1),\ N_{A\cap B}(2) = (0.6, 0.5, 0.5),\ N_{A\cap B}(2) = (0.4, 0.5, 0.3);$$
$$N_{A\cup B}(1) = (0.9, 0.8, 0.1),\ N_{A\cup B}(2) = (0.8, 0.6, 0.3),\ N_{A\cup B}(2) = (0.6, 0.5, 0.3)$$

are also single valued neutrosophic ideals of S.

Definition 3.17. Let $(S_1, \cdot_1, \leq_1)$ and $(S_2, \cdot_2, \leq_2)$ be two ordered semigroups, then $(S_1 \times S_2, \cdot, \leq)$ is an ordered semigroup where $(x, y) \cdot (z, w) = (x \cdot_1 z, y \cdot_2 w)$ and $(x, y) \leq (z, w)$ if and only if $x \leq_1 z$ and $y \leq_2 w$.

Definition 3.18. Let $(S_1, \cdot_1, \leq_1)$ and $(S_2, \cdot_2, \leq_2)$ be two ordered semigroups, and let A and B be two single valued neutrosophic sets over $S_1 \times S_2$ defined as follows:

$$N_{A\times B}(x, y) = (T_{A\times B}(x, y), I_{A\times B}(x, y), F_{A\times B}(x, y))$$

where $T_{A\times B}(x, y) = T_A(x) \wedge T_B(y)$, $I_{A\times B}(x, y) = I_A(x) \wedge I_B(y)$ and $F_{A\times B}(x, y) = F_A(x) \vee F_B(y)$.

Theorem 3.19. *Let $(S_1, \cdot_1, \leq_1)$ and $(S_2, \cdot_2, \leq_2)$ be two ordered semigroups, and let A and B be two single valued neutrosophic right (left) ideal of S_1 and S_2 respectively. Then $A \times B$ is a single valued neutrosophic right (left) ideal of $S_1 \times S_2$.*

Proof. Let A and B be single valued neutrosophic left ideal of S_1 and S_2 respectively. Then for all $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$, we have:

(1) $T_A(x_1 \cdot y_1) \geq T_A(y_1)$ and $T_B(x_2 \cdot y_2) \geq T_B(y_2)$;
(2) $I_A(x_1 \cdot y_1) \geq I_A(y_1)$ and $I_B(x_2 \cdot y_2) \geq I_B(y_2)$;
(3) $F_A(x_1 \cdot y_1) \leq F_A(y_1)$ and $F_B(x_2 \cdot y_2) \leq F_B(y_2)$;
(4) If $(x_1, x_2) \leq (y_1, y_2)$. This latter implies that $T_A(x_1) \geq T_A(y_1)$, $T_B(x_2) \geq T_B(y_2)$, $I_A(x_1) \geq I_A(y_1)$, $I_B(x_2) \geq I_B(y_2)$, $F_A(x_1) \leq F_A(y_1)$ and $F_B(x_2) \leq F_B(y_2)$.

We get that, $T_{A\times B}((x_1, x_2) \cdot (y_1, y_2)) = T_{A\times B}(x_1 \cdot y_1, x_2 \cdot y_2) = T_A(x_1 \cdot y_1) \wedge T_B(x_2 \cdot y_2) \geq T_A(y_1) \wedge T_B(y_2) \geq T_{A\times B}(y_1, y_2)$,
$I_{A\times B}((x_1, x_2) \cdot (y_1, y_2)) = I_{A\times B}(x_1 \cdot y_1, x_2 \cdot y_2) = I_A(x_1 \cdot y_1) \wedge I_B(x_2 \cdot y_2) \geq I_A(y_1) \wedge I_B(y_2) \geq I_{A\times B}(y_1, y_2)$,
$F_{A\times B}((x_1, x_2) \cdot (y_1, y_2)) = F_{A\times B}(x_1 \cdot y_1, x_2 \cdot y_2) = F_A(x_1 \cdot y_1) \vee F_B(x_2 \cdot y_2) \leq F_A(y_1) \vee F_B(y_2) \leq F_{A\times B}(y_1, y_2)$,
and if $(x_1, x_2) \leq (y_1, y_2)$, then $x_1 \leq_1 y_1$ and $x_2 \leq_2 y_2$, so easily we can see that $T_{A\times B}(x_1, x_2) \geq T_{A\times B}(y_1, y_2)$, $I_{A\times B}(x_1, x_2) \geq I_{A\times B}(y_1, y_2)$, and $F_{A\times B}(x_1, x_2) \leq F_{A\times B}(y_1, y_2)$.
Therefore, $A \times B$ is a single valued neutrosophic left ideal of $S_1 \times S_2$.
Similarly, we can prove the case of single valued neutrosophic right ideal. □

4. Single valued neutrosophic bi-ideals in ordered semigroups

In this section, we define single valued neutrosophic bi-ideals in ordered semigroups, study some of their properties, and provide several examples. The results of this section can be considered as a generalization of fuzzy bi-ideals in ordered semigroups [10].

Definition 4.1. Let $(S, \cdot, \leq)$ be an ordered semigroup and A be a single valued neutrosophic set over S. Then A is said to be a single valued neutrosophic bi-ideal of S if it is a single valued neutrosophic subsemigroup of S and if for all $x, y, z \in S$, $N_A(x \cdot y \cdot z) \geq N_A(x) \wedge N_A(y)$(i.e. $T_A(x \cdot y \cdot z) \geq T_A(x) \wedge T_A(y)$, $I_A(x \cdot y \cdot z) \geq I_A(x) \wedge I_A(y)$ and $F_A(x \cdot y \cdot z) \leq F_A(x) \vee F_A(y)$).

Theorem 4.2. *Let $(S, \cdot, \leq)$ be an ordered semigroup and A be a single valued neutrosophic left(right) ideal over S . Then A is a single valued neutrosophic bi-ideal of S.*

Proof. Let A be a single valued neutrosophic left ideal of S, then A is a single valued neutrosophic subsemigroup and $T_A(x \cdot y) \geq T_A(y), I_A(x \cdot y) \geq I_A(y), F_A(x \cdot y) \leq F_A(y)$ and if $x \leq y, T_A(x) \geq T_A(y), I_A(x) \geq I_A(y), F_A(x) \leq F_A(y)$.
Let $x, y, z \in S$. Then $T_A(x \cdot y \cdot z) \geq T_A(y \cdot z) \geq T_A(z) \geq T_A(x) \wedge T_A(z)$;
$I_A(x \cdot y \cdot z) \geq I_A(y \cdot z) \geq I_A(z) \geq I_A(x) \wedge I_A(z)$, and $F_A(x \cdot y \cdot z) \leq F_A(y \cdot z) \leq F_A(z) \leq F_A(x) \vee F_A(z)$
Therefore, A is a single valued neutrosophic bi-ideal of S. □

Remark 4.3. Not every single valued neutrosophic bi-ideal is a single valued neutrosophic left or right ideal.

Example 4.4. Let $(S, \cdot)$ be an ordered semigroup defined by the following table:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	c	a

Let " $\leq$ " be defined as follows: $\leq = \{(a,a),(a,b),(a,c),(b,b),(c,c)\}$ and A be an SVNS on S defined by: $N_A(a) = (0.9, 0.8, 0.1)$, $N_A(b) = (0.8, 0.5, 0.4)$ and $N_A(c) = (0.7, 0.6, 0.2)$. Then A is SVN bi-ideal of S but it is neither SVN right nor left ideal of S since $T_A(b \cdot c) = T_A(c \cdot b) = T_A(c) \ngeq T_A(b)$.

Theorem 4.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup and A_α a single valued neutrosophic bi-ideal of S. Then $\bigcap_\alpha A_\alpha$ is a single valued neutrosophic bi-ideal of S.*

Proof. Let A_α be a single valued neutrosophic bi-ideal of S for all α. Then A_α is a single valued neutrosophic subsemigroup of S. Hence by Lemma 3.10 $\bigcap_\alpha A_\alpha$ is a single valued neutrosophic subsemigroup of S.

Also, we have that $T_{A_\alpha}(x \cdot y \cdot z) \geq T_{A_\alpha}(x) \wedge T_{A_\alpha}(z)$, $I_{A_\alpha}(x \cdot y \cdot z) \geq I_{A_\alpha}(x) \wedge I_{A_\alpha}(z)$ and $F_{A_\alpha}(x \cdot y \cdot z) \leq F_{A_\alpha}(x) \vee F_{A_\alpha}(z)$. This latter implies that

$T_{\bigcap_\alpha A_\alpha}(x \cdot y \cdot z) = \inf_\alpha T_{A_\alpha}(x \cdot y \cdot z) \geq \inf_\alpha\{T_{A_\alpha}(x) \wedge T_{A_\alpha}(z)\} = \inf_\alpha T_{A_\alpha}(x) \wedge \inf_\alpha T_{A_\alpha}(z) = T_{\bigcap_\alpha A_\alpha}(x) \wedge T_{\bigcap_\alpha A_\alpha}(z)$;

$I_{\bigcap_\alpha A_\alpha}(x \cdot y \cdot z) = \inf_\alpha I_{A_\alpha}(x \cdot y \cdot z) \geq \inf_\alpha\{I_{A_\alpha}(x) \wedge I_{A_\alpha}(z)\} = \inf_\alpha I_{A_\alpha}(x) \wedge \inf_\alpha I_{A_\alpha}(z) = I_{\bigcap_\alpha A_\alpha}(x) \wedge I_{\bigcap_\alpha A_\alpha}(z)$;

$F_{\bigcap_\alpha A_\alpha}(x \cdot y \cdot z) = \sup_\alpha F_{A_\alpha}(x \cdot y \cdot z) \leq \sup_\alpha\{F_{A_\alpha}(x) \vee F_{A_\alpha}(z)\} = \sup_\alpha F_{A_\alpha}(x) \vee \sup_\alpha F_{A_\alpha}(z) = F_{\bigcap_\alpha A_\alpha}(x) \vee F_{\bigcap_\alpha A_\alpha}(z)$.

Therefore, $\bigcap_\alpha A_\alpha$ is a single valued neutrosophic bi-ideal of S. □

Theorem 4.6. *Let $(S_1, \cdot_1, \leq_1)$ and $(S_2, \cdot_2, \leq_2)$ be two ordered semigroups, and let A and B be two single valued neutrosophic bi-ideals of S_1 and S_2 respectively. Then $A \times B$ is a single valued neutrosophic bi-ideals of $S_1 \times S_2$.*

Proof. Let A and B be single valued neutrosophic bi-ideals of S_1 and S_2 respectively. Then for all $x_1, y_1, z_1 \in S_1$, $x_2, y_2, z_2 \in S_2$, A and B are single valued neutrosophic bi-ideals of S, $T_A(x_1 \cdot y_1 \cdot z_1) \geq T_A(x_1) \wedge T_A(z_1)$, $T_B(x_2 \cdot y_2 \cdot z_2) \geq T_B(x_2) \wedge T_B(z_2)$, $I_A(x_1 \cdot y_1 \cdot z_1) \geq I_A(x_1) \wedge I_A(z_1)$, $I_B(x_2 \cdot y_2 \cdot z_2) \geq I_B(x_2) \wedge I_B(z_2)$, $F_A(x_1 \cdot y_1 \cdot z_1) \leq F_A(x_1) \vee F_A(z_1)$ and

$F_B(x_2 \cdot y_2 \cdot z_2) \leq F_B(x_2) \vee F_B(z_2)$.

So, we get that

$T_{A\times B}((x_1,x_2).(y_1,y_2).(z_1,z_2)) = T_{A\times B}(x_1\cdot y_1\cdot z_1, x_2\cdot y_2\cdot z_2) = T_A(x_1\cdot y_1\cdot z_1) \wedge T_B(x_2\cdot y_2\cdot z_2) \geq T_A(x_1) \wedge T_A(z_1) \wedge T_B(x_2) \wedge T_B(z_2) = T_{A\times B}(x_1,x_2) \wedge T_{A\times B}(z_1,z_2)$;

$I_{A\times B}((x_1,x_2)\cdot(y_1,y_2).(z_1,z_2)) = I_{A\times B}(x_1\cdot y_1\cdot z_1, x_2\cdot y_2\cdot z_2) = I_A(x_1\cdot y_1\cdot z_1) \wedge I_B(x_2\cdot y_2\cdot z_2) \geq I_A(x_1) \wedge I_A(z_1) \wedge I_B(x_2) \wedge I_B(z_2) = I_{A\times B}(x_1,x_2) \wedge I_{A\times B}(z_1,z_2)$;

$F_{A\times B}((x_1,x_2).(y_1,y_2).(z_1,z_2)) = F_{A\times B}(x_1\cdot y_1\cdot z_1, x_2\cdot y_2\cdot z_2) = F_A(x_1\cdot y_1]\cdot z_1) \vee F_B(x_2\cdot y_2\cdot z_2) \leq F_A(x_1) \vee F_A(z_1) \vee F_B(x_2) \vee F_B(z_2) = F_{A\times B}(x_1,x_2) \vee F_{A\times B}(z_1,z_2)$.

And as A and B are single valued neutrosophic subsemigroup of S_1 and S_2 respectively. Then by Theorem 3.19, we get that $A \times B$ is a single valued neutrosophic subsemigroup of $S_1 \times S_2$.

□

Example 4.7. Let $(S, \star)$ be the semigroup defined by the following table:

$\star$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

and let " $\leq$ " be defined as follows: $\leq = \{(0,0),(0,1),(2,2),(2,1),(0,2)\}$. Then $(S,\star,\leq)$ is an ordered semigroup. Let A be an SVNS on S defined by N_A as follows:

$$N_A(0) = (0.9, 0.3, 0.1),\ N_A(1) = (0.9, 0.2, 0.2) \text{ and } N_A(2) = (0.9, 0.2, 0.2).$$

Then A is a single valued neutrosophic ideal of S since

$T_A(0\star 0) = T_A(0) = 0.9 \geq T_A(0) \vee T_A(0) = 0.9$;

$T_A(0\star 1) = T_A(0) = 0.9 \geq T_A(0) \vee T_A(1) = 0.9$;

$T_A(0\star 2) = T_A(0) = 0.9 \geq T_A(0) \vee T_A(2) = 0.9$;

$T_A(1\star 0) = T_A(0) = 0.9 \geq T_A(1) \vee T_A(0) = 0.9$;

$T_A(1\star 1) = T_A(1) = 0.9 \geq T_A(1) \vee T_A(1) = 0.9$;

$T_A(1\star 2) = T_A(2) = 0.9 \geq T_A(1) \vee T_A(2) = 0.9$;

$T_A(2\star 0) = T_A(0) = 0.9 \geq T_A(2) \vee T_A(0) = 0.9$;

$T_A(2\star 1) = T_A(1) = 0.9 \geq T_A(2) \vee T_A(1) = 0.9$;

$T_A(2\star 2) = T_A(2) = 0.9 \geq T_A(2) \vee T_A(2) = 0.9$;

$I_A(0\star 0) = T_A(0) = 0.3 \geq I_A(0) \vee I_A(0) = 0.3$;

$I_A(0\star 1) = I_A(0) = 0.3 \geq I_A(0) \vee I_A(1) = 0.3$;

$I_A(0\star 2) = I_A(0) = 0.3 \geq I_A(0) \vee T_A(2) = 0.3$;

$I_A(1\star 0) = I_A(0) = 0.3 \geq I_A(1) \vee I_A(0) = 0.3$;

$I_A(1\star 1) = I_A(1) = 0.2 \geq I_A(1) \vee I_A(1) = 0.2$;

$I_A(1 \star 2) = I_A(2) = 0.2 \geq I_A(1) \vee I_A(2) = 0.2;$
$I_A(2 \star 0) = I_A(0) = 0.3 \geq I_A(2) \vee I_A(0) = 0.3;$
$I_A(2 \star 1) = I_A(1) = 0.2 \geq I_A(2) \vee I_A(1) = 0.2;$
$I_A(2 \star 2) = I_A(2) = 0.2 \geq I_A(2) \vee I_A(2) = 0.2;$
$F_A(0 \star 0) = F_A(0) = 0.1 \leq F_A(0) \wedge F_A(0) = 0.1;$
$F_A(0 \star 1) = F_A(0) = 0.1 \leq F_A(0) \wedge F_A(1) = 0.1;$
$F_A(0 \star 2) = F_A(0) = 0.1 \leq F_A(0) \wedge F_A(2) = 0.1;$
$F_A(1 \star 0) = F_A(0) = 0.1 \leq F_A(1) \wedge F_A(0) = 0.1;$
$F_A(1 \star 1) = F_A(1) = 0.2 \leq F_A(1) \wedge F_A(1) = 0.2;$
$F_A(1 \star 2) = F_A(2) = 0.2 \leq F_A(1) \wedge F_A(2) = 0.2;$
$F_A(2 \star 0) = F_A(0) = 0.1 \leq T_A(2) \wedge F_A(0) = 0.2;$
$F_A(2 \star 1) = F_A(1) = 0.2 \leq F_A(2) \wedge F_A(1) = 0.2;$
$F_A(2 \star 2) = F_A(2) = 0.2 \leq F_A(2) \wedge F_A(2) = 0.2.$

Moreover, $0 \leq 1 \leq 2$ implies that $T_A(0) \geq T_A(1) \geq T_A(2)$, $I_A(0) \geq I_A(1) \geq I_A(2)$ and $F_A(0) \leq F_A(1) \leq F_A(2)$.
Therefore, A is a single valued neutrosophic ideal of S.

Example 4.8. Let $(S, \star, \leq)$ be the semigroup defined in Example 4.7 and B a single valued neutrosophic set over S defined by N_B as follows

$$N_B(0) = (0.9, 0.2, 0.1),\ N_B(1) = (0.8, 0.1, 0.3) \text{ and } N_B(2) = (0.7, 0.1, 0.4).$$

Then B is a single valued neutrosophic left ideal of S since
$T_B(0 \star 0) = T_B(0) = 0.9 \geq T_B(0) = 0.9;$
$T_B(0 \star 1) = T_B(0) = 0.9 \geq T_B(1) = 0.8;$
$T_B(0 \star 2) = T_B(0) = 0.9 \geq T_B(2) = 0.7;$
$T_B(1 \star 0) = T_B(0) = 0.9 \geq T_B(0) = 0.9;$
$T_B(1 \star 1) = T_B(1) = 0.8 \geq T_B(1) = 0.8;$
$T_B(1 \star 2) = T_B(2) = 0.7 \geq T_B(2) = 0.7;$
$T_B(2 \star 0) = T_B(0) = 0.9 \geq T_B(0) = 0.9;$
$T_B(2 \star 1) = T_B(1) = 0.8 \geq T_B(1) = 0.8;$
$T_B(2 \star 2) = T_B(2) = 0.7 \geq T_B(2) = 0.7;$
$I_B(0 \star 0) = T_B(0) = 0.2 \geq I_B(0) = 0.2;$
$I_B(0 \star 1) = I_B(0) = 0.2 \geq I_B(1) = 0.1;$
$I_B(0 \star 2) = I_B(0) = 0.2 \geq I_B(2) = 0.1;$
$I_B(1 \star 0) = I_B(0) = 0.2 \geq I_B(0) = 0.2;$
$I_B(1 \star 1) = I_B(1) = 0.1 \geq I_B(1) = 0.1;$
$I_B(1 \star 2) = I_B(2) = 0.1 \geq I_B(2) = 0.1;$

$I_B(2 \star 0) = I_B(0) = 0.2 \geq I_B(0) = 0.2;$
$I_B(2 \star 1) = I_B(1) = 0.1 \geq I_B(1) = 0.1;$
$I_B(2 \star 2) = I_B(2) = 0.1 \geq I_B(2) = 0.1;$
$F_B(0 \star 0) = F_B(0) = 0.1 \leq F_B(0) = 0.1;$
$F_B(0 \star 1) = F_B(0) = 0.1 \leq F_B(1) = 0.3;$
$F_B(0 \star 2) = F_B(0) = 0.1 \leq F_B(2) = 0.4;$
$F_B(1 \star 0) = F_B(0) = 0.1 \leq F_B(0) = 0.1;$
$F_B(1 \star 1) = F_B(1) = 0.3 \leq F_B(1) = 0.3;$
$F_B(1 \star 2) = F_B(2) = 0.4 \leq F_B(2) = 0.4;$
$F_B(2 \star 0) = F_B(0) = 0.1 \leq F_B(0) = 0.1;$
$F_B(2 \star 1) = F_B(1) = 0.3 \leq F_B(1) = 0.3;$
$F_B(2 \star 2) = F_B(2) = 0.4 \leq F_B(2) = 0.4.$

Moreover, $0 \leq 1 \leq 2$ implies that $T_B(0) \geq T_B(1) \geq T_B(2)$, $I_B(0) \geq I_B(1) \geq I_B(2)$ and $F_B(0) \leq F_B(1) \leq F_B(2)$.
Moreover, since B is an SVN left ideal of S, it follows by Theorem 4.2 that B is a single valued neutrosophic bi-ideal of S.

Example 4.9. Let $M_2(\mathbb{N})$ be the set of 2×2 matrices (i.e. $M_2(\mathbb{N}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{N}\}$).
And let A be an SVNS on $M_2(\mathbb{N})$ defined by N_A as follows

$$N_A(X) = \begin{cases} (0.8, 0.4, 0.2) & \text{if } X \in I; \\ (0.6, 0.3, 0.5) & \text{if } X \notin I. \end{cases}$$

where $I = \{\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}; k \in \mathbb{N}\}$.
Then A is neither a single valued neutrosophic left ideal nor single valued neutrosophic right ideal of $M_2(\mathbb{N})$. Moreover, it is a single valued neutrosophic bi-ideal of $M_2(\mathbb{N})$.

Proof. First we show that A is neither a single valued neutrosophic right ideal nor a single valued neutrosophic left ideal of $M_2(\mathbb{N})$.
Let $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in I$ and $Y = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \notin I$
So, we have, $X.Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \notin I$.
Then $N_A(X.Y) = (0.6, 0.3, 0.5)$. But $T_A(X.Y) = 0.6 \ngeq T_A(X) = 0.8$. So, A is not an SVN right ideal of $M_2(\mathbb{N})$.
Also we have, $Y.X = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \notin I$.
Then $N_A(Y.X) = (0.6, 0.3, 0.5)$. But $T_A(Y.X) = 0.6 \ngeq T_A(X) = 0.8$. So, A is not an SVN left

ideal of $M_2(\mathbb{N})$.

Next, we show that A is a single valued neutrosophic subsemigroup of $M_2(\mathbb{N})$.
Let $X, Y \in M_2(\mathbb{N})$. We consider the following cases.

- **Case** $X, Y \in I$, then we have $N_A(X) = N_A(Y) = (0.8, 0.4, 0.2)$. Then $X.Y \in I$, then $N_A(X.Y) = (0.8, 0.4, 0.2)$.
  So, easily we can see that, $T_A(X.Y) = 0.8 \geq T_A(X) \wedge T_A(Y) = 0.8$, $I_A(X.Y) = 0.4 \geq I_A(X) \wedge I_A(Y) = 0.4$, $F_A(X.Y) = 0.2 \leq F_A(X) \vee F_A(Y) = 0.2$, and if $X \leq Y$, then $T_A(X) = 0.8 \geq T_A(Y) = 0.8$, $I_A(X) = 0.4 \geq I_A(Y) = 0.4$ and $F_A(X) = 0.2 \leq F_A(Y) = 0.2$.
- **Case** $X, Y \notin I$, then we have $N_A(X) = N_A(Y) = (0.6, 0.3, 0.5)$. So, easily we can see that, $T_A(X.Y) \geq 0.6 = T_A(X) \wedge T_A(Y)$, $I_A(X.Y) \geq 0.3 = I_A(X) \wedge I_A(Y)$, $F_A(X.Y) \leq 0.5 = F_A(X) \vee F_A(Y)$.
- **Case** $X \in I$**,** $Y \notin I$, then we have $N_A(X) = (0.8, 0.4, 0.2)$ and $N_A(Y) = (0.6, 0.3, 0.5)$. So, easily we can see that, $T_A(X.Y) \geq 0.6 = T_A(X) \wedge T_A(Y)$, $I_A(X.Y) \geq 0.3 = I_A(X) \wedge I_A(Y)$, $F_A(X.Y) \leq 0.5 = F_A(X) \vee F_A(Y)$.

Therefore, A is an SVN subsemigroup of $M_2(\mathbb{N})$.
Simple computations show that A is an SVN bi-ideal of $M_2(\mathbb{N})$. □

5. Conclusion

This paper dealt with single valued neutrosophic sets in ordered semigroups where several concepts about single valued neutrosophic ideals and single valued neutrosophic bi-ideals were defined and studied with several examples. The results in this paper are generalization of fuzzy ideals (bi-ideals) in ordered semigroups.

For future research, it will be interesting to discuss single valued neutrosophic sets in other ordered algebraic structures.

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Matrix Games with Single-Valued Triangular Neutrosophic Numbers as Pay-offs

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Abstract. Game theory is commonly used in competitive situations because of its significance in decision-making. Different types of fuzzy sets can handle uncertainty in matrix games. Neutrosophic set theory plays a vital role in analyzing complexity, ambiguity, incompleteness, and inconsistency in real-world problems. This study develops a novel approach to solve neutrosophic matrix games using linear programming problems with single-valued triangular neutrosophic numbers as pay-offs. This paper establishes some theoretical aspects of game theory in a neutrosophic environment. A numerical example verifies the theoretical results using the traditional simplex approach to achieve the strategy and value of the game. The proposed work is useful to model and solve conflict situations in decision-making problems with partial knowledge as data in a simple manner.

Keywords: Matrix game; Neutrosophic set; Single valued triangular neutrosophic number; Neutrosophic matrix game

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1. **Introduction**

Real-world conflict scenarios are often investigated using game theory. It is difficult to collect the right data from decision-makers in today's situations. Fuzzy set theory is based on unreliable information and vagueness due to a lack of some pieces of information and accurate data. Previous research investigated complexity in game theory using fuzzy sets, intuitionistic fuzzy sets, and rough fuzzy sets. The concept of neutrosophic set theory in games is new at the moment, and it is a common research subject all over the world for dealing with competitive situations.

Neumann and Morgenstern [1] established the notion of game theory. Although, the classical game theory has exact data and factual information about the players. In uncertain situations, the notion of the fuzzy set theory proposed by Zadeh [2] is applied to many fields. Campus [5] introduced a model based on a linear programming approach to interpreting fuzzy matrix games. Sakawa and Nishizaki [6, 10] investigated max-min solution methods for multi-objective conflict resolution problems. Bector et al. [11, 12] determined the matrix games with fuzzy goals and fuzzy payoffs. The concept of dual linear programming approach employed by Vijay et al. [13]. Several researchers [14, 15, 25, 27, 37] developed fuzzy matrix games. To determine the uncertainty about non-membership degrees Atanassov [4, 8] inducted intuitionistic fuzzy set theory. Further, intuitionistic fuzzy concept applied by [16–19, 21–24, 26, 38] to study game-theoretic models using linear programming approach. After that, Intuitionistic fuzzy sets were extended to interval-valued intuitionistic fuzzy sets and hesitant fuzzy sets. Kumar and Garg [28] suggested the TOPSIS method under interval-valued intuitionistic fuzzy environment. Xue et al. [45] applied the Ambika method to determine the matrix games with hesitant fuzzy knowledge and investigated the counter-terrorism problem. A methodology based on the linear programming approach was applied to solve the matrix games with triangular dual hesitant fuzzy numbers as payoffs by Yang and Song [39].

The intuitionistic fuzzy sets can not successfully deal in the circumstances of good, unacceptable, and uncertain decision-making problems. Therefore a novel theory was necessary. Smarandache [7, 9] filled the gap and introduced the concept of neutrosophic set theory, which deals with incomplete, inconsistent, and indeterminate situations. Single valued neutrosophic sets as an extension of neutrosophic sets were presented by Wang et al. [20]. A de-neutrosophication idea for linear and non-linear generalized triangular neutrosophic numbers was performed by Chakraborty et al. [30]. The concept of neutrosophic set and number has been successfully applied by Abdel-basset et al. [31–33], and developed methods for sustainable supplier selection problems. [34–36] investigated decision making models based on neutrosophic sets. A similar study of neutrosophic sets and numbers was provided by Broumi et al. [29]. Khalil et al. [40] suggested a new idea for the single-valued neutrosophic fuzzy soft set. Neutrosophic soft, rough topology and its applications to multicriteria decision-making problems were proposed by Riaz et al. [41]. Based on the neutrosophic fuzzy approach, an economical production quantity model was suggested by De et al. [42] for imperfect production processes under game. Du et al. [44] in neutrosophic Z-numbers conditions investigated a multicriteria decision-making approach. In contemporary situations to handle the conflicting political circumstances, a neutrosophic model for non-cooperative games was inducted by Arias et al. [43] using single-valued triangular neutrosophic numbers. Bhaumik et al. [46] introduced a new ranking approach to solve bi-matrix games based on (α, β, γ) -cut set of a single-valued triangular neutrosophic number.

Game theory is widely used in competitive scenarios due to its importance in decision-making. In real-world problems, the concept of neutrosophic set theory is useful for analyzing complexity, uncertainty, incompleteness, and inconsistency. In matrix games with single-valued triangular neutrosophic numbers as pay-offs, we developed a novel approach focused on linear programming using de-neutrosophication as values and ambiguities. The standard simplex approach is used to accomplish the strategy and value of the game for the individual player by providing a numerical representation. The proposed work is capable of quickly resolving conflict situations in decision-making problems using partial information as data.

The main novelties of this work are pointed as:

- A new class of matrix game, namely neutrosophic matrix game, is defined under partial informative situations.
- A mathematical model of neutrosophic matrix game is developed.
- Values and ambiguities are derived for single-valued triangular neutrosophic numbers, and some new theorems are provided.
- The theoretical results are verified by a numerical example arising in conflict situations in decision-making problems with partial knowledge as data.

The research paper is designed as: Section 2 contains preliminaries and definitions. Values and ambiguities are determined in Section 3. Section 4 deals with value index and ambiguity index. Section 5 describes a mathematical model of a matrix game. A numerical example is demonstrated in Section 6. Section 7 concludes the results of the paper.

2. Preliminaries and definitions

In this section we recall some basic definitions and notations which are useful throughout the paper.

Definition 2.1. Let $X = \{X_1, X_2, X_3, \ldots, X_n\}$ be the universal set. A neutrosophic set $\tilde{A}$ in the universal set X, is characterized by its truth membership function $\mu_{\tilde{A}}$, indeterminacy membership function $\pi_{\tilde{A}}$ and falsity membership function $\nu_{\tilde{A}}$ which associates with $X_i \in X$ to a real number in the interval $[0, 1]$ and defined as

$$\tilde{A} = \{\left\langle X_i, \mu_{\tilde{A}}(X_i), \pi_{\tilde{A}}(X_i), \nu_{\tilde{A}}(X_i)\right\rangle |X_i \in X\}. \tag{1}$$

Definition 2.2. A single valued triangular neutrosophic number defined on the set of real numbers is a neutrosophic set, denoted by $\tilde{A}^{TNN} = \langle(\xi, \eta, \zeta)\, ; \sigma, \rho, \tau\rangle$ whose truth membership, indeterminacy membership and falsity membership functions respectively are given as follows:

$$\mu_{\tilde{A}^{TNN}}(x) = \begin{cases} \frac{(x-\xi)\sigma}{(\eta-\xi)} & ;\xi \le x \le \eta \\ \sigma & ;x = \eta \\ \frac{(\zeta-x)\sigma}{(\zeta-\eta)} & ;\eta \le x \le \zeta \\ 0 & ;otherwise \end{cases} \tag{2}$$

$$\pi_{\tilde{A}^{TNN}}(x) = \begin{cases} \frac{(\eta-x)+\rho(x-\xi)}{(\eta-\xi)} & ;\xi \le x \le \eta \\ \rho & ;x = \eta \\ \frac{(x-\eta)+\rho(\zeta-x)}{(\zeta-\eta)} & ;\eta \le x \le \zeta \\ 1 & ;otherwise \end{cases} \tag{3}$$

$$\nu_{\tilde{A}^{TNN}}(x) = \begin{cases} \frac{(\eta-x)+\tau(x-\xi)}{(\eta-\xi)} & ;\xi \le x \le \eta \\ \tau & ;x = \eta \\ \frac{(x-\eta)+\tau(\zeta-x)}{(\zeta-\eta)} & ;\eta \le x \le \zeta \\ 1 & ;otherwise \end{cases} \tag{4}$$

where $0 \le \sigma \le 1$, $0 \le \rho \le 1$, $0 \le \tau \le 1$ and $0 \le \sigma + \rho + \tau \le 3$. σ represents the maximum degree of truth membership, ρ represents the minimum degree of indeterminacy membership and τ represents the minimum degree of falsity membership.

Definition 2.3. Let $\tilde{A}^{TNN} = \langle(\xi_1, \eta_1, \zeta_1)\,; \sigma_1, \rho_1, \tau_1\rangle$ and $\tilde{B}^{TNN} = \langle(\xi_2, \eta_2, \zeta_2)\,; \sigma_2, \rho_2, \tau_2\rangle$ be two single valued triangular neutrosophic numbers and λ be a real number, then some arithmetical operations are stipulated as follows:

- Addition

$$\tilde{A}^{TNN} + \tilde{B}^{TNN} = \langle(\xi_1 + \xi_2, \eta_1 + \eta_2, \zeta_1 + \zeta_2)\,; \min(\sigma_1, \sigma_2), \max(\rho_1, \rho_2), \max(\tau_1, \tau_2)\rangle\,. \tag{5}$$

- Symmetric Image

$$-\tilde{A}^{TNN} = \langle(-\zeta_1, -\eta_1, -\xi_1)\,; \sigma_1, \rho_1, \tau_1\rangle\,. \tag{6}$$

- Subtraction

$$\tilde{A}^{TNN} - \tilde{B}^{TNN} = \langle(\xi_1 - \zeta_2, \eta_1 - \eta_2, \zeta_1 - \xi_2)\,; \min(\sigma_1, \sigma_2), \max(\rho_1, \rho_2), \max(\tau_1, \tau_2)\rangle\,. \tag{7}$$

- Multiplication

$$\tilde{A}^{TNN} \times \tilde{B}^{TNN} = \langle(\xi_1\xi_2, \eta_1\eta_2, \zeta_1\zeta_2)\,; \min(\sigma_1, \sigma_2), \max(\rho_1, \rho_2), \max(\tau_1, \tau_2)\rangle\,. \tag{8}$$

- Scalar Product

$$\lambda\tilde{A}^{TNN} = \begin{cases} \langle(\lambda\xi_1, \lambda\eta_1, \lambda\zeta_1)\,; \sigma_1, \rho_1, \tau_1\rangle & ;\lambda > 0 \\ \langle(\lambda\zeta_1, \lambda\eta_1, \lambda\xi_1)\,; \sigma_1, \rho_1, \tau_1\rangle & ;\lambda < 0 \end{cases} \tag{9}$$

Definition 2.4. The (α, β, γ)- cut of a single valued triangular neutrosophic number $\tilde{A}^{TNN} = \langle (\xi, \eta, \zeta); \sigma, \rho, \tau \rangle$ is a closed crisp interval of real numbers denoted by $\tilde{A}^{TNN}_{(\alpha,\beta,\gamma)}$ and defined as

$$\tilde{A}^{TNN}_{(\alpha,\beta,\gamma)} = \{x | \mu_{\tilde{A}^{TNN}}(x) \geq \alpha, \pi_{\tilde{A}^{TNN}}(x) \leq \beta, \nu_{\tilde{A}^{TNN}}(x) \leq \gamma\}. \tag{10}$$

where $0 \leq \alpha \leq \sigma$, $\rho \leq \beta \leq 1$, $\tau \leq \gamma \leq 1$ and $0 \leq \alpha + \beta + \gamma \leq 3$.

Definition 2.5. The α- cut of a single valued triangular neutrosophic number $\tilde{A}^{TNN} = \langle (\xi, \eta, \zeta); \sigma, \rho, \tau \rangle$ is a closed crisp interval of real numbers denoted by $\tilde{A}^{TNN}_{\alpha} = \left[L_{\tilde{A}^{TNN}_{\alpha}}, R_{\tilde{A}^{TNN}_{\alpha}}\right]$ and defined as

$$\begin{aligned} \tilde{A}^{TNN}_{\alpha} &= \{x | \mu_{\tilde{A}^{TNN}}(x) \geq \alpha\} \\ &= \left[L_{\tilde{A}^{TNN}_{\alpha}}, R_{\tilde{A}^{TNN}_{\alpha}}\right] \\ &= \left[\xi + \frac{\alpha(\eta - \xi)}{\sigma}, \zeta - \frac{\alpha(\zeta - \eta)}{\sigma}\right]. \end{aligned} \tag{11}$$

Definition 2.6. The β- cut of a single valued triangular neutrosophic number $\tilde{A}^{TNN} = \langle (\xi, \eta, \zeta); \sigma, \rho, \tau \rangle$ is a closed crisp interval of real numbers denoted by $\tilde{A}^{TNN}_{\beta} = \left[L_{\tilde{A}^{TNN}_{\beta}}, R_{\tilde{A}^{TNN}_{\beta}}\right]$ and defined as

$$\begin{aligned} \tilde{A}^{TNN}_{\beta} &= \{x | \pi_{\tilde{A}^{TNN}}(x) \leq \beta\} \\ &= \left[L_{\tilde{A}^{TNN}_{\beta}}, R_{\tilde{A}^{TNN}_{\beta}}\right] \\ &= \left[\frac{(1-\beta)\eta + (\beta - \rho)\xi}{(1-\rho)}, \frac{(1-\beta)\eta + (\beta - \rho)\zeta}{(1-\rho)}\right]. \end{aligned} \tag{12}$$

Definition 2.7. The γ- cut of a single valued triangular neutrosophic number $\tilde{A}^{TNN} = \langle (\xi, \eta, \zeta); \sigma, \rho, \tau \rangle$ is a closed crisp interval of real numbers denoted by $\tilde{A}^{TNN}_{\gamma} = \left[L_{\tilde{A}^{TNN}_{\gamma}}, R_{\tilde{A}^{TNN}_{\gamma}}\right]$ and defined as

$$\begin{aligned} \tilde{A}^{TNN}_{\gamma} &= \{x | \nu_{\tilde{A}^{TNN}}(x) \leq \gamma\} \\ &= \left[L_{\tilde{A}^{TNN}_{\gamma}}, R_{\tilde{A}^{TNN}_{\gamma}}\right] \\ &= \left[\frac{(1-\gamma)\eta + (\gamma - \tau)\xi}{(1-\tau)}, \frac{(1-\gamma)\eta + (\gamma - \tau)\zeta}{(1-\tau)}\right]. \end{aligned} \tag{13}$$

Theorem 2.8. *Let $\tilde{A}^{TNN} = \langle (\xi, \eta, \zeta); \sigma, \rho, \tau \rangle$ be any single valued triangular neutrosophic number then for any $\alpha \in [0, \sigma]$, $\beta \in [\rho, 1]$ and $\gamma \in [\tau, 1]$ the following equality hold*

$$\tilde{A}^{TNN}_{(\alpha,\beta,\gamma)} = \tilde{A}^{TNN}_{\alpha} \cap \tilde{A}^{TNN}_{\beta} \cap \tilde{A}^{TNN}_{\gamma}.$$

3. Values and ambiguities for the membership functions of $\tilde{A}^{TNN}$

Let $\tilde{A}^{TNN}_{\alpha}$, $\tilde{A}^{TNN}_{\beta}$ and $\tilde{A}^{TNN}_{\gamma}$ be the α-cut, β-cut and γ-cut of a single valued triangular neutrosophic number $\tilde{A}^{TNN}$ respectively, then the values and ambiguities for different membership functions of $\tilde{A}^{TNN}$ are defined as follows:

Value of the true membership function:

$$V_{\mu}(\tilde{A}^{TNN}) = \int_{0}^{\sigma} \frac{\left(L_{\tilde{A}_{\alpha}^{TNN}} + R_{\tilde{A}_{\alpha}^{TNN}}\right)}{2} \phi(\alpha)\, d\alpha. \tag{14}$$

Value of the indeterminacy membership function:

$$V_{\pi}(\tilde{A}^{TNN}) = \int_{\rho}^{1} \frac{\left(L_{\tilde{A}_{\beta}^{TNN}} + R_{\tilde{A}_{\beta}^{TNN}}\right)}{2} \psi(\beta)\, d\beta. \tag{15}$$

Value of the falsity membership function:

$$V_{\nu}(\tilde{A}^{TNN}) = \int_{\tau}^{1} \frac{\left(L_{\tilde{A}_{\gamma}^{TNN}} + R_{\tilde{A}_{\gamma}^{TNN}}\right)}{2} \chi(\gamma)\, d\gamma. \tag{16}$$

Ambiguity of the true membership function:

$$Amb_{\mu}(\tilde{A}^{TNN}) = \int_{0}^{\sigma} \left(R_{\tilde{A}_{\alpha}^{TNN}} - L_{\tilde{A}_{\alpha}^{TNN}}\right) \phi(\alpha)\, d\alpha. \tag{17}$$

Ambiguity of the indeterminacy membership function:

$$Amb_{\pi}(\tilde{A}^{TNN}) = \int_{\rho}^{1} \left(R_{\tilde{A}_{\beta}^{TNN}} - L_{\tilde{A}_{\beta}^{TNN}}\right) \psi(\beta)\, d\beta. \tag{18}$$

Ambiguity of the falsity membership function:

$$Amb_{\nu}(\tilde{A}^{TNN}) = \int_{\tau}^{1} \left(R_{\tilde{A}_{\gamma}^{TNN}} - L_{\tilde{A}_{\gamma}^{TNN}}\right) \chi(\gamma)\, d\gamma. \tag{19}$$

Here $\phi(\alpha)$ is a nonnegative increasing function defined on $[0, \sigma]$ with $\phi(0) = 0$ and $\int_0^{\sigma} \phi(\alpha)\, d\alpha = \sigma$. $\psi(\beta)$ is a nonnegative decreasing function defined on $[\rho, 1]$ with $\psi(1) = 0$ and $\int_{\rho}^{1} \psi(\beta)\, d\beta = 1 - \rho$ and $\chi(\gamma)$ is a nonnegative decreasing function defined on $[\tau, 1]$ with $\chi(1) = 0$ and $\int_{\tau}^{1} \chi(\gamma)\, d\gamma = 1 - \tau$.

According to the equations (11), (14) and suitable nonnegative functions $\phi(\alpha)$, $\psi(\beta)$ and $\chi(\gamma)$ as $\phi(\alpha) = \frac{2\alpha}{\sigma}$, $\psi(\beta) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi(\gamma) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the value of true membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned} V_{\mu}(\tilde{A}^{TNN}) &= \int_{0}^{\sigma} \frac{\alpha\left(\xi + \frac{\alpha(\eta-\xi)}{\sigma} + \zeta - \frac{\alpha(\zeta-\eta)}{\sigma}\right)}{\sigma} d\alpha \\ &= \frac{1}{\sigma^2} \int_{0}^{\sigma} \left[\sigma(\xi+\zeta) + \alpha(2\eta - \xi - \zeta)\right] \alpha d\alpha \\ &= \frac{1}{\sigma^2} \left[\frac{\sigma^3(\xi+\zeta)}{2} + \frac{\sigma^3(2\eta-\xi-\zeta)}{3}\right] \\ &= \frac{(\xi + 4\eta + \zeta)\sigma}{6}. \end{aligned} \tag{20}$$

According to the equations (12), (15) and suitable nonnegative functions $\phi(\alpha)$, $\psi(\beta)$ and $\chi(\gamma)$ as $\phi(\alpha) = \frac{2\alpha}{\sigma}$, $\psi(\beta) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi(\gamma) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the value of indeterminacy membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned} V_\pi(\tilde{A}^{TNN}) = & \int_\rho^1 \frac{(1-\beta)\left(\frac{(1-\beta)\eta+(\beta-\rho)\xi}{(1-\rho)} + \frac{(1-\beta)\eta+(\beta-\rho)\zeta}{(1-\rho)}\right)}{(1-\rho)} d\beta \\ = & \frac{1}{(1-\rho)^2} \int_\rho^1 \left[2\eta(1-\beta) + (\beta-\rho)(\xi+\zeta)\right](1-\beta)\, d\beta \\ = & \frac{1}{(1-\rho)^2} \int_\rho^1 \left[(1-\beta)^2(2\eta-\xi-\zeta) + (1-\rho)(\xi+\zeta)(1-\beta)\right] d\beta \\ = & \frac{1}{(1-\rho)^2} \left[\frac{(1-\rho)^3(2\eta-\xi-\zeta)}{3} + \frac{(1-\rho)^3(\xi+\zeta)}{2}\right] \\ = & \frac{(\xi+4\eta+\zeta)(1-\rho)}{6}. \end{aligned} \tag{21}$$

According to the equations (13), (16) and suitable nonnegative functions $\phi(\alpha)$, $\psi(\beta)$ and $\chi(\gamma)$ as $\phi(\alpha) = \frac{2\alpha}{\sigma}$, $\psi(\beta) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi(\gamma) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the value of falsity membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned} V_\nu(\tilde{A}^{TNN}) = & \int_\tau^1 \frac{(1-\gamma)\left(\frac{(1-\gamma)\eta+(\gamma-\tau)\xi}{(1-\tau)} + \frac{(1-\gamma)\eta+(\gamma-\tau)\zeta}{(1-\tau)}\right)}{(1-\tau)} d\gamma \\ = & \frac{1}{(1-\tau)^2} \int_\tau^1 \left[2\eta(1-\gamma) + (\gamma-\tau)(\xi+\zeta)\right](1-\gamma)\, d\gamma \\ = & \frac{1}{(1-\tau)^2} \int_\tau^1 \left[(1-\gamma)^2(2\eta-\xi-\zeta) + (1-\tau)(\xi+\zeta)(1-\gamma)\right] d\gamma \\ = & \frac{1}{(1-\tau)^2} \left[\frac{(1-\tau)^3(2\eta-\xi-\zeta)}{3} + \frac{(1-\tau)^3(\xi+\zeta)}{2}\right] \\ = & \frac{(\xi+4\eta+\zeta)(1-\tau)}{6}. \end{aligned} \tag{22}$$

According to the equations (11), (17) and suitable nonnegative functions $\phi(\alpha)$, $\psi(\beta)$ and $\chi(\gamma)$ as $\phi(\alpha) = \frac{2\alpha}{\sigma}$, $\psi(\beta) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi(\gamma) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the ambiguity of true membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned} Amb_\mu(\tilde{A}^{TNN}) = & \int_0^\sigma \frac{2\alpha\left(\zeta - \frac{\alpha(\zeta-\eta)}{\sigma} - \xi - \frac{\alpha(\eta-\xi)}{\sigma} +\right)}{\sigma} d\alpha \\ = & \frac{1}{\sigma^2} \int_0^\sigma 2\alpha(\zeta-\xi)(\sigma-\alpha)\, d\alpha \\ = & \frac{2}{\sigma^2}(\zeta-\xi)\sigma^3\left(\frac{1}{2}-\frac{1}{3}\right) \\ = & \frac{(\zeta-\xi)\sigma}{3}. \end{aligned} \tag{23}$$

According to the equations (12), (18) and suitable nonnegative functions $\phi(\alpha)$, $\psi(\beta)$ and $\chi(\gamma)$ as $\phi(\alpha) = \frac{2\alpha}{\sigma}$, $\psi(\beta) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi(\gamma) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the ambiguity of indeterminacy membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned}Amb_\pi(\tilde{A}^{TNN}) &= \int_\rho^1 \frac{2\left(1-\beta\right)\left[\frac{(1-\beta)\eta+(\beta-\rho)\zeta}{(1-\rho)} - \frac{(1-\beta)\eta+(\beta-\rho)\xi}{(1-\rho)}\right]}{(1-\rho)} d\beta \\ &= \frac{2}{\left(1-\rho\right)^2}\int_\rho^1 \left(\beta-\rho\right)\left(\zeta-\xi\right)\left(1-\beta\right) d\beta \\ &= \frac{2\left(\zeta-\xi\right)}{\left(1-\rho\right)^2}\int_\rho^1 \left[\left(1-\rho\right)\left(1-\beta\right) - \left(1-\beta\right)^2\right] d\beta \\ &= \frac{2\left(\zeta-\xi\right)\left(1-\rho\right)^3}{\left(1-\rho\right)^2}\left(\frac{1}{2}-\frac{1}{3}\right) \\ &= \frac{\left(\zeta-\xi\right)\left(1-\rho\right)}{3}.\end{aligned} \tag{24}$$

According to the equations (13), (19) and suitable nonnegative functions $\phi\left(\alpha\right)$, $\psi\left(\beta\right)$ and $\chi\left(\gamma\right)$ as $\phi\left(\alpha\right) = \frac{2\alpha}{\sigma}$, $\psi\left(\beta\right) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi\left(\gamma\right) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the ambiguity of falsity membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned}Amb_\nu(\tilde{A}^{TNN}) &= \int_\tau^1 \frac{2\left(1-\gamma\right)\left[\frac{(1-\gamma)\eta+(\gamma-\tau)\zeta}{(1-\tau)} - \frac{(1-\gamma)\eta+(\gamma-\tau)\xi}{(1-\tau)}\right]}{(1-\tau)} d\gamma \\ &= \frac{2}{\left(1-\tau\right)^2}\int_\tau^1 \left(\gamma-\tau\right)\left(\zeta-\xi\right)\left(1-\gamma\right) d\gamma \\ &= \frac{2\left(\zeta-\xi\right)}{\left(1-\tau\right)^2}\int_\tau^1 \left[\left(1-\tau\right)\left(1-\gamma\right) - \left(1-\gamma\right)^2\right] d\gamma \\ &= \frac{2\left(\zeta-\xi\right)\left(1-\tau\right)^3}{\left(1-\tau\right)^2}\left(\frac{1}{2}-\frac{1}{3}\right) \\ &= \frac{\left(\zeta-\xi\right)\left(1-\tau\right)}{3}.\end{aligned} \tag{25}$$

Theorem 3.1. *Let $\tilde{A}^{TNN} = \langle\left(\xi_1, \eta_1, \zeta_1\right); \sigma_1, \rho_1, \tau_1\rangle$ and $\tilde{B}^{TNN} = \langle\left(\xi_2, \eta_2, \zeta_2\right); \sigma_2, \rho_2, \tau_2\rangle$ be two single valued triangular neutrosophic numbers with $\sigma_1 = \sigma_2$, $\rho_1 = \rho_2$ and $\tau_1 = \tau_2$ then the following equalities hold*

(1) $V_\mu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) = V_\mu\left(\tilde{A}^{TNN}\right) + V_\mu\left(\tilde{B}^{TNN}\right)$.
(2) $V_\pi\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) = V_\pi\left(\tilde{A}^{TNN}\right) + V_\pi\left(\tilde{B}^{TNN}\right)$.
(3) $V_\nu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) = V_\nu\left(\tilde{A}^{TNN}\right) + V_\nu\left(\tilde{B}^{TNN}\right)$.

Proof. Using Definition 2.3 and according to the given statement, we have

$$\tilde{A}^{TNN} + \tilde{B}^{TNN} = \langle\left(\xi_1+\xi_2, \eta_1+\eta_2, \zeta_1+\zeta_2\right); \sigma_1, \rho_1, \tau_1\rangle.$$

Thus by the definition of value of true membership function, we obtain

$$\begin{aligned}V_\mu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) &= \frac{\left[\left(\xi_1+\xi_2\right) + 4\left(\eta_1+\eta_2\right) + \left(\zeta_1+\zeta_2\right)\right]\sigma_1}{6} \\ &= \frac{\left(\xi_1+4\eta_1+\zeta_1\right)\sigma_1}{6} + \frac{\left(\xi_2+4\eta_2+\zeta_2\right)\sigma_2}{6}\end{aligned}$$

Therefore,

$$V_\mu\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)=V_\mu\left(\tilde{A}^{TNN}\right)+V_\mu\left(\tilde{B}^{TNN}\right).$$

In a similar manner the remaining results of the theorem can also be proved. □

Theorem 3.2. *Let* $\tilde{A}^{TNN}=\langle(\xi_1,\eta_1,\zeta_1);\sigma_1,\rho_1,\tau_1\rangle$ *and* $\tilde{B}^{TNN}=\langle(\xi_2,\eta_2,\zeta_2);\sigma_2,\rho_2,\tau_2\rangle$ *be two single valued triangular neutrosophic numbers with* $\sigma_1=\sigma_2$, $\rho_1=\rho_2$ *and* $\tau_1=\tau_2$ *then the following equalities hold*

(1) $Amb_\mu\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)=Amb_\mu\left(\tilde{A}^{TNN}\right)+Amb_\mu\left(\tilde{B}^{TNN}\right)$.
(2) $Amb_\pi\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)=Amb_\pi\left(\tilde{A}^{TNN}\right)+Amb_\pi\left(\tilde{B}^{TNN}\right)$.
(3) $Amb_\nu\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)=Amb_\nu\left(\tilde{A}^{TNN}\right)+Amb_\nu\left(\tilde{B}^{TNN}\right)$.

Proof. Using Definition 2.3 and according to the given statement, we have

$$\tilde{A}^{TNN}+\tilde{B}^{TNN}=\langle(\xi_1+\xi_2,\eta_1+\eta_2,\zeta_1+\zeta_2);\sigma_1,\rho_1,\tau_1\rangle.$$

Thus by the definition of ambiguity of true membership function, we obtain

$$\begin{aligned}Amb_\mu\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)&=\frac{[(\zeta_1+\zeta_2)-(\xi_1+\xi_2)]\,\sigma_1}{3}\\&=\frac{(\zeta_1-\xi_1)\,\sigma_1}{3}+\frac{(\zeta_2-\xi_2)\,\sigma_2}{3}\end{aligned}$$

Therefore,

$$Amb_\mu\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)=Amb_\mu\left(\tilde{A}^{TNN}\right)+Amb_\mu\left(\tilde{B}^{TNN}\right).$$

In a similar manner the remaining results of the theorem can also be proved. □

4. Value index and ambiguity index of $\tilde{A}^{TNN}$

Let $\tilde{A}^{TNN}=\langle(\xi,\eta,\zeta);\sigma,\rho,\tau\rangle$ be a single valued triangular neutrosophic number then the value index and the ambiguity index for $\tilde{A}^{TNN}$ are defined as follows:

(1) Value Index:

$$V\left(\tilde{A}^{TNN},\lambda\right)=V_\pi(\tilde{A}^{TNN})+V_\mu(\tilde{A}^{TNN})+\lambda\left[V_\nu(\tilde{A}^{TNN})-V_\mu(\tilde{A}^{TNN})\right]. \quad (26)$$

(2) Ambiguity Index:

$$A\left(\tilde{A}^{TNN},\lambda\right)=Amb_\pi(\tilde{A}^{TNN})+Amb_\nu(\tilde{A}^{TNN})-\lambda\left[Amb_\nu(\tilde{A}^{TNN})-Amb_\mu(\tilde{A}^{TNN})\right]. \quad (27)$$

which are continuous non decreasing and non increasing functions of the parameter λ respectively. Here $\lambda\in[0,1]$ represents the decision maker's preference informations. $\lambda\in[0,\frac{1}{2})$ represents that the decision maker prefer uncertainty or negative feeling. $\lambda\in(\frac{1}{2},1]$ represents that the decision maker prefer certainty or positive feeling. $\lambda=\frac{1}{2}$ represents that the decision maker is indifferent between positive and negative feeling.

Theorem 4.1. *Let* $\tilde{A}^{TNN} = \langle (\xi_1, \eta_1, \zeta_1); \sigma_1, \rho_1, \tau_1 \rangle$ *and* $\tilde{B}^{TNN} = \langle (\xi_2, \eta_2, \zeta_2); \sigma_2, \rho_2, \tau_2 \rangle$ *be two single valued triangular neutrosophic numbers with* $\sigma_1 = \sigma_2$, $\rho_1 = \rho_2$ *and* $\tau_1 = \tau_2$ *then the following equalities hold*

(1) $V\left(\tilde{A}^{TNFN} + \tilde{B}^{TNN}, \lambda\right) = V\left(\tilde{A}^{TNN}, \lambda\right) + V\left(\tilde{B}^{TNN}, \lambda\right)$.
(2) $A\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}, \lambda\right) = A\left(\tilde{A}^{TNN}, \lambda\right) + A\left(\tilde{B}^{TNN}, \lambda\right)$.

Proof. According to equation (26), we can write

$$\begin{aligned} V\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}, \lambda\right) =& V_\pi\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) + V_\mu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) \\ &+ \lambda\left[V_\nu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) - V_\mu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right)\right]. \end{aligned}$$

Using Theorem 3.2, we obtain

$$\begin{aligned} V\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}, \lambda\right) =& V_\pi\left(\tilde{A}^{TNN}\right) + V_\pi\left(\tilde{B}^{TNN}\right) + V_\mu\left(\tilde{A}^{TNN}\right) + V_\mu\left(\tilde{B}^{TNN}\right) \\ &+ \lambda\left[V_\nu\left(\tilde{A}^{TNN}\right) + V_\nu\left(\tilde{B}^{TNN}\right) - V_\mu\left(\tilde{A}^{TNN}\right) - V_\mu\left(\tilde{B}^{TNN}\right)\right] \\ =& \left[V_\pi\left(\tilde{A}^{TNN}\right) + V_\mu\left(\tilde{A}^{TNN}\right) + \lambda\left(V_\nu\left(\tilde{A}^{TNN}\right) - V_\mu\left(\tilde{A}^{TNN}\right)\right)\right] \\ &+ \left[V_\pi\left(\tilde{B}^{TNN}\right) + V_\mu\left(\tilde{B}^{TNN}\right) + \lambda\left(V_\nu\left(\tilde{B}^{TNN}\right) - V_\mu\left(\tilde{B}^{TNN}\right)\right)\right] \\ =& V\left(\tilde{A}^{TNN}, \lambda\right) + V\left(\tilde{B}^{TNN}, \lambda\right). \end{aligned}$$

This completes the first part of the theorem.

Now, according to equation (27), we can write

$$\begin{aligned} A\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}, \lambda\right) =& Amb_\pi\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) + Amb_\nu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) \\ &- \lambda\left[Amb_\nu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) - Amb_\mu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right)\right]. \end{aligned}$$

Using Theorem 3.3, we have

$$\begin{aligned} A\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}, \lambda\right) =& Amb_\pi\left(\tilde{A}^{TNN}\right) + Amb_\pi\left(\tilde{B}^{TNN}\right) + Amb_\nu\left(\tilde{A}^{TNN}\right) + Amb_\nu\left(\tilde{B}^{TNN}\right) \\ &- \lambda\left[Amb_\nu\left(\tilde{A}^{TNN}\right) + Amb_\nu\left(\tilde{B}^{TNN}\right) - Amb_\mu\left(\tilde{A}^{TNN}\right) - Amb_\mu\left(\tilde{B}^{TNN}\right)\right] \\ =& \left[Amb_\pi\left(\tilde{A}^{TNN}\right) + Amb_\nu\left(\tilde{A}^{TNN}\right) - \lambda\left(Amb_\nu\left(\tilde{A}^{TNN}\right) - Amb_\mu\left(\tilde{A}^{TNN}\right)\right)\right] \\ &+ \left[Amb_\pi\left(\tilde{B}^{TNN}\right) + Amb_\nu\left(\tilde{B}^{TNN}\right) - \lambda\left(Amb_\nu\left(\tilde{B}^{TNN}\right) - Amb_\mu\left(\tilde{B}^{TNN}\right)\right)\right] \\ =& A\left(\tilde{A}^{TNN}, \lambda\right) + A\left(\tilde{B}^{TNN}, \lambda\right). \end{aligned}$$

This completes the second part of the theorem. □

Remark 4.2. It is easily seen that the value index and the ambiguity index are nonnegative for a nonnegative single valued triangular neutrosophic number, i.e., $V\left(\tilde{A}^{TNN}, \lambda\right) \geq 0$

and $A\left(\tilde{A}^{TNN}, \lambda\right) \geq 0$. Also the value index should be maximized and the ambiguity index should be minimized, furthermore as a summarized result we can easily seen that the relations $\max V\left(\tilde{A}^{TNN}, \lambda\right) = V_{\pi}(\tilde{A}^{TNN}) + V_{\nu}(\tilde{A}^{TNN})$ and $\min A\left(\tilde{A}^{TNN}, \lambda\right) = Amb_{\pi}(\tilde{A}^{TNN}) + Amb_{\mu}(\tilde{A}^{TNN})$ holds.

Remark 4.3. If we assume that the decision maker is indifferent between the certainty and uncertainty, i.e., $\lambda = \frac{1}{2}$, then the value index and ambiguity index are given by

$$V\left(\tilde{A}^{TNN}, \frac{1}{2}\right) \equiv V\left(\tilde{A}^{TNN}\right) = V_{\pi}(\tilde{A}^{TNN}) + \frac{1}{2}\left[V_{\mu}(\tilde{A}^{TNN}) + V_{\nu}(\tilde{A}^{TNN})\right]. \tag{28}$$

$$A\left(\tilde{A}^{TNN}, \frac{1}{2}\right) \equiv A\left(\tilde{A}^{TNN}\right) = Amb_{\pi}(\tilde{A}^{TNN}) + \frac{1}{2}\left[Amb_{\mu}(\tilde{A}^{TNN}) + Amb_{\nu}(\tilde{A}^{TNN})\right]. \tag{29}$$

Theorem 4.4. *Let $\tilde{A}^{TNN} = \langle(\xi_1, \eta_1, \zeta_1); \sigma_1, \rho_1, \tau_1\rangle$ and $\tilde{B}^{TNN} = \langle(\xi_2, \eta_2, \zeta_2); \sigma_2, \rho_2, \tau_2\rangle$ be two single valued triangular neutrosophic numbers and λ_1, λ_2 be any two nonnegative real numbers then the following equalities hold*

(1) $V_{\mu}\left(\lambda_1\tilde{A}^{TNN} + \lambda_2\tilde{B}^{TNN}\right) = \min(\sigma_1, \sigma_2)\left[\lambda_1\frac{V_{\mu}(\tilde{A}^{TNN})}{\sigma_1} + \lambda_2\frac{V_{\mu}(\tilde{B}^{TNN})}{\sigma_2}\right]$.

(2) $V_{\pi}\left(\lambda_1\tilde{A}^{TNN} + \lambda_2\tilde{B}^{TNN}\right) = \min(1-\rho_1, 1-\rho_2)\left[\lambda_1\frac{V_{\pi}(\tilde{A}^{TNN})}{(1-\rho_1)} + \lambda_2\frac{V_{\pi}(\tilde{B}^{TNN})}{(1-\rho_2)}\right]$.

(3) $V_{\nu}\left(\lambda_1\tilde{A}^{TNN} + \lambda_2\tilde{B}^{TNN}\right) = \min(1-\tau_1, 1-\tau_2)\left[\lambda_1\frac{V_{\nu}(\tilde{A}^{TNN})}{(1-\tau_1)} + \lambda_2\frac{V_{\nu}(\tilde{B}^{TNN})}{(1-\tau_2)}\right]$.

(4) $Amb_{\mu}\left(\lambda_1\tilde{A}^{TNN} + \lambda_2\tilde{B}^{TNN}\right) = \min(\sigma_1, \sigma_2)\left[\lambda_1\frac{Amb_{\mu}(\tilde{A}^{TNN})}{\sigma_1} + \lambda_2\frac{Amb_{\mu}(\tilde{B}^{TNN})}{\sigma_2}\right]$.

(5) $Amb_{\pi}\left(\lambda_1\tilde{A}^{TNN} + \lambda_2\tilde{B}^{TNN}\right)$
$= \min(1-\rho_1, 1-\rho_2)\left[\lambda_1\frac{Amb_{\pi}(\tilde{A}^{TNN})}{(1-\rho_1)} + \lambda_2\frac{Amb_{\pi}(\tilde{B}^{TNN})}{(1-\rho_2)}\right]$.

(6) $Amb_{\nu}\left(\lambda_1\tilde{A}^{TNN} + \lambda_2\tilde{B}^{TNN}\right)$
$= \min(1-\tau_1, 1-\tau_2)\left[\lambda_1\frac{Amb_{\nu}(\tilde{A}^{TNN})}{(1-\tau_1)} + \lambda_2\frac{Amb_{\nu}(\tilde{B}^{TNN})}{(1-\tau_2)}\right]$.

Proof. The above results can be easily proven by using Definition 2.3 and the equations (20) to (25). □

4.1. *De-neutrosophication*

Let $N(R)$ be the set of all single valued triangular neutrosophic numbers defined on the set of real numbers, then a linear de-neutrosophication function $F : N(R) \to R$ for single valued triangular neutrosophic numbers in terms of value index and ambiguity index can be defined as follows

$$F\left(\tilde{A}^{TNN}\right) = V\left(\tilde{A}^{TNN}\right) - A\left(\tilde{A}^{TNN}\right). \tag{30}$$

5. Mathematical model of a matrix game

A two person zero sum matrix game played by a maximizing player (as player I) and a minimizing player (as player II) having the pure strategy $i = \{1, 2, \ldots, m\}$ and $j = \{1, 2, \ldots, n\}$ respectively is denoted by $[a_{ij}]_{m\times n}$. Here a_{ij} is the pay-off value for the player I and its opposite is the pay-off value for player II, when they choose the strategies i and j respectively such that there exists the saddle point of the game. If the matrix game $[a_{ij}]_{m\times n}$ has no saddle point, i.e., $\max_i\{\min_j\{a_{ij}\}\} \neq \min_j\{\max_i\{a_{ij}\}\}$, then to solve such matrix games we adopt the mixed strategy sets S_1 and S_2 for the player I and II respectively, as $S_1 = \{X = (x_1, x_2, \ldots, x_m) \in R^m : x_i \geq 0, \forall i = 1, 2, \ldots m, and \sum_{i=1}^{m} x_i = 1\}$ and $S_2 = \{Y = (y_1, y_2, \ldots, y_n) \in R^n : y_j \geq 0, \forall j = 1, 2, \ldots n, and \sum_{j=1}^{n} y_j = 1\}$.

5.1. *Mathematical model of a neutrosophic matrix game*

The maximin and minimax principal for matrix games states that the player I choose such a strategy which maximize his minimum expected gain and the player II choose such a strategy which minimizes his maximum expected loss, thus for the neutrosophic matrix game, we have as

For player I

$$\begin{cases} \max\limits_{x_i}\{\min\{\sum_{i=1}^{m} \tilde{a}_{i1}^{TNN} x_i, \sum_{i=1}^{m} \tilde{a}_{i2}^{TNN} x_i, \ldots, \sum_{i=1}^{m} \tilde{a}_{in}^{TNN} x_i\}\} \\ \text{s.t.}, \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \ldots, m \end{cases} \tag{31}$$

For player II

$$\begin{cases} \min\limits_{y_j}\{\max\{\sum_{j=1}^{n} \tilde{a}_{1j}^{TNN} y_j, \sum_{j=1}^{n} \tilde{a}_{2j}^{TNN} y_j, \ldots, \sum_{j=1}^{n} \tilde{a}_{mj}^{TNN} y_j\}\} \\ \text{s.t.}, \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \geq 0, \forall j = 1, 2, \ldots, n \end{cases} \tag{32}$$

Now, let $\min\{\sum_{i=1}^{m} \tilde{a}_{i1}^{TNN} x_i, \sum_{i=1}^{m} \tilde{a}_{i2}^{TNN} x_i, \ldots, \sum_{i=1}^{m} \tilde{a}_{in}^{TNN} x_i\} = \tilde{u}^{TNN}$ be the expected minimum gain for player I and $\max\{\sum_{j=1}^{n} \tilde{a}_{1j}^{TNN} y_j, \sum_{j=1}^{n} \tilde{a}_{2j}^{TNN} y_j, \ldots, \sum_{j=1}^{n} \tilde{a}_{mj}^{TNN} y_j\} = \tilde{v}^{TNN}$ be the expected maximum loss for player II. Then the problems (31) and (32) can be written as

For player I

$$\begin{cases} \max \tilde{u}^{TNN} \\ \text{s.t.}, \sum_{i=1}^{m} \tilde{a}_{i1}^{TNN} x_i \succeq \tilde{u}^{TNN} \\ \sum_{i=1}^{m} \tilde{a}_{i2}^{TNN} x_i \succeq \tilde{u}^{TNN} \\ \ldots \\ \sum_{i=1}^{m} \tilde{a}_{in}^{TNN} x_i \succeq \tilde{u}^{TNN} \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \ldots, m \end{cases} \tag{33}$$

For player II

$$\begin{cases} \min \tilde{v}^{TNN} \\ \text{s.t.}, \sum_{j=1}^{n} \tilde{a}_{1j}^{TNN} y_j \preceq \tilde{v}^{TNN} \\ \sum_{j=1}^{n} \tilde{a}_{2j}^{TNN} y_j \preceq \tilde{v}^{TNN} \\ \dots \\ \sum_{j=1}^{n} \tilde{a}_{mj}^{TNN} y_j \preceq \tilde{v}^{TNN} \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \geq 0, \forall j = 1, 2, \dots, n \end{cases} \tag{34}$$

Here $\tilde{u}^{TNN} = \langle (u_1, u_2, u_3); \sigma, \rho, \tau \rangle$ and $\tilde{v}^{TNN} = \langle (v_1, v_2, v_3); \sigma', \rho', \tau' \rangle$ are the single valued triangular neutrosophic numbers as expected minimum gain and expected maximum loss respectively. And $\succeq$ and $\preceq$ denotes the neutrosophic versions of the order relation $\geq$ and $\leq$ on the set of real numbers and has linguistic interpretation as 'essentially greater than or equal' and 'essentially less than or equal' respectively. The problems (33) and (34) are known as the neutrosophic linear programming problems for the player I and II respectively and can be written in the standard form as

For player I

$$\begin{cases} \max \tilde{u}^{TNN} \\ \text{s.t.}, \sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i \succeq \tilde{u}^{TNN}, \forall j = 1, 2, \dots, n \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \dots, m \end{cases} \tag{35}$$

For player II

$$\begin{cases} \min \tilde{v}^{TNN} \\ \text{s.t.}, \sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j \preceq \tilde{v}^{TNN}, \forall i = 1, 2, \dots, m \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \geq 0, \forall j = 1, 2, \dots, n \end{cases} \tag{36}$$

Now, utilizing the de-neutrosophication function $F : N(R) \to R$ defined by the equation (30), the above neutrosophic linear programming problems (35) and (36) can be transformed into the crisp linear programming problems for the player I and II respectively as follows

For player I

$$\begin{cases} \max F\left(\tilde{u}^{TNN}\right) \\ \text{s.t.}, F\left(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i\right) \geq F\left(\tilde{u}^{TNN}\right), \forall j = 1, 2, \dots, n \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \dots, m \end{cases} \tag{37}$$

For Player II

$$\begin{cases} \min F\left(\tilde{v}^{TNN}\right) \\ \text{s.t.}, F\left(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j\right) \leq F\left(\tilde{v}^{TNN}\right), \forall i = 1, 2, \ldots, m \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \geq 0, \forall j = 1, 2, \ldots, n \end{cases} \tag{38}$$

Using equations (28) to (30) the above crisp linear programming problems (37) and (38) for player I and II respectively, can be written as

For player I

$$\begin{cases} \max V_\pi(\tilde{u}^{TNN}) + \frac{1}{2}\left[V_\mu(\tilde{u}^{TNN}) + V_\nu(\tilde{u}^{TNN})\right] - Amb_\pi(\tilde{u}^{TNN}) \\ -\frac{1}{2}\left[Amb_\mu(\tilde{u}^{TNN}) + Amb_\nu(\tilde{u}^{TNN})\right] \\ \text{s.t.}, V_\pi(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) + \frac{1}{2}\left[V_\mu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) + V_\nu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i)\right] \\ -Amb_\pi(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) - \frac{1}{2}\left[Amb_\mu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) + Amb_\nu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i)\right] \\ \geq V_\pi(\tilde{u}^{TNN}) + \frac{1}{2}\left[V_\mu(\tilde{u}^{TNN}) + V_\nu(\tilde{u}^{TNN})\right] - Amb_\pi(\tilde{u}^{TNN}) \\ -\frac{1}{2}\left[Amb_\mu(\tilde{u}^{TNN}) + Amb_\nu(\tilde{u}^{TNN})\right], \forall j = 1, 2, \ldots, n \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \ldots, m \end{cases} \tag{39}$$

For player II

$$\begin{cases} \min V_\pi(\tilde{v}^{TNN}) + \frac{1}{2}\left[V_\mu(\tilde{v}^{TNN}) + V_\nu(\tilde{v}^{TNN})\right] - Amb_\pi(\tilde{v}^{TNN}) \\ -\frac{1}{2}\left[Amb_\mu(\tilde{v}^{TNN}) + Amb_\nu(\tilde{v}^{TNN})\right] \\ \text{s.t.}, V_\pi(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) + \frac{1}{2}\left[V_\mu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) + V_\nu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j)\right] \\ -Amb_\pi(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) - \frac{1}{2}\left[Amb_\mu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) + Amb_\nu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j)\right] \\ \leq V_\pi(\tilde{v}^{TNN}) + \frac{1}{2}\left[V_\mu(\tilde{v}^{TNN}) + V_\nu(\tilde{v}^{TNN})\right] - Amb_\pi(\tilde{v}^{TNN}) \\ -\frac{1}{2}\left[Amb_\mu(\tilde{v}^{TNN}) + Amb_\nu(\tilde{v}^{TNN})\right], \forall i = 1, 2, \ldots, m \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \geq 0, \forall j = 1, 2, \ldots, n \end{cases} \tag{40}$$

The problems (39) and (40) can also be reformed as

For player I

$$\begin{cases} \max V_\pi(\tilde{u}^{TNN}) - Amb_\pi(\tilde{u}^{TNN}) \\ +\frac{1}{2}\left[V_\mu(\tilde{u}^{TNN}) - Amb_\mu(\tilde{u}^{TNN}) + V_\nu(\tilde{u}^{TNN}) - Amb_\nu(\tilde{u}^{TNN})\right] \\ \text{s.t.}, V_\pi(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) - Amb_\pi(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) \\ +\frac{1}{2}\left[V_\mu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) - Amb_\mu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i)\right] \\ +\frac{1}{2}\left[V_\nu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) - Amb_\nu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i)\right] \geq V_\pi(\tilde{u}^{TNN}) - Amb_\pi(\tilde{u}^{TNN}) \\ +\frac{1}{2}\left[V_\mu(\tilde{u}^{TNN}) - Amb_\mu(\tilde{u}^{TNN}) + V_\nu(\tilde{u}^{TNN}) - Amb_\nu(\tilde{u}^{TNN})\right], \forall j = 1, 2, \ldots, n \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \ldots, m \end{cases} \tag{41}$$

For player II

$$\begin{cases} \min V_\pi(\tilde{v}^{TNN}) - Amb_\pi(\tilde{v}^{TNN}) \\ +\frac{1}{2}\left[V_\mu(\tilde{v}^{TNN}) - Amb_\mu(\tilde{v}^{TNN}) + V_\nu(\tilde{v}^{TNN}) - Amb_\nu(\tilde{v}^{TNN})\right] \\ \text{s.t.}, V_\pi(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) - Amb_\pi(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) \\ +\frac{1}{2}\left[V_\mu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) - Amb_\mu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j)\right] \\ +\frac{1}{2}\left[V_\nu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) - Amb_\nu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j)\right] \leq V_\pi(\tilde{v}^{TNN}) - Amb_\pi(\tilde{v}^{TNN}) \\ +\frac{1}{2}\left[V_\mu(\tilde{v}^{TNN}) - Amb_\mu(\tilde{v}^{TNN}) + V_\nu(\tilde{v}^{TNN}) - Amb_\nu(\tilde{v}^{TNN})\right], \forall i = 1, 2, \ldots, m \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \geq 0, \forall j = 1, 2, \ldots, n \end{cases} \tag{42}$$

The problems (41) and (42) further can be written in the following manner by using the expected minimum gain and expected maximum loss $\tilde{u}^{TNN} = \langle (u_1, u_2, u_3); \sigma, \rho, \tau \rangle$ and $\tilde{v}^{TNN} = \langle (v_1, v_2, v_3); \sigma', \rho', \tau' \rangle$ as

For player I

$$\begin{cases} \max \frac{(u_1+4u_2+u_3)(1-\rho)}{6} - \frac{(u_3-u_1)(1-\rho)}{3} \\ +\frac{1}{2}\left[\frac{(u_1+4u_2+u_3)\sigma}{6} - \frac{(u_3-u_1)\sigma}{3} + \frac{(u_1+4u_2+u_3)(1-\tau)}{6} - \frac{(u_3-u_1)(1-\tau)}{3}\right] \\ \text{s.t.}, \min_i (1-\rho_{ij})\left(\sum_{i=1}^{m} \frac{V_\pi(\tilde{a}_{ij}^{TNN})x_i}{(1-\rho_{ij})} - \sum_{i=1}^{m} \frac{Amb_\pi(\tilde{a}_{ij}^{TNN})x_i}{(1-\rho_{ij})}\right) \\ +\frac{\min_i(\sigma_{ij})}{2}\left(\sum_{i=1}^{m} \frac{V_\mu(\tilde{a}_{ij}^{TNN})x_i}{(\sigma_{ij})} - \sum_{i=1}^{m} \frac{Amb_\mu(\tilde{a}_{ij}^{TNN})x_i}{(\sigma_{ij})}\right) \\ +\frac{\min_i(1-\tau_{ij})}{2}\left(\sum_{i=1}^{m} \frac{V_\nu(\tilde{a}_{ij}^{TNN})x_i}{(1-\tau_{ij})} - \sum_{i=1}^{m} \frac{Amb_\nu(\tilde{a}_{ij}^{TNN})x_i}{(1-\tau_{ij})}\right) \geq \frac{(u_1+4u_2+u_3)(1-\rho)}{6} - \frac{(u_3-u_1)(1-\rho)}{3} \\ +\frac{1}{2}\left[\frac{(u_1+4u_2+u_3)\sigma}{6} - \frac{(u_3-u_1)\sigma}{3} + \frac{(u_1+4u_2+u_3)(1-\tau)}{6} - \frac{(u_3-u_1)(1-\tau)}{3}\right], \forall j = 1, 2, \ldots, n \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \ldots, m \end{cases} \tag{43}$$

For player II

$$\begin{cases} \min \frac{(v_1+4v_2+v_3)(1-\rho')}{6} - \frac{(v_3-v_1)(1-\rho')}{3} \\ +\frac{1}{2}\left[\frac{(v_1+4v_2+v_3)\sigma'}{6} - \frac{(v_3-v_1)\sigma'}{3} + \frac{(v_1+4v_2+v_3)(1-\tau')}{6} - \frac{(v_3-v_1)(1-\tau')}{3}\right] \\ \text{s.t.}, \min_j (1-\rho_{ij}')\left(\sum_{j=1}^{n} \frac{V_\pi(\tilde{a}_{ij}^{TNN})y_j}{(1-\rho_{ij}')} - \sum_{j=1}^{n} \frac{Amb_\pi(\tilde{a}_{ij}^{TNN})y_j}{(1-\rho_{ij}')}\right) \\ +\frac{\min_j(\sigma_{ij}')}{2}\left(\sum_{j=1}^{n} \frac{V_\mu(\tilde{a}_{ij}^{TNN})y_j}{(\sigma_{ij}')} - \sum_{j=1}^{n} \frac{Amb_\mu(\tilde{a}_{ij}^{TNN})y_j}{(\sigma_{ij}')}\right) \\ +\frac{\min_j(1-\tau_{ij}')}{2}\left(\sum_{j=1}^{n} \frac{V_\nu(\tilde{a}_{ij}^{TNN})y_j}{(1-\tau_{ij}')} - \sum_{j=1}^{n} \frac{Amb_\nu(\tilde{a}_{ij}^{TNN})y_j}{(1-\tau_{ij}')}\right) \leq \frac{(v_1+4v_2+v_3)(1-\rho')}{6} - \frac{(v_3-v_1)(1-\rho')}{3} \\ +\frac{1}{2}\left[\frac{(v_1+4v_2+v_3)\sigma'}{6} - \frac{(v_3-v_1)\sigma'}{3} + \frac{(v_1+4v_2+v_3)(1-\tau')}{6} - \frac{(v_3-v_1)(1-\tau')}{3}\right], \forall i = 1, 2, \ldots, m \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \geq 0, \forall j = 1, 2, \ldots, n \end{cases} \tag{44}$$

For convenience, let

$$\frac{(u_1+4u_2+u_3)\,\sigma}{6}-\frac{(u_3-u_1)\,\sigma}{3}=L_1 \tag{45}$$

$$\frac{(u_1+4u_2+u_3)\,(1-\rho)}{6}-\frac{(u_3-u_1)\,(1-\rho)}{3}=M_1 \tag{46}$$

$$\frac{(u_1+4u_2+u_3)\,(1-\tau)}{6}-\frac{(u_3-u_1)\,(1-\tau)}{3}=N_1 \tag{47}$$

$$\frac{(v_1+4v_2+v_3)\,\sigma'}{6}-\frac{(v_3-v_1)\,\sigma'}{3}=L_2 \tag{48}$$

$$\frac{(v_1+4v_2+v_3)\,(1-\rho_{ij}{}')}{6}-\frac{(v_3-v_1)\,(1-\rho_{ij}{}')}{3}=M_2 \tag{49}$$

$$\frac{(v_1+4v_2+v_3)\,(1-\tau_{ij}{}')}{6}-\frac{(v_3-v_1)\,(1-\tau_{ij}{}')}{3}=N_2 \tag{50}$$

Then the problems (43) and (44) reduces as

For player I

$$\begin{cases} \max \frac{1}{2}L_1+M_1+\frac{1}{2}N_1 \\ \text{s.t.}, \min\limits_i (1-\rho_{ij})\left(\sum_{i=1}^m \frac{V_\pi(\tilde{a}_{ij}^{TNN})x_i}{(1-\rho_{ij})}-\sum_{i=1}^m \frac{Amb_\pi(\tilde{a}_{ij}^{TNN})x_i}{(1-\rho_{ij})}\right) \\ +\frac{\min\limits_i(\sigma_{ij})}{2}\left(\sum_{i=1}^m \frac{V_\mu(\tilde{a}_{ij}^{TNN})x_i}{(\sigma_{ij})}-\sum_{i=1}^m \frac{Amb_\mu(\tilde{a}_{ij}^{TNN})x_i}{(\sigma_{ij})}\right) \\ +\frac{\min\limits_i(1-\tau_{ij})}{2}\left(\sum_{i=1}^m \frac{V_\nu(\tilde{a}_{ij}^{TNN})x_i}{(1-\tau_{ij})}-\sum_{i=1}^m \frac{Amb_\nu(\tilde{a}_{ij}^{TNN})x_i}{(1-\tau_{ij})}\right) \geq \frac{1}{2}L_1+M_1+\frac{1}{2}N_1, \forall j=1,2,\ldots,n \\ \sum_{i=1}^m x_i=1 \\ \text{and}; x_i \geq 0, \forall i=1,2,\ldots,m \end{cases} \tag{51}$$

For player II

$$\begin{cases} \min \frac{1}{2}L_2+M_2+\frac{1}{2}N_2 \\ \text{s.t.}, \min\limits_j (1-\rho_{ij}{}')\left(\sum_{j=1}^n \frac{V_\pi(\tilde{a}_{ij}^{TNN})y_j}{(1-\rho_{ij}{}')}-\sum_{j=1}^n \frac{Amb_\pi(\tilde{a}_{ij}^{TNN})y_j}{(1-\rho_{ij}{}')}\right) \\ +\frac{\min\limits_j(\sigma_{ij}{}')}{2}\left(\sum_{j=1}^n \frac{V_\mu(\tilde{a}_{ij}^{TNN})y_j}{(\sigma_{ij}{}')}-\sum_{j=1}^n \frac{Amb_\mu(\tilde{a}_{ij}^{TNN})y_j}{(\sigma_{ij}{}')}\right) \\ +\frac{\min\limits_j(1-\tau_{ij}{}')}{2}\left(\sum_{j=1}^n \frac{V_\nu(\tilde{a}_{ij}^{TNN})y_j}{(1-\tau_{ij}{}')}-\sum_{j=1}^n \frac{Amb_\nu(\tilde{a}_{ij}^{TNN})y_j}{(1-\tau_{ij}{}')}\right) \leq \frac{1}{2}L_2+M_2+\frac{1}{2}N_2, \forall i=1,2,\ldots,m \\ \sum_{j=1}^n y_j=1 \\ \text{and}; y_j \geq 0, \forall j=1,2,\ldots,n \end{cases} \tag{52}$$

6. **Numerical example**

Consider a two person zero sum matrix game whose pay-offS are single valued triangular neutrosophic numbers as follows

$$\tilde{A}^{TNN}=\begin{bmatrix} \tilde{a}_{11}^{TNN} & \tilde{a}_{12}^{TNN} \\ \tilde{a}_{21}^{TNN} & \tilde{a}_{22}^{TNN} \end{bmatrix}$$

Here $\tilde{a}_{11}^{TNN} = \langle (175, 180, 190)\,; 0.6, 0.4, 0.2\rangle$, $\tilde{a}_{12}^{TNN} = \langle (150, 156, 158)\,; 0.6, 0.35, 0.1\rangle$, $\tilde{a}_{21}^{TNN} = \langle (80, 90, 100)\,; 0.9, 0.5, 0.1\rangle$, $\tilde{a}_{22}^{TNN} = \langle (175, 180, 190)\,; 0.6, 0.4, 0.2\rangle$. According to the problems (51) and (52) as explained in the mathematical procedure for a two person zero sum neutrosophic matrix game, we have

For player I

$$\begin{cases} \max \frac{1}{2}L_1 + M_1 + \frac{1}{2}N_1 \\ \text{s.t.}, \min\left(1-\rho_{11}, 1-\rho_{21}\right)\left(\frac{V_\pi(\tilde{a}_{11}^{TNN})x_1}{(1-\rho_{11})} + \frac{V_\pi(\tilde{a}_{21}^{TNN})x_2}{(1-\rho_{21})} - \frac{Amb_\pi(\tilde{a}_{11}^{TNN})x_1}{(1-\rho_{11})} - \frac{Amb_\pi(\tilde{a}_{21}^{TNN})x_2}{(1-\rho_{21})}\right) \\ +\frac{\min(\sigma_{11},\sigma_{21})}{2}\left(\frac{V_\mu(\tilde{a}_{11}^{TNN})x_1}{(\sigma_{11})} + \frac{V_\mu(\tilde{a}_{21}^{TNN})x_2}{(\sigma_{21})} - \frac{Amb_\mu(\tilde{a}_{11}^{TNN})x_1}{(\sigma_{11})} - \frac{Amb_\mu(\tilde{a}_{21}^{TNN})x_2}{(\sigma_{21})}\right) \\ +\frac{\min(1-\tau_{11},1-\tau_{21})}{2}\left(\frac{V_\nu(\tilde{a}_{11}^{TNN})x_1}{(1-\tau_{11})} + \frac{V_\nu(\tilde{a}_{21}^{TNN})x_2}{(1-\tau_{21})} - \frac{Amb_\nu(\tilde{a}_{11}^{TNN})x_1}{(1-\tau_{11})} - \frac{Amb_\nu(\tilde{a}_{21}^{TNN})x_2}{(1-\tau_{21})}\right) \\ \geq \frac{1}{2}L_1 + M_1 + \frac{1}{2}N_1 \\ \min\left(1-\rho_{12}, 1-\rho_{22}\right)\left(\frac{V_\pi(\tilde{a}_{12}^{TNN})x_1}{(1-\rho_{12})} + \frac{V_\pi(\tilde{a}_{22}^{TNN})x_2}{(1-\rho_{22})} - \frac{Amb_\pi(\tilde{a}_{12}^{TNN})x_1}{(1-\rho_{12})} - \frac{Amb_\pi(\tilde{a}_{22}^{TNN})x_2}{(1-\rho_{22})}\right) \\ +\frac{\min(\sigma_{12},\sigma_{22})}{2}\left(\frac{V_\mu(\tilde{a}_{12}^{TNN})x_1}{(\sigma_{12})} + \frac{V_\mu(\tilde{a}_{22}^{TNN})x_2}{(\sigma_{22})} - \frac{Amb_\mu(\tilde{a}_{12}^{TNN})x_1}{(\sigma_{12})} - \frac{Amb_\mu(\tilde{a}_{22}^{TNN})x_2}{(\sigma_{22})}\right) \\ +\frac{\min(1-\tau_{12},1-\tau_{22})}{2}\left(\frac{V_\nu(\tilde{a}_{12}^{TNN})x_1}{(1-\tau_{12})} + \frac{V_\nu(\tilde{a}_{22}^{TNN})x_2}{(1-\tau_{22})} - \frac{Amb_\nu(\tilde{a}_{12}^{TNN})x_1}{(1-\tau_{12})} - \frac{Amb_\nu(\tilde{a}_{22}^{TNN})x_2}{(1-\tau_{22})}\right) \\ \geq \frac{1}{2}L_1 + M_1 + \frac{1}{2}N_1 \\ x_1 + x_2 = 1 \\ \text{and}; x_1, x_2, L_1, M_1, N_1 \geq 0 \end{cases}$$

For player II

$$\begin{cases} \min \frac{1}{2}L_2 + M_2 + \frac{1}{2}N_2 \\ \text{s.t.}, \min\left(1-\rho_{11}', 1-\rho_{12}'\right)\left(\frac{V_\pi(\tilde{a}_{11}^{TNN})y_1}{(1-\rho_{11}')} + \frac{V_\pi(\tilde{a}_{12}^{TNN})y_2}{(1-\rho_{12}')} - \frac{Amb_\pi(\tilde{a}_{11}^{TNN})y_1}{(1-\rho_{11}')} - \frac{Amb_\pi(\tilde{a}_{12}^{TNN})y_2}{(1-\rho_{12}')}\right) \\ +\frac{\min(\sigma_{11}',\sigma_{12}')}{2}\left(\frac{V_\mu(\tilde{a}_{11}^{TNN})y_1}{(\sigma_{11}')} + \frac{V_\mu(\tilde{a}_{12}^{TNN})y_2}{(\sigma_{12}')} - \frac{Amb_\mu(\tilde{a}_{11}^{TNN})y_1}{(\sigma_{11}')} - \frac{Amb_\mu(\tilde{a}_{12}^{TNN})y_2}{(\sigma_{12}')}\right) \\ +\frac{\min(1-\tau_{11}',1-\tau_{12}')}{2}\left(\frac{V_\nu(\tilde{a}_{11}^{TNN})y_1}{(1-\tau_{11}')} + \frac{V_\nu(\tilde{a}_{12}^{TNN})y_2}{(1-\tau_{12}')} - \frac{Amb_\nu(\tilde{a}_{11}^{TNN})y_1}{(1-\tau_{11}')} - \frac{Amb_\nu(\tilde{a}_{12}^{TNN})y_2}{(1-\tau_{12}')}\right) \\ \leq \frac{1}{2}L_2 + M_2 + \frac{1}{2}N_2 \\ \min\left(1-\rho_{21}', 1-\rho_{22}'\right)\left(\frac{V_\pi(\tilde{a}_{21}^{TNN})y_1}{(1-\rho_{21}')} + \frac{V_\pi(\tilde{a}_{22}^{TNN})y_2}{(1-\rho_{22}')} - \frac{Amb_\pi(\tilde{a}_{21}^{TNN})y_1}{(1-\rho_{21}')} - \frac{Amb_\pi(\tilde{a}_{22}^{TNN})y_2}{(1-\rho_{22}')}\right) \\ +\frac{\min(\sigma_{21}',\sigma_{22}')}{2}\left(\frac{V_\mu(\tilde{a}_{21}^{TNN})y_1}{(\sigma_{21}')} + \frac{V_\mu(\tilde{a}_{22}^{TNN})y_2}{(\sigma_{22}')} - \frac{Amb_\mu(\tilde{a}_{21}^{TNN})y_1}{(\sigma_{21}')} - \frac{Amb_\mu(\tilde{a}_{22}^{TNN})y_2}{(\sigma_{22}')}\right) \\ +\frac{\min(1-\tau_{21}',1-\tau_{22}')}{2}\left(\frac{V_\nu(\tilde{a}_{21}^{TNN})y_1}{(1-\tau_{21}')} + \frac{V_\nu(\tilde{a}_{22}^{TNN})y_2}{(1-\tau_{22}')} - \frac{Amb_\nu(\tilde{a}_{21}^{TNN})y_1}{(1-\tau_{21}')} - \frac{Amb_\nu(\tilde{a}_{22}^{TNN})y_2}{(1-\tau_{22}')}\right) \\ \leq \frac{1}{2}L_2 + M_2 + \frac{1}{2}N_2 \\ y_1 + y_2 = 1 \\ \text{and}; y_1, y_2, L_2, M_2, N_2 \geq 0 \end{cases}$$

Hence, we obtain

For player I

$$\begin{cases} \max \frac{1}{2}L_1 + M_1 + \frac{1}{2}N_1 \\ \text{s.t.}, 211x_1 + 100x_2 \geq 0.5L_1 + M_1 + 0.5N_1 \\ 198.4667x_1 + 228.5833x_2 \geq 0.5L_1 + M_1 + 0.5N_1 \\ x_1 + x_2 = 1 \\ \text{and}; x_1, x_2, L_1, M_1, N_1 \geq 0 \end{cases}$$

For player II

$$\begin{cases} \min \frac{1}{2}L_2 + M_2 + \frac{1}{2}N_2 \\ \text{s.t.}, 228.5833y_1 + 198.4667y_2 \leq 0.5L_2 + M_2 + 0.5N_2 \\ 100y_1 + 211y_2 \leq 0.5L_2 + M_2 + 0.5N_2 \\ y_1 + y_2 = 1 \\ \text{and}; y_1, y_2, L_2, M_2, N_2 \geq 0 \end{cases}$$

Using standard simplex method we obtain that the optimal strategies for the player I and II are $X = (0.9112, 0.0888)^T$ and $Y = (0.0888, 0.9112)^T$ respectively. The minimum expected gain as single valued triangular neutrosophic number for player I is $\langle (152.2200, 158.1312, 160.8416)\,; 0.6, 0.4, 0.2\rangle$, while the maximum expected loss as single valued triangular neutrosophic number for player II is $\langle (166.5640, 172.0080, 180.0080)\,; 0.6, 0.5, 0.2\rangle$, when they choose the optimal strategies as $X = (0.9112, 0.0888)^T$ and $Y = (0.0888, 0.9112)^T$ respectively.

7. Conclusion

We have investigated a two-person zero-sum matrix game in a neutrosophic environment with single-valued triangular neutrosophic numbers as pay-offs. A ranking or de-neutrosophication, based on value and ambiguity index using α- cut, β- cut, and γ- cut is developed. A pair of neutrosophic linear programming problems estimated by the max-min approach of optimality of the two-person zero-sum matrix game is converted into another pair of crisp linear programming problems. Strategies and values of the matrix game are obtained by providing a numerical example.

The primary results of this study are pointed as:

- The relative properties and cut sets are developed for single-valued triangular neutrosophic numbers.
- Expressions for values and ambiguities are derived for single-valued triangular neutrosophic numbers.
- Related theorems for value and ambiguity indices are stated and proved.
- De-neutrosophication concept based on value and ambiguity index is derived.
- Established a mathematical model corresponding to neutrosophic matrix game.
- A numerical example is provided and verified to illustrate the theoretical establishments.

In the future, we can extend the recommended method for different types of neutrosophic numbers as an interval-valued neutrosophic number, bipolar neutrosophic number, and single-valued trapezoidal neutrosophic numbers.

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Impact of Complex Interval Neutrosophic Soft Set Theory in Decision making By Using Aggregate Operator

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Abstract: The main purpose of the paper is to introduce the notion of complex interval neutrosophic soft set (CIVNSS) theory, which is the generalization of the soft set, fuzzy soft set, interval-valued fuzzy soft set, interval neutrosophic soft set, etc. to describe the uncertain time-periodic phenomena in the form of an interval. After that, some important properties, and operations on CIVNSSs have been discussed. Also, we study the similarity measures on CIVNSSs. Then, an algorithm has been constructed by using the CIVNSS aggregate operator. Finally, to show the impact of CIVNSS in solving real decision-making problems, an example, which is suitable to the current theory, is chosen, which ensures the effectiveness of the proposed theory in group decision-making problem.

Keywords: Fuzzy set; soft set; complex fuzzy soft set; complex neutrosophic soft set; aggregate operator.

1. Introduction

Complex interval neutrosophic soft set (CIVNSS) is a new kind of soft set where the truth-membership function, indeterminacy-membership function, and the falsity-membership function are replaced by complex-valued functions in the form of an interval. It is the new way to handle parametric data in which time-phase plays an important role to describe the incomplete, indeterminate, inconsistent, or contradictory information systematically. The main feature in CIVNSS is the presence of phase and its membership in the form of an interval. In the group decision-making problem, researchers realized that the time period is an important factor along with the membership value so that decision-makers can make the real decision and it is more reliable and more acceptable than the other existing theories in which there is no scope of considering time-period. So, this new concept provides more scope for the decision-makers to make the real decisions with more feasibility.

In a crisp set, there are only two choices for the belongingness of an object, and, for this, we use two bits i.e., if an object belongs to a set, we assign 1, and for not belongs to we assign 0 for that particular object. There is no other option regarding the belongingness of an object. But due to the uncertainty involved in real life, we cannot restrict ourselves with only two values. This leads to the introduction of the fuzzy set theory, by Zadeh [1] in 1965. By fuzzy set theory, we represent the uncertainty with the help of a membership function. Later on, we realize that the non-membership value is also equally as important as the membership value to design the vagueness. So, Atanassov[2] introduced another mathematical tool known as the intuitionistic fuzzy set. In an intuitionistic fuzzy set, each object has membership value as well as non-membership value and they depend on each other. Researchers use the concept of fuzzy set theory in different application areas and introduce new theories and results. Later on, the fuzzy set has been extended by introducing an interval-valued intuitionistic fuzzy set [3], an interval-valued fuzzy set [4], a Pythagorean fuzzy set and its application [5], multi fuzzy set[6] ,etc.

In the fuzzy set theory, the vague concept is handled by a membership function and its nature is extremely individual. In reality, to measure the uncertainty there exist different possibilities so, setting up a membership function is a difficult task. For example, to represent the concept of 'middle-aged person' we define the membership function via a triangular form and a trapezoidal form of fuzzy membership function by setting up the age limit in different ways. So, there is a problem to choose the best criteria fit for the middle-aged person. So, there is a chance of getting different membership values for a single person. This difficulty of membership function has been removed by introducing a soft set by Russian mathematician Molodtsov[7] in 1999. Soft set theory handled uncertainty or vagueness differently by using the notion of parameterization. In soft set theory, to define an object, no need to introduce a membership function. It can be applied in different fields including game theory, social science, medical science, operation research, decision-making, pattern recognition, algebra, etc. Parameters may not be always crisp, but maybe in fuzzy words so, such types of vagueness demand several kinds of extensions of soft set theory which leads to the introduction of rough soft sets and fuzzy soft sets [8], fuzzy soft set theory, and its application [9], intuitionistic fuzzy parameterized soft set theory and its decision making [10], bipolar soft sets [11], hypersoft set[12] ,etc.

Incomplete information can be handled by the intuitionistic fuzzy set. But it cannot represent the indeterminacy involve in the data. So, there is a demand for another tool that is capable of representing incomplete, indeterminate, and uncertain information in an organized manner. This purpose is solved by introducing the neutrosophic set proposed by Smarandache [13]. The nature of indeterminacy is different as it depends on the problem so, researchers use this concept in various ways to tackle different essence of indeterminacy present in real life. Neutrosophic set is the extension of fuzzy set, intuitionistic fuzzy set, interval-valued fuzzy set, and interval-valued intuitionistic fuzzy set. For scientific implementation, we use a single-valued neutrosophic set introduced by Wang et al. [14]. The neutrosophic set has several extensions and applications among which some significant works are neutrosophic soft set [15], aggregate operators of neutrosophic hypersoft sets[16], rough neutrosophic sets[17], interval neutrosophic sets[18], interval neutrosophic tangent similarity measure based MADM strategy[19], bipolar neutrosophic sets and their application[20] , neutrosophic refined sets in medical diagnosis[21], distance-based similarity measure for refined neutrosophic sets and its application[22], an approach of TOPSIS technique for developing supplier selection under type-2 neutrosophic number[23], an integrated neutrosophic ANP and VIKOR method for supplier selection[24], neutrosophic approach for evaluation of the green supply chain management[25], group decision making model based on neutrosophic sets for heart disease diagnosis[26], bipolar neutrosophic multi-criteria decision making framework for professional selection[27], a novel intelligent medical decision support model based on soft computing IoT[28] etc.

But in all the above discussions, there is one information gap. To make it clear we consider an example. In a medical diagnosis problem, one person may have a variety of symptoms or attributes, or criteria. But in that case, we do not consider the information 'time duration of the symptom' though, it is also necessary information and should be considered together with the information's, 'belongingness level of a symptom' or 'non-belongingness level of a symptom' or 'indeterminacy level of a symptom' for proper diagnosis of a patient. To cover up such problem complex fuzzy set [29], complex fuzzy soft set [30-31], complex intuitionistic fuzzy soft set [32], complex fuzzy logic [33], interval-valued complex fuzzy soft sets [34], complex neutrosophic set [35], complex neutrosophic soft set [36],etc. are introduced.

Ramot et al. introduced complex fuzzy sets (CFSs) to ensure the accurate time-periodic representation of the fuzziness behavior of the attributes to generalize the membership structure. The problems that are intrinsic in CFSs can be handled with the help of complex intuitionistic fuzzy soft sets (CIFSSs) and complex vague soft sets(CVSSs). Selvachandran et al. generalize the CFS model by introducing the interval-valued complex fuzzy soft set(IV-CFSS). By combining the complex fuzzy sets and neutrosophic sets, Ali et al. developed complex neutrosophic sets (CNSs). In 2018, Ali et al. [37] formulate an interval complex neutrosophic set (ICNS) and apply it in decision making. To handle the parametric data, Broumi et al. [36] introduced complex neutrosophic soft sets (CNSSs).

The main objective of this paper is to introduce the notion of complex interval neutrosophic soft sets (CIVNSSs). CIVNSSs are formed by combining the interval-valued fuzzy sets (IV-FSs) and the complex neutrosophic soft sets (CNSSs). CIVNSS is the extension of CNSS. The main objective behind the modeling of CIVNSS is to provide a more general framework for time-periodic phenomena to ensure a more accurate representation of uncertainty of three-dimensional information about the problem parameters and an interval-based truth-membership, falsity-membership, and indeterminacy-membership structure. Moreover, we study some operations and distance measures on CIVNSSs. Finally, we use complex interval neutrosophic set aggregate operators to solve real-life problems in real decision-making.

The main motivation behind the introduction of complex interval neutrosophic soft set has been furnished below point wise:

- A soft set has been introduced to tackle parametric data in which the attributes associated with the parameter attain only the values 0 or 1.
- To overcome the issues which cannot be explained by a soft set, a fuzzy soft set is introduced where an attribute can take any values that belong to the unit closed interval [0, 1].
- The fuzzy soft set has been further extended by introducing an interval-valued fuzzy soft set and intuitionistic fuzzy soft set. In interval-valued fuzzy soft set, a decision-maker may take the membership value as a subset of [0,1] and in the intuitionistic fuzzy soft set, a decision-maker has a scope to assign non-membership value along with the non-membership value with the condition that their sum cannot exceed 1.
- Interval-valued fuzzy soft set and the intuitionistic fuzzy soft set has been extended further by introducing interval-valued intuitionistic fuzzy soft set where the value of an attribute can be represented by a pairwise interval in which the first interval is for membership degree and the second

interval for the non-membership degree with the condition that the sum of their supremum cannot exceed 1.

- A neutrosophic soft set has been introduced in which every attribute has three membership values and each belongs to the interval [0,1].
- Interval neutrosophic soft set has been introduced to extend the notion of the neutrosophic soft set where each membership value is a subset of [0,1].
- Sometimes time-period is an issue while solving decision making problem in real-world and such problem cannot be solved by soft set, fuzzy soft set, intuitionistic fuzzy soft set, interval-valued fuzzy soft set, neutrosophic soft set, interval-neutrosophic soft set ,etc. To eradicate such an issue, a complex interval neutrosophic soft set has been introduced in the present literature. So, complex interval neutrosophic soft set can be viewed as follows:

soft set $\subseteq$ fuzzy soft set $\subseteq$ intuitionistic fuzzy soft set / interval-valued fuzzy soft set $\subseteq$ interval-valued intuitionistic fuzzy soft set $\subseteq$ neutrosophic soft set $\subseteq$ interval neutrosophic soft set $\subseteq$ complex interval neutrosophic soft set.

The paper is organized in the following manner:
In section 2, we give a brief literature review that is relevant to the subsequent sections. In section 3, some operations on CIVNSSs have been proposed. In section 4, similarity measures on CIVNSSs have been discussed. In section 5, aggregation of CIVNSSs has been discussed. In section 6, an algorithm has been constructed by using CIVNSS aggregate operators. In section 7, an application of the proposed algorithm has been suggested. Finally, the paper is concluded in section 8.

2. Literature Review

2.1 Definition (Zadeh, 1965) Let X be a set of the universe. A fuzzy set on X can be defined as a set of ordered pairs of the form given by,

$A^{*} = \{(x, \mu_A(x)) : x \in X\}$, where μ_A denotes the membership function, and $\mu_A : X \to [0,1]$.

2.2 Definition (Molodtsov, 1999) Let X be the initial universe set and E be the set of parameters and $P(X)$ denotes the power set of X . Then the pair (F, A) is called a soft set over X , where $A \subseteq E$, and $F : A \to P(X)$.

2.3 Definition (Atanassov, 1986) Let X be a fixed set and A be a subset of X . Then an intuitionistic fuzzy set on X can be defined as a set of an ordered triplet of the form given by,

$A^* = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where μ_A and γ_A denote the membership function and non-membership function respectively such that $\mu_A, \gamma_A : X \to [0,1]$,and $0 \le \mu_A(x) + \gamma_A(x) \le 1$.

2.4 Definition (Bustince, 2010) An interval-valued fuzzy set A^* on a universe X is a mapping such that, $A^* : X \to Int[0,1]$, where $Int[0,1]$ denotes the set of all closed subintervals of $[0,1]$ and the membership of an element $x \in X$ is defined as $\mu_X(x) = [\mu_X^l(x), \mu_X^u(x)]$.

2.5 Definition (Cagman et al., 2011) Let U be an initial universe and E be a set of parameters which are in fuzzy words. Then the pair $\left(\tilde{F}, E\right)$ is called a fuzzy soft set (FSS) over U if $\tilde{F} : E \to \tilde{P}(U)$, where $\tilde{P}(U)$ denotes the set of all fuzzy subsets over U .

2.6 Definition (Smarandache, 2005) Let U be an initial universe. A neutrosophic set $\tilde{N}$ is an object having the form $\tilde{N} = \left\{ \left\langle x, T_{\tilde{N}}(x), I_{\tilde{N}}(x), F_{\tilde{N}}(x) \right\rangle : x \in U \right\}$, where the functions $T, I, F : U \to]^-0,1^+[$, denote the truth, indeterminacy, and falsity membership functions, respectively and they must satisfy the condition, $^-0 \le T_{\tilde{N}}(x) + I_{\tilde{N}}(x) + F_{\tilde{N}}(x) \le 3^+$. For practical application, it is difficult to apply. So we define its special form, called single-valued neutrosophic set (SVNS).

2.7 Definition (Wang et al., 2010) Let U denotes the space of objects with generic elements $x \in U$. Then, an SVNS on U is denoted by $\hat{N}$ and it is defined as $\hat{N} = \left\{ \left\langle x, T_{\hat{N}}(x), I_{\hat{N}}(x), F_{\hat{N}}(x) \right\rangle : x \in U \right\}$, where $T, I, F : U \to [0,1]$.

2.8 Definition (Ramot et al., 2002) A complex fuzzy set(CFS) C^* over a universal set U is defined by taking complex fuzzy-valued membership degree $(\mu_\varsigma(x))$ to each of the elements of U where,

$\mu_\varsigma(x) = a_\varsigma(x) e^{ib_\varsigma(x)}, \ i \equiv \sqrt{-1}, \ \forall x \in U$.

$a_{\varsigma}(x) \in [0,1]$ is called the amplitude part and $b_{\varsigma}(x) \in [0,2\pi]$ is called the phase part in the complex fuzzy-valued membership degree $\mu_{\varsigma}(x)$ of x.

2.9 Definition (Thirunavukarasu et al., 2017) A complex fuzzy soft set (CFSS) over a universal set U is defined as an ordered pair $\left(\overset{\approx}{F}, E\right)$ where, $\overset{\approx}{F}$ is a mapping defined as, $\overset{\approx}{F}: E \to \overset{\approx}{P}(U)$; $\overset{\approx}{P}(U)$ denotes the set of all complex fuzzy subsets of the set U.

Let, $U = \{x_1, x_2, ..., x_m\}$ be the set of the universe and $E = \{e_1, e_2, ..., e_n\}$ be the set of complex fuzzy-valued parameters then, a complex fuzzy soft set $\left(\overset{\approx}{F}, E\right)$ can be defined as follows

$\left(\overset{\approx}{F}, E\right) = \left\{\left(e_j, \overset{\approx}{F}(e_j)\right) : \forall e_j \in E\right\}$,where $\overset{\approx}{F}(e_j) = \{x_1 / x_{1j}, x_2 / x_{2j}, ..., x_m / x_{sj} : \forall e_j \in E\}$

x_{sj} is a complex fuzzy evaluation of an alternative x_s over a parameter e_j as, $x_{sj} = p_{sj} e^{iu_{sj}}$, where $p_{sj} \in [0,1]$ is the amplitude part and $u_{sj} \in [0, 2\pi]$ is the periodic part; $s = 1, 2, ..., m$ and $j = 1, 2, ..., \text{n}$. So, CFSS is a combination of a soft set (SS) with CFS by taking all the parameters in the complex fuzzy sense in a soft set.

2.10 Definition (Broumi et al., 2017) Let U be an initial universe and E be a set of parameters, $A \subseteq E$, and ψ_A be a complex neutrosophic set over U for all $x \in U$. Then, a complex neutrosophic soft set (CNSS) τ_A over U is defined as a mapping $\tau_A : E \to CN(U)$, where $CN(U)$ denotes the set of complex neutrosophic sets in U and it is defined as

$\tau_A = \left\{\left(x, \psi_A(x)\right) : x \in E, \psi_A(x) \in CN(U)\right\}$,

Where $\psi_A(x) = \left(\alpha_A(x) e^{i\mu_A(x)}, \beta_A(x) e^{i\nu_A(x)}, \delta_A(x) e^{i\omega_A(x)}\right)$, $\alpha_A, \beta_A, \delta_A \in [0,1]$, and $\mu_A, \nu_A, \omega_A \in (0, 2\pi]$.

2.11 Definition Let U be an initial universe and E be a set of parameters, $A \subseteq E$, and $\overset{*}{\psi}_A$ be a complex interval neutrosophic set over U for all $x \in U$. Then, a complex interval neutrosophic soft set (CIVNSS) $\overset{*}{\tau}_A$ over U is defined as a mapping $\overset{*}{\tau} A : E \to CIVN(U)$, where $CIVN(U)$ denotes the set of complex interval neutrosophic sets in U and it is defined as

$$\overset{*}{\tau} A = \left\{ \left(x, \overset{*}{\psi} A(x) \right) : x \in E, \overset{*}{\psi} A(x) \in CIVN(U) \right\},$$

Where

$$\overset{*}{\psi}_A(x) = \left(\alpha_A(x) e^{i\mu_A(x)}, \beta_A(x) e^{i\nu_A(x)}, \delta_A(x) e^{i\omega_A(x)} \right) \quad , \quad \alpha_A, \beta_A, \delta_A \subseteq [0,1] \quad ,$$

$\alpha_A(x) = \left[\alpha^l{}_A(x), \alpha^u{}_A(x) \right]$, $\beta_A(x) = \left[\beta^l{}_A(x), \beta^u{}_A(x) \right]$, $\delta_A(x) = \left[\delta^l{}_A(x), \delta^u{}_A(x) \right]$ and $\mu_A, \nu_A, \omega_A \in (0, 2\pi]$

For more clarity we consider the following example:

2.11.1 Example Let, $U = \{x_1, x_2, x_3, x_4\}$ be the set of developing countries under consideration, E be a set of parameters that signifies a country's time-dependent population indicators, and $A = \{e_1, e_2, e_3\} \subseteq E$, where the parameters stand for e_1 =birth rate, e_2 =death rate ,and e_3 =immigration rate. Then we define the CIVNSs as follows

$$\overset{*}{\psi} A(e_1) = \left\{ \begin{array}{l} \left\langle \dfrac{[0.3, 0.4] e^{i0.6\pi}, [0.5, 0.6] e^{i0.8\pi}, [0.3, 0.5] e^{i0.4\pi}}{x_1} \right\rangle, \left\langle \dfrac{[0.5, 0.8] e^{i0.4\pi}, [0.3, 0.4] e^{i\frac{\pi}{3}}, [0.25, 0.55] e^{i0.2\pi}}{x_2} \right\rangle, \\ \left\langle \dfrac{[0.2, 0.5] e^{i0.4\pi}, [0.1, 0.2] e^{i\frac{2\pi}{3}}, [0.6, 0.7] e^{i\frac{4\pi}{3}}}{x_3} \right\rangle, \left\langle \dfrac{[0.6, 0.7] e^{i\frac{5\pi}{4}}, [0.45, 0.65] e^{i0.4\pi}, [0.7, 0.8] e^{i0.5\pi}}{x_4} \right\rangle \end{array} \right\}$$

$$\overset{*}{\psi}_A(e_2)=\left\{\begin{array}{l}\left\langle\frac{[0.25,0.75]e^{i0.1\pi},[0.4,0.6]e^{i0.6\pi},[0.8,0.9]e^{i\frac{\pi}{3}}}{x_1}\right\rangle,\left\langle\frac{[0.1,0.3]e^{i0.2\pi},[0.35,0.65]e^{i\frac{2\pi}{3}},[0.6,0.7]e^{i0.5\pi}}{x_2}\right\rangle,\\ \left\langle\frac{[0.3,0.5]e^{i\frac{2\pi}{3}},[0.1,0.3]e^{i\frac{4\pi}{3}},[0.6,0.8]e^{i\frac{2\pi}{3}}}{x_3}\right\rangle,\left\langle\frac{[0.4,0.6]e^{i\frac{3\pi}{4}},[0.65,0.75]e^{i0.5\pi},[0.25,0.45]e^{i0.45\pi}}{x_4}\right\rangle\end{array}\right\}$$

$$\overset{*}{\psi}_A(e_3)=\left\{\begin{array}{l}\left\langle\frac{[0.25,0.35]e^{i0.3\pi},[0.4,0.6]e^{i0.2\pi},[0.3,0.5]e^{i0.1\pi}}{x_1}\right\rangle,\left\langle\frac{[0.45,0.65]e^{i\frac{5\pi}{6}},[0.3,0.6]e^{i\frac{3\pi}{4}},[0.65,0.85]e^{i0.4\pi}}{x_2}\right\rangle,\\ \left\langle\frac{[0.7,0.8]e^{i0.3\pi},[0.2,0.3]e^{i\frac{\pi}{3}},[0.8,0.9]e^{i\frac{2\pi}{3}}}{x_3}\right\rangle,\left\langle\frac{[0.5,0.6]e^{i\frac{3\pi}{4}},[0.35,0.65]e^{i0.3\pi},[0.6,0.7]e^{i0.6\pi}}{x_4}\right\rangle\end{array}\right\}$$

Then the complex interval neutrosophic soft set $\overset{*}{\tau}_A$ can be written as a collection of complex interval neutrosophic sets of the form

$$\overset{*}{\tau}_A=\left\{\overset{*}{\psi}_A(e_1),\overset{*}{\psi}_A(e_2),\overset{*}{\psi}_A(e_3)\right\}$$

2.12 Definition Let us consider the two CIVNSSs over the set of the universe U as follows:

$$\overset{*}{\tau}_A=\left\{\left(x,\overset{*}{\psi}_A(x)\right):x\in E,\overset{*}{\psi}_A(x)\in CIVN(U)\right\},$$

Where

$\overset{*}{\psi}_A(x)=\left(\alpha_A(x)e^{i\mu_A(x)},\beta_A(x)e^{i\nu_A(x)},\delta_A(x)e^{i\omega_A(x)}\right)$, $\alpha_A,\beta_A,\delta_A\subseteq[0,1]$, $\alpha_A(x)=\left[\alpha^l{}_A(x),\alpha^u{}_A(x)\right]$, $\beta_A(x)=\left[\beta^l{}_A(x),\beta^u{}_A(x)\right]$, $\delta_A(x)=\left[\delta^l{}_A(x),\delta^u{}_A(x)\right]$ and $\mu_A,\nu_A,\omega_A\in(0,2\pi]$

and $\overset{*}{\tau}_B=\left\{\left(x,\overset{*}{\psi}_B(x)\right):x\in E,\overset{*}{\psi}_B(x)\in CIVN(U)\right\},$

where

$\overset{*}{\psi}_B(x)=\left(\alpha_B(x)e^{i\mu_B(x)},\beta_B(x)e^{i\nu_B(x)},\delta_B(x)e^{i\omega_B(x)}\right)$, $\alpha_B,\beta_B,\delta_B\subseteq[0,1]$, $\alpha_B(x)=\left[\alpha^l{}_B(x),\alpha^u{}_B(x)\right]$, $\beta_B(x)=\left[\beta^l{}_B(x),\beta^u{}_B(x)\right]$, $\delta_B(x)=\left[\delta^l{}_B(x),\delta^u{}_B(x)\right]$ and $\mu_B,\nu_B,\omega_B\in(0,2\pi]$

Then we consider the following:

(i) $\overset{*}{\tau}_A$ is said to be an empty CIVNSS, denoted by $\overset{*}{\tau}_{A_\varnothing}$, if $\overset{*}{\psi}_A(x)=\varnothing$, for all $x\in U$.

(ii) $\overset{*}{\tau}_A$ is said to be an absolute CIVNSS, denoted by $\overset{*}{\tau}_{A_U}$, if $\overset{*}{\psi}_A(x)=U$, for all $x\in U$.

(iii) $\overset{*}{\tau}_A$ is said to be a normal CIVNSS, denoted by $\overset{*}{\tau}_{A_N}$, if $\alpha_A(x)=[1,1],\beta_A(x)=[1,1],\delta_A(x)=[1,1]$ and $\mu_A,\nu_A,\omega_A=2\pi,$ for all $x\in U$.

(iv) $\overset{*}{\tau}_A$ is said to be a CIVNS-subset of $\overset{*}{\tau}_B$, denoted by $\overset{*}{\tau}_A\subseteq\overset{*}{\tau}_B$, if for all $x\in U$, $\overset{*}{\psi}_A(e)\subseteq\overset{*}{\psi}_B(e)$, that is the following conditions are satisfied:

$\alpha_A(e)\subseteq\alpha_B(e),\ \beta_A(e)\subseteq\beta_B(e),\ \delta_A(x)\subseteq\delta_B(x)$

,and $\mu_A(e)\le\mu_B(e),\nu_A(e)\le\nu_B(e),\ \omega_A(e)\le\omega_B(e)$.

(v) $\overset{*}{\tau}_A$ is said to be equal to $\overset{*}{\tau}_B$, denoted by $\overset{*}{\tau}_A=\overset{*}{\tau}_B$, if for all $x\in U$, $\overset{*}{\psi}_A(e)=\overset{*}{\psi}_B(e)$, that is the following conditions are satisfied:

$\alpha_A(e)=\alpha_B(e),\ \beta_A(e)=\beta_B(e),\ \delta_A(x)=\delta_B(x)$

,and $\mu_A(e)=\mu_B(e),\nu_A(e)=\nu_B(e),\ \omega_A(e)=\omega_B(e)$

3. Operations on Complex Interval Neutrosophic Soft Sets

In this section, we discuss different sorts of set-theoretic operations on CIVNSSs.

Let $\overset{*}{\tau}_A$ and $\overset{*}{\tau}_B$ be two CIVNSSs over the common universal set U. Then we define the following operations:

3.1 Definition Complement of $\overset{*}{\tau}_A$ is denoted by $\left(\overset{*}{\tau}_A\right)^c$ and it is defined as:

$$\left(\overset{*}{\psi}_A(x)\right)^c = \left(\delta_A(x)e^{i(2\pi-\mu_A(x))}, \left(1-\beta_A(x)\right)e^{i(2\pi-\nu_A(x))}, \alpha_A(x)e^{i(2\pi-\omega_A(x))}\right)$$

It is to be noted that $\left(\left(\overset{*}{\psi}_A(x)\right)^c\right)^c = \overset{*}{\psi}_A(x)$

3.2 Definition

Let ,

$$\overset{*}{\tau}_A = \left\{\left(x, \overset{*}{\psi}_A(x)\right): x \in E, \overset{*}{\psi}_A(x) \in CIVN(U)\right\}, \text{ and}$$

$$\overset{*}{\tau}_B = \left\{\left(x, \overset{*}{\psi}_B(x)\right): x \in E, \overset{*}{\psi}_B(x) \in CIVN(U)\right\}$$

be two IVNSSs over the common universe U . Then, their union is denoted by $\overset{*}{\tau}_A \tilde{\bigcup} \overset{*}{\tau}_B$ and is defined as:

$$\overset{*}{\tau}_C = \overset{*}{\tau}_A \tilde{\bigcup} \overset{*}{\tau}_B = = \left\{\left(x, \overset{*}{\psi}_A(x) \tilde{\cup} \overset{*}{\psi}_B(x)\right): x \in U\right\}, \text{ where } C = A \cup B$$

$$\overset{*}{\tau}_C(e) = \begin{cases} \left(x, \overset{*}{\psi}_A(x)\right) & if\ e \in A - B \\ \left(x, \overset{*}{\psi}_B(x)\right) & if\ e \in B - A \\ \left(x, \overset{*}{\psi}_A(x) \tilde{\cup} \overset{*}{\psi}_B(x)\right) & if\ e \in A \cap B \end{cases}$$

Where

$$\overset{*}{\psi}_A(x) \tilde{\cup} \overset{*}{\psi}_B(x) = \begin{Bmatrix} \left(\left[\alpha_A^l(x) \vee \alpha_B^l(x), \alpha_A^u(x) \vee \alpha_B^u(x)\right]\right)e^{i(\mu_A(x) \cup \mu_B(x))} \\ \left(\left[\beta_A^l(x) \vee \beta_B^l(x), \beta_A^u(x) \vee \beta_B^u(x)\right]\right)e^{i(\nu_A(x) \cup \nu_B(x))} \\ \left(\left[\delta_A^l(x) \wedge \delta_B^l(x), \delta_A^u(x) \wedge \delta_B^u(x)\right]\right)e^{i(\omega_A(x) \cup \omega_B(x))} \end{Bmatrix}$$

3.3Definition

Let ,

$\overset{*}{\tau}_A = \left\{ \left(x, \overset{*}{\psi}_A(x) \right) : x \in E, \overset{*}{\psi}_A(x) \in CIVN(U) \right\}$ and

$\overset{*}{\tau}_B = \left\{ \left(x, \overset{*}{\psi}_B(x) \right) : x \in E, \overset{*}{\psi}_B(x) \in CIVN(U) \right\}$

be two IVNSSs over the common universe U . Then, their intersection is denoted by $\overset{*}{\tau}_A \tilde{\bigcap} \overset{*}{\tau}_B$ and is defined as:

$\overset{*}{\tau}_C = \overset{*}{\tau}_A \tilde{\bigcap} \overset{*}{\tau}_B = = \left\{ \left(x, \overset{*}{\psi}_A(x) \tilde{\bigcap} \overset{*}{\psi}_B(x) \right) : x \in U \right\}$, where $C = A \cap B$

$$\overset{*}{\tau}_C(e) = \begin{cases} \left(x, \overset{*}{\psi}_A(x) \right) & if\ e \in A - B \\ \left(x, \overset{*}{\psi}_B(x) \right) & if\ e \in B - A \\ \left(x, \overset{*}{\psi}_A(x) \tilde{\cap} \overset{*}{\psi}_B(x) \right) & if\ e \in A \cap B \end{cases}$$

Where

$$\overset{*}{\psi}_A(x) \tilde{\cap} \overset{*}{\psi}_B(x) = \left\{ \begin{matrix} \left(\left[\alpha_A^l(x) \wedge \alpha_B^l(x), \alpha_A^u(x) \wedge \alpha_B^u(x) \right] \right) e^{i(\mu_A(x) \cap \mu_B(x))} \\ \left(\left[\beta_A^l(x) \wedge \beta_B^l(x), \beta_A^u(x) \wedge \beta_B^u(x) \right] \right) e^{i(\nu_A(x) \cap \nu_B(x))} \\ \left(\left[\delta_A^l(x) \vee \delta_B^l(x), \delta_A^u(x) \vee \delta_B^u(x) \right] \right) e^{i(\omega_A(x) \cap \omega_B(x))} \end{matrix} \right\}$$

4. Similarity measure of complex interval neutrosophic soft sets

Nowadays the concept of similarity measure has been used in almost all scientific disciplines. The Similarity measure of two objects determines the degree of closeness or the degree of sameness between them. In many different fields like pattern recognition, decision-making, disease diagnosis, etc. it has been used quite successfully.

Now considering $U=\{x_1,x_2,x_3,.....,x_m\}$ be the set of the universe and $E=\{e_1,e_2,e_3,.....,e_n\}$ be the set of parameters where they are used in complex interval neutrosophic sense. Then, a mapping

$\xi:\chi_{CIVNSS}(U)\times\chi_{CIVNSS}(U)\rightarrow([0,1],[0,1],[0,1])$, where $\chi_{CIVNSS}(U)$ denotes the set of all complex interval neutrosophic soft set over the universe U, is said to be a similarity measure if it satisfies the following conditions:

For all $\left(\tilde{\tilde{P}},E\right),\left(\tilde{\tilde{Q}},E\right),\left(\tilde{\tilde{R}},E\right)\in\chi_{CIVNSS}(U)$

(i) $\overset{*}{S}\left(\left(\tilde{\tilde{P}},E\right),\left(\tilde{\tilde{Q}},E\right)\right)\in([0,1],[0,1],[0,1])$

(ii) $\overset{*}{S}\left(\left(\tilde{\tilde{P}},E\right),\left(\tilde{\tilde{Q}},E\right)\right)=\overset{*}{S}\left(\left(\tilde{\tilde{Q}},E\right),\left(\tilde{\tilde{P}},E\right)\right)$

(iii) $\overset{*}{S}\left(\left(\tilde{\tilde{P}},E\right),\left(\tilde{\tilde{Q}},E\right)\right)=(1,1,1)\Leftrightarrow\left(\tilde{\tilde{P}},E\right)=\left(\tilde{\tilde{Q}},E\right)$

(iv) If $\left(\tilde{\tilde{P}},E\right)\subseteq\left(\tilde{\tilde{Q}},E\right)\subseteq\left(\tilde{\tilde{R}},E\right)$ then, $\overset{*}{S}\left(\left(\tilde{\tilde{P}},E\right),\left(\tilde{\tilde{Q}},E\right)\right)\geq\overset{*}{S}\left(\left(\tilde{\tilde{P}},E\right),\left(\tilde{\tilde{R}},E\right)\right)$

, and $\overset{*}{S}\left(\left(\tilde{\tilde{Q}},E\right),\left(\tilde{\tilde{R}},E\right)\right)\geq\overset{*}{S}\left(\left(\tilde{\tilde{P}},E\right),\left(\tilde{\tilde{R}},E\right)\right)$

4.1 Ratio Similarity measure of two complex interval neutrosophic soft sets

Let $\left(\tilde{\tilde{F}},E\right)$ and $\left(\tilde{\tilde{G}},E\right)$ be two complex interval neutrosophic soft sets over U as follows:

$$\left(\tilde{\tilde{F}},E\right)=\left\{\left(e_j,\tilde{\tilde{F}}(e_j)\right):\forall e_j\in E\right\}=\left\{e_j,\left(x_s/\left(p^F_{s_j}e^{i\mu^F_{s_j}},q^F_{s_j}e^{i\nu^F_{s_j}},r^F_{s_j}e^{i\delta^F_{s_j}}\right)\right):\forall e_j\in E,x_s\in U\right\}\text{ and}$$

$$\left(\tilde{\tilde{G}},E\right)=\left\{\left(e_j,\tilde{\tilde{G}}(e_j)\right):\forall e_j\in E\right\}=\left\{e_j,\left(x_s/\left(p^G_{s_j}e^{i\mu^G_{s_j}},q^G_{s_j}e^{i\nu^G_{s_j}},r^G_{s_j}e^{i\delta^G_{s_j}}\right)\right):\forall e_j\in E,x_s\in U\right\}$$

Where $p_{s_j}^F, q_{s_j}^F, r_{s_j}^F \subseteq [0,1]$ are the amplitude parts of the truth-membership, indeterminacy-membership ,and falsity-membership values respectively and $\mu_{s_j}^F, \nu_{s_j}^F, \delta_{s_j}^F \in [0, 2\pi]$ are the respective phase parts of the evaluation of an alternative x_s concerning for to the parameter e_j over the CIVNSS $\left(\overset{\approx}{F}, E\right)$. Similarly, we can write for $\left(\overset{\approx}{G}, E\right)$.

Since in every evaluation there exist two decision information to each membership value, one is amplitude part another one is phase part. So, to measure the similarity degree between the two CIVNSSs $\left(\overset{\approx}{F}, E\right)$ and $\left(\overset{\approx}{G}, E\right)$, we have measured one similarity for the amplitude part and phase part individually and then added them for deriving the total similarity.

4.2 Definition

The ratio similarity between $\left(\overset{\approx}{F}, E\right)$ and $\left(\overset{\approx}{G}, E\right)$ is denoted by $\overset{\approx}{S}_R\left(\left(\overset{\approx}{F}, E\right), \left(\overset{\approx}{G}, E\right)\right)$ and is defined by the following equation:

$$\overset{\approx}{S}_R\left(\left(\overset{\approx}{F}, E\right), \left(\overset{\approx}{G}, E\right)\right) = \frac{\sum_{j=1}^{n} w_j \overset{\approx}{S}_R\left(\overset{\approx}{F}(e_j), \overset{\approx}{G}(e_j)\right)}{\sum_{j=1}^{n} w_j}$$

Where

$$\tilde{\tilde{S}}_R\left(\tilde{\tilde{F}}\left(e_j\right),\tilde{\tilde{G}}\left(e_j\right)\right) =$$

$$\frac{\sum_{j=1}^{n}\max\left(\min\left(p_{s_j}^{l^F},p_{s_j}^{l^G}\right),\min\left(p_{s_j}^{u^F},p_{s_j}^{u^G}\right)\right)+\max\left(\min\left(q_{s_j}^{l^F},q_{s_j}^{l^G}\right),\min\left(q_{s_j}^{u^F},q_{s_j}^{u^G}\right)\right)+\min\left(\max\left(r_{s_j}^{l^F},r_{s_j}^{l^G}\right),\max\left(r_{s_j}^{u^F},r_{s_j}^{u^G}\right)\right)}{\sum_{j=1}^{n}\left(\min\left(p_{s_j}^{l^F},p_{s_j}^{l^G}\right)+\min\left(p_{s_j}^{u^F},p_{s_j}^{u^G}\right)+\min\left(q_{s_j}^{l^F},q_{s_j}^{l^G}\right)+\min\left(q_{s_j}^{u^F},q_{s_j}^{u^G}\right)+\max\left(r_{s_j}^{l^F},r_{s_j}^{l^G}\right)+\max\left(r_{s_j}^{u^F},r_{s_j}^{u^G}\right)\right)}$$

,and $w=\{w_1,w_2,........,w_n\}$ are the weights of the parameters and each $w_j\in[0,1]$.

If , $\sum_{j=1}^{n} w_j=1$, then the above equation takes the form as,

$$\tilde{\tilde{S}}_R\left(\left(\tilde{\tilde{F}},E\right),\left(\tilde{\tilde{G}},E\right)\right)=\sum_{j=1}^{n} w_j\,\tilde{\tilde{S}}_R\left(\tilde{\tilde{F}}\left(e_j\right),\tilde{\tilde{G}}\left(e_j\right)\right)$$

5. Aggregation of complex interval neutrosophic soft sets

Let, $U=\{u_1,u_2,.....,u_m\}$ be the set of alternatives and $E=\{e_1,e_2,.....,e_n\}$ be the set of parameters which are in complex interval neutrosophic sense. Consider K-CIVNSSs $\left(\tilde{\tilde{F}}^1,E\right),\left(\tilde{\tilde{F}}^2,E\right),......,\left(\tilde{\tilde{F}}^k,E\right)$ from $\tau_{CIVN}(U)$ (set of all CIVNSSs over U. Then the mapping

$\tilde{\tilde{B}}:\tau_{CIVN}(U)\times\tau_{CIVN}(U)\times..........\times\tau_{CIVN}(U)\rightarrow\tau_{CIVN}(U)$ satisfies the following properties:

(a) $\tilde{\tilde{B}}\left(\left(\tilde{\tilde{F^1}},E\right)_{\tilde{1}},\left(\tilde{\tilde{F^2}},E\right)_{\tilde{1}},........,\left(\tilde{\tilde{F^k}},E\right)_{\tilde{1}}\right)=\left(\tilde{\tilde{F}},E\right)_{\tilde{1}}$, where $\left(\tilde{\tilde{F}},E\right)_{\tilde{1}}$ is the absolute complex interval neutrosophic soft set over U.

(b) $\tilde{\tilde{B}}\left(\left(\tilde{\tilde{F^1}},E\right)_{\tilde{0}},\left(\tilde{\tilde{F^2}},E\right)_{\tilde{0}},........,\left(\tilde{\tilde{F^k}},E\right)_{\tilde{0}}\right)=\left(\tilde{\tilde{F}},E\right)_{\tilde{0}}$, where $\left(\tilde{\tilde{F}},E\right)_{\tilde{0}}$ is the null complex interval neutrosophic soft set over U.

(c) If, for all $i = 1, 2, 3, ..., k$, $\left(\overset{\approx}{F^{i}}, E\right) \subseteq \left(\overset{\approx}{G^{i}}, E\right)$ then,

$\overset{\approx}{B}\left(\left(\overset{\approx}{F^{1}}, E\right), \left(\overset{\approx}{F^{2}}, E\right), ..., \left(\overset{\approx}{F^{k}}, E\right)\right) \leq \overset{\approx}{B}\left(\left(\overset{\approx}{G^{1}}, E\right), \left(\overset{\approx}{G^{2}}, E\right), ..., \left(\overset{\approx}{G^{k}}, E\right)\right)$, where

$\left(\overset{\approx}{G^{1}}, E\right), \left(\overset{\approx}{G^{2}}, E\right), ..., \left(\overset{\approx}{G^{k}}, E\right)$ be another k-CIVNSSs over U .

(d) The aggregate operator satisfies the inequality

$\left(\overset{\approx}{F^{-}}, E\right) \leq \overset{\approx}{B}\left(\left(\overset{\approx}{F^{1}}, E\right), \left(\overset{\approx}{F^{2}}, E\right), ..., \left(\overset{\approx}{F^{k}}, E\right)\right) \leq \left(\overset{\approx}{F^{+}}, E\right)$, where $\left(\overset{\approx}{F^{-}}, E\right)$ is the worst(min-min-max-valued for amplitude part and min-valued for phase part) CIVNSS and $\left(\overset{\approx}{F^{+}}, E\right)$ is the best(max-max-min-valued for amplitude part and max-valued for phase part)CIVNSS over k-CIVNSSs.

Now we consider the following tables for better understanding:

Table1 for absolute CIVNSS $\left(\overset{\approx}{F}, E\right)_{1^{\prec}}$

	e_1	e_2	……………	e_n
x_1	$([1,1][1,1][1,1])e^{i2\pi}$	$([1,1][1,1][1,1])e^{i2\pi}$	………………	$([1,1][1,1][1,1])e^{i2\pi}$
x_2	$([1,1][1,1][1,1])e^{i2\pi}$	$([1,1][1,1][1,1])e^{i2\pi}$	……………	$([1,1][1,1][1,1])e^{i2\pi}$
………….	………….	…………………	…………….	
x_m	$([1,1][1,1][1,1])e^{i2\pi}$	$([1,1][1,1][1,1])e^{i2\pi}$	…………………	$([1,1][1,1][1,1])e^{i2\pi}$

Table1. Absolute CIVNSS $\left(\overset{\approx}{F}, E\right)_{1^{\prec}}$

Table2 for null CIVNSS $\left(\overset{\approx}{F}, E\right)_{0^{\prec}}$

	e_1	e_2	…………….	e_n
x_1	$([0,0][0,0][0,0])e^{i0\pi}$	$([0,0][0,0][0,0])e^{i0\pi}$	………………	$([0,0][0,0][0,0])e^{i0\pi}$
x_2	$([0,0][0,0][0,0])e^{i0\pi}$	$([0,0][0,0][0,0])e^{i0\pi}$	……………	$([0,0][0,0][0,0])e^{i0\pi}$
………… .	………….	…………………	…………….	
x_m	$([0,0][0,0][0,0])e^{i0\pi}$	$([0,0][0,0][0,0])e^{i0\pi}$	……………… …	$([0,0][0,0][0,0])e^{i0\pi}$

Table2. Null CIVNSS $\left(\overset{\approx}{F}, E\right)_{1^{\prec}}$

Table 3 for K-CIVNSSs

	e_1	e_2	…… …	e_n
x_1	$\begin{pmatrix}\left[p_{11}^l, p_{11}^u\right]e^{i\alpha_{11}^1}, \left[q_{11}^l, q_{11}^u\right]e^{i\beta_{11}^1}, \\ \left[r_{11}^l, r_{11}^u\right]e^{i\delta_{11}^1}\end{pmatrix}$	$\begin{pmatrix}\left[p_{12}^l, p_{12}^u\right]e^{i\alpha_{12}^1}, \left[q_{12}^l, q_{12}^u\right]e^{i\beta_{12}^1}, \\ \left[r_{12}^l, r_{12}^u\right]e^{i\delta_{12}^1}\end{pmatrix}$	…… …	$\begin{pmatrix}\left[p_{1n}^l, p_{1n}^u\right]e^{i\alpha_{1n}^1}, \left[q_{1n}^l, q_{1n}^u\right]e^{i\beta_{1n}^1}, \\ \left[r_{1n}^l, r_{1n}^u\right]e^{i\delta_{1n}^1}\end{pmatrix}$
x_2	$\begin{pmatrix}\left[p_{21}^l, p_{21}^u\right]e^{i\alpha_{21}^2}, \left[q_{21}^l, q_{21}^u\right]e^{i\beta_{21}^2}, \\ \left[r_{21}^l, r_{21}^u\right]e^{i\delta_{21}^2}\end{pmatrix}$	$\begin{pmatrix}\left[p_{22}^l, p_{22}^u\right]e^{i\alpha_{22}^2}, \left[q_{22}^l, q_{22}^u\right]e^{i\beta_{22}^2}, \\ \left[r_{22}^l, r_{22}^u\right]e^{i\delta_{22}^2}\end{pmatrix}$	…… …	$\begin{pmatrix}\left[p_{2n}^l, p_{2n}^u\right]e^{i\alpha_{2n}^2}, \left[q_{2n}^l, q_{2n}^u\right]e^{i\beta_{2n}^2}, \\ \left[r_{2n}^l, r_{2n}^u\right]e^{i\delta_{2n}^2}\end{pmatrix}$
…….	………….	…………………	…… …	……………

x_m	$\left(\begin{array}{l} [p^l_{m1}, p^u_{m1}] e^{i\alpha^k_{m1}}, [q^l_{m1}, q^u_{m1}] e^{i\beta^k_{m1}}, \\ [r^l_{m1}, r^u_{m1}] e^{i\delta^k_{m1}} \end{array} \right)$	$\left(\begin{array}{l} [p^l_{m2}, p^u_{m2}] e^{i\alpha^k_{m2}}, [q^l_{m2}, q^u_{m2}] e^{i\beta^k_{m2}}, \\ [r^l_{m2}, r^u_{m2}] e^{i\delta^k_{m2}} \end{array} \right)$	....	$\left(\begin{array}{l} [p^l_{mn}, p^u_{mn}] e^{i\alpha^k_{mn}}, [q^l_{mn}, q^u_{mn}] e^{i\beta^k_{mn}}, \\ [r^l_{mn}, r^u_{mn}] e^{i\delta^k_{mn}} \end{array} \right)$

Table3. K- CIVNSSs

Table4 for the best CIVNSS $\left(\overset{\approx}{F^{+}}, E \right)$

	e_1	...	e_n
x_1	$\left\langle \begin{array}{l} [\max(p^{l1}_{11}, p^{l2}_{11}, ..., p^{lk}_{11}), \max(p^{u1}_{11}, p^{u2}_{11}, ..., p^{uk}_{11})] e^{i\max(\alpha^1_{11}, \alpha^2_{11}, ..., \alpha^k_{11})}, \\ [\max(q^{l1}_{11}, q^{l2}_{11}, ..., q^{lk}_{11}), \max(q^{u1}_{11}, q^{u2}_{11}, ..., q^{uk}_{11})] e^{i\max(\beta^1_{11}, \beta^2_{11}, ..., \beta^k_{11})}, \\ [\min(r^{l1}_{11}, r^{l2}_{11}, ..., r^{lk}_{11}), \min(r^{u1}_{11}, r^{u2}_{11}, ..., r^{uk}_{11})] e^{i\max(\delta^1_{11}, \delta^2_{11}, ..., \delta^k_{11})} \end{array} \right\rangle$	....	$\left\langle \begin{array}{l} [\max(p^{l1}_{1n}, p^{l2}_{1n}, ..., p^{lk}_{1n}), \max(p^{u1}_{1n}, p^{u2}_{1n}, ..., p^{uk}_{1n})] e^{i\max(\alpha^1_{1n}, \alpha^2_{1n}, ..., \alpha^k_{1n})}, \\ [\max(q^{l1}_{1n}, q^{l2}_{1n}, ..., q^{lk}_{1n}), \max(q^{u1}_{1n}, q^{u2}_{1n}, ..., q^{uk}_{1n})] e^{i\max(\beta^1_{1n}, \beta^2_{1n}, ..., \beta^k_{1n})}, \\ [\min(r^{l1}_{1n}, r^{l2}_{1n}, ..., r^{lk}_{1n}), \min(r^{u1}_{1n}, r^{u2}_{1n}, ..., r^{uk}_{1n})] e^{i\max(\delta^1_{1n}, \delta^2_{1n}, ..., \delta^k_{1n})} \end{array} \right\rangle$
x_2	$\left\langle \begin{array}{l} [\max(p^{l1}_{21}, p^{l2}_{21}, ..., p^{lk}_{21}), \max(p^{u1}_{21}, p^{u2}_{21}, ..., p^{uk}_{21})] e^{i\max(\alpha^1_{21}, \alpha^2_{21}, ..., \alpha^k_{21})}, \\ [\max(q^{l1}_{21}, q^{l2}_{21}, ..., q^{lk}_{21}), \max(q^{u1}_{21}, q^{u2}_{21}, ..., q^{uk}_{21})] e^{i\max(\beta^1_{21}, \beta^2_{21}, ..., \beta^k_{21})}, \\ [\min(r^{l1}_{21}, r^{l2}_{21}, ..., r^{lk}_{21}), \min(r^{u1}_{21}, r^{u2}_{21}, ..., r^{uk}_{21})] e^{i\max(\delta^1_{21}, \delta^2_{21}, ..., \delta^k_{21})} \end{array} \right\rangle$	....	$\left\langle \begin{array}{l} [\max(p^{l1}_{2n}, p^{l2}_{2n}, ..., p^{lk}_{2n}), \max(p^{u1}_{2n}, p^{u2}_{2n}, ..., p^{uk}_{2n})] e^{i\max(\alpha^1_{2n}, \alpha^2_{2n}, ..., \alpha^k_{2n})}, \\ [\max(q^{l1}_{2n}, q^{l2}_{2n}, ..., q^{lk}_{2n}), \max(q^{u1}_{2n}, q^{u2}_{2n}, ..., q^{uk}_{2n})] e^{i\max(\beta^1_{2n}, \beta^2_{2n}, ..., \beta^k_{2n})}, \\ [\min(r^{l1}_{2n}, r^{l2}_{2n}, ..., r^{lk}_{2n}), \min(r^{u1}_{2n}, r^{u2}_{2n}, ..., r^{uk}_{2n})] e^{i\max(\delta^1_{2n}, \delta^2_{2n}, ..., \delta^k_{2n})} \end{array} \right\rangle$
...		...	
x_m	$\left\langle \begin{array}{l} [\max(p^{l1}_{m1}, p^{l2}_{m1}, ..., p^{lk}_{m1}), \max(p^{u1}_{m1}, p^{u2}_{m1}, ..., p^{uk}_{m1})] e^{i\max(\alpha^1_{m1}, \alpha^2_{m1}, ..., \alpha^k_{m1})}, \\ [\max(q^{l1}_{m1}, q^{l2}_{m1}, ..., q^{lk}_{m1}), \max(q^{u1}_{m1}, q^{u2}_{m1}, ..., q^{uk}_{m1})] e^{i\max(\beta^1_{m1}, \beta^2_{m1}, ..., \beta^k_{m1})}, \\ [\min(r^{l1}_{m1}, r^{l2}_{m1}, ..., r^{lk}_{m1}), \min(r^{u1}_{m1}, r^{u2}_{m1}, ..., r^{uk}_{m1})] e^{i\max(\delta^1_{m1}, \delta^2_{m1}, ..., \delta^k_{m1})} \end{array} \right\rangle$	.... .	$\left\langle \begin{array}{l} [\max(p^{l1}_{mn}, p^{l2}_{mn}, ..., p^{lk}_{mn}), \max(p^{u1}_{mn}, p^{u2}_{mn}, ..., p^{uk}_{mn})] e^{i\max(\alpha^1_{mn}, \alpha^2_{mn}, ..., \alpha^k_{mn})}, \\ [\max(q^{l1}_{mn}, q^{l2}_{mn}, ..., q^{lk}_{mn}), \max(q^{u1}_{mn}, q^{u2}_{mn}, ..., q^{uk}_{mn})] e^{i\max(\beta^1_{mn}, \beta^2_{mn}, ..., \beta^k_{mn})}, \\ [\min(r^{l1}_{mn}, r^{l2}_{mn}, ..., r^{lk}_{mn}), \min(r^{u1}_{mn}, r^{u2}_{mn}, ..., r^{uk}_{mn})] e^{i\max(\delta^1_{mn}, \delta^2_{mn}, ..., \delta^k_{mn})} \end{array} \right\rangle$

Table4. Best CIVNSS $\left(\overset{\approx}{F^{+}}, E \right)$

Table5 for the worst CIVNSS $\left(\overset{\approx}{F^{-}}, E \right)$

	e_1	...	e_n
x_1	$\left\langle \left[\min(p_{11}^{l1}, p_{11}^{l2}, ..., p_{11}^{lk}), \min(p_{11}^{u1}, p_{11}^{u2}, ..., p_{11}^{uk})\right] e^{i\min(\alpha_{11}^{1},\alpha_{11}^{2},.....,\alpha_{11}^{k})}, \left[\min(q_{11}^{l1}, q_{11}^{l2}, ..., q_{11}^{lk}), \min(q_{11}^{u1}, q_{11}^{u2}, ..., q_{11}^{uk})\right] e^{i\min(\beta_{11}^{1},\beta_{11}^{2},.....,\beta_{11}^{k})}, \left[\max(r_{11}^{l1}, r_{11}^{l2}, ..., r_{11}^{lk}), \max(r_{11}^{u1}, r_{11}^{u2}, ..., r_{11}^{uk})\right] e^{i\min(\delta_{11}^{1},\delta_{11}^{2},.....,\delta_{11}^{k})} \right\rangle$	....	$\left\langle \left[\min(p_{1n}^{l1}, p_{1n}^{l2}, ..., p_{1n}^{lk}), \min(p_{1n}^{u1}, p_{1n}^{u2}, ..., p_{1n}^{uk})\right] e^{i\min(\alpha_{1n}^{1},\alpha_{1n}^{2},.....,\alpha_{1n}^{k})}, \left[\min(q_{1n}^{l1}, q_{1n}^{l2}, ..., q_{1n}^{lk}), \min(q_{1n}^{u1}, q_{1n}^{u2}, ..., q_{1n}^{uk})\right] e^{i\min(\beta_{1n}^{1},\beta_{1n}^{2},.....,\beta_{1n}^{k})}, \left[\max(r_{1n}^{l1}, r_{1n}^{l2}, ..., r_{1n}^{lk}), \max(r_{1n}^{u1}, r_{1n}^{u2}, ..., r_{1n}^{uk})\right] e^{i\min(\delta_{1n}^{1},\delta_{1n}^{2},.....,\delta_{1n}^{k})} \right\rangle$
x_2	$\left\langle \left[\min(p_{21}^{l1}, p_{21}^{l2}, ..., p_{21}^{lk}), \min(p_{21}^{u1}, p_{21}^{u2}, ..., p_{21}^{uk})\right] e^{i\min(\alpha_{21}^{1},\alpha_{21}^{2},.....,\alpha_{21}^{k})}, \left[\min(q_{21}^{l1}, q_{21}^{l2}, ..., q_{21}^{lk}), \min(q_{21}^{u1}, q_{21}^{u2}, ..., q_{21}^{uk})\right] e^{i\min(\beta_{21}^{1},\beta_{21}^{2},.....,\beta_{21}^{k})}, \left[\max(r_{21}^{l1}, r_{21}^{l2}, ..., r_{21}^{lk}), \max(r_{21}^{u1}, r_{21}^{u2}, ..., r_{21}^{uk})\right] e^{i\min(\delta_{21}^{1},\delta_{21}^{2},.....,\delta_{21}^{k})} \right\rangle$	....	$\left\langle \left[\min(p_{2n}^{l1}, p_{2n}^{l2}, ..., p_{2n}^{lk}), \min(p_{2n}^{u1}, p_{2n}^{u2}, ..., p_{2n}^{uk})\right] e^{i\min(\alpha_{2n}^{1},\alpha_{2n}^{2},.....,\alpha_{2n}^{k})}, \left[\min(q_{2n}^{l1}, q_{2n}^{l2}, ..., q_{2n}^{lk}), \min(q_{2n}^{u1}, q_{2n}^{u2}, ..., q_{2n}^{uk})\right] e^{i\min(\beta_{2n}^{1},\beta_{2n}^{2},.....,\beta_{2n}^{k})}, \left[\max(r_{2n}^{l1}, r_{2n}^{l2}, ..., r_{2n}^{lk}), \max(r_{2n}^{u1}, r_{2n}^{u2}, ..., r_{2n}^{uk})\right] e^{i\min(\delta_{2n}^{1},\delta_{2n}^{2},.....,\delta_{2n}^{k})} \right\rangle$
...		...	
x_m	$\left\langle \left[\min(p_{m1}^{l1}, p_{m1}^{l2}, ..., p_{m1}^{lk}), \min(p_{m1}^{u1}, p_{m1}^{u2}, ..., p_{m1}^{uk})\right] e^{i\min(\alpha_{m1}^{1},\alpha_{m1}^{2},.....,\alpha_{m1}^{k})}, \left[\min(q_{m1}^{l1}, q_{m1}^{l2}, ..., q_{m1}^{lk}), \min(q_{m1}^{u1}, q_{m1}^{u2}, ..., q_{m1}^{uk})\right] e^{i\min(\beta_{m1}^{1},\beta_{m1}^{2},.....,\beta_{m1}^{k})}, \left[\max(r_{m1}^{l1}, r_{m1}^{l2}, ..., r_{m1}^{lk}), \max(r_{m1}^{u1}, r_{m1}^{u2}, ..., r_{m1}^{uk})\right] e^{i\min(\delta_{m1}^{1},\delta_{m1}^{2},.....,\delta_{m1}^{k})} \right\rangle$	.....	$\left\langle \left[\min(p_{mn}^{l1}, p_{mn}^{l2}, ..., p_{mn}^{lk}), \min(p_{mn}^{u1}, p_{mn}^{u2}, ..., p_{mn}^{uk})\right] e^{i\min(\alpha_{mn}^{1},\alpha_{mn}^{2},.....,\alpha_{mn}^{k})}, \left[\min(q_{mn}^{l1}, q_{mn}^{l2}, ..., q_{mn}^{lk}), \min(q_{mn}^{u1}, q_{mn}^{u2}, ..., q_{mn}^{uk})\right] e^{i\min(\beta_{mn}^{1},\beta_{mn}^{2},.....,\beta_{mn}^{k})}, \left[\max(r_{mn}^{l1}, r_{mn}^{l2}, ..., r_{mn}^{lk}), \max(r_{mn}^{u1}, r_{mn}^{u2}, ..., r_{mn}^{uk})\right] e^{i\min(\delta_{mn}^{1},\delta_{mn}^{2},.....,\delta_{mn}^{k})} \right\rangle$

Table5. Worst CIVNSS $\left(\overset{\approx}{F}^{-}, E\right)$

5.1 Complex interval neutrosophic soft geometric mean aggregation operator

Aggregation of some CIVNSSs produces a CIVNSS. We have introduced the geometric mean aggregation of CIVNSSs.

Let $\left(\overset{\approx}{F}^{1}, E\right), \left(\overset{\approx}{F}^{2}, E\right), ..., \left(\overset{\approx}{F}^{k}, E\right)$ be k-CIVNSSs over a universe U and $w = \{w_1, w_2, ..., w_k\}$ be k real numbers such that $w_i \in [0,1]$, and $\sum_{i=1}^{k} w_i = 1$. Then, the complex interval neutrosophic soft geometric aggregation of k-CIVNSSs is denoted by,

$\overset{\approx}{B}_{GM}(w_1, w_2, ..., w_k)\left(\left(\overset{\approx}{F}^{1}, E\right), \left(\overset{\approx}{F}^{2}, E\right), ..., \left(\overset{\approx}{F}^{k}, E\right)\right)$, and is defined as follows:

$$\tilde{\tilde{B}}_{GM}\left(w_1,w_2,.....,w_k\right)\left(\left(\tilde{\tilde{F^1}},E\right),\left(\tilde{\tilde{F^2}},E\right),........,\left(\tilde{\tilde{F^k}},E\right)\right)=\left(\tilde{\tilde{F}},E\right)$$

$$=$$

$$\left\{\left(e_j,\left(\frac{x_s}{\left(\left(\prod_{i=1}^{k}\left(p_{s_j}^{ui}-p_{s_j}^{li}\right)e^{i\alpha_{s_j}^i}\right)\left(\prod_{i=1}^{k}\left(q_{s_j}^{ui}-q_{s_j}^{li}\right)e^{i\beta_{s_j}^i}\right)\left(\prod_{i=1}^{k}\left(r_{s_j}^{ui}-r_{s_j}^{li}\right)e^{i\delta_{s_j}^i}\right)\right)^{w_i}}\right)\right):\forall e_j\in E,x_s\in U\right\}$$

We have the following properties:

(a) $\tilde{\tilde{B}}_{GM}\left(w_1,w_2,.....,w_k\right)\left(\left(\tilde{\tilde{F^1}},E\right)_{1^{\prec}},\left(\tilde{\tilde{F^2}},E\right)_{1^{\prec}},........,\left(\tilde{\tilde{F^k}},E\right)_{1^{\prec}}\right)=\left(\tilde{\tilde{F}},E\right)_{1^{\prec}}$, where $\left(\tilde{\tilde{F}},E\right)_{1^{\prec}}$ denotes the absolute complex interval neutrosophic soft set over the universe U.

(b) $\tilde{\tilde{B}}_{GM}\left(w_1,w_2,.....,w_k\right)\left(\left(\tilde{\tilde{F^1}},E\right)_{0^{\prec}},\left(\tilde{\tilde{F^2}},E\right)_{0^{\prec}},........,\left(\tilde{\tilde{F^k}},E\right)_{0^{\prec}}\right)=\left(\tilde{\tilde{F}},E\right)_{0^{\prec}}$, where $\left(\tilde{\tilde{F}},E\right)_{0^{\prec}}$ denotes the null complex interval neutrosophic soft set over the universe U.

(c) If, for all $i=1,2,....,k$, $\left(\tilde{\tilde{F^i}},E\right)\le\left(\tilde{\tilde{G^i}},E\right)$ then

$$\tilde{\tilde{B}}_{GM}\left(w_1,w_2,.....,w_k\right)\left(\left(\tilde{\tilde{F^1}},E\right),\left(\tilde{\tilde{F^2}},E\right),........,\left(\tilde{\tilde{F^k}},E\right)\right)\le\tilde{\tilde{B}}_{GM}\left(w_1,w_2,.....,w_k\right)\left(\left(\tilde{\tilde{G^1}},E\right),\left(\tilde{\tilde{G^2}},E\right),........,\left(\tilde{\tilde{G^k}},E\right)\right);$$

where $\left(\tilde{\tilde{G^1}},E\right),\left(\tilde{\tilde{G^2}},E\right),........,\left(\tilde{\tilde{G^k}},E\right)$ be another set of K-CIVNSSs over U.

(d) $\left(\tilde{\tilde{F}}^{-},E\right)\le\tilde{\tilde{B}}_{GM}\left(w_1,w_2,.....,w_k\right)\left(\left(\tilde{\tilde{F^1}},E\right),\left(\tilde{\tilde{F^2}},E\right),........,\left(\tilde{\tilde{F^k}},E\right)\right)\le\left(\tilde{\tilde{F}}^{+},E\right)$.

6. Construction of an algorithm by using complex interval neutrosophic soft sets aggregate operator

In this section, a step-wise method is described by using complex interval neutrosophic soft sets aggregate operators and it is useful and effective to deal with real-life decision-making problems.

Step1: Input a set of m alternatives $U=\{u_1,u_2,.....,u_m\}$ that have been defined by k-experts $D=\{d_1,d_2,.....,d_k\}$ concerning for to n complex interval neutrosophic parameters $E=\{e_1,e_2,.....,e_n\}$.

Step2: Opinions of K-experts have been described by K-CIVNSSs $\left(\overset{\approx}{F^1},E\right),\left(\overset{\approx}{F^2},E\right),......,\left(\overset{\approx}{F^k},E\right)$ defined as

$$\left(\overset{\approx}{F^l},E\right)=\left\{\left(e_j,\overset{\approx}{F}\left(e_j\right)\right):\forall e_j\in E\right\}=\left\{e_j,\left(x_s/\left(p^{lF}_{s_j}e^{il\mu^F_{s_j}},q^{lF}_{s_j}e^{il\nu^F_{s_j}},r^{lF}_{s_j}e^{il\delta^F_{s_j}}\right)\right):\forall e_j\in E,x_s\in U\right\}$$

Where $p^{lF}_{s_j},q^{lF}_{s_j},r^{lF}_{s_j}\subseteq[0,1]$ are the amplitude parts of the truth-membership, indeterminacy-membership ,and falsity-membership values respectively and $\mu^{lF}_{s_j},\nu^{lF}_{s_j},\delta^{lF}_{s_j}\in[0,2\pi]$ are the respective phase parts of the evaluation of an alternative x_s concerning for to the parameter e_j over the CIVNSS $\left(\overset{\approx}{F},E\right)$.

Step3: Construct the best $\left(\overset{\approx}{F^+},E\right)$ and the worst $\left(\overset{\approx}{F^-},E\right)$ CIVNSS over K-CIVNSSs.

Step4: Evaluate the approximate index $P_l\left(d_l\right)$ of an expert d_l is given by,

$$P_l\left(d_l\right)=\frac{\hat{S}\left(\left(\overset{\approx}{F^+},E\right),\left(\overset{\approx}{F^l},E\right)\right)}{\hat{S}\left(\left(\overset{\approx}{F^+},E\right),\left(\overset{\approx}{F^l},E\right)\right)+\hat{S}\left(\left(\overset{\approx}{F^-},E\right),\left(\overset{\approx}{F^l},E\right)\right)}$$
, where $\hat{S}$ indicates similarity measure.

Step5: Measure the nearness index $C_l\left(d_l\right)$ of an expert d_l is given by,

$$C_l\left(d_l\right)=\sum_{l\neq l',l'=1}^{k}\frac{\hat{S}\left(\left(\overset{\approx}{F^l},E\right),\left(\overset{\approx}{F^{l'}},E\right)\right)}{l+1}$$

Step6: Derive the preference rate $\varpi\left(d_l\right)$ of an expert d_l as,

$\varpi(d_l)=\frac{(P_l(d_l)\otimes C_l(d_l))}{\sum_{l=1}^{k}(P_l(d_l)\otimes C_l(d_l))}$, where $\otimes$ denotes the linear product.

Step7: Construct the collective CIVNSS $\left(\overset{\approx}{F},E\right)$ from K-CIVNSSs which is derived by using complex interval neutrosophic soft geometric mean aggregation operators as follows:

$$\overset{\approx}{B}_{GM}(w_1,w_2,.....,w_k)\left(\left(\overset{\approx}{F^1},E\right),\left(\overset{\approx}{F^2},E\right),........,\left(\overset{\approx}{F^k},E\right)\right)$$

Step8: Determined the combined weight of a parameter e_j as follows:

$$W_j=\frac{\lambda(w_j^*)\otimes(1-\lambda)(w_j^{\#})}{\sum_{j=1}^{n}\left(\lambda(w_j^*)\otimes(1-\lambda)(w_j^{\#})\right)}\text{, where } w_j^*=\frac{1}{k}\left(w_j^1\otimes w_j^2\otimes........\otimes w_j^k\right);$$

$$w_j^{\#}=\frac{1}{k(k-1)}\times\frac{\sum_{S'=1,\,S\neq S'}^{k}\left\{\frac{1}{2}\left(\left(\left(p_{s_j}^u-p_{s_j}^l\right)\mu_{s_j}\right)^2+\left(\left(q_{s_j}^u-q_{s_j}^l\right)\nu_{s_j}\right)^2+\left(\left(r_{s_j}^u-r_{s_j}^l\right)\delta_{s_j}\right)^2\right)\right\}^{\frac{1}{2}}}{\sum_{j=1}^{k}\sum_{S'=1,S\neq S'}^{k}\left\{\frac{1}{2}\left(\left(\left(p_{s_j}^u-p_{s_j}^l\right)\mu_{s_j}\right)^2+\left(\left(q_{s_j}^u-q_{s_j}^l\right)\nu_{s_j}\right)^2+\left(\left(r_{s_j}^u-r_{s_j}^l\right)\delta_{s_j}\right)^2\right)\right\}^{\frac{1}{2}}}$$

,and λ is the influence parameter such that $\lambda\in[0,1]$.

Since, experts came from different environments along with different specialization, judgment powers, and knowledge, so they may impose different weights on the associated parameters. If $w^l=\{w_1^l,w_2^l,w_3^l,.....,w_n^l\}$ be the associated weights of the parameters given by an expert d_l such that $\sum_{j=1}^{n}w_j^l=1$,and $w_j^l\in[0,1]$.

Step9: Select the best alternative by determining the upper-alternative $\left(\tilde{X}\right)$ and the lower-alternative $\left(\underset{\sim}{X}\right)$ as follows:

$$\tilde{X} = \left\{ e_j, \left(\begin{array}{l} \left(\left(p_{1j}^u - p_{1j}^l \right) e^{i\mu_{1j}} \otimes \left(q_{1j}^u - q_{1j}^l \right) e^{i\nu_{1j}} \otimes \left(r_{1j}^u - r_{1j}^l \right) e^{i\delta_{1j}} \right) \cup \left(\left(p_{2j}^u - p_{2j}^l \right) e^{i\mu_{2j}} \otimes \left(q_{2j}^u - q_{2j}^l \right) e^{i\nu_{2j}} \otimes \left(r_{2j}^u - r_{2j}^l \right) e^{i\delta_{2j}} \right) \ldots\ldots \\ \cup \left(\left(p_{mj}^u - p_{mj}^l \right) e^{i\mu_{mj}} \otimes \left(q_{mj}^u - q_{mj}^l \right) e^{i\nu_{mj}} \otimes \left(r_{mj}^u - r_{mj}^l \right) e^{i\delta_{mj}} \right) \end{array} \right) : \forall e_j \in E \right\}$$

$$= \left\{ \left(e_j, \tilde{P} e^{i\tilde{\mu}_j}, \tilde{Q} e^{i\tilde{\nu}_j}, \tilde{R} e^{i\tilde{\delta}_j} \right) : \forall e_j \in E \right\}$$

$$\underset{\sim}{X} = \left\{ e_j, \left(\begin{array}{l} \left(\left(p_{1j}^u - p_{1j}^l \right) e^{i\mu_{1j}} \otimes \left(q_{1j}^u - q_{1j}^l \right) e^{i\nu_{1j}} \otimes \left(r_{1j}^u - r_{1j}^l \right) e^{i\delta_{1j}} \right) \cap \left(\left(p_{2j}^u - p_{2j}^l \right) e^{i\mu_{2j}} \otimes \left(q_{2j}^u - q_{2j}^l \right) e^{i\nu_{2j}} \otimes \left(r_{2j}^u - r_{2j}^l \right) e^{i\delta_{2j}} \right) \ldots\ldots \\ \cap \left(\left(p_{mj}^u - p_{mj}^l \right) e^{i\mu_{mj}} \otimes \left(q_{mj}^u - q_{mj}^l \right) e^{i\nu_{mj}} \otimes \left(r_{mj}^u - r_{mj}^l \right) e^{i\delta_{mj}} \right) \end{array} \right) : \forall e_j \in E \right\}$$

$$= \left\{ \left(e_j, \underset{\sim}{P} e^{i\underset{\sim}{\mu}_j}, \underset{\sim}{Q} e^{i\underset{\sim}{\nu}_j}, \underset{\sim}{R} e^{i\underset{\sim}{\delta}_j} \right) : \forall e_j \in E \right\}$$

Step10: Determine the separation level of $\left(\tilde{X}\right)$ and $\left(\underset{\sim}{X}\right)$ as follows:

$$D\left(\frac{x_s}{\tilde{X}}\right) = \left(\frac{1}{2n} \left\{ \begin{array}{l} \left(w_1^2 \otimes \left(\left(\frac{p_{s_1}^l + p_{s_1}^u}{2} - \tilde{p}_1 \right) \left(\mu_{s_1} - \tilde{\mu}_{s_1} \right) \right)^2 \otimes \left(\left(\frac{q_{s_1}^l + q_{s_1}^u}{2} - \tilde{q}_1 \right) \left(\nu_{s_1} - \tilde{\nu}_{s_1} \right) \right)^2 \otimes \left(\left(\frac{r_{s_1}^l + r_{s_1}^u}{2} - \tilde{r}_1 \right) \left(\delta_{s_1} - \tilde{\delta}_{s_1} \right) \right)^2 \right) + \\ \left(w_2^2 \otimes \left(\left(\frac{p_{s_2}^l + p_{s_2}^u}{2} - \tilde{p}_2 \right) \left(\mu_{s_2} - \tilde{\mu}_{s_2} \right) \right)^2 \otimes \left(\left(\frac{q_{s_2}^l + q_{s_2}^u}{2} - \tilde{q}_2 \right) \left(\nu_{s_2} - \tilde{\nu}_{s_2} \right) \right)^2 \otimes \left(\left(\frac{r_{s_2}^l + r_{s_2}^u}{2} - \tilde{r}_2 \right) \left(\delta_{s_2} - \tilde{\delta}_{s_2} \right) \right)^2 \right) + \\ \ldots\ldots\ldots\ldots\ldots + \left(w_n^2 \otimes \left(\left(\frac{p_{s_n}^l + p_{s_n}^u}{2} - \tilde{p}_n \right) \left(\mu_{s_n} - \tilde{\mu}_{s_n} \right) \right)^2 \otimes \left(\left(\frac{q_{s_n}^l + q_{s_n}^u}{2} - \tilde{q}_n \right) \left(\nu_{s_n} - \tilde{\nu}_{s_n} \right) \right)^2 \otimes \left(\left(\frac{r_{s_n}^l + r_{s_n}^u}{2} - \tilde{r}_n \right) \left(\delta_{s_n} - \tilde{\delta}_{s_n} \right) \right)^2 \right) \end{array} \right\} \right)^{\frac{1}{2}}$$

$$D\left(\frac{x_s}{\underset{\sim}{X}}\right) = \left(\frac{1}{2n} \left\{ \begin{array}{l} \left(w_1^2 \otimes \left(\left(\frac{p_{s_1}^l + p_{s_1}^u}{2} - \underset{\sim}{p}_1 \right) \left(\mu_{s_1} - \underset{\sim}{\mu}_{s_1} \right) \right)^2 \otimes \left(\left(\frac{q_{s_1}^l + q_{s_1}^u}{2} - \underset{\sim}{q}_1 \right) \left(\nu_{s_1} - \underset{\sim}{\nu}_{s_1} \right) \right)^2 \otimes \left(\left(\frac{r_{s_1}^l + r_{s_1}^u}{2} - \underset{\sim}{r}_1 \right) \left(\delta_{s_1} - \underset{\sim}{\delta}_{s_1} \right) \right)^2 \right) + \\ \left(w_2^2 \otimes \left(\left(\frac{p_{s_2}^l + p_{s_2}^u}{2} - \underset{\sim}{p}_2 \right) \left(\mu_{s_2} - \underset{\sim}{\mu}_{s_2} \right) \right)^2 \otimes \left(\left(\frac{q_{s_2}^l + q_{s_2}^u}{2} - \underset{\sim}{q}_2 \right) \left(\nu_{s_2} - \underset{\sim}{\nu}_{s_2} \right) \right)^2 \otimes \left(\left(\frac{r_{s_2}^l + r_{s_2}^u}{2} - \underset{\sim}{r}_2 \right) \left(\delta_{s_2} - \underset{\sim}{\delta}_{s_2} \right) \right)^2 \right) + \\ \ldots\ldots\ldots\ldots\ldots + \left(w_n^2 \otimes \left(\left(\frac{p_{s_n}^l + p_{s_n}^u}{2} - \underset{\sim}{p}_n \right) \left(\mu_{s_n} - \underset{\sim}{\mu}_{s_n} \right) \right)^2 \otimes \left(\left(\frac{q_{s_n}^l + q_{s_n}^u}{2} - \underset{\sim}{q}_n \right) \left(\nu_{s_n} - \underset{\sim}{\nu}_{s_n} \right) \right)^2 \otimes \left(\left(\frac{r_{s_n}^l + r_{s_n}^u}{2} - \underset{\sim}{r}_n \right) \left(\delta_{s_n} - \underset{\sim}{\delta}_{s_n} \right) \right)^2 \right) \end{array} \right\} \right)^{\frac{1}{2}}$$

Step11: Obtain the ranking index of an alternative x_s by using the formula

$$\tilde{R}(x_s) = \frac{D\left(\frac{x_s}{\tilde{X}}\right)}{D\left(\frac{x_s}{\tilde{X}}\right) + D\left(\frac{x_s}{\underset{\sim}{X}}\right)}; s = 1, 2, \ldots, m$$

The alternative having a maximum ranking index $\tilde{R}$ will be selected as the best or optimal alternative for this multi-expert decision-making. If more than one alternative has the same maximum ranking index, then, we will select any one of them as an optimal solution.

7. An application on a financial problem

A trader wants to set up a car manufacturing company and the set of alternatives $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ represent a set of six cars among which he or she has to choose any one of the alternatives which fulfilled all the pre-assigned criteria. Selection of any one of the alternatives influenced by the set of parameters $E = \{e_1, e_2, e_3, e_4, e_5\}$. Here, the parameters stand for land, labor, capital, entrepreneurship, and raw material cost respectively. Now, a set of four experts denoted by $D = \{d_1, d_2, d_3, d_4\}$ have been assigned for monitoring the parameters to reach a common decision about which a car manufacturing company is more likely to choose which have these parametric characters. Here, the belongingness level of a parameter has been taken through the amplitude part (interval form due to the more complexity involved in the problem which has neutrosophic nature) and the time duration of a parameter has been taken through the phase part. All the data has been collected by the decision-makers on 20 consecutive days. To express this data in the interval $[0, 2\pi]$, 2π has been taken here instead of 20 days.

About of on this idea and by using the algorithm discussed in section6, the trader will be able to choose the best alternatives and it is possible when all the decision-makers come to a common solution and it is possible due to the aggregate operators used in the said algorithm. The calculation part is left for the readers as an exercise.

8. Conclusions

In this article, we first give the basic definition of complex interval neutrosophic soft sets and some basic operations on them. We then discuss similarity measures on complex interval neutrosophic soft sets and their aggregation. An algorithm has been introduced by using complex interval neutrosophic soft sets aggregate operators. To apply the algorithm to the decision-making problem we give an application that shows the algorithm can be successfully applied in financial problems. In the future, there is a scope to extend the notion of complex interval neutrosophic soft set by introducing hypersoft set introduced by Smarandache [12] in 2018. Also, the comlex interval neutrosophic soft set may be applied comprehensively in different fields such as engineering, medical science, finance, game theory, computer science, decision-making,etc.

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A Model Describing the Neutrosophic Differential Equation and Its Application On Mine Safety

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Abstract. In the theory of uncertainty and approximation neutrosophy plays a significant role. Neutrosophy is tool emerged on standard or non-standard to measured the mathematical model of uncertainty, vagueness, ambiguity etc. In light of these major issues, the paper outlines of Neutrosophic Set, Single Valued Neutrosophic Set, Triangular Single Valued Neutrosophic Number and Trapezoidal Single Valued Neutrosophic Number. It also propose Neutrosophic Differential Equation and shown its solution in different conditions. Thereafter mining safety model via Single Valued neutrosophic number be epitomized. At last a mathematical experiment is done to exhibit its reality and use fullness of this Number.

Keywords: neutrosophic set(NS); single valued neutrosophic set(SVNS); triangular single valued neutrosophic number(TSVNNs); trapezoidal single valued neutrosophic number(TrSVNNs); neutrosophic differential equation(NDE); mining safety model

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1. **Introduction**

NS highlights the origin and nature of neutralise in different fields which is the generalization of classical set, fuzzy set(FS), intuitionistic fuzzy set(IFS) etc. Gradually varying value is used in FS theory rather than precise or sharp value. In 1965 [1], a famous paper was published by Prof. L.A. Zadeh as "Fuzzy sets" in "Information and Control" that provided some new mathematical tool which enable us to described and handle dubious or unclear notions. FS theory, only shows membership degree and do not provide any idea about non-membership degree. In reality, this linguistic statement don't fulfill the logical statement. When choosing the membership degree there may exist some types of doubtfulness or absence of information are present while defining the membership. Due to this doubtfulness, an idea of IFS as generalization of FS was introduced by Atanassov in 1983 [2]. IFS consider both membership and non-membership function. IFS only pick up incomplete information. In 2003 [3], A new concept, say, NS was innovated by Smarandache. It deals with the study of origin, nature and scope of neutralise, as well as their interaction with different idealism spectra. NS is the generalization of CS, FS, IFS and so on. A NS can be distinguish by a truth membership function 'μ_T', an indeterminacy membership function 'ν_I', and a falsity membership function 'σ_F'. In NS: μ_T, ν_I and σ_F are not dependent, which is useful in situations such as information fusion. In NS μ_T, ν_I, σ_F being the real standard or non standard subset of $^{-}]0,1[^{+}$,moreover in SNVS, μ_T, ν_I, σ_F be the subset of [0,1]. From philosophical point of view, NS generalised the above mentioned sets but from scientific or engineering point of view, need to be defined. Else it's difficult to apply in many real application.

It is much noticed that when modeling some problems related to physical science and engineering, where the parameters are unknown but performed in an interval. Before, the application of interval arithmetic managed such circumstances, where mathematical calculation is done on intervals to get the estimate of target quantities in respective intervals. Fuzzy arithmetic is the generalization of the intervals arithmetic. As the principle definition of FS which approve gradation of membership for an element of the Universal set. So the situation of the modeling based on fuzzy arithmetic is awaited to publish more realistically. There are several types of fuzzy number are exist. These are applied in Decision-making problem and so on [4]. But it is not efficient for any application where the knowledge about membership degree is lacking. Latter generalization it to intuitionistic fuzzy number [5] were developed. In these paper we define several types of neutrosophic numbers and their cuts.

In the field of science & engineering, differentiation takes on an evidential role. Many problems stand up with uncertain or imprecise parameters. Due to this naiveness, we bear upon the differential equation with imprecise parameters. Fuzzy differential equation [6] has been proposed to model this uncertainty. However, it consider only membership value. Later, intuitionistic fuzzy differential [7] equation was founded with degree of membership and non-membership function. However, the term indeterminacy is absent in the above logic's. Hence, neutrosophic differential eqaution(NDE) [8–10] was developed to model indeterminacy. In this paper, a mining safety model describe [11], this model consist of three differential equations, those differential equations describe via Single Valued Neutrosophic Number(SVNNs). The solution of the equation is describe later.

In reality, the collected data, in many situations, it was observed that is insufficient and transmit some misinformation. As a result, the solution obtained from these data suffers with insufficiency and inconsistency. In these situations, the neutrosophic sets offer better result.

We have designed the paper in the following way: Section-2 gives some preliminaries concept and definition. Section-3 contains definition of NDE. Section-4 contains solution of NDE with numerical example. Section-5 contains Mining Safety model. Section-6 contains Mining safety model formulation. Section-7 described solution mode of the model. Section-8 contains numerical experiment and consequently, conclusions are discussed in Section-9. The references are shown in Section-10.

2. Preliminaries

2.1. *Definition of NS* [12]

Let $\mathfrak{U}$ be a Universal set. A NS $\tilde{\mathcal{A}}^{NS}$ of $\mathfrak{U}$ be defined by $\tilde{\mathcal{A}}^{NS} = \langle(\mathfrak{u}; \mu_T(\mathfrak{u}), \nu_I(\mathfrak{u}), \sigma_F(\mathfrak{u})) : \mathfrak{u} \in \mathfrak{U}\rangle$ where $\mu_T(\mathfrak{u}), \nu_I(\mathfrak{u}), \sigma_F(\mathfrak{u})$ be outlined as the truth membership, indeterminacy membership, falsity membership grade of $\mathfrak{u}$ in $\tilde{\mathcal{A}}^{NS}$ which are real standard or non-standard subsets of $^{-}]0, 1[^{+}$ & $\mu_T(\mathfrak{u}) + \nu_I(\mathfrak{u}) + \sigma_F(\mathfrak{u}) \leqslant 3^{+}$.

2.2. *Definition of SVNS* [12]

Let $\mathfrak{U}$ be a Universal set. A SVNS $\tilde{\mathcal{A}}^{Ne}$ of $\mathfrak{U}$ be defined by $\tilde{\mathcal{A}}^{Ne} = \langle(\mathfrak{u}; \mu_T(\mathfrak{u}), \nu_I(\mathfrak{u}), \sigma_F(\mathfrak{u})) : \mathfrak{u} \in \mathfrak{U}\rangle$ where $\mu_T(\mathfrak{u}), \nu_I(\mathfrak{u}), \sigma_F(\mathfrak{u})$ be outlined as the truth membership, indeterminacy membership, falsity membership grade of $\mathfrak{u}$ in $\tilde{\mathcal{A}}^{Ne}$ which are subset of $[0, 1]$ & $\mu_T(\mathfrak{u}) + \nu_I(\mathfrak{u}) + \sigma_F(\mathfrak{u}) \leqslant 3$.

2.3. *Definition of TSVNNs* [8]

A TSVNNs is denoted by $\tilde{\mathcal{A}}^{Ne} = \langle \mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{a}'_3; w_\mu, w_\nu, w_\sigma \rangle$ whose truth, indeterminacy and falsity membership functions are defined by

$$\mu_T(\mathfrak{u}) = \begin{cases} (\frac{\mathfrak{u}-\mathfrak{a}'_1}{\mathfrak{a}'_2-\mathfrak{a}'_1})w_\mu & \text{when } \mathfrak{a}'_1 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_2 \\ w_\mu & \text{when } \mathfrak{u} = \mathfrak{a}'_2 \\ (\frac{\mathfrak{a}'_3-\mathfrak{u}}{\mathfrak{a}'_3-\mathfrak{a}'_2})w_\mu & \text{when } a'_2 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_3 \\ 0 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}'_1 \text{ or } \mathfrak{u} \geqslant \mathfrak{a}'_3 \end{cases}$$

$$\nu_I(\mathfrak{u}) = \begin{cases} \frac{(\mathfrak{a}'_2-\mathfrak{u})+(\mathfrak{u}-\mathfrak{a}'_1)w_\nu}{\mathfrak{a}'_2-\mathfrak{a}'_1} & \text{when } \mathfrak{a}'_1 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_2 \\ w_\nu & \text{when } \mathfrak{u} = \mathfrak{a}'_2 \\ \frac{(\mathfrak{u}-\mathfrak{a}'_2)+(\mathfrak{a}'_3-\mathfrak{u})w_\nu}{\mathfrak{a}'_3-\mathfrak{a}'_2} & \text{when } \mathfrak{a}'_2 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_3 \\ 1 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}'_1 \text{ or } \mathfrak{u} \geqslant \mathfrak{a}'_3 \end{cases}$$

$$\sigma_F(\mathfrak{u}) = \begin{cases} \frac{(\mathfrak{a}_2'-\mathfrak{u})+(\mathfrak{u}-\mathfrak{a}_1')w_\sigma}{\mathfrak{a}_2'-\mathfrak{a}_1'} & \text{when } \mathfrak{a}_1' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_2' \\ w_\sigma & \text{when } \mathfrak{u} = \mathfrak{a}_2' \\ \frac{(\mathfrak{u}-\mathfrak{a}_2')+(\mathfrak{a}_3'-\mathfrak{u})w_\sigma}{\mathfrak{a}_3'-\mathfrak{a}_2'} & \text{when } \mathfrak{a}_2' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_3' \\ 1 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}_1' \text{ or } \mathfrak{u} \geqslant \mathfrak{a}_3' \end{cases}$$

where $\mu_T(\mathfrak{u}) + \nu_I(\mathfrak{u}) + \sigma_F(\mathfrak{u}) \leqslant 3$ & $w_\mu \in (0,1]$, $w_\nu, w_\sigma \in [0,1)$.

2.4. ***Definition of TrSVNNs*** *[9]*

A TrSVNNs is denoted by $\tilde{\mathcal{A}}^{Ne} = \langle \mathfrak{a}_1', \mathfrak{a}_2', \mathfrak{a}_3', \mathfrak{a}_4'; w_\mu, w_\nu, w_\sigma \rangle$ whose truth, indeterminacy and falsity membership functions are defined by

$$\mu_T(\mathfrak{u}) = \begin{cases} (\frac{\mathfrak{u}-\mathfrak{a}_1'}{\mathfrak{a}_2'-\mathfrak{a}_1'})w_\mu & \text{when } \mathfrak{a}_1' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_2' \\ w_\mu & \text{when } \mathfrak{a}_2' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_3' \\ (\frac{\mathfrak{a}_4'-\mathfrak{u}}{\mathfrak{a}_4'-\mathfrak{a}_3'})w_\mu & \text{when } a_3' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_4' \\ 0 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}_1' \text{ or } \mathfrak{u} \geqslant \mathfrak{a}_4' \end{cases}$$

$$\nu_I(\mathfrak{u}) = \begin{cases} \frac{(\mathfrak{a}_2'-\mathfrak{u})+(\mathfrak{u}-\mathfrak{a}_1')w_\nu}{\mathfrak{a}_2'-\mathfrak{a}_1'} & \text{when } \mathfrak{a}_1' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_2' \\ w_\nu & \text{when } \mathfrak{a}_2' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_3' \\ \frac{(\mathfrak{u}-\mathfrak{a}_3')+(\mathfrak{a}_4'-\mathfrak{u})w_\nu}{\mathfrak{a}_4'-\mathfrak{a}_3'} & \text{when } \mathfrak{a}_3' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_4' \\ 1 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}_1' \text{ or } \mathfrak{u} \geqslant \mathfrak{a}_4' \end{cases}$$

$$\sigma_F(\mathfrak{u}) = \begin{cases} \frac{(\mathfrak{a}_2'-\mathfrak{u})+(\mathfrak{u}-\mathfrak{a}_1')w_\sigma}{\mathfrak{a}_2'-\mathfrak{a}_1'} & \text{when } \mathfrak{a}_1' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_2' \\ w_\sigma & \text{when } \mathfrak{a}_2' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_3' \\ \frac{(\mathfrak{u}-\mathfrak{a}_3')+(\mathfrak{a}_4'-\mathfrak{u})w_\sigma}{\mathfrak{a}_4'-\mathfrak{a}_3'} & \text{when } \mathfrak{a}_3' \leqslant \mathfrak{u} \leqslant \mathfrak{a}_4' \\ 1 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}_1' \text{ or } \mathfrak{u} \geqslant \mathfrak{a}_4' \end{cases}$$

where $\mu_T(\mathfrak{u}) + \nu_I(\mathfrak{u}) + \sigma_F(\mathfrak{u}) \leqslant 3$ & $w_\mu \in (0,1]$, $w_\nu, w_\sigma \in [0,1)$.

2.5. ***Cut Set*** *[8]*

Let $\tilde{\mathcal{A}}^{Ne}$ be any SVNS, then (r, β, γ)-cut of SVNS is denoted by $\tilde{\mathcal{A}}^{Ne}(r, \beta, \gamma)$and it is defined by $\tilde{\mathcal{A}}^{Ne}(r, \beta, \gamma)$=$\langle \mathfrak{u} \in \mathfrak{U} : \mu_T(\mathfrak{u}) \geqslant r, \nu_I(\mathfrak{u}) \leqslant \beta, \sigma_F(\mathfrak{u}) \leqslant \gamma; 0 < r \leqslant 1, 0 \leqslant \beta < 1, 0 \leqslant \gamma < 1 \rangle$.

2.6. ***Operation Using SVNNs:*** *[13]*

Consider two TSVNNs, $\tilde{\mathcal{A}}^{Ne}$=$\langle \mathfrak{a}_1', \mathfrak{a}_2', \mathfrak{a}_3'; w_\mu, w_\nu, w_\sigma \rangle$; $\tilde{\mathcal{B}}^{Ne}$=$\langle \mathfrak{b}_1', \mathfrak{b}_2', \mathfrak{b}_3'; u_\mu, u_\nu, u_\sigma \rangle$, the following operation are:

• **Addition:**

$\tilde{\mathcal{A}}^{Ne}+\tilde{\mathcal{B}}^{Ne} = \langle[(\mathfrak{a}_1'+\mathfrak{b}_1',\mathfrak{a}_2'+\mathfrak{b}_2',\mathfrak{a}_3'+\mathfrak{b}_3');w_\mu\wedge u_\mu,w_\nu\vee u_\nu,w_\sigma\vee u_\sigma]\rangle$

• **Substraction:**

$\tilde{\mathcal{A}}^{Ne}$-$\tilde{\mathcal{B}}^{Ne} = \langle[(\mathfrak{a}_1'-\mathfrak{b}_3',\mathfrak{a}_2'-\mathfrak{b}_2',\mathfrak{a}_3'-\mathfrak{b}_1');w_\mu\wedge u_\mu,w_\nu\vee u_\nu,w_\sigma\vee u_\sigma]\rangle$

• **Multiplication:**

$\tilde{\mathcal{A}}^{Ne}.\tilde{\mathcal{B}}^{Ne} = \langle[(\mathfrak{a}_1'\mathfrak{b}_1',\mathfrak{a}_2'\mathfrak{b}_2',\mathfrak{a}_3'\mathfrak{b}_3');w_\mu\wedge u_\mu,w_\nu\vee u_\nu,w_\sigma\vee u_\sigma]\rangle$

• **Division:**

$\frac{\tilde{\mathcal{A}}^{Ne}}{\tilde{\mathcal{B}}^{Ne}}=\langle[(\frac{\mathfrak{a}_1'}{\mathfrak{b}_3'},\frac{\mathfrak{a}_2'}{\mathfrak{b}_2'},\frac{\mathfrak{a}_3'}{\mathfrak{b}_1'});w_\mu\wedge u_\mu,w_\nu\vee u_\nu,w_\sigma\vee u_\sigma]\rangle$

Where $\wedge = Min, \vee = Max$

3. Definition of NDE: [8]

Consider an Ordinary differential equation $\frac{d\mathbb{Y}}{d\mathfrak{t}} = \mathcal{K}\mathbb{Y}$, $\mathfrak{t}\in[0,\infty)$ with initial condition(IC) $\mathbb{Y}(\mathfrak{t}_0)=\mathbb{Y}_0$. The above ODE is called NDE if any one of the following three cases hold:

(i) $\tilde{\mathcal{K}}^{Ne}$ is SVNNs & $\mathbb{Y}_0$ is Crisp number.

(ii) $\mathcal{K}$ is Crisp number & $\tilde{\mathbb{Y}_0}^{Ne}$ is SVNNs.

(iii) Both $\tilde{\mathcal{K}}^{Ne}$ & $\tilde{\mathbb{Y}_0}^{Ne}$ are SVNNs.

Let the classical solution [14] be $\tilde{\mathbb{Y}}^{Ne}(\mathfrak{t})$ and its Cut be $\mathbb{Y}(\mathfrak{t},r,\beta,\gamma)$=$\langle[\mathbb{Y}_1(\mathfrak{t},r),\mathbb{Y}_2(\mathfrak{t},r)],[\mathbb{Y}_1'(\mathfrak{t},\beta),\mathbb{Y}_2'(\mathfrak{t},\beta)],[\mathbb{Y}_1''(\mathfrak{t},\gamma),\mathbb{Y}_2''(\mathfrak{t},\gamma)]\rangle$.

The solution is strong if

(i) $\frac{d\mathbb{Y}_1(\mathfrak{t},r)}{dr}>0$, $\frac{d\mathbb{Y}_2(\mathfrak{t},r)}{dr}<0\ \forall\ r\in(0,1]$, $\mathbb{Y}_1(\mathfrak{t},1)\leqslant\mathbb{Y}_2(\mathfrak{t},1)$

(ii) $\frac{d\mathbb{Y}_1'(\mathfrak{t},\beta)}{d\beta}<0$, $\frac{d\mathbb{Y}_2'(\mathfrak{t},\beta)}{d\beta}>0\ \forall\ \beta\in[0,1)$, $\mathbb{Y}_1'(\mathfrak{t},0)\leqslant\mathbb{Y}_2'(\mathfrak{t},0)$

(iii) $\frac{d\mathbb{Y}_1''(\mathfrak{t},\gamma)}{d\gamma}<0$, $\frac{d\mathbb{Y}_2''(\mathfrak{t},\gamma)}{d\gamma}>0\ \forall\ \gamma\in[0,1)$, $\mathbb{Y}_1''(\mathfrak{t},0)\leqslant\mathbb{Y}_2''(\mathfrak{t},0)$

Otherwise the solution is weak solution.

4. Solution of NDE

(i) $\tilde{\mathcal{K}}^{Ne}$ is SVNNs & $\mathbb{Y}_0$ is Crisp number.

Case 1 When Sign of $\tilde{\mathcal{K}}^{Ne}$ is positive.

Therefore required solutions are

$\mathbb{Y}_1(\mathfrak{t},r)=\mathbb{Y}_0e^{\mathbb{K}_1(r)(\mathfrak{t}-\mathfrak{t}_o)}$; $\mathbb{Y}_2(\mathfrak{t},r)=\mathbb{Y}_0e^{\mathbb{K}_2(r)(\mathfrak{t}-\mathfrak{t}_o)}$

$\mathbb{Y}_1'(\mathfrak{t},\beta)=\mathbb{Y}_0e^{\mathbb{K}_1'(\beta)(\mathfrak{t}-\mathfrak{t}_o)}$; $\mathbb{Y}_2'(\mathfrak{t},\beta)=\mathbb{Y}_0e^{\mathbb{K}_2'(\beta)(\mathfrak{t}-\mathfrak{t}_o)}$

$\mathbb{Y}_1''(\mathfrak{t},\gamma)=\mathbb{Y}_0e^{\mathbb{K}_1''(\gamma)(\mathfrak{t}-\mathfrak{t}_o)}$; $\mathbb{Y}_2''(\mathfrak{t},\gamma)=\mathbb{Y}_0e^{\mathbb{K}_2''(\gamma)(\mathfrak{t}-\mathfrak{t}_o)}$

Case 2 When Sign of $\tilde{\mathcal{K}}^{Ne}$ is negative.

Therefore required solutions are

$$\mathbb{Y}_1(\mathfrak{t},r)=\frac{\mathbb{Y}_0}{2}[(1+\sqrt{\frac{\mathbb{K}_2(r)}{\mathbb{K}_1(r)}})e^{-\sqrt{\mathbb{K}_1(r)\mathbb{K}_2(r)}(\mathfrak{t}-\mathfrak{t}_\circ)}+(1-\sqrt{\frac{\mathbb{K}_2(r)}{\mathbb{K}_1(r)}})e^{\sqrt{\mathbb{K}_1(r)\mathbb{K}_2(r)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

$$\mathbb{Y}_2(\mathfrak{t},r)=\frac{\mathbb{Y}_0}{2}[(\sqrt{\frac{\mathbb{K}_1(r)}{\mathbb{K}_2(r)}}+1)e^{-\sqrt{\mathbb{K}_1(r)\mathbb{K}_2(r)}(\mathfrak{t}-\mathfrak{t}_\circ)}-(\sqrt{\frac{\mathbb{K}_1(r)}{\mathbb{K}_2(r)}}-1)e^{\sqrt{\mathbb{K}_1(r)\mathbb{K}_2(r)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

$$\mathbb{Y}'_1(\mathfrak{t},\beta)=\frac{\mathbb{Y}_0}{2}[(1+\sqrt{\frac{\mathbb{K}'_2(\beta)}{\mathbb{K}'_1(\beta)}})e^{-\sqrt{\mathbb{K}'_1(\beta)\mathbb{K}'_2(\beta)}(\mathfrak{t}-\mathfrak{t}_\circ)}+(1-\sqrt{\frac{\mathbb{K}'_2(\beta)}{\mathbb{K}'_1(\beta)}})e^{\sqrt{\mathbb{K}'_1(\beta)\mathbb{K}'_2(\beta)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

$$\mathbb{Y}'_2(\mathfrak{t},\beta)=\frac{\mathbb{Y}_0}{2}[(\sqrt{\frac{\mathbb{K}'_1(\beta)}{\mathbb{K}'_2(\beta)}}+1)e^{-\sqrt{\mathbb{K}'_1(\beta)\mathbb{K}'_2(\beta)}(\mathfrak{t}-\mathfrak{t}_\circ)}-(\sqrt{\frac{\mathbb{K}'_1(\beta)}{\mathbb{K}'_2(\beta)}}-1)e^{\sqrt{\mathbb{K}'_1(\beta)\mathbb{K}'_2(\beta)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

$$\mathbb{Y}''_1(\mathfrak{t},\gamma)=\frac{\mathbb{Y}_0}{2}[(1+\sqrt{\frac{\mathbb{K}''_2(\gamma)}{\mathbb{K}''_1(\gamma)}})e^{-\sqrt{\mathbb{K}''_1(\gamma)\mathbb{K}''_2(\gamma)}(\mathfrak{t}-\mathfrak{t}_\circ)}+(1-\sqrt{\frac{\mathbb{K}''_2(\gamma)}{\mathbb{K}''_1(\gamma)}})e^{\sqrt{\mathbb{K}''_1(\gamma)\mathbb{K}''_2(\gamma)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

$$\mathbb{Y}''_2(\mathfrak{t},\gamma)=\frac{\mathbb{Y}_0}{2}[(\sqrt{\frac{\mathbb{K}''_1(\gamma)}{\mathbb{K}''_2(\gamma)}}+1)e^{-\sqrt{\mathbb{K}''_1(\gamma)\mathbb{K}''_2(\gamma)}(\mathfrak{t}-\mathfrak{t}_\circ)}-(\sqrt{\frac{\mathbb{K}''_1(\gamma)}{\mathbb{K}''_2(\gamma)}}-1)e^{\sqrt{\mathbb{K}''_1(\gamma)\mathbb{K}''_2(\gamma)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

Where $\langle[\mathbb{K}_1(r),\mathbb{K}_2(r)],[\mathbb{K}'_1(\beta),\mathbb{K}'_2(\beta)],[\mathbb{K}''_1(\gamma),\mathbb{K}''_2(\gamma)]\rangle$ is the cut set of $\tilde{\mathcal{K}}^{Ne}$. Solutions are strong or week if it satisfies the condition of NDE.

Similarly, we can get the solution of other two cases.

Numerical Example: Let us consider NDE $\frac{d\mathbb{Y}}{d\mathfrak{t}}=\mathcal{K}\mathbb{Y}$, with IC $\tilde{\mathbb{Y}}^{Ne}(0)=\langle 3,4,5;0.8,0.2,0.3\rangle$, $\mathcal{K}=\frac{1}{3}$.

Solution: Required (r,β,γ)-cut solution at $\mathfrak{t}=2$ we get $\mathbb{Y}_1(\mathfrak{t},r)=[3+1.25r]e^{\frac{2}{3}}$; $\mathbb{Y}_2(\mathfrak{t},r)=[5-1.25r]e^{\frac{2}{3}}$; $\mathbb{Y}'_1(\mathfrak{t},\beta)=[\frac{3.4-\beta}{0.8}]e^{\frac{2}{3}}$; $\mathbb{Y}'_2(\mathfrak{t},\beta)=[\frac{3+\beta}{0.8}]e^{\frac{2}{3}}$; $\mathbb{Y}''_1(\mathfrak{t},\gamma)=[\frac{3.1-\gamma}{0.7}]e^{\frac{2}{3}}$; $\mathbb{Y}''_2(\mathfrak{t},\gamma)=[\frac{2.5+\gamma}{0.7}]e^{\frac{2}{3}}$.

When we take $\mathfrak{t}=2$ and for different values of r,β,γ the solution is given in Table 1. The graphical interpretation of the table is also shown in the form of membership function in the Figure. 1.

5. Mining Safety Model

The miming industry has played an important role in development in the human civilization. Extraction of minerals from the underground system of work has involved a considerable amount of risks like roof fall over the workplace, inundation of the workplace due to the influx of water from the old working, explosion, influx of poisonous gases in the workplace, etc. Similarly, the opencast system of work has involved chances of runway of dumpers, sliding of benches in the workplace, striking by the fly rocks blasting, etc. These phenomenon's not only

r, β, γ	$\mathbb{Y}_1(\mathsf{t}, r)$	$\mathbb{Y}_2(\mathsf{t}, r)$	$\mathbb{Y}'_1(\mathsf{t}, \beta)$	$\mathbb{Y}'_2(\mathsf{t}, \beta)$	$\mathbb{Y}''_1(\mathsf{t}, \gamma)$	$\mathbb{Y}''_2(\mathsf{t}, \gamma)$
0	5.8432	9.7387	8.2778	7.3040	8.6257	6.9561
0.1	6.0866	9.4952	8.0344	7.5474	8.3474	7.2344
0.2	6.3301	9.2517	7.7909	7.7909	8.0692	7.5127
0.3	6.5736	9.0083	7.5475	8.0344	7.7909	7.7909
0.4	6.8171	8.7648	7.3040	8.2779	7.5127	8.0692
0.5	7.0605	8.5213	7.0605	8.5213	7.2344	8.3474
0.6	7.3040	8.2779	6.8171	8.7648	6.9562	8.6257
0.7	7.5475	8.0344	6.5736	9.0082	6.6779	8.9039
0.8	7.7909	7.7909	6.3301	9.2517	6.3997	9.1822
0.9	8.0344	7.5475	6.0867	9.4952	6.1214	9.4604
1.0	8.2779	7.3040	5.8432	9.7387	5.8432	9.7387

TABLE 1. Solution for $\mathsf{t} = 2$

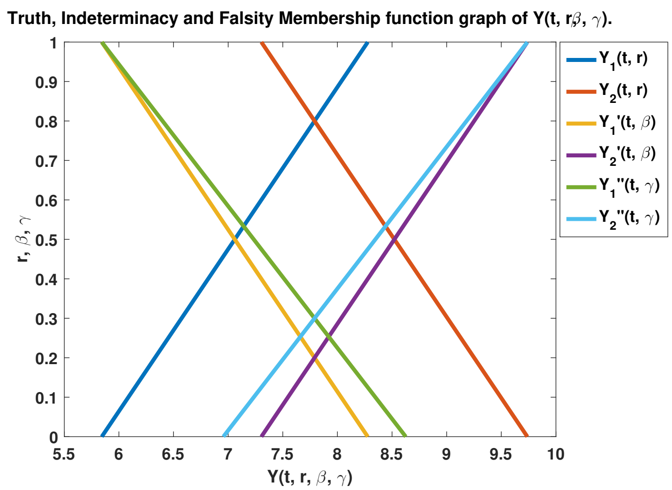

FIGURE 1. Membership Function Graph (at t=2).

causes injury to the workmen, sometimes lead to fatal. Improper used and malfunctioning mining equipment or system also results an accident.

The system fails safely is denoted by λ_1 and system fails unsafely is denoted by λ_2 for the mining safety model used here. Either λ_1 or λ_2 or both λ_1 and λ_2 are imprecise in nature. Our main interest in this paper are given below:

• Formulate Mining Safety model.

• Observe solution of the model in Crisp environment.

• Observe solution of the mining model in three ways:

(i) when λ_1 is SVNNs and λ_2 is crisp number.

(ii) when λ_1 is crisp number and λ_2 is SVNNs.

(iii) both λ_1 and λ_2 are SVNNs.

• Observe cut value in table form of the solution of the mining model in each of the cases mention above and show its graphical representation.

5.1. ***Acceptation***

(I) All events are not dependent to one another.

(II) The probability of progression from one condition to another is $\Psi\delta\mathfrak{t}$; $\delta\mathfrak{t}$ indicates finite time interval, Ψ indicate the progression rate from one condition to another.

(III) $(\Psi\delta\mathfrak{t})(\Psi\delta\mathfrak{t}) \to 0$.

(IV) $\mathcal{P}\{\eta(\delta\mathfrak{t}) \geqslant 2\} = o(\delta\mathfrak{t})$, where $\eta(\delta\mathfrak{t})$ be the number of event that occur in $\delta\mathfrak{t}$.

(V) $\mathcal{P}\{\eta(\delta\mathfrak{t}) = 1\} = \Psi\delta\mathfrak{t} + o(\delta\mathfrak{t})$, where $\Psi > 0$.

(VI) $\lim_{\delta\mathfrak{t}\to 0} \frac{o(\delta\mathfrak{t})}{\delta\mathfrak{t}} = 0$.

5.2. ***Input data***

$\mathfrak{t}$= time.

λ_1= mining system safe failure rate.

λ_2= mining system unsafe failure rate.

5.3. ***Output data***

$\mathcal{P}_0(\mathfrak{t})$= Probability of Mining system operating normally.

$\mathcal{P}_1(\mathfrak{t})$= Probability of Mining system failed safely.

$\mathcal{P}_2(\mathfrak{t})$= Probability of Mining system failed unsafely.

5.4. *Modulator*

$\mathfrak{t}$ time.

$\delta\mathfrak{t}$ finite time intervall.

$\mathcal{P}_0(\mathfrak{t}+\delta\mathfrak{t})$ operating probability in state 0 at time $\mathfrak{t}+\delta\mathfrak{t}$.

$\mathcal{P}_1(\mathfrak{t}+\delta\mathfrak{t})$ safe fail probability in state 1 at time $\mathfrak{t}+\delta\mathfrak{t}$.

$\mathcal{P}_2(\mathfrak{t}+\delta\mathfrak{t})$ unsafe fail probability in state 2 at time $\mathfrak{t}+\delta\mathfrak{t}$.

j=0 state operating normal.

j=1 state fail safe.

j=2 state fail unsafe.

$\mathcal{P}_j(\mathfrak{t})$ probability in state j at time $\mathfrak{t}$.

$\lambda_1\delta\mathfrak{t}$ safe fail probability in finite time interval $\delta\mathfrak{t}$

$\lambda_2\delta\mathfrak{t}$ unsafe fail probability in $\delta\mathfrak{t}$

(1 - $\lambda_1\delta\mathfrak{t}$) no safe fail probability in $\delta\mathfrak{t}$

(1 - $\lambda_1\delta\mathfrak{t}$) no unsafe fail probability in $\delta\mathfrak{t}$

6. Model Formulation

Consider a mining system, the state space diagram is shown in Figure-2.

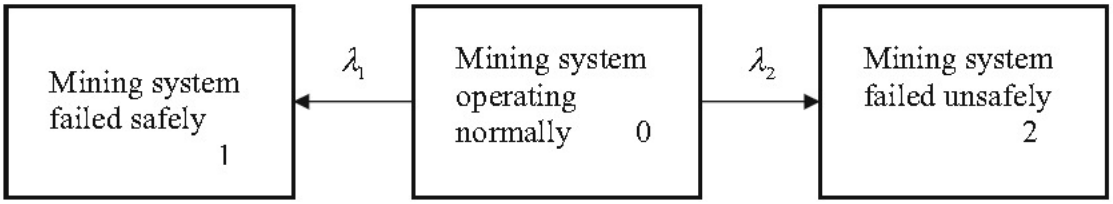

FIGURE 2. Mining system state space diagram

From Fig.2,we get following three equations

$$\mathcal{P}_0(\mathfrak{t}+\delta\mathfrak{t}) = \mathcal{P}_0(\mathfrak{t})(1-\lambda_1\delta\mathfrak{t})(1-\lambda_2\delta\mathfrak{t}) \tag{1}$$

$$\mathcal{P}_1(\mathfrak{t}+\delta\mathfrak{t}) = \mathcal{P}_1(\mathfrak{t})(1-o(\delta\mathfrak{t})) + \mathcal{P}_0(\mathfrak{t})\lambda_1\delta\mathfrak{t} \tag{2}$$

$$\mathcal{P}_2(\mathfrak{t}+\delta\mathfrak{t}) = \mathcal{P}_2(\mathfrak{t})(1-o(\delta\mathfrak{t})) + \mathcal{P}_0(\mathfrak{t})\lambda_2\delta\mathfrak{t} \tag{3}$$

From (1), (2), (3) we get

$$\therefore \frac{d\mathcal{P}_0(\mathfrak{t})}{d\mathfrak{t}} = -(\lambda_1+\lambda_2)\mathcal{P}_0(\mathfrak{t}) \tag{4}$$

$$\frac{d\mathcal{P}_1(\mathfrak{t})}{d\mathfrak{t}} = \lambda_1\mathcal{P}_0(\mathfrak{t}) \tag{5}$$

$$\frac{d\mathcal{P}_2(\mathfrak{t})}{d\mathfrak{t}} = \lambda_2\mathcal{P}_0(\mathfrak{t}) \tag{6}$$

with IC: $\mathcal{P}_j(0) = 1$ for j=0 & $\mathcal{P}_j(0) = 0$ for j=1,2.

7. Solution mode

7.1. ***Crisp Solution:***

Input data: Both λ_1 and λ_2 are Crisp number..

Output data: We get the values of $\mathcal{P}_0(\mathfrak{t})$, $\mathcal{P}_1(\mathfrak{t})$, $\mathcal{P}_2(\mathfrak{t})$.

7.2. ***Neutrosophic Solution:***

Input data: Three cases arise

Case-1: $\tilde{\lambda_1}^{Ne}$=$\langle \mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{a}'_3; w_\mu, w_\nu, w_\sigma\rangle$ & λ_2 is Crisp number.

Case-2: λ_1 is Crisp number & $\tilde{\lambda_2}^{Ne}$=$\langle \mathfrak{b}'_1, \mathfrak{b}'_2, \mathfrak{b}'_3; u_\mu, u_\nu, u_\sigma\rangle$

Case-3: $\tilde{\lambda_1}^{Ne}$=$\langle \mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{a}'_3; w_\mu, w_\nu, w_\sigma\rangle$ & $\tilde{\lambda_2}^{Ne}$=$\langle \mathfrak{b}'_1, \mathfrak{b}'_2, \mathfrak{b}'_3; u_\mu, u_\nu, u_\sigma\rangle$

Output data:

Let, $\tilde{\mathcal{P}_0}(\mathfrak{t})^{Ne}$,$\tilde{\mathcal{P}_1}(\mathfrak{t})^{Ne}$,$\tilde{\mathcal{P}_2}(\mathfrak{t})^{Ne}$ be the solution of the modified model with Cut
$\mathcal{P}_0(\mathfrak{t},r,\beta,\gamma)$=$\langle[\mathcal{P}_{01}(\mathfrak{t},r),\mathcal{P}_{02}(\mathfrak{t},r)],[\mathcal{P}'_{01}(\mathfrak{t},\beta),\mathcal{P}'_{02}(\mathfrak{t},\beta)],[\mathcal{P}''_{01}(\mathfrak{t},\gamma),\mathcal{P}''_{02}(\mathfrak{t},\gamma)]\rangle$

$\mathcal{P}_1(\mathfrak{t},r,\beta,\gamma)$=$\langle[\mathcal{P}_{11}(\mathfrak{t},r),\mathcal{P}_{12}(\mathfrak{t},r)],[\mathcal{P}'_{11}(\mathfrak{t},\beta),\mathcal{P}'_{12}(\mathfrak{t},\beta)],[\mathcal{P}''_{11}(\mathfrak{t},\gamma),\mathcal{P}''_{12}(\mathfrak{t},\gamma)]\rangle$

$\mathcal{P}_2(\mathfrak{t},r,\beta,\gamma)$=$\langle[\mathcal{P}_{21}(\mathfrak{t},r),\mathcal{P}_{22}(\mathfrak{t},r)],[\mathcal{P}'_{21}(\mathfrak{t},\beta),\mathcal{P}'_{22}(\mathfrak{t},\beta)],[\mathcal{P}''_{21}(\mathfrak{t},\gamma),\mathcal{P}''_{22}(\mathfrak{t},\gamma)]\rangle$

Solution is strong or weak if it satisfies the condition of NDE.

8. Numerical Experiment

8.1. *Crisp Solution*

Input data: $\lambda_1 = 0.009; \lambda_2 = 0.001$; $\mathfrak{t}$=20-h.

Output: $\mathcal{P}_2(20)$=0.018127

8.2. *NS Solution*

Case: 1

Input data: $\tilde{\lambda_1}^{Ne}$=$\langle 0.007, 0.009, 0.011; 0.5, 0.3, 0.2\rangle; \lambda_2 = 0.001$; $\mathfrak{t}$=20-h.

Output: When we take the value $\mathfrak{t}$=20-h the output of λ_1^{Ne} is TSVNNs & λ_2 is crisp number are shown in Table-2 and the corresponding membership function shown in Figure-3.

Case: 2

Input data: λ_1=0.009; $\tilde{\lambda_2}^{Ne} = \langle 0.0007, 0.001, 0.0013; 0.7, 0.5, 0.4\rangle$; $\mathfrak{t}$=20-h.

Output: When we take the value $\mathfrak{t}$=20-h the output of λ_1 is Crisp number and $\tilde{\lambda_2}^{Ne}$ is TSVNNs are shown in Table-3 and the corresponding membership function shown in Figure-4.

Case: 3

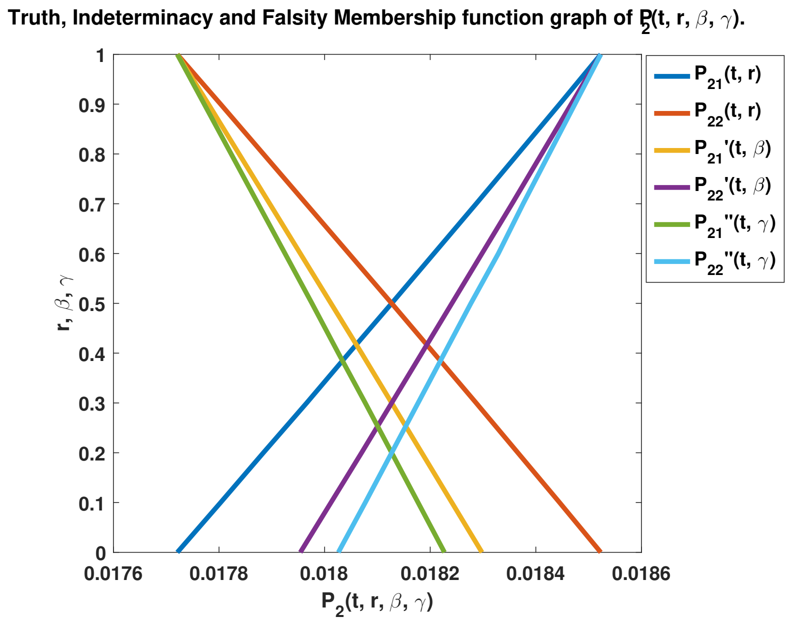

FIGURE 3. Membership Function Graph (at t=20).

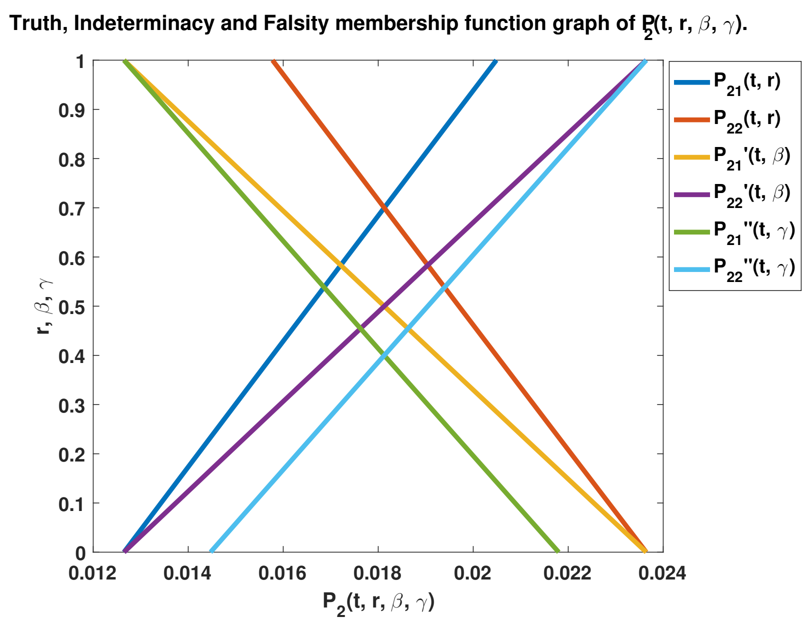

FIGURE 4. Membership Function Graph (at t=20).

r,β,γ	$\mathcal{P}_{21}(\mathfrak{t},r)$	$\mathcal{P}_{22}(\mathfrak{t},r)$	$\mathcal{P}'_{21}(\mathfrak{t},\beta)$	$\mathcal{P}'_{22}(\mathfrak{t},\beta)$	$\mathcal{P}''_{21}(\mathfrak{t},\gamma)$	$\mathcal{P}''_{22}(\mathfrak{t},\gamma)$
0	0.017721	0.018523	0.018298	0.017954	0.018227	0.018026
0.1	0.017803	0.018445	0.018241	0.018012	0.018177	0.018077
0.2	0.017884	0.018366	0.018184	0.018070	0.018127	0.018127
0.3	0.017966	0.018287	0.018127	0.018127	0.018077	0.018177
0.4	0.018046	0.018207	0.018070	0.018184	0.018026	0.018227
0.5	0.018127	0.018127	0.018012	0.018241	0.017976	0.018277
0.6	0.018207	0.018046	0.017954	0.018298	0.017925	0.018329
0.7	0.018287	0.017965	0.017896	0.018355	0.017874	0.018376
0.8	0.018366	0.017884	0.017838	0.018411	0.017823	0.018425
0.9	0.018445	0.017803	0.017780	0.018467	0.017772	0.018474
1.0	0.018523	0.017721	0.017721	0.018523	0.017721	0.018523

TABLE 2. $\tilde{\lambda_1}^{Ne}$ is TSVNNs & λ_2 is Crisp number.

r,β,γ	$\mathcal{P}_{21}(\mathfrak{t},r)$	$\mathcal{P}_{22}(\mathfrak{t},r)$	$\mathcal{P}'_{21}(\mathfrak{t},\beta)$	$\mathcal{P}'_{22}(\mathfrak{t},\beta)$	$\mathcal{P}''_{21}(\mathfrak{t},\gamma)$	$\mathcal{P}''_{22}(\mathfrak{t},\gamma)$
0	0.012647	0.023643	0.023643	0.012647	0.021800	0.014470
0.1	0.013427	0.022853	0.022537	0.013740	0.020881	0.015382
0.2	0.014209	0.022063	0.021432	0.014834	0.019962	0.016296
0.3	0.014991	0.021275	0.020329	0.015930	0.019044	0.017211
0.4	0.015774	0.020487	0.019227	0.017028	0.018127	0.018127
0.5	0.016557	0.019699	0.018127	0.018127	0.017211	0.019044
0.6	0.017342	0.018913	0.017028	0.019227	0.016296	0.019962
0.7	0.018127	0.018127	0.015930	0.020329	0.015382	0.020881
0.8	0.018913	0.017342	0.014834	0.021432	0.014470	0.021800
0.9	0.019699	0.016557	0.013740	0.022537	0.013558	0.022721
1.0	0.020487	0.015774	0.012647	0.023643	0.012647	0.023643

TABLE 3. λ_1 is Crisp number & $\tilde{\lambda_2}^{Ne}$ is TSVNNs

Input data:

$\tilde{\lambda_1}^{Ne}=\langle 0.007, 0.009, 0.011; 0.5, 0.3, 0.2\rangle$; $\tilde{\lambda_2}^{Ne} = \langle 0.0007, 0.001, 0.0013; 0.7, 0.5, 0.4\rangle$; $\mathfrak{t}$=20-h.

Output: When we take the value $\mathfrak{t}$=20-h the output of $\tilde{\lambda_1}^{Ne}$ & $\tilde{\lambda_2}^{Ne}$ are TSVNNs are shown in Table-4 and the corresponding membership function shown in Figure-5.

From the table values and graph, we see that

$\mathcal{P}_1(\mathfrak{t},r)$ is increasing function and

$\mathcal{P}_2(\mathfrak{t},r)$ is decreasing function, whereas

r, β, γ	$\mathcal{P}_{21}(t, r)$	$\mathcal{P}_{22}(t, r)$	$\mathcal{P}'_{21}(t, \beta)$	$\mathcal{P}'_{22}(t, \beta)$	$\mathcal{P}''_{21}(t, \gamma)$	$\mathcal{P}''_{22}(t, \gamma)$
0	0.012361	0.024156	0.024156	0.012361	0.022118	0.014253
0.1	0.013188	0.023247	0.022930	0.013493	0.021109	0.015210
0.2	0.014023	0.022345	0.021714	0.014635	0.020108	0.016175
0.3	0.014866	0.021449	0.020508	0.015788	0.019165	0.017147
0.4	0.015715	0.020561	0.019312	0.016952	0.018127	0.018127
0.5	0.016573	0.019681	0.018127	0.018127	0.017147	0.019114
0.6	0.017438	0.018807	0.016952	0.019312	0.016175	0.020108
0.7	0.018310	0.017941	0.015788	0.020508	0.015210	0.021109
0.8	0.019190	0.017083	0.014635	0.021714	0.014253	0.022118
0.9	0.020077	0.016232	0.013493	0.022930	0.013303	0.023134
1.0	0.020971	0.015388	0.012361	0.024156	0.012361	0.024156

TABLE 4. Both $\tilde{\lambda}_1^{Ne}$ & $\tilde{\lambda}_2^{Ne}$ are TSVNNs

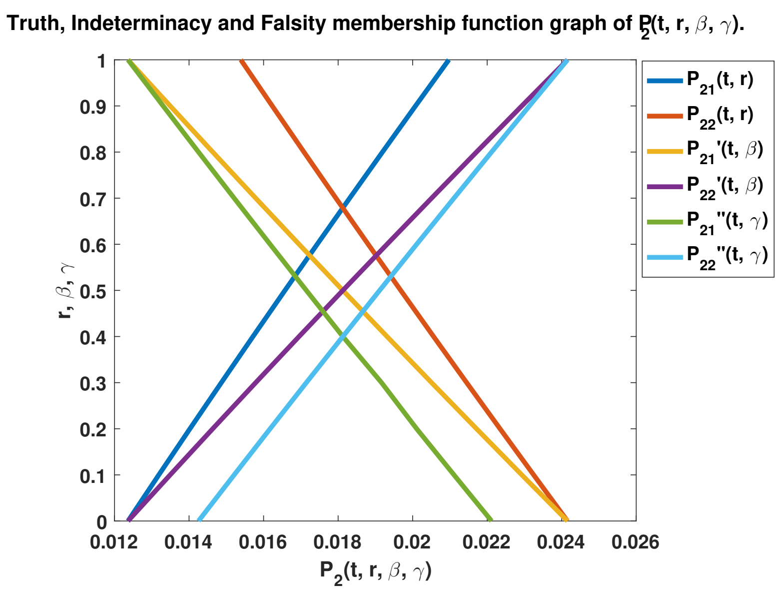

FIGURE 5. Membership Function Graph (at t=20).

$\mathcal{P}'_1(t, \beta)$, $\mathcal{P}''_1(t, \gamma)$ are decreasing functions and
$\mathcal{P}'_2(t, \beta)$, $\mathcal{P}''_2(t, \gamma)$ are increasing functions. Hence, the solution is strong solution.

9. Conclusion

• NS is a hot research topic and can be applied for solving the mathematical model of uncertainty, vagueness, ambiguity, etc.
• The mining safety model described in this paper with two parameters which satisfies the condition of NDE has got strong solutions.
• The solutions of the three differential equations of the mining safety model have been described via TSVNNs.
• The paper has also proposed numerical experiment and graphical representation of truth, indeterminacy and falsity membership function.

This will promote the future study of trapezoidal single valued neutrosophic numbers.

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Some operations on rough bipolar interval neutrosophic sets

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Abstract. In this study we introduce the concept of rough bipolar interval neutrosophic sets which is a combination of rough sets and bipolar interval neutrosophi sets. Also we define union, complement, intersection and some interesting properties of this set.

Keywords: Rough sets, Bipolar neutrosophic sets, interval neutrosophic sets , rough bipolar neutrosophic sets.

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1. **Introduction**

The notion of fuzzy set theory studied by Zadeh [9] in 1965 to deal with uncertainty. This theory has been applied in many real life applications to handle uncertainty. After Zadeh [10] introduced interval valued fuzzy sets. Attanasov [1] extended the fuzzy sets to intutionistic fuzzy set. In 1998, Smarandache [7] studied the concept of neutrosohic set. Lee [6] introduced the concept of bipolar fuzzy sets, as an extension of fuzzy sets. In bipolar fuzzy sets the degree of membership is extended from $[0, 1]$to $[-1, 1]$. In a bipolar fuzzy set, if the degree of membership of an element is zero, then we say the element is unrelated to the corresponding property, the membership degree (0,1] of an element specifies that the element somewhat satisfies the property, and the membership degree [-1,0) of an element implies that the element somewhat satisfies the implicit counter property. In 2014, Broumi et al. [2] , [3] presented the concept of rough neutrosophic set to deal indeterminacy in more flexible way. The rough set theory familiarized by Pawlak is an excellent mathematical tool for the analysis of uncertain, inconsistency and vague description of objects. Deli et al. [4] defined bipolar neutrosophic set and showed numerical example for multi-criteria decision making problem. Gong et al. [5]

introduced interval valued rough fuzzy set. Subha et al. [8] applied interval valued rough fuzzy sets in many real life applications.

2. Preliminaries

For basic concepts related to this paper refer [1], [2], [3], [4], [5], [6], [7], [9] , [10].

3. Rough Bipolar Interval Neutrosophic sets

Let H be the universe and R be an equivalence relation on H. Let I be a bipolar interval neutrosophic set in H. Then the lower and upper approximation of I in (H, R) is defined by
$L(I) = \left\{\left\langle x, L(a_I^p), L(b_I^p), L(c_I^p), L(a_I^n), L(b_I^n), L(c_I^n)\right\rangle, x \in H\right\}$
$U(I) = \left\{\left\langle x, U(a_I^p), U(b_I^p), U(c_I^p), U(a_I^n), U(b_I^n), U(c_I^n)\right\rangle, x \in H\right\}$
where
$L(a_I^p)(x) = \bigwedge_{z\in[x]_R} a^p(z)$, $L(b_I^p)(x) = \bigvee_{z\in[x]_R} b^p(z)$, $L(c_I^p)(x) = \bigvee_{z\in[x]_R} c^p(z)$
$L(a_I^n)(x) = \bigwedge_{z\in[x]_R} a^n(z)$, $L(b_I^n)(x) = \bigvee_{z\in[x]_R} b^n(z)$, $L(c_I^n)(x) = \bigvee_{z\in[x]_R} c^n(z)$
$U(a_I^p)(x) = \bigvee_{z\in[x]_R} a^p(z)$, $U(b_I^p)(x) = \bigwedge_{z\in[x]_R} b^p(z)$, $U(c_I^p)(x) = \bigwedge_{z\in[x]_R} c^p(z)$
$U(a_I^n)(x) = \bigvee_{z\in[x]_R} a^n(z)$, $U(b_I^n)(x) = \bigwedge_{z\in[x]_R} b^n(z)$, $U(c_I^n)(x) = \bigwedge_{z\in[x]_R} c^n(z)$ for all $x \in H$.
Then $R(I) = (L(I), U(I))$ is called a rough bipolar interval neutrosophic set in (H, R). Here $L(I)$ and $U(I)$ are also bipolar interval neutrosophic sets.

Example 3.1. Let $H = \{i, j, k, l, m\}$ be the universe. Let I be the bipolar interval neutrosophic set defined by,
$i = ([0.60, 0.70], [0.40, 0.50], [0.10, 0.20], [-0.90, -0.80], [-0.70, -0.60], [-0.30, -0.20])$
$j = ([0.40, 0.50], [0.10, 0.20], [0.01, 0.20], [-0.70, -0.50], [-0.40, -0.30], [-0.80, -0.70])$
$k = ([0.50, 0.40], [0.10, 0.30], [0.60, 0.70], [-0.60, -0.50], [-0.30, -0.20], [-0.70, -0.60])$
$l = ([0.65, 0.75], [0.58, 0.68], [0.51, 0.61], [-0.85, -0.75], [-0.81, -0.71], [-0.68, -0.58])$
$m = ([0.81, 0.91], [0.62, 0.72], [0.34, 0.44], [-0.85, -0.75], [-0.65, -0.55], [-0.30, -0.20])$
Then the equivalence classes of H are defined by $\{\{i, j, m\}, \{k, l\}\}$. The lower approximation of I is
$L(a_I^p)(x) = \{(i, [.40, .50]), (j, [.40, .50]), (k, [.50, .40]), (l, [.50, .40]), (m, [.40, .50])\}$
$L(b_I^p)(x) = \{(i, [.62, .72]), (j, [.62, .72]), (k, [.58, .68]), (l, [.58, .48]), (m, [.62, .72])\}$
$L(c_I^p)(x) = \{(i, [.34, .44]), (j, [.34, .44]), (k, [.60, .70]), (l, [.60, .70]), (m, [.34, .44])\}$
$L(a_I^n)(x) =$
$\{(i, [-.90, -.80]), (j, [-.90, -.80]), (k, [-.85, -.75]), (l, [-.85, -.75]), (m, [-.90, -.80])\}$
$L(b_I^n)(x) =$
$\{(i, [-.40, -.30]), (j, [-.40, -.30]), (k, [-.30, -.20]), (l, [-.30, -.20]), (m, [-.40, -.30])\}$
$L(c_I^n)(x) =$

$\{(i, [-.30, -.20]), (j, [-.30, -.20]), (k, [-.68, -.58]), (l, [-.68, -.58]), (m, [-.30, -.20])\}$

Also

$U(a_I^p)(x) = \{(i, [.81, .91]), (j, [.81, .91]), (k, [.65, .75]), (l, [.65, .75]), (m, [.81, .91])\}$

$U(b_I^p)(x) = \{(i, [.10, .20]), (j, [.10, .20]), (k, [.10, .30]), (l, [.10, .30]), (m, [.10, .20])\}$

$U(c_I^p)(x) = \{(i, [.01, .20]), (j, [.01, .20]), (k, [.51, .61]), (l, [.51, .61]), (m, [.01, .20])\}$

$U(a_I^n)(x) =$

$\{(i, [-.70, -.50]), (j, [-.70, -.50]), (k, [-.60, -.50]), (l, [-.60, -.50]), (m, [-.70, -.50])\}$

$U(b_I^n)(x) =$

$\{(i, [-.70, -.60]), (j, [-.70, -.60]), (k, [-.81, -.71]), (l, [-.81, -.71]), (m, [-.70, -.60])\}$

$U(c_I^n)(x) =$

$\{(i, [-.80, -.70]), (j, [-.80, -.70]), (k, [-.70, -.60]), (l, [-.70, -.60]), (m, [-.80, -.70])\}$

Example 3.2. Let $H = \{p, q, r, s, t\}$ be the universe. Let j be the bipolar interval neutrosophic set defined by,

$p = ([0.50, 0.60], [0.20, 0.30], [0.10, 0.20], [-0.80, -0.70], [-0.70, -0.50], [-0.30, -0.20])$

$q = ([0.30, 0.50], [0.10, 0.20], [0.03, 0.40], [-0.70, -0.60], [-0.50, -0.40], [-0.90, -0.80])$

$r = ([0.40, 0.30], [0.01, 0.30], [0.50, 0.40], [-0.70, -0.50], [-0.80, -0.60], [-0.70, -0.50])$

$s = ([0.71, 0.81], [0.52, 0.62], [0.44, 0.54], [-0.75, -0.65], [-0.45, -0.35], [-0.14, -0.01])$

$t = ([0.50, 0.60], [0.40, 0.50], [0.60, 0.80], [-0.85, -0.75], [-0.71, -0.61], [-0.58, -0.40])$

Then the equivalence classes of H are defined by $\{\{p, q, t\}, \{r, s\}\}$. The lower approximation of I is

$L(a_I^p)(x) = \{(p, [.30, .50]), (q, [.30, .50]), (r, [.40, .30]), (s, [.40, .30]), (t, [.30, .50])\}$

$L(b_I^p)(x) = \{(p, [.40, .50]), (q, [.40, .50]), (r, [.52, .62]), (s, [.52, .62]), (t, [.40, .50])\}$

$L(c_I^p)(x) = \{(p, [.60, .80]), (q, [.60, .80]), (r, [.50, .54]), (s, [.50, .54]), (t, [.60, .80])\}$

$L(a_I^n)(x) =$

$\{(p, [-.85, -.75]), (q, [-.84, -.75]), (r, [-.75, -.65]), (s, [-.75, -.65]), (t, [-.85, -.75])\}$

$L(b_I^n)(x) =$

$\{(p, [-.50, -.40]), (q, [-.50, -.40]), (r, [-.45, -.35]), (s, [-.45, -.35]), (m, [-.50, -.40])\}$

$L(c_I^n)(x) =$

$\{(p, [-.30, -.20]), (q, [-.30, -.20]), (r, [-.14, -.01]), (s, [-.14, -.01]), (t, [-.30, -.20])\}$

Also

$U(a_I^p)(x) = \{(p, [.50, .60]), (q, [.50, .60]), (r, [.71, .81]), (s, [.71, .81]), (t, [.50, .60])\}$

$U(b_I^p)(x) = \{(p, [.10, .20]), (q, [.10, .20]), (r, [.01, .30]), (s, [.01, .30]), (t, [.10, .20])\}$

$U(c_I^p)(x) = \{(p, [.03, .20]), (q, [.03, .20]), (r, [.44, .40]), (s, [.44, .40]), (t, [.03, .20])\}$

$U(a_I^n)(x) =$

$\{(p, [-.70, -.60]), (q, [-.70, -.60]), (r, [-.70, -.50]), (s, [-.70, -.50]), (t, [-.70, -.60])\}$

$U(b_I^n)(x) =$
$\{(p, [-.71, -.61]), (q, [-.71, -.61]), (r, [-.80, -.60]), (s, [-.80, -.60]), (t, [-.71, -.61])\}$
$U(c_I^n)(x) =$
$\{(p, [-.90, -.80]), (q, [-.90, -.80]), (r, [-.70, -.50]), (s, [-.70, -.50]), (t, [-.90, -.80])\}$

Definition 3.3. Let $R(I)$ and $R(J)$ be two rough bipolar interval neutrosophic sets. Then for all $l \in H$ $R(I) \subseteq R(J) \Leftrightarrow L(a_I^p)(l) \leq U(a_J^p)(l)$, $L(b_I^p)(l) \leq U(b_J^p)(l)$, $L(c_I^p)(l) \geq U(b_J^p)(l)$ and $L(a_I^n)(l) \geq U(a_J^n)(l)$, $L(b_I^n)(l) \geq U(b_J^n)(l)$, $L(c_I^n)(l) \leq U(b_J^n)(l)$.

Definition 3.4. Union of two rough bipolar interval neutrosophic sets $R(I)$ and $R(J)$, is defined as

$$L(I) \cup L(J)(l) = max\left(L(a_I^p)(l), L(a_J^p)(l)\right), \frac{L(b_I^p)(l)+L(b_J^p)(l)}{2}, min\left(L(c_I^p)(l), L(c_J^p)(l)\right), \\ min\left(L(a_I^n)(l), L(a_J^n)(l)\right), \frac{L(b_I^n)(l)+L(b_J^n)(l)}{2}, max\left(L(c_I^n)(l), L(c_J^n)(l)\right)$$

$$U(I) \cup U(J)(l) = max\left(U(a_I^p)(l), U(a_J^p)(l)\right), \frac{U(b_I^p)(l)+U(b_J^p)(l)}{2}, min\left(U(c_I^p)(l), U(c_J^p)(l)\right), \\ min\left(U(a_I^n)(l), U(a_J^n)(l)\right), \frac{U(b_I^n)(l)+U(b_J^n)(l)}{2}, max\left(U(c_I^n)(l), U(c_J^n)(l)\right)$$

for every $l \in H$.

Example 3.5. Consider two rough bipolar interval neutrosophic sets as in Example 3.1 and 3.2 then

$$L(I) \cup L(J) = \begin{cases} i, [.40..50], [.51, .61], [.34, .44], [-.90, -.80], [-.45, -.35], [-.30, -.20] \\ j, [.40..50], [.51, .61], [.34, .44], [-.90, -.80], [-.45, -.35], [-.30, -.20] \\ k, [.50, .40], [.55, .65], [.50, .54], [-.85, -.75], [-.38, -.28], [-.14, -.01] \\ l, [.50, .40], [.55, .65], [.50, .54], [-.85, -.75], [-.38, -.28], [-.14, -.01] \\ m, [.40..50], [.51, .61], [.34, .44], [-.90, -.80], [-.45, -.35], [-.30, -.20] \end{cases}$$

also

$$U(I) \cup U(J) = \begin{cases} i, [.81, .91], [.10, .20], [.01, .20], [-.70, -.60], [-.71, -.61], [-.90, -.80] \\ j, [.81, .91], [.10, .20], [.01, .20], [-.70, -.60], [-.71, -.61], [-.90, -.80] \\ k, [.71, .81], [.16, .30], [.51, .61], [-.60, -.50], [-.81, -.66], [-.70, -.50] \\ l, [.71, .81], [.16, .30], [.51, .61], [-.60, -.50], [-.81, -.66], [-.70, -.50] \\ m, [.81, .91], [.10, .20], [.01, .20], [-.70, -.60], [-.71, -.61], [-.90, -.80] \end{cases}$$

Definition 3.6. Intersection of two rough bipolar interval neutrosophic sets I and J, is defined as

$$L(I) \cap L(J)(l) = min\left(L(a_I^p)(l), L(a_J^p)(l)\right), \frac{L(b_I^p)(l)+L(b_J^p)(l)}{2}, max\left(L(c_I^p)(l), L(c_J^p)(l)\right), \\ max\left(L(a_I^n)(l), L(a_J^n)(l)\right), \frac{L(b_I^n)(l)+L(b_J^n)(l)}{2}, min\left(L(c_I^n)(l), L(c_J^n)(l)\right)$$

$$U(I) \cap U(J)(l) = min\left(U(a_I^p)(l), U(a_J^p)(l)\right), \frac{U(b_I^p)(l)+U(b_J^p)(l)}{2}, max\left(U(c_I^p)(l), U(c_J^p)(l)\right),$$

$$max\left(U(a_I^n)(l), U(a_J^n)(l)\right), \frac{U(b_I^n)(l)+U(b_J^n)(l)}{2}, min\left(U(c_I^n)(l), U(c_J^n)(l)\right)$$

for every $l \in H$.

Example 3.7. Consider two rough bipolar interval neutrosophic sets as in Example 3.1 and 3.2 then

$$L(I) \cap L(J) = \begin{cases} i, [.30..50], [.51, .61], [.60, .80], [-.85, -.75], [-.45, -.35], [-.30, -.20] \\ j, [.30..50], [.51, .61], [.60, .80], [-.85, -.75], [-.45, -.35], [-.30, -.20] \\ k, [.40, .30], [.55, .65], [.60, .70], [-.75, -.65], [-.38, -.28], [-.68, -.58] \\ l, [.40, .30], [.55, .65], [.60, .70], [-.75, -.65], [-.38, -.28], [-.68, -.58] \\ m, [.30..50], [.51, .61], [.60, .80], [-.85, -.75], [-.45, -.35], [-.30, -.20] \end{cases}$$

also

$$U(I) \cap U(J) = \begin{cases} i, [.50, .60], [.10, .20], [.03, .20], [-.70, -.50], [-.71, -.61], [-.90, -.80] \\ j, [.50, .60], [.10, .20], [.03, .20], [-.70, -.50], [-.71, -.61], [-.90, -.80] \\ k, [.65, .75], [.14, .30], [.51, .61], [-.60, -.50], [-.81, -.66], [-.70, -.60] \\ l, [.65, .75], [.14, .30], [.51, .61], [-.60, -.50], [-.81, -.66], [-.70, -.60] \\ m, [.50, .60], [.10, .20], [.03, .20], [-.70, -.50], [-.71, -.61], [-.90, -.80] \end{cases}$$

Definition 3.8. The complement of a rough bipolar interval neutrosophic set $R(I)$ is defined as $R(I)^c = (L(I)^c, U(I)^c)$ where $L(I)^c$and $U(I)^c$ are the lower and upper approximations of $R(I)^c$.
$L(a_I^p)^c(l) = 1 - L(a_I^p)(l)$, $L(b_I^p)^c(l) = 1 - L(b_I^p)(l)$ and $L(c_I^p)^c(l) = 1 - L(c_I^p)(l)$
$L(a_I^n)^c(l) = -1 - L(a_I^p)(l)$, $L(b_I^n)^c(l) = -1 - L(b_I^p)(l)$ and $L(c_I^n)^c(l) = -1 - L(c_I^p)(l)$.
Also, $U(a_I^p)^c(l) = 1 - U(a_I^p)(l)$, $U(b_I^p)^c(l) = 1 - U(b_I^p)(l)$ and $U(c_I^p)^c(l) = 1 - U(c_I^p)(l)$
$U(a_I^n)^c(l) = -1 - U(a_I^p)(l)$, $U(b_I^n)^c(l) = -1 - U(b_I^p)(l)$ and $U(c_I^n)^c(l) = -1 - U(c_I^p)(l)$.
for all $l \in H$.

Definition 3.9. If $R(I)$ and $R(J)$ are two rough bipolar interval neutrosophic sets in H, then

(1) $R(I) = R(J) \Leftrightarrow L(I) = L(J), U(I) = U(J)$
(2) $R(I) \subseteq R(J) \Leftrightarrow L(I) \subseteq L(J), U(I) \subseteq U(J)$
(3) $R(I) \cup R(J) \Leftrightarrow \langle L(I) \cup L(J), U(I) \cup U(J)\rangle$
(4) $R(I) \cap R(J) \Leftrightarrow \langle L(I) \cap L(J), U(I) \cap U(J)\rangle$
(5) $R(I) + R(J) \Leftrightarrow \langle L(I) + L(J), U(I) + U(J)\rangle$
(6) $R(I) \circ R(J) \Leftrightarrow \langle L(I) \circ L(J), U(I) \circ U(J)\rangle$

Proposition 3.10. *Let I and J are rough bipolar interval neutrosophic sets in (H, R) then*

(1) $I^c(I^c) = I$

(2) $L(I) \subseteq U(I)$

(3) $(L(I) \cup L(J))^c = L(I)^c \cap L(J)^c$

(4) $(L(I) \cap L(J))^c = L(I)^c \cup L(J)^c$

(5) $(U(I) \cup U(J))^c = U(I)^c \cap U(J)^c$

(6) $(U(I) \cap U(J))^c = U(I)^c \cup U(J)^c$

Proposition 3.11. *If $R(I)$ and $R(J)$ are rough bipolar interval neutrosophic sets then*

(1) $(R(I) \cup R(J))^c = (R(I))^c \cap (R(J))^c$

(2) $(R(I) \cap R(J))^c = (R(I))^c \cup (R(J))^c$

Proposition 3.12. *If I and J are bipolar interval neutrosophic sets such that $I \subseteq J$ implies $R(I) \subseteq R(J)$*

(1) $R(I \cup J) \supseteq R(I) \cup R(J)$

(2) $R(I \cap J) \supseteq R(I) \cap R(J)$

Proof : Let $l \in H$ **then**

$$\begin{aligned} L(a^n_{I\cup J})(l) &= \bigwedge_{z\in[l]_R} a^n_{I\cup J}(z) \\ &= \bigwedge_{z\in[l]_R} \max\{a^n_I, a^n_J\} \\ &\geq \max\left\{\bigwedge_{z\in[l]_R} a^n_I, \bigwedge_{z\in[l]_R} a^n_J\right\} \\ &= \max\{L(a^n_I), L(a^n_J)\} \\ &= L(a^n_I) \cup L(a^n_J) \end{aligned}$$

for all $l \in H$**.**

Similarly we can prove $L(b^n_{I\cup J})(l) \leq L(b^n_I) \cup L(b^n_J)$

$L(c^n_{I\cup J})(l) \leq L(c^n_I) \cup L(c^n_J)$

$L(a^p_{I\cup J})(l) \leq L(a^p_I) \cup L(a^p_J)$

$L(b^p_{I\cup J})(l) \geq L(b^p_I) \cup L(b^p_J)$

$L(c^p_{I\cup J})(l) \geq L(c^p_I) \cup L(c^p_J)$

Hence, $L(I \cup J) \supseteq L(I) \cup L(J)$

4. Conclusions

In this paper we introduce the notion of rough bipolar interval neutrosophic set. We also study some properties of this set and prove some propositions. The rough bipolar interval neutrosophic set is a combination of rough bipolar set and interval neutrosophic set. The proposed concept can be used in many aplications such as decision making problem, recognition pattern etc.

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A Theoretical and Analytical Approach for Fundamental Framework of Composite Mappings on Fuzzy Hypersoft Classes

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Abstract. Mapping is a fundamental mathematical concept that is used in many elementary areas of science and mathematics and has numerous applications. The core purpose of this study is to provide a theoretical and analytical approach for carving out a basic structure of composite mappings on the classes of Fuzzy Hypersoft (FHS) sets. It is a comprehensive study of existing concepts regarding mappings on fuzzy soft, soft and hesitant fuzzy soft classes through characterizing of composite mappings on FHS classes. Moreover, certain generalized properties of mappings on FHS classes like FHS images and FHS inverse images, are established. Some related results are verified with the help of illustrative examples.

Keywords: Fuzzy soft, Soft classes, Hesitant fuzzy soft classes, Composite mappings, Fuzzy hypersoft set.

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1. Introduction

In 1965, Zadeh introduced the theory of fuzzy sets [24]. It has been utilized in different decision making problem [20]- [21]- [22]. There are some theories, theory of likelihood, theory of intuitionistic fuzzy sets [2], [5], theory of vague sets [10], the theory of interval mathematics [2], [11], and theory of rough sets [13] which can be considered as scientific apparatuses for dealing with uncertainties and ambiguous. Despite that, every one of these speculations has their innate challenges as brought up in [12]. The main reason behind these troubles is potentially the inadequacy of the parametrization device of the hypothesis.

Therefore, Molodtsov [12] started the idea of a soft set (SS) as a numerical device for dealing with uncertainties which are liberated from the above challenges (We know about the SS characterized by Pawlak [14], which is an alternate idea and helpful to understand some other kind of issues).

Karaaslan [8] introduced soft class and its pertinent activities. Athar et al. [3], [4] introduced the concept of mappings on fuzzy soft classes and mappings on soft classes in 2009 and 2011 individually. They considered the properties of the soft image and soft inverse image. They also characterized the properties of fuzzy SS, fuzzy S-image, fuzzy S-inverse image of fuzzy S-sets and supported them with examples. Manash et al. [7] gave the idea of composite mappings on hesitant fuzzy soft classes in 2016 and discussed some interesting properties of this idea.

In a diversity of real-life applications, the attributes should be further sub-partitioned into attribute values for more clear understanding. Samarandache [9] fulfilled this need and developed the concept of the HSS as a generalization of the SS. He opened numerous fields in this way of thinking and generalized SS to the hyper-soft set by changing the planning F into a multi-contention function. At that point, he made the differentiation between the sorts of initial universes, crisp, fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic respectively. Thus, he also showed that a HS set can be crisp, fuzzy, intuitionistic fuzzy, neutrosophic and plithogenic respectively. Saeed et al. [19, 25] explained some basic concepts like HS subset, HS complement, not HS set, absolute set, union, intersection, AND, OR, restricted union, extended intersection, relevant complement, restricted difference, restricted symmetric difference, HS set relation, sub relation, complement relation, HS representation in matrices form, and different operations on matrices. Saeed et al. [23] characterized mapping under a hypersoft set environment, then some of its essential properties like HS images, HS inverse images were also discussed.

The core purpose of this study is to provide a theoretical and analytical approach for carving out a basic structure of composite mappings on the classes of FHS sets. It is an comprehensive study of existing concepts regarding mappings on fuzzy soft, soft classes and hesitant fuzzy soft classes through characterizing of composite mappings on FHS classes. Moreover, certain generalized properties of mappings on FHS classes like FHS images and FHS inverse images, are established. Some related results are proved with the help of illustrative examples. The ordering of the following portion is working out as follows.

In Section 2, some pivotal regarding fuzzy set, SS, fuzzy soft class, soft class, hypersoft set, and fuzzy hypersoft set (FHSS) are re-imagined. In Section 3, composite mappings on FHS classes, FHS image, FHS inverse image, and its relevant theorems with their essential properties are considered. In the last section, some concluding remarks are described.

1.1. *Motivation*

In a diversity of real-life applications, the attributes should be further sub-partitioned into attribute values for more clear understanding. Samarandache [9] fulfilled this need and developed the concept of the FHSS as a generalization of the fuzzy soft set. Now, It will be a question that how do we define composite mappings for FHSS classes? FHSS set is significant? To answers these questions and getting inspiration from the above writing, it is relevant to broaden the idea of mappings for those sets managing disjoint arrangements of attributed values, i.e FHSS. In this investigation, an extension is made in existing theories with respect to mappings on fuzzy soft, soft classes and hesitant fuzzy soft classes by characterizing composite mappings on FHS classes. The striking component of composite mappings on FHS classes is that it can mirror the interrelationship between the multi-input contentions. Moreover, certain generalized properties of mappings on FHS classes like FHS images and FHS inverse images, are established. Some related results are proved with the help of illustrative examples.

2. **Preliminaries**

Throughout the following, let $L = F_1 \times F_2 \times F_3 \times ... \times F_n$, $M = F'_1 \times F'_2 \times F'_3 \times ... \times F'_n$, $N = F''_1 \times F''_2 \times F''_3 \times ... \times F''_n$, $O = F'''_1 \times F'''_2 \times F'''_3 \times ... \times F'''_n$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, \beta_3, ..., \beta_n)$.

Definition 2.1. [21] The fuzzy set $X = \{(x, \xi_X(x))|x \in X\}$ such that

$$\xi_X : X \to [0, 1]$$

where $\xi_X(x)$ describes the membership percentage of $x \in X$.

Definition 2.2. [12] A pair (F, A) is said to be soft set over X, where F is a mapping given as

$$F : A \to P(X)$$

On the other way, a SS is the parameterized family of subsets of the universe X. In other words, a soft set over X is a parameterized family of subsets of the universe. For $\epsilon \in A$. $F(\epsilon)$ may be considered as the set of ϵ approximate elements of the soft set (F, A).

Definition 2.3. [6] Let X be an initial universe, indexed class of fuzzy sets $\{f_i : f_i : X \to [0, 1]), i = 1, 2, ..., n\}$ is called a fuzzy class.

Definition 2.4. [1] Let X be a universe and E a set of attributes. Then the pair (X, E) denotes the collection of all fuzzy soft sets on X with attributes from E and is called a fuzzy soft class.

Definition 2.5. [9] Let $a_1, a_2, a_3, \cdots, a_n$ be the distinct attributes whose corresponding attribute values belongs to the sets $F_1, F_2, F_3, \cdots, F_n$ respectively, where $F_i \cap F_j = \Phi$ for $i \neq j$. A pair (Υ, L) is called a hypersoft set over the universal set X, where Υ is the mapping given by $\Upsilon : L \longrightarrow P(X)$.
For more definition see [15–19].

Definition 2.6. [19] Suppose X and $F(X)$ be the universal set and all fuzzy subsets of X respectively. Let $a_1, a_2, a_3, \cdots, a_n$ be the distinct attributes whose corresponding attribute values belongs to the sets $F_1, F_2, F_3, ..., F_n$ respectively, where $F_i \cap F_j = \Phi$ for $i \neq j$ and i, j $\in \{1, 2, 3, ..., n\}$. Then the FHSS is the pair (Σ_L, L) over X defined by a map $\Sigma_L : L \to F(X)$.

3. Main results

Definition 3.1. Suppose X and $F(X)$ be the universal set and all fuzzy subsets of X respectively, let $a_1, a_2, a_3, \cdots, a_n$ be the distinct attributes whose corresponding attribute values belongs to the sets $F_1, F_2, F_3, \cdots, F_n$ respectively, where $F_i \cap F_j = \Phi$ for $i \neq j$, let $\digamma = \{\varsigma_i : i = 1, 2, ..., n\}$ be a collection of decision makers. Indexed class of FHSS $\{\vartheta_{\varsigma_i} : \vartheta_{\varsigma_i} : L \to F(X), \varsigma_i \in \digamma\}$, is said to be fuzzy hypersoft class and it can be symbolized in such a form $\vartheta_\digamma$. If for any $\varsigma_i \in \digamma$, $\vartheta_{\varsigma_i} = \Phi$, the FHSS $\vartheta_{\varsigma_i} \notin \vartheta_\digamma$.

Example 3.2. Let $X = \{a =$ Holstein, $b =$ Angus, $c =$ Charolais $\}$ be the set of cow categories. Peter decide to purchase a cow for milk to get vitamin B12 and iodine of doctor instruction. They visit Bos tarus (a Europeon Cattle) to buy such cow which fulfills his requirements. Let $a_1 =$ vision and hearing, $a_2 =$ Cost, $a_3 =$ Colour, distinct attributes whose attribute values belong to the sets F_1, F_2, F_3. Let $F_1 = \{f_1 =$ Excellent peripheral vision, $f_2 =$ Low peripheral vision $\}$, $F_2 = \{f_3 =$High, $f_4 = Low\}$, $F_3 = \{f_5 =$ White $\}$ and let $\digamma = \{\varsigma_1, \varsigma_1, \varsigma_1\}$ be a set of decision makers. If we consider FHS sets given as

$$\vartheta_{\varsigma_1} = \left\{ \begin{array}{l} ((f_1, f_3, f_5), \{0.5/a, 0.7/b\}), ((f_1, f_4, f_5), \{0.1/c\}), \\ ((f_2, f_3, f_5), \{0.4/b, 0.2/c\}), ((f_2, f_4, f_5), \{0.02/b\}) \end{array} \right\}$$

$$\vartheta_{\varsigma_2} = \left\{ \begin{array}{l} ((f_1, f_3, f_5), \{0.05/b, 0.006/c\}), ((f_1, f_4, f_5), \{0.08/a\}), \\ ((f_2, f_3, f_5), \{0.55/a, 0.75/c\}), ((f_2, f_4, f_5), \{0.52/b\}) \end{array} \right\}$$

$$\vartheta_{\varsigma_3} = \left\{ \begin{array}{l} (f_1, f_3, f_5)\{0.008/c, 0.25/a\}), ((f_1, f_4, f_5), \{0.12/b\}), \\ ((f_2, f_3, f_5), \{0.05/b, 0.64/c\}), ((f_2, f_4, f_5), \{0.28/c\}) \end{array} \right\}$$

and

$$g_{\varsigma_1} = \left\{ \begin{array}{l} ((f_1, f_3, f_5), \{0.87/a\}), ((f_1, f_4, f_5), \{0.23/a\}), \\ ((f_2, f_3, f_5), \{0.09/c, 0.54/a\}), ((f_2, f_4, f_5), \{0.53/a\}) \end{array} \right\}$$

$$g_{\varsigma_2} = \left\{ \begin{array}{l} ((f_1, f_3, f_5), \{0.05/c\}), ((f_1, f_4, f_5), \{0.34/b\}), \\ ((f_2, f_3, f_5), \{0.32/b, 0.27/c\}), ((f_2, f_4, f_5), \{0.08/c\}) \end{array} \right\}$$

$$g_{\varsigma_3} = \left\{ \begin{array}{l} ((f_1, f_3, f_5), \{0.27/b\}), ((f_1, f_4, f_5), \{0.52/b\}), \\ ((f_2, f_3, f_5), \{0.37/a, 0.38/c\}), ((f_2, f_4, f_5), \{0.001/a\}) \end{array} \right\}$$

Then FHS classes can be written as $\{\vartheta_{\varsigma_1}, \vartheta_{\varsigma_2}, \vartheta_{\varsigma_3}\}$, $\{g_{\varsigma_1}, g_{\varsigma_2}, g_{\varsigma_3}\}$.

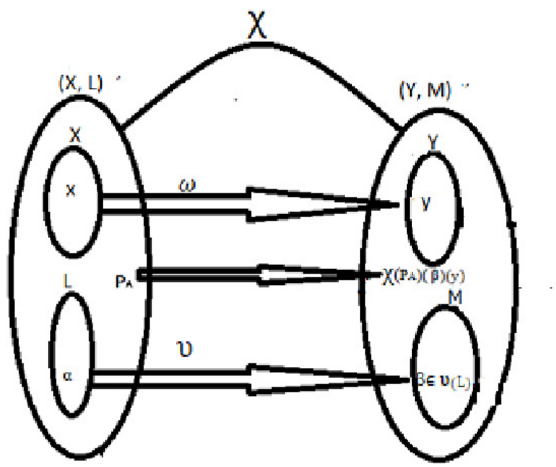

FIGURE 1. Representation of Fuzzy Hypersoft Mapping

Definition 3.3. Let $\overline{(X, L)}$ and $\overline{(Y, M)}$ be classes of FHS sets over X and Y with attributes from L and M respectively. Let $\omega : X \to Y$ and $\upsilon : L \to M$ be mappings. Then a FHS mappings $\chi = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(Y, M)}$ is defined as follows, for a FHSS P_A in $\overline{(X, L)}$, $\chi(P_A)$ is a FHSS in $\overline{(Y, M)}$ obtained as follows, for $\beta \in \upsilon(L) \subseteq M$ and $y \in Y$, $\chi(P_A)(\beta)(y) = \cup_{\alpha \in \upsilon^{-1}(\beta) \cap A,\ s \in \omega^{-1}(y)} (\alpha)\mu_s \chi(P_A)$ is called a fuzzy hypersoft image of a FHSS P_A. Hence $(P_A, \chi(P_A)) \in \chi$, where $P_A \subseteq \overline{(X, L)}$, $\chi(P_A) \subseteq \overline{(Y, M)}$. For more detail see fig 1.

Definition 3.4. If $\chi : \overline{(X,L)} \to \overline{(Y,M)}$ be a FHS mapping, then FHS class $\overline{(X,L)}$ is called the domain of χ and the FHS class $G_B \in \overline{(Y,M)} : G_B = \chi(H_A)$, for some $H_A \in \overline{(X,L)}$ is called the range of χ. The FHS class $\overline{(Y,M)}$ is called co-domain of χ.

Definition 3.5. Let $\chi = (\omega, \upsilon) : \overline{(X,L)} \to \overline{(Y,M)}$ be a FHS mapping and G_B a FHSS in $\overline{(Y,M)}$, where $\omega : X \to Y$, $\upsilon : L \to M$ and $B \subseteq M$. Then $\chi^{-1}(G_B)$ is a FHSS in $\overline{(X,L)}$ defined as follows, for $\alpha \in \upsilon^{-1}(B) \subseteq L$ and $x \in X$, $\chi^{-1}(G_B)(\alpha)(x) = (\upsilon(\alpha))\mu_{p(x)}\chi^{-1}(G_B)$ is called a FHS inverse image of G_B. For more detail see fig 2

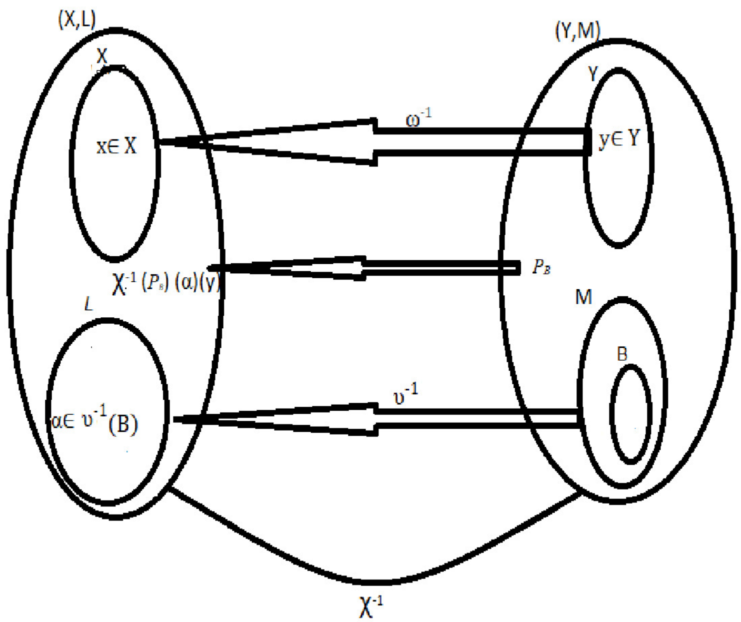

FIGURE 2. Representation of Fuzzy Hypersoft Inverse Mapping

Example 3.6. Let $X = \{a =$ Holstein, $b =$ Angus, $c =$ Charolais $\}$ and $Y = \{x =$ Murrah, $y =$ Siamese, $z =$ Surti $\}$ be the two universal sets of Cows and Buffalo categories respectively. Peter plans his dairy farm business and wants to understand the difference between a strategy and a tactic. For this purpose, he creates the relationship between these two types of cattle categories which is helpful regrading his dairy farm business. Let $a_1 =$ vision and hearing , $a_2 =$ Cost, $a_3 =$ Colour, and $b_1 =$ appearance, $b_2 =$ colour, $b_3 =$ price be the two types of distinct attributes whose corresponding attribute values belong to the sets F_1, F_2, F_3 and F_1', F_2', F_3' respectively. Let $F_1 = \{f_1 =$ Excellent peripheral vision, $f_2 =$ Low peripheral vision $\}$, $F_2 = \{f_3 =$High $\}$, $F_3 = \{f_4 =$ White, $f_5 =$ brown $\}$. Similarly, $F_1' = \{f_1' =$ Good, $f_2' =$ massive $\}$, $F_2' = \{f_3' =$ black $\}$, $F_3' = \{f_4' =$ low price $\}$ and $\overline{(X,L)}$, $\overline{(Y,M)}$ be two classes of FHS sets, where $L = F_1 \times F_2 \times F_3$ and $M = F'_1 \times F'_2 \times F'_3$. Let $\omega : X \to Y$, $\upsilon : F_1 \times F_2 \times F_3 \to F'_1 \times F'_2 \times F'_3$ be mappings as follows

$\omega(a) = y, \omega(b) = x, \omega(c) = y$ and

$\upsilon(f_1, f_3, f_4) = (f'_2, f'_3, f'_4), \upsilon(f_1, f_3, f_5) = (f'_1, f'_3, f'_4)$,

$\upsilon(f_2, f_3, f_4) = (f'_2, f'_3, f'_4), \upsilon(f_2, f_3, f_5) = (f'_1, f'_3, f'_4)$.

Consider a FHSS P_A in $\overline{(X, L)}$ as

$$P_A = \left\{ \begin{array}{l} \left\{ \begin{array}{l} (f_1, f_3, f_4) = \{< a, \{0.5, 0.3\} >, \\ < b, \{0.9, 0.3, 0.5\} >, < c, \{0.3\} > \end{array} \right\}, \\ \left\{ \begin{array}{l} (f_1, f_3, f_5) = \{< a, \{0.3, 0.9\} >, \\ < b, \{0.5, 0.1\} >, < c, \{0.1\} >\} \end{array} \right\} \end{array} \right\}$$

Then the FHS image of P_A under $\chi = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(Y, M)}$ is obtained as

$\chi(P_A)(f_{1'}, f_{3'}, f_{4'})(x) = \cup_{\alpha \in \upsilon^{-1}(f_{1'}, f_{3'}, f_{4'}) \cap A, s \in \omega^{-1}(x)} (\alpha)\mu_s = \cup_{\alpha \in (f_1, f_3, f_5) \cap A, s \in b} (\alpha)\mu_s$

$= (f_1, f_3, f_5)\mu_b = \{0.5, 0.1\}$,

$\chi(P_A)(f_{1'}, f_{3'}, f_{4'})(y) = \{0.3, 0.9\}$,

$\chi(P_A)(f_{1'}, f_{3'}, f_{4'})(z) = \{0\}$

$\chi(P_A)(f_{2'}, f_{3'}, f_{4'})(x) = \{0.9, 0.3, 0.5\}$

$\chi(P_A)(f_{2'}, f_{3'}, f_{4'})(y) = \{0.3, 0.5\}$

$\chi(P_A)(f_{2'}, f_{3'}, f_{4'})(z) = \{0\}$

$$\chi(P_A) = \left\{ \begin{array}{l} \left\{ \begin{array}{l} (f'_1, f'_3, f'_4) = \{< x, \{0.5, 0.1\} >, \\ < y, \{0.3, 0.9\} >, < z, \{0\} > \end{array} \right\}, \\ \left\{ \begin{array}{l} (f'_2, f'_3, f'_4) = \{< x, \{0.9, 0.3, 0.5\} >, \\ < y, \{0.3, 0.5\} >, < z, \{0\} >\} \end{array} \right\} \end{array} \right\}$$

Again consider a FHSS P'_B in $\overline{(Y, M)}$ as

$$P'_B = \left\{ \begin{array}{l} \left\{ \begin{array}{l} (f'_1, f'_3, f'_4) = \{< x, \{0.3, 0.4\} >, \\ < y, \{0.4, 0.5, 0.1\} >, < z, \{0.9, 0.3\} > \end{array} \right\}, \\ \left\{ \begin{array}{l} (f'_2, f'_3, f'_4) = \{< x, \{0.4, 0.5\} >, \\ < y, \{0.9, 0.3\} >, < z, \{0.5, 0.4\} >\} \end{array} \right\} \end{array} \right\}$$

Therefore,

$\chi^{-1}(P'_B)(f_1, f_3, f_4)(a) = \upsilon((f_1, f_3, f_4))\mu_{\omega(a)}$

$= (f'_2, f'_3, f'_4)\mu_{\omega(a)} = (f'_2, f'_3, f'_4)\mu_y = \{0.9, 0.3\}$

$\chi^{-1}(P'_B)(f_1, f_3, f_4)(b) = \{0.4, 0.5\}$

$\chi^{-1}(P'_B)(f_1, f_3, f_4)(c) = \{0.9, 0.3\}$

$\chi^{-1}(P'_B)(f_1, f_3, f_5)(a) = \{0.4, 0.5, 0.1\}$

$\chi^{-1}(P'_B)(f_1, f_3, f_5)(b) = \{0.3, 0.4\}$

$\chi^{-1}(P'_B)(f_1, f_3, f_5)(c) = \{0.4, 0.5, 0.1\}$

$\chi^{-1}(P'_B)(f_2, f_3, f_4)(a) = \{0.9, 0.3\}$

$\chi^{-1}(P'_B)(f_2, f_3, f_4)(b) = \{0.4, 0.5\}$

$\chi^{-1}(P'_B)(f_2, f_3, f_4)(c) = \{0.9, 0.3\}$

$\chi^{-1}(P'_B)(f_2, f_3, f_5)(a) = \{0.4, 0.5, 0.1\}$
$\chi^{-1}(P'_B)(f_2, f_3, f_5)(b) = \{0.3, 0.4\}$
$\chi^{-1}(P'_B)(f_2, f_3, f_5)(c) = \{0.4, 0.5, 0.1\}$
Similarly,

$$\chi^{-1}(P'_B) = \left\{ \begin{array}{l} \left\{ \begin{array}{l} \{(f_1, f_3, f_4) = \{< a, \{0.9, 0.3\} >, \\ < b, \{0.4, 0.5\} >, < c, \{0.9, 0.3\} >\} \end{array} \right\}, \\ \left\{ \begin{array}{l} \{(f_1, f_3, f_5) = \{< a, \{0.4, 0.5, 0.1\} >, \\ < b, \{0.3, 0.4\} >, < c, \{0.4, 0.5, 0.1\} >\} \end{array} \right\}, \\ \left\{ \begin{array}{l} \{(f_2, f_3, f_4) = \{< a, \{0.9, 0.3\} >, \\ < b, \{0.4, 0.5\} >, < c, \{0.9, 0.3\} >\} \end{array} \right\}, \\ \left\{ \begin{array}{l} \{(f_2, f_3, f_5) = \{< a, \{0.4, 0.5, 0.1\} >, \\ < b, \{0.3, 0.4\} >, < c, \{0.4, 0.5, 0.1\} >\} \end{array} \right\} \end{array} \right\}.$$

In case of $\chi : \overline{(X, L)} \to \overline{(Y, M)}$, Peter should purchase those buffalo having attribute values (f_1 = Excellent peripheral vision,f_3 = High,f_4 = White) or can purchase those cow having this attributes values (f'_1 = Good, f'_3 = black,f'_4 = low price) because both of the categories are interrelated and fulfills the requirements individually. For more clarity see 3.

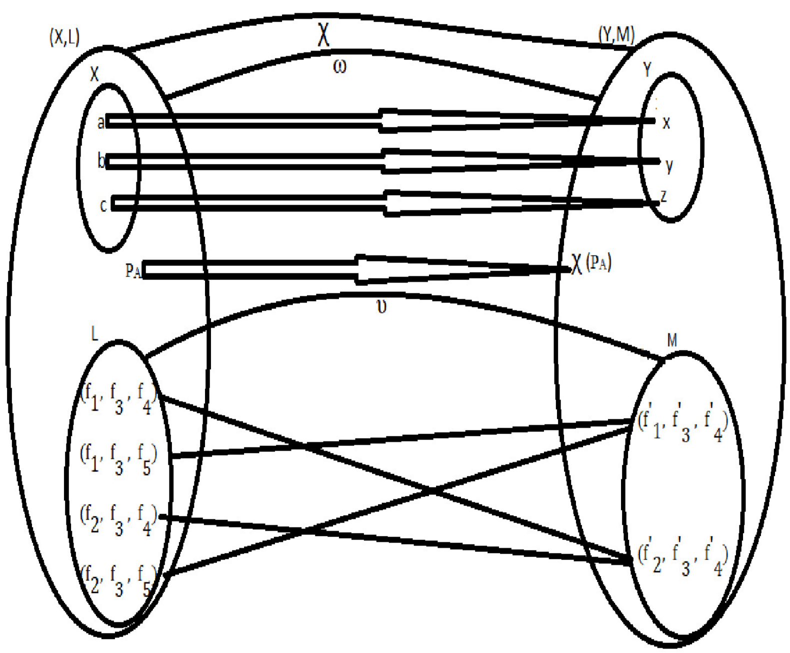

Figure 3. Representation of Fuzzy Hypersoft Mapping

Similarly, for inverse FHSS see fig 4.

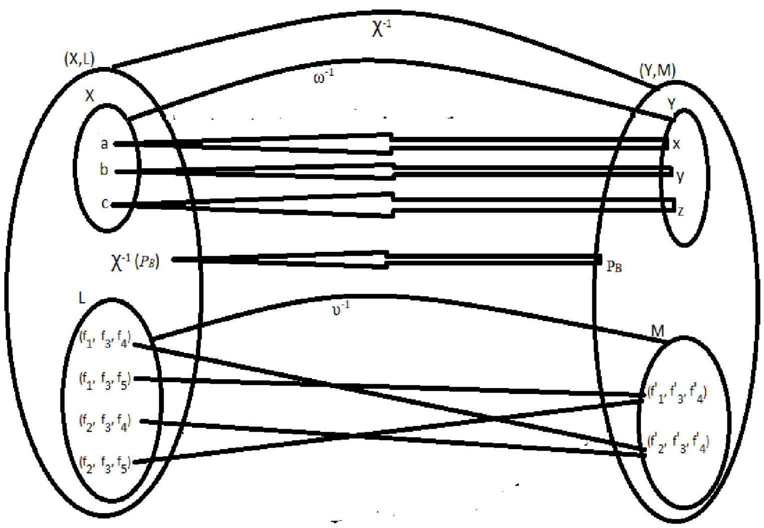

FIGURE 4. Representation of Fuzzy Hypersoft Inverse Mapping

Definition 3.7. Let $\chi = (\omega, \upsilon)$ be a FHS mapping of a FHS class $\overline{(X, L)}$ into a FHS class $\overline{(Y, M)}$. Then

(1) χ is said to be a one-one (or injection) FHS mapping if for both $\omega : X \to Y$ and $\upsilon : L \to M$ are one-one.
(2) χ is said to be a onto (or surjection) FHS mapping if for both $\omega : X \to Y$ and $\upsilon : L \to M$ are onto.

If χ is both one-one and onto then χ is called a one-one onto (or bijective) correspondence of FHS mapping.

Theorem 3.8. *Let $\chi = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(X, M)}$ and $\phi = (m, n) : \overline{(X, L)} \to \overline{(X, M)}$ are two FHS mappings. Then χ and ϕ are equal if and only if $\omega = m$ and $\upsilon = n$.*

Proof. Obvious.

Theorem 3.9. *Two FHS mappings χ and ϕ of a FHS class $\overline{(X, L)}$ into a FHS class $\overline{(Y, M)}$ are equal if and only if $\chi(P_A) = \phi(P_A)$, for all $P_A \in \overline{(X, L)}$.*

Proof. Let $\chi = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(X, M)}$ and $\phi = (m, n) : \overline{(X, L)} \to \overline{(X, M)}$ are two FHS mappings. Since ω and υ are equal, this implies $\omega = m$ and $\upsilon = n$, let $\beta \in \upsilon(L) \subseteq M$ and $y \in Y$, $\chi(P_A)(\beta)(y) = \cup_{\alpha \in \upsilon^{-1}(\beta) \cap A,\ s \in \omega^{-1}(y)}(\alpha)\mu_s = \cup_{\alpha \in n^{-1}(\beta) \cap A,\ s \in m^{-1}(y)}(\alpha)\mu_s$. Hence $\chi(P_A) = \phi(P_A)$.
Conversely,
let $\chi(P_A) = \phi(P_A)$, for all $P_A \in \overline{(X, L)}$, let $(P, Q) \in \chi$, where $P \in \overline{(X, L)}$ and $Q \in \overline{(Y, M)}$.

Therefore $Q = \chi(P) = \phi(P)$, this gives $(P, Q) \in \phi$. Therefore $\chi \subseteq \phi$. Similarly, it can be proved that $\phi \subseteq \chi$. Hence $\phi = \chi$.

Definition 3.10. If $\chi = (\omega, \upsilon) : \overline{(X, L)} \rightarrow \overline{(Y, M)}$ and $\phi = (m, n) : \overline{(Y, M)} \rightarrow \overline{(Z, N)}$ are two FHS mappings, then their composite $\phi o \chi$ is a FHS mapping of $\overline{(X, L)}$ into $\overline{(Z, N)}$ such that for every $P_A \in \overline{(X, L)}$, $(\phi o \chi)(P_A) = \phi(\chi(P_A))$. We defined as for $\beta \in n(M) \subseteq N$ and $y \in Z$, $\phi(\chi(P_A))(\beta)(y) = \cup_{\alpha \in n^{-1}(\beta) \cap \chi(A),\ s \in m^{-1}(y)}(\alpha)\mu_s$. For more detail see fig 5.

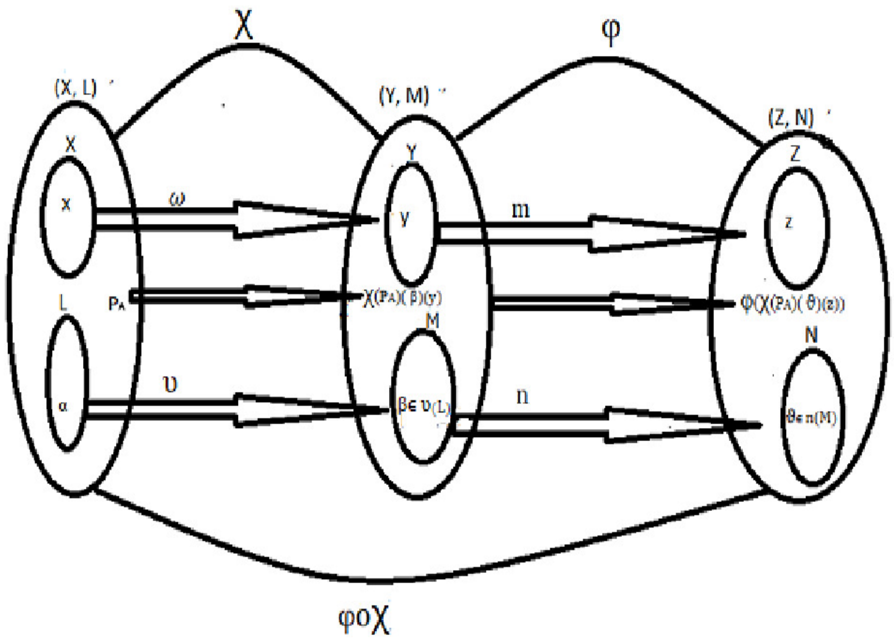

FIGURE 5. Representation of Composite Fuzzy Hypersoft Mapping

Example 3.11. From 3.6, consider the FHS mapping $\phi = (m, n) : \overline{(Y, M)} \rightarrow \overline{(Z, N)}$ in such a way $m(x) = h_2, m(y) = h_3, m(z) = h_2$,
and

$$n(f'_1, f'_3, f'_4) = (f''_1, f''_2, f''_3),$$
$$n(f'_2, f'_3, f'_4) = (f''_1, f''_2, f''_4)$$

where $Z = \{h_1, h_2, h_3\}$, $N = F''_1 \times F''_2 \times F''_3 = \{(f''_1, f''_2, f''_3), (f''_1, f''_2, f''_4)\}$. Therefore

$$\phi(\chi(P_A))(f''_1, f''_2, f''_3)(h_1) = \cup_{\alpha \in n^{-1}(f''_1, f''_2, f''_3) \cap A,\ s \in m^{-1}(h_1)}(\alpha)\mu_s$$

$$= \cup_{\alpha \in (f'_1, f'_3, f'_4) \cap \chi(A),\ s \in \Phi}(\alpha)\mu_s = (f'_1, f'_3, f'_4)\mu_\Phi = \{0\},$$

$$\phi(\chi(P_A))(f''_1, f''_2, f''_3)(h_2) = \{0.3, 0.4, 0.9\},$$

$$\phi(\chi(P_A))(f''_1, f''_2, f''_3)(h_3) = \{0.4, 0.5, 0.1\},$$

So,

$$\phi(\chi(P_A))(f''_1, f''_2, f''_4)(h_1) = \{0\}$$
$$\phi(\chi(P_A))(f''_1, f''_2, f''_4)(h_2) = \{0.4, 0.5\},$$
$$\phi(\chi(P_A))(f''_1, f''_2, f''_4)(h_3) = \{0.9, 0.3\}$$

$$(\phi o \chi)(P_A) = \phi(\chi(P_A) = \left\{ \begin{array}{l} \left\{ \begin{array}{l} (f''_1, f''_2, f''_4) = \{< h_1, \{0\} >, \\ < h_2, \{0.3, 0.4, 0.9\} >, < h_3, \{0.4, 0.5, 0.1\} > \end{array} \right\} \\ \left\{ \begin{array}{l} (f''_1, f''_2, f''_4) = \{< h_1, \{0\} >, \\ < h_2, \{0.4, 0.5\} >, < h_3, \{0.9, 0.3\} > \end{array} \right\} \end{array} \right\}$$

Theorem 3.12. *Let $\chi = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(Y, M)}$ and $\phi = (m, n) : \overline{(Y, M)} \to \overline{(Z, N)}$ are two FHS mappings.Then*

(1) *if χ and ϕ are one-one then so is $\phi o \chi$.*
(2) *if ϕ and χ are onto then so is $\chi o \phi$.*
(3) *if χ and ϕ are both bijections then so is $\phi o \chi$.*

Example 3.13. Fom 3.6, 3.11, let $\omega(a) = z, \omega(b) = x, \omega(c) = y$,
$\upsilon(f_1, f_3, f_4) = (f'_2, f'_3, f'_5)$,
$\upsilon(f_1, f_3, f_5) = (f'_1, f'_3, f'_4)$
and
$m(x) = h_2, m(y) = h_3, m(z) = h_1$

$$n(f'_1, f'_3, f'_4) = (f''_1, f''_2, f''_3),$$
$$n(f'_2, f'_3, f'_4) = (f''_1, f''_2, f''_4).$$

Also we consider a FHSSP_A in $\overline{(X, L)}$ as

$$P_A = \left\{ \begin{array}{l} \{(f_1, f_3, f_4) = \left\{ \begin{array}{l} < a, \{0.5, 0.3\} >, \\ < b, \{0.9, 0.3, 0.5\} >, \\ < c, \{0.3\} > \end{array} \right\}, \\ \{(f_1, f_3, f_5) = \left\{ \begin{array}{l} < a, \{0.3, 0.9\} >, \\ < b, \{0.5, 0.1\} >, \\ < c, \{0.1\} >\} \end{array} \right\} \end{array} \right\}$$

Then the FHS image of P_A under $\chi = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(Y, M)}$ is obtained as

$$\chi(P_A)(f'_1, f'_3, f'_4)(x) = \cup_{\alpha \in \upsilon^{-1}(f'_1, f'_3, f'_4) \cap A,\ s \in \omega^{-1}(x)} (\alpha)\mu_s$$
$$= \cup_{\alpha \in (f_1, f_3, f_5) \cap A,\ s \in b} (\alpha)\mu_s = (f_1, f_3, f_5)\mu_b = \{0.5, 0.1\},$$
$$\chi(P_A)(f'_1, f'_3, f'_4)(y) = \{0.1\},$$
$$\chi(P_A)(f'_1, f'_3, f'_4)(z) = \{0.3, 0.9\},$$

$$\chi(P_A)(f_2', f_3', f_4')(x) = \{0.9, 0.3, 0.5\},$$

$$\chi(P_A)(f_2', f_3', f_4')(y) = \{0.5, 0.3\},$$

$$\chi(P_A)(f_2', f_3', f_4')(z) = \{0.3\}$$

$$\chi(P_A) = \left\{ \begin{array}{l} \left\{ (f'_1, f'_3, f'_4) = \left\{ \begin{array}{l} < x, \{0.5, 0.1\} >, \\ < y, \{0.1\} >, \\ < z, \{0.3, 0.9\} > \end{array} \right\} \right\}, \\ \left\{ (f'_2, f'_3, f'_5) = \left\{ \begin{array}{l} < x, \{0.9, 0.3, 0.5\} >, \\ < y, \{0.3, 0.5\} >, \\ < z, \{0.3\} > \end{array} \right\} \right\} \end{array} \right\}$$

Again,

$$\phi(\chi(P_A))(f_1'', f_2'', f_3'')(h_1) = \cup_{\alpha \in n^{-1}(f_1'', f_2'', f_3'') \cap A,\ s \in m^{-1}(h_1)}(\alpha)\mu_s$$

$$= \cup_{\alpha \in (f_1', f_3', f_4') \cap \chi(A),\ s \in z}(\alpha)\mu_s = (f_1', f_3', f_4')\mu_z = \{0.9, 0.3\},$$

$$\phi(\chi(P_A))(f_1'', f_2'', f_3'')(h_2) = \{0.5, 0.1\},$$

$$\phi(\chi(P_A))(f_1'', f_2'', f_3'')(h_3) = \{0.1\},$$

$$\phi(\chi(P_A))(f_1'', f_2'', f_4'')(h_1) = \{0.3\},$$

$$\phi(\chi(P_A))(f_1'', f_2'', f_4'')(h_2) = \{0.9, 0.3, 0.5\},$$

$$\phi(\chi(P_A))(f_1'', f_2'', f_4'')(h_3) = \{0.3, 0.5\},$$

$$(\phi o \chi)(P_A) = \phi(\chi(P_A) = \left\{ \begin{array}{l} \left\{ \begin{array}{l} (f''_1, f''_2, f''_4) = \{< h_1, \{0.9, 0.3\} >, \\ < h_2, \{0.5, 0.1\} >, < h_3, \{0.1\} > \end{array} \right\} \\ \left\{ \begin{array}{l} (f''_1, f''_2, f''_4) = \{< h_1, \{0.3\} >, \\ < h_2, \{0.9, 0.3, 0.5\} >, < h_3, \{0.3, 0.5\} > \end{array} \right\} \end{array} \right\}$$

Therefore, from above example we see that, if χ and ϕ are one-one then so is $\phi o \chi$. Similarly, for onto as well as bijections.

Theorem 3.14. *Let us consider three FHS mappings* $\chi : \overline{(X, L)} \to \overline{(Y, M)}$, $\phi : \overline{(Y, M)} \to \overline{(Z, N)}$ *and* $\varphi : \overline{(Z, N)} \to \overline{(A, O)}$. *Then* $\varphi o(\phi o \chi) = (\varphi o \phi) o \chi$.

Proof. Let $P_A \in \overline{(X, L)}$, now from definition we have, $[\varphi o(\phi o \chi)](P_A) = (\varphi o \phi) o \chi(P_A)] = \varphi[\phi(\chi(P_A))]$, also $[(\varphi o \phi) o \chi](P_A) = (\varphi o \phi)(\chi(P_A)) = \varphi[\phi(\chi(P_A))]$. Hence $\varphi o(\phi o \chi) = (\varphi o \phi) o \chi$.

Theorem 3.15. *A FHS mapping* $\chi : \overline{(X, L)} \to \overline{(Y, M)}$ *is said to be many one FHS mapping if two (or more than two) FHS sets in* $\overline{(X, L)}$ *have the same FHS image in* $\overline{(Y, M)}$.

Example 3.16. From example 3.6, consider the FHSS $Q_A \in \overline{(X, L)}$,

$$Q_A = \left\{ \begin{array}{l} \left\{ (f_1, f_3, f_4) = \left\{ \begin{array}{l} < a, \{0.5, 0.1\} >, \\ < b, \{0.9, 0.3, 0.5\} >, \\ < c, \{0.3\} > \end{array} \right\} \right\}, \\ \left\{ (f_1, f_3, f_5) = \left\{ \begin{array}{l} < a, \{0.3, 0.9\} >, \\ < b, \{0.5, 0.1\} >, \\ < c, \{0.1\} > \end{array} \right\} \right\} \end{array} \right\}$$

Then the FHS image of P_A under $\chi = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(Y, M)}$ can be written as

$$\chi(Q_A) = \left\{ \begin{array}{l} \left\{ (f'_1, f'_3, f'_4) = \left\{ \begin{array}{l} \{< x, \{0.5, 0.1\} >, \\ < y, \{0.3, 0.9\} >, \\ < z, \{0\} >\} \end{array} \right\} \right\}, \\ \left\{ (f'_2, f'_3, f'_5) = \left\{ \begin{array}{l} < x, \{0.9, 0.3, 0.5\} >, \\ < y, \{0.3, 0.5\} >, \\ < z, \{0\} > \end{array} \right\} \right\} \end{array} \right\}$$

Therefore $\chi(P_A) = \chi(Q_A)$. Hence χ is many one FHS mapping.

Definition 3.17. Let $i = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(X, L)}$ be a FHS mapping, where $\omega : X \to X$ and $\upsilon : L \to L$. Then χ is said to be a FHS identity mapping if both ω and υ are identity mappings.

Remark 3.18. $i = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(X, L)}$ be a FHS identity mapping, then $i(P_A) = P_A$, where $P_A \in \overline{(X, L)}$.

Definition 3.19. Let $\chi = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(Y, M)}$ be a FHS mapping and let $i = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(X, L)}$ and $j = (r, t) : \overline{(Y, M)} \to \overline{(Y, M)}$ are FHS identity mappings then $\chi o i = \chi$ and $j o \chi = \chi$.

Example 3.20. Consider the following example, we consider P_A from example 3.6 and consider the FHS mappings $i = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(X, L)}$, where $\omega : X \to X$ and $\upsilon : L \to L$, such that $\omega(a) = a, \omega(b) = b, \omega(c) = c$, and

$$\upsilon(f_1, f_3, f_4) = (f_1, f_3, f_4), \upsilon(f_1, f_3, f_5) = (f_1, f_3, f_5)$$

$$\upsilon(f_2, f_3, f_4) = (f_2, f_3, f_4), \upsilon(f_2, f_3, f_5) = (f_2, f_3, f_4)$$

Therefore,

$$i(P_A)(f_1, f_3, f_4)(a) = \cup_{\alpha \in \upsilon^{-1}(f_1, f_3, f_4) \cap A, s \in \omega^{-1}(a)}(\alpha)\mu_s$$

$$\cup_{\alpha \in (f_1, f_3, f_4), s \in \{a\}}(\alpha)\mu_s = (f_1, f_3, f_4)\mu_a = \{0.5, 0.1\}$$

$$i(P_A)(f_1, f_3, f_4)(b) = \{0.9, 0.3, 0.5\},$$

$$i(P_A)(f_1, f_3, f_4)(c) = \{0.3\},$$

So,

$$i(P_A)(f_1, f_3, f_5)(a) = \{0.3, 0.9\},$$

$$i(P_A)(f_1, f_3, f_5)(b) = \{0.5, 0.1\},$$

$$i(P_A)(f_1, f_3, f_5)(c) = \{0.1\}$$

we get,

$$i(P_A) = \left\{ \begin{array}{l} \left\{ (f_1, f_3, f_4) = \left\{ \begin{array}{l} < a, \{0.5, 0.3\} >, \\ < b, \{0.9, 0.3, 0.5\} >, \\ < c, \{0.3\} > \end{array} \right\} \right\}, \\ \left\{ (f_1, f_3, f_5) = \left\{ \begin{array}{l} < a, \{0.3, 0.9\} >, \\ < b, \{0.5, 0.1\} >, \\ < c, \{0.1\} > \end{array} \right\} \right\} \end{array} \right\}$$

Hence $i(P_A) = P_A \Rightarrow \chi(i(P_A)) = \chi(P_A) \Rightarrow (\chi o i)(P_A) = \chi(P_A) \Rightarrow \chi o i = \chi$.
Similarly, we get $\chi(P_A) \in \overline{(X, L)}$ and $j(\chi(P_A)) = (\chi(P_A)) \Rightarrow (j o \chi)(P_A) = \chi(P_A)$.
Hence $j o \chi = \chi$.

Definition 3.21. A one-one onto FHS mapping $\chi = (\omega, \upsilon) : \overline{(X, L)} \to \overline{(Y, M)}$ is called FHS invertable mapping. Its FHS inverse mapping is denoted by $\chi^{-1} = (\omega^{-1}, \upsilon^{-1}) : \overline{(Y, M)} \to \overline{(X, L)}$.

Remark 3.22. In a FHS invertable mapping $\chi : \overline{(X, L)} \to \overline{(Y, M)}$, for $P_A \in \overline{(X, L)}, P_B \in \overline{(Y, M)}, \chi^{-1}(P_B) = P_A$, whenever $\chi(P_A) = P_B$.

Example 3.23. We consider $\chi(L_A) = P_B$ (see 3.13). Therefore,

$$\chi^{-1}(P_B)(f_1, f_3, f_4)(a) = (\upsilon(f_1, f_3, f_4)\mu_{\omega(a)} = (f_1', f_3', f_4')\mu_z = \{0.3, 0.9\}$$

$$\chi^{-1}(P_B)(f_1, f_3, f_4)(b) = \{0.5, 0.1\},$$

$$\chi^{-1}(P_B)(f_1, f_3, f_4)(c) = \{0.1\},$$

So,

$$\chi^{-1}(P_B)(f_1, f_3, f_5)(a) = \{0.3\},$$

$$\chi^{-1}(P_B)(f_1, f_3, f_5)(b) = \{0.9, 0.3, 0.5\},$$

$$\chi^{-1}(P_B)(f_1, f_3, f_5)(c) = \{0.3, 0.5\}$$

we get,

$$\chi^{-1}(P_B) = \left\{ \begin{array}{l} \left\{ (f_1, f_3, f_4) = \left\{ \begin{array}{l} < a, \{0.9, 0.3\} >, \\ < b, \{0.5, 0.1\} >, \\ < c, \{0.1\} > \end{array} \right\} \right\}, \\ \left\{ (f_1, f_3, f_5) = \left\{ \begin{array}{l} < a, \{0.3\} >, \\ < b, \{0.9, 0.3, 0.5\} >, \\ < c, \{0.3, 0.5, \} > \end{array} \right\} \right\} \end{array} \right\}$$

Hence, $\chi^{-1}(P_B) = L_A$.

Theorem 3.24. *Let $\chi : \overline{(X, L)} \to \overline{(Y, M)}$ be a FHS invertable mapping. Therefore its FHS inverse mapping is unique.*

Proof. Let χ^{-1} and ϕ^{-1} are two FHS inverse mappings of χ. Therefore, $\chi^{-1}(P_B) = P_A$, whenever $\chi(P_A) = P_B, P_A \in \overline{(X, L)}$, $P_B \in \overline{(Y, M)}$, and $\phi^{-1}(P_B) = H_A$, whenever $\phi(H_A) = P_B, H_A \in \overline{(X, L)}, P_B \in \overline{(Y, M)}$. Thus $\chi(P_A) = \phi(H_A)$. Since χ is one-one, therefore $P_A = H_A$. Hence $\chi^{-1}(P_B) = \phi^{-1}(P_B)$ i.e $\chi^{-1} = \phi^{-1}$.

3.1. *Theorem*

Let $\chi : \overline{(X, L)} \to \overline{(Y, M)}$ and $\phi : \overline{(Y, M)} \to \overline{(Z, N)}$ are two one-one onto FHS mappings. If $\chi^{-1} : \overline{(Y, M)} \to \overline{(X, L)}$ and $\phi^{-1} : \overline{(Z, N)} \to \overline{(Y, M)}$ are FHS inverse mappings of χ and ϕ, respectively, then the inverse of the mapping $\phi o \chi : \overline{(X, L)} \to \overline{(Z, N)}$ is the FHS mapping $\chi^{-1} o \phi^{-1} : \overline{(Z, N)} \to \overline{(X, L)}$. For more detail see fig 6

Proof. Obvious.

Theorem 3.25. *A FHS mapping $\chi : \overline{(X, L)} \to \overline{(Y, M)}$ is invertable if and only if there exists a FHS inverse mapping $\chi^{-1} : \overline{(Y, M)} \to \overline{(X, L)}$ such that $\chi^{-1} o \chi = i_{\overline{(X,L)}}$ and $\chi o \chi^{-1} = i_{\overline{(Y,M)}}$, where $i_{\overline{(X,L)}}$ and $i_{\overline{(Y,M)}}$ is FHS identity mapping on $\overline{(X, L)}$ and $\overline{(Y, M)}$, respectively.*

Proof. Let $\chi : \overline{(X, L)} \to \overline{(Y, M)}$ be a FHS invertable mapping. Therfore, by definition we have $\chi^{-1}(P_B) = P_A$, whenever $\chi(P_A) = P_B, P_A \in \overline{(X, L)}, P_B \in \overline{(Y, M)}$. Since $(\chi^{-1} o \chi)(P_A) = \chi^{-1}(\chi(P_A)) = \chi^{-1}(P_B) = P_A$. Therefore, $\chi^{-1} o \chi = i_{\overline{(X,L)}}$. Similarly, we prove that $\chi o \chi^{-1} = i_{\overline{(Y,M)}}$.

Theorem 3.26. *If $\chi : \overline{(X, L)} \to \overline{(Y, M)}$ and $\phi : \overline{(Y, M)} \to \overline{(Z, N)}$ are two one-one onto FHS mapping then $(\phi o \chi)^{-1} = \chi^{-1} o \phi^{-1}$.*

Proof. Since χ and ϕ are one-one onto FHS mapping, then there exists $\chi^{-1} : \overline{(Y, M)} \to \overline{(X, L)}$ and $\phi^{-1} : \overline{(Z, N)} \to \overline{(Y, M)}$ such that $\chi^{-1}(P_B) = P_A$, whenever $\chi(P_A) = P_B, P_A \in \overline{(X, L)}, P_B \in \overline{(Y, M)}$, and $\phi^{-1}(H_C) = P_B$, whenever $\phi(P_B) = H_C, H_C \in \overline{(Z, N)}, P_B \in \overline{(Y, M)}$. Therefore, $(\phi o \chi)(P_A) = \phi[\chi(P_A)] = \phi(P_B) = H_C$. As $\phi o \chi$ is one-one onto, therefore

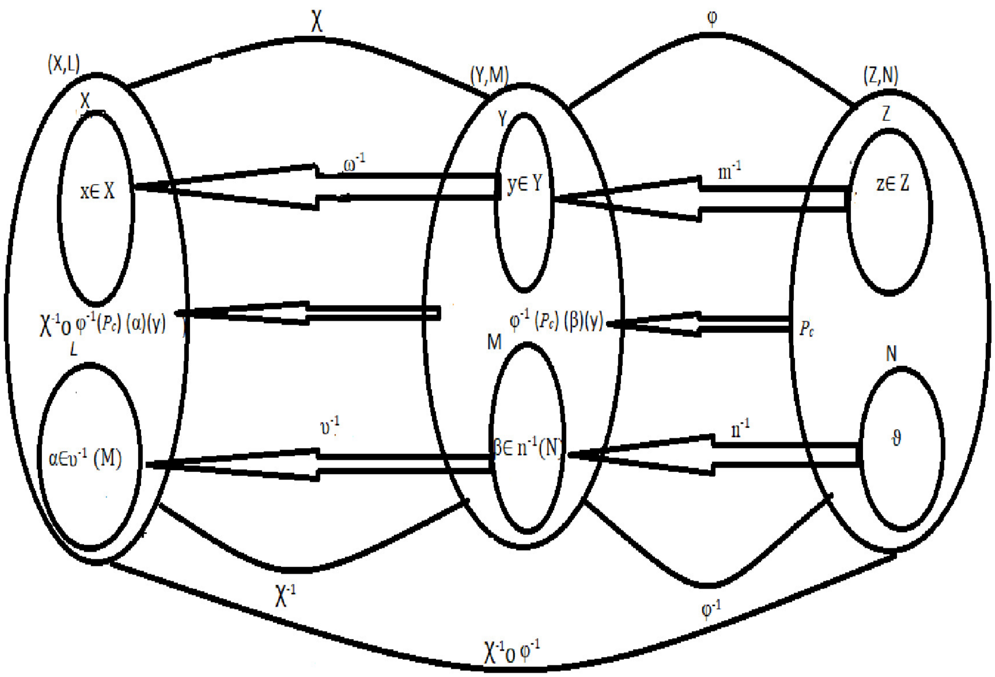

FIGURE 6. Representation of Composite Fuzzy Hypersoft Inverse Mapping

$(\phi o \chi)^{-1}$ exists such that $(\phi o \chi)(P_A) = H_C \Rightarrow (\phi o \chi)^{-1}(H_C) = P_A$. Also $(\phi^{-1} o \phi^{-1})(H_C) = \chi^{-1}[\phi^{-1}(H_C)] = \chi^{-1}(P_B) = P_A$. Hence $(\phi o \chi)^{-1}(H_C) = (\chi^{-1} o \phi^{-1})(H_C) \Rightarrow (\phi o \chi)^{-1} = \chi^{-1} o \phi^{-1}$.

4. **Conclusion**

A basic structure of composite mapping of FHS classes is established along with generalization of certain properties and results. It is very helpful for solving problems involving uncertainty and vagueness. In the future, we will expand our exploration in the domain of Neutrosophic Hypersoft Set, Plithogenic Crisp Hypersoft Set, Plithogenic Fuzzy Hypersoft Set, Plithogenic Intuitionistic Fuzzy Hypersoft Set, Plithogenic Neutrosophic Hypersoft Set, complex Intuitionistic Fuzzy Hypersoft Set, complex Neutrosophic Hypersoft Set, and their hybrid structures. We will apply them in medical imaging issues, pattern recognition, recommender frameworks, social, the monetary framework estimated thinking, image processing, and game theory.

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An Efficient Framework for Drug Product Selection – DPS according to Neutrosophic BWM, MABAC and PROMETHEE II Methods

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Abstract: Developments of systems in healthcare and medical sector have greatly influenced the way we shape our life. Several successful techniques, algorithms and systems have been proposed to solve small version of the change state of each drug according to specific patient. Traditional algorithms and techniques are faced by many difficulties such as (Large Scale, Continuous change of both drug set and patient state, and lack of information). In this study, we propose a methodology for Drug Products Selection - DPS according to every patient individually based on a real data set of US drug bank. A Best Worst Method (BWM), Multi-Attributive Border Approximation Area Comparison (MABAC). And Preference Ranking Organization Method for Enrichment Evaluation (PROMETHEE II) are suggested as a systematic procedure for assessing drug products under the canopy of Neutrosophic theory. Bipolar Neutrosophic Linguistic Numbers (BNLNs) handles the ambiguity, and uncertainty by bipolar Neutrosophic scale, BWM calculates the significance weights of assessment criteria, MABAC as an accurate method assesses drug products, and PROMETHEE II presents effectiveness arrangements of the possible alternatives. A case of 7 real drug products of a real patient against 7 criteria are assessed by 3 doctors to measure the accuracy of the suggested methodology.

Keywords: Drug Product Selection; Neutrosophic Sets; Bipolar; BWM; MABAC; PROMETHEE II

1. Introduction

According to data from Food & Drug Administration of US government – FDA [1], a patient may face some serious situations which led to a sensitivity of some drug product's component, and gradients and of course no one needs to reach out that level of high sensitivity issues when comes to the front because of validation failure. The importance of the validation process before taking a drug product costs nothing compared to the treatment, the one needs when a sensitivity issue comes. The same happens about drug products and their interactions on a patient that has already been taking a set of some other drugs' products. A Drug-Drug Interaction – DDI, and Food-Drug Interaction - FDI lead to serious issues the one may avoid because of validation process. According to a 2007 report on medication safety issued by the Institute for Safe Medication Practices, close to 40 percent of the U.S. population receive prescriptions for four or more medications. And the rate of adverse drug reactions increases dramatically after a patient is on four or more medications [2]. While using a real up-to-date data set of drug bank [3] of US, it is important to analyze a real patient profiles with reviewing their historical records to validate and solve the mysterious and uncertainty of adding one more drug product to their daily routine.

A health-care service provided for doctors, and patients together to prevent or minimize Medical Errors – MEs that harm patients [4]. Measuring how a drug product affect a patient is a critical process which requires a validation. Not only a general validation but it should focus on every patient's situation. Validation on both sides, drug product level and patient level with avoiding any data limitation. Not all drug products the one may take are descripted by a specialist or a doctor, there are many over-the-counter - OTC drug products which a patient can buy and add it manually to his daily drug products set as a valid medicine [5]. The importance of the validation process must be available to both doctors as specialists, and public.

The importance of applying such methodology not limited to doctors and patients but also includes pharmacists. In US, state pharmacy Drug Product Selection – DPS laws allow pharmacists to more easily switch prescriptions from brand-to-generic drugs [6]. Since the objective of the healthcare improved applications is to make it simpler for patients to remain linked to their providers, and for their providers to transfer responsible, value-founded care to their populations [7]. Validation process is the basic concept to transform the healthcare daily actions from novelty to actuality [8]. Five different real cases are reviewed and validated their newly added drug products to their current drug product set with respect of 7 criteria (sex, age, preferred dosage form, sensitivity, DDI, FDI, and price).

The Drug Product Selection – DPS is a problem of Multi-Criteria Decision Making (MCDM) with multiple criteria, alternatives, and decision makers as it can be described according to various criterions rather quantitative or qualitative. Multiple methodologies were illustrated and evaluated the Drug Product Selection – DPS [9,10]. In this study, a proposed methodology of Best Worst Method (BWM), Multi-Attributive Border Approximation Area Comparison (MABAC), and Preference Ranking Organization Method for Enrichment Evaluations (PROMETHEE II) are suggested as an effective integration in multi-criteria decision for assessing the Drug Product Selection – DPS. The Drug Product Selection – DPS challenges of ambiguity, inconsistent information, imprecision, and uncertainty are handled with linguistic variables parameterized by bipolar Neutrosophic scale. Hence, the hybrid methodology of Bipolar Neutrosophic Linguistic Numbers (BNLNs) of BWM [11,12] is suggested to calculate the significance weights of assessment criteria, and MABAC as an accurate method is presented to assess Drug Product Selection – DPS [13]. In addition to consider the qualitative criteria compensation in Drug Product Selection – DPS in MABAC in order to overcome drawbacks PROMETHEE II of non-compensation to reinforce the serving effectiveness arrangements of the possible alternatives of drug products. An experiential case including 7 assessment criteria, assessed against 7 products of different drugs' components to prove validity of the suggested methodology.

The article is planned as follows: Section 2 presents the literature review. Section 3 presents the hybrid methodology of decision making for selecting appropriate drug product under specific conditions using Neutrosophic theory by the integration of the BWM, MABAC and PROMETHEE II. Section 4 presents a case study to validate the proposed model and achieve to a final efficient rank. Section 5 summarizes the aim of the proposed study and the future work.

2. Materials and Methods

2.1. Related Studies and Materials

A review of literature will be displayed about the Drug Product Selection – DPS assessment of selecting the appropriate drug product. BWM and its extended BNLNs are applied to various areas, from manufacturing to drug product selection [14]. Although plenty of papers have been published in these areas [15-17], there are few contributions applied to the evaluation of drug product selection – DPS against multiple criteria all together. The MABAC been extended under various fuzzy environments [18]. E.g. combined interval fuzzy rough sets with the MABAC method to rank the firefighting chopper [19]. Hence, to beat limitations of MABAC method the concept of PROMETHE II has been presented. Many of studies have been enhanced the PROMETHEE II method to solve decision making issues under ambiguous contexts [20]. In [21], presented the PROMETHEE II method under hesitant fuzzy linguistic circumstances to choose green logistic suppliers. Due to conditions of uncertainty and incomplete information, a novel PROMETHEE II method is proposed to solve decision making issues under probability multi-valued Neutrosophic situation [22]. Usually, it is hard for DMs to straight allocate the weight values of assessment criteria in advance. [16] presented a novel weights calculation method, the BWM approach. In [23], combined the BWM method with grey system to calculate the weights of criteria. In [24], the BWM method enhanced with applying hesitant fuzzy numbers to explain criteria relative significance grades. In real life situations decisions, alternatives, criterions are taken in conditions of ambiguity, vagueness, inconsistent information, qualitative information, imprecision, subjectivity and uncertainty. The Bipolar Neutrosophic is used to enhance MCDM [25]. LNNs based on descriptive expressions to describe the judgments of decision makers, criterions, and alternatives is used widely in different MCDM domain e.g. IoT [26,27], medical [28,29], supply chain management [30, 31]. We propose to build a hybrid methodology of BNLNs based on BWM, MABAC, and PROMETHEE II.

2.2. Methodology

We propose a hybrid methodology for assessment of Drug Product Selection – DPS according to specific conditions of individual patient through a given historical record based on BNLNs. A descriptive BNLNs is associated with traditional BWM for prioritizing the problem's criterion. The uncertainty of a drug product against criteria may be presented and hence; we propose MABAC for handling the complexity and uncertainty. Then we evaluate the results of each drug product and solve the non-compensation using PROMETHEE II. Combining the mentioned methods together enabled us to build a robust and hybrid methodology using BWM, MABAC and PROMETHEE II in

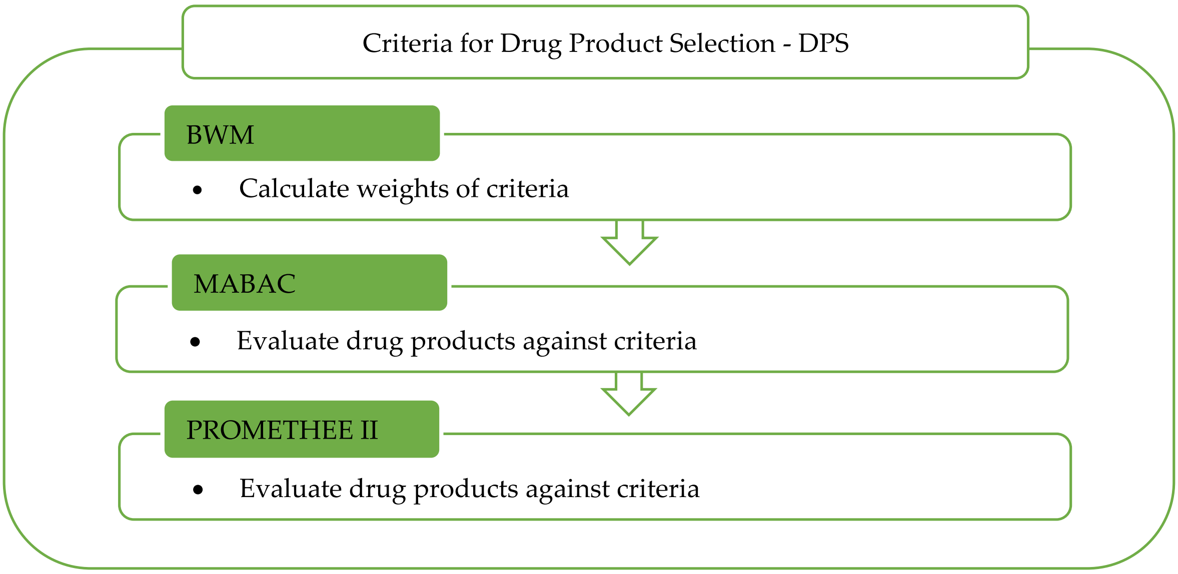

Figure 1. Proposed Approach Conceptualization

a row. Illustred in Figure 1.

A hybrid decision making framework has been designed and built on the integration by extending BWM, MABAC, PROMETHEE II methods to priorities the drug products that have no conflicts, or have less effect on the patient according to his/her historical records with respect of a patient preferred drug product form as well. The drug products evaluation goes through a (13) connected process and the drug product that achieves the requirements and meets the expectation is the best choice and suggested by the system for its compatibility against the selection criterions. The evaluation process is analyzed and compared against a real data of both patients and drug data set. The Steps of the proposed bipolar Neutrosophic with BWM, MABAC, PROMETHEE II and the connected process of selecting a compatible drug product is modeled in Figure 2. and illustrated in details as following:

Phase (I)
Obtain Assessment Matrix

1. Construct original decision making matrix
2. Convert BNLNs into crisp value
3. Standardize the hybrid assessment matrix

Phase (II)
Calculate the Criteria Weights Based on Extended BWM

4. Select the best and worst criteria
5. Acquire the linguistic Best-to-Others vector
6. Obtain the linguistic Others-to-Worst vector
7. Acquire criteria weights

Phase (III)
Build the Difference Matrix Based on MABAC method

8. Calculate the weighted normalized assessment matrix
9. Determine the border approximation area vector
10. Obtain the difference matrix

Phase (IV)
Get Ranking Results based on PROMETHEE II

11. Compute the full preference degree
12. Calculate the positive and negative flows of alternatives
13. Attain the final ranking results of alternatives

Figure 2. Hybrid Decision Making Framework

Phase (I): Obtain Assessment Information

The goal from this phase is to obtain the assessment information:

Step 1: Construct original decision-maker assessment matrix

The linguistic term - LTS provided by decision makers using descriptive expressions such as: (Extremely important, Very important, Midst important, Perfect, Approximately similar, Poor, Midst poor, Very poor, Extremely poor. The BNLNS is an extension of fuzzy set, bipolar fuzzy set, intuitionistic fuzzy set, LTS, and Neutrosophic sets is introduced by [35]. Bipolar Neutrosophic is [T^+, I^+, F^+, T^-, I^-, F^-] where T^+, I^+, F^+ range in [1,0] and T^-, I^-, F^- range in [-1,0]. T^+, I^+, F^+ is the positive membership degree indicating the truth membership, indeterminacy membership and falsity membership and T^-, I^-, F^- is the negative membership degree indicates the truth membership, indeterminacy membership, and falsity membership. E.g. [0.3, 0.2, 0.6, -0.2, -0.1, -0.5] is a bipolar Neutrosophic number.

For BNLNS qualitive criteria values can be calculated by decision makers under a predefined the LTS. Then, an initial hybrid decision making matrix is built as [26]

$$G^D = \begin{array}{c} \\ H_1 \\ \vdots \\ H_0 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & \dots & C_p \end{array} \\ \begin{bmatrix} b_{11}^D & \cdots & b_{1p}^D \\ \vdots & \ddots & \vdots \\ b_{01}^D & \cdots & b_{0p}^D \end{bmatrix} \end{array} \tag{1}$$

Where b_{sr}^D is a BNLNS, representing the assessment result of alternative H_s (s = 1,2, … o) with respect to criterion C_r(r = 1,2, … p) and D = 1,2,3 represent number of decision maker.

Step 2: Convert BNLNs into crisp value using score function mentioned as [28]

$$s(b_{op}) = \left(\frac{1}{6}\right) * (T^+ + 1 - I^+ + 1 - F^+ + 1 + T^- - I^- - F^-) \tag{2}$$

Step 3: Standardize the hybrid assessment matrix.

Normalize the positive and negative criteria of the decision matrix as follows:

For crisp value, the assessment data b_{sr}(s = 1,2, … … o, r = 1,2, … … p) can be normalized with [13]:

$$N_{sr} = \begin{cases} \dfrac{b_{sr} - \min\limits_r(b_{sr})}{\max\limits_r(b_{sr}) - \min\limits_r(b_{sr})}, & for\ beneficial\ criteria \\ \dfrac{\max\limits_r(b_{sr}) - b_{sr}}{\max\limits_r(b_{sr}) - \min\limits_r(b_{sr})}, & for\ non-beneficial \end{cases} \tag{3}$$

Then, a normalized hybrid assessment matrix is formed as

$$N = \begin{array}{c} \\ H_1 \\ \vdots \\ H_0 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & \dots & C_p \end{array} \\ \begin{bmatrix} N_{11} & \cdots & N_{1p} \\ \vdots & \ddots & \vdots \\ N_{o1} & \cdots & N_{op} \end{bmatrix} \end{array} \tag{4}$$

Where N_{sr} shows the normalized value of the decision matrix of S^{th} alternative in R^{th} criteria.

Phase (II): Calculate the Criteria Weights Based on Extended BWM

In this study, the BWM is extended with LTS to obtain the weights of criteria given linguistic expressions.

Step 4: Select the best and the worst criteria

When calculated the assessment criteria $\{C_1 \ldots Cp\}$, decision makers need to choose the best (namely, the most significant) criterion, denoted as C_B. Meanwhile the worst (namely, the least significant) criterion should be selected and represented as C_W.

Step 5: Acquire the linguistic Best-to-Others vector

Make pairwise comparison between the most important criterion CB and the other criteria, then a linguistic Best to-Others vector is obtained with [11]:

$$LC_B = (C_{B1}, C_{B2} \ldots \ldots \ldots \ldots C_{Bp}) \quad \textbf{(5)}$$

Where C_{Br} is a linguistic term within a certain LTS, representing the preference degree of the best criterion CB over criterion c_r ($r = 1,2, \ldots p$) In specific, $C_{BB} = 1$.

Step 6: Obtain the linguistic Others-to-Worst vector

Similarly, make pairwise comparison between the other criteria and the worst criterion C_W, then a linguistic Others-to-Worst vector is obtained with [11]:

$$LC_W = (C_{1W}, C_{2W} \ldots \ldots C_{pW}) \quad \textbf{(6)}$$

Where C_{rW} is a linguistic term within a certain LTS, representing the preference degree of criterion c_r ($r = 1,2, \ldots p$) over the worst criterion C_W in precise, $C_{WW} = 1$.

Step 7: Acquire the weights of criteria

The goal from this step to obtain optimal weights of criteria so that the BWM is extended with crisp number for nonlinear programming model as mentioned [11]:

- min ε is subject to

$$\begin{cases} \frac{w_B}{w_r} - C_{Br} \mid \leq \varepsilon \; For \; all \; r \\ \frac{w_r}{w_W} - C_{rW} \mid \leq \varepsilon \; For \; all \; r \end{cases} \quad \textbf{(7)}$$

Where w_r is the weight of criterion C_r, w_B is the weight of the best criteria C_B and, w_W is the weight of the worst criteria C_W. when solving model **(7)** the weight of w_r and optimal consistency index ε can be computed.

Phase (III): Build the Difference Matrix Based on MABAC method

Build difference matrix built on the idea of MABAC method.

Step 8: Calculate the weighted normalized assessment matrix

Given the normalized values of assessment and the weights of criteria. The weighted normalized values of each criterion are got as follow [13]:

$$\widehat{N}_{sr} = (w_r + N_{sr} * w_r, \; s = 1,2, \ldots o, \; r = 1,2, \ldots p \; p \quad \textbf{(8)}$$

Where w_r is a weight of criteria r and N_{sr} is a normalized value of s and r.

Step 9: Determine the border approximation area vector

The border approximation area vector X is computed as [13]:

$$X_r = \frac{1}{p} \sum_{s=1}^{p} \widehat{N}\text{sr} \quad s = 1,2, \ldots o, \; r = 1,2, \ldots p \quad \textbf{(9)}$$

By calculating the values of the border approximation area matrix, a [0 × 1] matrix is obtained.

Step 10: Obtain the difference matrix

The difference degree between the border approximation area X_r and each element $\widehat{N}_{sr}$ in the weighted normalized assessment matrix can be calculated with [13]:

$$S_{sr} = \widehat{N}_{sr} - X_r\, p \quad \textbf{(10)}$$

Hence, the difference matrix S = $(S_{sr})_{oxp}$ is accomplished.

Phase (IV): Get the Ranking Results Based on PROMETHEE II

Attain the rank of hospitals based on PROMETHEE II method

Step 11: Compute the full preference degree

Compute the alternative difference of s^{th} alternative with respect to other alternative. the preference function is used in this study as follows [32]:

$$P_r(H_s, H_t) = \begin{cases} 0 & if\ S_{sr} - S_{tr} \le 0 \\ S_{sr} - S_{tr} & if\ S_{sr} - S_{tr} > 0 \end{cases}, s, t = 1, 2, \dots o \quad \textbf{(11)}$$

Then the aggregated preference function can be computed as:

$$P(H_s, H_t) = \sum\nolimits_p^o W_r * P_r(H_s, H_t) \Big/ \sum\nolimits_p^o W_r \quad \textbf{(12)}$$

Step 12: Calculate the positive and negative flows of alternatives

The positive flow (namely, the outgoing flow) $\psi^+(Hi)$ [32]:

$$\psi^+(H_i) = \frac{1}{n-1} \sum\nolimits_{t=1, t\ne s}^{o} P(H_s, H_t)\ \ s = 1,2, \dots o \quad \textbf{(13)}$$

The negative flow (namely, the entering flow) $\psi^-(Hi)$ [32]:

$$\psi^-(H_i) = \frac{1}{n-1} \sum\nolimits_{t=1, t\ne s}^{o} P(H_t, H_s)\ \ s = 1,2, \dots o \quad \textbf{(14)}$$

Step 13: Attain the final ranking result of alternatives

The net outranking $\psi(Hi)$ of alternative H_i [32]:

$$\psi(H_i) = \psi^+(Hi) - \psi^-(H_i)\ s = 1,2, \dots o \quad \textbf{(15)}$$

Hence, the final ranking order can be achieved according to the net outranking flow value of each alternative. The larger the value of $\psi(H_i)$, the better the alternative H_i.

3. Results

A case of selecting the appropriate drug product according to real information about patients and drug bank data set is presented to verify the applicability of the integrated method. (5) different real cases (p1, p2, p3, p4, and p5) are reviewed and validated their newly added drug products to their current drug products set with respect to (7) categories: patient sex, drug age-restricted, patient's preferred drug form, Drug-Drug Interactions - DDI, Food-Drug Interactions - FDI, patient sensitivity-list against drugs, and price of a drug product.

Selected patients are real and suffer from the same symptoms, fatigue and are followed up from the Cardiology Department of Zagazig University Hospital - Governmental Hospital - with each of them differs in health status and patient history.

The gathered data is real in both sides, Patients' profiles are real cases and the drugs information come from a **DRUG BANK** which provides up-to-date information regarding drugs and all information needed to apply our study. Some information is hidden under the policy and privacy of sharing patient's data like name, age, and sex. The sample data of patient 1 (p1) required by the algorithm is mentioned in Table 1. Full patient's data, and drug products list are listed in Appendix (A). we refer to drugs and its products by their Drug Bank IDs.

Table 1. Sample data of patients

	Name**	Age**	Sex**	Form list	Sensitivity list	Current drug list
P_1	-	-	-	Tablet Capsule Injection	DB00758 DB01069	DB00199

** hidden data due to privacy

The hybrid method aims to provide the best-suitable drug product selection for patients. Our system studies every patient state carefully, putting the historical records into consideration so that it plays the role of an evaluator for every newly added drug into a patient drug list. The suggested approach integrates the BWM, MABAC, and PROMETHEE II with BNLNs in order to assess drug selection.

The main criteria and sub-criteria of drug selection service are selected by evaluating the historical records and preferred data provided through a patient profile to the requested added drug. Therefore, the study considers 2 main criteria and 7 sub-criteria as shown in Figure 3, and described in Table 2;

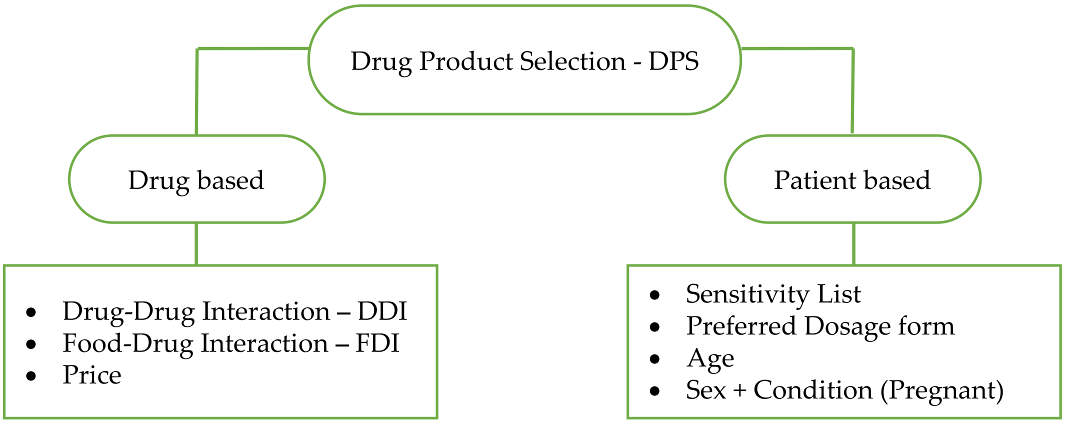

Figure 3. Structure for Drug Product Selection service.

Table 2. Drug Product Selection criteria

Factor	Criteria	Description
Drug based	C_1	Drug-Drug Interaction - DDI
	C_2	Food-Drug Interaction – FDI
	C_3	Price
Patient based	C_4	Sensitivity list
	C_5	Preferred Form
	C_6	Age
	C_7	Sex + Condition

In phase 1. Experts make assessment with respect to the evaluation criteria in Table 2. As criteria C_1 to C_7 are qualitative factors, evaluation information of these subjective criteria is by means of BNLNs. However, 6 criteria belong to non-beneficial type, their scopes are different. Only preferred Form criteria is a beneficial criterion.

Step 1: Construct an original decision maker assessment matrix

Calculate the suitable LTS for weights of criteria and alternatives with respect to every criterion. Each LTS is extended by bipolar Neutrosophic sets to be BNLNs as mentioned in table (3). The BNLNs is described as follows [28]: Extremely important = [0.9, 0.1, 0.0, 0.0, -0.8, -0.9] Where the first three numbers present the positive membership degree. ($T^+(x)$, $I^+(x)$, $F^+(x)$) 0.9, 0.1 and 0.0, such that $T^+(x)$ the truth degree in positive membership. $I^+(x)$ the indeterminacy degree and $F^+(x)$ the falsity degree. The last three numbers present the negative membership degree. ($T^-(x)$, $I^-(x)$, $F^-(x)$) 0.0, -0.8, and -0.9, $T^-(x)$ the truth degree in negative membership, such that $I^-(x)$ the indeterminacy degree and $F^-(x)$ the falsity degree.

Table 3. Bipolar Neutrosophic numbers scale

LTS	Bipolar Neutrosophic numbers scale [$T^+(x)$, $I^+(x)$, $F^+(x)$, $T^-(x)$, $I^-(x)$, $F^-(x)$]	Crisp Value
Extremely high	[0.9,0.1,0.0,0.0, -0.8, -0.9]	0.92
Very high	[1.0,0.0,0.1, -0.3, -0.8, -0.9]	0.73
Midst high	[1.0,0.0,0.1, -0.3, -0.8, -0.9]	0.72
High	[0.7,0.6,0.5, -0.2, -0.5, -0.6]	0.58
Approximately Similar	[0.5,0.2,0.3, -0.3, -0.1, -0.3]	0.52
Low	[0.2,0.3,0.4, -0.8, -0.6, -0.4]	0.45
Midst low	[0.4,0.4,0.3, -0.5, -0.2, -0.1]	0.42
Very low	[0.3,0.1,0.9, -0.4, -0.2, -0.1]	0.36
Extremely low	[0.1,0.9,0.8, -0.9, -0.2, -0.1]	0.13

Step 2: Convert BNLNs into crisp value using score function

Convert BNLNs to crisp value in Table 3. by using score function in Eq. 2.

Table 4., and Table 5. represent the assessments for the original decision maker and the system sequentially using Eq. 1.

Table 4. Original decision making matrix

	C_1*	C_2**	C_3***	C_4*	C_5****	C_6*****	C_7*****
D_1	T	0	9.23	T	tablet	-	-
D_2	T	0	14.78	F	tablet	-	-
D_3	T	4	5.28	F	capsule	-	-
D_4	T	1	3.84	T	injection	-	-
D_5	T	0	143.5	F	injection	-	-
D_6	F	1	2.61	F	tablet	-	-
D_7	F	0	144	F	injection	-	-

***DDI, and Sensitivity: T is given Extremely high, where F is given Extremely low.**

**** FDI: system converts (0) very low, (1-2) low, (3:5) high, and (+5) very high.**

*** Price: system converts (-10 USD) very low, (+10: 25 USD) low, (+25: 50 USD) high, (+50 USD) very high.

**** FORM: given values for every patient and prioritize (very high, high, very low) as the same list order.

***** Age, and Sex: excluded for privacy and policy terms.

Table 5. Assessment of DPS by the system

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
D_1	0.92	0.36	0.36	0.92	0.73	-	-
D_2	0.92	0.36	0.45	0.13	0.73	-	-
D_3	0.92	0.58	0.36	0.13	0.58	-	-
D_4	0.92	0.45	0.36	0.92	0.36	-	-
D_5	0.92	0.36	0.73	0.13	0.36	-	-
D_6	0.13	0.45	0.36	0.13	0.73	-	-
D_7	0.13	0.36	0.73	0.13	0.36	-	-

Step 3: Standardize the hybrid assessment matrix

Normalize the decision matrix, given the positive or negative type of the criteria using Eq. 3, the normalized values of the decision matrix using Eq. 4 are shown as in Table 6.

Table 6. Normalized values of the decision matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
D_1	0	1	1	0	1	-	-
D_2	0	1	0.76	1	1	-	-
D_3	0	0	1	1	0.59	-	-
D_4	0	0.59	1	0	0	-	-
D_5	0	1	0	1	0	-	-
D_6	1	0.59	1	1	1	-	-
D_7	1	1	0	1	0	-	-
max	0.92	0.58	0.73	0.92	0.73	-	-
min	0.13	0.36	0.36	0.13	0.36	-	-

In Phase 2. The goal from this phase determine the weights of criteria based on evaluation of decision maker. Use BWM to calculate weights of criteria.

Step 4: Select the best and the worst criteria

At the beginning C_4 is the best criteria and the C_7 is the worst criteria.

Step 5: Acquire the linguistic Best-to-Others vector

Construct pairwise comparison vector for the best criteria using Eq. 5 in Table 7.

Table 7. Pairwise comparison vector for the best criterion

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
	very low	low	high	same	high	ext. high	ext. high
C_4	2	4	6	1	6	9	9
	0.36	0.45	0.58	1	0.58	0.92	0.92

Step 6: Obtain the linguistic Others-to-Worst vector

Construct pairwise comparison vector for the worst criteria using Eq. 6 in Table 8.

Table 8. Pairwise comparison vector for the worst criterion

		C_7	
C_1	very high	8	0.73
C_2	midst high	7	0.72
C_3	Approx. similar	5	0.52
C_4	ext. high	9	0.92
C_5	high	6	0.58
C_6	midst low	3	0.42
C_7	same	1	1

Step 7: Acquire the weights of criteria

By applying best to others and worst to others using Eq. 7 the weights are computed in Table 9. Figure 4 shows priority of criteria. The consistency ratio ksi = 0.1049 which indicates a desirable consistency.

Table 9. Criteria weights based on BWM

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7
Weights	0.2447	0.1223	0.0816	0.3845	0.0816	0.0544	0.0311

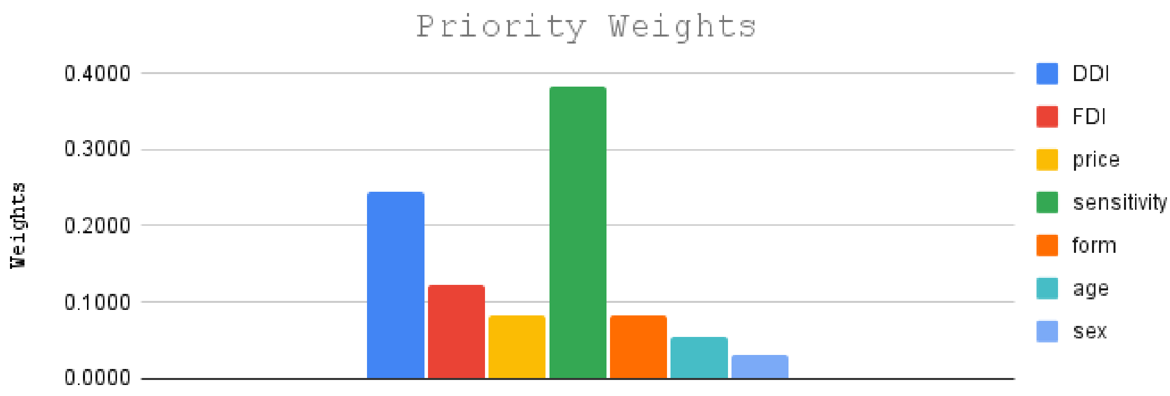

Figure 4. Priority weights of criteria

In Phase 3.

Build the difference matrix based on MABAC method:

Step 8: Calculate the weighted normalized assessment matrix

Construct the weighted normalized decision matrix using Eq. 8. E.g. the weighted normalized values of the first criteria are as follows:

$\widehat{N}_{11} = w_1 + N_{11} * w_1 = 0.2447 * (0 + 0.2447) = 0.2447$

$\widehat{N}_{21} = w_1 + N_{21} * w_1 = 0.2447 * (0 + 0.2447) = 0.2447$

..

$\widehat{N}_{71} = w_1 + N_{71} * w_1 = 0.2447 * (1 + 0.2447) = 0.4893$

The other weighted normalized values of the criteria are calculated in Table 10.

Table 10. Values of the weighted normalized decision matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
D_1	0.2447	0.2447	0.1631	0.384	0.1631	-	-
D_2	0.2447	0.2447	0.1433	0.7689	0.1631	-	-
D_3	0.2447	0.1223	0.1631	0.7689	0.1300	-	-
D_4	0.2447	0.1946	0.1631	0.384	0.0816	-	-
D_5	0.2447	0.2447	0.0816	0.7689	0.0816	-	-
D_6	0.4893	0.1946	0.1631	0.7689	0.1631	-	-
D_7	0.4893	0.2447	0.0816	0.7689	0.0816	-	-

Step 9: Determine the border approximation area vector

Compute the border approximate area matrix using Eq. 9. The amounts of the border approximate area matrix are as shown in Table 11. Figure 5. a scatter chart shows the amount of the border approximate area.

Table 11. Approximation area amounts

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7
Approximation area	0.3146	0.2129	0.1370	0.6591	0.1234	0.0000	0.0000

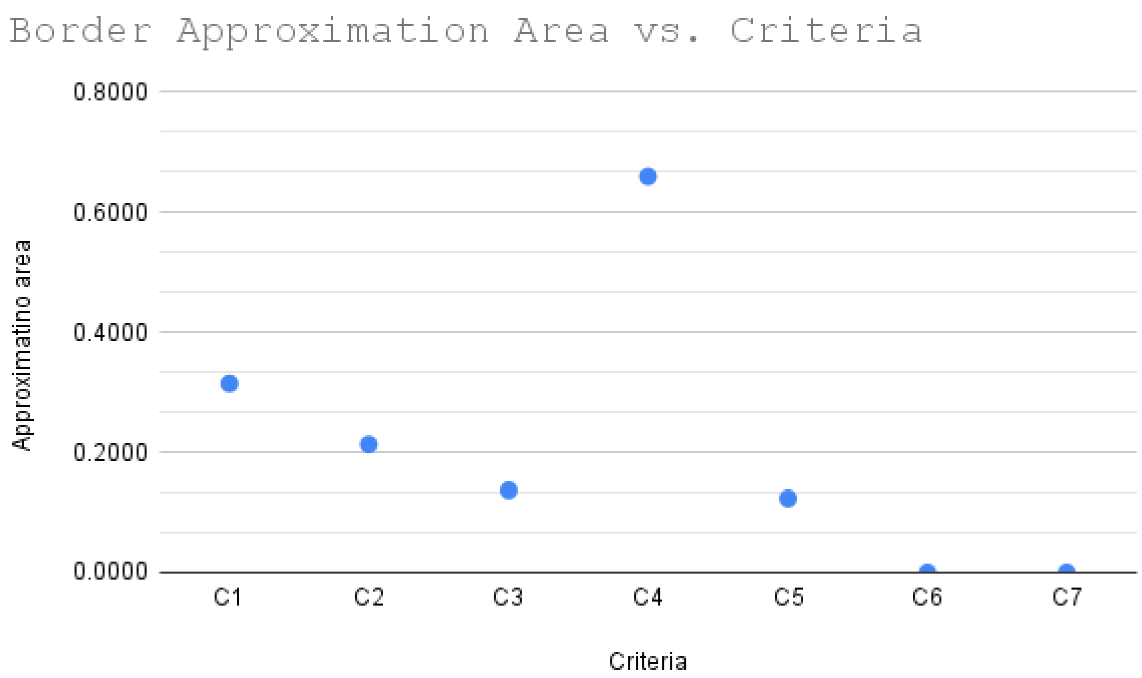

Figure 5. Amount of border approximation area of criteria

Step 10: Obtain the difference matrix

Compute The distance from the border approximate area using Eq. 10. The distance of each alternative from the border approximate area, is shown in
Table 12.

Table 12. Distance from the border approximate area

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
D_1	-0.070	0.032	0.026	-0.275	0.040	-	-
D_2	-0.070	0.032	0.006	0.110	0.040	-	-
D_3	-0.070	-0.091	0.026	0.110	0.007	-	-
D_4	-0.070	-0.018	0.026	-0.275	-0.042	-	-
D_5	-0.070	0.032	-0.055	0.110	-0.042	-	-
D_6	0.175	-0.018	0.026	0.110	0.040	-	-
D_7	0.175	0.032	-0.055	0.110	-0.042	-	-

In phase 4:

Get the ranking results based on PROMETHEE II.

Step 11: Compute the full preference degree

Calculate the evaluative differences of s^{th} alternative with respect to other alternatives. Compute the preference function using Eq. 11. Calculate the aggregated preference function using Eq. 12 in Table 13.

Table 13. Preference values and aggregated preference values

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	Aggregated Pref.
D_{12}	0.000	0.000	0.020	0.000	0.000	0.000	0.000	0.0016178
D_{13}	0.000	0.122	0.000	0.000	0.033	0.000	0.000	0.0176610
D_{14}	0.000	0.050	0.000	0.000	0.082	0.000	0.000	0.0127729
..								
D_{21}	0.000	0.000	0.000	0.384	0.000	0.000	0.000	0.1478141
D_{23}	0.000	0.122	0.000	0.000	0.033	0.000	0.000	0.0176610
D_{24}	0.000	0.050	0.000	0.384	0.082	0.000	0.000	0.1605870
..								
D_{76}	0.000	0.050	0.000	0.000	0.000	0.000	0.000	0.0061219

***Full calculation in Appendix B.**

Step 12: Calculate the positive and negative flows of alternatives

Calculate the positive and negative flows of alternatives using Eq. 13 Eq. 14. Calculate the net outranking flow of each alternative using Eq. 15. Indicates that $\psi(D_6) > \psi(D_7) > \psi(D_2) > \psi(D_5) > \psi(D_3) > \psi(D_1) > \psi(D_4)$. Table 14. shows all the calculations' results.

Table 14. Positive, negative, and net flow of alternatives

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	$\psi^+(H_i)$	$\psi^-(H_i)$	Net Flow
D_1	0.0000	0.0016	0.0177	0.0128	0.0133	0.0061	0.0133	0.0648	0.8588	-0.7940
D_2	0.1478	0.0000	0.0177	0.1606	0.0117	0.0061	0.0117	0.3556	0.1262	0.2294
D_3	0.1478	0.0016	0.0000	0.1518	0.0106	0.0000	0.0106	0.3224	0.2054	0.1171
D_4	0.0000	0.0016	0.0088	0.0000	0.0067	0.0000	0.0067	0.0238	0.9072	-0.8834
D_5	0.1478	0.0000	0.0150	0.1539	0.0000	0.0061	0.0000	0.3228	0.1753	0.1476
D_6	0.2077	0.0615	0.0714	0.2143	0.0732	0.0000	0.0133	0.6413	0.0245	0.6168
D_7	0.2077	0.0599	0.0748	0.2138	0.0599	0.0061	0.0000	0.6221	0.0555	0.5666

Step 13: Attain the final ranking result of alternatives

Determine the ranking of all the considered alternatives in Table 15 depending on the values of net flow calculated in the previous step. The ranking order is $D_6 > D_7 > D_2 > D_5 > D_3 > D_1 > D_4$. Hence, the best drug product alternative is D_6. Figure 6. shows the order of drug products against p_1 profile resulted from our methodology and compared to real doctors' recommendations.

Table 15. Priority of Alternatives - ranking

Alternatives	Rank-SYS	Doctor-1	Doctor-2	Doctor-3
D_1	6	6	6	6
D_2	3	4	3	4
D_3	5	3	5	5
D_4	7	7	7	7
D_5	4	5	4	3
D_6	1	1	2	1
D_7	2	2	1	2

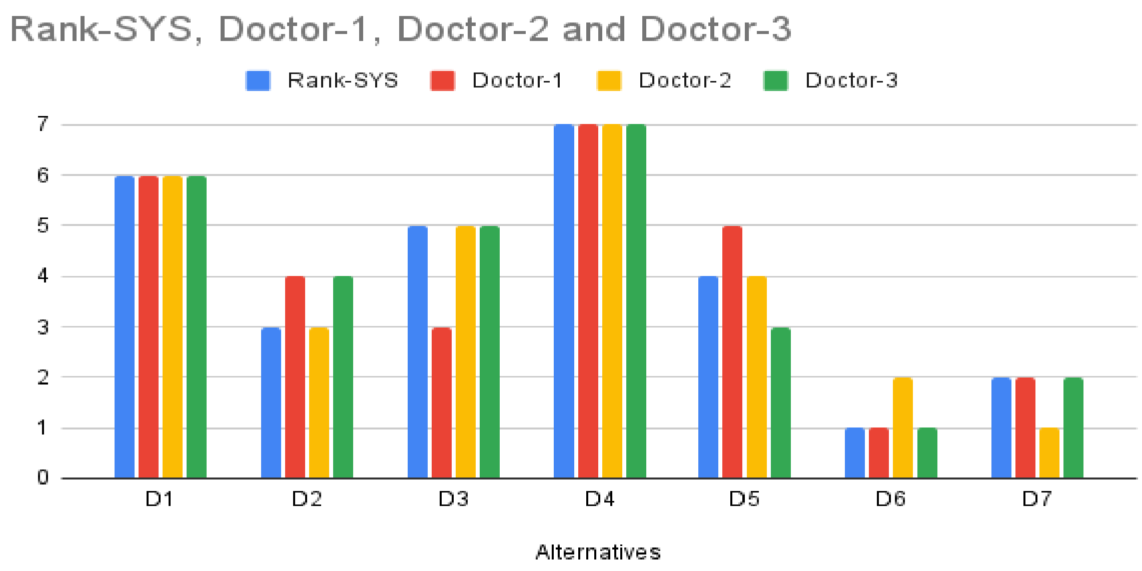

Figure 6. Alternatives order – final rank

Figure 6. shows the difference in drug products order as a result from the methodology and other doctors. We can notice that they all avoid products that p1 has a sensitivity against while the products that p1 has nothing against come in first 2 places. From this point of view. The most affected criterion is sensitivity then DDI, and we must validate any newly added drug product against sensitivity and DDI into our daily drug products list if exist.

4. Applications

The study presents a hybrid methodology of extended BNLNs with Neutrosophic set of BWM, MABAC, and PROMETHEE II to facilitate the Drug Products Selection – DPS process among set of alternatives drug products to prioritize them against every patient's case individually. The real data of both drugs products and patient's profiles are gathered and assessed by the Neutrosophic BWM, MABAC, and PROMETHEE II to evaluate the alternatives products effectively and present a reference of sorted products according to the patient profile. We discuss the outcomes with real doctors after studying every patient's profile carefully and we found the recommended sort is slightly different while taking all the aspects, and criteria into deep study. They are matching the basic concept of excluding drug products that a patient has sensitivity against so it comes in the tail while, drug products that has no conflict against the criteria present in the head and drug products that have any criteria calculations present in the middle. The study presents that the most significant criteria that affect the results is the patient's sensitivity of some drugs. It should be prevented or set out of the scope of the resulting rank. In real life, a process of selecting a drug product should be validated against every patient's condition using our methodology so that a drug product with no conflicts, preferred dosage form, and price is recommended.

5. Conclusions

The study shows the effectiveness of using a system in aim to validate a drug product among same category products. The real data used present a strong point to measure with a real results assessed by real doctors on real patients. The accuracy presented is accepted and we are planning to integrating other advanced methods to enhance the accuracy of such results. The future work includes updated algorithm that excludes and alerts the drug products against sensitivity and handles multiple drugs of patient's current drugs list – CDL that present DDI to the newly added drug product. The future algorithm may use another applicable methodology like TOPSIS and present the comparative studies that might affect the accuracy of resulting rank.

Acknowledgments:

We would like to express our special thanks to the Drug Bank team who gave us the golden opportunity to do this study on real up-to-date drug data all the research period along with patients' profiles.

Appendix A. Drug products, and patients' profiles analysis

Table A1 Drug products data

	Drug Name	Drug Bank ID	Product Name	Dosage Form	Strength	Route	Country
D_1	Clopidogrel	DB00758	Plavix	Tablet, film coated	75 mg/1	Oral	US
D_2	Ticagrelor	DB08816	Brilinta	Tablet	90 mg/1	Oral	US
D_3	Ciclosporin	DB00091	Cyclosporine	Capsule	100 mg/1	Oral	US
D_4	Promethazine	DB01069	Phenergan	Injection	25 mg/1mL	Intramuscular; Intravenous	US
D_5	Voriconazole	DB00582	Voriconazole	Injection, powder, for solution	10 mg/1mL	Intravenous	US
D_6	Ticlopidine	DB00208	Ticlid	Tablet, film coated	250 mg/1	Oral	US
D_7	Floxuridine	DB00322	Floxuridine	Injection, powder, lyophilized, for solution	100 mg/1mL	Intra-arterial	US

All products of the same category, meshID = D065688

Table A2 Patients' profiles analysis

	Name**	Age**	Sex**	Form list	Sensitivity list	Current drug list
P_1	-	-	-	Tablet Capsule Injection	DB00758 DB01069	DB00199
P2	-	-	-	Injection Capsule Tablet	DB00758	DB06777
P3	-	-	-	Capsule Tablet Injection	-	DB01238 DB01595
P4	-	-	-	Injection Capsule Tablet	-	DB06779
P5	-	-	-	Injection Capsule Tablet	DB01069 DB08816	DB01032

**** hidden data due to privacy**

Appendix B. Preference, and aggregated preference values

Table B1 Preference values and aggregated preference values

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	Aggregated Pref.
D_{12}	0.000	0.000	0.020	0.000	0.000	0.000	0.000	0.0016178
D_{13}	0.000	0.122	0.000	0.000	0.033	0.000	0.000	0.0176610
D_{14}	0.000	0.050	0.000	0.000	0.082	0.000	0.000	0.0127729
D_{15}	0.000	0.000	0.082	0.000	0.082	0.000	0.000	0.0133019
D_{16}	0.000	0.050	0.000	0.000	0.000	0.000	0.000	0.0061219
D_{17}	0.000	0.000	0.082	0.000	0.082	0.000	0.000	0.0133019
D_{21}	0.000	0.000	0.000	0.384	0.000	0.000	0.000	0.1478141
D_{23}	0.000	0.122	0.000	0.000	0.033	0.000	0.000	0.0176610
D_{24}	0.000	0.050	0.000	0.384	0.082	0.000	0.000	0.1605870
D_{25}	0.000	0.000	0.062	0.000	0.082	0.000	0.000	0.0116841
D_{26}	0.000	0.050	0.000	0.000	0.000	0.000	0.000	0.0061219
D_{27}	0.000	0.000	0.062	0.000	0.082	0.000	0.000	0.0116841
D_{31}	0.000	0.000	0.000	0.384	0.000	0.000	0.000	0.1478141
D_{32}	0.000	0.000	0.020	0.000	0.000	0.000	0.000	0.0016178
D_{34}	0.000	0.000	0.000	0.384	0.048	0.000	0.000	0.1517687
D_{35}	0.000	0.000	0.082	0.000	0.048	0.000	0.000	0.0106056
D_{36}	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.0000000
D_{37}	0.000	0.000	0.082	0.000	0.048	0.000	0.000	0.0106056
D_{41}	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.0000000
D_{42}	0.000	0.000	0.020	0.000	0.000	0.000	0.000	0.0016178
D_{43}	0.000	0.072	0.000	0.000	0.000	0.000	0.000	0.0088427
D_{45}	0.000	0.000	0.082	0.000	0.000	0.000	0.000	0.0066510
D_{46}	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.0000000
D_{47}	0.000	0.000	0.082	0.000	0.000	0.000	0.000	0.0066510
D_{51}	0.000	0.000	0.000	0.384	0.000	0.000	0.000	0.1478141
D_{52}	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.0000000
D_{53}	0.000	0.122	0.000	0.000	0.000	0.000	0.000	0.0149647
D_{54}	0.000	0.050	0.000	0.384	0.000	0.000	0.000	0.1539360
D_{56}	0.000	0.050	0.000	0.000	0.000	0.000	0.000	0.0061219
D_{57}	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.0000000
D_{61}	0.245	0.000	0.000	0.384	0.000	0.000	0.000	0.2076727
D_{62}	0.245	0.000	0.020	0.000	0.000	0.000	0.000	0.0614764
D_{63}	0.245	0.072	0.000	0.000	0.033	0.000	0.000	0.0713977
D_{64}	0.245	0.000	0.000	0.384	0.082	0.000	0.000	0.2143237
D_{65}	0.245	0.000	0.082	0.000	0.082	0.000	0.000	0.0731605
D_{67}	0.000	0.000	0.082	0.000	0.082	0.000	0.000	0.0133019
D_{71}	0.245	0.000	0.000	0.384	0.000	0.000	0.000	0.2076727

D_{72}	0.245	0.000	0.000	0.000	0.000	0.000	0.000	0.0598586
D_{73}	0.245	0.122	0.000	0.000	0.000	0.000	0.000	0.0748233
D_{74}	0.245	0.050	0.000	0.384	0.000	0.000	0.000	0.2137946
D_{75}	0.245	0.000	0.000	0.000	0.000	0.000	0.000	0.0598586
D_{76}	0.000	0.050	0.000	0.000	0.000	0.000	0.000	0.0061219

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The neutrosophic integrals by parts

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Abstract: The purpose of this article is to study the neutrosophic integrals by parts, all cases in which we can apply integration by parts are discussed, including solve the repeated and non-terminating functions like the product of trigonometric and exponential using rotary integrals. In addition, the Tabular method has been introduced in the computation of neutrosophic integrals, where the Tabular method is considered to be easier than the neutrosophic integrals by Parts method in finding the neutrosophic integrals. Where detailed examples were given to clarify each case.

Keywords: neutrosophic integrals by parts; tabular method; indeterminacy.

1. Introduction

As an alternative to the existing logics, Smarandache proposed the Neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3-13]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist, he defined the standard form of neutrosophic complex number, and found root index $n \geq 2$ of a neutrosophic real and complex number [2-4], studying the concept of the Neutrosophic probability [3-5], the Neutrosophic statistics [4][6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1-8]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [9]. Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side represented with triangular neutrosophic numbers [10]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [11-12]. Y.Alhasan studied the concepts of neutrosophic complex numbers, the general exponential form of a neutrosophic complex, and the neutrosophic integrals and integration methods [7-14-24]. On the other hand, M.Abdel-Basset presented study in the science of neutrosophic about an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number [15]. Also, neutrosophic sets played an important role in applied science such as health care, industry, and optimization [16-17-18-19]. Recently, there are increasing efforts to study the neutrosophic generalized structures and spaces such as refined neutrosophic modules, spaces, equations, and rings [21-22-23].

Integration is important in human life, and one of its most important applications is the calculation of area, size and arc length. In our reality we find things that cannot be precisely defined, and that contain an indeterminacy part.

Paper consists of 5 sections. In 1th section, provides an introduction, in which neutrosophic science review has given. In 2th section, some definitions and examples of neutrosophic real number neutrosophic indefinite integral and are discussed. The 3th section frames neutrosophic integration by parts, in which three cases were discussed, including solve the repeated and non-terminating functions like the product of trigonometric and exponential using rotary integrals. The 4th section The 4th section introduces the Tabular method to find the integrals by parts in the stats 1 and 2. In 5th section, a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic integration by substitution method [24]

Definition2.1.1

$Let\ \ f: D_f \subseteq R \to R_f \cup \{I\}$, to evaluate $\int f(x)dx$

Put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

By substitution, we get:

$$\int f(x)dx = \int f(u)\acute{g}(u)du$$

then we can directly integral it.

Theorme2.1.1:

If $\int f(x,I)dx = \varphi(x,I)\ \ then,$

$$\int f\big((a+bI)x+c+dI\big)\,dx = \left(\frac{1}{a} - \frac{b}{a(a+b)}I\right)\varphi\big((a+bI)x+c+dI\big) + C$$

where C is an indeterminate real constant, $a \neq 0\,, a \neq -b\ and\ b, c, d$ are real numbers, while $I\ =$ indeterminacy.

Theorme2.1.2:

Let $\ \ f: D_f \subseteq R \to R_f \cup \{I\}\ $ then:

$$\int \frac{\acute{f}(x,I)}{f(x,I)}dx = ln|f(x,I)| + C$$

where C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while $I\ =$ indeterminacy).

Theorme2.1.3:

Let $\ \ f: D_f \subseteq R \to R_f \cup \{I\}$, then:

$$\int \frac{\acute{f}(x,I)}{\sqrt{f(x,I)}}dx = 2\sqrt{f(x,I)} + C$$

where C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while $I\ =$ indeterminacy).

Theorme2.1.4:
$f: D_f \subseteq R \to R_f \cup \{I\}$, then:

$$\int [f(x,I)]^n \acute{f}(x)\, dx = \frac{[f(x,I)]^{n+1}}{n+1} + C$$

Where n is any rational number. C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while I = indeterminacy).

2.2. Integrating products of neutrosophic trigonometric function [24]

I. $\int \sin^m(a+bI)x \cos^n(a+bI)x\ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following two cases:

- Case $n\ is$ odd:
  - Split of $\cos(a+bI)x$
  - Apply $\cos^2(a+bI)x = 1 - \sin^2(a+bI)x$
  - We substitution $u = \sin(a+bI)x$
- Case $m\ is$ odd:
  - Split of $\sin(a+bI)x$
  - Apply $\sin^2(a+bI)x = 1 - \cos^2(a+bI)x$
  - We substitution $u = \cos(a+bI)x$

II. $\int \tan^m(a+bI)x \sec^n(a+bI)x\ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following cases:

- Case $n\ is$ even:
  - Split of $\sec^2(a+bI)x$
  - Apply $\sec^2(a+bI)x = 1 + \tan^2(a+bI)x$
  - We substitution $u = \tan(a+bI)x$
- Case $m\ is$ odd:
  - Split of $\sec(a+bI)x \tan(a+bI)x$
  - Apply $\tan^2(a+bI)x = \mathrm{ses}^2(a+bI)x - 1$
  - We substitution $u = \mathrm{ses}(a+bI)x$
- Case m even and n odd:
  - Apply $\tan^2(a+bI)x = \mathrm{ses}^2(a+bI)x - 1$
  - We substitution $u = \mathrm{ses}(a+bI)x$ or $u = \tan(a+bI)x$, depending on the case.

III. $\int \cot^m(a+bI)x \csc^n(a+bI)x\ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following cases:

- Case $n\ is$ even:
  - Split of $\csc^2(a+bI)x$
  - Apply $\csc^2(a+bI)x = 1 + \cot^2(a+bI)x$
  - We substitution $u = \cot(a+bI)x$
- Case $m\ is$ odd:

- Split of $\csc(a+bI)x \cot(a+bI)x$
- Apply $\cot^2(a+bI)x = \csc^2(a+bI)x - 1$
- We substitution $u = \csc(a+bI)x$

➢ Case m even and n odd:
- Apply $\cot^2(a+bI)x = \csc^2(a+bI)x - 1$
- We substitution $u = \csc(a+bI)x$ or $u = \cot(a+bI)x$, depending on the case.

2.3. Neutrosophic trigonometric identities [24]

1) $sin(a+bI)x \cos(c+dI)\, x = \frac{1}{2}\,[\sin(a+bI+c+dI) + \sin(a+bI-c-dI)]$

2) $cos(a+bI)x \sin(c+dI)\, x = \frac{1}{2}\,[\sin(a+bI+c+dI) - \sin(a+bI-c-dI)]$

3) $cos(a+bI)x \cos(c+dI)\, x = \frac{1}{2}\,[\cos(a+bI+c+dI) + \cos(a+bI-c-dI)]$

4) $sin(a+bI)x \sin(c+dI)\, x = \frac{-1}{2}\,[\cos(a+bI+c+dI) - \cos(a+bI-c-dI)]$

Where $a \neq c\ (not\ zero)\ and\ b, d$ are real numbers, while I = indeterminacy.

3. Neutrosophic integration by parts

There are integrals that cannot be evaluated by direct integration methods or by substitution, so in this current section we present a powerful tool called neutrosophic integration by parts. We have observed that every differentiation rule gives rise to a corresponding itegration rule. So, let:

$f: D_f \subseteq R \rightarrow R_f \cup \{I\}$ and $g: D_g \subseteq R \rightarrow R_g \cup \{I\}$

then, for the product rule:

$$\frac{d}{dx}[f(x)\, g(x)] = \acute{f}(x)g(x) + f(x)\acute{g}(x)$$

integrating both sides of this equation gives us:

$$\int \frac{d}{dx}[f(x)\, g(x)]\, dx = \int \acute{f}(x)g(x)\ dx + \int f(x)\acute{g}(x)\, dx$$

$$\int f(x)\acute{g}(x)\, dx = f(x)\, g(x) - \int \acute{f}(x)g(x)\ dx$$

it is usually convenient to write this using the notation:

$$u = f(x) \quad \Rightarrow \quad du = \acute{f}(x)\, dx$$

$$v = g(x) \quad \Rightarrow \quad dv = \acute{g}(x)\, dx$$

so the neutrosophic integration by parts algorithm becomes

$$\int u\, dv = u.v - \int v\, du$$

There are three cases of the neutrosophic integration by parts:

➢ state1: neutrosophic integration from the form:

$$\int (a+bI)x^n\, e^{(c+dI)x}\, dx$$

Where $c \neq 0\ and\ c \neq -d$

To find this integral, we do the following:

Put $u = (a+bI)x^n \quad \Rightarrow \quad du = n(a+bI)x^{n-1}\, dx$

$dv = e^{(c+dI)x}\, dx \quad \Rightarrow \quad v = \frac{1}{c+dI} e^{(c+dI)x}$

Then apply:

$$\int u\, dv = u.v - \int v\, du$$

We get:

$$\int (a+bI)x^n\, e^{(c+dI)x}\, dx = \left(\frac{a}{c} + \frac{\text{cb} - \text{ad}}{\text{c}(\text{c}+\text{d})}.I\right)\left(x^n e^{(c+dI)x} - \int nx^{n-1}\, e^{(c+dI)x} dx\right) + C$$

We find the required integral by repeated the integration.

where C is an indeterminate real constant (i.e. constant of the form $e + kI$, where e, k are real numbers, while I = indeterminacy).

Example3.1

Find:

$$\int (3+2I)x\, e^{(2+4I)x}\, dx$$

Solution:

$$u = (3+2I)x \quad \Rightarrow \quad du = (3+2I)dx$$

$$dv = e^{(2+4I)x}\, dx \quad \Rightarrow \quad v = \frac{1}{2+4I}\, e^{(2+4I)x}$$

Then apply:

$$\int u\, dv = u.v - \int v\, du$$

We get:

$$\int (3+2I)x\, e^{(2+4I)x}\, dx = \left(\frac{3}{2} + \frac{4-12}{12}.I\right)\left(xe^{(2+4I)x} - \int e^{(2+4I)x}\, dx\right)$$

$$= \left(\frac{3}{2} - \frac{8}{12}.I\right)\left(xe^{(2+4I)x} - \frac{1}{2+4I} e^{(2+4I)x}\right)$$

$$= \left(\frac{3}{2} - \frac{2}{3}.I\right)\left(x - \frac{1}{2} + \frac{1}{3}I\right) e^{(2+4I)x} + C$$

- state2: neutrosophic integration from the form:

$$\int (a+bI)x^n \sin(c+dI)x\, dx \quad or \quad \int (a+bI)x^n \cos(c+dI)x\, dx$$

Where $c \neq 0\ and\ c \neq -d$

To find the first integral, we do the following:

Put $u = (a+bI)x^n \quad \Rightarrow \quad du = n(a+bI)x^{n-1}\, dx$

$dv = \sin(c+dI)x\, dx \quad \Rightarrow \quad v = \frac{-1}{c+dI} \cos(c+dI)x$

Then apply:

$$\int u\,dv = u.v - \int v\,du$$

We get:

$$\int (a+bI)x^n \sin(c+dI)x\,dx$$
$$= \left(\frac{a}{c} + \frac{\mathrm{cb}-\mathrm{ad}}{\mathrm{c(c+d)}}.I\right)\left(-x^n \sin(c+dI)x + \int nx^{n-1}\cos(c+dI)x\,dx\right) + C$$

We find the required integral by repeated the integration. By the same method we evaluate the second integral:

$$\int (a+bI)x^n \cos(c+dI)x\,dx$$

Example3.2

Find:

$$\int (1+I)x\ \sin(2-3I)x\,dx$$

Solution:

$$u = (1+I)x \quad \Rightarrow \quad du = (1+I)dx$$

$$dv = \sin(2-3I)x\,dx \quad \Rightarrow \quad v = \tfrac{-1}{2-3I}\ \cos(2-3I)x$$

Then apply:

$$\int u\,dv = u.v - \int v\,du$$

We get:

$$\int (1+I)x\sin(2-3I)x\ \ dx = \left(\frac{1}{2} + \frac{-3-2}{-2}.I\right)\left(-x\cos(2-3I)x + \int \cos(2-3I)x\ \ dx\right)$$

$$= \left(\frac{3}{2} + \frac{1}{2}.I\right)\left(-x\cos(2-3I)x + \frac{1}{2-3I}\sin(2-3I)x\right)$$

$$= \left(\frac{3}{2} + \frac{1}{2}.I\right)\left(-x\cos(2-3I)x + \left(\frac{1}{2} - \frac{3}{2}I\right)\sin(2-3I)x\right) + C$$

- state3: neutrosophic integration from the form:

$$\int e^{(a+bI)x}\sin(c+dI)x\,dx \quad or \quad \int e^{(a+bI)x}\cos(c+dI)x\,dx$$

Where $c \neq 0\ and\ c \neq -d$

To find the first integral, we do the following:

Put $\quad u = e^{(a+bI)x} \quad \Rightarrow \quad du = (a+bI)e^{(a+bI)x}\,dx$

$$dv = \sin(c+dI)x\,dx \quad \Rightarrow \quad v = \tfrac{-1}{c+dI}\cos(c+dI)x$$

Then apply:

$$\int u\,dv = u.v - \int v\,du$$

We get:

$$\int e^{(a+bI)x}\sin(c+dI)x\,dx$$

$$= \left(\frac{-1}{c+dI}\right) e^{(a+bI)x} \cos(c+dI)x + \left(\frac{a+bI}{c+dI}\right) \int e^{(a+bI)x} \cos(c+dI)x \, dx \quad (*)$$

By using integration by parts again to evaluate:

$$\int e^{(a+bI)x} \cos(c+dI)x \, dx$$

Put $u = e^{(a+bI)x} \quad \Longrightarrow \quad du = (a+bI)e^{(a+bI)x} \, dx$

$dv = \cos(c+dI)x \, dx \quad \Longrightarrow \quad v = \frac{1}{c+dI} \sin(c+dI)x$

We get:

$$\int e^{(a+bI)x} \cos(c+dI)x \, dx = \left(\frac{-1}{c+dI}\right) e^{(a+bI)x} \sin(c+dI)x + \left(\frac{a+bI}{c+dI}\right) \int e^{(a+bI)x} \sin(c+dI)x \, dx$$

By substitution in $(*)$:

$$\int e^{(a+bI)x} \sin(c+dI)x \, dx = \left(\frac{-1}{c+dI}\right) e^{(a+bI)x} \cos(c+dI)x + \left(\frac{a+bI}{c+dI}\right) \left(\left(\frac{-1}{c+dI}\right) e^{(a+bI)x} \sin(c+dI)x + \left(\frac{a+bI}{c+dI}\right) \int e^{(a+bI)x} \sin(c+dI)x \, dx \right)$$

$$= \left(\frac{-1}{c+dI}\right) e^{(a+bI)x} \cos(c+dI)x - \left(\frac{a+bI}{(c+dI)^2}\right) e^{(a+bI)x} \sin(c+dI)x + \left(\frac{a+bI}{c+dI}\right)^2 \int e^{(a+bI)x} \sin(c+dI)x \, dx$$

$$\Longrightarrow \left(1 - \left(\frac{a+bI}{c+dI}\right)^2\right) \int e^{(a+bI)x} \sin(c+dI)x \, dx = \left(\frac{-1}{c+dI}\right) e^{(a+bI)x} \cos(c+dI)x - \left(\frac{a+bI}{(c+dI)^2}\right) e^{(a+bI)x} \sin(c+dI)x$$

$$\Longrightarrow \int e^{(a+bI)x} \sin(c+dI)x \, dx = \left(\frac{(c+dI)^2}{(c+dI)^2 - (a+bI)^2}\right) \left(\left(\frac{-1}{c+dI}\right) e^{(a+bI)x} \cos(c+dI)x - \left(\frac{a+bI}{(c+dI)^2}\right) e^{(a+bI)x} \sin(c+dI)x + C \right)$$

By the same method we evaluate the second integral:

$$\int e^{(a+bI)x} \cos(c+dI)x \, dx$$

Example3.2

Find:

$$\int e^{(1+I)x} \cos(2+I)x \, dx$$

Solution:

Put $u = e^{(1+I)x} \implies du = (1+I)e^{(1+I)x}\,dx$

$dv = \cos(2+I)x dx \implies v = \frac{1}{2+I}\sin(2+I)x$

T hen apply:

$$\int u\,dv = u.v - \int v\,du$$

We get:

$$\int e^{(1+I)x}\cos(2+I)x\,dx = \frac{1}{2+I}e^{(1+I)x}\sin(2+I)x - \left(\frac{1+I}{2+I}\right)\int e^{(1+I)x}\sin(2+I)x\,\,dx \quad (*)$$

By using integration by parts again to evaluate:

$$\int e^{(1+I)x}\sin(2+I)x\,\,dx$$

Put $u = e^{(1+I)x} \implies du = (1+I)e^{(1+I)x}\,dx$

$dv = \sin(2+I)x\,dx \implies v = \frac{-1}{2+I}\cos(2+I)x$

We get:

$$\int e^{(1+I)x}\sin(2+I)x\,dx = \frac{-1}{2+I}e^{(1+I)x}\cos(2+I)x + \left(\frac{1+I}{2+I}\right)\int e^{(1+I)x}\cos(2+I)x\,dx$$

By substitution in $(*)$:

$$\int e^{(1+I)x}\cos(2+I)x\,dx = \frac{1}{2+I}e^{(1+I)x}\sin(2+I)x - \left(\frac{1+I}{2+I}\right)\left(\frac{-1}{2+I}e^{(1+I)x}\cos(2+I)x + \left(\frac{1+I}{2+I}\right)\int e^{(1+I)x}\cos(2+I)x\,dx\right)$$

$$= \frac{1}{2+I}e^{(1+I)x}\sin(2+I)x + \frac{1+I}{(2+I)^2}e^{(1+I)x}\cos(2+I)x - \left(\frac{1+I}{2+I}\right)^2\int e^{(1+I)x}\cos(2+I)x\,dx$$

$$\implies \left(1 + \left(\frac{1+I}{2+I}\right)^2\right)\int e^{(1+I)x}\cos(2+I)x\,dx = \frac{1}{2+I}e^{(1+I)x}\sin(2+I)x + \frac{1+I}{4+5I}e^{(1+I)x}\cos(2+I)x$$

$$\implies \int e^{(1+I)x}\cos(2+I)x\,dx = \left(\frac{4+5I}{5+8I}\right)\left(\frac{1}{2+I}e^{(1+I)x}\sin(2+I)x + \frac{1+I}{4+5I}e^{(1+I)x}\cos(2+I)x\right)$$

$$= \left(\frac{4}{5} - \frac{7}{65}I\right)\left(\left(\frac{1}{2} - \frac{1}{6}I\right)e^{(1+I)x}\sin(2+I)x + \left(\frac{1}{4} - \frac{1}{36}I\right)e^{(1+I)x}\cos(2+I)x\right) + C$$

- state3: neutrosophic integration from the form:

$$\int (a+bI)x^n\sin(c+dI)x\,dx \quad or \quad \int (a+bI)x^n\cos(c+dI)x\,dx$$

Where $c \neq 0$ *and* $c \neq -d$

To find the first integral, we do the following:

Put $u = (a + bI)x^n \quad \Longrightarrow \quad du = n(a + bI)x^{n-1}\, dx$

$dv = \sin(c + dI)x\, dx \quad \Longrightarrow \quad v = \frac{-1}{c+dI}\cos(c + dI)x$

Then apply:

$$\int u\, dv = u.v - \int v\, du$$

We get:

$$\int (a + bI)x^n \sin(c + dI)x\, dx$$
$$= \left(\frac{a}{c} + \frac{\mathrm{cb} - \mathrm{ad}}{\mathrm{c(c + d)}}.I\right)\left(-x^n \sin(c + dI)x + \int nx^{n-1}\cos(c + dI)x\, dx\right) + C$$

We find the required integral by repeated the integration. By the same method we evaluate the second integral:

$$\int (a + bI)x^n \cos(c + dI)x\, dx$$

Example3.3

Find:

$$\int (1 + I)x\ \sin(2 - 3I)x\, dx$$

Solution:

$$u = (1 + I)x \quad \Longrightarrow \quad du = (1 + I)dx$$

$$dv = \sin(2 - 3I)x\, dx \quad \Longrightarrow \quad v = \frac{-1}{2-3I}\ \cos(2 - 3I)x$$

Then apply:

$$\int u\, dv = u.v - \int v\, du$$

We get:

$$\int (1 + I)x \sin(2 - 3I)x\ dx = \left(\frac{1}{2} + \frac{-3 - 2}{-2}.I\right)\left(-x\cos(2 - 3I)x + \int\ \cos(2 - 3I)x\ dx\right)$$

$$= \left(\frac{3}{2} + \frac{1}{2}.I\right)\left(-x\cos(2 - 3I)x + \frac{1}{2 - 3I}\sin(2 - 3I)x\right)$$

$$= \left(\frac{3}{2} + \frac{1}{2}.I\right)\left(-x\cos(2 - 3I)x(c + dI)x + \left(\frac{1}{2} - \frac{3}{2}I\right)\sin(2 - 3I)x\right) + C$$

- state4: neutrosophic integration from the form:

$$\int (a + bI)x^n \ln(c + dI)x\, dx\,,\ \ n \neq 1$$

To find the first integral, we do the following:

Put $u = ln(c + dI)x \quad \Longrightarrow \quad du = \frac{1}{x}\, dx$

$dv = (a+bI)x^n\,dx \quad \Rightarrow \quad v = \frac{a+bI}{n+1}x^{n+1}$

Then apply:

$$\int u\,dv = u.v - \int v\,du$$

We get:

$$\int (a+bI)x^n\, ln(c+dI)x\,dx = \frac{a+bI}{n+1}x^{n+1}\ln|(c+dI)|x + \int \frac{a+bI}{n+1}x^n\,dx$$

$$= \frac{a+bI}{n+1}x^{n+1}\ln(c+dI)x + \frac{a+bI}{(n+1)^2}x^{n+1} + C$$

Example3.4

Find:

$$\int (7+4I)x\;\; ln(6+3I)x\,dx$$

Solution:

Put $u = ln(6+3I)x \quad \Rightarrow \quad du = \frac{1}{x}\,dx$

$dv = (7+4I)x\,dx \quad \Rightarrow \quad v = \frac{7+4I}{2}x^2$

Then apply:

$$\int u\,dv = u.v - \int v\,du$$

We get:

$$\int (7+4I)x\;\; ln\,(6+3I)x\,dx = \frac{7+4I}{2}x^2 ln(6+3I)x - \frac{7+4I}{2}\int x\,dx$$
$$= \frac{7+4I}{2}x^2 ln(6+3I)x - \frac{7+4I}{2}\frac{x^2}{2}$$

$$= \left(\frac{7}{2}+2I\right)\left(x^2 ln(6+3I)x - \frac{x^2}{2}\right) + C$$

Remark:

To find the following integrals:

$$\int (a+bI)x^n\, sin^{-1}(c+dI)x\,dx,\;\; \int (a+bI)x^n\, cos^{-1}(c+dI)x\,dx,$$
$$\int (a+bI)x^n\, tan^{-1}(c+dI)x\,dx \quad n \neq 1$$

We are following the same state4, whereas:

Put

$u = \; sin^{-1}(c+dI)x \;\; or \;\; cos^{-1}(c+dI)x \;\; or \;\; tan^{-1}(c+dI)x, \quad and \;\; dv = (a+bI)x^n\,dx$

Example3.5

Find:

$$\int (4+I)x\, tan^{-1}(2+5I)x\,dx$$

Solution:

Put

$$u = \; tan^{-1}(2+5I)x \quad \Rightarrow \quad du = \frac{2+5I}{1+(2+5I)^2x^2}\,dx$$

$$dv = (4+I)x\,dx \qquad \Rightarrow \qquad v = \frac{4+I}{2}x^2$$

We get:

$$\int (4+I)x\,tan^{-1}(2+5I)x\,dx = \frac{4+I}{2}x^2\,tan^{-1}(2+5I)x - \frac{8+27I}{2}\int \frac{x^2}{1+(4+45I)x^2}\,dx$$

$$= \frac{4+I}{2}x^2\,tan^{-1}(2+5I)x - \frac{8+27I}{2}\int \left(\frac{1}{4+45I} - \frac{1}{4+45I}\;\frac{1}{1+(4+45I)x^2}\right)dx$$

$$= \frac{4+I}{2}x^2\,tan^{-1}(2+5I)x - \frac{8+27I}{2}\left(\frac{1}{4+45I}x - \frac{2+5I}{4+45I}\;\;tan^{-1}(2+5I)x\right) + C$$

$$= \left(2+\frac{1}{2}I\right)x^2\,tan^{-1}(2+5I)x - \frac{8+27I}{2}\left(\left(\frac{1}{4}-\frac{45}{196}\right)x - \left(\frac{2}{4}-\frac{70}{196}I\right)\;\;tan^{-1}(2+5I)x\right) + C$$

4.Tabular method to find the integrals by parts in the stats 1 and 2

- Differentiate the polynomial function, and we repeat that until we get to zero.
- Integral the second function, and repeat that until we get to the zero that we got from the differentiation.
- Arrange the products of the derivatives in one column, and the products of the integrals in another column corresponding to it.
- Draw an arrow from each entry in the first column to the entry that is one row down in the second column.
- Label the arrows with alternating + and − signs, starting with a +.
- For each arrow, form the product of the expressions at its tip and tail and then multiply that product by + or − in accordance with the sign on the arrow.

Example4.1

Find the following integral by using tabular method:

$$\int \left((3+I)x^2 + 2x\right) e^{(2-4I)x}\,dx$$

Solution:

derivation	integration
$(+)\ (3+I)x^2 + 2x$	$e^{(2-4I)x}$
$(-)\ (3+I)x + 2$	$\frac{1}{2-4I}e^{(2-4I)x}$
$(+)\ (3+I)$	$\frac{1}{4}e^{(2-4I)x}$
0	$\frac{1}{8-16I}e^{(2-4I)x}$

Hence:

$$\int \left((3+I)x^2 + 2x\right) e^{(2-4I)x}\,dx$$

$$= \left((3+I)x^2 + 2x\right)\frac{1}{2-4I}e^{(2-4I)x} - \left((3+I)x + 2\right)\,\frac{1}{2}e^{(2-4I)x} + \frac{3+I}{8-16I}e^{(2-4I)x}$$

$$= \left((3+I)x^2+2x\right)\left(\frac{1}{2}-I\right)e^{(2-4I)x} - \left((3+I)x+2\right)\frac{1}{4}e^{(2-4I)x} + \left(\frac{3}{8}-\frac{5}{8}\right)e^{(2-4I)x} + C$$

Example4.2

Find the following integral by using tabular method:

$$\int \left((3+I)x^2+2x\right)cos(2-4I)x\,dx$$

Solution:

derivation	integration
$(+)\ (3+I)x^2+2x$	$cos(2-4I)x$
$(-)\ (3+I)x+2$	$\frac{1}{2-4I}sin(2-4I)x$
$(+)\ (3+I)$	$\frac{-1}{4}cos(2-4I)x$
0	$\frac{-1}{8-16I}sin(2-4I)x$

Hence:

$$\int \left((3+I)x^2+2x\right)e^{(2-4I)x}\,dx$$

$$= \left((3+I)x^2+2x\right)\frac{1}{2-4I}e^{(2-4I)x} - \left((3+I)x+2\right)\frac{1}{2}e^{(2-4I)x} - \frac{3-I}{8-16I}e^{(2-4I)x}$$

$$= \left((3+I)x^2+2x\right)\left(\frac{1}{2}-I\right)sin(2-4I)x + \left((3+I)x+2\right)\frac{1}{4}cos(2-4I)x - \left(\frac{3}{8}-\frac{5}{8}\right)sin(2-4I)x + C$$

5. Conclusions

Integrals are important in our life, as they facilitate many mathematical operations in our reality, and this is what led us to study the neutrosophic integrals by parts, and the tabular method, which is considered easier than the neutrosophic integrals by parts for some neutrosophic integrals. This paper is considered an introduction to the applications in neutrosophic integrals.

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Solving Neutrosophic Linear Programming Problems Using Exterior Point Simplex Algorithm

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Abstract: In this manuscript, three contributions are proposed. First contribution is proposing a good evaluation between the fuzzy and neutrosophic approaches using a novel fuzzy-neutrosophic transfer. Second contribution is introducing a general framework for solving the neutrosphic linear programming problems using the advantages of the method of Abdel-Basset et al. and the advantages of Singh et al.'s method. Third contribution is proposing a new neutrosophic exterior point simplex algorithm NEPSA and its fuzzy version FEPSA. NEPSA has two paths to get optimal solutions. One path consists of basic not feasible solutions but the other path is feasible. Finally, the numerical examples and results analysis show that NEPSA more than accurate FEPSA.

Keywords: Fuzzy Linear Programming; Ranking Function; Trapezoidal Fuzzy Number; Trapezoidal Neutrosophic Number; Exterior Point Simplex Algorithm.

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1. Introduction

Fuzzy sets were introduced by Zadeh [20] to handle vague and imprecise information. But also fuzzy set does not represent vague and imprecise information efficiently, because it considers only the truthiness function. After then, Atanassov [3] introduced the concept of intuitionistic fuzzy set to handle vague and imprecise information, by considering both the truth and falsity function. But also intuitionistic fuzzy set does not simulate human decision making process. Because the proper decision is fundamentally a problem of arranging and explicate facts the concept of neutrosophic set theory was presented by Smarandache, to handle vague, imprecise and inconsistent information [9,10,11,12]. Neutrosophic set theory simulates decision-making process of humans, by considering all aspects of decision-making process. Neutrosophic set is a popularization of fuzzy and intuitionistic fuzzy sets; each element of set had a truth, indeterminacy and falsity membership function. So, neutrosophic set can assimilate inaccurate, vague and maladjusted information efficiently and effectively [18, 19].

The first EPSA was developed by Paparrizos for the assignment problem [27]. Later, Paparrizos generalized EPSA to the general LP [28]. Primal-dual versions of the algorithm are discussed in [29,30]. From the geometry of EPSA, In particular, EPSA proved to be up to ten times faster than simplex algorithm on randomly generated optimal LPs of medium size.

EPSA constructs two paths to the optimal solution. One path consists of basic but not feasible solutions; so this is an “exterior path”. The second path is feasible. It consists of line segments, the endpoints of which lie on the boundary of the feasible region. EPSA relies on the idea that making steps in directions that are linear combinations of attractive descent directions which can lead to faster practical convergence than that achievable by simplex algorithm. Although EPSA outperforms clearly the original simplex algorithm (on randomly generated dense and sparse LPs) it has two computational disadvantages. Firstly, it is difficult to construct "good moving directions". We use the term "good moving direction" loosely. A good moving direction is a direction that makes the algorithm efficient in practice. Geometrically a good moving direction is a direction that comes close to the optimal solution. In fact the two paths depend on the initial feasible segment (direction) and the initial feasible vertex. Secondly, there is no known way of moving into the interior of the feasible region. This movement will provide more flexibility in the search for computationally good directions.

Badr *et al* [8] proposed a new method to solve the fuzzy linear programming problem. It is called fuzzy exterior point simplex algorithm (FEPSA). It constructs two ways to get the optimal solution. One path consists of basic not feasible solutions. The second way is feasible.

For more details about the linear programming, the reader can refer to [13,5,4,6]. On the other hand, for more details about the fuzzy linear programming, the reader is referred to [7]. Finally, for more details about the neutrosophic linear programming, the reader may refer to [2,14,15,16,17,24,25,26,31].

The remaining parts of this research are organized as follows: In sect. 2, we introduce the basic concepts of fuzzy and neutrosophic sets and a new technique which converts the fuzzy representation to the neutrosophic representation. The fuzzy rank functions and it corresponding neutrosophic rank functions are proposed in Sec. 3. In Sec. 4, we propose Singh et al.'s modifications [32] and the proposed modification for primal neutrosophic simplex method and a new neutrosophic exterior point simplex algorithm NEPSA. In Sec. 5, we propose two numerical examples that show the importance of the proposed modification for primal neutrosophic simplex method and they show the superiority of the proposed algorithm NEPSA. Finally, we introduce the future work and conclusions in Sec. 6.

2. Preliminaries

In this section, we introduce three subsections. First one is representation of the fuzzy numbers. Second is the representation of the neutrosophic numbers. Finally, we show that how to move from fuzzy representation to neutrosophic representation. In other words, how do to convert the fuzzy numbers to neutrosophic numbers.

2.1 Fuzzy Representation

We review the fundamental notions of fuzzy set theory, initiated by Bellman and Zadeh [20].

2.1.1 Definition: A convex fuzzy set $\widetilde{A}$ on $\mathbb{R}$ is a fuzzy number if the following conditions hold:

- Its membership function is piecewise continuous.
- There exist three intervals [*a, b*], [*b, c*], [*c, d*] such that μ_a is increasing on [*a, b*], equal to 1 on [*b, c*], decreasing on [*c, d*] and equal to 0 elsewhere.

2.1.2 Definition: Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ denote the trapezoidal fuzzy number, where $(a^L - \alpha, a^U + \beta)$ is the support of ã and $[a^L, a^U]$ its core.

Remark 1*:* We denote the set of all trapezoidal fuzzy numbers by *F(*$\mathbb{R}$*).*

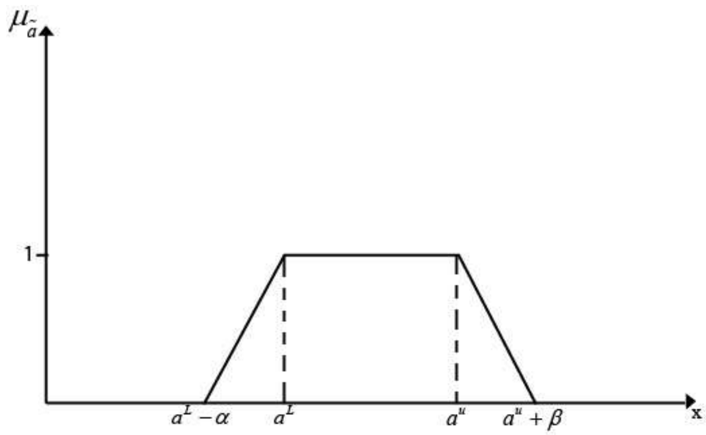

Figure 1. Truth membership function of trapezoidal fuzzy numbers

We next define arithmetic on trapezoidal fuzzy numbers. Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$be two trapezoidal fuzzy numbers. Define:

$x\tilde{a} = (xa^L, xa^U, x\alpha, x\beta) : \ x > 0, \quad x \in \mathbb{R};$

$x\tilde{a} = (xa^U, xa^L, -x\beta, -x\alpha): x < 0, \quad x \in \mathbb{R};$

$\tilde{a}+\tilde{b}=(a^L, a^U, \alpha, \beta)+ (b^L, b^U, \gamma, \theta) = [a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta]$

We point out that the arithmetic on trapezoidal fuzzy numbers follows the Extension Principle which is discussed in [22].

2.2 Neutrosophic Representation

In this subsection, some of basic definitions in the neutrosophic set theory are introduced:

2.2.1 Definition [1]: A single-valued neutrosophic set N which is a subset of X is defined as follows: $N = \{< x, T_N(x), I_N(x), F_N(x) >: x \in X\}$where X is a universe of discourse, $T_N(x): X \rightarrow [0,1]$, $I_N(x): X \rightarrow [0,1]$ and $F_N(x): X \rightarrow [0,1]$ with $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3$ for all$x \in X$, $T_N(x), I_N(x)$ and $F_N(x)$represent truth membership, indeterminacy membership and falsity membership degrees of x to N.

2.2.2 Definition [1]: The trapezoidal neutrosophic number $\tilde{A}$ is a neutrosophic set in R with the following truth, indeterminacy and falsity membership functions:

$$T_{\tilde{A}}(x) = \begin{cases} \frac{\alpha_{\tilde{A}}(x-a\ \ )}{a_2-a_1}: a_1 \leq x \leq a_2 \\ \alpha_{\tilde{A}} \quad : a_2 \leq x \leq a_3 \\ \alpha_{\tilde{A}}(\frac{x-a_3}{a_4-a_3}) : a_3 \leq x \leq a_4 \\ 0 \quad otherwise \end{cases} \quad I_{\tilde{A}}(x) = \begin{cases} \frac{(a_2-x+\theta_{\tilde{A}}(x-a_1'))}{a_2-a_1'}: a_1' \leq x \leq a_2 \\ \theta_{\tilde{A}} \quad : a_2 \leq x \leq a_3 \\ \frac{(x-a_3+\theta_{\tilde{A}}(a_4'-x))}{a_4'-a_3}: a_3 \leq x \leq a_4' \\ 1 \quad otherwise \end{cases}$$

$$F_{\tilde{A}}(x)=\begin{cases}\dfrac{(a_2-x+\beta_{\tilde{A}}(x-a_1^{"}))}{a_2-a_1^{"}} & : a_1^{"}\le x\le a_2\\ \beta_{\tilde{A}} & : a_2\le x\le a_3\\ \dfrac{(x-a_3+\beta_{\tilde{A}}(a_4^{"}-x))}{a_4^{"}-a_3} & : a_3\le x\le a_4^{"}\\ 1 & otherwise\end{cases}$$

Where $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and $\beta_{\tilde{A}}$ represent the maximum degree of truthiness, minimum degree of indeterminacy and minimum degree of falsity, respectively, $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and $\beta_{\tilde{A}} \in [0,1]$. The membership functions of trapezoidal neutrosophic number are shown in Fig. 2. It is clear that$a_1^{"} < a_1 < a_1' < a_2 < a_3 < a_4' < a_4 < a_4^{"}$.

Remark 2: Here $T_{\tilde{A}}(x)$ increases with a constant rate for [a_1, a_2] and decreases with a constant rate for [a_3, a_4]. $F_{\tilde{A}}(x)$decreases with a constant rate for [a_1'', a_2] and increases with a constant rate for [a_3, a_4'']. $I_{\tilde{A}}(x)$increases and decreases with a constant rate for [a_1', a_2] simultaneously, and it decreases and increases with a constant rate for [a_3, a_4'] simultaneously.

Remark 3: If $a_2 - a_1 = a_4 - a_3$ the trapezoidal neutrosophic number is called the symmetric trapezoidal neutrosophic number.

2.2.3 Definition [1]: Let $\tilde{A} = \ < a_1, a_2, a_3, a_4; \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} >$ and $\tilde{B} = \ < b_1, b_2, b_3, b_4; \alpha_{\tilde{B}}, \theta_{\tilde{B}}, \beta_{\tilde{B}} >$ are two trapezoidal neutrosophic numbers, then the mathematical operations are presented as follows:

$$\tilde{A}+\tilde{B} = \ < (a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} >$$

$$\tilde{A}-\tilde{B} = \ < (a_1-b_4, a_2-b_3, a_3-b_2, a_4-b_1); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} >$$

$$\tilde{A}^{-1} = \ < (\tfrac{1}{a_4}, \tfrac{1}{a_3}, \tfrac{1}{a_2}, \tfrac{1}{a_1}); \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} > \text{where}(\tilde{A} \neq 0)$$

$$\lambda\tilde{A} = \begin{cases} < \lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4; \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} >: & \lambda > 0\\ < \lambda a_4, \lambda a_3, \lambda a_2, \lambda a_1; \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} >: & \lambda < 0\end{cases}$$

$$\tilde{A}\tilde{B} = \begin{cases} < (a_1b_1, a_2b_2, a_3b_3, a_4b_4); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > & if\ (a_4 > 0, b_4 > 0)\\ < (a_1b_4, a_2b_3, a_3b_2, a_4b_1); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > & if\ (a_4 < 0, b_4 > 0)\\ < (a_4b_4, a_3b_3, a_2b_2, a_1b_1); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > & if\ (a_4 < 0, b_4 < 0)\end{cases}$$

$$\frac{\tilde{A}}{\tilde{B}} = \begin{cases} < (\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > & if\ (a_4 > 0, b_4 > 0)\\ < (\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > & if\ (a_4 < 0, b_4 > 0)\\ < (\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4}); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > & if\ (a_4 < 0, b_4 < 0)\end{cases}$$

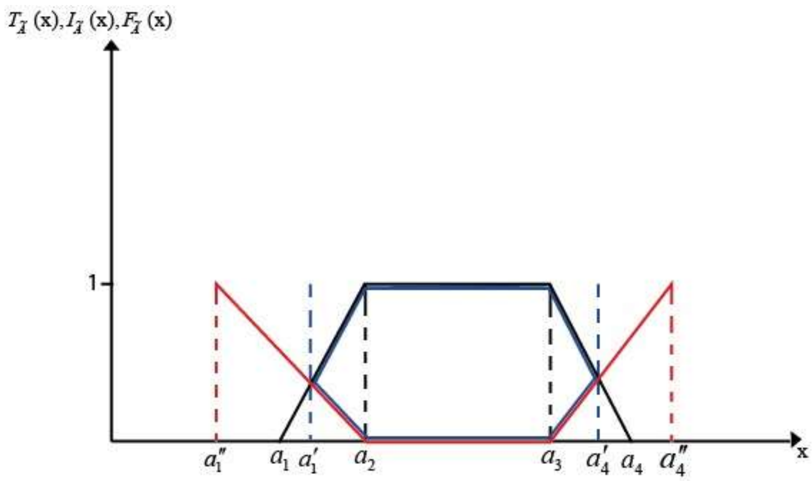

Figure 2. Truth, indeterminacy and falsity membership functions of trapezoidal neutrosophic numbers

2.3 Fuzzy-Neutrosophic Transformation

The main goal of this subsection is to explain how to convert fuzzy numbers representation into neutrosophic numbers representation. This transformation is used for simplicity and fair comparison between them. It is known that there are many rank functions for ordering the fuzzy and neutrosophic numbers. We emphasize using the same function for both fuzzy numbers and neutrosophic numbers to obtain a fair comparison between them. Here we also explain how to apply this technique.

From Figure 1 and Figure 2 we can illustrate the following relations between the two representations:

$$a_1 = a_2 - \alpha,\ a_2 = a^L,\ a_3 = a^U \text{ and } a_4 = a_3 + \beta \quad (1)$$

Assuming that the rank function is used for ordering the fuzzy numbers as follows:

$$R(\tilde{a}) = a^l + a^u + \frac{\beta - \alpha}{2} \quad (2)$$

From relations (1) we express the rank function to be used for ordering the neutrosophic numbers as follows:

$$R(\tilde{a}) = \frac{1}{2}\sum_{i=1}^{4} \tilde{a}_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}}) \quad (3)$$

From (1), we can convert fuzzy numbers representation into neutrosophic numbers representation. On the other hand from (2) and (3), we can use the same function for both fuzzy numbers and neutrosophic numbers to obtain a fair comparison between them.

3. Rank Functions

Assuming that $T_{\tilde{A}} = 1$, $I_{\tilde{A}} = 0$, $\tilde{F}_A = 0$,then the TrNN $\tilde{a} =< a_1, a_2, a_3, a_4; T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}} >$ will be transformed into a trapezoidal fuzzy number $\tilde{a} =< a_1, a_2, a_3, a_4; 1,0,0 >$ and hence, in this case:

- The expression $R(\tilde{a}) = \frac{1}{2}\sum_{j=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ is equivalent to the expression $R(\tilde{a}) = \frac{1}{2}\sum_{j=1}^{4} a_i + 1$

Furthermore, it well be known that if $a_1 = a_2 = a_3 = a_4$ then the trapezoidal fuzzy number $\tilde{A} =< a_1, a_2, a_3, a_4; 1,0,0 >$ will be transformed into a real number *A = (a, a, a, a; 1, 0, 0)* and hence, in this case:

- The expression $R(\tilde{a}) = \frac{1}{2}\sum_{j=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ is equivalent to the expression $R(A) = 2a + 1 \neq a$

Table 1. The rank function and it corresponding neutrosophic rank function

No	Fuzzy Rank Function	Corresponding Neutrosophic Rank Function	Rank function of constraints
1	$R(\tilde{a}) = (a^l + a^u + \frac{\beta - \alpha}{2})$	$R(\tilde{a}) = \frac{1}{2}\sum_{j=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$	$R(a) = 2a + 1$
2	$R(\tilde{a}) = (\frac{a^l + a^u}{2})$	$R(\tilde{a}) = \left(\frac{a_2 + a_3}{2}\right) + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$	$R(a) = a + 1$
3	$R(\tilde{a}) = (\frac{a^l + a^u}{2} + \frac{\beta - \alpha}{4})$	$R(\tilde{a}) = \frac{1}{4}\sum_{j=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$	$R(a) = a + 1$

4. Algorithms

In this section; we first present Singh *et al*.'s modifications [32] and the proposed modification about the mathematical incorrect assumptions, considered by Abdel-Basset *et al*. [1] in their proposed method to convert from neutrosophic numbers into real numbers**.** Second, we propose a new Exterior point simplex algorithm. Finally, we develop this algorithm in order to solve linear programming with neutrosophic numbers.

4.1. General Framework for Solving Neutrosphic Linear Programming Problems

The main objective of this section is to remove the confusion among readers regarding the contributions of Abdel-Basset et al. and the contributions of Singh et al. In this paper, we present a general framework for solving neotrosophic linear programming problems using the advantages of the method of Abdel-Basset et al. and the advantages of Singh et al.'s method.

In 2019, Abdel-Basset et al. [1] presented a simple and effective model for solving neutrosophic linear programming problems supported by a set of numerous examples and a comparison between their approaches presented and solving these examples using the fuzzy method. Consequently, Abdel-Basset et al were able to prove the effectiveness of his approach in solving neutrosophic linear programming problems. On the other hand, Singh et al, 2019 [32] introduced modifications to Abdel-Basset model. These modifications summarized in how neutrosophic numbers are converted into real numbers.

In order to illustrate the method of each of them in solving neutrosophic linear programming problems, we assume the general form of neutrosophic linear programming problems as follows:

$$max \setminus \ min \ \left[\tilde{z} = \sum_{j=1}^{n} \tilde{c}_j x_j\right]$$
$$\text{s.t.}$$
$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \ \leq, \geq, = \tilde{b}_j \ , \ \ i = 1,2, \dots\dots, m; \ \ x_j \geq 0 \ , \ j = 1,2, \dots\dots, n.$$

Model (1) illustrates the method of Abdel-Basset et al in converting neutrosophic numbers into deterministic numbers (in the objective function)

$$max \setminus \ min \ \left[R(\tilde{z}) = \sum_{j=1}^{n} R(\tilde{c}_j) x_j\right]$$

While model (2) illustrates the method of Singh et al. in converting neutrosophic numbers into deterministic numbers (in the objective function)

$$max \setminus \ min \ \left[R(\tilde{z}) = R(\sum_{j=1}^{n} \tilde{c}_j x_j\right]$$

In fact, there is a complete match between the method presented by Abdel-Basset et al and the method presented by Singh et al. In the case of converting fuzzy numbers to real numbers because $R(\tilde{A}_1 \oplus \tilde{A}_2) = R(\tilde{A}_1) + R(\tilde{A}_2)$ where $\tilde{A}_1$ *and* $\tilde{A}_2$ are fuzzy numbers.

On the other hand, when converting neutrosophic numbers to real numbers, the proposed method presented by Singh et al. is more accurate than the method suggested by Abdel-Basset et al. mathematically, because $R(\tilde{A}_1 \oplus \tilde{A}_2) \neq R(\tilde{A}_1) + R(\tilde{A}_2)$ where $\tilde{A}_1 and \tilde{A}_2$ are neutrosophic numbers.

Lemma 1: Let $\tilde{A}_1$ *and* $\tilde{A}_2$ are fuzzy numbers then $\boldsymbol{R(\tilde{A}_1 \oplus \tilde{A}_2) = R(\tilde{A}_1) + R(\tilde{A}_2)}$

Proof:

Suppose that $\tilde{A}_1 = (a_1^l, a_1^u, \propto_1, \beta_1)$ and $\tilde{A}_2 = (a_2^l, a_2^u, \propto_2, \beta_2)$ are two Trapezoidal fuzzy numbers as shown in Figure 1, and the used rank function is defined as follows: $R(\tilde{A}) = \frac{a^L+a^U}{2} + \frac{\beta-\alpha}{4}$

$$R(\tilde{A}_1 \oplus \tilde{A}_2) = \mathrm{R}\,((a_1^l + a_2^l), (a_1^u + a_2^u), (\propto_1 + \propto_2), (\beta_1 + \beta_2)) = \frac{a_1^l+a_2^l+a_1^u+a_2^u}{2} + \frac{\beta_1+\beta_2-\propto_1-\propto_2}{4} \quad (4)$$

While,

$$R(\tilde{A}_1) + R(\tilde{A}_2) = \frac{a_1^l+a_1^u}{2} + \frac{\beta_1+\propto_1}{4} + \frac{a_2^l+a_2^u}{2} + \frac{\beta_2+\propto_2}{4} = \frac{a_1^l+a_2^l+a_1^u+a_2^u}{2} + \frac{\beta_1+\beta_2-\propto_1-\propto_2}{4} \quad (5)$$

It is obvious from (4) and (5) that $R(\tilde{A}_1 \oplus \tilde{A}_2) = R(\tilde{A}_1) + R(\tilde{A}_2)$

Lemma 2: Let $\widetilde{\mathrm{A}}$ *and* $\widetilde{\mathrm{B}}$ are neutrosophic numbers then $\boldsymbol{R(\widetilde{\mathrm{A}} \oplus \widetilde{\mathrm{B}}) \neq R(\widetilde{\mathrm{A}}) + R(\widetilde{\mathrm{B}})}$

Proof:

Suppose that $\widetilde{\mathrm{A}} = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4, \mathrm{T}_{\widetilde{\mathrm{A}}}, \mathrm{I}_{\widetilde{\mathrm{A}}}, \mathrm{F}_{\widetilde{\mathrm{A}}})$ and $\widetilde{\mathrm{B}} = (\mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3, \mathrm{b}_4, \mathrm{T}_{\widetilde{\mathrm{B}}}, \mathrm{I}_{\widetilde{\mathrm{B}}}, \mathrm{F}_{\widetilde{\mathrm{B}}})$ are two Trapezoidal neutrosophic numbers as shown in Fig. 2 and the used rank function is defined as follows:

$$\mathrm{R}(\widetilde{\mathrm{A}}) = \frac{\mathrm{a}_1 + \mathrm{a}_4 + 2\,(\mathrm{a}_2 + \mathrm{a}_3)}{2}$$

$$\mathrm{R}(\widetilde{\mathrm{A}} \oplus \widetilde{\mathrm{B}}) = R\,((\mathrm{a}_1 + \mathrm{b}_1), (\mathrm{a}_2 + \mathrm{b}_2), (\mathrm{a}_3 + \mathrm{b}_3), (\mathrm{a}_4 + \mathrm{b}_4); \min\{\mathrm{T}_{\widetilde{\mathrm{A}}}, \mathrm{T}_{\widetilde{\mathrm{B}}}\}, \max\{\mathrm{I}_{\widetilde{\mathrm{A}}}, \mathrm{I}_{\widetilde{\mathrm{B}}}\}, \max\{\mathrm{F}_{\widetilde{\mathrm{A}}}, \mathrm{F}_{\widetilde{\mathrm{B}}}\}$$

$$= \frac{\mathrm{a}_1+\mathrm{b}_1+\mathrm{a}_4+\mathrm{b}_4+2(\mathrm{a}_2+\mathrm{b}_2+\mathrm{a}_3+\mathrm{b}_3)}{2} + \min\{\mathrm{T}_{\widetilde{\mathrm{A}}}, \mathrm{T}_{\widetilde{\mathrm{B}}}\} - \max\{\mathrm{I}_{\widetilde{\mathrm{A}}}, \mathrm{I}_{\widetilde{\mathrm{B}}}\} - \max\{\mathrm{F}_{\widetilde{\mathrm{A}}}, \mathrm{F}_{\widetilde{\mathrm{B}}}\}) \quad (6)$$

On the other hand,

$$\mathrm{R}(\widetilde{\mathrm{A}}) + \mathrm{R}(\widetilde{\mathrm{B}}) = \frac{\mathrm{a}_1+\mathrm{a}_4+2\,(\mathrm{a}_2+\mathrm{a}_3)}{2} + (\mathrm{T}_{\widetilde{\mathrm{A}}} - \mathrm{I}_{\widetilde{\mathrm{A}}} - \mathrm{F}_{\widetilde{\mathrm{A}}}) + \frac{\mathrm{b}_1+\mathrm{b}_4+2\,(\mathrm{b}_2+\mathrm{b}_3)}{2} + (\mathrm{T}_{\widetilde{\mathrm{B}}} - \mathrm{I}_{\widetilde{\mathrm{B}}} - \mathrm{F}_{\widetilde{\mathrm{B}}})$$

$$= \frac{\mathrm{a}_1+\mathrm{b}_1+\mathrm{a}_4+\mathrm{b}_4+2(\mathrm{a}_2+\mathrm{b}_2+\mathrm{a}_3+\mathrm{b}_3)}{2} + \min\{\mathrm{T}_{\widetilde{\mathrm{A}}}, \mathrm{T}_{\widetilde{\mathrm{B}}}\} - \max\{\mathrm{I}_{\widetilde{\mathrm{A}}}, \mathrm{I}_{\widetilde{\mathrm{B}}}\} - \max\{\mathrm{F}_{\widetilde{\mathrm{A}}}, \mathrm{F}_{\widetilde{\mathrm{B}}}\}) \quad (7)$$

It is obvious from (6) and (7) that $\mathrm{R}(\widetilde{\mathrm{A}} \oplus \widetilde{\mathrm{B}}) \neq \mathrm{R}(\widetilde{\mathrm{A}}) + \mathrm{R}(\widetilde{\mathrm{B}})$

Remark 4:

Other considerations were not discussed by Singh et al. such as:

1. Abdel-Basset et al. [1] used the rank function for the maximization problems of NLP, and used another rank function for the minimization problems, which means that he used the two rank functions in his proposed model.

2. Abdel-Basset et al [1], compared his proposed model with other models, using different rank functions, thus the comparison is unfair.

Section 2.3 addressed these considerations by finding a relationship between the representation of fuzzy numbers and the representation of neutrosophic numbers.

Now, we can introduce a general framework for solving the linear programming problems using neutrosophic numbers as follows:

Step 1: neutrosophic or uncertain information is generally processed by transforming into an accurate or crisp number by using the same ranking function for maximization and minimization problem for both fuzzy numbers and neutrosophic numbers to obtain a fair comparison between them using the method suggested by Singh et al. [32].

All parameters are represented by trapezoidal neutrosophic numbers, except variables are exemplified only by real values.

$$\begin{aligned} & max \ \backslash \ min\left[\textstyle\sum_{j=1}^{n} = \tilde{c}_j x_j\right] \\ & \text{s.t.} \\ & \sum_{j=1}^{n} \tilde{a}_{ij} x_j \ \leq, \geq, = \tilde{b}_j \\ & i = 1,2,\ldots\ldots, m; \ x_j \geq 0, \ j = 1,2,\ldots\ldots, n \qquad (8) \end{aligned}$$

The Equation (8) can be transformed into Exact crisp linear programming problem

$$Max\ /Min\ [\sum_{j=1}^{n} R(\tilde{c}_j x_j) - \sum_{j=1}^{n} T_{\tilde{c}_j} x_j + \sum_{j=1}^{n} I_{\tilde{c}_j} x_j + \sum_{j=1}^{n} F_{\tilde{c}_j} x_j + \min_{1\leq j\leq n}\left\{T_{\tilde{c}_j} x_j\right\} - \max_{1\leq j\leq n}\left\{I_{\tilde{c}_j} x_j\right\} - \max_{1\leq j\leq n}\left\{F_{\tilde{c}_j} x_j\right\}]$$

s.t.

$$(\sum_{j=1}^{n} R(\tilde{a}_{ij}) x_j) + 1 \ \leq, \geq, = R(\tilde{b}_j)\ , \ \ i = 1,2,\ldots\ldots, m;$$

$$x_j \geq 0\ , \ j = 1,2,\ldots\ldots, n.(2)$$

This transformation can happen at the beginning of the decision process, or in the middle or final stage.

Step 2: Let $\tilde{A} = (a_1, a_2, a_3, a_4, T_{\tilde{A}},\ I_{\tilde{A}},\ F_{\tilde{A}})$ be a trapezoidal neutrosophic number, where a_1, a_2, a_3, a_4; are lower bound, first, second median value and upper bound for trapezoidal neutrosophic number, respectively. Also $T_{\tilde{A}},\ I_{\tilde{A}},\ F_{\tilde{A}}$ are the truth, indeterminacy and falsity degree of trapezoidal neutrosophic number. Ranking function for this trapezoidal neutrosophic number is as follows:

$$R(\tilde{A}) = \frac{a_1 + a_4 + 2\ (a_2 + a_3)}{2} + \text{Confirmation degree}$$

Mathematically, this function can be written as follows:

$$R(\tilde{A}) = \frac{a_1 + a_4 + 2\ (a_2 + a_3)}{2} + (T_{\tilde{A}} - I_{\tilde{A}} - F_{\tilde{A}})$$

Step 3: Solve the crisp model using the standard method and obtain the optimal solution of problem

Table 2. Singh et al.'s modifications.

no	NLPP- (Type)	NLPP- (Form)	Exact Crisp LPP
1	The coefficients of the objective function are represented by trapezoidal neutrosophic numbers	$max\backslash\min\ z = [\sum_{j=1}^{n} \tilde{c}_j x_j]$ s.t $\sum_{j=1}^{n} a_{ij}x_j \le, \ge, = b_j,\ i = 1,2,\dots\dots,m;\ x_j \ge 0,\ j = 1,2,\dots\dots,n.$	$Max\ /Min\ z = [\sum_{j=1}^{n} R(\tilde{c}_j x_j) - \sum_{j=1}^{n} T_{\tilde{c}_j}x_j + \sum_{j=1}^{n} I_{\tilde{c}_j}x_j + \sum_{j=1}^{n} F_{\tilde{c}_j}x_j + \min_{1\le j\le n}\{T_{\tilde{c}_j}x_j\} - \max_{1\le j\le n}\{I_{\tilde{c}_j}x_j\} - \max_{1\le j\le n}\{F_{\tilde{c}_j}x_j\}]$ $s.t.\ \sum_{j=1}^{n} a_{ij}x_j \le, \ge, = b_j,\ i = 1,2,\dots\dots,m;\ x_j \ge 0,\ j = 1,2,\dots\dots,n.$
2	The coefficients of constraints variables and right hand side are represented by trapezoidal neutrosophic numbers	$\max\backslash\ \min\ z = [\sum_{j=1}^{n} c_j x_j]$ s.t. $\sum_{j=1}^{n} \tilde{a}_{ij}x_j \le, \ge, = \tilde{b}_j,\ i = 1,2,\dots\dots,m;\ x_j \ge 0,\ j = 1,2,\dots\dots,n.$	$\mathrm{Max\ /\ min\ z} = \sum_{j=1}^{n} c_j x_j$ $s.t.\ [\sum_{j=1}^{n} R(\tilde{a}_{ij}x_j) - \sum_{j=1}^{n} T_{\tilde{a}_{ij}}x_j + \sum_{j=1}^{n} I_{\tilde{a}_{ij}}x_j + \sum_{j=1}^{n} F_{\tilde{a}_{ij}}x_j + \min_{1\le j\le n}\{T_{\tilde{a}_{ij}}x_j\} - \max_{1\le j\le n}\{I_{\tilde{a}_{ij}}x_j\} - \max_{1\le j\le n}\{F_{\tilde{a}_{ij}}x_j\}] \le, \ge, = R(\tilde{b}_i)$ $x_j \ge 0,\ j = 1,2,\dots\dots,n.$
3	All parameters are represented by trapezoidal neutrosophic numbers, except variables are exemplified only by real values	$max\ \backslash\ \min z = [\sum_{j=1}^{n} \tilde{c}_j x_j]$ s.t. $\sum_{j=1}^{n} \tilde{a}_{ij}x_j \le, \ge, = \tilde{b}_j,\ i = 1,2,\dots\dots,m;\ x_j \ge 0,\ j = 1,2,\dots\dots,n.$	$Max\ /Min\ z = [\sum_{j=1}^{n} R(\tilde{c}_j x_j) - \sum_{j=1}^{n} T_{\tilde{c}_j}x_j + \sum_{j=1}^{n} I_{\tilde{c}_j}x_j + \sum_{j=1}^{n} F_{\tilde{c}_j}x_j + \min_{1\le j\le n}\{T_{\tilde{c}_j}x_j\} - \max_{1\le j\le n}\{I_{\tilde{c}_j}x_j\} - \max_{1\le j\le n}\{F_{\tilde{c}_j}x_j\}]$ s.t. $(\sum_{j=1}^{n} R(\tilde{a}_{ij})x_j) + 1 \le, \ge, = R(\tilde{b}_j),\ i = 1,2,\dots\dots,m;$ $x_j \ge 0,\ j = 1,2,\dots\dots,n.$
4	The coefficients of objective function and constraints variables are represented by real numbers and right hand side are represented by trapezoidal neutrosophic numbers	$\max\backslash\ \min z = [\sum_{j=1}^{n} c_j x_j]$ s.t. $\sum_{j=1}^{n} a_{ij}x_j \le, \ge, = \tilde{b}_j,\ i = 1,2,\dots\dots,m;\ x_j \ge 0,\ j = 1,2,\dots\dots,n.$	$\mathrm{Max\ /\ min} \sum_{j=1}^{n} c_j x_j$ s.t. $R\left[\sum_{j=1}^{n}(a_{ij}x_j)\right] \le, \ge, = R(\tilde{b}_i)$ $x_j \ge 0,\ j = 1,2,\dots\dots,n.$

Remark 5: If $R(a) = a + 1$ and the coefficients of the objective function & constraints variables are real, then the fuzzy linear programming problem is equivalent to the neutrosophic linear programming problem.

- NLPP: neutrosophic linear programming problem.

4.2 A novel neutrosophic Exterior Point Simplex Algorithm (NEPSA)

Badr *et al* [8] proposed a fuzzy exterior point simplex algorithm (FEPSA) for solving the linear programming problems with fuzzy numbers. In this section, we propose a new algorithm which solves linear programming with neutrosophic numbers (Neutrosophic exterior point simplex algorithm NEPSA).

Neutrosophic Exterior Point Simplex Algorithm (NEPSA)

Step0: (Initialization)

- Transfer fuzzy numbers into neutrosophic numbers (see section 3)
- Apply the general framework (see section 4)
- Start with a feasible basic point and construct the corresponding tableau exterior simplex.

Step1: (Test of termination)

Find the set $J_- = \{j: \tilde{a}_{0j} \underset{R}{<} \tilde{0}\}$. If $J_- = \Phi$, STOP. The problem is optimal.

Otherwise, calculate $\tilde{a}_{00} = \sum_{j\in J_-} \tilde{a}_{ij}$ and $a_{i0} = \sum_{j\in J_-} a_{ij}$ where $i = 1, 2, \ldots, m$

Step2: (Choice of entering variable)

Find the set $I_+ = \{i: a_{i0} > 0\}$. If $I_+ = \Phi$, STOP. The problem is unbounded.

Otherwise, determine the index of entering variable r from the relation :

$$\frac{b_r}{a_{r0}} = min\left\{\frac{b_j}{a_{r0}}: i \in I_+\right\}$$

Step3: (Choice of leaving variable)

Put $J_+ = \{j: \tilde{a}_{0j} > \tilde{0}\}$ and calculate

$$\theta_1 = \frac{-\tilde{a}_{0k}}{a_{rk}} = min\left\{\frac{-\tilde{a}_{0j}}{a_{rj}}: j \in J_-, a_{rj} > 0\right\}$$

$$\theta_2 = \frac{-\tilde{a}_{0l}}{a_{rl}} = min\left\{\frac{-\tilde{a}_{0j}}{a_{rj}}: j \in J_+, a_{rj} < 0\right\}$$

Find the index of the leaving variable s, if $\theta_1 \leq \theta_2$ put $s = k$ otherwise $s = l$.

Step4: (Pivoting)

Form the next tableau by the pivoting variable a_{rs} and go to Step1

5. Numerical Examples and Results Analysis

In this section, two benchmark examples (P1 and P2) are proposed to compare between the proposed algorithm NEPSA and its fuzzy version FEPSA.

Table 3. Special fuzzy linear programming from different references

Problem No.	Problem object function and constrained	Reference
P1	$Max\ \tilde{z} = (2,4,2,6)x_1 + (2,6,1,3)x_2 + (1,3,1,3)x_3$ s.t $x_1 + x_2 + 2x_3 \leq 2$ $2x_1 + 3x_2 + 4x_3 \leq 3$ $6x_1 + 6x_2 + 2x_3 \leq 8$ $x_1, x_2, x_3 \geq 0$	[8]
P2	$Max\ \tilde{z} = (13,15,2,2)x_1 + (12,14,3,3)x_2 + (15,17,2,2)x_3$ s.t. $12x_1 + 13x_2 + 12x_3 \leq (475,505,6,6)$ $14x_1 + 13x_3 \leq (460,480,8,8)$ $12x_1 + 15x_2 \leq (465,495,5,5)$ $x_1, x_2, x_3 \geq 0$	[21]

5.1 Example 1 (P1) [8] :

Consider the following linear programming problem

$$Max\ \tilde{z} = (2,4,2,6)x_1 + (2,6,1,3)x_2 + (1,3,1,3)x_3$$
s.t
$$x_1 + x_2 + 2x_3 \leq 2$$
$$2x_1 + 3x_2 + 4x_3 \leq 3$$
$$6x_1 + 6x_2 + 2x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

First: We will convert the fuzzy numbers into neutrosophic numbers

Then, using the following rank function:

$$R(\breve{a}) = \frac{1}{2}\sum_{i=1}^{4} \tilde{a}_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$$
$$R(a) = 2a + 1$$

$$Max\ z = R[(0,2,4,10)]x_1 + R[(1,2,6,9)]x_2 + R[(0,1,3,6)]x_3$$
s.t.
$$x_1 + x_2 + 2x_3 \leq 2$$
$$2x_1 + 3x_2 + 4x_3 \leq 3$$
$$6x_1 + 6x_2 + 2x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

Putting the last formula into the standard form, we have:

$$Max\ z = 9x_1 + 10x_2 + 6x_3$$
s.t.
$$x_1 + x_2 + 2x_3 + x_4 = 2$$
$$2x_1 + 3x_2 + 4x_3 + x_5 = 3$$
$$6x_1 + 6x_2 + 2x_3 + x_6 = 8$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Step (0): we construct the initial tableau of exterior simplex:

		x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z		-9	-10	-6	0	0	0	2
x_4	4	1	1	2	1	0	0	2
x_5	9	2	3	4	0	1	0	3
x_6	14	6	6	2	0	0	1	8

Step (1): $J_- = \{j: a_{0j} < 0\} = \{1, 2, 3\} \neq \emptyset$ the algorithm does not stop.

Step (2): $I_+ = \{i: a_{i0} > 0\} = \{1, 2, 3\} \neq \Phi$ the problem is not unbounded

$$\frac{br}{a_{r0}} = min\left\{\frac{b_i}{a_{i0}}, i \in I_+\right\} = min\left\{\frac{b_1}{a_{10}}, \frac{b_2}{a_{20}}, \frac{b_3}{a_{30}}\right\} = min\left\{\frac{2}{4}, \frac{3}{9}, \frac{8}{14}\right\} = \frac{3}{9} \Rightarrow r = 2$$

Then, the leaving variable is x_5

Step (3): $J_+ = \{j: a_{0j} \underset{R}{>} 0\} = \Phi$

$$\theta_1 = \frac{-a_{0k}}{a_{rk}} = min\left\{\frac{-a_{0j}}{a_{rj}} = j \in J-, a_{rj} > 0\right\} = min\left\{\frac{-a_{0l}}{a_{21}}, \frac{-a_{02}}{a_{22}}, \frac{-a_{03}}{a_{23}}\right\} = min\left\{\frac{9}{2}, \frac{10}{3}, \frac{6}{4}\right\} = \frac{6}{4} \Rightarrow k = 3$$

Then, the entering variable is x_3

$\theta_2 \underset{R}{=} \frac{-a_{0l}}{a_{rl}} \underset{R}{=} min\left\{\frac{-a_{0j}}{a_{rj}} : j \in J_+, a_{rj} < 0\right\} \Rightarrow \theta_2 = min\{\Phi\} = \infty \Rightarrow \theta_1 < \theta_2 \Rightarrow s = k = 3$, the pivot element is a_{23}

Step (4): the next tableau by pivot element:

		x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z		-6	$\frac{-11}{2}$	0	0	$\frac{3}{2}$	0	$\frac{13}{2}$
x_4	$\frac{-1}{2}$	0	$\frac{-1}{2}$	0	1	$\frac{-1}{2}$	0	$\frac{1}{2}$
x_3	$\frac{5}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	0	$\frac{1}{4}$	0	$\frac{3}{4}$
x_6	$\frac{19}{2}$	5	$\frac{9}{2}$	0	0	$\frac{-1}{2}$	1	$\frac{13}{2}$

Step (1): $J_- = \{j: a_{0j} < 0\} = \{1, 2\} \neq \emptyset$ the algorithm does not stop.

Step (2): $I_+ = \{i: a_{i0} > 0\} = \{2, 3\} \neq \Phi$ the problem is not unbounded

$$\frac{br}{a_{r0}} = min\left\{\frac{b_i}{a_{i0}}, i \in I_+\right\} = min\left\{\frac{b_2}{a_{20}}, \frac{b_3}{a_{30}}\right\} = min\left\{\frac{3}{5}, \frac{13}{19}\right\} = \frac{3}{5} \Rightarrow r = 2$$

Then, the leaving variable is x_3

Step (3): $J_+ = \{j: a_{0j} \underset{R}{>} 0\} = \{5\}$

$$\theta_1 = \frac{-a_{0k}}{a_{rk}} = min\left\{\frac{-a_{0j}}{a_{rj}} = j \in J-, a_{rj} > 0\right\} = min\left\{\frac{-a_{01}}{a_{21}}, \frac{-a_{02}}{a_{22}}\right\} = min\left\{12, \frac{22}{3}\right\} = \frac{22}{3} \Rightarrow k = 2$$

Then, the entering variable is x_2

$\theta_2 \underset{R}{=} \frac{-a_{0l}}{a_{rl}} \underset{R}{=} min\left\{\frac{-a_{0j}}{a_{rj}} : j \in J_+, a_{rj} < 0\right\} \Rightarrow \theta_2 = min\{\Phi\} = \infty \Rightarrow \theta_1 < \theta_2 \Rightarrow s = k = 2$, the pivot element is a_{22}

Step (4): the next tableau by pivot element:

		x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z		$\frac{-7}{3}$	0	$\frac{22}{3}$	0	$\frac{10}{3}$	0	12
x_4	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	1	$\frac{-1}{3}$	0	1
x_2	$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	0	$\frac{1}{3}$	0	1
x_6	2	2	0	-6	0	-2	1	2

Step (1): $J_- = \{j: a_{0j} < 0\} = \{1\} \neq \emptyset$ the algorithm does not stop.

Step (2): $I_+ = \{i: a_{i0} > 0\} = \{1,2,3\} \neq \Phi$ the problem is not unbounded

$$\frac{br}{a_{r0}} = min\left\{\frac{b_i}{a_{i0}}, i \in I_+\right\} = min\left\{\frac{b_1}{a_{10}}, \frac{b_2}{a_{20}}, \frac{b_3}{a_{30}}\right\} = min\left\{3, \frac{3}{2}, 1\right\} = 1 \Rightarrow r = 3$$

Then, the leaving variable is x_6

Step (3): $J_+ = \{j: a_{0j} \underset{R}{>} 0\} = \{3,5\}$

$\theta_1 = \frac{-a_{0k}}{a_{rk}} = min\left\{\frac{-a_{0j}}{a_{rj}} = j \in J-, a_{rj} > 0\right\} = min\left\{\frac{-a_{01}}{a_{31}}\right\} = min\left\{\frac{7}{6}\right\} \Rightarrow k = 1$

Then, the entering variable is x_1

$\theta_2 \underset{R}{=} \frac{-a_{0l}}{a_{rl}} \underset{R}{=} min\left\{\frac{-a_{0j}}{a_{rj}} : j \in J_+, a_{rj} < 0\right\} \Rightarrow \theta_2 = min\{\Phi\} = \infty \Rightarrow \theta_1 < \theta_2 \Rightarrow s = k = 1$, the pivot element is a_{31}

Step (4): the next tableau by pivot element:

	x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z	0	0	$\frac{1}{3}$	0	1	$\frac{7}{6}$	$\frac{43}{3}$
x_4	0	0	$\frac{5}{3}$	1	0	$\frac{-1}{6}$	$\frac{2}{3}$
x_2	0	1	$\frac{10}{3}$	0	1	$\frac{-1}{3}$	$\frac{1}{3}$
x_1	1	0	-3	0	-1	$\frac{1}{2}$	1

Step (1): $J_- : \left\{j: a_{0j} \underset{R}{<} 0\right\} = \Phi$, the algorithm stops .

The solution is: $z = \frac{43}{3}, x_1 = 1, x_2 = \frac{1}{3}, x_3 = 0$

Table 4. A comparison between fuzzy EPSA & Neutrosophic EPSA

	FEPSA[7]	NEPSA
$Iteration\ no.$	3	3
Z	11	14.33
x_1	1	1
x_2	$\frac{1}{3}$	$\frac{1}{3}$
x_3	0	0

In Table 4, we make a comparison between FEPSA and NEPSA. It is clear that the neutrosophic approach NEPSA is more accurate than the fuzzy approach FEPSA according to the value of objective function. The value of objective function of NEPSA is 14.33 while FEPSA has 11 where the type of this problem is maximization. From Table 4, we deduce that NEPSA is more accurate than FEPSA.

5.1 Case study (P_2) [21]:

A company produces three products P1, P2 and P3. These products are processed on three different machines M1, M2 and M3. The time required to manufacture one unit of each product and the daily capacity of the machines are given below:

	Time per unit(minutes)			
Machines	p_1	p_2	p_3	**Machine Capacity (min/day)**
M1	12	13	12	490
M2	14	-	13	470
M3	12	15	-	480

Note that the time availability can vary from day to day due to break down of machines, overtime work etc. Finally the profit for each product can also vary due to variations in price. At the same time the company wants to keep the profit somewhat close to 14 for P1, 13 for P2 and 16 for P3. The company wants to determine the range of each product to be produced per day to maximize its profit. It is assumed that all the amounts produced are consumed in the market.

Since the profit from each product and the time availability on each machine are uncertain, the number of units to be produced on each product will also be uncertain. So we will model the problem as a fuzzy linear programming problem. We use symmetric trapezoidal fuzzy numbers for each uncertain value. Profit for P1 which is close to 14 is modelled as [13, 15, 2, 2]. Similarly the other parameters are also modelled as symmetric trapezoidal fuzzy numbers taking into account the nature of the problem and other requirements. So we formulate the given fuzzy linear programming problem as:

$$Max\ \tilde{z} = (13,15,2,2)x_1 + (12,14,3,3)x_2 + (15,17,2,2)x_3$$
$$s.t.$$
$$12x_1 + 13x_2 + 12x_3 \leq (475,505,6,6)$$
$$14x_1 + \qquad 13x_3 \leq (460,480,8,8)$$
$$12x_1 + 15x_2 \leq (465,495,5,5)$$
$$x_1, x_2, x_3 \geq 0$$

5.1.1 Solving case study using fuzzy exterior point simplex method

Putting the formula into the standard form, we have:

$$Max\ \tilde{z} = (13,15,2,2)x_1 + (12,14,3,3)x_2 + (15,17,2,2)x_3$$
s.t.
$$12x_1 + 13x_2 + 12x_3 + x_4 = (475,505,6,6)$$
$$14x_1 + 13x_3 + x_5 = (460,480,8,8)$$
$$12x_1 + 15x_2 + x_6 = (465,495,5,5)$$
$$x_1, x_2, x_3 \geq 0$$

Step (0): we construct the initial tableau of fuzzy exterior simplex:

		x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z		$-(13,15,2,2)$	$-(12,14,3,3)$	$-(15,17,2,2)$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
x_4	37	12	13	12	1	0	0	(475,505,6,6)
x_5	27	14	0	13	0	1	0	(460,480,8,8)
x_6	27	12	15	0	0	0	1	(465,495,5,5)

Step (1): $J- = \{j: a_{0j} \underset{R}{<} 0\} = \{1, 2,3\} \neq \emptyset$ the algorithm does not stop.

Step (2): $I_+ = \{i: a_{i0} > 0\} = \{1, 2,3\} \neq \Phi$ the problem is not unbounded

$$\frac{br}{a_{r0}} = min\left\{\frac{b_i}{a_{i0}}, i \in I_+\right\} = min\left\{\frac{b_1}{a_{10}}, \frac{b_2}{a_{20}}, \frac{b_3}{a_{30}}\right\} = min\left\{\frac{R(475,505,6,6)}{37}, \frac{R(460,480,8,8)}{27}, \frac{R(465,495,5,5)}{27}\right\}$$
$$= \frac{490}{37} \Rightarrow r = 1$$

Then, the leaving variable is x_4

Step (3): $J_+ = \{j: a_{0j} \underset{R}{>} 0\} = \Phi$

$\theta_1 = \frac{-a_{0k}}{a_{rk}} = min\left\{\frac{-a_{0j}}{a_{rj}} = j \in J-, a_{rj} > 0\right\} = min\left\{\frac{-a_{0l}}{a_{11}}, \frac{-a_{02}}{a_{12}}, \frac{-a_{03}}{a_{13}}\right\} = min\left\{\frac{R(13,15,2,2)}{12}, \frac{R(12,14,3,3)}{13}, \frac{R(15,17,2,2)}{12}\right\} = \frac{13}{13} = 1 \Rightarrow k = 2$

Then, the entering variable is x_2

$\theta_2 = \frac{-a_{0l}}{a_{rl}} = min\left\{\frac{-a_{0j}}{a_{rj}} : j \in J_+, a_{rj} < 0\right\} \Rightarrow \theta_2 = min\{\Phi\} = \infty \Rightarrow \theta_1 < \theta_2 \Rightarrow s = k = 2$, the pivot element is a_{12}

Step (4): the next tableau by pivot element:

		x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z		$(\frac{-25}{13}, \frac{-27}{13}, \frac{10}{13}, \frac{10}{13})$	0	$(\frac{-51}{13}, \frac{-53}{13}, \frac{10}{13}, \frac{10}{13})$	$(\frac{12}{13}, \frac{14}{13}, \frac{3}{13}, \frac{3}{13})$	$\tilde{0}$	$\tilde{0}$	(475,505,6,6)
x_2	$\frac{24}{13}$	$\frac{12}{13}$	1	$\frac{12}{13}$	$\frac{1}{13}$	0	0	$(\frac{475}{13}, \frac{505}{13}, \frac{6}{13}, \frac{6}{13})$
x_5	27	14	0	13	0	1	0	(460,480,8,8)
x_6	$\frac{-204}{13}$	$\frac{-24}{13}$	0	$\frac{-180}{13}$	$\frac{15}{13}$	0	1	$(\frac{-1080}{13}, \frac{-1140}{13}, \frac{-25}{13}, \frac{-25}{13})$

Step (1): $J_- = \{j: a_{0j} \underset{R}{<} 0\} = \{1,3\} \neq \emptyset$ the algorithm does not stop.

Step (2): $I_+ = \{i: a_{i0} > 0\} = \{1,2\} \neq \Phi$ the problem is not unbounded

$$\frac{br}{a_{r0}} = min\left\{\frac{b_i}{a_{i0}}, i \in I_+\right\} = min\left\{\frac{b_1}{a_{10}}, \frac{b_2}{a_{20}}\right\} = min\left\{\frac{R(\frac{475}{13}, \frac{505}{13}, \frac{6}{13}, \frac{6}{13})}{\frac{24}{13}}, \frac{R(460,480,8,8)}{27}\right\}$$
$$= min\{\frac{245}{12}, \frac{470}{27}\} = \frac{470}{27} \Rightarrow r = 2$$

Then, the leaving variable is x_5

Step (3): $J_+ = \{j: a_{0j} \underset{R}{>} 0\} = \{4,5\}$

$\theta_1 = \frac{-a_{0k}}{a_{rk}} = min\left\{\frac{-a_{0j}}{a_{rj}} = j \in J-, a_{rj} > 0\right\} = min\left\{\frac{-a_{03}}{a_{23}}\right\} = min\{\frac{R(\frac{389}{182}, \frac{391}{182}, \frac{-5}{91}, \frac{-5}{91})}{\frac{13}{14}}\} = \frac{30}{13} \Rightarrow k = 3$

Then, the entering variable is x_3

$\theta_2 = \frac{-a_{0l}}{a_{rl}} = min\left\{\frac{-a_{0j}}{a_{rj}} : j \in J_+, a_{rj} < 0\right\} \Rightarrow \theta_2 = min\{\Phi\} = \infty \Rightarrow \theta_1 < \theta_2 \Rightarrow s = k = 3$, the pivot element is a_{23}

Step (4): the next tableau by pivot element:

	x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z	$(\frac{389}{169},\frac{391}{169},\frac{-10}{169},\frac{-10}{169})$	0	0	$(\frac{12}{13},\frac{14}{13},\frac{3}{13},\frac{3}{13})$	$(\frac{51}{169},\frac{53}{169},\frac{-10}{169},\frac{-10}{169})$	$\tilde{0}$	$(\frac{8015}{13},\frac{8485}{13},\frac{110}{13},\frac{110}{13})$
x_2	$\frac{-12}{169}$	1	0	$\frac{1}{14}$	$\frac{-12}{169}$	0	$(\frac{655}{169},\frac{805}{169},\frac{-18}{169},\frac{-18}{169})$
x_3	$\frac{14}{13}$	0	1	0	$\frac{1}{13}$	0	$(\frac{460}{13},\frac{480}{13},\frac{8}{13},\frac{8}{13})$
x_6	$\frac{2208}{169}$	0	0	$\frac{-15}{13}$	$\frac{180}{169}$	1	$(\frac{481320}{1183},\frac{501060}{1183},\frac{1115}{169},\frac{1115}{169})$

The solution is: $z = 634.6$, $x_1 = 0, x_2 = \frac{731}{169}, x_3 = \frac{471}{13}$

5.1.2 Solving case study using neutrosophic exterior point simplex method

First: We will convert the fuzzy numbers into neutrosophic numbers

Then, using the following rank function:

$$R(\breve{a}) = \frac{a_2 + a_3}{2} + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$$

$$R(a) = a + 1$$

$Max\ \tilde{z} = R[(13,15,2,2)]x_1 + R[(12,14,3,3)]x_2 + R[(15,17,2,2)]x_3$

s.t.

$$12x_1 + 13x_2 + 12x_3 \le R[(475,505,6,6)]$$
$$14x_1 + \quad 13x_3 \le R[(460,480,8,8)]$$
$$12x_1 + 15x_2 \le R[(465,495,5,5)]$$
$$x_1, x_2, x_3 \ge 0$$

Putting the last formula into the standard form, we have:

$Max\ z = 15x_1 + 14x_2 + 17x_3 - 2$

s.t.

$12x_1 + 13x_2 + 12x_3 + x_4 = 491$

$14x_1 + 13x_3 + x_5 = 471$

$12x_1 + 15x_2 + x_6 = 481$

$x_1, x_2, x_3, x_4 \ge 0$

Step (0): we construct the initial tableau of exterior simplex:

		x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z		−15	−14	−17	0	0	0	2
x_4	37	12	13	12	1	0	0	491
x_5	27	14	0	13	0	1	0	471
x_6	27	12	15	0	0	0	1	481

Step (1): $J- = \{j: a_{0j} < 0\} = \{1, 2,3\} \neq \emptyset$ the algorithm does not stop.

Step (2): $I_+ = \{i: a_{i0} > 0\} = \{1, 2,3\} \neq \Phi$ the problem is not unbounded

$$\frac{br}{a_{r0}} = min\left\{\frac{b_i}{a_{i0}}, i \in I_+\right\} = min\left\{\frac{b_1}{a_{10}}, \frac{b_2}{a_{20}}, \frac{b_3}{a_{30}}\right\} = min\left\{\frac{491}{37}, \frac{471}{27}, \frac{481}{27}\right\} = \frac{491}{37} \Rightarrow r = 1$$

Then, the leaving variable is x_4

Step (3): $J_+ = \{j: a_{0j} \underset{R}{>} 0\} = \Phi$

$$\theta_1 = \frac{-a_{0k}}{a_{rk}} = min\left\{\frac{-a_{0j}}{a_{rj}} = j \in J-, a_{rj} > 0\right\} = min\left\{\frac{-a_{0l}}{a_{11}}, \frac{-a_{02}}{a_{12}}, \frac{-a_{03}}{a_{13}}\right\} = min\left\{\frac{15}{12}, \frac{14}{13}, \frac{17}{12}\right\} = \frac{14}{13} \Rightarrow k = 2$$

Then, the entering variable is x_2

$\theta_2 = \frac{-a_{0l}}{a_{rl}} = mi\left\{\frac{-a_{0j}}{a_{rj}} : j \in J_+, a_{rj} < 0\right\} \Rightarrow \theta_2 = min\{\Phi\} = \infty \Rightarrow \theta_1 < \theta_2 \Rightarrow s = k = 2$, the pivot element is a_{12}

Step (4): the next tableau by pivot element:

		x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z		$\frac{-27}{13}$	0	$\frac{-53}{13}$	$\frac{14}{13}$	0	0	$\frac{6900}{13}$
x_2	$\frac{24}{13}$	$\frac{12}{13}$	1	$\frac{12}{13}$	$\frac{1}{13}$	0	0	$\frac{491}{13}$
x_5	27	14	0	13	0	1	0	471
x_6	$\frac{-204}{13}$	$\frac{-24}{13}$	0	$\frac{-180}{13}$	$\frac{-15}{13}$	0	1	$\frac{-1112}{13}$

Step (1): $J_- = \{j: a_{0j} < 0\} = \{1,3\} \neq \emptyset$ the algorithm does not stop.

Step (2): $I_+ = \{i: a_{i0} > 0\} = \{1,2\} \neq \Phi$ the problem is not unbounded

$$\frac{br}{a_{r0}} = min\left\{\frac{b_i}{a_{i0}}, i \in I_+\right\} = min\left\{\frac{b_1}{a_{10}}, \frac{b_2}{a_{20}}\right\} = min\left\{\frac{491}{24}, \frac{471}{27}\right\} = \frac{471}{27} \Rightarrow r = 2$$

Then, the leaving variable is x_5

Step (3): $J_+ = \{j: a_{0j} \underset{R}{>} 0\} = \{4\}$

$\theta_1 = \frac{-a_{0k}}{a_{rk}} = min\left\{\frac{-a_{0j}}{a_{rj}} = j \in J-, a_{rj} > 0\right\} = min\left\{\frac{-a_{01}}{a_{21}}, \frac{-a_{03}}{a_{23}}\right\} = min\left\{\frac{27}{182}, \frac{53}{169}\right\} = \frac{27}{182} \Rightarrow k = 1$

Then, the entering variable is x_2

$\theta_2 \underset{R}{=} \frac{-a_{0l}}{a_{rl}} \underset{R}{=} min\left\{\frac{-a_{0j}}{a_{rj}} : j \in J_+, a_{rj} < 0\right\} \Rightarrow \theta_2 = min\{\emptyset\} = \infty \Rightarrow \theta_1 < \theta_2 \Rightarrow s = k = 1$, the pivot element is a_{21}

Step (4): the next tableau by pivot element:

		x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z		0	0	$\frac{-391}{182}$	$\frac{14}{13}$	$\frac{27}{182}$	0	$\frac{8409}{14}$
x_2	$\frac{6}{91}$	0	1	$\frac{6}{91}$	$\frac{1}{13}$	$\frac{-6}{91}$	0	$\frac{47}{7}$
x_1	$\frac{13}{14}$	1	0	$\frac{13}{14}$	0	$\frac{1}{14}$	0	$\frac{471}{14}$
x_6	$\frac{-1104}{91}$	0	0	$\frac{-1104}{91}$	$\frac{-15}{13}$	$\frac{12}{91}$	1	$\frac{-164}{7}$

Step (1): $J_- = \{j: a_{0j} < 0\} = \{3\} \neq \emptyset$ the algorithm does not stop.

Step (2): $I_+ = \{i: a_{i0} > 0\} = \{1,2\} \neq \Phi$ the problem is not unbounded

$$\frac{br}{a_{r0}} = min\left\{\frac{b_i}{a_{i0}}, i \in I_+\right\} = min\left\{\frac{b_1}{a_{10}}, \frac{b_2}{a_{20}}\right\} = min\left\{\frac{611}{6}, \frac{471}{13}\right\} = \frac{471}{13} \Rightarrow r = 2$$

Then, the leaving variable is x_1

Step (3): $J_+ = \{j: a_{0j} \underset{R}{>} 0\} = \{4,5\}$

$\theta_1 = \frac{-a_{0k}}{a_{rk}} = min\left\{\frac{-a_{0j}}{a_{rj}} = j \in J-, a_{rj} > 0\right\} = min\left\{\frac{-a_{03}}{a_{23}}\right\} = min\left\{\frac{391}{169}\right\} = 2.3 \Rightarrow k = 3$

Then, the entering variable is x_3

$\theta_2 \underset{R}{=} \frac{-a_{0l}}{a_{rl}} \underset{R}{=} min\left\{\frac{-a_{0j}}{a_{rj}} : j \in J_+, a_{rj} < 0\right\} \Rightarrow \theta_2 = min\{\emptyset\} = \infty \Rightarrow \theta_1 < \theta_2 \Rightarrow s = k = 3$, the pivot element is a_{23}

Step (4): the next tableau by pivot element:

	x_1	x_2	x_3	x_4	x_5	x_6	$R.H.S$
z	$\frac{391}{169}$	0	0	$\frac{14}{13}$	$\frac{53}{169}$	0	$\frac{114663}{169}$
x_2	$\frac{-12}{169}$	1	0	$\frac{1}{13}$	$\frac{-12}{169}$	0	$\frac{731}{169}$
x_3	$\frac{14}{13}$	0	1	0	$\frac{1}{13}$	0	$\frac{471}{13}$
x_6	$\frac{2208}{169}$	0	0	$\frac{-15}{13}$	$\frac{180}{169}$	1	$\frac{70324}{169}$

Step (1): $J_- : \left\{ j : a_{0j} \underset{R}{<} 0 \right\} = \Phi$, the algorithm stops .

The solution is: $z = 678.4 , x_1 = 0, x_2 = \frac{731}{169}, x_3 = \frac{471}{13}$

Table 5. A comparison between Fuzzy EPSA & Neutrosophic EPSA

	FEPSA	**NEPSA**
$Iter.no.$	3	3
Z	634.6	678.4
$\mathbf{x_1}$	0	0
$\mathbf{x_2}$	$\frac{730}{169}$	$\frac{731}{169}$
$\mathbf{x_3}$	$\frac{470}{13}$	$\frac{471}{13}$

In Table 5, we make a comparison between FEPSA and NEPSA. It is clear that the neutrosophic approach NEPSA is more accurate than the fuzzy approach FEPSA according to the value of objective function. The value of objective function of NEPSA is 678.4 while FEPSA has 634.6 where the type of this problem is maximization. From Table 5, we deduce that NEPSA is more accurate than FEPSA.

6. Conclusion

Three contributions were proposed. First contribution was proposing a good evaluation between the fuzzy and neutrosophic approaches using a novel fuzzy-neutrosophic transfer. Second contribution was introducing a general framework for solving the neutrosphic linear programming problems using the advantages of the method of Abdel-Basset et al. and the advantages of Singh et al.'s method. Third contribution was proposing a new neutrosophic exterior point simplex algorithm NEPSA and its fuzzy version FEPSA. NEPSA has two paths to get optimal solutions. One path consists of basic not feasible solutions but the other path is feasible. Finally, the numerical examples and results analysis showed that NEPSA more than accurate FEPSA.

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On neutrosophic uninorms

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Abstract. Uninorm generalizes the notion of t-norm and t-conorm in fuzzy logic theory. They are three increasing, commutative and associate operators having one neutral element. However, such specific value identifies the kind of operator it is; t-norms have the 1 as neutral element, t-conorms have the 0 and uninorms have every number lying between 0 and 1. Uninorms have been applied as aggregators in many fields of Artificial Intelligence and Decision Making. This theory has also been extended to the framework of interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and L-fuzzy sets. This paper aims to explore neutrosophic uninorms. We demonstrate that it is possible to define uninorms operators from neutrosophic logic. Additionally, we define neutrosophic implicators induced by neutrosophic uninorms. The combination of both, Neutrosophy and uninorms, enriches the applicability of uninorms operators due to the possibility of incorporating indeterminancy as part of the Neutrosophy contribution.

Keywords: neutrosophic uninorm, uninorm, neutrosophic logic, neutrosophic implicator.

1 Introduction

Uninorms generalize the concepts of t-norm and t-conorm in fuzzy set theory, see [17]. Uninorm operators fulfill commutativity, associativity, increasing monotonicity and the existence of a neutral element *e*, in the same way that t-norm and t-conorm do, see [21]. When *e* is 1, the uninorm is a t-norm, when *e* is 0, it is a t-conorm. The generalization consists in widening to [0, 1] the range of values where the neutral element can lie.

Uninorms are not only used to extend theoretically the other aforementioned fuzzy operators, furthermore we can find in literature many fields where they are applied as aggregators, for example, in expert systems, image processing, neural networks, classifiers, among others, see [4, 10, 13, 16, 19, 22, 27]. Moreover, there exists a fuzzy implicator theory based on uninorms, [7].

G. Deschrijver and E. Kerre in [15], extend fuzzy uninorms concepts to interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and L-fuzzy sets, see [5-6, 14, 18]. They proved in [14], that these four kind of fuzzy sets are isomorphic each another, therefore, it is sufficient to prove uninorm properties in the framework of the L^*-fuzzy set theory.

On the other hand, "Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra", [23-24, 26]. The novelty of this theory is that it includes for the first time the notion of indeterminacy in fuzzy set theory, that is to say, this approach admits the membership and non membership of elements or objects to a set, akin to intuitionistic fuzzy set theory does, as well as a third function which represents indeterminacy. This theory acknowledges that ignorance, contradiction, paradox and other knowledge representation conditions, which are often considered undesirable from the classic logic viewpoint, also should be taken into account.

Neutrosophy has been applied in wide-ranging kinds of areas, e.g., image processing, decision making, clustering, among others. This is due to the nature of this theory, which allows representing and calculating with indeterminacies.

This paper is devoted to introducing neutrosophic uninorms or N-uninorms, for generalizing uninorm operators to the neutrosopic framework. It is worthily to remark that N-uninorms are used to denote neutrosophic uninorms, not n-uninorms, see [2]. To our knowledge, this seems to be the first approach to neutrosophic uninorms. In neutrosophic logic, neutrosophic norms generalize t-norms and neutrosophic conorms generalize t-conorms, hence, N-uninorms extend fuzzy uninorms, uninorms on L^*-fuzzy sets, n-norms and n-conorms.

N-uninorms could replace fuzzy uninorms in the mathematical models where usually the latter one are employed, because this new approach keeps the advantages of uninorms as an esteemed aggregator, which is here improved with the appropriateness of neutrosophy to deal with human reasoning, knowledge representation, vagueness and uncertainty, when indeterminacy is present.

The present paper is organized as follows; the preliminary definitions and results necessaries to develop our work will be given in Section 2. Section 3 is dedicated to exposing the N-uninorm theory, including N-uninorm implicators. Finally, Section 4 draws the conclusions.

2 Preliminaries

This section is devoted to exposing the preliminary definitions and results necessaries to develop the proposed theory of N-uninorms. The first subsection is dedicated to summarizing the basic definitions and results on uninorms. In the second one we recall the definition and aspects concerning neutrosophic logic theory.

2.1 Basic notions of uninorm theory

Definition 2.1. A *uninorm* is a commutative, associative and increasing mapping $U: [0,1]^2 \to [0,1]$, where there exists $e \in [0,1]$, called neutral element, such that $\forall x \in [0,1], U(e,x) = x$, [17].

If *e* = 1, U is a t-norm and if *e* = 0, U is a t-conorm.

Deschrijver and Kerre in [15] extend this definition to the framework of interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and L-fuzzy sets, which are pairwise isomorphics, therefore they restrict their theory to the set $L^* = \{(x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\}$.

Let us recall two well-known algebraic definitions that we explicitly write for the sake of being self-contained. They are namely, *Partially Ordered Set* or *poset* and *Lattice*, [1, 9, 20].

Definition 2.2. A *Partially Ordered Set* or *poset* is a pair (P, ≤), where P is a set and ≤ is a binary relation over P, which satisfies for every x, y, z ∈ P, the three following conditions:

1. x≤x (Reflexive).
2. If x≤y and y≤x, then x = y (Antisymmetry).
3. If x≤y and y≤z, then x≤z (Transitivity).

An *upper bound* of X, X⊆P, is an element *a*∈P, such that ∀x∈X it holds x≤*a*. Equivalently, a *lower bound* is an element *b*∈P, such that ∀x∈X, *b*≤x. The *supremum* of X is the least upper bound and the *infimum* is the greater lower bound.

Definition 2.3. A *lattice* $(L, \leq_L)$ is a poset, where every pair of elements x and y in L have an infimum or 'meet', denoted by x∧y and a supremum or 'join' denoted by x∨y.

L is a *complete lattice* if every of its subsets has an infimum and a supremum in L.

The lattice $(L^*, \leq_{L^*})$ is defined by the following poset:

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, $\forall (x_1, x_2), (y_1, y_2) \in L^*$. The units of $L^*$are $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. See that $x = (x_1, x_2)$ and $y = (y_1, y_2)$ can be incomparable with regard to $\leq_{L^*}$, where either $x_1 < y_1$ and $x_2 < y_2$, or $x_1 > y_1$ and $x_2 > y_2$. It is denoted by $x \parallel_{L^*} y$.

Evidently, $(x_1, x_2) \geq_{L^*} (y_1, y_2)$ if and only if $(y_1, y_2) \leq_{L^*} (x_1, x_2)$. If $(x_1, x_2) \leq_{L^*} (y_1, y_2)$ and $(x_1, x_2) \geq_{L^*} (y_1, y_2)$ then $(x_1, x_2) =_{L^*} (y_1, y_2)$.

Formally, the uninorm on L^* is defined as follows:

Definition 2.4. A *uninorm on* L^* is a commutative, associative and increasing mapping $\mathbf{U}: {L^*}^2 \to L^*$, where there exists $e \in L^*$, called neutral element, such that $\forall x \in L^*, \mathbf{U}(e,x) = x$, [15].

Here, if $e = 1_{L^*}$, **U** defines a t-norm on $L^*$and if $e = 0_{L^*}$, it is a t-conorm on L^*. Nevertheless, the most interesting cases of uninorms are those where *e* satisfies $0_{L^*} <_{L^*} e <_{L^*} 1_{L^*}$.

In [15] we can find properties and their demonstrations concerning uninorms on L^* that generalize the properties of fuzzy uninorms, including those of the uninorm-based R-implicators and S-implicators. Further, we shall guide the exposition of N-uninorms theory through the theory developed in that paper. Our goal is to prove that N-uninorms extend uninorms on L^*.

2.2 Basic notions of neutrosophic logic

Definition 2.5. Given X, a universe of discourse containing elements or objects. A is a *neutrosophic set* ([25-26]) if it has the form: $A = \{(x: T_A(x), I_A(x), F_A(x)),\ x \in X\}$, where $T_A(x), I_A(x), F_A(x) \subseteq]^-0, 1^+[$, i.e., they are three functions over either the standard or nonstandard subsets of $]^-0, 1^+[$. $T_A(x)$ represents the degree of membership of x to A, $I_A(x)$ represents its degree of indeterminacy and $F_A(x)$ its degree of non-membership. They do not satisfy any restriction, i.e., $\forall x \in X,\ ^-0 \leq \inf T_A(x) + \inf I_A(x) + \inf F_A(x) \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Another particular definition is that of *Single-valued Neutrosophic set*, which is formally defined as follows:

Definition 2.6. Given X, a universe of discourse which contains elements or objects. A is a *single-valued neutrosophic set* (*SVNS*) [25] if it has the form: $A = \{(x: T_A(x), I_A(x), F_A(x)), x \in X\}$, where $T_A(x), I_A(x), F_A(x) \in [0, 1]$. $T_A(x)$ represents the degree of membership of x to A, $I_A(x)$ represents its degree of indeterminacy and $F_A(x)$ its degree of non-membership. $\forall x \in X, 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

See that SVNS is derived from the definition of neutrosophic sets. In the present paper we prefer to use the former one.

In neutrosophic set theory a lattice can be defined as follows:

Given the universe of discourse X and $x(T_x, I_x, F_x)$, $y(T_y, I_y, F_y)$ two SVNS, we say that $x \leq_N y$ if and only if $T_x \leq T_y$, $I_x \geq I_y$ and $F_x \geq F_y$, $(X, \leq_N)$ is a poset. Whereas, $(L, \wedge, \vee)$ is a lattice, because it is a triple direct product of lattices, see [9]. $x \wedge y = (\min\{T_x, T_y\}, \max\{I_x, I_y\}, \max\{F_x, F_y\})$ and $x \vee y = (\max\{T_x, T_y\}, \min\{I_x, I_y\}, \min\{F_x, F_y\})$. Moreover, it is easy to prove that it is complete.

Let us remark that this definition is valid for interval-valued neutrosophic sets, when we substitute their operators by interval-valued operators.

See also that there exist two special elements, viz., $0_N = (0, 1, 1)$ and $1_N = (1, 0, 0)$, which are the infimum and the supremum respectively, of every SVNS with regard to $\leq_N$.

Given two neutrosophic sets, A and B, three basic operations over them are the following [25]:

1. $A \cap B = A \wedge B$ (Conjunction).

2. $A \cup B = A \vee B$ (Disjunction).

3. $\bar{A} = (F_A, 1 - I_A, T_A)$ (Complement).

Definition 2.7. A *neutrosophic norm* or *n-norm* N_n [25], is a mapping $N_n: (]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[)^2 \rightarrow]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, such that $N_n\left(x(T_x, I_x, F_x), y(T_y, I_y, F_y)\right) = (N_nT(x, y), N_nI(x, y), N_nF(x, y))$, where N_nT means the degree of membership, N_nI the degree of indeterminacy and N_nF the degree of non-membership of the conjunction of both, x and y.

For every x, y and z belonging to the universe of discourse, N_n must satisfy the following axioms:

1. $N_n(x, 0_N) = 0_N$ and $N_n(x, 1_N) = x$ (Boundary conditions).
2. $N_n(x, y) = N_n(y, x)$ (Commutativity).
3. If $x \leq_N y$, then $N_n(x, z) \leq_N N_n(y, z)$ (Monotonicity).
4. $N_n(N_n(x, y), z) = N_n(x, N_n(y, z))$ (Associativity).

Definition 2.8. A *neutrosophic conorm* or *n-conorm* N_c [25], is a mapping $N_c: (]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[)^2 \rightarrow]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, such that $N_c\left(x(T_x, I_x, F_x), y(T_y, I_y, F_y)\right) = (N_cT(x, y), N_cI(x, y), N_cF(x, y))$, where N_cT means the degree of membership, N_cI the degree of indeterminacy and N_cF the degree of non-membership of the disjunction of x with y.

For every x, y and z belonging to the universe of discourse, N_c must satisfy the following axioms:

1. $N_c(x, 0_N) = x$ and $N_c(x, 1_N) = 1_N$ (Boundary conditions).
2. $N_c(x, y) = N_c(y, x)$ (Commutativity).
3. If $x \leq_N y$, then $N_c(x, z) \leq_N N_c(y, z)$ (Monotonicity).
4. $N_c(N_c(x, y), z) = N_c(x, N_c(y, z))$ (Associativity).

According to [8] a Singled-valued neutrosophic negator is defined as follows:

Definition 2.9. a *singled-valued neutrosophic negator* is a decreasing unary neutrosophic operator $N_N: [0, 1]^3 \longrightarrow [0, 1]^3$, satisfying the following boundary conditions:

1. $N_N(0_N) = 1_N$.
2. $N_N(1_N) = 0_N$.

It is called *involutive* if and only if $N_N(N_N(x)) = x$ for every $x \in [0, 1]^3$.

In the following, we show the neutrosophic negators that we shall consider hereunder, extracted from the literature, see [25]. Given a SVNS $A(T_A, I_A, F_A)$, we have:

1. $N_N((T_A, I_A, F_A)) = (1 - T_A, 1 - I_A, 1 - F_A)$, $N_N((T_A, I_A, F_A)) = (1 - T_A, I_A, 1 - F_A)$, $N_N((T_A, I_A, F_A)) = (F_A, I_A, T_A)$ and $N_N((T_A, I_A, F_A)) = (F_A, 1 - I_A, T_A)$ (Involutive negators).
2. $N_N((T_A, I_A, F_A)) = \left(F_A, \frac{F_A + I_A + T_A}{3}, T_A\right)$ and $N_N((T_A, I_A, F_A)) = \left(1 - T_A, \frac{F_A + I_A + T_A}{3}, 1 - F_A\right)$ (Non-involutive negators).

In literature, we found neutrosophic implicators, which extend only the notion of S-implications [11]. Moreover, we did not find a general definition on neutrosophic implications except in [8]. In the following, we conclude this section with such definition and properties.

Definition 2.10. A *singled-valued neutrosophic implicator* is an operator $I_N: [0,1]^3 \times [0,1]^3 \rightarrow [0,1]^3$ which satisfies the following conditions, for all $x, x', y, y' \in [0,1]^3$:

1. If $x' \leq_N x$, then $I_N(x,y) \leq_N I_N(x',y)$.
2. If $y \leq_N y'$, then $I_N(x,y) \leq_N I_N(x,y')$.
3. $I_N(0_N,0_N) = I_N(0_N,1_N) = I_N(1_N,1_N) = 1_N$.
4. $I_N(1_N,0_N) = 0_N$.

Herein we use the term neutrosophic implicator or n-implicator to mean singled-valued neutrosophic implicator.

It can satisfy the following properties for every $x, y, z \in [0,1]^3$:

1. $I_N(1_N, x) = x$ (Neutrality principle)
2. $I_N(x,y) = I_N(N_{IN}(y), N_{IN}(x))$, where $N_{IN}(x) = I_N(x, 0_N)$ is an n-negator (Contrapositivity).
3. $I_N(x, I_N(y,z)) = I_N(y, I_N(x,z))$ (Interchangeability principle).
4. $x \leq_N y$ if and only if $I_N(x,y) = 1_N$ (Confinement principle).
5. I_N is a continuous mapping (Continuity).

3 Neutrosophic uninorms

This section is the core of the present paper, because here we explain the neutrosophic uninorm theory. We start defining this concept formally.

3.1 N-uninorms

Definition 3.1. A *neutrosophic uninorm* or *N-uninorm* $\mathbf{U}_N$, is a commutative, increasing and associative mapping, $\mathbf{U}_N: (]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[)^2 \rightarrow]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[$, such that:
$\mathbf{U}_N\left(x(T_x,I_x,F_x), y(T_y,I_y,F_y)\right) = (\mathbf{U}_N T(x,y), \mathbf{U}_N I(x,y), \mathbf{U}_N F(x,y))$, where $\mathbf{U}_N T$ means the degree of membership, $\mathbf{U}_N I$ the degree of indeterminacy and $\mathbf{U}_N F$ the degree of non-membership of both, x and y. Additionally, there exists a neutral element $e \in]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[$, where $\forall x \in]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[$, $\mathbf{U}_N(e,x) = x$.

Remark 3.1. See that Def. 3.1, extends Def. 2.4 in two ways, according to the differences between L^* fuzzy sets and neutrosophic sets. First, $\mathbf{U}_N$ includes the third function representing indeterminacy and secondly, there not exists constraints in the relationship among T, I and F. In addition, Def. 3.1 extends Def. 2.7 when $e = 1_N$ and Def 2.8., when $e = 0_N$.

Remark 3.2. For the sake of simplicity, we shall develop the theory only for singled-valued neutrosophic uninorms.

A trivial consequence of Def. 3.1 is that the neutral element is unique, which is a uninorm property in Def. 2.1 and Def. 2.4.

In the following, we explore the formulas of N-uninorms related to those corresponding to n-norms and n-conorms. For this end, first we need to describe two kinds of sets, namely, $E_1 = \{x \in [0,1]^3 : x \leq_N e\}$ and $E_2 = \{x \in [0,1]^3 : x \geq_N e\}$.

Lemma 3.1. Let $e \in]0,1] \times [0,1[\times [0,1[$. The mapping $\phi_e: [0,1]^3 \rightarrow [0,1]^3$, defined by:

$$\phi_e(x) = \left(e_1 x_1, x_2 + e_2(1-x_2), x_3 + e_3(1-x_3)\right) \quad (1)$$

for every $x \in [0,1]^3$ is an increasing bijection from $[0,1]^3$ to E_1 and ϕ_e^{-1} is increasing as well.

Proof. To prove ϕ_e is injective, let $x, y \in [0,1]^3$ and suppose $\phi_e(x) = \phi_e(y)$.Then, clearly the equation $\left(e_1x_1, x_2 + e_2(1-x_2), x_3 + e_3(1-x_3)\right) = \left(e_1y_1, y_2 + e_2(1-y_2), y_3 + e_3(1-y_3)\right)$ is fulfilled only if x = y, and the injection is proved, also taking into account that we excluded the cases $e_1 = 0$, $e_2 = 1$ and $e_3 = 1$.

Let us take any $y \in E_1$ and define $x = (x_1, x_2, x_3)$, such that $x_1 = \frac{y_1}{e_1}$, $x_2 = \frac{y_2 - e_2}{1 - e_2}$ and $x_3 = \frac{y_3 - e_3}{1 - e_3}$. Then, $\phi_e(x) = y$ and $x_1, x_2, x_3 \in [0,1]$, which can be proved applying $y \leq_N e$. Therefore, ϕ_e is surjective and evidently it is increasing. The equation of the inverse is the following:

$$\phi_e^{-1}(x) = \left(\frac{x_1}{e_1}, \frac{x_2 - e_2}{1 - e_2}, \frac{x_3 - e_3}{1 - e_3}\right) \quad (2)$$

□

Lemma 3.2. Let $e \in [0,1[\times]0,1]\times]0,1]$. The mapping $\psi_e: [0,1]^3 \to [0,1]^3$, defined by:

$$\psi_e(x) = (e_1 + x_1 - e_1x_1, e_2x_2, e_3x_3) \quad (3)$$

for every $x \in [0,1]^3$ is an increasing bijection from $[0,1]^3$ to E_2 as well as ψ_e^{-1} is increasing.

Proof. This lemma can be proved similarly to the proof carried out in the Lemma 3.1. The equation of the inverse is as follows:

$$\psi_e^{-1}(x) = \left(\frac{x_1 - e_1}{1 - e_1}, \frac{x_2}{e_2}, \frac{x_3}{e_3}\right) \quad (4)$$

□

Theorem 3.3. Given $\mathbf{U}_N$ an N-uninorm with neutral element $e \in]0, 1[^3$. Then the following two conditions are satisfied:

i. The mapping $N_{n,\mathbf{U}_N}: [0,1]^3 \times [0,1]^3 \to [0,1]^3$ defined for all $x, y \in [0,1]^3$ by the equation:

$$N_{n,\mathbf{U}_N}(x,y) = \phi_e^{-1}\left(\mathbf{U}_N(\phi_e(x), \phi_e(y))\right) \quad (5)$$

is an n-norm.

ii. The mapping $N_{c,\mathbf{U}_N}: [0,1]^3 \times [0,1]^3 \to [0,1]^3$ defined for all $x, y \in [0,1]^3$ by the equation:

$$N_{c,\mathbf{U}_N}(x,y) = \psi_e^{-1}\left(\mathbf{U}_N(\psi_e(x), \psi_e(y))\right) \quad (6)$$

is an n-conorm.

Proof. This theorem is a consequence of Lemmas 3.1 and 3.2.□

Remark 3.3. Some cases of e were excluded in Lemmas 3.1, 3.2 and Theorem 3.3, for instance, $e = (0, \beta, \gamma)$, where $0 \leq \beta, \gamma \leq 1$ in Lemma 3.1. It is easy to prove that when e is one of them, there not exist any increasing bijection from $[0,1]^3$ to E_1 or E_2, because E_1 or E_2 have one constant component, and therefore they only depend on at most two components, however, $[0,1]^3$ depends on three, and that contradicts the injection. For example, if $e = (0, \beta, \gamma)$, then $E_1 = \{0\} \times [\beta, 1] \times [\gamma, 1]$, and there not exists a bijective mapping from $[0,1]^3$ to E_1.

Corollary 3.4. Given $\mathbf{U}_N$ an N-uninorm with neutral element $e \in]0, 1[^3$. Then the following two conditions are satisfied:

i. For every $x, y \in E_1$, $\mathbf{U}_N(x,y) = \phi_e\left(N_{n,\mathbf{U}_N}(\phi_e^{-1}(x), \phi_e^{-1}(y))\right)$.

ii. For every $x, y \in E_2$, $\mathbf{U}_N(x,y) = \psi_e\left(N_{c,\mathbf{U}_N}(\psi_e^{-1}(x), \psi_e^{-1}(y))\right)$.

Proof. The proof is obtained immediately from Theorem 3.3.□

Remark 3.4. See that Theorem 3.3 and Corollary 3.4 mean that we can define N-uninorms from n-norms and n-conorms, and vice versa.

Remark 3.5. Comparing the precedent issues with their similar ones appeared in [15], we can find few differences and numerous similarities. Indeed, so far we have proved that N-uninorms extend the approach to structures of uninorms on L^* fuzzy sets, which is valid to interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and Goguen's L-fuzzy sets.

Definition 3.2. We say that $N_n(x,y)$ is an *Archimedean n-norm* respect to $<_N$ if for every $x \in [0,1]^3$ it satisfies: $N_n(x,x) <_N x$.

Definition 3.3. We say that $N_c(x,y)$ is an *Archimedean n-conorm* respect to $<_N$ if for every $x \in [0,1]^3$ it satisfies: $N_c(x,x) >_N x$.

Definition 3.4. $\mathbf{U}_N(x,y)$ is an *Archimedean N-uninorm* respect to $<_N$ if it satisfies the following conditions:

1. $\mathbf{U}_N(x,x) <_N x$ for every $0 <_N x <_N e$.
2. $\mathbf{U}_N(x,x) >_N x$ for every $e <_N x <_N 1_N$.

Proposition 3.5. Given $\mathbf{U}_N$ an N-uninorm with neutral element $e \in]0, 1[^3$. It is Archimedean if and only if the n-norm and n-conorm defined in Eq. 5 and 6, respectively, are Archimedean.

Proof Let $0 <_N x <_N e$, and $\mathbf{U}_N(x,y)$ an Archimedean N-uninorm, i.e., $\mathbf{U}_N(x,x) <_N x$, then taking into account that ϕ_e and ϕ_e^{-1} are increasing bijections, we have $N_{n,\mathbf{U}_N}(x,x) = \phi_e^{-1}\left(\mathbf{U}_N(\phi_e(x), \phi_e(x))\right) <_N \phi_e^{-1}(\phi_e(x)) = x$. Equivalently, it is easy to prove that $N_{n,\mathbf{U}_N}(x,x) <_N x$ implies $\mathbf{U}_N(x,x) <_N x$. The proof for the n-conorm is similar.□

Proposition 3.6. Given $\mathbf{U}_N$ an N-uninorm with neutral element e, and $x, y \in [0,1]^3$ are two elements such that either $x \le_N e \le_N y$ or $y \le_N e \le_N x$, then the following two inequalities hold:

$\min(x,y) \le_N \mathbf{U}_N(x,y) \le_N \max(x,y)$.

Proof. Without loss of generality, suppose $x \le_N e \le_N y$, then because of the monotonicity of the N-uninorms $\mathbf{U}_N(x,y) \le_N \mathbf{U}_N(e,y) = y = \max(x,y)$ and $\mathbf{U}_N(x,y) \ge_N \mathbf{U}_N(x,e) = x = \min(x,y)$.□

The proposition above means that there exists a domain where $\mathbf{U}_N$ is compensatory with regard to $\le_N$.

Let us note that there exists other sets where $x \parallel_{\le_N} y$ or $x \parallel_{\le_N} e$.

Example 3.1. Two examples of N-uninorms are the following:

Recalling the well-known weakest and strongest fuzzy uninorms, respectively, defined as follows:

$$\underline{U}_{e_1}(x_1,y_1) := \begin{cases} 0 & \text{if } 0 \le x_1, y_1 < e_1 \\ \max\{x_1,y_1\} & \text{if } e_1 \le x_1, y_1 \le 1 \\ \min\{x_1,y_1\} & \text{otherwise} \end{cases} \text{ and } \overline{U}_{e_1}(x_1,y_1) := \begin{cases} \min\{x_1,y_1\} & \text{if } 0 \le x_1, y_1 \le e_1 \\ 1 & \text{if } e_1 < x_1, y_1 \le 1 \\ \max\{x_1,y_1\} & \text{otherwise} \end{cases}$$

For every $x_1, y_1 \in [0,1]$ and $e_1 \in]0,1[$.

Let us define two N-uninorms as follows: for every $x, y \in [0,1]^3$ and $e \in [0,1]^3$ is the neutral element:

$$\underline{\mathbf{U}}_e(x,y) := \left(\underline{U}_{e_1}(x_1,y_1), \overline{U}_{e_2}(x_2,y_2), \overline{U}_{e_3}(x_3,y_3)\right) \quad (7)$$

and

$$\overline{\mathbf{U}}_e(x,y) := \left(\overline{U}_{e_1}(x_1,y_1), \underline{U}_{e_2}(x_2,y_2), \underline{U}_{e_3}(x_3,y_3)\right) \quad (8)$$

Both $\underline{\mathbf{U}}_e(x,y)$ and $\overline{\mathbf{U}}_e(x,y)$, are N-uninorms, because every one of the components are uninorms, thus, they are commutative, associative and increasing. The neutral element components are formed by the neutral elements of every individual uninorm.

Moreover, $\underline{\mathbf{U}}_e(x,y)$ is a conjunctive N-uninorm and $\overline{\mathbf{U}}_e(x,y)$ is a disjunctive N-uninorm, i.e., $\underline{\mathbf{U}}_e(0_N, 1_N) = 0_N$ and $\overline{\mathbf{U}}_e(0_N, 1_N) = 1_N$.

See that $\mathbf{U}_e(x,y) = \left(\underline{U}_{e_1}(x_1,y_1), \underline{U}_{e_2}(x_2,y_2), \underline{U}_{e_3}(x_3,y_3)\right)$ is also an N-uninorm, nevertheless, it is neither conjunctive nor disjunctive, $\mathbf{U}_e(0_N, 1_N) = (0,0,0)$.

Definition 3.5. An N-uninorm $\mathbf{U}_N$ is said to be *t-representable* if there exist three fuzzy uninorms, $U_{e_1}(x_1,y_1)$, $U_{e_2}(x_2,y_2)$ and $U_{e_3}(x_3,y_3)$, such that for all $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ it has the form $\mathbf{U}_N(x,y) = \left(U_{e_1}(x_1,y_1), U_{e_2}(x_2,y_2), U_{e_3}(x_3,y_3)\right)$.

Proposition 3.7. Let $\mathbf{U}_N$ be an N-uninorm with neutral element e and $x \in [0,1]^3$, then the following properties hold:

i. $\mathbf{U}_N(0_N, 0_N) = 0_N$ and $\mathbf{U}_N(1_N, 1_N) = 1_N$.
ii. If $e \in [0,1]^3 \setminus \{0_N, 1_N\}$, we have $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$, for every $x \in [0,1]^3$.
iii. If $e \in [0,1]^3 \setminus \{0_N, 1_N\}$, then either $\mathbf{U}_N(0_N, 1_N) = 0_N$ or $\mathbf{U}_N(0_N, 1_N) = 1_N$ or $\mathbf{U}_N(0_N, 1_N) \parallel_{\le_N} e$.

Proof.

i. See that $\mathbf{U}_N(e, 0_N) = 0_N$, $\mathbf{U}_N(e, 1_N) = 1_N$ and apply the increasing axiom of N-uninorm.
ii. If $x \le_N e$ then because $\mathbf{U}_N$ is increasing, we have $\mathbf{U}_N(0_N, x) \le_N \mathbf{U}_N(0_N, e) = 0_N$, thus, $\mathbf{U}_N(0_N, x) = 0_N$ and $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, x), 1_N)$. Because of the commutativity and the associativity, $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$.
If $x \ge_N e$ then $\mathbf{U}_N(1_N, x) \ge_N \mathbf{U}_N(1_N, e) = 1_N$ and therefore, $\mathbf{U}_N(1_N, x) = 1_N$. $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(0_N, \mathbf{U}_N(1_N, x))$, and finally due to the commutativity and associativity, we obtain $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$.
If $x \parallel_{\le_N} e$ then $x \wedge e \le_N x \le_N x \vee e$. We have $x \wedge e \le_N e$ and $e \le_N x \vee e$, thus according to the precedent results $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x \wedge e) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x \vee e)$. Applying the increasing axiom of N-uninorms we obtain $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$.
iii. Suppose $\mathbf{U}_N(0_N, 1_N) \nparallel_{\le_N} e$, that implies either $\mathbf{U}_N(0_N, 1_N) \le_N e$ or $e \le_N \mathbf{U}_N(0_N, 1_N)$.
If $\mathbf{U}_N(0_N, 1_N) \le_N e$, then $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), 0_N) = 0_N$, according to ii.
If $\mathbf{U}_N(0_N, 1_N) \ge_N e$, then $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), 1_N) = 1_N$, according to ii.□

Let us note that the precedent issues are similar to the ones obtained in [15].

3.2 Implicators induced by N-uninorms

This subsection is dedicated to explore the notion of n-implicators induced by N-uninorms. First of all we define the concept of neutrosophic R-implicator, which is new in this framework, at least in the scope of our knowledge.

Definition 3.6. A *neutrosophic R- implicator* or *n-R-implicator* is an n-implicator defined as follows:

Given N_n an n-norm, for every $x, y \in [0,1]^3$, $RI_N(x,y) = \sup\{t \in [0,1]^3 : N_n(x,t) \leq_N y\}$.

Let us note that this definition extends both, the definition of fuzzy R-implicator, see [7], and that of L* fuzzy implicator, [15]. As well as others appeared in [3, 12].

Indeed, it is an actual n-implicator. Taking into account the properties of $\leq_N$, and the increasing property of n-norms with regard to $\leq_N$, we have that $RI_N(x,\cdot)$ is decreasing and $RI_N(\cdot,y)$ is increasing. Additionally, the satisfaction of the boundary conditions by RI_N can be verified straightforwardly.

Example 3.2. Let $a = (0.6,\ 0.2,\ 0.4)$, $b = (0.7,\ 0.1,\ 0.3)$ and $c = (0.5,\ 0.3,\ 0.5)$ be three SVNS. Observe that $c \leq_N a \leq_N b$. Consider the n-norm, $N_{n-min}(x,y) = (\min\{T_x, T_y\}, \max\{I_x, I_y\}, \max\{F_x, F_y\})$.

Then, $RI_N(a,b) = 1_N$, $RI_N(a,c) = (0.5,\ 0.3,\ 0.5)$, $RI_N(b,a) = (0.6, 0.2, 0.4)$ and $RI_N(c,a) = 1_N$. See that $RI_N(a,c) \leq_N RI_N(a,b)$ and $RI_N(b,a) \leq_N RI_N(c,a)$.

Proposition 3.8. Let RI_N be an n-R-implicator induced by the n-norm N_n, then the two following properties hold:

i. $RI_N(1_N, y) = y$ for every $y \in [0,1]^3$ (Neutrality principle).
ii. $RI_N(x,x) = 1_N$ for every $x \in [0,1]^3$ (Identity principle).
iii. $x, y \in [0,1]^3$ and $x \leq_N y$ if and only if $RI_N(x,y) = 1_N$ (Confinement principle).

Proof.

i. For $y \in [0,1]^3$, we have $RI_N(1_N, y) = \sup\{t \in [0,1]^3 : N_n(1_N, t) = t \leq_N y\} = y$.
ii. For $x \in [0,1]^3$, we have $RI_N(x,x) = \sup\{t \in [0,1]^3 : N_n(x,t) \leq_N x\} = 1_N$, because N_n is increasing and $N_n(x, 1_N) = x$.
iii. For $x, y \in [0,1]^3$ and $x \leq_N y$, taking into account the inequalities $N_n(x,t) \leq_N N_n(x, 1_N) = x \leq_N y$ for every $t \in [0,1]^3$, we have $RI_N(x,y) = 1_N$. On the other hand, $RI_N(x,y) = 1_N$ evidently implies $x \leq_N y$, from the definition.

Theorem 3.9. Let $\mathbf{U}_N$ be an N-uninorm with neutral element $e \in]0,\ 1[^3$. Let us establish the mapping $RI_{\mathbf{U}_N} : [0,1]^3 \times [0,1]^3 \to [0,1]^3$ defined as follows:

$RI_{\mathbf{U}_N}(x,y) = \sup\{t \in [0,1]^3 : \mathbf{U}_N(x,t) \leq_N y\}$ for every $x, y \in [0,1]^3$.

It is an n-implicator if and only if there exists $\tilde{x} >_N 0_N$ such that every $x \geq_N \tilde{x}$ satisfies $\mathbf{U}_N(0_N, x) = 0_N$.

Proof. It is easy to verify that $RI_{\mathbf{U}_N}(x,\cdot)$ is decreasing and $RI_{\mathbf{U}_N}(\cdot,y)$ is increasing.

On the other hand, $RI_{\mathbf{U}_N}(0_N, 1_N) = RI_{\mathbf{U}_N}(1_N, 1_N) = 1_N$, because $\mathbf{U}_N$ is increasing and 1_N is the supremum.

See that for every $t \in [0,1]^3$, $\mathbf{U}_N(1_N, t) \geq_N \mathbf{U}_N(e, t) = t$, then $\mathbf{U}_N(1_N, t) >_N 0_N$ if and only if $t >_N 0_N$, therefore $RI_{\mathbf{U}_N}(1_N, 0_N) = 0_N$.

Additionally, if there exists $\tilde{x} >_N 0_N$ such that every $x \geq_N \tilde{x}$ satisfies $\mathbf{U}_N(0_N, x) = 0_N$, then because $\mathbf{U}_N$ is increasing and 1_N is the supremum of that set, $\mathbf{U}_N(0_N, 1_N) = 0_N$ and $RI_{\mathbf{U}_N}(0_N, 0_N) = 1_N$.□

Remark 3.6. The Theorem 3.9 is valid when $\mathbf{U}_N$ is a conjuctive N-uninorm.

Example 3.3. Given again $a = (0.6,\ 0.2,\ 0.4)$, $b = (0.7,\ 0.1,\ 0.3)$ and $c = (0.5,\ 0.3,\ 0.5)$, three SVNS, as in Example 3.2. Let us consider $\underline{\mathbf{U}}_e$ of the Example 3.1, where $e = (0.5, 0.5, 0.5)$. Recall that $\underline{\mathbf{U}}_e(0_N, 1_N) = 0_N$. Then, $RI_{\underline{\mathbf{U}}_e}(a,b) = (0.7, 0.1, 0.3)$, $RI_{\underline{\mathbf{U}}_e}(a,c) = (0.5, 0.5, 0.5)$, $RI_{\underline{\mathbf{U}}_e}(b,a) = (0.5, 0.5, 0.5)$ and $RI_{\underline{\mathbf{U}}_e}(c,a) = (0.6, 0.2, 0.4)$.

Proposition 3.10. Given $\mathbf{U}_N$ an N-uninorm with $e \in [0,1]^3 \setminus \{0_N, 1_N\}$. Then, $RI_{\mathbf{U}_N}(e, x) = x$, for every $x \in [0,1]^3$.

Proof. Let us fix $x \in [0,1]^3$, $RI_{\mathbf{U}_N}(e,x) = \sup\{t \in [0,1]^3 : \mathbf{U}_N(e,t) = t \leq_N x\} = x$.□

Proposition 3.11. Given $\mathbf{U}_N$ an N-uninorm with $e \in [0,1]^3 \setminus \{0_N, 1_N\}$. $RI_{\mathbf{U}_N}(x, 1_N) = 1_N$, for every $x \in [0,1]^3$ (Right boundary condition).

Proof. Taking into account $\mathbf{U}_N$ is increasing and 1_N is the supremum of the elements of the lattice, then, $RI_{\mathbf{U}_N}(x, 1_N) = \sup\{t \in [0,1]^3 : \mathbf{U}_N(x,t) \leq_N 1_N\} = 1_N$.□

Proposition 3.12. Given $\mathbf{U}_N$ an N-uninorm with $e \in [0,1]^3 \setminus \{0_N, 1_N\}$. If it is contrapositive respect to a negator N_N, which satisfies $N_N(e) = e$, then $N_N(x) = N_{NI_{\mathbf{U}_N}}(x) = RI_{\mathbf{U}_N}(x, e)$ for every $x \in [0,1]^3$ and $N_{NI_{\mathbf{U}_N}}$ is involutive.

Proof. Reproduce the similar proof in [15] adapted to N-uninorms.□

Proposition 3.13. Given $\mathbf{U}_\mathrm{N}$ an N-uninorm and N_N an n-negator. The mapping $\mathrm{SI}_{\mathbf{U}_\mathrm{N}}(\mathrm{x},\mathrm{y}) = \mathbf{U}_\mathrm{N}(\mathrm{N}_\mathrm{N}(\mathrm{x}),\mathrm{y})$ is an n-implicator if and only if $\mathbf{U}_\mathrm{N}$ is disjunctive.

Proof. Reproduce the similar proof in [15] adapted to N-uninorms.□

Example 3.4. Revisiting Examples 3.2 and 3.3, where $\mathrm{a} = (0.6,\ 0.2,\ 0.4)$, $\mathrm{b} = (0.7,\ 0.1,\ 0.3)$ and $\mathrm{c} = (0.5,\ 0.3,\ 0.5)$. Now we consider the n-negator $\mathrm{N}_\mathrm{N}\big((\mathrm{T}_\mathrm{x},\mathrm{I}_\mathrm{x},\mathrm{F}_\mathrm{x})\big) = (\mathrm{F}_\mathrm{x},\mathrm{I}_\mathrm{x},\mathrm{T}_\mathrm{x})$ and from the Example 3.1, $\overline{\mathbf{U}}_e(\mathrm{x},\mathrm{y})$ with $e = (0.5, 0.5, 0.5)$. There, we proved it is disjunctive.

Then, we have $\mathrm{SI}_{\overline{\mathbf{U}}_e}(\mathrm{a},\mathrm{b}) = (0.7, 0, 0.3)$, $\mathrm{SI}_{\overline{\mathbf{U}}_e}(\mathrm{a},\mathrm{c}) = (0.4,\ 0, 0.6)$, $\mathrm{SI}_{\overline{\mathbf{U}}_e}(\mathrm{b},\mathrm{a}) = (0.6,\ 0, 0.4)$ and $\mathrm{SI}_{\overline{\mathbf{U}}_e}(\mathrm{c},\mathrm{a}) = (0.6, 0, 0.4)$.

Proposition 3.14. Given $\mathbf{U}_\mathrm{N}$ an N-uninorm and N_N an n-negator. The mapping $\mathrm{SI}_{\mathbf{U}_\mathrm{N}}$ satisfies the Interchangeability Principle:

$\mathrm{SI}_{\mathbf{U}_\mathrm{N}}\left(\mathrm{x}, \mathrm{SI}_{\mathbf{U}_\mathrm{N}}(\mathrm{y},\mathrm{z})\right) = \mathrm{SI}_{\mathbf{U}_\mathrm{N}}\left(\mathrm{y}, \mathrm{SI}_{\mathbf{U}_\mathrm{N}}(\mathrm{x},\mathrm{z})\right)$ for every $\mathrm{x},\mathrm{y},\mathrm{z} \in [0,1]^3$.

Proof. It is proved by using the commutativity and associativity of N-uninorms.□

Conclusion

The proposed paper was devoted to define and study a new operator called neutrosophic uninorm or N-uninorm. We demonstrated that it is possible to extend the notion of uninorm to the framework of neutrosophy logic theory. In addition, we defined new neutrosophic implicators induced by N-uninorms. Moreover, we introduced a new neutrosophic implicator which generalizes the fuzzy notion of R-implicator. The importance of this new theory is that the appreciated quality of fuzzy uninorms as aggregators is enriched with the capacity of neutrosophy to deal with indeterminacy.

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On Refined Neutrosophic Hyperrings

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Abstract. This paper presents the refinement of a type of neutrosophic hyperring in which $+'$ and $\cdot'$ are hyperopraetions and studied some of its properties. Several interesting results and examples are presented.

Keywords: .

Neutrosophic, neutrosophic hyperring, neutrosophic hypersubring, refined neutrosophic hyperring, refined neutrosophic hypersubhyperring , refined neutrosophic hyperring homomorphism.

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1. Introduction

In a general sense the triple $(R, +, \cdot)$ is an hyperring if the hyperoperations $+$ and $\cdot$ are such that $(R, +)$ is a hypergroup, $(R, \cdot)$ is semihypergroup and $\cdot$ is distributive with respect to $+$. These structures are essentially rings with approximately modified axioms. Different notions of hyperrings have been investigated by researchers in the field of algebraic hyperstructures. For example, Krasner in [20] introduced a type of hyperring in which $''+''$ is an hyperoperation and $''\cdot''$ is a binary operation. This type of hyperring is referred to as a Krasner hyperring. In [24] a type of hyperring called multiplicatve hyperring was introduced by Rota. In this hyperring $''+''$ is considered as an ordinary addition and $''\cdot''$ as an hyperoperation. The type of hyperring in which $''+''$ and $''\cdot''$ were hyperoperations was studied by De Salvo in [14]. These classes of hyperrings were further studied by Barghi [12] , Asokkumar and Velrajan [9–11] .

In 1995, Smarandache generalized fuzzy logic/set and intuitionistic fuzzy logic/set by introducing a new branch of philosophy called Neutrosophy, which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I) and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]-0,1+[$ as can be seen in [22,23] . Ever since the introduction of this theory, several neutrosophic structures have been introduced, some of which includes; neutrosophic group, neutrosophic rings, neutrosophic modules, neutrosophic hypergroups, neutrosophic hyperrings, neutrosophic loops and many more. Smarandache in [22] introduced the concept of refined neutrosophic logic and neutrosophic set which is basically the splitting of the components $< T, I, F >$ into subcomponents of the form $< T_1, T_2, \cdots, T_p; I_1, I_2, \cdots, I_r; F_1, F_2, \cdots, F_s >$. This concept inspired the work of Agboola in [5] where he introduced refined neutrosophic algebraic structures. A lot of results have been published on the refinement of some of the known neutrosophic algebraic structures/hyperstructures ever since the work of Agboola. A comprehensive review of refined neutrosophic structures/hyperstrutures, can be found in [1,2,8,15–19] .

In this paper, the refinement of neutrosophic hyperring is studied and several interesting results and examples are presented.

2. Preliminaries

In this section, we will give some definitions, examples and results that will be used in the sequel.

Definition 2.1. [13] Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ x = A \circ \{x\} \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Definition 2.2. [13] Let H be a non-empty set and let $+$ be a hyperoperation on H. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

(1) $x + y = y + x$, for all $x, y \in H$,
(2) $x + (y + z) = (x + y) + z$, for all $x, y, z \in H$,
(3) there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, for all $x \in H$,
(4) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
(5) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in H$. A nonempty subset A of H is called a subcanonical hypergroup if A is a canonical hypergroup under the same hyperaddition as that of H that is, for every $a, b \in A$, $a - b \in A$. If in addition $a + A - a \subseteq A$ for all $a \in H$, A is said to be normal.

Definition 2.3. A hyperring is a triple $(R, +, \cdot)$ satisfying the following axioms:

(1) $(R, +)$ is a canonical hypergroup.
(2) $(R, \cdot)$ is a semihypergroup such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$, that is, 0 is a bilaterally absorbing element,
(3) For all $x, y, z \in R$,
   (a) $x \cdot (y + z) = x \cdot y + x \cdot z$ and

(b) $(x+y)\cdot z = x\cdot z+y\cdot z$. That is, the hyperoperation . is distributive over the hyperoperation +.

Definition 2.4. Let $(R,+,\cdot)$ be a hyperring and let A be a nonempty subset of R. A is said to be a subhyperring of R if $(A,+,\cdot)$ is itself a hyperring.

Definition 2.5. Let A be a subhyperring of a hyperring R. Then,

(1) A is called a left hyperideal of R if $r\cdot a \subseteq A$ for all $r \in R, a \in A$.

(2) A is called a right hyperideal of R if $a\cdot r \subseteq A$ for all $r \in R, a \in A$. A is called a hyperideal of R if A is both left and right hyperideal of R.

Definition 2.6. Let A be a hyperideal of a hyperring R. A is said to be normal in R if $r + A - r \subseteq A$ for all $r \in R$.

It will be assumed that I splits into two sub-indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. With the properties that:

$$\begin{aligned} I_1I_1 &= I_1^2 &&= I_1, \\ I_2I_2 &= I_2^2 &&= I_2 \text{ and} \\ I_1I_2 &= I_2I_1 &&= I_1. \end{aligned}$$

Definition 2.7. [4] If $*: X(I_1,I_2)\times X(I_1,I_2) \mapsto X(I_1,I_2)$ is a binary operation defined on $X(I_1,I_2)$, then the couple $(X(I_1,I_2),*)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$.

Definition 2.8. [4] Let $(X(I_1,I_2),+,\cdot)$ be any refined neutrosophic algebraic structure where + and . are ordinary addition and multiplication respectively.
For any two elements $(a,bI_1,cI_2),(d,eI_1,fI_2) \in X(I_1,I_2)$, we define

$$(a,bI_1,cI_2)+(d,eI_1,fI_2) = (a+d,(b+e)I_1,(c+f)I_2),$$

$$(a,bI_1,cI_2)\cdot(d,eI_1,fI_2) = (ad,(ae+bd+be+bf+ce)I_1,(af+cd+cf)I_2).$$

Definition 2.9. [4] If + and . are ordinary addition and multiplication, I_k with $k = 1,2$ have the following properties:

(1) $I_k + I_k + \cdots + I_k = nI_k$.

(2) $I_k + (-I_k) = 0$.

(3) $I_k \cdot I_k \cdot \cdots \cdot I_k = I_k^n = I_k$ for all positive integers $n > 1$.

(4) $0 \cdot I_k = 0$.

(5) I_k^{-1} is undefined and therefore does not exist.

Definition 2.10. [4] Let $(G, *)$ be any group. The couple $(G(I_1, I_2), *)$ is called a refined neutrosophic group generated by G, I_1 and I_2. $(G(I_1, I_2), *)$ is said to be commutative if for all $x, y \in G(I_1, I_2)$, we have $x * y = y * x$. Otherwise, we call $(G(I_1, I_2), *)$ a non -commutative refined neutrosophic group.

Definition 2.11. [4] If $(X(I_1, I_2), *)$ and $(Y(I_1, I_2), *')$ are two refined neutrosophic algebraic structures, the mapping

$$\phi : (X(I_1, I_2), *) \longrightarrow (Y(I_1, I_2), *')$$

is called a neutrosophic homomorphism if the following conditions hold:

(1) $\phi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \phi((a, bI_1, cI_2)) *' \phi((d, eI_1, fI_2))$.
(2) $\phi(I_k) = I_k$ for all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$ and $k = 1, 2$.

Example 2.12. [4] Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2)(0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 2.13. [4] Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *')$ be two refined neutrosophic groups. Let $\phi : G(I_1, I_2) \times H(I_1, I_2) \to G(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let $\psi : G(I_1, I_2) \times H(I_1, I_2) \to H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

Definition 2.14. [6] Let $(H, +)$ be any canonical hypergroup and let I be an indeterminate.
Let $H(I) =< H \cup I >= \{(a, bI) : a, b \in H\}$ be a set generated by H and I. The hyperstructure $(H(I), +)$ is called a neutrosophic canonical hypergroup . For all $(a, bI), (c, dI) \in H(I)$ with $b \neq 0$ or $d \neq 0$, we define $(a, bI) + (c, dI) = \{(x, yI) : x \in a + c,\ y \in a + d \cup b + c \cup b + d\}$. An element $I \in H(I)$ is represented by $(0, I)$ in $H(I)$ and any element $x \in H$ is represented by $(x, 0)$ in $H(I)$. For any nonempty subset $A(I)$ of $H(I)$, we define $-A(I) = \{-(a, bI) = (-a, -bI) : a, b \in H\}$.

Definition 2.15. [6] Let $(H(I), +)$ be a neutrosophic canonical hypergroup .

(1) A nonempty subset $A(I)$ of $H(I)$ is called a neutrosophic subcanonical hypergroup of $H(I)$ if $(A(I), +)$ is itself a neutrosophic canonical hypergroup . It is essential that $A(I)$ must contain a proper subset which is a subcanonical hypergroup of H.
If $A(I)$ does not contain a proper subset which is a subcanonical hypergroup of H, then it is called a pseudo neutrosophic subcanonical hypergroup of $H(I)$.
(2) If $A(I)$ is a neutrosophic subcanonical hypergroup (pseudo neutrosophic subcanonical hypergroup), $A(I)$ is said to be normal in $H(I)$ if for all $(a, bI) \in H(I)$, $(a, bI) + A(I) - (a, bI) \subseteq A(I)$.

Definition 2.16. [6] Let $(R, +, \cdot)$ be any hyperring and let I be an indeterminate. The hyperstructure $(R(I), +, \cdot)$ generated by R and I, that is, $R(I) =< R \cup I >$, is called a neutrosophic hyperring. For all $(a, bI), (c, dI) \in R(I)$ with $b \neq 0$ or $d \neq 0$, we define

$$(a, bI) \cdot (c, dI) = \{(x, yI) : x \in a \cdot c, y \in a \cdot d \cup b \cdot c \cup b \cdot d\}.$$

Definition 2.17. [6] Let $(R(I), +, \cdot)$ be a neutrosophic hyperring and let $A(I)$ be a nonempty subset of $R(I)$. $A(I)$ is called a neutrosophic subhyperring of $R(I)$ if $(A(I), +, \cdot)$ is itself a neutrosophic hyperring . It is essential that $A(I)$ must contain a proper subset which is a hyperring. Otherwise, $A(I)$ is called a pseudo neutrosophic subhyperring of $R(I)$.

Definition 2.18. [6] Let $(R(I), +, \cdot)$ be a neutrosophic hyperring and let $A(I)$ be a neutrosophic subhyperring of $R(I)$.

(1) $A(I)$ is called a left neutrosophic hyperideal if $(r, sI) \cdot (a, bI) \subseteq A(I)$ for all $(r, sI) \in R(I)$ and $(a, bI) \in A(I)$.
(2) $A(I)$ is called a right neutrosophic hyperideal if $(a, bI) \cdot (r, sI) \subseteq A(I)$ for all $(r, sI) \in R(I)$ and $(a, bI) \in A(I)$.
(3) $A(I)$ is called a neutrosophic hyperideal if $A(I)$ is both a left and right neutrosophic hyperideal .

A neutrosophic hyperideal $A(I)$ of $R(I)$ is said to be normal in $R(I)$ if for all $(r, sI) \in R(I)$

$$(r, sI) + A(I) - (r, sI) \subseteq A(I).$$

Definition 2.19. [6] Let $(R_1(I), +, \cdot)$ and $(R_2(I), +, \cdot)$ be two neutrosophic hyperring and let $\phi : R_1(I) \longrightarrow R_2(I)$ be a mapping from $R_1(I)$ into $R_2(I)$.

(1) ϕ is called a homomorphism if :
   (a) ϕ is a hyperring homomorphism,
   (b) $\phi((0, I)) = (0, I)$.
(2) ϕ is called a good or strong homomorphism if:
   (a) ϕ is a good or strong hyperring homomorphism,
   (b) $\phi((0, I)) = (0, I)$.
(3) ϕ is called an isomorphism (strong isomorphism) if ϕ is a bijective homomorphism (strong homomorphism).

3. Formulation of a refined neutrosophic hyperrings

In this section, we study and present the development of refined neutrosophic hyperring $(R(I_1, I_2), +, \cdot)$ generated by R, I_1 and I_2 where the operations $''+''$ and $''\cdot''$ are hyperoperations. i.e.,

$$+, \cdot : R(I_1, I_2) \times R(I_1, I_2) \longrightarrow 2^{R(I_1, I_2)}.$$

For all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in R(I_1, I_2)$ with $a, b, c, d, e, f \in R$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = \{(p, qI_1, rI_2) : p \in a + d, q \in (b + e), r \in (c + f)\},$$

$$(a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = \{(p, qI_1, rI_2) : p \in ad,\ q \in ae + bd + be + bf + ce, r \in af + cd + cf\}.$$

Definition 3.1. A refined neutrosophic hyperring is a tripple $(R(I_1, I_2), +, \cdot)$ satisfying the following axioms:

(1) $(R(I_1, I_2), +)$ is a refined neutrosophic canonical hypergroup .
(2) $(R(I_1, I_2), \cdot)$ is a refined neutrosophic semihypergroup.
(3) For all $(a, bI_1, cI_2), (d, eI_1, fI_2), (g, hI_1, jI_2) \in R(I_1, I_2)$,
   (a) $(a, bI_1, cI_2)\cdot((d, eI_1, fI_2)+(g, hI_1, jI_2)) = (a, bI_1, cI_2)\cdot(d, eI_1, fI_2)+(a, bI_1, cI_2)\cdot(g, hI_1, jI_2)$ and
   (b) $((d, eI_1, fI_2) + (g, hI_1, jI_2)) \cdot (a, bI_1, cI_2) = (d, eI_1, fI_2) \cdot (a, bI_1, cI_2) + (g, hI_1, jI_2) \cdot (a, bI_1, cI_2)$.

Definition 3.2. Let $(R(I_1, I_2), +, \cdot)$ be a refined neutrosophic hyperring. A non-empty subset $M(I_1, I_2)$ of $R(I_1, I_2)$ is called a refined neutrosophic subhyperring of $R(I_1, I_2)$ if $(M(I_1, I_2), +, \cdot)$ is itself a neutrosophic hyperring. It is essential that $M(I_1, I_2)$ must contain a proper subset which is a hyperring. Otherwise, $M(I_1, I_2)$ is called a refined pseudo neutrosophic subhyperring of $R(I_1, I_2)$.

Definition 3.3. Let $R(I_1, I_2)$ be a refined neutrosophic hyperring. The refined neutrosophic subhyperring $M(I_1, I_2)$ is said to be normal in $R(I_1, I_2)$ if and only if $(a, bI_1, cI_2) + M(I_1, I_2) - (a, bI_1, cI_2) \subseteq M(I_1, I_2)$ for all $(a, bI_1, cI_2) \in R(I_1, I_2)$.

Definition 3.4. Let $(R(I_1, I_2)), +, \cdot)$ be a refined neutrosophic hyperring and let $M(I_1, I_2)$ be a refined neutrosophic subhyperring of $R(I_1, I_2)$. $(M(I_1, I_2), +, \cdot)$ is a left(right) refined neutrosophic hyperideal of $R(I_1, I_2)$ if $x \cdot m \in M(I_1, I_2)[m \cdot x \in M(I_1, I_2)]$ for all $x = (a, bI_1, cI_2) \in R(I_1, I_2)$ and $m = (p, qI_1, sI_2) \in M(I_1, I_2)$. $M(I_1, I_2)$ is a refined neutrosophic hyperideal if $M(I_1, I_2)$ is both left and right refined neutrosophic hyperideal.

Remark 3.5. It should be noted that a refined neutrosophic hyperideal $H(I_1, I_2)$ of a refined neutrosophic hyperring $R(I_1, I_2)$ is normal in $R(I_1, I_2)$ only if hyperideal H is normal in hyperring R.

Proposition 3.6. *Let $(R(I_1, I_2), +, \cdot)$ be any refined neutrosophic hyperring. $(R(I_1, I_2), +, \cdot)$ is a hyperring.*

Proof. (1) That $(R(I_1, I_2), +)$ is a canonical hypergroup follows from Proposition 2.3 in [19].

(2) We show that $(R(I_1, I_2), \cdot)$ is a semihypergroup.

$$\begin{aligned} x \cdot (y \cdot z) &= (a, bI_1, cI_2) \cdot ((d, eI_1, fI_2) \cdot (g, hI_1, kI_2)) \\ &= (a, bI_1, cI_2) \cdot ((dg, (dh + eg + eh + ek + fh)I_1, (dk + fg + fk)I_2) \\ &= (a(dg), (a(dh) + a(eg) + a(eh) + a(ek) + a(fh) + b(dg) + b(dh) + b(eg) + b(eh) \\ &\quad +b(ek) + b(fh) + b(dk) + b(fg) + b(fk) + c(dh) + c(eg) + c(eh) + c(ek) + c(fh))I_1, \\ &\quad (a(dk) + a(fg) + a(fk) + c(dg) + c(dk) + c(fg) + c(fk))I_2) \\ &= (ad)g, ((aI_1, ((ad)k + (af)g + (af)k + (cd)g + (cd)k + (cf)g + (cf)k)I_2) \\ &= ((a, bI_1, cI_2) \cdot (d, eI_1, fI_2)) \cdot (g, hI_1, kI_2) \\ &= (x \cdot y) \cdot z. \end{aligned}$$

Accordingly, $(R(I_1, I_2), \cdot)$ is a semihypergroup. Also, for all $(a, bI_1, cI_2) \in R(I_1, I_2)$,

$$(a, bI_1, cI_2) \cdot (0, 0I_1, 0I_2) = \{(x, yI_1, zI_2) : x \in a \cdot 0,\ y \in a \cdot 0 + b \cdot 0 + c \cdot 0, z \in a \cdot 0 + c \cdot 0\} = \{(0, 0I_1, 0I_2)\}.$$

Similarly, it can be shown that $(0, 0I_1, 0I_2) \cdot (a, bI_1, cI_2) = \{(0, 0I_1, 0I_2)\}$. Hence, $(0, 0I_1, 0I_2)$ is a bilaterally absorbing element.

(3) For the distributivity of $\cdot$ over $+$.

Let $a = (x, yI_1, zI_2), b = (u, vI_1, sI_2), c = (k, mI_1, nI_2)$ be arbitrary elements in $R(I_1, I_2)$ with $x, y, z, u, v, s, k, m, n \in R$.

$$\begin{aligned} a \cdot (b + c) &= a \cdot \{(h_1, h_2I_1, h_3I_2) : h_1 \in u + k, h_2 \in v + m, h_3 \in s + n\} \\ &= \{(x, yI_1, zI_2) \cdot (h_1, h_2I_1, h_3I_2) : h_1 \in u + k, h_2 \in v + m, h_3 \in s + n\} \\ &= \{(p_1, p_2I_1, p_3I_2) : p_1 \in xh_1, p_2 \in xh_2 + yh_1 + yh_2 + yh_3 + zh_2, p_3 \in xh_3 + zh_1 + zh_3\} \\ &= \{(p_1, p_2I_1, p_3I_2) : p_1 \in xu + xk, p_2 \in xv + xm + yu + yk + yv + ym + ys + yn+ \\ &\quad zv + zm, p_3 \in xs + xn + zu + zk + zs + zn\}. \end{aligned}$$

Now if we take $p_1 = t_1 + t_1',\ p_2 = t_2 + t_2',\ p_3 = t_3 + t_3',$ then we have

$$\begin{aligned} a \cdot (b + c) &= \{(t_1 + t_1', (t_2 + t_2')I_1, (t_3 + t_3')I_2) : t_1 + t_1' \in xu + xk, \\ &\quad t_2 + t_2' \in xv + xm + yu + yk + yv + ym + ys + yn + zv + zm, \\ &\quad t_3 + t_3' \in xs + xn + zu + zk + zs + zn\} \\ &= \{(t_1, t_2I_1, t_3I_2) : t_1 \in xu,\ t_2 \in xv + yu + yv + ys + zv,\ t_3 \in xs + zu + zs\}+ \\ &\quad \{(t_1', t_2'I_1, t_3'I_2) : t_1' \in xk,\ t_2' \in xm + yk + ym + yn + zm,\ t_3' \in xn + zk + zn\} \\ &= (x, yI_1, zI2) \cdot (u, vI_1, sI_2) + (x, yI_1, zI_2) \cdot (k, mI_1, nI_2) \\ &= a \cdot b + a \cdot c. \end{aligned}$$

Similarly, we can show that $(b + c) \cdot a = b \cdot a + c \cdot a$. Therefore $\cdot$ is distributive over $+$.

Hence $R(I_1, I_2)$ is a hyperring. □

Example 3.7. Let $R(I_1, I_2) = \{a_1 = (s, sI_1, sI_2), a_2 = (s, sI_1, tI_2), a_3 = (s, tI_1, sI_2),$ $a_4 = (s, tI_1, tI_2),\ b_1 = (t, tI_1, tI_2), b_2 = (t, tI_1, sI_2), b_3 = (t, sI_1, tI_2), b_4 = (t, sI_1, sI_2)\}$ be a refined neutrosophic set and let $+$ be the hyperoperation on $R(I_1, I_2)$ defined as in the tables below. Let $a = \{a_1, a_2, a_3, a_4\}$ and $b = \{b_1, b_2, b_3, b_4\}$.

It is clear from Table 1 and 2 that $(R(I_1, I_2), +, \cdot)$ is a refined neutrosophic hyperring.

TABLE 1. Cayley table for the binary operation " + "

$+$	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_1	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_2	a_2	$\{a_1, a_2\}$	a_4	$\{a_3, a_4\}$	$\{b_1, b_2\}$	b_1	$\{b_3, b_4\}$	b_3
a_3	a_3	a_4	$\{a_1, a_3\}$	$\{a_2, a_4\}$	$\{b_1, b_3\}$	$\{b_2, b_4\}$	b_1	b_2
a_4	a_4	$\{a_3, a_4\}$	$\{a_2, a_4\}$	a	b	$\{b_1, b_3\}$	$\{b_1, b_2\}$	b_1
b_1	b_1	$\{b_1, b_2\}$	$\{b_1, b_3\}$	b	$R(I_1, I_2)$	$\{a_2, a_4, b_1, b_3\}$	$\{a_3, a_4, b_1, b_2\}$	$\{a_4, b_1\}$
b_2	b_2	b_1	$\{b_2, b_4\}$	$\{b_1, b_3\}$	$\{a_2, a_4, b_1, b_3\}$	$\{a_1, a_3, b_2, b_4\}$	$\{a_4, b_1\}$	$\{a_3, b_2\}$
b_3	b_3	$\{b_3, b_4\}$	b_1	$\{b_1, b_2\}$	$\{a_3, a_4, b_1, b_2\}$	$\{a_4, b_1\}$	$\{a_1, a_2, b_3, b_4\}$	$\{a_2, b_3\}$
b_4	b_4	b_3	b_2	b_1	$\{a_4, b_1\}$	$\{a_3, b_2\}$	$\{a_2, b_3\}$	$\{a_1, b_4\}$

Proposition 3.8. *Let $(R(I_1, I_2), +_1, \cdot_1)$ be a refined neutrosophic hyperring and let $(K, +_2, \cdot_2)$ be a hyperring. Define for all $(x_1, k_1), (x_2, k_2) \in R(I_1, I_2) \times K$ the hyperoperations $''+''$ and $''\cdot''$ by*

$$(x_1, k_1) + (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 +_1 x_2,\ k_3 \in k_1 +_2 k_2\}$$

and

$$(x_1, k_1) \cdot (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 \cdot_1 x_2,\ k_3 \in k_1 \cdot_2 k_2\}.$$

Then $(R(I_1, I_2) \times K, +, \cdot)$ is a refined neutrosophic hyperring.

Proof. (1) That $(R(I_1, I_2) \times K, +)$ is a canonical hypergroup follows from the proof of Proposition 2.6 in [19] .

(2) We shall show that $(R(I_1, I_2) \times K, \cdot)$ is a refined neutrosophic semihypergroup.

Let $(r_1, k_1), (r_2, k_2), (r_3, k_3) \in R(I_1, I_2) \times K$ where $r = (a, bI_1, cI_2)$.

$(r_1, k_1) \cdot ((r_2, k_2) \cdot (r_3, k_3)) =$

$((a_1, b_1I_1, c_1I_2), k_1) \cdot [((a_2, b_2I_1, c_2I_2), k_2) \cdot ((a_3, b_3I_1, c_3I_2), k_3)]$

$= ((a_1, b_1I_1, c_1I_2), k_1) \cdot \{((p, qI_1, sI_2), k_4) :\ p \in a_2 \cdot_1 a_3,$

$q \in a_2 \cdot_1 b_3 +_1 b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3, s \in a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3,\ k_4 \in k_2 \cdot_2 k_3\}$

$= \{((x, yI_1, zI_2), k_5) : x \in a_1 \cdot_1 p,$

TABLE 2. Cayley table for the binary operation $''\cdot''$

$\cdot$	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	$\{a_1, a_2\}$	$\{a_1, a_3\}$	a	a	a	$\{a_1, a_2\}$	$\{a_1, a_2\}$
a_3	a_1	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$
a_4	a_1	a	$\{a_1, a_3\}$	a	a	a	a	a
b_1	a_1	a	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$R(I_1, I_2)$	$R(I_1, I_2)$	$R(I_1, I_2)$
b_2	a_1	a	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$\{a_1, a_3, b_2, b_4\}$	$R(I_1, I_2)$	$\{a_1, a_3, b_2, b_4\}$
b_3	a_1	$\{a_1, a_2\}$	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$R(I_1, I_2)$	$\{a_1, a_2, b_3, b_4\}$	$\{a_1, a_2, b_3, b_4\}$
b_4	a_1	$\{a_1, a_2\}$	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$\{a_1, a_3, b_2, b_4\}$	$\{a_1, a_2, b_3, b_4\}$	$\{a_1, b_4\}$

$y \in a_1 \cdot_1 q +_1 b_1 \cdot_1 p +_1 b_1 \cdot_1 q +_1 b_1 \cdot_1 s +_1 c_1 \cdot_1 q, z \in a_1 \cdot_1 s +_1 c_1 \cdot_1 p +_1 c_1 \cdot_1 s,\ k_5 \in k_1 \cdot_2 k_4\}$

$= \{((x, yI_1, zI_2), k_5) : x \in a_1 \cdot_1 (a_2 \cdot_1 a_3),$

$y \in a_1 \cdot_1 (a_2 \cdot_1 b_3 +_1 b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3) +_1 b_1 \cdot_1 (a_2 \cdot_1 a_3) +_1 b_1 \cdot_1 (a_2 \cdot_1 b_3 +_1$
$b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3) +_1 b_1 \cdot_1 (a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3) +_1 c_1 \cdot_1 (a_2 \cdot_1 b_3 +_1$
$b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3),$

$z \in a_1 \cdot_1 (a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3) +_1 c_1 \cdot_1 (a_2 \cdot_1 a_3) +_1 c_1 \cdot_1 (a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3),$
$k_5 \in k_1 \cdot_2 (k_2 \cdot_2 k_3)\}$

$= \{((x, yI_1, zI_2), k_5) : x \in a_1 \cdot_1 a_2 \cdot_1 a_3,$

$y \in a_1 \cdot_1 a_2 \cdot_1 b_3 +_1 a_1 \cdot_1 b_2 \cdot_1 a_3 +_1 a_1 \cdot_1 b_2 \cdot_1 b_3 +_1 a_1 \cdot_1 b_2 \cdot_1 c_3 +_1 a_1 \cdot_1 c_2 \cdot_1 b_3 +_1 b_1 \cdot_1 a_2 \cdot_1 a_3 +_1$
$b_1 \cdot_1 a_2 \cdot_1 b_3 +_1 b_1 \cdot_1 b_2 \cdot_1 a_3 +_1 b_1 \cdot_1 b_2 \cdot_1 b_3 +_1 b_1 \cdot_1 b_2 \cdot_1 c_3 +_1 b_1 \cdot_1 c_2 \cdot_1 b_3 +_1 b_1 \cdot_1 a_2 \cdot_1 c_3 +_1 b_1 \cdot_1$
$c_2 \cdot_1 a_3 +_1 b_1 \cdot_1 c_2 \cdot_1 c_3 +_1 c_1 \cdot_1 a_2 \cdot_1 b_3 +_1 c_1 \cdot_1 b_2 \cdot_1 a_3 +_1 c_1 \cdot_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3,$

$z \in a_1 \cdot_1 a_2 \cdot_1 c_3 +_1 a_1 \cdot_1 c_2 \cdot_1 a_3 +_1 a_1 \cdot_1 c_2 \cdot_1 c_3 +_1 c_1 \cdot_1 a_2 \cdot a_3 +_1 c_1 \cdot_1 a_2 \cdot_1 c_3 +_1 c_1 \cdot_1 c_2 \cdot_1 a_3 +_1 c_1 \cdot_1 c_2 \cdot_1 c_3, k_5 \in$
$k_1 \cdot_2 k_2 \cdot_2 k_3\}$

$= \{((x, yI_1, zI_2), k_5) : x \in (a_1 \cdot_1 a_2) \cdot_1 a_3,$

$y \in (a_1 \cdot_1 a_2) \cdot_1 b_3 +_1 (a_1 \cdot_1 b_2 +_1 b_1 \cdot_1 a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2) \cdot_1 a_3 +_1 (a_1 \cdot_1 b_2 +_1 b_1 \cdot_1$
$a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2) \cdot_1 b_3 +_1 (a_1 \cdot_1 b_2 +_1 b_1 \cdot_1 a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2) \cdot_1$

$c_3 +_1 (a_1c_2 +_1 c_1a_2 +_1 c_1c_2) \cdot_1 b_3,$

$z \in (a_1 \cdot_1 a_2) \cdot_1 c_3 +_1 (a_1 \cdot_1 c_2 +_1 c_1 \cdot_1 a_2 +_1 c_1 \cdot_1 c_2) \cdot_1 a_3 +_1 (a_1 \cdot_1 c_2 +_1 c_1 \cdot_1 a_2 +_1 c_1 \cdot_1 c_2) \cdot_1 c_3, k_5 \in$

$(k_1 \cdot_2 k_2) \cdot_2 k_3\}$

$= \{((m, nI_1, hI_2), k) : m \in a_1 \cdot_1 a_2,\ n \in a_1 \cdot_1 b_2 +_1 b_1 \cdot_1 a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2, h \in$

$a_1c_2 +_1 c_1a_2 +_1 c_1c_2, k \in k_1 \cdot_2 k_2\} \cdot ((a_3, b_3I_1, c_3I_2), k_3)$

$= [((a_1, b_1I_1, c_1I_2), k_1) \cdot ((a_2, b_2I_1, c_2I_2), k_2)] \cdot ((a_3, b_3I_1, c_3I_2))$

$= ((r_1, k_1) \cdot (r_2, k_2)) \cdot (r_3, k_3).$

Accordingly, $(R(I_1, I_2) \times K, \cdot)$ is a refined neutrosophic semihypergroup.

Also, for all $((a, bI_1, cI_2), k) \in R(I_1, I_2) \times K$,

$$\begin{aligned} ((a, bI_1, cI_2), k) \cdot ((0, 0I_1, 0I_2), 0) &= \{((x, yI_1, zI_2), k_1) : x \in a \cdot_1 0, y \in a \cdot_1 0 +_1 b \cdot_1 0 +_1 c \cdot_1 0, \\ &\quad z \in a \cdot_1 0 +_1 c \cdot 0,\ k_1 \in k \cdot_2 0\} \\ &= \{((0, 0I_1, 0I_2), 0)\}. \end{aligned}$$

Similarly, it can be shown that $((0, 0I_1, 0I_2), 0) \cdot ((a, bI_1, cI_2), k) = \{((0, 0I_1, 0I_2), 0)\}$.

Hence, $((0, 0I_1, 0I_2), 0)$ is a bilaterally absorbing element.

(3) For the distributivity of $\cdot$ over $+$.

Let $a = ((x, yI_1, zI_2), t_1),\ b = ((u, vI_1, sI_2), t_2),\ c = ((k, mI_1, nI_2), t_3)$ be arbitrary elements in $R(I_1, I_2) \times K$ with $x, y, z, u, v, s, k, m, n \in R$ and $t_1, t_2, t_3 \in K$.

$$\begin{aligned} a \cdot (b + c) &= a \cdot \{((h_1, h_2I_1, h_3I_2), t_4) : h_1 \in u +_1 k, h_2 \in v +_1 m, h_3 \in s +_1 n, t_4 \in t_2 +_2 t_3\} \\ &= \{((x, yI_1, zI_2), t_1) \cdot ((h_1, h_2I_1, h_3I_2), t_4) : h_1 \in u +_1 k, h_2 \in v +_1 m, h_3 \in s +_1 n, \\ &\quad t_4 \in t_2 +_2 t_3\} \\ &= \{((p_1, p_2I_1, p_3I_2), t_5) : p_1 \in x \cdot_1 h_1, p_2 \in x \cdot_1 h_2 +_1 y \cdot_1 h_1 +_1 y \cdot_1 h_2 +_1 y \cdot_1 h_3 \\ &\quad +_1 z \cdot_1 h_2, p_3 \in x \cdot_1 h_3 +_1 z \cdot_1 h_1 +_1 z \cdot_1 h_3, t_5 \in t_1 \cdot_2 t_4\} \\ &= \{((p_1, p_2I_1, p_3I_2), t_5) : p_1 \in x \cdot_1 u +_1 x \cdot_1 k, \\ &\quad p_2 \in x \cdot_1 v +_1 x \cdot_1 m +_1 y \cdot_1 u +_1 y \cdot_1 k +_1 y \cdot_1 v +_1 y \cdot_1 m +_1 y \cdot_1 s +_1 y \cdot_1 n +_1 \\ &\quad z \cdot_1 v +_1 z \cdot_1 m,\ p_3 \in x \cdot_1 s +_1 x \cdot_1 n +_1 z \cdot_1 u +_1 z \cdot_1 k +_1 z \cdot_1 s +_1 z \cdot_1 n, \\ &\quad t_5 \in t_1 \cdot_2 t_2 +_2 t_1 \cdot_2 t_3\}. \end{aligned}$$

Now if we take $p_1 = g_1 +_1 g_1',\ p_2 = g_2 +_1 g_2',\ p_3 = g_3 +_1 g_3',\ t_5 = h_1 +_2 h_1'$ then we have

$$\begin{aligned} a \cdot (b + c) &= \{((g_1 +_1 g_1', (g_2 +_1 g_2')I_1, (g_3 +_1 g_3')I_2), (h_1 +_2 h_1')) : g_1 +_1 g_1' \in x \cdot_1 u +_1 x \cdot_1 k, \\ &\quad g_2 +_1 g_2' \in x \cdot_1 v +_1 x \cdot_1 m +_1 y \cdot_1 u +_1 y \cdot_1 k +_1 y \cdot_1 v +_1 y \cdot_1 m +_1 y \cdot_1 s +_1 y \cdot_1 n +_1 \\ &\quad z \cdot_1 v +_1 z \cdot_1 m,\ g_3 +_1 g_3' \in x \cdot_1 s +_1 x \cdot_1 n +_1 z \cdot_1 u +_1 z \cdot_1 k +_1 z \cdot_1 s +_1 z \cdot_1 n, \\ &\quad h_1 +_2 h_1' \in t_1 \cdot_2 t_2 +_2 t_1 \cdot_2 t_3\} \\ &= \{((g_1, g_2I_1, g_3I_2), h_1) : g_1 \in x \cdot_1 u, g_2 \in x \cdot_1 v +_1 y \cdot_1 u +_1 y \cdot_1 v +_1 y \cdot_1 s +_1 z \cdot_1 v, \\ &\quad g_3 \in x \cdot_1 s +_1 z \cdot_1 u +_1 z \cdot_1 s, h_1 \in t_1 \cdot_2 t_2\} + \\ &\quad \{((g_1', g_2'I_1, g_3'I_2), h_1') : g_1' \in x \cdot_1 k, g_2' \in x \cdot_1 m +_1 y \cdot_1 k +_1 y \cdot_1 m +_1 y \cdot_1 n +_1 z \cdot_1 m, \\ &\quad g_3' \in x \cdot_1 n +_1 z \cdot_1 k +_1 z \cdot_1 n, h_1' \in t_1 \cdot_2 t_3\} \\ &= a \cdot b + a \cdot c. \end{aligned}$$

Similarly, we can show that $(b + c) \cdot a = b \cdot a + c \cdot a$.

Therefore $\cdot$ is distributive over $+$. Hence $(R(I_1, I_2), \times K, +, \cdot)$ is a refined neutrosophic Hyperring. □

Proposition 3.9. *Let $(R(I_1,I_2),+_1,\cdot_1)$ and $(K(I_1,I_2),+_2,\cdot_2)$ be any two refined neutrosophic hyperring. Define for all $(x_1,k_1),(x_2,k_2)\in R(I_1,I_2)\times K(I_1,I_2)$ the hyperoperations $''+''$ and $''\cdot''$ by*

$$(x_1,k_1)+(x_2,k_2)=\{(x_3,k_3):x_3\in x_1+_1x_2,\ k_3\in k_1+_2k_2\}$$

and

$$(x_1,k_1)\cdot(x_2,k_2)=\{(x_3,k_3):x_3\in x_1\cdot_1x_2,\ k_3\in k_1\cdot_2k_2\}.$$

Then $(R(I_1,I_2)\times K(I_1,I_2),+,\cdot)$ is a refined neutrosophic hyperring.

Proof. The proof is similar to the proof of Proposition 3.8. □

Lemma 3.10. *Let $R(I_1,I_2)$ be a refined neutrosophic hyperring. A non-empty subset $M(I_1,I_2)$ of $R(I_1,I_2)$ is a left(right) refined neutrosophic hyperideal if and only if for $m_1=(p_1,q_1I_1,s_1I_1),m_2=(p_2,q_2I_1,s_2I_1)\in M(I_1,I_2)$ and $x=(a,bI_1,cI_2)\in R(I_1,I_2)$*

(1) $m_1-m_2\subseteq M(I_1,I_2)$,

(2) $x\cdot m_1\in M(I_1,I_2)$ $[m_1\cdot x\in M(I_1,I_2)]$.

Definition 3.11. Let $H(I_1,I_2)$ and $J(I_1,I_2)$ be any two nonempty subsets of a refined neutrosophic hyperring $R(I_1,I_2)$.

(1) The sum $H(I_1,I_2)+J(I_1,I_2)=\{(x,yI_1,zI_2):x\in x_1+x_2,\ y\in y_1+y_2,\ z\in z_1+z_2\}$.
For some $x_1,y_1,z_1\in H,\ x_2,y_2,z_2\in J$.

(2) The product

$$H(I_1,I_2)J(I_1,I_2)=\{(x,yI_1,zI_2):(x,yI_1,zI_2)\in\sum_{i=1}^{n}(a_i,b_iI_1,c_iI_2)\cdot(d_i,e_iI_1,f_iI_1),\ n\in\mathbb{Z}^+\}.$$

Proposition 3.12. *Let $R(I_1,I_2)$ be a refined neutrosophic hyperring. Let $H(I_1,I_2)$ and $J(I_1,I_2)$ be refined neutrosophic hyperideals of $R(I_1,I_2)$ then :*

(1) *$H(I_1,I_2)+J(I_1,I_2)$ is a refined neutrosophic hyperideal.*

(2) *$H(I_1,I_2)J(I_1,I_2)$ is a refined neutrosophic hyperideal.*

Proof. (1) Let $x=(a,bI_1,cI_2),y=(d,eI_1,fI_2)\in H(I_1,I_2)+J(I_1,I_2)$ and let $r=(g,hI_1,kI_2)\in R(I_1,I_2)$.

$$\begin{aligned}(i)\ x-y &= (a,bI_1,cI_2)-(d,eI_1,fI_2)=(a,bI_1,cI_2)+(-d,-eI_1,-fI_2)\\
&= \{(p,qI_1,rI_2):p\in a+(-d),q\in b+(-e),\ r\in c+(-f)\}\\
&= \{(p_1+p_2,(q_1+q_2)I_1,(r_1+r_2)I_2):p_1+p_2\in(a_1+a_2)+(-d_1+(-d_2)),\\
&\quad q_1+q_2\in(b_1+b_2)+(-e_1+(-e_2)),\ r_1+r_2\in(c_1+c_2)+(-f_1+(-f_2))\}\\
&= \{(p_1,q_1I_1,r_1I_2):p_1\in a_1+(-d_1),\ q_1\in b_1+(-e_1),\ r_1\in c_1+(-f_1)\}+\\
&\quad \{(p_2,q_2I_1,r_2I_2):p_2\in a_2+(-d_2),\ q_2\in b_2+(-e_2),\ r_2\in c_2+(-f_2)\}\\
&= \{(p_1,q_1I_1,r_1I_2):p_1\in a_1-d_1,\ q_1\in b_1-e_1,r_1\in c_1-f_1\}+\\
&\quad \{(p_2,q_2I_1,r_2I_2):p_2\in a_2-d_2,\ q_2\in b_2-e_2,r_2\in c_2-f_2\}\\
&= (x_1-y_1)+(x_2-y_2)\\
&\subseteq H(I_1,I_2)+J(I_1,I_2).\end{aligned}$$

$$
\begin{aligned}
(ii)\ r\cdot x &= (g,hI_1,kI_2)\cdot(a,bI_1,cI_2)\\
&= \{(u,vI_1,mI_2): u\in ga,\ v\in gb+ha+hb+hc+kb,\ m\in gc+ka+kc\}\\
&= \{(u_1+u_2,(v_1+v_2)I_1,(m_1+m_2)I_2): u_1+u_2\in g(a_1+a_2),\\
&\quad v_1+v_2\in g(b_1+b_2)+h(a_1+a_2)+h(b_1+b_2)+h(c_1+c_2)+k(b_1+b_2),\\
&\quad m_1+m_2\in g(c_1+c_2)+k(a_1+a_2)+k(c_1+c_2)\}\\
&= \{(u_1,v_1I_1,m_1I_2): u_1\in ga_1, v_1\in gb_1+ha_1+hb_1+hc_1+kb_1, m\in gc_1+ka_1+kc_1\}+\\
&\quad \{(u_2,v_2I_1,m_2I_2): u_2\in ga_2, v_2\in gb_2+ha_2+hb_2+hc_2+kb_2, m_2\in gc_2+ka_2+kc_2\}\\
&= r\cdot x_1+r\cdot x_2\\
&\subseteq H(I_1,I_2)+J(I_1,I_2).
\end{aligned}
$$

$$
\begin{aligned}
(iii)\ x\cdot r &= (a,bI_1,cI_2)\cdot(g,hI_1,kI_2)\\
&= \{(u,vI_1,mI_2): u\in ag,\ v\in ah+bg+bh+bk+ch,\ m\in ak+cg+ck\}\\
&= \{(u_1+u_2,(v_1+v_2)I_1,(m_1+m_2)I_2): u_1+u_2\in (a_1+a_2)g,\\
&\quad v_1+v_2\in (a_1+a_2)h+(b_1+b_2)g+(b_1+b_2)h+(b_1+b_2)k+(c_1+c_2)h,\\
&\quad m_1+m_2\in (a_1+a_2)k+(c_1+c_2)g+(c_1+c_2)k\}\\
&= \{(u_1,v_1I_1,m_1I_2): u_1\in a_1g, v_1\in a_1h+b_1g+b_1h+b_1k+c_1h, m_1\in a_1k+c_1g+c_1k\}+\\
&\quad \{(u_2,v_2I_1,m_2I_2): u_2\in a_2g, v_2\in a_2h+b_2g+b_2h+b_2k+c_2h, m_2\in a_2k+c_2g+c_2k\}\\
&= x_1\cdot r+x_2\cdot r\\
&\subseteq H(I_1,I_2)+J(I_1,I_2).
\end{aligned}
$$

(2) Let $x=(a,bI_1,cI_2), y=(d,eI_1,fI_2)\in H(I_1,I_2)J(I_1,I_2)$ and let $r=(g,hI_1,kI_2)\in R(I_1,I_2)$.

Here

$$(a,bI_1,cI_2)\in\sum_{i=1}^{n}(a_i,b_iI,c_iI)\cdot(a_i',b_i'I_1,c_i'I_2) \text{ and } (d,eI_1,fI_2)\in\sum_{i=1}^{n}(d_i,e_iI_1,f_iI_2)\cdot(d_i',e_i'I_1,f_i'I_2).$$

For $(a_i,b_iI_1,c_iI_2),(d_i,e_iI_1,f_iI_2)\in H(I_1,I_2)$, $(a_i',b_i'I_1,c_i'I_2),(d_i',e_i'I_1,f_i'I_2)\in J(I_1,I_2)$, $a_i,b_i,c_i,d_i,e_i,f_i\in H$ and $a_i',b_i',c_i',d_i',e_i',f_i'\in J$.

So we have

$$a\in\sum_{i=1}^{n}a_ia_i',\ b\in\sum_{i=1}^{n}(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i'),\ c\in\sum_{i=1}^{n}(a_ic_i'+c_ia_i'+c_ic_i')$$

and

$$d\in\sum_{i=1}^{n}d_id_i',\ e\in\sum_{i=1}^{n}(d_ie_i'+e_id_i'+e_ie_i'+e_if_i'+f_ie_i'),\ f\in\sum_{i=1}^{n}(d_if_i'+f_id_i'+f_if_i').$$

$$
\begin{aligned}
(i)\ x-y &= (a,bI_1,cI_2)-(d,eI_1,fI_2)=(a,bI_1,cI_2)+(-d,-eI_1,-fI_2)\\
&= \{(u,vI_1,mI_2): u\in a-d,\ v\in b-e,\ m\in c-f\}\\
&= \{(u,vI_1,mI_2): u\in \textstyle\sum_{i=1}^{n}a_ia_i'-\sum_{i=1}^{n}d_id_i',\\
&\quad v\in\textstyle\sum_{i=1}^{n}(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')-\sum_{i=1}^{n}(d_ie_i'+e_id_i'+e_ie_i'+e_if_i'+f_ie_i'),\\
&\quad m\in\textstyle\sum_{i=1}^{n}(a_ic_i'+c_ia_i'+c_ic_i')-\sum_{i=1}^{n}(d_if_i'+f_id_i'+f_if_i')\}\\
&= \{(u,vI_1,mI_2): u\in\textstyle\sum_{i=1}^{n}(a_ia_i'+(-d_id_i')),\\
&\quad v\in\textstyle\sum_{i=1}^{n}(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i'+(-d_ie_i')+(-e_id_i')+(-e_ie_i')+(-e_if_i')\\
&\quad +(-f_ie_i')),\ m\in\textstyle\sum_{i=1}^{n}(a_ic_i'+c_ia_i'+c_ic_i'+(-d_if_i')+(-f_id_i')+(-f_if_i'))\}\\
&\subseteq H(I_1,I_2)J(I_1,I_2).
\end{aligned}
$$

$$\begin{aligned}
(ii)\ r\cdot x &= (g,hI_1,kI_2)\cdot(a,bI_1,cI_2)\\
&= \{(u,vI_1,mI_2): u\in ga,\ v\in gb+ha+hb+hc+kb,\ m\in gc+ka+kc\}\\
&= \{(u,vI_1,mI_2): u\in g\textstyle\sum_{i=1}^n a_ia_i',\\
&\quad v\in g\textstyle\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')+h\sum_{i=1}^n a_ia_i'+\\
&\quad h\textstyle\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')+h\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')+\\
&\quad k\textstyle\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i'),\\
&\quad m\in g\textstyle\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')+k\sum_{i=1}^n a_ia_i'+k\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')\}\\
&= \{(u,vI_1,mI_2): u\in\textstyle\sum_{i=1}^n ga_ia_i',\\
&\quad v\in\textstyle\sum_{i=1}^n(ga_ib_i'+gb_ia_i'+gb_ib_i'+gb_ic_i'+gc_ib_i'+ha_ia_i'+ha_ib_i'+hb_ia_i'+hb_ib_i'+\\
&\quad hb_ic_i'+hc_ib_i'+ha_ic_i'+hc_ia_i'+hc_ic_i'+ka_ib_i'+kb_ia_i'+kb_ib_i'+kb_ic_i'+kc_ib_i'),\\
&\quad m\in\textstyle\sum_{i=1}^n(ga_ic_i'+gc_ia_i'+gc_ic_i'+ka_ia_i'+ka_ic_i'+kc_ia_i'+kc_ic_i')\}\\
&\subseteq H(I_1,I_2)J(I_1,I_2).
\end{aligned}$$

$$\begin{aligned}
(ii)\ x\cdot r &= (a,bI_1,cI_2)\cdot(g,hI_1,kI_2)\\
&= \{(u,vI_1,mI_2): u\in ag,\ v\in ah+bg+bh+bk+ch,\ m\in ak+cg+ck\}\\
&= \{(u,vI_1,mI_2): u\in\textstyle\sum_{i=1}^n a_ia_i'g,\\
&\quad v\in\textstyle\sum_{i=1}^n a_ia_i'h+\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')g+\\
&\quad \textstyle\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')h+\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')k+\\
&\quad \textstyle\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')h,\\
&\quad m\in\textstyle\sum_{i=1}^n a_ia_i'k+\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')g+\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')k\}\\
&= \{(u,vI_1,mI_2): u\in\textstyle\sum_{i=1}^n a_ia_i'g,\\
&\quad v\in\textstyle\sum_{i=1}^n(a_ia_i'h+a_ib_i'g+b_ia_i'g+b_ib_i'g+b_ic_i'g+c_ib_i'g+a_ib_i'h+b_ia_i'h+b_ib_i'h+\\
&\quad b_ic_i'h+c_ib_i'h+a_ib_i'k+b_ia_i'k+b_ib_i'k+b_ic_i'k+c_ib_i'k+a_ic_i'h+c_ia_i'h+c_ic_i'h),\\
&\quad m\in\textstyle\sum_{i=1}^n(a_ia_i'k+a_ic_i'g+c_ia_i'g+c_ic_i'g+a_ic_i'k+c_ia_i'k+c_ic_i'k)\}\\
&\subseteq H(I_1,I_2)J(I_1,I_2).
\end{aligned}$$

Hence $H(I_1,I_2)J(I_1,I_2)$ is a refined neutrosophic hyperideal of $R(I_1,I_2)$. □

Proposition 3.13. *Let $R(I_1,I_2)$ be a refined neutrosophic hyperrings and $J_i(I_1,I_2)_{i\in\Lambda}$ be a family of refined neutrosophic hyperideals of $R(I_1,I_2)$, then $\bigcap_{i\in\Lambda} J_i(I_1,I_2)$ is a refined neutrosophic hyperideal of $R(I_1,I_2)$.*

Proof. The proof is the same as the proof in classical case. □

Proposition 3.14. *Let $H(I_1,I_2)$ and $J(I_1,I_2)$ be a refined neutrosophic hyperideals of a refined neutrosophic hyperring $R(I_1,I_2)$ such that $J(I_1,I_2)$ is normal in $R(I_1,I_2)$. Then*

(1) *$H(I_1,I_2)\cap J(I_1,I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1,I_2)$.*

(2) *$J(I_1,I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1,I_2)+J(I_1,I_2)$.*

Proof. (1) That $H(I_1,I_2)\cap J(I_1,I_2)$ is a refined neutrosophic hyperideal of $H(I_1,I_2)$ can be easily established. So, it remains to show that $H(I_1,I_2)\cap J(I_1,I_2)$ is normal in $H(I_1,I_2)$.

Let $x = (a, bI_1, cI_2) \in H(I_1, I_2) \cap J(I_1, I_2)$, $h = (u, vI_1, tI_2) \in H(I_1, I_2)$ with $a, b, c \in H \cap J$ and $u, v, t \in H$. Then

$$\begin{aligned}
h + H(I_1, I_2) \cap J(I_1, I_2) - h &= h + x - h \ for \ x \in H(I_1, I_2) \cap J(I_1, I_2) \\
&= (u, vI_1, tI_2) + (a, bI_1, cI_2) - (u, vI_1, tI_2) \\
&= \{(p, qI_1, rI_2) : p \in u + a - u, \ q \in v + b - v, \ r \in t + c - t\} \\
&= \{(p, qI_1, rI_2) : p \in u + (H \cap J) - u, \ q \in v + (H \cap J) - v, \\
&\quad r \in t + (H \cap J) - t\} \\
&= \{(p, qI_1, rI_2) : p \in u + (H \cap J) - u \subseteq H \cap J, \\
&\quad q \in v + (H \cap J) - v \subseteq H \cap J, \ r \in t + (H \cap J) - t \subseteq H \cap J\} \\
&= \{(p, qI_1, rI_2) : p \in H \cap J, \ q \in H \cap J, \ r \in H \cap J\} \\
&\subseteq H(I_1, I_2) \cap J(I_1, I_2).
\end{aligned}$$

Accordingly, $H(I_1, I_2) \cap J(I_1, I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1, I_2)$.

(2) That $J(I_1, I_2)$ is a refined neutrosophic hyperideal of $H(I_1, I_2) + J(I_1, I_2)$ can be easily established. So, it remains to show that $J(I_1, I_2)$ is normal in $H(I_1, I_2) + J(I_1, I_2)$. Let $x = (a, bI_1, cI_2) \in J(I_1, I_2)$, $h = (u, vI_1, tI_2) = (u_1 + u_2, (v_1 + v_2)I_1, (t_1 + t_2)I_2) \in H(I_1, I_2) + J(I_1, I_2)$ with $a, b, c, u_2, v_2, t_2 \in J$ and $u_1, v_1, t_2 \in H$. Then

$$\begin{aligned}
h + J(I_1, I_2) - h &= h + x - h \ for \ x \in J(I_1, I_2) \\
&= (u, vI_1, tI_2) + (a, bI_1, cI_2) - (u, vI_1, tI_2) \\
&= ((u_1 + u_2), (v_1 + v_2)I_1, (t_1 + t_2)I_2) + (a, bI_1, cI_2) \\
&\quad -((u_1 + u_2), (v_1 + v_2)I_1, (t_1 + t_2)I_2) \\
&= \{(p, qI_1, rI_2) : p \in (u_1 + u_2) + a - (u_1 + u_2), \ q \in (v_1 + v_2) + b - (v_1 + v_2), \\
&\quad r \in (t_1 + t_2) + c - (t_1 + t_2) \\
&= \{(p, qI_1, rI_2) : p \in (u_1 + u_2) + J - (u_1 + u_2), \ q \in (v_1 + v_2) + J - (v_1 + v_2), \\
&\quad r \in (t_1 + t_2) + J - (t_1 + t_2)\} \\
&= \{(p, qI_1, rI_2) : p \in u_1 + (u_2 + J - u_2) - u_1, \ q \in v_1 + (v_2 + J - v_2) - v_1, \\
&\quad r \in t_1 + (t_2 + J - t_2) - t_1\} \\
&\subseteq \{(p, qI_1, rI_2) : p \in u_1 + J - u_1, \ q \in v_1 + J - v_1, r \in t_1 + J - t_1\} \\
&= \{(p, qI_1, rI_2) : p \in u_1 + J - u_1 \subseteq J, \ q \in v_1 + J - v_1 \subseteq J, \ r \in t_1 + J - t_1 \subseteq J\} \\
&= \{(p, qI_1, rI_2) : p \in J, \ q \in J, \ r \in J\} \\
&\subseteq J(I_1, I_2).
\end{aligned}$$

Accordingly, $J(I_1, I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1, I_2) + J(I_1, I_2)$. □

Let $R(I_1, I_2)$ be a refined neutrosophic hyperring, and let $H(I_1, I_2)$ be a refined neutrosophic hyperideal of $R(I_1, I_2)$. Since $H(I_1, I_2)$ is a refined neutrosophic subcanonical hypergroup of $R(I_1, I_2)$, if $(R/H, +)$ is a canonical hypergroup then

$$R(I_1, I_2)/H(I_1, I_2) = \{\bar(x, yI_1, zI_2) : (x, yI_1, zI_2) \in R(I_1, I_2)\}$$

is a refined neutrosophic canonical hypergroup under the hyperaddition $+'$ defined for $r_1 + H(I_1, I_2), r_2 + H(I_1, I_2) \in R(I_1, I_2)/H(I_1, I_2)$ with $r_1 = (x_1, y_1 I_1, z_i I_2), r_2 = (x_2, y_2 I_1, z_2 I_2)$, by

$$r_1 + H(I_1, I_2) +' r_2 + H(I_1, I_2) = (r_1 +' r_2) + H(I_1, I_2).$$

Define on $R(I_1, I_2)/H(I_1, I_2)$ a hypermultiplication $\cdot'$ by

$$r_1 + H(I_1, I_2) \cdot' r_2 + H(I_1, I_2) = (r_1 r_2) + H(I_1, I_2).$$

It can be shown that $(R(I_1, I_2)/H(I_1, I_2), +', \cdot')$ is a refined neutrosophic hyperring if $(R/H, +, \cdot)$ is a hyperring.

Definition 3.15. Let $(R(I_1, I_2), +_1, \cdot_1)$ and $(P(I_1, I_2), +_2, \cdot_2)$ be any two refined neutrosophic hypergrings and let

$$\phi : R(I_1, I_2) \longrightarrow P(I_1, I_2)$$

be a mapping from $R(I_1, I_2)$ into $P(I_1, I_2)$.

(1) ϕ is called a refined neutrosophic hyperring homomorphism if:
  (a) ϕ is hyperring homomorphism,
  (b) $\phi(I_k) = I_k$ for $k = 1, 2$.
(2) ϕ is called a good refined neutrosophic hyperring homomorphism if:
  (a) ϕ is good hyperring homomorphism,
  (b) $\phi(I_k) = I_k$ for $k = 1, 2$.
(3) ϕ is called a refined neutrosophic hyperring isomorphism if ϕ is a refined neutrosophic hyperring homomorphism and ϕ^{-1} is also a refined neutrosophic hyperring homomorphism.

Definition 3.16. Let $\phi : R(I_1, I_2) \longrightarrow P(I_1, I_2)$ be a refined neutrosophic hyperring homomorphism from a refined neutrosophic hyperring $R(I_1, I_2)$ into a refined neutrosophic hyperring $P(I_1, I_2)$.

(1) The $Ker\phi = \{(u, vI_1, wI_2) \in R(I_1, I_2) : \phi((u, vI_1, wI_2)) = (0, 0I_1, 0I_2)\}$.
(2) The $Im\phi = \{\phi((u, vI_1, wI_2)) : (u, vI_1, wI_2) \in R(I_1, I_2)\}$.

Proposition 3.17. *Let $\phi : R(I_1, I_2) \longrightarrow P(I_1, I_2)$ be a refined neutrosophic homomorphism.*

(1) *The kernel of ϕ is not a neutrosophic subhyperring of $R(I_1, I_2)$.*
(2) *The kernel of ϕ is not a neutrosophic hyper ideal of $R(I_1, I_2)$.*
(3) *The image of ϕ is a neutrosophic subhyperring of $P(I_1, I_2)$.*

Proof. (1) It follows easily from 1 of definition 3.16.
(2) It follows from the Proof of 1.
(3) The proof is similar to the proof in classical case.

It can be shown that $ker\phi$ is just a subhyperrings of $R(I_1, I_2)$. □

4. Conclusions

This paper studied the refinement of a type of neutrosophic hyperrings in which $''+''$ and $''\cdot''$ are hyperoperations and presented their basic properties. It was established that every refined neutrosophic hyperring is a hyperring. It was also shown that the kernel of a refined neutrosophic hyperring homomorphism is not a refined neutrosophic hyperideal but the image is a refined neutrosophic subhyperring.

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Neutrosophic Tri-Topological Space

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Abstract: In this article, we present the notion of neutrosophic tri-topological space as a generalization of neutrosophic bi-topological space. Besides, we study the different types of open sets and closed sets namely neutrosophic tri-open sets, neutrosophic tri-closed sets, neutrosophic tri-semi-open sets, neutrosophic tri-pre-closed sets, etc. via neutrosophic tri-topological spaces. Further, we investigate several properties, and prove some propositions, theorems on neutrosophic tri-topological spaces.

Keywords: Tri-open set; Tri-closed set; Tri-semi-open set; Tri-pre-open set; Neutrosophic crisp tri-topology; Neutrosophic tri-topology.

1. Introduction

The concept of Neutrosophic Set (NS) was grounded by Smarandache [1] by extending the concept of Fuzzy Set 2[] and intuitionistic FS [3]. The notion of Neutrosophic Topological Space (NTS) was developed by Salama and Alblowi [4] in 2012. Afterwards, Arokiarani et al. [5] studied the neutrosophic semi-open functions and established a relation between them. Iswaraya and Bageerathi [6] introduced the notion of neutrosophic semi-closed set and neutrosophic semi-open set via NTSs. Later on, Dhavaseelan, and Jafari [7] introduced the generalized neutrosophic closed sets. Thereafter, Pushpalatha and Nandhini [8] studied the neutrosophic generalized closed sets in NTS. Shanthi et al. [9] introduced the concept of neutrosophic generalized semi closed sets in NTS. Ebenanjar et al. [10] presented the neutrosophic *b*-open sets in NTS. Maheswari et al. [11] introduced the concept of neutrosophic generalized *b*-closed sets in NTS. Afterwards, the concept of generalized neutrosophic *b*-open set via NTS was introduced by Das and Pramanik [12] in 2020. Thereafter, the concept of neutrosophic Φ-open sets and neutrosophic Φ-continuous functions was presented by Das and Pramanik [13].

The notion of neutrosophic crisp topology on neutrosophic crisp set was introduced by Salama and Alblowi [14]. Later on, the notion of neutrosophic crisp tri-topological space was introduced by Al-Hamido and Gharibah [15] in 2018.

In 1963, Kelly [16] introduced the notion of bi-topological space. Thereafter, the concept of neutrosophic bi-topological space was presented by Ozturk and Ozkan [17] in 2019. Later on, Das and Tripathy [18] introduced the pairwise neutrosophic *b*-open sets via neutrosophic bi-topological spaces. Recently, Tripathy and Das [19] studied the concept of pairwise neutrosophic *b*-continuous functions via neutrosophic bi-topological spaces.

So, we received enough motivation to do research on neutrosophic tri-topological space to extend the concept of neutrosophic bi-topological space.

In this study, we procure the notion of neutrosophic tri-topological space as a generalization of the neutrosophic bi-topological space. Besides, we introduce the different types of open sets and closed sets namely, neutrosophic tri-open sets, neutrosophic tri-closed sets, neutrosophic tri-semi-open sets, neutrosophic tri-pre-closed sets, etc. via neutrosophic tri-topological spaces. Further, we investigate several properties of these kinds of neutrosophic tri-open sets.

Research Gap: No investigation on neutrosophic tri-topological space has been reported in the recent literature.

Motivation: To reduce the research gap, we present the notion and different properties of neutrosophic tri-topological space.

The remaining part of this article is divided into the following sections:

Section-2 is on preliminaries and definitions. In this section, we give some definitions and theorems, which are relevant to this article. In section-3, we present the notion of neutrosophic tri-topology and neutrosophic tri-topological space and also we give proofs of some theorems on neutrosophic tri-topological space. In section-4, we give the concluding remarks of the work done in the present article.

Throughout this article, we use the following short terms for the clarity of the presentation.

Short Terms	
Neutrosophic Set	NS
Neutrosophic Topology	NT
Neutrosophic Topological Space	NTS
Neutrosophic Open Set	N-O-S
Neutrosophic Closed Set	N-O-S
Neutrosophic Semi-Open	NSO
Neutrosophic Pre-Open	NPO
Neutrosophic Bi-Topological Space	NBTS
Neutrosophic Tri-Topological Space	N-Tri-TS

Neutrosophic Tri-Open Set	N-tri-OS
Neutrosophic Tri-Closed Set	N-tri-CS

2. Some Relevant Definitions:

Definition 2.1.[1] A neutrosophic set L over a universe of discourse Ψ is defined as follows:

$$L=\{(n, T_L(n), I_L(n), F_L(n)): n\in\Psi\},$$

where $T_L(n)$, $I_L(n)$, $F_L(n)$ ($\in]^{-}0,1^{+}[$) are respectively denotes the truth, indeterminacy and falsity membership values of $n\in\Psi$, and so $^{-}0 \le T_L(n) + I_L(n) + F_L(n) \le 3^{+}$ for all $n\in\Psi$.

Definition 2.2.[1] The neutrosophic null set (0_N) and neutrosophic whole set (1_N) over a universe of discourse Ψ are defined as follows:

(*i*) $0_N =\{(n,0,0,1): n\in\Psi\}$;

(*ii*) $1_N =\{(n,1,0,0): n\in\Psi\}$.

Obviously, $0_N \subseteq 1_N$.

Definition 2.3.[20] Assume that Ψ be a universe of discourse. Then, a neutrosophic crisp set Q is defined by $Q=\{Q_1, Q_2, Q_3\}$, where Q_i (i=1,2,3) is a subset of Ψ such that $Q_i \cap Q_j =\phi$ (i, j= 1,2,3 and $i\neq j$)

Definition 2.4.[4] Assume that Ψ be a universe of discourse, and τ be a set of some NSs over Ψ. Then, τ is called a Neutrosophic Topology (NT) on Ψ if the following axioms hold:

(*i*) $0_N, 1_N \in \tau$;

(*ii*) $X_1, X_2 \in\tau \Rightarrow X_1\cap X_2 \in\tau$;

(*iii*) $\{X_i : i\in\Delta\} \subseteq \tau \Rightarrow \cup X_i \in\tau$.

The pair (Ψ, τ) is said to be an NTS. If $X\in\tau$, then X is called a neutrosophic-open-set (N-O-S) and its complement X^c is called a neutrosophic-closed-set (N-C-S).

Definition 2.5.[17] Assume that (Ψ, τ_1) and (Ψ, τ_2) be any two different NTSs. Then, we can call the triplet (Ψ, τ_1, τ_2) as a Neutrosophic Bi-Topological Space (NBTS).

Definition 2.6.[17] Assume that (Ψ, τ_1, τ_2) be an NBTS. Then, a neutrosophic subset X of Ψ is said to be a pairwise neutrosophic open set in (Ψ, τ_1, τ_2) if there exists an N-O-S T_1 in (Ψ, τ_1) and an N-O-S T_2 in (Ψ, τ_2) such that $X = T_1\cup T_2$.

Theorem 3.1.[18] Let (Ψ, τ_1, τ_2) be an NBTS. Then, a neutrosophic subset X of Ψ is called as

(*i*) τ_{ij} –neutrosophic-semi-open if and only if $X\subseteq N_{cl}^{i}N_{int}^{j}(X)$;

(*ii*) τ_{ij} –neutrosophic-pre-open if and only if $X\subseteq N_{int}^{j}N_{cl}^{i}(X)$;

(*iii*) τ_{ij} –neutrosophic-*b*-open if and only if $X\subseteq N_{cl}^{i}N_{int}^{j}(X) \cup N_{int}^{j}N_{cl}^{i}(X)$.

Theorem 3.2.[18] Assume that (Ψ, τ_1, τ_2) be an NBTS. Then, every τ_{ij}-neutrosophic-pre-open (τ_{ij}-neutrosophic-semi-open) set is a τ_{ij}-neutrosophic-*b*-open.

Definition 2.7.[18] Assume that (Ψ, τ_1, τ_2) be an NBTS. Then X, an NS over Ψ is said to be a

(*i*) pairwise τ_{ij}-neutrosophic-semi-open set (pairwise τ_{ij}-neutrosophic-pre-open set) in an NBTS (Ψ, τ_1, τ_2) if and only if $X=T\cup K$, where T is a τ_{ij}-neutrosophic-semi-open set (τ_{ij}-neutrosophic-pre-open set) and K is a τ_{ji}-neutrosophic-semi-open set (τ_{ji}-neutrosophic-pre-open set) in (Ψ, τ_1, τ_2).

(*ii*) pairwise τ_{ij}-neutrosophic-*b*-open set in an NBTS (Ψ, τ_1, τ_2) if $X=T\cup K$, where T is a τ_{ij}-neutrosophic-*b*-open set and K is a τ_{ji}-neutrosophic-*b*-open set in (Ψ, τ_1, τ_2).

Theorem 3.3.[18] Assume that (Ψ, τ_1, τ_2) be an NBTS. If X is a pairwise τ_{ij}-neutrosophic-semi open (pairwise τ_{ij}-neutrosophic-pre-open) set, then it is also a pairwise τ_{ij}-neutrosophic-*b*-open set.

Theorem 3.4.[18] In an NBTS (Ψ, τ_1, τ_2), the union of any two pairwise τ_{ij}-neutrosophic-*b*-open sets is a pairwise τ_{ij}-neutrosophic-*b*-open set.

Definition 2.8.[15] Assume that τ_1, τ_2 and τ_3 be any three neutrosophic crisp topology on a universe of discourse *W*. Then, (*W*, τ_1, τ_2, τ_3) is called a neutrosophic crisp tri-topological space.

3. Neutrosophic Tri-Topological Space:

In this section, we introduce the notion of neutrosophic tri-topological space, and formulate several results on it.

Definition 3.1. Suppose that (Ψ, τ_1), (Ψ, τ_2) and (Ψ, τ_3) be any three different NTSs. Then, the structure (Ψ, τ_1, τ_2, τ_3) is called a neutrosophic tri-topological space (N-Tri-TS).

Example 3.1. Let Ψ={*u*, *v*, *w*} be a universe of discourse. Let X_1, X_2, Y_1, Y_2, Y_3, Z_1 and Z_2 be seven NSs over Ψ such that:

X_1={(*u*,0.9,0.3,0.7), (*v*,0.5,0.6,0.8), (*w*,0.3,0.5,0.2)};

X_2={(*u*,0.7,0.4,0.7), (*v*,0.5,0.9,0.9), (*w*,0.1,0.8,0.4)};

Y_1= {(*u*,0.9,0.3,0.7), (*v*,0.5,0.6,0.8), (*w*,0.3,0.5,0.2)};

Y_2= {(*u*,1.0,0.2,0.5), (*v*,0.9,0.5,0.8), (*w*,0.5,0.2,0.1)};

Y_3= {(*u*,1.0,0.2,0.3), (*v*,1.0,0.1,0.8), (*w*,0.9,0.1,0.1)};

Z_1= {(*u*,0.5,0.2,0.5), (*v*,0.9,0.4,0.3), (*w*,0.7,0.2,0.5)};

Z_2= {(*u*,0.4,0.5,0.6), (*v*,0.5,0.6,0.4), (*w*,0.7,0.6,0.6)};

Then, clearly τ_1={0_N, 1_N, X_1, X_2}, τ_2={0_N, 1_N, Y_1, Y_2, Y_3} and τ_3={0_N, 1_N, Z_1, Z_2} are three different NTs on Ψ. So, (Ψ, τ_1, τ_2, τ_3) is a neutrosophic tri-topological space.

Definition 3.2. Assume that (Ψ, τ_1, τ_2, τ_3) be an N-Tri-TS. Then *R*, an NS over Ψ is called a neutrosophic tri-open set (N-tri-OS) if $R\in\tau_1\cup\tau_2\cup\tau_3$. A neutrosophic set *R* is called a neutrosophic tri-closed set (N-Tri-CS) if and only if R^c is an N-Tri-OS in (Ψ, τ_1, τ_2, τ_3).

Remark 3.1. The collection of all neutrosophic tri-open sets and neutrosophic tri-closed sets in (Ψ, τ_1, τ_2, τ_3) are denoted by N-Tri-O(Ψ) and N-Tri-C(Ψ) respectively.

Theorem 3.1. Every N-O-S in (Ψ, τ_i), *i*=1,2,3 is a neutrosophic tri-open set in (Ψ, τ_1, τ_2, τ_3).

Proof. Assume that *W* be an N-O-S in (Ψ, τ_i), *i*=1,2,3. Therefore, $W\in\tau_i$, *i*=1,2,3. This implies, $W\in \bigcup_{i\in\{1,2,3\}}\tau_i$. Hence, *W* is a neutrosophic tri-open set in (Ψ, τ_1, τ_2, τ_3). Therefore, every N-O-S in (Ψ, τ_i), *i* =1,2,3 is a neutrosophic tri-open set in (Ψ, τ_1, τ_2, τ_3).

Example 3.2. Let us consider an N-Tri-TS (Ψ, τ_1, τ_2, τ_3) as shown in Example 3.1. Then, the neutrosophic open sets X_1, X_2 (in (Ψ, τ_1)), Y_1, Y_2, Y_3 (in (Ψ, τ_2)), Z_1, Z_2 (in (Ψ, τ_3)) are neutrosophic tri-open sets in (Ψ, τ_1, τ_2, τ_3).

Remark 3.2. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), the union of two neutrosophic tri-open sets may not be a neutrosophic tri-open set. This follows from the following example.

Example 3.3. Consider the N-Tri-TS (Ψ, τ_1, τ_2, τ_3) which has been shown in Example 3.1. Then, clearly Y_3 and Z_1 are two neutrosophic tri-open sets in (Ψ, τ_1, τ_2, τ_3). But their union $Y_3\cup Z_1$={(*u*,1.0,0.2,0.3),

(v,1.0,0.1,0.3), (w,0.9,0.1,0.1)} is not a neutrosophic tri-open set, because $Y_3 \cup Z_1 \notin \bigcup_{i\in\{1,2,3\}} \tau_i$. Hence, the union of two neutrosophic tri-open sets may not be a neutrosophic tri-open set.

Remark 3.3. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), the intersection of two neutrosophic tri-open sets may not be a neutrosophic tri-open set. This follows from the following example.

Example 3.4. Consider the N-Tri-TS (Ψ, τ_1, τ_2, τ_3) which is shown in Example 3.1. Then, clearly X_1 and Z_2 are two neutrosophic tri-open sets in (Ψ, τ_1, τ_2, τ_3). But their union $X_1 \cap Z_2$={(u,0.4,0.5,0.7), (v,0.5,0.6,0.8), (w,0.3,0.6,0.6)} is not a neutrosophic tri-open set, because $X_1 \cap Z_2 \notin \bigcup_{i\in\{1,2,3\}} \tau_i$. Hence, the intersection of two neutrosophic tri-open sets may not be a neutrosophic tri-open set.

Theorem 3.2. Every N-C-S in (Ψ, τ_i) (i=1,2,3) is a neutrosophic tri-closed set in (Ψ, τ_1, τ_2, τ_3).

Proof. Assume that W be an N-C-S in (Ψ, τ_i). So W^c is an N-O-S in (Ψ, τ_i) (i=1,2,3). Therefore, $W^c \in \tau_i$, i=1,2,3. This implies, $W^c \in \bigcup_{i\in\{1,2,3\}} \tau_i$. This implies, W^c is a neutrosophic tri-open set in (Ψ, τ_1, τ_2, τ_3). Therefore, W is a neutrosophic tri-closed set in (Ψ, τ_1, τ_2, τ_3). Hence, every N-C-S in (Ψ, τ_i), i=1,2,3 is a neutrosophic tri-open set in (Ψ, τ_1, τ_2, τ_3).

Example 3.5. Suppose that (Ψ, τ_1, τ_2, τ_3) be an N-Tri-TS as shown in Example 3.1. Clearly, X_1^c = {(u,0.1,0.7,0.3), (v,0.5,0.4,0.2), (w,0.7,0.5,0.8)} , X_2^c = {(u,0.3,0.6,0.3), (v,0.5,0.1,0.1), (w,0.9,0.2,0.6)} are NCSs in (Ψ, τ_1), Y_1^c = {(u,0.1,0.7,0.3), (v,0.1,0.5,0.2), (w,0.5,0.8,0.9)}, Y_2^c = {(u,0.0,0.8,0.5), (v,0.1,0.5,0.2), (w,0.5,0.8,0.9)}, Y_3^c = {(u,0.0,0.8,0.7), (v,0.0,0.9,0.2), (w,0.1,0.9,0.9)} are NCSs in (Ψ, τ_2), and Z_1^c = {(u,0.5,0.8,0.5), (v,0.1,0.6,0.7), (w,0.3,0.8,0.5)}, Z_2^c = {(u,0.6,0.5,0.4), (v,0.5,0.4,0.6), (w,0.3,0.4,0.4)} are NCSs in (Ψ, τ_3). Therefore, X_1, X_2 are NOSs in (Ψ, τ_1), Y_1, Y_2, Y_3 are NOSs in (Ψ, τ_2), and Z_1, Z_2 are NOSs in (Ψ, τ_3). By Example 3.2, X_1, X_2, Y_1, Y_2, Y_3, Z_1 and Z_2 are neutrosophic tri-open sets in (Ψ, τ_1, τ_2, τ_3). Therefore, by definition 3.2, X_1^c, X_2^c, Y_1^c, Y_2^c, Y_3^c, Z_1^c and Z_2^c are neutrosophic tri-closed sets in (Ψ, τ_1, τ_2, τ_3).

Definition 3.3. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), an NS G over Ψ is called a neutrosophic tri-semi-open set if G is a neutrosophic semi open set in at least one of three NTSs (Ψ, τ_1), (Ψ, τ_2), and (Ψ, τ_3).

Example 3.6. Consider the N-Tri-TS (Ψ, τ_1, τ_2, τ_3) which is shown in Example 3.1. Then, W={(u,1.0,0.2,0.5), (v,0.7,0.5,0.7), (w,0.9,0.8,0.3)} is a neutrosophic tri-semi-open set in (Ψ, τ_1, τ_2, τ_3), because W is a neutrosophic semi open set in (Ψ, τ_1).

Definition 3.4. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), a neutrosophic set G over Ψ is called a neutrosophic tri-pre-open set if G is a neutrosophic pre-open set in at least one of three NTSs (Ψ, τ_1), (Ψ, τ_2) and (Ψ, τ_3).

Example 3.7. Consider the N-Tri-TS (Ψ, τ_1, τ_2, τ_3) which is shown in Example 3.1. Then, W={(u,0.5,0.3,0.2), (v,0.6,0.3,0.2), (w,0.8,0.4,0.2)} is a neutrosophic tri-pre-open set in (Ψ, τ_1, τ_2, τ_3), because W is a neutrosophic pre- open set in (Ψ, τ_1).

Definition 3.5. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), an NS G over Ψ is called a neutrosophic tri-b-open set if G is a neutrosophic b-open set in at least one of three NTSs (Ψ, τ_1), (Ψ, τ_2), and (Ψ, τ_3).

Example 3.8. Consider the N-Tri-TS (Ψ, τ_1, τ_2, τ_3) which is shown in Example 3.1. Then, W={(u,0.9,0.3,0.6), (v,0.8,0.6,0.4), (w,0.9,0.9,0.9)} is a neutrosophic tri-b-open set in (Ψ, τ_1, τ_2, τ_3), because W is a neutrosophic b-open set in (Ψ, τ_1).

Remark 3.4. Assume that (Ψ, τ_1, τ_2, τ_3) be an N-Tri-TS. Let $\tau_{1,2,3}$= $\tau_1 \cup \tau_2 \cup \tau_3$. Then, $\tau_{1,2,3}$ may not be a neutrosophic topology on Ψ in general. This follows from the following example.

Example 3.9. Suppose that (Ψ, τ_1, τ_2, τ_3) is a neutrosophic tri-topological space, where τ_1={0_N, 1_N, {(u,0.6,0.3,0.6), (v,0.5,0.4,0.5), (w,0.8,0.5,0.8)}, {(u,0.7,0.1,0.4), (v,0.9,0.3,0.3), (w,0.9,0.1,0.5)}}, τ_2={0_N, 1_N, {(u,1.0,0.3,0.8), (v,0.8,0.4,0.7), (w,0.8,0.6,0.8)}, {(u,0.8,0.4,0.9), (v,0.5,0.5,1.0), (w,0.5,0.8,1.0)}}, τ_3={0_N, 1_N, {(u,0.5,0.2,0.5), (v,0.9,0.4,0.3), (w,0.7,0.2,0.5)}, {(u,0.4,0.5,0.6), (v,0.5,0.6,0.4), (w,0.7,0.6,0.6)}} are three different NTs on Ψ. Clearly, {(u,0.7,0.1,0.4), (v,0.9,0.3,0.3), (w,0.9,0.1,0.5)} and {(u,0.8,0.4,0.9), (v,0.5, 0.5,1.0), (w,0.5,0.8,1.0)} Ï $\tau_{1,2,3}$, but their intersection {(u,0.7,0.4,0.9), (v,0.5,0.5,1.0), (w,0.5,0.8,1.0)} Ï $\tau_{1,2,3}$. Hence, $\tau_{1,2,3}$ does not form a topology on Ψ.

Definition 3.6. Assume that (Ψ, τ_1, τ_2, τ_3) is an N-Tri-TS. Then R, an NS over Ψ is said to be a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3) if and only if there exist neutrosophic open sets R_1 in τ_1, R_2 in τ_2, and R_3 in τ_3 such that $R = R_1 \cup R_2 \cup R_3$.

Example 3.10. Consider the N-Tri-TS (Ψ, τ_1, τ_2, τ_3) as shown in Example 3.7. Then, W={(u,1.0,0.2,0.5), (v,0.9,0.4,0.3), (w,0.8,0.2,0.5)} is a neutrosophic tri-t-open set, since there exist NOSs R_1={(u,0.6,0.3,0.6), (v,0.5,0.4,0.5), (w,0.8,0.5,0.8)} in τ_1, R_2={(u,1.0,0.3,0.8), (v,0.8,0.4,0.7), (w,0.8,0.6,0.8)} in τ_2, and R_3={(u,0.5,0.2,0.5), (v,0.9,0.4,0.3), (w,0.7,0.2,0.5)} in τ_3 such that $W= R_1 \cup R_2 \cup R_3$.

Remark 3.5. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), an NS G is called a neutrosophic tri-t-closed set if and only if G^c is a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3).

Theorem 3.3. Assume that (Ψ, τ_1, τ_2, τ_3) be an N-Tri-TS.

(i) The neutrosophic null set (0_N) and the neutrosophic whole set (1_N) are always a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3);

(ii) Every NOS in (Ψ, τ_1), (Ψ, τ_2) and (Ψ, τ_3) are neutrosophic tri-t-open sets in (Ψ, τ_1, τ_2, τ_3);

(iii) Every NCS in (Ψ, τ_1), (Ψ, τ_2) and (Ψ, τ_3) are neutrosophic tri-t-closed sets in (Ψ, τ_1, τ_2, τ_3).

Proof. (i) We can write the neutrosophic null set (0_N) as $0_N = W \cup M \cup N$, where $W = 0_N$, $M = 0_N$, $N = 0_N$ are NOSs in (Ψ, τ_1), (Ψ, τ_2) and (Ψ, τ_3) respectively. Hence, 0_N is a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3).

Similarly, we can write the neutrosophic whole set (1_N) as $1_N = W \cup M \cup N$, where $W=1_N$, $M =1_N$, $N =1_N$ are NOSs in (Ψ, τ_1), (Ψ, τ_2) and (Ψ, τ_3) respectively. Hence, 1_N is a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3).

(ii) Suppose that W be an NOS in (Ψ, τ_1). Now, we can write $W = W \cup 0_N \cup 0_N$. Therefore, there exist NOSs W, 0_N, 0_N in (Ψ, τ_1), (Ψ, τ_2) and (Ψ, τ_3) respectively such that $W = W \cup 0_N \cup 0_N$. Hence, W is a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3).

Suppose that W be an NOS in (Ψ, τ_2). Now, we can write $W = 0_N \cup W \cup 0_N$. Therefore, there exist NOSs 0_N, W, 0_N in (Ψ, τ_1), (Ψ, τ_2) and (Ψ, τ_3) respectively such that $W = 0_N \cup W \cup 0_N$. Hence, W is a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3).

Suppose that W be an NOS in (Ψ, τ_3). Now, we can write $W = 0_N \cup 0_N \cup W$. Therefore, there exist NOSs 0_N, 0_N, W in (Ψ, τ_1), (Ψ, τ_2), and (Ψ, τ_3) respectively such that $W = 0_N \cup 0_N \cup W$. Hence, W is a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3).

(iii) Suppose that W be an NCS in (Ψ, τ_1). So W^c is an NOS in (Ψ, τ_1). By the second part of this theorem, W^c is a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3). Hence, W is a neutrosophic tri-t-closed set in (Ψ, τ_1, τ_2, τ_3).

Suppose that W be an NCS in (Ψ, τ_2). So W^c is an NOS in (Ψ, τ_2). By the second part of this theorem, W^c is a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3). Hence, W is a neutrosophic tri-t-closed set in (Ψ, τ_1, τ_2, τ_3).

Suppose that W be an NCS in (Ψ, τ_3). So W^c is an NOS in (Ψ, τ_3). By the second part of this theorem, W^c is a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3). Hence, W is a neutrosophic tri-t-closed set in (Ψ, τ_1, τ_2, τ_3).

Theorem 3.4. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), the union of two neutrosophic tri-t-open sets is a neutrosophic tri-t-open set.

Proof. Assume that X and Y are any two neutrosophic tri-t-open sets in an N-Tri-TS (Ψ, τ_1, τ_2, τ_3). So there exist NOSs X_1, Y_1 in (Ψ, τ_1), X_2, Y_2 in (Ψ, τ_2), and X_3, Y_3 in (Ψ, τ_3), such that $X = X_1 \cup X_2 \cup X_3$ and $Y = Y_1 \cup Y_2 \cup Y_3$. Now $X \cup Y = (X_1 \cup X_2 \cup X_3) \cup (Y_1 \cup Y_2 \cup Y_3) = (X_1 \cup Y_1) \cup (X_2 \cup Y_2) \cup (X_3 \cup Y_3)$. Since X_1, Y_1 are NOSs in (Ψ, τ_1), so $X_1 \cup Y_1$ is an NOS in (Ψ, τ_1). Since X_2, Y_2 are NOSs in (Ψ, τ_2), so $X_2 \cup Y_2$ is an NOS in (Ψ, τ_2). Since X_3, Y_3 are NOSs in (Ψ, τ_3), so $X_3 \cup Y_3$ is an NOS in (Ψ, τ_3). Therefore, $X \cup Y$ is a neutrosophic tri-t-open set in (Ψ, τ_1, τ_2, τ_3).

Remark 3.6. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), the intersection of any two neutrosophic tri-t-open sets may not be a neutrosophic tri-t-open set.

Definition 3.7. Assume that (Ψ, τ_1, τ_2, τ_3) be an N-Tri-TS. Then Q, an NS over Ψ is said to be a neutrosophic tri-t-semi-open set in (Ψ, τ_1, τ_2, τ_3) if and only if there exists a neutrosophic semi open sets Q_1 in (W, τ_1), Q_2 in (W, τ_2), and Q_3 in (W, τ_3) such that $Q=Q_1 \cup Q_2 \cup Q_3$.

Theorem 3.5. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), every neutrosophic tri-semi-open set is a neutrosophic tri-t-semi-open set.

Proof. Assume that X be a neutrosophic tri-semi-open set in an N-Tri-TS (Ψ, τ_1, τ_2, τ_3). So, X must be an NSO set in at least one of the NTS (Ψ, τ_1), (Ψ, τ_2), (Ψ, τ_3). So, there will be seven cases.

Case-1: X is an NSO set in (Ψ, τ_1);

Case-2: X is an NSO set in (Ψ, τ_2);

Case-3: X is an NSO set in (Ψ, τ_3);

Case-4: X is an NSO set in (Ψ, τ_1), and (Ψ, τ_2);

Case-5: X is an NSO set in (Ψ, τ_1), and (Ψ, τ_3);

Case-6: X is an NSO set in (Ψ, τ_2), and (Ψ, τ_3);

Case-7: X is an NSO set in (Ψ, τ_1), (Ψ, τ_2), and (Ψ, τ_3).

In case-1, we can express, $X = X \cup 0_N \cup 0_N$, that is X is the union of NSO sets X (in (W, τ_1)), 0_N (in (W, τ_2)), and 0_N (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-semi-open set in (Ψ, τ_1, τ_2, τ_3).

In case-2, we can express, $X = 0_N \cup X \cup 0_N$, that is X is the union of NSO sets 0_N (in (W, τ_1)), X (in (W, τ_2)), and 0_N (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-semi-open set in (Ψ, τ_1, τ_2, τ_3).

In case-3, we can express, $X = 0_N \cup 0_N \cup X$, that is X is the union of NSO sets 0_N (in (W, τ_1)), 0_N (in (W, τ_2)), and X (in (W,τ_3)). Therefore, X is a neutrosophic tri-t-semi-open set in ((Ψ, τ_1, τ_2, τ_3).

In case-4, we can express, $X = X \cup X \cup 0_N$, that is X is the union of NSO sets X (in (W, τ_1)), X (in (W, τ_2)), and 0_N (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-semi-open set in (Ψ, τ_1, τ_2, τ_3).

In case-5, we can express, $X = X \cup 0_N \cup X$, that is X is the union of NSO sets X (in (W, τ_1)), 0_N (in (W, τ_2)), and X (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-semi-open set in (Ψ, τ_1, τ_2, τ_3).

In case-6, we can express, $X = 0_N \cup X \cup X$, that is X is the union of NSO sets 0_N (in (W, τ_1)), X (in (W, τ_2)), and X (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-semi-open set in (Ψ, τ_1, τ_2, τ_3).

In case-7, we can express, $X = X \cup X \cup X$, that is X is the union of NSO sets X (in (W, τ_1)), X (in (W, τ_2)), and X (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-semi-open set in (Ψ, τ_1, τ_2, τ_3).

Hence, every neutrosophic tri-semi-open set is a neutrosophic tri-t-semi-open set.

Definition 3.8. Assume that (Ψ, τ_1, τ_2, τ_3) be an N-Tri-TS. Then Q, an NS over Ψ is said to be a neutrosophic tri-t-pre-open set in (Ψ, τ_1, τ_2, τ_3) iff there exist a neutrosophic-pre-open sets Q_1 in τ_1, Q_2 in τ_2, and Q_3 in τ_3 such that $Q = Q_1 \cup Q_2 \cup Q_3$.

Theorem 3.6. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), every neutrosophic tri-pre-open set is a neutrosophic tri-t-pre-open set.

Proof. Assume that X is a neutrosophic tri-pre-open set in an N-Tri-TS (Ψ, τ_1, τ_2, τ_3). So, X must be an NPO set in at least one of the NTS (Ψ, τ_1), (Ψ, τ_2), (Ψ, τ_3). So, there will be seven cases.

Case-1: X is an NPO set in (Ψ, τ_1);

Case-2: X is an NPO set in (Ψ, τ_2);

Case-3: X is an NPO set in (Ψ, τ_3);

Case-4: X is an NPO set in (Ψ, τ_1), and (Ψ, τ_2);

Case-5: X is an NPO set in (Ψ, τ_1), and (Ψ, τ_3);

Case-6: X is an NPO set in (Ψ, τ_2), and (Ψ, τ_3);

Case-7: X is an NPO set in (Ψ, τ_1), (Ψ, τ_2), and (Ψ, τ_3).

In case-1, we can express, $X = X \cup 0_N \cup 0_N$, that is X is the union of NPO sets X (in (W, τ_1)), 0_N (in (W, τ_2)), and 0_N (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-pre-open set in (Ψ, τ_1, τ_2, τ_3).

In case-2, we can express, $X = 0_N \cup X \cup 0_N$, that is X is the union of NPO sets 0_N (in (W, τ_1)), X (in (W, τ_2)), and 0_N (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-pre-open set in (Ψ, τ_1, τ_2, τ_3).

In case-3, we can express, $X = 0_N \cup 0_N \cup X$, that is X is the union of NPO sets 0_N (in (W, τ_1)), 0_N (in (W, τ_2)), and X (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-pre-open set in (Ψ, τ_1, τ_2, τ_3).

In case-4, we can express, $X = X \cup X \cup 0_N$, that is X is the union of NPO sets X (in (W, τ_1)), X (in (W, τ_2)), and 0_N (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-pre-open set in (Ψ,τ_1,τ_2,τ_3).

In case-5, we can express, $X = X \cup 0_N \cup X$, that is X is the union of NPO sets X (in (W, τ_1)), 0_N (in (W, τ_2)), and X (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-pre-open set in (Ψ, τ_1, τ_2, τ_3).

In case-6, we can express, $X = 0_N \cup X \cup X$, that is X is the union of NPO sets 0_N (in (W, τ_1)), X (in (W, τ_2)), and X (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-pre-open set in (Ψ, τ_1, τ_2, τ_3).

In case-7, we can express, $X = X \cup X \cup X$, that is X is the union of NPO sets X (in (W, τ_1)), X (in (W, τ_2)), and X (in (W, τ_3)) . Therefore, X is a neutrosophic tri-t-pre-open set in (Ψ, τ_1, τ_2, τ_3).

Hence, every neutrosophic tri-pre-open set is a neutrosophic tri-t-pre-open set.

Definition 3.9. Assume that (Ψ, τ_1, τ_2, τ_3) is an N-Tri-TS. Then Q, an NS over Ψ is said to be a neutrosophic tri-t-b-open set in (Ψ, τ_1, τ_2, τ_3) if and only if there exist three neutrosophic-b-open sets, namely Q_1 in τ_1, Q_2 in τ_2, and Q_3 in τ_3 such that $Q = Q_1 \cup Q_2 \cup Q_3$.

Theorem 3.7. In an N-Tri-TS (Ψ, τ_1, τ_2, τ_3), every neutrosophic tri-b-open set is a neutrosophic tri-t-b-open set.

Proof. Assume that X be a neutrosophic tri-b-open set in an N-Tri-TS (Ψ, τ_1, τ_2, τ_3). So, X must be a neutrosophic b-open set in at least one of the NTS (Ψ, τ_1), (Ψ, τ_2), (Ψ, τ_3). So there will be seven cases.

Case-1: X is a neutrosophic b-open set in (Ψ, τ_1);

Case-2: X is a neutrosophic b-open set in (Ψ, τ_2);

Case-3: X is a neutrosophic b-open set in (Ψ, τ_3);

Case-4: X is a neutrosophic b-open set in (Ψ, τ_1), and (Ψ, τ_2);

Case-5: X is a neutrosophic b-open set in (Ψ, τ_1), and (Ψ, τ_3);

Case-6: X is a neutrosophic b-open set in (Ψ, τ_2), and (Ψ, τ_3);

Case-7: X is a neutrosophic b-open set in (Ψ, τ_1), (Ψ, τ_2), and (Ψ, τ_3).

In case-1, we can express, $X = X \cup 0_N \cup 0_N$, that is X is the union of neutrosophic b-open sets X (in (W, τ_1)), 0_N (in (W, τ_2)), and 0_N (in (W, τ_3)) . Therefore, X is a neutrosophic tri-t-b-open set in (Ψ, τ_1, τ_2, τ_3).

In case-2, we can express, $X = 0_N \cup X \cup 0_N$, that is X is the union of neutrosophic b-open sets 0_N (in (W, τ_1)), X (in (W, τ_2)), and 0_N (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-b-open set in (Ψ, τ_1, τ_2, τ_3).

In case-3, we can express, $X = 0_N \cup 0_N \cup X$, that is X is the union of neutrosophic b-open sets 0_N (in (W, τ_1)), 0_N (in (W, τ_2)), and X (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-b-open set in (Ψ, τ_1, τ_2, τ_3).

In case-4, we can express, $X = X \cup X \cup 0_N$, that is X is the union of neutrosophic b-open sets X (in (W, τ_1)), X (in (W, τ_2)), and 0_N (in (W, τ_3)) . Therefore, X is a neutrosophic tri-t-b-open set in (Ψ, τ_1, τ_2, τ_3).

In case-5, we can express, $X = X \cup 0_N \cup X$, that is X is the union of neutrosophic b-open sets X (in (W, τ_1)), 0_N (in (W, τ_2)), and X (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-b-open set in (Ψ, τ_1, τ_2, τ_3).

In case-6, we can express, $X = 0_N \cup X \cup X$, that is X is the union of neutrosophic b-open sets 0_N (in (W, τ_1)), X (in (W, τ_2)), and X (in (W, τ_3)). Therefore, X is a neutrosophic tri-t-b-open set in (Ψ, τ_1, τ_2, τ_3).

In case-7, we can express, $X = X \cup X \cup X$, that is X is the union of neutrosophic b-open sets X (in (W, τ_1)), X (in (W, τ_2)), and X (in (W, τ_3)) . Therefore, X is a neutrosophic tri-t-b-open set in (Ψ, τ_1, τ_2, τ_3).

Hence, every neutrosophic tri-b-open set is a neutrosophic tri-t-b-open set.

Definition 3.10. Assume that (Ψ, τ_1, τ_2, τ_3) be an N-Tri-TS. Let X be an NS over Ψ. The neutrosophic tri-t-interior ($N\text{-}tri\text{-}t_{int}$) and neutrosophic tri-$t$-closure ($N\text{-}tri\text{-}t_{cl}$) of X is defined as follows:

$N\text{-}tri\text{-}t_{int}(X) = \cup\{Y$: Y is a neutrosophic tri-t-open set and $Y \subseteq X\}$;

$N\text{-}tri\text{-}t_{cl}(X) = \cap\{Y$: Y is a neutrosophic tri-t-closed set and $X \subseteq Y\}$.

It is clearly observed that $N\text{-}tri\text{-}t_{int}(X)$ is the largest neutrosophic tri-t-open set which is contained in X and $N\text{-}tri\text{-}t_{cl}(X)$ is the smallest neutrosophic tri-t-closed set which contains X.

Theorem 3.8. Let (Ψ, τ_1, τ_2, τ_3) be an N-Tri-TS. Let X and Y be two neutrosophic sets over Ψ. Then,

(*i*) $N\text{-}tri\text{-}t_{int}(X) \subseteq X$;

(*ii*) $X \subseteq Y \Rightarrow N\text{-}tri\text{-}t_{int}(X) \subseteq N\text{-}tri\text{-}t_{int}(Y)$;

(*iii*) If X is a neutrosophic tri-t-open set, then $N\text{-}tri\text{-}t_{int}(X)=X$;

(*iv*) $N\text{-}tri\text{-}t_{int}(0_N)=0_N$, and $N\text{-}tri\text{-}t_{int}(1_N)=1_N$.

Proof.

(i) From the definition 3.10, we see that $N\text{-}tri\text{-}t_{int}(X) = \cup\{B$: B is a neutrosophic tri-t-open set and $B \subseteq X\}$. Since $B \subseteq X$, so $\cup\{B$: B is a neutrosophic tri-t-open set and $B \subseteq X\} \subseteq X$. Therefore, $N\text{-}tri\text{-}t_{int}(X) \subseteq X$.

(ii) Suppose that X and Y are two neutrosophic sets over Ψ such that $X \subseteq Y$. Then,

$N\text{-}tri\text{-}t_{int}(X)$

$= \cup\{B: B$ is a neutrosophic tri-t-open set and $B \subseteq X\}$;

$\subseteq \cup\{B: B$ is a neutrosophic tri-t-open set and $B \subseteq Y\}$ [since $X \subseteq Y$]

$= N\text{-}tri\text{-}t_{int}(Y)$

$\Rightarrow N\text{-}tri\text{-}t_{int}(X) \subseteq N\text{-}tri\text{-}t_{int}(Y)$.

Therefore, $X \subseteq Y \Rightarrow N\text{-}tri\text{-}t_{int}(X) \subseteq N\text{-}tri\text{-}t_{int}(Y)$.

(iii) Assume that X be a neutrosophic tri-t-open set in an N-Tri-TS $(\Psi, \tau_1, \tau_2, \tau_3)$. Now, $N\text{-}tri\text{-}t_{int}(X) = \cup\{B: B$ is a neutrosophic tri-t-open set and $B \subseteq X\}$. Since X is a neutrosophic tri-t-open set in $(\Psi, \tau_1, \tau_2, \tau_3)$, so X is the largest neutrosophic tri-t-open set in $(\Psi, \tau_1, \tau_2, \tau_3)$, which is contained in X. Hence $\cup\{B: B$ is a neutrosophic tri-t-open set and $B \subseteq X\} = X$. Therefore, $N\text{-}tri\text{-}t_{int}(X) = X$.

(iv) We know that 0_N, and 1_N are neutrosophic tri-t-open sets in $(\Psi, \tau_1, \tau_2, \tau_3)$, so by the third part of this theorem, we have $N\text{-}tri\text{-}t_{int}(0_N) = 0_N$, $N\text{-}tri\text{-}t_{int}(1_N) = 1_N$.

Theorem 3.9. Let $(\Psi, \tau_1, \tau_2, \tau_3)$ be an N-Tri-TS. Let X and Y be two neutrosophic sets over Ψ. Then,

(*i*) $X \subseteq N\text{-}tri\text{-}t_{cl}(X)$;

(*ii*) $X \subseteq Y \Rightarrow N\text{-}tri\text{-}t_{cl}(X) \subseteq N\text{-}tri\text{-}t_{cl}(Y)$;

(*iii*) X is a neutrosophic tri-t-closed set iff $N\text{-}tri\text{-}t_{cl}(X) = X$;

(*iv*) $N\text{-}tri\text{-}t_{cl}(0_N)=0_N$, and $N\text{-}tri\text{-}t_{cl}(1_N)=1_N$;

Proof. (i) From the definition 3.10, we see that $N\text{-}tri\text{-}t_{cl}(X)= \cap\{B: B$ is a neutrosophic tri-t-closed set and $X \subseteq B\}$. Since each $X \subseteq B$, so $X \subseteq \cap\{B: B$ is a neutrosophic tri-t-closed set and $X \subseteq B\}$. Therefore, $X \subseteq N\text{-}tri\text{-}t_{cl}(X)$.

(ii) Suppose that X and Y are two neutrosophic sets over Ψ such that $X \subseteq Y$. Then,

$N\text{-}tri\text{-}t_{cl}(X)$

$= \cap\{B: B$ is a neutrosophic tri-t-closed set and $X \subseteq B\}$.

$\subseteq \cap\{B: B$ is a neutrosophic tri-t-closed set and $Y \subseteq B\}$ [since $X \subseteq Y$]

$= N\text{-}tri\text{-}t_{cl}(Y)$.

Therefore, $X \subseteq Y \Rightarrow N\text{-}tri\text{-}t_{cl}(X) \subseteq N\text{-}tri\text{-}t_{cl}(Y)$.

(iii) Assume that X be a neutrosophic tri-t-closed set in $(\Psi, \tau_1, \tau_2, \tau_3)$. Now, $N\text{-}tri\text{-}t_{cl}(X) = \cap\{B: B$ is a neutrosophic tri-t-closed set and $X \subseteq B\}$. Since X is a neutrosophic tri-t-closed set in $(\Psi, \tau_1, \tau_2, \tau_3)$, so X is the smallest neutrosophic tri-t-closed set in $(\Psi, \tau_1, \tau_2, \tau_3)$, which contains X. Therefore, $\cap\{B: B$ is a neutrosophic tri-t-closed set and $X \subseteq B\} = X$. Therefore, $N\text{-}tri\text{-}t_{cl}(X) = X$.

(iv) It is known that, 0_N and 1_N are neutrosophic tri-t-closed sets in $(\Psi, \tau_1, \tau_2, \tau_3)$. So, by the third part of this theorem, we have $N\text{-}tri\text{-}t_{cl}(0_N) = 0_N$, $N\text{-}tri\text{-}t_{cl}(1_N) = 1_N$.

Theorem 3.11. Let $(\Psi, \tau_1, \tau_2, \tau_3)$ be an N-Tri-TS. Let X be an NS over Ψ. Then, $\tau_i\text{-}N_{int}(X) = N\text{-}tri\text{-}t_{int}(X)$.

Proof. Assume that X be a neutrosophic subset of an N-Tri-TS $(\Psi, \tau_1, \tau_2, \tau_3)$. Now, $\tau_i\text{-}N_{int}(X) = \cup\{Y: Y$ is an NOS in (Ψ, τ_i) and $Y \subseteq X\}$. Since Y is an NOS in (Ψ, τ_i), so by second part of Theorem 3.3., Y is a neutrosophic tri-t-open set in $(\Psi, \tau_1, \tau_2, \tau_3)$.

Therefore, $\tau_i\text{-}N_{int}(X)$

$= \cup\{Y: Y$ is an NOS in (Ψ, τ_i) and $Y \subseteq X\}$

$= \cup\{Y: Y$ is a neutrosophic tri-t-open set in $(\Psi, \tau_1, \tau_2, \tau_3)$, and $Y \subseteq X\}$

$= N\text{-}tri\text{-}t_{int}(X)$.

Hence, in an N-Tri-TS $(\Psi, \tau_1, \tau_2, \tau_3)$, $\tau_i\text{-}N_{int}(X) = N\text{-}tri\text{-}t_{int}(X)$ for any neutrosophic set X.

Theorem 3.12. Let (Ψ, τ_1, τ_2, τ_3) be an N-Tri-TS. Let X be an NS over Ψ. Then, τ_i-$N_{cl}(X) \subseteq N\text{-}tri\text{-}t_{cl}(X)$.

Proof. Assume that X be a neutrosophic subset of an N-Tri-TS (Ψ, τ_1, τ_2, τ_3). Now, τ_i-$N_{cl}(X) = \cap\{Y: Y$ is an NCS in (Ψ, τ_i) and $X \subseteq Y\}$. Since Y is an NCS in (Ψ, τ_i), so by third part of Theorem 3.3., Y is a neutrosophic tri-*t*-closed set in (Ψ, τ_1, τ_2, τ_3).

Therefore, τ_i-$N_{cl}(X)$

$= \cap\{Y: Y$ is an NCS in (Ψ, τ_i) and $X \subseteq Y\}$

$= \cap\{Y: Y$ is a neutrosophic tri-*t*-closed set in (Ψ, τ_1, τ_2, τ_3) and $X \subseteq Y\}$

$= N\text{-}tri\text{-}t_{cl}(X)$.

Hence, τ_i-$N_{cl}(X) = N\text{-}tri\text{-}t_{cl}(X)$, for any neutrosophic subset X of (Ψ, τ_1, τ_2, τ_3).

4. Conclusions

In this study, we introduce the notion neutrosophic tri-topological spaces. Also, we establish some of their basic properties. By defining neutrosophic tri-topology and neutrosophic tri-topological space, we present well described examples and proofs of some theorems on neutrosophic tri-topological spaces.

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Hybridization between deep learning algorithms and neutrosophic theory in medical image processing: A survey

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Abstract: Deep learning can successfully extract data features based on dealing greatly with non-linear problems. Deep learning has the highest performance in medical image analysis and diagnosis. Additionally, deep learning performance is affected by insufficient medical image data such as fuzziness or incompleteness. The neutrosophic approach can enhance deep learning performance with its great dealing with inconsistency and ambiguity information in medical data. This survey investigates the various ways in which deep learning is enhanced with neutrosophic systems and provides an overview and concept on each other. The hybrid techniques are classified based on different medical image modalities in different medical image processing stages such as preprocessing, segmentation, classification, and clustering. Finally, future works are also explored. In this study the highest accuracy was achieved by hybridization between neutrosophic and LASTM to classify the cardio views. While the highest capability to precisely detect those with the disease (sensitivity) is achieved by integration between neutrosophic, convolution neural network and support vector machine. Best specificity was obtained by neutrosophic and LSTM.

Keywords: Medical image; Neutrosophic; Deep learning; denoising; classification; segmentation; clustering; image modalities.

1. Introduction

Recently, rapid diagnosis, and treatment of diseases becomes a major area in computer science using different medical image modalities such as computed tomography (CT), magnetic resonance imaging (MRI), Microscopic image analysis (MIA), ultrasound (US), and X-ray [1]. Usually, radiologists and physicians perform the interpretation of a medical image. However, Computer-aided systems can help human experts and doctors from potential fatigue and individual variations in pathology reading. Deep learning (DL) and neutrosophic techniques can help to improve the rate of computational medical image analysis [2].

In the last few decades, many automatic analysis systems have been implemented from scanned and loaded medical images. Between the 1970s and 1990s, low-level pixel and mathematical processing (edge detection, region growing, fitting lines) were the main techniques for doing medical image analysis. It was common in that period there are same as if-then-else statements in expert systems. Haugeland, 1985 named these systems GOFAI (good old-fashioned artificial intelligence) like rule-based image processing systems [3, 4].

Supervised learning (SL) develops systems using training data at the end of the 1990s. These systems become more and more common in medical image analysis such as atlas approaches which fit new data from the training data, feature extraction, and use of statistical classifiers such as computer-aided diagnosis (CAD) and active shape models. These methods have become successful in many medical image analysis tasks. So, systems are trained using extracted data vectors instead of data designed by humans. The optimal decision boundary is done using computer algorithms in high dimensional space. Feature extraction is a vital step in model deployment. This process is named the hand-crafted feature [4].

Logical progress is learning optimal features that represent data for a complex problem. This concept lies at the foundation of many DL models. So, DL techniques permit the machine to make data interpretation by learning complex mathematical problems. These models consist of linear and/or non-linear functions as input data and weighted model parameters. These functions treat hierarchy as a layer, so the name of DL is inspired by a larger number of such layers. Usually, Training data tasks such as denoising, segmentation, and classification help the computational model to learn its parameter. The basic idea of DL is an Artificial Neural Network (ANN) that contains multiple layers of neurons, while its parameters (weights) identify the parameter of the connections between the neurons and layers [5].

DL uses testing data to perform the same task accurately, which makes DL more generalizable than other different machine learning (ML) techniques. DL learn parameter using a back-propagation strategy which iteratively attains the desired parameter value using the Gradient Descent technique. The single epoch is the terminology of update the model parameter using whole training data once. Usually, modern DL models are trained for hundreds of epochs before utilization [5].

Convolutional neural networks (CNNs) are the most popular network in DL. A CNN does a mathematical operation called convolution [6]. CNN was introduces to the world in handwriting digit recognition in LeNet [7]. After those novel approaches were implemented for effectively training deep networks, and improvements were produced in main computing systems. Krizhevsky et al. (2012) proposed the AlexNet based on CNNs which trained on ImageNet data in December 2012 [8].

The medical image analysis society observes these crucial improvements. later than, other DL architecture has deployed recurrent neural networks (RNNs), autoencoders (AEs), restricted Boltzmann machines (RBMs) [9], and deep belief networks (DBN). All these contributions take attention to the medical image analysis community [4].

On the other hand, many problems in medical image analysis have been shown such as impression and uncertainty, incomplete, fuzziness, and inconsistent, which derive from acquisition errors, incomplete knowledge, or stochasticity. These problems make denoising, segmentation, clustering, and classification are difficult operations to perform on medical image analysis [10]. So soft computing combined with medical applications and DL architecture to get solutions for unsolvable problems. Many theories can deal with ambiguous information such as para consistence logic theory [11], intuitionistic fuzzy set (IFS) theory [12], fuzzy set (FS) theory [13], probability theory [14]. One of these methods is the FS introduced by Zadeh (1965) which solves fuzziness and ambiguity problems that exist in medical data [15]. One disadvantage of FS that it doesn't take indeterminacy in its consideration independently [16].

Neutrosophy is a novel philosophy branch adopted by Smarandache that is a scope of neutralities. Neutrosophy can specify classical logic, fuzzy logic (FL), and imprecise probability. Human rational can deal with the ambiguity of knowledge linguistic mistakes and so neutrosophy can deal with this ambiguity. Usually, neutrosophy includes a neutrosophic set (NS) which can deal with neutralities along with their relations. Neutrosophy has truth, falsity, and an indeterminacy degree, which are independent of each other [17].

Medical data is unpredictable, partial, inaccurate, incomplete, and vague. Expert systems vary in inputs because of not existence of the specified policy in treatment or drug usage. Normally, a large scale of information process is needed in a medical image, a significant part is unconscious and rapid processing and computation. Therefore, indeterminacy, irregularity, ambiguities, and vagueness must be solved. Intelligent diagnosis system has been observed by computer science and applicable mathematics field. So evolutionary neural network in breast cancer detection has been proposed by [18]. Also, Tan et al. [19] classified the hepatitis rule and breast cancer cases using a hybrid evolutionary algorithm (EA) and genetic programming (GP).

A new neuro-fuzzy technique for segmentation of the MRI data to brain tumor [20]. Lately, NL and NS have major importance in medical applications. Neutrosophic systems prove its successful effect than fuzzy counterparts. It can deal with major processes in medical systems such as data acquisition, data generation, indeterminacy, truth, and falsity. Hence, NS comprises truth, indeterminate, and false membership functions. In2015, Ye [21] proposed the improved cosine similarity measure for simplified NS (SNS) and applied it to medical diagnosis. The single-valued NS (SVNS) is the crucial usage set in most applications introduced by Wang et al. [22].

There are lots of surveys in medical image analysis discuss the use of NS in different levels of image analysis such as (denoising, segmentation, and classification) or using NS in different medical image modalities on different organs, but all of these surveys did not cover the last researchs and algorithms of using DL under NS theory [23], [24], [25], [26], [27], [28], [29]. Also (Elhassouny et al.,2019) discuss the integration of NS and ML algorithms, but not cover the medical image analysis domain [30]. On the other hand, many surveys on DL in a medical image, but these studies ignore the dealing of inconsistency and fuzzification information in the medical image [2], [31], [4], [32].

This study aims to introduce a survey on DL algorithms in medical image processing under a NS theory. The following sections are arranged as follows. Section 2 provides an overview of different image modalities. Section 3 provides a general concept of NS and NS algorithms that are used in medical image processing stages such as denoising, segmentation, and classification in different image modalities such as MRI, CT, US, CT, and MIA. In section 4 we introduce an overview of the DL concept and previous work on techniques that commonly used different medical image processing stages. In section 5 explores hybridization between NS and DL and how this integration can affect the performance in medical image processing and analysis.

2. Medical image modalities

The medical image modalities help for more medical image analysis and diagnosis. Medical image modalities differ in characterization and applications that assist the study organs and diagnosis and treatment follow-up. These modalities can be categorized into five types: MIA, MRI, US, X-ray, CT [33]. A short discussion about these modalities is introduced in Table 1.

Table 1. Medical Imaging Modalities.

Medical image modalities	Famous issue
MIA	Fuzziness, inconsistency, weak robustness[29].

MRI	Gaussian, Rician, Rayleigh[27].
US	Gaussian, Speckle noise[27].
X-Ray	Gaussian, Poisson[34].
CT	Speckle noise, gaussian noise, salt and pepper noise[35].

2.1 Microscopic image analysis

MIA is a good effect on converting to completely automatic analysis instead of the human observer. MIA Can carry a blood smear or any tissue of the human body [36]. MIA can help improve the diagnosis performance in various diseases. Many applications use MIA in processing techniques such as image enhancement, microscopic segmentation, cell classification, and White blood cell detection.

2.2 Magnetic resonance imaging

MRI is essential in non-invasive diagnosis and is one of the most commonly and reliably used clinical situations. The most characterization of MRI that it can pick up soft tissue such as blood vessels, organs in the pelvis, the abdomen of (heart, liver, kidney). These cross-sectional images are taken by magnets and radio waves to form a slice of the human body.

MRI is safe for children and pregnant women because no radiation exposure also has high accuracy. On the other hand, MRI has a great sensitivity to a movement which affects the organs that involve movement. Additionally, MRI suffers from magnetic field distribution and patients cannot wear metallic devices such as pacemakers. Many applications of computer science using MRI such as tissue classification, liver diagnosis,3D tumor visualization [33].

2.3 Ultrasound

The US transformed the echoes retrieving sounds into images, So the US cannot detect bone organs. Advantage of US is low cost, high resolution, no radiation, and widely available scan. But it is difficult to image lungs and bones resolution to be affected easily. A lot of applications are using US images such as pregnancy, breast cancer detection, liver and tumor diagnosis [33].

2.4 X-ray image

X-rays are considered the most and oldest imaging types. X-ray images formed using electromagnetic radiation. Most X-ray applications are detecting problems with the skeletal system, diagnose cancer via mammography, gastric concerns, and dental problems detection [27].

2.5 Computed tomography

Series of Cross-sectional images are formed using x-ray sensors. So, it can detect hard tissue such as bones. CT has a wide scan area so it can pick up blood vessels and brain, liver. CT has an advantage over MRI in that it has a short time of scan with high resolution. But it has limitations in tissue characterization, sensitivity, high cost, and high radiation. Many applications are using the CT image such as covide-19 detection, chest diagnostic, brain simulation, tumor detection [33].

3. Neutrosophic set in medical image analysis

Samarandache introduced neutrosophic in 1995 which is the generalization form of fuzzy [37]. Neutrosophy is the fundamental of neutrosophic probability, neutrosophic statistics, neutrosophic logic (NL), and (NS) [37]. The NS is the general concept of the classical sets, FS, interval-valued fuzzy set [38], IFS [39]. NS concept (A) in relation to its opposite (Anti-A) and the neutrality (Neut-A), which is not (A) nor (Anti-A). The (Neut-A) and (Anti-A) are mentioned as (Non-A). So concept (A) is neutralized and balanced by (Anti-A) and (Non-A) concepts [40].

NL defines three neutrosophic components: T, I, F for truth, false, and indeterminacy membership degree in <A>.NL can handle the uncertainty by providing extra domain I which increases the efficiency of dealing with uncertainty unlike FL [41]. NL is able to perform the difference between relative truth and absolute truth as well as between relative falsity and absolute falsity, so NL component (T, I, F) can be over-flooded over 1 or under-dried 0 [23].Commonly, some definitions are given for the NS as follows: [40, 42]

Definition 1. T, I and F are real standard or non-standard of $]-0,1+[$ with

$$supT = t\_sup, infT = t\_inf,$$
$$supI = i - sup, infI = i_{inf}, supF = f_{sup}, infF = f_{inf and n_{sup}} = t_{sup} + i_{sup} + f_{sup}, n_{inf} = t_{inf} + \cdots$$
$$i_{inf} + f\_inf$$

Definition 2. (Neutrosophic image) for U is universe, a neutrosophic image P_{NS} is characterized by subsets T, I and F. A pixel P in the image is defined as P(T, I,F).Then, the pixel P(n,m) in the image domain is converted to NS image using the following equations:

$$H_{NS}(n,m) = \{\mathrm{T}(n,m), \mathrm{I}(n,m), \mathrm{F}(n,m)\} \tag{1}$$

Where *T(n,m),I(n,m) and F(n,m)* probabilities of white, indeterminate, non-white sets:

$$T(n,m) = \frac{\bar{g}(n,m) - \bar{g}_{min}}{\bar{g}_{max} - \bar{g}_{min}} \tag{2}$$

$$\bar{g}(n,m) = \frac{1}{w * w} \sum_{x=n-w/2}^{n+w/2} \sum_{y=m-w/2}^{m+w/2} g(x,y) \tag{3}$$

$$I(n,m) = \frac{\delta(n,m) - \delta_{min}}{\delta_{max} - \delta_{min}} \tag{4}$$

$$\delta(n,m) = abs(g(n,m) - \bar{g}(n,m)) \tag{5}$$

$$F(n,m) = 1 - T(n,m) \tag{6}$$

Where *g(n,m)* is the intensity value of the pixel (n,m) ,$\bar{g}(n,m)$ is the local mean value of $g(n,m)$,$\delta(n,m)$ is the absolute value of the difference between intensity $g(n,m)$and its local mean value $\bar{g}(n,m)$ [23].

Definition 3. (Neutrosophic image entropy) The gray image entropy measures the distribution of intensities. Maximum entropy value implies for equal intensities probability and small entropy implies for non-uniform intensity distribution [23]. The NS image entropy defined as the summation of the entropies of three subsets T, I and F, which is given by:

$$E_{NS} = E_T + E_I + E_F \tag{7}$$

$$E_T = -\sum_{i=\min\{T\}}^{\max\{T\}} P_T(i) \ln P_T(i) \tag{8}$$

$$E_I = -\sum_{i=\min\{I\}}^{\max\{I\}} P_I(i) \ln P_I(i) \tag{9}$$

$$E_F = -\sum_{i=min\{F\}}^{\max\{F\}} P_F(i) \ln P_F(i) \tag{10}$$

where, E_T, E_I and E_F are the entropies of sets *T, I* and *F*, respectively. Also, $P_T(i), P_I(i), P_F(i)$ are the probabilities in *T, I* and *F;* respectively. Commonly, E_T and E_F are used to measure the distribution of the elements in NS, and E_I is evaluated to measure the indeterminacy distribution.

Recently NL and NS had major importance in the medical domains. Neutrosophic systems are noticed to be more successful than fuzzy systems. NS can deal with indeterminacy in medical information which makes it more generalization than FS [23]. NS provides approximate the connection between modern medical image analysis and fuzzy approaches. It improves performance in different medical systems processes such as acquisition, generation, sorting. Therefore, the NS has an independent (T, I, F) membership function. Lately, Ye [21] used the cosine function, SVNS, and interval neutrosophic cosine similarity to propose cosine similarity measures for SNSs for medical analysis issues. Afterward, the weighted cosine similarity measures of SNSs were used. Wang et al. [22] proposed a SVNS, which is the main example of NS for most applicable applications. Research of NS in medical diagnosis cover many problems are in different image modalities and different tasks such as denoising, clustering, classification.

3.1. Neutrosophic set in medical image denoising

Generally, the medical image consists of noises, these noises are kind of intermediate information. Removing noises from the medical images is an important research area in computer science. Dealing with indeterminacy in images under the NS theory helping in reaching better performance during the image preprocessing stage [43]. Many approaches rely on NS for reducing salt and pepper, speckle, rician, and gaussian noise are listed in Table 2.

In 2011 Mohan et al. [44] proposed a filter to remove noise from MRI image by converting it to NS domain. Then obtain the membership values of T, I, F. The γ-median filter used to decrease the indeterminacy entropy. This approach compared with the median filter and NLM filer and shows superior results. An extension for this study applies ω – wiener filter on MRI image [45]. Also, the same author expanded the study on nonlocal NS (NLNS) [46]. The results show superior result for ω – wiener with higher PSNR.

In 2012 Koundal et al. [47] applied Kuan filter and Lee filter on US image. An extension to this study [48] use Gamma variation on neutrosophic domain to improve image quality. The same author proposed a Nakagami distribution method based on NS. The results show superior results to Nakagami distribution approach based on NS.

Another contribution on RGB image in [49] aimed to improve the quality of image based on NSS. Another study on NSS in [50] on liver image. the results shows higher PSNR to [50] but [49] improve the contrast of image more better. Another contribution improved the NLM using the weighted function based on NL to enhance US image. Furthermore Ashour et al. [26] introduce a novel method for dermoscopic image denoising based on OIF which aims to optimize the indeterminacy filter using GA. Also, Nasef et al. [51] improve the dark area in skeletal image based on NS and SSA under multi-criteria.

Table 2. Different medical image denoising methods using the NS theory.

Authors	Modality	Data	Noise	Gray /RGB	Denoising Method	Metric
Mohan et al. (2011) [44]	MRI	MRI brain (axial, Sagittal)	Rician	Gray	γ -median	PSNR* (Axial, Sagittal)= (19.90,19.11)
Mohan et al. (2012) [45]	MRI	Axial MRI brain	Rician	Gray	ω – wiener	SSIM†=0.9682 PSNR=24.08 QILV‡=0.9882
Koundal et al. (2012) [47]	US	Thyroid	Speckle	Gray	Kuan filter and Lee filter	SNR§ (Lee)=17.5375 SNR (Kuan) =17.0408 EPI¶ (Lee) = 0.7858 EPI (Kaun)=0.7599
Mohan et al. (2013) [46]	MRI	Axial MRI brain	Rician	Gray	nonlocal NS (NLNS) and ω – wiener	PSNR=23.92 SSIM= 0.9254
Koundal et al. (2016) [48]	US	Thyroid image	Synthetic Speckle	Gray	Based on gamma distribution	UQI‖ = 0.8606 FSIM# =0.8790 EPI = 0.8718 MSSIM**=0.8099 VIF& = 0.3565
Koundal et al.2018 [52]	US	Thyroid image	Synthetic Speckle	Gray	Based on Nakagami distribution	UQI = 0.8606 EPI = 0.8813 MSSIM=0.8139 VIF = 0.3771
Shahin et al. (2018a) [49]	MIA	blood smear images (3327 of different types of WBCs)	illumination, contrast, and color balance problems	RGB	Neutrosophic similarity score(NSS) scaling	Cost Time (Sec per image) from 0.276 s to 0.96 s
Rahimizadeh et al. (2019) [53]	US	Different organs	Speckle	Gray	Weighted function + (NLM) filter	SNR in noise level 0.4=53.36 SSIM for noise level 0.4=0.9514
Bharti et al.(2020) [50]	US	Liver	Speckle	Gray	(NSS)	PSNR= 34.18±1.80×10-1

						AMBE&&= 0.0326±9.87×10 -3 EPI= 0.9687±2.1×10-3 UIQI^= 0.9757±1.17×10 -2 MS-SSIM^^= 0.9996±1.63×10 -4
Ashour et al. (2019)[26]	Dermoscopic	20 randomly selected images (International Skin Imaging Collaboration)	synthetic Gaussian	Gray	Optimized indeterminacy filter (OIF) + genetic algorithm (GA)	SNR=27.75 PSNR=31.47 MSE++=57.86 RMSE°=7.18
Nasef et al. (2020) [51]	Skeletal scintigraphy	Data collected from Menoufia University Hospital in Egypt	Dark regions	Gray level	Salp Swarm algorithm (SSA) under multi-criteria	Results show that implementation achieves better performance in most criteria

Key of Table 2: * Peak-signal-to-noise ratio, † Structural Similarity index, ‡ Quality Index based on local variance, § Signal-to-noise ratio, ¶ Edge preservation index, | | Universal quality index, # Visual information fidelity, ** Multi-scale structural similarity index metric , & Visual information fidelity , &Absolute mean brightness error , ^ universal image quality index , ^^ Multi-scale structural similarity, ++ Mean squared error , ° The root of mean squared error

3.2. *Neutrosophic sets in medical image clustering and segmentation*

In computer vision Clustering means several grouped objects that are related in common members. On the other hand, segmentation in medical image separate object from the background so the image is separated into non-overlapping various regions. Clustering and segmentation are crucial processes in medical image diagnosis [23]. There are various approaches that use to segment and cluster the medical image using the NS theory in many medical applications as in Table 3.

Shahin et al. [54] proposed a new method for WBC segmentation using multi-scale similarity measure based on NS domain. This approach is obtained on RGB public dataset. Another proposed on RGB image, Ashour et al. [55] proposed a segmentation method on WBC image. This approach aims to detect blood cell by first using canny edge detector and then circular Hough transform (CHT) based on NS domain. Finally, K-means detect nuclei in blood cell image.

A study on dental image segmentation aimed to increase the accuracy in x-ray image by introducing a new fuzzy clustering methods based on NS orthogonal matrix [56]. Furthermore, Ashour et al. [57] proposed an approach on dermoscopic grayscale image based on clustering histogram. The histogram clustering method aims to determine the needed number of clusters in NS c-means.

On breast cancer segmentation, Lotfollahi et al. [58] proposed a new method based on active contour to detect the tumor outline. Then the initial contour is defined depending on intensity and NS feature. Another study on breast cancer CT image, by first transform the CT image to NS domain. Then apply RGI segmentation algorithm for lesion segmentation [59]. Guo et al. [60] introduced an approach for Skin lesion segmentation based on NCM and KGC. This approach shows superior result than using KGC only or traditional GC.

In brain tumor segmentation Palanisamy et al. [61] introduced the integration between NS and FCM and optimizing the clustering using modified PSO. Another contribution on brain tumor segmentation, by Singh [62] proposed a T2NS method for selecting multi adaptive threshold to segment brain lesions. This approach shows superior results on neutrosophic-based adaptive threshold. Also, Tufail et al. [63] proposed a method to extract ROI under NS domain based on modified s-function.

In Parkinson's disease, Singh [64] proposed methods based on neutrosophic based adaptive threshold for segmentation. This approach aimed to solve two problems, first is the gray and white boundaries, second unclear gray regions. An expand for this work, which composed of two parts NEBCA for segmentation and HSV color system for better representation.

Table 3. Different medical image segmentation/clustering methods using the NS theory.

Authors	Modality	Organ	Gray/ RGB	Methods	Evaluation metric
Shahin et al. (2018b) [54]	MIA	WBC	RGB	multi-scale similarity measure	Precision= 97.2%
Ali et al. (2018) [56]	x-ray	Dental	Gray level	NS orthogonal principle	Davies-Bouldin =10.562 Simplified silhouete width criterion (SSWC)=0.941 Visibility metric (VM)=484.002
Ashour et al. (2018) [57]	Dermoscopic	skin lesion	Gray level	Histogram-based clustering estimation (HBCE) and neutrosophic c-means (NCM)	Accuracy= 96.3%
Lotfollahi et al. (2018) [58]	US	Breast	Gray level	Active contour models	True positives (TP)=95% False positives (FP)=6% Similarity scores=90%

. Lee et al. (2018) [59]	CT	Breast lesion	Gray level	RGI integrated with NS	Dice coefficient (DC) =0.82 AUC = 0.8
Guo et al. (2019) [60]	Dermoscopic images	Skin lesion segmentation	Gray level	(NCM) and kernel graph cut(KGC)	Accuracy= 97.41% DC = 93.27% Jaccard similarity coefficients (JAC) = 87.78% Sensitivity=99.21%
Ashour et al. (2019) [55]	MIA	WBC	RGB	Canny detector, circular Hough transform (CHT) and k-means	Accuracy=98.44% DC=93.10% JAC= 87.14% Sensitivity=95.08%
Palanisamy et al. (2019) [61]	MRI	Brain tumor	Gray level	Fuzzy C-mean (FCM) clustering guided with a modified particle swarm optimization (PSO)	Sensitivity=95.43% Specificity=98.58% JAC= 87.56% DC= 94.32%
Singh (2020) [64]	MRI	Parkinson's disease	Gray level	Adaptive threshold and neutrosophic entropy based	Result for testing data PSNR=62.99 MSE= 0.10 SSIM= 0.7006
Singh (2020) [65]	MRI	Parkinson's disease	Gray level	neutrosophic-entropy based clustering algorithm (NEBCA), and HSV color system.	**(Result of HSV color system based on testing set)** standard deviation (SD)= 13885000 total neutrosophic entropy information (TNEI) =323670
Singh (2021) [62]	MRI	Brain tumors	Gray level	Type-2 NS (T2NS) entropy and multiple threshold	**Result of sets (Set I, Set II and Set III)** JAC=97.07%, 97.92% ,97.13% Correlation coefficients=0.9638, 0.9698 ,0.9610 Uniformity measures = 0.9624, 0.9633 and 0.9660

Tufail et al. (2021) [63]	MRI	Detect ROI Brain tumor	Gray level	ROI in tumor images extract using S-function	Sensitivity=98%, false negative (FN) =1.5%

3.3. Neutrosophic sets in medical image classification

NS classification is achieving success because of its use of simple procedures. The NS classifier utilizes NL to manage the gain noise and indeterminacy. We summarized all studies in Table 4.

Xian et al.[66] introduced neutrosophic subset and neutrosophic connectedness. This approach showed it's results on breast US images with superior performance than fuzzy connectedness. Moreover, Gaber et al.[67] proposed a segmentation and classification approach for thermogram image. in segmentation, an integration between NS and FCM was introduced. In classification, a SVM is used to normal or abnormal regions. Another study proposed a SVM as classifier by combining texture and morphological features.

Table 4. Different medical image classification methods using the NS theory.

Author	Year	Modality	Task	Method using NS
Xian et al.[66]	(2014)	US	Breast	Neutro-Connectedness
Gaber et al.[67]	(2015)	Thermogram	Breast	Fast fuzzy c-means (FFCM)
Amin et al. [68]	(2016)	US	Breast	NS score

4. Deep learning in medical image analysis

DL has great success over traditional ML algorithms. Neural networks (NN) are the key foundation for DL. It is implemented with more than two layers to permit non-linear operations, Which makes it widely used in medical image Denoising and clustering, segmentation, and classification phases [23].

Learning strategies are divided into supervised, semi-supervised, and unsupervised algorithms. SL has a labeled data that intends to learn the function of given data, Semi-supervised learning (SSL) intends to learn unlabeled data points using knowledge learned from labeled data. Unsupervised learning (USL) has not any label data, so its objective is to deduce the real structure present with a group of the data point.

At Present, CNN and RNN are widely utilized in medical image diagnosis as (SL). On the other hand, the Auto-encoder (AE), Restricted boltzmann machine (RBM), and DBN are widely used as (USL)or (SSL). In the following Table 5. there is a brief on Different DL architectures [23]. Koundal et al.2018 [52].

Table 5. Different DL architecture in SL, SSL, and USL

SL			
Architecture	**Variants**	**Main feature**	**Main problem**
CNN	LeNet Alexnet VGGnet Resnet GoogleNet	-parameter sharing -spatial relationship	Require labeled data and large scale of data
RNN	LSTM	-parameter sharing -recurrent connections	-Vanishing /exploding gradient problem

USL or SSL		
Architecture	**Main features**	**Main problem**
AE	-Has unidirectional connection - Greedy strategy is implemented in each layer	Require a pretrained phase
RBM	More general than DBN where all edges are indirect	Cannot Optimizing parameters during large scale of data
DBN	Is probabilistic generative model with an RBM.	High computation Training process results from initialization

4.1. DL in medical image denoising

One of famous medical noisy image is additive white noisy images which can be salt and pepper, gaussian, and poison noisy images. Also, real noisy image is considered one of denoising problem as it comprises blurry and false image artifacts.in the other hand, there is a need for DL technique to overcome the real complex and noise which results from image corruption. Sometimes the images consist of hybrid types of noises [69].Some of DL techniques to solve noisy image problem follows in Table 6.

Table 6. DL techniques in medical image denoising.

Ref.	Method	Modalities	Noise
Ma et al. (2018) [70]	conditional Generative adversarial network (cGAN)	Retinal OCT	Speckle (Edge preservation)
Meng et al.(2020)[71]	CNN	CT	Low-dose CT imaging
Chai et al.(2019)[72]	hierarchical deep generative adversarial networks (HD-GANs)	CT	Low-dose CT imaging
Jifara et al. (2019)[73]	CNN using residual learning	x-ray	Poison

Gholizadeh-Ansari et al.(2018)[74]	CNN using residual learning	CT, x-ray	Low dose

4.2. Deep learning in medical image clustering and segmentation

Lesions and organs segmentation is an important area in DL. Some semantic segmentation research works have been done over various segmentation modalities and organs are shown in Table 7.

Table 7. DL techniques in medical segmentation.

Authors	Modality	Organ	Methods
Konstantinos et al. (2017) [75]	MRI	brain	3D FCNN, CRF
Zilly et al. (2017) [76]	Retinal image	Glaucoma	Simple CNN, sequential learned to use boosting.
Abdel-Basset et al. (2021) [77]	CT	Covid-19	Implement FSS-2019-nCov architecture based on Few-shot learning
Chen et al. (2021) [78]	X-ray	Dental	(MSLPNet) multi-scale location perception network
Ding et al. (2021) [79]	MRI	Brain	Informed DL segmentation (FI-DL-Seg) network

4.3. Deep learning in medical image classification

DL shows precedence success in image classification. CNN is the most used architecture since the propositions of Alexnet 2012 by [8]. Which becomes the beginning of many architectures depending on CNN such as GoogleNet, VGG, and Resnet. Many studies show its effort in DL in medical image classification shows in Table 8.

Table 8. DL techniques in medical image classification.

Author	Year	Modality	Task	Method
Pinaya et al. [80]	2016	Brain morphography	Schizophrenia	DBN
Fu et al. [81]	2018	Retinal	Glaucoma	M-Net
Zhang et al.[82]	2019	Different modalities	Multi-classification task	Multiple DCNNs
Wang et al. [83]	2020	CT	Liver	CNN

5. Deep learning in medical image analysis using NS theory

Some studies show results of NS integration with DL algorithms in medical images analysis with explores in this section and in the following Table 9.

Table 9: Summarization of different studies in medical image analysis using NS-based DL models.

Author	Organ/modality	Task	DL architecture
Guo et al. (2019) [84]	Skin lesion	classification	DCNN
Özyurt et al.(2019) [85]	Brain tumor	Segmentation, classification	CNN with neutrosophic expert maximum fuzzy (NS-CNN) sure entropy
Khalifa et al. (2020) [86]	X-ray Covid-19	Classification	Alexnet, GoogleNet, Resnet
Cai et al. (2019) [87]	mammogram Breast cancer	Classification	DCNN
Shain et al. (2020) [88]	cardio location	Classification	CNN architectures, LSTM

MDCNN is proposed by Guo et al.(2019) [84] for skin dermoscopic image classification between melanoma(malignant) and nevi(benign). First, Guo et al. implement DCNN architecture consists of convolution, ReLU (rectified linear unit), pooling, softmax, and classification layer. For speeding up the training phase, multiple pre-trained CNN models were applied with transfer learning (TL), where a neutrosophic reinforcement sample learning (NRSL) strategy is introduced in the MDCNN. Usually, the NRSL is not used in a single DCNN training; the NRSL is used in the model to develop the next DCNN samples in the MDCNN model.

The ISIC2016 dataset was joined to assess the proposed NMDCNN model as in Figure 1. For every DCNN, the implemented NRSL TL-based was introduced to train each DCNN model on the different samples. The NSS was introduced to define the reinforced training time, which differs based on sample performance. This procedure was replicated for every DCNN in the MDCNN, selection criteria based on the previous model previous score.

An MDCNN architecture is constructed by follow multiple networks of the same implementation. For Q, the total number of DCNNs samples, the MDCNN can be expressed as:

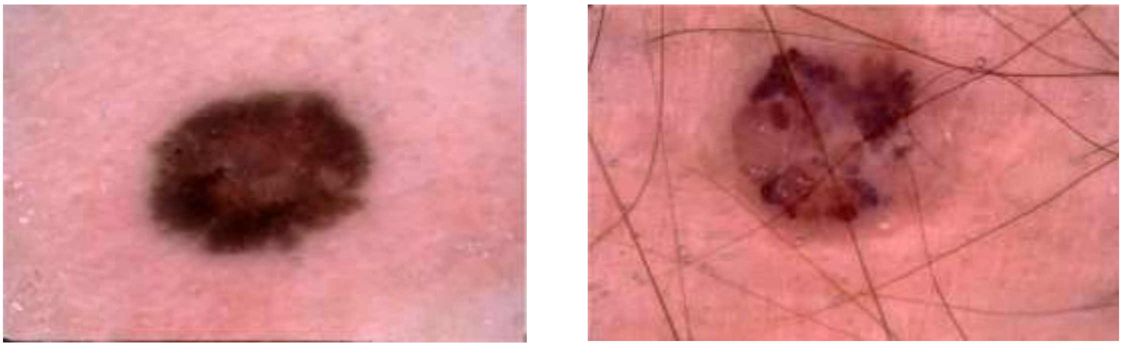
$$MDCNN = \{CNN_1, CNN_2, \ldots, CNN_q\}$$

Normally, every DCNN, usual training was used while the incremental learning was used to generate samples for the next DCNN as follows:

$$SP_{q+1} = \{SP_q, ReInSP_q\}$$

where SP_q and SP_{q+1} are the sample for the qth DCNN and (q+1)th DCNN, respectively, and $ReInSP_q$ is the reinforcement sample for the (q+1)th DCNN.

The voting scheme is used for evaluating the classification results. The results show the effect of the NMDCNN model on testing, training accuracies with 97.78%, 85.22% respectively.



(a) (b)

Figure 1. Samples from demoscopic dataset: (A) benign (nevus) (B) malignant.

Cai et al. [87] introduced an NS-DL technique to detect breast cancer in the mammogram. The proposed consists of five stages as in Figure 2. The test is done using data gathered from Nanfang Hospital (NFH), Guangzhou, China, and the publicly available [89].

In the coarse stage detection, the thresholding method is used for image binarization, and the connected component analysis classifies the binary image regions to (TP) and (FP). The DCNN1 is the training phase based on DCNN architecture. In Traditional DCNN, all samples are trained in fixed time so, adding NRSL strategy through the training phase trains samples in adaptive time.

The Microcalcifications (MC) clustering phase done using density-based spatial clustering of applications with a noise (DBSCAN) algorithm which has high adaptability with noise. The second classifier (DCNN2) is used to train the data set while DCNN1 is only for MC detection and FP deduction.

The final stage is diagnosis and testing for data to give the probability of malignancy using a bounding box. The results of MC detection stage 92% sensitivity and0.50 FP per image in cluster evaluation. After 40 epochs, training, validation, and testing accuracies are 99.87%, 95.12%, and 93.68% respectively while, 98.03%, 93.49%, and 92.36% for comparative. Methods AUC for validation and testing 0.908 and 0.872 for DCNN2.

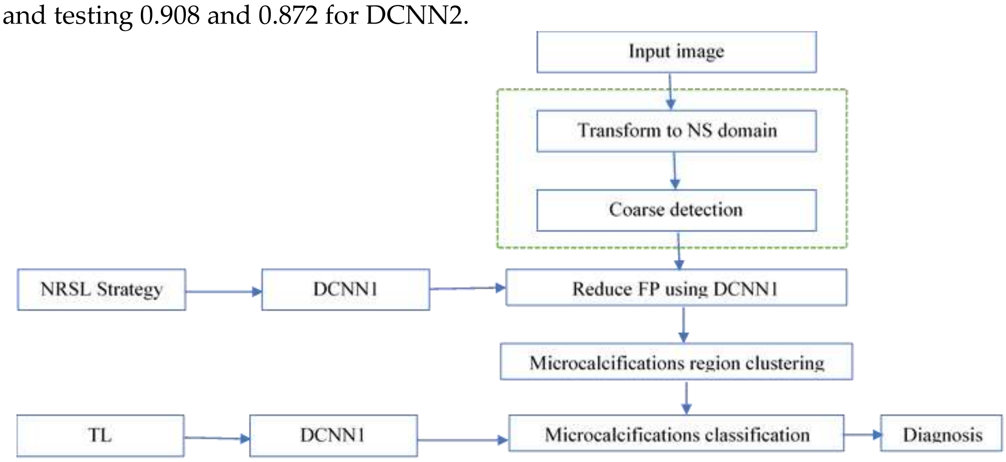


Figure 2. Breast cancer detection architecture based on NS and DL.

Özyurt et al.(2019) [85] classified the segmented brain tumor using hybridization between NS and CNN(NS-CNN). To test the proposed approach The Cancer Genome Atlas Glioblastoma Multiforme (TCGA-GBM) data collection in The Cancer Imaging Archive (TCIA) was used. The proposed (NS-EMFSE-CNN) is consists of 3 stages: segmentation, feature extraction, and classification as in Figure 3.

The first stage segments the MRI brain using the NS-EMFSE algorithm. This algorithm consists of three steps. In the first step, some pre-processing techniques are applied to (T1-GD sequence) MR. In the second step the NS- EMFSE method transform the filtered image NS image then converts it to a binary image. The last step is cleaning residual pixels and fill gaps in the edge image. This process is reviewed in Algorithm 1.

Algorithm 1: Image Segmentation using NS-EMFSE	
1.Getting brain MRI from dataset.	
2. Convert the image to grayscale form.	Step 1
3.Apply adaptive winner filter to gray scale image.	
4.Using NS-EMFSE method to convert filtered image into binary image.	Step 2
5.Cleannig residual pixels in binary image.	Step 3
6.Filling gaps.	
The finally segmented image is obtained in algorithm 2 for training and testing preparation	

Algorithm 2: Image preparation for training and testing
1. Get the corresponding white colored points in the segmented image.
2. Get the corresponding points in the original image.

The second stage in NS-EMFSE-CNN is feature extraction which is done using Alexnet architecture based on CNN to avoid manual feature extraction. The fully connected layer (FC7) in Alextnet obtain 4096 features which are given to the next stage of classification as in Figure 3.

The final stage is a classification which is done using Support vector machine (SVM) and K-nearest neighbor (KNN) classifiers. The results show higher accuracy when using SVM classifier than KNN. Also shows high sensitivity for both classifiers which indicates that feature is more judicial to benign tumors. At the same time, specificity rates were lower which indicates that malignant tumor features were less notable.

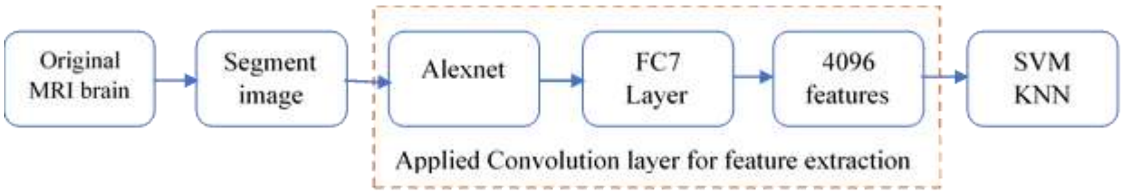


Figure 3. (NS-EMFSE-CNN) architecture

Khalifa et al. [86] provided a study that shows the effect of hybrid NS and DTL on classification. The study work on Covid-19 x-ray dataset images. It was gathered from various websites such as the Italian Society of Medical, online publications, and the Radiopaedia web. The formed dataset is arranged into four categories normal, pneumonia bacterial, pneumonia virus, and COVID-19 with several images 79, 79, and 79, and 69, respectively.

The NS-DTL model first converted the original image to the NS image So every pixel in the image has been divided into three subsets (T, I, F). Then applied different DTL model with DL strategies under specific hyperparameter for training and testing phases such as in Table 10. More than 36 trails had been conducting to assess the performance of the NS conversion. Four domains of images are tested, and they are the original images T, I, and F images such as in Figure 4.

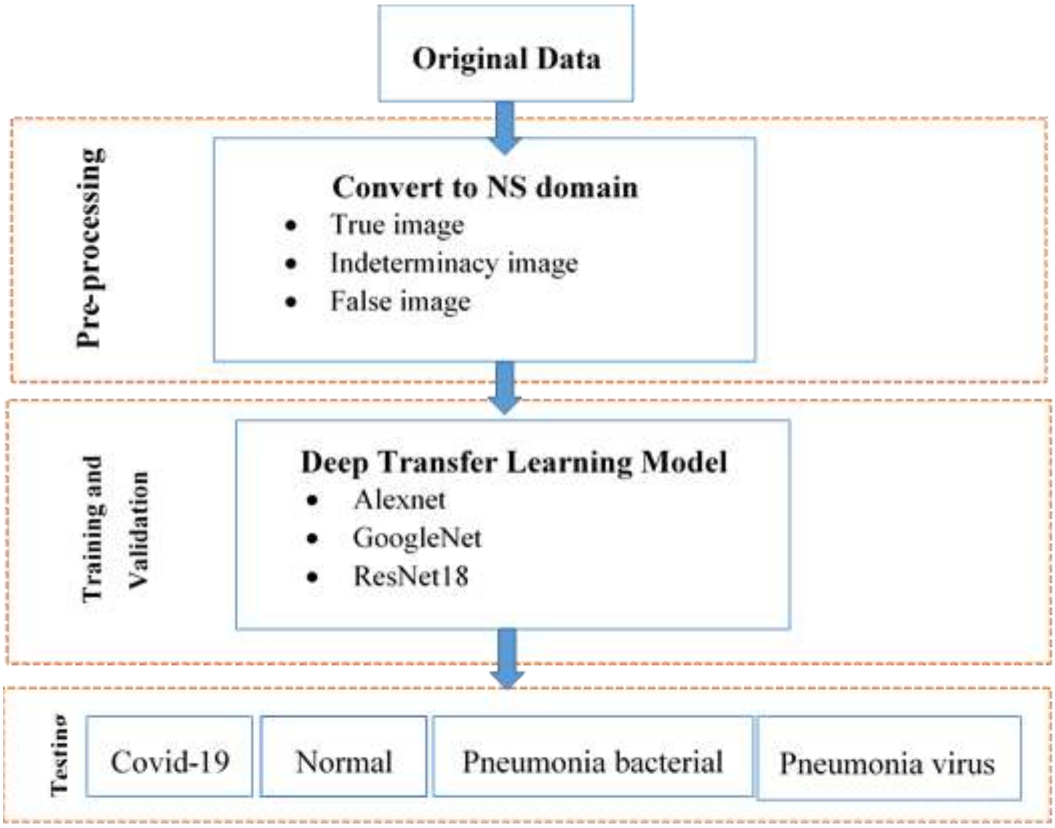


Figure 4. NS/DTL model

Table 10. Ns-DTL learning strategies, models and hyperparameters.

Different training and testing strategies
Training-Testing
70% – 30%.
80% – 20%.
90% – 10%.
Different DT models
Resnet18
Googlenet
Alexnet
Hyperparameters
Optimizer: Adaboost
Momentum: 0.9
Epochs: 50
Early stopping: 10 epochs
Batch size: 32
Learning Rate: 0.001

According to the experimental results, the maximum accuracy possible in the testing accuracy and performance metrics such as F1Score, recall, precision was achieved by the Indeterminacy (I) NS domain.

Shain et al. [88] proposed a classification framework to classify the 3-location of cardio view. The proposed integrated the LSTM and CNN architecture as in Figure 5. Also, the proposed use of NS to extract the temporal descriptor to combine the spatial and NS temporal. Then Using CNN as a pre-trained model. Lastly, utilize LSTM to classify each echo clip into eight cardio-views.

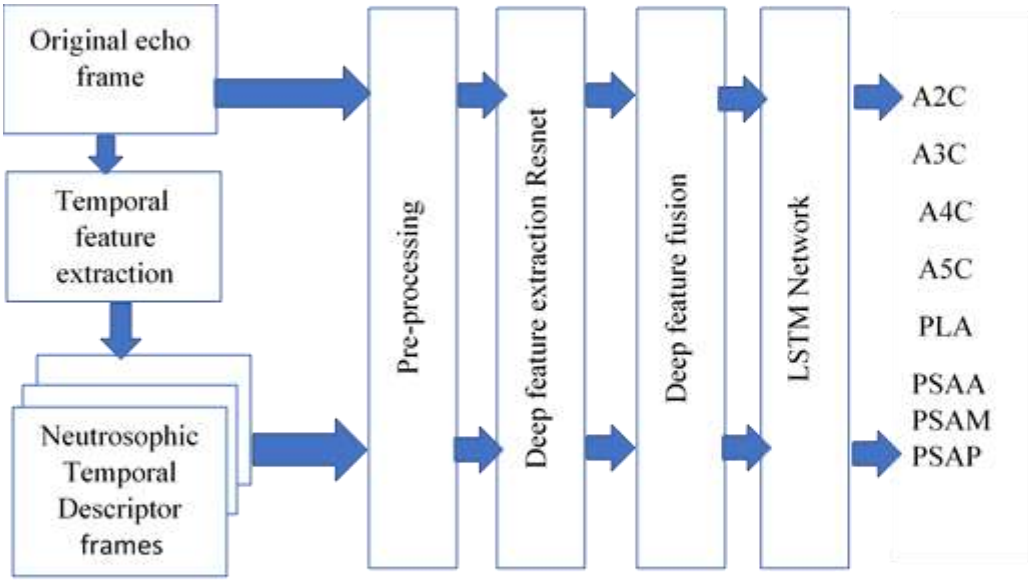


Figure 5. Eight cardio-views classification system.

Echocardiography clips consist of frames each frame has a spatial descriptor, which is converted to a temporal descriptor using the NS theory. Then a preprocessing stage of resizing spatial and temporal frames to fit the pre-trained network.

The proposed use of both spatial and temporal descriptors for CNN feature extraction. The feature extraction phase is done using CNN architectures such as Alexnet, VGGNets, GoogleNet, Densenet, and ResNets. The next stage is feature fusion which gathers the latest information of cascading spatial and temporal descriptors from FC layers of both model's streams. Finally, LSTM is used to classify the fused CNN features. LSTM the implementation is done using Quad-Core 2.9 GHz Intel i5 with 16 GB of memory, and moderate graphic processing unit NVIDIA TITAN-Xp GPU with 12 GB RAM.

The results show that ResNet101 achieves the highest performance in Spatial-temporal and fusion features with an accuracy score of 96.3% and 99.1% for cardio location classification.

6. Conclusion and future works

Recent progress in the deep learning research area shows a successful impact on medical image analysis. Deep learning performance can be improved via integration with neutrosophic systems. Recently, deep learning performance was affected by noise, ambiguity, or incomplete data, which are the major problems in medical data images.

At the time being, many researchers aim to tackle these issues by using neutrosophic systems. Some studies show that the hybridization of neutrosophic theory and deep learning can enhance the performance of medical image analysis where data are noisy, fuzzy, incomplete, or ambiguous. Neutrosophic systems can be used as an essential part of deep learning models by using neutrosophic reinforcement sample learning to speed up the training procedure and reinforce training to the poor performance samples with more times according to their performance. Neutrosophic image transformation (T, I, F) for each pixel can increase the computational complexity, but it provides great resistance to noise. The availability of software platforms such as Intel MKL, AMD ROCm, and Nvidia CUDA can speed up deep learning processes.

In almost all neutrosophic sets in medical image analysis or in hybridization between Neutrosophic sets and deep learning, image conversion to the neutrosophic image has been carried using the same equations, which used in all medical image modalities, all different stages in image analysis, and with any deep learning architecture integration. Every case in medical image analysis requires convenient define of membership function.

At present, some research efforts show the results of deep learning and neutrosophic set integration. But there is an essential need to show studies in neutrosophic in deep learning parameter optimizing, neutrosophic with medical big data analysis, and various types of medical image modalities and applications. So, more comprehensive studies should be developed, such as studies on fuzzy neural networks. Enhancing the performance of neutrosophic deep learning models can be explored in the future.

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Types of System of the neutrosophic linear equations and Cramer's rule

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Abstract: The purpose of this article is studying the types of system of neutrosophic linear equations, where the neutrosophic linear equation and the solution to the neutrosophic linear equation are defined. Also, finding a solution to a linear equation with two variables, the general situation of the solution. In addition to studying the n-variable neutrosophic linear equation and the system of neutrosophic homogeneous linear equations. The most important is the introduction of the concept of Cramer's rule to solve the system of neutrosophic linear equations. Provide enough examples for each case to enhance understanding.

Keywords: neutrosophic liner equation; Cramer's rule; solution of the neutrosophic linear equation.

1. Introduction

As an alternative to the existing logics, Smarandache proposed the Neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3-13]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist, he defined the standard form of neutrosophic complex number, and found root index $n \geq 2$ of a neutrosophic real and complex number [2-4], studying the concept of the Neutrosophic probability [3-5], the Neutrosophic statistics [4][6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1-8]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [9].Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side represented with triangular neutrosophic numbers [10]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [11-12]. Y.Alhasan studied the concepts of neutrosophic complex numbers, the general exponential form of a neutrosophic complex, and the neutrosophic integrals and integration methods [7-14-17]. On the other hand, M.Abdel-Basset presented study in the science of neutrosophic about an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number [15]. Also, neutrosophic sets played an important role in applied science such as health care, industry, and optimization [16]. Recently, Alhasan,Y., and Alfahal, A presented study in the neutrosophic differential equations that translate into linear[18].

Mathematical equations are used to solve real problems in our daily life, for example, mathematical equations are used in electronic chips used in all modern machines and devices, such as washing machines and dryers, cars, airplanes, ships, cell phones, computers, space programs and so on. One may be surprised when he learns that there are about 2 million algorithms and mathematical equations in mobile and computer devices.

The paper consists of 5 sections. In 1th section, provides an introduction, in which neutrosophic science review has given. In 2th section, some definitions and examples of neutrosophic real number. The 3th section frames studying types of system of the neutrosophic linear equations. The 4th section introduces the concept of Cramer's rule to solve the system of neutrosophic linear equations. In 5th section, a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic Real Number [4]

Suppose that w is a neutrosophic number, then it takes the following standard form: $w = a + bI$ where a, b are real coefficients, and I represent indeterminacy, such $0.I = 0$ and $I^n = I$, for all positive integers n.

2.2. Division of neutrosophic real numbers [4]

Suppose that w_1, w_2 are two neutrosophic numbers, where

$$w_1 = a_1 + b_1 I \ , \qquad w_2 = a_2 + b_2 I$$

To find $(a_1 + b_1 I) \div (a_2 + b_2 I)$, we can write:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} \equiv x + yI$$

where x and y are real unknowns.

$$a_1 + b_1 I \equiv (a_2 + b_2 I)(x + yI)$$

$$a_1 + b_1 I \equiv a_2 x + (b_2 x + a_2 y + b_2 y)I$$

by identifying the coefficients, we get

$$a_1 = a_2 x$$

$$b_1 = b_2 x + (a_2 + b_{2)} y$$

We obtain unique one solution only, provided that:

$$\begin{vmatrix} a_2 & 0 \\ b_2 & a_2 + b_2 \end{vmatrix} \neq 0 \quad \Rightarrow \quad a_2(a_2 + b_2) \neq 0$$

Hence: $a_2 \neq 0$ *and* $a_2 \neq -b_2$ are the conditions for the division of two neutrosophic real numbers to exist.

Then:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} . I$$

3. The neutrosophic linear equation

Defination3.1:
The neutrosophic linear equation of n variables $x_1, x_2, x_3, \dots\dots x_n$, is each equation that takes the form:

$$(a_1 + b_1 I)x_1 + (a_2 + b_2 I)x_2 + (a_3 + b_3 I)x_3 + \cdots + (a_n + b_n I)x_n = c + dI$$

Where:
$a_1, a_2, a_3, \dots, a_n, b_1, b_2, b_3, \dots, b_n, c, d$ are real coefficients, and I represent indeterminacy.
We call $(a_1 + b_1 I), (a_2 + b_2 I), (a_3 + b_3 I), \dots, (a_n + b_n I)$ neutrosophic coefficients of the borders of the equation, and $c + dI$ constant neutrosophic border of the equation.

Remarks3.1:
We call each equation of the form:

$$(a_1 + b_1 I)x + (a_2 + b_2 I)y = c + dI$$

the two-variable neutrosophic linear equation, where:
$a_1, a_2, a_3, \dots, a_n, b_1, b_2, b_3, \dots, b_n, c, d$ are real coefficients, and I represent indeterminacy.

Remarks3.2:
We call each equation of the form:

$$(a_1 + b_1 I)x + (a_2 + b_2 I)y + (a_3 + b_3 I)z = c + dI$$

the three-variable neutrosophic linear equation, where:
$a_1, a_2, a_3, \dots, a_n, b_1, b_2, b_3, \dots, b_n, c, d$ are real coefficients, and I represent indeterminacy.

Example3.1:

- ✓ $(2 + 3I)x + (4 - 5I)y + (-1 + I)z = 5 + 4I$
- ✓ $(3 - I)x + (9 - 5I)y + (-1 + I)z + (3 - 2I)w = 7 - I$
- ✓ $(1 + I)x + (2 - 5I)y = 6 - 2I$

Defination3.2:
Solution of the neutrosophic linear equation:

$$(a_1 + b_1 I)x_1 + (a_2 + b_2 I)x_2 + (a_3 + b_3 I)x_3 + \cdots + (a_n + b_n I)x_n = c + dI$$

is finding the values of the variables $x_1, x_2, x_3, \dots, x_n$ that make its first term equal to its second term. where a_1, a_2, b_1, b_2, c, d are real coefficients.

Example3.2:
The equation: $(1 + I)x + (2 - 5I)y = 6 - 2I$ accepts the solution

$$x = 1 - I,\ \ y = \frac{5}{2} - \frac{23}{6} I$$

because fulfills the equation:

$$L_1 = (1 + I)(1 - I) + (2 - 5I)\left(\frac{5}{2} - \frac{23}{6} I\right) = 1 - I + 5 - \frac{46}{6} I - \frac{25}{2} I + \frac{115}{6} I = 6 - 2I = L_2$$

3.1 Solution of the two-variables neutrosophic linear equation

For the neutrosophic linear equation:

$$(a_1 + b_1 I)x + (a_2 + b_2 I)y = c + dI$$

unlimited number of solutions defined by form:

$$S = \left\{(x, y) \in R^2 \cup \{I\}: y = \left(\frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)}.I\right)x + \frac{\mathrm{c}}{a_2} + \frac{a_2 \mathrm{d} - \mathrm{c} b_2}{a_2(a_2 + b_2)}.I\right\}$$

where a_1, a_2, b_1, b_2, c, d are real coefficients, $a_2 \neq 0$ and $a_2 \neq -b_2$, by given a value for one of the two variables, we obtain a value for the other variable.

Example3.1.1:
Find solution of the equation:

$$(1 + I)x + (2 - 5I)y = 6 - 2I$$

Solution:

$$y = \frac{-(1+I)}{(2-5I)}x + \frac{6-2I}{(2-5I)}$$

$$y = \left(\frac{-1}{2} + \frac{7}{6}I\right)x + 3 + \frac{13}{3}I$$

Then the set of solutions is:

$$S = \left\{(x,y) \in R^2 \cup \{I\}: y = \left(\frac{-1}{2} + \frac{7}{6}I\right)x + 3 + \frac{13}{3}I\right\}$$

By given any value for the variables x, we obtain a value of the variable y.

3.2 General situation: Solution of the n-variable neutrosophic linear equation

For the neutrosophic linear equation:

$$(a_1 + b_1 I)x_1 + (a_2 + b_2 I)x_2 + (a_3 + b_3 I)x_3 + \cdots + (a_n + b_n I)x_n = c + dI$$

unlimited number of solutions, where $a_1, a_2, a_3, \dots, a_n, b_1, b_2, b_3, \dots, b_n, c, d$ are real coefficients.

4. System of the neutrosophic linear equations

It is a system of neutrosophic linear equations that given by the form:

$$\begin{cases} (a_{11} + b_{11} I)x_1 + (a_{12} + b_{12} I)x_2 + (a_{13} + b_{13} I)x_3 + \cdots + (a_{1n} + b_{1n} I)x_n = c_1 + d_1 I \\ (a_{21} + b_{21} I)x_1 + (a_{22} + b_{22} I)x_2 + (a_{23} + b_{23} I)x_3 + \cdots + (a_{2n} + b_{2n} I)x_n = c_2 + d_2 I \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \dots = \cdots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots = \cdots \\ (a_{m1} + b_{m1} I)x_1 + (a_{m2} + b_{m2} I)x_2 + (a_{m3} + b_{m3} I)x_3 + \cdots + (a_{mn} + b_{mn} I)x_n = c_m + d_m I \end{cases}$$

Where:

a_{ij}, b_{ij}, c_j, d_j are real coefficients, $i = 1, \dots, n$, $j = 1, \dots, m$, and I represent indeterminacy.

4.1 Solution of system of the neutrosophic linear equations

Solution of system of the neutrosophic linear equation:

$$\begin{cases} (a_{11} + b_{11} I)x_1 + (a_{12} + b_{12} I)x_2 + (a_{13} + b_{13} I)x_3 + \cdots + (a_{1n} + b_{1n} I)x_n = c_1 + d_1 I \\ (a_{21} + b_{21} I)x_1 + (a_{22} + b_{22} I)x_2 + (a_{23} + b_{23} I)x_3 + \cdots + (a_{2n} + b_{2n} I)x_n = c_2 + d_2 I \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \dots = \cdots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots = \cdots \\ (a_{m1} + b_{m1} I)x_1 + (a_{m2} + b_{m2} I)x_2 + (a_{m3} + b_{m3} I)x_3 + \cdots + (a_{mn} + b_{mn} I)x_n = c_m + d_m I \end{cases} \quad (*)$$

Where:

a_{ij}, b_{ij}, c_j, d_j are real coefficients, $i = 1, \dots, n$, $j = 1, \dots, m$, and I represent indeterminacy.

is finding the values of the variables $x_1, x_2, x_3, \dots, x_n$ that fulfill each of the system equations. We call this solution the system solution.

Remarks4.1.1:

We distinguish three cases to solve the system$(*)$:

1. It may have unique solution
2. It may be impossible to solve
3. It may have Unlimited number of solutions

4.2 Cramer's rule to solve system of the neutrosophic linear equations

Let the system of the neutrosophic linear equations:

$$\begin{cases} (a_{11}+b_{11}I)x_1+(a_{12}+b_{12}I)x_2+(a_{13}+b_{13}I)x_3+\cdots+(a_{1n}+b_{1n}I)x_n=c_1+d_1I \\ (a_{21}+b_{21}I)x_1+(a_{22}+b_{22}I)x_2+(a_{23}+b_{23}I)x_3+\cdots+(a_{2n}+b_{2n}I)x_n=c_2+d_2I \\ \quad .\dots \quad \dots. \quad \dots \quad \dots. \quad \dots \quad \dots = \cdots \\ \quad .\dots \quad \dots. \quad \dots \quad \dots. \quad \dots \quad \dots = \cdots \\ (a_{n1}+b_{n1}I)x_1+(a_{n2}+b_{n2}I)x_2+(a_{n3}+b_{n3}I)x_3+\cdots+(a_{nn}+b_{nn}I)x_n=c_n+d_nI \end{cases} \quad (*)$$

with the n variables and the n equations, A the neutrosophic coefficient matrix of the system, and suppose $\det(A)=a+bI$, We distinguish the following cases:

1. If $\det(A)\neq 0+0I$ or $a\neq 0$ or $a\neq -b$, then the system has unique solution given by the formula:

$$x_i=\frac{\det(x_i)}{\det(A)}\ ;\ i=1,2,\dots,n$$

Where $\det(x_i)$ is the determinant produced by the determinant $\det(A)$ by replacing the column of constants with the i -order column.

2. If $a=0$ or $a=-b$, then the system is impossible to solve.
3. If $\det(A)=0+0I$, then we distinguish tow cases:
   - If one of the determinants $\det(x_1)$, $\det(x_2),\dots,\det(x_n)$, is not equal to zero, then the system is impossible to solve.
   - If $\det(x_i)=0+0I;\ i=1,2,\dots,n$, then the system is impossible to solve or it have unlimited number of solutions.

4.2.1 Solve the system of two linear equations by two variables

Let:

$$(a_{11}+b_{11}I)x+(a_{12}+b_{12}I)\mathrm{y}=c_1+d_1I$$
$$(a_{21}+b_{21}I)x+(a_{22}+b_{22}I)\mathrm{y}=c_2+d_2I$$

$$A=\begin{bmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \end{bmatrix}$$

$$\det(A)=\begin{vmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \end{vmatrix}=$$

$$\det(x)=\begin{vmatrix} c_1+d_1I & a_{12}+b_{12}I \\ c_2+d_2I & a_{22}+b_{22}I \end{vmatrix}\ ,\ \det(y)=\begin{vmatrix} a_{11}+b_{11}I & c_1+d_1I \\ a_{21}+b_{21}I & c_2+d_2I \end{vmatrix}$$

Suppose $\det(A)=a+bI$, We distinguish the following cases:

1. If $\det(A)\neq 0+0I$ or $a\neq 0$ or $a\neq -b$, then the system has unique solution given by the formulas:

$$x=\frac{\det(x)}{\det(A)}\quad and\quad y=\frac{\det(y)}{\det(A)}$$

2. If $a=0$ or $a=-b$, then the system is impossible to solve.
3. If $\det(A)=0+0I$, then we distinguish tow cases:
   - If one of the determinants $\det(x)$, $\det(y)$, is not equal to zero, then the system is impossible to solve.

- If $\det(x_i) = 0 + 0I;\ i = 1,2,\dots,n$, then the system has unlimited number of solutions.

Note: To verify the solution, we substitute the value of x and y into one of the equations.

Example4.2.1:

Let:

$$(2+I)x + 3y = 5 + I$$
$$(3-2I)x + 2y = I$$

Find solution of the system.

Solution:

$$A = \begin{bmatrix} 2+I & 3 \\ 3-2I & 2 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2+I & 3 \\ 3-2I & 2 \end{vmatrix} = 4 + 2I - 9 + 6I = -5 + 8I \neq 0 + 0I$$

Then the system has one solution, is:

$$\det x = \begin{vmatrix} 5+I & 3 \\ I & 2 \end{vmatrix} = 10 - I \quad , \quad \det y = \begin{vmatrix} 2+I & 5+I \\ 3-2I & I \end{vmatrix} = -15 + 12I$$

$$x = \frac{\det x}{\det A} = \frac{10 - I}{-5 + 8I} = -2 + 5I \quad and \quad y = \frac{\det y}{\det A} = \frac{-15 + 12I}{-5 + 8I} = 3 - 4I$$

$$(x, y) = (-2 + 5I, 3 - 4I)$$

To verify the solution, we substitute the value of x and y into the first equation:

$$(2+I)x + 3y = (2+I)(-2+5I) + 3(3-4I)$$
$$= -4 - 2I + 10I + 5I + 9 - 12I = 5 + I$$

Example4.2.2:

Let:

$$2Ix + 7y = I$$
$$3Ix + y = 2I$$

Find solution of the system.

Solution:

$$A = \begin{bmatrix} 2I & 7 \\ 3I & 1 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2I & 7 \\ 3I & 1 \end{vmatrix} = 2I - 21I = 0 - 19I$$

This does not fulfill the condition: $\det A = \mathrm{a} + \mathrm{b}I$, $\mathrm{a} \neq 0$ and $\mathrm{a} \neq -\mathrm{b}$

Because:

$$\det x = \begin{vmatrix} I & 7 \\ 2I & 1 \end{vmatrix} = I - 14I = 0 - 13I$$

Then:

$$x = \frac{\det x}{\det A} = \frac{0 - 13I}{0 - 19I}\ (undefined)$$

So, the system is impossible to solve.

Example4.2.3:

Let:

$$(2+I)x + 3y = 5 + I$$
$$(3-2I)x + y = I$$

Find solution of the system.

Solution:

$$A = \begin{bmatrix} 2+I & 3 \\ 3-2I & 1 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2+I & 3 \\ 3-2I & 1 \end{vmatrix} = 2 + I - 9 + 6I = -7 + 7I$$

W This does not fulfill the condition: $\det A = \mathrm{a} + \mathrm{b}I$, $\mathrm{a} \neq 0$ and $\mathrm{a} \neq -\mathrm{b}$, where: $(a = -7, b = 7 \; and \; a = -b)$.

So, the system is impossible to solve.

Example4.2.4:

Let:

$$(1 + I)x + (3 - I)y = 2 - 3I$$

$$(2 + 2I)x + (6 - 2I)y = 4 - 6I$$

Find solution of the system.

Solution:

$$A = \begin{bmatrix} 1+I & 3-I \\ 2+2I & 6-2I \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1+I & 3-I \\ 2+2I & 6-2I \end{vmatrix} = 0 + 0I$$

$$\det x = \begin{vmatrix} 2-3I & 3-I \\ 4-6I & 6-2I \end{vmatrix} = 0 + 0I \;, \quad \det y = \begin{vmatrix} 1+I & 2-3I \\ 2+2I & 4-6I \end{vmatrix} = 0 + 0I$$

As: $\det A = \det x = \det y = 0 + 0I$, so the system has Unlimited number of solutions defined by form:

$$S = \left\{(x, y) \in R^2 \cup \{I\} : y = \left(\frac{1}{3} + \frac{2}{3}.I\right)x + \frac{2}{3} - \frac{7}{6}.I\right\}$$

By given any value for the variable x, we obtain a value of the variable y.

4.2.2 Solve the system of three linear equations by three variables

Let:

$$(a_{11} + b_{11}I)x + (a_{12} + b_{12}I)\mathrm{y} + (a_{13} + b_{13}I)z = c_1 + d_1I \quad (1)$$
$$(a_{21} + b_{21}I)x + (a_{22} + b_{22}I)\mathrm{y} + (a_{23} + b_{23}I)z = c_2 + d_2I \quad (2)$$
$$(a_{33} + b_{33}I)x + (a_{33} + b_{33}I)\mathrm{y} + (a_{33} + b_{33}I)x = c_3 + d_3I \quad (3)$$

$$A = \begin{bmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I & a_{13} + b_{13}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I & a_{23} + b_{23}I \\ a_{33} + b_{33}I & a_{33} + b_{33}I & a_{33} + b_{33}I \end{bmatrix}$$

$$\det A = \begin{vmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I & a_{13} + b_{13}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I & a_{23} + b_{23}I \\ a_{33} + b_{33}I & a_{33} + b_{33}I & a_{33} + b_{33}I \end{vmatrix}$$

$$\det x = \begin{vmatrix} c_1 + d_1I & a_{12} + b_{12}I & a_{13} + b_{13}I \\ c_2 + d_2I & a_{22} + b_{22}I & a_{23} + b_{23}I \\ c_3 + d_3I & a_{33} + b_{33}I & a_{33} + b_{33}I \end{vmatrix} \;, \quad \det y = \begin{vmatrix} a_{11} + b_{11}I & c_1 + d_1I & a_{13} + b_{13}I \\ a_{21} + b_{21}I & c_2 + d_2I & a_{23} + b_{23}I \\ a_{33} + b_{33}I & c_3 + d_3I & a_{33} + b_{33}I \end{vmatrix}$$

$$\det z = \begin{vmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I & c_1 + d_1I \\ a_{21} + b_{21}I & a_{22} + b_{22}I & c_2 + d_2I \\ a_{33} + b_{33}I & a_{33} + b_{33}I & c_3 + d_3I \end{vmatrix}$$

We will discuss the second case of (3):

If $\det(A) = 0 + 0I$ *and* $\det(x_i) = 0 + 0I$; $i = 1,2, \dots, n$, then we are looking for the solutions of two of the system of equations, such as {(1), (2)}:

- ✓ If the system {(1), (2)} is impossible to solve, then the system {(1), (2), (3)} is impossible to solve.

- ✓ If the system {(1), (2)} has unlimited number of solutions from the form $x = g(z), y = h(z)$, (maybe $g(z)$ *or* $h(z)$ is constant), then we substitution in (3) to obtain on the equation of the form:

$$0.z = \beta;\ \beta \in R \cup \{I\}$$

This equation is impossible to solve or it have unlimited number of solutions (According to β value).

Example4.2.5:

Let:

$$(2 + 2I)x + (3 + 3I)y + (-5 - 5I)z = 1 \qquad (1)$$

$$(1 + I)x + (-1 - I)y + (1 + I)z = 2 \qquad (2)$$

$$(5 + 5I)x + (5 + 5I)y + (-9 - 9I)z = 4 \qquad (3)$$

Find solution of the system.

Solution:

$$A = \begin{bmatrix} 2+2I & 3+3I & -5-5I \\ 1+I & -1-I & 1+I \\ 5+5I & 5+5I & -9-9I \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2+2I & 3+3I & -5-5I \\ 1+I & -1-I & 1+I \\ 5+5I & 5+5I & -9-9I \end{vmatrix} = 0 + 0I$$

$$\det x = \begin{vmatrix} 1 & 3+3I & -5-5I \\ 2 & -1-I & 1+I \\ 4 & 5+5I & -9-9I \end{vmatrix} = 0 + 0I \quad , \quad \det y = \begin{vmatrix} 2+2I & 1 & -5-5I \\ 1+I & 2 & 1+I \\ 5+5I & 4 & -9-9I \end{vmatrix} = 0 + 0I$$

$$\det z = \begin{vmatrix} 2+2I & 3+3I & 1 \\ 1+I & -1-I & 2 \\ 5+5I & 5+5I & 4 \end{vmatrix} = 0 + 0I$$

As: $\det A = 0 + 0I = \det x = \det y = \det z = 0 + 0I$, so we are looking for a system solution {(1), (2)}, then:

$$x = \frac{1}{5}\left(2z + 7 - \frac{7}{2}I\right)$$

$$y = \frac{1}{5}\left(7z - 3 + \frac{3}{2}I\right)$$

By substitution in (3), we get:

$$(0 + 0I)z = 0 + 0I$$

This equation has unlimited number of solutions, so the system {(1), (2), (3)} has unlimited number of solutions given by:

$$S = \left\{(x, y, z) \in R^3 \cup \{I\}: x = \frac{1}{5}\left(2z + 7 - \frac{7}{2}I\right), y = \frac{1}{5}\left(7z - 3 + \frac{3}{2}I\right)\right\}$$

Or:

$$S = \left\{\left(\frac{1}{5}\left(2z + 7 - \frac{7}{2}I\right), \frac{1}{5}\left(7z - 3 + \frac{3}{2}I\right), z\right); z \in R \cup \{I\}\right\}$$

Example4.2.6:

Let:

$$(2 + I)x + (1 + I)y + (3 - I)Z = 2 + I$$

$$(-1+I)x+(3-2I)y+(1+3I)Z=4+2I$$
$$(3+2I)x+(4-I)y+(2-3I)Z=5-I$$

Find solution of the system.

Solution:

$$A=\begin{bmatrix}2+I & 1+I & 3-I\\ -1+I & 3-2I & 1+3I\\ 3+2I & 4-I & 2-3I\end{bmatrix}$$

$$\det A=\begin{vmatrix}2+I & 1+I & 3-I\\ -1+I & 3-2I & 1+3I\\ 3+2I & 4-I & 2-3I\end{vmatrix}=-30+21I$$

$$\det x=\begin{vmatrix}2+I & 1+I & 3-I\\ 4+2I & 3-2I & 1+3I\\ 2-3I & 4-I & 2-3I\end{vmatrix}=4+29I \quad , \quad \det y=\begin{vmatrix}2+I & 2+I & 3-I\\ -1+I & 4+2I & 1+3I\\ 3+2I & 2-3I & 2-3I\end{vmatrix}=-35-31I$$

$$\det z=\begin{vmatrix}2+I & 1+I & 2+I\\ -1+I & 3-2I & 4+2I\\ 3+2I & 4-I & 2-3I\end{vmatrix}=-11+14I$$

Then:

$$x=\frac{\det x}{\det A}=\frac{4+29I}{-30+21I}=\frac{-2}{15}-\frac{477}{135}I$$

$$y=\frac{\det y}{\det A}=\frac{-35-31I}{-30+21I}=\frac{7}{6}+\frac{37}{6}I$$

$$z=\frac{\det z}{\det A}=\frac{-11+14I}{-30+21I}=\frac{11}{30}-\frac{7}{10}I$$

$$(x,y,z)=\left(\frac{-2}{15}-\frac{477}{135}I,\frac{7}{6}+\frac{37}{6}I,\frac{11}{30}-\frac{7}{10}I\right)$$

To verify the solution, we substitute the value of x and y into the first equation:

$$(2+I)(\frac{-2}{15}-\frac{477}{135}I)+(1+I)(\frac{7}{6}+\frac{37}{6}I)+(3-I)(\frac{11}{30}-\frac{7}{10}I)=2+I$$

Example4.2.7:

Let:

$$(2+I)x+(1+I)y+(3-I)z=1+I \qquad (1)$$
$$(4+2I)x+(2+2I)y+(6-2I)z=1+2I \qquad (2)$$
$$(6+3I)x+(3+3I)y+(9-3I)z=-1+3I \qquad (3)$$

Find solution of the system.

Solution:

$$A=\begin{bmatrix}2+I & 1+I & 3-I\\ 4+2I & 2+2I & 6-2I\\ 6+3I & 3+3I & 9-3I\end{bmatrix}$$

$$\det A=\begin{vmatrix}2+I & 1+I & 3-I\\ 4+2I & 2+2I & 6-2I\\ 6+3I & 3+3I & 9-3I\end{vmatrix}=0+0I$$

$$\det x = \begin{vmatrix} 1+I & 1+I & 3-I \\ 1+2I & 2+2I & 6-2I \\ -1+3I & 3+3I & 9-3I \end{vmatrix} = 20I$$

As: $\det A = 0 + 0I$ *and* $\det x \neq 0 + 0I$, so the system is impossible to solve.

4.3 System of the homogeneous neutrosophic linear equations

Defination4.3.1:

It is a system of homogeneous neutrosophic linear equations that given by the form:

$$\begin{cases} (a_{11}+b_{11}I)x_1 + (a_{12}+b_{12}I)x_2 + (a_{13}+b_{13}I)x_3 + \cdots + (a_{1n}+b_{1n}I)x_n = 0+0I \\ (a_{21}+b_{21}I)x_1 + (a_{22}+b_{22}I)x_2 + (a_{23}+b_{23}I)x_3 + \cdots + (a_{2n}+b_{2n}I)x_n = 0+0I \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad = \cdots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad = \cdots \\ (a_{m1}+b_{m1}I)x_1 + (a_{m2}+b_{m2}I)x_2 + (a_{m3}+b_{m3}I)x_3 + \cdots + (a_{mn}+b_{mn}I)x_n = 0+0I \end{cases}$$

Where:

a_{ij}, b_{ij}, c_j, d_j are real coefficients, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and I represent indeterminacy.

Remarks4.3.1:

We distinguish two cases to solve the system of the homogeneous neutrosophic linear equations:

1. It may have one solution
2. It may have Unlimited number of solutions

Hence it is always a solvable system (It is not impossible to solve).

Example4.3.1:

Let:

$$(2+I)x + (1+I)y + (3-I)z = 0$$

$$(-1+I)x + (3-2I)y + (1+3I)z = 0$$

$$(3+2I)x + (4-I)y + (2-3I)z = 0$$

Find solution of the system.

Solution:

$$A = \begin{bmatrix} 2+I & 1+I & 3-I \\ -1+I & 3-2I & 1+3I \\ 3+2I & 4-I & 2-3I \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2+I & 1+I & 3-I \\ -1+I & 3-2I & 1+3I \\ 3+2I & 4-I & 2-3I \end{vmatrix} = -30 + 21I$$

$$\det x = \begin{vmatrix} 0 & 1+I & 3-I \\ 0 & 3-2I & 1+3I \\ 0 & 4-I & 2-3I \end{vmatrix} = 0 \quad , \quad \det y = \begin{vmatrix} 2+I & 0 & 3-I \\ -1+I & 0 & 1+3I \\ 3+2I & 0 & 2-3I \end{vmatrix} = 0$$

$$\det z = \begin{vmatrix} 2+I & 1+I & 0 \\ -1+I & 3-2I & 0 \\ 3+2I & 4-I & 0 \end{vmatrix} = 0$$

Then:

$$x = \frac{\det x}{\det A} = 0, \quad y = \frac{\det y}{\det A} = 0 , \quad z = \frac{\det z}{\det A} = 0$$

$$(x, y, z) = (0,0,0)$$

5. Conclusions

This study was presented based on the importance of linear equations in our lives, where we introduced the concept of neutrosophic linear equation and its types. In addition to the introduction of Cramer's rule to solve the neutrosophic system of equations. This study can be generalized and applied in several fields of application in our current reality, traffic as an example.

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On Refined Neutrosophic Canonical Hypergroups

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Abstract. Refinement of neutrosophic algebraic structure or hyperstructure allows for the splitting of the indeterminate factor into different sub-indeterminate and gives a detailed information about the neutrosophic structure/hyperstructure considered. This paper is concerned with the development of a refined neutrosophic canonical hypergroup from a canonical hypergroup R and sub-indeterminate I_1 and I_2. Several interesting results and examples are presented. The paper also studies refined neutrosophic homomorphisms and establishes the existence of a good homomorphism between a refined neutrosophic canonical hypergroup $R(I_1, I_2)$ and a neutrosophic canonical hypergroup $R(I)$.

Keywords: Neutrosophic, neutrosophic canonical hypergroup, neutrosophic subcanonical hypergroup, refined neutrosophic canonical hypergroup, refined neutrosophic subcanonical hypergroup, refined neutrosophic canonical hypergroup homomorphism.

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1. **Introduction**

The refinement of neutrosophic components of the form $< T_1, T_2, \cdots, T_p; I_1, I_2, \cdots, I_r; F_1, F_2, \cdots, F_s >$ was introduced by Florentine Smarandache in [17] . The birth of this refinement led to the extension of neutrosophic numbers $a + bI$ into refined neutrosophic numbers of the form $(a + b_1I_1 + b_2I_2 + \cdots + b_nI_n)$ where $a, b_1, b_2, \cdots, b_n$ are real or complex numbers. The concept of refined neutrosophic set was later studied using refined neutrosophic number and this paved way for the introduction of refined neutrosophic algebraic structures by Agboola in [5]. Ever since he studied and introduced this structure, several researchers in this field have studied this concept and a great deal of results have been published. For example recently, Ibrahim et al., published in [11–14] their results on refined neutrosophic vector spaces, refined neutrosophic hypergroup and refined neutrosophic hypervector spaces. Also, Adeleke et al., in [1,2] studied refined neutrosophic rings, refined neutrosophic subrings, refined neutrosophic ideals and refined neutrosophic ring homomorphisms. And in [8] Agboola et al., refined neutrosophic algebraic hyperstructures and presented some of its elementary properties. For details about algebraic hyperstructure, neutrosophic structures/hyperstructure and new trends in neutrosophic theory the readers should see [3, 6, 7, 9, 10, 15, 16].

2. Preliminaries

In this section, we will give some definitions, examples and results that will be used in the sequel.

Definition 2.1. [4] If $* : X(I_1, I_2) \times X(I_1, I_2) \mapsto X(I_1, I_2)$ is a binary operation defined on $X(I_1, I_2)$, then the couple $(X(I_1, I_2), *)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$.

For the purposes of this paper, it will be assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that:

$$\begin{aligned} I_1I_1 &= I_1^2 &&= I_1, \\ I_2I_2 &= I_2^2 &&= I_2 \text{ and} \\ I_1I_2 &= I_2I_1 &&= I_1. \end{aligned}$$

Definition 2.2. [4] Let $(X(I_1, I_2), +, \cdot)$ be any refined neutrosophic algebraic structure where + and . are ordinary addition and multiplication respectively.
For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2),$$

$$(a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1, (af + cd + cf)I_2).$$

Definition 2.3. [4] If + and . are ordinary addition and multiplication, I_k with $k = 1, 2$ have the following properties:

(1) $I_k + I_k + \cdots + I_k = nI_k$.
(2) $I_k + (-I_k) = 0$.
(3) $I_k \cdot I_k \cdot \cdots \cdot I_k = I_k^n = I_k$ for all positive integers $n > 1$.
(4) $0 \cdot I_k = 0$.
(5) I_k^{-1} is undefined and therefore does not exist.

Definition 2.4. [4] Let $(G, *)$ be any group. The couple $(G(I_1, I_2), *)$ is called a refined neutrosophic group generated by G, I_1 and I_2. $(G(I_1, I_2), *)$ is said to be commutative if for all $x, y \in G(I_1, I_2)$, we have $x * y = y * x$. Otherwise, we call $(G(I_1, I_2), *)$ a non -commutative refined neutrosophic group.

Definition 2.5. [4] If $(X(I_1, I_2), *)$ and $(Y(I_1, I_2), *')$ are two refined neutrosophic algebraic structures, the mapping

$$\phi : (X(I_1, I_2), *) \longrightarrow (Y(I_1, I_2), *')$$

is called a neutrosophic homomorphism if the following conditions hold:

(1) $\phi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \phi((a, bI_1, cI_2)) *' \phi((d, eI_1, fI_2))$.

(2) $\phi(I_k) = I_k$ for all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$ and $k = 1, 2$.

Example 2.6. [4] Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 2.7. [4] Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *')$ be two refined neutrosophic groups. Let $\phi : G(I_1, I_2) \times H(I_1, I_2) \to G(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let $\psi : G(I_1, I_2) \times H(I_1, I_2) \to H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

Definition 2.8. [10] Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ x = A \circ \{x\} \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Definition 2.9. [10] Let H be a non-empty set and let $+$ be a hyperoperation on H. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

(1) $x + y = y + x$, for all $x, y \in H$,

(2) $x + (y + z) = (x + y) + z$, for all $x, y, z \in H$,

(3) there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, for all $x \in H$,

(4) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,

(5) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in H$. A nonempty subset A of H is called a subcanonical hypergroup if A is a canonical hypergroup under the same hyperaddition as that of H that is, for every $a, b \in A$, $a - b \in A$. If in addition $a + A - a \subseteq A$ for all $a \in H$, A is said to be normal.

Definition 2.10. [6] Let $(H, +)$ be any canonical hypergroup and let I be an indeterminate. Let $H(I) =< H \cup I >= \{(a, bI) : a, b \in H\}$ be a set generated by H and I. The hyperstructure $(H(I), +)$ is called a neutrosophic canonical hypergroup . For all $(a, bI), (c, dI) \in H(I)$ with $b \neq 0$ or $d \neq 0$, we define

$$(a, bI) + (c, dI) = \{(x, yI) : x \in a + c, \ y \in a + d \cup b + c \cup b + d\}.$$

An element $I \in H(I)$ is represented by $(0, I)$ in $H(I)$ and any element $x \in H$ is represented by $(x, 0)$ in $H(I)$. For any nonempty subset $A(I)$ of $H(I)$, we define $-A(I) = \{-(a, bI) = (-a, -bI) : a, b \in H\}$.

Definition 2.11. [6] Let $(H(I), +)$ be a neutrosophic canonical hypergroup .

(1) A nonempty subset $A(I)$ of $H(I)$ is called a neutrosophic subcanonical hypergroup of $H(I)$ if $(A(I), +)$ is itself a neutrosophic canonical hypergroup . It is essential that $A(I)$ must contain a proper subset which is a subcanonical hypergroup of H.
If $A(I)$ does not contain a proper subset which is a subcanonical hypergroup of H, then it is called a pseudo neutrosophic subcanonical hypergroup of $H(I)$.

(2) If $A(I)$ is a neutrosophic subcanonical hypergroup (pseudo neutrosophic subcanonical hypergroup), $A(I)$ is said to be normal in $H(I)$ if for all $(a, bI) \in H(I)$, $(a, bI) + A(I) - (a, bI) \subseteq A(I)$.

Definition 2.12. [6] Let $(H_1(I), +)$ and $(H_2(I), +)$ be two neutrosophic canonical hyper- groups and let
$\phi : H_1(I) \longrightarrow H_2(I)$ be a mapping from $H_1(I)$ into $H_2(I)$.

(1) ϕ is called a homomorphism if :
  (a) ϕ is a canonical hypergroup homomorphism,
  (b) $\phi((0, I)) = (0, I)$.
(2) ϕ is called a good or strong homomorphism if:
  (a) ϕ is a good or strong canonical hypergroup homomorphism,
  (b) $\phi((0, I)) = (0, I)$.
(3) ϕ is called an isomorphism (strong isomorphism) if ϕ is a bijective homomorphism (strong homomorphism).

3. Development of a refined neutrosophic canonical hypergroup

In this section, we study and present the development of refined neutrosophic canonical hypergroup and some of their basic properties.

Definition 3.1. Let $(R, +)$ be any canonical hypergroup. The couple $(R(I_1, I_2), +)$ is a neutrosophic canonical hypergroup generated by R, I_1 and I_2, where $+$ hyperoperations. i.e.,

$$+ : R(I_1, I_2) \times R(I_1, I_2) \longrightarrow 2^{R(I_1,I_2)}.$$

For all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in R(I_1, I_2)$ with $a, b, c, d, e, f \in R$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = \{(p, qI_1, rI_2) : p \in a + d, q \in (b + e), r \in (c + f)\}.$$

Lemma 3.2. *Let $(R(I_1, I_2), +)$ be any neutrosophic canonical hypergroup. Let $h_1 = (u, vI_1, tI_2)$, $h_2 = (m, nI_1, kI_2) \in R(I_1, I_2)$ with $u, v, t, m, n, k \in R$. For all $h_1, h_2 \in R(I_1, I_2)$ we have*

(1) $-(-h_1) = -(-u, -vI_1, -tI_2) = (-(-u), -(-v)I_1, -(-t)I_2) = (u, vI_1, tI_2)$.

(2) $(0,0I_1,0I_2)$ *is the unique element such that for every* $h_1 \in R(I_1,I_2)$, *there is an element* $-h_1 \in R(I_1,I_2)$ *with the property* $(0,0I_1,0I_2) \in h_1 - h_2$.

(3) $-(0,0I_1,0I_2) = (0,0I_1,0I_2)$.

(4) $-(h_1+h_2) = -h_1 - h_2$.

Proof. The proof is similar to the proof in classical case. □

Definition 3.3. Let $R(I_1,I_2)$ be a refined neutrosophic canonical hypergroup and let $K(I_1,I_2)$ be a proper subset of $R(I_1,I_2)$. Then

(1) $K(I_1,I_2)$ is said to be a refined neutrosophic subcanonical hypergroup of $R(I_1,I_2)$ if $K(I_1,I_2)$ is a refined neutrosophic canonical hypergroup. It is essential that $K(I_1,I_2)$ contains a proper subset which is a canonical hypergroup.

(2) $K(I_1,I_2)$ is said to be a refined pseudo neutrosophic subcanonical hypergroup of $R(I_1,I_2)$ if $K(I_1,I_2)$ is a refined neutrosophic canonical hypergroup which contains no proper subset which is a canonical hypergroup.

Proposition 3.4. *Every refined neutrosophic canonical hypergroup is a canonical hypergroup.*

Proof. Let $(R(I_1,I_2),+)$ be a refined neutrosophic canonical hypergroup and let $x=(a,bI_1,cI_2), y=(d,eI_1,fI_2), z=(g,hI_1,kI_2) \in R(I_1,I_2)$. Then :

$$\begin{aligned}(i)\ x+y &= (a,bI_1,cI_2)+(d,eI_1,fI_2)\\ &= \{(p,qI_1,sI_2): p\in a+d, q\in b+e, s\in c+f\}\\ &= \{(p,qI_1,sI_2): p\in d+a, q\in e+b, s\in f+c\}\\ &= (d,eI_1,fI_2)+(a,bI_1,cI_2)\\ &= y+x.\end{aligned}$$

$$\begin{aligned}(ii)\ (x+y)+z &= ((a,bI_1,cI_2)+(d,eI_1,fI_2))+(g,hI_1,kI_2)\\ &= \{(p,qI_1,sI_2): p\in a+d, q\in b+e, s\in c+f\}+(g,hI_1,kI_2)\\ &= \{(p',q'I_1,s'I_2): p'\in p+g, q'\in q+h, s'\in s+k\}\\ &= \{(p',q'I_1,s'I_2): p'\in (a+d)+g,\ q'\in (b+e)+h,\ s'\in (c+f)+k\}\\ &= \{(p',q'I_1,s'I_2): p'\in a+(d+g),\ q'\in b+(e+h),\ s'\in c+(f+k)\}\\ &= (a,bI_1,cI_2)+((d,eI_1,fI_2)+(g,hI_1,kI_2))\\ &= x+(y+z).\end{aligned}$$

$$\begin{aligned}(iii)\ (0,0I_1,0I_2)+(a,bI_1,cI_2) &= \{(p,qI_1,tI_2): p\in 0+a, q\in 0+b, t\in 0+c\}\\ &= \{(p,qI_1,tI_2): p\in \{a\}, q\in \{b\}, t\in \{c\}\}\\ &= \{(a,bI_1,cI_2)\}.\end{aligned}$$

Also, it can be shown that $(a, bI_1, cI_2) + (0, 0I_1, 0I_1) = \{(a, bI_1, cI_2)\}$. Hence, there exists a neutral element $(0, 0I_1, 0I_2) \in R(I_1, I_2)$.

$$\begin{aligned} (iv)\ ((a, bI_1, cI_2) + (-a, -bI_1, cI_2)) \quad \cap \quad & ((-a, -bI_1, cI_2) + (a, bI_1, cI_2)) \\ = \quad & \{(p, qI_1, tI_2) : p \in a + (-a), q \in b + (-b), t \in c + (-c)\} \\ & \cap \{(m, nI_1, uI_2) : m \in (-a) + a, n \in (-b) + b, u \in (-c) + c\} \\ = \quad & \{(p, qI_1, tI_2) : p \in \{0\},\ q \in \{0\},\ t \in \{0\}\} \\ & \cap \{(m, nI_1 tI_2) : m \in \{0\},\ n \in \{0\},\ t \in \{0\}\}. \end{aligned}$$

Then we can say that $(0, 0I_1, 0I_2) \in ((a, bI_1, cI_2) + (-a, -bI_1, -cI_2)) \cap ((-a, -bI_1, -cI_2) + (a, bI_1, cI_2))$ and therefore, $-(a, bI_1, cI_2)$ is the unique inverse of any $(a, bI_1, cI_2) \in R(I_1, I_2)$.

$$\begin{aligned} (v)\ \text{Let } z \in x + y, \quad & i.e \quad (g, hI_1, kI_2) \in (a, bI_1, cI_2) + (d, eI_1, fI_2).\ Then \\ (g, hI_1, kI_2) \quad & \in \quad \{(p, qI_1, tI_2) : p \in a + d, q \in b + e, t \in c + f\} \\ & = \quad \{(p, qI_1, tI_2) : d \in -a + p, e \in -b + q, f \in -c + t\} \\ & = \quad \{(d, eI_1, fI_2) : d \in -a + p,\ e \in -b + q,\ f \in -c + t\}. \end{aligned}$$

So, we have $(d, eI_1, fI_2) \in -(a, bI_1, cI_2) + (g, hI_1, kI_2)$.

Also we can show that $(a, bI_1, cI_2) \in (g, hI_1, kI_2) - (d, eI_1, fI_2)$. Hence, $z \in x + y$ implies that $x \in z - y$ and $y \in -x + z$. Accordingly, $R(I_1, I_2)$ is a canonical hypergroup. □

Example 3.5. Let $R(I_1, I_2) = \{a_1 = (s, sI_1, sI_2), a_2 = (s, sI_1, tI_2), a_3 = (s, tI_1, sI_2),$ $a_4 = (s, tI_1, tI_2),\ b_1 = (t, tI_1, tI_2), b_2 = (t, tI_1, sI_2), b_3 = (t, sI_1, tI_2), b_4 = (t, sI_1, sI_2)\}$ be a refined neutrosophic set and let $+$ be the hyperoperation on $R(I_1, I_2)$ defined as in the tables below. Let $a = \{a_1, a_2, a_3, a_4\}$ and $b = \{b_1, b_2, b_3, b_4\}$.

Table 1. Cayley table for the binary operation " + "

$+$	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_1	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_2	a_2	$\{a_1, a_2\}$	a_4	$\{a_3, a_4\}$	$\{b_1, b_2\}$	b_1	$\{b_3, b_4\}$	b_3
a_3	a_3	a_4	$\{a_1, a_3\}$	$\{a_2, a_4\}$	$\{b_1, b_3\}$	$\{b_2, b_4\}$	b_1	b_2
a_4	a_4	$\{a_3, a_4\}$	$\{a_2, a_4\}$	a	b	$\{b_1, b_3\}$	$\{b_1, b_2\}$	b_1
b_1	b_1	$\{b_1, b_2\}$	$\{b_1, b_3\}$	b	$R(I_1, I_2)$	$\{a_2, a_4, b_1, b_3\}$	$\{a_3, a_4, b_1, b_2\}$	$\{a_4, b_1\}$
b_2	b_2	b_1	$\{b_2, b_4\}$	$\{b_1, b_3\}$	$\{a_2, a_4, b_1, b_3\}$	$\{a_1, a_3, b_2, b_4\}$	$\{a_4, b_1\}$	$\{a_3, b_2\}$
b_3	b_3	$\{b_3, b_4\}$	b_1	$\{b_1, b_2\}$	$\{a_3, a_4, b_1, b_2\}$	$\{a_4, b_1\}$	$\{a_1, a_2, b_3, b_4\}$	$\{a_2, b_3\}$
b_4	b_4	b_3	b_2	b_1	$\{a_4, b_1\}$	$\{a_3, b_2\}$	$\{a_2, b_3\}$	$\{a_1, b_4\}$

It is clear from the table that $(R(I_1, I_2), +)$ is a refined neutrosophic canonical hypergroups.

Example 3.6. Let $R = \{0, u, v\}$ and define " + " on R as follows

Table 2. Cayley table for the hyper operation " + "

$+$	0	u	v
0	0	u	v
u	u	$\{0, u\}$	v
v	v	v	$\{0, u, v\}$

Let $R(I_1, I_2) = \{a_1 = (0, 0I_1, 0I_2), a_2 = (0, 0I_1, uI_2), a_3 = (0, 0I_1, vI_2), a_4 = (0, uI_1, 0I_2),$ $a_5 = (0, vI_1, 0I_2), a_6 = (0, uI_1, vI_2), a_7 = (0, vI_1, uI_2), a_8 = (0, uI_1, uI_2), a_9 = (0, vI_1, vI_2),$ $b_1 = (u, uI_1, uI_2), b_2 = (u, uI_1, 0I_2), b_3 = (u, uI_1, vI_2), b_4 = (u, 0I_1, uI_2), b_5 = (u, vI_1, uI_2),$ $b_6 = (u, 0I_1, vI_2), b_7 = (u, vI_1, 0I_2), b_8 = (u, 0I_1, 0I_2), b_9 = (u, vI_1, vI_2), c_1 = (v, vI_1, vI_2),$ $c_2 = (v, vI_1, 0I_2), c_3 = (v, vI_1, uI_2), c_4 = (v, 0I_1, vI_2), c_5 = (v, uI_1, vI_2), c_6 = (v, 0I_1, uI_2),$ $c_7 = (v, uI_1, 0I_2), c_8 = (v, 0I_1, 0I_2), c_9 = (v, uI_1, uI_2)\}$ be a refined neutrosophic set.

For any $(x, yI_1, zI_1), (p, qI_1, rI_2) \in R(I_1, I_2)$ define the hyperoperation $+'$ by

$$(x, yI_1, zI_1) +' (p, qI_1, rI_2) = \{(m, nI_1, sI_2) : m \in x + p, n \in y + q, s \in z + r\}.$$

Then $(R(I_1, I_2), +')$ is a refined neutrosophic canonical hypergroup.

Proposition 3.7. *Let $(R(I_1, I_2), +_1)$ be a refined neutrosophic canonical hypergroup and let $(K, +_2)$ be a canonical hypergroup. Define for all $(x_1, k_1), (x_2, k_2) \in R(I_1, I_2) \times K$ the hyperoperation $''+''$ by*

$$(x_1, k_1) + (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 +_1 x_2,\ k_3 \in k_1 +_2 k_2\}.$$

Where $x_i = (a_i, b_iI_1, c_iI_2)$ for $i = 1, 2 \cdots n$. Then $(R(I_1, I_2) \times K, +)$ is a refined neutrosophic canonical hypergroup.

Proof. Let $(x_1, k_1), (x_2, k_2), (x_3, k_3) \in R(I_1, I_2) \times K$ for $x_1, x_2, x_3 \in R(I_1, I_2)$ and $k_1, k_2, k_3 \in K$. Then :

(1) For commutativity;

$$\begin{aligned}
(x_1, k_1) + (x_2, k_2) &= ((a_1, b_1I_1, c_1I_2), k_1) + ((a_2, b_2I_1, c_2I_2), k_2) \\
&= \{((p, qI_1, sI_2), k) : p \in a_1 +_1 a_2,\ q \in b_1 +_1 b_2,\ s \in c_1 +_1 f_2,\ k \in k_1 +_2 k_2\} \\
&= \{((p, qI_1, sI_2), k) : p \in a_2 +_1 a_1,\ q \in b_2 +_1 b_1,\ s \in c_2 +_1 c_1,\ k \in k_2 +_2 k_1\} \\
&= ((a_2, b_2I_1, c_2I_2), k_2) + ((a_1, b_1I_1, c_1I_2), k_1) \\
&= (x_2, k_2) + (x_1, k_2).
\end{aligned}$$

(2) For associativity;

$$\begin{aligned}
[(x_1, k_1) + (x_2, k_2)] + (x_3, k_3) &= [((a_1, b_1I_1, c_1I_2), k_1) + ((a_2, b_2I_1, c_2I_2), k_2)] + ((a_3, b_3I_1, c_3I_2), k_3) \\
&= \{((p, qI_1, sI_2), k) : p \in a_1 +_1 a_2,\ q \in b_1 +_1 b_2,\ s \in c_1 +_1 c_2, \\
&\quad\ k \in k_1 +_2 k_2\} + ((a_3, b_3I_1, c_3I_2), k_3) \\
&= \{((p', q'I_1, s'I_2), k') : p' \in p +_1 a_3,\ q' \in q +_1 b_3,\ s' \in s +_1 c_3, \\
&\quad k' \in k +_2 k_3\} \\
&= \{((p', q'I_1, s'I_2), k') : p' \in (a_1 +_1 a_2) +_1 a_3,\ q' \in (b_1 +_1 b_2) +_1 b_3, \\
&\quad s' \in (c_1 +_1 c_2) +_1 c_3,\ k' \in (k_1 +_2 k_2) +_2 k_3\} \\
&= \{((p', q'I_1, s'I_2), k') : p' \in a_1 +_1 (a_2 +_1 a_3),\ q' \in b_1 +_1 (b_2 +_1 b_3), \\
&\quad s' \in c_1 +_1 (c_2 +_1 c_3),\ k' \in k_1 +_2 (k_2 +_2 k_3)\} \\
&= ((a_1, bI_1, c_1I_2), k_1) + [((a_2, b_2I_1, c_2I_2), k_2) + ((a_3, b_3I_1, c_3I_2), k_3)] \\
&= (x_1, k_1) + [(x_2, k_2) + (x_3, k_3)].
\end{aligned}$$

(3) Existence of inverse element:

We want to show that the element $((0, 0I_1, 0I_2), 0_k)$ is the neutral element in $R(I_1, I_2) \times K$. Where 0_k is the neutral element in K. Now, consider

$$\begin{aligned}
((0, 0I_1, 0I_2), 0_K) + ((a_1, b_1I_1, c_1I_2), k_1) &= \{((p, qI_1, tI_2), k) : p \in 0 +_1 a_1, q \in 0 +_1 b_1,\ t \in 0 +_1 c_1, \\
&\quad k \in 0_K +_2 k_1\} \\
&= \{((p, qI_1, tI_2), k)) : p \in \{a_1\}, q \in \{b_1\}, t \in \{c_1\},\ k \in \{k_1\} \\
&= \{((a_1, b_1I_1, c_1I_2), k_1\}.
\end{aligned}$$

Similarly, we can show that

$$((a_1, b_1I_1, c_1I_2), k_1) + ((0, 0I_1, 0I_1), 0_K) = \{((a_1, b_1I_1, c_1I_2), k_1)\}.$$

Hence, we can conclude that there exists a neutral element $((0, 0I_1, 0I_2), 0_K) \in R(I_1, I_2) \times K$.

(4) Existence of unique inverse:

We want to show that there exist a unique inverse for any $((a_1, b_1I_1, c_1I_2), k_1) \in R(I_1, I_2) \times K$.

Now, consider

$$\begin{aligned}
[((a_1, b_1I_1, c_1I_2), k_1) &+ ((-a_1, -b_1I_1, c_1I_2), -k_1)] \cap [((-a, -bI_1, -cI_2), -k_1) + ((a_1, bI_1, c_1I_2), k_1)] \\
&= \{((p, qI_1, tI_2), k) : p \in a_1 +_1 (-a_1), q \in b_1 +_1 (-b_1), t \in c_1 +_1 (-c_1), \\
&\quad k \in k_1 +_2 (-k_1)\} \\
&\quad \cap \{((m, nI_1, uI_2), k') : m \in -a_1 +_1 a_1, n \in -b_1 +_1 b_1, u \in -c +_1 c, \\
&\quad k' \in -k_1 +_2 k_1\} \\
&= \{((p, qI_1, tI_2), k) : p \in \{0\},\ q \in \{0\},\ t \in \{0\},\ k \in \{0_K\} \\
&\quad \cap \{((m, nI_1uI_2), k') : m \in \{0\},\ n \in \{0\},\ u \in \{0\},\ k' \in \{0_K\}.
\end{aligned}$$

Then we can say that

$((0, 0I_1, 0I_2), 0_K) \in (((a_1, b_1I_1, c_1I_2), k_1) + ((-a_1, -b_1I_1, -c_1I_2), -k_1) \cap (((-a_1, -b_1I_1, -c_1I_2), -k_1) + ((a_1, b_1I_1, c_1I_2), k_1)$ and therefore, $-((a_1, b_1I_1, c_1I_2), k_1)$ is the unique inverse of any $((a_1, b_1I_1, c_1I_2), k_1) \in R(I_1, I_2) \times K$.

(5) Let $(x_3, k_3) \in (x_1, k_1) + (x_2, k_2)$, *i.e.*, $((a_3, b_3I_1, c_3I_2), k_3) \in ((a_1, b_1I_1, c_1I_2), k_1) + ((a_2, b_2I_1, c_2I_2), k_2)$. Then

$$\begin{aligned}
((a_3, b_3I_1, c_3I_2), k_3) &\in \{((p, qI_1, tI_2), k) : p \in a_1 +_1 a_2,\ q \in b_1 +_1 b_2,\ t \in c_1 +_1 c_2,\ k \in k_1 +_2 k_2\} \\
&= \{((p, qI_1, tI_2), k) : a_2 \in -a_1 +_1 p, b_2 \in -b_1 +_1 q,\ c_2 \in -c_1 +_1 t, \\
&\quad k_2 \in -k_1 +_2 k\} \\
&= \{((a_2, b_2I_1, c_2I_2), k_2) : a_2 \in -a_1 +_1 p,\ b_2 \in -b_1 +_1 q,\ c_2 \in -c_1 +_1 t, \\
&\quad k_2 \in -k_1 +_2 k\}.
\end{aligned}$$

So, we have $((a_2, b_2I_1, c_2I_2), k_2) \in -((a_1, b_1I_1, c_1I_2), k_1) + ((a_3, b_3I_1, c_3I_2), k_3)$.

Also, we can show that $((a_1, b_1I_1, c_1I_2), k_1) \in ((a_3, b_3I_1, c_3I_2), k_3) - ((a_2, b_2I_1, c_2I_2), k_2)$. Hence, $(x_3, k_3) \in (x_1, k_1) + (x_2, k_2)$ implies that $(x_1, k_1) \in (x_3, k_3) - (x_2, k_2)$ and $(x_2, k_2) \in -(x_1, k_1) + (x_3, k_3)$.

Accordingly, $R(I_1, I_2)$ is a refined neutrosophic canonical hypergroup. □

Proposition 3.8. *Let $(R(I_1, I_2), +_1)$ and $(K(I_1, I_2), +_2)$ be two refined neutrosophic canonical hypergroup. Define for all $(x_1, k_1), (x_2, k_2) \in R(I_1, I_2) \times K(I_1, I_2)$ the hyperoperations $''+''$ by*

$$(x_1, k_1) + (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 +_1 x_2,\ k_3 \in k_1 +_2 k_2\}.$$

Where $x_i = (a_i, b_iI_1, c_iI_2)$ and $k_i = (u_i, v_iI_1, s_iI_2)$ for $i = 1, 2 \cdots n$.

Then $(R(I_1, I_2) \times K(I_1, I_2), +)$ is a refined neutrosophic canonical hypergroup.

Proof. The proof is similar to the prof of Proposition 3.7 . □

Lemma 3.9. *Let $R(I_1, I_2)$ be a refined neutrosophic canonical hypergroup. A non-empty subset $K(I_1, I_2)$ of $R(I_1, I_2)$ is a refined neutrosophic subcanonical hypergroup if and only if for $k_1 = (p_1, q_1I_1, s_1I_1), k_2 = (p_2, q_2I_1, s_2I_1) \in K(I_1, I_2)$ the following conditions hold:*

(1) $k_1 - k_2 \subseteq K(I_1, I_2)$,

(2) *$K(I_1, I_2)$ contains a proper subset which is a canonical hypergroup.*

Proposition 3.10. *Let $M(I_1, I_2)$ and $N(I_1, I_2)$ be any two refined neutrosophic subcanonical hypergroups of a refined neutrosophic canonical hypergroup $R(I_1, I_2)$ and let K be a subcanonical hypergroup of R. Then,*

(1) *$M(I_1, I_2) + N(I_1, I_2)$ is a refined neutrosophic subcanonical hypergroup of $R(I_1, I_2)$.*

(2) *$M(I_1, I_2) \cap N(I_1, I_2)$ is a refined neutrosophic subcanonical hypergroup of $R(I_1, I_2)$.*

(3) *$M(I_1, I_2) + K$ is a refined neutrosophic subcanonical hypergroup of $R(I_1, I_2)$.*

Proof. (1) It is clear that $(0, 0I_1, 0I_2) \in M(I_1, I_2) + N(I_1, I_2)$ since $M(I_1, I_2)$ and $N(I_1, I_2)$ are refined neutrosophic subcanonical hypergroup.

Let $(x, yI_1, zI_2), (u, vI_1wI_2) \in M(I_1, I_2) + N(I_1, I_2)$. Where $x = x_1 + x_2, y = y_1 + y_2$, $z = z_1 + z_2, u = u_1 + u_2, v = v_1 + v_2$ and $w = w_1 + w_2$. With $x_1, y_1, z_1, u_1, v_1, w_1 \in M$ and $x_2, y_2, z_2, u_2, v_2, w_2 \in N$. Then

$$\begin{aligned}
(x, yI_1, zI_2) - (u, vI_1wI_2) &= ((x_1 + x_2), (y_1 + y_2)I_1, (z_1 + z_2)I_2) - \\
&\quad ((u_1 + u_2), (v_1 + v_2)I_1, (w_1 + w_2)I_2) \\
&= ((x_1 + x_2) - (u_1 + u_2), ((y_1 + y_2) - (v_1 + v_2))I_1, \\
&\quad ((z_1 + z_2) - (w_1 + w_2)I_2) \\
&= \{(p, qI_1, rI_2) : p \in (x_1 - u_1) + (x_2 - u_2), q \in (y_1 - v_1) + (y_2 - v_2), \\
&\quad r \in (z_1 - w_1) + (z_2 - w_2)\} \\
&\subseteq M(I_1, I_2) + N(I_1, I_2).
\end{aligned}$$

Now, since the refined neutrosophic subcanonical hypergroups $M(I_1, I_2)$ and $N(I_1, I_2)$ contain a proper subset M and N respectively, which are canonical hypergroups. Then, $M + N$ is a canonical hypergroup which is contained in $M(I_1, I_2) + N(I_1, I_2)$. Hence $M(I_1, I_2) + N(I_1, I_2)$ is a refined neutrosophic subcanonical hypergroup of $R(I_1, I_2)$.

(2) The proof is similar to the proof in classical case.

(3) The proof follows the same approach as the proof of 1. □

Remark 3.11. It should be noted that if $M(I_1, I_2)$ is a refined pseudo neutrosophic subcanonical hypergroup of a refined neutrosophic canonical hypergroup $R(I_1, I_2)$ and K is a subcanonical hypergroup of a canonical hypergroup R. Then $M(I_1, I_2) + K$ is a refined neutrosophic subcanonical hypergroup of $R(I_1, I_2)$.

Definition 3.12. Let $R(I_1, I_2)$ be a refined neutrosophic canonical hypergroup. The refined neutrosophic subcanonical hypergroup $M(I_1, I_2)$ is said to be normal in $R(I_1, I_2)$ if

$$(a, bI_1, cI_2) + M(I_1, I_2) - (a, bI_1, cI_2) \subseteq M(I_1, I_2) \text{ for all } (a, bI_1, cI_2) \in R(I_1, I_2).$$

Definition 3.13. Let $M(I_1, I_2)$ be a normal refined neutrosophic subcanonical hypergroup of a refined neutrosophic canonical hypergroup $R(I_1, I_2)$. The quotient $R(I_1, I_2)/M(I_1, I_2)$ is defined by the set

$$\{r + M(I_1, I_2) : r = (x, yI_1, zI_2) \in R(I_1, I_2)\}.$$

Proposition 3.14. *Let $R(I_1, I_2)/M(I_1, I_2) = \{r + M(I_1, I_2) : r = (x, yI_1, zI_2) \in R(I_1, I_2)\}$. For $r_1 + M(I_1, I_2), r_2 + M(I_1, I_2) \in R(I_1, I_2)/M(I_1, I_2)$, if $r_1 + M(I_1, I_2) \cap r_2 + M(I_1, I_2) \neq \emptyset$ then $r_1 + M(I_1, I_2) = r_2 + M(I_1, I_2)$.*

Proof. Let $r_3 \in r_1 + M(I_1, I_2) \cap r_2 + M(I_1, I_2)$ i.e.,

$$(x_3, y_3I_1, z_3I_2) \in (x_1, y_1I_1, z_1I_2) + M(I_1, I_2) \cap (x_2, y_2I_1, z_2I_2) + M(I_1, I_2).$$

Obviously,

$$(x_3, y_3I_1, z_3I_2) \in (x_1, y_1I_1, z_1I_2) + M(I_1, I_2) \text{ and } (x_3, y_3I_1, z_3I_2) \in (x_2, y_2I_1, z_2I_2) + M(I_1, I_2).$$

So, for $m_1 = (u_1, v_1I_1, t_1I_2), m_2 = (u_2, v_2I_1, t_2I_2) \in M(I_1, I_2)$, with $u_1, u_2, u_3, v_1, v_2, v_3, t_1, t_2,$ $t_3 \in M$, we have

$$(x_3, y_3I_1, z_3I_2) \in (x_1, y_1I_1, z_1I_2) + (u_1, v_1I_1, t_1I_2) \text{ and } (x_3, y_3I_1, z_3I_2) \in (x_2, y_2I_1, z_2I_2) + (u_2, v_2I_1, t_2I_2).$$

$$(x_3, y_3I_1, z_3I_2) \in (x_1 + u_1, (y_1 + v_1)I_1, (z_1 + t_1)I_2) \text{ and } (x_3, y_3I_1, z_3I_2) \in (x_2 + u_2, (y_2 + v_2)I_1, (z_2 + t_2)I_2),$$

$$\Longrightarrow x_3 \in x_1 + u_1, y_3 \in y_1 + v_1, z_3 \in z_1 + t_1 \text{ and } x_3 \in x_2 + u_2, y_3 \in y_2 + v_2, z_3 \in z_2 + t_2.$$

Since $x_3 \in x_1 + u_1, y_3 \in y_1 + v_1, z_3 \in z_1 + t_1$ implies $x_1 \in x_3 - u_1, y_1 \in y_3 - v_1, z_1 \in z_3 - t_1$.
Then we have

$$x_1 \in x_3 - u_1 \subseteq (x_2 + u_2) - u_1 = x_2 + (u_2 - u_1) \subseteq x_2 + M,$$

$$y_1 \in y_3 - v_1 \subseteq (y_2 + v_2) - v_1 = y_2 + (v_2 - v_1) \subseteq y_2 + M,$$

$$z_1 \in z_3 - t_1 \subseteq (z_2 + t_2) - t_1 = z_2 + (t_2 - t_1) \subseteq z_2 + M,$$

$$\Longrightarrow (x_1, y_1I_1, z_1I_2) \subseteq (x_2, y_2I_1, z_2I_2) + M(I_1, I_2).$$

$$\therefore (x_1, y_1I_1, z_1I_2) + M(I_1, I_2) \subseteq (x_2, y_2I_1, z_2I_2) + M(I_1, I_2) + M(I_1, I_2) = (x_2, y_2I_1, z_2I_2) + M(I_1, I_2).$$

□

Similarly it can be shown that $(x_2, y_2I_1, z_2I_2) + M(I_1, I_2) \subseteq (x_2, y_2I_1, z_2I_2) + M(I_1, I_2)$. Hence the proof.

Proposition 3.15. *Let $M(I_1, I_2)$ be a normal refined neutrosophic subcanonical hypergroup of a refined neutrosophic canonical hypergroup $R(I_1, I_2)$. Let $R(I_1, I_2)/M(I_1, I_2)$ be as defined in Proposition 3.14. For all $r_1 + M(I_1, I_2), r_2 + M(I_1, I_2) \in R(I_1, I_2)/M(I_1, I_2)$ define the hyperoperation $+'$ by*

$$r_1 + M(I_1, I_2) +' r_2 + M(I_1, I_2) = (r_1 +' r_2) + M(I_1, I_2).$$

Then, $(R(I_1, I_2)/M(I_1, I_2), +')$ is a neutrosophic canonical hypergroup if R/M is a canonical hypergroup.

Definition 3.16. Let $(R(I_1, I_2), +_1)$ and $(M(I_1, I_2), +_2)$ be any two refined neutrosophic canonical hypergroups and let

$$\phi : R(I_1, I_2) \longrightarrow M(I_1, I_2)$$

be a mapping from $R(I_1, I_2)$ into $M(I_1, I_2)$.

(1) ϕ is called a refined neutrosophic canonical hypergroup homomorphism if:
   (a) for all x, y of $R(I_1, I_2)$, $\phi(x +_1 y) \subseteq \phi(x) +_2 \phi(y)$,
   (b) $\phi(0, 0I_1, 0I_2) = (0, 0I_1, 0I_2)$,
   (c) $\phi(I_k) = I_k$ for $k = 1, 2$.

(2) ϕ is called a good refined neutrosophic canonical hypergroup homomorphism if:
   (a) for all x, y of $R(I_1, I_2)$, $\phi(x +_1 y) = \phi(x) +_2 \phi(y)$,
   (b) $\phi(0, 0I_1, 0I_2) = (0, 0I_1, 0I_2)$,
   (c) $\phi(I_k) = I_k$ for $k = 1, 2$.

(3) ϕ is called a refined neutrosophic isomorphism if ϕ is a refined neutrosophic homomorphism and ϕ^{-1} is also a refined neutrosophic homomorphism.

Definition 3.17. Let $\phi : R(I_1, I_2) \longrightarrow M(I_1, I_2)$ be a refined neutrosophic canonical hypergroup homomorphism from a refined neutrosophic canonical hypergroup $R(I_1, I_2)$ into a refined neutrosophic canonical hypergroup $M(I_1, I_2)$.

(1) The kernel of ϕ denoted by $Ker\phi$ is the set $\{(u, vI_1, wI_2) \in R(I_1, I_2) : \phi((u, vI_1, wI_2)) = (0, 0I_1, 0I_2)\}$.

(2) The image of ϕ denoted by $Im\phi$ is the set $\{\phi((u, vI_1, wI_2)) : (u, vI_1, wI_2) \in R(I_1, I_2)\}$.

Proposition 3.18. *Let $\phi : R(I_1, I_2) \longrightarrow M(I_1, I_2)$ be a refined neutrosophic canonical hypergroup homomorphism from a refined neutrosophic canonical hypergroup $R(I_1, I_2)$ into a refined neutrosophic canonical hypergroup $M(I_1, I_2)$.*

(1) *The kernel of ϕ is not a neutrosophic subcanonical hypergroup of $R(I_1, I_2)$.*

(2) *The image of ϕ is a neutrosophic subcanonical hypergroup of $M(I_1, I_2)$.*

Proof. (1) It can be seen from the definition of Kernel that $Ker\phi$ is a subcanonical hypergroup and not a neutrosophic subcanonical hypergroup.

(2) The proof is similar to the proof in classical case. □

Remark 3.19. If ϕ in Proposition 3.18 is a good refined neutrosophic canonical hypergroup homomorphism and $P(I_1, I_2)$ is a normal refined neutrosophic subcanonical hypergroup of $R(I_1, I_2)$ then $\phi(P(I_1, I_2))$ is normal in $M(I_1, I_2)$. Also, if $Q(I_1, I_2)$ is a normal refined neutrosophic subcanonical hypergroup of $M(I_1, I_2)$, then $\phi^{-1}(Q(I_1, I_2))$ is normal in $R(I_1, I_2)$.

In what follows we shall establish the relationship between the refined neutrosophic canonical hypergroups and the parent or any neutrosophic canonical hypergroups. Since every neutrosophic (refined neutrosophic) canonical hypergroup is a canonical hypergroup. Then, our task will be to find a classical map ψ say, such that

$$\psi : R(I_1, I_2) \longrightarrow R(I).$$

And for all $(u, vI_1, wI_2) \in R(I_1, I_2)$ we define ψ by

$$\psi((u, vI_1, wI_2)) = (u, (v + w)I).$$

Proposition 3.20. *Let $(R(I_1, I_2), +')$ be a refined neutrosophic canonical hypergroup and let $(R(I), +)$ be a neutrosophic canonical hypergroup. The mapping ψ defined above is a good homomorphism.*

Proof. It can be easily shown that ψ is well defined.
Now, for $(u, vI_1, wI_2), (p, qI_1, tI_2) \in R(I_1, I_2)$ then

$$\begin{aligned} \psi((u, vI_1, wI_2) +' (p, qI_1, tI_2)) &= \psi((u + p), (v + q)I_1, (w + t)I_2) \\ &= ((u + p), (v + q + w + t)I) \\ &= ((u + p), (v + w)I + (q + t)I) \\ &= (u, (v + w)I) + (p, (q + t)I) \\ &= \psi((u, vI_1, wI_2)) +' \psi((p, qI_1, tI_2)). \end{aligned}$$

Hence ψ is a good homomorphism. □

Remark 3.21. The kernel of this map is given by

$$\begin{aligned} ker\psi &= \{(u, vI_1, wI_2) : \psi((u, vI_1, wI_2)) = (0, 0I_1, 0I_2)\} \\ &= \{(0, vI_1, (-v)I_2)\}. \end{aligned}$$

It can be shown that $ker\psi$ is a subcanonical hypergroup of $R(I_1, I_2)$.

4. **Conclusions**

This paper studied refinement of neutrosophic canonical hypergroup and presented some of its basic properties. Also, the existence of a good homomorphism between a refined neutrosophic canonical hypergroup $R(I_1, I_2)$ and a neutrosophic canonical hypergroup $R(I)$ was established. We hope to present and study more advance properties of refined neutrosophic canonical hypergroup and its substructures in our future papers.

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Neutrosophic Sets Integrated with Metaheuristic Algorithms: A survey

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Abstract: Neutrosophic set is a branch of neutrosophy concerned with nature, the genesis, and breadth of impartialities, and also their interaction with various mental spectra. Neutrosophic sets constitute relatively new expansions of intuitionistic fuzzy. In a short period of time, numerous researchers have accepted neutrosophic reasoning. Many researchers have linked neutrosophic science and metaheuristics in various ways over the last ten years. Metaheuristic research has attracted a great attention throughout the literature, which covers methodologies, apps, comparative analysis; due to its higher intensities and fruitful implementations. Metaheuristic algorithms are used to introduce the best or the optimum solutions to a lot of optimization problems due to the behavior of these algorithms inspired by Nature and its ability to adapt to problems, as well as the possibility of integrating more than one algorithm to reach the best solutions. Based on the previous reason, many researchers used these algorithms with neutrosophic science to present many platforms in the recent years, which was the motivation to introduce this survey paper. This paper is introduced to cover the publications from 2010 to 2021 in order to draw a comprehensive picture of metaheuristic research integrated with neutrosophic theory.

Keywords: Neutrosophic Set; Neutrosophy; Metaheuristic; Optimization

1. Introduction

Massive amounts of incomplete, vague, fuzzy, and inaccurate data are provided by real-world applications. Errors possession, lack of knowledge, or randomness can all contribute to uncertainty [1]. Several theories and methodologies have been proposed to deal with such ambiguous data, including probability theory[2], Theory of Para-consistent logic[3], Set theory with a fuzziness[4], and Fuzzy set theory with intuition[5]. Furthermore, such theories only can deal through one incorrect problem element at a time rather than the entire problem inside one paradigm; for instance, the fuzzy set theory can just handle imprecise and fuzzy data, not inconstant or unfinished issues in the same data. As a result, in attempt to settle such concerns in a unified framework, the neutrosophic methodology was introduced [6]. As a result, the neutrosophic methodology[6], which really is a philosophical subdivision incorporating philosophic awareness, set theory, intuitions, and probabilistic, can then be used to resolve these issues in a single framework. Neutrosophic methodology is the foundation of neutrosophic logic, which indicates ambiguity and uses a new platform named Neut-A to tackle problems which fuzzy logic can't [7]. Fuzzy logic is typically known as a two-valued logic extension in which statements do not have to be true or untrue, and might have a truth degree in range of 0 and 1. In comparison to all other logics, neutrosophic reasoning and Fuzzy logic with intuition have a higher percentage of "indeterminacy". That's also owing to unexpected criteria which can also be concealed for some unknowingness or proposals, but a neutrosophic logic allows for every item (T, I, F) to be flooded (stirring) over 1, in other words be '1+', or dehydrated, e.g. be '0', in order to distinguish among relative truth and actual truth, along with comparative falseness and ultimate untruth.

Neutrosophic is considered a new area of study which investigates the origin, natural world, and context of impartialities, and also their ability to interact with various mental spectra, according to Smarandache. Traditional logic, probabilistic reasoning, and inaccurate probability are all specified by Neutrosophic, multiple value logic. Neutrosophic is more human-like in that it describes knowledge inaccuracy or linguistic inconsistency as determined by multiple observers. A neutrosophic set is considered a subset in neutrosophic that investigates neutrality's essence, origin, or context of impartialities, as well as its relationships. Each incident within neutrosophic set theory owns a degree of truth, falsehood, and ambiguity that must also be analyzed separately from one another [8].

A neutrosophic set has recently emerged as a general, prominent, and comprehensive strategy. Many researchers have submitted many research papers using neutrosophic science to solve many problems. Looking at the recent years, we find that a link has been made between neutrosophic set theory and metaheuristic science so as to produce the best proposed solutions for many research problems. Integration between the two previous sciences has been based on the importance of metaheuristic.

The concept "metaheuristic" refers to higher-level methodologies that have been proposed for solving a wide range of optimization issues. A number of metaheuristic algorithms have recently proven successful in solving critical situations. The advantage of employing such algorithms to solve tremendous challenges would be that they produce a desired or optimum solution in a short time, even for problem sizes are large scale.

In the last twenty years, new and innovative evolutionary approaches have emerged successfully, despite the progress of classical metaheuristic algorithms. During this period of metaheuristic algorithm research, a large number of new metaheuristic algorithms inspired by evolutionary or behavioral processes are introduced.

Many of metaheuristic algorithms have been used integrated with neutrosophic science to answer a wide range of research issues. For instance, forest fires[9], document-level sentiment analysis[10],image segmentation[11, 12],breast cancer detection[13, 14], time series forecasting[15], Relief distribution and victim evacuation[16],modeling neutrosophic variables[17],CT image segmentation and two[18]… etc.

The goal of this essay is to provide a detailed insight of the major metaheuristic algorithms that have been combined with neutrosophic set theory to introduce a number of efficient solutions or platforms to a variety of problems over the last decade, as well as a clear explanation of NS and metaheuristic concepts.

The following is the structure of the survey on integration between meta- heuristics and neutrosophic. Sect. 2 introduces the concept and model of neutrosophic sets. Sect. 3 introduces the concept meta-heuristic algorithms. Sect. 4 introduces a global review on neutrosophic set incorporated with metaheuristic and its applications and platforms in different models. Finally, Section 5 concludes with a conclusion and recommendations for the future.

2. Theory of Information

In the neutrosophic scientific theory, every proposal is simulated to get the rate of reality $\underset{u}{\mu}(x)$, indeterminacy rate $\underset{u}{\sigma}(x)$, and negation rate $\underset{u}{\nu}(x)$. The theory of neutrosophy is a broadening of fuzziness and intuitionistic fuzzy sets as well as rational thinking. Neutrosophic theory is gaining momentum as a solution to a variety of real-world problems involving ambiguity, imprecision, vagueness, incompleteness, inconsistency, and indeterminacy[19, 20]The neutrosophic logic is used to deal with information that has a lot of uncertainty and irregularity. As a result, the neutrosophic theory is used in a variety of fields to address issues related to indeterminacy. To deal with uncertainty, we need some concepts to define the neutrosophic variable. The triple supports any value of a variable in a neutrosophic universe:

$$u = \{\underset{u}{\mu}(x), \underset{u}{\sigma}(x), \underset{u}{\nu}(x)\} \tag{1}$$

Where $\underset{u}{\mu}(x)$ denotes fact membership, $\underset{u}{\sigma}(x)$ denotes indeterminacy membership, and $\underset{u}{\nu}(x)$ denotes untruth membership. Such three aspects are self-contained and quantifiable. According to the neutrosophic set description[21] ,every element $x \in X$ inside set u represented in Eq. 1 is falling under the upcoming constraints:

$$0^{-} \le \underset{u}{\mu}(x), \underset{u}{\sigma}(x), \underset{u}{\nu}(x) \le 1^{+} \tag{2}$$

$$0^{-} \le \underset{u}{\mu}(x) + \underset{u}{\sigma}(x) + \underset{u}{\nu}(x) \le 3^{+} \tag{3}$$

The following equations limit a type 1 neutrosophic fuzzy set:

$$0^{-} \le \underset{u}{\mu}(x), \underset{u}{\sigma}(x), \underset{u}{\nu}(x) \le 1^{+} \tag{4}$$

$$\underset{u}{\mu}(x) \wedge \underset{u}{\sigma}(x) \wedge \underset{u}{\nu}(x) \le 0.5 \tag{5}$$

$$0^{-} \le \underset{u}{\mu}(x) + \underset{u}{\sigma}(x) + \underset{u}{\nu}(x) \le 3^{+} \tag{6}$$

The third classification, a neutrosophic intuitionistic set with type 2, is compelled with the upcoming formula:

$$0.5 \le \underset{u}{\mu}(x), \underset{u}{\sigma}(x), \underset{u}{\nu}(x) \tag{7}$$

$$\underset{u}{\mu}(x) \wedge \underset{u}{\sigma}(x) \le 0.5, \underset{u}{\mu}(x) \wedge \underset{u}{\nu}(x) \le 0.5, \underset{u}{\sigma}(x) \wedge \underset{u}{\nu}(x) \le 0.5 \tag{8}$$

$$0^{-} \le \underset{u}{\mu}(x) + \underset{u}{\sigma}(x) + \underset{u}{\nu}(x) \le 2^{+} \tag{9}$$

In order to make certainty science an open field for working in environments of degree of truth, uncertainty, and error, a set of definitions of neutrosophic science is provided in the literature.

Figure 1[9] depicts a typical neutrosophic decision-making aid. The system starts with a prepping process that converts traditional data from various repositories into rough set theory data that is then transferred via the neutrosophic platform. Aim of providing neutrosophic data 'B,' the neutrosophic system applies system formula 'U o R= B' toward a data of neutrosophic 'U,' during which R considers the group of neutrosophic laws pertaining that apply to the framework so as to provide the data 'B.'. In addition, o represents the intermediate representation. The wisdom of neutrosophic is obtained by extracting knowledge from neutrosophic data. As a result of applying the decision support system procedure to neutrosophic data, the neutrosophic decision seems to be the ultimate destination.

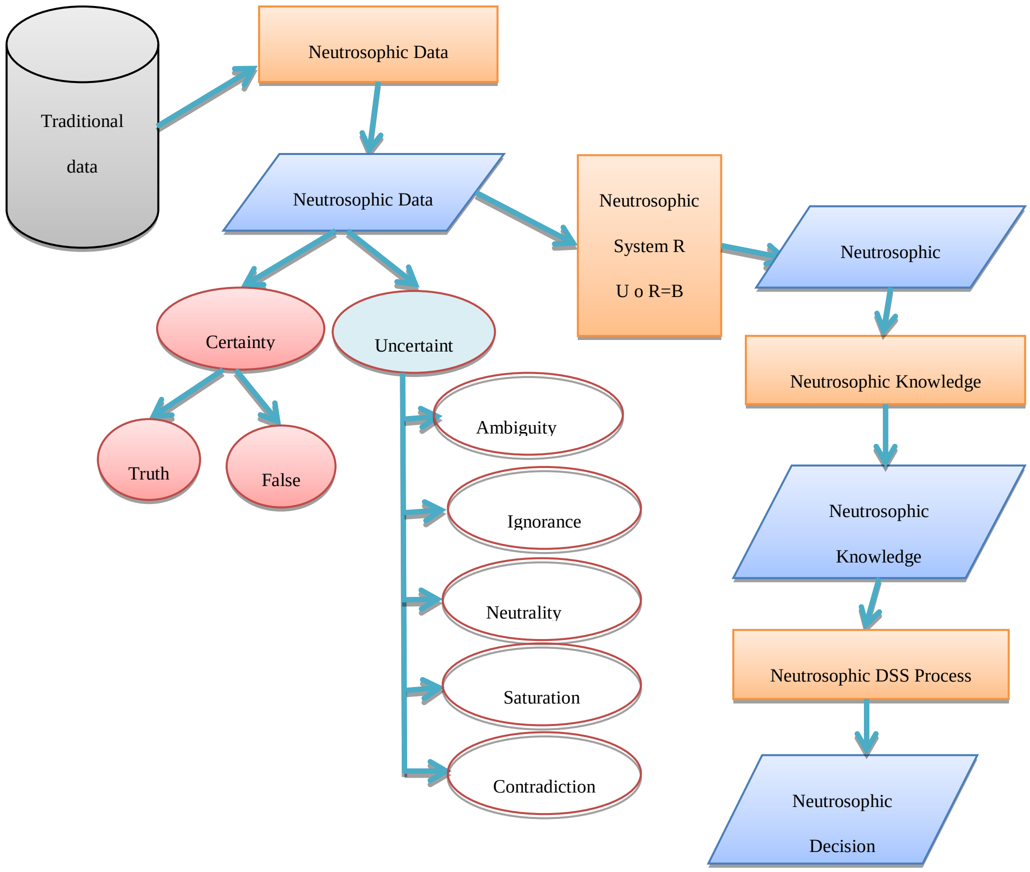

Figure 1. a typical neutrosophic decision-making aid

3. Overview on Meta-heuristic Algorithms

A metaheuristic is an algorithmic structure that covers a lot of optimization issues with only a few tweaks to adapt to the specific problem. By studying the nature of the work of any metaheuristic algorithm, we find that, the harmonization of two search archetypes: the exploration and the exploitation is the reason for metaheuristics robust searching mechanism. Metaheuristics can be used with a variety of classification criteria. Consider how metaheuristics are classified according to the path they take, whether they use memory, what form of neighborhood exploration they use, or how many existing solutions they carry from one iteration to the next. A lot of researchers introduced various metaheuristic categories.

4. Integration of Neutrosophic Set and Met-heuristic Algorithms

According to F Smarandache [6], neutrosophic set is basic paradigm which extrapolates with neutrosophic set and its variants, such as simplistic neutrosophic sets, single valued neutrosophic sets, fuzzy intuitionistic fuzzy sets, Interval-valued neutrosophic set, ragged neutrosophic set, intuitionistic neutrosophic set, interval neutrosophic set, neutrosophic soft set, neutrosophic hesitant fuzzy set. These variants have recently been integrated with meta-heuristic optimization algorithms and employed in a broad variety of topics, including computer applications, medical applications, image segmentation, clustering, text analysis, time series forecasting and more.

4.1. Neutrosophic Set and Met-heuristic Integration on Image Segmentation

Throughout the fields of photo processing and computer vision, accurately and efficiently segmenting images has always been critical. Biomedical image segmentation is a critical step in picture processing and style recognition that distinguishes objects from the background, determining the analysis' quality. The image is frequently segmented to non-overlapping pieces during this process. Fuzzy theory, when applied to image segmentation, retains more information than strict segmentation techniques. Segmenting the data can potentially be a part of different clusters using FCM. The indeterminacy of every object in the series, however, cannot be described or assessed using the traditional set techniques. As a result, fuzzy sight has been used to deal with uncertainty. Neutrosophic integrated with meta-heuristic has recently been a popular tool for dealing with this problem.

Image segmentation algorithms can also be divided into three categories according to the resemblance and incompleteness of gray levels: Integrating region-based segmenting, border segmenting, and segmentation approaches with specialized theoretical tools algorithms. The innate fuzziness of images, as well as the ambiguity during segmentation, is added to the complexity of image segmentation. Classical segmentation methods have a hard time keeping up with modern demands. For example, when a single monolithic sill is also utilized to segregate the objective from the surroundings in the limit picture segmentation approach, the impact cannot be optimal. In picture segmentation of region-based, it is generally more-segmented and babble-sensitive. If the image segmentation method is ineffective, the segmentation process will be of poor quality. In-depth image segmentation research is beneficial to enhance image processing follow-up performance. In recent decades, numerous researchers have conducted considerable picture segmentation research; however there isn’t presently absolutely clear a segmentation technique which is adequate over all images. Numerous image segmentation methodologies have been introduced that incorporate some particular theories and methods, like the FCM algorithm relies on cluster analysis, as a result of the emergence of several concepts and approaches in numerous sectors. The initial parameters of the FCM algorithm are extremely sensitive, and it may be necessary to manually adjust them to estimate the global optimum and strengthen segmentation speed. Furthermore, the conventional FCM methodology is vulnerable to noise or gray - scale discontinuities because it ignores spatial information. The region based segmentation method only considers information such as pixel intensities, image boundary, and so on, and ignores the image's inherent indeterminacy, which can lead to erroneous picture segmentation findings. Although, because turbulence is unavoidable throughout images acquired, transportation, and storage, de-noising is emerged as a key research for image processing. A lot of academics have suggested many de-noising techniques.

Benaichouche et al [22] boosted fuzzy c-means in 3 stages for solving image segmentation problems. First, particle swarm optimization algorithm (PSO) was incorporated to improve the initialization steps of fuzzy c-means method. Second, during cluster segmentation, the Mahalanobis space was also employed to limit the impact of geometric pattern upon the locative gray information incorporation of pictures. Ultimately, the clustered mistake had been rectified via redistributing potentially misclassified pixels, allowing the segmentation results to be refined. Canayaz et al [11] presented a segmentation method which could be applied to image processing. Image segmentation algorithms such as the Neutrosophic Set (NS), which is used to evaluate indeterminacy information, and metaheuristic algorithms have become popular. Both of these methods were used in this research. After transforming a picture into the neutrosophic domains, that has 3 subsets (Truth, indeterminacy, and Falsehood), after that, image pixels' indeterminacy is removed, and meaningful features of the image are acquired. Then, using T, I subset, the coefficient matrix is found, and the thresholds which coincide to the values optimizing the entropy objective function are dictated using coefficient matrices. This is accomplished using the Cricket Algorithm. The picture will be represented by these thresholds, and indeed the picture will be segregated as a result.

A new bandwidth image retrieval scheme is proposed by Rashno et al [23]. RGB images are first turned into two subgroups in the NS sector and then segmented for this job. Color features such as the dominant color descriptor (DCD), distribution, and statistical components are retrieved for each segment of an image. Wavelet characteristics are also retrieved from the entire image as texture features. A feature vector is created by combining all retrieved characteristics either from a fragmented or entire image. Feature vectors are offered for feature selection in ant colony optimization, which picks its most important features. For the final retrieval process, only a few features are used.

Gehad et al [24] presented a composite segmentation strategy depending on a variant of watershed algorithm and Neutrosophic reasoning. Pre - processing stage, CT image translation to the Neutrosophic space, and post-processing are the three aspects of the proposed technique. Normalization and the median filter are employed in the preprocessing step so as to improve the clarity and brightness ratios of the hepatic CT picture while also reducing noise. Three membership sets transform and depict the improved CT liver picture in the Neutrosophic set domain. Finally, in the last phase, morphology with mathematical formula and a variant of watershed method are utilized to improve the generated truthful image out from the previous step and recover the liver of the CT image.

Image segmentation of ultrasound breast cancer is a critical point; different studies were introduced to cover this area. For example, Zhang article [7] demonstrated segmenting of ultrasonography breast cancer imaging by defining a neutrosophic range, that is split into 3 subsets: T, I, and F, neutrosophic may be applied to image processing. The image is then segmented in the neutrosophic domain with the watershed technique. M Zhang et al [25]also introduced an approach for segmenting breast ultrasound pictures (BUS) using a neutrosophic approach and watershed algorithm. According to M Zhang, First, a BUS picture is tied with the domain of neutrosophic. The image is then converted into a binary one using neutrosophic logic, and the resultant image is segmented using the watershed technique. The tumor is finally found in the segmental area. TS Umamaheswari et al [14] addressed the problem of simultaneously gene selection and robust breast cancer (BC) test categorization by displaying two crossbred algorithms, namely the enhanced firefly algorithm (EFA) and the adaptive neuro neutrosophic inference system (ANNIS), both of which have chosen attributes for CTC detection. It is divided into 3 distinct phases: The main phase is to eliminate fineness markers associated with paired cell composition separation. The second step proposes a new meta-heuristic method based on EFA to discern prescient features for BC prediction. FAs have been changed in the EFA algorithm by using the discontinuous domain as a continuous domain. The EFA flexibly balances research and abuse to swiftly identify the optimum solution EFA is a new calculation method based on the blazing lighting technique used by fireflies. In the gene space, the EFA can quickly determine the best or relatively close gene subset amplifying a given fitness work.

Moving to histopathology, GI Sayed et al [13] presented an approach to histopathology slide imaging that uses neutrosophic sets (NS) and metaheuristic optimization algorithm called moth-flame (MFO) to detect mitosis automatically. The suggested method is divided into two primary phases: candidate extraction and candidate categorization. A Gaussian filtering is applied to the histopathology slide image during the candidate extraction stage, and the enhanced picture was transferred into the NS domain. The truth subset image was then subjected to morphological treatments in order to improve the image and focus on mitotic cells. Several statistical, form, textural, and echoes were retrieved from each candidate during the candidate categorization step. The greatest distinguishing properties of mitotic cells were then chosen using a meta-heuristic MFO algorithm principle. Finally, the characteristics that were chosen were supplied into the classification and regression trees (CART).

In the field of image segmentation, Nondestructive testing (NDT) is a method of detecting a flaw in metal without destroying it. To detect the flaw from an NDT image by using an image segmentation-based technique, it is a difficult task. The problem arises as a result of uncertainties in the NDT image pattern. The uncertainty should be addressed effectively when segmenting an NDT picture. S Dhar [26] described a novel technique for segmenting an NDT picture while dealing with uncertainties using a neutrosophic set in this paper. The NS handles the uncertainty by dividing the image to three subsets: truth, falsehood, and indeterminacy. Two procedures - mean and augmentation - are required for appropriate NS value representation. The bat algorithm is integrated to identify the right values of and based on statistics of the image (BA). The method determines the best values for and in order to adequately manage the uncertainty. In comparison to the most recent methods, we found the proposed method to be pretty satisfactory in terms of performance.

4.2. Neutrosophic Set and Meta-heuristic Integration on Time –Series-Forecasting

Recent years, various time series forecasting models were introduced based on neutrosophic integrated with meta-heuristic. For example, P Singh research paper [27] introduced a new method for forecasting time series datasets that uses a neutrosophic-quantum optimization strategy. The inherited uncertainty of a time series set of data was represented in this paper using neutrosophic set (NS) theory, which has three memberships: truthful, ambiguity, and falsehood. The term "neutrosophic time series" refers to these kinds of forms of time series datasets (NTS). NTS has also been used to model and forecast time series dataset. The success of the NTS molding technique is strongly reliant on the ideal picking of the discourse space and its related periods, according to the findings. The paper uses the quantitative optimization algorithm called QOA and aggregates, as well as the NTS molding technique to tackle this problem. By picking the global optimum universe of speech and its accompanying periods from a collection of local optimum solution, the NTS molding approach performs better with QOA. A new time - series model was suggested by P Singh et al [15] based on neutrosophic theory and the PSO algorithm. This suggested framework started with a time series set of data being represented in NS utilizing three separate NS subscriptions: truth, indeterminacy, and falsity. This NS representation of time series was given the label neutrosophic time - series data (NTS). The suggested model's predicting performance was discovered to be strongly dependent on the optimal universe of discussion of the time series set of data selection. In this work, the image segmentation problem was solved using the PSO method.

P Singh et al [28]also presented another research that focused on three primary issues with time series datasets: depiction of time series datasets using NS, a number of three membership degrees of NS combined, and predicting outcomes production. This study recommended using a neutrosophic-neuro-gradient technique to overcome these three domain-specific issues. The uncertainty associated with time series datasets was represented using NS theory. Numerous decision rules with the style of IF-THEN principles were generated in NTS and dubbed neutrosophic entropy decision rules (NEDRs). The forecasting findings were evolved using an ANN-based structure with NEDRs as an input. This study also used the gradient descent approach to reduce the disparities among of computed and targeted outcomes values in during experiment in order to enhance the effectiveness of the ANN and create optimal prediction performance.

For forest fires forecasting, M Gamal et al [9] introduced a platform that combines the measures of information theory with PSO to predict a neutrosophic parameter using empirical data. PSO is a meta-heuristic methodology for determining the best partitions for truth membership, indeterminacy, and falsity. For the wildfire temperature variable, the suggested methodology produced relatively similar function subsets, whereas the Fuzzy C-Mean obviously altered the function subsets. Estimating actual temperature vagueness in wildfire data will help to accurately forecast these fires.

4.3. Classifying MANET's Attacks Based on Neutrosophic and Meta-heuristic Integration

A mobile ad hoc network (MANET) is an ad hoc system made up of mobile wireless servers with no permanent telecommunications infrastructure. This platform's evolution may be more rapid and unpredictable.. Because of MANETs' unique characteristics, an adversary can launch several attacks on ad hoc networks. The most pressing concern with MANETs is security, which is critical to the system's overall utility. Accessibility of system administrations, privacy, and data integrity can all be achieved by ensuring that security concerns are addressed. MANETs are vulnerable to security attacks on accounts due to their characteristics such as open medium, powerful topology change, lack of central monitoring and management, pleasant computations, and no obvious protection mechanism. The battleground for MANETs vs. the security threat has altered as a result of these causes. Because of such traits, MANETs are more vulnerable to attacks from within the network. Remote connections also render MANETs more vulnerable to attacks, making it easier for an attacker to break into the system and gain access to the ongoing conversation.

Routing table overflows, flooding assault, wormhole attack, Mitm attacks, and greedy node misbehaving are some of the risks that MANET can face. Nodes in MANETs are vulnerable to intrusion and assault because they lack specified architecture and mobility. Intruder Detection Learning Technology is used by designers to protect a computer system from unauthorized access such as hackers. It is a learning challenge to use an intrusion detector to generate a classifier. The detectors should really have the ability to tell the difference between "abnormal" connections, also known as invasions or threats, and "normal" contacts.

Elwahsh et al [29] proposed an Intelligent System for Categorizing MANETs Attacks based on Neutrosophic and Genetic Algorithm (GA), which is a challenging step for categorizing MANETs threats. This framework is relying on two phases: the first is pre-processing and the second is classification for network invasions. In the preprocessing step, network characteristics are formatted in a classifying format. The data from the KDD network [30] is transformed to take the format of neutrosophic (Membership, Indeterminacy, Non-membership). The genetic algorithms (GA) searching technique is used to find a series of neutrosophic constraints to categorize MANETs assaults after transforming traditional KDD data to a neutrosophic data form. The GA starting population is made up of individuals who were chosen at random. A neutrosophic (if-then) categorization rule is represented by each individual.

H Elwahsh et al [31]proposed another method for classifying MANET’s threats using a composite neutrosophic intelligent system with genetic algorithm. This study presents a hybrid framework of Self-Organized Features Maps (SOFM) and evolutionary algorithms for MANETs attack inference (GA). To construct the MANET's neutrosophic conditioned parameters, the hybrid uses the SOFM's unsupervised learning capabilities. The neutrosophic variables, as well as the training set of data, are given to the GA, which uses the fitness function to discover the most suitable neutrosophic set of rules from a series of original sub threats. This approach is intended to identify unknown MANET assaults.

4.4. Job Shop Scheduling Based on Neutrosophic and Meta-heuristic Integration.

Scheduling module schedules machines work for reducing the maximum completion time (make span) or meeting other criteria. The flexible job-shop scheduling problem (FJSP) with routing flexibility seems to be more challenging, and can be thought of it as an integrated making plans and job shop scheduling (IPPS) problem, in which the two significant roles of process planning and task shop scheduling are regarded as a whole and streamlined simultaneously in order to take advantage of versatility in a production system. Because of their intricacy, meta-heuristic methods are frequently used to tackle scheduling difficulties. L Jin et al [32] presented a study on the modeling and optimization strategy for the problem of IPPS with unpredictable process time. To describe unknown processing times, NS is initially presented. They created an enhanced teaching-learning-based optimizing (TLBO) methodology to handle more resilient scheduling strategies owing to the complicated of the math modelThe scoring values of the unknown execution per each device are assessed and enhanced in the proposed optimization approach to achieve the most effective alternative. In [33] L Jin et al proposed a research focused the problem of IPPS with unpredictable process time in order to mitigate the inconsistency in make span in [36]. To simulate the unknown processing times, the innovative neutrosophic numbers are first presented. A mixed-integer linear programming (MILP) framework based on neutrosophic numbers is evolved; regarding the NP hardness and difficulty of estimating the model, the variable neighborhood search (VNS) embedded mimetic algorithm (MA) is formed to expedite extra efficient systems. The nominal make span criteria and the robustness requirement has been weighted summed in the suggested algorithm.

4.5. Image Clustering Based on Neutrosophic Set and Meta-heuristic Integration

Clustering is the division of a group of samples or items into a number of clusters with comparable common components. The fuzzy c-means (FCM) method is one of the most often used fuzzy clustering approaches. To acquire the data membership degrees in FCM, an iterative reduction of a cost function is done. This objective functions are subjected to a constraint for each data set, namely that combination of degrees of membership across bunches must equal one. The FCM approach, on the other hand, has some downsides, including the fact that, firstly its sensitivity to noise, secondly, it strives to reduce intra-cluster variance, thirdly, having a local minima, and fourth the outcomes are dependent on the beginning values. A group's number of noise points could be rather large. As a result, academics are interested in finding new approaches to address these issues. The NS was introduced to handle the uncertainties connected with the clustering based methodologies' parameters since it is a formidable strategy to cope with indeterminacy.

Based on NS, PSO, and the fast fuzzy c-means (FFCM) approach, Anter et al [18] suggested an upgraded segmentation method for abdominal CT liver tumors. To reduce the noise and modify the image, the median filter method was used first. The abdominal CT image was then processed using the neutrosophic domain. The PSO algorithm developed to enhance the FFCM algorithm before utilizing the approach to fragment the neutrosophic image. Subsequently, using the abdominal CT, a segmentation image of the liver was acquired.

Relying on PSO and FCM, Watershed image segmentation method was proposed by Yu et al [34], who used a new variant of PSO algorithm to obtain the accurate clustering core. They also segmented a tiny section of the original image with an accurate clustering core and enhanced fitness function, and acquired a segmentation results for the full image.

To tackle the drawbacks of FCM, J Zhao et al [35] introduced a technique called an Innovative Neutrosophic Image Segmentation Improved Fuzzy C-Means Methodology (NIS-IFCM), in which they first de-noise the image by transforming it into a neutrosophic image. Then the study proposed a new method that combines the PSO and FCM algorithms to decrease the FCM algorithm's reliance on the initial value introduced an innovative methodology for tackling the problem of image segmentation. Another study presented by F Zhao et al [36], he provided an innovative solution for this problem of poor image border segmenting. The proposed study combined FCM with PSO algorithm to enhance the capability of global search to tackle the issue of neutrosophic image clustering. Hanuman et al [37]presented a hybrid FCM-PSO approach. FCM-PSO is a hybrid method that combines the best aspects of FCM and PSO algorithms to solve the problem of local minima in FCM.

4.6. Image Thresholding Based on Neutrosophic Set and Meta-heuristic Integration

Image thresholding is a crucial step in segmenting and extracting objects from photos. In this area, a set of techniques have been offered. Typically, the indetermination every element in a crisp set, cannot usually be specified and assessed. The membership of fuzzy set N specified on universe A is traditionally represented by an actual number inside the traditional fuzzy set. The fuzzy sets methods, on the other hand, only evaluate the truthful membership that is substantiated and ignore the falsehood membership that is contradictory to the proof that is problematic in various situations. On the other hand, the NS combines the concepts of a variety of sets like the classic, interval valued fuzzy, fuzzy, interval valued intuitionistic fuzzy, and intuitionistic fuzzy into a single idea. In the NS, indeterminism is clearly measured, and truth (T), indeterminacy (I), and falsity (F) memberships are all independent. As a result, many experiments were conducted to increase the thresholding efficiency of NS.

Based on NS and improved artificial bee colony (I-ABC) algorithms, K Hanbay et al [38] introduced a new synthetic aperture radar (SAR) algorithm for picture segmentation. Threshold value estimation is viewed in this approach as a search technique for a suitable value within a gray scale period. For getting the best threshold value, the I-ABC optimized procedure is provided. To develop an efficient and powerful fitness function for the I-ABC approach, the input SAR picture is translated into the domain of NS. After that, images of the neutrosophic T and I subsets are obtained. A co-occurrence matrices relying on neutrosophic T and I subset pictures is created, and the objective functions of the I-ABC algorithm is represented using a two-dimensional gray entropy function. Consequently, in the I-ABC algorithm, the occupied, bystanders, and scouting bees rapidly explore the best threshold value.

M Nasef et al [39] introduced a study that is provided a multi-criteria adaptive strategy for brightening the dark parts of musculoskeletal scintigraphy images (NS) using the algorithm of Salp Swarm and the NS. Firstly, the task of improving the dark areas is turned into an optimization issue. The SSA is then used to identify the optimal enhancement for any image independently, and the neutrosophic algorithm is then used to calculate the similarity score for each image using adaptive weight coefficients produced by the SSA algorithm.

On the other hand, because conventional image segmentation techniques for side scan sonar (SSS) images are typically inefficient or inaccurate, Jianhu et al [40]work proposes a new image thresholding segmentation approach called SSS relying on the NS and quantitate-behaved particle swarm optimization (QPSO) algorithms. To begin, the image grayscale co-occurrence matrix is built with respect to the NS space, expressing the precise texture of the SSS picture, which can increase the accuracy of SSS image segmentation. The optimal two-dimensional segmentation threshold vector is then rapidly and precisely generated using the QPSO method, based on the two-dimensional maximum entropy theory, which can increases the effectiveness and reliability of SSS picture segmentation. Ultimately, target segmentation of SSS images with high noise levels is accomplished with precision and efficiency. The algorithm's efficiency is demonstrated by segmenting SSS images including various objectives.

4.7. COVID-19 Diagnosing Based on Neutrosophic set integration with metaheuristic

COVID-19, a rapidly spreading virus, created a tremendous demand for an accurate and quick testing approach. The well-known RT-PCR test is expensive and unavailable throughout many suspect instances. SH Basha et al [41] suggests a neutrosophic framework for diagnosing COVID-19 patients. The suggested framework consists of five phases. The speeded up robust features called SURF methodology is first performed for every X-ray image in order to obtain resilient invariance in features. Secondly, three selecting sampling techniques are used to deal with the dataset's imbalance. Thirdly, a neutrosophic rule-based categorization scheme is presented, which generates a set of rules depending on three neutrosophic quantities <T; I; F>, which represent the truthful, indeterminism, and falsehood degrees. Fourth, improve the classification's performance, a genetic algorithm is used to pick the best neutrosophic set of rules. The classification-based neutrosophic logic is proposed in the fifth step.

4.8. Integration of Neutrosophic Set and Meta-heuristic in other fields

Document, sentence, and aspect level sentiment analysis are the three layers of sentiment analysis. The text gives polarity at the document and sentence levels, respectively, on the basis of the entire document and sentence. The text gives positive polarity for some aspects but negative polarity for others at the aspect level. A Jain et al [10] proposed a composite framework, which is concerned with document level analysis called "Senti-NSetPSO" to evaluate large text document. Senti-NSetPSO consists of two binary and ternary classifiers based on PSO and Neutrosophic Set hybridization. This approach is appropriate for classifying large text document with a file size of more than 25 kilobytes. The size of the swarm is created from large text can valuable metric for implementing PSO convergence.

Another topic of study is the cloud environment. A feast of research papers was just given. Because the cloud environment is made up of distributed resources that are used in a dynamic manner, it is necessary to design optimal scheduling in the cloud environment to ensure that cloud consumers get the quality of service they want while cloud providers make the most money. However, the occurrence of inefficiencies when scheduling cloud resources poses a challenge to typical scheduling rules. The major goal of K Gurumurthy et al [42] research was to address ambiguity in cloud resource scheduling by developing a neutrosophic inference system for prioritizing incoming tasks and optimizing resource utilization using quantum cuckoo search cache management. The suggested study used neutrosophic representation to express the parameters involved in resource scheduling with the goal of reducing response and execution time while increasing throughput, which benefits the cloud service provider's profitability.

6. Conclusion and Future Work

Because many real-life decision-making problems entail imprecision, imprecision, ambiguity, inconsistencies, incomplete data, and indeterminacy, NS, meta-heuristics, as well as logic are becoming more prominent as answers. The research and applications of the neutrosophic- set, logic, measure, and probability are referred to as neutrosophic. The neutrosophic logic (NL) has traditionally been used to denote a mathematical formula of ambiguity, inconsistency, ambiguity, incompleteness, and redundancy inconsistency based on non-standard analysis. It is regarded as a framework for assessing indeterminacy, truth, and falsity. In contrast, the NS quantifies indeterminacy directly, while T, I, and F memberships are all independent. This property is critical in a variety of applications, including data fusion to merge data from many sensors and other biomedical diagnosis scenarios. The NS concept is an innovative mathematical technique to dealing with uncertainty that has a lot of potential in a lot of different ways. Recently, NS has been combined with meta-heuristics to create decision schemes for a variety of applications such as processes on images as thresholding, clustering, segmentation and classification, medical image processing, cloud computing, job-shop scheduling, time series forecasting, forest fires forecasting, document level sentiment analysis, modeling neutrosophic variables, breast cancer detection, and other uses. Because no study has been done on the use of the NS and meta-heuristic integration in picture registration, compression, or restoration, this will be the future direction. As a result, it is recommended to use NS techniques in such activities rather than existing procedures. Furthermore, the FCM is the most clustering technique that can be coupled with the NS and meta-heuristic to reduce uncertainty. As a result, it is recommended that the NS and meta-heuristic be combined with other clustering algorithms.

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H-Max Distance Measure of Bipolar Neutrosophic Sets and an Application to Medical Diagnosis

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Abstract: A single-valued neutrosophic set is one of the advanced fuzzy sets that is capable of handling complex real-world information satisfactorily. A development of single-valued neutrosophic set and fuzzy bipolar set, called a bipolar neutrosophic set, was introduced. Distance measures between fuzzy sets and advanced fuzzy sets are important tools in diagnostics and prediction problems. Sometimes they are defined without considering the condition of the inclusion relation on sets. In decision-making applications, this condition should be required (here it is called the inference of the measure). Moreover, in many cases, a distance measure capable of discriminating between two nearly identical objects is considered an effective measure. Motivated by these observations, in this paper, a new distance measure is proposed in a bipolar neutrosophic environment. Furthermore, an entropy measure is also developed by the similarity between two sets of mutual negation. Finally, an application to medical diagnosis is presented to illustrate the effective applicability of the proposed distance measure, where entropy values are used to characterize noises of different attributes.

Keywords: neutrosophic distance; similarity measure; bipolar neutrosophic sets; entropy measure; medical diagnosis

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1. Introduction

In 1965, the concept of a fuzzy set (FS) was introduced by Zadeh [1] to handle uncertainty of information in real-world inference systems. According to him, the degree of membership (positivity) of an element u to a FS on a universe U is one value $\mu(u)$, where $\mu(u) \in [0,1]$. The theory of FSs has reached a huge amount of achievements in a variety of application areas. However, in many real-life problems, the presence of negativity cannot be ignored. In 1983, Atanassov [2] proposed the concept of an intuitionistic fuzzy set (IFS) by considering the membership degree $\mu(u)$ as well as the non-membership degree $\nu(u)$ with the condition on their sum which is $\mu(u)+\nu(u) \le 1$. The theory and applications of IFSs have been strongly developed such as studies on logical operators [3-5] and applications in decision making [6-10].

From a philosophical perspective on the existence of the field of neutrosophy, Smarandache considers that using IFSs to treat indeterminate and inconsistent is not satisfactory enough. In 1999, Smarandache [11] introduced the concept of neutrosophic set (NS). He named its three characteristic functions the truth membership function, the indeterminacy-membership function, and falsity-membership function, denoted by $T(u)$, $I(u)$, and $F(u)$, respectively. Their outputs are real standard or nonstandard subsets of $]^{-}0,1^{+}[$. From the requirement of practical applications about representing the featured degrees by real values, Wang et al [12] provided the definition of single-valued neutrosophic sets (SVNSs). Cuong [13] also proposed the concept of picture fuzzy set (PFS) as a particular case of NSs. Some results on PFSs can be found in [14-19]. Because of the independent existence between the considered property and its corresponding implicit antagonist, Deli et al. [20] introduced the concept of bipolar neutrosophic sets (BNSs). This is a generalization of SVNSs and bipolar fuzzy sets [21]. In a BNS X, $T^{\square}(u), I^{\square}(u), F^{\square}(u)$ represent the characteristic degrees of an element $u \in U$ corresponding to X and $T^{\approx}(u), I^{\approx}(u), F^{\approx}(u)$ represent characteristic degrees of u to some implicit counter-property corresponding to X. Some research on NSs and BNSs and their applications can be found in [22-36].

The advanced fuzzy distance measures are known as effective tools for solving decision-making problems [6-10, 13, 37]. Some of distance measures of SVNSs were proposed such as Hausdorff distance [38], Cosine similarity measures [39], and the distance measures of Ye [40], Aydoğdu [41], Huang [26], and Ngan et al. [42]. In 2018, Vakkas [43] et al. introduced similarity measures of BNSs and their application to decision-making problems. Vakkas's measure was defined without considering the condition of the inclusion relation on sets. In decision-making applications, this condition (in this paper, it is called the inference of the measure) should be required. Moreover, Vakkas's proposal does not imply cross-evaluation, which is necessary to distinguish the differences and was discussed in intuitionistic fuzzy and single-value neutrosophic environments [7,10,42]. Motivated by these observations, in this paper, a new distance measure set that includes cross-evaluation and the inference of the measure is first proposed in a bipolar neutrosophic environment. Furthermore, an entropy measure is also developed by the similarity between two sets of mutual negation. Finally, an application to medical diagnosis on the UCI dataset is presented to illustrate the effective applicability of the proposed distance measure, where entropy values of different attribute sets are used to characterize their noises.

The next sections of the paper are distributed content as follows. Some basic concepts and the related measure formulas are presented in Section 2. In Section 3, the proposals on the distance measure, the similarity measure, and the entropy measure on BNSs are introduced. In Section 4, an application to medical diagnosis given to show the effectiveness of the proposed distance measure. Finally, Section 5 shows the conclusions of the study.

2. Preliminaries

Definition 1. [25] A NS X on a universe set U is characterized by three feature functions including a truth-membership function, $T_X : U \to]^{-}0,1^{+}[$, an indeterminacy-membership function, $I_X : U \to]^{-}0,1^{+}[$, and a falsity-membership function, $F_X : U \to]^{-}0,1^{+}[$, where

$$^{-}0 \le \sup_{U} T_X(z) + \sup_{U} I_X(z) + \sup_{U} F_X(z) \le 3^{+},\ z \in U. \tag{1}$$

Definition 2. [20] A BNS X on U is defined by the form as follows:

$$X = \left\{ < z, T_X^{\square}(z), I_X^{\square}(z), F_X^{\square}(z), T_X^{\approx}(z), I_X^{\approx}(z), F_X^{\approx}(z) > \mid z \in U \right\} \text{ or}$$

$$X = < T_X^{\square}, I_X^{\square}, F_X^{\square}, T_X^{\approx}, I_X^{\approx}, F_X^{\approx} >, \tag{2}$$

where $T_X^{\square}, I_X^{\square}, F_X^{\square} : U \to [0,1]$, and $T_X^{\approx}, I_X^{\approx}, F_X^{\approx} : U \to [-1,0]$.

Denoted by $BNS(U)$ the set of all BNSs on U.

Definition 3. [20] Let X_1 and X_2 be two BNSs on U, then

- $X_1 \subseteq X_2$ if and only if $T_1^{\square}(z) \le T_2^{\square}(z), I_1^{\square}(z) \ge I_2^{\square}(z), F_1^{\square}(z) \ge F_2^{\square}(z),\ T_1^{\approx}(z) \ge T_2^{\approx}(z),\ I_1^{\approx}(z) \le I_2^{\approx}(z),$ and $F_1^{\approx}(z) \le F_2^{\approx}(z)$.
- $X_1 = X_2$ if and only if $T_1^{\square}(z) = T_2^{\square}(z), I_1^{\square}(z) = I_2^{\square}(z), F_1^{\square}(z) = F_2^{\square}(z),\ T_1^{\approx}(z) = T_2^{\approx}(z), I_1^{\approx}(z) = I_2^{\approx}(z),$ and $F_1^{\approx}(z) = F_2^{\approx}(z)$.
- $X^c = \left\{ < z, F^{\square}(z), 1 - I^{\square}(z), T^{\square}(z), F^{\approx}(z), -1 - I^{\approx}(z), T^{\approx}(z) > \mid z \in U \right\}.$

Definition 4. [43] A similarity measure of BNSs is a $S : \left(BNS(U)\right)^2 \to \left[0,1\right]$ mapping satisfying

1. $0 \le S\left(X_1, X_2\right) \le 1$,
2. $S\left(X_1, X_2\right) = S\left(X_2, X_1\right)$,
3. $S\left(X_1, X_2\right) = 1$ for $X_1 = X_2$, where $X_1, X_2 \in BNS(U)$.

In 2018, Vakkas et al. [43] proposed a similarity measure of BNSs as follows:

$$S_V\left(X_1, X_2\right) = \alpha S_{V1}\left(X_1, X_2\right) + \left(1-\alpha\right) S_{V2}\left(X_1, X_2\right), \tag{3}$$

where $\alpha \in \left[0,1\right]$,

$$S_{V1}\left(X_1, X_2\right) = \sum_{i=1}^{n} \omega_i \left(\frac{\left[\begin{array}{l} \left(T_{X_1}^{\square}(z_i) T_{X_2}^{\square}(z_i) + I_{X_1}^{\square}(z_i) I_{X_2}^{\square}(z_i) + F_{X_1}^{\square}(z_i) F_{X_2}^{\square}(z_i) \right) \\ -\left(T_{X_1}^{\approx}(z_i) T_{X_2}^{\approx}(z_i) + I_{X_1}^{\approx}(z_i) I_{X_2}^{\approx}(z_i) + F_{X_1}^{\approx}(z_i) F_{X_2}^{\approx}(z_i) \right) \end{array} \right]}{\begin{array}{l} 2\Big[\left(T_{X_1}^{\square\,2}(z_i) + I_{X_1}^{\square\,2}(z_i) + F_{X_1}^{\square\,2}(z_i) \right) + \left(T_{X_2}^{\square\,2}(z_i) + I_{X_2}^{\square\,2}(z_i) + F_{X_2}^{\square\,2}(z_i) \right) \\ -\left(T_{X_1}^{\approx\,2}(z_i) + I_{X_1}^{\approx\,2}(z_i) + F_{X_1}^{\approx\,2}(z_i) \right) - \left(T_{X_2}^{\approx\,2}(z_i) + I_{X_2}^{\approx\,2}(z_i) + F_{X_2}^{\approx\,2}(z_i) \right) \Big] \end{array}} \right),$$

and

$$S_{V2}\left(X_1, X_2\right) = \sum_{i=1}^{n} \omega_i \left(\frac{\left[\begin{array}{l} \left(T_{X_1}^{\square}(x_i) T_{X_2}^{\square}(x_i) + I_{X_1}^{\square}(x_i) I_{X_2}^{\square}(x_i) + F_{X_1}^{\square}(x_i) F_{X_2}^{\square}(x_i) \right) \\ -\left(T_{X_1}^{\approx}(x_i) T_{X_2}^{\approx}(x_i) + I_{X_1}^{\approx}(x_i) I_{X_2}^{\approx}(x_i) + F_{X_1}^{\approx}(x_i) F_{X_2}^{\approx}(x_i) \right) \end{array} \right]}{\begin{array}{l} 2\Big[\sqrt{T_{X_1}^{\square\,2}(x_i) + I_{X_1}^{\square\,2}(x_i) + F_{X_1}^{\square\,2}(x_i)} \times \sqrt{T_{X_2}^{\square\,2}(x_i) + I_{X_2}^{\square\,2}(x_i) + F_{X_2}^{\square\,2}(x_i)} \\ -\sqrt{T_{X_1}^{\approx\,2}(x_i) + I_{X_1}^{\approx\,2}(x_i) + F_{X_1}^{\approx\,2}(x_i)} \times \sqrt{T_{X_2}^{\approx\,2}(x_i) + I_{X_2}^{\approx\,2}(x_i) + F_{X_2}^{\approx\,2}(x_i)} \Big] \end{array}} \right).$$

Note that: Vakkas's proposal is without considering the condition related to the inclusion relation on sets. Some other measures are built based on the triangle inequality condition instead of the condition related to the inclusion relation on sets, such as the Hamming distance and the Euclidean distance [44, 45].

In 2021, by reasoning about the need for the cross-evaluation, Ngan et al. [42] defined the H-max distance measure on SVNSs by

$$\begin{aligned} d_{HN}\left(X_1, X_2\right) = \sum_{i=1}^{n} \chi_i \Big(& \alpha_1 \left| T_{X_1}(z_i) - T_{X_2}(z_i) \right| + \alpha_2 \left| I_{X_1}(z_i) - I_{X_2}(z_i) \right| + \alpha_3 \left| F_{X_1}(z_i) - F_{X_2}(z_i) \right| \\ & + \alpha_4 \left| \max\left\{ T_{X_1}(z_i), I_{X_2}(z_i) \right\} - \max\left\{ I_{X_1}(z_i), T_{X_2}(z_i) \right\} \right| \\ & + \alpha_5 \left| \max\left\{ T_{X_1}(z_i), F_{X_2}(z_i) \right\} - \max\left\{ F_{X_1}(z_i), T_{X_2}(z_i) \right\} \right| \Big) \end{aligned} \tag{4}$$

where $\alpha_k \in \left(0,1\right)$, $\sum_{k=1}^{5} \alpha_k = 1$, $\chi_i \in \left[0,1\right]$, $\sum_{i=1}^{n} \chi_i = 1.$

3. H-max bipolar neutrosophic weighted measure

Now, the provided definition of distance measures of BNSs includes the inference condition. Furthermore, a specific distance measure, called H-max bipolar neutrosophic weighted measure, is introduced based on the formula of d_{HN} proposed by Ngan et al. [42].

Definition 5. For all $X_1, X_2, X_3 \in BNS(U)$ where $U = \{z_1, ..., z_n\}$, then a distance measure of BNSs is $d : (BNS(U))^2 \to [0,1]$ mapping satisfying

1. $d(X_1, X_2) = d(X_2, X_1)$,
2. $d(X_1, X_2) = 0$ if and only if $X_1 = X_2$,
3. If $X_1 \subseteq X_2 \subseteq X_3$, then $d(X_1, X_2) \le d(X_1, X_3)$ and $d(X_2, X_3) \le d(X_1, X_3)$.

Definition 6. Let $X_1, X_2 \in BNS(U)$ where $U = \{z_1, ..., z_n\}$ and

$$X_1 = \{< z, T_{X_1}^{\square}(z), I_{X_1}^{\square}(z), F_{X_1}^{\square}(z), T_{X_1}^{\approx}(z), I_{X_1}^{\approx}(z), F_{X_1}^{\approx}(z) > | z \in U\},$$

$$X_2 = \{< z, T_{X_2}^{\square}(z), I_{X_2}^{\square}(z), F_{X_2}^{\square}(z), T_{X_2}^{\approx}(z), I_{X_2}^{\approx}(z), F_{X_2}^{\approx}(z) > | z \in U\}.$$

Then, the formula of H-max bipolar neutrosophic weighted distance measure between X_1 and X_2 is as follows

$$d_{H-BN}(X_1, X_2) = \lambda d_{H-BN1}(X_1, X_2) + (1-\lambda) d_{H-BN2}(X_1, X_2), \quad (5)$$

where

$$\begin{aligned} d_{H-BN1}(X_1, X_2) = \sum_{i=1}^{n} \chi_i^{\square} \Big(& \alpha_1^{\square} \left| T_{X_1}^{\square}(z_i) - T_{X_2}^{\square}(z_i) \right| + \alpha_2^{\square} \left| I_{X_1}^{\square}(z_i) - I_{X_2}^{\square}(z_i) \right| + \alpha_3^{\square} \left| F_{X_1}^{\square}(z_i) - F_{X_2}^{\square}(z_i) \right| \\ & + \alpha_4^{\square} \left| \max\{T_{X_1}^{\square}(z_i), I_{X_2}^{\square}(z_i)\} - \max\{I_{X_1}^{\square}(z_i), T_{X_2}^{\square}(z_i)\} \right| \\ & + \alpha_5^{\square} \left| \max\{T_{X_1}^{\square}(z_i), F_{X_2}^{\square}(z_i)\} - \max\{F_{X_1}^{\square}(z_i), T_{X_2}^{\square}(z_i)\} \right| \Big), \end{aligned}$$

$$\begin{aligned} d_{H-BN2}(X_1, X_2) = \sum_{i=1}^{n} \chi_i^{\approx} \Big(& \alpha_1^{\approx} \left| T_{X_1}^{\approx}(z_i) - T_{X_2}^{\approx}(z_i) \right| + \alpha_2^{\approx} \left| I_{X_1}^{\approx}(z_i) - I_{X_2}^{\approx}(z_i) \right| + \alpha_3^{\approx} \left| F_{X_1}^{\approx}(z_i) - F_{X_2}^{\approx}(z_i) \right| \\ & + \alpha_4^{\approx} \left| \min\{T_{X_1}^{\approx}(z_i), I_{X_2}^{\approx}(z_i)\} - \min\{I_{X_1}^{\approx}(z_i), T_{X_2}^{\approx}(z_i)\} \right| \\ & + \alpha_5^{\approx} \left| \min\{T_{X_1}^{\approx}(z_i), F_{X_2}^{\approx}(z_i)\} - \min\{F_{X_1}^{\approx}(z_i), T_{X_2}^{\approx}(z_i)\} \right| \Big), \end{aligned}$$

where $\alpha_k^{\square}, \alpha_k^{\approx} \in (0,1)$, $\sum_{k=1}^{5} \alpha_k^{\square} = 1$, $\sum_{k=1}^{5} \alpha_k^{\square} = 1$, $\chi_i^{\square}, \chi_i^{\approx} \in [0,1]$, $\sum_{i=1}^{n} \chi_i^{\approx} = 1$, and $\lambda \in (0,1)$.

Proposition 1. d_{H-BN} satisfies the following properties for all $X_1, X_2, X_3 \in BNS(U)$.

1. $0 \le d_{H-BN}(X_1, X_2) \le 1$,
2. $d_{H-BN}(X_1, X_2) = 0$ if and only if $X_1 = X_2$,
3. $d_{H-BN}(X_1, X_2) = d_{H-BN}(X_2, X_1)$,
4. $d_{H-BN}(X_1, X_2) \le d_{H-BN}(X_1, X_3)$ and $d_{H-BN}(X_2, X_3) \le d_{H-BN}(X_1, X_3)$ if $X_1 \subseteq X_2 \subseteq X_3$.

Proof

1. Apparently, for all $i = 1, ..., n$,

$$\left| T_{X_1}^{\square}(z_i) - T_{X_2}^{\square}(z_i) \right|, \ \left| I_{X_1}^{\square}(z_i) - I_{X_2}^{\square}(z_i) \right|, \ \left| F_{X_1}^{\square}(z_i) - F_{X_2}^{\square}(z_i) \right| \in [0,1],$$

$$\left|\max\left\{T_{X_1}^{\square}(z_i),I_{X_2}^{\square}(z_i)\right\}-\max\left\{I_{X_1}^{\square}(z_i),T_{X_2}^{\square}(z_i)\right\}\right|\in\left[0,1\right],$$
$$\left|\max\left\{T_{X_1}^{\square}(z_i),F_{X_2}^{\square}(z_i)\right\}-\max\left\{F_{X_1}^{\square}(z_i),T_{X_2}^{\square}(z_i)\right\}\right|\in\left[0,1\right],$$

and

$$\left|T_{X_1}^{\approx}(z_i)-T_{X_2}^{\approx}(z_i)\right|,\left|I_{X_1}^{\approx}(z_i)-I_{X_2}^{\approx}(z_i)\right|,\left|F_{X_1}^{\approx}(z_i)-F_{X_2}^{\approx}(z_i)\right|\in\left[0,1\right],$$
$$\left|\min\left\{T_{X_1}^{\approx}(z_i),I_{X_2}^{\approx}(z_i)\right\}-\min\left\{I_{X_1}^{\approx}(z_i),T_{X_2}^{\approx}(z_i)\right\}\right|\in\left[0,1\right],$$
$$\left|\min\left\{T_{X_1}^{\approx}(z_i),F_{X_2}^{\approx}(z_i)\right\}-\min\left\{F_{X_1}^{\approx}(z_i),T_{X_2}^{\approx}(z_i)\right\}\right|\in\left[0,1\right].$$

Hence, $0\le d_{H-BN}\left(X_1,X_2\right)\le 1.$

2. Clearly, $d_{H-BN}\left(X_1,X_2\right)=0\Leftrightarrow\begin{cases}T_{X_1}^{\square}=T_{X_2}^{\square},T_{X_1}^{\approx}=T_{X_2}^{\approx}\\ I_{X_1}^{\square}=I_{X_2}^{\square},I_{X_1}^{\approx}=I_{X_2}^{\approx}\\ F_{X_1}^{\square}=F_{X_2}^{\square},F_{X_1}^{\approx}=F_{X_2}^{\approx}\end{cases}\Leftrightarrow X_1=X_2.$

3. It can be seen that d_{H-BN} has the symmetry property.

4. Let $X_1\subseteq X_2\subseteq X_3$ then for all $i=1,...,n$,

$$T_{X_1}^{\square}(z_i)\le T_{X_2}^{\square}(z_i)\le T_{X_3}^{\square}(z_i),I_{X_1}^{\square}(z_i)\ge I_{X_2}^{\square}(z_i)\ge I_{X_3}^{\square}(z_i),$$
$$F_{X_1}^{\square}(z_i)\ge F_{X_2}^{\square}(z_i)\ge F_{X_3}^{\square}(z_i),T_{X_1}^{\approx}(z_i)\ge T_{X_2}^{\approx}(z_i)\ge T_{X_3}^{\approx}(z_i),$$
$$I_{X_1}^{\approx}(z_i)\le I_{X_2}^{\approx}(z_i)\le I_{X_3}^{\approx}(z_i),\text{ and }F_{X_1}^{\approx}(z_i)\le F_{X_2}^{\approx}(z_i)\le F_{X_3}^{\approx}(z_i).$$

These lead to

$$\left|T_{X_1}^{\square}-T_{X_2}^{\square}\right|\le\left|T_{X_1}^{\square}-T_{X_3}^{\square}\right|,\left|I_{X_1}^{\square}-I_{X_2}^{\square}\right|\le\left|I_{X_1}^{\square}-I_{X_3}^{\square}\right|,\left|F_{X_1}^{\square}-F_{X_2}^{\square}\right|\le\left|F_{X_1}^{\square}-F_{X_3}^{\square}\right|,$$
$$\left|T_{X_1}^{\approx}-T_{X_2}^{\approx}\right|\le\left|T_{X_1}^{\approx}-T_{X_3}^{\approx}\right|,\left|I_{X_1}^{\approx}-I_{X_2}^{\approx}\right|\le\left|I_{X_1}^{\approx}-I_{X_3}^{\approx}\right|,\left|F_{X_1}^{\approx}-F_{X_2}^{\approx}\right|\le\left|F_{X_1}^{\approx}-F_{X_3}^{\approx}\right|.$$

Moreover,

$$\max\left\{T_{X_3}^{\square},I_{X_1}^{\square}\right\}\ge\max\left\{T_{X_2}^{\square},I_{X_1}^{\square}\right\}\ge\max\left\{T_{X_1}^{\square},I_{X_2}^{\square}\right\}\ge\max\left\{T_{X_1}^{\square},I_{X_3}^{\square}\right\},$$
$$\min\left\{T_{X_3}^{\approx},I_{X_1}^{\approx}\right\}\le\min\left\{T_{X_2}^{\approx},I_{X_1}^{\approx}\right\}\le\min\left\{T_{X_1}^{\approx},I_{X_2}^{\approx}\right\}\le\min\left\{T_{X_1}^{\approx},I_{X_3}^{\approx}\right\},$$

$$\max\left\{T_{X_3}^{\square},F_{X_1}^{\square}\right\}\ge\max\left\{T_{X_2}^{\square},F_{X_1}^{\square}\right\}\ge\max\left\{T_{X_1}^{\square},F_{X_2}^{\square}\right\}\ge\max\left\{T_{X_1}^{\square},F_{X_3}^{\square}\right\},$$
$$\min\left\{T_{X_3}^{\approx},F_{X_1}^{\approx}\right\}\le\min\left\{T_{X_2}^{\approx},F_{X_1}^{\approx}\right\}\le\min\left\{T_{X_1}^{\approx},F_{X_2}^{\approx}\right\}\le\min\left\{T_{X_1}^{\approx},F_{X_3}^{\approx}\right\}.$$

Hence,

$$\left|\max\left\{T_{X_2}^{\square},I_{X_1}^{\square}\right\}-\max\left\{T_{X_1}^{\square},I_{X_2}^{\square}\right\}\right|\le\left|\max\left\{T_{X_3}^{\square},I_{X_1}^{\square}\right\}-\max\left\{T_{X_1}^{\square},I_{X_3}^{\square}\right\}\right|,$$
$$\left|\min\left\{T_{X_2}^{\approx},I_{X_1}^{\approx}\right\}-\min\left\{T_{X_1}^{\approx},I_{X_2}^{\approx}\right\}\right|\le\left|\min\left\{T_{X_3}^{\approx},I_{X_1}^{\approx}\right\}-\min\left\{T_{X_1}^{\approx},I_{X_3}^{\approx}\right\}\right|,$$

$$\left|\max\left\{T_{X_2}^{\square},F_{X_1}^{\square}\right\}-\max\left\{T_{X_1}^{\square},F_{X_2}^{\square}\right\}\right|\le\left|\max\left\{T_{X_3}^{\square},F_{X_1}^{\square}\right\}-\max\left\{T_{X_1}^{\square},F_{X_3}^{\square}\right\}\right|,$$
$$\left|\min\left\{T_{X_2}^{\approx},F_{X_1}^{\approx}\right\}-\min\left\{T_{X_1}^{\approx},F_{X_2}^{\approx}\right\}\right|\le\left|\min\left\{T_{X_3}^{\approx},F_{X_1}^{\approx}\right\}-\min\left\{T_{X_1}^{\approx},F_{X_3}^{\approx}\right\}\right|.$$

Thus, $d_{H-BN}\left(X_1,X_2\right)\le d_{H-BN}\left(X_1,X_3\right)$. Similarly, $d_{H-BN}\left(X_2,X_3\right)\le d_{H-BN}\left(X_1,X_3\right)$ is proven. □

Definition 7. Let $X_1, X_2 \in BNS(U)$ where $U = \{z_1, ..., z_n\}$. Then, the formula of H-max bipolar neutrosophic weighted similarity measure between X_1 and X_2 is as follows

$$s_{H-BN}\left(X_1, X_2\right) = 1 - d_{H-BN}\left(X_1, X_2\right). \tag{6}$$

Proposition 2. s_{H-BN} satisfies the following properties, for all $X_1, X_2, X_3 \in BNS(U)$:

1. $0 \le s_{H-BN}\left(X_1, X_2\right) \le 1$,
2. $s_{H-BN}\left(X_1, X_2\right) = 1$ if and only if $X_1 = X_2$,
3. $s_{H-BN}\left(X_1, X_2\right) = s_{H-BN}\left(X_2, X_1\right)$,
4. $s_{H-BN}\left(X_1, X_2\right) \ge s_{H-BN}\left(X_1, X_3\right)$ and $s_{H-BN}\left(X_2, X_3\right) \ge s_{H-BN}\left(X_1, X_3\right)$ if $X_1 \subseteq X_2 \subseteq X_3$.

Remark 1. The proposed distance measure overcomes the limitations of the Hamming distance, the Euclidean distance [44, 45], and Vakkas's proposal [43]. Specifically,

- The proposed measure d_{H-BN} includes cross-evaluations:

$$\left|\max\left\{T_{X_1}^{\square}(z_i), I_{X_2}^{\square}(z_i)\right\} - \max\left\{I_{X_1}^{\square}(z_i), T_{X_2}^{\square}(z_i)\right\}\right|,$$
$$\left|\max\left\{T_{X_1}^{\square}(z_i), F_{X_2}^{\square}(z_i)\right\} - \max\left\{F_{X_1}^{\square}(z_i), T_{X_2}^{\square}(z_i)\right\}\right|,$$
$$\left|\min\left\{T_{X_1}^{\approx}(z_i), I_{X_2}^{\approx}(z_i)\right\} - \min\left\{I_{X_1}^{\approx}(z_i), T_{X_2}^{\approx}(z_i)\right\}\right|,$$
$$\left|\min\left\{T_{X_1}^{\approx}(z_i), F_{X_2}^{\approx}(z_i)\right\} - \min\left\{F_{X_1}^{\approx}(z_i), T_{X_2}^{\approx}(z_i)\right\}\right|.$$

- The proposed measure satisfies the property related to the inclusion relation, i.e., the property 4 in Proposition 1.

Example 1. Let $U = \{z_1, ..., z_n\}$. Put

$$X_1 = \, < 0_U, 0.01_U, 1_U, -0.15_U, 0_U, -0.8_U >,$$
$$X_2 = \, < 0.79_U, 0.01_U, 0.61_U, -0.79_U, 0_U, -0.61_U >,$$
$$X_3 = \, < 0.8_U, 0_U, 0.6_U, -0.8_U, 0_U, -0.6_U >.$$

Then, $X_1, X_2, X_3 \in BNS(U)$ and $X_1 \subset X_2 \subset X_3$. By the similarity measure of Vakkas et al. [43] and choosing specific values for the parameters, we have

$$S_V\left(X_1, X_3\right) = \frac{1}{2} S_{V1}\left(X_1, X_3\right) + \frac{1}{2} S_{V2}\left(X_1, X_3\right),$$

$$S_V\left(X_2, X_3\right) = \frac{1}{2} S_{V1}\left(X_2, X_3\right) + \frac{1}{2} S_{V2}\left(X_2, X_3\right),$$

where,

$$S_{V1}\left(X_1, X_3\right) = \frac{\left(0\times 0.8 + 0.01\times 0 + 1\times 0.6\right) - \left(\left(-0.15\right)\left(-0.8\right) + 0 + \left(-0.8\right)\times\left(-0.6\right)\right)}{\begin{array}{l}2\Big[\left(0^2 + 0.01^2 + 1^2\right) + \left(0.8^2 + 0^2 + 0.6^2\right) \\ -\left(\left(-0.15\right)^2 + 0^2 + \left(-0.8\right)^2\right) - \left(\left(-0.8\right)^2 + 0^2 + \left(-0.6\right)^2\right)\Big]\end{array}} = 0,$$

$$S_{V2}\left(X_1, X_3\right) = \frac{\left(0\times 0.8 + 0.01\times 0 + 1\times 0.6\right) - \left(\left(-0.15\right)\left(-0.8\right) + 0 + \left(-0.8\right)\times\left(-0.6\right)\right)}{\begin{array}{l}2\Big[\sqrt{0^2 + 0.01^2 + 1^2} \times \sqrt{0.8^2 + 0^2 + 0.6^2} \\ -\sqrt{\left(-0.15\right)^2 + 0^2 + \left(-0.8\right)^2} \times \sqrt{\left(-0.8\right)^2 + 0^2 + \left(-0.6\right)^2}\Big]\end{array}} = 0,$$

$$S_{V1}\left(X_2,X_3\right)=\frac{\left(0.79\times0.8+0.01\times0+0.61\times0.6\right)-\left(\left(-0.79\right)\times\left(-0.8\right)+0+\left(-0.61\right)\left(-0.6\right)\right)}{2\left[\left(0.79^2+0.01^2+0.61^2\right)+\left(0.8^2+0^2+0.6^2\right)-\left(\left(-0.79\right)^2+0^2+\left(-0.61\right)^2\right)-\left(\left(-0.8\right)^2+0^2+\left(-0.6\right)^2\right)\right]}=0,$$

$$S_{V2}\left(X_2,X_3\right)=\frac{\left(0.79\times0.8+0.01\times0+0.61\times0.6\right)-\left(\left(-0.79\right)\times\left(-0.8\right)+0+\left(-0.61\right)\left(-0.6\right)\right)}{2\left[\sqrt{0.79^2+0.01^2+0.61^2}\times\sqrt{0.8^2+0^2+0.6^2}-\sqrt{\left(-0.79\right)^2+0^2+\left(-0.61\right)^2}\times\sqrt{\left(-0.8\right)^2+0^2+\left(-0.6\right)^2}\right]}=0.$$

The obtained calculation results are $S_V\left(X_1,X_3\right)=0$ and $S_V\left(X_2,X_3\right)=0$.

Now, from Definition 6 and choosing specific values for the parameters, we have

$$d_{H-BN}\left(X_1,X_3\right)=\frac{1}{2}d_{H-BN1}\left(X_1,X_3\right)+\frac{1}{2}d_{H-BN2}\left(X_1,X_3\right),$$

$$d_{H-BN}\left(X_2,X_3\right)=\frac{1}{2}d_{H-BN1}\left(X_2,X_3\right)+\frac{1}{2}d_{H-BN2}\left(X_2,X_3\right),$$

where

$$d_{H-BN1}\left(X_1,X_3\right)=\frac{1}{5}\Big(\left|0-0.8\right|+\left|0.01-0\right|+\left|1-0.6\right|+\left|\max\{0,0\}-\max\{0.01,0.8\}\right|+\left|\max\{0,0.6\}-\max\{1,0.8\}\right|\Big)=0.482,$$

$$d_{H-BN2}\left(X_1,X_3\right)=\frac{1}{5}\Big(\left|0.15-0.8\right|+\left|0-0\right|+\left|0.8-0.6\right|+\left|\min\{-0.15,0\}-\min\{0,-0.8\}\right|+\left|\min\{-0.15,-0.6\}-\min\{-0.8,-0.8\}\right|\Big)=0.34,$$

$$d_{H-BN1}\left(X_2,X_3\right)=\frac{1}{5}\Big(\left|0.79-0.8\right|+\left|0.01-0\right|+\left|0.61-0.6\right|+\left|\max\{0.79,0\}-\max\{0.01,0.8\}\right|+\left|\max\{0.79,0.6\}-\max\{0.61,0.8\}\right|\Big)=0.01,$$

$$d_{H-BN2}\left(X_2,X_3\right)=\frac{1}{5}\Big(\left|0.79-0.8\right|+\left|0-0\right|+\left|0.61-0.6\right|+\left|\min\{-0.79,0\}-\min\{0,-0.8\}\right|+\left|\min\{-0.79,-0.6\}-\min\{-0.61,-0.8\}\right|\Big)=0.008.$$

Hence,

$$d_{H-BN}\left(X_1,X_3\right)=0.411>d_{H-BN}\left(X_2,X_3\right)=0.009$$
$$\left(s_{H-BN}\left(X_1,X_3\right)=0.589<s_{H-BN}\left(X_2,X_3\right)=0.991\right).$$

In this case, by observation we can also see that X_2 and X_3 are almost the same. In addition, since $X_1\subset X_2\subset X_3$, it can be deduced that the difference between X_1 and X_3 is greater than the that between X_2 and X_3. The proposed distance measure is likely to properly represent this assessment and inference and overcomes the limitation of the proposal of Vakkas et al. [43].

Example 2. Let $U=\{z_1,...,z_n\}$. Put

$$X_1=<0.4_U,0_U,0.4_U,-0.8_U,0_U,-0.8_U>,$$
$$X_2=<0.5_U,0_U,0.5_U,-0.7_U,0_U,-0.7_U>,$$
$$X_3=<0.4_U,0_U,0.6_U,-0.6_U,0_U,-0.8_U>.$$

Then, $X_1,X_2,X_3\in BNS(U)$, $X_1\not\subseteq X_2, X_2\not\subseteq X_1$, and $X_3\subset X_2$.

The Hamming distance [44, 45] on $BNS(U)$ can be defined as follows:

$$d_{Ham}\left(X_1,X_2\right)=\frac{1}{6}\sum_{i=1}^{n}\Big(\left|T^{\square}_{X_1}(z_i)-T^{\square}_{X_2}(z_i)\right|+\left|I^{\square}_{X_1}(z_i)-I^{\square}_{X_2}(z_i)\right|+\left|F^{\square}_{X_1}(z_i)-F^{\square}_{X_2}(z_i)\right|$$
$$+\left|T^{\approx}_{X_1}(z_i)-T^{\approx}_{X_2}(z_i)\right|+\left|I^{\approx}_{X_1}(z_i)-I^{\approx}_{X_2}(z_i)\right|+\left|F^{\approx}_{X_1}(z_i)-F^{\approx}_{X_2}(z_i)\right|\Big).$$

The Euclidean distance [44, 45] on $BNS(U)$ can be defined as follows:

$$d_{Eucl}\left(X_1,X_2\right)=\sum_{i=1}^{n}\Bigg(\frac{1}{6}\Big(\left|T^{\square}_{X_1}(z_i)-T^{\square}_{X_2}(z_i)\right|^2+\left|I^{\square}_{X_1}(z_i)-I^{\square}_{X_2}(z_i)\right|^2+\left|F^{\square}_{X_1}(z_i)-F^{\square}_{X_2}(z_i)\right|^2$$
$$+\left|T^{\approx}_{X_1}(z_i)-T^{\approx}_{X_2}(z_i)\right|^2+\left|I^{\approx}_{X_1}(z_i)-I^{\approx}_{X_2}(z_i)\right|^2+\left|F^{\approx}_{X_1}(z_i)-F^{\approx}_{X_2}(z_i)\right|^2\Big)\Bigg)^{\frac{1}{2}}.$$

Some of the calculation results obtained are as follows:

$$d_{Ham}\left(X_1,X_2\right)=d_{Ham}\left(X_3,X_2\right)=\frac{4}{6},$$

$$d_{Eucl}\left(X_1,X_2\right)=d_{Eucl}\left(X_3,X_2\right)=\frac{\sqrt{6}}{30},$$

$$d_{H-BN}\left(X_1,X_2\right)=0.06<d_{H-BN}\left(X_3,X_2\right)=0.08$$

Clearly, in this case, because of cross-evaluations, the proposed measure can distinguish the difference better than two related measures.

Definition 8. For $E:BNS(U)\to[0,1]$ mapping, if the following conditions are satisfied then E is an entropy measure of BNSs.

1. $E(X)=0$ if and only if X or X^c is a crisp set,
2. $E(X)=E(X^c)$; $E(X)=1$ if and only if $X=X^c$,
3. $E(X_1)\le E(X_2)$ if X_1 Ð X_2, i.e., if $T^{\square}_{X_1}\le T^{\square}_{X_2}$, $F^{\square}_{X_1}\ge F^{\square}_{X_2}$, $T^{\approx}_{X_1}\ge T^{\approx}_{X_2}$, $F^{\approx}_{X_1}\le F^{\approx}_{X_2}$ for $T^{\square}_{X_2}\le F^{\square}_{X_2}$, $T^{\approx}_{X_2}\ge F^{\approx}_{X_2}$, $I^{\square}_{X_1}=I^{\square}_{X_2}=0.5_U$, $I^{\approx}_{X_1}=I^{\approx}_{X_2}=-0.5_U$; and $T^{\square}_{X_1}\ge T^{\square}_{X_2}$, $F^{\square}_{X_1}\le F^{\square}_{X_2}$, $T^{\approx}_{X_1}\le T^{\approx}_{X_2}$, $F^{\approx}_{X_1}\ge F^{\approx}_{X_2}$ for $T^{\square}_{X_2}\ge F^{\square}_{X_2}$, $T^{\approx}_{X_2}\le F^{\approx}_{X_2}$, $I^{\square}_{X_1}=I^{\square}_{X_2}=0.5_U$, $I^{\approx}_{X_1}=I^{\approx}_{X_2}=-0.5_U$.

Proposition 3. Let $X\in BNS(U)$, where $U=\{z_1,...,z_n\}$, then $s_{H-BN}(X,X^c)$ is an entropy measure of X.

Proof.

1. If X be a crisp set, i.e., $T^{\square}_X=1_U, I^{\square}_X=F^{\square}_X=0_U, T^{\approx}_X=I^{\approx}_X=0_U, F^{\approx}_X=-1_U$, or $T^{\square}_X=I^{\square}_X=0_U, F^{\square}_X=1_U, T^{\approx}_X=-1_U, I^{\approx}_X=F^{\approx}_X=0_U$, then, $s_{H-BN}(X,X^c)=0$. Similarly, if X^c is a crisp set, then $s_{H-BN}(X,X^c)=0$. If $s_{H-BN}(X,X^c)=0$, then it's not hard to show that X or X^c is a crisp set.
2. From Proposition 2, we obtain that $E(X)=E(X^c)$; $s_{H-BN}(X,X^c)=1$ if and only if $X=X^c$.
3. Let X_1 Ð X_2, assume that $T^{\square}_{X_1}\le T^{\square}_{X_2}$, $F^{\square}_{X_1}\ge F^{\square}_{X_2}$, $T^{\approx}_{X_1}\ge T^{\approx}_{X_2}$, $F^{\approx}_{X_1}\le F^{\approx}_{X_2}$ for $T^{\square}_{X_2}\le F^{\square}_{X_2}$, $T^{\approx}_{X_2}\ge F^{\approx}_{X_2}$, $I^{\square}_{X_1}=I^{\square}_{X_2}=0.5_U$, $I^{\approx}_{X_1}=I^{\approx}_{X_2}=-0.5_U$, then

$$T^{\square}_{X_1}\le T^{\square}_{X_2}\le F^{\square}_{X_2}\le F^{\square}_{X_1},$$
$$T^{\approx}_{X_1}\ge T^{\approx}_{X_2}\ge F^{\approx}_{X_2}\ge F^{\approx}_{X_1},$$

$$\max\left\{T^{\square}_{X_1},0.5_U\right\}\le\max\left\{T^{\square}_{X_2},0.5_U\right\}\le\max\left\{F^{\square}_{X_2},0.5_U\right\}\le\max\left\{F^{\square}_{X_1},0.5_U\right\},$$

$$\min\left\{T^{\approx}_{X_1},-0.5_U\right\}\ge\min\left\{T^{\approx}_{X_2},-0.5_U\right\}\ge\min\left\{F^{\approx}_{X_2},-0.5_U\right\}\ge\min\left\{F^{\approx}_{X_1},-0.5_U\right\},$$

$$d_{H-BN}\left(X_t, X_t^{\,c}\right) = \lambda \sum_{i=1}^{n} \chi_i^{\square} \left(\left(\omega_1^{\square} + \omega_3^{\square} + \omega_5^{\square}\right) \left| T_{X_t}^{\square}(z_i) - F_{X_t}^{\square}(z_i) \right| + \omega_4^{\square} \left| \begin{matrix} \max\left\{T_{X_t}^{\square}(z_i), 0.5\right\} \\ -\max\left\{0.5, F_{X_t}^{\square}(z_i)\right\} \end{matrix} \right| \right)$$

$$+\left(1-\lambda\right) \sum_{i=1}^{n} \chi_i^{\approx} \left(\left(\omega_1^{\approx} + \omega_3^{\approx} + \omega_5^{\approx}\right) \left| T_{X_t}^{\approx}(z_i) - F_{X_t}^{\approx}(z_i) \right| + \omega_4^{\approx} \left| \begin{matrix} \min\left\{T_{X_t}^{\approx}(z_i), -0.5\right\} \\ -\min\left\{-0.5, F_{X_t}^{\approx}(z_i)\right\} \end{matrix} \right| \right), \; t = 1, 2.$$

Therefore, $d_{H-BN}\left(X_1, X_1^{\,c}\right) \geq d_{H-BN}\left(X_2, X_2^{\,c}\right)$ and then $s_{H-BN}\left(X_1, X_1^{\,c}\right) \leq s_{H-BN}\left(X_2, X_2^{\,c}\right)$.

Similarly, the remaining case is proved. □

4. An application of the H-Max Bipolar Neutrosophic Distance Measure to medical diagnosis

4.1. The H-BN method

A diagnostic problem is stated as follows:

- A medical dataset includes
  - m records of m corresponding patients P_i, $i = 1, 2, ..., m$,
  - n attributes (symptoms) A_j, $j = 1, 2..., n$, of a disease D,
  - k disease classes labeled C_t, $t = 1, 2, ..., k$, of D.
- The problem is to build a diagnostic system with
  - the inputs are the symptoms of a patient,
  - the output is a disease label.

The proposed method:

Inspired by the diagnostic method introduced in [42] by Ngan et al, the proposed method (H-BN) includes four steps as follows.

- **Step 1.** Built two relation matrices in the bipolar neutrosophic environment:
  - Matrix 1 (M1) presents the relations between the symptoms and patients (P_i and A_j are the i[th] row and the j[th] column of M1, respectively, $i = 1, ..., m; j = 1, ..., n$),
  - Matrix 2 (M2) shows the relations between the symptoms and the disease or the classification results. Specifically, M2 is a $k \times n$ matrix (C_t is the t[th] row of M2, $t = 1, ..., k$).
- **Step 2.** Find the entropies $E\left(A_j\right)$ of the symptoms A_j.
- **Step 3.** Calculate the similarity $s_{H-BN}\left(P_i, C_t\right)$ between the symptoms of P_i and the disease classes C_t, where $E\left(A_j\right)$ is put in the weight of A_j.
- **Step 4.** Diagnose the i[th] patient by finding the highest similarity value $\hat{s}_{H-BN}\left(P_i, C_t\right) = s_{H-BN}\left(P_i, C_{t_0}\right)$, $t_0 \in \left[1, k\right]$. The output is t_0.

4.2. Numeric example

In this section, we use the data in the numerical example in [42] on 5 male patients (aged about 30) of Indian Liver Patient Dataset (ILPD) taken from UCI. In the dataset described in Table 1, there are 2 diagnosis labels which are La-I (liver patient) and La-II (non-liver patient). In Table 1, the considered attributes (A_1 - A_7) are Alkaline Phosphotase, Alamine Aminotransferase, Aspartate Aminotransferase, Total Bilirubin, Direct Bilirubin, Albumin, and Albumin and Globulin Ratio.

Table 1. Data of 5 male patients of the ILPD dataset.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	**Class**
P_1	1.3	0.4	482	102	80	3.3	0.9	La-I
P_2	0.8	0.2	198	26	23	4	1	La-II
P_3	0.9	0.2	518	189	17	2.3	0.7	La-I
P_4	3.8	1.5	298	102	630	3.3	0.8	La-I
P_5	0.8	0.2	156	12	15	3.7	1.1	La-II

The steps of the proposed algorithm are implemented as follows:

- **Step 1:** Input data is fuzzified by the following fuzzification functions selected by experts.

$$R^{\prec}_{\alpha,\beta}(z)=\begin{cases}0 & z\le\alpha\\ \dfrac{z-\alpha}{\beta-\alpha} & \alpha<z\le\beta\\ 1 & \beta<z\end{cases},\qquad L^{\prec}_{\alpha',\beta'}(z)=\begin{cases}1 & z\le\alpha'\\ \dfrac{z-\beta'}{\alpha'-\beta'} & \alpha'<z\le\beta'\\ 0 & \beta'<z\end{cases}$$

$$R^{\succ}_{\rho,\ell}(z)=\begin{cases}-1 & z\le\rho\\ \dfrac{z-\ell}{\ell-\rho} & \rho<z\le\ell\\ 0 & \ell<z\end{cases},\qquad L^{\succ}_{\rho',\ell'}(z)=\begin{cases}0 & z\le\rho'\\ \dfrac{z-\rho'}{\rho'-\ell'} & \rho'<z\le\ell'\\ -1 & \ell'<z\end{cases}$$

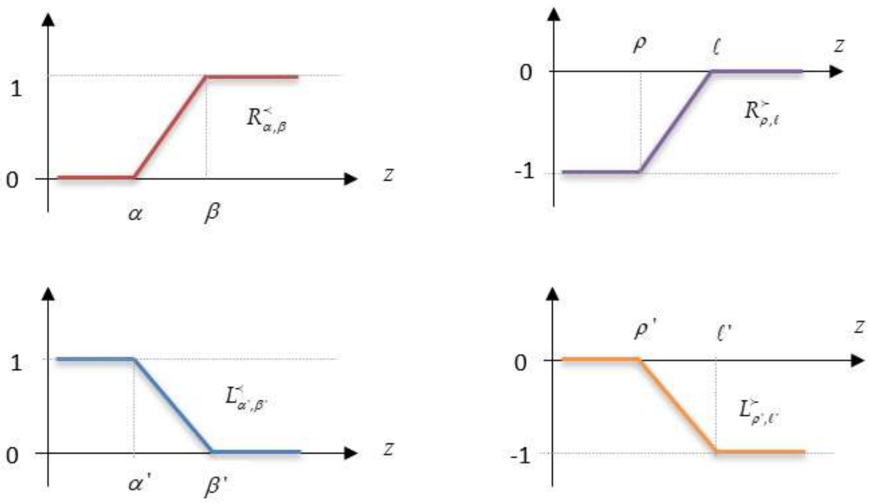

Figure 1. The fuzzification functions are illustrated by graphs.

Specifically, the symptoms on patients are represented as the following BNSs.

$$A_1=<T_1^{\prec}(z),I_1^{\prec}(z),F_1^{\prec}(z),T_1^{\approx}(z),I_1^{\approx}(z),F_1^{\approx}(z)>=$$
$$=<R^{\prec}_{1.2,5.3}(z),L^{\prec}_{0.2,3}(z),L^{\prec}_{0.6,4}(z),R^{\succ}_{0.9,5}(z),L^{\succ}_{0.5,3.5}(z),L^{\succ}_{0.3,4.5}(z)>$$

$$A_2=<T_2^{\prec}(z),I_2^{\prec}(z),F_2^{\prec}(z),T_2^{\approx}(z),I_2^{\approx}(z),F_2^{\approx}(z)>=$$
$$=<R^{\prec}_{0.4,2.3}(z),L^{\prec}_{0.1,1}(z),L^{\prec}_{0.15,1.5}(z),R^{\succ}_{0.2,2}(z),L^{\succ}_{0.2,1.2}(z),L^{\succ}_{0.3,2}(z)>$$

$$A_3 = < T_3^{\square}(z), I_3^{\square}(z), F_3^{\square}(z), T_3^{\approx}(z), I_3^{\approx}(z), F_3^{\approx}(z) >=$$
$$= < R_{140,486}^{\prec}(z), L_{80,250}^{\prec}(z), L_{100,400}^{\prec}(z), R_{110,450}^{\succ}(z), L_{90,300}^{\succ}(z), L_{110,420}^{\succ}(z) >$$

$$A_4 = < T_4^{\square}(z), I_4^{\square}(z), F_4^{\square}(z), T_4^{\approx}(z), I_4^{\approx}(z), F_4^{\approx}(z) > =$$
$$= < R_{33,119}^{\prec}(z), L_{5,60}^{\prec}(z), L_{30,100}^{\prec}(z), R_{25,90}^{\succ}(z), L_{10,70}^{\succ}(z), L_{40,95}^{\succ}(z) >$$

$$A_5 = < T_5^{\square}(z), I_5^{\square}(z), F_5^{\square}(z), T_5^{\approx}(z), I_5^{\approx}(z), F_5^{\approx}(z) > =$$
$$= < R_{33,100}^{\prec}(z), L_{10,90}^{\prec}(z), L_{23,95}^{\prec}(z), R_{33,100}^{\succ}(z), L_{10,90}^{\succ}(z), L_{23,95}^{\succ}(z) >$$

$$A_6 = < T_6^{\square}(z), I_6^{\square}(z), F_6^{\square}(z), T_6^{\approx}(z), I_6^{\approx}(z), F_6^{\approx}(z) > =$$
$$= < L_{2.2,3.5}^{\prec}(z), R_{2,4}^{\prec}(z), R_{3,5}^{\prec}(z), L_{2.3,3.3}^{\succ}(z), R_{2.2,4.1}^{\succ}(z), R_{2.8,5.2}^{\succ}(z) >$$

$$A_7 = < T_7^{\square}(z), I_7^{\square}(z), F_7^{\square}(z), T_7^{\approx}(z), I_7^{\approx}(z), F_7^{\approx}(z) > =$$
$$= < L_{0.5,1}^{\prec}(z), R_{0.3,1.5}^{\prec}(z), R_{0.8,2.5}^{\prec}(z), L_{0.6,1.1}^{\succ}(z), R_{0.2,1}^{\succ}(z), R_{0.7,2.8}^{\succ}(z) >$$

Two bipolar neutrosophic relation matrices M1 and M2 are placed in Tables 2 and 3.

Table 2. The relations between the symptoms and patients are presented.

(M1)	A_1	A_2	A_3	A_4	A_5	A_6	A_7
P_1	<0.02,0.6, 0.8,-0.9, -0.3,-0.2>	<0,0.6, 0.8,-0.9, -0.2,-0.06>	<0.9,0, 0,0, -1,-1>	<0.8,0, 0,0, -1,-1>	<0.7,0.1, 0.2,-0.3, -0.9,-0.8>	<0.1,0.6, 0.1,-1, -0.4,-0.8>	<0.2,0.5, 0.08,-0.6, -0.1,-0.9>
P_2	<0,0.7, 0.9,-1, -0.1,-0.1>	<0,0.8, 0.9,-1, 0,0>	<0.1,0.3, 0.6,-0.7, -0.5,-0.3>	<0,0.6, 1,-1, -0.3,0>	<0,0.8, 1,-1, -0.2,0>	<0,1, 0.5,-1, -0.05,-0.5>	<0,0.5, 0.1,-0.8, 0,-0.85>
P_3	<0,0.7, 0.9,-1, -0.1,-0.1>	<0,0.8, 0.9,-1, 0,0>	<1,0, 0,0, -1,-1>	<1,0, 0,0, -1,-1>	<0,0.9, 1,-1, -0.09,0>	<0.9,0.1, 0,0, -0.9,-1>	<0.6,0.3, 0,-0.2, -0.4,-1>
P_4	<0.6,0, 0.05,-0.3, -1,-0.8>	<0.5,0, 0,-0.3, -1,-0.7>	<0.4,0, 0.3,-0.4, -1,-0.6>	<0.8,0, 0,0, -1,-1>	<1,0, 0,0, -1,-1>	<0.1,0.6, 0.1,-1, -0.4,-0.8>	<0.4,0.4, 0,-0.4, -0.25,-0.95>
P_5	<0,0.7, 0.9,-1, -0.1,-0.1>	<0,0.8, 0.9,-1, 0,0>	<0.04,0.5, 0.8,-0.9, -0.3,-0.1>	<0,0.8, 1,-1, -0.03,0>	<0,0.9, 1,-1, -0.06,0>	<0,0.8, 0.3,-1, -0.2,-0.6>	<0,0.6, 0.2,-1, 0,-0.8>

Table 3. The relations between the symptoms and the classification results are shown.

(M2)	A_1	A_2	A_3	A_4	A_5	A_6	A_7
La-I	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>
La-II	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>

- **Step 2:** Finding the entropies $E\left(A_j\right)=s_{H-BN}(A_j,A_j^c)=1-d_{H-BN}(A_j,A_j^c)$ with $\chi_i^{\sim}=\chi_i^{\approx}=\omega_j^{\sim}=\omega_j^{\approx}=\frac{1}{5}\ (i,j=1,...,5)$ and $\lambda=\frac{1}{2}$:

$$E\left(A_1\right)=0.27,\quad E\left(A_2\right)=0.2,\ E\left(A_3\right)=0.33,\quad E\left(A_4\right)=0.08,$$

$$E\left(A_5\right)=0.13,\quad E\left(A_6\right)=0.55,\ E\left(A_7\right)=0.68\ .$$

- **Step 3:** Calculating the similarities $S(i-I)=s_{H-BN}\left(P_i,(\text{La - I})\right)$ and $S\left(i-II\right)=s_{H-BN}\left(P_i,(\text{La - II})\right)$ with $\omega_j^{\sim}=\omega_j^{\approx}=\frac{1}{5}\ (i,j=1,...,5)$, $\lambda=\frac{1}{2}$, and $\chi_j^{\sim}=\chi_j^{\approx}=\frac{E\left(A_j\right)}{\sum_{j=1}^{5}E\left(A_j\right)}$. The obtained results include: $\chi_1^{\sim}=0.12,\chi_2^{\sim}=0.09,\chi_3^{\sim}=0.15,\chi_4^{\sim}=0.035,\chi_5^{\sim}=0.06,\chi_6^{\sim}=0.245,\chi_7^{\sim}=0.3,$

$$S(1-I)=0.49>S(1-II)=0.475,\quad S(2-I)=0.2<S(2-II)=0.75,$$
$$S(3-I)=0.642>S(3-II)=0.327,\quad S(4-I)=0.63>S(4-II)=0.33,$$
$$S(5-I)=0.186<S(5-II)=0.788.$$

- **Step 4.** The outputs are decided as follows: The outputs of P_1,P_2,P_3,P_4, and P_5 are La-I, La-II, La-I, La-I, and La-II, respectively. These decisions and the last column of Table 1 are the same.

4.3. Experiment

In this part, we test the proposed method on the ILPD dataset on Matlab programming with the evaluation criteria on accuracy is Mean Absolute Error (MAE) and the speed of the algorithms is measured in seconds (sec). Also on this data, Ngan et al. [8] tested 14 other diagnostic methods, denoted by M_{SK1-1}, M_{SK1-2}, M_{SK1-3}, M_{SK1-4}, M_{SK2}, M_{WX}, M_{VS}, M_{ZJ}, M_W, M_J, M_{SA}, $M_{H-\max}$, M_{C-QDM}, and M_{P-QDM}, based on the considered intuitionistic fuzzy distance measures (see Table 4).

Table 4. MAEs and Sec of the considered methods on the ILPD dataset.

Methods	MAE	Sec
M_{SK1-1}	0.3195	0.6177
M_{SK1-2}	0.3158	0.4427
M_{SK1-3}	0.3316	0.4827
M_{SK1-4}	0.2918	0.4602
M_{SK2}	0.2902	0.6527
M_{WX}	0.3227	0.4427
M_{VS}	0.2893	0.5552
M_{ZJ}	0.3096	0.5602
M_W	0.2915	0.8452
M_J	0.289	1.2077
M_{SA},	0.3031	0.8102
$M_{H-\max}$	0.2848	0.51
M_{C-QDM}	0.2836	0.155
M_{P-QDM}	0.2831	0.469

H-BN	0.2729	0.559770

In Table 4, it can be observed that the MAE value of the proposed method (H-BN), which is 0.2729, is less (better) than those of the other algorithms on the ILPD datasets. Figure 2 shows the MAE values of the considered methods on the ILPD dataset, where the heights of the vertical bars present the MAE values of the corresponding algorithms. The heights of the H-BN method (green bars) are lower than those of the remaining bars, that means, it is the best algorithm in terms of accuracy of the considered algorithms on the ILPD dataset. We note that the computation time of our algorithms is very close to the computation time of the other methods.

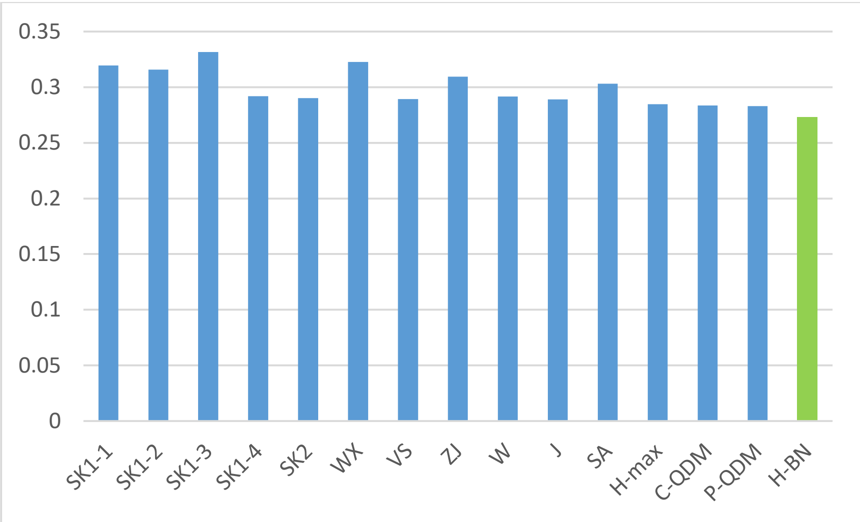

Figure 2. MAEs of the considered methods on the ILPD dataset.

5. Conclusions

In this paper, based on the H-max distance measure on IFSs and SVNSs, a new distance measure on BNSs is introduced to overcome the limitations of the related measures by including cross-evaluations and satisfying the condition of inference of a distance measure. Furthermore, a bipolar neutrosophic entropy measure and its basic properties are presented and proven. In addition, an application to medical diagnosis is shown to illustrate the effective applicability of the measures. There, the proposed diagnostic method called H-BN, a numerical example and real experiment are clarified in detail. In the future, we will test the proposed diagnostic method on other real datasets taken from UCI. Furthermore, we will develop the distance measure for interval-valued bipolar neutrosophic sets.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix

Source code and dataset of this paper can be found at this link: https://sourceforge.net/projects/hbn-datasets-code/.

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Some Neutrosophic Triplet Subgroup Properties and Homomorphism Theorems in Singular Weak Commutative Neutrosophic Extended Triplet Group

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Abstract. In 2018, the study of neutrosophic triplet cosets and neutrosophic triplet quotient group of a neutrosophic extended triplet group was initiated with a follow up of the establishment of fundamental homomorphism theorems for neutrosophic extended triplet group. But some lapses in these earlier results were identified and revised through the introduction of special kind of weak commutative neutrosophic extended triplet group (WCNETG) called perfect neutrosophic extended triplet group. Furthermore, neutro-homomorphism basic theorem has been established for commutative neutrosophic extended triplet group. In this current work, the generalization and extention of the above results was done by investigating neutro-homomorphism in singular WCNETG. This was achieved with the introduction and study of some new types of NT-subgroups that are right (left) cancellative, semi-strong, and maximally normal in a singular WCNETG. For any given non-empty subset S and NT-subgroup H of a singular WCNETG X, some of these new NT-subgroups were shown to exist as non-empty neutrosophic triplet normalizer, generated subset and centralizer of S, closure of H, derived subset of X and center of X. With these, the first, second and third neutro-isomorphism and neutro-correspondence theorems were established. This finally led to the proof of the neutro-Zassenhaus Lemma (Neutro-Butterfly Theorem).

Keywords: Group; Neutrosophic Extended Triplet Group; Weakly Commutative Neutrosophic Extended Triplet Group; Isomorphism Theorems

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1. Introduction

. After the emergence of generalized group (completely simple semigroup), which is an algebraic structure with deep physical background in the unified guage theory and also has direct relation with isotopies (Adeniran et al. [1]), some other algebraic structures which generalize generalized groups have evolved and have been studied alongside with their applications. Among these are neutrosophic triplet group (NTG); Smarandache and Ali [7] and Jaiyéọlá and Smarandache [11], neutrosophic extended triplet group (NETG); Zhang et al. [10], neutrosophic triplet loop (NTL); Jaiyéọlá and Smarandache [3], Quasi neutrosophic triplet loops; Zhang et al. [8], Jaiyéọlá [12, 13] and generalized neutrosophic extended triplet group; Ma et al. [14]. A summary account of these past efforts was compiled and reported by Smarandache et al. [15].

Smarandache and Ali [7] introduced neutrosophic triplets in 2016 while Smarandache [16–19] introduced neutrosophic extended triplets in between 2016 and 2017. The studies of neutrosophic extended triplet group and neutrosophic extended triplet loop became more fascinating with the recent studies of Abel-Grassmann neutrosophic triplet group (loop) and Bol-Moufang types of quasi neutrosophic triplet loops (Fenyves BCI-algebras) by Zhang et al. [20], Wu and Zhang [21] and Jaiyéọlá [12, 13]. The captivating discoveries in these studies are the facts that:

(1) a groupoid is a neutrosophic extended triplet group if and only if it is a completely regular semigroup;
(2) a groupoid is a weak commutative neutrosophic extended triplet group if and only if it is a Clifford semigroup (a type of completely regular semigroup);
(3) there are 540 varieties of Bol-Moufang type quasi neutrosophic triplet loops.

These discoveries established that: the theory of neutrosophic extended triplet group is associated with the theory of semigroup, the theory of weak commutative neutrosophic extended triplet group is associated with the theory of clifford semigroup and the theory of quasi neutrosophic triplet loops is expansive. Shalla and Olgun [5, 6] studied neutrosophic extended triplet group action and the Burnside's lemma, and their direct and Semi-direct products.

We now switch to the definition of a neutrosophic extended triplet group and related structures.

Definition 1.1. (Neutrosophic Extended Triplet Set-NETS)

Let X be a set together with a binary operation $*$ defined on it. Then, X is called a neutrosophic extended triplet set if for any $x \in X$, there exist a neutral of 'x' denoted by $neut(x)$ and an opposite of 'x' denoted by $anti(x)$, with $neut(x), anti(x) \in X$ such that:

$$x * neut(x) = neut(x) * x = x \quad \text{and} \quad x * anti(x) = anti(x) * x = neut(x).$$

The elements $x, neut(x)$ and $anti(x)$ are collectively referred to as neutrosophic triplet, and denote by $(x, neut(x), anti(x))$.

Remark 1.2. In a NETS X, for any $x \in X$, each of $neut(x)$ and $anti(x)$ may not be unique. This is because, in a neutrosophic triplet set $(X, *)$, an element y (resp. z) is the second (resp. third) component of a neutrosophic triplet if there exist $x, z \in X$ $(x, y \in X)$ such that $x * y = y * x = x$ and $x * z = z * x = y$. Thus, (x, y, z) is the neutrosophic triplet.

Definition 1.3. (Neutrosophic Extended Triplet Group-NETG)

Let $(X, *)$ be a neutrosophic extended triplet set. Then, $(X, *)$ is called a neutrosophic extended triplet group if $(X, *)$ is a semigroup. If in addition, $(X, *)$ obeys the commutativity law, then $(X, *)$ is called a commutative extended neutrosophic triplet group (CNETG).

Remark 1.4. In a NETG X, it was shown by Zhang et al. [9] that $neut(x)$ is unique for each $x \in X$. But, the same is not necessarily true for $anti(x)$. Thus, the set of opposites for $x \in X$ is usually denoted by $\{anti(x)\}$.

Definition 1.5. (Weak Commutative Neutrosophic Extended Triplet Group-WCNETG, Definition 4, Zhang et al. [9]; Singular NETG, Definition 6, Zhang et al. [10])

Let $(X, *)$ be a neutrosophic extended triplet group. $(X, *)$ is called a weak commutative neutrosophic extended triplet group (WCNETG) if $a * neut(b) = neut(b) * a$ for all $a, b \in X$.

A NETG is said to be singular if $|\{anti(x)\}| = 1$ for all $x \in X$.

Definition 1.6. (Neutrosophic Triplet Subgroup or NT-Subgroup)

Let $(X, *)$ be a neutrosophic extended triplet group and let $H \subseteq X$. H is called a neutrosophic triplet subgroup (NTSG) of X if $(H, *)$ is a neutrosophic extended triplet group and this is expressed as $H \leq X$. Furthermore, for any fixed $x \in X$, H is called x-normal NTSG of X, written $H \lhd_x X$ if $xy\ anti(x) \in H$ for all $y \in H$.

Lemma 1.7. *(Proposition 2, Zhang et al. [10])*

Let $(X, *)$ be a neutrosophic triplet group and let $H \subseteq X$. H is a neutrosophic triplet subgroup of X if and only if the following conditions are true.

(1) $(H, *)$ is a groupoid;

(2) $anti(x) \in H$ for all $x \in H$.

We now state some important results on singular NETG and WCNETG which are of importance to this work.

Theorem 1.8. *(Proposition 2, 3, Zhang et al. [9])*

Let $(X, *)$ be a NETG. Then $(X, *)$ is a WCNETG if and only the following conditions are true.

(1) $neut(x) * neut(y) = neut(y) * neut(x)$ *for all* $x, y \in X$.

(2) $neut(x) * neut(y) * x = x * neut(y)$ *for all* $x, y \in X$.

Hence, $neut(x) * neut(y) = neut(y * x)$ *and* $anti(x) * anti(y) \in \{anti(y * x)\}$ *for all* $x, y \in X$.

Theorem 1.9. *(Theorem 6, Zhang et al. [10])*

Let $(X, *)$ *be a singular NETG. Then*

(1) $neut(x) * anti(x) = anti(x) * neut(x) = anti(x)$ *for all* $x \in X$.

(2) $anti\big(neut(x)\big) = neut(x)$ *for all* $x \in X$.

(3) $anti\big(anti(x)\big) = x$ *for all* $x \in X$.

(4) $neut\big(anti(x)\big) = neut(x)$ *for all* $x \in X$.

Hence, $neut(x) * neut(y) = neut(y * x)$ *and* $anti(x) * anti(y) \in \{anti(y * x)\}$ *for all* $x, y \in X$.

Here are two methods of constructing a WCNETG as recently described. These new constructions will be of judicious use for illustrations and as examples in order to justify some of the results in this study.

Theorem 1.10. *(First WCNETG, Zhang et al. [20])*

Let $(G_1, *_1)$ *and* $(G_2, *_2)$ *be two groups, with identity elements* e_1 *and* e_2 *respectively, such that* $G_1 \cap G_2 = \emptyset$. *Let* $G = G_1 \cup G_2$, *and define the binary operation* $*$ *on* G *as follows:*

$$x * y = \begin{cases} x *_1 y, & \text{if } x, y \in G_1; \\ x *_2 y, & \text{if } x, y \in G_2; \\ x, & \text{if } x \in G_1,\ y \in G_2; \\ y, & \text{if } x \in G_2,\ y \in G_1 \end{cases}$$

Then, $(G, *)$ *is a WCNETG.*

Theorem 1.11. *(Second WCNETG, Zhang et al. [20])*

Let $(G_1, *_1)$ *and* $(G_2, *_2)$ *be two groups, with identity elements* e_1 *and* e_2 *respectively, such that* $G_1 \cap G_2 = \emptyset$. *Let* $G = G_1 \cup G_2$, *and define the binary operation* $*$ *on* G *as follows:*

$$x * y = \begin{cases} x *_1 y, & \text{if } x, y \in G_1; \\ x *_2 y, & \text{if } x, y \in G_2; \\ y, & \text{if } x \in G_1,\ y \in G_2; \\ x, & \text{if } x \in G_2,\ y \in G_1 \end{cases}$$

Then, $(G, *)$ *is a WCNETG.*

Remark 1.12. For easy reference, the WCNETG in Theorem 1.10 and WCNETG in Theorem 1.11 for any chosen pairs of groups will be called first WCNETG and second WCNETG respectively. It must be noted that both are singular WCNETGs.

Table 1. Group $(G_1, *_1)$

$*_1$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Table 2. Group $(G_2, *_2)$

$*_2$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	5	4	3
3	3	5	1	6	2	4
4	4	6	5	1	3	2
5	5	3	4	2	6	1
6	6	4	2	3	1	5

Table 3. First WCNETG $(G, *)$ of $(G_1, *_1)$ and $(G_2, *_2)$

$*$	e	a	b	c	1	2	3	4	5	6
e	e	a	b	c	e	e	e	e	e	e
a	a	e	c	b	a	a	a	a	a	a
b	b	c	e	a	b	b	b	b	b	b
c	c	b	a	e	c	c	c	c	c	c
1	e	a	b	c	1	2	3	4	5	6
2	e	a	b	c	2	1	6	5	4	3
3	e	a	b	c	3	5	1	6	2	4
4	e	a	b	c	4	6	5	1	3	2
5	e	a	b	c	5	3	4	2	6	1
6	e	a	b	c	6	4	2	3	1	5

Using the groups $(G_1, *_1)$ and $(G_2, *2)$ with multiplication Table 1 and Table 2, Zhang et al. [20] constructed a WCNETG $(G, *)$ with multiplication Table 3.

Bal et al. [22] initated the study of neutrosophic triplet cosets and neutrosophic triplet quotient group of a neutrosophic extended triplet group. This work was then followed up with the establishment of fundamental homomorphism theorems for neutrosophic extended triplet group by Celik et al. [23]. But, Zhang et al. [24] identified some lapses in these earlier articles and revised the results in question by introducing special kind of WCNETG called perfect NETG. On the other hand, Jaiyéọlá and Smarandache [11] also established an homomorphism for NETG which they jointly revised with some other authors in Zhang et al. [10] based on some observations in Zhang et al. [9]. By using a neutrosophic triplet subgroup of a commutative neutrosophic triplet group, Zhang et al. [25] established a new congruence relation, and then constructed the quotient structure induced by neutrosophic triplet subgroup to establish the neutro-homomorphism basic theorem.

. The aim of this current work is to generalize and extend the results in Zhang et al. [24, 25] by investigating neutro-homomorphism in singular WCNETG. This will be done with the introduction and study of some new types of NT-subgroups that are right (left) cancellative, semi-strong, and maximally normal in a singular WCNETG. For any given non-empty subset S and NT-subgroup H of a singular WCNETG X, some of these new NT-subgroups are shown to exist as non-empty neutrosophic triplet normalizer, generated subset and centralizer of S, closure of H, derived subset of X and center of X. With these, the first, second and third neutro-isomorphism and neutro-correspondence theorems are established. And finally, the neutro-Zassenhaus Lemma is established.

2. Main Results

2.1. *Some new results on first and second WCNETGs*

In this subsection, we shall discuss some results associated with the first and second WCNETGs, introduced in Theorem 1.10 and Theorem 1.11, which shall be found useful as examples for illustrations in latter subsections.

Lemma 2.1. *Let $(G,*)$ be the WCNETG of the groups $(G_1,*_1,e_1)$ and $(G_2,*_2,e_2)$ in Theorem 1.10 or Theorem 1.11. Let $h_i : G_i \to G_i,\ i = 1,2$ be mappings and let $h : G \to G$ be defined as*

$$h(x) = \begin{cases} h_1(x), & \text{if } x \in G_1; \\ h_2(x), & \text{if } x \in G_2 \end{cases}$$

1. *If $h_i,\ i = 1,2,$ are endomorphisms of $(G_i,*_i,e_i),\ i = 1,2,$ then h is an neutro-endomorphism of $(G,*)$.*
2. *h is a neuto-monomorphism (neutro-epimorphism) of $(G,*)$ if and only if $h_i,\ i = 1,2$ are monomorphisms (epimorphisms) of $(G_i,*_i,e_i),\ i = 1,2$.*
3. *h is a neuto-automorphism of $(G,*)$ if and only if $h_i,\ i = 1,2$ are automorphisms of $(G_i,*_i,e_i),\ i = 1,2$.*

Proof. This is easy. □

Lemma 2.2. *Let $(G,*)$ and $(G,\circ)$ be the WCNETGs of the pair of groups $(G_1,*_1)$ and $(G_2,*_2)$, and pair of groups $(G_1,\circ_1)$ and $(G_2,\circ_2)$ respectively in Theorem 1.10 or Theorem 1.11. Let $h_i : G_i \to G_i,\ i = 1,2$ be mappings and let $h : G \to G$ be defined as*

$$h(x) = \begin{cases} h_1(x), & \text{if } x \in G_1; \\ h_2(x), & \text{if } x \in G_2 \end{cases}$$

1. *If $h_i,\ i = 1,2,$ are homomorphisms of $(G_i,*_i),\ i = 1,2$ to $(G_i,\circ_i),\ i = 1,2,$ then h is a neutro-homomorphism of $(G,*)$ to $(G,\circ)$.*
2. *h is a neuto-monomorphism (neutro-epimorphism) of $(G,*)$ to $(G,\circ)$ if and only if $h_i,\ i = 1,2$ are monomorphisms (epimorphisms) of $(G_i,*_i),\ i = 1,2$ to $(G_i,\circ_i),\ i = 1,2$.*
3. *h is a neuto-isomorphism of $(G,*)$ to $(G,\circ)$ if and only if $h_i,\ i = 1,2$ are isomorphisms of $(G_i,*_i),\ i = 1,2$ to $(G_i,\circ_i),\ i = 1,2$.*
4. $\ker h = \ker h_1 \cup \ker h_2$ *and* $\mathrm{Im}(h) = \mathrm{Im}(h_1) \cup \mathrm{Im}(h_2)$.

Proof. The proof of this is a generalization of the proof of Lemma 2.1. □

2.2. *Some new subgroupoids and NT-subgroups of a WCNETG*

We shall now introduce some new NT-subgroups of a NETG and study them in singular WCNETG.

Definition 2.3. (Neutrosophic Triplet (Lormalizer, Mormalizer, Normalizer)-NTL, NTM, NTN)

Let X be a NETG and let $\emptyset \neq S \subseteq X$.

(1) The neutrosophic triplet lormalizer (NTL) of S in X is the set defined as $L(S) = \{x \in X | xS\ anti(x) = S\}$.
(2) The neutrosophic triplet mormalizer (NTM) of S in X is the set defined as $M(S) = \{x \in X | neut(x)\ S = S\}$.
(3) The neutrosophic triplet normalizer (NTN) of S in X is the set defined as $N(S) = L(S) \cap M(S)$.

Lemma 2.4. *Let X be a singular WCNETG and $\emptyset \neq S \subseteq X$.*

(1) If $L(S) \neq \emptyset$, then $L(S)$ is a subgroupoid of X.
(2) If $L(S) \neq \emptyset$, then for any $x \in L(S)$, $neut(x) \in L(S) \Leftrightarrow anti(x) \in L(S) \Leftrightarrow neut(x)\ S = S$.
(3) If $M(S) \neq \emptyset$, then $M(S)$ is a NT-subgroup of X.

Proof.

(1) Let $x, y \in L(S)$. Then,
$$(xy)S\ anti(xy) = (xy)S\ anti(y)anti(x) = x\big(yS\ anti(y)\big)anti(x) = xS\ anti(x) = S.$$
So, $xy \in L(S)$.

(2) $neut(x)S\ anti\big(neut(x)\big) = neut(x)S\ neut(x) = neut(x)neut(x)S = neut(x)S$ while
$$anti(x)S\ anti\big(anti(x)\big) = anti(x)xS\ anti(x)anti\big(anti(x)\big) = neut(x)S\ neut\big(anti(x)\big)$$
$$= neut(x)neut\big(anti(x)\big)S = neut\big(anti(x)x\big)S = neut\big(neut(x)\big)S = neut(x)S.$$
By these two arguments, $neut(x) \in L(S) \Leftrightarrow anti(x) \in L(S) \Leftrightarrow neut(x)\ S = S$.

(3) Let $x, y \in M(S)$. Then, $neut(xy)S = neut(y)neut(x)S = neut(y)S = S \Rightarrow xy \in M(S)$. If $x \in M(S)$, then $neut\big(anti(x)\big)S = neut(x)S = S$. So, going by Lemma 1.7 , $M(S)$ is a NT-subgroup of X. □

Example 2.5. In the singular WCNETG $(G, *)$ represented by Table 3, let $S = G_0 = \{e, 1\}$. Then, $L(G_0) = \{1, 2, 3, 4, 5, 6\} = G_2$ and $(G_2, *)$ is a subgroupoid of $(G, *)$. Furthermore, $M(G_0) = \{1, 2, 3, 4, 5, 6\} = G_2$ and $(G_2, *)$ is a NT-subgroup of $(G, *)$.

Theorem 2.6. *Let H and K be NT-subgroups of a singular WCNETG X. Then, HK is a NT-subgroup of X if and only if $HK = KH$.*

Proof. Let $HK = KH$ and let $a, b \in HK$. Then, $a = h_1k_1, b = h_2k_2$ for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$. So, $ab = h_1k_1h_2k_2 = h_1h_3k_3k_2 = h_4k_4 \in HK$ where $h_3k_3 = k_1h_2$. Let $a = hk$, then, $anti(a) = anti(hk) = anti(k)anti(h) = k'h' = h''k'' \in HK$. So, HK is a NT-subgroup of X going by Lemma 1.7.

Conversely, let HK be a NT-subgroup of X and let $a \in KH$. Then, $a = kh$ for some $k \in K$ and $h \in H$. So, $anti(a) = anti(kh) = anti(h)anti(k) = h'k' \in HK \Rightarrow KH \subseteq HK$. Let $b \in HK$, then $anti(b) \in HK$. Thus, $anti(b) = hk$, $h \in H, k \in K$, and so $b = anti\big(anti(b)\big) = anti(hk) = anti(k)anti(h) = k'h' \in KH \Rightarrow HK \subseteq KH$. $\therefore$ $HK = KH$. □

Example 2.7. Consider the singular WCNETG $(G, *)$ represented by Table 3.

(1) $(G_1, *_1)$ and $(G_2, *_2)$ are groups represented by Table 1 and Table 2 respectively. Hence, they are NT-subgroups $(G_1, *)$ and $(G_2, *)$ of $(G, *)$. Now, take $H = G_1$ and $K = G_2$, then $G_1G_2 = G_1 = G_2G_1$, and hence, Theorem 2.6 is true.
(2) $G_0 = \{e, 1\}$ is a NT-subgroup but not a subgroup of G. Now, take $H = G_0$ and $K = G_1$, then $G_0G_1 = G_1 = G_1G_0$, and hence, Theorem 2.6 is true.
(3) $G_2^e = \{e, 1, 2, 3, 4, 5, 6\}$ is a NT-subgroup but not a subgroup of G. Now, take $H = G_0$ and $K = G_2$, then $G_0G_2 = G_2^e = G_2G_0$, and hence, Theorem 2.6 is true.

Theorem 2.8. *Let X be a singular WCNETG, $\emptyset \neq S \subseteq X$ and H a NT-subgroup of X.*

(1) If $N(S) \neq \emptyset$, then $N(S)$ is a NT-subgroup of X.
(2) $N(H)$ is the largest NT-subgroup of X in which H is a x-normal NT-subgroup.
(3) If K is a NT-subgroup of $N(H)$, then $H \lhd_x HK$.

Proof.

(1) $N(S) \neq \emptyset \Leftrightarrow L(S), M(S) \neq \emptyset$. Since $N(S) = L(S) \cap M(S)$, then the fact that $N(S)$ is a NT-subgroup of X follows from Lemma 2.4
(2) Let H be a NT-subgroup of X. Then, $hH\ anti(h) = H$ for all $h \in H$. Thus, $H \subseteq N(H)$ and H is a NT-subgroup of $N(H)$. By definition, $xH\ anti(x) = H$ for all $x \in N(H)$. Hence, $H \lhd_x N(H)$. Let K be an arbitrary NT-subgroup of X such that $H \lhd_x K$. Then, $kH\ anti(k) = H$ for all $k \in K$, which implies that $K \subset N(H)$. Thus, $N(H)$ is the largest NT-subgroup of X in which H is a x-normal NT-subgroup.

(3) Let K be a NT-subgroup of $N(H)$, then for all $k \in K$, $kH\ anti(k) = H$. Hence, $kH\ anti(k)k = Hk \Rightarrow kH\ neut(k) = Hk \Rightarrow k\ neut(k)H = Hk \Rightarrow kH = Hk \Rightarrow HK = KH$. Hence, by Theorem 2.6, KH is a NT-subgroup of $N(H)$ and $H \subset KH$ (since $neut(k)\ H = H,\ k \in K \subset M(H)$). Consequently, $H \lhd_x HK$. □

Example 2.9. By Example 2.5, with $S = G_0 = \{e, 1\}$, $N(G_0) = L(G_0) \cap M(G_0) = \{1, 2, 3, 4, 5, 6\} = G_2$ and $(G_2, *)$ is a NT-subgroup of $(G, *)$.

Definition 2.10. (Normal Neutrosophic Triplet Subgroup)

Let X be a NETG and let N be a NT-subgroup of X. Let $neut(x)N = N$ for all $x \in X$, then N is said to be a normal NT-subgroup of X if $xN\ anti(x) \subset N$ and this represented by $N \lhd X$.

Lemma 2.11. *Let X be a singular WCNETG, $\emptyset \neq S \subseteq X$. If $< S >$ is generated by S in X, i.e.*

$$< S > = \left\{ \prod_{i=1}^{n} x_i = x_1 x_2 \cdots x_n \mid x_i \in S \text{ or } anti(x_i) \in S,\ 1 \leq i \leq n \right\},$$

then $< S >$ is a NT-subgroup of X which contains S.

Proof. $S \subset < S >$. So, $< S > \neq \emptyset$. If $a, b \in < S >$, then $a = \prod_{i=1}^{m} x_i$ and $b = \prod_{i=1}^{n} y_i$. So,

$$ab = \prod_{i=1}^{m} x_i \prod_{i=1}^{n} y_i \in < S > \text{ and } anti(a) = anti\left(\prod_{i=1}^{m} x_i\right) = \prod_{i=1}^{m} anti\big(x_{m-i+1}\big) \in < S >$$

Let Y be any NT-subgroup of X containing S; then for all $x \in S$, $x \in Y$. So, $anti(x) \in Y$, and Y contains all finite product $\prod_{i=1}^{n} x_i$ such that $x_i \in S$ or $anti(x_i) \in S$, $1 \leq i \leq n$. Hence, $< S > \subset Y$. □

Theorem 2.12. *Let X be a singular WCNETG and N a be NT-subgroup of X. If $neut(x)N = N$ for all $x \in X$, then the following are equivalent:*

(1) $N \lhd X$.

(2) $xN\ anti(x) = N$ for all $x \in X$.

(3) $xN = Nx$ for all $x \in X$.

(4) $xNyN = (xy)N$ for all $x, y \in X$.

Proof.

1⇒2: Let $N \lhd X$ and $x \in X$. Then, $xN\ anti(x) \subset N$. Since $anti(x) \in X$, then $anti(x)N\ anti\big(anti(x)\big) \subset N \Rightarrow anti(x)N\ x \subset N$. Now, $x(anti(x)N\ x)anti(x) = (x\ anti(x))N(x\ anti(x)) = neut(x)N\ neut(x) = N\ neut(x) = neut(x)N = N$. So, $N = x(anti(x)N\ x)anti(x) \subset xN\ anti(x) \Rightarrow N \subset xN\ anti(x)$. Hence, $xN\ anti(x) = N$.

2⇒3: $xN\ anti(x) = N \Rightarrow Nx = (xN\ anti(x))x = xN\ anti(x)x = xN\ neut(x) = xN \Rightarrow Nx = xN$.

3⇒4: $xNyN = x(Ny)N = x(yN)N = (xy)NN$. Now, $NN \subset N$ since N is a groupoid. On the other hand, $N = e(n)N \subset NN$ for some $n \in N$. Hence, $NN = N$. $\therefore\ xNyN = (xy)N$.

4⇒1: $xN\ anti(x) = xN\ neut(n)anti(x) = xN\ anti(x)neut(n) \subset xN\ anti(x)N = (x\ anti(x))N = neut(x)N = N \Rightarrow xN\ anti(x) \subset N \Rightarrow N \lhd X$.□

Remark 2.13. Note that $neut(x) \in N$ for all $x \in X \Rightarrow neut(x)N \subseteq N$ but the converse is not necessarily true. For example, in the first WNCETG of Table 3, $neut(x)G_1 = neut(x)\{e, a, b, c\} = G_1$, but $neut(x) \notin G_1$ for all $x \in G_2$.

Definition 2.14. (Closure of a set)

Let X be a NETG and $\emptyset \neq S \subseteq X$ and $Y \leq X$. The closure of S in H will be defined by $Cl_H(S) = \{x \in H | xS = S\}$. If $H = X$, then this will simply be expressed as $Cl(S)$.

Lemma 2.15. *Let X be a singular WCNETG and H a NT-subgroup of X. Then*

(1) $Cl(H) \neq \emptyset$ and $Cl(H)$ is a NT-subgroup of X.

(2) $Cl(H)$ is a NT-subgroup of $N(H)$.

Proof.

(1) $Cl(H) \neq \emptyset\ \because H \subseteq Cl(H)$. Let $x, y \in Cl(H)$, then $(xy)H = x(yH) = xH = H \Rightarrow xy \in Cl(H)$.

Let $x \in Cl(H)$, then $xH = H \Rightarrow \big(neut(x)x\big)H = H \Rightarrow neut(x)(xH) = H \Rightarrow neut(x)H = H \Rightarrow neut(x) \in Cl(H)$. Furthermore, $neut(x)H = H \Rightarrow \big(anti(x)x\big)H = H \Rightarrow anti(x)(xH) = H \Rightarrow anti(x)H = H \Rightarrow anti(x) \in Cl(H)$ and so, $Cl(H)$ is a NT-subgroup of X.

(2) Let $x \in Cl(H)$, then by (1), $neut(x)H = H$. More so, $H = neut(x)H = H\ neut(x) = Hx\ anti(x) = H\ anti(x) \Rightarrow H = H\ anti(x)$. Thence, $xH\ anti(x) = H\ anti(x) = H$. $\therefore\ Cl(H)$ is a NT-subgroup of $N(H)$.□

Example 2.16. For the singular WCNETG $(G, *)$ in Table 3, $G_0 = \{e, 1\} \leq G$, even though G_0 is not a subgroup in $(G, *)$. $Cl(G_0) = \{1\} \leq G$. Furthermore, by Example 2.9, with $H = G_0 = \{e, 1\}$, $N(G_0) = L(G_0) \cap M(G_0) = \{1, 2, 3, 4, 5, 6\} = G_2$ and $(G_2, *)$ is a NT-subgroup of $(G, *)$. So, $Cl(G_0) \leq (G_2, *)$.

Definition 2.17. Let X be a NETG.

(1) If $\emptyset \neq S \subseteq X$, the set $C_X(S) = \{x \in X | xs = sx \ \forall \ s \in S\}$ will be called the centralizer of S in X.
(2) The set $Z(X) = \{x \in X | xy = yx \ \forall \ y \in X\}$ will be called the center X.
(3) Let $Y \leq X$. Then, Y is called a complete NT-subgroup of X if $neut(g)y \in Y$ for all $g \in X$ and $y \in Y$.

Lemma 2.18. *Let X be a singular WCNETG.*

(1) For any $\emptyset \neq S \subseteq X$, $C_X(S) \neq \emptyset$ and $C_X(S)$ is a complete NT-subgroup of X for which $neut(g) \in C_X(S)$ for all $g \in X$. Furthermore, $neut(g) \in Cl\big(C_X(S)\big)$ if and only if $C_X(S) \subseteq neut(g)C_X(S)$ for all $g \in X$.
(2) $C_X(X) = Z(X) \lhd X \Leftrightarrow Z(X) \subseteq neut(g)Z(X)$ for all $g \in X$.

Proof.

(1) Consider $neut(g) \in X$, for any $g \in X$. Observe that $neut(g)s = s\ neut(g)$ for all $s \in S$ implies that $neut(g) \in C_X(S)$ for any $g \in X$. So, $C_X(S) \neq \emptyset$. Furthermore, $neut(g)C_X(S) \subseteq C_X(S)$ for any $g \in X$. So, $neut(g) \in Cl\big(C_X(S)\big) \Leftrightarrow C_X(S) \subseteq neut(g)C_X(S)$ for all $g \in X$.

Let $x, y \in C_X(S)$, then $xs = sx$ and $ys = sy$ for all $s \in S$.

$$(xy)s = x(ys) = x(sy) = (xs)y = (sx)y = s(xy) \Rightarrow xy \in C_X(S).$$

$$anti(x)s = anti(x)neut\big(anti(x)\big)s = anti(x)neut(x)s = anti(x)s\ neut(x) =$$
$$anti(x)sx\ anti(x) = anti(x)xs\ anti(x) = neut(x)s\ anti(x) = s\ neut(x)anti(x) =$$
$$s\ neut(x)anti(x) \Rightarrow anti(x) \in C_X(S).$$

So, $C_X(S)$ is a complete NT-subgroup of X.

(2) $C_X(X) = \{x \in X | xg = gx \ \forall \ g \in X\} = Z(X)$. Let $x \in Z(X)$ and $g \in X$, then $gx\ anti(g) = xg\ anti(g) = x\ neut(x) \in Z(X)$. So, $C_X(X) = Z(X) \lhd X \Leftrightarrow Z(X) \subseteq neut(g)Z(X)$ based on 1.□

Example 2.19. Consider the singular WCNETG $(G, *)$ represented by Table 3.

(1) Let $S = G_0 = \{e, 1\} \leq G$. $C_G(G_0) = G \leq G$ and so, $neut(g) \in C_G(G_0)$ for all $g \in G$. $Cl(C_G(G_0)) = Cl(G) = G_2 \leq G$. Observe that $neut(g) \in Cl(C_G(G_0))$ for some $g \in G$ and so, $neut(g) \notin Cl(C_G(G_0))$ for all $g \in G$.
(2) Furthermore, $Z(G) = \{1\} \cup G_1 \leq G$, Now, $xZ(G)\ anti(x) \subset Z(G)$ for all $x \in G$. For all $x \in G_2$, note that $neut(x)Z(G) = 1 \cdot Z(G) = Z(G)$ but for all $x \in G_1$, $neut(x)Z(G) = e \cdot Z(G) \subset Z(G)$. So, $neut(x)Z(G) \neq Z(G)$ for all $x \in G$. Hence, $Z(G) \ntriangleleft G$.

(3) Given any group G with subgroup H and normal subgroup K, G is a WCNETG with complete NT-subgroup H and normal NT-subgroup K.

Definition 2.20. Let X be a NETG.

(1) If $neut(a)b = neut(a)c$ implies that $b = c$ for all $a, b \in X$, then X is said to be neutro-left cancellative.
(2) If $b\ neut(a) = c\ neut(a)$ implies that $b = c$ for all $a, b \in X$, then X is said to be neutro-right cancellative.
(3) Let H be a NT-subgroup of X. H is said to be right self cancellative in X if $xH = H$ implies $x \in H$ for all $x \in X$. This will sometimes be represented as $H \leq_{\text{rsc}} X$.
(4) Let H be a NT-subgroup of X. H is said to be left self cancellative in X if $Hx = H$ implies $x \in H$ for all $x \in X$. This will sometimes be represented as $H \leq_{\text{lsc}} X$.
(5) Let H be a NT-subgroup of X. H is said to be a semi-strong NT-subgroup of X if $neut(x) \in H$ for all $x \in X$. This will sometimes be represented as $H \leq_{\text{ss}} X$.
(6) For $Y, Z \leq X$, Y will be said to be Z-neutro-solvable in X if for any $x \in X$ and $y \in Y$, $neut(x)y \in Z \Rightarrow y \in Z$.

Remark 2.21. In a WCNETG, neutro-left cancellation and neutro-right cancellation are equivalent. In a NETG, left self cancellation and right self cancellation are equivalent for any given normal NT-subgroup. The use of 'semi-strong' in Definition 2.20 is based on the use of 'strong' in Definition 5 of [9].

Example 2.22. Consider the singular WCNETG $(G, *)$ represented by Table 3.

(1) Based on Table 1 representing $(G_1, *_1)$, $(G_1, *)$ is a subgroup (hence, NT-subgroup)of $(G, *)$ but $G_1 \nleq_{\text{rsc}} G$ because $xG_1 = G_1 \nRightarrow x \in G_1$ for all $x \in G$. Similarly, $G_1 \nleq_{\text{lsc}} G$. On the hand, based on Table 2 representing $(G_2, *_2)$, $(G_2, *)$ is a subgroup (hence, NT-subgroup) of $(G, *)$. Whereas, $G_2 \leq_{\text{rsc}} G$ and $G_2 \leq_{\text{lsc}} G$. These difference between G_1 and G_2 shows that the notions of right self cancellation and left self cancellation NT-subgroup is peculiar in NETG and not trivial from the point of view classical group. This is because, even though, $G_0 = \{e, 1\}$ is not a subgroup of G, it is right self cancellative and left self cancellative.
(2) G_1 and G_2 are subgroups (hence, NT-subgroup) of $(G, *)$, but they are not semi-strong NT-subgroup of G because $neut(x) \notin G_1$ for all $x \in G_2$ and $neut(x) \notin G_2$ for all $x \in G_1$. Thus, the concept semi-strong NT-subgroup is peculiar in NETG and not trivial from the point of view classical group. This is because, even though $G_0 = \{e, 1\}$ is not a subgroup of G, it is a semi-strong NT-subgroup of G. In addition, despite the fact that $G_1^1 = \{1\} \cup G_1$ and $G_2^e = \{e\} \cup G_2$ are not subgroups of G, $G_1^1 \leq_{\text{ss}} G$ and $G_2^e \leq_{\text{ss}} G$.

(3) Since $G_2 \leq_{\rm rsc} G$, then it can be observed that $Cl(G_2) = G_2$.

(4) We shall now see that the notion of 'neutro solvability' in NETG is not subgroup biased as the case is in classical groups.

(a) Even though $G_0 = \{e, 1\}$ is a NT-subgroup of X and not a subgroup of X, it is both G_2^e-neutro solvable and G_1^1-neutro solvable in G.

(b) G_1 and G_2 are subgroups of G: G_2 is not G_1-neutro solvable in X, but G_1 is G_2-neutro solvable in G.

(c) G_1^1 and G_2^e are not subgroups of G: G_2^e is not G_1^1-neutro solvable in G, but G_1^1 is G_2^e-neutro solvable in G.

Lemma 2.23. *Let X be a NETG such that $Y, Z \leq X$.*

(1) $Y \leq_{rsc} X$ if and only if $Cl(Y) \subseteq Y$.

(2) $Cl(Y) \cap Cl(Z) \subseteq Cl(Y \cap Z) \Leftrightarrow xY \cap xZ = x(Y \cap Z)$ for all $x \in X$.

(3) Let X be a singular NETG. If any of the following is true:

(a) Y is Z-neutro-solvable in X and $Z \lhd X$ or $Z \leq_{ss} X$ or $neut(x) \in Cl(Z)$ for all $x \in X$;

(b) Z is Y-neutro-solvable in X and $Y \lhd X$ or $Y \leq_{ss} X$ or $neut(x) \in Cl(Y)$ for all $x \in X$;

then, $xY \cap xZ = x(Y \cap Z)$ for all $x \in X$ and $Cl(Y) \cap Cl(Z) \subseteq Cl(Y \cap Z)$.

Proof.

(1) Let $Y \leq_{\rm rsc} X$, then for any $x \in X$, $xY = Y \Rightarrow x \in Y$. Let $x \in Cl(Y)$, then $xY = Y \Rightarrow x \in Y$. So, $Cl(Y) \subseteq Y$.

Conversely, let $x \in Cl(Y)$, then $xY = Y$. Since $Cl(Y) \subseteq Y$, then, $x \in Y$. Thus, for any $x \in X$, $xY = Y \Rightarrow x \in Cl(Y) \Rightarrow x \in Y$. Thence, $Cl(Y) \subseteq Y$.

(2) If $Cl(Y) \cap Cl(Z) \subseteq Cl(Y \cap Z)$, then $x \in Cl(Y) \cap Cl(Z) \Rightarrow x \in Cl(Y \cap Z)$. So, $x \in Cl(Y) \Rightarrow xY = Y$ and $x \in Cl(Z) \Rightarrow xZ = Z$ for all $x \in X$ and $x(Y \cap Z) = Y \cap Z$ for all $x \in X$. Thus, $x(Y \cap Z) = Y \cap Z = xY \cap xZ = Y \cap Z$ for all $x \in X$.

Conversely, let $xY \cap xZ = x(Y \cap Z)$ for all $x \in X$, then $x \in Cl(Y) \cap Cl(Z) \Rightarrow Y \cap xZ = Y \cap Z$ for all $x \in X$ will give $x(Y \cap Z) = Y \cap Z$ for all $x \in X \Rightarrow x \in Cl(Y \cap Z)$. Therefore, $Cl(Y) \cap Cl(Z) \subseteq Cl(Y \cap Z)$.

(3) The proof of $x(Y \cap Z) \subseteq xY \cap xZ$ is routine while the proof of $xY \cap xZ \subseteq x(Y \cap Z)$ requires the conditions in (a) or (b). The last part follows from 2.□

Example 2.24.

(1) As mentioned in Example 2.22, $G_0 = \{1, e\} \leq_{\rm rsc} X$ and $Cl(G_0) = \{1\} \subset G_0$.

(2) $Cl(G_2^e) = G_2$ and $Cl(G_1^1) = \{1\}$, so $Cl(G_1^1) \cap Cl(G_2^e) = \{1\} = Cl(G_0) = Cl\big(G_1^1 \cap G_2^e\big)$.

(3) By Example 2.22(2)(4): $G_0 = \{e, 1\}$ is both G_2^e-neutro solvable and G_1^1-neutro solvable in G, and, $G_1^1 \leq_{ss} G$ and $G_2^e \leq_{ss} G$. So, $xY \cap xZ = x(Y \cap Z)$ for all $x \in X$ and $Cl(Y) \cap Cl(Z) \subseteq Cl(Y \cap Z)$ for the pairings: $Y = G_0$ and $Z = G_1^1$; $Y = G_0$ and $Z = G_2^e$.

2.3. *Neutrosophic Triplet Group Homomorphism*

Let X and Y be NETGs and let $\phi : X \to Y$. Then, ϕ is called a neutro-homomorphism if $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in X$. If a neutro-homomorphism is a mono (epi), then , it is called a neutro-monomorphism (neutro-epimorphism). If a neutro-homomorphism is a bijection, then , it is called a neutro-isomorphism. In such a case, X and Y are said to be neutro-isomorphic (or simply isomorphic) and this will be written as $X \cong Y$.

$\ker \phi = \{x \in X | \phi(x) = neut(y) \text{ for some } y \in Y\}$ and $\mathrm{Im}(\phi) = \{y \in Y | \phi(x) = y \text{ for some } x \in X\}$.

Theorem 2.25. *Let X be a singular WCNETG and $N \lhd X$. Then*

(1) $X/N = \{xN | x \in X\}$ is a group.

(2) The mapping $\phi : X \to X/N \uparrow x \mapsto xN$ is a neutro-epimorphism.

(3) Let $NT(X)$ and $NT(X/N)$ represent the set of all NTs of X and X/N respectively, i.e.

$$NT(X) = \big\{\big(x, neut(x), anti(x)\big) \mid x \in X\big\} \text{ and}$$

$$NT(X/N) = \big\{\big(xN, neut(xN), anti(xN)\big) \mid xN \in X/N\big\}.$$

Then, there exists a binary operation $\odot$ on $NT(X)$ and $NT(X/N)$, and a mapping $\alpha : NT(X) \to NT(X/N)$ such that

(a) $NT(X)$ is a singular WCNETG and $NT(X/N)$ is a group.

(b) α is a neutro-epimorphism if X/N is an abelian group.

(4) $\ker \phi = Cl(N)$ *and*

$$\ker \alpha = \Big(Cl(N), neut\big(Cl(N)\big), anti\big(Cl(N)\big)\Big) = \Big(\ker \phi, neut\big(\ker \phi\big), anti\big(\ker \phi\big)\Big).$$

Proof.

(1) **Closure:** By Theorem 2.12(4), $xNyN = (xy)N$ for all $x, y \in X$.

Associativity: By repeated use of Theorem 2.12(4), $(xNyN)zN = xN(yNzN)$ for all $x, y, z \in X$.

Identity: Let $neut(xN) = neut(x)N = N$. Then, $neut(xN)xN = neut(x)NxN = (neut(x)x)N = xN$ and $xN\ neut(xN) = xN\ neut(x)N = (x\ neut(x))N = xN$.

Inverse: Let $anti(xN) = anti(x)N$. Then, $anti(xN)xN = anti(x)NxN = (anti(x)x)N = neut(x)N = N$ and $xN\ anti(xN) = xN\ anti(x)N = (x\ anti(x))N = neut(x)N = N$.

$\therefore X/N$ is a group.

(2) By definition, ϕ is onto and for all $x, y \in X$, $\phi(xy) = (xy)N = xNyN = \phi(x)\phi(y)$. Thus, ϕ is a neutro-epimorphism.

(3) Define $\odot$ on $NT(X)$ as follows:

$$\big(x, neut(x), anti(x)\big) \odot \big(y, neut(y), anti(y)\big) = \big(xy, neut(y)neut(x), anti(y)anti(x)\big).$$

Closure: $\big(x, neut(x), anti(x)\big) \odot \big(y, neut(y), anti(y)\big) = \big(xy, neut(xy), anti(xy)\big) \in NT(X)$.

Neutral and Opposite: Define the neutral of $\big(x, neut(x), anti(x)\big)$ as follows:

$$neut\big(x, neut(x), anti(x)\big) = \big(neut(x), neut(x), neut(x)\big).\text{ Then}$$

$$neut\big(x, neut(x), anti(x)\big) = \big(neut(x), neut(neut(x)), anti(neut(x))\big) \in NT(X).$$

On the other hand, define the opposite of $\big(x, neut(x), anti(x)\big)$ as follows:

$$anti\big(x, neut(x), anti(x)\big) = \big(anti(x), neut(x), x\big).\text{ Then,}$$

$$anti\big(x, neut(x), anti(x)\big) = \big(anti(x), neut(anti(x)), anti(anti(x))\big) \in NT(X).\text{ Now}$$

$$LHS = \big(x, neut(x), anti(x)\big) \odot\ neut\big(x, neut(x), anti(x)\big) = \big(x, neut(x), anti(x)\big) \odot$$
$$\big(neut(x), neut(neut(x)), anti(neut(x))\big) = \big(x\ neut(x), neut(x\ neut(x)), anti(x\ neut(x))\big)$$
$$= \big(x, neut(x), anti(x)\big).\text{ Similarly,}$$
$$RHS = neut\big(x, neut(x), anti(x)\big) \odot \big(x, neut(x), anti(x)\big) = \big(x, neut(x), anti(x)\big).$$
$$LHS = \big(x, neut(x), anti(x)\big) \odot\ anti\big(x, neut(x), anti(x)\big) = \big(x, neut(x), anti(x)\big) \odot$$
$$\big(anti(x), neut(anti(x)), anti(anti(x))\big) = \big(x\ anti(x), neut(x\ anti(x)), anti(x\ anti(x))\big)$$
$$= \big(neut(x), neut(neut(x)), anti(neut(x))\big) = neut\big(x, neut(x), anti(x)\big).\text{ Similarly,}$$
$$RHS = anti\big(x, neut(x), anti(x)\big) \odot \big(x, neut(x), anti(x)\big) = neut\big(x, neut(x), anti(x)\big).$$
$$\therefore \Big(\big(x, neut(x), anti(x)\big), neut\big(x, neut(x), anti(x)\big), anti\big(x, neut(x), anti(x)\big)\Big)$$

forms a neutrosophic triplet for $\big(x, neut(x), anti(x)\big) \in NT(X)$ and so, $NT(X)$ is a neurotrophic triplet set.

Associativity: $LHS = \Big(\big(x, neut(x), anti(x)\big) \odot \big(y, neut(y), anti(y)\big)\Big) \odot$

$$\big(z, neut(z), anti(z)\big) = \big(xy, neut(xy), anti(xy)\big) \odot \big(z, neut(z), anti(z)\big) =$$
$$\big(xy \cdot z, neut(xy \cdot z), anti(xy \cdot z)\big).\text{ Similarly, } RHS = \big(x, neut(x), anti(x)\big) \odot$$
$$\Big(\big(y, neut(y), anti(y)\big) \odot \big(z, neut(z), anti(z)\big)\Big) = \big(x \cdot yz, neut(x \cdot yz), anti(x \cdot yz)\big)$$

So, $NT(X)$ is a NETG.

Weak Commutativity:

$$LHS = neut\big(x, neut(x), anti(x)\big) \odot \big(y, neut(y), anti(y)\big) =$$

$$\big(neut(x), neut(neut(x)), anti(neut(x))\big) \odot \big(y, neut(y), anti(y)\big) =$$

$$\big(neut(x)y, neut(neut(x)y), anti(neut(x)y)\big) =$$

$$\big(y\ neut(x), neut(y\ neut(x)), anti(y\ neut(x))\big) = \big(y, neut(y), anti(y)\big)\odot$$

$$neut\big(x, neut(x), anti(x)\big) = RHS.$$

Singularity: $anti\big(x, neut(x), anti(x)\big)$ is unique for each $\big(x, neut(x), anti(x)\big) \in NT(X)$.

$\therefore\ NT(X)$ is a singular WCNETG.

(4) $\ker\phi = \{x \in X|\phi(x) = neut(yN),\ yN \in X/N\} = \{x \in X|\phi(x) = neut(y)N = N,\ y \in X\} = Cl(N)$.

$$\ker\alpha = \Big\{\big(x, neut(x), anti(x)\big) \in NT(X)|\big(x, neut(x), anti(x)\big) = neut\big(x, neut(x), anti(x)\big)\Big\}$$

$$= \Big\{\big(x, neut(x), anti(x)\big) \in NT(X)|\big(xN, N, anti(xN)\big) = \big(N, N, N\big)\Big\}$$

$$= \Big\{\big(x, neut(x), anti(x)\big) \in NT(X)|xN = N \text{ and } anti(xN) = N\Big\}$$

$$= \Big\{\big(x, neut(x), anti(x)\big) \in NT(X)|x \in Cl(N) \text{ or } x \in \ker\phi\Big\}$$

$$= \Big(Cl(N), neut\big(Cl(N)\big), anti\big(Cl(N)\big)\Big) = \Big(\ker\phi, neut\big(\ker\phi\big), anti\big(\ker\phi\big)\Big). \square$$

2.4. *Isomorphism Theorems for Singular WCNETG*

We are now ready to establish the first, second and third neutro-isomorphism theorems, neutro-correspondence theorem and the neutro-Zassenhaus Lemma (Neutro-Butterfly Theorem).

Theorem 2.26. *(First Neutro-Isomorphism Theorem for Singular WCNETG)*

Let X and Y be singular WCNETGs and let $\phi : X \to Y$ be a a neutro-homomorphism.

(1) (a) $\ker\phi$ is a complete NT-subgroup of X.

(b) $\ker\phi \lhd_x X$ for all $x \in X$.

(c) $\ker\phi \lhd X \Leftrightarrow \ker\phi \subset neut(x)\ker\phi$ for all $x \in X$.

(2) $\mathrm{Im}(\phi)$ is a NT-subgroup of Y and if K is a NT-subgroup of Y, then $\emptyset \neq \phi^{-1}(K)$ is a NT-subgroup of X.

(3) If Y is neutro-left (neutro-right) cancellative and $\ker\phi \subset neut(x)\ker\phi$ for all $x \in X$, then $X/\ker\phi \cong \mathrm{Im}(\phi)$. Hence, if in addition, ϕ is a neutro-epimorphism, then $X/\ker\phi \cong Y$.

Proof. Let $\phi : X \to Y$ be a neutro-homomorphism, then $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in X$.

(1) Put $y = neut(x)$ in $\phi(xy) = \phi(x)\phi(y)$ to get $\phi\big(x\ neut(x)\big) = \phi(x)\phi(neut(x)) \Rightarrow \phi(x) = \phi(x)\phi(neut(x))$. Also, put $y = neut(x)$ in $\phi(yx) = \phi(y)\phi(x)$ to get $\phi\big(neut(x)x\big) = \phi(neut(x))\phi(x) \Rightarrow \phi(x) = \phi(neut(x))\phi(x)$. Thus, $\phi(neut(x)) = neut(\phi(x))$ for all $x \in X$. So, $\ker\phi \neq \emptyset$.

Let $a, b \in \ker\phi$, then $\phi(a) = neut(g)$ and $\phi(b) = neut(h)$ for some $g, h \in Y$. Then, $\phi(ab) = \phi(a)\phi(b) = neut(g)neut(h) = neut(gh) \Rightarrow ab \in \ker\phi$.

Put $y = anti(x)$ in $\phi(xy) = \phi(x)\phi(y)$ to get $\phi\big(x\ anti(x)\big) = \phi(x)\phi(anti(x)) \Rightarrow \phi(neut(x)) = \phi(x)\phi(anti(x)) \Rightarrow neut(\phi(x)) = \phi(x)\phi(anti(x))$. Also, put $y = anti(x)$ in $\phi(yx) = \phi(y)\phi(x)$ to get $\phi\big(anti(x)x\big) = \phi(anti(x))\phi(x) \Rightarrow \phi(neut(x)) = \phi(anti(x))\phi(x) \Rightarrow neut(\phi(x)) = \phi(anti(x))\phi(x)$. Thus, $\phi(anti(x)) = anti(\phi(x))$ for all $x \in X$.

Now, let $x \in \ker\phi$, then $\phi(x) = neut(y)$ for some $y \in Y$. Using the above result, $\phi(anti(x)) = anti(\phi(x)) = anti(neut(y)) = neut(y) \Rightarrow anti(x) \in \ker\phi$ for all $x \in X$. Thus, $\ker\phi$ is a NT-subgroup of X. Furthermore, for any $g \in X$ and $x \in \ker\phi$,

$$\phi\big(gx\ anti(g)\big) = \phi(g)\phi(x)\phi(anti(g)) = \phi(g)neut(y)anti(\phi(g)) = neut(y)\phi(g)\ anti(\phi(g))$$

$$= neut(y)neut(\phi(g)) = neut\big(y\phi(g)\big) \Rightarrow gx\ anti(g) \in \ker\phi.$$

Also, for any $g \in X$, $\phi(neut(g)) = neut\big(\phi(g)\big) \Rightarrow neut(g) \in \ker\phi$. Thus, $\ker\phi$ is a complete NT-subgroup of X, $\ker\phi \lhd_x X$ for all $x \in X$ and therefore, $\ker\phi \lhd X \Leftrightarrow \ker\phi \subset neut(x)\ker\phi$ for all $g \in X$.

(2) For any $g \in X$, $\phi(neut(g)) = neut\big(\phi(g)\big) \in \mathrm{Im}(\phi)$. So, $\mathrm{Im}(\phi) \neq \emptyset$. Let $x', y' \in \mathrm{Im}(\phi)$, then $x' = \phi(x)$ and $y' = \phi(y)$. Thus, $x'\ anti(y') = \phi(x)anti(\phi(y)) = \phi(x)\phi(anti(y)) = \phi(x\ anti(y)) \in \mathrm{Im}(\phi)$. So, $\mathrm{Im}(\phi)$ is a NT-subgroup of Y.

If K is a NT-subgroup of Y, then $\emptyset \neq \phi^{-1}(K) = \{x \in X : \phi(x) \in K\}$.

Let $x, y \in \phi^{-1}(K)$, then there exist $x', y' \in K$ such that $x' = \phi(x)$ and $y' = \phi(y)$. Thus, $x'\ anti(y') = \phi(x)anti(\phi(y)) = \phi(x)\phi(anti(y)) = \phi(x\ anti(y)) \in K \Rightarrow x\ anti(y) \in \phi^{-1}(K)$. So, $\phi^{-1}(K)$ is a NT-subgroup of X.

(3) Let $\psi : X/\ker\phi \to \mathrm{Im}(\phi)\ \uparrow\ \psi\big(x\ker\phi\big) = \phi(x)$ for each $x \in X$.

Well Defined: For any $x, y \in X$,

$$x\ker\phi = y\ker\phi \Rightarrow anti\big(y\ker\phi\big)x\ker\phi = anti\big(y\ker\phi\big)y\ker\phi \Rightarrow$$

$$\big(anti(y)x\big)\ker\phi = \ker\phi \Rightarrow anti(y)xr = s,\ r, s \in \ker\phi \Rightarrow \phi\big(anti(y)xr\big) = \phi(s) \Rightarrow$$

$$\phi\big(anti(y)x\big)\phi(r) = \phi(s) \Rightarrow \phi\big(anti(y)x\big)neut(r') = neut(s'),\ r', s' \in Y \Rightarrow$$

$$\phi\big(anti(y)x\big)neut(r')anti\big(neut(r')\big) = neut(s')anti\big(neut(r')\big) \Rightarrow$$

$$\phi\big(anti(y)x\big)neut(r') = neut(s')neut(r') \Rightarrow \phi\big(anti(y)x\big)neut(r') = neut(s'r') \Rightarrow$$

$$anti\big(\phi(y)\big)\phi(x)neut(r') = neut(s'r') \Rightarrow \phi(y)anti\big(\phi(y)\big)\phi(x)neut(r') =$$

$$\phi(y)neut(s'r') \Rightarrow neut\big(\phi(y)\big)\phi(x)neut(r') = \phi(y)neut(s'r') \Rightarrow$$

$$\phi(x)neut\big(\phi(y)\big)neut(r') = \phi(y)neut(s'r') \Rightarrow \phi(x)neut\big(\phi(y)r'\big) = \phi(y)neut(s'r')$$

$$\Rightarrow \phi(x) = \phi(y) \Rightarrow \psi\big(x\ker\phi\big) = \psi\big(y\ker\phi\big).$$

without out loss of generality, we take $neut\big(\phi(y)r'\big) = neut(s'r')$ and because H is neutro-left(or neutro-right) cancellative.

One to one:

$$\psi\big(x\ker\phi\big) = \psi\big(y\ker\phi\big) \Rightarrow \phi(x) = \phi(y) \Rightarrow \phi(x)anti\big(\phi(y)\big) = \phi(y)anti\big(\phi(y)\big) \Rightarrow$$

$$\phi\big(x\ anti(y)\big) = neut\big(\phi(y)\big) \Rightarrow x\ anti(y) \in \ker\phi \Rightarrow x\ anti(y)y \in y\ker\phi \Rightarrow$$

$$x\ neut(y) \in y\ker\phi \Rightarrow x\ neut(y)\ker\phi = x\ker\phi \subseteq y\ker\phi\ker\phi = y\ker\phi \Rightarrow$$

$$x\ker\phi \subseteq y\ker\phi$$

Similarly, it can be shown that $y\ker\phi \subseteq x\ker\phi$. Thus, $x\ker\phi = y\ker\phi$.

Onto: This is obvious.

neutro-homomorphism:

$$\psi\big(x\ker\phi\cdot y\ker\phi\big) = \psi\big((xy)\ker\phi\big) = \phi(xy) = \phi(x)\phi(y) = \psi\big(x\ker\phi\big)\psi\big(y\ker\phi\big)$$

$\therefore\ X/\ker\phi \cong \mathrm{Im}(\phi)$ and if ϕ is a neutro-epimorphism, then $X/\ker\phi \cong Y$.□

Example 2.27. In Lemma 2.2, consider the WCNETGs $(G,*)$ and $(G,\circ)$ of the pair of groups $(G_1,*_1,e_1)$ and $(G_2,*_2,e_2)$, and pair of groups $(G_1,\circ_1)$ and $(G_2,\circ_2)$ respectively. Let $h_i : (G_i,*_i) \to (G_i,\circ_i),\ i = 1,2$ be homomorphisms, then $h : (G,*) \to (G,\circ)$ is a neutro-homomorphism.

(1) Recall that $\ker h = \ker h_1 \cup \ker h_2$. So, $\ker h \leq (G,*)$ since $\ker h_1$ and $\ker h_2$ are subgroups of $(G_1,*_1)$ and $(G_2,*_2)$ respectively. We need the facts that $\ker h_i = \{g \in G_i | h_i(g) = e_i\}$ for $i = 1,2$ and $\{e_1,e_2\} \leq \ker h$. Let $Y = \ker h$, then for all $g \in G$ and any $y \in Y$:

$$h\big(neut(g)y\big) = \begin{cases} e_i \in \ker h, & \text{if } g \in G_i,\ y \in \ker h_i,\ i = 1,2; \\ e_i \in \ker h \text{ or } e_j \in \ker h, & \text{if } g \in G_i,\ y \in \ker h_j,\ i,j \in \{1,2\},\ i \neq j \end{cases}$$

Then, $neut(g)y \in \ker h$ for all $g \in G$ and any $y \in Y$. Whence, $\ker h$ is a complete NT-subgroup of $(G,*)$.

(2) $\ker h \lhd G \Leftrightarrow \ker h \subset neut(g)\ker h\ \forall\ g \in G$ if and only if $\ker h \subset neut(g)\ker h\ \forall\ g \in G_1$ and $\ker h \subset neut(g)\ker h\ \forall\ g \in G_2$ if and only if $\ker h \subset e_1 * \ker h\ \forall\ g \in G_1$ and $\ker h \subset e_2 * \ker h$.

(3) Recall that $\mathrm{Im}(h) = \mathrm{Im}(h_1) \cup \mathrm{Im}(h_2)$. So, $\mathrm{Im}(h) \leq (G,\circ)$ since $\mathrm{Im}(h_1)$ and $\mathrm{Im}(h_2)$ are subgroups of $(G_1,\circ_1)$ and $(G_2,\circ_2)$ respectively.

Theorem 2.28. *(Second Neutro-Isomorphism Theorem for Singular WCNETG)*

Let X be a singular WCNETG with NT-subgroups H and K such that K is right self cancellative in H, $hK = Kh$ and $neut(h) \in Cl(H), Cl(K)$ for all $h \in H$, and $neut(k) \in Cl(K)$ for all $k \in K$. Then,

(1) $K \lhd HK \leq X$.
(2) $H \cap K, K \lhd H$.
(3) $H/H \cap K \cong HK/K$.

Proof.

(1) $hK = Kh$ for all $h \in H$ implies that $HK = KH$. So, by Theorem 2.6, HK is a NT-subgroup of X. Let $hk \in HK$, $h \in H$ and $k \in K$. Then, for any $k_1 \in K$, $(hk)k_1\ anti(hk) = h(kk_1\ anti(k))anti(h) = hk_2\ anti(h) = h\ anti(h)k_3 = neut(h)k_3 \in K$ since $neut(h) \in Cl(K)$ for all $h \in H$. So, $(hk)k_1\ anti(hk) \in K$. Also, $neut(hk)K = neut(h)neut(k)K = neut(h)K = K$. Thus, $K \lhd HK \leq X$.

(2) Let $x \in H \cap K$, then $x \in H$ and $x \in K$. So, for all $h \in H$: $hx\ anti(h) = yh\ anti(h) = y\ neut(h) = neut(h)y \in K$ and $hx\ anti(h) \in H$. Furthermore, $neut(h)(H \cap K) = neut(h)H \cap neut(h)K = H \cap K$ since $neut(h) \in Cl(H)$ for all $h \in H$. Consequently, $H \cap K \lhd H$.

For all $k \in K, h \in H$, $hk\ anti(h) = k'h\ anti(h) = k'\ neut(h) = neut(h)k' \in K$ and $neut(h)K = K$. Thence, $K \lhd H$.

(3) Let $\phi : H \to HK/K \uparrow \phi(h) = (hk)K$ for all $h \in H$ and $k \in K$. K is rsc in H implies that $k \in Cl(K)$, and so, $\phi(h) = hK$ for all $h \in H$. So, ϕ is obviously well defined. By Theorem 2.12,

$$\phi(h_1h_2) = (h_1h_2)K = h_1Kh_2K = \phi(h_1)\phi(h_2)\ \forall\ h_1, h_2 \in H.$$

Also, ϕ is onto. Thus, ϕ is a neutro-epimorphism. HK/K is neutro-right (neutro-left) cancellative by Theorem 2.25(1).

$$\ker\phi = \{h \in H | \phi(h) = neut(xK) \text{ for some } xK \in HK/K\} = \{h \in H | hK = K\} =$$
$$\{h \in H | h \in K\} = H \cap K.$$

Therefore, by Theorem 2.26(3), $H/H \cap K \cong HK/K$. □

Remark 2.29. Theorem 2.28 can be visualized as diamond lattice structure and termed the Diamond Neutro-Isomorphism Theorem for singular WCNETG.

Theorem 2.30. *(Third Neutro-Isomorphism Theorem for Singular WCNETG)*

Let X be a singular WCNETG and let $H, K \lhd X$ be right self cancellative in X such that $K \subset H$. Then, $(X/K)/(H/K) \cong X/H$.

Proof. Consider the map $\phi : X/K \to X/H \ \uparrow \ \phi(xK) = xH$ for all $x \in X$. ϕ is well defined since K is right self cancellative:

$$xK = yH \Rightarrow anti\big(xK\big)xK = anti\big(xK\big)yK \Rightarrow \big(anti(x)y\big)K = K \Rightarrow \big(anti(x)y\big) \in K \Rightarrow$$

$$\big(anti(x)y\big) \in H \Rightarrow x\big(anti(x)y\big) \in xH \Rightarrow \big(neut(x)y\big) \in xH \Rightarrow \big(neut(x)y\big)H \subseteq xHH \Rightarrow$$

$$\big(y\ neut(x)\big)H \subseteq xH \Rightarrow yH \subseteq xH.$$

Similarly, it can be shown that $xH \subseteq yH$. So, $xH = yH \Rightarrow \phi(xK) = \phi(yK)$. By Theorem 2.12,

$$\phi\big(xKyK\big) = \phi\big((xy)K\big) = (xy)H = xHyH = \phi(xK)\phi(yK)$$

and ϕ is surjective. Hence, ϕ is a neutro-homomorphism. Since H is right self cancellative, then

$$\ker\phi = \{xK \in X/K | \phi(xK) = neut(xH) \text{ for some } xH \in X/H\} = \{xK \in X/K | xH = H\} =$$

$$\{xK \in X/K | xH = H\} = \{xK \in X/K | x \in H\} = H/K.$$

For any $x \in H$ and based on the fact that K is rsc in X implies that $k \in Cl(K)$,

$$neut(xK)H/K = K \cdot H/K = K\{hK | h \in H\} = \{k(hK) | h \in H\} = \{k(Kh) | h \in H\} =$$

$$\{(kK)h | h \in H\} = \{Kh | h \in H\} = \{hK | h \in H\} = H/K.$$

Therefore, by Theorem 2.26(3), $(X/K)/(H/K) \cong X/H$. □

Remark 2.31. Theorem 2.30 is termed the double quotient Neutro-Isomorphism Theorem for singular WCNETG.

Lemma 2.32. *Let X_1 and X_2 be singular WCNETGs and let $N_1 \lhd X_1$, $N_2 \lhd X_2$ such that N_1 and N_2 are right self cancellative in X_1 and X_2 respectively. Then, $\big(X_1 \times X_2\big)/\big(N_1 \times N_2\big) \cong \big(X_1/N_1\big) \times \big(X_2/N_2\big)$.*

Proof. $X_1 \times X_2$ is a singular WCNETG since X_1 and X_2 are singular WCNETGs. Since $N_1 \lhd X_1$, $N_2 \lhd X_2$, then $N_1 \times N_2 \lhd X_1 \times X_2$. By Theorem 2.25, X_1/N_1 and X_2/N_2 are neutro-right (neutro-left) cancellative singular WCNETGs. Thus, $\big(X_1/N_1\big) \times \big(X_2/N_2\big)$ is a neutro-right (neutro-left) cancellative singular WCNETG.

Let $\phi : X_1 \times X_2 \to \big(X_1/N_1\big) \times \big(X_2/N_2\big)$. Based on Theorem 2.12, ϕ is a neutro-epimorphism and $\ker\phi = N_1 \times N_2$ using the hypothesis that N_1 and N_2 are right self cancellative in X_1 and X_2 respectively. For any $(x_1, x_2) \in X_1 \times X_2$,

$$neut((x_1, x_2))N_1 \times N_2 = \big(neut(x_1), neut(x_2)\big)N_1 \times N_2 = neut(x_1)N_1 \times neut(x_2)N_2 = N_1 \times N_2.$$

Therefore, by Theorem 2.26(3), $\big(X_1 \times X_2\big)/\big(N_1 \times N_2\big) \cong \big(X_1/N_1\big) \times \big(X_2/N_2\big)$. □

Corollary 2.33. *Let $\{X_i\}_{i=1}^n$ be a family of singular WCNETGs and let $N_i \lhd X_i$ be right self cancellative in X_i, $1 \leq i \leq n$. Then, $\prod_{i=1}^n X_i/\prod^n i = 1N_i \cong \prod^n i = 1X_i/N_i$.*

Proof. This is the generalization of Lemma 2.32. □

Theorem 2.34. *(Neutro-Correspondence Theorem for Singular WCNETGs)*

Let X and Y be singular WCNETGs and let $\phi : X \to Y$ be a neutro-epimorphism.

(1) $G \leq X$ implies $\phi(G) \leq Y$.

(2) $H \leq Y$ implies $\phi^{-1}(H) \leq X$.

(3) $G \lhd X$ implies $\phi(G) \lhd Y$.

(4) $H \lhd Y$ implies $\phi^{-1}(H) \lhd X$.

(5) $G \leq_{rsc} X$ and $\ker\phi \subset G$ implies $\phi^{-1}\Big(\phi(G)\Big) = G$.

(6) There is a 1-1 correspondence between the set of right self cancellative NT-subgroups of X that contain $\ker\phi$, and the NT-subgroups of Y.

(7) Normal NT-subgroups of X correspond to normal NT-subgroups of Y.

Proof.

(1) Let $G \leq X$. Then, for all $a, b \in G$, $\phi(a)\phi(b) = \phi(ab) \in \phi(G)$ and $anti\big(\phi(a)\big) = \phi(anti(a)) \in \phi(G)$. Thus, $\phi(G) \leq Y$.

(2) Let $H \leq Y$. Then, for all $a, b \in G$, $\phi(ab) = \phi(a)\phi(b) \in H \Rightarrow ab \in \phi^{-1}(H)$ and $\phi(anti(a)) = anti\big(\phi(a)\big) \in H \Rightarrow anti(a) \in \phi^{-1}(H)$. So, $\phi^{-1}(H) \leq X$.

(3) Let $G \lhd X$, then $neut(x)G = G$ for all $x \in X$. Thus, $\phi\big(neut(x)\big)\phi(G) = neut\big(\phi(x)\big)\phi(G) = \phi(G) \Rightarrow neut(y)\phi(G) = \phi(G)$ for all $y \in Y$, where $y = \phi(x)$.

For each $y \in Y$ there exists $x \in X$ such that $y = \phi(x)$. Let $\phi(g) \in \phi(G)$. Then, $y\phi(g)\ anti(y) = \phi(x)\phi(g)anti\big(\phi(x)\big) = \phi\big(xg\ anti(x)\big) \in \phi(G)$ since $xg\ anti(x) \in G$. From these two arguments, $\phi(G) \lhd Y$.

(4) Let $H \lhd Y$. Then, $neut(y)H = H$ for all $y \in Y$. For each $y \in Y$, there exists $x \in X$ such that $y = \phi(x)$. So,

$$neut\big(\phi(x)\big)H = H \Rightarrow \phi\big(neut(x)\big)\phi\Big(\phi^{-1}(H)\Big) = H \Rightarrow \phi\Big(neut(x)\phi^{-1}(H)\Big) = H \Rightarrow$$

$$neut(x)\phi^{-1}(H) = \phi^{-1}(H).$$

Let $g \in \phi^{-1}(H) \Rightarrow \phi(g) \in H$. Let $x \in X$, then $\phi\big(xg\ anti(x)\big) = \phi(x)\phi(g)anti\big(\phi(x)\big) \in H$ since $H \lhd Y$. Thus, $\phi\big(xg\ anti(g)\big) \in H \Rightarrow xg\ anti(x) \in \phi^{-1}(H)$. Whence, $\phi^{-1}(H) \lhd X$.

(5) Trivially, $G \subset \phi^{-1}\Big(\phi(G)\Big)$. Let $G \leq_{\text{rsc}} X$ and $\ker\phi \subset G$.

If $x \in \phi^{-1}\Big(\phi(G)\Big)$, then $\phi(x) \in \phi(G) \Rightarrow \phi(x) = \phi(g)$ for some $g \in G$. So,

$$\phi(x)anti\big(\phi(g)\big) = \phi(g)anti\big(\phi(g)\big) = neut\big(\phi(g)\big) \Rightarrow \phi\big(x\ anti(g)\big) = neut\big(\phi(g)\big) \Rightarrow$$

$$x\ anti(g) \in \ker\phi \Rightarrow x\ anti(g) \in G \Rightarrow x\ anti(g)g \in Gg \subset G \Rightarrow x\ neut(g) \in G \Rightarrow x \in G.$$

Hence, $\phi^{-1}\Big(\phi(G)\Big) \subset G$ and therefore, $\phi^{-1}\Big(\phi(G)\Big) = G$.

(6) Let $\psi : V = \{G \leq X : \ker\phi \subset G \leq_{\text{rsc}} X\} \to W = \{H \leq Y\}$ be define as $\psi(G) = \phi(G)$. Let $H \in W \Rightarrow H \leq Y$, so that $\psi(G) = H \Rightarrow G = \phi^{-1}(H) \in V$ i.e. $\ker\phi \subset \phi^{-1}(H) \leq G$. Going by (5), $\phi\Big(\phi^{-1}(H)\Big) = H$. So, ψ is surjective.

$\psi(G_1) = \psi(G_2) \Rightarrow \phi(G_1) = \phi(G_2) \Rightarrow \phi^{-1}\big(\phi(G_1)\big) = \phi^{-1}\big(\phi(G_2)\big) \Rightarrow G_1 = G_2$. So, ψ is a bijection. Therefore, there is a 1-1 correspondence between the set of right self cancellative NT-subgroups of X containing $\ker\phi$, and the NT-subgroups of Y.

(7) This follows from (3).□

Corollary 2.35. *Let X be a singular WCNETG and let $N \lhd X$. Given any $Y \leq X/N$, there exists a unique $G \leq_{rsc} X$ such that $Y = G/N$. Furthermore, $G \lhd X$ if and only if $G/N \lhd X/N$.*

Proof. By Theorem 2.25, $\phi : X \to X/N$ defined by $\phi(x) = xN$ is a neutro canonical homomorphism. By Theorem 2.34(5),(6), there is a unique $G \leq_{\text{rsc}} X$ containing

$$\ker\phi = \{x \in X | \phi(x) = neut(xN)\} = \{x \in X | xN = neut(xN)\} =$$

$$\{x \in X | xN = N\} = \{x \in X | x \in N\} = N$$

such that $Y = \phi(G) = G/N$.

Furthermore, by Theorem 2.34(3), $G \lhd X \Rightarrow \phi(G) \lhd X/N \Rightarrow G/N \lhd X/N$. Conversely, by Theorem 2.34(4), $G/N \lhd X/N \Rightarrow \phi^{-1}(G/N) = G \lhd X$. □

Definition 2.36. Let X be a NETG.

(1) The neutral of X i.e. $NEUT(X) = X^{\text{neut}}$ will be called the set of the neutrals of elements in X: $NEUT(X) = X^{\text{neut}} = \{neut(x) : x \in X\}$.

(2) The neutral set, relative to $x \in X$ i.e. $NEUT(x) = X_x^{\text{neut}}$ will be the set of the neutral of $x \in X$: $NEUT(x) = X_x^{\text{neut}} = \{neut(x)\}$. Note that $|NEUT(x)| = 1$ for al $x \in X$.

(3) A normal NT-subgroup N of X will be called a maximal normal NT-subgroup if

(a) $N \neq X$

(b) $Y \leq_{\text{rsc}} X$ and $Y \supset N \Rightarrow Y = N$ or $Y = X$.

(4) A singular NETG X will be said to be neutro-simple if X has no proper normal NT-subgroup; i.e. X has no normal NT-subgroup except $NEUT(x)$ for any $x \in X$ or $NEUT(X)$ and X.

Lemma 2.37.

(1) Let X be a singular NETG, then $NEUT(x) \leq X$ for each $x \in X$.

(2) Let X be a singular WCNETG.

(a) $NEUT(x) \leq NEUT(X)$ for each $x \in X$.

(b) $NEUT(X)$ is commutative and $NEUT(X) \leq_{ss} X$.

(c) $NEUT(X) \lhd X$ and $NEUT(X)$ is a NT-subgroup of any semi-strong NT-subgroup of X.

(d) $NEUT(X)$ is the smallest semi-strong NT-subgroup of X i.e. $NEUT(X) = \bigcap_{H \leq_{ss} X} H$.

Proof. This is easy. □

Corollary 2.38. *Let X be a singular WCNETG and let $N \lhd X$. N is a maximal normal NT-subgroup of X if and only if X/N is neutro-simple.*

Proof. Let X be a singular WCNETG and let $N \lhd X$. If N is a maximal normal NT-subgroup of X, then $N \neq X$ and, $Y \leq_{\text{rsc}} X$ and $Y \supset N \Rightarrow Y = N$ or $Y = X$. Thus, by Corollary 2.35, $Y \lhd X \Rightarrow Y/N \lhd X/N \Rightarrow N/N \lhd X/N$ or $X/N \lhd X/N \Rightarrow \{N\} \lhd X/N$ or $X/N \lhd X/N \Rightarrow \{neut(xN)|x \in X\}$ or $X/N \lhd X/N \Rightarrow X/N$ is neutro-simple.

Conversely, if X/N is neutro-simple, then X/N has no normal NT-subgroup other than $\{N\}$ and X/N. Thus, going by Corollary 2.35, if $Y \lhd_{\text{rsc,ss}} X$ and $Y \supset N$ such that $Y/N \lhd X/N$, then $Y \lhd X$. Now, $Y/N \lhd X/N$ implies that $Y/N = \{N\} = N/N$ or $Y/N = X/N \Rightarrow Y = N$ or $Y = X$. So, N is a maximal normal NT-subgroup. □

Corollary 2.39. *Let X be a singular WCNETG and let Y, Z be maximal normal NT-subgroups of X such that $Y, Z \leq_{rsc,ss} X$. Then*

(1) $YZ \lhd_{rsc,ss} X$.

(2) $Y \cap Z$ is a maximal normal NT-subgroup of Y and of Z.

Proof.

(1) By Theorem 2.12, $yZ = Zy$ for all $y \in Y$ implies that $YZ = ZY$. Thus, by Theorem 2.6, $YZ \leq X$. Now, since $Y, Z \lhd X$, then, for all $x \in X,\ y \in Y,\ z \in Z$,

$$x(yz)\ anti(x) = x\ neut(x)yz\ anti(x) = xy\ neut(x)z\ anti(x) =$$

$$\big(xy\ anti(x)\big)\big(xz\ anti(x)\big) \in YZ \text{ and } neut(x)YZ = YZ.$$

$$\therefore\ Y, Z \lhd YZ \lhd X \tag{1}$$

Now, for any $x \in X$, $neut(x) = neut(x)neut(x) \in YZ \Rightarrow YZ \lhd_{\text{ss}} X$. For all $x, y \in X$, we already know that $xY = Y$ is equivalent to $xY = YY$ and $yZ = Z$ is equivalent to $yZ = ZZ$. So, $xYZ = YZ \Rightarrow xY = Y \Rightarrow x \in Y \subset YZ \Rightarrow x \in YZ$. So, $YZ \lhd_{\text{rsc}} X$. Therefore, $YZ \lhd_{\text{rsc,ss}} X$.

(2) Since Z is a maximal normal NT-subgroup of X, then $YZ = Z$ or $YZ = X$. But, $YZ = Z \Rightarrow Y \subset Z$, a contradiction to the fact that Y is a maximal normal NT-subgroup of X. Hence, $YZ = X$. Similarly, since Z is a maximal normal NT-subgroup of X, this also leads us to $YZ = X$.

From Theorem 2.28, $Y/Y \cap Z \cong YZ/Z$. So, $Y/Y \cap Z \cong X/Z$ and $Z/Y \cap Z \cong X/Z$. Hence, by Corollary 2.38, since Y and Z are maximal normal NT-subgroups of X, then, X/Z and X/Y are neutro-simple, whence, $Y/Y \cap Z$ and $Z/Y \cap Z$ are neutro-simple. Thus, $Y \cap Z$ is a maximal normal NT-subgroup of Y and Z. □

Definition 2.40. Let X be a singular NETG.

(1) A neutro-isomorphism $\alpha : X \to X$ will be called a neutro-automorphsim of X and the set of such mappings will be denoted by $Aut(X)$.

(2) For any fixed $g \in X$, the mapping $\alpha : X \to X$ defined by $I_g(x) = gx\ anti(g)$ for all $x \in X$ will be called a neutro-inner mapping of X at $g \in X$ and the set of such mappings will be denoted by $Inn(X)$.

Theorem 2.41.

(1) Let X be a singular NETG. Then, $Aut(X)$ is a group

(2) Let X be a singular WCNETG that is neutro-right (neutro-left) cancellative.

  (a) $Inn(X) \lhd_{rsc,ss} Aut(X)$.

  (b) $Inn(X)$ is a subgroup of $Aut(X)$ if and only if X is a group.

  (c) If $Z(X) \subset neut(x)Z(X)$ for all $x \in X$, then $X/Z(X) \cong Inn(X)$.

Proof.

(1) This is routine.

(2) (a) For any fixed $g \in X$ and for all $x, y \in X$, the following shows that I_g is an neutro-homomorphism.

$$I_g(xy) = g(xy)\ anti(g) = g\ neut(g)xy\ anti(g) = gx\ neut(g)y\ anti(g) =$$

$$gx\ anti(g)gy\ anti(g) = I_g(x)I_g(y).$$

I_g is 1-1 based on the following arguments.

$$I_g(x) = I_g(y) \Rightarrow gx\ anti(g) = gy\ anti(g) \Rightarrow gx\ anti(g)g = gy\ anti(g)g \Rightarrow$$

$$gx\ neut(g) = gy\ neut(g) \Rightarrow g\ neut(g)x = g\ neut(g)y \Rightarrow gx = gy \Rightarrow anti(g)gx =$$

$$anti(g)gy \Rightarrow neut(g)x = neut(g)y \Rightarrow x = y.$$

Using a similar argument, it can be shown that I_g is onto. So, $Inn(X) \subseteq Aut(X)$. For any fixed $g_1, g_2 \in X$ and for all $x \in X$, the following shows that $Inn(X)$ is a groupoid.

$$I_{g_1}I_{g_2}(x) = I_{g_1}\big(g_2x\ anti(g_2)\big) = g_1g_2x\ anti(g_2)anti(g_1) = g_1g_2x\ anti\big(g_1g_2\big) =$$

$$I_{g_1g_2}(x) \Rightarrow I_{g_1}I_{g_2} = I_{g_1g_2} \in Inn(X).$$

So, $neut(I_g) = I_{neut(g)} \in Inn(X)$ for each $g \in X$. Thus, $Inn(X) \neq \emptyset$. Now,

$$I_gI_{anti(g)}(x) = g\ anti(g)x\ anti\big(anti(g)\big)anti(g) = neut(g)x\ anti\big(neut(g)\big) =$$

$$I_{neut(g)}(x) \Rightarrow I_gI_{anti(g)} = I_{neut(g)}.$$

Similarly, $I_{anti(g)}I_g = I_{neut(g)}$ and so, $anti\big(I_g\big) = I_{anti(g)} \in Inn(X)$. Hence, $Inn(X) \leq Aut(x)$.

Let $\sigma \in Aut(X)$ and let $I_g \in Inn(X)$. Then,

$$\sigma I_g\sigma(x) = \sigma\big(g\sigma^{-1}(x)anti(g)\big) = \sigma(g)x\ anti\big(\sigma(g)\big) =$$

$$I_{\sigma(g)}(x) \Rightarrow \sigma I_g\sigma = I_{\sigma(g)} \in Inn(X) \text{ and } II_g(x) = I_g(x) \Rightarrow II_g(x) = I_g \in Inn(X).$$

So, $Inn(X) \lhd_{\text{rsc,ss}} Aut(X)$.

(b) $Inn(X)$ is a subgroup of $Aut(X)$ if and only if $I_{neut(g)} = I$. Now, $I_{neut(g)} = I \Rightarrow I_{neut(g)}(x) = I(x)\ \forall\ x \in X \Rightarrow neut(g)x\ anti(neut(g)) = x \Rightarrow neut(g)x = x$ and $x\ neut(g) = x \Rightarrow neut(g) = neut(x)\ \forall\ x, g \in X \Rightarrow X$ is a group. Conversely, if X is a group, then $neut(g)x\ anti(neut(g)) = x \Rightarrow I_{neut(g)} = I$. So, $Inn(X)$ is a subgroup of $Aut(X)$ if and only if X is a group.

(c) Let $\phi : X \to Aut(X)$ with $\phi(x) = I_x$. For any $x_1, x_2, x \in X$, ϕ is a neutro-homomorphism because

$$\phi(x_1x_2)(x) = I_{x_1x_2}(x) = x_1x_2x\ anti(x_1x_2) = x_1x_2x\ anti(x_2)anti(x_1) =$$

$$x_1I_{x_2}(x)anti(x_1) = I_{x_1}I_{x_2}(x) \Rightarrow \phi(x_1x_2) = I_{x_1}I_{x_2}.$$

$$\ker\phi = \{g \in X|\phi(g) = I\} = \{g \in X|\phi(g)(x) = x\ forall\ x \in X\} =$$

$$\{g \in X|\phi(g)(x) = x\ forall\ x \in X\} = \{g \in X|gx\ anti(g) = x\ forall\ x \in X\} =$$

$$\{g \in X|gx\ neut(g) = xg\ forall\ x \in X\} = \{g \in X|gx = xg\ forall\ x \in X\} = Z(X).$$

Going by Theorem 2.26(3), $X/Z(X) \cong Inn(X)$. □

Theorem 2.42. *(Neutro-Zassenhaus' Lemma for Singular WCNETG)*

Let X be a singular WCNETG such that

$B, C \leq_{rsc} X,\ B_0 \lhd B,\ C_0 \lhd C,\ B_0(B\cap C_0), C_0(C\cap B_0) \leq_{rsc} B\cap C$ *and* $B\cap C_0, C\cap B_0 \leq Cl(B\cap C)$.

If $neut(x) \in Cl(B), Cl(C)$ for all $x \in B\cap C$, then

$$\frac{B_0(B\cap C)}{B_0(B\cap C_0)} \cong \frac{C_0(C\cap B)}{C_0(C\cap B_0)}.$$

Proof. Let $K = B\cap C$ and $H = B_0(B\cap C_0)$. Since $B_0 \lhd B$, $bB_0 = B_0 b$ for all $b \in B$. So, since $K \subseteq B$, then $kB_0 = B_0 k$ for all $k \in K$. Also, $C_0 \lhd C \Rightarrow B\cap C_0 \lhd B\cap C = K$ since $neut(b)(B\cap C_0) = neut(b)B \cap neut(b)C_0 = B\cap C_0$ for all $b \in B\cap C = K$. Hence, $k(B\cap C_0) \subseteq (B\cap C_0)k$ for all $k \in K$. Thus,

$$Hk = B_0(B\cap C_0)k = B_0 k(B\cap C_0) = kB_0(B\cap C_0) = kH \Rightarrow Hk = kH\ \forall\ k \in K.$$

Let us now find HK and $H\cap K$. Thus, $HK = KH$ based on the following argument.

Since $B\cap C_0 \leq Cl(B\cap C)$, then $(B\cap C_0)(B\cap C) = B\cap C$, and so, $HK = B_0(B\cap C_0)(B\cap C) = B_0(B\cap C)$.

Let $y \in H\cap K \Rightarrow y \in H$ and $y \in K$. Now, $y \in H = B_0(B\cap C_0) \Rightarrow y = b_0 b,\ b_0 \in B_0,\ b \in B\cap C_0$. Let $b_0 b = d \in B\cap C = K$. Then, $d \in C$. Since $B\cap C_0 \subseteq C$, then $b \in C$. Now, $b_0 b = d \Rightarrow b_0\ neut(b) = d\ anti(b) \in C \Rightarrow b_0 \in C$ since $b \in B\cap C_0 \Rightarrow b \in B \Rightarrow neut(b) \in B$ and C is right self cancellative. Hence, $b_0 \in B_0\cap C \Rightarrow b_0 b \in (B_0\cap C)(B\cap C_0) \Rightarrow H\cap K \subseteq (B_0\cap C)(B\cap C_0)$.

On the other hand, $B_0\cap C \subset K,\ B\cap C_0 \subset K \Rightarrow (B_0\cap C)(B\cap C_0) \subset K$. Since $B_0\cap C \subseteq B_0$, then $(B_0\cap C)(B\cap C_0) \subset H\cap K$. Thus, $H\cap K = (B_0\cap C)(B\cap C_0)$.

Going by Theorem 2.28, if X is a singular WCNETG, with $H, K \leq X,\ H \leq_{\rm rsc} K$ and $Hk = kH,\ neut(k) \in Cl(K), Cl(H)$ for all $k \in K$, and $neut(h) \in Cl(H)$ for all $h \in H$, then

$$HK/H \cong K/H\cap K \qquad (2)$$

Substituting H, K, HK and $H\cap K$ in (2) we get

$$\frac{B_0(B\cap C)}{B_0(B\cap C_0)} \cong \frac{B\cap C}{(B_0\cap C)(B\cap C_0)} \qquad (3)$$

On interchanging the roles of B and C in (3), we get

$$\frac{C_0(C\cap B)}{C_0(C\cap B_0)} \cong \frac{C\cap B}{(C_0\cap B)(C\cap B_0)} \qquad (4)$$

Since $B_0\cap C, B\cap C_0 \lhd B\cap C$, then $(B_0\cap C)(B\cap C_0) = (B\cap C_0)(B_0\cap C)$. So, the right hand sides of (3) and (4) are equal. Thus, $\dfrac{B_0(B\cap C)}{B_0(B\cap C_0)} \cong \dfrac{C_0(C\cap B)}{C_0(C\cap B_0)}$. □

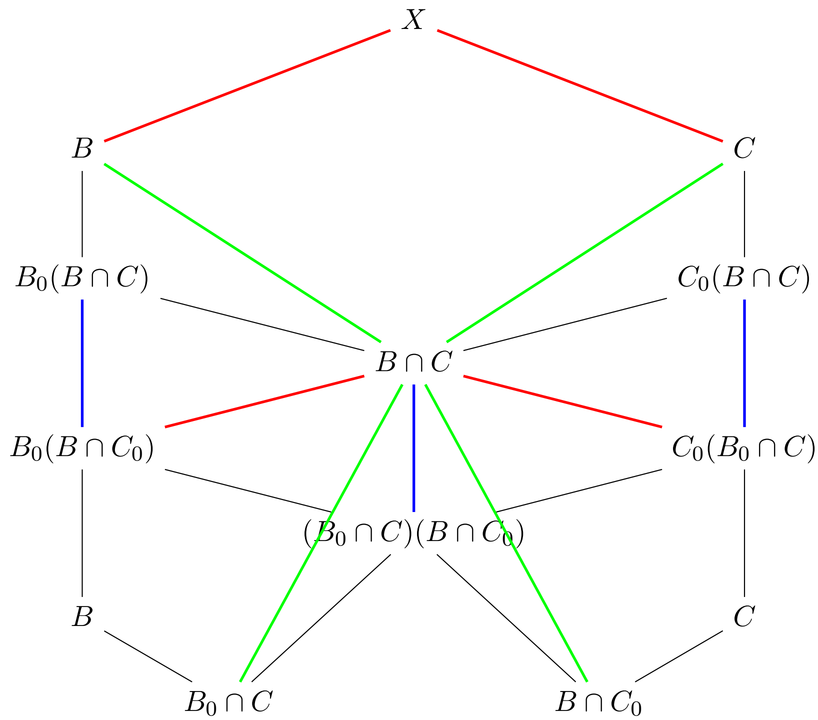

FIGURE 1. The Neutro-Butterfly

Remark 2.43. In Figure 1, the quotients given by the blue lines (by pairing) are neutro-isomorphic to each other based on (3) and (4), thus proving the Neutro Zassenhaus' Lemma. A black line indicates that the NT-subgroup that lie below is NT-normal in the NETG connected to it above in the plane of the figure. Also, the red (green) line indicates that the NT-subgroup that lie below is right self cancellative (closure-contained) respectively, in the NETG connected to it above in the plane of the figure. We acknowledge Kannappan Sampath [4] for adapting his LaTeX codes for Zassenhaus' Lemma for groups to generate Figure 1.

3. Conclusion

In this paper, we have been able to establish the homomorphism theorems (first, second and third neutro-isomorphism and neutro-corresponding theorems) and some other associated theorems (neutro-Zassenhaus Lemma) in singular WCNETG with the aid of newly introduced NT-subgroups such as: right cancellative, semi-strong, and maximally normal NT-subgroups. These results generalize their classical forms in group theory.

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