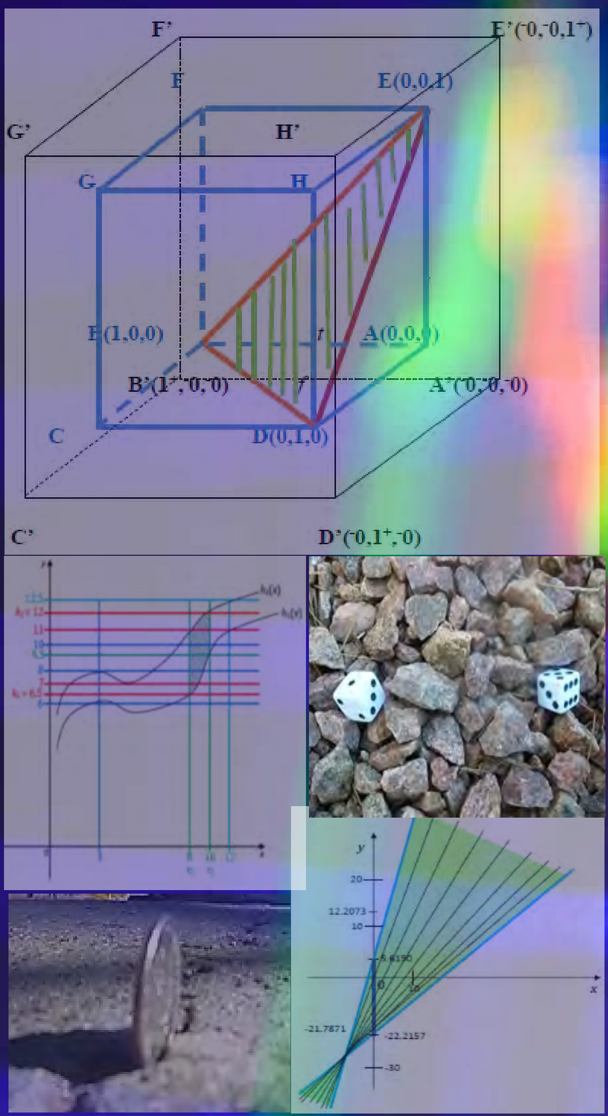
Volume 46, 2021

(updated)

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering



<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
Editors-in-Chief

ISSN 2331-6055 (Print)
ISSN 2331-608X (Online)

Neutrosophic Science
International Association (NSIA)

ISSN 2331-6055 (print) *ISSN 2331-608X (online)*

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

University of New Mexico

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

NSS

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

Information for Authors and Subscribers

*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

All submissions should be designed in MS Word format using our template file:

http://fs.unm.edu/NSS/NSS-paper-template.doc.

A variety of scientific books in many languages can be downloaded freely from the Digital Library of Science:

http://fs.unm.edu/ScienceLibrary.htm.

To submit a paper, mail the file to the Editor-in-Chief. To order printed issues, contact the Editor-in-Chief. This journal is non-commercial, academic edition. It is printed from private donations.

Information about the neutrosophics you get from the UNM website:

http://fs.unm.edu/neutrosophy.htm. The

home page of the journal is accessed on

http://fs.unm.edu/NSS.

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

NSS

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

**** NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, the NSS articles are indexed in Scopus.**

NSS ABSTRACTED/INDEXED IN

SCOPUS,
Google Scholar,
Google Plus,
Google Books,
EBSCO,
Cengage Thompson Gale (USA),
Cengage Learning (USA),
ProQuest (USA),
Amazon Kindle (USA),
University Grants Commission (UGC) - India,
DOAJ (Sweden),
International Society for Research Activity (ISRA),
Scientific Index Services (SIS),
Academic Research Index (ResearchBib),
Index Copernicus (European Union),
CNKI (Tongfang Knowledge Network Technology Co.,
Beijing, China),
Baidu Scholar (China),
Redalyc - Universidad Autonoma del Estado de Mexico (IberoAmerica),
Publons,
Scimago, etc.

Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

.

ISSN 2331-6055 (print) NSS ISSN 2331-608X (online)
University of New Mexico

Contents

Statistical Similarity Analysis Based on Neutrosophic Interval Probability

Jiaming Song[1], Rui Yong[1], Yingying Zhang[1], Minna Xu[2], Jun Ye[1*]

[1] School of Civil and Environmental Engineering, Ningbo University, Ningbo, 315211, P.R. China;
[2] Key Laboratory of Rock Mechanics and Geohazards of Zhejiang Province, Shaoxing University, Shaoxing, Zhejiang, 312000, P.R. China; 806474580@qq.com

* Correspondence: yejun1@nbu.edu.cn

Abstract: The roughness of the rock joints has a significant influence on the mechanical properties and deformation behavior of the rock masses. Due to the significant heterogeneity of the surface roughness of rock joints, there may be considerable variabilities, which makes it difficult to accurately determine the roughness. The evaluation of the anisotropic characteristics of joint roughness based on classical probability and statistics cannot reflect the vague, incomplete, imprecise, and indeterminate information of the roughness in different orientations. In this original study, we first propose the generalized Dice similarity measures based on neutrosophic interval statistical number (NISN) for evaluating the similarity of the roughness in different orientations. This method was applied to determine the similarity between the roughness along the sliding direction and each measurement direction. The research results show that this method can effectively determine the similarity between the roughness in different orientations. It may help determine the roughness along the potential sliding direction based on the roughness obtained in the direction with better measurement conditions.

Keywords: neutrosophic interval probability (NIP); neutrosophic interval statistical number (NISN); joint roughness coefficient (JRC); similarity measure

1. Introduction

The joint roughness coefficient (JRC) is one of the important parameters for evaluating the shear strength of rock joints and the stability of rock mass [1-5]. The effect of anisotropy on surface roughness has been proven to be an inherent characteristic of rock joints [6]. In addition, anisotropy can be seen everywhere in rock engineering, and the roughness of rock joints changes directionally, which is a crucial source of anisotropy behavior. Du et al. [7] measured 2180 joint profiles and statistically analyzed the roughness coefficients. The result indicates that the roughness coefficients of type I fractures (joints) and type III fractures (faults) are anisotropic. The anisotropy classification method proposed by Belen [8] showed that the roughness has a strong anisotropy. At present, many studies have been proposed for investigating the anisotropy of rock joints, and the anisotropy in roughness was proved to have a significant effect on the anisotropic shear strength of joints [9-13].

To describe the complex joint morphology, it is necessary to effectively quantify the anisotropic characteristics of the roughness of the joint surface. Chen et al. [14] proposed a geostatistical method to analyze the anisotropy and scale effect of rock joints. Ge et al. [15] believed that the roughness evaluation results in different directions of rock joints are very different, and it is recommended to select reasonable parameters in practice in combination with the sliding direction or the seepage direction. The anisotropy of the joint roughness is an important subject in the field of geotechnical engineering. In many practical situations, because the rock joints are located in different positions of the rock mass, and the empirical estimates obtained by experts along a certain direction are usually inaccurate, the potential sliding directions are different. Therefore, it is always difficult to calculate or provide a definite JRC value. While the neutrosophic number originally proposed by Smarandache [16-18] can express the incomplete and indeterminate information. It can better express the JRC with incomplete and uncertain information under an anisotropy environment. Recently, Ye et al. [19] proposed an approach to study the JRC with incomplete and indeterminate information using neutrosophic functions. Chen et al. [20] introduced a neutrosophic statistical method of JRC neutrosophic numbers for effectively analyzing the scale effect and anisotropy of JRC values, which was also applied by Muhammad [21] for JRC investigation. Smarandache [22] provided the interval function and some basic definitions of neutrosophic probability. Then, the neutrosophic interval statistical number (NISN) and NIP proposed by Chen et al. [23] are made up of both neutrosophic numbers and interval probability. However, there are few studies on the anisotropy of JRC based on the neutrosophy theory.

In the field, the situation of the rock joint surface is very complicated. Taking Laxiwa Hydropower Station as an example [24], there are many joints filled with calcite, but only a few countertops are exposed. Thus, it may be very difficult to measure the roughness of rock joints along some orientations under some conditions. It is a good remedy to calculate the properties of the unknown direction through the properties of rock joints obtained in the direction with better measurement conditions. The similarity measurement method is an important mathematical tool to determine the similarity between two objects [5,25-26]. The Dice similarity measure can be effectively used in the evaluation of the degree of similarity between the studied objects. But this similarity measure has not been applied to the JRC anisotropy assessment. Therefore, this paper extends the generalized Dice similarity measure by NIP, which can solve the problem of not being able to accurately obtain the roughness along the potential sliding direction. The main advantage of the similarity measure method is that it can effectively handle indeterminate information in anisotropic environments.

This paper is formed by the following parts. First, we will introduce the generalized Dice similarity measures between NISNs. Second, insight will be gained into the statistical measurement results of the similarity of joint roughness coefficients in different orientations by the generalized Dice similarity methods. Lastly, it gives conclusions and future study of this paper. These findings are of great significance for solving the problem that the joint roughness along the potential sliding direction cannot be obtained accurately on site.

2. Neutrosophic interval probability and Neutrosophic interval statistical number

Chen et al. (2017) defined the NIP in an interesting range $[x^L, x^U]$ of all individuals in the sample. The form of a NIP was expressed as $P = \langle [x^L, x^U], (P_T, P_I, P_F) \rangle$, where P_T, P_I, P_F are the true, indeterminate, and false probabilities belonging to the determinate, indeterminate, and failure ranges, respectively. For each trial data, the neutrosophic interval probability by the following equations:

$$\begin{cases} P_T = n_T / n \\ P_I = n_I / n \\ P_F = n_F / n \end{cases}, \quad (1)$$

where n is the total number of the individuals; n_T is the number of samples that fall in the interval $[x_m - \sigma, x_m + \sigma]$; n_I is the number of samples that fall in the interval $[x_m - 3\sigma, x_m - \sigma]$ and $(x_m + \sigma, x_m + 3\sigma)$; n_F is the number of the rest samples. Here, x_m denotes the statistical mean value and σ for standard deviation. The sum of true, indeterminate, and false probabilities is equal to 1.

A NISN R, which is combined NN with the expected value of NIP (confidence level λ), could be expressed as follows:

$$\begin{cases} R = x_m + (1-\lambda)I = x_m + (1 - \dfrac{P_T}{\sqrt{{P_T}^2 + P_I^2 + {P_F}^2}})I, & \text{for } I \in [\inf I, \sup I] \\ \lambda = \dfrac{P_T}{\sqrt{{P_T}^2 + P_I^2 + {P_F}^2}}, & \text{for } \lambda \in [0,1] \end{cases} \quad (2)$$

where *I* denote the indeterminacy and it ranges in the robust interval [−*σ*, *σ*].

The NISN is very suitable to express the interval value under indeterminate environments.

3. Generalized Dice similarity measures between NISNs

Definition 1. Let $R_A = a_A + b_A I$ and $R_B = a_B + b_B I$ be two neutrosophic numbers, where a_A, b_A, a_B, $b_B \geq 0$. A generalized Dice similarity measure between R_A and R_B is defined as follows:

$$\begin{aligned} D(R_A, R_B) &= \frac{2R_A \cdot R_B}{|R_A|^2 + |R_B|^2} \\ &= 2 \times \frac{(a_A + \inf(b_A I))(a_B + \inf(b_B I)) + (a_A + \sup(b_A I))(a_B + \sup(b_B I))}{(a_A + \inf(b_A I))^2 + (a_A + \sup(b_A I))^2 + (a_B + \inf(b_B I))^2 + (a_B + \sup(b_B I))^2} \end{aligned} \quad (3)$$

The generalized Dice similarity measure should satisfy the following properties (P1-P3):

(P1) $0 \leq D(R_A, R_B) \leq 1$;

(P2) $D(R_A, R_B) = 1$ if $R_A = R_B$;

(P3) $D(R_A, R_B) = D(R_B, R_A)$.

Definition 2. Let $A = \{R_{A1}, R_{A2}, \cdots, R_{An}\}$ and $B = \{R_{B1}, R_{B2}, \cdots, R_{Bn}\}$ be two sets of neutrosophic numbers, where $R_{Ak} = a_{Ak} + b_{Ak} I$ and $R_{Bk} = a_{Bk} + b_{Bk} I$, and (*k*=1, 2, ···, *n*) a_{Aj}, b_{Aj}, a_{Bj}, $b_{Bj} \geq 0$. Then, the generalized Dice similarity measure between *A* and *B* can be calculated by

$$\begin{aligned} D(A, B) &= \sum_{j=1}^{n} w_j \frac{2R_{Aj} \cdot R_{Bj}}{|R_{Aj}|^2 + |R_{Bj}|^2} \\ &= 2 \times \sum_{j=1}^{n} w_j \frac{(a_{Aj} + \inf(b_{Aj} I))(a_{Bj} + \inf(b_{Bj} I)) + (a_{Aj} + \sup(b_{Aj} I))(a_{Bj} + \sup(b_{Bj} I))}{(a_{Aj} + \inf(b_{Aj} I))^2 + (a_{Aj} + \sup(b_{Aj} I))^2 + (a_{Bj} + \inf(b_{Bj} I))^2 + (a_{Bj} + \sup(b_{Bj} I))^2} \end{aligned} \quad (4)$$

The generalized Dice similarity measure of two sets of neutrosophic numbers is satisfied the properties (P4-P5):

(P4) $0 \leq D(A, B) \leq 1$;

(P5) $D(A, B) = 1$ if $A = B$;

(P6) $D(A, B) = D(B, A)$.

According to **Definition 1**, the generalized Dice similarity measure between two neutrosophic intervals statistical number R_A and R_B can be expressed as

$$\begin{aligned} D(R_A, R_B) &= \frac{2R_A \cdot R_B}{|R_A|^2 + |R_B|^2} \\ &= 2 \times \frac{(x_{mA} - \sigma_A(1-\lambda_A))(x_{mB} - \sigma_B(1-\lambda_B)) + (x_{mA} + \sigma_A(1-\lambda_A))(x_{mB} + \sigma_B(1-\lambda_B))}{(x_{mA} - \sigma_A(1-\lambda_A))^2 + (x_{mA} + \sigma_A(1-\lambda_A))^2 + (x_{mB} - \sigma_B(1-\lambda_B))^2 + (x_{mB} - \sigma_B(1-\lambda_B))^2} \end{aligned} \quad (5)$$

4. Applications

Anisotropy is one of the basic characteristics of rock joints. The degree of anisotropy of JRC depends on the contact condition between the upper and lower joint surfaces, which controls the shear strength of the rock joint in different directions and the internal hydraulic transmission mechanism. In this study, for the evaluation of the anisotropic characteristics of JRC, statistical analysis is carried out based on classical mathematical methods. In this section, we use the NISNs proposed by Chen et al. (2017) to express the indeterminacy of JRC and then adopt the generalized

Dice similarity measure method to illustrate the similarity of the statistical results of JRC in all directions. To verify the validity and rationality of the proposed statistical similarity analysis based on the NIP, we selected the calcareous slate rock joint collected from Changshan County, Zhejiang Province. The joint surface is hard and complete, the joint wall is dense and slightly weathered, and the joint surface is smooth to rough, fully meets the requirements of this statistical analysis.

In this study, the generalized Dice similarity measure method is developed to estimate the similarity between the potential slip direction and the remaining directions of the structural plane, as described below.

Step1. According to the NIP statistical method, NISNs are utilized to represent the JRC values in each direction.

Table 1. Related value of JRC and NISNs.

Orientation θ (°)	x_m	σ	λ	R
0	10.5425	2.2385	0.921	[10.3651,10.7199]
15	10.0131	2.8392	0.857	[9.6084,10.4178]
30	10.5944	2.3528	0.918	[10.4014,10.7874]
45	9.9244	2.5120	0.884	[9.6321,10.2166]
60	9.0253	2.4592	0.919	[8.8250,9.2255]
75	7.9352	2.1063	0.901	[7.7260,8.1444]
90	7.0467	2.4054	0.728	[6.3935,.6998]
105	7.7766	2.4105	0.930	[7.6080,7.9452]
120	9.1324	2.3250	0.937	[8.9858,9.2790]
135	9.2258	1.9104	0.927	[9.0857,9.3659]
150	10.4673	2.4365	0.866	[9.9325,10.6021]
165	10.6035	2.2090	0.912	[9.9005,10.3066]
180	9.8501	2.1439	0.906	[9.6486,10.0515]
195	9.9383	2.2254	0.916	[9.7522,10.1245]
210	9.5903	1.9444	0.896	[9.3872,9.7933]
225	8.9167	1.9764	0.906	[8.7299,9.1034]
240	7.8582	1.8456	0.921	[7.7130,8.0034]
255	7.2166	1.9341	0.907	[7.0362,7.3970]
270	6.8025	2.1165	0.671	[6.1059,7.4990]
285	7.0061	1.5474	0.916	[6.8767,7.1355]
300	8.4720	1.7448	0.923	[8.3373,8.6068]
315	10.1428	2.4790	0.883	[9.9868,10.2988]
330	9.8295	2.2844	0.910	[9.6230,10.0360]
345	9.6831	2.0192	0.918	[9.5183,9.8479]

First, we conduct a statistical analysis of the JRC values, giving the mean value x_m, the standard deviation σ and the confidence level λ of each orientation θ, and then using equations (1) and (2) to obtain the results of NISNs in each direction, which are shown in Table 1.

Step 2. According to the generalized Dice similarity measure approach, determine the similarit**ies** between the potential slip direction and other directions**,** respectively.

Here, we take an azimuth angle of 0° as the reference object and assume that it is the potential slip direction of this case. Based on the obtained NISNs data, we calculate the similarity between 0° orientation and other orientations by equation (5).

Step 3. For determining the similarity between the potential slip direction and other directions, the range of the obtained similarity value is normalized from [0,1] to [−1,1].

To obtain the normalized correlation coefficient χ_k, the results of the similarity measure need to be changed as follows:

$$\chi_k = \frac{2\phi_k - \phi_{\min} - \phi_{\max}}{\phi_{\max} - \phi_{\min}} \tag{6}$$

where $\phi_{\max}$ and $\phi_{\min}$ represent the maximum and minimum values of similarity measure results, respectively.

If the correlation coefficient χ_k is negative 1, it means that this direction does not have the same JRC statistical characteristics as the potential slip direction. However, if the correlation coefficient χ_k of positive 1, the direction has the same JRC statistical characteristics as the potential slip direction.

For example, taking the azimuth angle $\theta = 0^o$ and $\theta = 30^o$ as a set of data to be measured, and the calculation process is as follows.

First, according to Equation (5), the generalized Dice similarity measure value is calculated as

$$\begin{aligned} D(R_A, R_B) &= \frac{2R_A \cdot R_B}{|R_A|^2 + |R_B|^2} \\ &= 2\times\frac{\left(x_{mA}-\sigma_A(1-\lambda_A)\right)\left(x_{mB}-\sigma_B(1-\lambda_B)\right)+\left(x_{mA}+\sigma_A(1-\lambda_A)\right)\left(x_{mB}+\sigma_B(1-\lambda_B)\right)}{\left(x_{mA}-\sigma_A(1-\lambda_A)\right)^2+\left(x_{mA}+\sigma_A(1-\lambda_A)\right)^2+\left(x_{mB}-\sigma_B(1-\lambda_B)\right)^2+\left(x_{mB}-\sigma_B(1-\lambda_B)\right)^2} \\ &= 1.0000 \end{aligned}$$

Then, obtaining the normalized correlation coefficient by using equation (6):

$$\begin{aligned} \chi_k &= \frac{2\phi_k - \phi_{\min} - \phi_{\max}}{\phi_{\max} - \phi_{\min}} \\ &= 0.9997 \end{aligned}$$

For other orientations of the data, we also calculated according to the above steps, the results are shown in Table 2.

It can be seen in Figure 1 that when the sample length is constant, as the orientation θ changes, the generalized similarity measure of JRC changes but without any special rules to follow. When the measurement direction is 30°, the generalized Dice similarity measure value reaches the highest value. While, when the measurement direction is 270°, it shows the smallest similarity. By calculating the variation of similarity in different directions, it can be concluded that the generalized Dice similarity measure value varies due to different measurement directions. In this case, for the orientations range within 60° to 135° and 225°to 300°, the similarity of JRC is small. When the measurement orientations are in the ranges of 15° to 45° and 150° to 195°, the similarity is large.

Through the above analysis, it can be seen that JRC has a significant anisotropy, and the evaluation results of similarity measures obtained in different directions are quite different. Therefore, in practice, according to the similarity measures of JRC, the roughness along the potential slip direction that is difficult to measure can be predicted based on the obtained roughness in the direction with better exposure conditions.

Table 2. Table of normalized values corresponding to each orientation

Orientation (°)	$D(R_A, R_B)$	χ_k
0	1.0000	1.0000
15	0.9984	0.9652
30	1.0000	0.9997
45	0.9981	0.9582
60	0.9880	0.7352
75	0.9610	0.1352
90	0.9228	–0.7101
105	0.9554	0.0128
120	0.9898	0.7735
135	0.9912	0.8042
150	0.9999	0.9897
165	0.9991	0.9799
180	0.9977	0.9489
195	0.9983	0.9615
210	0.9955	0.9011
225	0.9861	0.6929
240	0.9583	0.0768
255	0.9323	–0.5008
270	0.9097	–1.0000
285	0.9220	–0.7290
300	0.9766	0.4807
315	0.9992	0.9834
330	0.9975	0.9457
345	0.9964	0.9201

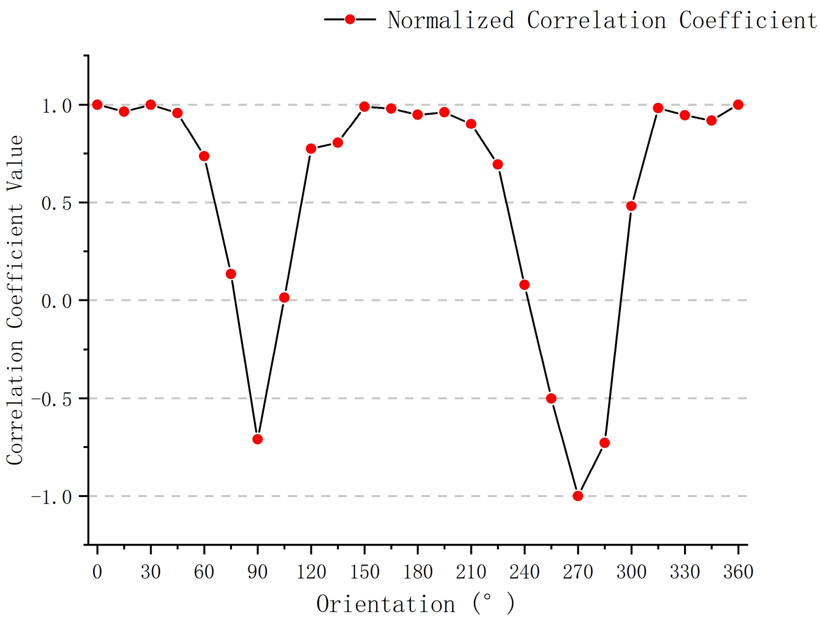


Figure 1. The relationship between orientation and normalized correlation coefficient value

5. Conclusion

In this study, the generalized Dice measures of NISNs are presented to estimate the similarity of the JRC values in different orientations. A quantitative evaluation of the similarity between the potential slip direction and other measurement directions was made. It could be applied to predict the properties of the potential slip direction based on the obtained roughness in the direction with better exposure conditions. Compared to classic statistical methods, for the processing of JRC data, not only are the average value and standard deviation considered, but also the confidence level is introduced, so that some uncertain information can be displayed more specifically. The JRC value of the potential slip direction is taken as the characteristic interval of the reference scale, and the JRC values in the remaining directions are used as the estimated characteristic intervals. Statistical similarity analysis based on neutrosophic interval probability overcomes the deficiency of the existing exposed rock joint area is small, and the required effectiveness information cannot be obtained on the potential slip surface, indicating its necessity in the JRC evaluation.

For future work, we can do more research on the anisotropy of joint surface roughness. For instance, we can propose more statistical analysis models and can also apply the proposed method to different aspects, such as extending the similarity measure to the scale effect of rock joints.

References

1. Ghazvinian, A.; Azinfar, M.J.; Norozi, P. Mechanical response of discontinuities of different joint wall contact strengths. *Arabian Journal of Geosciences* **2013**, *6*, 4419–4426.
2. Sanei, M.; Faramarzi, L.; Goli, S.; Fahimifar, A.; Rahmati, A.; Mehinrad, A. Development of a new equation for joint roughness coefficient (JRC) with fractal dimension: a case study of Bakhtiary Dam site in Iran. *Arabian Journal of Geosciences* **2015**, *8*, 465–475.
3. Cao, R.H.; Cao, P.; Lin, H.; Ma, G.; Chen, Y. Failure characteristics of intermittent fissures under a compressive shear test: experimental and numerical analyses. *Theoretical and Applied Fracture Mechanics* **2018**, *96*, 740–757.
4. Lin, H.; Wang, H.; Fan, X.; Cao, P.; Zhou, K. Particle size distribution effects on deformation properties of graded aggregate base under cyclic loading. *European Journal of Environmental and Civil Engineering* **2017**, 1–18.

5. Yong, R.; Ye, J.; Liang, Q.F.; Huang, M.; Du, S.G. Estimation of the joint roughness coefficient (JRC) of rock joints by vector similarity measures. *Bulletin of Engineering Geology and the Environment* **2018**, *77*, 735–749.
6. Du, S.G. Engineering Behavior of Discontinuities in Rock Mass. *Seismological Press* **1999**.
7. Du, S.G.; Tang, H.M. Research on Anisotropy of Rock Mass Fracture Roughness Coefficient. *Journal of Engineering Geology* **1993**, *1*, 32–41.
8. Belem, T.; Homand-Etienne, F.; Souley, M. Quantitative parameters for rock joint surface roughness. *Rock Mech Rock Eng* **2000**, *33*, 217–242.
9. Ramamurthy, T. Strength, modulus responses of anisotropic rocks. *Compressive Rock Engineering* **1993**, *1*, 313–329.
10. Nasseri, M.H.B.; Rao, K.S.; Ramamurthy, T. Anisotropic strength and deformational behavior of Himalayan schists. *International Journal of Rock Mechanics and Mining Sciences* **2003**, *40*, 3–23.
11. Cho, J.W.; Kim, H.; Jeon, S. Deformation and strength anisotropy of Asan gneiss, Boryeong shale, and Yeoncheon schist. *International Journal of Rock Mechanics and Mining Sciences* **2012**, *50*, 158–169.
12. Jing, L.; Nordlund, E.; Stephansson, O. An experimental study on the anisotropy and stress-dependency of the strength and deformability of rock joints. *Int. J. Rock Mech. Min. Sci. Geomech. Abstr* **1992**, *29*, 535–542.
13. Zhang, C.; Liu, N.; Chu, W. Multi-scale fine description on the anisotropic characteristics of quartz-mica schist. *Chinese Journal of Rock Mechanics and Engineering* **2018**, *37*, 573–582.
14. Chen, S.; Zhu, W.; Liu, S.; Zhang, F.; Guo, L. Anisotropy and size effects of surface roughness of rock joints. *Chinese Journal of Rock Mechanics and Engineering* **2015**, *34*, 57–66.
15. Ge, Y.F.; Tang, H.M.; Wang, Q.L. Anisotropy, scale and interval effects of natural rock discontinuity surface roughness. *Chinese Journal of Geotechnical Engineering* **2016**, *38*, 170–179.
16. Smarandache, F. Neutrosophy: Neutrosophic probability, set, and logic. American Research *Press*, *Rehoboth*, *USA* **1998**.
17. Smarandache, F. Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic Probability. *Sitech & Education Publisher*, Craiova – Columbus **2013**.
18. Smarandache, F. Introduction to neutrosophic statistics. *Computer ence* **2014**.
19. Ye, J.; Yong, R.; Liang, Q.F.; Huang, M.; Du, S.G. Neutrosophic functions of the joint roughness coefficient and the shear strength: a case study from the pyroclastic rock mass in Shaoxing City, China. *Math Problems Eng* **2016**.
20. Chen, J.Q.; Ye, J.; Du, S.G. Scale Effect and Anisotropy Analyzed for Neutrosophic Numbers of Rock Joint Roughness Coefficient Based on Neutrosophic Statistics. *Symmetry* **2017**, *9*, 14.
21. Aslam, M. A new method to analyze rock joint roughness coefficient based on neutrosophic statistics. *Measurement* **2019**, *146*, 65–71.
22. Smarandache, F. Neutrosophic Precalculus and Neutrosophic Calculus. *Europa-Nova, Brussels* **2015**.
23. Chen, J.Q.; Ye, J.; Du, S.G.; Yong, R. Expressions of Rock Joint Roughness Coefficient Using Neutrosophic Interval Statistical Numbers. *Symmetry* **2017**, *9*, 123.
24. Wu, Z.Y. Information processing and engineering application of rock mass structure. *Chengdu University of Technology* **2003**.
25. Ye, J.; Fang, Z.B.; Cui, W.H. Vector Similarity Measures of Q-Linguistic Neutrosophic Variable Sets and Their Multi-Attribute Decision Making Method. *Symmetry* **2018**, *10*, 531.
26. Ye, S.; Ye, J. Dice Similarity Measure between Single Valued Neutrosophic Multisets and Its Application in Medical Diagnosis, *Neutrosophic Sets and Systems* **2014**, *6*, 49–54.

Received: May 29, 2021. Accepted: October 1, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

The Duality Approach of the Neutrosophic Linear Programming

Huda E. Khalid [1*], Ahmed K. Essa[2]

[1*] Telafer University, The Administration Assistant for the President of the Telafer University, Telafer, Iraq. ; https://orcid.org/0000-0002-0968-5611

[2] Telafer University, Statistics Division, Telafer, Iraq. E-mail: ahmed.ahhu@gmail.com ; https://orcid.org/0000-0001-6153-9964

*Corresponding Author: Huda E. Khalid. Email: dr.huda-ismael@uotelafer.edu.iq

Abstract. The neutrosophic mathematical linear programming in its duality fashion is originally exhibited in this manuscript. In accordance with this concept, the relationship of the duality between the neutrosophic objective functions and neutrosophic constraints is given, three versions of linear programming related to $\mu_D(x)$, $\sigma_D(x)$, and $v_D(x)$ have been originated respectively, some important propositions have been discussed, two numerical examples were considered in the economic interpretation and in the hybrid renewable energy production.

Keyword: Neutrosophic Linear Programming Related to $\mu_D(x)$; Neutrosophic Linear Programming Related to $\sigma_D(x)$; Neutrosophic Linear Programming Related to $v_D(x)$; Parametric Dual Neutrosophic Linear Programming.

1. Introduction

It is well known that, when the region of the feasible solution of any mathematical problem is convex, then the optimality will be traditionally performed. For many years the dominant distinction in applied mathematics between problem types has rested upon linearity, or lack thereof. Our assignment here is to serve more than a half-century of work in convex analysis that has played a fundamental role in the development of computational in every branch of application whether the problem is in economical fields, industrial fields, or agricultural fields ... etc. Supposing that the problem is a neutrosophic linear programming problem and the established problem is taken from the economical point of view.

In any selling product, there are three assigned statuses:

S1- Profit situations.

S2- Loss situations.

S3- Indeterminate status which there are not clear criteria enables the decision maker to determine getting profit nor loss.

There are many factors that affecting on the degree of accomplishment for the above S1, S2, and S3 such as the season of the year, quality of the competition in each product, the spread of public pandemic as COVID 19… etc.

For a wise decision, the selling product should not mark with maximum profit often along time, since in such a case the competition, for instance, could obtain a profit of firm's policy be reducing their prices for the same articles. Therefore, and by the authors' opinion, it is very important to make an adaptation for the classical linear programming or the fuzzy linear programming into the neutrosophic linear programming with three versions related to their truth, indeterminate, and falsity membership functions, this kind of programming will parametric classical or fuzzy linear programming into three decision factors: -

1- The neutrosophic linear programming related to the truth membership function.

2- The neutrosophic linear programming related to the indeterminate membership function.

3- The neutrosophic linear programming related to the falsity membership functions.

Bellow two comparisons between Fuzzy Linear Programming and Neutrosophic Linear Programming:

1- In Fuzzy Linear Programming Problems (FLP) [1,15], as the optimal solution has depended on a limited number of constraints, therefore, much of the information that should be collected and having a good impact on the solution are absent, this is exactly what Neutrosophic Linear Programming (NLP) provides.
2- Given the power of LP, one could have expected even more applications. This might be because LP requires many well-defined and precise data which involves high information costs. In real-world applications certainty, reliability, and precision of data are often illusory. Being able to deal with vague and imprecise data may greatly contribute to the diffusion and application of LP. Neutrosophic Linear Programming problems have the ability to reformulate the soft linear programming problems

through three membership functions which are truth membership function, indeterminacy membership function, and falsity membership functions, while the Fuzzy Linear Programming deals with just one membership function.

This essay aims to advance a new way for analyzing linear programming containing three different and related faces in which the same problem can be approached from three various corners, it is neutrosophic linear programming of three membership functions. This wide insight can be appeared depending upon the neutrosophic logic, this kind of problem has established firstly in 1995 by Florentin Smarandache [2,3], the neutrosophic logic and theory have widespread since the NSS journal has been released in 2013. Dozens of papers were issued, and new mathematical concepts have been originated, such as, neutrosophic geometric programming has been established and modified at 2015-2020 by Huda et al [9,10,12-14], also presented another concept of geometric programming with neutrosophic less than or equal [6,8], neutrosophic (sleeves, Anti-sleeves, Neut-sleeves), and the neutrosophic convex set has been set up [11], the excluded middle law with the perspective of neutrosophic geometric programming [4,5,11].

The new type of linear programming that presented in this article will be defined in the triplet $(X = [0,1], N(X), c)$ corresponding to the case in which the expert- mathematician exactly knows his objective function, but the constraints set is of type neutrosophic linear programming, the upcoming preliminaries are necessary to build the mathematical structure of such problems.

Call the classical linear programming problems

$$\left.\begin{aligned} &Max \quad f = cx \\ &s.t. \quad Ax \leq b \\ &\quad x \geq 0 \end{aligned}\right\} \tag{1}$$

Which defined by the triplet (R^n, X, c), the goal of this problem is to find the optimal solution $x^* \in X \subset R^n$ such that $\forall\, x \in X: cx^* \geq cx$ with $X = \{x \in R^n | Ax \leq b: x \geq 0\}$.

The above classical linear programming can be redefined as a Neutrosophic Linear Programming (NLP) related to its truth, indeterminacy, and falsity membership functions. The following section contains some new definitions that coined and for the first time in this essay beside to some preliminaries which are necessary to build the mathematical formulas.

The upcoming sections of this paper have been organized as: section two contains three basic definitions that are necessary tools to follow up the mathematical requirements of this article, as well as, three new definitions were originally coined by the authors to extend the fuzzy linear programming to the neutrosophic linear programming. Section three was dedicated to two important propositions that regarded as the new mathematical vision for a new procedure that proves any neutrosophic linear programming related to its truth membership function can be regarded as the dual form for the neutrosophic linear programming related to its falsity membership function. In section four, two practical examples have been presented that assures the theoretical directions of the paper. Concluding section was the fifth section of this article.

2. Basic Concepts

2.1 Definition [16]

A neutrosophic set $\mathrm{D} \in \mathrm{N}(\mathrm{X})$ is defined as $\mathrm{D} = \{< \mu_{\mathrm{D}}(\mathrm{x}), \sigma_{\mathrm{D}}(\mathrm{x}), \mathrm{v}_{\mathrm{D}}(\mathrm{x}) >: \mathrm{x} \in \mathrm{X}\}$ where $\mu_{\mathrm{D}}(\mathrm{x}), \sigma_{\mathrm{D}}(\mathrm{x}), \mathrm{v}_{\mathrm{D}}(\mathrm{x})$ represent the membership function, the indeterminacy function, the non-membership function respectively.

2.2 Definition [11]

Let $D \in N(X), \forall\ (\alpha, \gamma, \beta) \in [0,1]$, written $D_{(\alpha,\gamma,\beta)} = \{x: \mu_{\mathrm{D}}(\mathrm{x}) \geq \alpha, \sigma_{\mathrm{D}}(\mathrm{x}) \geq \gamma, \mathrm{v}_{\mathrm{D}}(\mathrm{x}) \leq \beta\}$, $D_{(\alpha,\beta,\gamma)}$ is said to be an $(\alpha, \beta, \gamma) - cut$ set of a neutrosophic set D . Again, $D_{(\alpha,\gamma,\beta)^{+}} = \{\mathrm{x}: \mu_{\mathrm{D}}(\mathrm{x}) > \alpha, \sigma_{\mathrm{D}}(\mathrm{x}) > \gamma, \mathrm{v}_{\mathrm{D}}(\mathrm{x}) < \beta\}$, $D_{(\alpha,\gamma,\beta)^{+}}$ is said to be a strong $(\alpha, \gamma, \beta) - cut$ set of a neutrosophic set D, (α, γ, β) are confidence levels and $\alpha + \gamma + \beta \leq 3$.

2.3 Definition [7]

A mapping $D\colon \mathrm{X} \to [0,1], \mathrm{x} \to \mu_{\mathrm{D}}(\mathrm{x}), \mathrm{x} \to \sigma_{\mathrm{D}}(\mathrm{x})\ , \mathrm{x} \to \mathrm{v}_{\mathrm{D}}(\mathrm{x})$ is called a collection of neutrosophic elements, where μ_D a membership x corresponding to a neutrosophic set D, $\sigma_{\mathrm{D}}(\mathrm{x})$ an indeterminacy membership x corresponding to a neutrosophic set D, $\mathrm{v}_{\mathrm{D}}(\mathrm{x})$ a non-membership x corresponding to a neutrosophic set D.

The upcoming definitions are essentially requirements for completing the duties of this article, so the authors originally coined them as follow:

2.4 Definition

$$\left.\begin{array}{ll} & Max \quad f = cx \\ s.t. & \mu(A(x)_i, b) \geq \alpha \quad i = 1,2, \dots, m \\ & \alpha \in [0,1], \quad x \geq 0 \end{array}\right\} \quad (2)$$

Here $\alpha \in [0,1]$ is the respective $\alpha - cut$ of the neutrosophic constraint set related to the truth membership function μ .

The above problem has been defined in the neutrosophic triplet $(X, N(X), c)$, here $X = [0,1], c \in N(X^n), b \in N(X^m), A_{mn}$ is a matrix of neutrosophic values, where $N(X) = \{x \in X^n; f: X^n \to N(X^n)\}$, $\mu = \{\mu_1, \mu_2, \dots, \mu_m\}$ is an m- vector of truth membership functions.

2.5 Definition

Depending upon the structure of the mathematical formula of neutrosophic linear programming (2), one can define a new concept named neutrosophic linear programming related to the falsity membership function as follow: -

$$\left.\begin{array}{ll} & Min \quad f = cx \\ s.t. & V(A(x)_i, b) \leq \beta \\ & \beta \in [0, 1], \quad x \geq 0 \end{array}\right\} \quad (3)$$

Here $\beta \in [0, 1]$ is the respective $\beta - cut$ of the neutrosophic constraint set in the case of neutrosophic linear programming regarded to its falsity membership function.

One can base on the intuitive idea to conclude that the inequality $V(A(x)_i, b) \leq \beta$ is equivalent to the inequality

$$1 - \mu(A(x)_i, b) \leq \beta \quad (4)$$

$$\Rightarrow 1 - \beta \leq \mu(A(x)_i, b)$$

So, the inequality (4) can be rewrite as

$$\mu(A(x)_i, b) \geq 1 - \beta \quad (5)$$

Comparing (2) & (5), we conclude that $\alpha \equiv 1-\beta$

Note that the difference between the two problems (2) & (3) is that the problem (2) gives the optimal solution for the neutrosophic linear programming with respect to the truth membership function, while the problem (3) gives the optimal solution for the neutrosophic linear programming with respect to the falsity membership function.

2.6 Definition

It is well known for any mathematical programmer who has the tools for reformulating any classical mathematical programming problems into neutrosophic programming problems, that the neutrosophic linear programming related to its indeterminacy membership function has well defined when it can be defined as:

$$\left.\begin{array}{ll} & Max \quad f = cx \\ s.t. & \sigma_D(x) \geq \gamma \\ & x \geq 0 \end{array}\right\} \tag{6}$$

Here $\gamma \in [0,1]$ is the respective $\gamma - cut$ of the neutrosophic constrain set in the case of neutrosophic linear programming with respect to its indeterminacy membership function.

The problem (6) is equivalent to

$$\left.\begin{array}{ll} & Max \quad f = cx \\ s.t. & \mu(A(x)_i, b) \cap V(A(x)_i, b) \geq \gamma \end{array}\right\} \tag{7}$$

3. The Duality Approach of Neutrosophic Linear Programming

3. 1 Proposition

Given a neutrosophic linear programming problem (2), there always exist a corresponding dual problem which is exactly the neutrosophic linear programming problem (3), and they have the same neutrosophic solution.

Proof

Consider the following neutrosophic linear programming

$$\left.\begin{array}{ll} & Max \quad f = cx \\ s.t. & \mu(A(x)_i, b) \geq \alpha \quad i = 1,2,\dots,m \\ & \alpha \in [0,1], \quad x \geq 0 \end{array}\right\} \tag{8}$$

Where the m-vector of membership functions $\mu = \{\mu_1, \mu_2, \dots, \mu_m\}$ such that

$$\forall x \in X: \ \mu_j(x) = \begin{cases} 1 & x < b_j \\ \frac{(b_j+d_j)-x}{d_j} & b_j \leq x \leq b_j + d_j \\ 0 & x > b_j + d_j \end{cases} \tag{9}$$

Where the values of $d_j \in X \ \ (j = 1,2,\dots,m)$ expressing the admissible violations of the economic-expert allows in the accomplishment of the neutrosophic linear constraints of (9), it is obvious that the neutrosophic solution of (9) is found by obtaining the optimal solution of the linear neutrosophic problem

$$\left.\begin{array}{ll} & Max \quad f = cx \\ s.t. & \mu(A(x), b) \geq \alpha \\ & \alpha \in [0,1], \quad x \geq 0 \end{array}\right\} \tag{10}$$

Depending upon (9) we have

$$\frac{b+d-Ax}{d} \geq \alpha \Leftrightarrow b + d - Ax \geq d\alpha \ \Leftrightarrow Ax - b - d \leq -d\alpha \ \Leftrightarrow Ax \leq b + d(1-\alpha).$$

Therefore, we have

$$\left.\begin{array}{ll} & Max \quad f = cx \\ s.t. & Ax \leq b + d(1-\alpha) \\ & \alpha \in [0,1], \quad x \geq 0 \end{array}\right\} \tag{11}$$

As (11) is a classical parametric linear programming problem, its dual is given by

$$\left.\begin{array}{l} Min \ [b + d(1-\alpha)]u \\ s.t. \qquad uA^T \geq c \\ u \geq 0, \ \ \alpha \in [0,1] \end{array}\right\} \tag{12}$$

Let $Y = \{u \in N(X^m) | uA^T \geq c, \ \ u \geq 0\}$

So, we have

$$\left.\begin{array}{ll} & Min \ \ au \\ s.t. & a = b + d(1-\alpha) \\ & u \in Y, \ \alpha \in [0,1] \end{array}\right\} \tag{13}$$

Consider a as m-variable vectors and taking $\beta = 1 - \alpha$, this problem is equivalent to

$$\left.\begin{array}{c} Min\ \ au \\ s.t.\quad a \le\ b + d\beta \\ u \in Y,\ \beta \in [0,1] \end{array}\right\} \tag{14}$$

Understanding the equivalence in the sense that any optimal solution of (13) is also an optimal solution of (14), but as $\frac{(b_j+d_j)-a_j}{d_j} \ge \alpha$ which implies that

$$\left(b_j + d_j\right) - a_j \ge d_j\alpha \Leftrightarrow -a_j \ge -\left(b_j + d_j\right) + d_j\alpha \ \Leftrightarrow a_j \le b_j + d_j - d_j\alpha \Leftrightarrow a_j \le b_j + d_j(1 - \alpha)$$
$$\Leftrightarrow a_j \le b_j + d_j\beta \quad for\ j = 1,2, \dots, m$$

So (14) may be rewritten as

$$\left.\begin{array}{c} Min\ \ au \\ s.t.\quad \mu_j\left(a_j\right) \ge 1 - \beta \\ u \in Y,\ \beta \in [0,1] \end{array}\right\} \tag{15}$$

Which implies to the following formula

$$\left.\begin{array}{c} Min\ \ au \\ s.t.\quad 1 - \mu_j\left(a_j\right) \le \beta \\ u \in Y,\ \beta \in [0,1] \end{array}\right\} \tag{16}$$

Consequently

$$\left.\begin{array}{c} Min\ \ au \\ s.t.\quad V_j\left(a_j\right) \le \beta \\ u \in Y,\ \beta \in [0,1] \end{array}\right\} \tag{17}$$

With $\mu_j(.)$ is given by (9), $V_j\left(a_j\right)$ is a non-membership function that stated in def. (2.3), programming (17) is exactly represented a neutrosophic linear programming with respect to its falsity membership function. Since, in the optimum, (11) and (12) have the same parametric solution, the problem (17) has the same neutrosophic solution as (8) by taking $\beta \equiv 1 - \alpha$.

If we had initially started from the neutrosophic linear programming (2), we would by the same development, in a parallel way, have come to a neutrosophic linear programming (3) with the same neutrosophic solution.

3.2 Proposition

Given a neutrosophic linear programming (2) or a neutrosophic linear programming (3), with continues and strictly monotone membership function for the economic restrictions (costs or benefits), there exists a dual neutrosophic linear programming (3), or a dual neutrosophic linear programming (2) respectively of the former in such a way that both have the same neutrosophic solution.

Proof

Let $\mu_j: X \longrightarrow N(X), j = 1,2, \dots, m$ be continuous and strictly increasing function for the neutrosophic linear programming problem (2).

Given a classical linear programming with a neutrosophic inequality in its constraint

$$\left.\begin{array}{lll} & Max & cx \\ s.t. & Ax \leq ₦\ b & x \geq 0 \end{array}\right\} \tag{18}$$

Where ($\leq ₦$) is the neutrosophic version of the (less than or equal) inequality. We shall find its neutrosophic solution with respect to $\mu(.)$, and for every $\alpha \in [0,1]$ of the neutrosophic constraint set

$$\mu(Ax, b) \geq \alpha \qquad \alpha \in [0,1]$$

But according to the hypotheses as μ is continuous and strictly monotone, μ^{-1} exists, and $\mu(Ax, b) \geq \alpha \Leftrightarrow Ax \leq \emptyset(\alpha) = \mu^{-1}(\alpha)$, and the proof follows as in proposition (3.1).

4 Numerical Examples:

4.1 Example 1

Suppose we have a neutrosophic linear programming problem with neutrosophic less than or equal in its constraints and as follows:

$$\left.\begin{array}{c} Max \ \ f(x_1, x_2) = x_1 + x_2 \\ 4x_1 - x_2 \leq ₦\ 10 \\ x_1 + 2x_2 \leq ₦\ 15 \\ 5x_1 + 2x_2 \leq ₦\ 20 \\ x_i \geq 0 \end{array}\right\} \tag{19}$$

With membership functions as follow:

$$\mu_1(4x_1 - x_2, 10) \geq \alpha$$

Here $b_1 = 10$, if we take $d_1 = 5$ as an admissible violation of the first constraint.

So, $\mu_1(4x_1 - 2x_2, 10) = \frac{(15-4x_1+x_2)^2}{25} \geq \alpha,$

$$(15 - 4x_1 + x_2)^2 \geq 25\alpha \tag{20}$$

The optimal solution of the inequality (20) is equivalent to the optimal solution of

$$15 - 4x_1 + x_2 = 5\sqrt{\alpha}\ ,$$

$$-4x_1 + x_2 = 5\sqrt{\alpha} - 15 \tag{21}$$

Also we have,

$$\mu_2(x_1 + 2x_2, 15) \geq \alpha$$

Here $b_2 = 15$, if we take $d_2 = 8$ as an admissible violation of the second constraint.

$$\mu_2(x_1 + 2x_2, 15) = \frac{(23-x_1-2x_2)^2}{64} \geq \alpha,$$

$$23 - x_1 - 2x_2 = \sqrt{64\alpha},$$

$$-x_1 - 2x_2 = 8\sqrt{\alpha} - 23$$

$$x_1 + 2x_2 = 23 - 8\sqrt{\alpha} \tag{22}$$

Finally, the membership of the third constraint is

$$\mu_3(5x_1 + 2x_2, 20) \geq \alpha$$

It is obviously that $b_3 = 20$, and if we take the admissible violation for the third constraint as $d_3 = 10$, so we have

$$-5x_1 - 2x_2 = 10\sqrt{\alpha} - 30 \longrightarrow\ 5x_1 + 2x_2 = 30 - 10\sqrt{\alpha} \tag{23}$$

From (23) we have,

$$2x_2 = 30 - 10\sqrt{\alpha} - 5x_1 \tag{24}$$

Substitute (24) in (22),

$x_1 + 30 - 10\sqrt{\alpha} - 5x_1 = 23 - 8\sqrt{\alpha}$

$\therefore\ x_1 = \frac{7-2\sqrt{\alpha}}{4}$ (25)

Substitute (25) in (23) we get,

$[\frac{5}{4}(7 - 2\sqrt{\alpha}) + 2x_2 = 30 - 10\sqrt{\alpha}]$, simplify this formula by multiplying it by 4 getting the following formula:

$35 - 10\sqrt{\alpha} + 2x_2 = 120 - 40\sqrt{\alpha}$

$x_2 = \frac{85-30\sqrt{\alpha}}{8}$ (26)

Consequently, $\alpha \in [0,1], x_1^* = \frac{7-2\sqrt{\alpha}}{4}\ , x_2^* = \frac{85-30\sqrt{\alpha}}{8}$.

Thus,

$$f^*(x_1^*, x_2^*) = \frac{(99 - 34\sqrt{\alpha})}{8} \in [\frac{65}{8}, \frac{99}{8}]$$

And the neutrosophic solution for (19) with respect to its membership function μ becomes the neutrosophic set

$\left\{f(x), \mu(x): f(x) \in \left[\frac{65}{8}, \frac{99}{8}\right],\ \mu(x) = [\frac{99-8x}{34}]^2\right\}$ (27)

On the other hand, if we solve (19) by means of its dual (i.e., the corresponding of its neutrosophic linear programming (3)), we should have

$$\left.\begin{array}{c} Min\ w = (10 + 5\beta)u_1 + (15 + 8\beta)u_2 + (20 + 10\beta)u_3 \\ s.t.\ \ 4u_1 + u_2 + 5u_3 \geq 1 \\ -u_1 + 2u_2 + 2u_3 \geq 1 \\ \beta \in [0,1],\quad u_i \geq 0 \end{array}\right\}$$

Which is equivalent to the following program

$$\left.\begin{array}{c} Min\ w = (15 - 5\alpha)u_1 + (23 - 8\alpha)u_2 + (30 - 10\alpha)u_3 \\ s.t.\ \ 4u_1 + u_2 + 5u_3 \geq 1 \\ -u_1 + 2u_2 + 2u_3 \geq 1 \\ \alpha \in [0,1],\quad u_i \geq 0 \end{array}\right\}$$

Which, when solved in the same way as parametric neutrosophic linear programming problem (2), has an optimal solution $u_1 = 0, u_2 = \frac{3}{8}, u_3 = \frac{1}{8}$.

Therefore,

$w^* = (15 - 5\alpha)u_1 + (23 - 8\alpha)u_2 + (30 - 10\alpha)u_3 = \frac{(99-34\alpha)}{8} \in \left[\frac{65}{8}, \frac{99}{8}\right], \alpha \in [0,1],$

And the corresponding neutrosophic solution with respect to its falsity membership function is the neutrosophic set

$\left\{w, \mu(x): w \in \left[\frac{65}{8}, \frac{99}{8}\right],\ \mu(x) = [\frac{99-8x}{34}]^2\right\}$ which coincides with (27).

4.2 Case Study (Hybrid Renewable Energy Systems in View of Neutrosophic Linear Programming)

As a result of the increasing demand for renewable energy, which is an inexhaustible and generally inexpensive source compared to traditional energy sources such as oil, natural gas, and coal...etc. Recently, an urgent need has emerged to integrate renewable energy sources in order to build up an economical hybrid energetic system in the case where each type of energy is only available as of specific units. The case study that we will shed the light on it was appeared in the essay [17] where the authors presented the capability of estimates of annual power production by combining photovoltaic panels with wind turbines (PV/Wind system) having specific capacities to meet energy demand in a specific site with the lowest cost. The purpose of this paper is different from the Zaatri and Allab [17] wherein they tried to estimate annual power production, also they formulated their problem as standard integer linear programming where the objective function to be minimized is the initial capital investment and where the decision variables are the numbers of units which should be pure integer numbers. Really, in this essay we aim to adapt the same problem in a different approach using neutrosophic linear programming (2), so our goal in this section is to show that the extension of the classical linear programming in uncertainty modeling gives more analytical information to be studied, the field of the power production of PV/Wind system with respect to uncertainties is unfathomable field and there are little discussions in the literature on it. It is well known that the intermittency is a part of Indeterminacy which is in between: interruption and non- interruption, so the neutrosophic theory will be a strong tool in getting and analyzing the optimal solutions.

Let $c_1 = 130\$$ represents the unit cost of a photovoltaic panel, $c_2 = 100\$$ is the reduced cost of the type of wind turbine. The investment for capital cost of the hybrid system which may involve the number of photovoltaic panels (N_1), and the number of wind turbine (N_2). This capital cost which is the objective function Z_T has to be minimized, therefore: -

$$\left.\begin{array}{l} MinZ_T = 130\, n_1 + 100\, n_2 \\ s.t.\;\; 66\, n_1 + 84n_2 \geq 3000 \\ \qquad\quad n_2 \geq 6 \\ \qquad\quad n_1, n_2 \;\geq 0 \end{array}\right\} \tag{28}$$

The following solution depends on the neutrosophic linear programming (2), the membership functions related to the two constraints of the program (28) are:

$\mu_1(\, 66\, n_1 + 84n_2, 3000) \geq \alpha$

$\mu_2(\, n_2, 6) \geq \alpha$

Where, $d_1 = 1500\; and\; d_2 = 3$ are the admissible violations of these constraints,

$$\mu_1(\, 66\, n_1 + 84n_2, 3000) = \frac{(4500-66n_1-84n_2)^2}{2250000} = \alpha, \tag{29}$$

$$\mu_2(\, n_2, 6) = \frac{(9-n_2)^2}{9} = \alpha, \tag{30}$$

The formula (29) implies to $4500 - 66n_1 - 84n_2 = 1500\sqrt{\alpha},$

$$4500 - 1500\sqrt{\alpha} = 66n_1 + 84n_2 \tag{31}$$

While the formula (30) implies to

$$9 - n_2 = 3\sqrt{\alpha} \;\rightarrow n_2 = 9 - 3\sqrt{\alpha}, \tag{32}$$

As $\alpha \in [0,1] \rightarrow n_2 \in [6,9]$

Substituting (32) in (31), $4500 - 1500\sqrt{\alpha} = 66n_1 + 84\,(9 - 3\sqrt{\alpha}),\; 4500 - 1500\sqrt{\alpha} = 66n_1 + 756 - 252\sqrt{\alpha} \;\Longrightarrow 66n_1 = 3744 - 1248\,\sqrt{\alpha},$

$$n_1 = \frac{3744}{66} - \frac{1248}{66}\sqrt{\alpha},\;\; \alpha \in [0,1] \tag{33}$$

$$n_1 \in \left[\frac{2496}{66}, \frac{3744}{66}\right] = [37.8, 56.7],$$

As n_1 *and* n_2 should be pure integer numbers, we will approximate the interval $[37.8, 56.7] \cong [38, 57]$, $n_1 \in [38,57]$.

The optimal value for the objective function Z_T

$$Z_T^* = 130\left(\frac{3744}{66} - \frac{1248}{66}\sqrt{\alpha}\right) + 100\left(9 - 3\sqrt{\alpha}\right) = 8274.5 - 2758.18\sqrt{\alpha}$$

$\alpha \in [0,1] \rightarrow Z_T^* \in [5517\$, 8275\$]$.

5 Conclusion

In this article, the classical linear programming has been redefined for the new type of neutrosophic linear programming with respect to its membership function, indeterminacy membership function, and non-membership function, with neutrosophic less than or equal in its constraint. Three new definitions have been posited, and two propositions were presented and proved. Two numerical examples were necessary to illustrate the theoretical direction practically.

Acknowledgement: This research is supported by the Neutrosophic Science International Association (NSIA) in both of its headquarter in New Mexico University and its Iraqi branch at Telafer University, for more details about (NSIA) see the URL http://neutrosophicassociation.org/ .

Funding Statement: This research received no external funding.

Conflicts of Interest: The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Reference

[1] D. Dubois, H. Prade. ''Fuzzy Sets and Systems. Theory and Applications''. Academic Press (1980).

[2] F. Smarandache, (1995). ''Neutrosophic Logic and Set''.

[3] F. Smarandache, (1998), Neutrosophy. Neutrosophic Probability, Set, and Logic, Amer. Res. Press, Rehoboth, USA, 105.

[4] F. Smarandache "Law of Included Multiple-Middle & Principle of Dynamic Neutrosophic Opposition". EuropaNova & Education Publisher Brussels-Columbus, 2014.

[5] F. Smarandache, "Unmatter as a Consequence of the Law of Included Multiple-Middles", Annual Meeting of the APS Four Corners Section, Vol. 60, Number 11, Friday–Saturday, October 16–17, 2015; Tempe, Arizona, http://meetings.aps.org/Meeting/4CF15/Session/F1.66

[6] F. Smarandache, H. E. Khalid, A. K. Essa, M. Ali, "The Concept of Neutrosophic Less Than or Equal To: A New Insight in Unconstrained Geometric Programming", Critical Review, Volume XII, 2016, pp. 72-80.
[7] F. Smarandache, Huda E. Khalid "Neutrosophic Precalculus and Neutrosophic Calculus". Second enlarged edition, Pons asbl 5, Quai du Batelage, Brussels, Belgium, European Union, 2018.
[8] F. Smarandache, H. E. Khalid, A. K. Essa, "Neutrosophic Logic: The Revolutionary Logic in Science and Philosophy", Proceedings of the National Symposium, EuropaNova, Brussels, 2018.
[9] H. E. Khalid, "An Original Notion to Find Maximal Solution in the Fuzzy Neutrosophic Relation Equations (FNRE) with Geometric Programming (GP)", Neutrosophic Sets and Systems, vol. 7, 2015, pp. 3-7.
[10] H. E. Khalid, "The Novel Attempt for Finding Minimum Solution in Fuzzy Neutrosophic Relational Geometric Programming (FNRGP) with (max, min) Composition", Neutrosophic Sets and Systems, vol. 11, 2016, pp. 107-111.
[11] H. E. Khalid, F. Smarandache, A. K. Essa, (2018). The Basic Notions for (over, off, under) Neutrosophic Geometric Programming Problems. Neutrosophic Sets and Systems, 22, 50-62.
[12] H. E. Khalid, (2020). Geometric Programming Dealt with a Neutrosophic Relational Equations Under the ($\boldsymbol{max - min}$) Operation. Neutrosophic Sets in Decision Analysis and Operations Research, chapter four. IGI Global Publishing House.
[13] H. E. Khalid, "Neutrosophic Geometric Programming (NGP) with (max-product) Operator, An Innovative Model", Neutrosophic Sets and Systems, vol. 32, 2020.
[14] H. E. Khalid, F. Smarandache, A. K. Essa, (2016). A Neutrosophic Binomial Factorial Theorem with their Refrains. Neutrosophic Sets and Systems, 14, 50-62.
[15] H. Tanaka, K. Asai,. ''Fuzzy linear programming problems with fuzzy numbers'', Fuzzy sets and systems 13, (1984) , 108-110.
[16] V. Kandasamy, F. Smarandache, "Fuzzy Relational Maps and Neutrosophic Relational Maps", American Research Press, Rehoboth,2004.
[17] Zaatri A., Allab K., (2012), Analysis of PV/Wind systems by integer linear programming. Revue des Energies Renouvelables Vol. 15 N°1 (2012) 111 – 120.

Received: May 2, 2021. Accepted: October 2, 2021

Neutrosophic Set Approach to Study the Characteristic Behavior of Left Almost Rings

Inayatur Rehman[1], Asima Razzaque[2,*] and Muhammad Iftikhar Faraz[3]

[1]Department of Mathematics and Sciences, College of Arts and Applied Sciences, Dhofar University, Salalah, Oman

[1]irehman@du.edu.om

[2]Department of Basic Science, Deanship of Preparatory year, King Faisal University, Hofuf 31982, Al Hasa, Saudi Arabia

[2]arazzaque@kfu.edu.sa

[3]Department of Mechanical Engineering, College of Engineering, King Faisal University, Hofuf 31982, Al Hasa, Saudi Arabia

[3]mfaraz@kfu.edu.sa

* Correspondence: arazzaque@kfu.edu.sa; Tel.: (+966562744993)

Abstract: In this paper, the concept of Neutrosophic LA-rings is introduced. Furthermore, we investigate their algebraic structures. We discuss various types of ideals and establish a number of results to better understand the characteristic behavior of Neutrosophic LA-rings. In addition, we investigate the properties of the Neutrosophic M-system, Neutrosophic P-system, and Neutrosophic I-system in order to characterize the Neutrosophic LA-ring.

Keywords: Neutrosophic Sets; Neutrosophic LA-rings; Neutrosophic ideals; Neutrosophic M-systems; Neutrosophic P-systems and Neutrosophic I-systems.

---

1. Introduction

Different researchers have defined algebraic structures which were based on the crisp set. But the real-life problems could not be solved by crisp set theory. The crisp set deals with yes or no only and it never tells about in between yes and no. In 1965, Zadeh [1] introduced a fuzzy set theory to address the vagueness of various real-life problems. The fuzzy sets deals with membership in between 0 and 1. Later, Atanassov [2] in 1986, initiated intuitionistic fuzzy set. However, these theories have remained unsuccessful in finding a solution to many real-life mathematical challenges.

In 1999, Smarandache [3] gave the notion of Neutrosophic set. Nowadays, Neutrosophic set attains more attention of researchers due to its characteristic behavior to solve the indeterminate situations in the different fields of life. In 2006, Smarandache et al., [4] were the first ones who applied the concept of Neutrosophic sets on some algebraic structure and in their work, they introduced the Neutrosophic rings. Later, in 2011 Agboola et al., [5], discussed Neutrosophic rings-I. Neutrosophic groups and Neutrosophic sub-groups were introduced in 2012 by Agboola et al., [6]. Ali et al., [7-10] have used Neutrosophic set approach for different algebraic structures. In 2016, Khan et al.,[11] briefly discussed the characterization of Neutrosophic left almost semigroups.

The tremendous application of Neutrosophic sets is the main motivator for us to work in this field. Intuitively, Neutrosophic sets are gaining popularity among researchers. To investigate the application aspect of Neutrosophic sets, readers are directed to the most recent research work of Abdel Basset et al., [15-19], as well as [12-14].

LA-rings is generalized form of the commutative rings. LA-rings is non-commutative and non-associative algebraic structure. In recent times, a lot of research work has been done by different researches on this area of study. No doubt, LA-rings has a remarkable contribution in the development of non-associative theory in the current decade. Shah, Rehman, Asima and many other researchers have done noteworthy work in this ring structure. And they have published articles. The readers are referred to study [20-29] for comprehensive study of LA-rings.

We used Neutrosophic set approach to give the notion of Neutrosophic LA-rings. This may be a useful contribution to the non-associative field of mathematics. It may provide a new direction for future researchers to extend the non-associative area of mathematics. We discussed characteristic properties of substructures of Neutrosophic LA-rings. We gave the concept of different type of Neutrosophic ideals. We defined Neutrosophic prime ideals, Neutrosophic quasi ideals and Neutrosophic bi-ideals and established some results. One of the main results is: If $\boldsymbol{e}$ the left identity in Neutrosophic LA-ring $\boldsymbol{N(LR)}$, then $\boldsymbol{N(LR)}$ is fully Neutrosophic prime iff set ideal$(\boldsymbol{N(LR)})$ becomes totally ordered under inclusion and every ideal becomes idempotent. In last section, we discussed the characterizations of neutrosophic LA-ring by exploring the Neutrosophic $\boldsymbol{M}$-system, Neutrosophic $\boldsymbol{P}$-system and Neutrosophic $\boldsymbol{I}$-system. It is shown that: If $\boldsymbol{e}$ the left identity in $\boldsymbol{N(LR)}$, then a neutrosophic left ideal $\boldsymbol{N(LI)}$ is neutrosophic quasi-prime iff $\boldsymbol{N(LR)} \setminus \boldsymbol{N(LI)}$ is neutrosophic $\boldsymbol{M}$-system. Also, a relation is developed between neutrosophic $\boldsymbol{M}$-system and neutrosophic $\boldsymbol{P}$-system i.e., In a neutrosophic LA-ring $\boldsymbol{N(LR)}$, every neutrosophic $\boldsymbol{M}$-system is a neutrosophic $\boldsymbol{P}$-system.

2. Neutrosophic LA-rings

As preliminary, we recall the following definitions from refrences [3], [24] and [25].

Definition 2.1. [24] Let R be a set with at least two elements and two binary operations `+' and `·' defined on R. Suppose $(R, +)$ is an LA-group and $(R, .)$ is an LA-semigroup satisfying both left and right distributive laws: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$. Then $(R, +, .)$ is called an LA-ring.

Definition 2.2. [24] Let $(R, +, .)$ be an LA-ring. If S is a non-empty subset of R and S is itself an LA-ring under the binary operation induced by R, then S is called an LA-subring of R.

Definition 2.3. [25] If A is an LA-subring of an LA-ring $(R, +, .)$, then A is called a left ideal if $RA \subseteq A$. Right ideal and two sided ideal are defined in the usual manner.

Definition 2.4. [25] A nonempty subset S of an LA-ring R is called an M-system if for $a, b \in S$, there exists r in R such that $a(rb) \in S$.

Definition 2.5. [25] A nonempty subset Q of an LA-ring R with left identity e is called P-system if for all $a \in Q$, there exists $r \in R$ such that $a(ra) \in Q$.

Definition 2.6. [25] A nonempty subset S of an LA-ring R with left identity e is called an I-system if for all $a, b \in S$, $(\langle a\rangle \cap \langle b\rangle) \cap S \neq \varphi$.

Definition 2.7. [3] A Neurosophics set is define as $A = \{\langle x, T(x), I(x), F(x)\rangle : x \in X\}$, where X is a universe of discoveries and A is characterized by a truth-membership function $T: X \to \left]0^-, 1^+\right[$, an indeterminacy-membership function $I: X \to \left]0^-, 1^+\right[$ and a falsity-membership function $F: X \to \left]0^-, 1^+\right[$ and $0 \leq T(x) + I(x) + F(x) \leq 3$.

We initaiate our work with the following definition.

Definition 2.8. If R is a LA-ring, I is a neutrosophic element with the property $I^2 = I$. Then a non-empty set $\langle R \cup I \rangle = \{r + sI : r, s \in R\}$ under the "$\boxplus$" and "$\boxdot$" is a Neutrosophic LA- ring if:

i) $(\langle R \cup I \rangle, \boxplus)$ is Left Almost group

ii) $(\langle R \cup I \rangle, \boxdot)$ is Left Almost semigroup

iii) $\boxdot$ is distributive over $\boxplus$ from both sides

Throughout this paper we denote Neutrosophic Left Almost ring by $N(LR)$.

Example 2.9. Following are the Cayley tables (1 and 2) for an LA-ring $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ under the binary operations '$+$' and '.'

Cayley Table 1

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	0	3	1	6	4	7	5
2	1	3	0	2	5	7	4	6
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	6	4	7	5	2	0	3	1
6	5	7	4	6	1	3	0	2
7	7	6	5	4	3	2	1	0

Cayley Table 2

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	4	4	0	0	4	4	0
2	0	4	4	0	0	4	4	0
3	0	0	0	0	0	0	0	0
4	0	3	3	0	0	3	3	0
5	0	7	7	0	0	7	7	0
6	0	7	7	0	0	7	7	0
7	0	3	3	0	0	3	3	0

Then $N(LR) = \langle R \cup I \rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 0I, 1I, 2I, 3I, 4I, 5I, 6I, 7I\}$ becomes neutrosophic LA-ring under "$\boxplus$" and "$\boxdot$" as defined in Cayley tables (3 and 4):

Cayley Table 3

⊞	0	1	2	3	4	5	6	7	0I	1I	2I	3I	4I	5I	6I	7I
0	0	1	2	3	4	5	6	7	0I	1I	2I	3I	4I	5I	6I	7I
1	2	0	3	1	6	4	7	5	2I	0I	3I	1I	6I	4I	7I	5I
2	1	3	0	2	5	7	4	6	1I	3I	0I	2I	5I	7I	4I	6I
3	3	2	1	0	7	6	5	4	3I	2I	1I	0I	7I	6I	5I	4I
4	4	5	6	7	0	1	2	3	4I	5I	6I	7I	0I	1I	2I	3I
5	6	4	7	5	2	0	3	1	6I	4I	7I	5I	2I	0I	3I	1I
6	5	7	4	6	1	3	0	2	5I	7I	4I	6I	1I	3I	0I	2I
7	7	6	5	4	3	2	1	0	7I	6I	5I	4I	3I	2I	1I	0I
0I	0I	1I	2I	3I	4I	5I	6I	7I	0I	1I	2I	3I	4I	5I	6I	7I
1I	2I	0I	3I	1I	6I	4I	7I	5I	2I	0I	3I	1I	6I	4I	7I	5I
2I	1I	3I	0I	2I	5I	7I	4I	6I	1I	3I	0I	2I	5I	7I	4I	6I
3I	3I	2I	1I	0I	7I	6I	5I	4I	3I	2I	1I	0I	7I	6I	5I	4I
4I	4I	5I	6I	7I	0I	1I	2I	3I	4I	5I	6I	7I	0I	1I	2I	3I
5I	6I	4I	7I	5I	2I	0I	3I	1I	6I	4I	7I	5I	2I	0I	3I	1I
6I	5I	7I	4I	6I	1I	3I	0I	2I	5I	7I	4I	6I	1I	3I	0I	2I
7I	7I	6I	5I	4I	3I	2I	1I	0I	7I	6I	5I	4I	3I	2I	1I	0I

.

Cayley Table 4

⊡	0	1	2	3	4	5	6	7	0I	1I	2I	3I	4I	5I	6I	7I
0	0	0	0	0	0	0	0	0	0I	0I	0I	0I	0I	0I	0I	0I
1	0	4	4	0	0	4	4	0	0I	4I	4I	0I	0I	4I	4I	0I
2	0	4	4	0	0	4	4	0	0I	4I	4I	0I	0I	4I	4I	0I
3	0	0	0	0	0	0	0	0	0I	0I	0I	0I	0I	0I	0I	0I
4	0	3	3	0	0	3	3	0	0I	3I	3I	0I	0I	3I	3I	0I
5	0	7	7	0	0	7	7	0	0I	7I	7I	0I	0I	7I	7I	0I
6	0	7	7	0	0	7	7	0	0I	7I	7I	0I	0I	7I	7I	0I
7	0	3	3	0	0	3	3	0	0I	3I	3I	0I	0I	3I	3I	0I
0I	0I	0I	0I	0I	0I	0I	0I	0I	0I	0I	0I	0I	0I	0I	0I	0I
1I	0I	4I	4I	0I	0I	4I	4I	0I	0I	4I	4I	0I	0I	4I	4I	0I
2I	0I	4I	4I	0I	0I	4I	4I	0I	0I	4I	4I	0I	0I	4I	4I	0I

$3I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$	$0I$
$4I$	$0I$	$3I$	$3I$	$0I$	$0I$	$3I$	$3I$	$0I$	$0I$	$3I$	$3I$	$0I$	$0I$	$3I$	$3I$	$0I$
$5I$	$0I$	$7I$	$7I$	$0I$	$0I$	$7I$	$7I$	$0I$	$0I$	$7I$	$7I$	$0I$	$0I$	$7I$	$7I$	$0I$
$6I$	$0I$	$7I$	$7I$	$0I$	$0I$	$7I$	$7I$	$0I$	$0I$	$7I$	$7I$	$0I$	$0I$	$7I$	$7I$	$0I$
$7I$	$0I$	$3I$	$3I$	$0I$	$0I$	$3I$	$3I$	$0I$	$0I$	$3I$	$3I$	$0I$	$0I$	$3I$	$3I$	$0I$

Definition 2.10. Let $N(LR)$ be a neutrosophic LA-ring under the binary operations $\boxplus$ and $\boxdot$. A non-empty proper subset $N(SLR)$ of $N(LR)$ is said to be a neutrosophic subLA-ring if $N(SLR)$ is itself a neutrosophic LA-ring under " $\boxplus$ " and " $\boxdot$ " defined in $N(LR)$.

Lemma 2.11. Let $N(LR)$ be a Neutrosophic LA-ring. Then the proper subset $N(SLR)$ of $N(LR)$ is a Neutrosophic subLA-ring iff, every $(r' + s'I)$, $(p' + q'I) \in N(SLR)$ satisfies the following conditions:

(i) $(r' + s'I) \boxplus (p' + q'I)$ belongs $N(SLR)$
(ii) $(r' + s'I) \boxdot (p' + q'I)$ belongs $N(SLR)$

Proof. If $N(SLR)$ is neutrosophic subLA-ring, then it is clear from definition that $(N(SLR), \boxplus))$ becomes LA-group as well as $(N(SLR), \boxdot)$ becomes LA-semigroup. Consequently, the closure property holds for $N(SLR)$. Hence (i) and (ii) hold.

Conversely, suppose that (i) and (ii) is true for all $(r' + s'I)$, $(p' + q'I) \in N(SLR)$. Since the binary operations " $\boxplus$ " and " $\boxdot$ " are closed, so $(N(SLR), \boxplus))$ being the subset of $N(LR)$ will be LA-group, likewise $(N(SLR), \boxdot))$ will be LA-semigroup. Moreover, inheritably $\boxdot$ is distributive over $\boxplus$ from both sides. Hence, $N(SLR)$ is a neutrosophic subLA-ring.

Lemma 2.12. If $\{(N(SLR))_i, i \in J\}$ is the collection of neutrosophic subLA-rings of $N(LR)$. Then the intersection of this collection is either empty or again a neutrosophic subLA-ring.

Proof. Let $\{(N(SLR))_i, i \in J\}$ be a collection of neutrosophic subLA-rings of $N(LR)$. Assume that $\cap (N(SLR))_i$ is not empty. Let $(r' + s'I)$, $(p' + q'I) \in \cap(N(SLR))_i$. This implies $(r' + s'I) \in (N(SLR))_i$ and $(p' + q'I) \in (N(SLR))_i$ where $i \in J$. Since $(N(SLR))_i$ is the collection of neutrosophic subLA-rings. Therefore, each $(N(SLR))_i$, $\boxplus)$ will be LA-group, $(N(SLR))_i$, $\boxdot)$ will be LA-semigroup. Also $\boxdot$ is distributive over $\boxplus$ from both sides. Consequently, $(r' + s'I) \boxplus (p' + q'I) \in (N(SLR))_i$ for all $i \in J$ and likewise $(r' + s'I) \boxdot (p' + q'I) \in (N(SLR))_i$ for all $i \in J$. Therefore, $(r' + s'I) \boxplus (p' + q'I)$ and $(r' + s'I) \boxdot (p' + q'I) \in \cap(N(SLR))_i$ for all $i \in J$.

Definition 2.13. If $e \in N(LR)$, then e is left identity if $e \boxdot N(LR) = N(LR)$.

Example 2.14. The following Cayley tables (5 and 6) form a neutrosophic LA-ring $N(LR) = \langle R \cup I \rangle = \{0,1,2,3,4,5,6,7,8,0I,1I,2I,3I,4I,5I,6I,7I,8I\}$ and it can be easily observed that the element $7 \in N(LR)$ is the left identity.

Cayley Table 5

$\boxplus$	0	1	2	3	4	5	6	7	8	$0I$	$1I$	$2I$	$3I$	$4I$	$5I$	$6I$	$7I$	$8I$
0	3	4	6	8	7	2	5	1	0	$3I$	$4I$	$6I$	$8I$	$7I$	$2I$	$5I$	$1I$	$0I$
1	2	3	7	6	8	4	1	0	5	$2I$	$3I$	$7I$	$6I$	$8I$	$4I$	$1I$	$0I$	$5I$
2	1	5	3	4	2	0	8	6	7	$1I$	$5I$	$3I$	$4I$	$2I$	$0I$	$8I$	$6I$	$7I$
3	0	1	2	3	4	5	6	7	8	$0I$	$1I$	$2I$	$3I$	$4I$	$5I$	$6I$	$7I$	$8I$
4	5	0	4	2	3	1	7	8	6	$5I$	$0I$	$4I$	$2I$	$3I$	$1I$	$7I$	$8I$	$6I$
5	4	2	8	7	6	3	0	5	1	$4I$	$2I$	$8I$	$7I$	$6I$	$3I$	$0I$	$5I$	$1I$
6	7	6	0	1	5	8	3	2	4	$7I$	$6I$	$0I$	$1I$	$5I$	$8I$	$3I$	$2I$	$4I$
7	6	8	1	5	0	7	4	3	2	$6I$	$8I$	$1I$	$5I$	$0I$	$7I$	$4I$	$3I$	$2I$
8	8	7	5	0	1	6	2	4	3	$8I$	$7I$	$5I$	$0I$	$1I$	$6I$	$2I$	$4I$	$3I$

0I	3I	4I	6I	8I	7I	2I	5I	1I	0I	3I	4I	6I	8I	7I	2I	5I	1I	0I
1I	2I	3I	7I	6I	8I	4I	1I	0I	5I	2I	3I	7I	6I	8I	4I	1I	0I	5I
2I	1I	5I	3I	4I	2I	0I	8I	6I	7I	1I	5I	3I	4I	2I	0I	8I	6I	7I
3I	0I	1I	2I	3I	4I	5I	6I	7I	8I	0I	1I	2I	3I	4I	5I	6I	7I	8I
4I	5I	0I	4I	2I	3I	1I	7I	8I	6I	5I	0I	4I	2I	3I	1I	7I	8I	6I
5I	4I	2I	8I	7I	6I	3I	0I	5I	1I	4I	2I	8I	7I	6I	3I	0I	5I	1I
6I	7I	6I	0I	1I	5I	8I	3I	2I	4I	7I	6I	0I	1I	5I	8I	3I	2I	4I
7I	6I	8I	1I	5I	0I	7I	4I	3I	2I	6I	8I	1I	5I	0I	7I	4I	3I	2I
8I	8I	7I	5I	0I	1I	6I	2I	4I	3I	8I	7I	5I	0I	1I	6I	2I	4I	3I

Cayley Table 6

$\boxdot$	0	1	2	3	4	5	6	7	8	0I	1I	2I	3I	4I	5I	6I	7I	8I
0	3	1	6	3	1	6	6	1	3	3I	1I	6I	3I	1I	6I	6I	1I	3I
1	0	3	0	3	8	8	3	0	8	0I	3I	0I	3I	8I	8I	3I	0I	8I
2	8	1	5	3	7	2	6	4	0	8I	1I	5I	3I	7I	2I	6I	4I	0I
3	3	3	3	3	3	3	3	3	3	3I	3I	3I	3I	3I	3I	3I	3I	3I
4	0	6	7	3	5	4	1	2	8	0I	6I	7I	3I	5I	4I	1I	2I	8I
5	8	6	4	3	2	7	1	5	0	8I	6I	4I	3I	2I	7I	1I	5I	0I
6	8	3	8	3	0	0	3	8	0	8I	3I	8I	3I	0I	0I	3I	8I	0I
7	0	1	2	3	4	5	6	7	8	0I	1I	2I	3I	4I	5I	6I	7I	8I
8	3	6	1	3	6	1	1	6	3	3I	6I	1I	3I	6I	1I	1I	6I	3I
0I	3I	1I	6I	3I	1I	6I	6I	1I	3I	3I	1I	6I	3I	1I	6I	6I	1I	3I
1I	0I	3I	0I	3I	8I	8I	3I	0I	8I	0I	3I	0I	3I	8I	8I	3I	0I	8I
2I	8I	1I	5I	3I	7I	2I	6I	4I	0I	8I	1I	5I	3I	7I	2I	6I	4I	0I
3I	3I	3I	3I	3I	3I	3I	3I	3I	3I	3I	3I	3I	3I	3I	3I	3I	3I	3I
4I	0I	6I	7I	3I	5I	4I	1I	2I	8I	0I	6I	7I	3I	5I	4I	1I	2I	8I
5I	8I	6I	4I	3I	2I	7I	1I	5I	0I	8I	6I	4I	3I	2I	7I	1I	5I	0I
6I	8I	3I	8I	3I	0I	0I	3I	8I	0I	8I	3I	8I	3I	0I	0I	3I	8I	0I
7I	0I	1I	2I	3I	4I	5I	6I	7I	8I	0I	1I	2I	3I	4I	5I	6I	7I	8I
8I	3I	6I	1I	3I	6I	1I	1I	6I	3I	3I	6I	1I	3I	6I	1I	1I	6I	3I

3. Neutrosophic Ideals

Definition 3.1. A neutrosophic subLA-ring $N(SLR)$ of $N(LR)$ is known as a neutrosophic left ideal if $N(LR) \boxdot N(SLR) \subseteq N(SLR)$. Likewise, the right ideal and two sided ideal of $N(LR)$ can be easily defined.

We denote neutrosophic left ideal by $N(LI)$, neutrosophic right ideal by $N(RI)$ and two sided neutrosophic ideal will be denoted by $N(I)$.

Lemma 3.2. If e is the left identity in neutrosophic LA-ring $N(LR)$, then $N(RI)$ will be neutrosophic left ideal.

Proof. Suppose $r' + s'I \in N(LR)$, $m' + n'I \in N(RI)$. Then $r' + s'I \boxdot (m' + n'I) = (e \boxdot (r' + s'I)) \boxdot (m' + n'I)$=$((m' + n'I) \boxdot (r' + s'I)) \boxdot e \in N(RI)$. Therefore, $N(RI)$ becomes a neutrosophic left ideal also.

Remark 3.3. From **Lemma 3.2**, it is concluded a neutrosophic LA-rings having e the left identity, the neutrosophic ideal means the neutrosophic right ideal.

Proposition 3.4. Let $N(LR)$ a neutrosophic LA-ring having left identity. Then:

(i) $N(LR) \boxdot N(LI) = N(LI)$, $N(LI)$ neutrosophic left ideal of $N(LR)$.

(ii) $N(RI) \boxdot N(LR) = N(RI)$, $N(RI)$ neutrosophic right ideal of $N(LR)$.

Proof. (i) By definition, if $N(LI)$ is neutrosophic left ideal of $N(LR)$, then $N(LR) \boxdot N(LI) \subseteq N(LI)$. Let $p' + q'I \in N(LI)$. Then $p' + q'I = e \boxdot (p' + q'I) \in N(LR) \boxdot N(LI)$. Consequently, $N(LI) \subseteq N(LR) \boxdot N(LI)$ and hence $N(LR) \boxdot N(LI) = N(LI)$.

(ii) By definition, if $N(RI)$ is neutrosophic right ideal of $N(LR)$, then $N(RI) \boxdot N(LR) \subseteq N(RI)$. Let $m' + n'I \in N(RI)$. Then

$$\begin{aligned} m' + n'I &= e \boxdot (m' + n'I) \\ &= (e \boxdot e) \boxdot (m' + n'I) \\ &= ((m' + n'I) \boxdot e) \boxdot e \\ &\in (N(RI) \boxdot N(LR)) \boxdot N(LR) \\ &\subseteq N(RI) \boxdot N(LR). \end{aligned}$$

This implies $N(RI) \subseteq N(RI) \boxdot N(LR)$. Thus, $N(RI) \boxdot N(LR) = N(RI)$.

Lemma 3.5. If e is a left identity and $N(RI)$ is the neutrosophic right ideal of $N(LR)$, then $(N(RI))^2$ is a neutrosophic ideal of $N(LR)$.

Proof. If an element $l' + k'I \in \left(N(RI)\right)^2$, then $l' + k'I = (m' + n'I) \boxdot (p' + q'I)$, where $(m' + n'I)$, $(p' + q'I) \in N(RI)$. Let $(r' + s'I)$ be any element of $N(LR)$.

Now consider
$$\begin{aligned} (l' + k'I) \boxdot (r' + s'I) &= \left((m' + n'I) \boxdot (p' + q'I)\right) \boxdot (r' + s'I) \\ &= \left((r' + s'I) \boxdot (p' + q'I)\right) \boxdot (m' + n'I) \in N(RI) \boxdot N(RI) \\ &= \left(N(RI)\right)^2. \end{aligned}$$

This means $\left(N(RI)\right)^2$ is neutrosophic right ideal. Therefore, from **Lemma 3.2**, $\left(N(RI)\right)^2$becomes neutrosophic left ideal. Thus $\left(N(RI)\right)^2$ is neutrosophic ideal.

Remark 3.6. It is interesting to note that in a neutrosophic LA-ring $N(LR)$ having left identity, $\left(N(LI)\right)^2$ becomes neutrosophic ideal, where $N(LI)$ is neutrosophic ideal.

Lemma 3.7. If $N(LR)$ is a neutrosophic LA-rings having left identity. Let $N(I')$ is proper neutrosophic ideal of $N(LR)$. Then the left identity e does not belong to $N(I')$.

Proof. Contrarily, let $e \in N(I')$ and $r' + s'I \in N(LR)$. Now consider

$$\begin{aligned} r' + s'I &= e \boxdot (r' + s'I) \\ &\in N(I') \boxdot N(LR) \\ &\subseteq N(I'). \end{aligned}$$

This implies $N(LR) \subseteq N(I')$. But $N(I') \subseteq N(LR)$. This means $N(I') = N(LR)$. Hence a contradiction. Thus $e \notin N(I')$.

Definition 3.8. $N(PI)$ is neutrosophic ideal of $N(LR)$. $N(PI)$ is called neutrosophic prime ideal iff for any neutrosophic ideals $N(AI)$ and $N(BI)$, $N(AI) \boxdot N(BI) \subseteq N(PI)$ then either $N(AI) \subseteq N(PI)$ or $N(BI) \subseteq N(PI)$. $N(P)$ is called neutrosophic semi-prime if $N(I')^2 \subseteq N(PI)$ implies that $N(I') \subseteq N(PI)$, where $N(I')$ is any neutrosophic ideal of $N(LR)$.

If each neutrosophic ideal of $N(LR)$ is neutrosophic prime ideal, then $N(LR)$ is called fully neutrosophic prime and if all the neutrosophic ideals are neutrosophic semi-prime ideals than $N(LR)$ is called fully neutrosophic semi-prime.

Definition 3.9. If for all neutrosophic ideals $N(AI)$, $N(BI)$, either $N(AI) \subseteq N(BI)$ or $N(BI) \subseteq N(AI)$, then $N(LR)$ is called totally ordered under inclusion. It is symbolized by a set ideal$(N(LR))$.

Theorem 3.10. If e left identity in neutrosophic LA-rings $N(LR)$, then $N(LR)$ is fully neutrosophic prime iff set ideal$(N(LR))$ becomes totally ordered under inclusion and every ideal becomes idempotent.

Proof. Suppose $N(LR)$ is fully neutrosophic prime and $N(AI)$, $N(BI)$ be any neutrosophic ideals in $N(LR)$. Since $N(AI) \boxdot N(BI) \subseteq N(AI)$ and $N(AI) \boxdot N(BI) \subseteq N(BI)$, therefore $N(AI) \boxdot N(BI) \subseteq$

$N(AI) \cap N(BI)$. Since the intersection of neutrosophic prime ideals is prime. This implies that $N(AI) \cap N(BI)$ is prime and hence by definition, $N(AI) \subseteq N(AI) \cap N(BI)$ or $N(BI) \subseteq N(AI) \cap N(BI)$. This further implies that either $N(AI) \subseteq N(BI)$ or $N(BI) \subseteq N(AI)$. Thus set ideal$(N(LR))$ is totally ordered under the inclusion. Assume $N(I^{'})$ a neutrosophic ideal of $N(LR)$, where $N(LR)$ is fully neutrosophic prime. Then from **Lemma 3.5**, it is proved that $(N(I))^2$ is neutrosophic ideal in $N(LR)$, therefore $(N(I'))^2 \subseteq N(I')$. Also, $N(I') \subseteq (N(I'))^2$. Consequently, $(N(I'))^2 = N(I')$ this implies $N(I')$ is idempotent. Conversely, assume that set ideal$(N(LR))$ is totally ordered under the inclusion and each ideal becomes idempotent. Consider $N(UI)$, $N(VI)$ and $N(WI)$ be neutrosophic ideals in $N(LR)$. Let $N(UI) \boxdot N(VI) \subseteq N(WI)$ where $N(UI) \subseteq N(VI)$. As $N(UI)$ is an idempotent neutrosophic ideal in $N(LR)$, so $N(UI) = (N(UI))^2 = N(UI) \boxdot N(UI) \subseteq N(UI) \boxdot N(VI) \subseteq N(WI)$. Hence $N(VI) \subseteq N(WI)$. This $N(WI)$ is neutrosophic prime ideal. Similarly, on the same lines it can be proved that $N(UI)$ and $N(VI)$ are prime ideals in $N(LR)$. Hence $N(LR)$ is fully neutrosophic prime.

Definition 3.11. Let $N(LR)$ be a neutrosophic LA-ring. $N(QI)$ a non-empty subset is called neutrosophic quasi ideal if $N(QI) \boxdot N(LR) \cap N(LR) \boxdot N(QI) \subseteq N(QI)$.

Lemma 3.12. If $N(LR)$ is neutrosophic LA-ring. Let $N(RI)$, $N(LI)$ be the neutrosophic right and left ideal respectively. Then the intersection of $N(RI)$ and $N(LI)$ is a neutrosophic quasi ideal in $N(LR)$.

Proof. From the properties of neutrosophic right and left ideals it can be written that $N(LI) \cap N(RI) \subseteq N(RI)$ and $N(LI) \cap N(RI) \subseteq N(LI)$. Also $N(LR) \boxdot N(LI) \subseteq N(LI)$ and $N(RI) \boxdot N(LR) \subseteq N(RI)$. Now consider,

$$
\begin{aligned}
(N(LI) \cap N(RI)) \boxdot N(LR) \cap N(LR) \boxdot (N(LI) \cap N(RI)) \\
&\subseteq N(RI) \boxdot N(LR) \cap N(LR) \boxdot N(LI) \\
&\subseteq N(RI) \cap N(LI) \\
&= N(LI) \cap N(RI).
\end{aligned}
$$

Result proved.

Definition 3.13. Let $N(LR)$ be neutrosophic LA-rings. $N(BI)$ is neutrosophic generalized bi-ideal, if $(N(BI) \boxdot N(LR)) \boxdot N(BI) \subseteq N(BI)$. It is important to note that if the non-empty subset of $N(LR)$ is a neutrosophic subLA-ring then $N(B)$ is called neutrosophic bi-ideal of $N(LR)$.

Proposition 3.14. Let e be left identity in $N(LR)$. Then each idempotent neutrosophic quasi ideal $N(QI)$ becomes a neutrosophic bi-ideal in $N(LR)$.

Proof. $N(QI)$ being a neutrosophic quasi ideal is a neutrosophic subLA-ring. Consider,

$$
\begin{aligned}
(N(QI) \boxdot N(LR)) \boxdot N(QI) &\subseteq (N(QI) \boxdot N(LR)) \boxdot N(LR) \\
&= \big(N(LR) \boxdot N(LR)\big) \boxdot N(QI) \\
&= N(LR) \boxdot N(QI)
\end{aligned}
$$

Again consider,

$$
\begin{aligned}
\big(N(QI) \boxdot N(LR)\big) \boxdot N(QI) &\subseteq \big(N(LR) \boxdot N(LR)\big) \boxdot N(QI) \\
&= \big(N(LR) \boxdot N(LR)\big) \boxdot \big(N(QI) \boxdot N(QI)\big) \\
&= \big(N(QI) \boxdot N(QI)\big) \boxdot \big(N(LR) \boxdot N(QLR)\big) \\
&= N(QI) \boxdot N(LR).
\end{aligned}
$$

Therefore, $\big(N(QI) \boxdot N(LR)\big) \boxdot N(QI) \subseteq (N(QI) \boxdot N(LR)) \cap (N(LR) \boxdot N(QI)) \subseteq N(QI)$. Hence proved.

Theorem 3.15. $N(QI^{'})$, $N(QI^{''})$ be neutrosophic quasi ideals in neutrosophic LA-rings $N(LR)$. Then the intersection of $N(Q^{'}) \cap N(Q^{''})$ is empty or neutrosophic quasi ideal of $N(LR)$.

Proof. Consider $N(LR) \boxdot [N(QI^{'}) \cap N(QI^{''})] \cap [N(QI^{'}) \cap N(QI^{''})] \boxdot N(LR)$

$= [N(LR) \boxdot N(QI^{'}) \cap N(LR) \boxdot N(QI^{''})] \cap [N(QI^{'}) \boxdot N(LR) \cap N(QI^{''}) \boxdot N(LR)]$

$= [N(LR) \boxdot N(QI^{'}) \cap N(QI^{'}) \boxdot N(LR)] \cap [N(LR) \boxdot N(QI^{''}) \cap N(QI^{''}) \boxdot N(LR)]$

$\subseteq N(QI^{'}) \cap N(QI^{''})$.

This complete the result.

Remark 3.16. It can be concluded from **Theorem 3.15**, that the intersection of neutrosophic quasi ideals in $N(LR)$ is empty or neutrosophic quasi ideal.

4. Neutrosophic Systems in Neutrosophic LA-ring

In this section of paper, we explore Neutrosophic M-system, Neutrosophic P-system and Neutrosophic I-system in neutrosophic LA-rings.

Definition 4.1. For $m' + n'I, p' + q'I \in M'$, an element $r' + s'I \in N(LR)$ and if $m' + n'I \boxdot (r' + s'I \boxdot p' + q'I) \subseteq M'$, then M' is called a Neutrosophic M-system, where M' is non-empty subset of $N(LR)$.

Example 4.2. One can easily check that in a neutrosophic LA-rings having left identity, $(N(LR),\boxdot)$ being a neutrosophic LA-semigroup becomes a neutrosophic M-system.

Definition 4.3. A neutrosophic left ideal $N(LI)$ of neutrosophic LA-rings is neutrosophic quasi-prime if for any neutrosophic left ideals $S(LI)$ and $T(LI)$, $S(LI) \boxdot T(LI) \subseteq N(LI)$ gives either $S(LI) \subseteq N(LI)$ or $T(LI) \subseteq N(LI)$. While $N(LI)$ is said to be a neutrosophic quasi-semiprime if for any neutrosophic left ideal $S(LI)$, $(S(LI))^2 \subseteq N(LI) \Rightarrow S(LI) \subseteq N(LI)$.

Proposition 4.4. In a neutrosophic LA-ring $N(LR)$ having left identity, following claims are equivalent:

(i) $N(LI)$ is a neutrosophic quasi-prime.

(ii) $S(LI) \boxdot T(LI) = \langle S(LI) \boxdot T(LI) \rangle \subseteq N(LI)$ means either $S(LI) \subseteq N(LI)$ or $T(LI) \subseteq N(LI)$.

(iii) If $S(LI) \nsubseteq N(LI)$ and $T(LI) \nsubseteq N(LI)$, then $S(LI) \boxdot T(LI) \nsubseteq N(LI)$.

(iv) If $r' + s'I,\ l' + m'I \in N(LR)$ but $r' + s'I,\ l' + m'I \notin N(LI)$ then $\langle r' + s'I \rangle \boxdot \langle l' + m'I \rangle \nsubseteq N(LI)$, then either $r' + s'I \in N(LI)$ or $l' + m'I \in N(LI)$.

(v) If $r' + s'I,\ l' + m'I \in N(LR)$ such that $r' + s'I \boxdot (N(LR) \boxdot l' + m'I) \subseteq N(LI)$, then either $r' + s'I \in N(LI)$ or $l' + m'I \in N(LI)$.

Proof. (i)⟺(ii) Suppose $N(LI)$ is a neutrosophic quasi-prime. Then it is quite clear from definition that if $S(LI) \boxdot T(LI) = \langle S(LI) \boxdot T(LI) \rangle \subseteq N(LI)$, then either $S(LI) \subseteq N(LI)$or $T(LI) \subseteq N(LI)$. Converse can be proved directly.

(ii)⟺(iii) obvious from given information.

(i) ⟹ (iv) Assume $\langle r' + s'I \rangle \boxdot \langle l' + m'I \rangle \subseteq N(LI)$, this means $\langle r' + s'I \rangle \subseteq N(LI)$ or $\langle l' + m'I \rangle \subseteq N(LI)$, which further means $r' + s'I \in N(LI)$ or $l' + m'I \in N(LI)$.

(iv)⟹(ii) Assume $S(LI) \boxdot T(LI) \subseteq N(LI)$. If $r' + s'I \in S(LI)$ and $l' + m'I \in T(LI)$, then $\langle r' + s'I \rangle \boxdot \langle l' + m'I \rangle \subseteq N(LI)$. Thus from hypothesis $r' + s'I \in N(LI)$ or $l' + m'I \in N(LI)$. Hence either

$S(LI) \subseteq N(LI)$ or $T(LI) \subseteq N(LI)$.

(i)⟺(iv) Assume $r' + s'I \boxdot (N(LR) \boxdot l' + m''I) \subseteq N(LI)$, then $N(LR) \boxdot [r' + s'I \boxdot (N(LR) \boxdot l' + m'I)] \subseteq N(LR) \boxdot N(LI) \subseteq N(LI)$. Now applying medial law and paramedical law, we conclude that $N(LR) \boxdot [r' + s'I \boxdot (N(LR) \boxdot l' + m'I)] = (N(LR) \boxdot r' + s'I) \boxdot (N(LR) \boxdot l' + m'I) \subseteq N(LI)$. As $N(LR) \boxdot r' + s'I$ and $N(LR) \boxdot l' + m'I$ are neutrosophic left ideals, this means $r' + s'I \in N(LI)$ or $l' + m'I \in N(LI)$. Converse is trivial.

Theorem 4.5. If e is a left identity in $N(LR)$, then a neutrosophic left ideal $N(LI)$ is neutrosophic quasi-prime iff $N(LR)\backslash N(LI)$ is neutrosophic M-system.

Proof. Suppose $N(LI)$ is neutrosophic quasi-prime. Assume $r' + s'I,\ l' + m'I \in N(LR)\backslash N(LI)$. Which means $r' + s'I \notin N(LI)$ and $l' + m'I \notin N(LI)$. Therefore by **Proposition 4.4,** $r' + s'I \boxdot (N(LR) \boxdot l' + m'I) \nsubseteq N(LI)$. It means there is some element $(p' + q'I) \in N(LR)$ such that $r' + s'I \boxdot (p' + q'I \boxdot l' + m'I) \nsubseteq N(LI)$ which further implies $r' + s'I \boxdot (p' + q'I \boxdot l' + m'I) \subseteq N(LR)\backslash N(LI)$. Therefore $N(LR)\backslash N(LI)$ is a neutrosophic M-system. Conversely assume that $N(LR)\backslash N(LI)$ is a neutrosophic M-system. Let $r' + s'I \boxdot (N(LR) \boxdot l' + m'I) \subseteq N(LI)$ and $r' + s'I \notin N(LI)$ and $l' + m'I \notin N(LI)$. This means $r' + s'I,\ l' + m'I \in N(LR)\backslash N(LI)$. As by hypothesis $N(LR)\backslash N(LI)$ is a neutrosophic M-system, so $(p' + q'I) \in N(LR)$ and $r' + s'I \boxdot (p' + q'I \boxdot l' + m'I) \subseteq N(LR)\backslash N(LI)$. This implies $r' + s'I \boxdot (p' + q'I \boxdot l' + m'I) \nsubseteq N(LI)$, which is a contradiction. Thus $r' + s'I \in N(LI)$ or $l' + m'I \in N(LI)$. Hence $N(LI)$ is neutrosophic quasi-prime.

Definition 4.6. A subset $P'$of $N(LR)$ is a neutrosophic P-system, if for any $p' + q'I \in P'$, there is $r' + s'I \in N(LR)$ such that $p' + q'I \boxdot (r' + s'I \boxdot p' + q'I) \subseteq P'$.

Proposition 4.7. In a neutrosophic LA-ring $N(LR)$ having left identity, given claims are equivalent:

(i) $N(LI)$ is neutrosophic quasi-semiprime.

(ii) $(S(LI))^2 = \langle (S(LI))^2 \rangle \subseteq N(LI)$ implies that $S(LI) \subseteq N(LI)$, where $S(LI)$ is a neutrosophic left ideal.

(iii) For any neutrosophic left ideal $S(LI)$, if $S(LI) \nsubseteq N(LI)$, then $(S(LI))^2 \nsubseteq N(LI)$.

(iv) For any element $r' + s'I \in N(LR)$, if $\langle r' + s'I \rangle^2 \subseteq N(LI)$, then $r' + s'I \in N(LI)$.

(v) For any element $r' + s'I \in N(LR)$, $p' + q'I \boxdot (r' + s'I \boxdot p' + q'I) \subseteq N(LI) \Rightarrow r' + s'I \in N(LI)$.

Proof. (i)⟺(ii) ⟺(iii) obviously by definition true.

(i)⟹(iv). Let $\langle r' + s'I \rangle^2 \subseteq N(LI)$. From hypothesis as $N(LI)$ is quasi-semiprime, so $r' + s'I \subseteq N(LI)$ and it means $r' + s'I \in N(LI)$.

(i)⟹(iv). Assume $(S(LI))^2 = \langle (S(LI))^2 \rangle \subseteq N(LI)$. Let $r' + s'I \in N(LI)$. Then by given condition $\langle r' + s'I \rangle^2 \subseteq N(LI)$ and it means $r' + s'I \in N(LI)$. Hence $S(LI) \subseteq N(LI)$.

(i)⟺(v) Obvious.

Theorem 4.8. If e is a left identity in $N(LR)$, then a neutrosophic left ideal $N(LI)$ is neutrosophic quasi-semiprime iff $N(LR)\backslash N(LI)$ is neutrosophic P-system.

Proof. Suppose $N(LI)$ is neutrosophic quasi-semiprime. Let $p' + q'I \in N(LR)\backslash N(LI)$. On contrary, assume that $r' + s'I \in N(LR)$ and $p' + q'I \boxdot (r' + s'I \boxdot p' + q'I) \subseteq N(LR)\backslash N(LI)$. This means $p' + q'I \boxdot (r' + s'I \boxdot p' + q'I) \subseteq N(LI)$. But as $N(LI)$ is neutrosophic quasi-semiprime, so from **Proposition 4.4,** $p' + q'I \in N(LI)$ a contradiction arise. Therefore, $r' + s'I \in N(LR)$ and $p' + q'I \boxdot (r' + s'I \boxdot p' + q'I) \subseteq N(LR)\backslash N(LI)$. Thus $N(LR)\backslash N(LI)$ is a neutrosophic P-system. Now let for any element $p' + q'I \in N(LR) \backslash N(LI)$, there is $r' + s'I \in N(LR)$ and $p' + q'I \boxdot (r' + s'I \boxdot p' + q'I) \subseteq N(LR) \backslash N(LI)$. Suppose $p' + q'I \boxdot (N(LR) \boxdot p' + q'I) \subseteq N(LI)$. This means $r' + s'I \in N(LR)$ and $p' + q'I \boxdot (r' + s'I \boxdot p' + q'I) \subseteq N(LR) \backslash N(LI)$. This further means that $p' + q'I \in N(LI)$. Thus by **Proposition 4.7** $N(LI)$ is a neutrosophic quasi-semiprime.

Lemma 4.9. In a neutrosophic LA-ring $N(LR)$, every neutrosophic M-system is a neutrosophic P-system.

Proof. Obvious from definition.

Definition 4.10. In a neutrosophic LA-ring $N(LR)$, a neutrosophic ideal $N(I')$ is called strongly irreducible iff for any neutrosophic ideals $S(I')$ and $T(I')$, if $S(I') \cap T(I') \subseteq N(I')$ implies that either $S(I') \subseteq N(I')$ or $T(I') \subseteq N(I')$.

Definition 4.11. For $m' + n'I,\ p' + q'I \in I'$, if $(\langle m + nI \rangle \cap \langle p' + q'I \rangle) \cap I' \neq \emptyset$, then I' is called a neutrosophic I-system, where I' is a subset of $N(LR)$.

Proposition 4.12. For neutrosophic ideal $N(I')$ of neutrosophic LA-ring $N(LR)$, the below statements are equivalent:

(i) $N(I')$ is strongly irreducible.

(ii) For any elements $m' + n'I,\ p' + q'I \in N(LR)$ such that $\langle m' + n'I \rangle \cap \langle p' + q'I \rangle \subseteq N(I')$ implies that either $m' + n'I \in N(I')$ or $p' + q'I \in N(I')$.

(iii) $N(LR) \backslash N(I')$ is a neutrosophic I-system.

Proof. (i)⟹(ii). Clear from definition.

(ii)⟹(iii). Assume that $m' + n'I,\ p' + q'I \in N(LR) \backslash N(I')$ and let $(\langle m' + n'I \rangle \cap \langle p' + q'I \rangle) \cap N(LR) \backslash N(I') = \emptyset$. This means $\langle m' + n'I \rangle \cap \langle p' + q'I \rangle \subseteq N(I')$, hence either $m' + n'I \in N(I')$ or $p' + q'I \in N(I')$ which is a contradiction. Thus $(\langle m' + n'I \rangle \cap \langle p' + q'I \rangle) \cap N(LR) \backslash N(I') \neq \emptyset$.

(iii)⟹(i). Let $S(I')$ and $T(I')$ be neutrosophic ideals such that $S(I') \cap T(I') \subseteq N(I')$. Assume that Let $S(I')$ and $T(I')$ do not contained in $N(I')$. This means there will be elements $m' + n'I \in S(I') \backslash N(I')$ and $p' + q'I \in T(I') \backslash N(I')$. This further implies that $m' + n'I,\ p' + q'I \in N(LR) \backslash N(I')$. Hence by hypothesis $(\langle m' + n'I \rangle \cap \langle p' + q'I \rangle) \cap N(LR) \backslash N(I') \neq \emptyset$, which implies that there will be an element $r' + s'I \in \langle m' + n'I \rangle \cap \langle p' + q'I \rangle$ such that $r' + s'I \in N(LR) \backslash N(I')$. It means $r' + s'I \in \langle m' + n'I \rangle \cap \langle p' + q'I \rangle \subseteq S(I') \cap T(I') \subseteq N(I')$. Which further means $S(I') \cap T(I') \nsubseteq N(I')$. Here arise a contradiction, hence either $S(I') \subseteq N(I')$ or $T(I') \subseteq N(I')$. Thus $N(I')$ is strongly irreducible.

5. Conclusions

We initiated Neutrosophic LA-rings in this research work. Which will be a first attempt to enhance and develop non-associative area of mathematical sciences. It will open a new gateway for the upcoming researchers to extend this non-associative field of mathematics. In order to look at the algebraic characteristics of Neutrosophic LA-rings, we studied their ideals (Neutrosophic prime ideals, Neutrosophic semi-prime, Neutrosophic quasi ideals and Neutrosophic bi-ideals). We established number of results to study the characteristic properties of Neutrosophic LA-rings. We explored the characterizations of Neutrosophic LA-ring by the properties of Neutrosophic M-system, Neutrosophic P-system and Neutrosophic I-system and established number of results. Also, a relation is developed between neutrosophic M-system and neutrosophic P-system. In the light of our findings, we may conclude that our work is going to be a good and helpful contribution to the study of algebraic structures based on Neutrosophic sets. Further, we are planning to work out the structural study of Neutrosophic LA-rings by extending it to some theoretical applications in Neutrosophic fuzzy algebraic structures. Particularly, Neutrosophic soft LA-rings, Neutrosophic LA-semirings, Neutrosophic soft LA-semirings and related structures.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L. Fuzzy sets. *Information and Control* **1965**, 338-353.
2. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems* **1986**, 20, 87-96.
3. Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. *American Reserch Press, Rehoboth, NM,* **1999.**
4. Kandasamy, W. B. V.; Smarandache, F. Neutrosophic rings. *Phoenix, Arizona **2006.***
5. Agboola, A. A. A.; Akinola, A. D.; Oyebola, O. Y. Neutrosophic rings I. *International Journal of Mathematical Combinatorics* **2011,** 4, 1-14.
6. Agboola, A. A. A.; Akinola, A. D.; Oyebola, O. Y. Neutrosophic Groups and Neutrosophic Subgroups. *International Journal of Mathematical Combinatorics* **2012,** 3, 1-9.
7. Ali, M.; Smarandache, F.; Shabir, M.; Vladareanu, L. Generalization of Neutrosophic rings and Neutrosophic fields. *Neutrosophic Sets and Systems* **2014**, 5, 9-14.
8. Ali, M.; Shabir, M.; Naz, M.; Smarandache, F. Neutrosophic left almost semigroup. , *Neutrosophic Sets and Systems* **2014**, 3, 18-28.
9. Ali, M.; Smarandache, F.; Shabir, M.; Naz, M.. Neutrosophic bi-LA-semi group and Neutosophic N-LA-semi group. *Neutrosophic Sets and Systems* **2014**, 4, 19-24.
10. Ali, M.; Shabir, M.; Smarandache, F.; Vladareanu, L. Neutrosophic LA-semigroup rings. *Neutrosophic Sets and Systems* **2015**, 7, 81-88.
11. Khan, M.; Smarandache, F.; Afzal, S. Neutrosophic set approach for characterizations of left almost semi groups. *Neutrosophic Sets and Systems* **2016**, 11, 79-94.
12. Hashim, R. M.; Gulistan, M.; Rehman, I.; Hassan, N.; Nasruddin, A. M. Neutrosophic Bipolar Fuzzy Set and its Application in Medicines Preparation. *Neutrosophic Sets and Systems* **2020**, DOI. 10.5281/zenodo.3639213.
13. Hashim, R. M.; Gulistan, M.; Smarandache, F. Applications of Neutrosophic Bipolar fuzzy Sets in HOPE Foundation for Planning to Build a children Hospital with Different Types of similarity measures. Symmetry **2018**, 10-31.
14. Abdel-Basset, M. A novel and powerful framework based on Neutrosophic sets to aid patients with cancer. *Future Generation Computer Systems* **2019**, 98, 144-153.
15. Abdel-Basset, M.; Manogaran, G.; Gamal, A.; Smarandache, F. A group decision making framework based on Neutrosophic TOPSIS approach for smart medical device selection. *Journal of Medical Systems* **2019**, 43, 1-13.
16. Abdel-Basset, M.; Atef, A.; & Smarandache, F. A hybrid Neutrosophic multiple criteria group decision making approach for project selection. *Cognitive Systems Research* **2019**, *57*, 216-227.
17. Abdel-Basset, M.; Manogaran, G.; Mohamed, M; & Chilamkurti, N. Three-way decisions based on neutrosophic sets and AHP-QFD framework for supplier selection problem. *Future Generation Computer Systems* **2018**, *89*, 19-30.
18. Abdel-Basset, M., Gunasekaran, M., Mohamed, M., & Chilamkurti, N. A framework for risk assessment, management and evaluation: Economic tool for quantifying risks in supply chain. *Future Generation Computer Systems* **2019**, *90*, 489-502.

19. Abdel-Basset, M., & Mohamed, R. A novel plithogenic TOPSIS-CRITIC model for sustainable supply chain risk management. *Journal of Cleaner Production* **2020**, *247*, 119586.
20. Rehman, I.; Shah, M.; Shah, T.; Asima, R. On existence of non-associative LA-rings*. Analele Stiintifice ale Universitatii ovidius Constanta* **2013**, 21, 223-228.
21. Asima, R.; Rehman, I.; Shum, K. P. On Soft LA-modules and Exact Sequences. Italian Journal of Pure and Applied Mathematics 2017, 38, 797-814.
22. Shah, T.; Kausar, N.; Rehman, I. Intuitionistics fuzzy normal subring over a non-associative ring. *Analele Stiintifice ale Universitatii ovidius Constanta* **2012,** 20, 369-386.
23. Shah, T.; Raees, M.; Ali, G. On LA-Modules. *Int. J. Contemp. Math. Sciences* **2011**, 6, 999-1006.
24. Shah, T.; Rehman, I. On LA-rings of finitely non-zero functions*. Int. J. Contemp. Math. Sciences* **2010**, 5, 209-222.
25. Shah, T.; Rehman, I. On characterizations of LA-rings through some properties of their ideals. *Southeast Asian Bull. Math.* **2012**, 36, 695-705.
26. Shah, M.; Shah, T. Some basic properties of LA-rings. *Int. Math. Forum* **2011**, 6, 2195-2199.
27. Shah, T.; Asima, R.; Rehman, I. Application of soft sets to non-associative rings*. Journal of Intelligent & Fuzzy Systems* **2016**, 30, 1537-1546.
28. Shah, T.; Asima, R. Soft M-systems in a class of soft Non-associative rings*. U.P.B. Sci. Bull., Series A* **2015**, 77, 131-142.
29. Shah, T.; Asima, R.; Rehman, I.; Gondal, M. A; Faraz, M. I.; Shum, K. P. Literature Survey on Non-associative Rings and Developments. *European Journal of Pure and Applied Mathematics* **2019,** *12, 370-408.*

Received: May 4, 2021. Accepted: October 1, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

Dual Artificial Variable-Free Simplex Algorithm for Solving Neutrosophic Linear Programming Problems

Aya Rabie [1], Essam el Seidy [2], Amani Elrayes [3] and Elsayed Badr [4]

[1] Planning Techniques center, Institute of National Planning, Cairo, Egypt; E-mail: aya.ibrahim@inp.edu.eg
[2] Mathematics Department, Faculty of Science, Ain Shams University, Egypt; E-mail: essamelseidy@sci.asu.edu.eg
[3] Planning Techniques center, Institute of National Planning, Cairo, Egypt; E-mail: amani.elrayes@inp.edu.eg
4 Scientific Computing Department, Faculty of Computers and Informatics, Benha University, Egypt;
E-mail:alsayed.badr@fci.bu.edu.eg
* Correspondence: aya.ebrahim828@yahoo.com; Tel.:0201155380850

Abstract. This paper presents a simplified form of dual simplex algorithm for solving linear programming problems with fuzzy and neutrosophic numbers which supplies some great benefits over phase 1 of traditional dual simplex algorithm. For instance, it could start with any infeasible basis of linear programming problems; it doesn't need any kind of artificial variables or artificial constraints, so the number of variables of the proposed method is less than the number of variables in the traditional dual simplex algorithm, therefore; the run time for the proposed algorithm is also faster than the phase 1 of traditional dual simplex algorithm, and the proposed method overcomes the traditional dual simplex algorithm for both the fuzzy approach and the neutrosophic approach according to the iterations number. We also use numerical examples to compare between the fuzzy and the neutrosophic approaches, the results show that the neutrosophic approach is more accurate than the fuzzy approach. Furthermore, the proposed algorithm overcomes the phase 1 of traditional dual simplex algorithm for both the fuzzy and neutrosophic approach.

Keywords: Fuzzy Number; Neutrosophic Number; Rank Function; Dual Artificial Variable Free version of Simplex Method.

1. Introduction

Linear programming is the most frequently applied operations research technique. A linear programming model represents real world situations with some sets of parameters determined by experts and decision makers while in real world applications certainty, reliability and precision are often illusory concepts, therefore experts and decision makers cannot determine the exact value of parameters, or they may not be in a position to specify the objective functions or constraints precisely. By implying fuzzy and neutrosophic set theory to linear programming, which leads to fuzzy and neutrosophic linear programming, the so-called problems are being overcome. All of this causes us to resort to fuzzy and neutrosophic numbers that deal with uncertain information. Neutrosophic Set (NS) [26] is a generalization of the fuzzy set [27] and intuitionistic fuzzy set [3]; each element of set had a truth, indeterminacy and falsity membership functions. So, neutrosophic set can assimilate inaccurate, vague and maladjusted information efficiently and effectively.

After the pioneering work on fuzzy linear programming by Tanaka et al. [17,18] and Zimmermann [19], several kinds of fuzzy linear programming problems have appeared in the literatures and different methods have been proposed to solve such problems [16,23,29]. One important class of these methods that has been high-lighted by many researches is based on comparing of fuzzy numbers using ranking functions. Based on this idea, Maleki et al. [23] proposed a simple method for solving fuzzy number linear programming (FNLP) problems. After that, many various approaches appeared that deal with the vague and imprecise information such as intuitionistic fuzzy set and neutrosophic set.

Arsham [5] introduced the simplex method without using artificial variables. First, the basic feasible variable set (BVS) is determined to be the empty set. Then, the non-basic variable is chosen to be the basic variable one by one until the BVS is full. After the problem has the complete BVS, the simplex method is performed. However, this method has the mistake as shown by Enge and Huhn [21] in 1998.

Pan [24] presented the simplex algorithm by avoiding artificial variables. The algorithm starts when the initial basis gives primal and dual infeasible solutions by adjusting negative reduced costs to a single positive value. Then, the dual solution is feasible and the dual simplex method is performed. After the optimal solution is found in this step, the original reduced costs are restored and the simplex method is performed.

Abdel-Basset et.al [1] proposed the neutrosophic simplex algorithm that solves the neutrosophic linear programming (NLP). They introduced a comparison between fuzzy approach and neutrosophic approach by using numerical examples. On the other hand, their manuscript has some incorrect assumptions.

Akanksha Singh et.al [25] spotted some incorrect assumptions in Abdel-Basset's manuscript [1]. They suggested the required modifications in Abdel-Basset's method. On the other hand, [21] used different rank functions to compare between fuzzy approach and neutrosophic approach, which makes this comparison not fair. Therefore, in this essay, the authors emphasis use the same rank function.

Elsayed Badr et.al [2] proposed a novel method that deal with initial non basic solution. This method is called neutrosophic two-phase method and it solves the linear programming problems with neutrosophic numbers. They used the same rank function when they compared between fuzzy approach and neutrosophic approach, which makes the comparison is fair.

For more details about the linear programming, the reader can refer to [6,7,11-13,15]. On the other hand, for more details about the fuzzy linear programming, the reader is referred to [2,9,10,20]. Finally, for more details about the neutrosophic linear programming, the reader may refer to [8].

In this paper, we apply dual artificial variable-free simplex algorithm for solving linear programming problems with fuzzy and neutrosophic numbers, which has several advantages, for instance, it could start with any infeasible basis of linear programming problem. This algorithm follows the same pivoting sequence as of dual simplex phase 1 without showing any explicit description of artificial variables which also makes it space efficient. The proposed algorithm reduces the size of the problem and reduces the execution time to solve the problem. Then the CPU time for the proposed method is also faster than the phase 1 of traditional dual simplex method. So, the proposed method can reduce the computational time. We also compare between the neutrosophic approach and the fuzzy approach using numerical examples.

The remaining parts of this work are organized as follows: In sec. 2, the fundament concepts of fuzzy and neutrosophic sets have been presented, and a new technique which converts the fuzzy representation to the neutrosophic representation has been proposed. Akanksha Singh *et al.*'s modifications [25] and a new neutrosophic dual artificial variable-free simplex algorithm (NDAVFSA) are proposed in Sec. 3. In Sec. 4, a numerical example that shows the importance of the proposed modification for primal neutrosophic simplex method has been introduced, and the superiority of the proposed algorithm (NDAVFSA) on the primal neutrosophic simplex algorithm has been shown. Finally, we introduce conclusions and the future work in Sec. 5.

2. Preliminaries

In this section, three subsections have been introduced. First one is representation of the fuzzy numbers. Second, the representation of the neutrosophic numbers. Finally, we show that how do to convert the fuzzy numbers and neutrosophic numbers to crisp number.

2.1. Fuzzy Representation

We review the fundamental notions of fuzzy set theory, initiated by Bellman and Zadeh [22].

2.1.1. Definition

A convex fuzzy set $\tilde{A}$ on $\mathbb{R}$ is a fuzzy number if the following conditions hold:

(a) Its membership function is piecewise continuous.

(b) There exist three intervals [a, b], [b, c], [c, d] such that $\mu_{\tilde{a}}$ is increasing on [a, b], equal to 1 on $[b, c]$, decreasing on $[c, d]$ and equal to 0 elsewhere.

2.1.2. Definition

Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ denote the trapezoidal fuzzy number, where

$(a^L - \alpha, a^U + \beta)$ is the support of $\tilde{a}$and $[a^L, a^U]$ its core.

Remark 1: We denote the set of all trapezoidal fuzzy numbers by F ($\mathbb{R}$) as shown as in figure 1.

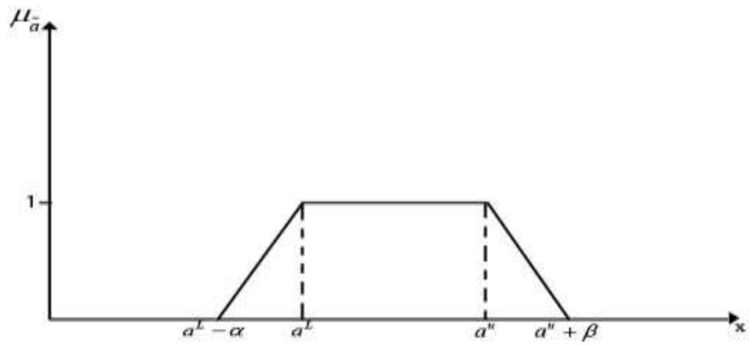


Figure 1: Truth membership function of trapezoidal fuzzy number ̃a

2.1.3. Definition

Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$ be two trapezoidal fuzzy numbers, the arithmetic operation on the trapezoidal fuzzy number are defined as:

$x\tilde{a} = (xa^L, xa^U, x\alpha, x\beta);\ x > 0,\quad x \in \mathbb{R}.$

$x\tilde{a} = (xa^U, xa^L, -x\beta, -x\alpha);\ x < 0,\quad x \in \mathbb{R}.$

$\tilde{a}+\tilde{b}=(a^L, a^U, \alpha, \beta)+ (b^L, b^U, \gamma, \theta)= [a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta]$

2.2. Neutrosophic Representation

In this subsection, some basic definitions in the neutrosophic set theory are introduced:

2.2.1. Definition [1]

the trapezoidal neutrosophic number $\tilde{A}$ is a neutrosophic set in $\mathbb{R}$ with the following truth (T), indeterminacy (I) and falsity (F) membership functions as shown in figure 2:

$$T_{\tilde{A}}(x) = \begin{cases} \frac{\alpha_{\tilde{A}}(x-a_1)}{a_2-a_1} : a_1 \le x \le a_2 \\ \alpha_{\tilde{A}} \quad : a_2 \le x \le a_3 \\ \alpha_{\tilde{A}}(\frac{x-a_3}{a_4-a_3}) : a_3 \le x \le a_4 \\ 0 \quad \text{otherwise} \end{cases} \qquad I_{\tilde{A}}(x) = \begin{cases} \frac{(a_2-x+\theta_{\tilde{A}}(x-a_1'))}{a_2-a_1'} : a_1' \le x \le a_2 \\ \theta_{\tilde{A}} \quad : a_2 \le x \le a_3 \\ \frac{(x-a_3+\theta_{\tilde{A}}(a_4'-x))}{a_4'-a_3} : a_3 \le x \le a_4' \\ 1 \quad \text{otherwise} \end{cases}$$

$$F_{\tilde{A}}(x) = \begin{cases} \frac{(a_2 - x + \beta_{\tilde{A}}(x - a_1''))}{a_2 - a_1''} : a_1'' \le x \le a_2 \\ \beta_{\tilde{A}} \quad : a_2 \le x \le a_3 \\ \frac{(x - a_3 + \beta_{\tilde{A}}(a_4'' - x))}{a_4'' - a_3} : a_3 \le x \le a_4'' \\ 1 \quad \text{otherwise} \end{cases}$$

where $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and $\beta_{\tilde{A}}$ represent the maximum degree of truthiness, minimum degree of indeterminacy and minimum degree of falsity, respectively, $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and $\beta_{\tilde{A}} \in [0,1]$

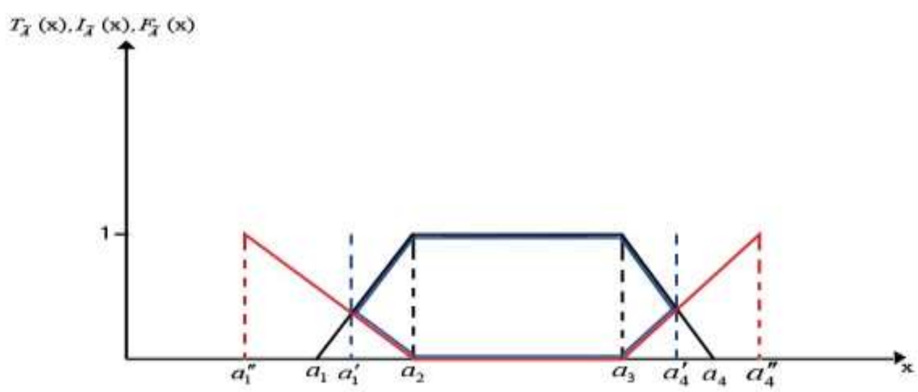


Figure 2: Truth, indeterminacy and falsity membership functions of trapezoidal neutrosophic number $\tilde{A}$

2.2.2. Definition [1]

the mathematical operations on two trapezoidal neutrosophic numbers. $\widetilde{A} =< a_1, a_2, a_3, a_4; \alpha_{\widetilde{A}}, \theta_{\widetilde{A}}, \beta_{\widetilde{A}} >$ and $\tilde{B} = < b_1, b_2, b_3, b_4; \alpha_{\widetilde{B}}, \theta_{\widetilde{B}}, \beta_{\widetilde{B}} >$ are as follows:

$$\widetilde{A} + \widetilde{B} = < (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} >$$

$$\widetilde{A} - \widetilde{B} = < (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} >$$

$$\widetilde{A}^{-1} = < (\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}); \alpha_{\widetilde{A}}, \theta_{\widetilde{A}}, \beta_{\widetilde{A}} > \text{where } \widetilde{A} \neq 0)$$

$$\lambda\widetilde{A} = \begin{cases} < \lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4; \alpha_{\widetilde{A}}, \theta_{\widetilde{A}}, \beta_{\widetilde{A}} > : \lambda > 0 \\ < \lambda a_4, \lambda a_3, \lambda a_2, \lambda a_1; \alpha_{\widetilde{A}}, \theta_{\widetilde{A}}, \beta_{\widetilde{A}} > : \lambda < 0 \end{cases}$$

$$\widetilde{A\widetilde{B}} = \begin{cases} < (a_1b_1, a_2b_2, a_3b_3, a_4b_4); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 > 0, b_4 > 0) \\ < (a_1b_4, a_2b_3, a_3b_2, a_4b_1); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 < 0, b_4 > 0) \\ < (a_4b_4, a_3b_3, a_2b_2, a_1b_1); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

$$\frac{\widetilde{A}}{\widetilde{B}} \begin{cases} < \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 > 0, b_4 > 0) \\ < \left(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}\right); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 < 0, b_4 > 0) \\ < \left(\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4}\right); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

2.3. Transfer from Fuzzy Representation to Neutrosophic Representation [25]

The goal of this section is to explain how to convert fuzzy numbers representation into neutrosophic numbers representation. This transformation is used for simplicity and to make the comparison fair. There are many types of techniques to transfer from fuzzy to neutrosophic representation such as, ranking functions and ∝-cut technique.

2.3.1. Definition.

Ranking function is a viable approach for ordering fuzzy numbers and neutrosophic numbers. It is known that there are many ranking functions for ordering the fuzzy numbers and neutrosophic numbers.

In this subsection, we explain how to apply technique to convert from fuzzy number to neutrosophic number:

From Figure 1 and Figure 2 we can illustrate the following relations between the two representations:

$$a_1 = a_2 - \alpha, a_2 = a^L, a_3 = a^U \text{ and } a_4 = a_3 + \beta \tag{1}$$

Assuming that the ranking function is used for ordering the fuzzy numbers as follows:

$$R(\tilde{a}) = \frac{a^L+a^U}{2} + \frac{\beta-\alpha}{4} \qquad (2)$$

$\beta - \alpha = a_4 - a_3 - (a_2 - a_1) = a_4 - a_3 - a_2 + a_1$

$R(\tilde{a}) = \frac{a_2+a_3}{2} + \frac{a_4-a_3-a_2+a_1}{4} = \frac{a_2+a_3+a_4+a_1}{4}$

From the relations (1) & (2) we can express the rank function is used for ordering the neutrosophic numbers as follows:

$$R(\tilde{a}) = \frac{1}{4}\sum_{i=1}^{4} \tilde{a}_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}}) \qquad (3)$$

From (1), we can convert fuzzy numbers representation into neutrosophic numbers representation. On the other hand from (2) and (3), we can use the same function for both fuzzy numbers and neutrosophic numbers to obtain a fair comparison between them.

(i) Assuming that $T_{\tilde{A}} = 1$, $I_{\tilde{A}} = 0$, $\tilde{F}_A = 0$, then the TrNN $\tilde{a} =< a_1, a_2, a_3, a_4; T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}} >$ equal to number $\tilde{a} =< a_1, a_2, a_3, a_4; 1,0,0 >$ and hence, in this case,

The expression $R(\tilde{a}) = \frac{1}{4}\sum_{i=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ is equivalent to the expression

$$R(\tilde{a}) = \frac{1}{4}\sum_{i=1}^{4} \tilde{a}_i + 1$$

(ii) Furthermore, it is well known the fact that if $a_1 = a_2 = a_3 = a_4$ then the trapezoidal neutrosophic number $\tilde{A} =< a_1, a_2, a_3, a_4; 1,0,0 >$ will be transformed into a real number A = (a, a, a, a; 1, 0, 0) and hence, in this case, the expression $R(\tilde{a}) = \frac{1}{4}\sum_{i=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ is equivalent to the expression $R(A) = a + 1 \neq a$

The following table represents the fuzzy ranking function, and the corresponding neutrosophic ranking function and the corresponding real ranking function.

Table 1: fuzzy ranking function into it's corresponding neutrosophic ranking function

Fuzzy Rank Function	Corresponding Neutrosophic Rank Function	Corresponding Real Rank function of constraints
$R(\tilde{a}) = (\frac{a^l + a^u}{2} + \frac{\beta - \alpha}{4})$	$R(\tilde{a}) = \frac{1}{4}\sum_{j=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$	$R(a) = a + 1$

3. Algorithms

In this section; firstly, we present Akanksha Singh *et al.*'s modifications [25] and the proposed modification about the mathematical incorrect assumptions, considered by Abdel-Basset *et al.* [1] in their proposed method to convert from neutrosophic numbers into real numbers. Secondly, we propose a new fuzzy dual artificial variable free simplex algorithm. Finally, we develop this algorithm in order to solve linear programming with neutrosophic numbers (neutrosophic dual artificial variable free simplex algorithm).

3.1. Akanksha Singh *et al.*'s modifications [25]

The following table presents Akanksha Singh et al.'s modifications to convert from neutrosophic number to crisp number.

Table 2: Akanksha Singh et al.'s modifications.

No	NLPP- (Type)	NLPP- (Form)	Exact Crisp LPP
1	The coefficients of the objective function are represented by trapezoidal neutrosophic numbers	$Max\backslash \text{Min } \left[\sum_{j=1}^{n} = \tilde{c}_j x_j\right]$ s.t $\sum_{j=1}^{n} a_{ij}x_j \le, \ge, b_j$, $i = 1,2,\ldots\ldots,m; x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$	$Max\ /Min\ [\sum_{j=1}^{n} R(\tilde{c}_j x_j) - \sum_{j=1}^{n} T_{\tilde{c}_j} x_j + \sum_{j=1}^{n} I_{\tilde{c}_j} x_j + \sum_{j=1}^{n} F_{\tilde{c}_j} x_j + \min_{1\le j\le n}\{T_{\tilde{c}_j} x_j\} - \max_{1\le j\le n}\{I_{\tilde{c}_j} x_j\} - \max_{1\le j\le n}\{F_{\tilde{c}_j} x_j\}]$ $s.t.$ $\sum_{j=1}^{n} a_{ij}x_j \le, \ge, = b_j$, $i = 1,2,\ldots\ldots,m;$ $x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$
2	The coefficients of constraints variables and right hand side are represented by trapezoidal neutrosophic numbers	$Max\backslash \text{Min } \left[\sum_{j=1}^{n} = c_j x_j\right]$ s.t. $\sum_{j=1}^{n} \tilde{a}_{ij}x_j \le, \ge, = \tilde{b}_j$, $i = 1,2,\ldots\ldots,m;\ x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$	$\text{Max / Min } \sum_{j=1}^{n} c_j x_j$ $s.t.$ $\left[\sum_{j=1}^{n} R(\tilde{a}_{ij}x_j) - \sum_{j=1}^{n} T_{\tilde{a}_{ij}} x_j + \sum_{j=1}^{n} I_{\tilde{a}_{ij}} x_j + \sum_{j=1}^{n} F_{\tilde{a}_{ij}} x_j + \min_{1\le j\le n}\{T_{\tilde{a}_{ij}} x_j\} - \max_{1\le j\le n}\{I_{\tilde{a}_{ij}} x_j\} - \max_{1\le j\le n}\{F_{\tilde{a}_{ij}} x_j\}\right] \le, \ge, = R(\tilde{b}_i)$ $x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$
3	All parameters are represented by trapezoidal neutrosophic numbers, except variables are exemplified only by real values	$Max\backslash \text{Min } \left[\sum_{j=1}^{n} = \tilde{c}_j x_j\right]$ s.t. $\sum_{j=1}^{n} \tilde{a}_{ij}x_j \le, \ge, = \tilde{b}_j$, $i = 1,2,\ldots\ldots,m; x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$	$Max\ /Min\ [\sum_{j=1}^{n} R(\tilde{c}_j x_j) - \sum_{j=1}^{n} T_{\tilde{c}_j} x_j + \sum_{j=1}^{n} I_{\tilde{c}_j} x_j + \sum_{j=1}^{n} F_{\tilde{c}_j} x_j + \min_{1\le j\le n}\{T_{\tilde{c}_j} x_j\} - \max_{1\le j\le n}\{I_{\tilde{c}_j} x_j\} - \max_{1\le j\le n}\{F_{\tilde{c}_j} x_j\}]$ s.t. $(\sum_{j=1}^{n} R(\tilde{a}_{ij})x_j) + 1 \le, \ge, = R(\tilde{b}_j)$, $i = 1,2,\ldots\ldots,m;$ $x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$
4	The coefficients of objective function and constraints variables are represented by real numbers and right hand side are represented by trapezoidal neutrosophic numbers	$Max\backslash \text{Min } \left[\sum_{j=1}^{n} = c_j x_j\right]$ s.t. $\sum_{j=1}^{n} a_{ij}x_j \le, \ge, = \tilde{b}_j$, $i = 1,2,\ldots\ldots,m; x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$	$\text{Max / Min } \sum_{j=1}^{n} c_j x_j$ $s.t.$ $R\left[\sum_{j=1}^{n}(a_{ij}x_j)\right] \le, \ge, = R(\tilde{b}_i)$ $x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$

Remark 2:

- If $R(a) = a + 1$ and the coefficients of the objective function & constraints variables are real, then the fuzzy linear programming problem is equivalent to the neutrosophic linear programming problem.
- $\sim$: represents the presence of neutrosophic numbers within the matrices or vectors.
- NLPP: neutrosophic linear programming problem.

3.2. A novel Neutrosophic Dual Artificial Variable Free Simplex Algorithm.

Nayatullah et al [22] proposed a streamlined artificial variable free version of simplex algorithm (AVFSA) for solving the linear programming problems with real numbers. In this section, we propose a new algorithm which solves linear programming with neutrosophic numbers (Neutrosophic Dual Artificial Variable-Free Simplex Algorithm NDAVFSA). The proposed algorithm overcame traditional neutrosophic dual simplex algorithms.

Algorithm 1: Neutrosophic Dual Artificial variable -Free Simplex Algorithm (NDAVFSA)

Step 0: (Initialization)

- Converting fuzzy numbers into neutrosophic numbers according to Section 2.3.1 [25]
- Apply Akanksha Singh et al.'s modifications according to Section 3.1

Step 1: Let $\tilde{K}$ be a maximal subset of $\tilde{B}$ such that $\tilde{B} = \{ j : d_{0j} < 0 , j \in \tilde{N}\}$. If $\tilde{K} = \varphi$ then $D(\tilde{B})$ is dual feasible. Exit.

Step 2: Denote the basic variables y_k by $-y_k^-$ and compute dual infeasibility objective vector $W'(\tilde{B}) \in R^{\tilde{B}}$ such that $w'_i = \sum_{j\in k} d_{ij},\ i \in \tilde{B}$. Place w' to the right of the dictionary $D(\tilde{B})$.

Step 3: Let $\tilde{L} \subseteq \tilde{B}$ such that $\tilde{L} = \{ i: w'_i < 0,\ i \in \tilde{B} \}$. If $\tilde{L} = \varphi$ then $D(\tilde{B})$ is dual inconsistent. Exit.

Step 4: (Choice of entering variable)

Choose $r \in \tilde{L}$ such that $w'_r \le w'_i\ \forall\, i \in \tilde{L}$ (Ties are broken arbitrarily)

Step 5: (Choice of leaving variable)

Choose $k_1 \in \tilde{K}$ and $k_2 \in \tilde{N} \setminus \tilde{K}$ such that:

k_1= arg max $\{\{\frac{d_{0j}}{d_{rj}} | \ (d_{0j} \le 0, d_{rj} > 0)\}, j \in \tilde{K} \}$

k_2= arg max $\{\{\frac{d_{0j}}{d_{rj}} | \ (d_{0j} \ge 0, d_{rj} < 0)\}, j \in \tilde{N} \setminus \tilde{K} \}$ Set $\tilde{K}$: = arg max $\{\frac{d_{0k_1}}{\tilde{d}_{rk_1}}, \frac{d_{0k_2}}{d_{rk_2}}\}$

Step 6: Make a pivot on (r,k) ($\Rightarrow$ Set $\tilde{B} := (\tilde{B} \cup \{k\})\setminus\{r\}, \tilde{N} := (\tilde{N} \cup \{r\})\setminus\{k\}$ and update $D(\tilde{B})$ along with the appended $w'(\tilde{B})$).

Step 7: If $k \in \tilde{K}$, set $\tilde{K} := \tilde{K} \setminus \{k\}$ and $w'_k := w'_k + 1$, replace notation of $-y_k^-$ by y_k

Step 8: Pivot operation

For $r \in \tilde{B}, k \in \tilde{N}$ and (r,k) being the position of the pivot element d_{rk} ($\neq 0$) of D, then one can obtain an updated equivalent short table $D(\tilde{B}')$ with a new basis $\tilde{B}' := (\tilde{B} \cup \{k\})\setminus\{r\}$ and the new non-basis $\tilde{N}' := (\tilde{N} \cup \{r\})\setminus\{k\}$ by performing the following operations on $D(\tilde{B})$.

$$d'_{rk} := \frac{1}{d_{rk}}$$

$$d'_{rj} := \frac{d_{rj}}{d_{rk}}, j \in \tilde{N}\setminus\{k\}$$

$$d'_{ik} := -\frac{d_{ik}}{d_{rk}}, i \in \tilde{B}\setminus\{r\}$$

$$d'_{ij} := d_{ij} - d_{rj} \times \frac{d_{ik}}{d_{rk}}, i \in \tilde{B}\setminus\{r\}, j \in \tilde{N}\setminus\{k\}$$

The above replacement is known as a pivot operation on (r, k).

Step 9: If $\widetilde{K} = \varphi$, then $D(\tilde{B})$ is dual feasible. Exit.
Otherwise, go to step 3.

Step 10: If phase 1 ends with an objective value equal to zero, it implies that all artificial variables have attained a value zero (means all infeasibilities have been removed) and our current basis is feasible to the original problem, then we return to the original objective and proceed with simplex phase 2. **Otherwise**, we conclude that the problem has no solution.

4. Numerical Examples and Results Analysis

In this study, we solve well-known fuzzy and neutrosophic linear programming problem that presented in [28] with the traditional and proposed method.

$$\begin{aligned} & Max\ \tilde{z} = (1,5,2,4)x_1 + (10,12,2,6)x_2 \\ & s.t. \\ & 3x_1 + 10x_2 \leq 10 \\ & x_1 - x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned} \qquad \boldsymbol{P_1}$$

In the upcoming two subsections, problem $\boldsymbol{P_1}$ will be solved using fuzzy & neutrosophic dual artificial variable-free simplex method respectively, uses the same rank function and compare between the results.

4.1. Solving $(\boldsymbol{P_1})$ using Fuzzy Dual Artificial Variable-Free Simplex Method

Putting $\boldsymbol{P_1}$ in the standard dual form, we have:

$$\begin{aligned} & Min\ \tilde{z} = (-5,-1,4,2)x_1 + (-12,-10,6,2)x_2 \\ & s.t. \\ & 3x_1 + 10x_2 + x_3 = 10 \\ & -x_1 + x_2 \qquad + x_4 = -2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned} \qquad \text{D1}$$

By adding non-negative slack variablesx_3, x_4, the associated short table of D1 can be constructed as shown below. The dual variables y_3, y_4 have been demonstrated explicitly as it is required to observe dual variables too.

Here y is the dual objective variable. Objective coefficients (z) of primal non basic variables are the values of dual basic variables, and values of primal basic variables are coefficients of dual non-basic variables in dual objective.

b	$\mathbf{x_1}$	$\mathbf{x_2}$

$$\begin{array}{c} \mathbf{z} \\ \mathbf{x_3} \\ \mathbf{x_4} \\ \\ \end{array}\left[\begin{array}{ccc} 0 & (-5,-1,4,2) & (-12,-10,6,2) \\ 10 & 3 & 10 \\ -2 & -1 & 1 \\ & -y_1' & -y_2' \end{array}\right]\begin{array}{c} \\ \mathbf{y_3} \\ \mathbf{y_4} \\ \\ \end{array}$$

Here k = {1, 2}, replace $-y_k^{-} \rightarrow y_k$

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_3} \\ \mathbf{x_4} \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} \mathbf{b} & \mathbf{x_1} & \mathbf{x_2} & \mathbf{w'} \end{array} \\ \left[\begin{array}{cccc} 0 & (-5,-1,4,2) & (-12,-10,6,2) & (11,17,4,10) \\ 10 & 3* & 10 & -13 \\ -2 & -1 & 1 & 0 \\ & -y_1' & -y_2' & \end{array}\right] \end{array}\begin{array}{c} \\ \\ \mathbf{y_3} \\ \mathbf{y_4} \\ \\ \end{array}$$

Initial table:

Here B = {3, 4} and N = {1, 2}, according to most negative dual coefficient rule k = 1, so leaving dual basic variable is y_1 and according to artificial variable free dual minimum ratio test r = 3, the entering dual basic variable is 'y_3'. Perform the pivot operation on (3, 1). Replace $-y_1^{-} \rightarrow y_1$, k = {1, 2}\ {1} = {2}, $w'_1 := w'_1 + 1$.

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_1} \\ \mathbf{x_4} \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} \mathbf{b} & \mathbf{x_3} & \mathbf{x_2} & \mathbf{w'} \end{array} \\ \left[\begin{array}{cccc} (5,4,6,104/3) & (-3,-5,6,80/3) & (-1,-3,4,32/3) & (-1,-3,4,32/3) \\ 10/3 & 1/3 & 10/3 & -10/3 \\ 4/3 & 1/3 & 13/3* & -13/3 \\ & y_3 & -y_2' & \end{array}\right] \end{array}\begin{array}{c} \\ \\ \mathbf{y_1} \\ \mathbf{y_4} \\ \\ \end{array}$$

Iteration 2:

Here, k = 2 and r = 4perform pivot operation on (4, 2). Since k ϵ k, replace$-y_2^{-} \rightarrow y_2$; k = {2}\ {2} = {}, $w'_2 := w'_2 + 1$.

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_1} \\ \mathbf{x_2} \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} \mathbf{b} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{w'} \end{array} \\ \left[\begin{array}{cccc} (4,16,-3,53/13) & (7,9,20,-94/13) & (1/5,1/3,-1/3,-71/65) & 0 \\ 30/13 & 1/13 & -10/13 & 0 \\ 4/13 & 1/13 & 3/13 & 0 \\ & y_3 & y_4 & - \end{array}\right] \end{array}\begin{array}{c} \\ \\ \mathbf{y_1} \\ \mathbf{y_2} \\ \\ \end{array}$$

Dual feasibility is achieved; the complementary dual feasible solution is (x_1, x_2) = (30/13, 4/13).

Resolve (P_1) using neutrosophic dual artificial variable-free simplex method uses the same rank function and we will compare between them.

4.2. Solving (P_1) using Neutrosophic Dual Artificial Variable-Free Simplex method

First: We will convert the fuzzy numbers into neutrosophic numbers. Then, using the following rank function:

$$R(\breve{a}) = \frac{1}{4}\sum_{i=1}^{4} \tilde{a}_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$$

$$\begin{aligned} &Min\ \tilde{z} = R[(-1,1,5,9)]x_1 + R[8,10,12,18]x_2 \\ &s.t. \\ &\quad 3x_1 + 10x_2 \leq 10 \\ &\quad x_1 - x_2 \geq 2 \\ &\quad\ x_1, x_2 \geq 0 \end{aligned} \qquad \text{D1}$$

$$\begin{aligned} &Min\ z = 9/2x_1 + 13x_2 - 1 \\ &s.t. \\ &\quad 3x_1 + 10x_2 \leq 10 \\ &\quad x_1 - x_2 \geq 2 \\ &\quad\ x_1, x_2 \geq 0 \end{aligned} \qquad \text{D2}$$

Putting (D2) in the standard form:

$$\begin{aligned} &Min\ z = 9/2x_1 + 13x_2 - 1 \\ &s.t. \\ &\quad 3x_1 + 10x_2 + x_3 = 10 \\ &\quad -x_1 + x_2 \qquad +x_4 = -2 \\ &\qquad\quad x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_3} \\ \mathbf{x_4} \\ \\ \end{array} \begin{array}{c} \begin{matrix} \mathbf{b} & \mathbf{x_1} & \mathbf{x_2} \end{matrix} \\ \begin{bmatrix} 0 & -9/2 & -13 \\ 10 & 3 & 10 \\ -2 & -1 & 1 \\ & -y'_1 & -y'_2 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \mathbf{y_3} \\ \mathbf{y_4} \\ \end{array}$$

Here k = {1, 2}, replace $-{y_k}^{-} \rightarrow y_k$

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_3} \\ \mathbf{x_4} \\ \\ \end{array} \begin{array}{c} \begin{matrix} \mathbf{b} & \mathbf{x_1} & \mathbf{x_2} & \mathbf{w'} \end{matrix} \\ \begin{bmatrix} 0 & -9/2 & -13 & 35/2 \\ 10 & 3 & 10* & -13 \\ -2 & -1 & 1 & 0 \\ & -y'_1 & -y'_2 & \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \mathbf{y_3} \\ \mathbf{y_4} \\ \end{array}$$

Initial table:

Here B = {3, 4} and N = {1, 2}, according to most negative dual coefficient rule k = 2, so leaving dual basic variable is y_2 and according to artificial variable-free dual minimum ratio test r = 3, the entering dual basic variable is 'y_3'. Perform the pivot operation on (3, 1). Replace $-{y_2}^{-} \rightarrow \mathbf{y_2}$, k = {1, 2}\ {2} = {1}, $w'_2 := w'_2 + 1$.

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_2} \\ \mathbf{x_4} \\ \\ \end{array} \begin{array}{c} \begin{matrix} \mathbf{b} & \mathbf{x_1} & \mathbf{x_3} & \mathbf{w'} \end{matrix} \\ \begin{bmatrix} 13 & -3/5 & 13/10 & 3/5 \\ 1 & 3/10* & 1/10 & -3/10 \\ -3 & -13/10 & -1/10 & 13/10 \\ & -y'_1 & y_3 & \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \mathbf{y_2} \\ \mathbf{y_4} \\ \end{array}$$

Iteration 2:

Here, k = 1 and r = 2 perform pivot operation on (2, 1). Since k ϵ k, replace$-x_1^- \rightarrow x_1$; k = {1}\ {1} = {}$w'_1 := w'_1 + 1$.

$$\begin{array}{c} \\ z \\ x_1 \\ x_4 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} b & x_1 & x_3 & w' \end{array} \\ \left[\begin{array}{cccc} 15 & 2 & 3/2 & 0 \\ 10/3 & 10/3 & 1/3 & 0 \\ 4/3 & 13/3 & 1/3 & 0 \\ & y_1 & y_3 & \end{array}\right] \end{array} \begin{array}{c} \\ \\ y_1 \\ y_4 \\ \\ \end{array}$$

Dual feasibility is achieved; the complementary dual feasible solution is (x_1,x_2)= (10/3, 0).

$Max\ z = 9/2\ x_1 + 13\ x_2 - 1 = 15 - 1 = 14$

Table 3*:* *A comparison between two-phase algorithm, Fuzzy and Neutrosophic D*AVFSA

	Two-Phase Simplex Algorithm	**Fuzzy Dual Art Simplex Algorithm**	**Neutrosophic Dual Art Simplex Algorithm**
no (iteration)	5	3	3
Z	11.8	11.8	14
$\mathbf{x_1}$	30/13	30/13	10/3
$\mathbf{x_2}$	4/13	4/13	0

In table 3, a good comparisons have been made between two-phase simplex algorithm, fuzzy dual artificial variable-free simplex algorithm and neutrosophic dual artificial variable-free simplex algorithm; we noticed that the neutrosophic approach is more accurate than the fuzzy approach. On the other hand, the proposed algorithm overcomes the traditional two phase simplex algorithm for both the fuzzy approach and the neutrosophic approach according to the iterations number.

Conclusion

In this work, a new algorithm (Dual Artificial Variable-Free Simplex Algorithm) has been proposed, which solves linear programming problems with fuzzy and neutrosophic numbers. In this algorithm, the artificial variables are virtually present but their presence is not revealed to the user in the form of extra columns in the simplex table. It also follows the same pivoting sequence as of simplex phase 1 without showing any explicit description of artificial variables or artificial constraints but it could be easily seen that numbers of computations are noticeably reduced and the proposed algorithm overcame the traditional simplex algorithm for both the neutrosophic approach and the fuzzy approach according to the iterations number. We also compared between the neutrosophic approach and the fuzzy approach using numerical examples. Finally, the numerical examples show that the neutrosophic approach is more accurate than the fuzzy approach. In future work, we propose new hybrid methods such as using the cosine simplex method for phase 2 or using a traditional simplex algorithm for phase 2 while phase 1 uses the proposed method was proposed in this paper. We expect that these hybrid methods may overcome the traditional method.

Acknowledgments
The authors are thankful to the anonymous referees for useful suggestions and valuable comments. Also, many grateful goes to the Neutrosophic Science International Association (NSIA)/ Iraqi branch for their efforts and valuable recommendations to improve this work.

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

References
[1] Abdel-Basset, M., Gunasekaran, M., Mohamed, M., & Smarandache, F. (2019). A novel method for solving the fully neutrosophic linear programming problems. Neural Computing and Applications, 31(5), 1595-1605.
[2] Alwakil, A. N., Badr, E. M., Elsisy, M. A., & Algendy, H. S. A Comparative Study between PSO and FA Algorithms for optimizing the underground water confined steady flow using a fuzzy approach.
[3] Angelov, P. P. (1997). Optimization in an intuitionistic fuzzy environment. Fuzzy sets and Systems, 86(3), 299-306.
[4] Atanassov, K. (1986). Intuitionistic fuzzy sets. fuzzy sets and systems 20 (1), 87-96.
[5] Arsham, H. (1997). An artificial-free simplex-type algorithm for general LP models. Mathematical and Computer modelling, 25(1), 107-123.
[6] Badr, E., Almotairi, S., Eirokh, A., Abdel-Hay, A., & Almutairi, B. (2020). An Integer Linear Programming Model for Solving Radio Mean Labeling Problem. IEEE Access, 8, 162343-162349.
[7] Badr, E., & Almotairi, S. (2020). On a dual direct cosine simplex type algorithm and its computational behavior. Mathematical Problems in Engineering, 2020.
[8] Badr, E. M., Salam, M. A., & Smarandache, F. (2020). Solving Neutrosophic Linear Programming Problems with Two-Phase Approach. In Neutrosophic Sets in Decision Analysis and Operations Research (pp. 391-412). IGI Global.
[9] Badr, E. M. (2014, May). On Fuzzy Primal Exterior Point Simplex Algorithm for Solving Linear Programming Problems with Fuzzy Numbers. In The International Conference on Mathematics and Engineering Physics (Vol. 7, No. International Conference on Mathematics and Engineering Physics (ICMEP-7), pp. 1-13). Military Technical College.
[10] Badr, E. M., & Elgendy, H. (2019). A Hybrid water cycle-particle swarm optimization for solving the fuzzy underground water confined steady flow. Indonesian J. Elect. Eng. Comput. Sci., 19(1), 492-504.
[11] Badr, E. S., Paparrizos, K., Thanasis, B., & Varkas, G. (2006, June). Some computational results on the efficiency of an exterior point algorithm. In Proceedings of the 18th national conference of Hellenic Operational Research Society (HELORS) (pp. 15-17).
[12] Badr, E. S., Moussa, M., Paparrizos, K., Samaras, N., & Sifaleras, A. (2006). Some computational results on MPI parallel implementation of dense simplex method. Transactions on Engineering, Computing and Technology, 17, 228-231.

[13] Badr, E. S., Paparrizos, K., Samaras, N., & Sifaleras, A. (2005, June). On the basis inverse of the exterior point simplex algorithm. In Proc. of the 17th National Conference of Hellenic Operational Research Society (HELORS) (pp. 16-18).
[14] Bellman, R. E., & Zadeh, L. A. (1970). Decision-making in a fuzzy environment. Management science, 17(4), B-141.
[15] Dantzig, G. (2020). Linear Programming and Extensions. Springer; 5th ed. 2020 edition.
[16] Deli, I., & Şubaş, Y. (2017). A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. International Journal of Machine Learning and Cybernetics, 8(4), 1309-1322.
[17] Ebrahimnejad, A., Nasseri, S. H., Lotfi, F. H., & Soltanifar, M. (2010). A primal-dual method for linear programming problems with fuzzy variables. European Journal of Industrial Engineering, 4(2), 189-209.
[18] Ebrahimnejad, A., Nasseri, S. H., & Lotfi, F. H. (2010). Bounded linear programs with trapezoidal fuzzy numbers. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 18(03), 269-286.
[19] Ebrahimnejad, A., & Nasseri, S. H. (2010). A dual simplex method for bounded linear programmes with fuzzy numbers. International Journal of Mathematics in Operational Research, 2(6), 762-779.
[20] Faizulkhakov, K. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20(1), 87-96.
[21] Huhn, P. (1998). A counterexample to H. Arsham:" Initialization of the simplex algorithm: An artificial free approach".
[22] Inayatullah, S., Touheed, N., & Imtiaz, M. (2015). A streamlined artificial variable free version of simplex method. PloS one, 10(3), e0116156.
[23] Maleki, H. R., Tata, M., & Mashinchi, M. (2000). Linear programming with fuzzy variables. Fuzzy sets and systems, 109(1), 21-33.
[24] Pan, P. Q. (2000). Primal perturbation simplex algorithms for linear programming. Journal of Computational Mathematics, 587-596.
[25] Singh, A., Kumar, A., & Appadoo, S. S. (2019). A novel method for solving the fully neutrosophic linear programming problems: Suggested modifications. Journal of Intelligent & Fuzzy Systems, 37(1), 885-895.
[26] Tanaka, H., & Asai, K. (1984). Fuzzy linear programming problems with fuzzy numbers. Fuzzy sets and systems, 13(1), 1-10.
[27] Verdegay, J. L. (1984). A dual approach to solve the fuzzy linear programming problem. Fuzzy sets and systems, 14(2), 131-141.
[28] Wutsqa, D. U., & Insani, N. (2016). The Trapezoidal Fuzzy Number Linear Programming. *Journal of Innovative Technology and Education*, *3*(1), 123-130.
[29] Zadeh, L. A. (1965). Information and control. Fuzzy sets, 8(3), 338-353.

Received: May 8, 2021. Accepted: October 3, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

A multi-criteria fuzzy neutrosophic decision-making model for solving the supply chain network problem

Ahteshamul Haq[1], Srikant Gupta[2,*] and Aquil Ahmed[3]

[1] Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh (UP), India;(ahteshamhps@gmail.com, a.haq@myamu.ac.in). https://orcid.org/0000-0003-1004-3429
[2] Jaipuria Institute of Management, Jaipur (Rajasthan), India; operation.srikant@hotmail.com
[3] Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh (UP), India; aquilstat@gmail.com
* Correspondence: operation.srikant@hotmail.com ;

Abstract: Supply Chain is a multi-objective decision-making problem with multiple conflicting objective functions related to each supply chain operation and its corresponding sub-criteria. The main focus of this paper is the development of a model that takes into account some important components of real-world supply chain planning. To do so, we proposed a supply chain model that involves multiple suppliers, multiple plants, multiple warehouses, and multiple distributors firms. This approach is designed to tackle a complex multi-site composite supply chain issue under uncertainty as a fuzzy multi-objective model with the primary objective to optimize the transportation cost and delivery time simultaneously. We have used neutrosophical set theory to tackle the ambiguity related to supply chain by using truth, indeterminacy and falsity membership functions and, finally neutrosophical compromise programming approach has been used for obtaining the desired solution. In order to demonstrate the efficiency of the developed models, an industrial design problems has been given. The findings reported is compared to other well-known approaches.

Keywords: Supply Chain; Multi-objective Optimization; Neutrosophic Set.

1. Introduction

Supply Chain (SC) network optimization plays a crucial role in assessing the performance of the whole SC. The challenge with the SC layout consists of determining when and how to distribute equipment (plants, factories, distribution centres) and how to transfer material (raw materials, components, finished products) through the network of organizations (suppliers, producers, sellers, retailers and customers) to maximize overall efficiency (Nurjanni et al. [1]). SC is a network of factories processing raw materials, converting them into intermediate products and then finished products, and supplying the products via a delivery chain to customers. SC's fundamental goal is to "optimize chain efficiency and provide as much benefit as possible with as little expense as possible". In other words, it seeks to unite all the representatives in the SC to work together within the organization as a way to optimize efficiency in the SC and provide the maximum value to all relevant parties. If a company buys raw materials for use in the manufacture of a product, it then sells them to customers, which means that the organization has an SC, which it must manage afterwards. Companies face difficulties in seeking solutions to satisfy ever-increasing consumer demands and stay successful in the markets while maintaining expenses controlled. SC includes handling of a number of tasks related to the arranging, scheduling and monitoring of the flow of supplies, components and products; maintaining inventories of acquired components and packaging issues; reasonable and cost-effective storage of products; and, ultimately, delivering them to the consumer (Khan et al. [2]). Effective governance of SC needs continuous enhancement at both the level at customer support and the internal operational efficiencies of the SC firms. In the most simplistic point, customer support involves reliably adequate order fill levels, strong on-time fulfilment levels and a relatively small number of goods returned by consumers for whatever reason. Internal productivity for SC companies ensures that such entities get an acceptable rate of return for their product and other resource expenditures (Hugos [3]). Mathematical programming frameworks have been commonly used to evaluate and improve SC efficiency, and it could play a significant role in the creation of alternatives to complex SC design.

The neutrosophic set is considered as a generalization of the intuitionistic fuzzy set. While fuzzy sets use true and false for express relationship, neutrosophic sets uses three different types of membership functions (Smarandache [4]). The neutrosophical set has three membership functions, i.e., maximizing truth (belonging), indeterminacy (partly belonging) and reducing falsity (nonbelonging) effectively. The neutrosophical programming approach was developed and widely utilized in real-life applications based on the neutrosophical set. Gamal et al. [5] used neutrosophic set theory in supplier selection to overcome the situation when the decision makers might have restricted knowledge or different opinions, and to specify deterministic valuation values to comparison judgments. Later on, Abdel-Baset et al. [6] proposed an advanced type of neutrosophic technique, called type 2 neutrosophic numbers for the supplier selection problem.

Motivated by different studies in supply chain and neutrosophic programming, which is being a new research area with the potential to capture the decision-makers truth, indeterminacy and falsity goals, we have formulated the mathematical model of supply chain under neutrosophic environment. The objective of this study is to offer SC with a more realistic context for achieving better results in the context of uncertainty. In addition, the neutrosophical compromise programming approach does not just focus on maximizing and minimizing the satisfaction and dissatisfaction of the decision makers, but also on optimizing the degree of satisfaction related to indeterminacy. Moreover, the developed approach is also compared with simple additive, weighted additive and pre-emptive goal programming approaches, to show the efficacy of the proposed methodology.

This paper consists of six sections: the current segment presenting an introduction to the study problem. Section 2 describes relevant work on this topic. Section 3 explains the structure of the SC model. The technique of the solution to solving the problem is discussed in Section 4. Section 5 describes the implementation of the theoretical model to a case study, and Section 6 concludes with the analysis and future directions of research in this area.

2. Literature Review

The literature review undertaken in the framework of this study allowed us to find out a gap in SC optimization. To the extent of our understanding, there is a limited number of research work discussing neutrosophicity utilizing a multiobjective optimization to tackle trade-offs between overall transportation cost and total delivery time in SC. The literature review discussing the issues of transportation and distribution planning constructed as a single and multiobjective model and solved using a complicated approach to optimization.

Badhotiya et al. [7] tackle the issue of distribution, manufacturing and delivery planning for a two-echelon SC, composed of several producers that supplying to different sales locations and formulated it as a multi-objective model. Further ambiguity and imprecision were regarded in the problem, and a fuzzy multi-objective optimization technique was applied that simultaneously optimizes three objectives; total cost, total delivery time, and backorder amount. Rabbani et al. [8] considered a closed-loop SC that involved a logistics supplier for a producer, a dealer and a third party. Three tri-level leader-follower Stackelberg game models have been introduced to explore how a producer can do remanufacturing or pay a product license charge for retailers and partner with them in remanufacturing. Modak and Kelle [9] identified the double-channel SC with contingent stochastic consumer demand under price and distribution period, and the findings indicated that market volatility influences the optimal price and lead time. Sharahi et al. [10] dealt with the issue of location-allocation and delivery of output in an SC of three echelons. Type-II fuzzy sets theory were used to model uncertainty in supply, operation, and demand. Gholamian et al. [11] proposed a mathematical model for production planning by considering the majority of SC expenses parameters, such as cost of shipping, cost of inventory holding, cost of shortage, cost of processing and associated human costs under uncertainty of demand, and formulated it using a complex multi-objective model of optimization. Kristianto et al. [12] suggested a two-stage model with the goal of improving product distribution and transportation when adjustments have disrupted the SC network as a consequence of a catastrophe or market shift. They implemented the methodology of decomposition to transform the problem into the shortest problem of the fuzzy path. Bilgen [13] tackled the problem of fuzzy centralized manufacturing and delivery plans underneath a packaged products company's SC network. Vagueness in the objective function and capacity restrictions is replicated by Zimmermann's [14] linear membership function approach. Three separate aggregation operators were introduced to transform the Fuzzy model into a crisp one.

Current neutrosophic literature shows that a limited number of authors have taken an interest in this framework, and this is expected to be a significant new area of research in the future. Kar et al. [15] proposed a neutrosophic optimization technique for a shortage-free inventory model where the cost of output is inversely proportional to the set-up costs and the volume of supply. A neutrosophical fuzzy programming method (NFPA) focused on the neutrosophic decision was suggested by Ahmad et al. [16] to solve the proposed SC design problem. The developed SC network has been built for various multi-product raw materials/parts, and multi-echelons together with single time horizons. To identify the activities contributing to improving the economic and environmental performance, Abdel-Baset et al. [17] tested green SC activities using the robust rating with neutrosophic set theory. The feasibility of the new approach is measured using the two different types of case studies, i.e., Egypt's petroleum sector and China's manufacturing company. As a technique to solve multi-criteria decision-making in green supplier selection problems, Liang et al. [18] suggested single-valued trapezoidal neutrosophic choice relations. In the neutrosophical framework, Thamaraiselvi and Santhi [19] developed the mathematical representation of a transportation problem. Abdel-Baset et al. [20] addressed the complexities of the issue, increasing awareness among healthcare sector experts, and assessing smart medical devices according to specific assessment requirements. In the decisionmaking process, neutrosophics with TOPSIS methodology was implemented to cope with the vagueness, and ambiguity, by taking into consideration the decision conditions in the evidence gathered by the decision-makers. Liang et al. [21] established a novel fuzzy-based method for assessing B2C e-commerce websites and defined interrelationships and prioritized orders within parameters through integrating single-valued neutrosophic trapezoidal numbers with DEMATEL methodology. Some recent works related to the use neutrosophic includes , Abdel-Basset et al. [22] suggested a novel hybrid methodology for the selection of the offshore wind power plant location integrating the two distinct forms of MCDM approaches in the neutrosophic environment. Also, by use of MCDM model, Abdel-Basset et al. [23] has conducted a comprehensive sustainability assessment of the hydrogen generation possibilities.

Practical alignment of transportation and distribution planning in SC frequently requires trade-offs with multiple conflicting priorities that need to be balanced by the decision-maker at the same time. Owing to many reasons such as variability in human behavior, shifting environmental circumstances, and unavailability or inappropriate knowledge, these objective roles are sometimes fuzzy or uncertain. This study introduces a complex multi-objective programming framework to address the SC problem including multiple locations and different time periods, then illustrates the same on a real-life manufacturing problem to validate the accuracy of the developed model. The benefit of implementing fuzzy set theory is that it helps the decision-maker to calculate an imprecise expectation.

3. Mathematical Model

According to Nurjanni et al. [1], SCM is "A set of approaches utilized to efficiently integrate suppliers, manufacturers, warehouses, and stores, so that merchandise is produced and distributed at the right quantities, to the right locations, and at the right time, in order to minimize system-wide costs while satisfying service level requirements." Charles et al. [24] presented the demand-and-supply-rooted concept of ambiguity with the constrained multiobjective optimization framework and established a fuzzy goal programming approach to solve it. To achieve the desired solution, the proposed model was solved through three separate approaches, including simple additive goal programming, weighted goal programming, and pre-emptive goal programming approaches respectively. Gupta et al. [25] presented an effective goal programming methodology to solve the SC problem in order to concurrently reduce overall shipping costs and total production period related to inventory volumes, initial stock available at each source, as well as customer demand and usable storage capacity at each destination, and restrictions on total expenditure in an uncertain environment. Gupta et al. [26] presented the problem of the SC network as a bi-level programming problem in which the primary goal is to decide the optimum order allocation of goods where the requirements of the consumer and the availability for the items are elastic. Motivated by such studies in SC, we have formulated the multi-objective SC model and the following notations have been used for the model formulation which are listed below:

The nomenclature for the notations and terms used in the design of the model is as follows:

Indices

$i-$ Multiple suppliers indices, *(i=1, 2, …,I);*

$j-$ Multiple plants indices, *(j=1, 2, …,J);*

$k-$ Multiple warehouses indices, *(k=1, 2, …,K);*

$l-$ Multiple distributors indices, *(l=1, 2, …,L);*

$t-$ Objective function indices, *(t=1, 2, …,T);*

Parameters

$SCS_i -$ Supply capacity of the i^{th} suppliers (in '000),

$PCP_j -$ Potential capacity of the j^{th} plants (in'000),

$PCW_k -$ Potential capacity of the k^{th} warehouses (in '000),

$ADR_l -$ Annual demand from the l^{th} distributors (in '000),

$CSP_{ij} -$ Cost of shipping one unit from the supply suppliers *i* to the plant *j,* (in '000),

$CPW_{jk} -$ Cost of producing and shipping one unit from the plant *j* to the warehouse *k,* (in '000),

$CPR_{jl} -$ Cost of producing and shipping one unit from the plant *j* to the distributors *l,* (in '000),

$CWR_{kl} -$ Cost of shipping one unit from the warehouse *k* to the distributors *l,* (in '000),

$TPW_{jk} -$ Delivery time of shipping one unit from the plant *j* to the warehouse *k* (in Hrs),

$TPR_{jl} -$ Delivery time of shipping one unit from the plant *j* to the distributors *l* (in Hrs),

$TWR_{kl} -$ Delivery time of shipping one unit from the warehouse *k* to the distributors *l* (in Hrs),

Decision variables

$W_{ij} -$ Quantity shipped from the supply suppliers *i* to the plant *j*

$X_{jk} -$ Quantity shipped from the plant *j* to the warehouse *k*

$Y_{jl} -$ Quantity shipped from the plant *j* to the distributors *i*

Z_{kl} − Quantity shipped from the warehouse *k* to the distributors *i*

The mathematical model of multi-objective SC problem formulated in the case of a deterministic situation by using the notations mentioned above as:

The 1st objective function, which helps in the optimization of the SC shipping costs, is given by:
Minimize

$$F_1 = \sum_{i=1}^{I}\sum_{j=1}^{J} CSP_{ij}W_{ij} + \sum_{j=1}^{J}\sum_{k=1}^{K} CPW_{jk}X_{jk} + \sum_{j=1}^{J}\sum_{l=1}^{L} CPR_{jl}Y_{jl} + \sum_{k=1}^{K}\sum_{l=1}^{L} CWR_{kl}Z_{kl} \quad (1)$$

The 2nd objective function, which helps in the optimization of the SC delivery time, is given by:
Minimize

$$F_2 = \sum_{j=1}^{J}\sum_{k=1}^{K} TPW_{jk}X_{jk} + \sum_{j=1}^{J}\sum_{l=1}^{L} TPR_{jl}Y_{jl} + \sum_{k=1}^{K}\sum_{l=1}^{L} TWR_{kl}Z_{kl} \quad (2)$$

Subject to

Constraint I is related to the overall volume of the product to be delivered from the supplier to the plant.

$$\sum_{j=1}^{J} W_{ij} \le SCS_i \quad (3)$$

Constraint II is concerned with the quantity produced at the plant, which cannot surpass its efficiency.

$$\sum_{l=1}^{L} Y_{jl} + \sum_{k=1}^{K} X_{jk} \le PCP_j \quad (4)$$

Constraint III is concerned with the volume to be delivered via the various warehouses that cannot surpass its efficiency.

$$\sum_{l=1}^{L} Z_{kl} \le PCW_k \quad (5)$$

Constraint IV is concerned about the volume to be delivered to the distributors, which will meet the demand of the consumer.

$$\sum_{k=1}^{K} Z_{kl} + \sum_{j=1}^{J} Y_{jl} \le ADR_l \quad (6)$$

Constraint V is concerned with the total quantity delivered to the warehouse and distributors from the plant, which cannot surpass the quantity of the obtained materials.

$$\sum_{i=1}^{I} W_{ij} \ge \sum_{k=1}^{K} X_{jk} + \sum_{l=1}^{L} Y_{jl} \quad (7)$$

Constraint VI is concerned with the volume delivered to the distributors from the warehouse, which cannot surpass their capacity.

$$\sum_{j=1}^{J} X_{jk} \ge \sum_{l=1}^{L} Z_{kl} \quad (8)$$

with non-negative restriction:

$$W_{ij} \ge 0, \forall i, j$$

$$X_{jk} \ge 0, \forall j, k$$

$$Y_{jl} \ge 0, \forall j, l$$

$$Z_{kl} \ge 0, \forall k, l$$

The multi-objective optimization model of SC can be mathematically formulated as follows by combining all the objective functions and constraints, are combined:

Model 1

$$\text{Minimize } F_1 = \sum_{i=1}^{I}\sum_{j=1}^{J} CSP_{ij}W_{ij} + \sum_{j=1}^{J}\sum_{k=1}^{K} CPW_{jk}X_{jk} + \sum_{j=1}^{J}\sum_{l=1}^{L} CPR_{jl}Y_{jl} + \sum_{k=1}^{K}\sum_{l=1}^{L} CWR_{kl}Z_{kl}$$

$$\text{Minimize } F_2 = \sum_{j=1}^{J}\sum_{k=1}^{K} TPW_{jk}X_{jk} + \sum_{j=1}^{J}\sum_{l=1}^{L} TPR_{jl}Y_{jl} + \sum_{k=1}^{K}\sum_{l=1}^{L} TWR_{kl}Z_{kl}$$

Subject to

$$\sum_{j=1}^{J} W_{ij} \le SCS_i$$

$$\sum_{l=1}^{L} Y_{jl} + \sum_{k=1}^{K} X_{jk} \le PCP_j$$

$$\sum_{l=1}^{L} Z_{kl} \le PCW_k$$

$$\sum_{k=1}^{K} Z_{kl} + \sum_{j=1}^{J} Y_{jl} \le ADR_l$$

$$\sum_{i=1}^{I} W_{ij} \ge \sum_{k=1}^{K} X_{jk} + \sum_{l=1}^{L} Y_{jl}$$

$$\sum_{j=1}^{J} X_{jk} \ge \sum_{l=1}^{L} Z_{kl}$$

$$W_{ij} \ge 0, \forall i, j$$

$$X_{jk} \ge 0, \forall j, k$$

$$Y_{jl} \ge 0, \forall j, l$$

$$Z_{kl} \ge 0, \forall k, l$$

3.1 **Uncertain Model**

The model formulated above has been developed when the decision-maker knows the exact value of each parameter being considered. Due to sudden increases in prices of raw materials, higher gasoline costs, higher deployment sites, fluctuating consumer behavior, rivalry amongst customer service policies of various firms, environmental factors, inability to supply requested goods in a timely manner, political and government decisions on specific taxes on purchase, development, delivery end-of-use stock management are the most influential factors creating uncertainty in SC. In the past many methods were suggested to cope with the environment of ambiguity. Zadeh's [27] fuzzy sets (FS) just allow membership function and can't accommodate certain vagueness parameters. In order to address this knowledge deficit, Atanassov [28] proposed an expansion to fuzzy sets called intuitionistic fuzzy sets (IFS). Though IFS theory can accommodate missing knowledge for specific real-world problems, it cannot solve all forms of ambiguity such as contradictory and indeterminate proof. Therefore, the neutrosophic set (NS) was developed by Smarandache [29] as a comprehensive composition that generalizes classical theory of all forms of FS. NS can handle indefinite, vague and conflicting information where the indeterminacy is explicitly quantified, and can separately identify the three forms of membership functions. Furthermore, with such assumptions of uncertainty, Model 1 with uncertain parameters could be reformulated as:

Model 2

$$\text{Minimize } F_1 = \sum_{i=1}^{I}\sum_{j=1}^{J}\tilde{CSP}_{ij} W_{ij} + \sum_{j=1}^{J}\sum_{k=1}^{K}\tilde{CPW}_{jk} X_{jk} + \sum_{j=1}^{J}\sum_{l=1}^{L}\tilde{CPR}_{jl} Y_{jl} + \sum_{k=1}^{K}\sum_{l=1}^{L}\tilde{CWR}_{kl} Z_{kl}$$

$$\text{Minimize } F_2 = \sum_{j=1}^{J}\sum_{k=1}^{K}\tilde{TPW}_{jk} X_{jk} + \sum_{j=1}^{J}\sum_{l=1}^{L}\tilde{TPR}_{jl} Y_{jl} + \sum_{k=1}^{K}\sum_{l=1}^{L}\tilde{TWR}_{kl} Z_{kl}$$

Subject to

$$\sum_{j=1}^{J} W_{ij} \le \tilde{SCS}_i$$

$$\sum_{l=1}^{L} Y_{jl} + \sum_{k=1}^{K} X_{jk} \le PCP_j$$

$$\sum_{l=1}^{L} Z_{kl} \le PCW_k$$

$$\sum_{k=1}^{K} Z_{kl} + \sum_{j=1}^{J} Y_{jl} \le \tilde{ADR}_l$$

$$\sum_{i=1}^{I} W_{ij} \ge \sum_{k=1}^{K} X_{jk} + \sum_{l=1}^{L} Y_{jl}$$

$$\sum_{j=1}^{J} X_{jk} \ge \sum_{l=1}^{L} Z_{kl}$$

$$W_{ij} \ge 0, \forall i, j$$

$$X_{jk} \ge 0, \forall j, k$$

$$Y_{jl} \ge 0, \forall j, l$$

$$Z_{kl} \ge 0, \forall k, l$$

where, the uncertain parameters $\tilde{CSP}, \tilde{CPW}, \tilde{CPR}, \tilde{CWR}, \tilde{TPW}, \tilde{TPR}, \tilde{TWR}, \tilde{SCS}$ and $\tilde{ADR}$ are assumed to hold the neutrosophic sets assumptions (detail see Liang et al. [18]). Let us assumed that $\delta_{\tilde{CSP}}, \varphi_{\tilde{CSP}}, \gamma_{\tilde{CSP}} \in [0,1]$ and $CSP_1, CSP_2, CSP_3 \in \Re$ such that $CSP_1 \le CSP_2 \le CSP_3$. Then a single-value triangular neutrosophic number $\tilde{CSP} = \left((CSP_1, CSP_2, CSP_3); \delta_{\tilde{CSP}}, \varphi_{\tilde{CSP}}, \gamma_{\tilde{CSP}}\right)$ is a special neutrosophic set on the real line set $\Re$, whose truth-membership, indeterminacy-membership, and falsity-membership functions are given as follows:

$$\mu_{\tilde{CSP}}(CSP) = \begin{cases} \delta_{\tilde{CSP}} \dfrac{(CSP - CSP_1)}{(CSP_2 - CSP_1)}, & CSP_1 \le CSP \le CSP_2 \\ \delta_{\tilde{CSP}}, & CSP = CSP_2 \\ \delta_{\tilde{CSP}} \dfrac{(CSP_3 - CSP)}{(CSP_3 - CSP_2)}, & CSP_2 \le CSP \le CSP_3 \\ 0, & otherwise \end{cases} \tag{9}$$

$$\theta_{\tilde{CSP}}(CSP) = \begin{cases} \dfrac{(CSP_2 - CSP + \varphi_{\tilde{CSP}}(CSP - CSP_1))}{(CSP_2 - CSP_1)}, & CSP_1 \le CSP \le CSP_2 \\ \varphi_{\tilde{CSP}}, & CSP = CSP_2 \\ \dfrac{(CSP - CSP_2 + \varphi_{\tilde{CSP}}(CSP_3 - CSP))}{(CSP_3 - CSP_2)}, & CSP_2 \le CSP \le CSP_3 \\ 0, & otherwise \end{cases} \tag{10}$$

$$\psi_{\tilde{CSP}}(CSP)=\begin{cases}\dfrac{(CSP_2-CSP+\gamma_{\tilde{CSP}}(CSP-CSP_1))}{(CSP_2-CSP_1)}, & CSP_1\le CSP\le CSP_2\\ \gamma_{\tilde{CSP}}, & CSP=CSP_2\\ \dfrac{(CSP-CSP_2+\gamma_{\tilde{CSP}}(CSP_3-CSP))}{(CSP_3-CSP_2)}, & CSP_2\le CSP\le CSP_3\\ 0, & otherwise\end{cases} \tag{11}$$

where $\delta_{\tilde{CSP}}, \varphi_{\tilde{CSP}}, \gamma_{\tilde{CSP}}$ denote the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree, respectively. A single-valued triangular neutrosophic number $\tilde{CSP}=\left((CSP_1,CSP_2,CSP_3);\delta_{\tilde{CSP}},\varphi_{\tilde{CSP}},\gamma_{\tilde{CSP}}\right)$ may express an ill-defined quantity about CSP, which is approximately equal to CSP. Then, the score function for the $\tilde{CSP}$ is obtained by using the equation (12), which is given below:

$$S(\tilde{CSP})=\frac{1}{16}(CSP_1+CSP_2+CSP_3)\times(2+\delta_{\tilde{CSP}}-\varphi_{\tilde{CSP}}-\gamma_{\tilde{CSP}}) \tag{12}$$

The same holds for other uncertain parameters.

3.2 Neutrosophic Compromise Programming

An approach to solving the multi-optimization problem has been implemented based on the NS principle. The neutrosophical compromise goal programming solution is based on the principle of NS, which consists of optimization of three membership functions such as optimizing the degree of truth and indeterminacy and decreasing the extent of falsity membership. Firstly, the bounds for each objective function have been defined to construct the three different types of membership functions for the formulated multi-objective SC problem. The upper $U_t, \forall t$ and lower $L_t, \forall t$ values for the neutrosophical problem for case minimization have therefore been determined as:

$$U_t^T=U_t,\ L_t^T=L_t,\ \forall t \qquad \text{for truth membership}$$

$$U_t^I=L_t^T+q_t^{'}(U_t^T-L_t^T), L_t^I=L_t^T,\ \forall t \qquad \text{for Indeterminacy membership}$$

$$U_t^F=U_t^T,\ L_t^F=L_t^T+q_t(U_t^T-L_t^T),\ \forall t \qquad \text{for falsity membership}$$

Where q_t and $q_t^{'}$ are sensitivity variables for falsity and indeterminacy membership functions shall be selected by the decision-maker, and based on these sensitivity variables, the three different types of membership function for the neutrosophical problem can be constructed as follows:

$$\mu_t^T=\begin{cases}1, & F_t\le L_t^T\\ \dfrac{U_t^T-F_t}{U_t^T-L_t^T}, & L_t^T\le F_t\le U_t^T\\ 0, & F_t\ge U_t^T\end{cases}$$

$$\sigma_t^I=\begin{cases}1, & F_t\le L_t^I\\ \dfrac{U_t^I-F_t}{U_t^I-L_t^I}, & L_t^I\le F_t\le U_t^I\\ 0, & F_t\ge U_t^I\end{cases}$$

$$\upsilon_t^F=\begin{cases}0, & F_t\le L_t^F\\ \dfrac{F_t-L_t^F}{U_t^F-L_t^F}, & L_t^F\le F_t\le U_t^F\\ 1, & F_t\ge U_t^F\end{cases}$$

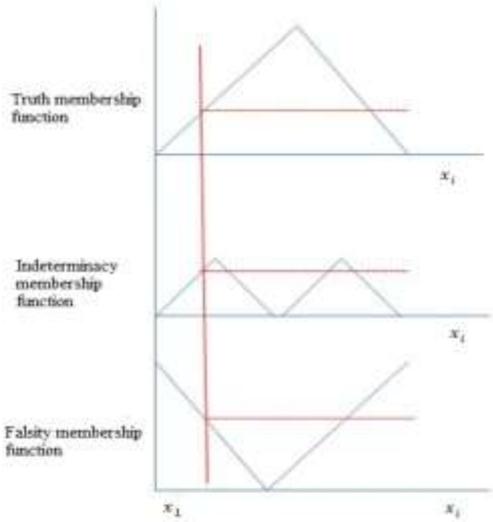


Fig. 1. Neutrosophication Process

Where, we trying to maximize the Truth (μ_t^T) and Indeterminacy (σ_t^I) membership functions; and also trying to minimize the falsity (υ_t^F) membership functions. Following the optimization process introduced by (Bellman and Zadeh, [30]; Rizk-Allah et al., [31]; Das et al. [32]; Khan et al. [33]), the multi-objective SCN neutrosophical optimization model can be formulated as follows:

Model 2(a)

$$\text{Max}\min \mu_t^T$$
$$\text{Max}\min \sigma_t^I$$
$$\text{Min}\max \upsilon_t^F$$

Subject to

$$\sum_{j=1}^{J} W_{ij} \le \tilde{SCS}_i$$
$$\sum_{l=1}^{L} Y_{jl} + \sum_{k=1}^{K} X_{jk} \le PCP_j$$
$$\sum_{l=1}^{L} Z_{kl} \le PCW_k$$
$$\sum_{k=1}^{K} Z_{kl} + \sum_{j=1}^{J} Y_{jl} \le \tilde{ADR}_l$$
$$\sum_{i=1}^{I} W_{ij} \ge \sum_{k=1}^{K} X_{jk} + \sum_{l=1}^{L} Y_{jl}$$
$$\sum_{j=1}^{J} X_{jk} \ge \sum_{l=1}^{L} Z_{kl}$$
$$W_{ij} \ge 0, \forall i, j$$
$$X_{jk} \ge 0, \forall j, k$$
$$Y_{jl} \ge 0, \forall j, l$$
$$Z_{kl} \ge 0, \forall k, l$$

It is not easy to solve the above model (2a) with the presence of three objective functions, therefore with the help of auxiliary parameters, the model (2a) can be transformed into a single objective model, given below:

Model 2(b)

$$\text{Maximize } \sum_{t=1}^{2}(\mu_t + \sigma_t - \upsilon_t)$$

Subject to

$$\mu_t^T \geq \mu_t,\ \forall\ t$$
$$\sigma_t^T \geq \sigma_t,\ \forall\ t$$
$$\upsilon_t^T \leq \upsilon_t,\ \forall\ t$$
$$\mu_t \geq \sigma_t,\ \forall t$$
$$\mu_t \geq \upsilon_t,\ \forall\ t$$
$$\mu_t + \sigma_t + \upsilon_t \leq 3,\ \forall t$$

constraints of model 2(a)

The above Model 2(b) has been used to get the compromise solution of the formulated problem.

4. Numerical Illustration

In view of demonstrating the method established, we considered the fictional scenario of modeling and optimizing a SC network situation, with some imprecise data being considered on it, described by neutrosophical triangular fuzzy numbers. We assumed a network consisting of multiple numbers of suppliers, multiple numbers of production plants, multiple numbers of warehouses and multiple numbers of distributors, in various regional areas or places. Five suppliers are assumed to distribute the raw resources to four manufacturing plants. The delivery network consists of six warehouses where, before being shipped out to eight distributors, goods are temporarily positioned and processed, and eventually, items are shipped out to many consumers. The imprecise information in Tables 1-8 are listed below:

Table 1. Uncertain Transportation Cost from the Supplier to the Manufacturing Plant.

Supplier	**Manufacturing Plant**			
	P_1	**P_2**	**P_3**	**P_4**
S_1	((196,199,202); 0.3,0.4,0.5)	((89,93,97); 0.6,0.7,0.8)	((146,148,150); 0.2,0.3,0.4)	((194,196,198); 0.1,0.2,0.3)
S_2	((294,306,312); 0.6,0.8,0.9)	((146,148,150); 0.2,0.3,0.4)	((194,196,198); 0.1,0.2,0.3)	(((196,199,202); 0.3,0.4,0.5)
S_3	((491,499,507); 0.1,0.2,0.3)	((119,121,123); 0.4,0.5,0.6)	((204,206,208); 0.1,0.2,0.3)	((202,205,208); 0.3,0.4.0.5)
S_4	((389,394,399); 0.7,0.8,0.9)	((296,300,304); 0.5,0.6,0.7)	((239,244,249); 0.7,0.8,0.9)	((296,300,304); 0.5,0.6,0.7)
S_5	((591,599,607); 0.3,0.4.0.5)	((689,691,693); 0.3,0.4,0.5)	((296,300,304); 0.5,0.6,0.7)	((339,341,345); 0.4,0.5,0.6)

Table 2. Uncertain Transportation Cost from the Plant to the Distributor.

Plant	**Distributor**							
	D_1	**D_2**	**D_3**	**D_4**	**D_5**	**D_6**	**D_7**	**D_8**
P_1	((296,300,304); 0.5,0.6,0.7)	((429,432,436); 0.4,0.5,0.6)	((341,344,347); 0.6,0.8,0.9)	((429,432,436); 0.4,0.5,0.6)	((204,206,208); 0.1,0.2,0.3)	((339,341,345); 0.4,0.5,0.6)	((391,394,396); 0.6,0.7,0.8)	((469,471,473); 0.1,0.2,0.3)
P_2	((339,341,345); 0.4,0.5,0.6)	((491,494,496); 0.2,0.3,0.4)	((294,300,306); 0.2,0.3,0.4)	((371,374,378); 0.5,0.6,0.7)	((269,272,275); 0.7,0.8,0.9)	((371,374,378); 0.5,0.6,0.7)	((469,471,473); 0.1,0.2,0.3)	((431,435,439); 0.7,0.8,0.9)

P_3	((431,435,439); 0.7,0.8,0.9)	((469,472,475); 0.1,0.2,0.3)	((341,343,346); 0.7,0.8,0.9)	((339,341,345); 0.4,0.5,0.6)	((296,298,300); 0.1,0.2,0.3)	((369,371,324); 0.3,0.4,0.5)	((431,435,439); 0.7,0.8,0.9)	((469,471,473); 0.1,0.2,0.3)
P_4	((489,492,495); 0.5,0.6,0.7)	((431,435,438); 0.3,0.4.0.5)	((319,321,324); 0.5,0.6,0.7)	((391,394,396); 0.6,0.7,0.8)	((319,321,323) 0.3,0.4,0.5)	((386,388,400); 0.3,0.4,0.5)	((319,321,323) 0.3,0.4,0.5)	((431,435,439); 0.7,0.8,0.9)

Table 3. Uncertain Transportation Cost from the Manufacturing Plant to the Warehouse.

Plant	**Warehouses**					
	W_1	**W_2**	**W_3**	**W_4**	**W_5**	**W_6**
P_1	((296,300,304); 0.5,0.6,0.7)	((144,148,152); 0.2,0.3,0.4)	((196,199,202); 0.3,0.4,0.5)	((196,199,202); 0.3,0.4,0.5)	((121,123,125); 0.3,0.4.0.5)	((296,300,304); 0.5,0.6,0.7)
P_2	((389,392,395); 0.2,0.3,0.4)	((121,123,125); 0.3,0.4.0.5)	((219,221,225); 0.4,0.5,0.6)	((241,244,247); 0.6,0.7,0.8)	((269,271,273); 0.3,0.4,0.5)	((311,313,317); 0.4,0.5,0.6)
P_3	((541,545,548); 0.7,0.8,0.9)	((144,148,152); 0.2,0.3,0.4)	((196,199,202); 0.3,0.4,0.5)	((296,300,304); 0.5,0.6,0.7)	((241,244,247); 0.6,0.7,0.8)	((296,300,304); 0.5,0.6,0.7)
P_4	((639,641,643); 0.6,0.7,0.8)	((341,344,347); 0.6,0.8,0.9)	((296,300,304); 0.5,0.6,0.7)	((121,123,125); 0.3,0.4.0.5)	((294,296,298); 0.4,0.5,0.6)	((301,303,307); 0.4,0.5,0.6)

Table 4. Uncertain Transportation Cost from the Warehouses to the Distributor.

Ware house	**Distributor**							
	D_1	**D_2**	**D_3**	**D_4**	**D_5**	**D_6**	**D_7**	**D_8**
W_1	((146,148,150); 0.3,0.4,0.5)	((179,182,183); 0.1,0.2,0.3)	((161,163,165); 0.5,0.6,0.7)	((169,171,173); 0.3,0.4,0.5)	((169,171,173); 0.3,0.4,0.5)	((194,196,198); 0.1,0.2,0.3)	((181,184,187); 0.1,0.2,0.3)	((164,166,168); 0.1,0.2,0.3)
W_2	((109,111,113); 0.2,0.3,0.4)	((191,193,195); 0.5,0.6,0.7)	((164,166,168); 0.1,0.2,0.3)	((166,168,170); 0.3,0.4,0.5)	((179,181,184); 0.2,0.4,0.5)	((181,184,187); 0.1,0.2,0.3)	((179,181,184); 0.2,0.4,0.5)	((172,174,176); 0.7,0.8,0.9)
W_3	((121,124,128); 0.5,0.6,0.7)	((189,191,194); 0.3,0.4,0.5)	((131,134,137); 0.5,0.6,0.7)	((176,179,183); 0.1,0.2,0.3)	((179,181,184); 0.2,0.4,0.5)	((181,184,187); 0.1,0.2,0.3)	((181,184,187); 0.1,0.2,0.3)	((169,171,173); 0.3,0.4,0.5)
W_4	((126,129,132); 0.7,0.8,0.9)	((169,171,173); 0.3,0.4,0.5)	((136,139,142); 0.1,0.2,0.3)	((181,184,187); 0.1,0.2,0.3)	((189,191,194); 0.3,0.4,0.5)	((171,175,180); 0.2,0.4,0.5)	((179,181,184); 0.2,0.4,0.5)	((171,174,176); 0.7,0.8,0.9)
W_5	((136,139,142); 0.1,0.2,0.3)	((169,171,173); 0.2,0.4,0.5)	((146,148,150); 0.3,0.4,0.5)	((179,181,184); 0.2,0.4,0.5)	((191,194,198); 0.7,0.8,0.9)	((159,161,164); 0.2,0.4,0.5)	((191,194,196); 0.3,0.4,0.5)	((169,171,173); 0.3,0.4,0.5)
W_6	((169,171,173); 0.3,0.4,0.5)	((151,153,155); 0.7,0.8,0.9)	((146,148,150); 0.3,0.4,0.5)	((191,193,194); 0.3,0.4,0.5)	((194,196,198); 0.1,0.2,0.3)	((181,184,187); 0.1,0.2,0.3)	((189,191,194); 0.3,0.4,0.5)	((164,166,168); 0.1,0.2,0.3)

Table 5. Uncertain Delivery Time of Item from the Plant to the Distributor.

Plant	**Distributor**							
	D_1	**D_2**	**D_3**	**D_4**	**D_5**	**D_6**	**D_7**	**D_8**
P_1	((46,49,52); 0.1,0.2,0.3)	((64,68,72); 0.1,0.2,0.3)	((51,56,59); 0.3,0.4,0.5)	((59,61,64); 0.1,0.2,0.3)	((36,40,44); 0.5,0.6,0.7)	((46,49,52); 0.1,0.2,0.3)	((71,74,78); 0.2,0.4,0.5)	((74,88,94); 0.4,0.5,0.6)
P_2	((29,33,37); 0.7,0.8,0.9)	((56,58,60); 0.2,0.4,0.5)	((39,41,45); 0.1,0.2,0.3)	((40,44,48); 0.7,0.8,0.9)	((19,21,24); 0.4,0.5,0.6)	((41,46,50); 0.3,0.4,0.5)	((64,76,1,3) 0.7,0.8,0.9)	((91,93,95); 0.1,0.2,0.3)

P_3	((71,74,78); 0.2,0.4,0.5)	((64,76,1,3) 0.7,0.8,0.9)	((71,74,78); 0.2,0.4,0.5)	((76,80,84); 0.2,0.4,0.5)	((56,60,62); 0.3,0.4,0.5)	((66,70,74); 0.1,0.2,0.3)	((71,74,78); 0.2,0.4,0.5)	((89,93,96); 0.3,0.4,0.5)
P_4	((89,93,96); 0.3,0.4,0.5)	((91,93,95); 0.1,0.2,0.3)	((74,78,82); 0.7,0.8,0.9)	((81,83,85); 0.7,0.8,0.9)	((54,58,60); 0.5,0.6,0.7)	((66,74,78); 0.5,0.6,0.7)	((74,88,94); 0.4,0.5,0.6)	((81,83,85); 0.7,0.8,0.9)

Table 6. Uncertain Delivery Time of Item from the Manufacturing Plant to the Warehouse.

Plant	**Warehouses**					
	W_1	**W_2**	**W_3**	**W_4**	**W_5**	**W_6**
P_1	((26,34,,42); 0.7,0.8,0.9)	((14,26,34); 0.2,0.4,0.5)	((16,24,32); 0.1,0.2,0.3)	((9,16,23); 0.7,0.8,0.9)	((26,29,34); 0.2,0.4,0.5)	((24,31,38); 0.7,0.8,0.9)
P_2	((34,46,54); 0.1,0.2,0.3)	((16,24,32); 0.1,0.2,0.3)	((19,31,35); 0.2,0.4,0.5)	((26,34,,42); 0.7,0.8,0.9)	((24,31,35); 0.1,0.2,0.3)	((36,39,44); 0.2,0.4,0.5)
P_3	((51,59,64); 0.2,0.4,0.5)	((54,66,72); 0.7,0.8,0.9)	((51,59,64); 0.2,0.4,0.5)	((54,66,72); 0.7,0.8,0.9)	((56,64,72); 0.5,0.6,0.7)	((66,74,78); 0.5,0.6,0.7)
P_4	((76,80,84); 0.2,0.4,0.5)	((54,66,72); 0.7,0.8,0.9)	((29,33,37); 0.7,0.8,0.9)	((51,59,64); 0.2,0.4,0.5)	((64,68,72); 0.1,0.2,0.3)	((71,74,77); 0.2,0.3,0.4)

Table 7. Uncertain Delivery Time of Item from the Warehouse to the Distributor

Warehouses	**Distributor**							
	D_1	**D_2**	**D_3**	**D_4**	**D_5**	**D_6**	**D_7**	**D_8**
W_1	((16,24,28); 0.7,0.8,0.9)	((14,26,34); 0.2,0.4,0.5)	((21,29,35); 0.2,0.4,0.5)	((26,29,34); 0.2,0.4,0.5)	((21,29,35); 0.2,0.4,0.5)	((19,22,25); 0.7,0.8,0.9)	((36,40,44); 0.5,0.6,0.7)	((29,33,37); 0.7,0.8,0.9)
W_2	((19,22,25); 0.7,0.8,0.9)	((16,24,28); 0.7,0.8,0.9)	((19,31,35); 0.2,0.4,0.5)	((21,24,28); 0.1,0.2,0.3)	((26,29,34); 0.2,0.4,0.5)	((26,29,34); 0.2,0.4,0.5)	(29,35,38); 0.2,0.4,0.5)	((21,24,28); 0.1,0.2,0.3)
W_3	((21,29,35); 0.2,0.4,0.5)	((14,26,34); 0.2,0.4,0.5)	((21,29,35); 0.2,0.4,0.5)	((29,33,37); 0.7,0.8,0.9)	((31,34,38); 0.7,0.8,0.9)	((34,46,54); 0.1,0.2,0.3)	((41,44,48); 0.3,0.4,0.5)	((34,46,54); 0.1,0.2,0.3)
W_4	((14,26,34); 0.2,0.4,0.5)	((24,31,35); 0.1,0.2,0.3)	((19,22,25); 0.7,0.8,0.9)	((24,31,35); 0.1,0.2,0.3)	((24,31,35); 0.1,0.2,0.3)	((26,29,34); 0.2,0.4,0.5)	((19,31,35); 0.2,0.4,0.5)	((21,29,35); 0.2,0.4,0.5)
W_5	((16,24,28); 0.7,0.8,0.9)	((19,22,25); 0.7,0.8,0.9))	((16,24,28); 0.7,0.8,0.9)	((16,24,28); 0.7,0.8,0.9)	((34,46,54); 0.1,0.2,0.3)	((34,46,54); 0.1,0.2,0.3)	((36,40,44); 0.5,0.6,0.7)	((41,44,48); 0.3,0.4,0.5)
W_6	((16,24,28); 0.7,0.8,0.9)	((19,31,35); 0.2,0.4,0.5)	((14,26,34); 0.2,0.4,0.5)	((14,26,34); 0.2,0.4,0.5)	((29,33,37); 0.7,0.8,0.9)	((31,34,36); 0.2,0.4,0.5)	((41,44,48); 0.3,0.4,0.5)	((64,68,72); 0.1,0.2,0.3)

Table 8. Right hand side parameters

Fuzzy demand	**Fuzzy supply**	**Fixed capacity of plant**	**Fixed capacity of warehouse**
((180,190,200); 0.7,0.8,0.9)	((90,95,100); 0.2,0.4,0.5)	470	150
((480,490,500); 0.1,0.2,0.3)	((50,55,60); 0.3,0.4,0.5)	300	180
((200,210,220); 0.2,0.4,0.5)	((85,90,95); 0.1,0.2,0.3)	330	160
((205,215,225); 0.3,0.4,0.5)	((65,70,75); 0.4,0.5,0.6)	320	200
((290,300,310); 0.4,0.5,0.6)	((60,65,70); 0.7,0.8,0.9)		180
	((105,110,115); 0.4,0.5,0.6)		220
	((110,115,120); 0.5,0.6,0.7)		
	((80,85,90); 0.3,0.4,0.5)		

By using all the information given in the table from 1 to 8, the multi-objective SC problem has been formulated. With the presence of uncertainty, the model cannot be solved directly; therefore, the crisp model has been obtained by using the equation (12). Before solving the formulated non-linear multi-objective SC model, the feasibility of the formulated is determined by using the LINGO software (LINGO software is a comprehensive tool designed to make building and solving Linear and Nonlinear (convex and non-convex) programming problem) by determining the lower and upper bound of both the objective functions. LINGO software includes identification of the infeasibility and unboundness of the formulated linear and non-linear model. The Solver Status box of LINGO software details the model classification (linear, non-linear or other), state of the current solution (whether local or global optimum, feasible or infeasible, etc.), the value of the objective function, the infeasibility of the model (amount constraints are violated by), and the number of iterations required to solve the model.

After checking the feasibility of the model construct, the next task is to solve the formulated multi-objective SC model by using the neutrosophic compromise programming. Neutrosophic compromise programming has the key advantage over the other techniques because it helps the decision-makers to consider three categories of membership functions (truth degree, falsity degree or degree of indeterminacy) and while other techniques employed for solving a multi-objective model only takes one membership function dependent on both upper and lower limits of the objective functions. For solving the formulated problem, decision-maker first solve the multiple objective optimization problem by considering a single objective at a time and ignoring the others objectives with the given set of constraints. The solution thus obtained is consider as the idle solution for each of the objective functions and helps in the determination of aspiration level to each of the objective functions. The bounds for the two objective functions are determined as:

The truth membership functions for the first and second objective functions are constructed as follows.

$$\mu_1^T(F_1(x)) = \begin{cases} 1 & \text{if } F_1(x) < 278631.5 \\ \dfrac{394228.8 - F_1(x)}{394228.8 - 278631.5} & \text{if } F_1(x) \in [278631.5, 394228.8] \\ 0 & \text{if } F_1(x) > 394228.8 \end{cases}$$

$$\mu_2^T(F_2(x)) = \begin{cases} 1 & \text{if } F_2(x) < 25273.76 \\ \dfrac{28307.68 - F_2(x)}{28307.68 - 25273.76} & \text{if } F_2(x) \in [25273.76, 28307.68] \\ 0 & \text{if } F_2(x) > 28307.68 \end{cases}$$

The Indeterminacy membership functions for the first and second objective functions are constructed as follows.

$$\sigma_1^I(F_1(x)) = \begin{cases} 1 & \text{if } F_1(x) < 278631.5 \\ \dfrac{382669.07 - F_1(x)}{382669.07 - 278631.5} & \text{if } F_1(x) \in [278631.5, 382669.07] \\ 0 & \text{if } F_1(x) > 382669.07 \end{cases}$$

$$\sigma_2^I(F_2(x)) = \begin{cases} 1 & \text{if } F_2(x) < 25273.76 \\ \dfrac{28004.288 - F_2(x)}{28004.288 - 25273.76} & \text{if } F_2(x) \in [25273.76, 28004.288] \\ 0 & \text{if } F_2(x) > 28004.288 \end{cases}$$

The falsity membership functions for the first and second objective functions are constructed as follows.

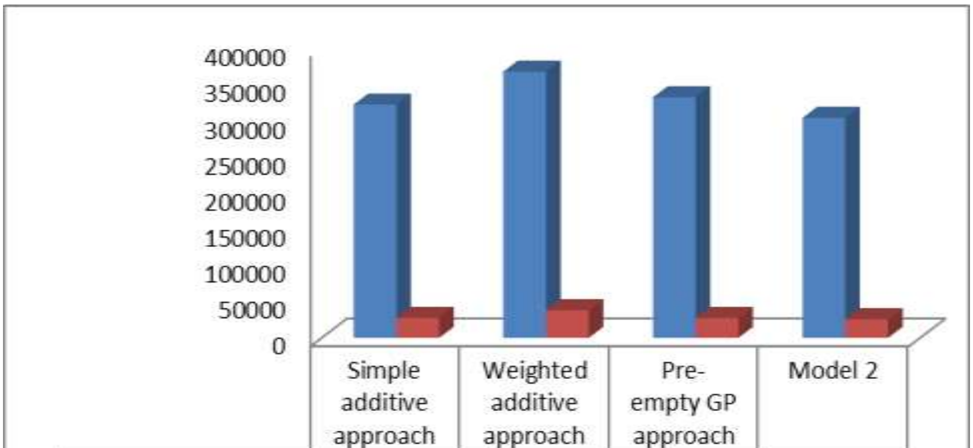
$$\upsilon_1^F(F_1(x)) = \begin{cases} 0 & \text{if } F_1(x) < 290191.23 \\ \dfrac{F_1(x) - 290191.23}{394228.8 - 290191.23} & \text{if } F_1(x) \in [290191.23, 394228.8] \\ 1 & \text{if } F_1(x) > 394228.8 \end{cases}$$

$$\upsilon_2^F(F_2(x)) = \begin{cases} 0 & \text{if } F_2(x) < 25577.152 \\ \dfrac{F_2(x) - 25577.152}{28307.68 - 25577.152} & \text{if } F_2(x) \in [25577.152, 28307.68] \\ 1 & \text{if } F_2(x) > 28307.68 \end{cases}$$

After combining all the membership function together, the compromise solution for the multi-objective SC neutrosophic model is obtained as:

$$F_1 = 304305.60, F_2 = 25742.69, \mu_1^T = 0.7779007, \sigma_1^I = 0.7532231, \upsilon_1^F = 0.1356658,$$
$$\mu_2^T = 0.8454378, \sigma_2^I = 0.8282642, \upsilon_2^F = 0.06062465, W_{11} = 135, W_{14} = 47, W_{22} = 275,$$
$$W_{24} = 9, W_{32} = 198, X_{14} = 135, X_{22} = 177, X_{44} = 56, Y_{21} = 93, Y_{24} = 31, Y_{25} = 63,$$
$$Y_{26} = 109, Z_{22} = 54, Z_{23} = 88, Z_{24} = 35, Z_{47} = 110, Z_{48} = 81$$

After using the neutrosophic compromise programming, the total minimum transportation cost incurred from various multiple sources to different distributors through multiple plants and warehouses is 304305:60; furthermore, the minimum delivery time taken from various multiple sources to different distributors through multiple plants and warehouses is 25742:69. The final finished goods quantity to be shipped from various multiple plants to various warehouses is 368 units; the quantity to be shipped from various multiple plants to various distributors is 296 units; the quantity to be shipped from various multiple warehouses to various distributors is 368 units. We have also compared the proposed work of neutrosophic compromise programming with other well-known techniques used to solve the multi-objective model. The used approach of neutrosophic compromise programming is based on three different types of membership functions, i.e., the degree of truth and indeterminacy and the extent of falsity membership that provides more flexibility in decision making process. To show the efficacy of the proposed work, the formulated model has been solved by using three different approaches namely, simple additive approach, simple weighted additive approach, and pre-emptive goal programming approach. The obtained result has been presented in below Fig. 2, shows the supremacy of the proposed work over other methods.



	Simple additive approach	Weighted additive approach	Pre-empty GP approach	Model 2
Transportation Cost	322788.3	368301.5	332788.3	304305.6
Delivery Time	27479.3	38185.27	27479.04	25742.69

Fig. 2 Result Comparison

After obtaining the deterministic form of each of neutrosophical triangular fuzzy number by using the equation number (12), and also after constructing the membership function of each of the objective functions (using lower and upper bound), different approaches namely, simple additive approach, simple weighted additive approach, and pre-emptive goal programming approach has been used over model (2a) for getting the compromise solution. Simple additive approach (Tiwari et al., [34]) is a method used to solve the problem of multi-attribute decision making. The basic concept simple additive approach is to find the sum of each alternative's performance rating on all attributes; simple weighted additive approach (Chou et al.[35]) is the method used in solving the problem of multi-attribute decision making The basic concept weighted additive approach is to find the sum of the weighted performance rating for each alternative on all attributes; and pre-emptive goal programming approach (Biswas and Pal [36]) is a hierarchy of priority levels for the goals, so the primary importance is to receive first-priority attention, secondary importance receives second-priority attention, and so forth (if there are more than two priority levels. The results indicated that, these approaches failed to optimize the objective function completely, but through neutrosophical compromise programming approach we are able to optimize the each objective functions efficiently that is very important for supply chain.

Conclusion

There are numerous causes of uncertainty, which can arise from the demand side, production side, manufacturing cycle, and scheduling and distribution processes, constantly endanger the quality and efficacy of the SC. Uncertainty can result in shortages with bottlenecks, and can also impact the SC's overall efficiency. Therefore, it is important to find the means of managing it. The well-known methods such as probability, fuzzy set, and multi-choices theory are not sufficient in certain real-world circumstances to cope with such conditions in which indeterminacy is involved. The main aim of this paper is to implement the novel neutrosophical compromise programming approach, that together optimizes the degrees of truth, indeterminacy and falsity of objectivity functions. The efficiency of the proposed work is also studied where the suggested approach produces improved results in compare to simple additive approach, simple weighted additive approach and a pre-emptive goal programming approach. This result demonstrates the efficiency or dominance on current strategies that the neutrosophic technique's is quite adequate, explanatory, and a good representative of real-life situations. Therefore, it is expected that the approach developed would open up new opportunities in the field of multi-criteria problems and can be applied in other realistic field problems, such as scheduling problems, transportation problems, project management, capital utilization planning, traveling salesman problems, etc.

Acknowledgements All the authors are very thankful to the Editor in Chief and the anonymous reviewers who helped improve the paper's quality and presentation substantially. The paper has been submitted with the consent of all authors. The paper is the joint effort of all authors. This paper has not been submitted anywhere else for publication.

Declarations The paper has been submitted with the consent of all authors. The paper is the joint efforts of all authors. This paper has not been submitted anywhere else for publication.

Funding This research received no external funding.

Conflicts of interest The authors declare no conflict of interest.

Authors' contributions The model's concept and supervised by the 1st author of the paper. The model is solved and made the initial draft by the 2nd and 3rd author of the Paper. Finally, the 2nd author corrected the English and revised the whole manuscript based on the journal's requirement.

References

1. Nurjanni, K. P., Carvalho, M. S., & Costa, L. (2017). Green supply chain design: A mathematical modeling approach based on a multi-objective optimization model. *International Journal of Production Economics*, *183*, 421-432. https://doi.org/10.1016/j.ijpe.2016.08.028
2. Khan, S. A., Chaabane, A., & Dweiri, F. T. (2018). Multi-criteria decision-making methods application in supply chain management: A systematic literature. *Multi-criteria methods and techniques applied to supply chain management*, *1*.
3. Hugos, M. H. (2018). *Essentials of supply chain management*. John Wiley & Sons.
4. Smarandache, F. (2010). Neutrosophic set–a generalization of the intuitionistic fuzzy set. *Journal of Defense Resources Management (JoDRM)*, *1*(1), 107-116.
5. Gamal, A., Ismail, M., & Smarandache, F. *A Novel Methodology Developing an Integrated ANP: A Neutrosophic Model for Supplier Selection*. Infinite Study.
6. Abdel-Basset, M., Saleh, M., Gamal, A., & Smarandache, F. (2019). An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. *Applied Soft Computing*, *77*, 438-452. https://doi.org/10.1016/j.asoc.2019.01.035
7. Badhotiya, G. K., Soni, G., & Mittal, M. L. (2019). Fuzzy multi-objective optimization for multi-site integrated production and distribution planning in two echelon supply chain. *The International Journal of Advanced Manufacturing Technology*, *102*(1), 635-645. https://doi.org/10.1007/s00170-018-3204-2
8. Rabbani, M., Ahmadzadeh, K., & Farrokhi-Asl, H. (2019). Remanufacturing models under technology licensing with consideration of environmental issues. *Process Integration and Optimization for Sustainability*, *3*(3), 383-401. https://doi.org/10.1007/s41660-019-00085-8
9. Modak, N. M., & Kelle, P. (2019). Managing a dual-channel supply chain under price and delivery-time dependent stochastic demand. *European Journal of Operational Research*, *272*(1), 147-161. https://doi.org/10.1016/j.ejor.2018.05.067
10. J.-Sharahi, S., Khalili-Damghani, K., Abtahi, A. R., & Rashidi-Komijan, A. (2018). Type-II fuzzy multi-product, multi-level, multi-period location–allocation, production–distribution problem in supply chains: modelling and optimisation approach. *Fuzzy Information and Engineering*, *10*(2), 260-283. https://doi.org/10.1080/16168658.2018.1517978
11. Gholamian, N., Mahdavi, I., & Tavakkoli-Moghaddam, R. (2016). Multi-objective multi-product multi-site aggregate production planning in a supply chain under uncertainty: fuzzy multi-objective optimisation. *International Journal of Computer Integrated Manufacturing*, *29*(2), 149-165. https://doi.org/10.1080/0951192X.2014.1002811
12. Kristianto, Y., Gunasekaran, A., Helo, P., & Hao, Y. (2014). A model of resilient supply chain network design: A two-stage programming with fuzzy shortest path. *Expert systems with applications*, *41*(1), 39-49. https://doi.org/10.1016/j.eswa.2013.07.009
13. Bilgen, B. (2010). Application of fuzzy mathematical programming approach to the production allocation and distribution supply chain network problem. *Expert Systems with Applications*, *37*(6), 4488-4495. https://doi.org/10.1016/j.eswa.2009.12.062
14. Zimmermann, H. J. (1975). Description and optimization of fuzzy systems. *International journal of general System*, *2*(1), 209-215. https://doi.org/10.1080/03081077508960870
15. Kar, C., Mondal, B., & Roy, T. K. (2018). An inventory model under space constraint in neutrosophic environment: a neutrosophic geometric programming approach. *Neutrosophic Sets and Systems: An International Book Series in Information Science and Engineering, vol. 21, 2018*, 201893.
16. Ahmad, F., Adhami, A. Y., & Smarandache, F. (2020). Modified neutrosophic fuzzy optimization model for optimal closed-loop supply chain management under uncertainty. In *Optimization theory based on neutrosophic and plithogenic sets* (pp. 343-403). Academic Press. https://doi.org/10.1016/B978-0-12-819670-0.00015-9
17. Abdel-Baset, M., Chang, V., & Gamal, A. (2019). Evaluation of the green supply chain management practices: A novel neutrosophic approach. *Computers in Industry*, *108*, 210-220. https://doi.org/10.1016/j.compind.2019.02.013
18. Liang, R. X., Wang, J. Q., & Zhang, H. Y. (2018). A multi-criteria decision-making method based on single-valued trapezoidal neutrosophic preference relations with complete weight information. *Neural Computing and Applications*, *30*(11), 3383-3398. https://doi.org/10.1007/s00521-017-2925-8

19. Thamaraiselvi, A., & Santhi, R. (2016). A new approach for optimization of real life transportation problem in neutrosophic environment. *Mathematical Problems in Engineering*, *2016*. https://doi.org/10.1155/2016/5950747
20. Abdel-Basset, M., Manogaran, G., Gamal, A., & Smarandache, F. (2019). A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection. *Journal of medical systems*, *43*(2), 38. https://doi.org/10.1007/s10916-019-1156-1
21. Liang, R., Wang, J., & Zhang, H. (2017). Evaluation of e-commerce websites: An integrated approach under a single-valued trapezoidal neutrosophic environment. *Knowledge-Based Systems*, *135*, 44-59. https://doi.org/10.1016/j.knosys.2017.08.002
22. Abdel-Basset, M., Gamal, A., Chakrabortty, R. K., & Ryan, M. J. (2021). Evaluation of sustainable hydrogen production options using an advanced hybrid MCDM approach: A case study. *International Journal of Hydrogen Energy*, *46*(5), 4567-4591. https://doi.org/10.1016/j.ijhydene.2020.10.232
23. Abdel-Basset, M., Gamal, A., Chakrabortty, R. K., & Ryan, M. (2021). A new hybrid multi-criteria decision-making approach for location selection of sustainable offshore wind energy stations: A case study. *Journal of Cleaner Production*, *280*, 124462. https://doi.org/10.1016/j.jclepro.2020.124462
24. Charles, V., Gupta, S., & Ali, I. (2019). A fuzzy goal programming approach for solving multi-objective supply chain network problems with pareto-distributed random variables. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *27*(04), 559-593. https://doi.org/10.1142/S0218488519500259
25. Gupta, S., Ali, I., & Ahmed, A. (2018). Efficient fuzzy goal programming model for multi-objective production distribution problem. *International journal of applied and computational mathematics*, *4*(2), 1-19. https://doi.org/10.1007/s40819-018-0511-0
26. Gupta, S., Ali, I., & Ahmed, A. (2018). Multi-objective bi-level supply chain network order allocation problem under fuzziness. *Opsearch*, *55*(3), 721-748. https://doi.org/10.1007/s12597-018-0340-2
27. Zadeh, L. A. (1996). Fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh* (pp. 394-432). https://doi.org/10.1142/9789814261302_0021
28. Atanassov, K. T. (1999). Intuitionistic fuzzy sets. In *Intuitionistic fuzzy sets* (pp. 1-137). Physica, Heidelberg. https://doi.org/10.1007/978-3-7908-1870-3_1
29. Smarandache, F. (1999). A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic.
30. Bellman, R. E., & Zadeh, L. A. (1970). Decision-making in a fuzzy environment. *Management science*, *17*(4), B-141. https://doi.org/10.1287/mnsc.17.4.B141
31. Rizk-Allah, R. M., Hassanien, A. E., & Elhoseny, M. (2018). A multi-objective transportation model under neutrosophic environment. *Computers & Electrical Engineering*, *69*, 705-719. https://doi.org/10.1016/j.compeleceng.2018.02.024
32. Das, S. K., & Roy, S. K. (2019). Effect of variable carbon emission in a multi-objective transportation-p-facility location problem under neutrosophic environment. *Computers & Industrial Engineering*, *132*, 311-324. https://doi.org/10.1016/j.cie.2019.04.037
33. Khan, M. F., Haq, A., Ahmed, A., & Ali, I. (2021). Multiobjective Multi-Product Production Planning Problem Using Intuitionistic and Neutrosophic Fuzzy Programming. IEEE Access, 9, 37466-37486. https://doi.org/10.1109/ACCESS.2021.3063725
34. Tiwari, R. N., Dharmar, S., & Rao, J. (1987). Fuzzy goal programming—an additive model. Fuzzy sets and systems, 24(1), 27-34. https://doi.org/10.1016/0165-0114(87)90111-4
35. Chou, S. Y., Chang, Y. H., & Shen, C. Y. (2008). A fuzzy simple additive weighting system under group decision-making for facility location selection with objective/subjective attributes. *European Journal of Operational Research*, *189*(1), 132-145. https://doi.org/10.1016/j.ejor.2007.05.006
36. Biswas, A., & Pal, B. B. (2005). Application of fuzzy goal programming technique to land use planning in agricultural system. *Omega*, *33*(5), 391-398. https://doi.org/10.1016/j.omega.2004.07.003

Received: May 7, 2021. Accepted: October 4, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

Fundamental Homomorphism Theorems for Neutrosophic Triplet Module

Mehmet Çelik[1] and Necati Olgun[1*]

[1] Department of Mathematics, Gaziantep University, Gaziantep 27310 , Turkey; mathcelik@gmail.com, olgun@gantep.edu.tr

* Correspondence: olgun@gantep.edu.tr ; Tel.: +905363214006

Abstract: In this chapter, our aim is to prove neutro-isomorphism theorems. We define the quotient NT quotient Module and prove the fundamental theorem of neutro-homomorphism. Also, we present and prove the first neutro-isomorphism theorem for neutrosopfic triplet Modules, the second neutro-isomorphism theorem for neutrosopfic triplet Modules, the third neutro-Isomorphism theorem for neutrosopfic triplet Modules and a few special cases.

Keywords: NT submodule, NT R – modüle, NT quotient Module, Neutro- homomorphism, neutro-isomorphism

---

1. Introduction

In 1980, Smarandache presented neutrosophy, a part of philosophy. Neutrosophy, which is neutrosophic logic, probability depend on the set in [1]. Neutrosophic logic is the logic of some general concepts such as fuzzy logic presented by Zadeh in [2] and Provided by Atanassov intuitive fuzzy logic in [3].Fuzzy sets membership function but has an intuitive fuzzy set membership function and non-function and does not define membership indeterminancy. But; neutrosophic set includes all the functions. Many researchers have studied the concept neutrosophic theory and its application to issue multiple-criteria decision analysis.in [4-11]. Sahin M., and Kargın A., investigated NT metric space and NT normed space in [12]. Lately, Olgun at al. introduced the neutrosophic module in [13]; Şahin at al. presented Neutrosophic soft lattices in [14]; soft normed rings in [15]; centroid single valued neutrosophic triangular number and its applications in [16]; centroid single valued neutrosophic number and its applications in [17]. Ji at al. searched multi – valued neutrosophic environments and its applications in [18]. Also, Smarandache at al. searched NT theory in [19] and NT groups in [20, 21]. A NT has a form <m, neut(m), anti(m)> where; neut(m) is neutral of "m" and anti(m) is opposite of "m". Moreover, neut(m) is different from the classical unitary element and NT group is different from the classical group as well. Lately, Smarandache at al. investigated the NT field [22] and the NT ring [23]. Şahin at al. presented NT metric space, NT vector space and NT normed space in [24] and NT inner product in [25]. Smarandache at al. searched NT G- Module in [26]. Bal at al. searched NT cosets and quotient groups in [27]. Şahin at al. presented fixed point theorem for NT partial metric space and Neutrosophic triplet v – generalized metric space in [28-29]. Çelik at al. searched fundamental homomorphism theorems for NETGs in [30] and Çelik at al. Searched neutrosophic triplet R-module in [31]

The concept of an R – module over a ring is a general term of the notion of vector space. The basic structure of Abelian rings, can be more common. Because modular theory is more complicated than the structure of a vector space. Lately, Ai at al. defined the irreducible modules and fusion rules for parafermion vertex operator algebras in [32] and Creutzig at al. introduced Braided tensor categories of admissible modules for affine lie algebras in [33].

In this study, we examine the concept of NT R-Modules. So we obtain a new algebraic structures on NT groups and NT ring. In section 2, we give basic definitions of NT sets, NT groups, NT ring, NT vector space, Neutro-Monomorphism, Neutro-Epimorphism, and Neutro-Isomorphism . In section 3, we define the quotient NT quotient Module and prove the fundamental theorem of neutro-homomorphism. Also, we present and prove the first neutro-isomorphism theorem for neutrosopfic triplet Modules, the second neutro-isomorphism theorem for neutrosopfic triplet Modules, the third neutro-Isomorphism theorem for neutrosopfic triplet Modules and a few special cases. Also, we explain the NT quotient R-module. Finally, in Chapter 4, we give some results.

2. Preliminaries

In this section, we present the basic definitions that are important for the development of the paper.

Definition 2.1: [21] Let N be a set together with a binary operation ∇. Then, N is called a NT set if for any $k \in N$ there exists a neutral of "k" called $neut(k)$ that is different from the classical algebraic unitary element and an opposite of "k" called $anti(k)$ with $neut(k)$ and $anti(k)$ belonging to N, such that

$$k \nabla neut(k) = neut(k) \nabla k = k,$$

and

$$k \nabla anti(k) = anti(k) \nabla k = neut(k).$$

Definition 2.2: [21] Let (N, ∇) be a NT set. Then, N is called a NT group if the following conditions hold.

(1) If (N, ∇) is well-defined, i.e., for any $k, l \in N$, one has $k \nabla l \in N$.

(2) If (N, ∇) is associative, i.e., $(k \nabla l)\nabla m = k \nabla (l \nabla m)$ for all $k, l, m \in N$.

Definition 2.3: [24] Let $(NTF, \nabla_1, \blacksquare_1)$ be a NT field, and let $(NTV, \nabla_2, \blacksquare_2)$ be a NT set together with binary operations "∇_2" and "$\blacksquare_2$". Then $(NTV, \nabla_2, \blacksquare_2)$ is called a NT vector space if the following conditions hold. For all $p, r \in NTV$, and for all $t \in NTF$, such that $p\nabla_2 r \in NTV$ and $p\blacksquare_2 t \in NTV$ [24];

(1) $(p\nabla_2 r)\nabla_2 s = p\nabla_2 (r\nabla_2 s);\ p, r, s \in NTV;$

(2) $p\nabla_2 r = r\nabla_2 p;\ p, r \in NTV;$

(3) $(r\nabla_2 p)\blacksquare_2 t = (r\blacksquare_2 t)\nabla_2(p\blacksquare_2 t);\ t \in NTF \ and\ p, r \in NTV;$

(4) $(t\nabla_1 c)\blacksquare_2 p = (t\blacksquare_2 p)\nabla_1(c\blacksquare_2 p);\ t, c \in NTF\ and\ p \in NTV;$

(5) $(t\blacksquare_1 c)\blacksquare_2 p = t\blacksquare_1(c\blacksquare_2 p);\ t, c \in NTF\ and\ p \in NTV;$

(6) There exists any $t \in NTF \ni p\blacksquare_2 neut(t) = neut(t)\blacksquare_2 p = p;\ p \in NTV.$

Definition 2.4: [26] Let (G, ∇) be a NT group, $(NTV, \nabla_1, \blacksquare_1)$ be a NT vector space on a NT field $(NTF, \nabla_2, \blacksquare_2)$, and $g\,\nabla\, l \in NTV$ for $g \in G,\ l \in NTV$. If the following conditions are satisfied, then $(NTV, \nabla_1, \blacksquare_1)$ is called NT G-module.

a) There exists $g \in G \ni k * neut(g) = neut(g) * k = k$, for every $k \in NTV;$

b) $l\nabla_1(g\nabla_1 h) = (l\nabla_1 g)\nabla_1 h,\ \ \forall\, l \in NTV;\ g, h \in G;$

c) $(r_1\blacksquare_1 s_1\nabla_1 r_2\blacksquare_1 s_2)\nabla g = x\blacksquare_1 (h\nabla g)\nabla_1 y\blacksquare_1 (l\nabla g), \forall\, x, y \in NTF;\ h, l \in NTV;\ g \in G.$

Definition 2.5: [23] The NT ring is a set endowed with two binary laws $(M, *, \#)$ such that,

a) $(M, *)$ is a abelian NT group; which means that:

- $(M,*)$ is a commutative NT with respect to the law * (i.e. if x belongs to M, then $neut(x)$ and $anti(x)$, defined with respect to the law *, also belong to M)

- The law * is well – defined, associative, and commutative on M (as in the classical sense);

b) $(M,*)$ is a set such that the law # on M is well-defined and associative (as in the classical sense);

c) The law # is distributive with respect to the law * (as in the classical sense)

Definition 2.6: Let (NTR,∇, ■) be a commutative NT ring and let (NTM, ∗) be a NT abelian group and ° be a binary operation such that °: NTR x NTM→ NTM. Then (NTM, ∗, °) is called a NT R-Module on (NTR,∇,■) if the following conditions are satisfied. Where,

1) p °(r∗s) =(p °r)∗ (p °s), ∀ r, s ∈ NTM and p ∈ NTR.

2) (p∇k)°r = (p∇r)°(k∇r), ∀ p, k ∈ NTR and ∀ r ∈ NTM

3) (p■k)°r = p■(k°r), ∀ r, s ∈ NTR and ∀ m ∈ NTM

4) For all m ∈ NTM; there exists at least a c ∈ NTR such that m°neut(c)= neut(c)°m = m. Where, neut(c) is neutral element of c for ■.

Definition 2.7: Let (NTM, ∗, °) be a NT R-Module on NT ring (NTR,∇,■) and NTSM ⊂ NTM. Then (NTSM, ∗, °) is called NT R - submodule of (NTM, ∗, °), if (NTSM, ∗, °) is a NT R – module on NT ring (NTR,∇,■).

Definition 2.7: (NTM_1, , $°_1$) be a NT R-module on NT ring (NTR,∇,■) and (NTM_2, $∗_2$, $°_2$) be a NT R-module on NT ring (NTR,∇,■). A mapping f: NTM_1 → NTM_2 is said to be NT R-module homomorphism when

$f((r^{\circ}{}_{1}m) *_1 (s^{\circ}{}_{1}n)) = (r^{\circ}{}_{2}f(m)) *_2 (s^{\circ}{}_{2}f(n))$, for all r, s ∈ NTR and m, n ∈ NTM_1.

Definition 2.8: Assume that *(N_1, ∗)* and *(N_2,∘)* be two *NETG's*. If a mapping $f : N_1 \rightarrow N_2$ of *NETG* is only one to one (injective) *f* is called neutro-monomorphism.

Definition 2.9: Let *(N_1, ∗)* and *(N_2,∘)* be two *NETG's*. If a mapping $f : N_1 \rightarrow N_2$ is only onto (surjective) *f* is called neutro-epimorphism.

Definition 2.9: Let *(N_1, ∗)* and *(N_2,∘)* be two *NETGs*. If a mapping $f : N_1 \rightarrow N_2$ neutro-homomorphism is one to one and onto *f* is called neutro-isomorphism. Here, N_1 and N_2 are called neutro-isomorphic and denoted as $N_1 \cong N_2$.

3. Quotient NTM and Neutro-Isomorphism

In this chapter, We prove neutro-isomorphism theorems. we define the quotient NTM and prove the fundamental theorem of neutro-homomorphism. We also prove the first neutro-isomorphism theorem for neutrosopfic triplet Modules, the second neutro-isomorphism theorem for neutrosopfic triplet Modules, the third neutro-Isomorphism theorem for neutrosopfic triplet Modules and a few special cases.

Definition 3.1: Let NTM, NTM′ be neutrosopfic triplet left modules over the neutrosopfic triplet ring R. A map ð : NTM → NTM′ is called a neutrosopfic triplet left R-module homomorphism if :

1. ð is a neutrosopfic triplet group neutro-homomorphism, that is if, for every m, n ∈ NTM we have ð (m + n) = ð (m) + ð (n);
2. For every r ∈ R and for every m ∈ M we have ð (r · m) = r · ð (m)

If ð : NTM → NTM′ is a neutrosopfic triplet R-module neutro-homomorphism we say that:

i) ð is a neutro-monomorphism if the map ð is injective ;

ii) ð is a neutro-epimorphism if the map ð is surjective ;

iii)ð is an isomorphism if the map ð is bijective.

We will say that NTM and NTM′ are neutro-isomorphic and we will write NTM ≅ NTM′ if there exists a neutro-isomorphism ð : NTM → NTM′. Observe that, in this case, the inverse map of ð, $ð^{-1}$: NTM′ → NTM is also a module isomorphism.

Example 3.2. Let R be a neutrosopfic triplet ring. Given an element a ∈ R the map

$$ð_a : R \rightarrow R$$

$$r \rightarrow r \cdot_R a$$

is a left NTM neutro-homomorphism from $_RR$ into $_RR$. Observe that, if a ≠ neut(a), then $ð_a$ is not a NTR neutro-homomorphism.

Theorem 3.3. Let *R* be a *NTR*, let *M* be a NTM and let *H* be a neutrosophic triplet *R* -Submodule. We define a left *NTM* structure on the neutrosophic triplet abelian group *M* / *H* by neutrosophic triplet setting, for every $\dot{r} \in R$ and for every $\dot{m} \in M$, $\dot{r} \cdot (\dot{m} + H) = (\dot{r} \cdot \dot{m}) + H$. Moreover, with respect to this structure, the canonical projection $\partial\ H : M \rightarrow M / H$ becomes a surjective neutrosophic triplet *R* -module homomorphism.

Proof. We have first to show that (1) is well defined, that is, given any $\dot{r} \in R$, $\dot{m}, \dot{m}' \in M$ such that $\dot{m} + H = \dot{n} + H$ (i.e. $\dot{m} - \dot{n} \in H$), we have that $(\dot{r} \cdot \dot{m}) + H = \dot{r} \cdot \dot{n} + H$ (i.e. $\dot{r} \cdot \dot{m} - \dot{r} \cdot \dot{n} \in H$). But $\dot{m} - \dot{n} \in H$ implies that $\dot{r} \cdot \dot{m} - \dot{r} \cdot \dot{n} = \dot{r} \cdot (\dot{m} - \dot{n}) \in H$ as H is a submodule of M. Let now $\underset{.}{k}, \underset{.}{l} \in R$, $\dot{m}$, $\dot{n} \in R$. We have:

$\underset{.}{k} \cdot [(\dot{m} + H) + (\dot{n} + H)] = \underset{.}{k} \cdot [(\dot{m} + \dot{n}) + H] = (\underset{.}{k} \cdot (\dot{m} + \dot{n})) + H = (\underset{.}{k} \cdot \dot{m} + \underset{.}{k} \cdot \dot{n}) + H = (\underset{.}{k} \cdot \dot{m} + H) + (\underset{.}{k} \cdot \dot{n} + H) = \underset{.}{k} \cdot (\dot{m} + H) + \underset{.}{k} \cdot (\dot{n} + H)$;

$(\underset{.}{k} + \underset{.}{l}) \cdot (\dot{m} + H) = ((\underset{.}{k} + \underset{.}{l}) \cdot \dot{m}) + H = (\underset{.}{k} \cdot \dot{m} + \underset{.}{l} \cdot \dot{m}) + H = (\underset{.}{k} \cdot \dot{m} + H) + (\underset{.}{l} \cdot \dot{m} + H) = \underset{.}{k} \cdot (\dot{m} + H) + \underset{.}{l} \cdot (\dot{m} + H)$; $(\underset{.}{k} \cdot \underset{.}{l})(\dot{m} + H) = ((\underset{.}{k} \cdot_R \underset{.}{l})\, \dot{m}) + H = (\underset{.}{k} \cdot (\underset{.}{l} \cdot \dot{m})) + H = \underset{.}{k} \cdot (\underset{.}{l} \cdot \dot{m} + H) = \underset{.}{k} \cdot (\underset{.}{l} \cdot (\dot{m} + H))$; $neut(\underset{.}{k}, \underset{.}{l})_R \cdot (\dot{m} + H) = (neut(\underset{.}{k}, \underset{.}{l})_R \cdot \dot{m}) + H = \dot{m} + H$.

Finally: $\partial H (\underset{.}{k} \cdot \dot{m}) = \underset{.}{k} \cdot \dot{m} + H = \underset{.}{k} \cdot (\dot{m} + H) = \underset{.}{k} \cdot \partial H (\dot{m})$.

Definition 3.4. Let NTM be a neutrosophic triplet left module over a neutrosophic triplet ring R and let H be a neutrosophic triplet submodule of M. The neutrosophic triplet left R -module having the neutrosophic triplet quotient group M / H for its underlying neutrosophic triplet abelian group is called the neutrosophic triplet quotient module (or a neutrosophic triplet factor module) of NTM modulo $NTSM$ and is denoted by $NTM/NTSM$.

Theorem 3.5. Let R be a neutrosophic triplet ring and let $\delta : NTM \rightarrow NTM'$ be a neutrosophic triplet left R-module neutro-homomorphism. If S is a $NTSM$ of NTM contained in $Ker(\delta)$, then there exists a NTM neutro-homomorphism $\bar{\delta}$: $NTM/NTSM \rightarrow NTM'$ such that the diagram commutes

i.e. $\delta = \bar{\delta} \circ \partial S$.

Moreover:

1. $\bar{\delta}$ is unique with respect to this property;

2. $Im(\delta) = Im(\bar{\delta})$ and $Ker(\bar{\delta}) = Ker(\delta)/S$;

3. $\bar{\delta}$ is injective $\Leftrightarrow S = Ker(\delta)$.

Proof. In view of the Fundamental Theorem for the a neutrosophic triplet quotient group there exists a a neutrosophic triplet group neutro-homomorphism $\bar{\delta}$: $NTM/NTSM \rightarrow NTM'$ such that $\delta = \bar{\delta} \circ \partial S$.

Moreover: 1) such a neutrosophic triplet group neutro homomorphism is unique;

2) $Im(\delta) = Im(\bar{\delta})$, $Ker(\bar{\delta}) = Ker(\delta)/S$;

3) $\bar{\delta}$is injective $\Leftrightarrow S = Ker(\delta)$.

Hence we only have to prove that, for every $m \in NTM$ and $r \in R$:

$\bar{\delta}\ (r(m + S)) = r \cdot \bar{\delta}\ (m + S)$.

It is now an easy calculation to arrive at:

$\bar{\tilde{\delta}}$ $(r\cdot(m+S)) = \bar{\tilde{\delta}}$ $(r\cdot m+S) = \bar{\tilde{\delta}}$ $(\partial S\,(r\cdot m)) = \delta\,(r\cdot m) = r\cdot\,\delta\,(x) = r\cdot\bar{\tilde{\delta}}$ $(\partial S\,(m)) = r\cdot(m+S).$

Corollary 3.6. (First neutro-Isomorphism Theorem for NTM).

Let R be a NTR and $\delta : NTM \rightarrow NTM'$ be a NTLM neutro-homomorphism. Then the assignment

$$m + Ker(\delta) \rightarrow \delta\,(m)$$

defines an neutro-isomorphism of neutrosophic triplet left R-modules

$$\tilde{\bar{\delta}} : NTM/Ker(\delta) \rightarrow Im(\delta)$$

In particular, if δ is surjective, then $\tilde{\bar{\delta}}$ is an neutro isomorphism and

$$NTM/Ker(\delta) \cong NTM'.$$

Theorem 3.7. (Second neutro-Isomorphism Theorem for NTM)

Let H and B be $NTSM$ of a NTM over a NTR. Then $H \cap B$ and $H + B$ are neutrosophic triplet submodules of NTM and the assignment $m + (H \cap B) \rightarrow m + B$ defines an neutrosophic triplet R-module neutro-isomorphism from $H/(H \cap B)$ into $H + B / B$. Therefore:

$$H/(H \cap B) \cong H + B / B$$

Proof. We know that $H \cap B$ is a $NTSM$ of NTM. Let r ∈ R, s ∈ $H \cap B$. Then rs ∈ H and rs ∈ B, as H and B are neutrosophic triplet submodules of NTM. Therefore r·s ∈ $H \cap B$. We know that $H + B$ is a neutrosophic triplet subgroup of NTM. Let r ∈ R, s ∈ $H + B$. Then there exist m ∈ H and n ∈ B such that s = m + n. Obviously rm ∈ H and rn ∈ B, and hence r·s = r·m + r·n ∈ $H + B$. In view of the Second neutro-Isomorphism Theorem for neutrosophic triplet groups, the assignment m + ($H \cap B$) → m + B defines a neutrosophic triplet group neutro-isomorphism $\delta : H/(H \cap B) \rightarrow H + B / B$. Let r ∈ R, m ∈ H, then we calculate:

$\delta\,(r(m + (H \cap B)) = \delta\,(rm + (H \cap B)) = rm + B = r(m + B) = r\,\delta\,(m + (H \cap B))$. Therefore δ is a neutrosophic triplet left R-module neutro-isomorphism.

Theorem 3.8. Let R be a NTR, $\delta : NTM \rightarrow NTM'$ be a neutrosophic triplet left R-module neutro-homomorphism. For every neutrosophic triplet submodule S of M containing $Ker(\delta)$ the assignment

$m + S \rightarrow \delta\,(m) + \delta\,(S)$ defines a neutro-isomorphism $\hat{\bar{\delta}}\ S : M/S \rightarrow Im(\delta)/\delta(S)$. Therefore

$M/S \cong Im(\delta)/\delta(S).$

Proof. We know that the assignment $m + S \rightarrow \delta\,(m) + \delta(S)$ defines a neutrosophic triplet group neutro-isomorphism $\pi = \hat{\bar{\delta}}_N : M/S \rightarrow Im(\delta)/S$.

Let $r \in R$, $m \in S$. We have :

$\pi\ (r(m+S)) = \pi\ (rm+S) = \delta\ (rm) + \delta\ (S) = (r\ \delta\ (m)) + \delta\ (S) = r(\delta\ (m) + \delta(S)) = r\ \pi\ (m+S)$ Therefore π is a neutrosophic triplet left R-module neutro-isomorphism.

Corollary 3.9. (Third neutro-Isomorphism Theorem for NTM)

Let H and B be neutrosophic triplet submodules of a NTM over a NTR and assume that $H \subseteq B$.

Then the assignment $m + B \rightarrow (m + H) + H/\ B$. Defines a neutrosophic triplet left R-module neutro-isomorphism from M/H into $M/H/\ B\ /H$. Therefore

$$M/B \cong M/H/\ B\ /H.$$

Proof. Apply Theorem 3.8 to $\partial_H : M \rightarrow M/H$, recalling that $\partial_H\ (B) = B\ /\ H$.

4. Conclusions

This article mainly focused on fundamental homomorphism theorems for neutrosophic R-modules. We gave and proved the fundamental theorem of neutro-homomorphism, as well as first,second and third neutro-isomorphism theorems explained for NTM. Furthermore, we define neutro-monomorphism, neutro-epimorphism. By applying them to neutrosophic algebraic structures. We looked at it as closely related as different systems. Using the concept of the fundamental theorem of neutro-Homomorphism and neutro-isomorphism theorems, the relationship between neutrosophic algebraic structures was studied.

Abbreviations

NT: Neutrosophic triplet

NTS:Neutrosophic triplet set

NETG: Neutrosophic extended triplet group

NTM: Neutrosophic triplet R-module

NTSM: Neutrosophic triplet R-submodule

NTLM: Neutrosophic triplet left R-module

References

1. F. Smarandache, Neutrosophy: Neutrosophic Probability, Set and Logic, Rehoboth, Amer. Research Press (1998).

2. A. L. Zadeh, Fuzzy sets, Information and control , (1965) 8.3 338-353,

3.T. K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst, (1986) 20:87–96

4. Liu P., Zhang L., Liu X., and Wang P., Multi-valued Neutrosophic Number Bonferroni mean Operators and Their Application in Multiple Attribute Group Decision Making, internal journal of information technology & decision making (2016), 15(5), pp. 1181-1210

5. Sahin M., Deli I., and Ulucay V.,Similarity measure of bipolar neutrosophic sets and their application to multiple criteria decision making, Neural Comput & Applic. (2016), DOI 10. 1007/S00521

6. Sahin M., Deli I., I, and Ulucay V., Jaccard vector similarity measure of bipolar neutrosophic set based on multi-criteria decision making, International conference on natural science and engineering (ICNASE'16) (2016), March 19–20, Kilis, Turkey

7. Liu P., The aggregation operators based on Archimedean t-conorm and t-norm for the single valued neutrosophic numbers and their application to Decision Making, International Journal of Fuzzy Systems (2016), 18(5) pp. 849-863

8. Liu C. and Luo Y., Power aggregation operators of simplifield neutrosophic sets and their use in multi-attribute group decision making, İEE/CAA Journal of Automatica Sinica (2017), DOI: 10.1109/JAS.2017.7510424

9. Sahin R. and Liu P., Some approaches to multi criteria decision making based on exponential operations of simplied neutrosophic numbers, Journal of Intelligent & Fuzzy Systems (2017), 32(3) pp. 2083-2099, DOI: 10.3233/JIFS-161695

10. Liu P. and Li H., Multi attribute decision-making method based on some normal neutrosophic bonferroni mean operators, Neural Computing and Applications (2017), 28(1), pp. 179-194, DOI 10.1007/s00521-015-2048-z

11. Broumi S., Bakali A., Talea M., Smarandache F., Decision-Making Method Based On the Interval Valued Neutrosophic Graph, Future Technologie, IEEE (2016), pp. 44-50

12. Sahin M., and Kargın A., Neutrosophic triplet metric space and neutrosophic triplet normed space, ICMME -2017, Şanlıurfa, Turkey

13. Olgun N., and Bal M., Neutrosophic modules, Neutrosophic Operational Research, (2017), 2(9), pp. 181-192

14. Şahin M., Uluçay V., Olgun N. and Kilicman Adem, On neutrosophic soft lattices, Afrika matematika (2017), 28(3) pp. 379-388

15. Şahin M., Uluçay V., and Olgun N., Soft normed rings, Springerplus, (2016), 5(1), pp. 1-6

16. Şahin M., Uluçay V., Olgun N. and Kilicman Adem, On neutrosophic soft lattices, Afrika matematika (2017), 28(3) pp. 379-388

17. Şahin M., Olgun N., Uluçay V., Kargın A., and Smarandache, F., A new similarity measure based on falsity value between single valued neutrosophic sets based on the centroid points of transformed single valued neutrosophic numbers with applications to pattern recognition, Neutrosophic Sets and Systems (2017), 15, pp. 31-48, doi: org/10.5281/zenodo570934

18. Ji P., Zang H., and Wang J., A projection – based TODIM method under multi-valued neutrosophic enviroments and its application in personnel selection, Neutral Computing and Applications (2018), 29, pp. 221-234

19. F. Smarandache and M. Ali, Neutrosophic triplet as extension of matter plasma, unmatter plasma and antimatter plasma, APS Gaseous Electronics Conference (2016), doi: 10.1103/BAPS.2016.GEC.HT6.110

20. F. Smarandache and M. Ali, The Neutrosophic Triplet Group and its Application to Physics, presented by F. S. to Universidad Nacional de Quilmes, Department of Science and Technology, Bernal, Buenos Aires, Argentina (02 June 2014)

21. F. Smarandache and M. Ali, Neutrosophic triplet group. Neural Computing and Applications, (2016) 1-7.

22. F. Smarandache and M. Ali, Neutrosophic Triplet Field Used in Physical Applications, (Log Number: NWS17-2017-000061), 18th Annual Meeting of the APS Northwest Section, Pacific University, Forest Grove, OR, USA (June 1-3, 2017)

23. F. Smarandache and M. Ali, Neutrosophic Triplet Ring and its Applications, (Log Number: NWS17-2017-000062), 18th Annual Meeting of the APS Northwest Section, Pacific University, Forest Grove, OR, USA (June 1-3, 2017).

24. M. Şahin and A. Kargın, Neutrosophic triplet normed space, Open Physics, (2017), 15:697-704

25. Şahin M. and Kargın A., Neutrosophic triplet inner product space, Neutrosophic Operational Research, (2017), 2(10), pp. 193-215,

26. Smarandache F., Şahin M., Kargın A. Neutrosophic Triplet G- Module, Mathematics – MDPI, (2018), 6, 53

27. Bal M., Shalla M. M., Olgun N. Neutrosophic triplet cosets and quotient groups, Symmetry – MDPI, (2018), 10, 126

28. Şahin M., Kargın A., Çoban M. A., Fixed point theorem for neutrosophic triplet partial metric space, Symmetry – MDPI,(2018), 10, 240

29. Şahin M. ve Kargın A., Neutrosophic triplet v – generalized metric space, Axioms – MDPI (2018)7, 67

30. Çelik M., M. M. Shalla, Olgun N., Fundamental homomorphism theorems for neutrosophic extended triplet groups, Symmetry- MDPI (2018) 10, 32

31. Çelik at al. Searched neutrosophic triplet R-module

32. Ai C., Dong C., Jiao X., Ren L., The irreducible modules and fusion rules for parafermion vertex operator algebras, Transactions of the American Mathematical Society (2018) 8, 5963-5981

33. Creutzig T., Huang Y., Yang J., Braided tensor categories of admissible modules for affine lie algebras, Communications in Mathematical Physics (2018)3, 827-854

Received: May 6, 2021. Accepted: October 5, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

Neutrosophic Pythagorean Sets with Dependent Neutrosophic Pythagorean Components and its Improved Correlation Coefficients

R. Radha [1],*, A. Stanis Arul Mary [2], R. Prema [3] and Said Broumi [4]

[1] Research Scholar; Department of Mathematics, Nirmala College for Women, Coimbatore (TN), India. radharmat2020@gmail.com
[2] Assistant Professor, Department of Mathematics, Nirmala College for Women, Coimbatore (TN), India. stanisarulmary@gmail.com
[3] Associate Professor, Department of Commerce, KPR college of Arts Science ,Coimbatore (TN), India. rprems6@gmail.com
[4] Laboratory of Information Processing, Faculty of Science, Ben M ' Sik, University Hassan II, Casablanca, Morocco, broumisaid78@gmail.com

* Correspondence: radharmat2020@gmail.com,kathirsubbu1987@gmail.com

Abstract: The concept of Neutrosophic Pythagorean set [NPS] with dependent Neutrosophic Pythagorean components was introduced and discussed the relationship between dependent neutrosophic and neutrosophic pythagorean components. The correlation coefficient is a statistical measure which contributes in deciding the degree to which changes in one variable predict changes in another. In this article, we analyze the characteristics of neutrosophic pythagorean sets with improved correlation coefficients. We've also used the same approach in multiple attribute decision-making methodologies including one with a neutrosophic pythagorean environment. Finally, we implemented for above technique to the problem of multiple attribute group decision making.

Keywords: neutrosophic pythagorean sets, neutrosophic Sets, Improved correlation coefficient.

1. Introduction

Fuzzy sets were introduced by Zadeh [20] in 1965 that permits the membership perform valued within the interval [0,1] and set theory it's an extension of classical pure mathematics. Fuzzy set helps to deal the thought of uncertainty, unclearness and impreciseness that isn't attainable within the cantorian set. As Associate in Nursing extension of Zadeh's fuzzy set theory intuitionistic fuzzy set(IFS) was introduced by Atanassov [1] in 1986, that consists of degree of membership and degree of non membership and lies within the interval of [0,1]. IFS theory wide utilized in the areas of logic programming, decision making issues, medical diagnosis etc.

Florentine Smarandache [12] introduced the idea of Neutrosophic set in 1995 that provides the information of neutral thought by introducing the new issue referred to as uncertainty within the set. thus neutrosophic set was framed and it includes the parts of truth membership function(T), indeterminacy membership function(I), and falsity membership function(F) severally. Neutrosophic sets deals with non normal interval of]−0 1+[. Since neutrosophic set deals the indeterminateness effectively it plays an very important role in several applications areas embrace info technology, decision web, electronic database systems, diagnosis, multicriteria higher cognitive process issues etc.,

To method the unfinished data or imperfect data to unclearness a brand new mathematical approach i.e., To deal the important world issues, Wang [13](2010) introduced the idea of single valued neutrosophic sets(SVNS) that is additionally referred to as an extension of intuitionistic fuzzy sets and it became a really new hot analysis topic currently. The concept of neutrosophic pythagorean sets with dependent neutrosophic components was introduced by R. Jhansi and K. Mohana[6].

Further, R. Radha and A. Stanis Arul Mary[7] outlined a brand new hybrid model of Pentapartitioned Neutrosophic Pythagorean sets (PNPS) and Quadripartioned neutrosophic pythagorean sets in 2021.Correlation coefficient may be a effective mathematical tool to live the strength of the link between 2 variables. such a lot of researchers pay the attention to the idea of varied correlation coefficients of the various sets like fuzzy set, IFS, SVNS, QSVNS. In 1999 D.A Chiang and N.P. Lin [3] projected the correlation of fuzzy sets underneath fuzzy setting. Later D.H. Hong [4] (2006) outlined fuzzy measures for a coefficient of correlation of fuzzy numbers below Tw (the weakest t-norm) based mostly fuzzy arithmetic operations.

Correlation coefficients plays a very important role in several universe issues like multiple attribute cluster higher cognitive process, cluster analysis, pattern recognition, diagnosis etc., therefore several authors targeted the idea of shaping correlation coefficients to resolve the important world issues in significantly multicriteria decision making strategies. Jun Ye [19] outlined the improved correlation coefficients of single valued neutrosophic sets and interval neutrosophic sets for multiple attribute higher cognitive process to beat the drawbacks of the correlation coefficients of single valued neutrosophic sets (SVNSs) that is outlined in [17].

In this paper, we have applying improved correlation coefficient on Neutrosophic Pythagorean sets and studied with an example. In the third section, the idea of Neutrosophic Pythagorean set was initiated and in fourth section, the improved correlation coefficient was applied to neutrosophic pythagorean sets. Finally, the decision making under improved correlation was illustrated by an example in the last section

2 Preliminaries

2.1 Definition [12]

Let X be a universe. A Neutrosophic set A on X can be defined as follows:

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}$$

Where T_A , $I_A, F_A: U \rightarrow [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$

Here, $T_A(x)$ is the degree of membership, $I_A(x)$ is the degree of inderminancy and $F_A(x)$ is the degree of non-membership.

He re, $T_A(x)$ and $F_A(x)$ are dependent neutrosophic components and $I_A(x)$ is an independent component.

2.2 Definition [6]

Let X be a universe. A Pythagorean Neutrosophic set A with T and F are dependent neutrosophic components and I as independent component for $A = \{< x, T_A, I_A, F_A >: x \in X\}$ on X is an object of the form

$$(T_A)^2 + (I_A)^2 + (F_A)^2 \leq 2$$

Here, $T_A(x)$ is the truth membership, $I_A(x)$ is indeterminancy membership and $F_A(x)$ is the false membership .

2.3 Definition [6]

The complement of a Pythagorean Neutrosophic set $A = \{< x, T_A, U_A, F_A >: r \in R\}$ with dependent Neutrosophic components is

$A^C = \{< x, F_A, U_A, T_A >: r \in R\}$.

2.4 Definition [6]

Let $A = \{< x, T_A, U_A, F_A >: r \in R\}$ and $B = \{< x, T_B, U_B, F_B >: r \in R\}$ are two Pythagorean Neutrosophic sets with dependent Neutrosophic components on the universe R. Then the union and intersection of two sets can be defined by

A $\cup B$ = {max (T_A, T_B), max (U_A, U_B),min (F_A, F_B)},

A $\cap B$ = {min (T_A, T_B), min (U_A, U_B),max (F_A, F_B)}.

3.Neutrosophic Pythagorean Set with Dependent Neutrosophic Pythagorean Components

3.1 Definition

Let R be a universe. A Neutrosophic pythagorean set A with T and F as dependent Neutrosophic Pythagorean components and U as independent component for A on R is an object of the form

$$A = \{< x, T_A, U_A, F_A >: r \in R\}$$

Where $(T_A)^2 + (F_A)^2 \leq 1$ and

$$(T_A)^2 + (U_A)^2 + (F_A)^2 \leq 2$$

Here, $T_A(x)$ is the truth membership, $U_A(x)$ is indeterminancy membership and $F_A(x)$ is the false membership .

Remark: When T and F as dependent Neutrosophic Components, then T + F ≤ 1.

3.2 Definition

The complement of a Neutrosophic Pythagorean set $A = \{< x, T_A, U_A, F_A >: r \in R\}$ with dependent Neutrosophic Pythagorean components is

$A^C = \{< x, F_A, 1 - U_A, T_A >: r \in R\}$.

3.3 Definition

Let $A = \{< x, T_A, U_A, F_A >: r \in R\}$ and $B = \{< x, T_B, U_B, F_B >: r \in R\}$ are two Neutrosophic Pythagorean sets with dependent Neutrosophic Pythagorean components on the universe R. Then the union and intersection of two sets can be defined by

A $\cup B$ = {max (T_A, T_B), min (U_A, U_B),min (F_A, F_B)},

A $\cap B$ = {min (T_A, T_B), max (U_A, U_B),max (F_A, F_B)}.

3.4 Example

Let R = {a, b} and A ={ (a, 0.4,0.3),(b,0.5,0.2)}.

Then τ = {0,1, A} is a topology on R. Then A is a Neutrosophic Pythagorean set.

3.5 Example

Let R = {a, b} and A ={ (a, 0.7,0.7),(b,0.7,0.7)}.

Then τ = {0,1, A} is a topology on R. Since $T_A + F_A > 1$,then A is a Neutrosophic Pythagorean set with dependent neutrosophic pythagorean components but not dependent neutrosophic components.

$But \ (T_A)^2 + (U_A)^2 + (F_A)^2 \leq 2$. Hence A is a Pythagorean Neutrosophic set.

3.6 Example

Let R = {a, b} and A ={ (a, 0.8,0.7),(b,0.7,0.7)}.

Then τ = {0,1, A} is a topology on R. Since $T_A + F_A > 1, (T_A)^2 + (F_A)^2 > 1$,then A is not a Neutrosophic Pythagorean set with dependent neutrosophic pythagorean components and dependent neutrosophic components.

$But \ (T_A)^2 + (U_A)^2 + (F_A)^2 \leq 2$. Hence A is a Pythagorean Neutrosophic set.

4. Improved Correlation Coefficients

Based on the concept of correlation coefficient of NPS s ,we have defined the improved correlation coefficients of NPS s in the following section.

4.1 Definition

Let P and Q be any two NPs s in the universe of discourse R = { $r_1, r_2, r_3, \ldots, r_n$ }, then the improved correlation coefficient between P and Q is defined as follows

$$K(P, Q) = \frac{1}{3n}\sum_{k=1}^{n} [\,\alpha_k(1 - \Delta T_k) + +\gamma_k(1 - \Delta U_k) + \mu_k(1 - \Delta F_k)\,] \quad (1)$$

Where

$$\alpha_k = \frac{2 - \Delta T_k - \Delta T_{max}}{2 - \Delta T_{min} - \Delta T_{max}},$$

$$\gamma_k = \frac{2 - \Delta U_k - \Delta U_{max}}{2 - \Delta U_{min} - \Delta U_{max}},$$

$$\mu_k = \frac{2 - \Delta F_k - \Delta F_{max}}{2 - \Delta F_{min} - \Delta F_{max}},$$

$$\Delta T_K = |T_P^2(r_k) - T_Q^2(r_k)\,|,$$

$$\Delta U_K = |U_P^2(r_k) - U_Q^2(r_k)\,|,$$

$$\Delta F_K = |F_P^2(r_k) - F_Q^2(r_k)\,|,$$

$$\Delta T_{min} = min_k|T_P^2(r_k) - T_Q^2(r_k)\,|,$$

$$\Delta U_{min} = min_k|U_P^2(r_k) - U_Q^2(r_k)|,$$

$$\Delta F_{min} = min_k|F_P^2(r_k) - F_Q^2(r_k)|,$$

$$\Delta T_{max} = max_k|T_P^2(r_k) - T_Q^2(r_k)|,$$

$$\Delta U_{max} = max_k|U_P^2(r_k) - U_Q^2(r_k)|,$$

$$\Delta F_{max} = max_k|F_P^2(r_k) - F_Q^2(r_k)|,$$

For any $r_k \in R$ and k = 1, 2, 3….. n.

4.2 Theorem

For any two NPS s P and Q in the universe of discourse R = { $r_1, r_2, r_3, \ldots, r_n$ }, the improved correlation coefficient K(P, Q) satisfies the following properties.

1) K (P, Q) = K (Q, P);
2) $0 \le K(P, Q) \le 1$;
3) K(P, Q) = 1 iff P =Q.

Proof

(1) It is obvious and straightforward.

(2) Here, $0 \le \alpha_k \le 1,\ 0 \le \gamma_k \le 1,\ 0 \le \mu_k \le 1,\ 0 \le 1 - \Delta T_k \le 1,$
$0 \le 1 - \Delta U_k \le 1,\ 0 \le 1 - \Delta F_k \le 1,$ Therefore the following inequation satisfies
$0 \le \alpha_k (1 - \Delta T_k) + +\gamma_k(1 - \Delta U_k) + \mu_k(1 - \Delta F_k) \le 3.$ Hence we have $0 \le K(P, Q) \le 1$

(3) If K (P, Q) = 1, then we get $\alpha_k (1 - \Delta T_k) + \gamma_k(1 - \Delta U_k) + \mu_k (1 - \Delta F_k) = 3.$

Since $0 \le \alpha_k (1 - \Delta T_k) \le 1, 0 \le \gamma_k(1 - \Delta U_k) \le 1$ and $0 \le \mu_k (1 - \Delta F_k) \le 1$, there are

$\alpha_k (1 - \Delta T_k) = 1, \gamma_k(1 - \Delta U_k) = 1$ and $\mu_k (1 - \Delta F_k) = 1$. And also since $0 \le \alpha_k \le 1,\ 0 \le \gamma_k \le 1$ and $0 \le \mu_k \le 1, 0 \le 1 - \Delta T_k \le 1,\ 0 \le 1 - \Delta U_k \le 1,\ 0 \le 1 - \Delta F_k \le 1$. We get $\alpha_k = = \gamma_k = \mu_k = 1$ and $1 - \Delta T_k = 1 - \Delta U_k = 1 - \Delta F_k = 1$. This implies, $\Delta T_k = \Delta T_{min} = \Delta T_{max} = 0,\ \Delta U_k = \Delta U_{min} = \Delta U_{max} = 0, \Delta F_k = \Delta F_{min} = \Delta F_{max} = 0$. Hence $T_P(r_k) = T_Q(r_k),\ U_P(r_k) = U_Q(r_k)$ and $F_P(r_k) = F_Q(r_k)$ for any $r_k \in R$ and k = 1,2,3….n.Hence P = Q.

Conversely, assume that P = Q, this implies $T_P(r_k) = T_Q(r_k),\ U_P(r_k) = U_Q(r_k)$ and $F_P(r_k) = F_Q(r_k)$ for any $r_k \in R$ and k = 1,2,3….n. Thus $\Delta T_k = \Delta T_{min} = \Delta T_{max} = 0, \Delta U_k = \Delta U_{min} = \Delta U_{max} = 0, \Delta F_k = \Delta F_{min} = \Delta F_{max} = 0$. Hence we get K (P, Q) = 1.

The improved correlation coefficient formula which is defined is correct and also satisfies these properties in the above theorem . When we use any constant $\varepsilon > 3$ in the following expressions

$$\alpha_k = \frac{\varepsilon - \Delta T_k - \Delta T_{max}}{\varepsilon - \Delta T_{min} - \Delta T_{max}},$$

$$\gamma_k = \frac{\varepsilon - \Delta U_k - \Delta U_{max}}{\varepsilon - \Delta U_{min} - \Delta U_{max}},$$

$$\mu_k = \frac{\varepsilon - \Delta F_k - \Delta F_{max}}{\varepsilon - \Delta F_{min} - \Delta F_{max}}$$

4.3 Example

Let A = { r, 0,0,0,0} and B = { r, 0.4,0.2,0.5} be any two NPS s in R. Therefore by equation (1) we get K(A, B) =0.15 . It shows that the above defined improved correlation coefficient overcome the disadvantages of the correlation coefficient .

In the following, we define a weighted correlation coefficient between NPS s since the differences in the elements are considered into an account,

Let w_k be the weight of each element r_k(k = 1,2…n), $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$, then the weighted correlation coefficient between the NPS s A and B

$$K_w(\text{A, B}) = \frac{1}{3}\sum_{k=1}^{n} w_k[\ \alpha_k(1 - \Delta T_k) + \gamma_k(1 - \Delta U_k) + \mu_k(1 - \Delta F_k)\ ] \tag{2}$$

If $w = (1/n,1/n,1/n,\ldots.1/n)^T$, then equation (2) reduces to equation (1). K_w(A, B) also satisfies the three properties in the above theorem.

4.4 Theorem

Let w_k be the weight for each element r_k(k = 1,2,…n), $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$, then the weighted correlation coefficient between the NPS s A and B which is denoted by K_w(A, B) defined in equation (3.2) satisfies the following properties.

1) K_w(A, B) = K_w(B, A);
2) $0 \leq K_w(\text{A, B}) \leq 1$;
3) K_w(A, B) = 1 iff A = B.

It is similar to prove the properties in theorem 3.1

5 Decision Making using the improved correlation coefficient of NPS s

Multiple attribute decision making (MADM) problems refers to make decisions when several attributes are involved in real -life problem. For example one may buy a vehicle by analysing the attributes which is given in tems of price, style,safety, comfort etc.,

Here we consider a multiple attribute decision making problem with Neutrosophic pythagorean information and the characteristic of an alternative A_i(i = 1,2,…m) on an attribute C_j(j = 1,2…n) is represented by the following NPS s:

$A_i = \{(C_j, T_{A_i}(C_j), U_{A_i}(C_j), F_{A_i}(C_j) \setminus C_j \in C, j = 1,2, \ldots n\}$,

Where $T_{A_i}(C_j), U_{A_i}(C_j), F_{A_i}(C_j) \in [0,1]$ and

$0 \leq T_{A_j}^2(C_j) + +U_{A_j}^2(C_j) + F_{A_j}^2(C_j) \leq 2$ for $C_j \in C, j = 1,2, \ldots n$ and I = 1,2,…m.

$$d_{ij} = (t_{ij}, u_{ij}, f_{ij})(i = 1,2, \ldots m; j = 1,2 \ldots n).$$

Here the values of d_{ij} are usually derived from the evaluation of an alternative A_i with respect to a criteria C_j by the expert or decision maker. Therefore we got a Neutrosophic pythagorean decision matrix $D = (d_{ij})_{m\times n}$.

In the caseof ideal alternative A^* an ideal PNP can be defined by

d_j^* = (t_j^*, u_j^*, f_j^*) = (1, 0,0)(j = 1,2…n) in the decision making method,

Hence the wighted correlation coefficient between an alternative A_i(i=1,2….m) and the ideal alternative A^* is given by,

$$K_w(A_i, A^*) = \frac{1}{3}\sum_{j=1}^{n} w_j[\alpha_k(1 - \Delta t_{ij}) + \gamma_{ij}(1 - \Delta u_{ij}) + \mu_k(1 - \Delta f_{ij})] \tag{3}$$

Where,

$$\alpha_k = \frac{2 - \Delta t_{ij} - \Delta t_{imax}}{2 - \Delta t_{imin} - \Delta t_{imax}},$$

$$\gamma_{ij} = \frac{2 - \Delta u_{ij} - \Delta u_{imax}}{2 - \Delta u_{imin} - \Delta u_{imax}},$$

$$\mu_k = \frac{2 - \Delta f_{ij} - \Delta f_{imax}}{2 - \Delta f_{imin} - \Delta f_{imax}},$$

$$\Delta t_{ij} = |t_{ij}^2 - t_j^*|,$$

$$\Delta u_{ij} = |u_{ij}^2 - u_j^*|,$$

$$\Delta f_{ij} = |f_{ij}^2 - f_j^*|,$$

$$\Delta t_{imin} = min_j\ |t_{ij}^2 - t_j^*|,$$

$$\Delta u_{imin} = min_j\ |u_{ij}^2 - u_j^*|,$$

$$\Delta f_{imin} = min_j\ |f_{ij}^2 - f_j^*|,$$

$$\Delta t_{imax} = max_j\ |t_{ij}^2 - t_j^*|,$$

$$\Delta u_{imax} = max_j\ |u_{ij}^2 - u_j^*|,$$

$$\Delta f_{imax} = max_j\ |f_{ij}^2 - f_j^*|,$$

For i = 1,2….m and j = 1,2….n.

By using the above weighted correlation coefficient We can derive the ranking order of all alternatives and we can choose the best one among those.

5.1 Example

This section deals the example for the multiple attribute decision making problem with the given alt ernatives corresponds to the criteria alloted under Neutrosophic pythagorean environment

For this example, the three potential alternatives are to be evaluated under the four different attributes. The types of intellectual property rights are the alternatives and the various cybercrimes are the attributes for this example. The three potential alternatives are A_1 − copyright, A_2 −patent right and A_3 − trademark and the four diffident attributes are C_1 − infringement, C_2 – piracy and C_3 − cybersquatting . For the evaluation of an alternative A_1 with respect to an attribute C_1 , it is obtained from the questionnaire of a domain expert. According to the attributes we will derive the ranking order of all alternatives and based on this ranking order customer will select the best one. The weight vector of the above attributes is given by w = (0.35,0.4,0.25),Here the alternatives are to be evaluated under the above three attributes by the form of NPS s, In general the evaluation of an alternative A_i with respect to the attributes C_j (i=1,2,3,j=1,2,3) will be done by the questionnaire of a domain expert. In particular, while asking the opinion about an alternative A_1 with respect to an attribute C_1 , the possibility he (or) she say that the statement true is 0.4 , the statement indeterminacy is 0.3 and the statement false is 0.4 . It can be denoted in neutrosophic notation as d_{11} = (0.4,0.3,0.4).

$A_i \setminus C_j$	C_1	C_2	C_3
A_1	[0.4,0.3,0.4]	[0.5.0.4,0.5]	[0.4,0.1,0.4]
A_2	[0.4,0.2,0.6]	[0.3,0.3,0.5]	[0.1,0.4,0.2]
A_3	[0.3,0.4,0.4]	[0.5,0.1,0.4]	[0.4,0.5,0.4]

Then by using the proposed method we will obtain the most desirable alternative. We can get the values of the improved correlation coefficient $M_w(A_i, A^*)$ (i = 1,2,3) by using Equation (3.3).

Hence $M_w(A_1, A^*)$ = 0.2069, $M_w(A_2, A^*)$ = 0.17738, $M_w(A_3, A^*)$ = 0.13516. Therefore

Thus ranking order of the three potential alternatives is $A_1 > A_2 > A_3$. Therefore we can say that A1 alternative copyright have more cyber problems subsists in original literary, dramatic, musical, artistic, cinematographic film, sound recording and computer programme as well than the other alternatives of intellectual property rights. The decision making method provided in this paper is more judicious and more vigorous.

6. Conclusion

In this paper, we've outlined the improved correlation coefficient of NP sets and this is often applicable for a few cases ,once the correlation coefficient of NP sets is undefined (or) unmeaningful and additionally studied its properties. Decision making could be a process that plays a significant role in real world issues. the most method in higher cognitive process is recognizing the matter (or) chance and deciding to deal with it. Here we've mentioned the decision making technique using the improved correlation of NP sets and in significantly an illustrative example is given in multiple attribute higher cognitive process issues that involves the many alternatives supported varied criteria. Therefore our projected improved correlation of NP sets helps to spot the foremost appropriate different to the client supported on the given criteria.

Funding: "This research received no external funding"

Acknowledgments: I would like to express my special thanks of gratitude to S.P. Rhea and R. Kathiresan for their guidance and constant support in completing my paper.

Conflicts of Interest: "The authors declare no conflict of interest."

References

1. K. Atanasov, Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems*.**1986**,volume 20 87-96.
2. Broumi S, Smarandache F (2014) Rough Neutrosophic sets, *Ital J Pure Appl Math* ,***2014***, *volume* 32:493-502.
3. D.A. Chiang and N.P. Lin, Correlation of fuzzy sets, *Fuzzy Sets and Systems* **1999**, volume102, 221-226.
4. D.H. Hong, Fuzzy measures for a correlation coefficient of fuzzy numbers under Tw (the weakest tnorm)-based fuzzy arithmetic operations, *Information Sciences* **2006** volume 176,150-160.
5. Rajashi Chatterjee, P. Majumdar and S.K. Samanta, On some similarity measures and entropy on quadripartitioned single valued neutrosophic sets, *Journal of Intelligent and Fuzzy Systems* ,**2016**, volume 302475-2485.
6. R. Jhansi, K. Mohana , Florentin Smarandache, Correlation Measure of Pythagorean Neutrosophic sets with T and F as Dependent Neutrosophic Components, *Neutrosophic sets and systems,* **2021,** vol 30,203-212
7. R. Radha, A. Stanis Arul Mary. Pentapartitioned Neutrosophic pythagorean Soft set, *IRJMETS*, **2021 ,** *Volume 3(2)*,905-914.
8. R. Radha, A. Stanis Arul Mary. Pentapartitioned Neutrosophic Pythagorean Set, *IRJASH*, **2021**, *volume 3*, 62-82.
9. R. Radha, A. Stanis Arul Mary. Heptapartitioned neutrosophic sets, *IRJCT*, ***2021*** ,*volume 2*,222-230.
10. R. Radha, A. Stanis Arul Mary, F. Smarandache. Quadripartitioned Neutrosophic Pythagorean soft set, *International journal of Neutrosophic Science,* ***2021,*** *volume14(1)*,9-23.
11. R. Radha, A. Stanis Arul Mary. Neutrosophic Pythagorean soft set, *Neutrosophic sets and systems,* **2021,** ***vol*** *42*,65-78.
12. R. Radha, A. Stanis Arul Mary, Pentapartitioned Neutrosophic Pythagorean Lie algebra, (Accepted)
13. R, Radha, A. Stanis Arul Mary, Improved Correlation Coefficients of Quadripartitioned Neutrosophic Pythagorean sets using MADM, *Journal of Computational Mathematica*, volume 5(1),**2021,** 142-153
14. R. Radha, A.Stanis Arul Mary, Pentapartitioned Neutrosophic Generalized semi-closed sets, *Journal of Computational Mathematica*,volume5(1),**2021,**123-131
15. Rama Malik, Surapati Pramanik. Pentapartitioned Neutrosophic set and its properties, *Neutrosophic Sets and Systems*, **2020,** *Vol 36*,184-192.
16. F.Smarandache, A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic, *American Research Press, Rehoboth*.

17. Wang H, Smarandache F, Zhang YQ, Sunderraman R Single valued neutrosophic sets, *Multispace Multistruct* ,***2010*** *,volume* 4:410-413.
18. G.W. Wei, H.J. Wang and R. Lin, Application of correlation coefficient to interval-valued intuitionistic fuzzy multiple attribute decision-making with incomplete weight information, *Knowledge and Information Systems* ***2011***. *volume* 26 , 337-349.
19. J. Ye, Multicriteria fuzzy decision-making method using entropy weights-based correlation coefficients of interval valued intuitionistic fuzzy sets, *Applied Mathematical Modelling* ***2020*** *,volume* 34 , 38643870.
20. Jun Ye, Another Form of Correlation Coefficient between Single Valued Neutrosophic Sets and its multiple Attribute Decision Making Method, *Neutrosophic Sets and Systems*, 1, 8-12. doi.org/10.5281/zenodo.571265
21. J. Ye, Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment, *International Journal of General Systems* ***2013***, *volume* 42(4) , 386394.
22. J. Ye, Correlation coefficient of dual hesitant fuzzy sets and its application to multiple attribute decision making, *Applied Mathematical Modelling* ***2014***, *volume* 38 , 659666.
23. J. Ye, Improved correlation coefficients of single valued neutrosophic sets and interval neutrosophic sets for multiple attribute decision making,*Journal of Intelligent and Fuzzy Systems* ***2014,*** *volume* 27 24532462.
24. L.Zadeh , Fuzzy sets, *Information and Control* **1965**, volume 8, 87-96.
25. Zulqarnain R M, Xin X.L, Siddique, I, Asghar Khan, W. Yousif M.A Topsis method based on correlation coefficient under Pythagorean fuzzy environment and its application towards Green supply chain Management, *Sustainability*,**2021**,13,1642.

Received: May 6, 2021. Accepted: October 2, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

Assessment of English Teaching Systems Using a Single-Valued Neutrosophic MACROS Method

Ning Tang[1,*] , Bing Li[2] and Mohamed Elhoseny[3]

[1]Based Teaching Department, Henan Polytechnic, Zhengzhou 450046, Henan, China
E-mail: 25015@hnzj.edu.cn
[2]College of Information Engineering, Henan Polytechnic, Zhengzhou 450046, Henan, China
E-mail: 22044@hnzj.edu.cn
[3]College of Computer Information Technology, American University of the Emirates, Dubai, UAE
E-mail: Mohamed.elhoseny@aue.ae

*Correspondence: E-mail: 25015@hnzj.edu.cn

Abstract: Multi-Criteria Decision Making (MCDM) approaches are an effective tool for dealing with decision-making in various areas. It is a very complex and challenging task for computing an admittable solution with different and conflicting criteria. This work developed a new Measurement of Alternatives and Ranking according to the Compromise Solution (MARCOS) approach for English Teaching System (ETS). The main advantage of using this method is using a cost and profit solution for starting the formulation matrix, calculating the utility degree in both solutions, new way for calculation a function of utility and combination method, employed a large set of criteria and alternatives while keeping stability. ETS is very important for organizations, countries, and governments. It is a very critical task for assessing ETS. This paper proposed an example for using the MARCOS method for Assessment ETS. This example contains five main criteria, twenty-two sub-criteria and six alternatives for assessment ETS. The MARCOS method is employed under Single Valued Neutrosophic Sets (SVNSs) because the assessment ETS contains incomplete and uncertain information. So, SVNSs are an effective tool for overcoming this uncertainty. Scale from 1-5 used for evaluated criteria and alternatives by three experts and decision-makers who have an expert in this field. This paper help organization and countries which want to build an ETS.

Keywords: MARCOS, English Teaching System, SVNSs, Uncertainty.

---

1. Introduction

Higher education plays a vital role in the assessment English Teaching System (ETS) by improving the quality of training senor talent[1], [2]. Many countries that do not have an English mother tongue are trying to improve the education process in science and the English language by training students. So this goal is very important to evaluate and enhance the quality of English teaching with the ability of English outstanding[3], [4]. Assessment ETS is a very complex task due to contains many various criteria and alternatives like teaching system, management system, research of scientific, teachers, students, innovation, system integrations and mechanism of teaching, course material, employment, resource utilization, self-study communications skills, various methods and technical skills. So many researchers move toward innovation to assess the ETS by using various methods and functions.

The process of evaluation ETS contains incomplete and vague information. So, we propose a Single Valued Neutrosophic Sets (SVNSs) to overcome this problem through introduce three values truth, indeterminacy and falsity membership degrees. SVNSs used to handle with the incomplete, inconsistent and uncertainty information. It used is this paper to deal with vague information in process assessment ETS. SVNS used in scientific and engineering fields. Due to this problem contains multiple and conflict criteria, the multi-criteria decision making (MCDM) methods were used for this evaluation. We select an MCDM method MARCOS for evaluation ETS. MARCOS method is used for calculation weight of criteria and rank alternatives. It is the best method for dealing with conflict and complex criteria and alternatives. It builds a relationship between criteria and alternatives through cost and benefit ideal solutions. Also, MARCOS is used for calculation the utility degree between cost and benefit ideal solutions. The main benefit of the utility function is to compute the position of alternatives regard cost and benefit ideal solutions. It used to present the anti-ideal and ideal solution and determine utility degree for two solution. It deal effectively with large dimension criteria and alternatives. The best alternative determined by nearest to benefit solution and farness of cost solution.

Stević et al.[5] used the MARCOS method for supplier selection in healthcare industries. They used twenty-one criteria and eight alternatives for their problem. They used fuzzy systems and scales from 1 to 5 to evaluate criteria and alternatives. The main limitations in their paper not considering the indeterminacy value in their calculations. They used only truth, and falsity membership degrees.

Puška et al. [6] used a MARCOS method for the selection of sustainable suppliers. They used fuzzy systems in their calculations. They were not consider the indeterminacy value in their calculations.

The main contributions in this paper, we proposed a hybrid model from SVNSs and the MARCOS method for overcoming the uncertainty in evaluating ETS. We use six alternatives with five main criteria and twenty-two criteria. The indeterminacy value considers in calculations to overcome incomplete information. This paper help decision makers and government to make a best decisions in process of English teaching. This paper aids many countries to develop process of English teaching by providing many criteria that impact in this process.

The rest of this paper presented section two for hybrid model and section three presented an example and results. Section 4 presented conclusions of this paper.

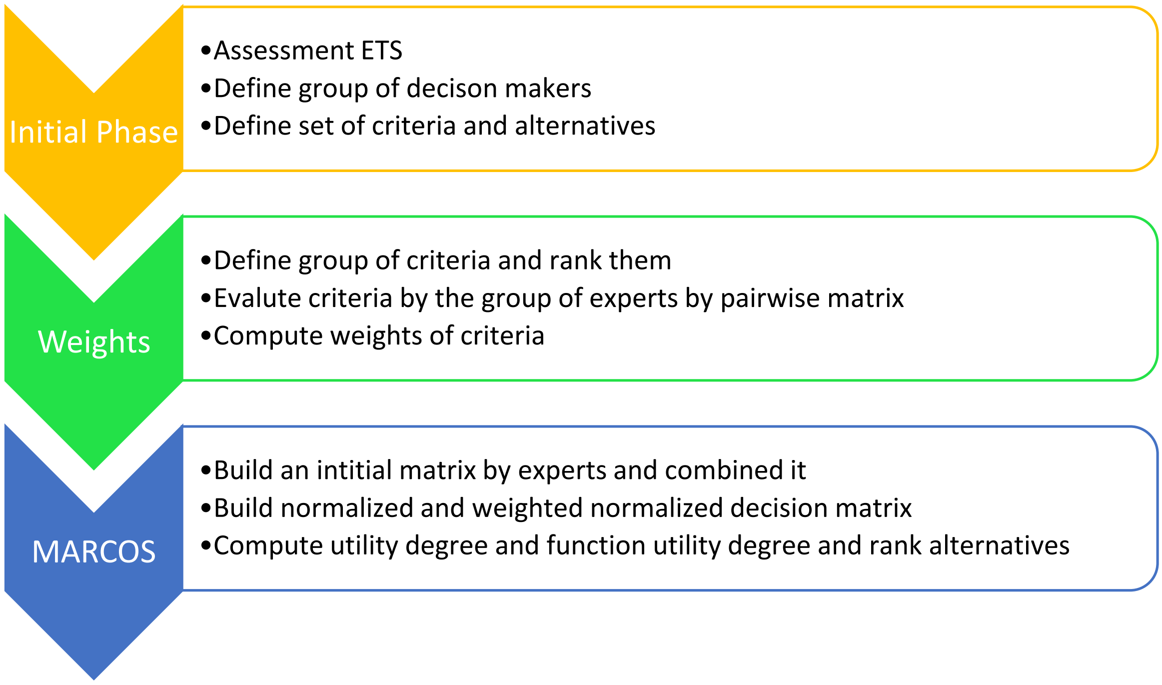


Fig 1. The methodology of this paper

2. Framework of this paper

This section consists form two-part. The first part is calculating the weights of criteria, and the second part rank alternatives and introduce neutrosophic equations. The neutrosophic sets created by Smarandache[7]–[14] . Fig 1. presented the methodology of this paper.

The following definitions with SVNNs.

Definition 1: let $K_1 = (T_1, I_1, F_1)$ $K_2 = (T_2, I_2, F_2)$two single-value neutrosophic numbers (T_1, I_1, F_1) present the Truth, Indeterminacy and Falsity and their operations presented as follow:

Complement $K_1^C = (F_1, 1 - I_1, T_1)$ (1)

Equality $K_1 = K_2$ *if and only if* $K_1 \subseteq K_2$ *and* $K_2 \subseteq K_1$ (2)

Union $K_1 \cup K_2 = (T_1 \vee T_2, I_1 \wedge I_2, F_1 \wedge F_2)$

(3)

Intersection$K_1 \cap K_2 = (T_1 \wedge T_2, I_1 \vee I_2, F_1 \vee F_2)$

(4)

Definition 2: The following addition and multiplication the two SVNSs:

$K_1 \oplus K_2 = (T_1 + T_2 - T_1T_2, I_1 I_2, F_1F_2)$ (5)

$K_1 \otimes K_2 = (T_1T_2, I_1 + I_2 - I_1 I_2, F_1 + F_2 - F_1F_2)$ (6)

Definition 3: The following subtraction and division

$K_1 \ominus K_2 = \left(\frac{T_1 - T_2}{1 - T_2} + \frac{I_1}{I_2}, \frac{F_1}{F_2}\right) k_1 > k_2, T_2 \neq 0, I_2 \neq 0, F_2 \neq 0,$ (7)

$K_1 \oslash K_2 = (\frac{T_1}{T_2} + \frac{I_1 - I_2}{1 - I_2}, \frac{F_1 - F_2}{1 - F_2}) k_2 > k_1, T_2 \neq 0, I_2 \neq 0, F_2 \neq 0,$ (8)

The steps of the MARCOS method are organized as follow:

Step 1: Build an initial decision matrix between criteria and alternatives. So, define the number of criteria, alternatives and experts who evaluate the decision matrix—then combined the initial matrix that includes opinions of various experts into one decision matrix. Then apply score function to obtain the single value instead of three values.

S(A) = $\frac{2+a-b-c}{3}$ (9) where a,b,c refers to Truth, Indeterminacy and Falsity value

Step 2: Define the cost (B) and benefit (A) ideal solution in the initial matrix. This matrix called the extended matrix. The ideal benefit solution computed by the maximum of criteria value considers the best characteristics. But ideal cost solution is the opposite benefit ideal solution. Cost ideal solution computed by the minimum value of each criterion.

Step 3: Build an extended normalized matrix.

$norm_{xy} = \frac{S_x}{A_x}$ for benefit criteria (10)

$norm_{xy} = \frac{A_x}{S_x}$ for cost criteria (11)

Where S_x presented value of decision matrix and A_x present value of benefit ideal solution. x refers to the number of criteria and y refers to number of alternatives

Step 4: Build a weighted normalized decision matrix by multiplying values of the extended normalized matrix by the value of criteria.

$Q_{xy} = norm_{xy} * E_y$ (12)

Where, value of E_y presented weights of criteria.

Step 5: Compute the utility degree of alternatives for benefit and cost ideal solution.

$H_x^+ = \frac{L_x}{L_{A_x}}$ for benefit criteria. (13)

$H_x^- = \frac{L_x}{L_{B_x}}$ for cost criteria. (14)

$L_x = \sum_{x=1}^{n} Q_{xy}$ where L_x summation values of weighted normalized decision matrix (15)

Step 6: Compute the utility function of alternative which show relationship between best and cost ideal solution.

$$f(H_x) = \frac{H_x^+ + H_x^-}{1 + \frac{1-f(H_x^+)}{f(H_x^+)} + \frac{1-H_x^-}{H_x^-}} \quad (16)$$

Step 6.1 The utility function for cost and benefit ideal solution can compute as:

$f(H)_x^+ = \frac{H_x^-}{H_x^+ + H_x^-}$ for benefit criteria (17)

$f(H)_x^- = \frac{H_x^+}{H_x^+ + H_x^-}$ for cost criteria (18)

Step 7: Rank alternatives according to the highest value of utility function.

Table 1. The five main and twenty-two sub criteria.

Main Criteria	**Sub Criteria**
Student's Learning (SL)	Interest of ET (SL.1)
	Learning initiative (SL.2)
	Self-study (SL.3)
	Ability find and solve problems (SL.4)
Innovation (I)	Intelligent educational technology (I.1)
	Excellent course (I2)
	Learning base (I3)
System Integration (SI)	Political Success (SI.1)
	Professional Compaction (SI2)

	Practices and exercises (SI.3)
	Transformation Rate (SI.4)
	Physical achievement (SI.5)
Management (M)	Ability ET Management (M.1)
	Reward and punishment (M2)
	Resource utilization (M.3)
Professional Teachers (PT)	Cognitive comprehension (PT.1)
	Technical skills (PT.2)
	Course materials (PT.3)
	Scientific research (PT.4)
	Communications skills (PT.5)
	Skilled Teachers (PT.6)
	Teaching effect (PT.7)

3. An Example and Results

In this section, we provide an example for a MARCOS method and introduce its results. First, the five main criteria, twenty-two sub-criteria and six alternatives, are used for an example. Table 1 presents five main criteria and twenty-two sub-criteria. The criteria proposed in this work collected for literature review [4], [15]. Fig 2. Present the alternatives proposed in this work.

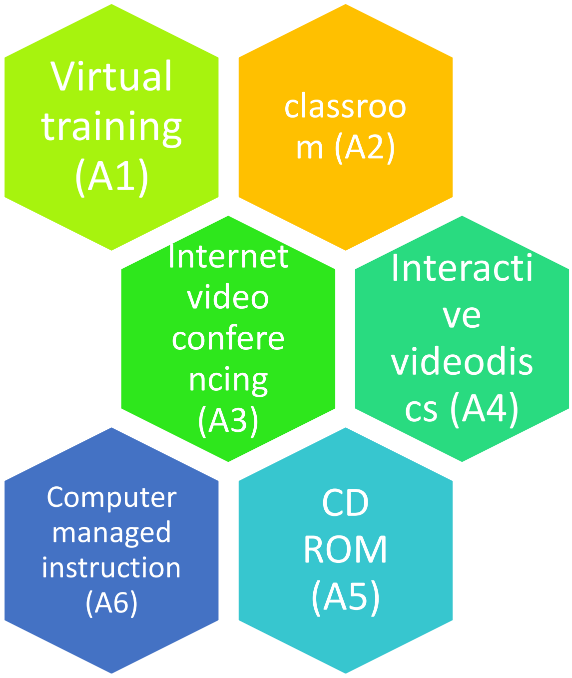


Fig 2. Six alternatives.

Table 2. SVNNS

Linguistics terms	SVNNS
Very Bad (VP)	<0.30,0.75,0.70>
Bad (P)	<0.40,0.65,0.60>
Medium (M)	<0.50,0.50,0.50>
Good (G)	<0.80,0.15,0.20>
Very Good (VG)	<0.90,0.10,0.10>

Three decision-makers and experts evaluated criteria and alternatives by Single-Valued Neutrosophic Numbers in Table 2. Where Very Bad presents the lowest rank and Very Moral presents the highest rank. First, experts evaluated criteria for calculating the weights of criteria. Table 3 presented the opinions of experts for evaluation criteria. The weights of criteria computed by the mean value of criteria for three criteria. Fig 3. presented the weights of five main criteria from Fig 3. Professional teachers are the highest weight of criteria, and system integration is the lowest weight of criteria.

Values in Table 3. Obtained from applying score function of SVNSs using Eq (10).

Table 3. Pairwise matrix for five main criteria

	SL	I	SI	M	PT	Sum
DM1	0.8167	0.383	0.283	0.9	0.8167	3.1994
DM2	0.383	0.9	0.383	0.283	0.8167	2.7657
DM3	0.9	0.9	0.8167	0.383	0.9	3.8997
DM1	0.255267	0.11971	0.088454	0.281303	0.255267	
DM2	0.138482	0.325415	0.138482	0.102325	0.295296	
DM3	0.230787	0.230787	0.209426	0.098213	0.230787	
Mean	0.208179	0.225304	0.145454	0.160613	0.26045	

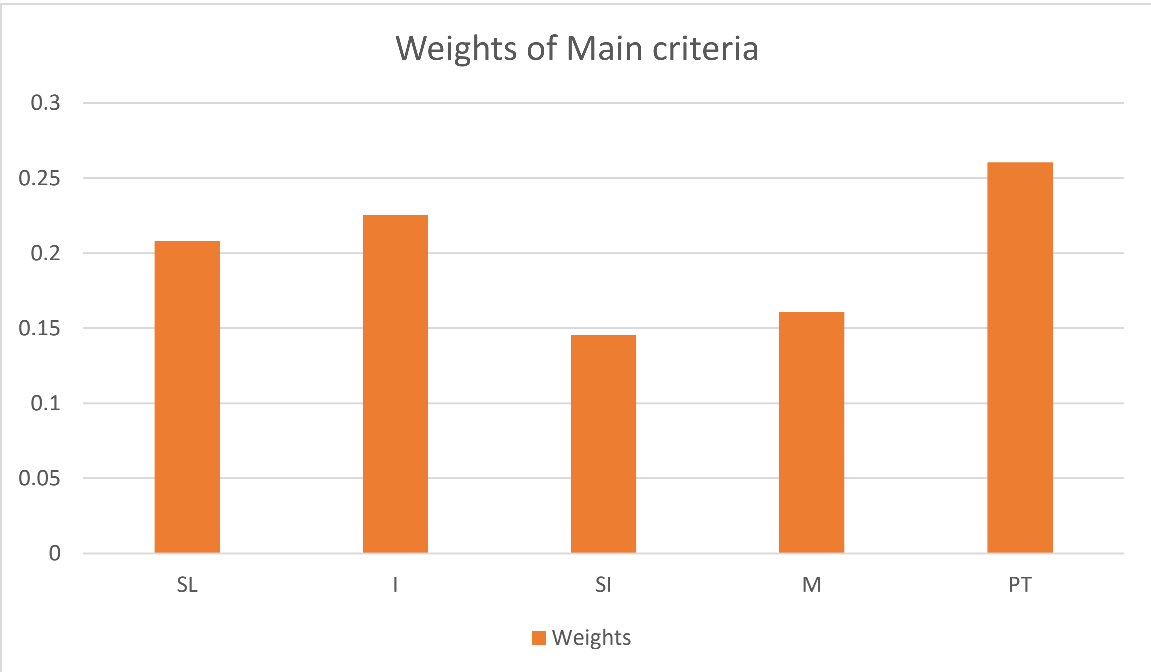


Fig 3. Weights of main criteria

Then evaluate sub-criteria by experts. Table 4-8 present the opinions of experts in twenty-two sub-criteria and weights of sub-criteria. Fig 4-8 present the weights of sub-criteria for five main criteria. Value in Table 4-8 obtained from score function is in Eq. (10).

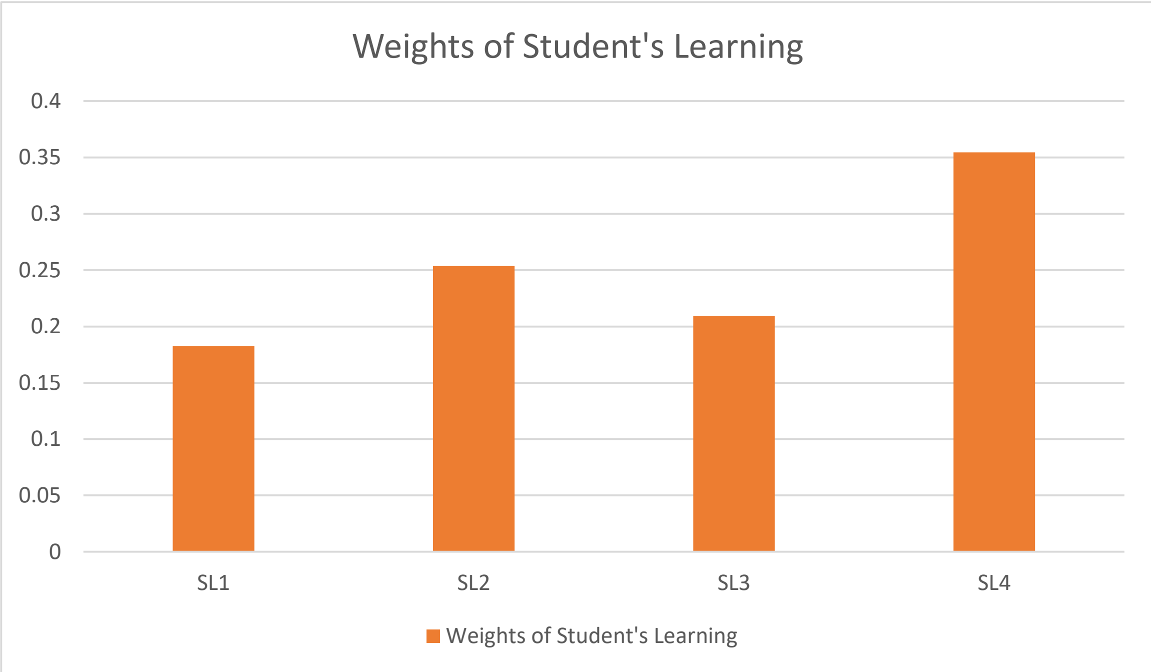


Fig 4. Weights of Student's Learning

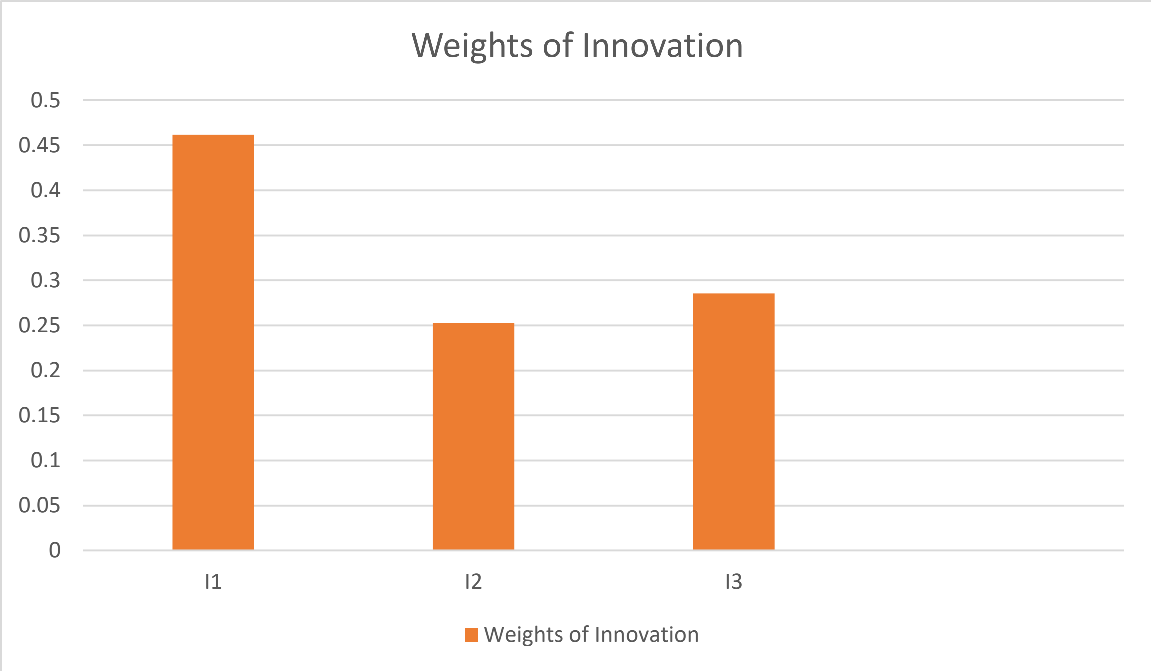


Fig 5. Weights of Innovation.

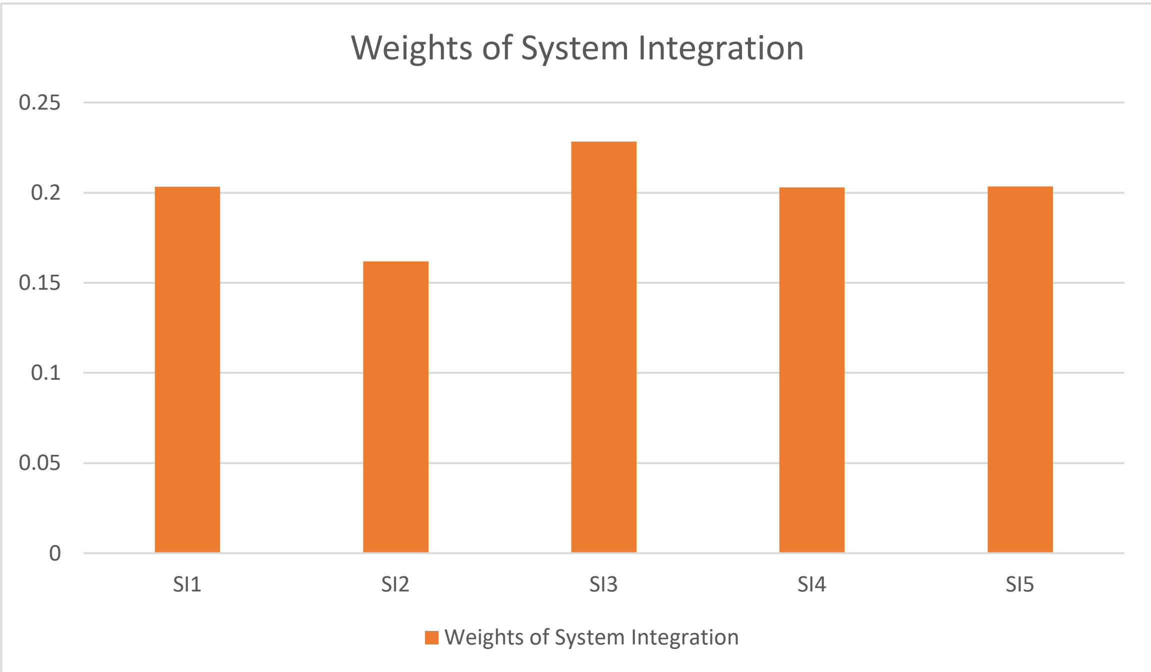


Fig 6. Weights of System Integration.

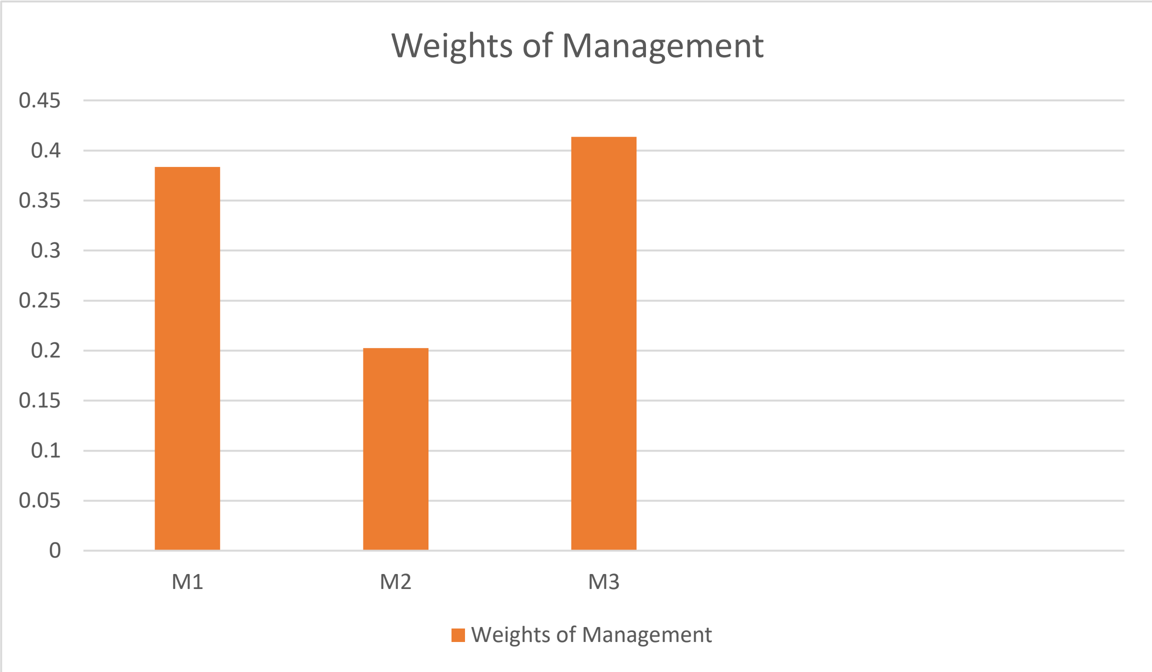


Fig 7. Weights of Management.

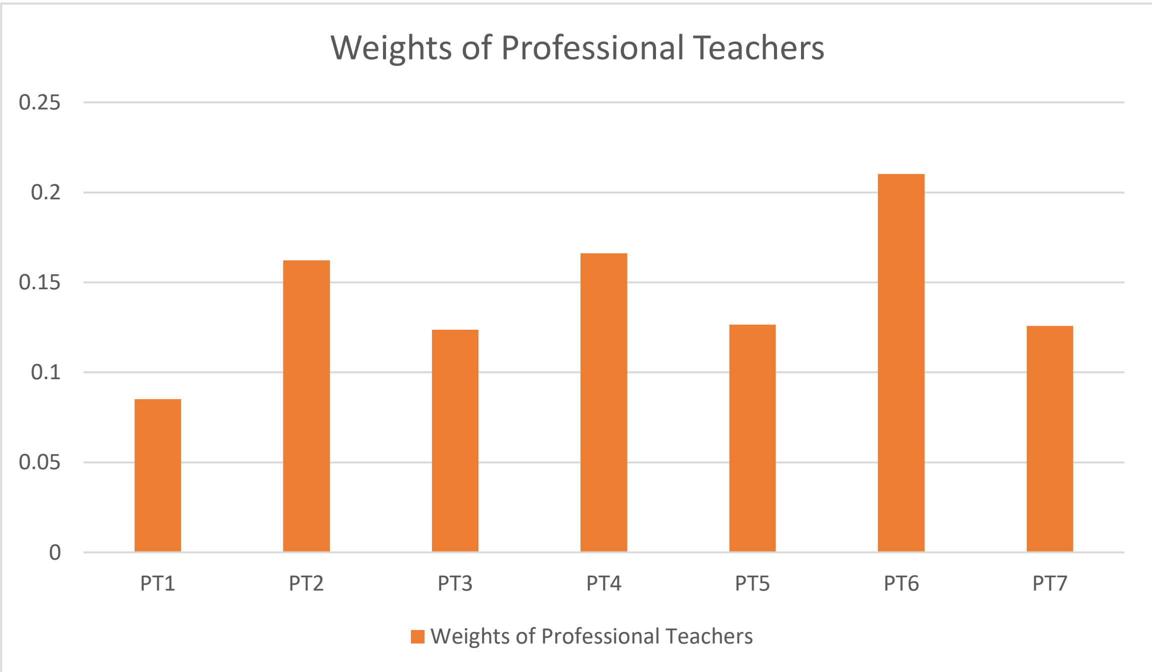


Fig 8. Weights of professional Teachers.

Table 4. Pairwise matrix for Student's Learning

	SL_1	SL_2	SL_3	SL_4	Sum
DM1	0.283	0.9	0.9	0.8167	2.8997
DM2	0.283	0.283	0.383	0.8167	1.7657
DM3	0.8167	0.8167	0.283	0.9	2.8164
DM1	0.097596	0.310377	0.310377	0.28165	
DM2	0.160276	0.160276	0.216911	0.462536	
DM3	0.28998	0.28998	0.100483	0.319557	
Mean	0.182618	0.253544	0.209257	0.354581	

Table 5. Pairwise matrix for Innovation

	I_1	I_2	I_3	Sum
DM1	0.8167	0.383	0.9	2.0997
DM2	0.8167	0.8167	0.5	2.1334
DM3	0.9	0.283	0.283	1.466
DM1	0.38896	0.182407	0.428633	
DM2	0.382816	0.382816	0.234368	
DM3	0.613915	0.193042	0.193042	
Mean	0.461897	0.252755	0.285348	

Table 6. Pairwise matrix for System Integration

	SI_1	SI_2	SI_3	SI_4	SI_5	Sum
DM1	0.9	0.283	0.5	0.9	0.8167	3.3997

DM2	0.8167	0.9	0.9	0.283	0.283	3.1827
DM3	0.283	0.383	0.8167	0.8167	0.9	3.1994
DM1	0.264729	0.083243	0.147072	0.264729	0.240227	
DM2	0.256606	0.282779	0.282779	0.088918	0.088918	
DM3	0.088454	0.11971	0.255267	0.255267	0.281303	
Mean	0.203263	0.16191	0.228372	0.202971	0.203483	

Table 7. Pairwise matrix for Management

	M_1	**M_2**	**M_3**	**Sum**
DM1	0.9	0.283	0.383	1.566
DM2	0.8167	0.5	0.8167	2.1334
DM3	0.283	0.283	0.9	1.466
DM1	0.574713	0.180715	0.244572	
DM2	0.382816	0.234368	0.382816	
DM3	0.193042	0.193042	0.613915	
Mean	0.383524	0.202708	0.413768	

Table 8. Pairwise matrix for Professional Teachers

	PT_1	**PT_2**	**PT_3**	**PT_4**	**PT_5**	**PT_6**	**PT_7**	**Sum**
DM1	0.383	0.8167	0.383	0.8167	0.9	0.8167	0.283	4.3991
DM2	0.283	0.8167	0.283	0.383	0.383	0.9	0.383	3.4317
DM3	0.383	0.283	0.9	0.9	0.283	0.8167	0.9	4.4657

DM1	0.087063	0.185652	0.087063	0.185652	0.204587	0.185652	0.064331	
DM2	0.082466	0.237987	0.082466	0.111606	0.111606	0.262261	0.111606	
DM3	0.085765	0.063372	0.201536	0.201536	0.063372	0.182883	0.201536	
Mean	0.085098	0.162337	0.123689	0.166265	0.126522	0.210265	0.125825	

Then compute the weights of global criteria by multiplying weights of main criteria by weights of sub-criteria. Table 9 presented the values of weights sub-criteria.

Table 9. Global weights for sub criteria.

	Weights
SL.1	0.038017
SL.2	0.052783
SL.3	0.043563
SL.4	0.073816
I.1	0.104067
I.2	0.056947
I.3	0.06429
SI.1	0.029565
SI.2	0.023551
SI.3	0.033218
SI.4	0.029523
SI.5	0.029597
M.1	0.061599
M.2	0.032558
M.3	0.066457
PT.1	0.022164

PT_2	0.042281
PT_3	0.032215
PT_4	0.043304
PT_5	0.032953
PT_6	0.054764
PT_7	0.032771

Then rank six alternatives according to values of the MARCOS method. First, build the three initial matrices that contain the opinions of three matrices with the first row presents the cost ideal solution, and the last row presents the ideal benefit solution. Table 10-12 presented the opinions of three experts. Then combined three decision matrices into one matrix in Table 13. Then normalize the decision matrix in Table 14. Then compute the weighted normalized decision matrix in Table 15. Then compute the utility degree and function utility in Table 16. Then rank alternatives according to the highest value in function utility in Table 16. Fig 9 presented rank of alternatives. The classroom is the highest rank, and CD-ROM is the lowest rank. Values in Table 10-12 obtained through Eq. (10). Column A and B refers to the cost and ideal solution as mentioned in step 2. And all criteria are benefit.

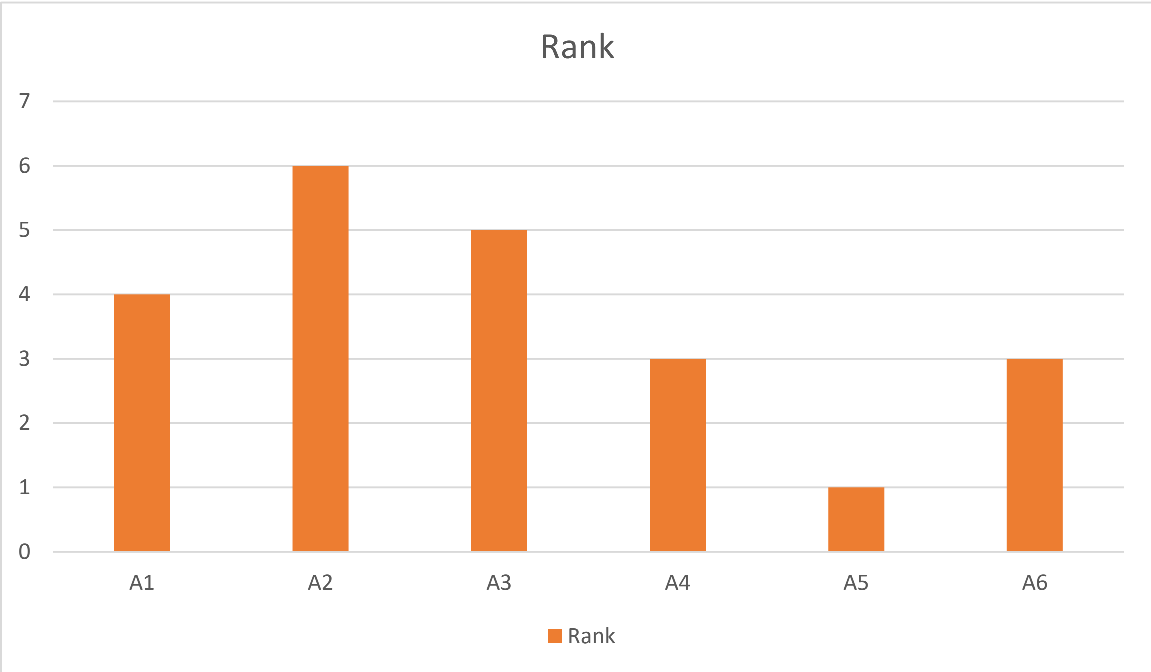


Fig 9. Rank alternatives

Table 10. The initial decision matrix by first expert

DM1	A	A1	A2	A3	A4	A5
SL_1	0.283	0.283	0.5	0.8167	0.8167	0.383
SL_2	0.383	0.383	0.9	0.5	0.9	0.5
SL_3	0.283	0.5	0.9	0.8167	0.8167	0.283
SL_4	0.283	0.8167	0.283	0.383	0.9	0.5
I_1	0.283	0.8167	0.9	0.9	0.283	0.5
I_2	0.283	0.9	0.8167	0.8167	0.9	0.9
I_3	0.283	0.9	0.8167	0.8167	0.283	0.283
SI_1	0.283	0.8167	0.9	0.5	0.5	0.283
SI_2	0.283	0.283	0.9	0.8167	0.8167	0.383
SI_3	0.283	0.383	0.9	0.383	0.283	0.5
SI_4	0.283	0.8167	0.9	0.9	0.5	0.8167
SI_5	0.283	0.5	0.5	0.9	0.283	0.383
M_1	0.5	0.9	0.5	0.5	0.8167	0.8167
M_2	0.283	0.283	0.383	0.283	0.383	0.5
M_3	0.283	0.9	0.9	0.8167	0.283	0.5
PT_1	0.283	0.383	0.283	0.8167	0.8167	0.9
PT_2	0.283	0.5	0.8167	0.283	0.383	0.5
PT_3	0.283	0.9	0.8167	0.9	0.5	0.8167
PT_4	0.283	0.383	0.283	0.9	0.5	0.8167
PT_5	0.283	0.5	0.8167	0.9	0.8167	0.283
PT_6	0.283	0.8167	0.5	0.8167	0.9	0.283
PT_7	0.283	0.8167	0.5	0.5	0.9	0.9

A6	B
0.283	0.8167
0.383	0.9
0.383	0.9
0.5	0.9
0.9	0.9
0.283	0.9
0.383	0.9
0.383	0.9
0.283	0.9
0.383	0.9
0.283	0.9
0.5	0.9
0.8167	0.9
0.5	0.5
0.283	0.9
0.5	0.9
0.9	0.9
0.283	0.9
0.283	0.9
0.5	0.9
0.383	0.9
0.283	0.9

Table 11. The initial decision matrix by second expert

DM2	A	A1	A2	A3	A4	A5	A6	B
SL.1	0.283	0.9	0.5	0.9	0.5	0.283	0.283	0.9
SL.2	0.5	0.8167	0.8167	0.5	0.5	0.5	0.8167	0.8167
SL.3	0.283	0.8167	0.8167	0.5	0.8167	0.283	0.9	0.9
SL.4	0.283	0.5	0.283	0.5	0.9	0.5	0.9	0.9
I1	0.283	0.283	0.9	0.5	0.283	0.5	0.8167	0.9
I2	0.283	0.5	0.8167	0.283	0.9	0.9	0.5	0.9
I3	0.283	0.383	0.8167	0.8167	0.283	0.283	0.9	0.9
SI.1	0.283	0.5	0.9	0.5	0.283	0.283	0.9	0.9
SI.2	0.383	0.8167	0.9	0.8167	0.8167	0.383	0.9	0.9
SI.3	0.283	0.8167	0.9	0.5	0.283	0.5	0.283	0.9
SI.4	0.5	0.9	0.9	0.8167	0.5	0.8167	0.5	0.9
SI.5	0.283	0.8167	0.5	0.9	0.283	0.383	0.8167	0.9
M.1	0.283	0.5	0.5	0.8167	0.5	0.8167	0.283	0.8167
M.2	0.283	0.5	0.383	0.283	0.383	0.5	0.9	0.9
M.3	0.283	0.8167	0.5	0.8167	0.283	0.5	0.383	0.8167
PT.1	0.283	0.8167	0.283	0.9	0.8167	0.9	0.383	0.9
PT2	0.283	0.9	0.8167	0.283	0.383	0.5	0.8167	0.9
PT3	0.5	0.8167	0.5	0.9	0.5	0.8167	0.5	0.9
PT4	0.283	0.5	0.283	0.9	0.5	0.5	0.283	0.9
PT5	0.283	0.5	0.8167	0.9	0.8167	0.283	0.5	0.9
PT6	0.383	0.5	0.5	0.8167	0.9	0.5	0.383	0.9
PT7	0.5	0.5	0.9	0.5	0.9	0.9	0.5	0.9

Table 12. The initial decision matrix by third expert

DM3	A	A1	A2	A3	A4	A5	A6	B
SL.$_1$	0.283	0.8167	0.283	0.9	0.5	0.283	0.383	0.9
SL.$_2$	0.283	0.283	0.8167	0.5	0.5	0.5	0.383	0.8167
SL.$_3$	0.283	0.5	0.8167	0.9	0.8167	0.283	0.283	0.9
SL.$_4$	0.283	0.9	0.283	0.9	0.9	0.5	0.383	0.9
I$_1$	0.283	0.383	0.9	0.8167	0.283	0.5	0.283	0.9
I$_2$	0.283	0.9	0.8167	0.283	0.8167	0.9	0.383	0.9
I$_3$	0.283	0.283	0.8167	0.5	0.283	0.283	0.5	0.8167
SI$_1$	0.283	0.8167	0.9	0.5	0.283	0.283	0.5	0.9
SI$_2$	0.383	0.5	0.9	0.8167	0.8167	0.383	0.5	0.9
SI$_3$	0.283	0.8167	0.9	0.5	0.283	0.5	0.5	0.9
SI$_4$	0.5	0.8167	0.9	0.9	0.5	0.8167	0.5	0.9
SI$_5$	0.283	0.9	0.5	0.9	0.283	0.383	0.9	0.9
M$_1$	0.283	0.283	0.5	0.383	0.5	0.8167	0.283	0.8167
M$_2$	0.283	0.8167	0.383	0.283	0.383	0.5	0.383	0.8167
M$_3$	0.283	0.9	0.5	0.8167	0.5	0.5	0.283	0.9
PT$_1$	0.283	0.9	0.283	0.9	0.8167	0.9	0.283	0.9
PT$_2$	0.283	0.8167	0.5	0.283	0.383	0.5	0.5	0.8167
PT$_3$	0.5	0.9	0.5	0.9	0.5	0.8167	0.8167	0.9
PT$_4$	0.283	0.9	0.283	0.283	0.5	0.5	0.383	0.9
PT$_5$	0.283	0.383	0.8167	0.283	0.8167	0.283	0.8167	0.8167
PT$_6$	0.5	0.8167	0.5	0.9	0.9	0.5	0.8167	0.9
PT$_7$	0.283	0.8167	0.9	0.5	0.283	0.9	0.383	0.9

Table 13. The Combined decision matrix.

	A	A1	A2	A3	A4	A5	A6	B
SL$_{.1}$	0.283	0.666567	0.427667	0.872233	0.605567	0.316333	0.316333	0.872233
SL$_{.2}$	0.388667	0.494233	0.844467	0.5	0.633333	0.5	0.527567	0.844467
SL$_{.3}$	0.283	0.605567	0.844467	0.7389	0.8167	0.283	0.522	0.9
SL$_{.4}$	0.283	0.7389	0.283	0.594333	0.9	0.5	0.594333	0.9
I$_1$	0.283	0.494233	0.9	0.7389	0.283	0.5	0.666567	0.9
I$_2$	0.283	0.766667	0.8167	0.4609	0.872233	0.9	0.388667	0.9
I$_3$	0.283	0.522	0.8167	0.711133	0.283	0.283	0.594333	0.872233
SI$_1$	0.283	0.711133	0.9	0.5	0.355333	0.283	0.594333	0.9
SI$_2$	0.349667	0.533233	0.9	0.8167	0.8167	0.383	0.561	0.9
SI$_3$	0.283	0.672133	0.9	0.461	0.283	0.5	0.388667	0.9
SI$_4$	0.427667	0.844467	0.9	0.872233	0.5	0.8167	0.427667	0.9
SI$_5$	0.283	0.7389	0.5	0.9	0.283	0.383	0.7389	0.9
M$_{.1}$	0.355333	0.561	0.5	0.566567	0.605567	0.8167	0.4609	0.844467
M$_{.2}$	0.283	0.533233	0.383	0.283	0.383	0.5	0.594333	0.7389
M$_{.3}$	0.283	0.872233	0.633333	0.8167	0.355333	0.5	0.316333	0.872233
PT$_{.1}$	0.283	0.6999	0.283	0.872233	0.8167	0.9	0.388667	0.9
PT$_2$	0.283	0.7389	0.711133	0.283	0.383	0.5	0.7389	0.872233
PT$_3$	0.427667	0.872233	0.605567	0.9	0.5	0.8167	0.533233	0.9
PT$_4$	0.283	0.594333	0.283	0.694333	0.5	0.605567	0.316333	0.9
PT$_5$	0.283	0.461	0.8167	0.694333	0.8167	0.283	0.605567	0.872233
PT$_6$	0.388667	0.711133	0.5	0.844467	0.9	0.427667	0.527567	0.9
PT$_7$	0.355333	0.711133	0.766667	0.5	0.694333	0.9	0.388667	0.9

Table 14. The Normalized decision matrix.

	SL$_{.1}$	SL$_{.2}$	SL$_{.3}$	SL$_{.4}$	I$_1$	I$_2$	I$_3$	SI$_1$	SI$_2$	SI$_3$	SI$_4$	SI$_5$	M$_{.1}$	M$_{.2}$	M$_{.3}$	PT$_{.1}$	PT$_2$	PT$_3$	PT$_4$	PT$_5$	PT$_6$	PT$_7$

B	A6	A5	A4	A3	A2	A1	A
1	0.362671	0.362671	0.694271	1	0.490312	0.764207	0.324454
1	0.624734	0.59209	0.74998	0.59209	1	0.585261	0.460251
1	0.58	0.314444	0.907444	0.821	0.938296	0.672852	0.314444
1	0.66037	0.555556	1	0.66037	0.314444	0.821	0.314444
1	0.74063	0.555556	0.314444	0.821	1	0.549148	0.314444
1	0.431852	1	0.969148	0.512111	0.907444	0.851852	0.314444
1	0.681393	0.324454	0.324454	0.815302	0.936332	0.598464	0.324454
1	0.66037	0.314444	0.394815	0.555556	1	0.790148	0.314444
1	0.623333	0.425556	0.907444	0.907444	1	0.592481	0.388519
1	0.431852	0.555556	0.314444	0.512222	1	0.746815	0.314444
1	0.475185	0.907444	0.555556	0.969148	1	0.938296	0.475185
1	0.821	0.425556	0.314444	1	0.555556	0.821	0.314444
1	0.545788	0.967119	0.7171	0.670917	0.59209	0.664325	0.420778
1	0.804349	0.676682	0.518338	0.383002	0.518338	0.721658	0.383002
1	0.362671	0.573241	0.407383	0.936332	0.726105	1	0.324454
1	0.431852	1	0.907444	0.969148	0.314444	0.777667	0.314444
1	0.847136	0.573241	0.439103	0.324454	0.815302	0.847136	0.324454
1	0.592481	0.907444	0.555556	1	0.672852	0.969148	0.475185
1	0.351481	0.672852	0.555556	0.771481	0.314444	0.66037	0.314444
1	0.694271	0.324454	0.936332	0.796041	0.936332	0.528528	0.324454
1	0.586185	0.475185	1	0.938296	0.555556	0.790148	0.431852
1	0.431852	1	0.771481	0.555556	0.851852	0.790148	0.394815

Table 15. The weighted normalized decision matrix.

	A
SL.1	0.012335
SL.2	0.024293
SL.3	0.013698
SL.4	0.023211
I.1	0.032723
I.2	0.017907
I.3	0.020859
SI.1	0.009297
SI.2	0.00915
SI.3	0.010445
SI.4	0.014029
SI.5	0.009307
M.1	0.02592
M.2	0.01247
M.3	0.021562
PT.1	0.006969
PT2	0.013718
PT3	0.015308
PT4	0.013617
PT5	0.010692
PT6	0.02365
PT7	0.012938

A1	A2	A3	A4	A5	A6	B
0.029053	0.01864	0.038017	0.026394	0.013788	0.013788	0.038017
0.030892	0.052783	0.031252	0.039586	0.031252	0.032975	0.052783
0.029311	0.040875	0.035765	0.039531	0.013698	0.025266	0.043563
0.060603	0.023211	0.048746	0.073816	0.041009	0.048746	0.073816
0.057148	0.104067	0.085439	0.032723	0.057815	0.077075	0.104067
0.04851	0.051676	0.029163	0.05519	0.056947	0.024593	0.056947
0.038475	0.060197	0.052416	0.020859	0.020859	0.043807	0.06429
0.023361	0.029565	0.016425	0.011673	0.009297	0.019524	0.029565
0.013953	0.023551	0.021371	0.021371	0.010022	0.01468	0.023551
0.024807	0.033218	0.017015	0.010445	0.018454	0.014345	0.033218
0.027701	0.029523	0.028612	0.016402	0.026791	0.014029	0.029523
0.024299	0.016443	0.029597	0.009307	0.012595	0.024299	0.029597
0.040922	0.036472	0.041328	0.044173	0.059574	0.03362	0.061599
0.023496	0.016876	0.01247	0.016876	0.022031	0.026188	0.032558
0.066457	0.048255	0.062226	0.027073	0.038096	0.024102	0.066457
0.017236	0.006969	0.02148	0.020112	0.022164	0.009571	0.022164
0.035817	0.034471	0.013718	0.018566	0.024237	0.035817	0.042281
0.031221	0.021676	0.032215	0.017897	0.029233	0.019087	0.032215
0.028596	0.013617	0.033408	0.024058	0.029137	0.01522	0.043304
0.017416	0.030855	0.026232	0.030855	0.010692	0.022878	0.032953
0.043271	0.030424	0.051384	0.054764	0.026023	0.032102	0.054764
0.025894	0.027916	0.018206	0.025282	0.032771	0.014152	0.032771

Table 16. Utility, function utility and rank alternatives.

	L_x	H_x^+	H_x^-	f(H_x^+)	f(H_x^-)	$(f(H_x)$	Rank
A	0.354097						
A1	0.738442	0.738442	2.085423	0.7385	0.2615	0.675859	3
A2	0.751279	0.751279	2.121677	0.7385	0.2615	0.687609	1
A3	0.746485	0.746485	2.108137	0.7385	0.2615	0.683221	2
A4	0.636951	0.636951	1.798806	0.7385	0.2615	0.58297	4

A5	0.606483	0.606483	0.354097	0.368628	0.631372	0.291384	6
A6	0.585865	0.585865	1.654533	0.7385	0.2615	0.536213	5
B	1						

4. Conclusions

This paper introduces assessment ETS through an MCDM method. A new method is an extension of SVNSs called MARCOS. The main idea of this method proposed a relationship between alternatives and cost, benefit ideal solutions. Also, MARCOS is used for computing utility degree and function utility degree for cost and benefit ideal solution. Then rank alternatives through the highest value of function utility degree. ETS was proposed through the MARCOS method with six alternatives, five main criteria and twenty-two sub-criteria. Three experts and decision-makers who have experience in this area evaluated criteria and alternatives. In this paper, the weights of criteria and rank of alternatives were determined.

From the outcome, professional teachers are the highest, and system integration is the lowest. While in alternatives, the CD-ROM is the lowest in alternatives, and the classroom is the highest in alternatives. MARCOS method is an effective tool for dealing with uncertain information.

References

[1] A. P. Ayala and H. G. V. Berumen, "Evaluation of the influence that higher education boosts on students' entrepreneurial proclivity: Evidence from Mexico and Spain: Article 100404," *Int. J. Manag. Educ.*, vol. 18, no. 3, pp. 1–21, 2020.

[2] R. Matsuoka, "Concerted cultivation developed in a standardized education system," *Soc. Sci. Res.*, vol. 77, pp. 161–178, 2019.

[3] K.-D. Vattøy and K. Smith, "Students' perceptions of teachers' feedback practice in teaching English as a foreign language," *Teach. Teach. Educ.*, vol. 85, pp. 260–268, 2019.

[4] N. Li, "A Fuzzy Evaluation Model of College English Teaching Quality Based on Analytic Hierarchy Process," *Int. J. Emerg. Technol. Learn.*, vol. 16, no. 2, pp. 17–30, 2021.

[5] Ž. Stević, D. Pamučar, A. Puška, and P. Chatterjee, "Sustainable supplier selection in healthcare industries using a new MCDM method: Measurement of alternatives and ranking according to COmpromise solution (MARCOS)," *Comput. Ind. Eng.*, vol. 140, p. 106231, 2020.

[6] A. Puška, Ž. Stević, and I. Stojanović, "Selection of Sustainable Suppliers Using the Fuzzy MARCOS Method," *Curr. Chinese Sci.*, vol. 1, no. 2, pp. 218–229, 2021.

[7] F. Smarandache, "Neutrosophic set-a generalization of the intuitionistic fuzzy set," *Int. J. pure Appl. Math.*, vol. 24, no. 3, p. 287, 2005.

[8] F. Smarandache, "Neutrosophic set–a generalization of the intuitionistic fuzzy set," *J. Def. Resour. Manag.*, vol. 1, no. 1, pp. 107–116, 2010.

[9] F. Smarandache, "Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set), pythagorean fuzzy set, spherical fuzzy set, and q-rung orthopair fuzzy set, while neutrosophication is a generalization of regret theory, grey system theory, and three-ways decision (revisited)," *J. New Theory*, no. 29, pp. 1–31, 2019.

[10] F. Smarandache, "Neutrosophy, neutrosophic probability, set and logic, American Res," *Press. Rehoboth, USA*, 1998.

[11] F. Smarandache, "Linguistic paradoxes and tautologies," *Collect. Pap. Vol. III*, vol. 3, p. 122, 2000.

[12] F. Smarandache, "A unifying field in Logics: Neutrosophic Logic.," in *Philosophy*,

American Research Press, 1999, pp. 1–141.

[13] S. Pramanik, R. Roy, T. K. Roy, and F. Smarandache, "Multi-attribute decision making based on several trigonometric hamming similarity measures under interval rough neutrosophic environment," *Neutrosophic Sets Syst.*, vol. 19, pp. 110–118, 2018.

[14] S. Broumi and F. Smarandache, "Interval neutrosophic rough set," *Neutrosophic Sets Syst.*, vol. 7, pp. 23–31, 2015.

[15] H. Li, "Improved fuzzy-assisted hierarchical neural network system for design of computer-aided English teaching system," *Comput. Intell.*, 2020.

Received: May 9, 2021. Accepted: October 1, 2021

Identification of the Most Significant Risk Factor of COVID-19 in Economy Using Cosine Similarity Measure under SVPNS-Environment

Priyanka Majumder[1], Suman Das[2,*], Rakhal Das[3] and Binod Chandra Tripathy[4]

[1]Department of Basic Science and Humanities (Mathematics), Techno College of Engineering Agartala, Agartala, Tripura, India

[1]E-mail: majumderpriyanka94@yahoo.com

[2]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

[2]E-mail: sumandas18843@gmail.com, dr.suman1995@yahoo.com

[3]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

[3]E-mail: rakhaldas95@gmail.com, rakhal.mathematics@tripurauniv.in

[4]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

[4]E-mail: tripathybc@gmail.com, tripathybc@yahoo.com

***Correspondence:** sumandas18843@gmail.com

Abstract: In this article, we procure the idea of single-valued pentapartitioned neutrosophic cosine similarity measure (SVPNCSM) and single-valued pentapartitioned neutrosophic weighted cosine similarity measure (SVPNWCSM) under the single-valued pentapartitioned neutrosophic set (SVPNS) environment. Besides, we formulate several interesting results on SVPNCSM and SVPNWCSM of similarities between two SVPNSs. Further, we present a multi-attribute decision-making (MADM) model under SVPNS environment using the SVPNCSM. Finally, we provide a numerical example to show the applicability and effectiveness of our proposed MADM technique.

Keywords: *Neutrosophic Set*; *Similarity Measure*; *SVPNS*; *COVID-19*.

---

1. Introduction:

In 1965, Late Prof. L.A. Zadeh grounded the concept of fuzzy set theory to deal with the problems having uncertainty. In a fuzzy set, every element has a degree of membership lies between 0 and 1.

In 1986, K. Atanassov presented the idea of intuitionistic fuzzy set by generalizing fuzzy set theory. In an intuitionistic fuzzy set, every element has both degree of membership and non-membership lies between 0 and 1. Many researchers around the globe applied the notion of fuzzy set, intuitionistic fuzzy set and their extensions in the area of theoretical and practical research. Smarandache [40] introduced the idea of neutrosophic set theory by extending the idea of fuzzy set and intuitionistic fuzzy set theory to deal with the events which cannot be easily express by the degree of membership and non-membership. In an neutrosophic set, every element has three independent memberships values namely truth, indeterminacy, and false membership values respectively lies between 0 and 1. The degree of indeterminacy of a mathematical expression plays a vital role in every MADM problem of this real world. Afterwards, Wang et al. [43] extended the idea of neutrosophic set, and grounded the notion of single-valued neutrosophic set (SVNS) in the year 2010, which is more effective in dealing with the situation having incomplete and indeterminate information. Till now, many mathematicians used SVNS and their extensions in several branches of this real world such as medical diagnosis [34, 35], fault diagnosis [46, 47], data mining [30], decision-making problems [5, 11-13, 15, 27-29, 32-33, 36, 48], etc.

In the year 2020, Mallick and Pramanik [25] grounded the notion of SVPNS by splitting the indeterminacy membership function into three independent membership function namely contradiction membership function, ignorance membership function and unknown membership function. Afterwards, the concept of pentapartitioned neutrosophic *Q*-ideals of pentapartitioned neutrosophic *Q*-algebra was introduced by Das et al. [10]. In 2021, Das et al. [13] proposed a MADM technique using tangent Similarity Measure under SVPNS environment. Recently, Das et al. [12] established a MADM strategy based on grey relational analysis under the SVPNS environment.

The rest of this article has been designed as follows:

Section 2 is on the preliminaries and relevant definitions. In section 3, we introduce the concept of SVPNCSM and SVPNWCSM of similarities between two SVPNSs. Further, we formulate some theorems and propositions on SVPNCSM and SVPNWCSM under the SVPNS environment. In section 4, we propose a MADM technique using the SVPNWCSM under the SVPNS environment. In section 5, we validate the proposed MADM technique by providing a real world numerical example. Section 6 presents the concluding remarks of our work done in this paper. In this section, we also state some future scope of research in this direction.

Throughout this article, we use the following short terms for the clarity of the presentation.

Short Terms	
Single-Valued Neutrosophic Set	SVNS

Multi-Attribute Decision Making	MADM
Single-Valued Pentapartitioned Neutrosophic Set	SVPNS
Single-Valued Pentapartitioned Neutrosophic Cosine Similarity Measure	SVPNCSM
Single-Valued Pentapartitioned Neutrosophic Weighted Cosine Similarity Measure	SVPNWCSM
Decision Matrix	DM
Positive Ideal Alternative	PIA
Accumulated Measure Function	AMF

2.

Preliminaries and Definitions:

In the year 2020, Mallick and Pramanik [25] presented the notion of SVPNS as follows:
Assume that U be a universe of discourse. Then L, a SVPNS over U is defined by:

$$L = \{(\alpha, \Delta_L(\alpha), \Gamma_L(\alpha), \Pi_L(\alpha), \Omega_L(\alpha), \Phi_L(\alpha)): \alpha \in U\}.$$

Here, Δ_L, Γ_L, Π_L, Ω_L and Φ_L are the truth, contradiction, ignorance, unknown, and false membership functions from U to the unit interval [0, 1] respectively i.e., $\Delta_L(\alpha)$, $\Gamma_L(\alpha)$, $\Pi_L(\alpha)$, $\Omega_L(\alpha)$, $\Phi_L(\alpha) \in [0, 1]$, for each $\alpha \in U$. So, $0 \leq \Delta_L(\alpha) + \Gamma_L(\alpha) + \Pi_L(\alpha) + \Omega_L(\alpha) + \Phi_L(\alpha) \leq 5$, for each $\alpha \in U$.

The absolute SVPNS (1_{PN}) and the null SVPNS (0_{PN}) over a fixed set U are defined as follows:

(*i*) $1_{PN} = \{(\alpha, 1, 1, 0, 0, 0): \alpha \in U\}$,

(*ii*) $0_{PN} = \{(\alpha, 0, 0, 1, 1, 1): \alpha \in U\}$.

Suppose that $L = \{(\alpha, \Delta_L(\alpha), \Gamma_L(\alpha), \Pi_L(\alpha), \Omega_L(\alpha), \Phi_L(\alpha)): \alpha \in U\}$ and $M = \{(\alpha, \Delta_M(\alpha), \Gamma_M(\alpha), \Pi_M(\alpha), \Omega_M(\alpha), \Phi_M(\alpha)): \alpha \in U\}$ be any two SVPNSs over U. Then,

(*i*) $L \subseteq M$ if and only if $\Delta_L(\alpha) \leq \Delta_M(\alpha)$, $\Gamma_L(\alpha) \leq \Gamma_M(\alpha)$, $\Pi_L(\alpha) \geq \Pi_M(\alpha)$, $\Omega_L(\alpha) \geq \Omega_M(\alpha)$, $\Phi_L(\alpha) \geq \Phi_M(\alpha)$, for all $\alpha \in U$;

(*ii*) $L^c = \{(\alpha, \Phi_L(\alpha), \Omega_L(\alpha), 1-\Pi_L(\alpha), \Gamma_L(\alpha), \Delta_L(\alpha)): \alpha \in U\}$;

(*iii*) $L \cup M = \{(\alpha, \max\{\Delta_L(\alpha), \Delta_M(\alpha)\}, \max\{\Gamma_L(\alpha), \Gamma_M(\alpha)\}, \min\{\Pi_L(\alpha), \Pi_M(\alpha)\}, \min\{\Omega_L(\alpha), \Omega_M(\alpha)\}, \min\{\Phi_L(\alpha), \Phi_M(\alpha)\}): \alpha \in U\}$;

(*iv*) $L \cap M = \{(\alpha, \min\{\Delta_L(\alpha), \Delta_M(\alpha)\}, \min\{\Gamma_L(\alpha), \Gamma_M(\alpha)\}, \max\{\Pi_L(\alpha), \Pi_M(\alpha)\}, \max\{\Omega_L(\alpha), \Omega_M(\alpha)\}, \max\{\Phi_L(\alpha), \Phi_M(\alpha)\}): \alpha \in U\}$.

Suppose that $L = \{(p, 0.6, 0.1, 0.3, 0.4, 0.5), (q, 0.9, 0.1, 0.2, 0.2, 0.1)\}$ and $M = \{(p, 0.9, 0.2, 0.2, 0.1, 0.4), (q, 1.0, 0.3, 0.1, 0.2, 0.1)\}$ be two SVPNSs over a universe of discourse $U = \{p, q\}$. Then,

(*i*) $L \subseteq M$;

(*ii*) $L^c = \{(p, 0.5, 0.4, 0.7, 0.1, 0.6), (q, 0.1, 0.2, 0.8, 0.1, 0.9)\}$ and $M^c = \{(p, 0.4, 0.1, 0.8, 0.2, 0.9), (q, 0.1, 0.2, 0.9, 0.3, 1.0)\}$;

(*iii*) $L \cup M$ = {(*p*, 0.9, 0.2, 0.2, 0.1, 0.4), (*q*, 1.0, 0.3, 0.1, 0.2, 0.1)};

(*iv*) $L \cap M$ = {(*p*, 0.6, 0.1, 0.3, 0.4, 0.5), (*q*, 0.9, 0.1, 0.2, 0.2, 0.1)}.

3. SVPNCSM and SVPNWCSM under the SVPNS Environment:

In this section, we propose two similarity measures namely single-valued pentapartitioned neutrosophic cosine similarity measure (SVPNCSM) and single-valued pentapartitioned neutrosophic weighted cosine similarity measure (SVPNWCSM) under the SVPNS environment. Further, we formulate several interesting results on them under the SVPNS environment.

Definition 3.1. Let $L = \{(\alpha, \Delta_L(\alpha), \Gamma_L(\alpha), \Pi_L(\alpha), \Omega_L(\alpha), \Phi_L(\alpha)) : \alpha \in U\}$ and $M = \{(\alpha, \Delta_M(\alpha), \Gamma_M(\alpha), \Pi_M(\alpha), \Omega_M(\alpha), \Phi_M(\alpha)) : \alpha \in U\}$ be two SVPNSs over a fixed set *U*. Then, the SVPNCSM of similarities between *L* and *M* is defined as follows:

$$P_{SVPNCSM}(L, M) = 1-\frac{1}{n}\sum_{\alpha \in U} cos\left[\frac{\pi}{10}\left[|\Delta_L(\alpha)-\Delta_M(\alpha)|+|\Gamma_L(\alpha)-\Gamma_M(\alpha)|+|\Pi_L(\alpha)-\Pi_M(\alpha)|+|\Omega_L(\alpha)-\Omega_M(\alpha)| + |\Phi_L(\alpha)-\Phi_M(\alpha)|\right]\right]. \quad (1)$$

Theorem 3.1. If $P_{SVPNCSM}(L, M)$ be the SVPNCSM of similarities between the SVPNSs *L* and *M*, then the following holds:

(*i*) $0 \le P_{SVPNCSM}(L, M) \le 1$;

(*ii*) $P_{SVPNCSM}(L, M) = P_{SVPNCSM}(M, L)$;

(*iii*) $L = M \Leftrightarrow P_{SVPNCSM}(L, M) = 0$.

Proof. (*i*) We know that, the cosine function is a monotonic decreasing function in the interval [0, π/2]. It is also lies in the interval [0, 1]. Hence, $0 \le P_{SVPNCSM}(L, M) \le 1$.

(*ii*) We have, $P_{SVPNCSM}(L, M)$

$$= 1-\frac{1}{n}\sum_{d \in U} cos\left[\frac{\pi}{10}\left[|\Delta_L(d)-\Delta_M(d)| + |\Gamma_L(d)-\Gamma_M(d)| + |\Pi_L(d)-\Pi_M(d)| + |\Omega_L(d)-\Omega_M(d)| + |\Phi_L(d)- \Phi_M(d)|\right]\right]$$

$$= 1-\frac{1}{n}\sum_{d \in U} cos\left[\frac{\pi}{10}\left[|\Delta_M(d)-\Delta_L(d)| + |\Gamma_M(d)-\Gamma_L(d)| + |\Pi_M(d)-\Pi_L(d)| + |\Omega_M(d)-\Omega_L(d)| + |\Phi_M(d)-\Phi_L(d)|\right]\right]$$

$= P_{SVPNCSM}(M, L)$

Therefore, $P_{SVPNCSM}(L, M) = P_{SVPNCSM}(M, L)$.

(*iii*) Suppose that *L* and *M* be two SVPNSs over *U* such that $L = M$.

Now, $L = M$

$\Rightarrow \Delta_L(d)=\Delta_M(d)$, $\Gamma_L(d)= \Gamma_M(d)$, $\Pi_L(d)=\Pi_M(d)$, $\Omega_L(d)=\Omega_M(d)$, and $\Phi_L(d)=\Phi_M(d)$, for all $d \in U$

$\Rightarrow |\Delta_L(d)-\Delta_M(d)|=0$, $|\Gamma_L(d)-\Gamma_M(d)|=0$, $|\Pi_L(d)- \Pi_M(d)|=0$, $|\Omega_L(d)-\Omega_M(d)|=0$ and $|\Phi_L(d)-\Phi_M(d)|=0$, for all $d \in U$

Hence, $P_{SVPNCSM}(L, M) = 1- \frac{1}{n}\sum_{d \in U} cos(0) = 0$.

Conversely, suppose that $P_{SVPNSSM}(W, M) = 0$.

Now, $P_{SVPNSSM}(W, M) = 0$

$\Rightarrow |\Delta_L(d)-\Delta_M(d)|=0, |\Gamma_L(d)-\Gamma_M(d)|=0, |\Pi_L(d)-\Pi_M(d)|=0, |\Omega_L(d)- \Omega_M(d)|=0, |\Phi_L(d)-\Phi_M(d)|=0$, for all $d \in U$

$\Rightarrow \Delta_L(d)=\Delta_M(d)$, $\Gamma_L(d)=\Gamma_M(d)$, $\Pi_L(d)=\Pi_M(d)$, $\Omega_L(d)=\Omega_M(d)$, and $\Phi_L(d)=\Phi_M(d)$, for all $d\in U$

Hence, $L = M$.

Theorem 3.2. If L, M and C be three SVPNSs over a fixed set U such that $L\subseteq M\subseteq C$, then $P_{SVPNCSM}(L, M) \leq P_{SVPNCSM}(L, C)$ and $P_{SVPNCSM}(M, C) \leq P_{SVPNCSM}(L, C)$.

Proof. Suppose that L, M and C be three SVPNSs over a fixed set U such that $L\subseteq M\subseteq C$. So, $\Delta_L(d) \leq \Delta_M(d)$, $\Gamma_L(d) \leq \Gamma_M(d)$, $\Pi_L(d) \geq \Pi_M(d)$, $\Omega_L(d) \geq \Omega_M(d)$, $\Phi_L(d) \geq \Phi_M(d)$, $\Delta_M(d) \leq \Delta_C(d)$, $\Gamma_M(d) \leq \Gamma_C(d)$, $\Pi_M(d) \geq \Pi_C(d)$, $\Omega_M(d) \geq \Omega_C(d)$, $\Phi_M(d) \geq \Phi_C(d)$, $\Delta_L(d) \leq \Delta_C(d)$, $\Gamma_L(d) \leq \Gamma_C(d)$, $\Pi_L(d) \geq \Pi_C(d)$, $\Omega_L(d) \geq \Omega_C(d)$, $\Phi_L(d) \geq \Phi_C(d)$, for all $d\in U$.

Therefore, $|\Delta_L(d)-\Delta_M(d)|\leq|\Delta_L(d)-\Delta_C(d)|$, $|\Gamma_L(d)-\Gamma_M(d)|\leq|\Gamma_L(d)-\Gamma_C(d)|$, $|\Pi_L(d)-\Pi_M(d)|\leq|\Pi_L(d)-\Pi_C(d)|$, $|\Omega_L(d)-\Omega_M(d)|\leq|\Omega_L(d)-\Omega_C(d)|$, $|\Phi_L(d)-\Phi_M(d)|\leq|\Phi_L(d)-\Phi_C(d)|$, for all $d\in U$.

Therefore, $P_{SVPNCSM}(L, M)$

$$= 1-\frac{1}{n}\sum_{d\in U} cos[\frac{\pi}{10}[|\Delta_L(d)-\Delta_M(d)|+|\Gamma_L(d)-\Gamma_M(d)|+|\Pi_L(d)-\Pi_M(d)|+|\Omega_L(d)-\Omega_M(d)| +|\Phi_L(d)-\Phi_M(d)|]]$$

$$\leq 1-\frac{1}{n}\sum_{d\in U} cos[\frac{\pi}{10}[|\Delta_L(d)-\Delta_C(d)|+|\Gamma_L(d)-\Gamma_C(d)|+|\Pi_L(d)-\Pi_C(d)|+|\Omega_L(d)-\Omega_C(d)| +|\Phi_L(d)-\Phi_C(d)|]]$$

$= P_{SVPNCSM}(L, C)$

Hence, $P_{SVPNCSM}(L, M) \leq P_{SVPNCSM}(L, C)$.

Further, we have $|\Delta_M(d)-\Delta_C(d)|\leq|\Delta_L(d)-\Delta_C(d)|$, $|\Gamma_M(d)-\Gamma_C(d)|\leq|\Gamma_L(d)-\Gamma_C(d)|$, $|\Pi_M(d)-\Pi_C(d)|\leq|\Pi_L(d)-\Pi_C(d)|$, $|\Omega_M(d)-\Omega_C(d)|\leq|\Omega_L(d)-\Omega_C(d)|$, $|\Phi_M(d)-\Phi_C(d)|\leq|\Phi_L(d)-\Phi_C(d)|$, for all $d\in U$.

Therefore, $P_{SVPNCSM}(M, C)$

$$=1-\frac{1}{n}\sum_{d\in U} cos[\frac{\pi}{10}[|\Delta_M(d)-\Delta_C(d)|+|\Gamma_M(d)-\Gamma_C(d)|+|\Pi_M(d)-\Pi_C(d)|+|\Omega_M(d)-\Omega_C(d)|+|\Phi_M(d)-\Phi_C(d)|]]$$

$$\leq 1-\frac{1}{n}\sum_{d\in U} cos[\frac{\pi}{10}[|\Delta_L(d)-\Delta_C(d)|+|\Gamma_L(d)-\Gamma_C(d)|+|\Pi_L(d)-\Pi_C(d)|+|\Omega_L(d)-\Omega_C(d)| +|\Phi_L(d)-\Phi_C(d)|]]$$

$= P_{SVPNCSM}(L, C)$

Hence, $P_{SVPNCSM}(M, C) \leq P_{SVPNCSM}(L, C)$.

Definition 3.2. Assume that, $L = \{(d, \Delta_L(d), \Gamma_L(d), \Pi_L(d), \Omega_L(d), \Phi_L(d)) : d\in U\}$ and $W= \{(d, \Delta_W(d), \Gamma_W(d), \Pi_W(d), \Omega_W(d), \Phi_W(d)) : d\in U\}$ be two SVPNSs over a universe of discourse U. Then, the single valued pentapartitioned neutrosophic weighted cosine similarity measure (in short SVPNWCSM) between L and W is defined by:

$$P_{SVPNWCSM}(L, W) = 1-\frac{1}{n}\sum_{d\in U} w_d.cos[\frac{\pi}{10}[|\Delta_L(d)-\Delta_W(d)|+|\Gamma_L(d)-\Gamma_W(d)|+|\Pi_L(d)-\Pi_W(d)|+|\Omega_L(d)-\Omega_W(d)|+|\Phi_L(d)-\Phi_W(d)|]], \quad (2)$$

where, $\sum_{d\in U} w_d = 1$.

Now, we formulate the following results in view of the above theorems:

Proposition 3.1. Let $P_{SVPNWCSM}(L, W)$ be the SVPNWCSM of similarities between the SVPNSs L and W. Then,

(*i*) $0 \le P_{SVPNWCSM}(L, W) \le 1$;

(*ii*) $P_{SVPNWCSM}(L, W) = P_{SVPNWCSM}(W, L)$;

(*iii*) $P_{SVPNWCSM}(L, W) = 0$ iff $L = W$.

Proposition 3.2. If L, M and W be three SVPNSs over a universe of discourse U such that $L \subseteq M \subseteq W$, then $P_{SVPNWCSM}(L, M) \le P_{SVPNWCSM}(L, W)$ and $P_{SVPNWCSM}(M, W) \le P_{SVPNWCSM}(L, W)$.

4. SVPNCSM Based MADM Strategy under SVPNS Environment:

In this section, an attempt is made to propose a MADM strategy under the SVPNS environment using the SVPNCSM of similarities between two SVPNSs.

Let us consider a MADM problem, where $U = \{U_1, U_2, ..., U_p\}$ and $A = \{A_1, A_2, ..., A_q\}$ denotes the set of all possible alternatives and attributes respectively. Then, a decision maker can give their evaluation information for each alternative U_i (i = 1, 2, ..., p) with respect to the each attribute A_j (j = 1, 2,......, q) by using a SVPNS. By using the decision maker's whole evaluation information, we can form a decision matrix (DM).

The steps of our proposed MADM strategy are discussed below. Figure 1 represents the flow chart of the proposed MADM strategy.

Step-1: Formation of the DM by using SVPNS

In this step, we can build a DM by using the decision maker's evaluation information $P_{U_i} = \{(A_j, \Delta_{ij}(U_i, A_j), \Gamma_{ij}(U_i, A_j), \Pi_{ij}(U_i, A_j), \Omega_{ij}(U_i, A_j), \Phi_{ij}(U_i, A_j)) : A_j \in A\}$ for each alternative U_i (i = 1, 2, ..., p) with respect to the attributes A_j(j = 1, 2, ..., q), where $(\Delta_{ij}(U_i, A_j), \Gamma_{ij}(U_i, A_j), \Pi_{ij}(U_i, A_j), \Omega_{ij}(U_i, A_j), \Phi_{ij}(U_i, A_j)) = (U_i, A_j)$ (say) (i = 1, 2, ..., p, and j = 1, 2, ..., q) indicates the evaluation information of alternatives U_i (i = 1, 2, ..., p) with respect to the attribute A_j (j = 1, 2, ..., q).

The DM can be expressed as follows:

DM	A_1	A_2	....	....	A_q
U_1	(U_1, A_1)	(U_1, A_2)	....	....	(U_1, A_q)
U_2	(U_2, A_1)	(U_2, A_2)	......	....	(U_2, A_q)
....	....	....	....	....	....
U_p	(U_p, A_1)	(U_p, A_2)	....	....	(U_p, A_q)

Step-2: Determination of the Weights for Each Attribute

In every MADM strategy, the determination of weights for every attributes is an important task. If the information of attributes' weight is completely unknown, then the decision maker can use the compromise function to calculate the weights for each attribute.

The compromise function of β_j for each U_j is defined as follows:

$\beta_j = \sum_{i=1}^{p} (3 + \Delta_{ij}(U_i, A_j) + \Gamma_{ij}(U_i, A_j) - \Pi_{ij}(U_i, A_j) - \Omega_{ij}(U_i, A_j) - \Phi_{ij}(U_i, A_j))/5.$ (3)

Then, the weight of the j-th attribute is defined by $w_j = \frac{\beta_j}{\sum_{j=1}^{q} \beta_j}$ (4)

Here, $\sum_{j=1}^{q} w_j = 1$.

Step-3: Selection of the Positive Ideal Alternative (PIA)

In this step, the decision maker can form the PIA by using the maximum operator for all the attributes.

The positive ideal alternative I is defined as follows:

$$I = (C_1, C_2, \ldots\ldots\ldots, C_q), \quad (5)$$

where C_j = (max $\{\Delta_{ij}(U_i, A_j): i=1, 2, 3, \ldots.., p\}$, max $\{\Gamma_{ij}(U_i, A_j): i=1, 2, 3, \ldots.., p\}$, min $\{\Pi_{ij}(U_i, A_j): i=1, 2, 3, \ldots.., p\}$, min $\{\Omega_{ij}(U_i, A_j): i=1, 2, 3, \ldots.., p\}$, min $\{\Phi_{ij}(U_i, A_j): i=1, 2, 3, \ldots.., p\}$), $j = 1, 2, \ldots., q$. (6)

Step-4: Determination of the Accumulated Measure Values

In this step, we use an accumulated measure function (AMF) to aggregate the SVPNCSM corresponding to each alternative.

The AMF is defined by $P_{AMF}(U_i) = \sum_{j=1}^{q} w_j \,.\, P_{SVPNCSM}((U_i, A_j), C_j)$, (7)

where, $(U_i, A_j) = (\Delta_{ij}(U_i, A_j), \Gamma_{ij}(U_i, A_j), \Pi_{ij}(U_i, A_j), \Omega_{ij}(U_i, A_j), \Phi_{ij}(U_i, A_j))$.

Step-5: Ranking of the Alternatives

Finally, we prepare the ranking order of alternatives based on the descending order of accumulated measure values. The alternative associated with the lowest accumulated measure value is the best suitable alternatives.

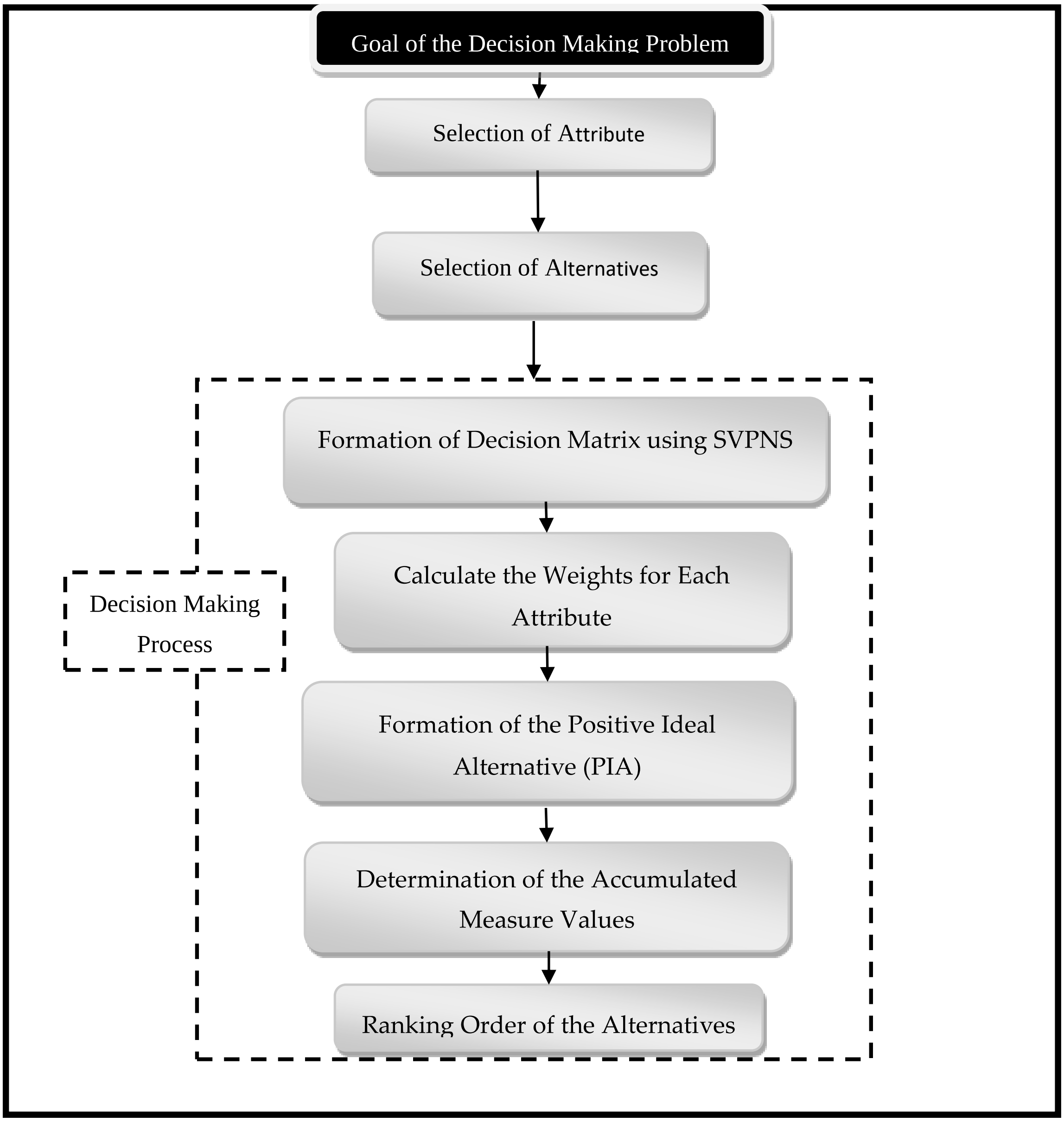


Figure-1: Proposed MADM-Strategy

5. Application of the Proposed MADM Strategy:

In this section, we present a numerical example to show the applicability of our proposed MADM strategy.

Example 5.1. "**Identification of the Most Significant Risk Factor of COVID-19 in the Economy**".

Presently, the world is under the threat of a deadly virus named Corona Virus. The COVID-19 makes an unbelievable provocation with very serious social and economic outcomes [17]. As per the World Health Organization (WHO), Corona Virus disease is a pandemic [44]. On 31st December 2019, it was reported to the World Health Organization that a number of pneumonia cases due to an unknown cause had happened in Wuhan city of China [45]. In January 2020 a new virus was spotted which was later named as the 2019 Novel Corona Virus responsible for the outbreak. In February 2020, WHO named the Virus as Corona Virus Disease 2019 as it had appeared in 2019 [26]. This virus influences immediately to a person's lungs and has symptoms similar to influenza and pneumonia [37]. Although it is not known correctly the process to transmit of COVID-19 from man to man, the method of transmission is same as other respiratory diseases [1, 36]. Environmental factors have a vital role in the movement of the virus [7]. Many researchers observed and established many techniques to cope up with medical and decision-making obstacles. Till 11th July 2021, the total number of confirmed case are 187,280,697 which have been reported worldwide, with 4,043,032 died and 171,255,731 recovered [8, 44]. The COVID-19 pandemic poses an immense threat to people's health and livelihood more specifically the employment [18]. A large number of countries have imposed lockdown and as a result, companies cannot afford to run smoothly. According to UNESCO, more than 188 countries have halted schools, colleges, and universities, responsible to affect the educations of nearly 90% of the world's students. The lockdown has caused the renewal of the environment, with the factories being closed and reduction in transportation vehicles use. COVID-19 improved the air quality in various parts of the world due to the imposing of lockdown [17].

Coronavirus has quickly influenced our everyday life, organizations, upset the world exchange and developments. Recognizable proof of the sickness at a beginning phase is essential to control the spread of the infection since it quickly spreads from one individual to another. The greater part of the nations has hindered their assembling of the items [17]. The different businesses and areas are influenced by the reason for this sickness; these incorporate the drugs business, sun based force area, the travel industry, Information and gadgets industry. This infection makes critical thump on impacts on the day by day life of residents, just as about the worldwide economy. So in this paper, we have focused on the bad impacts of COVID-19 on economy to survive in this COVID-19 phase. Some factors of economy are effected by COVID-19 [24], these are (i) Slowing of the manufacturing of essential goods (U_1), (ii) Disrupt the supply chain of products (U_2), (iii) Losses in national and international business (U_3) (iv) Poor cash flow in the market (U_4) and (v) Significant slowing down in the revenue growth (U_5). All these factors are selected from Literature review (A_1), Expert survey (A_2) and Media survey (A_3). Although, all these factors are effected by COVID-19 pandemic phase but all factors are not equally effected. So the main objective of the present investigation is to identify the most significant effected economical factor affected by COVID-19 pandemic phase. For identify the most significant indicators we use a novel similarity measure namely SVPNCSM under SVPNS environment. All the factors U_1, U_2, U_3, U_4, and U_5 are consider as alternatives and A_1, A_2 and A_3 are consider as attribute. Figure-2 depicts the decision hierarchy of the current problem. Recently Majumder et al. [24] used decision making techniques for select the most significant for speeding the COVID-19. Majumder [23] also used PNN (Polynomial Neural Network) model for predicting confirmed and death cases daily. Assessing the unemployment problem was studied by Nguyen [31] using decision making technique. Aydin and Seker [4] proposed a MADM model to choose the suitable location for isolation in a hospital. Alkan and Kahraman [3] developed a model to evaluate the significant strategies of government against in COVID-19 period with the help of TOPSIS method under the q-rung orthopair fuzzy environment. Ahmad et al. [2] proposed a technique to identifying affected cases globally using the fuzzy cloud based COVID-19.

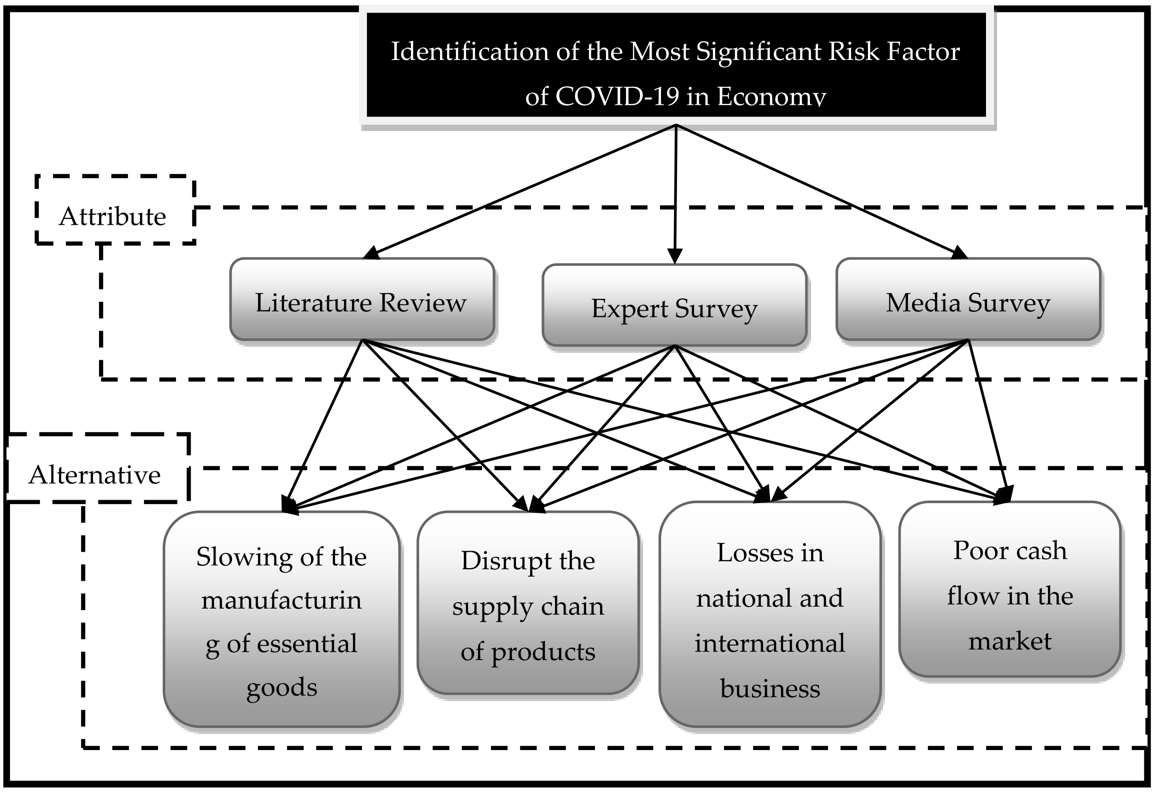


Figure- 2: Decision Hierarchy of the Current Problem.

The proposed MADM strategy is presented as follows:

By using the evaluation information for all alternatives given by the decision makers, we prepare the decision matrix as follows:

Table-1:

	A_1	A_2	A_3
U_1	(0.9,0.1,0.1,0.2,0.1)	(1.0,0.2,0.0,0.0,0.1)	(1.0,0.0,0.1,0.2,0.2)
U_2	(0.8,0.2,0.1,0.0,0.0)	(0.9,0.0,0.1,0.0,0.1)	(0.8,0.1,0.0,0.1,0.1)
U_3	(1.0,0.2,0.2,0.1,0.1)	(0.8,0.2,0.2,0.1,0.1)	(0.9,0.3,0.2,0.1,0.0)
U_4	(0.9,0.1,0.1,0.0,0.2)	(0.9,0.1,0.1,0.1,0.1)	(1.0,0.0,0.1,0.2,0.1)
U_5	(1.0,0.2,0.2,0.1,0.0)	(0.9,0.1,0.2,0.0,0.1)	(1.0,0.2,0.2,0.2,0.0)

Now, by using eq. (3) & eq. (4), we get the weights w_1 = 0.3362989, w_2 = 0.3345196, and w_3 = 0.3291815.

The positive ideal alternative I have been formed using eq. (5) & eq. (6), which was shown in Table-2.

Table-2:

	A_1	A_2	A_3
I	(1.0,0.2,0.1,0.0,0.0)	(1.0,0.2,0.0,0.0,0.1)	(1.0,0.3,0.0,0.1,0.0)

Now, by using eq. (7), we calculate the accumulated measure values of each alternative U_1, U_2, U_3, U_4, and U_5 as follows:

$P_{AMF}(U_1)$ = 0.0496014196;

$P_{AMF}(U_2)$ = 0.0040071176;

$P_{AMF}(U_3)$ = 0.0070106764;

$P_{AMF}(U_4)$ = 0.011291815;

$P_{AMF}(U_5)$ = 0.0059822066.

Here, the order of the accumulated measure values is $P_{AMF}(U_1) > P_{AMF}(U_4) > P_{AMF}(U_3) > P_{AMF}(U_5) > P_{AMF}(U_2)$. Therefore, the alternative U_2 i.e., "**disrupt the supply chain of products**" is the most significant risk factor of COVID-19 in the economy.

6. Conclusions:

In the article, we have established a MADM strategy based on SVPNCSM of similarities between two SVPNSs under the SVPNS environment. Further, we have validated our proposed MADM strategy by solving an illustrative real world numerical example to demonstrate the effectiveness and usefulness of the proposed MADM strategy.

Further, it is hoped that, the proposed MADM strategy can also be used to deal with other decision-making problems such as tender selection [11], weaver selection [15], logistic center location selection [32, 33], medical diagnosis [34, 35], fault diagnosis [46, 47], etc.

Conflict of Interest: The authors declare that they have no conflict of interest.

References:

1. Acter, T., Uddin, N., Das, J., Akhter, A., Choudhury, T.R., & Kim, S. (2020). Evolution of severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) as coronavirus disease 2019 (COVID-19) pandemic: A global health emergency. *Science of the Total Environment*, 138996.
2. Ahmad, S., Mehfuz, S., Beg, J., Khan, N.A., & Khan, A.H. (2021). Fuzzy Cloud Based COVID-19 Diagnosis Assistant for identifying affected cases globally using MCDM. *Materials Today: Proceedings*.
3. Alkan, N., & Kahraman, C. (2021). Evaluation of government strategies against COVID-19 pandemic using q-rung orthopair fuzzy TOPSIS method. *Applied Soft Computing*, 107653.
4. Aydin, N., & Seker, S. (2021). Determining the location of isolation hospitals for COVID-19 via Delphi-based MCDM method. *International Journal of Intelligent Systems*, 36(6), 3011-3034.
5. Biswas, P., Pramanik, S., & Giri, B.C. (2014). Entropy based grey relational analysis method for multi-attribute decision making under single valued neutrosophic assessments. *Neutrosophic Sets and Systems*, 2, 102-110.
6. Campbell D, Bannock. 'Unlike Anything Seen in Peacetime': NHS Prepares for Surge in Covid-19 Cases;2020.https://www.theguardian.com/world/2020/mar/13/-unlike-anything-seen-in-peacetime-nhs-prepares-for-surge-in-covid-19-coronavirus-cases.
7. Cássaro, F.A., & Pires, L.F. (2020). Can we predict the occurrence of COVID-19 cases? Considerations using a simple model of growth. *Science of the Total Environment*, 138834.
8. Chakraborty, I., & Maity, P. (2020). COVID-19 outbreak: Migration, effects on society, global environment and prevention. *Science of the Total Environment*, 138882.
9. Das, S. (2021). Neutrosophic Supra Simply Open Set and Neutrosophic Supra Simply Compact Space. *Neutrosophic Sets and Systems*, 43, 105-113.
10. Das, S., Das, R., Granados, C., & Mukherjee, A. (2021). Pentapartitioned Neutrosophic *Q*-Ideals of *Q*-Algebra. *Neutrosophic Sets and Systems*, 41, 52-63.

11. Das, S., Das, R., & Tripathy, B.C. (2020). Multi-criteria group decision making model using single-valued neutrosophic set. *LogForum*, 16(3), 421-429.
12. Das, S., Shil, B., & Pramanik, S. SVPNS-MADM strategy based on GRA in SVPNS Environment. *Neutrosophic Sets and Systems*, In Press.
13. Das, S., Shil, B., & Tripathy, B.C. (2021). Tangent Similarity Measure Based MADM-Strategy under SVPNS-Environment. *Neutrosophic Sets and Systems*, 43, 93-104.
14. Das, R., Smarandache, F., & Tripathy, B.C. (2020). Neutrosophic Fuzzy Matrices and Some Algebraic Operations. *Neutrosophic Sets and Systems*, 32, 401-409.
15. Dey, P.P., Pramanik, S., & Giri, B.C. (2015). An extended grey relational analysis based interval neutrosophic multi attribute decision making for weaver selection. *Journal of New Theory*, 9, 82-93.
16. European Council, Council of the European Union. Report on the comprehensive economic policy response to the COVID-19 pandemic, https://www.consilium.europa.eu/en/press/press-releases/2020/04/09/report-on-the-comprehensive-economic-policy-response-to-the-covid-19-pandemic/(2020, accessed 25 May 2020).
17. Ferguson, N., Laydon, D., Nedjati-Gilani, G., Imai, N., Ainslie, K., Baguelin, M., & Dighe, A. (2020). Report 9: Impact of non-pharmaceutical interventions (NPIs) to reduce COVID19 mortality and healthcare demand.
18. Haleem, A., Javaid, M., & Vaishya, R. (2020). Effects of COVID-19 pandemic in daily life. *Current medicine research and practice*, 10(2), 78.
19. Jin, Y.H., Cai, L., Cheng, Z.S., Cheng, H., Deng, T., Fan, Y.P., & Wang, X.H. (2020). A rapid advice guideline for the diagnosis and treatment of 2019 novel coronavirus (2019-nCoV) infected pneumonia (standard version). *Military Medical Research*, *7*(1), 1-23.
20. Mahapatra, R., Samanta, S., & Pal, M. (2021). Generalized neutrosophic planar graphs and its application. *Journal of Applied Mathematics and Computing*, 65, 693-712.
21. Mahapatra, R., Samanta, S., Pal, M., Lee, J.G., Khalid, S.K., Naseem, U., & Bhadoria, R.S. (2021). Colouring of COVID-19 Affected Region Based on Fuzzy Directed Graphs. *Computers, Materials & Continua*, 68(1), 1219-1233.
22. Mahapatra, R., Samanta, S., Pal, M., & Xin, Q. (2020). Link Prediction in Social Networks by Neutrosophic Graph. *International Journal of Computational Intelligence Systems*, 13(1), 1699-1713.
23. Majumder, P. (2021). Daily confirmed cases and deaths prediction of novel coronavirus in Asian continent Polynomial Neural Network. In *Biomedical Engineering Tools for Management for Patients with COVID-19* (pp. 163-172). Academic Press.
24. Majumder, P., Biswas, P., & Majumder, S. (2020). Application of New TOPSIS Approach to Identify the Most Significant Risk Factor and Continuous Monitoring of Death of COVID-19. *Electronic Journal of General Medicine*, 17(6).

25. Mallick, R., & Pramanik, S. (2020). Pentapartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 36, 184-192.
26. Ministry of Health, Covid 19 (2020), https://covid19.saglik.gov.tr/.
27. Mondal, K., & Pramanik, S. (2015). Neutrosophic tangent similarity measure and its application to multiple attribute decision making. *Neutrosophic sets and systems*, 9, 80-87.
28. Mondal, K., Pramanik, S., & Giri, B.C. (2018). Interval neutrosophic tangent similarity measure based MAMD strategy and its application to MAMD problems.*Neutrosophic sets and systems*, 19, 47-53.
29. Mondal, K., Pramanik, S., & Giri, B.C. (2018). Single valued neutrosophic hyperbolic sine similarity measure based MADM strategy. *Neutrosophic Sets and Systems*, 20, 3-11.
30. Mondal, K., Pramanik, S., & Smarandache, F. (2016). Role of neutrosophic logic in data mining. *New Trends in Neutrosophic Theory and Application*. Pons Editions, Brussels, 15-23.
31. Nguyen, P.H., Tsai, J.F., Nguyen, H.P., Nguyen, V.T., & Dao, T.K. (2020). Assessing the Unemployment Problem Using A Grey MCDM Model under COVID-19 Impacts: A Case Analysis from Vietnam. *The Journal of Asian Finance, Economics, and Business*, *7*(12), 53-62.
32. Pramanik, S., Dalapati, S., & Roy, T.K. (2016). Logistics center location selection approach based on neutrosophic multicriteria decision making. *New Trends in Neutrosophic Theory and Application*. Pons Editions, Brussels, 161-174.
33. Pramanik, S., Dalapati, S., & Roy, T.K. (2018). Neutrosophic multi-attribute group decision making strategy for logistic center location selection. *Neutrosophic Operational Research*, Vol. III. Pons Asbl, Brussels, 13-32.
34. Pramanik, S., & Mondal, K. (2015). Cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis. *Global Journal of Advanced Research*, 2(1), 212-220.
35. Pramanik, S., & Mondal, K. (2015). Cotangent similarity measure of rough neutrosophic sets and its application to medical diagnosis. *Journal of New Theory*, 4, 464-471.
36. Raza Hashmi, M., Riaz, M., & Smarandache, F. m-polar Neutrosophic Generalized Weighted and m-polar Neutrosophic Generalized Einstein Weighted Aggregation Operators to Diagnose Coronavirus (COVID-19). *Journal of Intelligent & Fuzzy Systems*, (Preprint), 1-21.
37. Read, J.M., Bridgen, J.R., Cummings, D.A., Ho, A., & Jewell, C.P. (2020). Novel coronavirus 2019-nCoV: early estimation of epidemiological parameters and epidemic predictions. *MedRxiv*.
38. Saadat, S., Rawtani, D., & Hussain, C.M. (2020). Environmental perspective of COVID-19. *Science of The Total Environment*, 138870.
39. Saglietto, A., D'Ascenzo, F., Zoccai, G.B., & De Ferrari, G.M. (2020). COVID-19 in Europe: the Italian lesson. *Lancet*, 395(10230), 1110-1.
40. Smarandache, F. (1998). A unifying field in logics, neutrosophy: neutrosophic probability, set and logic. Rehoboth, *American Research Press*.

41. Tobías, A., Carnerero, C., Reche, C., Massagué, J., Via, M., Minguillón, M.C., & Querol, X. (2020). Changes in air quality during the lockdown in Barcelona (Spain) one month into the SARS-CoV-2 epidemic. *Science of the Total Environment*, 138540.
42. Tripathy, B.C., & Das, S. (2021). Pairwise Neutrosophic *b*-Continuous Function in Neutrosophic Bitopological Spaces. *Neutrosophic Sets and Systems*, 43, 82-92.
43. Wang, H., Smarandache, F., Zhang, Y.Q., & Sunderraman, R. (2010). Single valued neutrosophic sets. *Multispace and Multistructure*, 4, 410-413.
44. World Health Organization. WHO Director-General's opening remarks at the media briefing on COVID-19- 11TH March 2020.
45. Worldometer, Coronavirus, https://www.worldometers.info/coronavirus/country/
46. Ye, J. (2016). Fault diagnoses of steam turbine using the exponential similarity measure of neutrosophic numbers. *Journal of Intelligent & Fuzzy Systems*, 30(4), 1927-1934.
47. Ye, J. (2017). Single-valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine. *Soft Computing*, 21(3), 817-825.
48. Ye, J., & Zhang, Q.S. (2014). Single valued neutrosophic similarity measures for multiple attribute decision making. *Neutrosophic Sets and Systems*, 2, 48-54.

Received: May 10, 2021. Accepted: October 4, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

An Algorithm Based on Correlation Coefficient Under Neutrosophic hypersoft set environment with its Application for Decision-Making

Rana Muhammad Zulqarnain[1*], Imran Siddique[2], Shahzad Ahmad[1], Sehrish Ayaz[2], Said Broumi[3]

[1] Department of Mathematics, University of Management and Technology, Lahore, Sialkot Campus, Pakistan. E-mail: shahzad.ahmed@skt.umt.edu.pk
[2] Department of Mathematics, University of Management and Technology, Lahore, Sialkot Campus, Pakistan. E-mail: imransiddique@umt.edu.pk
[3] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, Casablanca. E-mail: broumisaid78@gmail.com.

[1*] Correspondence: E-mail: ranazulqarnain7777@gmail.com

Abstract:

The correlation coefficient among the two parameters plays a significant part in statistics. Further, the exactness in the assessment of correlation depends upon information from the set of discourse. The data collected for various statistical studies is full of ambiguities. In this paper, we discuss some basic concepts which are helpful to build the structure of present research such as soft set, hypersoft set, and neutrosophic hypersoft set (NHSS). The neutrosophic hypersoft set is an extension of the neutrosophic soft set. In it, we establish the idea of correlation and weighted correlation coefficients with some desirable properties under NHSS. We also, propose a new decision-making technique and construct an algorithm based on developed correlation measures. Furthermore, To ensure the applicability of the proposed methods an illustrative example is given.

Keywords: Hypersoft set, NHSS, correlation coefficient, weighted correlation coefficient

1. Introduction

Ambiguity plays a dynamic role in many areas of life (such as modeling, medicine, engineering, etc.). However, people have raised a common question, that is, how do we express and use the concept of uncertainty in mathematical modeling. Many researchers in the world have proposed and recommended different methods of using uncertainty theory. First of all, Zadeh developed the concept of a fuzzy set (FS) [1] to solve problems that contain uncertainty and ambiguity. In some cases, we must carefully consider membership as a non-membership value to correctly represent objects that FS cannot handle. To overcome these difficulties, Atanasov proposed the idea of intuitionistic fuzzy sets (IFS) [2]. Atanassov's intuitionistic fuzzy sets only deal with insufficient data due to membership and non-membership values, but IFS cannot deal with incompatible and imprecise information. Molodtsov [3] proposed a general mathematical tool to deal with uncertain, ambiguous, and uncertain matters, called soft set (SS). Maji et al. [4] extended the concept of SS and developed some operations with properties and used the established concepts for decision-making [5]. By combining the FS and SS Maji et al. [6] established the fuzzy soft set (FSS) and intuitionistic fuzzy soft set (IFSS) and studied their operations and properties [7]. Zulqarnain et al. [8] established the correlation coefficient for interval-valued intuitionistic fuzzy soft set and developed the TOPSIS approach based on their presented correlation measures. Zulqarnain et al. [9, 10] discussed the Pythagorean fuzzy soft sets (PFSS) and established the aggregation operator and TOPSIS technique to solve the MCDM problem.

Maji [11] offered the idea of a neutrosophic soft set (NSS) with necessary operations and properties. The idea of the possibility NSS was developed by Karaaslan [12] and introduced a possibility of neutrosophic soft decision-making method to solve those problems which contain uncertainty based on And-product. Broumi [13] developed the generalized NSS with some operations and properties and used the proposed concept for decision making. To solve MCDM problems with PFSS, Zulqarnain et al. [14] presented the interaction aggregation operators for PFSS. Based on the correlation of IFS, the term CC of SVNSs [15] was introduced. In [16] the idea of simplified NSs introduced with some operational laws and aggregation operators such as weighted arithmetic and weighted geometric average operators. They constructed an MCDM method on the base of proposed aggregation operators. Masooma et al. [17] progressed a new concept through combining the multipolar fuzzy set and neutrosophic set which is known as the multipolar neutrosophic set, they also established various characterization and operations with examples. Zulqarnain et al. [18, 19] utilized the neutrosophic TOPSIS model to solve the MCDM problem and for the selection of suppliers in the production industry.

Correlation performs a significant part in statistics as well as engineering. By way of correlation analysis, the mixture of two variables can be utilized to compute the mutuality of the two variables. Although probabilistic methods have been applied to various practical engineering problems, there are still some obstacles to probabilistic strategies. For example, the probability of this process depends on the large amount of data collected, which is random. However, large complex systems have many fuzzy uncertainties, so it is difficult to obtain accurate probability events. Therefore, due to limited quantitative information, results based on probability theory do not always provide useful information for experts. In addition, in actual applications, sometimes there is not enough data to correctly process standard statistical data. Due to the aforementioned obstacles, results based on probability theory are not always available to experts. Therefore, probabilistic methods are usually insufficient to resolve such inherent uncertainties in the data. Many researchers in the world have proposed and proposed different methods to solve problems that contain uncertainty. To measure the relationship between fuzzy numbers, Yu [20] established the CC of fuzzy numbers.

Recently, Smarandache [21] extended the concept of the SS to hypersoft set (HSS) by replacing the single-parameter function F with a multi-parameter (sub-attribute) function defined on Cartesian products of n different attributes. The established HSS is more flexible than SS and is more suitable for the decision-making environment. He also introduced the further extension of HSS, such as crisp HSS, fuzzy HSS, intuitionistic fuzzy HSS, neutrosophic HSS, and plithogenic HSS. Nowadays, HSS theory and its extensions are developing rapidly. Many researchers have developed different operators and properties based on HSS and its extensions [22-36]. Abdel-Basset [37] uses a plithogenic set theory to resolve uncertain information and evaluate the financial performance of manufacturing. Then, they use VIKOR and TOPSIS methods to find the weight vector of financial ratios using the AHP method to achieve this goal. Abdel-basset et al. [38] recommended an efficient combination of plithogenic aggregation operations as well as quality feature deployment strategies. The advantage of this combination is that it can improve accuracy as well as assess the decision-makers.

The following research is organized as follows: In Section 2, we review some basic definitions used in the following sequels, such as SS, NSS, and NHSS, etc. In Section 3, the idea of CC and WCC is developed with some necessary properties. An algorithm and decision-making method will be developed in section 4. We also used the developed approach to solve decision making problems in an uncertain environment. Finally, the conclusion is made in section 5.

2. Preliminaries

In this section, we recollect some basic definitions which are helpful to build the structure of the following manuscript such as soft set, hypersoft set, and neutrosophic hypersoft set.

Definition 2.1 [3]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathsf{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathsf{A})$ is called a soft set over $\mathcal{U}$ and its mapping is given as

$$\mathcal{F}: \mathsf{A} \to \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, \mathsf{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \ if \ e \notin \mathsf{A}\}$$

Definition 2.2 [21]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of multi-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}})$ is said to be HSS over $\mathcal{U}$ and its mapping is defined as

$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \to \mathcal{P}(\mathcal{U})$.

It is also defined as

$(\mathcal{F}, \ddot{\mathsf{A}}) = \{\breve{a}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}): \breve{a} \in \ddot{\mathsf{A}}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) \in \mathcal{P}(\mathcal{U})\}$

Definition 2.3 [21]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $NS^{\mathcal{U}}$ be a collection of all neutrosophic subsets over $\mathcal{U}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}})$ is said to be NHSS over $\mathcal{U}$ and its mapping is defined as

$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \to NS^{\mathcal{U}}$.

It is also defined as

$(\mathcal{F}, \ddot{\mathsf{A}}) = \{(\breve{a}, \mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a})): \breve{a} \in \ddot{\mathsf{A}}, \mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a}) \in NS^{\mathcal{U}}\}$, where $\mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a}) = \{\langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta), \gamma_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}\}$, where $\sigma_{\mathcal{F}(\breve{a})}(\delta)$, $\tau_{\mathcal{F}(\breve{a})}(\delta)$, and $\gamma_{\mathcal{F}(\breve{a})}(\delta)$ represent the truth, indeterminacy, and falsity grades of the attributes such as $\sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta), \gamma_{\mathcal{F}(\breve{a})}(\delta) \in [0, 1]$, and $0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta) + \tau_{\mathcal{F}(\breve{a})}(\delta) + \gamma_{\mathcal{F}(\breve{a})}(\delta) \leq 3$. Simply a neutrosophic hypersoft number (NHSN) can be expressed as $\mathcal{F} = \{(\sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta), \gamma_{\mathcal{F}(\breve{a})}(\delta))\}$, where $0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta) + \tau_{\mathcal{F}(\breve{a})}(\delta) + \gamma_{\mathcal{F}(\breve{a})}(\delta) \leq 3$.

Example 2.4

Consider the universe of discourse $\mathcal{U} = \{\delta_1, \delta_2\}$ and $\mathfrak{L} = \{\ell_1 = Teaching\ methdology, \ell_2 = Subjects, \ell_3 = Classes\}$ be a collection of attributes with following their corresponding attribute values are given as teaching methodology = $L_1 = \{a_{11} = project\ base, a_{12} = class\ discussion\}$, Subjects = $L_2 = \{a_{21} = Mathematics, a_{22} = Computer\ Science, a_{23} = Statistics\}$, and Classes = $L_3 = \{a_{31} = Masters, a_{32} = Doctorol\}$. Let $\ddot{\mathsf{A}} = L_1 \times L_2 \times L_3$ be a set of attributes

$\ddot{\mathsf{A}} = L_1 \times L_2 \times L_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}, a_{23}\} \times \{a_{31}, a_{32}\}$

$= \begin{Bmatrix} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}), (a_{11}, a_{23}, a_{31}), (a_{11}, a_{23}, a_{32}), \\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{21}, a_{32}), (a_{12}, a_{22}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{12}, a_{23}, a_{31}), (a_{12}, a_{23}, a_{32}), \end{Bmatrix}$

$\ddot{\mathsf{A}} = \{\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5, \breve{a}_6, \breve{a}_7, \breve{a}_8, \breve{a}_9, \breve{a}_{10}, \breve{a}_{11}, \breve{a}_{12}\}$

Then the NHSS over $\mathcal{U}$ is given as follows

$$(\boldsymbol{\mathcal{F}},\ddot{A}) = \left\{\begin{array}{c} \left(\breve{\boldsymbol{a}}_1,(\boldsymbol{\delta}_1,(.6,.3,.8)),(\boldsymbol{\delta}_2,(.9,.3,.5))\right),\left(\breve{\boldsymbol{a}}_2,(\boldsymbol{\delta}_1,(.5,.2,.7)),(\boldsymbol{\delta}_2,(.7,.1,.5))\right),\left(\breve{\boldsymbol{a}}_3,(\boldsymbol{\delta}_1,(.5,.2,.8)),(\boldsymbol{\delta}_2,(.4,.3,.4))\right), \\ \left(\breve{\boldsymbol{a}}_4,(\boldsymbol{\delta}_1,(.2,.5,.6)),(\boldsymbol{\delta}_2,(.5,.1,.6))\right),\left(\breve{\boldsymbol{a}}_5,(\boldsymbol{\delta}_1,(.8,.4,.3)),(\boldsymbol{\delta}_2,(.2,.3,.5))\right),\left(\breve{\boldsymbol{a}}_6,(\boldsymbol{\delta}_1,(.9,.6,.4)),(\boldsymbol{\delta}_2,(.7,.6,.8))\right), \\ \left(\breve{\boldsymbol{a}}_7,(\boldsymbol{\delta}_1,(.6..5,.3)),(\boldsymbol{\delta}_2,(.4,.2,.8))\right),\left(\breve{\boldsymbol{a}}_8,(\boldsymbol{\delta}_1,(.8,.2,.5)),(\boldsymbol{\delta}_2,(.6,.8,.4))\right),\left(\breve{\boldsymbol{a}}_9,(\boldsymbol{\delta}_1,(.7,.4,.9)),(\boldsymbol{\delta}_2,(.7..3,.5))\right), \\ \left(\breve{\boldsymbol{a}}_{10},(\boldsymbol{\delta}_1,(.8,.4,.6)),(\boldsymbol{\delta}_2,(.7,.2,.9))\right),\left(\breve{\boldsymbol{a}}_{11},(\boldsymbol{\delta}_1,(.8,.4,.5)),(\boldsymbol{\delta}_2,(.4,.2,.5))\right),\left(\breve{\boldsymbol{a}}_5,(\boldsymbol{\delta}_1,(.7,.5,.8)),(\boldsymbol{\delta}_2,(.7,.5,.9))\right) \end{array}\right\}$$

3. Correlation Coefficient for Neutrosophic Hypersoft Set

In this section, the concept of correlation coefficient and weighted correlation coefficient on NHSS has been proposed with some basic properties.

Definition 3.1

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\dddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs defined over a universe of discourse $\mathcal{U}$. Then, the informational neutrosophic energies of $(\mathcal{F},\ddot{A})$ and $(\mathcal{G},\dddot{A})$ can be described as follows:

$$\varsigma_{NHSS}(\mathcal{F},\ddot{A}) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right) \quad (1)$$

$$\boldsymbol{\varsigma_{NHSS}(\mathcal{G},\dddot{A}) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)}. \quad (2)$$

Definition 3.2

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\dddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs defined over a universe of discourse $\mathcal{U}$. Then, the correlation measure between $(\mathcal{F},\ddot{A})$ and $(\mathcal{G},\dddot{A})$ can be described as follows:

$$\boldsymbol{\mathcal{C}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\dddot{A})) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)}. \quad (3)$$

Proposition 3.3

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\dddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs and $\mathcal{C}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\dddot{A}))$ be a correlation between them, then the following properties hold.

1. $\mathcal{C}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\dddot{A})) = \varsigma_{NHSS}(\mathcal{F},\ddot{A})$
2. $\mathcal{C}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\dddot{A})) = \varsigma_{NHSS}(\mathcal{G},\dddot{A})$

Proof: The proof is trivial.

Definition 3.4

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\dddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs, then correlation coefficient between them given as $\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\dddot{A}))$ and expressed as follows:

$$\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\dddot{A})) = \frac{\mathcal{C}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\dddot{A}))}{\sqrt{\varsigma_{NHSS}(\mathcal{F},\ddot{A})} * \sqrt{\varsigma_{NHSS}(\mathcal{G},\dddot{A})}} \quad (4)$$

$$\boldsymbol{\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\dddot{A}))} = \frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)}{\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2}} \quad (5)$$

Proposition 3.5

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\dddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs, then CC between them satisfies the following properties

1. $0 \leq \delta_{NHSS}((\mathcal{F},Ä),(\mathcal{G},Ä)) \leq 1$
2. $\delta_{NHSS}((\mathcal{F},Ä),(\mathcal{G},Ä)) = \delta_{NHSS}((\mathcal{G},Ä),(\mathcal{F},Ä))$
3. If $(\mathcal{F},Ä) = (\mathcal{G},Ä)$, that is $\forall\ i,\ k,\ \sigma_{\mathcal{F}(ă_k)}(\delta_i) = \sigma_{\mathcal{G}(ă_k)}(\delta_i),\ \tau_{\mathcal{F}(ă_k)}(\delta_i) = \tau_{\mathcal{G}(ă_k)}(\delta_i)$, and $\gamma_{\mathcal{F}(ă_k)}(\delta_i) = \gamma_{\mathcal{G}(ă_k)}(\delta_i)$ then $\delta_{NHSS}((\mathcal{F},Ä),(\mathcal{G},Ä)) = 1$.

Proof 1. $\delta_{NHSS}((\mathcal{F},Ä),(\mathcal{G},Ä)) \geq 0$ is trivial, here we only need to prove that $\delta_{NHSS}((\mathcal{F},Ä),(\mathcal{G},Ä)) \leq 1$. From equation 3, we have

$$\delta_{NHSS}((\mathcal{F},Ä),(\mathcal{G},Ä)) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sigma_{\mathcal{F}(ă_k)}(\delta_i) * \sigma_{\mathcal{G}(ă_k)}(\delta_i) + \tau_{\mathcal{F}(ă_k)}(\delta_i) * \tau_{\mathcal{G}(ă_k)}(\delta_i) + \gamma_{\mathcal{F}(ă_k)}(\delta_i) * \gamma_{\mathcal{G}(ă_k)}(\delta_i)\right)$$

$$= \sum_{k=1}^{m}\left(\sigma_{\mathcal{F}(ă_k)}(\delta_1) * \sigma_{\mathcal{G}(ă_k)}(\delta_1) + \tau_{\mathcal{F}(ă_k)}(\delta_1) * \tau_{\mathcal{G}(ă_k)}(\delta_1) + \gamma_{\mathcal{F}(ă_k)}(\delta_1) * \gamma_{\mathcal{G}(ă_k)}(\delta_1)\right)$$
$$+ \sum_{k=1}^{m}\left(\sigma_{\mathcal{F}(ă_k)}(\delta_2) * \sigma_{\mathcal{G}(ă_k)}(\delta_2) + \tau_{\mathcal{F}(ă_k)}(\delta_2) * \tau_{\mathcal{G}(ă_k)}(\delta_2) + \gamma_{\mathcal{F}(ă_k)}(\delta_2) * \gamma_{\mathcal{G}(ă_k)}(\delta_2)\right)$$
$$+ \\ \vdots \\ +$$
$$\sum_{k=1}^{m}\left(\sigma_{\mathcal{F}(ă_k)}(\delta_n) * \sigma_{\mathcal{G}(ă_k)}(\delta_n) + \tau_{\mathcal{F}(ă_k)}(\delta_n) * \tau_{\mathcal{G}(ă_k)}(\delta_n) + \gamma_{\mathcal{F}(ă_k)}(\delta_n) * \gamma_{\mathcal{G}(ă_k)}(\delta_n)\right)$$

$$\delta_{NHSS}((\mathcal{F},Ä),(\mathcal{G},Ä)) = \left\{\begin{matrix} \left(\sigma_{\mathcal{F}(ă_1)}(\delta_1) * \sigma_{\mathcal{G}(ă_1)}(\delta_1) + \tau_{\mathcal{F}(ă_1)}(\delta_1) * \tau_{\mathcal{G}(ă_1)}(\delta_1) + \gamma_{\mathcal{F}(ă_1)}(\delta_1) * \gamma_{\mathcal{G}(ă_1)}(\delta_1)\right) \\ \left(\sigma_{\mathcal{F}(ă_2)}(\delta_1) * \sigma_{\mathcal{G}(ă_2)}(\delta_1) + \tau_{\mathcal{F}(ă_2)}(\delta_1) * \tau_{\mathcal{G}(ă_2)}(\delta_1) + \gamma_{\mathcal{F}(ă_2)}(\delta_1) * \gamma_{\mathcal{G}(ă_2)}(\delta_1)\right) \\ \vdots \\ \left(\sigma_{\mathcal{F}(ă_m)}(\delta_1) * \sigma_{\mathcal{G}(ă_m)}(\delta_1) + \tau_{\mathcal{F}(ă_m)}(\delta_1) * \tau_{\mathcal{G}(ă_m)}(\delta_1) + \gamma_{\mathcal{F}(ă_m)}(\delta_1) * \gamma_{\mathcal{G}(ă_m)}(\delta_1)\right) \end{matrix}\right\}$$
$$+ \left\{\begin{matrix} \left(\sigma_{\mathcal{F}(ă_1)}(\delta_2) * \sigma_{\mathcal{G}(ă_1)}(\delta_2) + \tau_{\mathcal{F}(ă_1)}(\delta_2) * \tau_{\mathcal{G}(ă_1)}(\delta_2) + \gamma_{\mathcal{F}(ă_1)}(\delta_2) * \gamma_{\mathcal{G}(ă_1)}(\delta_2)\right) + \\ \left(\sigma_{\mathcal{F}(ă_2)}(\delta_2) * \sigma_{\mathcal{G}(ă_2)}(\delta_2) + \tau_{\mathcal{F}(ă_2)}(\delta_2) * \tau_{\mathcal{G}(ă_2)}(\delta_2) + \gamma_{\mathcal{F}(ă_2)}(\delta_2) * \gamma_{\mathcal{G}(ă_2)}(\delta_2)\right) + \\ \vdots \\ \left(\sigma_{\mathcal{F}(ă_m)}(\delta_2) * \sigma_{\mathcal{G}(ă_m)}(\delta_2) + \tau_{\mathcal{F}(ă_m)}(\delta_2) * \tau_{\mathcal{G}(ă_m)}(\delta_2) + \gamma_{\mathcal{F}(ă_m)}(\delta_2) * \gamma_{\mathcal{G}(ă_m)}(\delta_2)\right) \end{matrix}\right\}$$
$$+ \\ \vdots \\ +$$
$$\left\{\begin{matrix} \left(\sigma_{\mathcal{F}(ă_1)}(\delta_n) * \sigma_{\mathcal{G}(ă_1)}(\delta_n) + \tau_{\mathcal{F}(ă_1)}(\delta_n) * \tau_{\mathcal{G}(ă_1)}(\delta_n) + \gamma_{\mathcal{F}(ă_1)}(\delta_n) * \gamma_{\mathcal{G}(ă_1)}(\delta_n)\right) + \\ \left(\sigma_{\mathcal{F}(ă_2)}(\delta_n) * \sigma_{\mathcal{G}(ă_2)}(\delta_n) + \tau_{\mathcal{F}(ă_2)}(\delta_n) * \tau_{\mathcal{G}(ă_2)}(\delta_n) + \gamma_{\mathcal{F}(ă_2)}(\delta_n) * \gamma_{\mathcal{G}(ă_2)}(\delta_n)\right) + \\ \vdots \\ \left(\sigma_{\mathcal{F}(ă_m)}(\delta_n) * \sigma_{\mathcal{G}(ă_m)}(\delta_n) + \tau_{\mathcal{F}(ă_m)}(\delta_n) * \tau_{\mathcal{G}(ă_m)}(\delta_n) + \gamma_{\mathcal{F}(ă_m)}(\delta_n) * \gamma_{\mathcal{G}(ă_m)}(\delta_n)\right) \end{matrix}\right\}$$

$$= \sum_{k=1}^{m}\left(\left(\sigma_{\mathcal{F}(ă_k)}(\delta_1) * \sigma_{\mathcal{G}(ă_k)}(\delta_1)\right) + \left(\sigma_{\mathcal{F}(ă_k)}(\delta_2) * \sigma_{\mathcal{G}(ă_k)}(\delta_2)\right) + \cdots + \left(\sigma_{\mathcal{F}(ă_k)}(\delta_n) * \sigma_{\mathcal{G}(ă_k)}(\delta_n)\right)\right) +$$
$$\sum_{k=1}^{m}\left(\left(\tau_{\mathcal{F}(ă_k)}(\delta_1) * \tau_{\mathcal{G}(ă_k)}(\delta_1)\right) + \left(\tau_{\mathcal{F}(ă_k)}(\delta_2) * \tau_{\mathcal{G}(ă_k)}(\delta_2)\right) + \cdots + \left(\tau_{\mathcal{F}(ă_k)}(\delta_n) * \tau_{\mathcal{G}(ă_k)}(\delta_n)\right)\right) +$$
$$\sum_{k=1}^{m}\left(\left(\gamma_{\mathcal{F}(ă_k)}(\delta_1) * \gamma_{\mathcal{G}(ă_k)}(\delta_1)\right) + \left(\gamma_{\mathcal{F}(ă_k)}(\delta_2) * \gamma_{\mathcal{G}(ă_k)}(\delta_2)\right) + \cdots + \left(\gamma_{\mathcal{F}(ă_k)}(\delta_n) * \gamma_{\mathcal{G}(ă_k)}(\delta_n)\right)\right)$$

By using Cauchy-Schwarz inequality

$$\delta_{NHSS}((\mathcal{F},Ä),(\mathcal{G},Ä))^2 \leq$$
$$\sum_{k=1}^{m}\left\{\begin{matrix}\left(\left(\sigma_{\mathcal{F}(ă_k)}(\delta_1)\right)^2 + \left(\sigma_{\mathcal{F}(ă_k)}(\delta_2)\right)^2 + \cdots + \left(\sigma_{\mathcal{F}(ă_k)}(\delta_n)\right)^2\right) + \left(\left(\tau_{\mathcal{F}(ă_k)}(\delta_1)\right)^2 + \left(\tau_{\mathcal{F}(ă_k)}(\delta_2)\right)^2 + \cdots + \left(\tau_{\mathcal{F}(ă_k)}(\delta_n)\right)^2\right) \\ + \left(\left(\gamma_{\mathcal{F}(ă_k)}(\delta_1)\right)^2 + \left(\gamma_{\mathcal{F}(ă_k)}(\delta_2)\right)^2 + \cdots + \left(\gamma_{\mathcal{F}(ă_k)}(\delta_n)\right)^2\right)\end{matrix}\right\}$$
$$\times \sum_{k=1}^{m}\left\{\begin{matrix}\left(\left(\sigma_{\mathcal{G}(ă_k)}(\delta_1)\right)^2 + \left(\sigma_{\mathcal{G}(ă_k)}(\delta_2)\right)^2 + \cdots + \left(\sigma_{\mathcal{G}(ă_k)}(\delta_n)\right)^2\right) + \left(\left(\tau_{\mathcal{G}(ă_k)}(\delta_1)\right)^2 + \left(\tau_{\mathcal{G}(ă_k)}(\delta_2)\right)^2 + \cdots + \left(\tau_{\mathcal{G}(ă_k)}(\delta_n)\right)^2\right) \\ + \left(\left(\gamma_{\mathcal{G}(ă_k)}(\delta_1)\right)^2 + \left(\gamma_{\mathcal{G}(ă_k)}(\delta_2)\right)^2 + \cdots + \left(\gamma_{\mathcal{G}(ă_k)}(\delta_n)\right)^2\right)\end{matrix}\right\}$$

$$\delta_{NHSS}((\mathcal{F},Ä),(\mathcal{G},Ä))^2 \leq$$

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)$$
$$\times\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)$$

$\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))^2 \leq \varsigma_{NHSS}(\mathcal{F},\ddot{A}) \times \varsigma_{NHSS}(\mathcal{G},\ddot{\mathbb{A}})$.

Therefore, $\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))^2 \leq \varsigma_{NHSS}(\mathcal{F},\ddot{A}) \times \varsigma_{NHSS}(\mathcal{G},\ddot{\mathbb{A}})$. Hence, by using definition 3.4, we have $\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \leq 1$. So, $0 \leq \delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \leq 1$.

Proof 2. The proof is obvious.

Proof 3. From equation 5, we have

$\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) =$

$$\frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)}{\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2}}$$

As we know that

$\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, and $\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\ \forall\ i,\ k$. We get

$\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) =$

$$\frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}{\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}}$$

$\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = 1$

Thus, prove the required result.

Definition 3.6

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs. Then, their correlation coefficient is given as $\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$ and defined as follows:

$$\delta^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \frac{\mathcal{C}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))}{max\{\varsigma_{NHSS}(\mathcal{F},\ddot{A}),\varsigma_{NHSS}(\mathcal{G},\ddot{\mathbb{A}})\}} \quad (6)$$

$\delta^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) =$

$$\frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)}{max\left\{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right),\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\right\}} \quad (7)$$

Proposition 3.7

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs. Then, CC between them satisfies the following properties

1. $0 \leq \delta^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \leq 1$
2. $\delta^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \delta^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$
3. If $(\mathcal{F},\ddot{A}) = (\mathcal{G},\ddot{\mathbb{A}})$, that is $\forall\ i,\ k,\ \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, and $\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, then $\delta^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = 1$.

Proof 1. $\delta^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \geq 0$ is trivial, here we only need to prove that $\delta^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \leq 1$. From equation 3, we have

$\delta^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)$

$$= \sum_{k=1}^{m}\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_1) + \tau_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \tau_{\mathcal{G}(\breve{a}_k)}(\delta_1) + \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)$$
$$+ \sum_{k=1}^{m}\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_2) * \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_2) + \tau_{\mathcal{F}(\breve{a}_k)}(\delta_2) * \tau_{\mathcal{G}(\breve{a}_k)}(\delta_2) + \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_2) * \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)$$
$$+$$
$$\vdots$$

$$+\sum_{k=1}^{m}\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\tau_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\tau_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)$$

$$\delta_{NHSS}^{1}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))=\left\{\begin{array}{c}\left(\sigma_{\mathcal{F}(\breve{a}_1)}(\delta_1)*\sigma_{\mathcal{G}(\breve{a}_1)}(\delta_1)+\tau_{\mathcal{F}(\breve{a}_1)}(\delta_1)*\tau_{\mathcal{G}(\breve{a}_1)}(\delta_1)+\gamma_{\mathcal{F}(\breve{a}_1)}(\delta_1)*\gamma_{\mathcal{G}(\breve{a}_1)}(\delta_1)\right)\\ \left(\sigma_{\mathcal{F}(\breve{a}_2)}(\delta_1)*\sigma_{\mathcal{G}(\breve{a}_2)}(\delta_1)+\tau_{\mathcal{F}(\breve{a}_2)}(\delta_1)*\tau_{\mathcal{G}(\breve{a}_2)}(\delta_1)+\gamma_{\mathcal{F}(\breve{a}_2)}(\delta_1)*\gamma_{\mathcal{G}(\breve{a}_2)}(\delta_1)\right)\\ \vdots\\ \left(\sigma_{\mathcal{F}(\breve{a}_m)}(\delta_1)*\sigma_{\mathcal{G}(\breve{a}_m)}(\delta_1)+\tau_{\mathcal{F}(\breve{a}_m)}(\delta_1)*\tau_{\mathcal{G}(\breve{a}_m)}(\delta_1)+\gamma_{\mathcal{F}(\breve{a}_m)}(\delta_1)*\gamma_{\mathcal{G}(\breve{a}_m)}(\delta_1)\right)\end{array}\right\}$$

$$+\left\{\begin{array}{c}\left(\sigma_{\mathcal{F}(\breve{a}_1)}(\delta_2)*\sigma_{\mathcal{G}(\breve{a}_1)}(\delta_2)+\tau_{\mathcal{F}(\breve{a}_1)}(\delta_2)*\tau_{\mathcal{G}(\breve{a}_1)}(\delta_2)+\gamma_{\mathcal{F}(\breve{a}_1)}(\delta_2)*\gamma_{\mathcal{G}(\breve{a}_1)}(\delta_2)\right)+\\ \left(\sigma_{\mathcal{F}(\breve{a}_2)}(\delta_2)*\sigma_{\mathcal{G}(\breve{a}_2)}(\delta_2)+\tau_{\mathcal{F}(\breve{a}_2)}(\delta_2)*\tau_{\mathcal{G}(\breve{a}_2)}(\delta_2)+\gamma_{\mathcal{F}(\breve{a}_2)}(\delta_2)*\gamma_{\mathcal{G}(\breve{a}_2)}(\delta_2)\right)+\\ \vdots\\ \left(\sigma_{\mathcal{F}(\breve{a}_m)}(\delta_2)*\sigma_{\mathcal{G}(\breve{a}_m)}(\delta_2)+\tau_{\mathcal{F}(\breve{a}_m)}(\delta_2)*\tau_{\mathcal{G}(\breve{a}_m)}(\delta_2)+\gamma_{\mathcal{F}(\breve{a}_m)}(\delta_2)*\gamma_{\mathcal{G}(\breve{a}_m)}(\delta_2)\right)\end{array}\right\}$$

$$+\\ \vdots\\ +$$

$$\left\{\begin{array}{c}\left(\sigma_{\mathcal{F}(\breve{a}_1)}(\delta_n)*\sigma_{\mathcal{G}(\breve{a}_1)}(\delta_n)+\tau_{\mathcal{F}(\breve{a}_1)}(\delta_n)*\tau_{\mathcal{G}(\breve{a}_1)}(\delta_n)+\gamma_{\mathcal{F}(\breve{a}_1)}(\delta_n)*\gamma_{\mathcal{G}(\breve{a}_1)}(\delta_n)\right)+\\ \left(\sigma_{\mathcal{F}(\breve{a}_2)}(\delta_n)*\sigma_{\mathcal{G}(\breve{a}_2)}(\delta_n)+\tau_{\mathcal{F}(\breve{a}_2)}(\delta_n)*\tau_{\mathcal{G}(\breve{a}_2)}(\delta_n)+\gamma_{\mathcal{F}(\breve{a}_2)}(\delta_n)*\gamma_{\mathcal{G}(\breve{a}_2)}(\delta_n)\right)+\\ \vdots\\ \left(\sigma_{\mathcal{F}(\breve{a}_m)}(\delta_n)*\sigma_{\mathcal{G}(\breve{a}_m)}(\delta_n)+\tau_{\mathcal{F}(\breve{a}_m)}(\delta_n)*\tau_{\mathcal{G}(\breve{a}_m)}(\delta_n)+\gamma_{\mathcal{F}(\breve{a}_m)}(\delta_n)*\gamma_{\mathcal{G}(\breve{a}_m)}(\delta_n)\right)\end{array}\right\}$$

$$=\sum_{k=1}^{m}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)+\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)+\cdots+\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)\right)+$$
$$\sum_{k=1}^{m}\left(\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\tau_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\tau_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)+\cdots+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\tau_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)\right)+$$
$$\sum_{k=1}^{m}\left(\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)+\cdots+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)\right)$$

By using Cauchy-Schwarz inequality

$$\delta_{NHSS}^{1}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))\leq$$
$$\sum_{k=1}^{m}\left\{\begin{array}{c}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2+\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2+\cdots+\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2\right)+\left(\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2+\cdots+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2\right)\\ +\left(\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2+\cdots+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2\right)\end{array}\right\}$$
$$\times\sum_{k=1}^{m}\left\{\begin{array}{c}\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2+\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2+\cdots+\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2\right)+\left(\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2+\cdots+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2\right)\\ +\left(\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2+\cdots+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2\right)\end{array}\right\}$$

$$\delta_{NHSS}^{1}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))^2\leq$$
$$\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)$$
$$\times\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)$$

$\delta_{NHSS}^{1}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))^2\leq\varsigma_{NHSS}(\mathcal{F},\ddot{A})\times\varsigma_{IFHSS}(\mathcal{G},\ddot{\mathbb{A}})$.

Therefore, $\delta_{NHSS}^{1}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))^2\leq\varsigma_{NHSS}(\mathcal{F},\ddot{A})\times\varsigma_{NHSS}(\mathcal{G},\ddot{\mathbb{A}})$. Hence, by using definition 3.6, we have $\delta_{NHSS}^{1}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))\leq 1$. So, $0\leq\delta_{NHSS}^{1}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))\leq 1$.

Proof 2. The proof is obvious.

Proof 3. From equation 7, we have

$$\delta_{NHSS}^{1}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))=\frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)}{max\left\{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right),\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\right\}}$$

As we know that

$\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)=\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)=\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, and $\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)=\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\ \forall\ i,\ k$. We get

$\delta^{1}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}})) =$

$$\frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}{max\left\{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right),\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)\right\}}$$

$\delta^{1}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}})) = 1$

Thus, prove the required result.

Definition 3.8

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i,\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i),\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i),\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\text{Ѫ}}) = \left\{\left(\delta_i,\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i),\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i),\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs. Then, their weighted correlation coefficient is given as $\delta_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}}))$ and defined as follows:

$$\delta_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}})) = \frac{\mathcal{C}_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}}))}{\sqrt{\varsigma_{WNHSS}(\mathcal{G},\ddot{\text{Ѫ}})}*\sqrt{\varsigma_{WNHSS}(\mathcal{G},\ddot{\text{Ѫ}})}} \quad (8)$$

$\delta_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}})) =$

$$\frac{\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)\right)}{\sqrt{\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)}\sqrt{\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)}}$$

(9)

Definition 3.9

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i,\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i),\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i),\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\text{Ѫ}}) = \left\{\left(\delta_i,\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i),\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i),\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs. Then, their weighted correlation coefficient is given as $\delta^{1}_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}}))$ and defined as follows:

$$\delta^{1}_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}})) = \frac{\mathcal{C}_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}}))}{max\{\varsigma_{WNHSS}(\mathcal{F},\ddot{A}),\varsigma_{WNHSS}(\mathcal{G},\ddot{\text{Ѫ}})\}} \quad (10)$$

$\delta^{1}_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}})) =$

$$\frac{\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)\right)}{max\left\{\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)\right),\sum_{k=1}^{m}\acute{\omega}_k\left(\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)\right)\right\}}$$

(11)

If we consider $\Omega = \{\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\}$ and $\gamma = \{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\}$, then $\delta_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}}))$ and $\delta^{1}_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}}))$ are reduced to $\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}}))$ and $\delta^{1}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}}))$ respectively.

Proposition 3.10

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i,\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i),\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i),\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\text{Ѫ}}) = \left\{\left(\delta_i,\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i),\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i),\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs. Then, CC between them satisfies the following properties

1. $0 \le \delta_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}})) \le 1$
2. $\delta_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}})) = \delta_{WNHSS}((\mathcal{G},\ddot{\text{Ѫ}}),(\mathcal{F},\ddot{A}))$
3. If $(\mathcal{F},\ddot{A}) = (\mathcal{G},\ddot{\text{Ѫ}})$, that is $\forall\ i,\ k,\ \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, and $\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)$ then $\delta_{WNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\text{Ѫ}})) = 1$.

Proof Similar to proposition 3.5.

4. Application of Correlation Coefficient for Decision Making Under NHSS Environment

In this section, we proposed the algorithm based on CC under NHSS and utilize the proposed approach for decision making in real-life problems.

4.1 Algorithm for Correlation Coefficient under NHSS

Step 1. Pick out the set containing sub-attributes of parameters.

Step 2. Construct the NHSS according to experts in form of NHSNs.

Step 3. Find the informational neutrosophic energies of NHSS.

Step 4. Calculate the correlation between NHSSs by using the following formula

$$\mathcal{C}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\breve{A}})) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)$$

Step 5. Calculate the CC between NHSSs by using the following formula

$$\delta_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\breve{A}})) = \frac{\mathcal{C}_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\breve{A}}))}{\sqrt{\varsigma_{NHSS}(\mathcal{F},\ddot{A})} * \sqrt{\varsigma_{NHSS}(\mathcal{G},\ddot{\breve{A}})}}$$

Step 6. Choose the alternative with a maximum value of CC.

Step 7. Analyze the ranking of the alternatives.

A flowchart of the above-presented algorithm can be seen in figure 1.

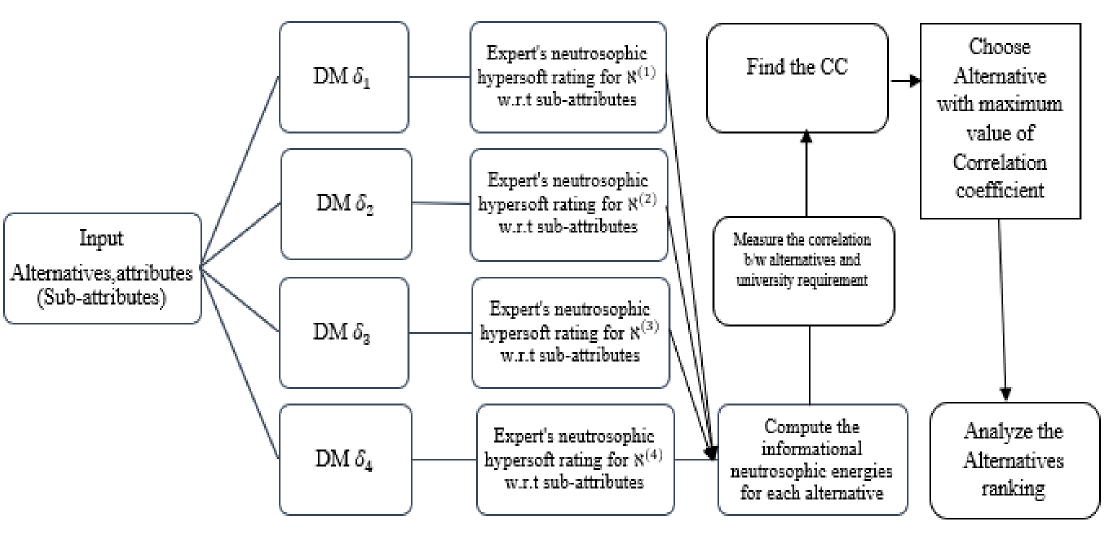


Figure 1: Flowchart for correlation coefficient under NHSS

4.1 Problem Formulation and Application of NHSS For Decision Making

Department of the scientific discipline of some university $\mathcal{U}$ will have one scholarship for the position of post-doctorate. Several scholars apply to get a scholarship but referable probabilistic along with CGPA (cumulative grade points average), simply four scholars call for enrolled for undervaluation such as $\aleph = \{\aleph^1, \aleph^2, \aleph^3, \aleph^4\}$ be a set of selected scholars call for the interview. The president of the university hires a committee of four decision-makers (DM) $\mathcal{U} = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ for the selection post-doctoral scholar. The team of DM decides the criteria (attributes) for the selection of post-doctorate position such as $\mathfrak{L} = \{\ell_1 = \text{Publications}, \ell_2 = \text{Subjects}, \ell_3 = \text{IF}\}$ be a collection of attributes and their corresponding sub-attribute are given as Publications = $\ell_1 = \{a_{11} = \textit{more than } 10, a_{12} = \textit{less than } 10\}$, Subjects = $\ell_2 = \{a_{21} = \textit{Mathematics}, a_{22} = \textit{Computer Science}\}$, and IF = $\ell_3 = \{a_{31} = 45, a_{32} = 47\}$. Let $\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3$ be a set of sub-attributes

$\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}\} \times \{a_{31}, a_{32}\}$

$= \begin{Bmatrix} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}), \\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{21}, a_{32}), (a_{12}, a_{22}, a_{31}), (a_{12}, a_{22}, a_{32}) \end{Bmatrix}$, $\mathfrak{L}' = \{\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5, \breve{a}_6, \breve{a}_7, \breve{a}_8\}$ be a set of all multi sub-attributes. Each DM will evaluate the ratings of each alternative in the form of NHSNs under the considered multi sub-attributes. The developed method to find the best alternative is as follows.

4.1.1. Application of NHSS For Decision Making

Assume $\aleph = \{\aleph^1, \aleph^2, \aleph^3, \aleph^4\}$ be a set of alternatives who are shortlisted for interview and $\mathfrak{L} = \{\ell_1 = \text{Publications}, \ell_2 = \text{Subjects}, \ell_3 = \text{Qualification}\}$ be a set of parameters for the selection of scholarship positions. Let the corresponding sub-attribute are given as Publications $= \ell_1 = \{a_{11} = \textit{more than } 10, a_{12} = \textit{less than } 10\}$, Subjects $= \ell_2 = \{a_{21} = \textit{Mathematics}, a_{22} = \textit{Computer Science}\}$, and IF $= \ell_3 = \{a_{31} = 45, a_{32} = 47\}$. Let $\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3$ be a set of sub-attributes. Development of decision matrix according to the requirement of the scientific discipline department in terms of NHSNs.

Table 1. Decision Matrix of Concerning Department

$\wp$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	(.2,.5,.9)	(.5,.7,.6)	(.5,.6,.9)	(.5,.8,.7)	(.4,.7,.6)	(.8,.6,.3)	(.5,.4,.7)	(.6,.4,.8)
δ_2	(.5,.9,.7)	(.6,.4,.7)	(.5,.8 .2)	(.7,.4 .2)	(.9,.5,.7)	(.4,.7,.9)	(.9,.2,.5)	(.2,.8,.5)
δ_3	(.7,.3,.5)	(.7,.4,.2)	(.8,.2,.6)	(.7,.3,.6)	(.8,.4 .9)	(.7,.5,.8)	(.9,.6,.8)	(.6,.3,.8)
δ_4	(.5,.4,.7)	(.4,.7,.3)	(.6,.3,.8)	(.5,.4,.6)	(.7,.3,.5)	(.8,.3,.2)	(.5,.4,.7)	(.6,.2,.7)

Table 2. Decision Matrix for Alternative $\aleph^{(1)}$

$\aleph^{(1)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	(.3,.5,.8)	(.2,.3,.6)	(.5,.1,.3)	(.8,.6,.7)	(.5,.9,.6)	(.8,.2,.6)	(.5,.4,.1)	(.9,.3,.5)
δ_2	(.5,.2,.7)	(.2,.4,.6)	(.3,.8 .4)	(.7,.5 .2)	(.9,.2,.6)	(.5,.2,.4)	(.9,.2,.5)	(.8,.4,.5)
δ_3	(.6,.2,.4)	(.4,.7,.5)	(.5,.1,.6)	(.7,.3,.4)	(.2,.6 .9)	(.9,.3,.5)	(.2,.3,.8)	(.6,.3,.8)
δ_4	(.2,.4,.7)	(.7,.2,.3)	(.6,.3,.8)	(.2,.4,.6)	(.7,.3,.5)	(.9,.3,.6)	(.3,.4,.5)	(.6,.2,.7)

Table 3. Decision Matrix for Alternative $\aleph^{(2)}$

$\aleph^{(2)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	(.8,.5,.6)	(.5,.4,.2)	(.4,.3,.6)	(.4,.8,.6)	(.7,.6,.5)	(.4,.1,.3)	(.7,.8,.5)	(.8,.4,.7)
δ_2	(.6,.5,.2)	(.5,.6,.5)	(.9,.5,.8)	(.6,.4,.5)	(.7,.5,.8)	(.7,.5,.7)	(.3,.5,.9)	(.6,.4,.9)
δ_3	(.2,.5,.2)	(.9,.4,.6)	(.2,.5,.4)	(.7,.3,.2)	(.6,.4,.5)	(.3,.5,.7)	(.4,.6,.2)	(.6,.7,.9)
δ_4	(.5,.2,.4)	(.7,.5,.9)	(.6,.3,.4)	(.9,.5,.1)	(.3,.4,.6)	(.6,.5,.2)	(.9,.5,.6)	(.3,.4,.3)

Table 4. Decision Matrix for Alternative $\aleph^{(3)}$

$\aleph^{(3)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	(.3,.5,.2)	(.8,.7,.3)	(.7,.2,.9)	(.9,.5,.1)	(.3,.4,.6)	(.1,.5,.2)	(.9,.5,.1)	(.7,.4,.3)
δ_2	(.6,.7,.2)	(.7,.8,.3)	(.2,.4,.6)	(.6,.1,.2)	(.9,.5,.6)	(.7,.2,.3)	(.4,.7,.6)	(.7,.2,.4)
δ_3	(.3,.9,.7)	(.5,.9,.1)	(.7,.3,.2)	(.2,.1,.2)	(.7,.9,.8)	(.7,.2,.1)	(.7,.4,.5)	(.1,.7,.9)

$\boldsymbol{\delta_4}$	(.7,.8,.6)	(.7,.2,.5)	(.7,.3,.2)	(.3,.2,.7)	(.4,.6,.8)	(.5,.6,.2)	(.7,.2,.6)	(.8,.6,.9)

Table 5. Decision Matrix for Alternative $\aleph^{(4)}$

$\aleph^{(4)}$	$\breve{\alpha}_1$	$\breve{\alpha}_2$	$\breve{\alpha}_3$	$\breve{\alpha}_4$	$\breve{\alpha}_5$	$\breve{\alpha}_6$	$\breve{\alpha}_7$	$\breve{\alpha}_8$
$\boldsymbol{\delta_1}$	(.7,.4,.1)	(.7,.3,.1)	(.7,.4,.6)	(.4,.9,.6)	(.7,.2,.5)	(.7,.3,.2)	(.7,.4,.6)	(.9,.4,.3)
$\boldsymbol{\delta_2}$	(.1,.4,.5)	(.6,.2,.3)	(.7,.4,.3)	(.6,.2,.5)	(.6,.2,.1)	(.5,.4,.7)	(.3,.5,.1)	(.6,.2,.7)
$\boldsymbol{\delta_3}$	(.5,.4,.3)	(.6,.4,.7)	(.6,.2,.1)	(.6,.3,.5)	(.4,.7,.9)	(.2,.7,.4)	(.5,.3,.9)	(.3,.6,.2)
$\boldsymbol{\delta_4}$	(.4,.2,.6)	(.7,.4,.3)	(.5,.4,.9)	(.4,.2,.3)	(.4,.1,.3)	(.4,.5,.2)	(.1,.6,.5)	(.1,.5,.2)

By using Tables 1-5, compute the correlation coefficient between $\delta_{NHSS}(\wp,\aleph^{(1)})$, $\delta_{NHSS}(\wp,\aleph^{(2)})$, $\delta_{NHSS}(\wp,\aleph^{(3)})$, $\delta_{NHSS}(\wp,\aleph^{(4)})$ by using equation 5 given as follows:
$\delta_{NHSS}(\wp,\aleph^{(1)}) = .99658$, $\delta_{NHSS}(\wp,\aleph^{(2)}) = .99732$, $\delta_{NHSS}(\wp,\aleph^{(3)}) = .99894$, and $\delta_{NHSS}(\wp,\aleph^{(4)}) = .99669$. This shows that $\delta_{IVNHSS}(\wp,\aleph^{(3)}) > \delta_{IVNHSS}(\wp,\aleph^{(2)}) > \delta_{IVNHSS}(\wp,\aleph^{(4)}) > \delta_{IVNHSS}(\wp,\aleph^{(1)})$. It can be seen from this ranking alternative $\aleph^{(3)}$ is the most suitable alternative. Therefore $\aleph^{(3)}$ is the best alternative, the ranking of other alternatives given as $\aleph^{(3)} > \aleph^{(2)} > \aleph^{(4)} > \aleph^{(1)}$. Graphical results of alternatives ratings can be seen in figure 2.

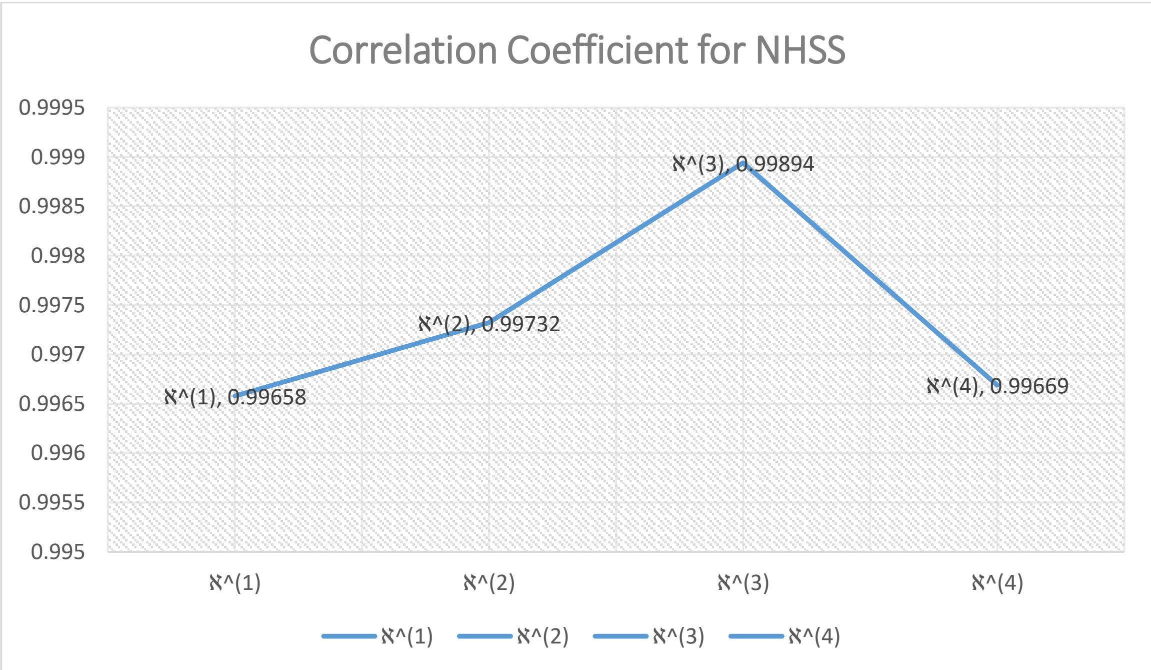


Figure 2: Alternative's rating based on correlation coefficient under NHSS

5. Conclusion

The neutrosophic hypersoft set is a novel concept that is an extension neutrosophic soft set. In this manuscript, we studied some basic concepts which were necessary to build the structure of the paper. We introduced the correlation and weighted correlation coefficient with some necessary properties under the NHSS environment. A decision-making approach has been developed based on the established correlation coefficient and presented an algorithm under NHSS. Finally, a numerical illustration has been described to solve the decision-making problem by using the proposed technique. In the future, anyone can extend the NHSS to interval valued NHSS, aggregation operators, TOPSIS technique based on developed CC.

References

1. L. A. Zadeh, Fuzzy Sets, Information and Control, 8(1965) 338–353.
2. K. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems, 20(1986) 87–96.
3. D. Molodtsov, Soft Set Theory First Results, Computers & Mathematics with Applications, 37(1999) 19–31.
4. P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, Computers and Mathematics with Applications, 45(4–5)(2003) 555–562.
5. P. K. Maji, A. R. Roy, R. Biswas, An Application of Soft Sets in A Decision Making Problem, Computers and Mathematics with Applications, 44(2002) 1077–1083.
6. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, Journal of Fuzzy Mathematics, 9 (2001), 589–602.
7. P. K. Maji, R. Biswas, A. Roy, Intuitionistic fuzzy soft sets, Journal of Fuzzy Mathematics, 9 (2001), 677–692.
8. R. M. Zulqarnain, X. L. Xin, M. Saqlain, W. A. Khan, TOPSIS Method Based on the Correlation Coefficient of Interval-Valued Intuitionistic Fuzzy Soft Sets and Aggregation Operators with Their Application in Decision-Making, Journal of Mathematics. 2021, Volume 2021, 1-16.
9. R. M. Zulqarnain, X. L. Xin, I. Siddique, W. Asghar Khan, M. A. Yousif, TOPSIS Method Based on Correlation Coefficient under Pythagorean Fuzzy Soft Environment and Its Application towards Green Supply Chain Management. Sustainability 2021, 13, 1642. https://doi.org/10.3390/su13041642.
10. R. M. Zulqarnain, X. L. Xin, H. Garg, W. A. Khan, Aggregation operators of pythagorean fuzzy soft sets with their application for green supplier chain management, Journal of Intelligent and Fuzzy Systems, (2021) 40 (3), 5545-5563. DOI: 10.3233/JIFS-202781.
11. P. K. Maji, Neutrosophic soft set. *Annals of Fuzzy Mathematics and Informatics*. **2013,** *5(1)*, 157–168.
12. F. Karaaslan, Possibility neutrosophic soft sets and PNS-decision making method. *Applied Soft Computing Journal*. **2016**, *54*, 403–414.
13. S. Broumi, Generalized Neutrosophic Soft Set. International Journal of Computer Science, Engineering and Information Technology. **2013,** 3(2), 17–30.
14. R. M. Zulqarnain, X. L. Xin, H. Garg, R. Ali, (2021), Interaction aggregation operators to solve multi criteria decision making problem under pythagorean fuzzy soft environment. Journal of Intelligent & Fuzzy Systems, 41 (1), 1151-1171.
15. H. Wang, F. Smarandache, Y. Zhang, . Single valued neutrosophic sets. *Int. J. Gen. Syst*, **2013**, *42*, 386–394.
16. J. Ye, A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *Journal of Intelligent and Fuzzy Systems.* **2014,** *26*, 2459–2466.
17. M. R. Hashmi, M. Riaz, F. Smarandache, m-Polar Neutrosophic Topology with Applications to Multi criteria Decision-Making in Medical Diagnosis and Clustering Analysis. *Int. J. Fuzzy Syst*, **2019,** Dec (2019).
18. R. M. Zulqarnain, X. L. Xin, M. Saqlain, F. Smarandache, M. I. Ahamad, An integrated model of Neutrosophic TOPSIS with application in Multi-Criteria Decision-Making Problem. Neutrosophic Sets and Systems, 40,118-133, (2021).
19. R. M. Zulqarnain, X. L. Xin, M. Saeed, F. Smarandache, N. Ahmad, Generalized Neutrosophic TOPSIS to Solve Multi-Criteria Decision-Making Problems, Neutrosophic Sets and Systems, 38, 276-292, (2020).
20. C. Yu, Correlation of fuzzy numbers. Fuzzy Sets and Systems. 1993, 55, 303-307.
21. F. Smarandache, 2018. Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set. Neutrosophic Sets and Systems, 22 (2018), 168-170.
22. R. M. Zulqarnain, X. L. Xin, M. Saeed, Extension of TOPSIS method under intuitionistic fuzzy hypersoft environment based on correlation coefficient and aggregation operators to solve decision making problem, AIMS Mathematics, 6 (3), 2732-2755, (2021).
23. R. M. Zulqarnain, X. L. Xin, B. Ali, S. Broumi, S. Abdal, M. I. Ahamad, Decision-Making Approach Based on Correlation Coefficient with its Properties Under Interval-Valued Neutrosophic hypersoft set environment. Neutrosophic Sets and Systems, 40,12-28, (2021).
24. S. Gayen, F. Smarandache, S. Jha, M. K. Singh, S. Broumi, R. Kumar, (2020). Introduction to Plithogenic Subgroup. In Neutrosophic Graph Theory and Algorithms (pp. 213-259). IGI Global.
25. R. M. Zulqarnain, X. L. Xin, M. Saqlain, F. Smarandache, (2020), Generalized Aggregate Operators on Neutrosophic Hypersoft Set, Neutrosophic Sets and Systems, 36, 271-281.

26. R. M. Zulqarnain, I. Siddique, R. Ali, F. Jarad, A. Samad, T. Abdeljawad, Neutrosophic Hypersoft Matrices with Application to Solve Multiattributive Decision-Making Problems, Volume 2021, Article ID 5589874, 1-17, https://doi.org/10.1155/2021/5589874.
27. R. M. Zulqarnain, X. L. Xin, M. Saqlain, M. Saeed, F. Smarandache, M. I. Ahamad, Some Fundamental Operations on Interval Valued Neutrosophic Hypersoft Set with Their Properties. Neutrosophic Sets and Systems, 40,134-148, (2021).
28. R. M. Zulqarnain, I. Siddique, R. Ali, D. Pamucar, D. Marinkovic, D. Bozanic, Robust Aggregation Operators for Intuitionistic Fuzzy Hyper-soft Set With Their Application to Solve MCDM Problem, Entropy 2021, 23, 688. https://doi.org/10.3390/e23060688.
29. M. Abbas, G. Murtaza, F. Smarandache, Basic operations on hypersoft sets and hypersoft point, Neutrosophic Sets and Systems, Vol. 35, 2020, 407-421.
30. R. M. Zulqarnain, I. Saddique, F. Jarad, R. Ali, T. Abdeljawad, Development of TOPSIS Technique Under Pythagorean Fuzzy Hypersoft Environment Based on Correlation Coefficient and Its Application Towards the Selection of Antivirus Mask in COVID-19 Pandemic. Complexity, Volume 2021, Article ID 6634991, 27 pages, https://doi.org/10.1155/2021/6634991.
31. N. Martin, F. Smarandache, Introduction to Combined Plithogenic Hypersoft Sets, Neutrosophic Sets and Systems, Vol. 35, 2020, 503-510.
32. N. Martin, F. Smarandache, Concentric Plithogenic Hypergraph based on Plithogenic Hypersoft sets – A Novel Outlook, Neutrosophic Sets and Systems, Vol. 33, 2020, 78-91.
33. S. Alkhazaleh, Plithogenic Soft Set, Neutrosophic Sets and Systems, Vol. 33, 2020, 256-274.
34. R. M. Zulqarnain, X. L. Xin, M. Saeed, A Development of Pythagorean fuzzy hypersoft set with basic operations and decision-making approach based on the correlation coefficient, Theory and Application of Hypersoft Set, Publisher: Pons Publishing House Brussels, 2021, 85-106.
35. A. Samad, R. M. Zulqarnain, E. Sermutlu, R. Ali, I. Siddique, F. Jarad , T. Abdeljawad, Selection of an Effective Hand Sanitizer to Reduce COVID-19 Effects and Extension of TOPSIS Technique Based on Correlation Coefficient under Neutrosophic Hypersoft Set, Complexity, 2021, Article ID 5531830, 1-22. https://doi.org/10.1155/2021/5531830.
36. R. M. Zulqarnain, I. Siddique, R. Ali, F. Jarad, A. Iampan, Multi Criteria Decision Making Approach For Pythagorean Fuzzy Hypersoft Sets Interaction Aggregation Operators, Mathematical Problems in Engineering, Volume 2021, https://doi.org/10.1155/2021/9964492.
37. M. Abdel-Basset, W. Ding, R. Mohamed, N. Metawa, An integrated plithogenic MCDM approach for financial performance evaluation of manufacturing industries. *Risk Manag*, 22(2020) 192–218.
38. M. Abdel-Basset, R. Mohamed, A. E. N. H. Zaied, F. Smarandache, (2019). A Hybrid Plithogenic Decision-Making Approach with Quality Function Deployment for Selecting Supply Chain Sustainability Metrics. Symmetry, 11(7), 903.

Received: May 7, 2021. Accepted: October 3, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

Medical Diagnosis via Distance-based Similarity Measure for Rough Neutrosophic Set

Suriana Alias[1]*, Daud Mohamad[2], Adibah Shuib[3,4], Nazhatul Sahima Mohd Yusoff[5], Norarida Abd Rhani[6] and Siti Nurul Fitriah Mohamad[7]

[1,5,6,7]Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA Cawangan Kelantan, Bukit Ilmu, 18500 Machang, Kelantan, Malaysia;
suria588@uitm.edu.my; nazha237@uitm.edu.my; norarida@uitm.edu.my; fitriah@uitm.edu.my

[2,3]Faculty of Computer and Mathematical Sciences, Kompleks Al-Khawarizmi, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia;
daud@tmsk.uitm.edu.my; adibah@tmsk.uitm.edu.my

[4]Malaysia Institute of Transport (MITRANS), Universiti Teknologi MARA (UiTM), Shah Alam, 40450 Shah Alam, Selangor, Malaysia.

*Correspondence: suria588@uitm.edu.my

Abstract: A rough neutrosophic set theory is a generalization of uncertainty set theory with a combination of upper and lower approximation and a pair of neutrosophic sets which are characterized by truth membership degree (T), indeterminacy membership degree (I), and falsity membership degree (F). This set theory is suitable for representing each criterion's relation in medical diagnoses, such as the relation of disease and symptom. This paper aims to propose a model of medical diagnosis via a distance-based similarity measure of a rough neutrosophic set. The first phase for the development model involves the roughness measure between information collected and a lower and upper approximation of rough neutrosophic set theory. Then, it is simultaneously used with extended Hausdorff distance measure to get the proper medical diagnosis. The result shows that each patient has a chest problem that contradicts the prior diagnosis. The finding shows that the roughness approximation is important to get the best result in a close distance-based similarity measure, especially for uncertainty information.

Keywords: Distance-based similarity measure, Medical diagnosis, Rough neutrosophic set, Roughness measure,

1. Introduction

The medical diagnosis contains lots of uncertainties and an increased volume of information. It becomes difficult to classify different symptoms under a single disease name. There is a possibility in some practical situations that each dimension has a different truth, indeterminacy, and falsity information. It is, therefore, important to use a more versatile method that can easily deal with unpredictable circumstances. Hence, a rough neutrosophic set (RNS) is a useful tool for dealing with uncertainty and incompleteness information for medical cases [1].

A rough neutrosophic set (RNS) is a generalization of rough set and neutrosophic set theory. Pawlak [2] introduced a rough set concept as a formal tool for modelling and processing incomplete information for information systems. The basic idea of the rough set is based upon the approximation of sets known as a lower approximation and an upper approximation of a set. Besides, the neutrosophic set proposed by Smarandache [3] is a generalization of a fuzzy set [4] and an intuitionistic fuzzy set [5]. Meanwhile, neutrosophic sets are characterized by truth membership function (T), indeterminacy membership function (I), and falsity membership function (F).

Since the rough neutrosophic set (RNS) involves a pair of approximation sets, then the roughness measure between them gives more chances for an informed decision. The study of this roughness measure is still not yet explored for RNS theory. Meanwhile, the study of distance-based similarity measures of RNS gives many measures, each representing specific properties and behavior in real-life decision making and pattern recognition works. Based on the relationship between distance and similarity measure, Pramanik et al. proposed several similarity measures: Cosine similarity measure [6] and Dice and Jaccard similarity measure [1]. Meanwhile, Pramanik et al. [7] used the Trigonometric Hamming similarity measure for multi decision-making in selecting laptops from a different company. Besides that, Mondal et al. [8] studied the similarity measure of RNS by introducing the variational coefficient for each similarity variable to solve the decision-making problem under-investment company option. Therefore, the application of RNS is widely explored, as discussed in the literature.

In this study, the roughness approximation of the rough set by Yao [10] is used to determine the roughness measure between the lower and upper approximations of RNS. Then, the extended Hausdorff distance measure is used in the first phase of implementation via medical diagnosis. Simultaneously, the roughness approximation is included in this dissimilarity measure. Therefore, this study aims to propose the new notion of roughness approximation for medical information via lower and upper approximations of RNS, and to determine the closeness of distance-based similarity measure between symptoms and diseases versus patients and symptoms for complete medical finding. The result is more accurate since the roughness of information is considered for the first term as a lower and upper approximation of RNS. For novelty, the roughness for a lower and upper approximation of RNS is not yet studied by other researchers. Following from there, medical information related to symptoms and diseases versus patients and symptoms is discussed thoroughly.

The rest of the paper is organized as follows. Section two is preliminaries for some important definition, while Section three introduces a new definition of the distance-based similarity measure. Section four presents the methodology involved in the medical diagnosis process, while Section five is the main implementation of medical findings. Lastly, Section six concludes the paper.

2. Preliminaries

This section recalled some important definitions of the rough neutrosophic set, extended Hausdorff distance of neutrosophic set, roughness approximation, and distance-based similarity measure. All the proof of the propositions may be referred to in [10 - 13].

2.1. Rough Neutrosophic Set

Definition 2.1.1 [11]**.** Let U be a non-null set and R be an equivalence relation on U. Let A be neutrosophic set in U with the truth membership function T_A, indeterminacy function I_A, and non-membership function F_A. The lower and the upper approximations of A in the approximation (U, R) denoted by $\underline{N}(A)$ and $\overline{N}(A)$ are respectively defined as follows:

$$\underline{N}(A) = \left(\langle x_j, T_{\underline{N}(A)}(x_j), I_{\underline{N}(A)}(x_j), F_{\underline{N}(A)}(x_j)\rangle \mid y \in [x_j]_R, j \in \mathbb{Z}^+, x_j \in U\right), \text{ and}$$

$$\overline{N}(A) = \left(\langle x_j, T_{\overline{N}(A)}(x_j), I_{\overline{N}(A)}(x_j), F_{\overline{N}(A)}(x_j)\rangle \mid y \in [x_j]_R, j \in \mathbb{Z}^+, x_j \in U\right)$$

where;

$j = 1,2,\ldots,q$ is a positive integer, $T_{\underline{N}(A)}(x_j) = \wedge_{y\in[x_j]_R} T_A(y_j)$, $I_{\underline{N}(A)}(x_j) = \vee_{y\in[x_j]_R} I_A(y_j)$, $F_{\underline{N}(A)}(x_j) = \vee_{y\in[x_j]_R} F_A(y_j)$, $T_{\overline{N}(A)}(x_j) = \vee_{y\in[x_j]_R} T_A(y_j)$, $I_{\overline{N}(A)}(x_j) = \wedge_{y\in[x_j]_R} I_A(y_j)$, and $F_{\overline{N}(A)}(x_j) = \wedge_{y\in[x_j]_R} F_A(y_j)$.

Here $\wedge$ and $\vee$ denote "min" and "max'' operators respectively and $[x_j]_R$ is the equivalence class of the x_j. The $T_A(y_j)$, $I_A(y_j)$and $F_A(y_j)$ are the truth membership, indeterminacy membership, and falsity membership of y concerning A.

Since $\underline{N}(A)$ and $\overline{N}(A)$ are two neutrosophic sets in U, thus the neutrosophic set mappings $\underline{N},\overline{N}: N(U) \to N(U)$ are respectively referred to as lower and upper rough neutrosophic set approximation operators, while the pair of $\left(\underline{N}(A),\overline{N}(A)\right)$ is called the rough neutrosophic set in (U,R), respectively. The rough neutrosophic set is denoted by:

$$N(A) = \left(\underline{N}(A),\overline{N}(A)\right) = \left(\langle x_j, \begin{pmatrix} [T_{\underline{N}(A)}(x_j), I_{\underline{N}(A)}(x_j), F_{\underline{N}(A)}(x_j)], \\ [T_{\overline{N}(A)}(x_j), I_{\overline{N}(A)}(x_j), F_{\overline{N}(A)}(x_j)] \end{pmatrix} \rangle \,|y \in [x_j]_R, j \in \mathbb{Z}^+, x_j \in U\right) \quad (1)$$

The truth membership set $[T_{\underline{N}(A)}(x_j), T_{\overline{N}(A)}(x_j)]$, indeterminacy membership set $[I_{\underline{N}(A)}(x_j), I_{\overline{N}(A)}(x_j)]$, and falsity membership $[F_{\underline{N}(A)}(x_j), F_{\overline{N}(A)}(x_j)]$ for lower and upper approximation of RNS may be in decreasing or increasing order.

Definition 2.1.2 [11]. If $N(A)$ is a rough neutrosophic set in (U,R), the rough complement of $N(A)$ is the rough neutrosophic set donated by $\sim N(A) = \left((\underline{N}(A))^c, (\overline{N}(A))^c\right)$, where $(\underline{N}(A))^c$ and $(\overline{N}(A))^c$ are the complements of neutrosophic set $\left(\underline{N}(A),\overline{N}(A)\right)$, respectively, given by

$$\sim N(A) = \left((\underline{N}(A))^c, (\overline{N}(A))^c\right) = \left\{\langle x_j, \begin{pmatrix} [F_{\underline{N}(A)}(x_j), 1 - I_{\underline{N}(A)}(x_j), T_{\underline{N}(A)}(x_j)], \\ [F_{\overline{N}(A)}(x_j), 1 - I_{\overline{N}(A)}(x_j), T_{\overline{N}(A)}(x_j)] \end{pmatrix} \rangle \,\middle|\, x_j \in U\right\} \quad (2)$$

2.2. Distance-based Similarity Measure

Definition 2.2.1 [12]. An extended Hausdorff Distance $d_N^{EH}(A,B)$ operator between neutrosophic set A and B is defined as follows:

$$d_N^{EH}(A,B) = \frac{1}{n}\sum_{i=1}^{n} max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_A(x_i)|, |F_A(x_i) - F_B(x_i)|\} \quad (3)$$

Definition 2.2.2 [13]. It is well known that similarity measures can be generated from distance measures. Therefore, the distance-based similarity measure based on extended Hausdorff distance between neutrosophic set A and B is defined as follows:

$$S_N(A,B) = 1 - d_N^{EH}(A,B) \quad (4)$$

where $d_N^{EH}(A,B)$ represents the extended Hausdorff distance between neutrosophic set A and B.

Proposition 1. The similarity measure $S_N(A,B)$ for neutrosophic set A and B satisfies the following properties:

(S1) $0 \le S_N(A,B) \le 1$;

(S2) $S_N(A,B) = 1$ if and only if $A = B$;

(S3) $S_N(A,B) = S_N(B,A)$;

(S4) $S_N(A,C) \le S_N(A,B)$ and $S_N(A,C) \le S_N(B,C)$ if C is neutrosophic set in X and $A \subseteq B \subseteq C$.

All proofs for these properties were discussed in [13] and [12].

2.3. Accuracy and roughness approximation

Definition 2.3.1 [10]**.** For a subset of object $X \subseteq U$, the accuracy measure is defined as:

$$\alpha_E(X) = \frac{\left|\underline{apr}_E(X)\right| + \left|(\overline{apr}_E(X))^c\right|}{|U|} \tag{5}$$

where X is a non-empty set, $E \in X$, $\underline{apr}_E(X)$ is a lower approximation of set E, $\overline{apr}_E(X)$ is an upper approximation of set E, $|.|$ denotes the cardinality of a set E, and $0 \leq \alpha_E(X) \leq 1$. Based on the accuracy measure, the roughness measure is defined by:

$$\rho_E(X) = 1 - \alpha_E(X) \tag{6}$$

3. Distance-based similarity measure with roughness approximation

This section introduces a distance-based similarity measure with roughness approximation, where the roughness approximation as in Equations 5 and 6 is defined simultaneously with an extended Hausdorff distance measure. The determination of the roughness measure is defined between a lower and upper approximation of rough neutrosophic set theory instead of the average measurement between them.

3.1. Distance-based Similarity Measure

Assume that A and B be any two rough neutrosophic sets in the universe of discourse U as follows:

$$A = \langle x_j, \begin{matrix} [T_{\underline{N}(A)}(x_j), I_{\underline{N}(A)}(x_j), F_{\underline{N}(A)}(x_j)], \\ [T_{\overline{N}(A)}(x_j), I_{\overline{N}(A)}(x_j), F_{\overline{N}(A)}(x_j)] \end{matrix} | x_j \in U \rangle \text{ and}$$

$$B = \langle x_j, \begin{matrix} [T_{\underline{N}(B)}(x_j), I_{\underline{N}(B)}(x_j), F_{\underline{N}(B)}(x_j)], \\ [T_{\overline{N}(B)}(x_j), I_{\overline{N}(B)}(x_j), F_{\overline{N}(B)}(x_j)] \end{matrix} | x_j \in U \rangle$$

Then, the distance-based similarity measure for RNS A and B is defined as:

Definition 3.1.1: Extended Hausdorff distance with roughness operator is given by

$$d_{RNS}^{EH}(A,B) = \frac{1}{k}\sum_{j=1}^{k} max \left\{ \begin{matrix} |\Delta T_{N(A)}(x_j) - \Delta T_{N(B)}(x_j)|, |\Delta I_{N(A)}(x_j) - \Delta I_{N(B)}(x_j)|, \\ |\Delta F_{N(A)}(x_j) - \Delta F_{N(B)}(x_j)| \end{matrix} \right\} \tag{7}$$

where;

$$\Delta T_{N(A)}(x_j) = 1 - \left(\frac{T_{\underline{N}(A)}(x_j) + \left(T_{\overline{N}(A)}(x_j)\right)^c}{|X|} \right),\ \Delta T_{N(B)}(x_j) = 1 - \left(\frac{T_{\underline{N}(B)}(x_j) + \left(T_{\overline{N}(B)}(x_j)\right)^c}{|X|} \right),$$

$$\Delta I_{N(A)}(x_j) = 1 - \left(\frac{I_{\underline{N}(A)}(x_j) + \left(I_{\overline{N}(A)}(x_j)\right)^c}{|X|} \right),\ \Delta I_{N(B)}(x_j) = 1 - \left(\frac{I_{\underline{N}(B)}(x_j) + \left(I_{\overline{N}(B)}(x_j)\right)^c}{|X|} \right),$$

$$\Delta F_{N(A)}(x_j) = 1 - \left(\frac{F_{\underline{N}(A)}(x_j) + \left(F_{\overline{N}(A)}(x_j)\right)^c}{|X|} \right), \text{ and } \Delta F_{N(B)}(x_j) = 1 - \left(\frac{F_{\underline{N}(B)}(x_j) + \left(F_{\overline{N}(B)}(x_j)\right)^c}{|X|} \right).$$

Here, Δ denotes the "roughness approximation" operator by rough approximation between the lower and upper approximation of RNS, while $|.|$ is the cardinality of the universal *X*.

Proposition 2. The extended Hausdorff distance $d_{RNS}^{EH}(A,B)$ between rough neutrosophic A and B satisfies the following properties:

(D1) $d_{RNS}^{EH}(A,B) \geq 0$. (non-negative)

(D2) $d_{RNS}^{EH}(A,B) = 0$ if and only if $A = B$, for all $A,B \in RNS$. (definiteness)

(D3) $d_{RNS}^{EH}(A,B) = d_{RNS}^{EH}(B,A)$. (symmetry)

(D4) If $A \subseteq B \subseteq C$, for $A,B,C \in RNS$, then $d_{RNS}^{EH}(A,C) \geq d_{RNS}^{EH}(A,B)$ and $d_{RNS}^{EH}(A,C) \geq d_{RNS}^{EH}(B,C)$. (triangle inequality)

Proof:

(D1) $d_{RNS}^{EH}(A,B) \geq 0$.

As $\Delta T_{N(A)}(x_j), \Delta I_{N(A)}(x_j), \Delta F_{N(A)}(x_j) \in [0,1]$, $\Delta T_{N(B)}(x_j), \Delta I_{N(B)}(x_j), \Delta F_{N(B)}(x_j) \in [0,1]$ for all $A,B \in RNS$, the distance measurement based on these functions also lies between $[0,1]$.

(D2) $d_{RNS}^{EH}(A,B) = 0$ if and only if $A = B$, for all $A,B \in RNS$.

For any two RNS A and B, if $A = B$, then the following relations hold for any $\Delta T_{N(A)}(x_j) = \Delta T_{N(B)}(x_j)$, $\Delta I_{N(A)}(x_j) = \Delta I_{N(B)}(x_j)$, $\Delta F_{N(A)}(x_j) = \Delta F_{N(B)}(x_j)$ which states that $|\Delta T_{N(A)}(x_j) - \Delta T_{N(B)}(x_j)| = 0$, $|\Delta I_{N(A)}(x_j) - \Delta I_{N(B)}(x_j)| = 0$, and $|\Delta F_{N(A)}(x_j) - \Delta F_{N(B)}(x_j)| = 0$. Thus, $d_{RNS}^{EH}(A,B) = 0$. Conversely, if $d_{RNS}^{EH}(A,B) = 0$, then the zero distance measure is possible only if $|\Delta T_{N(A)}(x_j) - \Delta T_{N(B)}(x_j)| = 0$, $|\Delta I_{N(A)}(x_j) - \Delta I_{N(B)}(x_j)| = 0$, and $|\Delta F_{N(A)}(x_j) - \Delta F_{N(B)}(x_j)| = 0$. This resulted from $\Delta T_{N(A)}(x_j) = \Delta T_{N(B)}(x_j)$, $\Delta I_{N(A)}(x_j) = \Delta I_{N(B)}(x_j)$, and $\Delta F_{N(A)}(x_j) = \Delta F_{N(B)}(x_j)$ for all i,j values. Hence $A = B$.

(D3) $d_{RNS}^{EH}(A,B) = d_{RNS}^{EH}(B,A)$. The proof is obvious.

(D4) If $A \subseteq B \subseteq C$, for $A,B,C \in RNS$, then $d_{RNS}^{EH}(A,C) \geq d_{RNS}^{EH}(A,B)$ and $d_{RNS}^{EH}(A,C) \geq d_{RNS}^{EH}(B,C)$.

Let $A \subseteq B \subseteq C$, which implies that; $\Delta T_{N(A)}(x_j) \leq \Delta T_{N(B)}(x_j) \leq \Delta T_{N(C)}(x_j)$, $\Delta I_{N(A)}(x_j) \geq \Delta I_{N(B)}(x_j) \geq \Delta I_{N(C)}(x_j)$, $\Delta F_{N(A)}(x_j) \geq \Delta F_{N(B)}(x_j) \geq \Delta F_{N(C)}(x_j)$ for every $x_j \in X$;

Then, we obtain the following relation:

a) $|\Delta T_{N(A)}(x_j) - \Delta T_{N(B)}(x_j)| \leq |\Delta T_{N(A)}(x_j) - \Delta T_{N(C)}(x_j)|$,
$|\Delta T_{N(B)}(x_j) - \Delta T_{N(C)}(x_j)| \leq |\Delta T_{N(A)}(x_j) - \Delta T_{N(C)}(x_j)|$,

b) $|\Delta I_{N(A)}(x_j) - \Delta I_{N(B)}(x_j)| \leq |\Delta I_{N(A)}(x_j) - \Delta I_{N(C)}(x_j)|$,
$|\Delta I_{N(B)}(x_j) - \Delta I_{N(C)}(x_j)| \leq |\Delta I_{N(A)}(x_j) - \Delta I_{N(C)}(x_j)|$,

c) $|\Delta F_{N(A)}(x_j) - \Delta F_{N(B)}(x_j)| \leq |\Delta F_{N(A)}(x_j) - \Delta F_{N(C)}(x_j)|$,
$|\Delta F_{N(B)}(x_j) - \Delta F_{N(C)}(x_j)| \leq |\Delta F_{N(A)}(x_j) - \Delta F_{N(C)}(x_j)|$,

Combining a), b), and (c), we obtain

$\frac{1}{k}\sum_{j=1}^{k} max\{|\Delta T_{N(A)}(x_j) - \Delta T_{N(B)}(x_j)|, |\Delta I_{N(A)}(x_j) - \Delta I_{N(B)}(x_j)|, |\Delta F_{N(A)}(x_j) - \Delta F_{N(B)}(x_j)|\} \leq \frac{1}{k}\sum_{j=1}^{k} max\{|\Delta T_{N(A)}(x_j) - \Delta T_{N(C)}(x_j)|, |I_{N(A)}(x_j) - \Delta I_{N(C)}(x_j)|, |\Delta F_{N(A)}(x_j) - \Delta F_{N(C)}(x_j)|\}$ and $\frac{1}{k}\sum_{j=1}^{k} max\{|\Delta T_{N(B)}(x_j) - \Delta T_{N(C)}(x_j)|, |\Delta I_{N(B)}(x_j) - \Delta I_{N(C)}(x_j)|, |\Delta F_{N(B)}(x_j) - \Delta F_{N(C)}(x_j)|\} \leq \frac{1}{k}\sum_{j=1}^{k} max\{|\Delta T_{N(A)}(x_j) - \Delta T_{N(C)}(x_j)|, |\Delta I_{N(A)}(x_j) - \Delta I_{N(C)}(x_j)|, |\Delta F_{N(A)}(x_j) - \Delta F_{N(C)}(x_j)|\}$. This implies that $d_{RNS}^{EH}(A,B) \leq d_{RNS}^{EH}(A,C)$ and $d_{RNS}^{EH}(B,C) \leq d_{RNS}^{EH}(A,C)$. Thus, the property (D4) is satisfied.

This completes the proof. ∎

4. Methodology

In this study, there are four phases to complete the medical diagnosis findings via distance-based similarity measure of a rough neutrosophic set (RNS).

Phase 1: Collection of data involving the information regarding the symptoms and diseases versus patients and symptoms from the medical report.

In this phase, the data collected on the relationship between symptoms and diseases as well as patients and symptoms are collected from the medical personnel.

Phase 2: Construct the RNS-set for the medical report.

The data collection is converted to RNS-set by using Definitions 2.1.1, as in Equation (1).

Phase 3: The determination of roughness approximation simultaneously with the distance-based similarity measure of RNS-set for medical findings.

RNS-set is used to determine the distance-based similarity measure of the relationship between symptoms and diseases as well as patients and symptoms using Definition 3.1.1 and Equations (2) and (7).

Phase 4: Discussion of a complete medical report.

Lastly, the complete medical report can be written to determine which patient's symptoms and diseases are related. If the distance measure is closer to zero, the conclusion is that the patient possibly suffers from the disease. Meanwhile, for similarity measure, if the measurement is greater than 0.5, then the conclusion is that the patient possibly suffers from the disease. On the other hand, if the similarity measure is less than 0.5, then the conclusion is that the patient may not possibly suffer from the disease.

5. Case Study: Implementation in Medical Diagnosis

In this section, the relationship between symptoms and diseases as well as patients and symptoms are considered in the same equivalence relation. Table 1 shows an example of the medical findings of patients represented in a tabular form. For diagnosis purpose, the patient is kept under supervision for a one-time inspection.

Table 1. Example of a medical finding of a patient

Temperature	Headache	Stomach pain	Cough	Chest pain
High	Yes (moderate)	Yes (moderate)	Yes (high)	Yes (high)

The main feature of this study is to consider the degree of truth membership, indeterminacy membership, and falsity membership for each element between two approximations. The data is adapted from Pramanik and Mondal [6]. Let $P = \{ p_1, p_2, p_3\}$ be a set of patients, $D = \{ d_1, d_2, d_3, d_4\}$ be a set of diseases, and $S = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of symptoms. The relation between patients and symptoms (see Table 2) and the relation between symptoms and diseases (see Table 3) are considered in the same equivalence relation.

Table 2. The relation between patients and symptoms

Relation, A	Temperature (x_1)	Headache (x_2)	Stomach pain (x_3)	Cough (x_4)	Chest pain (x_5)
Patient (p_1)	$\left\langle\begin{matrix}(0.6, 0.4, 0.3),\\(0.8, 0.2, 0.1)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.4, 0.4, 0.4),\\(0.6, 0.2, 0.2)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.5, 0.3, 0.2),\\(0.7, 0.1, 0.2)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.6, 0.2, 0.4),\\(0.8, 0.0, 0.2)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.4, 0.4, 0.4),\\(0.6, 0.2, 0.2)\end{matrix}\right\rangle$
Patient (p_2)	$\left\langle\begin{matrix}(0.5, 0.3, 0.4),\\(0.7, 0.3, 0.2)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.5, 0.5, 0.3),\\(0.7, 0.3, 0.3)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.5, 0.3, 0.4),\\(0.7, 0.1, 0.4)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.5, 0.3, 0.3),\\(0.9, 0.1, 0.3)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.5, 0.3, 0.3),\\(0.7, 0.1, 0.3)\end{matrix}\right\rangle$
Patient (p_3)	$\left\langle\begin{matrix}(0.6, 0.4, 0.4),\\(0.8, 0.2, 0.2)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.5, 0.2, 0.3),\\(0.7, 0.0, 0.1)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.4, 0.3, 0.4),\\(0.8, 0.1, 0.2)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.6, 0.1, 0.4),\\(0.8, 0.1, 0.2)\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(0.5, 0.3, 0.3),\\(0.7, 0.1, 0.1)\end{matrix}\right\rangle$

Table 3. The relation between symptoms and diseases

Relation, B	**Temperature** (x_1)	**Headache** (x_2)	**Stomach pain** (x_3)	**Cough** (x_4)	**Chest pain** (x_5)
Viral fever (d_1)	$\langle(0.6,0.5,0.4),(0.8,0.3,0.2)\rangle$	$\langle(0.5,0.3,0.4),(0.7,0.3,0.2)\rangle$	$\langle(0.2,0.3,0.4),(0.4,0.3,0.2)\rangle$	$\langle(0.4,0.3,0.3),(0.6,0.1,0.1)\rangle$	$\langle(0.2,0.4,0.4),(0.4,0.2,0.2)\rangle$
Malaria (d_2)	$\langle(0.1,0.4,0.4),(0.5,0.2,0.2)\rangle$	$\langle(0.2,0.3,0.4),(0.6,0.3,0.2)\rangle$	$\langle(0.1,0.4,0.4),(0.3,0.2,0.2)\rangle$	$\langle(0.3,0.3,0.3),(0.5,0.1,0.3)\rangle$	$\langle(0.1,0.3,0.3),(0.3,0.1,0.1)\rangle$
Stomach problem (d_3)	$\langle(0.3,0.4,0.4),(0.5,0.2,0.2)\rangle$	$\langle(0.2,0.3,0.3),(0.4,0.1,0.1)\rangle$	$\langle(0.4,0.3,0.4),(0.6,0.1,0.2)\rangle$	$\langle(0.1,0.6,0.6),(0.3,0.4,0.4)\rangle$	$\langle(0.1,0.4,0.4),(0.3,0.2,0.2)\rangle$
Chest problem (d_4)	$\langle(0.2,0.4,0.6),(0.4,0.4,0.4)\rangle$	$\langle(0.1,0.5,0.5),(0.5,0.3,0.3)\rangle$	$\langle(0.1,0.4,0.6),(0.3,0.2,0.4)\rangle$	$\langle(0.5,0.3,0.4),(0.7,0.3,0.2)\rangle$	$\langle(0.4,0.4,0.4),(0.6,0.2,0.2)\rangle$

Based on Pawlak [2], the lower approximation explains that the element set surely belongs to the object, while the upper approximation possibly belongs to the object. For example, based on the data collected in Table 2, the truth membership degree for temperature (x_1) that surely belongs to patient 1 (p_1) is equal to 0.6 and which possibly belongs to patient 1 (p_1) is equal to 0.8. The indeterminacy membership degree for temperature (x_1) that surely belongs to patient 1 (p_1) is equal to 0.4 and which possibly belongs to patient 1 (p_1) is equal to 0.2. Meanwhile, the falsity membership degree for temperature (x_1) which surely belongs to patient 1 (p_1) is equal to 0.3 and which possibly belongs to patient 1 (p_1) is equal to 0.2. The same description is indicated for each data in Table 2 and Table 3.

Next, the determination of roughness approximation simultaneously with the distance-based similarity measurement by extended Hausdorff distance is used to determine the proper medical diagnosis for model RNS for each patient. By using an Equation (2) and roughness operator in Definition 3.1.1, the truth roughness measure for relation A for patient (p_1) is calculated as follows:

$$\Delta T_{N(A)}(x_1) = 1 - \left(\frac{T_{\underline{N}(A)}(x_1) + \left(T_{\overline{N}(A)}(x_1)\right)^C}{|X|}\right) = 1 - \left(\frac{0.6+0.1}{|5|}\right) = 0.86.$$

Then, by using the same equation and definition, the roughness measure for all membership function for each relation A and relation B for patient (p_1), is presented as follows:

$\Delta T_{N(A)}(x_2) = 0.88$, $\Delta T_{N(A)}(x_3) = 0.86, \Delta T_{N(A)}(x_4) = 0.84$, and $\Delta T_{N(A)}(x_5) = 0.88$;
$\Delta I_{N(A)}(x_1) = 0.76, \Delta I_{N(A)}(x_2) = 0.76$, $\Delta I_{N(A)}(x_3) = 0.76, \Delta I_{N(A)}(x_4) = 0.76$, and $\Delta I_{N(A)}(x_5) = 0.76$;
$\Delta F_{N(A)}(x_1) = 0.78, \Delta F_{N(A)}(x_2) = 0.8$, $\Delta F_{N(A)}(x_3) = 0.82, \Delta F_{N(A)}(x_4) = 0.76$, and $\Delta F_{N(A)}(x_5) = 0.8$;
$\Delta T_{N(B)}(x_1) = 0.84, \Delta T_{N(B)}(x_2) = 0.86$, $\Delta T_{N(B)}(x_3) = 0.92, \Delta T_{N(B)}(x_4) = 0.9$, and
$\Delta T_{N(B)}(x_5) = 0.92$; $\Delta I_{N(B)}(x_1) = 0.76, \Delta I_{N(B)}(x_2) = 0.8$, $\Delta I_{N(B)}(x_3) = 0.8, \Delta I_{N(B)}(x_4) = 0.76$, and $\Delta I_{N(B)}(x_5) = 0.76$; $\Delta F_{N(B)}(x_1) = 0.76, \Delta F_{N(B)}(x_2) = 0.8$, $\Delta F_{N(B)}(x_3) = 0.84, \Delta F_{N(B)}(x_4) = 0.82$, and $\Delta F_{N(B)}(x_5) = 0.84$;

Then, simultaneously using Equation (7), the extended Hausdorff distance for medical diagnosis of patient 1 (p_1) with viral fever (d_1) symptom is calculated as follows:

$$d_{RNS}^{EH}(A,B) = \frac{1}{5}\sum_{j=1}^{5} max\left\{\begin{matrix}\left|\Delta T_{N(A)}(x_j) - \Delta T_{N(B)}(x_j)\right|, \left|\Delta I_{N(A)}(x_j) - \Delta I_{N(B)}(x_j)\right|,\\ \left|\Delta F_{N(A)}(x_j) - \Delta F_{N(B)}(x_j)\right|\end{matrix}\right\}$$

$= \frac{1}{5}(max\{|0.86 - 0.84|, |0.76 - 0.76|, |0.78 - 0.76|\} + max\{|0.88 - 0.86|, |0.76 - 0.8|, |0.8 - 0.8|\} + max\{|0.86 - 0.92|, |0.76 - 0.8|, |0.82 - 0.84|\} + max\{|0.84 - 0.9|, |0.76 - 0.76|, |0.76 - 0.82|\} + max\{|0.88 - 0.92|, |0.76 - 0.76|, |0.8 - 0.84|\}) = \frac{1}{5}(0.02 + 0.04 + 0.06 + 0.06 + 0.04) = \frac{1}{5}(0.22) = 0.044.$

Therefore, the extended Hausdorff distance for patient 1 (p_1) with viral fever (d_1) symptoms are 0.044. Then, a similar calculation will be repeated to obtain the result of medial finding for each patient by employing extended Hausdorff distance. The summary result for the proposed extended Hausdorff distance measure with roughness approximation is represented in Table 4.

Table 4. The proposed extended Hausdorff distance measure with roughness approximation

Proposed extended Hausdorff distance	Viral fever (d_1)	Malaria (d_2)	Stomach problem (d_3)	Chest problem (d_4)
Patient (p_1)	0.0440	0.0720	0.0560	**0.0280**
Patient (p_2)	0.0640	0.0960	0.0620	**0.0400**
Patient (p_3)	0.0440	0.0800	0.0600	**0.0360**

According to the result, all the proposed distance measure is close to zero. Here, the closest value to zero indicates the result is possibly "more suffering". Therefore, it shows that all patients are suffering from a chest problem.

By comparing the similarity measure as in Equation (4) with the previous result by Pramanik and Mondal [6] shown in Table 5, we can see that previously all patients were diagnosed with a viral fever. Therefore, a different diagnose result is determined for this study. However, all the similarity measure values are greater than 0.5, indicating that the patients possibly suffer from the disease. The closest similarity value to one indicates the highest possibility of diseases.

Table 5. The Cosine similarity measure and proposed distance-based similarity measure

Cosine similarity measure	Viral fever (d_1)	Malaria (d_2)	Stomach problem (d_3)	Chest problem (d_4)
Patient (p_1)	**0.9595**	0.9114	0.8498	0.8743
Patient (p_2)	**0.9624**	0.9320	0.8935	0.8307
Patient (p_3)	**0.9405**	0.8873	0.8487	0.8372
A proposed distance-based similarity measure				
Patient (p_1)	0.9560	0.9280	0.9440	**0.9720**
Patient (p_2)	0.9360	0.9040	0.9380	**0.9600**
Patient (p_3)	0.9560	0.9200	0.9400	**0.9640**

However, the proposed distance-based similarity result is more accurate since the roughness between the lower and upper approximations of RNS is considered simultaneously with the extended Hausdorff distance instead of only the mean operator between the lower and upper approximation of RNS. Even the other similarity measures led to the same final answer but extended Hausdorff distance shows the simplest and easiest way. Therefore, the chances to obtain the wrong answer are less than other similarity measures.

6. Conclusions

The complete medical diagnosis covered all the relation between the collection of medical information, such as the relationship between patients and symptoms as well as symptoms and diseases. This study successfully examines all the factors needed to complete the medical diagnosis where the distance-based similarity between the medical information is taken over for the first phase. The new notion of roughness approximation for medical information via lower and upper approximations of a rough neutrosophic set is successfully presented. In future work, it is valid to use the same method that involved data with upper and lower approximations. Besides that, distance-based similarity measures by extended Hausdorff distance can be applied in other fields such as the distance for a spatial object in Geographical Information Science (GIS), object recognition for multimedia application, and others.

Funding: "This research received no external funding"

Acknowledgments: Thank you to Universiti Teknologi MARA, Kelantan, for providing financial support. Thank you to our group members for sharing information and giving cooperation to complete this study. We are also grateful to the anonymous reviewer for their insightful comments and suggestions in improving the paper.

Conflicts of Interest: "The authors declare no conflict of interest."

References

1. Pramanik, S.; Mondal, K. Some rough neutrosophic similarity measures and their application to multi-attribute decision making, *Global Jo*, **2015**, *2(7)*, 61–74.
2. Pawlak, Z. Rough sets, *International Journal of Computer & Information Sciences*, **1982**, *11(5)*, 341–356.
3. Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability, In *American Research Press*, **1998**.
4. Zadeh, L. A. Fuzzy sets, *Information and Control,* **1965**, *8(3),* 338-353.
5. Atanassov, K. T. Intuitionistic fuzzy set, *Fuzzy Sets and Systems*, **1983**, *20*, 87–96.
6. Pramanik, S.; Mondal, K. Cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis, *Global Journal of Advanced Research*, **2015**, *2(1)*, 212–220.
7. Pramanik, S.; Roy, R.; Roy, T. K.; Smarandache, F. Multi-attribute decision making based on several trigonometric hamming similarity measures under interval rough neutrosophic environment, *Neutrosophic Sets and Systems Neutrosophic*, **2018**, *19*, 110–118.
8. Mondal, K.; Pramanik, S.; Smarandache, F. Multi-attribute decision making based on rough neutrosophic variational coefficient similarity measure, *Neutrosophic Sets and Systems*, **2016**, *13*, 3–17.
9. Majumdar, P.; Samanta, S. K. On similarity and entropy of neutrosophic sets, *Journal of Intelligent and Fuzzy Systems*, **2013**, *I*, 1–13.
10. Yao, Y.-Y. Notes on rough set approximations and associated measures, *J. Zhejiang Ocean Univ. (Natur. Sci.),* **2010**, *29(5)*, 399–410.
11. Broumi, S.; Smarandache, F.; Dhar, M. Rough neutrosophic sets, *Italian Journal of Pure and Applied Mathematics*, **2014**, *32*, 493–502.
12. Broumi, S.; Smarandache, F. Several similarity measures of neutrosophic sets, *Neutrosophic Sets and Systems*, **2013**, *1(1)*, 54–62.
13. Broumi, S.; Smarandache, F. Extended Hausdorff distance and similarity measures for neutrosophic refined sets and their application in medical diagnosis, *Journal of New Theory*, **2015**, *7*, 64–78.

Received: May 12, 2021. Accepted: October 6, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

A novel lexicographical-based method for trapezoidal neutrosophic linear programming problem

Sapan Kumar Das [1,*] S. A. Edalatpanah [2] and Jatindra Kumar Dash [3]

[1] Department of Revenue, Ministry of Finance, Govt. of India. (e-mail-cool.sapankumar@gmail.com)

[2] *Department of* Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran. (saedalatpanah@gmail.com)

3 Department of Computer Science and Engineering, SRM University-AP, Amravati, India (jkdashsrm@gmail.com)

* Correspondence:Tcool.sapankumar@gmail.com, Tel: +91-9853028740

Abstract: The aim of this paper is to introduce a simplified presentation of a new computing procedure for solving trapezoidal neutrosophic linear programming (TrNLP) problem under uncertainties. Therefore, we firstly define the concept of single valued neutrosophic (SVN) numbers and ranking functions. A new strategy is planned for solving the NLP problem without any ranking function. The planned strategy is depends on multi-objective LP (MOLP) issue and lexicographic order (LO). By following the means of planned strategy, the problem is changed into crisp LP (CLP) problem. In addition to this, a theoretical analysis is provided. Numerical examples are illustrated the proposed method and the consequences are in contrast with the distinct choice methods. The outcome shows that the proposed technique defeats the deficiencies and constraints of the existing method.

Keywords: Neutrosophic trapezoidal numbers; lexicographic method; linear programming, multi-objective linear programming

1. Introduction

Over the last few decades, LP has found numerous successful applications in diverse fields, including Operation Research (O.R), manufacturing, information technology, big science data, energy optimization, and the list goes on. LP has strongly influenced the mathematicians to develop various methods to handle this.

In traditional LP, all the parameters and decision variables are expected to take on exact numerical values. In actuality circumstances, the information is conflicting and undetermined. Due to uncertainty, the decision-maker cannot generally detail the issue in an all around characterized way and careful, nor can they in every case unequivocally anticipate the result of practical choices. For Example: In India, there are three candidates A, B, and C for M.P (Member of Parliament) contested in the election. If the probability is applied for the possible outcome, then the uncertainty can be known. Suppose that A wins is 45%, then there is a chance for loosing 55% too. In case of B, we say that 33% of winning chance, it does not mean that the probability that C wins is 22%, since there may be some NOTA votes (voters reject both all candidates) or not choosing any candidate. However, there is a chance of some error when calculating the possibility. From the above real examples, one may clear that the decision-makers (voters) cannot decide the outcome of the result accurately or precisely, because all the parameters are uncertain and imprecision. Therefore, the fuzzy set principle used to be delivered to handle such type of parameters by decision-makers. Firstly, the fuzzy set was once added by means of Zadeh[31]. The idea of selection making in fuzzy surrounding was proposed by Bellman and Zadeh[32]. Numerous researchers received this idea and stretched out it to take care of the linear programming issue. This problem is called fuzzy linear programming (FLP) problem.

There are two types of problems in the LP problem under a fuzzy environment: (i) symmetric (ii) non-symmetric, which was proposed by Zimmermann [1]. Many researchers [2,9,11-18,33-34] considered the problem of FLP and proposed various methods. The conception of the practical solution and α methodical solutions of the FLP problem was proposed by Ramik [19]. Ghanbari et al. [29] introduced a technique for tackling fuzzy LP issues with crisp formulations of the fuzzy problem. Using the ranking function for tackling the FLP issue was established by Maleki et al. [20]. In the study, Mahadevi-Amiri and Nasiri [13] introduced the duality approach for solving the FLP problem. The concept of sensitivity analysis for solving the FLP problem was proposed by Ebrahimnejad [35]. Jimenez et al. [30] considered the problem of FLP using a ranking function to rank with fuzzy in objectives and to deal with inequality constraints. Wan and Dong [22] considered the possibility of LP issues having trapezoidal fuzzy numbers using multi-objective programming problems and using membership function. A new type of fuzzy symmetric trapezoidal fuzzy number was considered by Ganesan, and Veeramani [21], and the technique was solved in absence of changing to crisp LP problems. Ebrahimnejad and Tavana [4] introduce another technique for tackling FLP issues, and the authors convert the problem into a parallel crisp LP issue and the issue was solved by primal simplex method.

If the parameters, variables, and constraints are taken fuzzy numbers, then it is called fully fuzzy problems, and the linear programming is known as a fully fuzzy LP (FFLP) problem. Lotfi et al. (7) considered an FFLP problem with equality constraints and solved by using lexicographic order (Lo) for ranking symmetric triangular fuzzy numbers. A problem of FFLP with equality constraints and gives a unique solution based on ranking function was proposed by Kumar et al. [6]. Followed by the method [6], a few revisions to make the model well, when all is said in done, was proposed by Najafi and Edalatpanah [8]. In the study, Khan et al. [5] proposed a technique for solving the FFLP problem with triangular fuzzy numbers, and the authors give a solution without transforming them into a classical problem. Dehghan et al. [3] introduced some realistic technique to understand a FF linear system (FFLS) that are related to the common techniques. At that point they broadened another strategy utilizing Linear Programming (LP) for illuminating close and non-close fuzzy frameworks. Veeramani and Duraisamy [10] proposed another methodology of taking care of the FFLP issue utilizing the idea of closest symmetric triangular fuzzy number estimate with save anticipated interim. Ezati et al. [14] put in lexicographic method on fuzzy triangular numbers, and the MOLP issue acquainted another calculation with illuminate FFLP. Das et al. [25] proposed a lexicographic strategy for taking care of FFLP issues with equality and inequality constraints having trapezoidal fuzzy numbers. Das [26] proposed another method for solving the FFLP problem having triangular fuzzy numbers by using lexicographic order (Lo). Ozkok et al. [36] presented a strategy for solving the FFLP problem having mixed constraints.

Due to some drawbacks of the fuzzy set, they can handle only membership function and cannot handle other parameters of vagueness. Therefore, Atanassov [39] established the concept of intuitionistic fuzzy sets (IFS), which is a hybrid of fuzzy sets. They considered both membership and non-membership functions. Here also, some researchers focusing on the use of IFSs in the LP problem; see [53-57].

Still, in practical conditions, it is facing some difficulty in case of decision making due to incomplete information. Therefore, a new set theory was introduced, which contains incomplete, inconsistent, and indeterminate information. This tool is called the neutrosophic set (NS). Neutrosophy was presented by Smarandache [58] as another speculation of fuzzy sets and IFSs. Neutrosophy set might be described by three autonomous degrees, i.e. (i) truth-membership degree (T), (ii) indeterminacy membership degree (I), (iii) falsity membership degree (F).

Wang et al., [52] introduced a single value neutrosophic set (SVNS) problem for solving a practical problem. There are also some scholars [43-46,59] considered the problem of SVNS and applied it in practical problems like the educational and social sectors. The basic definitions and notion of neutrosophic number (NN) was introduced by Samarandache [50]. Some researchers [37,40-42] considered various problems like optimization problems and gave some strategy to solve them. Recently, Abdel-Basset et al. [38] using some ranking functions for the trapezoidal neutrosophic numbers, presented a novel technique for neutrosophic LP. Currently, a direct model for solving the LP problem having triangular neutrosophic numbers was proposed by Edalatpanah [49]. Ye et al. [51] introduced to find the optimal solution of the LP problem in NNs environment.

For the best of our mind, fewer studies have used trapezoidal neutrosophic numbers in LP problems. Recently, an exciting method was proposed by Abdel-Basset et al. [38] for solving neutrosophic LP (NLP) having parameters are represented trapezoidal neutrosophic numbers. Following the method of [38], some modifications was suggested by Singh et al. [60]. The authors have used two ranking functions for both maximization and minimization separately. Now the problem of NP is transformed into a CLP problem and solved by the simplex method.

Using a ranking function in the solution strategies is a weak spot as the use of different ranking functions in a solution method might also result in obtaining different solutions. This weak spot is an inspiration for this investigation to present a solution method that is now not based on any ranking function. For this point, the LP with trapezoidal neutrosophic parameters is converted to a MOLP in which considering all the objective functions together gives an neutrosophic objective function value. By using the LO method, the obtained multi-objective crisp linear programming is changed into single CLP problem. As a preferred position of such methodology, it offers greater flexibility to decision maker. The obtained outcomes from the computational trials of the investigation exhibit the prevalence of the proposed multi-objective optimization method comparing to these of literature.

Contribution:

The main advantage of neutrosophic set is that it's help the decision-makers making by considering truth degree, falsity degree and indeterminacy degree. Here indeterminacy degree is for the most part considered as a free factor which has a significant commitment in decision-making. Due to some uncertainty in real world problem, it is better to use TrNLP problem instead of classical LP problem. For avoiding the unrealistic modeling we used TrNLP model in practical situations. In this article, a TrLP issue is thought of, where all the coefficients are thought to be a trapezoidal neutrosophic numbers. Along these lines, we are proposing a calculation for taking care of TrNLP issue with the assistance of the newly developed LO. To best of our knowledge, it would be the first method to solve the TrNLP problem with help of LO. Thus, for the approval of created technique, direct correlation with relative strategies doesn't emerge. Another Diet outline issue is delineated to show the effectiveness and utilization of our technique, in actuality, issue.

Motivation

Neutrosophic sets plays an important role in uncertainty modeling. The development of uncertainty theory plays a fundamental role in formulation of real-life scientific mathematical model, structural modeling in engineering field, medical diagnoses problem etc. Recently, some of researchers have introduced their model to solve TrNLP problem by using ranking function. How can we implement it in a linear programming based operation research problem? Is it possible to apply in real life problem? Still there is no method for applying LO technique in TrNLP problem. From this aspect we try to extend this research article.

Novelties

In this current decade, researchers have exposed their considerations to make progress with the theories related to neutrosophic area and constantly try to endorse its sufficient scope applications in dissimilar branches of neutrosophic domain. However, justifying all the views connecting to TrNLP theory our main objective is to support the theory efficiently with these following points.

1. Introduced LO function and its efficiency.
2. Application of TrNLP problem.
3. Compared the results with previous established results.

1.1 The rest of the paper is orchestrated in the accompanying way. Some basic definitions and notations are present in Section 2. In Section 3, the general form of FFLP with new method is presented. To show the applications of the proposed method, some real life problem and comparison analysis are discussed in Section 4. In Section 5, advantages of the proposed method over some existing methods are discussed. Finally, the conclusion is been drawn in the last section.

2. Preliminaries

In this area, we present some fundamental definitions and arithmetic operations on neutrosophic sets.

Definition 1 [28]. A set $\tilde{E}_{ne}$ in the universal discourse *X, which is denoted generically by x, is said to be a neutrosophic set if* $\tilde{E}_{ne} = \{< x : [\alpha_{\tilde{E}_{ne}}(x), \beta_{\tilde{E}_{ne}}(x), \delta_{\tilde{E}_{ne}}(x)] >: x \in X\}$. *The set is characterized by a* truth-membership function i.e. degree of confidence: $\alpha_{\tilde{E}_{ne}}(x) : X \to [0,1]$, an indeterminacy membership function i.e. degree of uncertainty: $\beta_{\tilde{E}_{ne}}(x) : X \to [0,1]$ and a falsity-membership function: degree of falsity: $\delta_{\tilde{E}_{ne}}(x) : X \to [0,1]$. SVN satisfies the condition:

$$0 \le \alpha_{\tilde{E}_{ne}}(x) + \beta_{\tilde{E}_{ne}}(x) + \delta_{\tilde{E}_{ne}}(x) \le 3.$$

Definition 2 [28]. For SVNSs *A* and *B*, *A* ⊆ *B* if and only if $\alpha_{\tilde{E}_{ne}A}(x) \le \alpha_{\tilde{E}_{ne}B}(x), \beta_{\tilde{E}_{ne}A}(x) \ge \beta_{\tilde{E}_{ne}B}(x)$ *and* $\delta_{\tilde{E}_{ne}A}(x) \ge \delta_{\tilde{E}_{ne}B}(x)$ for every *x* in *X*.

Definition 3 [38]. A trapezoidal neutrosophic number (TrNNs) is denoted by $\tilde{M}_{ne} =< (p^l, q^l, r^l, s^l), (\mu, \nu, \omega) >$ whose the three membership functions for the truth, indeterminacy, and falsity of x can be defined as follows:

$$\alpha_{\tilde{M}_{ne}}(x) = \begin{cases} \frac{(x - p^l)}{(q^l - p^l)}\mu & p^l \le x \le q^l, \\ \mu & q^l \le x \le r^l, \\ \frac{(s^l - x)}{(s^l - r^l)}\mu & r^l \le x \le s^l, \\ 0 & \textit{otherwise.} \end{cases}$$

$$\beta_{\tilde{M}_{ne}}(x)=\begin{cases}\frac{(q^l-x)}{(q^l-p^l)}\nu, & p^l\le x\le q^l,\\ \nu, & q^l\le x\le r^l,\\ \frac{(s^l-x)}{(s^l-r^l)}\nu, & r^l\le x\le s^l,\\ 1, & otherwise.\end{cases}$$

$$\delta_{\tilde{M}_{ne}}(x)=\begin{cases}\frac{(q^l-x)}{(q^l-p^l)}\omega, & p^l\le x\le q^l,\\ \omega, & q^l\le x\le r^l,\\ \frac{(s^l-x)}{(s^l-r^l)}\omega, & r^l\le x\le s^l,\\ 1, & otherwise.\end{cases}$$

Where, $0\le\alpha_{\tilde{E}_{ne}}(x)+\beta_{\tilde{E}_{ne}}(x)+\delta_{\tilde{E}_{ne}}(x)\le 3, x\in\tilde{M}_{ne}$. Additionally, when $p^l\ge 0,\ \tilde{M}_{ne}$ is called a nonnegative TrNN. Similarly, when $p^l<0,\ \tilde{M}_{ne}$ becomes a negative TrNN.

Definition 4 [38]. Suppose $\tilde{M}_{ne}=<(p_1^a,q_1^a,r_1^a,s_1^a),(\mu_1,\nu_1,\omega_1)>$ and $\tilde{N}_{ne}=<(p_2^a,q_2^a,r_2^a,s_2^a),(\mu_2,\nu_2,\omega_2)>$ be two TNNs. Then the arithmetic relations are defined as:

$$(i)\tilde{M}_{ne}\oplus\tilde{N}_{ne}=<(p_1^a+p_2^a,q_1^a+q_2^a,r_1^a+r_2^a,s_1^a+s_2^a),(\mu_1\wedge\mu_2,\nu_1\vee\nu_2,\omega_1\vee\omega_2)>$$
$$(ii)\tilde{M}_{ne}-\tilde{N}_{ne}=<(p_1^a-s_2^a,q_1^a-r_2^a,r_1^a-q_2^a,s_1^a-p_2^a),(\mu_1\wedge\mu_2,\nu_1\vee\nu_2,\omega_1\vee\omega_2)>$$
$$(iii)\tilde{M}_{ne}\otimes\tilde{N}_{ne}=<(p_1^ap_2^a,q_1^aq_2^a,r_1^ar_2^a,s_1^as_2^a),(\mu_1\wedge\mu_2,\nu_1\vee\nu_2,\omega_1\vee\omega_2)>,\ if\ p_1^a>0,q_1^a>0,$$
$$(iv)\lambda\tilde{M}_{ne}=\begin{cases}<(\lambda p_1^a,\lambda q_1^a,\lambda r_1^a,\lambda s_1^a),(\mu_1,\nu_1,\omega_1)>, & if\ \lambda>0\\ <(\lambda s_1^a,\lambda r_1^a,\lambda q_1^a,\lambda p_1^a),(\mu_1,\nu_1,\omega_1)>, & if\ \lambda<0\end{cases}$$

$$(v)\frac{\tilde{M}_{ne}}{\tilde{N}_{ne}}=\begin{cases}<(\frac{p_1^a}{s_2^a},\frac{q_1^a}{r_2^a},\frac{r_1^a}{q_2^a},\frac{s_1^a}{p_2^a});(\mu_1\wedge\mu_2,\nu_1\vee\nu_2,\omega_1\vee\omega_2)>\ if\ (s_1^a>0,s_2^a>0)\\ <(\frac{s_1^a}{s_2^a},\frac{r_1^a}{r_2^a},\frac{q_1^a}{q_2^a},\frac{p_1^a}{p_2^a});(\mu_1\wedge\mu_2,\nu_1\vee\nu_2,\omega_1\vee\omega_2)>\ if\ (s_1^a<0,s_2^a>0)\\ <(\frac{s_1^a}{p_2^a},\frac{r_1^a}{q_2^a},\frac{q_1^a}{r_2^a},\frac{p_1^a}{s_2^a});(\mu_1\wedge\mu_2,\nu_1\vee\nu_2,\omega_1\vee\omega_2)>\ if\ (s_1^a<0,s_2^a<0)\end{cases}$$

Definition 5. [38] A ranking function of neutrosophic numbers is a function $R:N(R)\to R$, where $N(R)$ is a set of neutrosophic numbers characterized on set of real numbers, which convert each neutrosophic number into the real line.

Let $\tilde{M}_{ne} =< (p_1^a, q_1^a, r_1^a, s_1^a), (\mu_1, \nu_1, \omega_1) >$ and $\tilde{N}_{ne} =< (p_2^a, q_2^a, r_2^a, s_2^a), (\mu_2, \nu_2, \omega_2) >$ be two trapezoidal neutrosophic numbers (TrNN), at that point:

1. If $R(\tilde{M}_{ne}) > R(\tilde{N}_{ne})$ then $\tilde{M}_{ne} > \tilde{N}_{ne}$.
2. If $R(\tilde{M}_{ne}) < R(\tilde{N}_{ne})$ then $\tilde{M}_{ne} < \tilde{N}_{ne}$.
3. If $R(\tilde{M}_{ne}) = R(\tilde{N}_{ne})$ then $\tilde{M}_{ne} = \tilde{N}_{ne}$.

Definition 6 Let $\tilde{M}_{ne} =< (p_1^a, q_1^a, r_1^a, s_1^a), (\mu_1, \nu_1, \omega_1) >$ and $\tilde{N}_{ne} =< (p_2^a, q_2^a, r_2^a, s_2^a), (\mu_2, \nu_2, \omega_2) >$ be any two neutrosophic trapezoidal numbers, then $p_1^a = p_2^a$, $q_1^a = q_2^a$, $r_1^a = r_2^a$, $s_1^a = s_2^a$, $\mu_1 = \mu_2, \nu_1 = \nu_2$, and $\omega_1 = \omega_2$.

3. Proposed method

Consider the standard form of neutrosophic linear programming (NLP) problem with *m* constraints and *n* variables having all coefficients and resources are represented trapezoidal neutrosophic numbers as follows:

$$\begin{aligned} &\text{maximize (minimize) } (\tilde{c}^t y) \\ &\text{s.t} \\ &\qquad \tilde{D}y \le \tilde{h}, \\ &\qquad\ y \ge 0. \end{aligned} \tag{1.1}$$

After all $\tilde{D} = \left[d_{ij} \right]_{m \times n}$ is the coefficient matrix, $\tilde{h} = \left[\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \ldots, \tilde{h}_m \right]^t$ is the trapezoidal neutrosophic available resource vector, $\tilde{c} = \left[\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \ldots, \tilde{c}_n \right]^t$ is the target coefficient and y is the selection variable vector.

Let $\tilde{r}, y^*$ be a possible location and an actual most effective answer of issue (1.1), individually. In the event that there exist an $y' \in \tilde{r}$ so as to fulfils the constraint $Dy \le h$, then, $\tilde{\mathbf{c}}^{\mathbf{T}} y' \ge \tilde{\mathbf{c}}^{\mathbf{T}} y^*$,then y' is also an novel optimal solution of problem (1.1) and is called an substitute exact optimal solution.

The objective function of model (1.1) is a TrNN as $\tilde{z}=\left(\left(\mathbf{c}_1^l\right)^T y,\left(\mathbf{c}_2^l\right)^T y,\left(\mathbf{c}_3^l\right)^T y,\left(\mathbf{c}_4^l\right)^T y;\mu_1 y,v_1 y,\omega_1 y\right)$. This objective function should be maximized as a TrNN. The constraints of model (1.1) is a TrNN consider as: $\tilde{D}=(d_1^l,d_2^l,d_3^l,d_4^l;\mu_d,v_d,\omega_d),\tilde{h}=(h_1^l,h_2^l,h_3^l,h_4^l;\mu_h,v_h,\omega_h)$.

The steps of the proposed method are as follows:

Step 1: By utilizing definition 3 and 4, the issue (1.1) might be composed as by the accompanying multi-objective structure for example

maximize (minimize) $\{(c_1^l)^t y,(c_2^l)^t y,(c_3^l)^t y,(c_4^l)^t y;(\mu_1)y,(v_1)y,(\omega_1 y)\}$ **(2.2)**

s.t

$$\{(d_1^l)y,(d_2^l)y,(d_3^l)y,(d_4^l)y;(\mu_d)y,(v_d)y,(\omega_d)y\}\le\{(h_1^l)y,(h_2^l)y,(h_3^l)y,(h_4^l)y;(\mu_h)y,(v_h)y,(\omega_h)y\}$$

$$y\ge 0.$$

Step-2. Now the issue (2.2) is changed into the accompanying MOLP issue:

$$\min imize\ (\max imize)\, Z_1=\left(\mathbf{c}_2^l\right)^T y-\left(\mathbf{c}_1^l\right)^T y$$

$$\max imize\ (\min imize)\, Z_2=\left(\mathbf{c}_2^l\right)^T y$$

$$\max imize(\min imize)\, Z_3=\frac{1}{2}\left(\left(\mathbf{c}_2^l\right)^T y+\left(\mathbf{c}_3^l\right)^T y\right)$$

$$\max imize(\min imize)\, Z_4=\left(\mathbf{c}_4^l\right)^T y-\left(\mathbf{c}_3^l\right)^T y$$

$$\max imize\ (\min imize)\, Z_5=\left(\mu_1\right)y+\left(\omega_1\right)y$$

$$\max imize\ (\min imize)\, Z_6=\left(v_1\right)y$$

$$\min imize\ (\max imize)\, Z_7=\left(\omega_1\right)y-\left(\mu_1\right)y$$

Subject to (3.3)

$$d_1^l y\le h_1^l$$

$$d_1^l y\le h_2^l$$

$$d_3^l y\le h_3^l$$

$$d_4^l y \le h_4^l$$

$$\mu_d y \le \mu_h$$

$$\nu_d y \le \nu_h$$

$$\omega_d y \le \omega_h$$

$$y \ge \mathbf{0}.$$

Step-3. Presently the issue (3.3) is likewise a MOLP issue. In objective functions, the lexicographic technique will be utilized to get lexicographic optimal solution of issue (3.3), we have:

$$\min imize\ (\max imize)\, Z_1 = \left(\mathbf{c}_2^l\right)^T y - \left(\mathbf{c}_1^l\right)^T y \tag{4.4}$$

Subject to

$$d_1^l y \le h_1^l$$

$$d_1^l y \le h_2^l$$

$$d_3^l y \le h_3^l$$

$$d_4^l y \le h_4^l$$

$$\mu_d y \le \mu_h$$

$$\nu_d y \le \nu_h$$

$$\omega_d y \le \omega_h$$

$$y \ge \mathbf{0}.$$

If (4.4) has a special best solution, at that point it is an ideal arrangement of (2.2). Else ourselves continue to following pace.

Step-4. Tackle the accompanying issue above ideal arrangement that is discovered in step-3 as succeed:

$$\max imize\ (\min imize)\, Z_2 = \left(\mathbf{c}_2^l\right)^T y \tag{5.5}$$

Subject to

$$\left(\mathbf{c}_2^l\right)^T y-\left(\mathbf{c}_1^l\right)^T y=J$$

all constraints of problem (4.4).

Where J is the optimal value of the Problem (4.4). In the event that (5.5) has a novel optimal result, at that point it is an ideal result of (2.2) and stop. In any case go to the following step.

Step-5. Tackle the accompanying issue above the ideal results that are observed in step-4 as succeed:

$$\max imize(\min imize)\frac{1}{2}((c_2^l)^t y+(c_3^l)^t y) \tag{6.6}$$

Subject to

$$(c_2^l)^t y=K$$

$$\left(\mathbf{c}_2^l\right)^T y-\left(\mathbf{c}_1^l\right)^T y=J$$

all constraints of problem (4.4).

where K is the optimal value of the problem (5.5). If (6.6) has a novel optimal solution, at that point it is an ideal result of (2.2) and stop. In any case go to the following step.

Step-6. Tackle the accompanying issue above the ideal results that are resolved in step-5 as succeed:

$$\max imize(\min imize)(c_4^l)^T y-(c_3^l)^T y \tag{7.7}$$

Subject to

$$\frac{1}{2}((c_2^l)^t y+(c_3^l)^t y)=L$$

$$(c_2^l)^t y=K$$

$$\left(\mathbf{c}_2^l\right)^T y-\left(\mathbf{c}_1^l\right)^T y=J$$

$$all\ constra\text{int}\,s\ of\ \ problem\,(4.4).$$

where L is the optimal value of the problem (6.6). If (7.7) has a novel optimal solution, at that point it is an ideal result of (2.2) and end. Else go to following step.

Step-7. Tackle the accompanying issue above the ideal results that are resolved in step-6 as succeed:

$$\max imize(\min imize)\,(\mu_1)y + (\omega_1)y \qquad (8.8)$$

Subject to

$$(c_4^l)^T y - (c_3^l)^T y = M$$
$$\frac{1}{2}((c_2^l)^t y + (c_3^l)^t y) = L$$
$$(c_2^l)^t y = K$$
$$\left(\mathbf{c}_2^l\right)^T y - \left(\mathbf{c}_1^l\right)^T y = J$$
$$all\ constra\text{int}\ s\ of\ \ problem\,(4.4).$$

where M is the optimal value of issue (7.7). If (8.8) has novel optimal solution, at that point it is ideal result of (2.2) and end. In any case go to following step.

Step-8. Tackle the accompanying issue above the ideal solutions that are resolved in step-7 as succeed:

$$\max imize(\min imize)(\nu_1 \qquad (9.9)$$

Subject to

$$(\mu_1)y + (\omega_1)y = N$$
$$(c_4^l)^T y - (c_3^l)^T y = M$$
$$\frac{1}{2}((c_2^l)^t y + (c_3^l)^t y) = L$$
$$(c_2^l)^t y = K$$
$$\left(\mathbf{c}_2^l\right)^T y - \left(\mathbf{c}_1^l\right)^T \mathbf{x} = J$$
$$all\ constra\text{int}\ s\ of\ \ problem\,(4.4).$$

where N is the optimal value of issue (8.8). If (9.9) has a novel optimal solution, at that point it is an ideal result of (2.2) and end. In other case go to the following step.

Step-9. Tackle the accompanying issue above the ideal solutions that are resolved in step-8 as succeed:

$$\min imize(\max imize)\,(\omega_1)y - (\mu_1)y \qquad (10.10)$$

Subject to

$$
\begin{aligned}
&(v_1)y = O \\
&(\mu_1)y + (\omega_1)y = N \\
&(c_4^l)^T y - (c_3^l)^T y = M \\
&\frac{1}{2}((c_2^l)^t y + (c_3^l)^t y) = L \\
&(c_2^l)^t y = K \\
&\left(\mathbf{c}_2^l\right)^T y - \left(\mathbf{c}_1^l\right)^T y = J \\
&all\, constra\operatorname{int} s\, of\;\; problem\,(4.4).
\end{aligned}
$$

where O is the ideal value of issue (9.9).

In Step-9, we get an precise ideal solution which is equal to the issue (2.2).

The stream outline depicts the technique of the proposed strategy as appeared in Fig. 1.

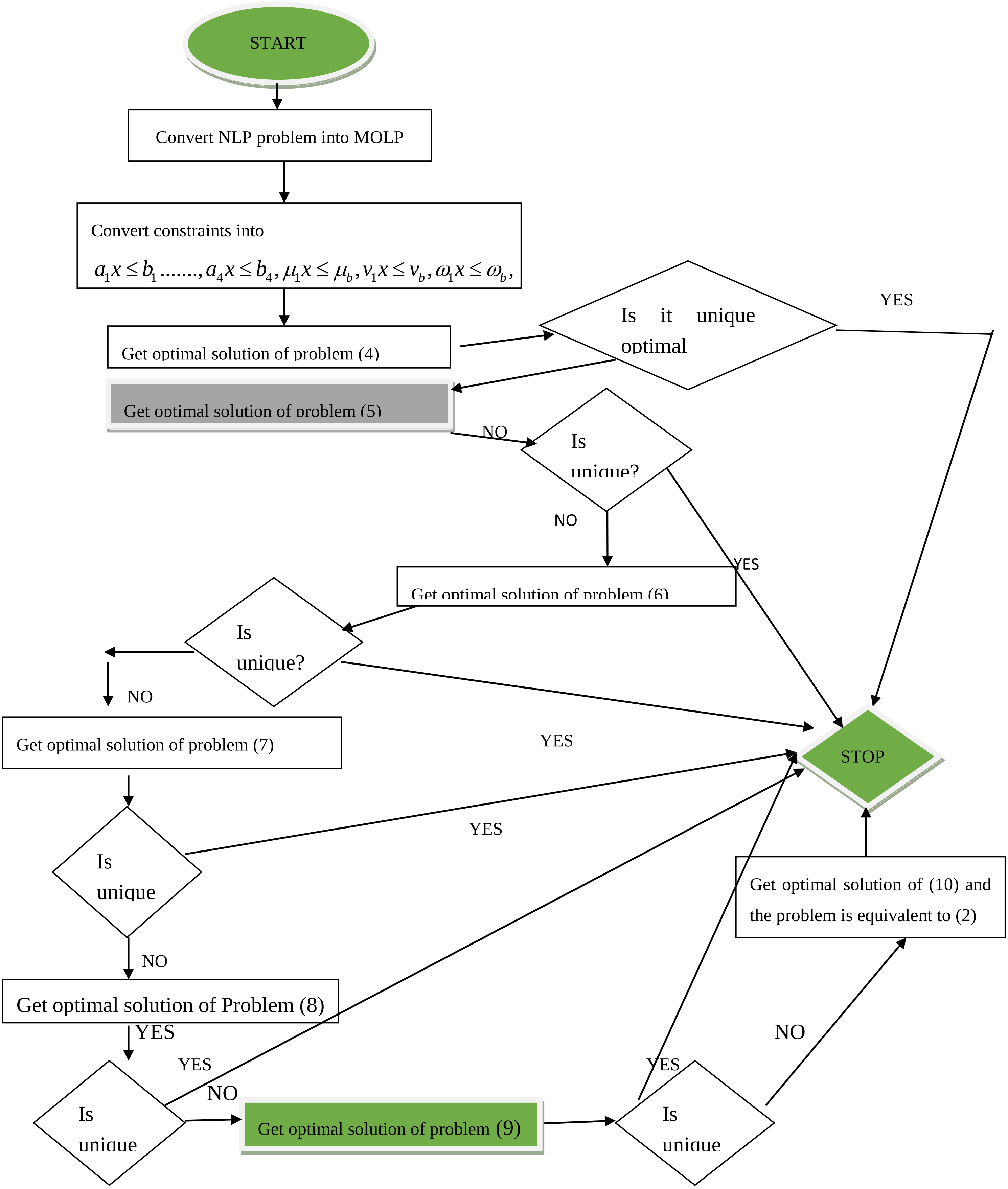


Fig-1. Flowchart depicting the proposed solution method

4. Numerical example

In this area, to demonstrate the pertinence and efficiency of our proposed model of NLP problems, we tackled the issue where the decision-makers always think the truth degree, indeterminacy, and falsity degree. Here the managers not fixed the conformation degree, and the confirmation degree may change as per real-life situation.

Example-1:

The information gathered from a proprietor of a provincial Electrical Cable maker (information is furnished with a legitimate understanding that the title of the organization won't occur unveiled) arranged with Bhubaneswar do appeared in desk-1.

An Electrical Cable maker makes two sorts of cable p1 and p2. These cable comprises of Metal and Plastic (R1, R2) utilized in per unit of cable. The accessibility of cables relies upon its creation however creation relies upon men, machine and so forth of cables are not known precisely because of electricity-failure, employment extra effort, surprising disappointments during instrument and so on. The shipping value of every day provide Metal with Plastic isn't familiar precisely because of varieties in paces of fuel, traffic issues and so on. In this way, all the parameters of the creation organization are unsure amounts with faltering. As per old incident of the proprietor the day by day provides of load is spoken to trapezoidal neutrosophic numbers in desk-1. The normal expense of per meter of p1 and p2 are (1,3,4,7;.8,0.2,0.4) and (4,6,8,10;0.9,0.3,0.5) units, individually. The most extreme every day supplies metals R1 and R2 are around (10,15,20,25;0.6,0.0,0.5) and (10,20,25,300.9,0.45,0.3) units individually. The maker needs to know so as to maximize the expense of Metal and Plastic what number of meters of Cables P1 and p2 he should deliver every day?

Table-1: The statistics of day to day provides of Metal along with Plastic

	Outcomes

Load	Metal (p1)	Plastic (p2)
R1	(2,4,6,8;0.8,0.2,0.4)	(3,5,9,12,0.7,0.2,0.1)
R2	(4,7,10,13;0.7,0.4,0.2)	(3,6,9,14;0.8,0.5,0.3)

Now the issue might be rewritten as

$$\max Z = (1,3,4,7;0.8,0.2,0.4)x_1 + (4,6,8,10;0.9,0.3,0.5)x_2$$

$$s.t.$$

$$(2,4,6,8;0.6,0.1,0.3)x_1 + (3,5,9,12;0.7,0.2,0.1)x_2 \leq (10,15,20,25;0.6,0.0,0.5)$$

$$(4,7,10,13;0.7,0.4,0.2)x_1 + (3,6,9,14;0.8,0.5,0.3)x_2 \leq (10,20,25,3;0.9,0.45,0.3)$$

$$x_1, x_2 \geq 0.$$

By sing Step-1, the problem may be rewritten as;

$$\max Z = (x_1, 3x_1, 4x_1, 7x_1; 0.8x_1, 0.2x_1, 0.4x_1) + (4x_2, 6x_2, 8x_2, 10x_2; 0.9x_2, 0.3x_2, 0.5x_2)$$

$$s.t.$$

$$(2x_1, 4x_1, 6x_1, 8x_1; 0.6x_1, 0.1x_1, 0.3x_1) + (3x_2, 5x_2, 9x_2, 12x_2; 0.7x_2, 0.2x_2, 0.1x_2)x_2$$

$$\leq (10,15,20,25;0.6,0.0,0.5)$$

$$(4x_1, 7x_1, 10x_1, 13x_1; 0.7x_1, 0.4x_1, 0.2x_1) + (3x_2, 6x_2, 9x_2, 14x_2; 0.8x_2, 0.5x_2, 0.3x_2)$$

$$\leq (10,20,25,30;0.9,0.45,0.3)$$

$$x_1, x_2 \geq 0.$$

Found on Step-2, the above model may be changed into MOLP model

$$\begin{aligned}
&\min Z_1 = 2x_1 + 2x_2 \\
&\max Z_2 = 3x_1 + 6x_2 \\
&\max Z_3 = 5.5x_1 + 9x_2 \\
&\max Z_4 = 3x_1 + 2x_2 \\
&\max Z_5 = 0.8x_1 \\
&\max Z_6 = 0.8x_1 - 0.4x_2 \\
&\min Z_7 = 0.8x_1 + 0.4x_2 \\
&s.t. \\
&\quad 2x_1 + 3x_2 \le 10 \\
&\quad 4x_1 + 5x_2 \le 15 \\
&\quad 6x_1 + 9x_2 \le 20 \\
&\quad 8x_1 + 25x_2 \le 25 \\
&\quad 0.6x_1 \le 0.6 \\
&\quad 0.2x_2 \le 0 \\
&\quad 0.3x_1 \le 0.5 \\
&\quad 4x_1 + 3x_2 \le 10 \\
&\quad 7x_1 + 6x_2 \le 20 \\
&\quad 10x_1 + 9x_2 \le 25 \\
&\quad 13x_1 + 14x_2 \le 30 \\
&\quad 0.7x_1 \le 0.9 \\
&\quad 0.5x_2 \le 0.45 \\
&\quad 0.3x_2 \le 0.3 \\
&\quad x_1, x_2 \ge 0.
\end{aligned} \qquad \textbf{(11)}$$

Using step3 to Step-9, the ideal answer of the problem is done as follows:

As a multi-objective formulation, the model (11) was solved by the proposed solution approach (steps 3-9). The obtained results are as : $x_1 = 1.25, x_2 = 0$ and the objective solution is: $Z = 3$.

The below table express the quality phase of our proposed method is that it off ers new perfect cost regards as differentiated and the present existing system. This is showed up in Table 2(Numerical assessment with existing procedures) exc lusively.

Table-2. Comparison of the proposed method with existing methods [38]

Approach	Optimal solution	Crisp objective value	Neutrosophic optimal value
Proposed Method	$x_1 = 1.25, x_2 = 0$	$Z = 3$	(1.25,3.75,5,8.75;0.8,0.2,0.4)
Existing Method [38]	$x_1 = 1.523, x_2 = 0$	$Z = 2.75$	(1.523,4.569,6.092,10.661;1,0,0)

Example-2

In this section, we consider the symmetric trapezoidal numbers in form of $(a^l, a^u, \alpha, \alpha)$. Where $(a^l, a^u, \alpha, \alpha)$ represented the lower, upper bound and first, second median value of trapezoidal number respectively. Additionally, here we consider the confirmation degree is (1,0,0). We shows the applicability of our proposed method, we consider the problem of Das et al. [25], Ganesan and Veeramani [21].

$$
\begin{aligned}
&\max Z = (13,15,2,2)x_1 + (12,14,3,3)x_2 + (15,17,2,2)x_3 \\
&s.t. \\
&\quad (11,13,2,2)x_1 + (12,14,1,1)x_2 + (11,13,2,2)x_3 \le (475,505,6,6) \\
&\quad (12,16,1,1)x_1 + (12,14,1,1)x_3 \le (460,480,8,8) \\
&\quad (11,13,2,2)x_1 + (14,16,3,3)x_2 \le (465,495,5,5) \\
&\qquad x_1, x_2, x_3 \ge 0.
\end{aligned}
$$

By using our proposed method (Steps 1 to 9), we get the results of Table-3.

Table-3. Comparison of the proposed method with existing methods [21,25,38]

Approach	Optimal solution	Crisp objective value	Neutrosophic optimal value
Proposed Method	$x_1 = 0, x_2 = 4.44, x_3 = 37.29$	$Z = 838.21$	(612.63,696.09,87.9,87.9;1,0,0)
Existing Method	$x_1 = 0, x_2 = 4.11, x_3 = 37.38$	$Z = 825.71$	(610.02,693,87.09,87.09;1,0,0)

[38]			
Existing method [25]	$x_1 = 0, x_2 = 4.23, x_3 = 34.28$	$Z = 622.97$	(564.96,680.98,80.98,80.98)

4. Result Analysis

At first, we examined the example-1, we compare our result with existing method Abdel-Basset[38], we conclude that,

1. In our method, the objective values equal to 3 and the existing method [38], the optimal value s 2.75. As the problem is maximization, therefore, our problem is better than the other method.
2. In our method, we do not use the slack variables in constraints. However, the existing method the authors have used the slack variables and solved in the simplex method.

For example-2, we consider the problem which was proposed by Das et al. [25] on the trapezoidal fuzzy number and Abdel-basset [38], and we compare it with our proposed method having trapezoidal neutrosophic numbers.

3. From table-3, it is clear that our objective values is maximized and equal to 838.21 as the problem is maximized given. In our comparison, neutrosophic is better handling than fuzzy in real-life situations.
4. In our lexicographic method is better than the ranking function of Abdel-Basset [38] method.The results are supported by the fact that the existing method [38] use a single ranking function, in this case the optimization criterion, whi ch can not guarantee the feasibility of the solution. However, proposed me thod in this paper is more concentrate regarding the indeterminacy and co nvert to MOLP problem by utilizing the LO.
5. In the existing method [38], the authors considered two types of ranking functions for handling different types of NLP problems. However, in our proposed method, we

propose only one type of method called the lexicographic method and that method can handle any type of NLP problem.

From the above Table-2 and Table-3, one can conclude that the optimum value of NLP problem is higher side of the present methods. Therefore, we can conclude that our proposed algorithm is a new way to handle the problem and its more effective.

All the problems are solved by LINGO version 18.0. Therefore, from the above real-life problem and the above discussion, we can conclude that our proposed m ethod ismore robust than the method proposed in [21,25,38].

4.1 Advantages and limitations of the proposed method

Here, in this paper, we proposed a new technique for trapezoidal neutrosophic fuzzy numbers based on lexicographic technique orders and the significant advantages of the proposed measure are given as follows.

- Trapezoidal fuzzy neutrosophic number is a simple design of arithmetic operations and easy and perceptive interpretation as well. Therefore the proposed measure is an easy and effective one under neutrosophic environment.
- Lexicographic Orders can be estimated with simple algorithm and significant level of accuracy can be acquired as well.
- While taking the inequality constraints convert to equality constraints we used LO technique instead of any slack variables.
- Also it can be applied in location planning, operations management, Neutrosophic Statistics,clustering, medical diagnosis, Optimization and image processing to get more accurate results without any computational complexity.

Limitations

- While used LO technique to convert multi-objective LP problem the numbers of constraints are bigger than original LP problem.
- Very slow response while applied in linear fractional problem and quadratic programming problem.

5.Conclusion

In a real-world environment, we handle imprecise, vague, and insufficient information by using the neutrosophic set. In this paper, we considered an NLP problem having trapezoidal neutrosophic numbers and transformed it into a MOLP problem. Based on the LO method, we solve the MOLP problem corresponding to the linear programming problem. It is believed that our method for the solution of NLP problem in the application of practical issue along with the simple issue might be adopted by scholars who are working in this field. Meantime, a numerical example was provided to show the efficiency of the proposed method and illustrate the solution process. The new model not only richens uncertain linear programming methods but also provides a new effective way for handling indeterminate optimization problems. Further, comparative analysis has been done with the existing methods to show the potential of the proposed LO method and various forms of trapezoidal fuzzy neutrosophic number have been listed and shown the uniqueness of the proposed tabular representation. Furthermore, advantages of the proposed measure are given. In future, the present work may be extended to other special types of neutrosophic set like pentagonal neutrosophic set, neutrosophic rough set, interval valued neutrosophic set and plithogeneic environments.

References

[1]. Zimmermann, H.-J., Fuzzy set theory and its applications, 2011:Springer Science & Business Media.

[2]. Baykasoğlu, A. and K. Subulan, An analysis of fully fuzzy linear programmi ng with fuzzy decision variables through logistics network design problem. Knowledge-Based Systems, 2015. **90**: p. 165-184.

[3]. Dehghan, M., B. Hashemi, and M. Ghatee, Computational methods for solvin g fully fuzzy linear systems. Applied Mathematics and Computation, 2006. **179**(1): p. 328-343.

[4]. Ebrahimnejad, A. and M. Tavana, A novel method for solving linear progra mming problems with symmetric trapezoidal fuzzy numbers. Applied Mathe matical Modelling, 2014. **38**(17): p. 4388-4395.

[5]. Khan, I.U., T. Ahmad, and N. Maan, A simplified novel technique for solvi ng fully fuzzy linear programming problems. Journal of Optimization Theor y and Applications, 2013. **159**(2): p. 536-546.

[6]. Kumar, A., J. Kaur, and P. Singh, A new method for solving fully fuzzy li near programming problems. Applied Mathematical Modelling, 2011. **35**(2): p. 817-823.

[7]. Lotfi, F.H., et al., Solving a full fuzzy linear programming using lexicograph y method and fuzzy approximate solution. Applied Mathematical Modelling, 2009. **33**(7): p. 3151-3156.

[8]. Saberi Najafi, H. and S.A. Edalatpanah, A Note on “A new method for sol ving fully fuzzy linear programming problems”. Applied Mathematical Mod elling, 2013. **37**(14): p. 7865-7867.

[9]. Shamooshaki, M., A. Hosseinzadeh, and S.A. Edalatpanah, A New Method f or Solving Fully Fuzzy Linear Programming with LR-type Fuzzy Numbers. International Journal of Data Envelopment Analysis and* Operations Resea rch*, 2014. **1**(3): p. 53-55.

[10]. Veeramani, C. and C. Duraisamy, Solving fuzzy linear programming proble m using symmetric fuzzy number approximation. International Journal of O perational Research, 2012. **15**(3): p. 321-336.

[11]. Wu, H.-C., Using the technique of scalarization to solve the multiobjective programming problems with fuzzy coefficients. Mathematical and Computer Modelling, 2008. **48**(1): p. 232-248.

[12]. Iskander, M., Using different dominance criteria in stochastic fuzzy linear m ultiobjective programming: A case of fuzzy weighted objective function. M athematical and computer modelling, 2003. **37**(1): p. 167-176.

[13]. Mahdavi-Amiri, N. and S. Nasseri, Duality in fuzzy number linear program ming by use of a certain linear ranking function. Applied Mathematics and Computation, 2006. **180**(1): p. 206-216.

[14]. Ezzati, R., E. Khorram, and R. Enayati, A new algorithm to solve fully fu zzy linear programming problems using the MOLP problem. Applied Math ematical Modelling, 2015. **39**(12): p. 3183–3193.

[15]. Dubois, D.J., Fuzzy sets and systems: theory and applications. Vol. 144. 19 80: Academic press.

[16]. Kauffman, A. and M.M. Gupta, Introduction to fuzzy arithmetic: theory and application. VanNostrand Reinhold, New York, 1991.

[17]. Liou, T.-S. and M.J.J. Wang, Ranking fuzzy numbers with integral value. F uzzy sets and systems, 1992. **50**(3): p. 247-255.

[18]. S. Das and T. Mandal, A single stage single constraints linear fractional pr ogramming problem: An approach,Opera. Rese. and Appl.: An Inter. J. 2 (2015) 1-5.

[19]. Ramik J (2005) Duality in fuzzy linear programming: some new concepts a nd results. Fuzzy Optim Decis Making 4:25–39

[20]. Maleki HR, Tata M, Mashinchi M (2000) Linear programming with fuzzy variables. Fuzzy Sets Syst 109:21–33

[21]. Ganesan K, Veeramani P (2006) Fuzzy linear programs with trapezoidal fuz zy numbers. Ann Oper Res 143:305–315.

[22]. Wan SP, Dong JY (2014) Possibility linear programming with trapezoidal f uzzy numbers. Appl Math Model 38:1660–1672.

[23]. J.J. Buckley, T. Feuring, Evolutionary algorithm solution to fuzzy problems: fuzzy linear programming, Fuzzy Sets and Systems, 2000, 109,35-53.

[24]. S Das, S.A. Edalatpanah and T. Mandal, A New Approach For Solving Fu lly Fuzzy Linear Fractional Programming Problems Using The Multi-Objecti ve Linear Programming, RAIRO-Operation Research, 2017, 51, 285-297.

[25]. S. Das, T. Mandal, S.A. Edalatpanah,A mathematical model for solving full y fuzzy linear programming problem with trapezoidal fuzzy numbers, Appli ed Intelligence, 2016, 46,509- 519.

[26]. S. Das, Modified method for solving fully fuzzy linear programming proble m with triangular fuzzy numbers, International Journal of research in indus trial engineering, 6 (2017), 293-311.

[27]. S. Das, T. Mandal, D. Behera, A new approach for solving fully fuzzy lin ear programming problem, International journal of mathematics in operation research, 15(2019), 296-309.

[28]. S. Das, J. K. Dash, Modified solution for Neutrosophic Linear Programming Problems with mixed constraints, International Journal of Research in Ind ustrial Engineering**. 9(2020), 13-24**.

[29]. R. Ghanbari, K. Ghorbani-Moghadam, N. Mahdavi Amiri, B. De Baets, Fuzzy linear programming problems: models and solut ions, Soft Computing, doi.org/10.1007/s00500-019-04519-w.

[30]. Jiménez, M., Arenas, M., Bilbao, A. and Rodríguez, M.V. (2007) 'Linear p rogramming with fuzzy parameters: an interactive method resolution', Europ ean Journal of Operational Research, Vol. 177, No. 3, pp.1599–1609.

[31]. L.A.Zadeh, Fuzzy Sets, Inf. And Cont., 1965, 8, 338-353.

[32]. R.E.Bellman, L.A. Zadeh, Decision making in a fuzzy environment, Manag. Sci. 17(1970) 141–164.

[33]. Edalatpanah, S. A., and S. Shahabi. "A new twophase method for the fuzz y primal simplex algorithm." International Review of Pure and Applied Ma thematics 8, no. 2 (2012): 157-164.

[34]. N. MahdaviAmiri, S.H. Nasseri, Duality in fuzzy number linear programmin g by use of a certain linear ranking function, Appl. Math. Comput. 180(2 006) 206–216.

[35]. A. Ebrahimnejad, Sensitivity analysis in fuzzy number linear programming p roblems,Math. Comput. Model. 53 (2011) 1878–1888.

[36]. B.A. Ozkok, I. Albayrak, H.G. Kocken and M. Ahlatcioglu, An approach f or finding fuzzy optimal and approximate fuzzy optimal solution of fully f uzzy linear programming problems with mixed constraints, Journal of Intell igent & Fuzzy Systems 31(1) (2016), 623–632.

[37]. I. Maiti, T. Mandal, S. Pramanik, Neutrosophic goal programming strategy f or multi-level multi-objective linear programming problem, Journal of Ambi ent Intelligence and Humanized Computing, 2019, 1-12.

[38]. M. AbdelBasset, M. Gunasekaran, M. Mohamed, & F. Smarandache, A nov el method for solving the fully neutrosophic linear programming problems. Neural Computi and Applications,2019, 31, 1595-1605.

[39]. KT. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst, 1986, 20, 87-96.

[40] S. Pramanik, PP Dey, Multi-level linear programming problem with neutrosophic numbers: A goal programmi ng strategy, Neutrosophic Sets Syst., 2019,29.

[41] S. Pramanik,PP. Dey, Bilevel linear programming problem with neutrosophic numbers, Neutrosophic Sets Syst,2018, 21, 110-121.

[42] S. Pramanik, D. Banerjee, Neutrosophic number goal programming for multi-objective linear programming problem in neutrosophic number environment, MOJ Res Rev,2018, 1, 135-141.

[43] P. Biswas, S. Pramanik, BC Giri, Entropy based grey relational analysis met hod for multiattribute decision making under single valued neutrosophic ass essments, Neutrosophic Sets System, 2014, 2, 105-113.

[44] P. Biswas, S. Pramanik, BC Giri, A new methodology for neutrosophic mul ti-attribute decision making with unknown weight information, Neutrosophic Sets System, 2014, 3, 42-50.

[45] S. Broumi, F. Smarandache, Single valued neutrosophic trapezoid linguistic a ggregation operators based multiattibute decision making, Bull PurApplSci Math Stat, 2014, 33, 135-155

[46] P.Biswas,S.Pramanik & B.C. Giri, TOPSIS method for multiattribute group d ecision making under singlevalued neutrosophic environment, Neural comput ing and Applications, 2018, 27,727-737.

[47]. M. Darehmiraki, A solution for the neutrosophic linear programming proble m with a new ranking function, Optimization theory based on Neutrosophi c and Plithogenic Sets,(2020).

[48]. A. Nafei et al, A new method for solving interval neutrosophic linear prog ramming problems,ActaPolytechnica Hungarica,2020, 1-17.

[49]. S.A.Edalatpanah, A Direct Model for Triangular Neutrosophic Linear Progra mming. International Journal of Neutrosophic Science,2020, 1, 15-24.

[50]. F. Smarandache, Introduction of neutrosophic statistics, Sitech and Education Publisher, Craivoa 2013.

[51]. J.Ye, Neutrosophic number linear programming method and its application u nder neutrosophic number environments, Soft Computing, 2018,22, 4639-464 6.

[52]. H. wang, F. Smarandache, YQ Zhang, R. Sunderraman, Single valued neutr osophic sets, Multispace and Multistruct, 2010,4, 410-413.

[53]. Parvathi, R., Malathi, C. (2012.a). Intuitionistic fuzzy linear optimization. No tes on Intuitionistic Fuzzy Sets, 18, 48–56.

[54]. Parvathi, R., Malathi, C. (2012.b). Intuitionistic fuzzy linear programming pr oblems. World applied Sciences Journal, 17, 1802–1807.

[55]. Dubey, D., Mehra, A. (2011). Linear programming with triangular intuitionis tic fuzzy numbers. Advances in Intelligent Systems Research, 1, 563–569.

[56]. Bharati SK, Singh SR (2015) A note on solving a fully intuitionistic fuzzy linear programming problem based on sign distance. Int J Comput Appl 119:30–35.

[57]. Bharati SK, Singh SR (2014) Solving multi objective linear programming pr oblems using intuitionistic fuzzy optimization method: a comparative study. Int J Model Optim 4:10.

[58]. Smarandache, F.(1998). Neutrosophy: Neutrosophic Probability, Set, and Logi c; American Research Press: Rehoboth, MA, USA.

[59]. Ji, P., Wang, J. Q., & Zhang, H. Y. (2018). Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party lo gistics providers. Neural Computing and Applications, 30(3), 799-823.

[60]. Singh, A., Kumar, A., & Appadoo, S. S. A novel method for solving the fully neutrosophic linear programming problems: Suggested modifications. Jo urnal of Intelligent & Fuzzy Systems, (Preprint), 1-12.

[61]. Wei Yang, Lulu Cai, S.A. Edalatpanah and F. Smarandache,(2020) Triangula r single neutrosophic data envelopment analysis: Application to hospital per formance management, Symmetry, 12,588-601.

[62]. S.Das, S.A.Edalatpanah,T.Mandal,(2020) Application of linear fractional programming problem with fuzzy nature in industry sector. FILOMAT, 34, 1-16.

[63] S. Das, (2020) Application of transportation problem under pentagonal neutrosophic environment, 1, 27-41.

Received: May 7, 2021. Accepted: October 3, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

Quadripartitioned Single Valued Neutrosophic Pythagorean Dombi Aggregate Operators in MCDM Problems

Somen Debnath[1,*]

[1]Department of Mathematics, Umakanta Academy, Agartala-799001, Tripura, India

*Correspondence: somen008@rediffmail.com; Tel.: (+918787301661)

Abstract: The quadripartitioned single-valued neutrosophic set (QSVNS) is developed to understand the concept of indeterminacy more clearly. It takes care of the diverse approaches while dealing with uncertainty under single-valued neutrosophic environment. To make the QSVNS more functional and logical, the notion of quadripartioned single-valued neutrosophic Pythagorean set (QSVNPS) is introduced. In QSVNPS, the components T, C, U, F are dependent in such a manner that $T+F \leq 1$, $C+U \leq 1$, and $T^2+C^2+U^2+F^2 \leq 2$. So, the QSVNPS is a powerful framework for modeling the imprecise human knowledge in a specific manner. To calculate the arithmetic operations, we consider the quadripartitioned single-valued neutrosophic Pythagorean numbers (QSVNPNs) associated with QSVNPSs. The main advantage of using QSVNPNs is that it allows the decision-makers to carry out the calculation on uncertain parameters. The present paper aims to study Dombi operators and to establish some new Dombi weight aggregate operators and develop some properties under QSVNPN environment for solving multi-criteria decision-making(MCDM) problems that we encounter in our day-to-day life process. Then we define the score and accuracy functions for ranking the QSVNPNs to choose the best-preferred alternative that goes through under a set of certain criteria. A model for MCDM problems based on Dombi operators under QSVNPNs has been introduced. To check the feasibility of the new approach, a numerical example is demonstrated that shows the effectiveness of the proposed model for multi-criteria decision-making. Finally, a comparative analysis between the rankings, obtained by using the proposed model, of the given set of alternatives under a certain set of criteria gives the optimal choice.

Keywords: Pythagorean fuzzy set; Quadripartitioned neutrosophic Pythagorean set; Dombi operator; MCDM.

1. Introduction

The introduction of the fuzzy set(FS)[1] proposed by Zadeh is a conceptual framework to be formed by replacing the two-valued characteristic function with the fuzzy membership function to define the imprecise information that we encounter in physical world phenomenon. It has the rich potential to address ambiguous issues. Due to the novelty of the FS, it has a wide range of applicability in information communication, pattern recognition, artificial intelligence, operation research, medical diagnosis, computer science, game theory, economics, environmental science, engineering, robotics, etc. In FS, every object of the universe is characterized by a membership function and the degree of membership is ranging between 0 and 1. Some contributed works related to fuzzy sets are proposed in the literature given in [2-6]. Later on, after critical investigation, it has been identified by the researchers that, the concept of hesitancy that is natural in human thinking cannot be described by FS due to its inherent difficulty. So, there is an information gap in FS and to eradicate such gap a new mathematical structure called intuitionistic fuzzy set (IFS) [7, 8] is introduced by adding a non-membership degree to the FS. In IFS, every object of the universe has a membership and a non-membership degree, and their sum cannot exceed 1. The hesitancy membership can be obtained by subtracting the sum of the membership and the non-membership degree from 1. Thus, the IFS provides incomplete information to the decision-maker and it can be viewed as an extension of FS. IFS can be reduced to an FS when its non-membership degree is 0. Some recent works on IFS are proposed in the literature given in [9-12].

However, researchers find the existence of an environment which cannot be addressed by FS and IFS. For example, suppose under a certain environment, the membership and the non-membership degrees of an object provided to the decision-maker are 0.5 and 0.7. Then, their sum is $0.5+0.7=1.2>1$. So, this type of phenomenon cannot be defined by FS and IFS. We need another powerful tool that can easily solve this problem. For the demand of the situation, the Pythagorean fuzzy set (PFS) [13] is introduced where the sum of the squares of the membership and the non-membership degree cannot exceed 1. With the help of PFS, the above problem can be easily defined, as $0.5^2+0.7^2=0.74<1$. Therefore, the Pythagorean environment is capable to accommodate both the fuzzy and the intuitionistic fuzzy environment to solve diverse problems. In PFS, with the help of Pythagorean fuzzy membership and non-membership function, we can enhance the applicability of FS and IFS. Different authors use PFS from different angles and develop some new operators which are capable to solve real decision-making problems. Some significant works related to PFSs are studied in the proposed literature given in [14-18].

There is another aspect of extending the notion of IFS and it is due to the unavailability of indeterminacy membership in IFS. The concept of indeterminacy is found relevant in human thinking. For example, in a certain class, a teacher asks a true-false type question to a group of 30 students in that class. The response of the students recorded as 20 says true, 5 say false and according to the remaining 5 students, it is neither true nor false. This gives a clear idea about how indeterminacy exists in our communication information. In 2005, Smarandache[19]introduced neutrosophic set(NS) as a generalization of the crisp set, FS, IFS, paraconsistent set, PFS, etc. Later on, for scientific and technical application, Wang et al. [20] introduced a single-valued neutrosophic set (SVNS) to develop the operators used in NS. In SVNS, each object of the universe is characterized by the three membership functions called truth-membership, indeterminacy-membership, and falsity-membership function in such a way that the sum of the three membership values cannot exceed 3. So, the SVNS is enabled to take care of the issues that contain uncertainty that contains indeterminacy. In [21], Jansi et al. introduced the correlation measures for Pythagorean neutrosophic sets (PNSs) with truth and falsity as dependent components. Furthermore, sometimes while working with NS, there is a doubt in the mind of the researchers that the indeterminacy to an element occurs due to the belongingness or non-belongingness. Such issues were presented by Chatterjee et al. [22] by introducing a quadripartitioned single-valued neutrosophic set (QSVNS). The QSVNS is a more generalized framework than SVNS and the motivation behind adopting this idea is due to Smarandache's four-valued neutrosophic logic and Belnap's four-valued logic. In QSNS, the indeterminacy component is being divided into two parts, namely, contradiction and unknown. By combining QSVNS and PFS, Radha et al. [23] introduced a new model known as a quadripartitioned single-valued neutrosophic Pythagorean set (QSVNPS). In QSVNPS, truth and falsity make one pair of a dependent component on the other hand contradiction and an unknown or ignorance make another pair of dependent components. Therefore, it looks quite logical to apply Pythagorean property on QSVNS.

The relationship of different types of sets in the context of the proposed study is exhibited in the form of an arrow diagram given as:

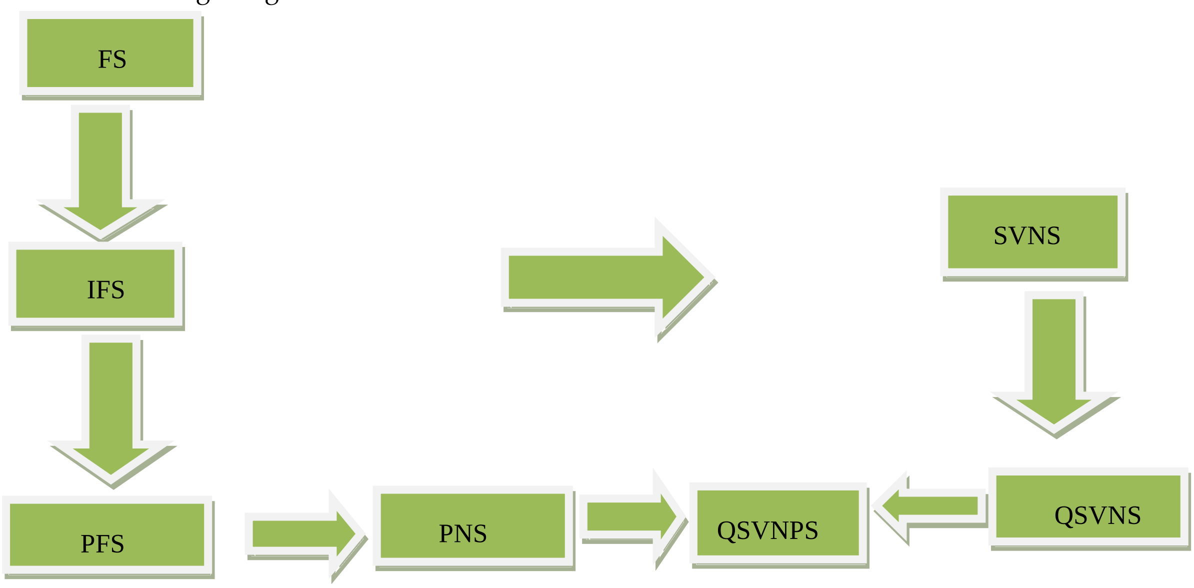


Fig 1. Arrow diagram to represent the relationship of different types of sets

Dombi operators have an excellent potential with operational parameters and due to these operational parameters, it is flexible to operate. In 1982, Dombi[24] introduced triangular t-norm and t-conorm operators. Roychoudhury et al. [25] generalize the Dombi class, intuitionistic fuzzy Dombi aggregation operator and their application to MADM proposed in [26], Dombi prioritized weighted aggregation operator on single-valued neutrosophic set for MADM is given by Wei et al. [27], Jana et al.[28]presented the bipolar fuzzy Dombi aggregation operators, Ashraf et al. [29] introduced the spherical fuzzy Dombi aggregation operator and their application in GDM, Garg et al. [30] initiated the neutrality operations based Pythagorean fuzzy aggregation operators and its application to MAGDM, Qiyas et al. [31] defined linguistic picture fuzzy aggregation operator. Furthermore, the Pythagorean Dombi fuzzy aggregation operator presented by Akram et al. [32], Khan et al. [33]

introduced the Pythagorean fuzzy Dombi aggregation operators and their application in decision support system, Jana et al. [34] described the Pythagorean fuzzy Dombi aggregation operators and their application in MADM, Akram et al. [35] extended the Dombi aggregation operator for DM under m-polar fuzzy information, bipolar neutrosophic Dombi aggregation operators with application in MADM problems are introduced by Mahmood et al. [36], etc.

Over the last decades, decision-making (DM) is an effective scientific approach for making decisions by assessing a set of alternatives and achieve the best results. There are various DM approaches or strategies that help to choose the optimal choice. In real-life scenarios, uncertainty plays an important role in decision-making and it captures considerable attention in various research areas. For getting more information about decision-making we discuss the following: correlation coefficient based TOPSIS method under interval-valued intuitionistic fuzzy soft environment and their aggregate operators for DM are defined in [37]. In [38], Zulqarnain et al. presented a correlation coefficient based TOPSIS method under Pythagorean fuzzy soft environment and apply it in green supply chain management. DM approach under interval-valued neutrosophic hypersoft set defined in [39]. Development of TOPSIS method under Pythagorean fuzzy hypersoft set for the selection of antivirus mask is given in [40]. In [41], an algorithm is introduced by using the generalized multipolar neutrosophic soft set for medical diagnosis DM problems. An extension of the TOPSIS technique based on the correlation coefficient under the neutrosophic hypersoft set is proposed for the selection of effective hand sanitizer to reduce the covid-19 effects as defined in [42]. Another extension of the TOPSIS method under intuitionistic fuzzy hypersoft environment is to solve the DM problem in [43]. Using the matrix representation of the neutrosophic hypersoft set, the MADM problems are solved in [44]. Some other popular DM approaches under different environments were studied in [45-51].

Motivated by the above discussion, we introduce QSVNPNs and studied various operational laws and properties on them. In addition, based on QSVNPNs, we define score function, accuracy function, and the operators QSVNPWA and QSVPWG. Furthermore, based on Dombi t-norm and t-co-norm operators, we introduce two new aggregate operators for the MCDM problem. Some properties based on these two new operators are also investigated in the study. A new model has been proposed by using the new aggregate operators. Also, we execute the model by taking a suitable example.

1.1 Motivation

The Dombi aggregate operators under quadripartitioned single-valued neutrosophic Pythagorean numbers environment has not yet been studied till date. This gives us the motivation to present the proposed study.

The rest of the paper is organized in the form: In section 2, we review some basic definitions that are useful for the subsequent sections. In section 3, we establish some Dombi operators on QSVNPNs. In section 4, we propose the Dombi weighted aggregation operators under QSVNPNs environment. Further, in section 5, we initiate an algorithm-based model for MCDM using QSVNP Information. A practical application based on the proposed model is discussed in section 6. Conclusion and the future scope are presented in section 7.

2. Preliminaries

In this section, we recall some basic definitions that are fundamental to the proposed topic.

Definition 2.1 [13-15] A PFS Ω over the set of the universe Γ is an object of the form $\Omega=\{\langle s,\mu_\Omega(s),\gamma_\Omega(s)\rangle : s\in\Gamma\}$, where $\mu_\Omega:\Gamma\to[0,1]$ and $\gamma_\Omega:\Gamma\to[0,1]$ are respectively the membership and the non-membership functions with the restriction that $0\le(\mu_\Omega(s))^2+(\gamma_\Omega(s))^2\le 1$ and the hesitancy is measured by $\Pi_\Omega(s)=\sqrt{1-(\mu_\Omega(s))^2-(\gamma_\Omega(s))^2}$. We represent the Pythagorean fuzzy number (PFN) as $\Omega=\langle\mu_\Omega,\gamma_\Omega\rangle$.

Definition 2.2 [20] A SVNS Ω over the set of the universe Γ is an object of the form $\Omega=\{\langle s,T_\Omega(s),I_\Omega(s),F_\Omega(s)\rangle : s\in\Gamma\}$, where $T_\Omega:\Gamma\to[0,1]$, $I_\Omega:\Gamma\to[0,1]$ and $F_\Omega:\Gamma\to[0,1]$ are respectively the truth, indeterminacy and falsity membership functions with the restriction $0\le T_\Omega(s)+I_\Omega(s)+F_\Omega(s)\le 3$. The SVNN is represented by $\Omega=\langle T_\Omega,I_\Omega,F_\Omega\rangle$.

Definition 2.3 [22] A QSVNS Ω over the set of the universe Γ is an object of the form $\Omega=\{\langle s,T_\Omega(s),C_\Omega(s),U_\Omega(s),F_\Omega(s)\rangle : s\in\Gamma\}$, where $T_\Omega(s),C_\Omega(s),U_\Omega(s)$, and $F_\Omega(s)$ are respectively the truth, contradiction, unknown and falsity membership values with the condition $0\le T_\Omega(s)+C_\Omega(s)+U_\Omega(s)+F_\Omega(s)\le 4$.

Definition 2.4 [23] A QSVNPS Ω with dependent neutrosophic components over the universe of discourse Γ is an object of the form $\Omega=\{\langle s,T_\Omega(s),C_\Omega(s),U_\Omega(s),F_\Omega(s)\rangle : s\in\Gamma\}$ where $T_\Omega(s),C_\Omega(s),U_\Omega(s)$, and $F_\Omega(s)$ are respectively the truth, contradiction, unknown and falsity membership values with the condition

$T_\Omega(s)+F_\Omega(s)\le 1, C_\Omega(s)+U_\Omega(s)\le 1$, and $0\le T^2{}_\Omega(s)+C^2{}_\Omega(s)+U^2{}_\Omega(s)+F^2{}_\Omega(s)\le 2$. The QSVNPN is denoted by $\Omega=\langle T_\Omega,C_\Omega,U_\Omega,F_\Omega\rangle$.

Definition 2.5 For any two QSVNPNs $\Omega_1=\langle T_{\Omega_1},C_{\Omega_1},U_{\Omega_1},F_{\Omega_1}\rangle$ and $\Omega_2=\langle T_{\Omega_2},C_{\Omega_2},U_{\Omega_2},F_{\Omega_2}\rangle$

We have the following properties:

1. $\Omega_1 \subseteq \Omega_2$ iff for any $\varepsilon \in \Gamma$,

$T_{\Omega_1}(\varepsilon) \le T_{\Omega_2}(\varepsilon), C_{\Omega_1}(\varepsilon) \le C_{\Omega_2}(\varepsilon)\ and\ U_{\Omega_1}(\varepsilon) \ge U_{\Omega_2}(\varepsilon), F_{\Omega_1}(\varepsilon) \ge F_{\Omega_2}(\varepsilon)$

2. $\Omega_1 = \Omega_2$ iff $\Omega_1 \subseteq \Omega_2$ and $\Omega_2 \subseteq \Omega_1$

3. $\Omega_1 \cup \Omega_2 = \left\langle \max\left(T_{\Omega_1}, T_{\Omega_2}\right), \max\left(C_{\Omega_1}, C_{\Omega_2}\right), \min\left(U_{\Omega_1}, U_{\Omega_2}\right), \min\left(F_{\Omega_1}, F_{\Omega_2}\right)\right\rangle$

4. $\Omega_1 \cap \Omega_2 = \left\langle \min\left(T_{\Omega_1}, T_{\Omega_2}\right), \min\left(C_{\Omega_1}, C_{\Omega_2}\right), \max\left(U_{\Omega_1}, U_{\Omega_2}\right), \max\left(F_{\Omega_1}, F_{\Omega_2}\right)\right\rangle$

5. $\Omega_1^c = \left\langle F_{\Omega_1}, U_{\Omega_1}, C_{\Omega_1}, T_{\Omega_1}\right\rangle$

Definition 2.6 For two QSVNPNs $\Omega_1 = \left\langle T_{\Omega_1}, C_{\Omega_1}, U_{\Omega_1}, F_{\Omega_1}\right\rangle$ and $\Omega_2 = \left\langle T_{\Omega_2}, C_{\Omega_2}, U_{\Omega_2}, F_{\Omega_2}\right\rangle$, the basic operational laws between them are given by:

1. $\Omega_1 \oplus \Omega_2 = \left\langle T_{\Omega_1}, C_{\Omega_1}, U_{\Omega_1}, F_{\Omega_1}\right\rangle \oplus \left\langle T_{\Omega_2}, C_{\Omega_2}, U_{\Omega_2}, F_{\Omega_2}\right\rangle = \left\langle \sqrt{T_{\Omega_1}^2 + T_{\Omega_2}^2 - T_{\Omega_1}^2.T_{\Omega_2}^2}, \sqrt{C_{\Omega_1}^2 + C_{\Omega_2}^2 - C_{\Omega_1}^2.C_{\Omega_2}^2}, U_{\Omega_1}.U_{\Omega_2}, F_{\Omega_1}.F_{\Omega_2}\right\rangle$

2. $\Omega_1 \otimes \Omega_2 = \left\langle T_{\Omega_1}, C_{\Omega_1}, U_{\Omega_1}, F_{\Omega_1}\right\rangle \otimes \left\langle T_{\Omega_2}, C_{\Omega_2}, U_{\Omega_2}, F_{\Omega_2}\right\rangle = \left\langle T_{\Omega_1}.T_{\Omega_2}, C_{\Omega_1}.C_{\Omega_2}, \sqrt{U_{\Omega_1}^2 + U_{\Omega_2}^2 - U_{\Omega_1}^2.U_{\Omega_2}^2}, \sqrt{F_{\Omega_1}^2 + F_{\Omega_2}^2 - F_{\Omega_1}^2.F_{\Omega_2}^2}\right\rangle$

3. For any $\kappa \ge 0$, $\kappa\Omega = \left\langle \sqrt{1-\left(1-T_\Omega^2\right)^\kappa}, \sqrt{1-\left(1-C_\Omega^2\right)^\kappa}, U_\Omega^k, F_\Omega^k \right\rangle$

4. For any $\kappa \ge 0$, $\Omega^\kappa = \left\langle T_\Omega^k, C_\Omega^k, \sqrt{1-\left(1-U_\Omega^2\right)^\kappa}, \sqrt{1-\left(1-F_\Omega^2\right)^\kappa} \right\rangle$

Definition 2.7 For any QSVNPN $\Omega = \left\langle T_\Omega, C_\Omega, U_\Omega, F_\Omega\right\rangle$ the score function $\Theta(\Omega)$ and the accuracy function $\Lambda(\Omega)$ can be defined as

$\Theta(\Omega) = T_\Omega^2 + C_\Omega^2 - U_\Omega^2 - I_\Omega^2$, where $\Theta(\Omega) \in [-1,1]$ and $\Lambda(\Omega) = T_\Omega^2 + C_\Omega^2 + U_\Omega^2 + I_\Omega^2$, where $\Lambda(\Omega) \in [0,1]$

Definition 2.8 Let $\Omega_1 = \left\langle T_{\Omega_1}, C_{\Omega_1}, U_{\Omega_1}, F_{\Omega_1}\right\rangle$ and $\Omega_2 = \left\langle T_{\Omega_2}, C_{\Omega_2}, U_{\Omega_2}, F_{\Omega_2}\right\rangle$ be two QSVNPNs over the common universe of discourse Γ and their corresponding score and accuracy functions are respectively $\Theta(\Omega_1), \Theta(\Omega_2)$ and $\Lambda(\Omega_1), \Lambda(\Omega_2)$. Then we consider the following:

1. If $\Theta\left(\Omega_1\right)<\Theta\left(\Omega_2\right)$, then $\Omega_1 \prec \Omega_2$

2. If $\Theta\left(\Omega_1\right)>\Theta\left(\Omega_2\right)$, then $\Omega_1 \succ \Omega_2$

3. If $\Theta\left(\Omega_1\right)=\Theta\left(\Omega_2\right)$, then we compare their accuracy function as:

(a)If $\Lambda\left(\Omega_1\right)<\Lambda\left(\Omega_2\right)$, then $\Lambda_1 \prec \Lambda_2$

(b)If $\Lambda\left(\Omega_1\right)>\Lambda\left(\Omega_2\right)$, then $\Lambda_1 \succ \Lambda_2$

(c) If $\Lambda\left(\Omega_1\right)=\Lambda\left(\Omega_2\right)$, then $\Lambda_1 \approx \Lambda_2$

Theorem 2.9

For any three QSVNPNs $\Omega_1=\left\langle T_{\Omega_1}, C_{\Omega_1}, U_{\Omega_1}, F_{\Omega_1}\right\rangle$, $\Omega_2=\left\langle T_{\Omega_2}, C_{\Omega_2}, U_{\Omega_2}, F_{\Omega_2}\right\rangle$ and Ω_3 = $\left\langle T_{\Omega_3}, C_{\Omega_3}, U_{\Omega_3}, F_{\Omega_3}\right\rangle$ over the universe of discourse Γ and $\kappa_1, \kappa_2 \geq 0$. Then,

1. $\Omega_1 \oplus \Omega_2=\Omega_2 \oplus \Omega_1$

2. $\Omega_1 \otimes \Omega_2=\Omega_2 \otimes \Omega_1$

3. $\left(\Omega_1 \oplus \Omega_2\right) \oplus \Omega_3=\Omega_1 \oplus\left(\Omega_2 \oplus \Omega_3\right)$

4. $\left(\Omega_1 \otimes \Omega_2\right) \otimes \Omega_3=\Omega_1 \otimes\left(\Omega_2 \otimes \Omega_3\right)$

5. $\kappa_1\left(\Omega_1 \oplus \Omega_2\right)=\kappa_1 \Omega_1 \oplus \kappa_1 \Omega_2, \kappa_1>0$

6. $\left(\Omega_1 \otimes \Omega_2\right)^{\kappa_1}=\Omega_1^{\kappa_1} \otimes \Omega_2^{\kappa_1}, \kappa_1>0$

7. $\kappa_1 \Omega_1 \oplus \kappa_2 \Omega_1=\left(\kappa_1+\kappa_2\right) \Omega_1, \kappa_1 \text{ and} \kappa_2>0$

8. $\Omega_1^{\kappa_1} \otimes \Omega_1^{\kappa_2}=\Omega_1^{\kappa_1+\kappa_2}, \kappa_1 \text{ and} \kappa_2>0$

Proof: All proofs are obvious.

Definition 2.10 Let Ω_m = $\left\langle T_{\Omega_m}, C_{\Omega_m}, U_{\Omega_m}, F_{\Omega_m}\right\rangle$ be a collection of QSVPNs in Γ where $m=1,2,3, \ldots, n$.

Then the quadripartitioned single-valued neutrosophic Pythagorean weighted averaging (QSVNPWA)operator with weight vector $\varpi_m(m=1,2, \ldots, n)$ where $\varpi_m \geq 0$ and $\sum_{m=1}^n \varpi_m=1$ is given by

$QSVNPWA(\Omega_1, \Omega_2, ..., \Omega_n) = \sum_{m=1}^{n} \Omega_m \varpi_m$.

Definition 2.11 Let $\Omega_m = \langle T_{\Omega_m}, C_{\Omega_m}, U_{\Omega_m}, F_{\Omega_m} \rangle$ be a collection of QSVPNs in Γ where $m = 1, 2, 3, ..., n$.

Then the quadripartitioned single-valued neutrosophic Pythagorean weighted geometric (QSVNPWG)operator with weight vector $\varpi_m (m = 1, 2, ..., n)$ where $\varpi_m \geq 0$ and $\sum_{m=1}^{n} \varpi_m = 1$ is given by

$$QSVNPWA(\Omega_1, \Omega_2, ..., \Omega_n) = \prod_{m=1}^{n} (\Omega_m)^{\varpi_m} .$$

3. Dombi Operations on QSVNPNs

Definition 3.1 [24] Let p and q be any two real numbers where $(p, q) \in (0,1) \times (0,1)$ with $\xi \geq 1$.

Then Dombi's t-norms and t-co-norms are defined as

$$\hat{H}(p, q) = \frac{1}{1 + \left[\left(\frac{1-p}{p} \right)^{\xi} + \left(\frac{1-q}{q} \right)^{\xi} \right]^{1/\xi}}$$ and

$$\hat{G}(p, q) = 1 - \frac{1}{1 + \left[\left(\frac{p}{1-p} \right)^{\xi} + \left(\frac{q}{1-q} \right)^{\xi} \right]^{1/\xi}}$$ respectively.

Based on definition **3.1**, we define the following Dombi's t-norms and t-co-norms operational laws on QSVNPNs:

Definition 3.2 Let $\Omega_1 = \langle T_{\Omega_1}, C_{\Omega_1}, U_{\Omega_1}, F_{\Omega_1} \rangle$ and $\Omega_2 = \langle T_{\Omega_2}, C_{\Omega_2}, U_{\Omega_2}, F_{\Omega_2} \rangle$ be two QSVNPNs over Γ with $\kappa \geq 0$ and $\xi \geq 1$. Then the Dombi's t-norms and t-co-norms operational laws defined on QSVNPNs are given by:

1. $$\Omega_1 \oplus \Omega_2 = \left\langle \begin{array}{c} \sqrt{1 - \frac{1}{1 + \left[\left(\frac{T_{\Omega_1}^2}{1 - T_{\Omega_1}^2} \right)^{\xi} + \left(\frac{T_{\Omega_2}^2}{1 - T_{\Omega_2}^2} \right)^{\xi} \right]^{1/\xi}}}, \sqrt{1 - \frac{1}{1 + \left[\left(\frac{C_{\Omega_1}^2}{1 - C_{\Omega_1}^2} \right)^{\xi} + \left(\frac{C_{\Omega_2}^2}{1 - C_{\Omega_2}^2} \right)^{\xi} \right]^{1/\xi}}}, \\ \frac{1}{1 + \left[\left(\frac{1 - U_{\Omega_1}}{U_{\Omega_1}} \right)^{\xi} + \left(\frac{1 - U_{\Omega_2}}{U_{\Omega_2}} \right)^{\xi} \right]^{1/\xi}}, \frac{1}{1 + \left[\left(\frac{1 - F_{\Omega_1}}{F_{\Omega_1}} \right)^{\xi} + \left(\frac{1 - F_{\Omega_2}}{F_{\Omega_2}} \right)^{\xi} \right]^{1/\xi}} \end{array} \right\rangle$$

2.

$$\Omega_1 \otimes \Omega_2 = \left\langle \begin{array}{l} \dfrac{1}{1+\left[\left(\dfrac{1-T_{\Omega_1}}{T_{\Omega_1}}\right)^{\xi}+\left(\dfrac{1-T_{\Omega_2}}{T_{\Omega_2}}\right)^{\xi}\right]^{1/\xi}}, \dfrac{1}{1+\left[\left(\dfrac{1-C_{\Omega_1}}{C_{\Omega_1}}\right)^{\xi}+\left(\dfrac{1-C_{\Omega_2}}{C_{\Omega_2}}\right)^{\xi}\right]^{1/\xi}}, \\ \sqrt{1-\dfrac{1}{1+\left[\left(\dfrac{U_{\Omega_1}^2}{1-U_{\Omega_1}^2}\right)^{\xi}+\left(\dfrac{U_{\Omega_2}^2}{1-U_{\Omega_2}^2}\right)^{\xi}\right]^{1/\xi}}}, \sqrt{1-\dfrac{1}{1+\left[\left(\dfrac{F_{\Omega_1}^2}{1-F_{\Omega_1}^2}\right)^{\xi}+\left(\dfrac{F_{\Omega_2}^2}{1-F_{\Omega_2}^2}\right)^{\xi}\right]^{1/\xi}}} \end{array} \right\rangle$$

3. $$\kappa\Omega = \left\langle \begin{array}{l} \sqrt{1-\dfrac{1}{1+\left[\kappa\left(\dfrac{T_{\Omega}^2}{1-T_{\Omega}^2}\right)^{\xi}\right]^{1/\xi}}}, \sqrt{1-\dfrac{1}{1+\left[\kappa\left(\dfrac{C_{\Omega}^2}{1-C_{\Omega}^2}\right)^{\xi}\right]^{1/\xi}}}, \\ \dfrac{1}{1+\left[\kappa\left(\dfrac{1-U_{\Omega}}{U_{\Omega}}\right)^{\xi}\right]^{1/\xi}}, \dfrac{1}{1+\left[\kappa\left(\dfrac{1-F_{\Omega}}{F_{\Omega}}\right)^{\xi}\right]^{1/\xi}} \end{array} \right\rangle$$

4. $$\Omega^{\kappa} = \left\langle \begin{array}{l} \dfrac{1}{1+\left[\kappa\left(\dfrac{1-T_{\Omega}}{T_{\Omega}}\right)^{\xi}\right]^{1/\xi}}, \dfrac{1}{1+\left[\kappa\left(\dfrac{1-C_{\Omega}}{C_{\Omega}}\right)^{\xi}\right]^{1/\xi}}, \\ \sqrt{1-\dfrac{1}{1+\left[\kappa\left(\dfrac{U_{\Omega}^2}{1-U_{\Omega}^2}\right)^{\xi}\right]^{1/\xi}}}, \sqrt{1-\dfrac{1}{1+\left[\kappa\left(\dfrac{F_{\Omega}^2}{1-F_{\Omega}^2}\right)^{\xi}\right]^{1/\xi}}}, \end{array} \right\rangle$$

Example 3.2.1 Let $\Omega_1 = \langle 0.6, 0.4, 0.6, 0.3 \rangle$ and $\Omega_2 = \langle 0.3, 0.6, 0.4, 0.5 \rangle$ be two QSVNPNs over Γ with $\kappa = 0.5$ and $\xi = 1.4$. Then we compute the above Dombi's t-norms and t-co-norms operators based on QSVNPNs as follows:

$$\Omega_1 \oplus \Omega_2 = \langle 0.611, 0.627, 0.372, 0.261 \rangle$$

$$\Omega_1 \otimes \Omega_2 = \langle 0.276, 0.353, 0.627, 0.522 \rangle$$

$$0.5\,\Omega_1 = \langle 0.505, 0.322, 0.711, 0.412 \rangle$$

$$\Omega_1^{0.5} = \langle 0.711, 0.522, 0.505, 0.238 \rangle$$

4. Dombi Weighted Aggregation Operators under QSVNPNs environment

In this section, based on Dombi operational laws on QSVNPNs, two Dombi ordered weighted aggregation operators, namely, quadripartitioned single-valued neutrosophic Pythagorean Dombi ordered weighted arithmetic aggregation(QSVNPDOWAA) operator and quadripartitioned single-valued neutrosophic Pythagorean Dombi ordered weighted geometric aggregation(QSVNPDOWGA) operator are formed. After that, some significant results based on these two operators are investigated.

Definition 4.1 Let $\Omega_m = \left\langle T_{\Omega_m}, C_{\Omega_m}, U_{\Omega_m}, F_{\Omega_m} \right\rangle$ be a collection of QSVNPNs with the weight vector $W = \left(\varpi_1, \varpi_2, ..., \varpi_m\right)^t$ where $l = 1, 2, ..., \mathrm{m}$ and $\varpi_l \geq 0$, $\sum_{l=1}^{m} \varpi_l = 1$ .Then the operator QSVNPDOWAA: $\Omega^m \to \Omega$ is defined as $QSVNPDOWAA\left(\Omega_1, \Omega_2, ..., \Omega_m\right) = \overset{m}{\underset{l=1}{\oplus}} \varpi_l \, \Omega_{\sigma(l)}$, where $\left(\Omega_{\sigma(1)}, \Omega_{\sigma(2)}, ..., \Omega_{\sigma(m)}\right)$ is the permutation of $\left(\Omega_1, \Omega_2, ..., \Omega_m\right)$ such that $\Omega_{\sigma(l-1)} \geq \Omega_{\sigma(l)}$ for all $l = 1, 2, ..., m$.

Theorem 4.2 If $\Omega_m = \left\langle T_{\Omega_m}, C_{\Omega_m}, U_{\Omega_m}, F_{\Omega_m} \right\rangle$ be a collection of QSVNPNs, then the resulting of these numbers by using QSVNPDOWAA operator defined above is again a QSVNPN and the resulting number can be obtained by using the following formula:

$$QSVNPDOWAA\left(\Omega_1, \Omega_2, ..., \Omega_m\right) = \overset{m}{\underset{l=1}{\oplus}} \varpi_l \, \Omega_{\sigma(l)} =$$

$$\left\langle \begin{array}{l} \sqrt{1 - \dfrac{1}{1 + \left[\sum_{l=1}^{m} \varpi_l \left(\dfrac{T_{\sigma(l)}^2}{1 - T_{\sigma(l)}^2} \right)^{\xi} \right]^{1/\xi}}}, \sqrt{1 - \dfrac{1}{1 + \left[\sum_{l=1}^{m} \varpi_l \left(\dfrac{C_{\sigma(l)}^2}{1 - C_{\sigma(l)}^2} \right)^{\xi} \right]^{1/\xi}}}, \\ \dfrac{1}{1 + \left[\sum_{l=1}^{m} \varpi_l \left(\dfrac{1 - U_{\sigma(l)}}{U_{\sigma(l)}} \right)^{\xi} \right]^{1/\xi}}, \dfrac{1}{1 + \left[\sum_{l=1}^{m} \varpi_l \left(\dfrac{1 - F_{\sigma(l)}}{F_{\sigma(l)}} \right)^{\xi} \right]^{1/\xi}} \end{array} \right\rangle ....(1)$$

Proof: By mathematical induction, we can prove the theorem given as.

Based on the Dombi operational laws of QSVNPNs for m=2, we have

$$QSVNPDOWAA(\Omega_1,\Omega_2)=\varpi_1\Omega_1\oplus\varpi_2\Omega_2=\left\langle \begin{array}{l}\sqrt{1-\frac{1}{1+\left[\varpi_1\left(\frac{T_1^2}{1-T_1^2}\right)^{\xi}+\varpi_2\left(\frac{T_2^2}{1-T_2^2}\right)^{\xi}\right]^{1/\xi}}},\ \sqrt{1-\frac{1}{1+\left[\varpi_1\left(\frac{C_1^2}{1-C_1^2}\right)^{\xi}+\varpi_2\left(\frac{C_2^2}{1-C_2^2}\right)^{\xi}\right]^{1/\xi}}},\\ \frac{1}{1+\left[\varpi_1\left(\frac{1-U_1}{U_1}\right)^{\xi}+\varpi_2\left(\frac{1-U_2}{U_2}\right)^{\xi}\right]^{1/\xi}},\frac{1}{1+\left[\varpi_1\left(\frac{1-F_1}{F_1}\right)^{\xi}+\varpi_2\left(\frac{1-F_2}{F_2}\right)^{\xi}\right]^{1/\xi}}\end{array}\right\rangle$$

$$=\left\langle \begin{array}{l}\sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{2}\varpi_l\left(\frac{T_l^2}{1-T_l^2}\right)^{\xi}\right]^{1/\xi}}},\ \sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{2}\varpi_l\left(\frac{C_l^2}{1-C_l^2}\right)^{\xi}\right]^{1/\xi}}},\\ \frac{1}{1+\left[\sum_{l=1}^{2}\varpi_l\left(\frac{1-U_l}{U_l}\right)^{\xi}\right]^{1/\xi}},\frac{1}{1+\left[\sum_{l=1}^{2}\varpi_l\left(\frac{1-F_l}{F_l}\right)^{\xi}\right]^{1/\xi}}\end{array}\right\rangle$$

Suppose the equation (1) holds for m=p, where $p\in N$. Then, we have

$$QSVNPDOWAA(\Omega_1,\Omega_2,.....,\Omega_p)=\left\langle \begin{array}{l}\sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\frac{T_l^2}{1-T_l^2}\right)^{\xi}\right]^{1/\xi}}},\ \sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\frac{C_l^2}{1-C_l^2}\right)^{\xi}\right]^{1/\xi}}},\\ \frac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\frac{1-U_l}{U_l}\right)^{\xi}\right]^{1/\xi}},\frac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\frac{1-F_l}{F_l}\right)^{\xi}\right]^{1/\xi}}\end{array}\right\rangle$$

Now, we shall have to show that the equation (1) is true for m=p+1 whenever it is already true for m=p

$$QSVNPDOWAA(\Omega_1,\Omega_2,.....,\Omega_p,\Omega_{p+1})$$

$$=\left\langle \begin{array}{l}\sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\frac{T_l^2}{1-T_l^2}\right)^{\xi}\right]^{1/\xi}}},\ \sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\frac{C_l^2}{1-C_l^2}\right)^{\xi}\right]^{1/\xi}}},\\ \frac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\frac{1-U_l}{U_l}\right)^{\xi}\right]^{1/\xi}},\frac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\frac{1-F_l}{F_l}\right)^{\xi}\right]^{1/\xi}}\end{array}\right\rangle\oplus\varpi_{p+1}\Omega_{P+1}$$

$$=\left\langle \begin{array}{l} \sqrt{1-\dfrac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\dfrac{T_l^2}{1-T_l^2}\right)^{\xi}\right]^{1/\xi}}},\ \sqrt{1-\dfrac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\dfrac{C_l^2}{1-C_l^2}\right)^{\xi}\right]^{1/\xi}}}, \\ \dfrac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\dfrac{1-U_l}{U_l}\right)^{\xi}\right]^{1/\xi}},\dfrac{1}{1+\left[\sum_{l=1}^{p}\varpi_l\left(\dfrac{1-F_l}{F_l}\right)^{\xi}\right]^{1/\xi}} \end{array}\right\rangle \oplus$$

$$\left\langle \begin{array}{l} \sqrt{1-\dfrac{1}{1+\left[\varpi_{p+1}\left(\dfrac{T_{p+1}^2}{1-T_{p+1}^2}\right)^{\xi}\right]^{1/\xi}}},\ \sqrt{1-\dfrac{1}{1+\left[\varpi_{p+1}\left(\dfrac{C_{p+1}^2}{1-C_{p+1}^2}\right)^{\xi}\right]^{1/\xi}}}, \\ \dfrac{1}{1+\left[\varpi_{p+1}\left(\dfrac{1-U_{p+1}}{U_{p+1}}\right)^{\xi}\right]^{1/\xi}},\dfrac{1}{1+\left[\varpi_{p+1}\left(\dfrac{1-F_{p+1}}{F_{p+1}}\right)^{\xi}\right]^{1/\xi}} \end{array}\right\rangle$$

$$=\left\langle \begin{array}{l} \sqrt{1-\dfrac{1}{1+\left[\sum_{l=1}^{p+1}\varpi_l\left(\dfrac{T_l^2}{1-T_l^2}\right)^{\xi}\right]^{1/\xi}}},\ \sqrt{1-\dfrac{1}{1+\left[\sum_{l=1}^{p+1}\varpi_l\left(\dfrac{C_l^2}{1-C_l^2}\right)^{\xi}\right]^{1/\xi}}}, \\ \dfrac{1}{1+\left[\sum_{l=1}^{p+1}\varpi_l\left(\dfrac{1-U_l}{U_l}\right)^{\xi}\right]^{1/\xi}},\dfrac{1}{1+\left[\sum_{l=1}^{p+1}\varpi_l\left(\dfrac{1-F_l}{F_l}\right)^{\xi}\right]^{1/\xi}} \end{array}\right\rangle$$

Thus, by the principle of mathematical induction, equation (1) holds for any natural number.

Example 4.2.1 Four farmers namely F_1, F_2, F_3 and F_4 want to check the expected fertility of a field for cultivation. The level of fertility of the field can be determined by using QSVNPNs under considering certain criteria by the decision-maker. According to the four farmers, the level of fertility of the soil is specified under the QSVNPN environment is given by $\Omega_1 = \langle 0.4, 0.3, 0.5, 0.3\rangle$, $\Omega_2 = \langle 0.6, 0.4, 0.2, 0.3\rangle$, $\Omega_3 = \langle 0.2, 0.3, 0.4, 0.6\rangle$ and $\Omega_4 = \langle 0.3, 0.4, 0.6, 0.4\rangle$ respectively and the corresponding weight vector of the farmers is given by $W = \langle 0.4, 0.25, 0.15, 0.2\rangle$.

First, we determine the score of each $\Omega_l\ (l = 1,2,3,4)$ given by,

$\Theta(\Omega_1) = 0.4^2 + 0.3^2 - 0.5^2 - 0.3^2 = -0.09$

Similarly, $\Theta(\Omega_2) = 0.39$, $\Theta(\Omega_3) = -0.39$ and $\Theta(\Omega_4) = -0.27$

Thus, $\Theta(\Omega_2) > \Theta(\Omega_1) > \Theta(\Omega_4) > \Theta(\Omega_3)$

Therefore, $\Omega_{\sigma(1)} = \Omega_2, \Omega_{\sigma(2)} = \Omega_1, \Omega_{\sigma(3)} = \Omega_4, \Omega_{\sigma(4)} = \Omega_3$

Thus, by using the QSVNPDOWAA operator with $\xi = 2$ we have,

$$QSVNPDOWAA(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$$

$$= \left\langle \begin{array}{l} \sqrt{1 - \dfrac{1}{1 + \left[0.25\left(\dfrac{0.6^2}{1-0.6^2}\right)^2 + 0.4\left(\dfrac{0.4^2}{1-0.4^2}\right)^2 + 0.2\left(\dfrac{0.3^2}{1-0.3^2}\right)^2 + 0.15\left(\dfrac{0.2^2}{1-0.2^2}\right)^2 \right]^{1/2}}}, \\ \sqrt{1 - \dfrac{1}{1 + \left[0.25\left(\dfrac{0.4^2}{1-0.4^2}\right)^2 + 0.4\left(\dfrac{0.3^2}{1-0.3^2}\right)^2 + 0.2\left(\dfrac{0.4^2}{1-0.4^2}\right)^2 + 0.15\left(\dfrac{0.3^2}{1-0.3^2}\right)^2 \right]^{1/2}}}, \\ \sqrt{\dfrac{1}{1 + \left[0.25\left(\dfrac{1-0.2}{0.2}\right)^2 + 0.4\left(\dfrac{1-0.5}{0.5}\right)^2 + 0.2\left(\dfrac{1-0.6}{0.6}\right)^2 + 0.15\left(\dfrac{1-0.4}{0.4}\right)^2 \right]^{1/2}}}, \\ \sqrt{\dfrac{1}{1 + \left[0.25\left(\dfrac{1-0.3}{0.3}\right)^2 + 0.4\left(\dfrac{1-0.3}{0.3}\right)^2 + 0.2\left(\dfrac{1-0.4}{0.4}\right)^2 + 0.15\left(\dfrac{1-0.6}{0.6}\right)^2 \right]^{1/2}}} \end{array} \right\rangle$$

$= \langle 0.486, 0.358, 0.312, 0.331 \rangle$

Therefore, $QSVNPDOWAA(\Omega_1, \Omega_2, \Omega_3, \Omega_4) = \langle 0.486, 0.358, 0.312, 0.331 \rangle$

So, the aggregate of four QSVNPNs under Dombi operation is again a QSVNPN. We can generalize it for any finite numbers.

Theorem 4.3 (Idempotency) For any QSVNPN $\Omega = \langle T_\Omega, C_\Omega, U_\Omega, F_\Omega \rangle$ we have

$$QSVNPDOWAA(\Omega, \Omega, \Omega, ..., \Omega) = \Omega.$$

Proof: Assuming $\Omega_l = \Omega \; (l = 1, 2, ..., m)$ and then applying equation (1), we have

$$QSVNPDOWAA(\Omega_1,\Omega_2,...,\Omega_m) = \overset{m}{\underset{l=1}{\oplus}} \varpi_l \ \Omega_l =$$

$$\left\langle \begin{array}{l} \sqrt{1-\dfrac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\dfrac{T_l^2}{1-T_l^2}\right)^{\xi}\right]^{1/\xi}}}, \sqrt{1-\dfrac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\dfrac{C_l^2}{1-C_l^2}\right)^{\xi}\right]^{1/\xi}}}, \\ \dfrac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\dfrac{1-U_l}{U_l}\right)^{\xi}\right]^{1/\xi}}, \dfrac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\dfrac{1-F_l}{F_l}\right)^{\xi}\right]^{1/\xi}} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \sqrt{1-\dfrac{1}{1+\left(\dfrac{T_l^2}{1-T_l^2}\right)}}, \sqrt{1-\dfrac{1}{1+\left(\dfrac{C_l^2}{1-C_l^2}\right)}}, \\ \dfrac{1}{1+\left[\left(\dfrac{1-U_l}{U_l}\right)\right]}, \dfrac{1}{1+\left(\dfrac{1-F_l}{F_l}\right)} \end{array} \right\rangle = \Omega$$

Hence proved.

Theorem 4.4 (Boundedness) Consider the collection of QSVNPNs $\{\Omega_1,\Omega_2,.....,\Omega_m\}$, where $\Omega_l = \langle T_{\Omega_l}, C_{\Omega_l}, U_{\Omega_l}, F_{\Omega_l} \rangle, l = 1,2,...,m$ in such a manner that $\Omega_{\min} = \min\{\Omega_1,\Omega_2,.....,\Omega_m\}$, and $\Omega_{\max} = \max\{\Omega_1,\Omega_2,.....,\Omega_m\}$. Then, $\Omega_{\min} \le QSVNPDOWAA(\Omega_1,\Omega_2,.....,\Omega_m) \le \Omega_{\max}$.

Proof: Suppose that

$\Omega_{\min} = \min\{\Omega_1,\Omega_2,.....,\Omega_m\} = \langle T_*, C_*, U_*, F_* \rangle$, and

$\Omega_{\max} = \max\{\Omega_1,\Omega_2,.....,\Omega_m\} = \langle T^*, C^*, U^*, F^* \rangle$.

Then, $T_* = \min\{T_l\}$, $C_* = \min\{C_l\}$, $U_* = \max\{U_l\}$, $F_* = \max\{F_l\}$ and $T^* = \max\{T_l\}$, $C^* = \max\{C_l\}, U^* = \min\{U_l\}, F^* = \min\{F_l\}$

Therefore, we have the following inequalities for the membership, contradictory, ignorance, and falsity membership respectively

$$\sqrt{1-\dfrac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\dfrac{T_*^2}{1-T_*^2}\right)^{\xi}\right]^{1/\xi}}} \le \sqrt{1-\dfrac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\dfrac{T_l^2}{1-T_l^2}\right)^{\xi}\right]^{1/\xi}}} \le \sqrt{1-\dfrac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\dfrac{T^{*2}}{1-T^{*2}}\right)^{\xi}\right]^{1/\xi}}}$$

$$\sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{C_*^2}{1-C_*^2}\right)^{\xi}\right]^{1/\xi}}} \le \sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{C_l^2}{1-C_l^2}\right)^{\xi}\right]^{1/\xi}}} \le \sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{C^{*2}}{1-C^{*2}}\right)^{\xi}\right]^{1/\xi}}}$$

$$\sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{U^{*2}}{1-U^{*2}}\right)^{\xi}\right]^{1/\xi}}} \le \sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{U_l^2}{1-U_l^2}\right)^{\xi}\right]^{1/\xi}}} \le \sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{U_*^2}{1-U_*^2}\right)^{\xi}\right]^{1/\xi}}}$$

$$\sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{C_*^2}{1-C_*^2}\right)^{\xi}\right]^{1/\xi}}} \le \sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{C_l^2}{1-C_l^2}\right)^{\xi}\right]^{1/\xi}}} \le \sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{C^{*2}}{1-C^{*2}}\right)^{\xi}\right]^{1/\xi}}}$$

This completes the proof.

Theorem 4.5 (Monotonicity) suppose the two collections of QSVNPNs are $\left\{\Omega'_1,\Omega'_2,.....,\Omega'_m\right\}$ and $\left\{\Omega_1,\Omega_2,.....,\Omega_m\right\}$ where $\Omega'_l=\left\langle T_{\Omega'_l},C_{\Omega'_l},U_{\Omega'_l},F_{\Omega'_l}\right\rangle,\Omega_l=\left\langle T_{\Omega_l},C_{\Omega_l},U_{\Omega_l},F_{\Omega_l}\right\rangle,l=1,2,...,m$

In such a manner that $T_{\Omega'_l}\le T_{\Omega_l}$, $C_{\Omega'_l}\le C_{\Omega_l}$, and $U_{\Omega'_l}\ge U_{\Omega_l}$, $F_{\Omega'_l}\ge F_{\Omega_l}$. Then

$$QSVNPDOWAA\left(\Omega'_1,\Omega'_2,.....,\Omega'_m\right)\le QSVNPDOWAA\left(\Omega_1,\Omega_2,.....,\Omega_m\right).$$

Proof: Suppose $QSVNPDOWAA\left(\Omega'_1,\Omega'_2,.....,\Omega'_m\right)=\left\langle T',C',U',F'\right\rangle$ and

$QSVNPDOWAA\left(\Omega_1,\Omega_2,.....,\Omega_m\right)=\left\langle T,C,U,F\right\rangle$. At first, we shall show that $T'\le T$.

Since, $T_{\Omega'_l}\le T_{\Omega_l}\Rightarrow\frac{T^2_{\Omega'_l}}{1-T^2_{\Omega'_l}}\le\frac{T^2_{\Omega_l}}{1-T^2_{\Omega_l}}$. Using this result we can write,

$$\left[\sum_{l=1}^{m}\varpi_l\left(\frac{T_l^{'2}}{1-T_l^{'2}}\right)^{\xi}\right]^{1/\xi} \leq \left[\sum_{l=1}^{m}\varpi_l\left(\frac{T_l^{2}}{1-T_l^{2}}\right)^{\xi}\right]^{1/\xi}$$

$$or\ 1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{T_l^{'2}}{1-T_l^{'2}}\right)^{\xi}\right]^{1/\xi} \leq 1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{T_l^{2}}{1-T_l^{2}}\right)^{\xi}\right]^{1/\xi}$$

$$or\ \frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{T_l^{'2}}{1-T_l^{'2}}\right)^{\xi}\right]^{1/\xi}} \geq \frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{T_l^{2}}{1-T_l^{2}}\right)^{\xi}\right]^{1/\xi}}$$

$$or\ 1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{T_l^{'2}}{1-T_l^{'2}}\right)^{\xi}\right]^{1/\xi}} \leq \frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{T_l^{2}}{1-T_l^{2}}\right)^{\xi}\right]^{1/\xi}}$$

$$or\ \sqrt{1-\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{T_l^{'2}}{1-T_l^{'2}}\right)^{\xi}\right]^{1/\xi}}} \leq \sqrt{\frac{1}{1+\left[\sum_{l=1}^{m}\varpi_l\left(\frac{T_l^{2}}{1-T_l^{2}}\right)^{\xi}\right]^{1/\xi}}}$$

Hence, $T^{'} \leq T$. Similarly, we can show that $C^{'} \leq C$, $U^{'} \geq U$ and $I^{'} \geq I$.

Theorem 4.6 (Reducibility) suppose $\{\Omega_1,\Omega_2,.....,\Omega_m\}$ be a collection of QSVNPNs in such a manner that $\Omega_l = \left\langle T_{\Omega_l}, C_{\Omega_l}, U_{\Omega_l}, F_{\Omega_l}\right\rangle, l=1,2,...,m$ with the corresponding weight vector $W=(\varpi_1,\varpi_2,....,\varpi_m)^t = \left(\frac{1}{m},\frac{1}{m},......,\frac{1}{m}\right)^t$. Then we can write

$$QSVNPDOWAA(\Omega_1,\Omega_2,.....,\Omega_m)=$$

$$\left\langle \sqrt{1-\frac{1}{1+\left[\frac{1}{m}\sum_{l=1}^{m}\left(\frac{T_l^2}{1-T_l^2}\right)^{\xi}\right]^{1/\xi}}}, \sqrt{1-\frac{1}{1+\left[\frac{1}{m}\sum_{l=1}^{m}\left(\frac{C_l^2}{1-C_l^2}\right)^{\xi}\right]^{1/\xi}}}, \frac{1}{1+\left[\frac{1}{m}\sum_{l=1}^{m}\left(\frac{1-U_l}{U_l}\right)^{\xi}\right]^{1/\xi}}, \frac{1}{1+\left[\frac{1}{m}\sum_{l=1}^{m}\left(\frac{1-F_l}{F_l}\right)^{\xi}\right]^{1/\xi}} \right\rangle$$

Definition 4.7 Let $\Omega_m = \left\langle T_{\Omega_m}, C_{\Omega_m}, U_{\Omega_m}, F_{\Omega_m} \right\rangle$ be a collection of QSVNPNs with the weight vector $W = (\varpi_1, \varpi_2, ..., \varpi_m)^t$ where $l = 1, 2, ..., \mathrm{m}$ and $\varpi_m \geq 0$, $\sum_{l=1}^{m} \varpi_l = 1$ .Then the operator QSVNPDOWGA: $\Omega^m \to \Omega$ is defined as

$QSVNPDOWGA(\Omega_1, \Omega_2, ..., \Omega_m) = \overset{m}{\underset{l=1}{\otimes}} \varpi_l \, \Omega_{\sigma(l)}$ Where $\left(\Omega_{\sigma(1)}, \Omega_{\sigma(2)}, ..., \Omega_{\sigma(m)}\right)$ is the permutation of $(\Omega_1, \Omega_2, ..., \Omega_m)$ such that $\Omega_{\sigma(l-1)} \geq \Omega_{\sigma(l)}$ for all $l = 1, 2, ..., m$.

Theorem 4.8 If $\Omega_m = \left\langle T_{\Omega_m}, C_{\Omega_m}, U_{\Omega_m}, F_{\Omega_m} \right\rangle$ be a collection of QSVNPNs, then the resulting of these numbers by using QSVNPDOWGA operator defined above is again a QSVNPN and the aggregate number can be obtained by using the following formula:

$$QSVNPDOWGA(\Omega_1, \Omega_2, ..., \Omega_m) = \overset{m}{\underset{l=1}{\otimes}} \varpi_l \, \Omega_{\sigma(l)} =$$

$$\left\langle \begin{array}{c} \dfrac{1}{1 + \left[\sum_{l=1}^{m} \varpi_l \left(\dfrac{1 - T_{\sigma(l)}}{T_{\sigma(l)}} \right)^{\xi} \right]^{1/\xi}}, \dfrac{1}{1 + \left[\sum_{l=1}^{m} \varpi_l \left(\dfrac{1 - C_{\sigma(l)}}{C_{\sigma(l)}} \right)^{\xi} \right]^{1/\xi}}, \\ \sqrt{1 - \dfrac{1}{1 + \left[\sum_{l=1}^{m} \varpi_l \left(\dfrac{U_{\sigma(l)}^2}{1 - U_{\sigma(l)}^2} \right)^{\xi} \right]^{1/\xi}}}, \sqrt{1 - \dfrac{1}{1 + \left[\sum_{l=1}^{m} \varpi_l \left(\dfrac{F_{\sigma(l)}^2}{1 - F_{\sigma(l)}^2} \right)^{\xi} \right]^{1/\xi}}} \end{array} \right\rangle$$

Proof: the proof is similar to the proof of Theorem **4.2**

Example 4.8.1 Revisiting example **4.2.1,** we can obtain the QSVNPDOWGA operator as follows

$$QSVNPDOWGA(\Omega_1, \Omega_2, \Omega_3, \Omega_4) = \overset{4}{\underset{l=1}{\otimes}} \varpi_l \, \Omega_{\sigma(l)} =$$

$$\left\langle \begin{array}{l} \dfrac{1}{1+\left[0.25\left(\dfrac{1-0.6}{0.6}\right)^2+0.4\left(\dfrac{1-0.4}{0.4}\right)^2+0.2\left(\dfrac{1-0.3}{0.3}\right)^2+0.15\left(\dfrac{1-0.2}{0.2}\right)^2\right]^{1/2}}, \\ \dfrac{1}{1+\left[0.25\left(\dfrac{1-0.4}{0.4}\right)^2+0.4\left(\dfrac{1-0.3}{0.3}\right)^2+0.2\left(\dfrac{1-0.4}{0.4}\right)^2+0.15\left(\dfrac{1-0.3}{0.3}\right)^2\right]^{1/2}}, \\ \sqrt{1-\dfrac{1}{1+\left[0.25\left(\dfrac{0.2^2}{1-0.2^2}\right)^2+0.4\left(\dfrac{0.5^2}{1-0.5^2}\right)^2+0.2\left(\dfrac{0.6^2}{1-0.6^2}\right)^2+0.15\left(\dfrac{0.4^2}{1-0.4^2}\right)^2\right]^{1/2}}}, \\ \sqrt{1-\dfrac{1}{1+\left[0.25\left(\dfrac{0.3^2}{1-0.3^2}\right)^2+0.4\left(\dfrac{0.3^2}{1-0.3^2}\right)^2+0.2\left(\dfrac{0.4^2}{1-0.4^2}\right)^2+0.15\left(\dfrac{0.6^2}{1-0.6^2}\right)^2\right]^{1/2}}} \end{array} \right\rangle$$

$$=\langle 0.320, 0.333, 0.502, 0.445\rangle$$

The aggregate is of four QSVNPNs under the QSVNPDOWGA operator is again a QSVNPN.

Theorem 4.9 (Idempotency) for any QSVNPN $\Omega=\langle T_\Omega, C_\Omega, U_\Omega, F_\Omega\rangle$ we have

$$QSVNPDOWGA(\Omega,\Omega,\Omega,.........,\Omega)=\Omega.$$

Theorem 4.10 (Boundedness) Consider the collection of QSVNPNs $\{\Omega_1,\Omega_2,.....,\Omega_m\}$, where $\Omega_l=\langle T_{\Omega_l}, C_{\Omega_l}, U_{\Omega_l}, F_{\Omega_l}\rangle, l=1,2,...,m$ in such a manner that $\Omega_{\min}=\min\{\Omega_1,\Omega_2,.....,\Omega_m\}$, and $\Omega_{\max}=\max\{\Omega_1,\Omega_2,.....,\Omega_m\}$.

Then, $\Omega_{\min}\le QSVNPDOWGA(\Omega_1,\Omega_2,.....,\Omega_m)\le\Omega_{\max}$.

Theorem 4.11 (Monotonicity) suppose the two collection of QSVNPNs are $\{\Omega'_1,\Omega'_2,.....,\Omega'_m\}$ and $\{\Omega_1,\Omega_2,.....,\Omega_m\}$ where $\Omega'_l=\langle T_{\Omega'_l}, C_{\Omega'_l}, U_{\Omega'_l}, F_{\Omega'_l}\rangle, \Omega_l=\langle T_{\Omega_l}, C_{\Omega_l}, U_{\Omega_l}, F_{\Omega_l}\rangle, l=1,2,...,m$

In such a manner that $T_{\Omega'_l}\le T_{\Omega_l}, C_{\Omega'_l}\le C_{\Omega_l}$ and $U_{\Omega'_l}\ge U_{\Omega_l}, F_{\Omega'_l}\ge F_{\Omega_l}$. Then

$$QSVNPDOWGA(\Omega'_1,\Omega'_2,.....,\Omega'_m)\le QSVNPDOWGA(\Omega_1,\Omega_2,.....,\Omega_m).$$

Theorem 4.12 (Reducibility) suppose $\{\Omega_1,\Omega_2,.....,\Omega_m\}$ be a collection of QSVNPNs in such a manner that $\Omega_l=\langle T_{\Omega_l},C_{\Omega_l},U_{\Omega_l},F_{\Omega_l}\rangle, l=1,2,...,m$ with the corresponding weight vector $W=(\varpi_1,\varpi_2,....,\varpi_m)^t=\left(\frac{1}{m},\frac{1}{m},......,\frac{1}{m}\right)^t$. Then we can write

$$QSVNPDOWGA(\Omega_1,\Omega_2,.....,\Omega_m)=\left\langle \begin{array}{l} \dfrac{1}{1+\left[\frac{1}{m}\sum\limits_{l=1}^{m}\left(\frac{1-T_l}{T_l}\right)^{\xi}\right]^{1/\xi}}, \dfrac{1}{1+\left[\frac{1}{m}\sum\limits_{l=1}^{m}\left(\frac{1-C_l}{C_l}\right)^{\xi}\right]^{1/\xi}} \\ \sqrt{1-\dfrac{1}{1+\left[\frac{1}{m}\sum\limits_{l=1}^{m}\left(\frac{U_l^2}{1-U_l^2}\right)^{\xi}\right]^{1/\xi}}}, \sqrt{1-\dfrac{1}{1+\left[\frac{1}{m}\sum\limits_{l=1}^{m}\left(\frac{F_l^2}{1-F_l^2}\right)^{\xi}\right]^{1/\xi}}} \end{array}\right\rangle$$

Theorem 4.13 (Commutativity)
suppose $\{\Omega_1,\Omega_2,.....,\Omega_m\},\Omega_l=\langle T_{\Omega_l},C_{\Omega_l},U_{\Omega_l},F_{\Omega_l}\rangle, l=1,2,...,m$ be a collection of QSVNPNs and $\{\Omega'_1,\Omega'_2,.....,\Omega'_m\},\Omega'_l=\langle T_{\Omega'_l},C_{\Omega'_l},U_{\Omega'_l},F_{\Omega'_l}\rangle, l=1,2,...,m$ be a permutation of $\{\Omega_1,\Omega_2,.....,\Omega_m\}$.

Then $QSVNPDOWGA\,(\Omega'_1,\Omega'_2,.....,\Omega'_m)=QSVNPDOWGA(\Omega_1,\Omega_2,.....,\Omega_m)$.

5. Model for MCDM Using Quadripartitioned Single-Valued Neutrosophic Pythagorean Information

In this section, a model for MCDM by using quadripartitioned single-valued Pythagorean information is proposed. Here the decision-maker gives the information in the form of quadripartitioned single-valued neutrosophic number form.

Let $A=\{a_1,a_2,.....,a_m\}$ denotes the set of attributes or alternatives denoted by a_i for $i=1,2,..,m$ and $C=\{c_1,c_2,.....,c_n\}$ indicates the set of criteria denoted by c_j for $j=1,2,..,\text{n}$.

An expert is engaged to provide his/her evaluation of an alternative a_i on a criterion c_j in the form of QSVNPN. The expert information is recorded in the form of a decision matrix denoted by $D^M=[\Omega_{ik}]_{m\times n}$ where $\Omega_{ij}=\langle T_{ij},C_{ij},U_{ij},F_{ij}\rangle$. Also, $W=(\varpi_1,\varpi_2,....,\varpi_n)$ is the weight vector of the decision-maker where $\sum_{p=1}^{n} \varpi_p = 1$ and $\varpi_p > 0$. The criteria can be of two types called benefit criteria and cost criteria. If in the decision matrix, there is any cost type criteria then it can be converted into the normalized decision matrix and it is given by:

$$ND^M = [s_{ik}] = \begin{cases} \Omega_{ik} = \langle T_{ik}, C_{ik}, U_{ik}, F_{ik} \rangle, \; \textit{for benefit criteria} \\ \Omega^c{}_{ik} = \langle F_{ik}, U_{ik}, C_{ik}, T_{ik} \rangle, \; \textit{for cost criteria} \end{cases}$$

The algorithm for the proposed model is given by:

Algorithm

Input:

Step1: Input the QSVNPNs given by the expert in the form of a decision matrix(DM).

Computations:

Step2: Normalize the decision matrix if it is required.

Step3: Calculate the collective information by using the proposed Dombi operators to evaluate the alternative preference values with associated weights.

Step4: Find the score $\Theta(A_p)$ and accuracy $\Lambda(A_p)$ values of the cumulative preference values.

Output:

Step5: Rank the alternatives and choose the best which has a maximum score value.

We utilize this algorithm in the following practical application.

6. An Application

Suppose Mr. X wants to buy a smartphone and for this, he has six available alternatives denoted by the set $M = \{M_1, M_2, M_3, M_4, M_5, M_6\}$. He wants to select the best alternative based on certain criteria denoted by the set

$C = \{C_1, C_2, C_3, C_4, C_5\}$, where C_1 =price, C_2 =battery capacity, C_3 =storage space, C_4 =camera quality, and C_5 =looks.

But there are certain issues with selecting the best alternative. All criteria have different units as price represents in a dollar, storage space in GB, battery capacity in MHA, camera quality in MP. So, we can't compare criteria with different units. Another issue is the use of linguistic terms. For example, we do not express the looks or appearance of mobile in units as it depends on the choice of the customers. Moreover, criteria are of two types, namely, non-beneficial and beneficial. Non-beneficial are those criteria whose lower value is desirable. For example, price or cost, we desire a product having a lower cost. On the other hand, beneficial criteria are those whose higher value is desirable. For example, we always desire a mobile with higher storage, having a high megapixel camera with excellent looks, and have high-quality battery. Because of this Mr. X engages a decision-maker or an expert. Firstly, the decision-maker assigns the weight vector associated with each criterion is given by $W = (0.1, 0.2, 0.3, 0.3, 0.1)$. The performance of all the alternatives based on the given criteria is determined by the decision-maker in the form of a decision matrix using the QSVNP information is given by:

Step1:

$$D^M = \begin{matrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{matrix} \begin{pmatrix} \overset{C_1}{\langle 0.3, 0.4, 0.5, 0.2 \rangle} & \overset{C_2}{\langle 0.3, 0.1, 0.4, 0.4 \rangle} & \overset{C_3}{\langle 0.3, 0.5, 0.3, 0.1 \rangle} & \overset{C_4}{\langle 0.2, 0.5, 0.4, 0.3 \rangle} & \overset{C_5}{\langle 0.3, 0.5, 0.1, 0.6 \rangle} \\ \langle 0.6, 0.7, 0.3, 0.2 \rangle & \langle 0.5, 0.3, 0.2, 0.4 \rangle & \langle 0.3, 0.4, 0.5, 0.3 \rangle & \langle 0.5, 0.7, 0.2, 0.3 \rangle & \langle 0.3, 0.2, 0.6, 0.7 \rangle \\ \langle 0.4, 0.3, 0.5, 0.4 \rangle & \langle 0.5, 0.3, 0.4, 0.4 \rangle & \langle 0.2, 0.4, 0.3, 0.4 \rangle & \langle 0.3, 0.5, 0.3, 0.4 \rangle & \langle 0.2, 0.1, 0.4, 0.6 \rangle \\ \langle 0.3, 0.6, 0.4, 0.3 \rangle & \langle 0.2, 0.1, 0.3, 0.4 \rangle & \langle 0.6, 0.3, 0.4, 0.3 \rangle & \langle 0.1, 0.2, 0.2, 0.4 \rangle & \langle 0.5, 0.6, 0.4, 0.3 \rangle \\ \langle 0.5, 0.5, 0.4, 0.3 \rangle & \langle 0.4, 0.3, 0.2, 0.2 \rangle & \langle 0.3, 0.3, 0.5, 0.3 \rangle & \langle 0.3, 0.5, 0.5, 0.3 \rangle & \langle 0.3, 0.2, 0.4, 0.6 \rangle \\ \langle 0.4, 0.6, 0.2, 0.4 \rangle & \langle 0.1, 0.4, 0.4, 0.6 \rangle & \langle 0.2, 0.4, 0.3, 0.6 \rangle & \langle 0.7, 0.3, 0.5, 0.1 \rangle & \langle 0.5, 0.3, 0.1, 0.3 \rangle \end{pmatrix}$$

As the criteria cost is a non-beneficiary criterion, therefore in the second step, we normalize the decision matrix:

Step2:

$$ND^M = \begin{matrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{matrix} \begin{pmatrix} \overset{C_1}{\langle 0.2, 0.5, 0.4, 0.3 \rangle} & \overset{C_2}{\langle 0.3, 0.1, 0.4, 0.4 \rangle} & \overset{C_3}{\langle 0.3, 0.5, 0.3, 0.1 \rangle} & \overset{C_4}{\langle 0.2, 0.5, 0.4, 0.3 \rangle} & \overset{C_5}{\langle 0.3, 0.5, 0.1, 0.6 \rangle} \\ \langle 0.2, 0.3, 0.7, 0.6 \rangle & \langle 0.5, 0.3, 0.2, 0.4 \rangle & \langle 0.3, 0.4, 0.5, 0.3 \rangle & \langle 0.5, 0.7, 0.2, 0.3 \rangle & \langle 0.3, 0.2, 0.6, 0.7 \rangle \\ \langle 0.4, 0.5, 0.3, 0.4 \rangle & \langle 0.5, 0.3, 0.4, 0.4 \rangle & \langle 0.2, 0.4, 0.3, 0.4 \rangle & \langle 0.3, 0.5, 0.3, 0.4 \rangle & \langle 0.2, 0.1, 0.4, 0.6 \rangle \\ \langle 0.3, 0.4, 0.6, 0.3 \rangle & \langle 0.2, 0.1, 0.3, 0.4 \rangle & \langle 0.6, 0.3, 0.4, 0.3 \rangle & \langle 0.1, 0.2, 0.2, 0.4 \rangle & \langle 0.5, 0.6, 0.4, 0.3 \rangle \\ \langle 0.3, 0.4, 0.5, 0.5 \rangle & \langle 0.4, 0.3, 0.2, 0.2 \rangle & \langle 0.3, 0.3, 0.5, 0.3 \rangle & \langle 0.3, 0.5, 0.5, 0.3 \rangle & \langle 0.3, 0.2, 0.4, 0.6 \rangle \\ \langle 0.4, 0.2, 0.6, 0.4 \rangle & \langle 0.1, 0.4, 0.4, 0.6 \rangle & \langle 0.2, 0.4, 0.3, 0.6 \rangle & \langle 0.7, 0.3, 0.5, 0.1 \rangle & \langle 0.5, 0.3, 0.1, 0.3 \rangle \end{pmatrix}$$

Step3: Compute the aggregate preference value of each alternative under all the criteria using the Dombi operator with $\xi = 3$ defined in **Theorem 4.2,** given as

$$\begin{matrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{matrix} \begin{pmatrix} \langle 0.279, 0.486, 0.189, 0.141 \rangle \\ \langle 0.458, 0.626, 0.238, 0.330 \rangle \\ \langle 0.411, 0.452, 0.317, 0.407 \rangle \\ \langle 0.527, 0.458, 0.259, 0.332 \rangle \\ \langle 0.338, 0.433, 0.294, 0.267 \rangle \\ \langle 0.626, 0.368, 0.189, 0.142 \rangle \end{pmatrix}$$

Step4: Next we calculate the score of each alternative given by:

$\Theta(M_1) = 0.258$, $\Theta(M_2) = 0.436$, $\Theta(M_3) = 0.107$, $\Theta(M_4) = 0.310$, $\Theta(M_5) = 0.144$ and $\Theta(M_6) = 0.471$

Step5: Rank the scores of all the alternatives, we have

$$\Theta(M_6) > \Theta(M_2) > \Theta(M_4) > \Theta(M_1) > \Theta(M_5) > \Theta(M_3)$$

Since the alternative M_6 is the best alternative as it has the highest rank among all. Therefore, Mr. X will buy M_6 mobile. If it is not available in the market then he will prefer the second-highest rank alternative i.e. M_2.

<u>Aliter:</u>

We repeat up to **step2**.

Step3: Compute the aggregate preference value of each alternative under all the criteria using the Dombi operator with $\xi = 3$ defined in **Theorem 4.8,** given as

$$\begin{matrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{matrix} \begin{pmatrix} \langle 0.237, 0.159, 0.375, 0.460 \rangle \\ \langle 0.304, 0.308, 0.573, 0.568 \rangle \\ \langle 0.243, 0.190, 0.350, 0.473 \rangle \\ \langle 0.139, 0.153, 0.464, 0.369 \rangle \\ \langle 0.311, 0.296, 0.479, 0.468 \rangle \\ \langle 0.154, 0.297, 0.488, 0.556 \rangle \end{pmatrix}$$

Step4: By using **definition 2.7,** the scores of each alternative are calculated as:

$\Theta(M_1) = -0.270$, $\Theta(M_2) = -0.463$, $\Theta(M_3) = -0.251$, $\Theta(M_4) = -0.308$, $\Theta(M_5) = -0.264$ and $\Theta(M_6) = -0.435$

Step5: Ranking of the alternatives is given by:

$$\Theta(M_3) > \Theta(M_5) > \Theta(M_1) > \Theta(M_4) > \Theta(M_6) > \Theta(M_2)$$

Here, M_3 is the best alternative.

To compare the results of the two methods, we consider the following table:

Table-1

Alternatives	**Ranking**		**Absolute Difference**	**Modified Rank**
	QSVNPDOWAA	**QSVNPDOWGA**		
M_1	**4**	**3**	**1**	**5.5**
M_2	**2**	**6**	**4**	**2.5**
M_3	**6**	**1**	**5**	**1**

M_4	**3**	**4**	**1**	**5.5**
M_5	**5**	**2**	**3**	**4**
M_6	**1**	**5**	**4**	**2.5**

As the two methods give two different optimal choices, therefore there is a hesitation for a decision-maker to choose the best alternative. To sort out such an issue, we consider the table-1, where we have determined the modified rank against each alternative. As M_3 has the highest modified rank, so M_3 is our preferred choice. In the case of a tie, there is more than one alternative under consideration.

Remark: Instead of taking $\xi = 3$ we can choose any value higher than or equal to 1. For each value of ξ we get the same rank of the alternatives.

7. Conclusion

In the present paper, we have studied the notion of QSVNPNs and their various operational laws based on Dombi operators. We also introduced two new Dombi operators, namely QSVNPDOWAA and QSVNPDOWGA in the quadripartitioned single-valued neutrosophic Pythagorean information. Based on these two operators we have studied different properties such as monotonicity, commutativity, reducibility, boundedness, and idempotency. Moreover, we have proposed a model for MCDM problems under the QSVNPN environment. Finally, for the practical application of the proposed model, a real-life based example is given by which we justify the feasibility and rationality of the model and shows that how it is effective in decision-making problem. In the future, the proposed model can be applied for risk management, disease diagnosis, control theory, MADM, game theory, and many other diverse fields for decision-making.

Acknowledgments: The Author is very much grateful to the anonymous reviewers for their valuable comments that help to improve the quality of the paper.

Conflicts of Interest: Declare no conflict of interest regarding the publication of the paper.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*, 338–353. doi:10.1016/s0019-9958(65)90241-x

[2] Klir, G., & Yuan, B. (1995). Fuzzy sets and fuzzy logic (Vol. 4). *New Jersey: Prentice hall*.

[3] Dubois, D. J. (1980). Fuzzy sets and systems: theory and applications (Vol. 144). *Academic press*.

[4] Zimmermann, H. J., Jones, A., & Kaufman, A. (Eds.). (1986). Fuzzy sets theory and applications. *D. Reidel*.

[5] Kahraman, C. (2008). Multi-criteria decision making methods and fuzzy sets. *Fuzzy Multi-Criteria Decision Making*, 1–18. doi:10.1007/978-0-387-76813-7_1

[6] Zimmermann, H. J. (1978). Fuzzy programming and linear programming with several objective functions. *Fuzzy sets and systems*, *1*, 45-55.

[7] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 20,* 87–96. doi:10.1016/s0165-0114(86)80034-3

[8] Yager, R. R. (2009). Some aspects of intuitionistic fuzzy sets. *Fuzzy Optimization and Decision Making, 8,* 67–90. doi:10.1007/s10700-009-9052-7

[9] De, S. K., Biswas, R., & Roy, A. R. (2001). An application of intuitionistic fuzzy sets in medical diagnosis. *Fuzzy sets and Systems*, *117*, 209-213.

[10] Zhang, H. (2014). Linguistic intuitionistic fuzzy sets and application in MAGDM. *Journal of Applied Mathematics, 2014,* 1–11. doi:10.1155/2014/432092

[11] Liang, Z., & Shi, P. (2003). Similarity measures on intuitionistic fuzzy sets. *Pattern recognition letters*, *24*, 2687-2693.

[12] Ye, J. (2011). Cosine similarity measures for intuitionistic fuzzy sets and their applications. *Mathematical and computer modelling*, *53*, 91-97.

[13] Yager, R. (2013). Pythagorean fuzzy subsets. *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS),* 57-61.

[14] Ejegwa, P. A. (2019). Pythagorean fuzzy set and its application in career placements based on academic performance using max–min–max composition. *Complex & Intelligent Systems.* doi:10.1007/s40747-019-0091-6

[15] Peng, X., & Selvachandran, G. (2019). Pythagorean fuzzy set: state of the art and future directions. *Artificial Intelligence Review*, *52,* 1873-1927.

[16] Garg, H. (2018). Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision-making process. *International Journal of Intelligent Systems*, *33*, 1234-1263.

[17] Zhang, X., & Xu, Z. (2014). Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *International Journal of Intelligent Systems*, *29*, 1061-1078.

[18] Garg, H. (2016). A novel correlation coefficients between Pythagorean fuzzy sets and its applications to decision-making processes. *International Journal of Intelligent Systems*, *31*, 1234-1252.

[19] Smarandache, F. (2005). Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, *24*, 287.

[20] Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). *Single valued neutrosophic sets*. Infinite study.

[21] Jansi, R., Mohana, . K., & Smarandache, F. (2019). Correlation measure for Pythagorean neutrosophic sets with T and F as dependent neutrosophic components. *Neutrosophic Sets and Systems*, *30*, 202-212.

[22] Chatterjee, R., Majumdar, P., & Samanta, S. K. (2016). On some similarity measures and entropy on quadripartitioned single valued neutrosophic sets. *Journal of Intelligent & Fuzzy Systems*, *30*, 2475-2485.

[23] Radha, R., & Mary, A. S. A. (2020). Quadri partitioned neutrosophic Pythagorean set. *Internation Journal of Research Publication and Review, 2,* 276-281.

[24] Dombi, J. (1982). A general class of fuzzy operators, the demorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. *Fuzzy Sets and Systems, 8,* 149–163. doi:10.1016/0165-0114(82)90005-7

[25] Roychowdhury, S., & Wang, B. H. (1994). Composite generalization of Dombi class and a new family of T-operators using additive-product connective generator. *Fuzzy Sets and Systems, 66,* 329–346. doi:10.1016/0165-0114(94)90100-7

[26] Seikh, M. R., & Mandal, U. (2021). Intuitionistic fuzzy Dombi aggregation operators and their application to multiple attribute decision-making. *Granular Computing*, *6*, 473-488.

[27] Wei, G., & Wei, Y. (2018). Some single-valued neutrosophic Dombi prioritized weighted aggregation operators in multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *35*, 2001-2013.

[28] Jana, C., Pal, M., & Wang, J. (2018). Bipolar fuzzy Dombi aggregation operators and its application in multiple-attribute decision-making process. *Journal of Ambient Intelligence and Humanized Computing.* doi:10.1007/s12652-018-1076-9

[29] Ashraf, S., Abdullah, S., & Mahmood, T. (2019). Spherical fuzzy Dombi aggregation operators and their application in group decision making problems. *Journal of Ambient Intelligence and Humanized Computing.* doi:10.1007/s12652-019-01333-y

[30] Garg, H. (2019). Neutrality operations-based Pythagorean fuzzy aggregation operators and its applications to multiple attribute group decision-making process. *Journal of Ambient Intelligence and Humanized Computing.* doi:10.1007/s12652-019-01448-2

[31] Qiyas, M., Abdullah, S., Ashraf, S., & Abdullah, L. (2019). Linguistic picture fuzzy Dombi aggregation operators and their application in multiple attribute group decision making problem. *Mathematics, 7, 764.* doi:10.3390/math7080764

[32] Akram, M., Dudek, W. A., & Dar, J. M. (2019). Pythagorean Dombi fuzzy aggregation operators with application in multicriteria decision-making. *International Journal of Intelligent Systems.* doi:10.1002/int.221 83

[33] Khan, A., Ashraf, S., Abdullah, S., Qiyas, M., Luo, J., & Khan, S. (2019). Pythagorean fuzzy Dombi aggregation operators and their application in decision support system. *Symmetry, 11, 383.* doi:10.3390/sym11030383

[34] Jana, C., Senapati, T., & Pal, M. (2019). Pythagorean fuzzy Dombi aggregation operators and its applications in multiple attribute decision-making. *International Journal of Intelligent Systems, 34,* 2019-2038. doi:10.1002/int.22125

[35] Akram, M., Yaqoob, N., Ali, G., & Chammam, W. (2020). Extensions of Dombi aggregation operators for decision making under m-Polar fuzzy information. *Journal of Mathematics, 2020, 1–20.* doi:10.1155/2020/4739567

[36] Mahmood, M. K., Zeng, S., Gulfam, M., Ali, S., & Jin, Y. (2020). Bipolar neutrosophic Dombi aggregation operators with application in multi-attribute decision making problems. *IEEE Access, 8, 156600–156614.* doi:10.1109/access.2020.3019485

[37] Zulqarnain, R. M., Xin, X. L., Saqlain, M., & Khan, W. A. (2021). TOPSIS method based on the correlation coefficient of interval-valued intuitionistic fuzzy soft sets and aggregation operators with their application in decision-making. *Journal of Mathematics*, *2021*. https://doi.org/10.1155/2021/6656858.

[38] Zulqarnain, R. M., Xin, X. L., Siddique, I., Khan, W. A., & Yousif, M. A. (2021). TOPSIS method based on correlation coefficient under pythagorean fuzzy soft environment and its application towards green supply chain management. *Sustainability*, *13*, 1642. https://doi.org/10.3390/su13041642.

[39] Zulqarnain, R. M., Xin, X. L., Ali, B., Broumi, S., Abda, S., & Ahamad, M. I. (2021). Decision-making approach based on correlation coefficient with its properties under interval-valued neutrosophic hypersoft set environment. *Neutrosophic Sets and Systems*, *40*, 12-28.

[40] Zulqarnain, R. M., Siddique, I., Jarad, F., Ali, R., & Abdeljawad, T. (2021). Development of TOPSIS technique under Pythagorean fuzzy hypersoft environment based on correlation coefficient and its application towards the selection of antivirus mask in covid-19 pandemic. *Complexity*, *2021*. https://doi.org/10.1155/2021/6634991.

[41] Zulqarnain, R. M., Garg, H., Siddique, I., Ali, R., Alsubie, A., Hamadneh, N. N., & Khan, I. (2021). Algorithms for a generalized multipolar neutrosophic soft set with information measures to solve medical diagnoses and decision-making problems. *Journal of Mathematics, 2021,* 1-30. https://doi.org/10.1155/2021/6654657.

[42] Samad, A., Zulqarnain, R. M., Sermutlu, E., Ali, R., Siddique, I., Jarad, F., & Abdeljawad, T. (2021). Selection of an effective hand sanitizer to reduce covid-19 effects and extension of TOPSIS technique based on correlation coefficient under neutrosophic hypersoft set. *Complexity*, *2021*, 1-22. https://doi.org/10.1155/2021/5531830.

[43] Zulqarnain, R. M., Xin, X. L., & Saeed, M. (2020). Extension of TOPSIS method under intuitionistic fuzzy hypersoft environment based on correlation coefficient and aggregation operators to solve decision making problem. *AIMS Mathematics*, *6*, 2732-2755.

[44] Zulqarnain, R. M., Siddique, I., Ali, R., Jarad, F., Samad, A., & Abdeljawad, T. (2021). Neutrosophic hypersoft matrices with application to solve multiattributive decision-making problems. *Complexity*, *2021*. https://doi.org/10.1155/2021/5589874.

[45] Zulqarnain, R. M., Xin, X. L., Garg, H., & Ali, R. Interaction aggregation operators to solve multi criteria decision making problem under pythagorean fuzzy soft environment. *Journal of Intelligent & Fuzzy Systems*, (Preprint), 1-21.

[46] Zulqarnain, R. M., Saeed, M., Bagh, A. L. İ., Abdal, S., Saqlain, M., Ahamad, M. I., & Zafar, Z. (2020). Generalized fuzzy TOPSIS to solve multi-criteria decision-making problems. *Journal of New Theory*, 32, 40-50.

[47] Zulqarnain, R. M., Xin, X. L., Saqlain, M., & Smarandache, F. (2020). Generalized aggregate operators on neutrosophic hypersoft set. *Neutrosophic Sets and Systems*, *36*, 271-281.

[48] Zulqarnain, R. M., Xin, X. L., Saeed, M., Smarandache, F. ., & Ahmad, N. (2020). Generalized neutrosophic TOPSIS to solve multi-criteria decision-making problems. *Neutrosophic Sets and Systems*, *38*, 276-292.

[49] Zulqarnain, R. M., Xin, X. L., Garg, H., & Khan, W. A. (2021). Aggregation operators of pythagorean fuzzy soft sets with their application for green supplier chain management. *Journal of Intelligent & Fuzzy Systems*, 40 , 5545-5563. doi: 10.3233/JIFS-202781

[50] Zulqarnain, R. M., Xin, X. L., Saqlain, M., Smarandache, F., & Ahamad, M. I. (2021). An integrated model of neutrosophic TOPSIS with application in multi-criteria decision-making problem. *Neutrosophic Sets and Systems*, *40*, 118-133.

[51] Zulqarnain, R. M., Siddique, I., Ali, R., Pamucar, D., Marinkovic, D., & Bozanic, D. (2021). Robust aggregation operators for intuitionistic fuzzy hypersoft set with their application to solve MCDM problem. *Entropy*, *23*, 688. https://doi.org/10.3390/e23060688

Received: May 4, 2021. Accepted: October 1, 2021

On β generalized α closed sets in Neutrosophic Topological spaces

M. Gomathi[1], V. Keerthika[2]

[1]Department of Science and Humanities, Sri Krishna College of Engineering and Technology, Coimbatore,Tamil Nadu,India.

gomathimathaiyan@gmail.com, gomathim@skcet.ac.in

[2]Department of Science and Humanities, Sri Krishna College of Engineering and Technology, Coimbatore,TamilNadu,India.

krt.keerthika@gmail.com, keerthikav@skcet.ac.in

Abstract: In this paper a new concept of neutrosophic closed sets called neutrosophic β generalized α closed set is introduced and their properties are thoroughly studied and analyzed and also discuss their relationship between basic closed sets in neutrosophic topological spaces. Some new interesting theorems are presented using newly introduced set.

Keywords: Neutrosophic β generalized α closed sets.

---

1. Introduction

The concept of intuitionistic fuzzy sets introduced by Atanassov(1), intuitionistic fuzzy topological spac by Coker(2),after that Floretin Smarandache(3) in 1999 extended the neutrosophic sets, neutrosophic topological spaces by A. A. Salama and S. A. Alblowi(9). Further the basic sets like neutrosophic open sets(N-OS), neutrosophic semi open sets(N-SOS), neutrosophic pre open sets(N-POS), neutrosophic α open sets(N- αOS), neutrosophic regular open sets(N-ROS), neutrosophic β open sets(N- βos), neutrosophic b open sets(N-bOS) are introduced in neutrosophic topological spaces and their properties are studied by various authors(8,10).

The main aim of this paper is to analyze a new concept of neutrosophic closed sets called neutrosophic β generalized α closed sets also specialized some of their basic properties with examples.

2. Preliminaries:

Here in this paper (X, τ) is the neutrosophic topological space .Also the neutrosophic interior is denoted by N-Int(A), neutrosophic closure is denoted by N-Cl(A) and

the complement of a neutrosophic set A is denoted by N-C(A) and the empty and whole sets are denoted by 0 and 1 respectively.

Definition 2.1: Let X be a non-empty fixed set. A neutrosophic set (NS) A is an object having the form A = $\{\langle x, \mu_{A(x)}, \sigma_{A(x)}, \nu_{A(x)}\rangle: x\in X\}$ where $\mu_A(x)$ represent the degree of membership , $\sigma_A(x)$ represent degree of indeterminacy and $\nu_A(x)$ represent the degree of nonmembership

Nonmembership respectively of each element x ∈ X to the set A.

A Neutrosophic set A = $\{\langle x, \mu_{A(x)}, \sigma_{A(x)}, \nu_{A(x)}\rangle: x\in X\}$ can be identified as an ordered triple $\langle \mu_A, \sigma_A, \nu_A \rangle$ in]-0, 1+[on X.

Definition 2.2: Let A = $\langle \mu_A, \sigma_A, \nu_A \rangle$ be a NS on X, then the complement Neutrosophic-C(A) may be defined as

1. Neutrosophic -C (A) = $\{\langle x, (1-\mu_A (x)), (1- \nu_A(x)) \rangle: x \in X\}$

2. Neutrosophic -C (A) = $\{\langle x, \nu_A(x), \sigma_A(x), \mu_A (x) \rangle: x \in X\}$

3. Neutrosophic -C (A) = $\{\langle x, \nu_A(x), (1-\sigma_A(x)), \mu_A (x) \rangle: x \in X\}$

Definition 2.3: For any two neutrosophic sets A = $\{\langle x, \mu_A (x), \sigma_A(x), \nu_A(x) \rangle: x\in X\}$ and

B = $\{\langle x, \mu_B (x), \sigma_B(x), \nu_B(x) \rangle: x\in X\}$ is

1. $(A \subseteq B) \Leftrightarrow \mu_A (x) \leq \mu_B (x), \sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x) \ \forall \ x \in X$

2. $(A \subseteq B) \Leftrightarrow \mu_A (x) \leq \mu_B (x), \sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x) \ \forall \ x \in X$

3. $(A \cap B) \Leftrightarrow \mu_A (x) \wedge \mu_B (x), \sigma_A(x) \wedge \sigma_B(x)$ and $\nu_A(x) \vee \nu_B(x)$

4. $(A \cap B) \Leftrightarrow \mu_A (x) \wedge \mu_B (x), \sigma_A(x) \vee \sigma_B(x)$ and $\nu_A(x) \vee \nu_B(x)$

5. $(A \cup B) \Leftrightarrow \mu_A (x) \vee \mu_B (x), \sigma_A(x) \vee \sigma_B(x)$ and $\nu_A(x) \wedge \nu_B(x)$

6. $(A \cup B) \Leftrightarrow \mu_A (x) \vee \mu_B (x), \sigma_A(x) \wedge \sigma_B(x)$ and $\nu_A(x) \wedge \nu_B(x)$

Definition 2.4: A neutrosophic topology (NT) on a non-empty set X is a family τ of neutrosophic subsets in X satisfies the following axioms:

(NT_1) 0_N, $1_N \in \tau$

(NT_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$

(NT_3) $\cup G_i \in \tau \ \forall \ \{G_i : i \in J\} \subseteq \tau$

In this case the pair (X, τ) is a neutrosophic topological space (NTS) and any neutrosophic set in τ is known as a neutrosophic open set (N-OS) in X. A neutrosophic set A is a neutrosophic closed set (NCS) if and only if its complement N-C(A) is a neutrosophic open set in X.

Definition 2.6: A neutrosophic set A of a NTS X is said to be

(i) A neutrosophic pre-open set (NP-OS) if A ⊆ NInt(NCl(A))

(ii) A neutrosophic semi-open set (NS-OS) if A ⊆ NCl(NInt(A))

(iii) A neutrosophic α-open set (Nα-OS) if A ⊆ NInt(NCl(NInt(A)))

(iv) A neutrosophic β-open set (N β-OS) if A ⊆ N-cl(N-int(N-cl(A))).

(v) A neutrosophic regular open set (N-ROS) if N-int(N-cl(A)) = A,

(vi) A neutrosophic b open set (N-bOS) if A ⊆ N-int(N-cl(A)) ∪ N-cl(N-int(A))

Definition 2.7: A neutrosophic set A of a NTS X is said to be

(i) A neutrosophic pre-closed set (NP-CS) if NCl(NInt(A)) ⊆ A

(ii) A neutrosophic semi-closed set (NS-CS) if NInt(NCl(A)) ⊆ A

(iii) A neutrosophic α-closed set (Nα-CS) if NCl(NInt(NCl(A))) ⊆ A

(iv) A neutrosophic β-closed set (Nα-CS) if Nint(Ncl(Nint(A))) ⊆ A

(v)A neutrosophic regular closed set (N-RCS) if N-cl(N-int(A)) = A,

(vi) A neutrosophic b closed set (N-bCS) if N-int(N-cl(A)) ∩ N-cl(N-int(A)) ⊆ A

Definition 2.8:

Consider a NS A in NTS. The Neutrosophic beta interior & Neutrosophic beta closure of A are defined as

Nβint(A)= ∪ { G, G is a N-βOS in X and G ⊆A}

Nβcl(A)= ∩ { G, G is a N-βOS in X and A ⊆ K}

Remark 2.9 :

Consider a NS A in NTS, then

(1) Nβcl(A)=A Nint(Ncl(Nint(A))),

(2) Nβint(A)=A∩Ncl(Nint(Ncl(A))).

Definition 2.10:

Consider a NS A in NTS. Then it is a Neutrosophic generalized beta closed set (N-GβCS) if N-βcl(A) ⊆ U whenever A ⊆ U and U is a NOS.

3. Neutrosophic β generalized α closed sets in Topological spaces

In this section we have introduced Neutrosophic β generalized α closed sets and studied some of their properties.

Definition 3.1: An Neutrosophic set A in an NTS (X, τ) is said to be an neutrosophic β generalized α closed set (NβGαCS) if Nβcl(A) ⊆ U whenever A ⊆ U and U is an N-αOS in (X, τ).

The family of all N-βGαCSs of an NTS (X, τ) is denoted by N-βGαC(X).

Example 3.2: Let X = {a, b} and G_1 = ⟨x, (0.5_a, 0.5_b), (0.5_a, 0.5_b), (0.5_a, 0.5_b)⟩ and G_2 = ⟨x, (0.4_a, 0.3_b), (0.6_a, 0.7_d), (0.6_a, 0.7_b)⟩. Then τ = {0, G_1, G_2, 1} is an NT on X. Here

Let S = ⟨x, (0.3_a, 0.2_b), (0.7_a, 0.8_b), (0.7_a, 0.8_b)⟩ then S is called an N-βGαCS in X.

Proposition 3.3: Every Neutrosophic-CS is an Neutrosophic -βGαCS in (X, τ) but reverse process is not true in general.

Proof: Let S be an N-CS in X. Let we take S ⊆ U where U is said to be an N-αOS in X. As N-βcl(S) ⊆ N-cl(S) = S ⊆ U by hypothesis, we have N-βcl(S) ⊆ U. Thus S is an N-βGαCS in (X, τ).

Example 3.4: Let X = {a, b}, G_1 = ⟨x, (0.5_a, 0.5_b), (0.5_a, 0.5_b), (0.5_a, 0.5_b)⟩ and G_2 = ⟨x, (0.4_a, 0.3_b), (0.6_a, 0.7_d), (0.6_a, 0.7_b)⟩. Then τ = {0, G_1, G_2, 1} is an NT on X.

Here we take S = ⟨x, (0.3a, 0.2b), (0.7a, 0.8b), (0.7a, 0.8b)⟩ is an (N)-βGαCS but it not an (N)-CS in (X, τ) since N-cl(S) = $G_1^c \neq$ S.

Proposition 3.5: Every Neutrosophic -SCS is an Neutrosophic -βGαCS in (X, τ) but reverse process is not true in general.

Proof: Let A be an N-SCS in X. Let we take A ⊆ U and U is said to be an N-αOS in X. As N-βeta closure(A) ⊆ N-semi closure(A) = A ⊆ U by hypothesis, we have N-βeta closure (A) ⊆ U. Then A is called an N-βGαCS in (X, τ).

Example 3.6: Let X = {a, b}, G_1 = ⟨x, (0.5a, 0.5b), (0.5a, 0.5b), (0.5a, 0.5b)⟩ and G_2 = ⟨x, (0.4a, 0.3b), (0.6a, 0.7b), (0.6a, 0.7b)⟩. Then τ = {0, G_1, G_2, 1} is an NT on X.
Here we take a point S = ⟨x, (0.3a, 0.2b), (0.7a, 0.8b), (0.7a, 0.8b)⟩ which satisfy N-βGαCS but does not satisfy N-SCS in (X, τ) since N-int(N-cl(S)) = N-int(G_1^c) = $G_1 \nsubseteq$ S.

Proposition 3.7: Every Neutrosophic-PCS is an Neutrosophic -βGαCS in (X, τ) but reverse process is not true in general.

Proof: Let A be an N-PCS in X. Let we take A ⊆ U and U is said to be an N-αOS in X. As N-βeta closure(A) ⊆ N-Pre closure(A) = A ⊆ U by hypothesis, we have N-βeta closure (A) ⊆ U. Then A is called an N-βGαCS in (X, τ).

Example 3.8: Let X = {a, b}, G_1 = ⟨x, (0.5a, 0.6b), (0.5a, 0.4b), (0.5a, 0.4b)⟩ and G_2 = ⟨x, (0.4a, 0.3b), (0.6a, 0.7b), (0.6a, 0.7b) ⟩. Then τ = {0, G_1, G_2, 1} is an NT on X. Here we take A = ⟨x, (0.4a, 0.3b), (0.6a, 0.7b), (0.6a, 0.7b)⟩ which satisfy N-βGαCS but does not satisfy N-PCS in (X, τ) as N-cl(N-int(A)) = N-cl(G_2) = $G_1^c \nsubseteq$ A.

Proposition 3.9: Every Neutrosophic-αCS is an Neutrosophic-βGαCS in (X, τ) but not conversely in general.

Proof: Let A be an N-αCS in X. Let as assume A ⊆ U and U is said to be an N-αOS in X. As N-βcl(A) ⊆ N-αcl(A) = A ⊆ U by hypothesis, we have N-βcl(A) ⊆ U. Then A is an N-βGαCS in (X, τ).

Example 3.10: Let X = {a, b}, G_1 = ⟨x, (0.5_a, 0.5_b), (0.5_a, 0.5_b), (0.5_a, 0.5_b)⟩ and G_2 = ⟨x, (0.4_a, 0.3_b), (0.6_a, 0.7_b), (0.6_a, 0.7_b) ⟩. Then τ = {0, G_1, G_2, 1} is an IFT on X. Here we take S = ⟨x, (0.3_a, 0.2_b), (0.7_a, 0.8_b), (0.7_a, 0.8_b)⟩ which satisfy N-βGαCS but does not satisfy N-αCS in (X, τ) since N-cl(N-int(N-cl(S))) = N-cl(N-int(G_1^c)) = N-cl(G_1) = $G_1^c \not\subseteq$ S.

Proposition 3.11: Every Neutrosophic-bCS is an Neutrosophic-βGαCS in (X, τ) but reverse process is not true in general.

Proof: Let A be an NbCS in X. Let as assume A ⊆ U and U is said to be an N-αOS in X. As N-βcl(A) ⊆ N-bcl(A) = A ⊆ U by hypothesis, we have N-βcl(A) ⊆ U. Then A is an N-βGαCS in (X, τ).

Example 3.12: Let X = {a, b}, G_1 = ⟨x, (0.5_a, 0.3_b), (0.5_a, 0.7_b), (0.5_a, 0.7_b)⟩ and G_2 = ⟨x, (0.4_a, 0.3_b), (0.6_a, 0.7_b), (0.6_a, 0.7_b)⟩. Then τ = {0, G_1, G_2, 1} is an NT on X. Here we assume A = ⟨x, (0.4_a, 0.6_b), (0.4_a, 0.4_b), (0.4_a, 0.4_b)⟩ which satisfy N-βGαCS but does not satisfy N-bCS in (X, τ) since N-int(N-cl(A)) ∩ N-cl(N-int(A)) = $G_1 \cap G_1^c$ = $G_1 \not\subseteq$ A.

Proposition 3.13: Every Neutrosophic-RCS is an Neutrosophic-βGαCS in (X, τ) but reverse process is not true in general.

Proof: Let A be an N-RCS in X. Then A is an N-CS as every N-RCS is an N-CS, A is an N-βGαCS in (X, τ).

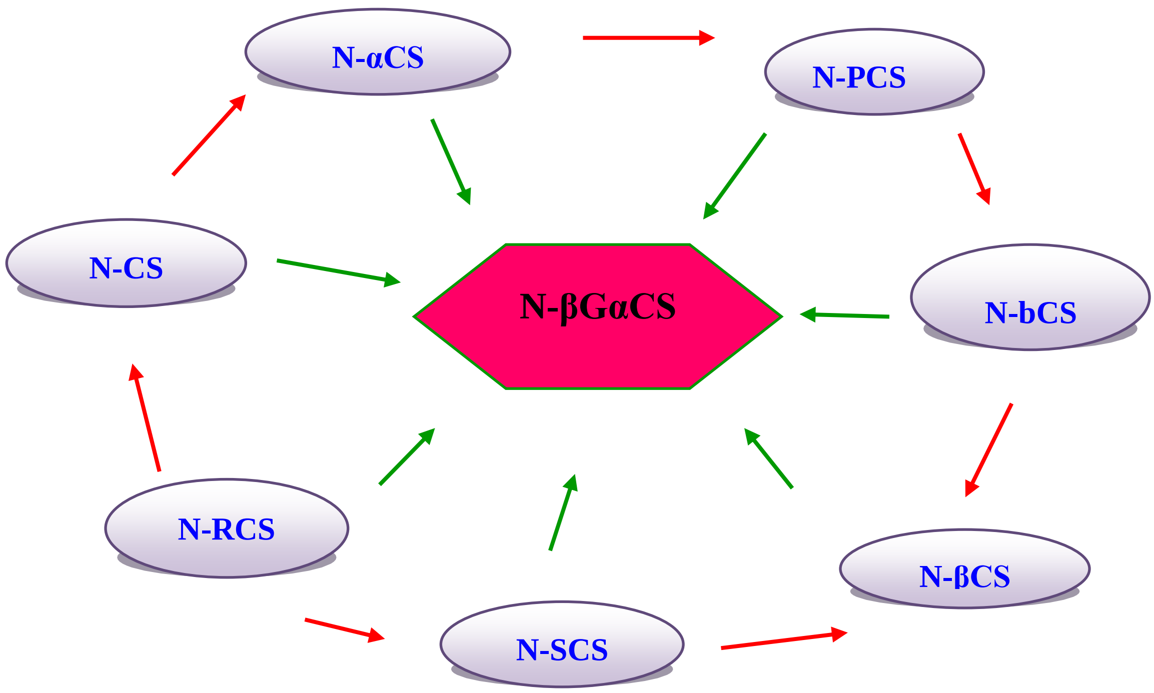
Example 3.14: Let X = {a, b}, G_1 = ⟨x, (0.5_a, 0.6_b), (0.5_a, 0.4_b), (0.5_a, 0.4_b)⟩ and G_2 = ⟨x, (0.4_a, 0.3_b), (0.6_a, 0.7_b), (0.6_a, 0.7_b)⟩. Then τ = {0, G_1, G_2, 1} is an NT on X. Here we take A = ⟨x, (0.4_a, 0.3_b), (0.6_a, 0.7_b)⟩ which satisfy N-βGαCS but does not satisfy N-RCS in (X, τ) as N-cl(N-int(A)) = N-cl(G_2) = $G_1^c \neq$ A.

Proposition 3.15: Every Neutrosophic-βCS is an Neutrosophic-βGαCS in (X, τ) but reverse process is not true in general.

Proof: Let A be an N-βCS in X. Let as assume A ⊆ U and U is said to be an N-αOS in X. Now N-βcl(A) = A ⊆ U, by hypothesis. Therefore we have N-βcl(A) ⊆ U. Hence A is an N-βGαCS in (X, τ).

Example 3.16: Let X = {a, b}, G_1 = ⟨x, (0.5a, 0.3b), (0.5a, 0.7b), (0.5a, 0.7b)⟩ and G_2 = ⟨x, (0.4a, 0.3b), (0.6a, 0.7b), (0.6a, 0.7b)⟩. Then τ = {0, G_1, G_2, 1} is an NT on X. Here A = ⟨x, (0.4a, 0.6b), (0.4a, 0.4b), (0.4a, 0.4b)⟩ which satisfy N-βGαCS but does not satisfy N-βCS in (X, τ) as N-int(N-cl(N-int(A))) = N-int(N-cl(G_2)) = N-int(G_1^c) = $G_1 \nsubseteq$ A.

In the following diagram, we have provided the relation between various types of neutrosophic closedness.



Remark 3.17: The union of any two N-βGαCSs is not an N-βGαCS in general as seen in the following example.

Example 3.18: Let us assume X = {a, b}, G_1 = ⟨x, (0.5a, 0.6b), (0.5a, 0.4b), (0.5a, 0.4b)⟩, G_2 = ⟨x, (0.2a, 0.3b), (0.8a, 0.7b), (0.8a, 0.7b)⟩ and G_3 = ⟨x, (0.6a, 0.7b), (0.4a, 0.3b), (0.4a, 0.3b)⟩. Then τ be {0, G_1, G_2, G_3, 1} is an Neutrosophic Topology on X.

The NSs A = ⟨x, (0.1_a, 0.5_b), (0.9_a, 0.5_b), (0.9_a, 0.5_b)⟩, B = ⟨x, (0.5_a, 0.2_b), (0.5_a, 0.8_b), (0.5_a, 0.8_b)⟩ are N-βGαCSs in (X, τ). But A ∪ B is not an N-βGαCS as A ∪ B = ⟨x, (0.5_a, 0.5_b), (0.5_a, 0.5_b), (0.5_a, 0.5_b)⟩ ⊆ G_1 but N-βcl(A ∪ B) = ⟨x, (0.6_a, 0.7_b), (0.4_a, 0.3_b), (0.4_a, 0.3_b)⟩ ⊈ G_1.

Remark 3.19: The intersection of any two N-βGαCSs is not an N-βGαCS in general as seen in the following example.

Example 3.20: Let X = {a, b}, G_1 = ⟨x, (0.5_a, 0.6_b), (0.5_a, 0.4_b), (0.5_a, 0.4_b)⟩, G_2 = ⟨x, (0.2_a, 0.3_b), (0.8_a, 0.7_b), (0.8_a, 0.7_b)⟩ and G_3 = ⟨x, (0.6_a, 0.7_b), (0.4_a, 0.3_b), (0.4_a, 0.3_b)⟩. Then τ = {0, G_1, G_2, G_3, 1} is an NT on X. The IFSs A = ⟨x, (0.5_a, 0.8_b), (0.5_a, 0.2_b), (0.5_a, 0.2_b)⟩, B = ⟨x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)⟩ are N-βGαCSs in (X, τ). But A ∩ B is not an N-βGαCS as A ∩ B = ⟨x, (0.5_a, 0.6_b), (0.5_a, 0.4_b), (0.5_a, 0.4_b)⟩ ⊆ G_1 but N-βcl(A ∩ B) = ⟨x, (0.6_a, 0.7_b), (0.4_a, 0.3_b), (0.4_a, 0.3_b)⟩ ⊈ G_1.

Proposition 3.21: Let (X, τ) be an NTS. Then for every A ∈ N-βGαC(X) and for every B ∈ NS(X), A ⊆ B ⊆ N-βcl(A) ⇒B ∈ N-βGαC(X).

Proof: Let B ⊆ U and also U be an N-α open set in X. Then since A ⊆ B, A ⊆ U. By hypothesis B ⊆ N-βcl(A). Therefore N-βcl(B) ⊆ N-βcl(N-βcl(A)) = N-βcl(A) ⊆ U, since A is an N-βGαCS in X. Hence B ∈ N-βGαC(X).

Proposition 3.22: If R is an N-αOS and an N-βGαCS in (X, τ), then R is an N-βCS in (X, τ).

Proof: Since R ⊆ R and R is an N-αOS in X, by hypothesis N-βcl(R) ⊆ R. But R ⊆ N-βcl(R). Therefore N-βcl(R) = R. Then R is an N-βCS in (X, τ).

Proposition 3.23: Let H ⊆ R ⊆ X where R is said to be an N-αOS and it is an N-βGαCS in X. Then H is an N-βGαCS in R if and only if H is an N-βGαCS in X.

Proof: Necessity: Let as assume J be an N-αOS in X and F ⊆ J. Also let H be an N-βGαCS in R. Then clearly H ⊆ R ∩ J and R ∩ J is an N-αOS in A. Hence neutrosophic beta closure of H in R, N-βcl$_R$(H) ⊆ R ∩ J and by Proposition 3.22, R is an N-βCS. Therefore N-βcl(R) = R. Now neutrosophic beta closure of H in X, N-βcl(H) ⊆ N-βcl(H) ∩ N-βcl(R) = N-βcl(H) ∩ R = N-βcl$_R$(H) ⊆ R ∩ J ⊆ J, that is N-βcl(H) ⊆ R, whenever H ⊆ R. Hence H is called N-βGαCS in X.

Sufficiency: Let V be an N-αOS in A, such that F ⊆ V. Since A is an N-αOS in X, V is an N-αOS in X. Therefore N-βcl(F) ⊆ V, as F is an N-βGαCS in X. Thus, N-βcl$_A$(F) = N-βcl(F) ∩ A ⊆ V ∩ A ⊆ V. Hence F is an N-βGαCS in A.

Proposition 3.24: An NS A is both an N-OS and an N-βGαCS if and only if A is an N-ROS in X.

Proof: Necessity: Let A be both an N-OS and an N-βGαCS in X. Then A is an N-αOS and an N-βGαCS. By Proposition 3.22, A is an N-βCS and N-int(N-cl(N-int(A))) ⊆ A. Since A is an N-OS, N-int(A) = A. Therefore N-int(N-cl(A)) ⊆ A. Since A is an N-OS, it is an N-POS. Hence A ⊆ N-int(N-cl(A)). Therefore A = N-int(N-cl(A)) and A is an N-ROS in X.

Sufficiency: Let A be an N-ROS in X then A= N-int(N-cl(A)). Since every N-ROS is an N-OS, A is an N-OS. We know N-int(N-cl(N-int(A))) = N-int(N-cl(A)) = A ⊆ A. Therefore A is an Neutrosophic β Closed set in X, and by Proposition 3.15, A is an N-βGαCS in X.

Proposition 3.25: Let (X, τ) be an NTS. Then N-βC(X) = N-βGαC(X) if every IFS in (X, τ) is an N-αOS in X.

Proof: Suppose that every NS in (X, τ) is an N-αOS in X. Let A ∈ N-βGαC(X). Then A is also an N-αOS by hypothesis. We know that A is an N-βCS. Therefore A ∈ N-βC(X).

Hence N-βGαC(X) ⊆ N-βC(X) ⟶ (i)

Let A ∈ N-βC(X). Then by Proposition 3.15, A is an N-βGαCS and A ∈ N-βGαC(X). Hence N-βC(X) ⊆ N-βGαC(X) ⟶ (ii). From (i) and (ii) N-βC(X) = N-βGαC(X).

Proposition 3.26: Let R be an N-αOS and an N-βGαCS of (X, τ). Then R ∩ F is an N-βGαCS of (X, τ) where F is an N-CS of X.

Proof: Suppose that R is an N-αOS and an N-βGαCS of (X, τ), then by Proposition 3.22, R is an N-βCS. But F is an N-CS in X. Hence R ∩ F is an N-βCS as every N-CS is an N-βCS. Then R ∩ F is an N-βGαCS in X.

References

1. Atanassov, K(1986)., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 87- 96.
2. Coker, D(1997)., An introduction to intuitionistic fuzzy topological spaces, Fuzzy sets and systems, 88, 81-89.
3. Floretin Smarandache(2010), Neutrosophic set:- A generalization of Intuitionistic fuzzy set, Jour. Of Defense Resources Management, 107-116.
4. Gomathi, M and Jayanthi, D(2017)., On intuitionistic fuzzy β generalized α closed sets, Global Journal of Pure and Applied Mathematics, 13(6), 2439-2455.
5. Jayanthi, D(2018), "α Generalized Closed sets in Neutrosophic Topological spaces", IJMTT, March-2018.
6. Narmatha. S, Glory Bebina. E, Vishnu Priyaa.R (2019).,"On πgβ–Closed Sets and Mappings in Neutrosophic Topological Spaces" International Journal of Innovative Technology and Exploring Engineering,Vol-8,pp. 505-510.
7. Nivetha, A. R. M. Vigneshwaran, N. M. Ali Abbas and S. M. Khalil, On N∗gα-contimuous in topological spaces of neutrosophy, J. Interdis. Math. to appear (2021).
8. Pusphpalatha, A and Nandhini(2019), T "Generalized closed sets via neutrosophic topological spaces", Malaya Journal of Mathematik", vol.7, pp.50-54.
9. Renu Thomas and Anila.S(2018), "On Neutrosophic semi-preopen sets and semi-preclosed sets in Neutrosophic Topological Spaces", International Journal of Scientific Research in Mathematical & Statistical Sciences, vol.5, pp.138-143.
10. Salama, A. A and Alblowi, S. A,(2012), "On Neutrosophic sets and Neutrosophic Topological Spaces," IOSR Jour.of Mathematics, 31-35.
11. Sreeja.D, Sarankumar.T, 2018, Generalized Alpha Closed Sets in Neutrosophic Topological Spaces, Journal of Applied Science and Computations, Vol.5(11), pp.1816-1823.
12. Smarandache F 2002 Neutrosophy and neutrosophic logic 1st Int. Conf. on Neutrosophy, Neutrosophic Logic, Set, Probability and Statistics University of New Mexico Gallup NM 87301 USA

13. Supriti Saha 1987 Fuzzy δ-continuous mappings Journal of Mathematical Analysis and Applications 126 pp 130-42
14. Saied S. M. and Khalil, S. M. Gamma ideal extension in gamma systems, J. Discrete Math. Sci. Crypt. to appear (2021).
15. Salama, A. A. Samarandache F. and K. Valeri, Neutrosophic closed set and neutrosophic continuous functions, Neutrosophic Sets Syst. 4 (2014), 4-8.
16. Venkateswara Rao, V. and Srinivasa Rao, Y(2017). "Neutrosophic Pre-open & Pre-closed sets in Neutosophic Topology",International Journal of chem. Tech Research, vol.10 pp. 449-458.
17. Zadeh, L.A(1965). "Fuzzy Sets", Informations and Control, vol.8, pp.338-353.

Received: May 20, 2021. Accepted: October 3, 2021

Pentapartitioned Neutrosophic Pythagorean Resolvable and Irresolvable Spaces

R. Radha [1],*, A. Stanis Arul Mary [2] and Said Broumi [3]

[1] Research Scholar, Department of Mathematics, Nirmala College for Women, Coimbatore, India (TN); radharmat2020@gmail.com
[2] Assistant Professor, Department of Mathematics, Nirmala College for Women, Coimbatore, India (TN); stanisarulmary@gmail.com
[3] Laboratory of Image Processing,Faculty of Science, Ben M ' Sik , University Hassan II, Casablanca, Morocco; broumisaid78@gmail.com

* Correspondence: radharmat2020@gmail.com

Abstract: In this paper, the concepts of Pentapartitioned Neutrosophic Pythagorean resolvable , Pentapartitioned Neutrosophic Pythagorean irresolvable, Pentapartitioned Neutrosophic Pythagorean open hereditarily irresolvable and maximally Pentapartitioned Neutrosophic Pythagorean irresolvable spaces are introduced. Also we investigated several properties of the Pentapartitioned Neutrosophic Pythagorean open hereditarily irresolvable spaces besides giving characterization of these spaces by means of somewhat Pentapartitioned Neutrosophic Pythagorean continuous functions and somewhat Pentapartitioned Neutrosophic Pythagorean open functions.

Keywords: Pentapartitioned neutrosophic pythagorean resolvable, pentapartitioned neutrosophic pythagorean open hereditarily irresolvable, somewhat pentapartitioned neutrosophic pythagorean irresolvable, pentapartitioned neutrosophic pythagorean continuous and open functions.

1. Introduction

Zadeh[16] introduced the important and useful concept of a fuzzy set which has invaded almost all branches of mathematics. The speculation of fuzzy topological space was studied and developed by C.L. Chang [4]. The paper of Chang sealed the approach for the following tremendous growth of the various fuzzy topological ideas. Since then a lot of attention has been paid to generalize the fundamental ideas of general topology in fuzzy setting and therefore a contemporary theory of fuzzy topology has been developed. Atanassov and plenty of researchers [1] worked on intuitionistic fuzzy sets within the literature. Florentin Smarandache [13] introduced the idea of Neutrosophic set in 1995 that provides the information of neutral thought by introducing the new issue referred to as uncertainty within the set. Thus neutrosophic set was framed and it includes the parts of truth membership function(T), indeterminacy membership function(I), and falsity membership function(F) severally. Neutrosophic sets deals with non normal interval of]−0 1+[. Pentapartitioned neutrosophic set and its properties were introduced by Rama Malik and Surpati Pramanik [12]. In this case, indeterminacy is divided into three components: contradiction, ignorance, and an unknown membership function. The concept of Pentapartitioned neutrosophic pythagorean sets was initiated by R. Radha and A. Stanis Arul Mary . The concept of neutrosophic fuzzy resolvable spaces and irresolvable spaces was introduced by M. Caldas et.al[3]. Now we extend the concepts to pentapartitioned neutrosophic pythagorean sets.

In this Paper we initiated the new concept of Pentapartitioned neutrosophic pythagorean resolvable, pentapartitioned neutrosophic pythagorean open hereditarily irresolvable, somewhat pentapartitioned neutrosophic pythagorean irresolvable, pentapartitioned neutrosophic pythagorean continuous and open functions and discussed some of its properties.

2. Preliminaries

2.1 Definition [13]

Let X be a universe. A Neutrosophic set A on X can be defined as follows:

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}$$

Where T_A , $I_A, F_A: U \to [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$

Here, $T_A(x)$ is the degree of membership, $I_A(x)$ is the degree of inderminancy and $F_A(x)$ is the degree of non-membership.

2.2 Definition [7]

Let X be a universe. A Pentapartitioned neutrosophic pythagorean [PNP] set A with T, F, C and U as dependent neutrosophic components and I as independent component for A on X is an object of the form

$$A = \{< x, T_A, C_A, I_A, U_A, F_A >: x \in X\}$$

Where $T_A + F_A \leq 1, C_A + U_A \leq 1$ and

$$(T_A)^2 + (C_A)^2 + (I_A)^2 + (U_A)^2 + (F_A)^2 \leq 3$$

Here, $T_A(x)$ is the truth membership, $C_A(x)$ is contradiction membership, $U_A(x)$ is ignorance membership, $F_A(x)$ is the false membership and $I_A(x)$ is an unknown membership.

2.3 Definition [12]

Let P be a non-empty set. A Pentapartitioned neutrosophic set A over P characterizes each element p in P a truth -membership function T_A , a contradiction membership function C_A, an ignorance membership function G_A, unknown membership function U_A and a false membership function F_A , such that for each p in P

$T_A + C_A + G_A + U_A + F_A \leq 5$.

2.4 Definition [7]

The complement of a pentapartitioned neutrosophic pythagorean set A on R is denoted by A^C or A^* and is defined as

$A^C = \{< x, F_A(x), U_A(x), 1 - G_A(x), C_A(x), T_A(x) > : x \in X\}$

2.5 Definition [7]

Let A = $< x, T_A(x), C_A(x), G_A(x), U_A(x), F_A(x) >$ and B = $< x, T_B(x), C_B(x), G_B(x), U_B(x), F_B(x) >$ are pentapartitioned neutrosophic pythagorean sets. Then

A ∪ B = $< x, max(T_A(x), T_B(x)), max(C_A(x), C_B(x)), \min(G_A(x), G_B(x)), \min(U_A(x), U_B(x)), \min(F_A(x), F_B(x)), >$

A ∩ B = $< x, min(T_A(x), T_B(x)), min(C_A(x), C_B(x)), \max(G_A(x), G_B(x)), max(U_A(x), U_B(x)), max(F_A(x), F_B(x)) >$

2.6 Definition[7]

A PNP topology τ on a nonempty set R is a family of a PNP sets in R satisfying the following axioms

1) 0,1∈ τ
2) $R_1 \cap R_2 \in \tau$ for any $R_1, R_2 \in \tau$
3) $\cup R_i \in \tau$ for any $R_i : i \in I \subseteq \tau$

The complement R* of PNP open set (PNPOS, in short) in PNP topological space [PNPTS] (R,$\boldsymbol{\tau}$), is called a PNP closed set [PNPCS].

2.7 Definition [7]

Let (R,$\boldsymbol{\tau}$) be a PNPTS and L be a PNPTS in R. Then the PNP interior and PNP Closure of R denoted by

Cl(L) = $\cap\{\boldsymbol{K}$: K is a PNPCS in R and L⊆ K}.

Int(L) = ∪{G: G is a PNPOS in R and G⊆ L}.

3. Pentapartitioned Neutrosophic Pythagorean Resolvable and Irresolvable Spaces

3.1 Definition

A Pentapartitioned neutrosophic pythagorean (PNP) set P in Pentapartitioned neutrosophic pythagorean topological space (PNPTS) (R, τ) is called pentapartitioned neutrosophic pythagorean dense if there exists no pentapartitioned neutrosophic pythagorean closed set Q in (R, τ) such that

P ⊂ Q ⊂ 1_R

Note: If P is a PNP open set , then the complement of PNP set P is a PNP closed set and it is denoted by P^*.

3.2 Example

Let R = { e, f} and define the pentapartitioned neutrosophic pythagorean set P as

$$P = \begin{cases} \{e, 0.4, 0.5, 0.7, 0.2, 0.3\} \\ \{f, 0.5, 0.3, 0.6, 0.1, 0.2\} \end{cases}$$

Then $\tau = \{0_R, 1_R, P\}$ is a pentapartitioned neutrosophic pythagorean topology on R. Hence P is a PNP dense set in (R, τ}.

3.3 Definition

A PNPTS (R, τ) is called PNP resolvable if there exists a PNP dense set P in (R, τ) such that PNPCl (P^*) = 1_R. Otherwise (R, τ) is called PNP irresolvable.

3.4 Example

Let R = { e, f} and define the pentapartitioned neutrosophic pythagorean set P, Q and R as

P = $\begin{cases}\{e, 0.3,0.4,0.3,0.3,0.1\} \\ \{f, 0.4,0.2,0.6,0.5,0.3\}\end{cases}$,

Q = $\begin{cases}\{e, 0.4,0.2,0.7,0.1,0.3\} \\ \{f, 0.6,0.1,0.3,0.2,0.2\}\end{cases}$ and

R = $\begin{cases}\{e, 0.1,0.2,0.4,0.3,0.4\} \\ \{f, 0.5,0.4,0.3,0.2,0.1\}\end{cases}$.

It can be seen that $\tau = \{0_R, 1_R, P\}$ is a pentapartitioned neutrosophic pythagorean topology on R. Then (R, τ) is a pentapartitioned neutrosophic topological space. Now PNPInt(Q) = 0_R, PNPInt(R) = 0_R, PNPCl (Q) = 1_R and PNPCl(R) = 1_R . Thus P and Q are PNP dense sets in (R, τ) such that PNPCl (Q^*) = 1_R and PNPCl(R^*) = 1_R. Hence the PNP topological space (R, τ) is PNP resolvable.

3.5 Example

Let R = { e, f} and define the pentapartitioned neutrosophic pythagorean set P, Q and R as

P = $\begin{cases}\{e, 0.2,0.3,0.5,0.4,0.5\} \\ \{f, 0.1,0.2,0.5,0.5,0.3\}\end{cases}$,

Q = $\begin{cases}\{e, 0.4,0.5,0.5,0.4,0.3\} \\ \{f, 0.5,0.4,0.4,0.3,0.2\}\end{cases}$ and

R = $\begin{cases}\{e, 0.3,0.4,0.4,0.3,0.2\} \\ \{f, 0.2,0.3,0.5\ ,0.2,0.1\}\end{cases}$.

It can be seen that $\tau = \{0_R, 1_R, P\}$ is a pentapartitioned neutrosophic pythagorean topology on R. Then (R, τ) is a pentapartitioned neutrosophic topological space. Now PNPInt(Q) = A, PNPInt(R) = A, PNPCl (Q) = 1_R and PNPCl(R) = 1_R . Thus P and Q are PNP dense sets in (R, τ) such that PNPCl (Q^*) = P^* and PNPCl(R^*) =P^*. Hence the PNP topological space (R, τ) is PNP irresolvable.

3.6 Theorem

A PNPTS (R, τ) is a PNP resolvable space iff (R, τ) has a pair of PNP dense set K_1 and K_2 such that $K_1 \subseteq K_2^*$.

Proof

Let (R, τ) be a PNPTS and (R, τ) be PNP resolvable space. Suppose that for all PNP dense sets K_i and K_j, we have $K_i \nsubseteq K_j^*$. Then $K_i \supset K_j^*$. Then PNPCl(K_i) ⊃ PNPCl(K_j^*) which implies that $1_R \supset$ PNPCl(K_j^*).Then PNPCl(K_j^*) ≠ 1_R. Also $K_j \supset K_i^*$, then PNPCl(K_j) ⊃ PNPCl(K_i^*) which implies that

$1_R \supset$ PNPCl(K_i^*). Therefore PNPCl (K_i^*) $\neq$ 1_R. Hence PNPCl(K_i^*) = 1_R, but PNPCl(K_i) $\neq$ 1_R for all PNP set K_i in (R, τ) which is a contradiction. Hence (R, τ) has a pair of PNP dense set K_1 and K_2 such that $K_1 \subseteq K_2^*$.

Conversely, suppose that the PNP topological space (R, τ) has a pair of PNP dense set K_1 and K_2 such that $K_1 \subseteq K_2^*$. Suppose that (R, τ) is a PNP irresolvable space, Then for all PNP dense sets K_1 and K_2 in (R, τ), we have PNPCl(K_1^*) $\neq 1_R$. Then PNPCl(K_2^*) $\neq 1_R$ implies that there exists a PNP closed set L in (R, τ) such that $K_2^* \subset$ L $\subset 1_R$. Then $K_1 \subset K_2^* \subset$ L $\subset 1_R$ implies that $K_1 \subset$ L $\subset 1_R$. But this is a contradiction. Hence (R, τ) is a PNP resolvable space.

3.7 Theorem

If (R, τ) is a PNP irresolvable space iff PNInt(P) $\neq$ 0 for all PNP dense set P in (R, τ).

Proof

Since (R, τ) is PNP irresolvable space for all PNP dense set P in (R, τ), PNPCl(P^*) $\neq 1_R$.Then (PNPInt(P)* $\neq 1_R$ which implies PNPInt(P) $\neq 0_R$.

Conversely PNPInt(P) $\neq 0_R$, for all PNP dense set P in (R, τ). Suppose that (R, τ) is PNP resolvable. Then there exists a PNP dense set P in (R, τ) such that PNPCl(P^*) = 1_R. This implies that (PNPInt(P)* = 1_R which again implies PNPInt(P) = 0_R. But this is a contradiction. Hence (R, τ) is PNP resolvable space.

3.8 Definition

A PNP topological space (R, τ) is called a PNP submaximal space if for each PNP set P in (R, τ), PNPCl(P) = 1_R.

3.9 Proposition

If the PNP topological space (R, τ) is PNP submaximal, then (R, τ) is PNP irresolvable.

Proof. Let (R, τ) be a PNP submaximal space. Assume that (R, τ) is a PNP resolvable space. Let P be a PNP dense set in (R, τ). Then PNPCl(P^*) = 1_R. Hence (PNPInt(P)* = 1_R which implies that PNPInt(P) = 0_R. Then P $\notin$ τ. This is a contradiction. Hence (R, τ) is PNP irresolvable space.

The converse of the above theorem is not true, which can be shown by the following example. See example 3.5.

3.10 Definition

A PNP topological space (R, τ) is called a maximal PNP irresolvable space if (R, τ) is PNP irresolvable and every PNP dense set P of (R, τ) is PNP open.

3.11 Example

Let R = { e, f} and define the pentapartitioned neutrosophic pythagorean set Q and R as

Q = $\begin{cases}\{e, 0.3,0.4,0.3,0.3,0.1\} \\ \{f, 0.4,0.2,0.6,0.5,0.3\}\end{cases}$ and

R = $\begin{cases}\{e, 0.1,0.2,0.4,0.3,0.4\} \\ \{f, 0.5,0.4,0.3,0.2,0.1\}\end{cases}$.

It can be seen that $\tau = \{0_R, 1_R, P\}$ is a pentapartitioned neutrosophic pythagorean topology on R. Then (R, τ) is a pentapartitioned neutrosophic topological space. Now PNPInt(Q^*) = 0_R, PNPInt(R^*) = 0_R, PNPCl (Q) = 1_R and PNPCl(R) = 1_R . Thus P and Q are PNP dense sets in (R, τ) such that PNPCl (Q^*) = Q^* and PNPCl(R^*) =R^*.Thus (R, τ) is PNP irresolvable and every PNP dense set of (R, τ) is PNP open. Therefore PNP topological space (R, τ) is maximally PNP irresolvable.

4 PNP open hereditarily Irresolvable space

4.1 Definition

A PNP topological space (R, τ) is said to be PNP open hereditarily irresolvable if PNPInt(PNPCl(P))≠ 0_R and PNPInt(P)≠ 0_R , for any PNP set P in (R, τ).

4.2 Example

Let R = { e, f} and define the pentapartitioned neutrosophic pythagorean set Q as

P = $\begin{cases}\{e, 0.2,0.1,0.5,0.4,0.5\} \\ \{f, 0.1,0.2,0.6,0.5,0.4\}\end{cases}$

It can be seen that $\tau = \{0_R, 1_R, P\}$ is a pentapartitioned neutrosophic pythagorean topology on R. Then (R, τ) is a pentapartitioned neutrosophic topological space. Now PNPInt(P) = P ≠ 0_R and PNPInt(PNPCl(P)) =PNPInt(P^*) = P ≠ 0_R . Thus (R, τ) is PNP open hereditarily irresolvable space.

4.3 Theorem

Let (R, τ) be a PNP topological space. If (R, τ) is PNP open hereditarily irresolvable, then (R, τ) is PNP irresolvable.

Proof

Let P be a PNP dense set in (R, τ). Then PNPCl(P) = 1_R which implies that PNPInt(PNPCl(P) = 1_R ≠ 0_R . Since (R, τ) is PNP open hereditarily irresolvable, we have PNPInt(P) ≠ 0_R. Therefore by theorem 3.7, PNPInt(P) ≠ 0_R for all PNP dense set in (R, τ) implies that (R, τ) is PNP irresolvable.

The converse of the above theorem is not true. See Example 4.4.

□

4.4 Example

Let R = { e, f} and define the pentapartitioned neutrosophic pythagorean set P, Q , R and S as

P = $\begin{cases}\{e, 0.1,0.5,0.5,0.2,0.6\} \\ \{f, 0.2,0.3,0.6,0.3,0.3\}'\end{cases}$

Q = $\begin{cases}\{e, 0.4,0.5,0.1,0.2,0.4\} \\ \{f, 0.3,0.2,0.7,0.2,0.1\}\end{cases}$

R = $\begin{cases} \{e, 0.4,0.5,0.1,0.2,0.4\} \\ \{f, 0.3,0.3,0.6\ \ ,0.2,0.1\} \end{cases}$ and

S = $\begin{cases} \{e, 0.2,0.1,0.7,0.6,0.2\} \\ \{f, 0.1,0.2,0.6\ \ ,0.5,0.4\} \end{cases}$.

It can be seen that $\tau = \{0_R, 1_R, P, Q, R\}$ is a pentapartitioned neutrosophic pythagorean topology on R. Then (R, τ) is a pentapartitioned neutrosophic topological space. Now PNPCl(P) = 1_R, PNPCl(Q) = 1_R, PNPCl(R) = 1_R and PNPCl (S) = 1_R . Thus P , Q,R and S are PNP dense sets in (R, τ) such that PNPCl (P^*) = P^* , PNPCl (Q^*) = Q^* and PNPCl(R^*) =R^* and PNPCl (S^*) = P^*. Hence the PNP topological space (R, τ) is PNP irresolvable. But PNPInt(PNPCl(S^*) = PNPInt(P^*) = 0_R. Therefore (R, τ) is not a PNP open hereditarily irresolvable space.

4.5 Theorem

Let (R, τ) be a PNP open hereditarily irresolvable. Then PNPInt(P) $\nsubseteq$ PNPInt(Q)* for any two PNP dense sets P and Q in (R, τ).

Proof.

Let P and Q be any two PNP dense sets in (R, τ). Then PNPCl(P) = 1_R and PNPCl(Q) =1_R implies that PNPInt(PNPCl(P))$\neq 0_R$ and PNPInt(PNPCl(Q)$\neq$ 0_R. Since (R, τ) is PNP open hereditarily irresolvable, PNPInt(P) $\neq 0_R$ and PNPInt(Q) $\neq$ 0_R . Hence by theorem 3.6, P $\nsubseteq Q^*$. Therefore PNPInt(P) $\subseteq P \nsubseteq Q^* \subseteq (PNPInt(Q))^*$. Hence we have PNPInt(P) $\subseteq (PNPInt(Q))^*$ for any two PNP dense sets P and Q in (R, τ).

4.6 Theorem

Let (R, τ) be a PNP topological space. If (R, τ) is PNP open hereditarily irresolvable, then PNPInt(P) = 0_R for any nonzero PNP dense set P in (R, τ) which implies that PNPInt(PNPCl(P)) = 0_R.

Proof:

Let P be a PNP set in (R, τ) such that PNPInt(P) = 0_R . We claim that PNPInt(PNPcl(P)) = 0_R. Suppose that PNPInt(PNPCl(P)) = 0_R. Since (R, τ) is PNP open hereditarily irresolvable, we have PNPInt(P) $\neq$ 0_R which is a contradiction to PNPInt(P) = 0_R. Hence PNPInt(PNPCl(P) = 0_R.

4.7 Theorem

Let (R, τ) be a PNP topological space. If (R, τ) is PNP open hereditarily irresolvable, then PNPCl(P) = 1_R for any nonzero PNP dense set P in (R, τ) which implies that PNPCl(PNPInt(P) = 0_R.

Proof

Let P be a PNP set in (R, τ) such that PNPCl(P) = 1_R.Then we have (PNPCl(P))* = 0_R which implies that PNPInt(P^*) = 0_R.Since (R, τ) is PNP open hereditarily irresolvable by theorem 4.6. We have PNPInt(PNPCl(P^*)) = 0_R. Therefore (PNPCl(PNPInt(P))* = 0_R implies that PNPCl(PNPInt(P)) = 1_R.

5 Somewhat PNP Continuous and PNP Somewhat PNP open

5.1 Definition

Let (R, τ) and (M,σ) be any two PNP topological spaces. A function f: (R, τ) → (M,σ) is called somewhat PNP continuous if for a P ∈ σ and $f^{-1}(P) \neq 0_R$, there exists a Q $\in \tau$ such that Q$\neq 0_R$ and Q ⊆ $f^{-1}(P)$.

5.2 Definition

Let (R, τ) and (M,σ) be any two PNP topological spaces. A function f: (R, τ) → (M,σ) is called somewhat PNP open if for a P ∈ σ and $P \neq 0_R$, there exists a Q $\in \tau$ such that Q$\neq 0_R$ and Q⊆ $f(P)$.

5.3 Theorem

Let (R, τ) and (M,σ) be any two PNP topological spaces. A function f: (R, τ) → (M,σ) is called somewhat PNP continuous and injective. If PNPInt(P) = 0_R for any non-zero PNP set P in (R, τ), then PNPInt(f(P)) = 0_M in (M,σ) .

Proof

Let P be a non-zero PNP set in (R, τ) such that PNPInt(P) = 0_R. Now we prove that PNPInt(f(P)) = 0_M . Suppose that PNPInt(f(P)) $\neq 0_M$ in (M,σ) . Then there exists a nonzero PNP set Q in (M,σ) such that Q ⊆ f(P). Thus , we have $f^{-1}(Q) \subseteq f^{-1}(f(P))$. Since f is somewhat PNP continuous, there exists a S $\in \tau$ such that S $\neq 0_R$ and S $\subseteq f^{-1}(Q)$. Hence S $\subseteq f^{-1}(Q) \subseteq P$ which implies that PNPInt(P) $\neq 0_R$ .This is a contradiction. Hence PNPInt(f(P)) = 0_M in (M,σ).

5.4 Theorem

Let (R, τ) and (M,σ) be any two PNP topological spaces. A function f: (R, τ) → (M,σ) is called somewhat PNP continuous , injective and PNPInt(PNPCl(P)) = 0_R for any non-zero PNP set P in (R, τ), then PNPInt(PNPCl(f(P))) = 0_M in (M,σ) .

Proof

Let P be a non-zero PNP set in (R, τ) such that PNPInt(PNPCl(P)) = 0_R. Now we claim that (PNPCl(f(P))) = 0_M . Suppose that PNPInt(PNPCl(f(P))) $\neq 0_M$ in (M,σ) .Then PNPCl(f(P) $\neq 0_M$ and PNPCl(f(P))* $\neq 0_M$. Now PNPCl(f(P))* $\neq 0_M$ ∈ M. Since f is somewhat PNP continuous, there exists a Q $\in \tau$ such that Q $\neq 0_R$ and Q $\subseteq f^{-1}((\mathrm{PNPCl(f(P))})^*)$. Observe that Q $\subseteq$ $f^{-1}((\mathrm{PNPCl(f(P))})^*)$ which implies that $f^{-1}(\mathrm{PNPCl(f(P))}) \subseteq Q^*$.

Since f is injective, thus P $\subseteq f^{-1}$(f(P) $\subseteq f^{-1}(\mathrm{PNPCl(f(P))}) \subseteq Q^*$ which implies that P $\subseteq Q^*$. Therefore Q $\subseteq P^*$. This implies that PNPInt(P^*) $\neq 0_R$.Let PNPInt(P^*) $= S \neq 0_R$. Then we have PNPCl(PNPInt(P^*)) = PNPCl(S) $\neq 1_R$ which implies that PNPInt(PNPCl(P)) $\neq 0_R$ .This is a contradiction. Hence PNPInt(PNPCl(f(P))) = 0_M in (M,σ).

5.5 Theorem

Let (R, τ) and (M, σ) be any two PNP topological spaces. If the function f: (R, τ) → (M, σ) is somewhat PNP open and PNPInt((P)) = 0_R for any non-zero PNP set P in (M, σ), then PNPInt(f^{-1}(P)) = 0_R in (R, τ).

Proof

Let P be a non-zero PNP set in (M, σ) such that PNPInt(P) = 0_R. Now we claim that PNPInt(f^{-1}(P)) = 0_R in (R, τ). Suppose that (PNPInt(f^{-1}(P)) $\neq 0_R$ in (R, τ). Then there exists a non-zero PNP open set Q in (R, τ) such that Q $\subseteq f^{-1}(P)$. Thus we have f(Q) $\subseteq f\left(f^{-1}(P)\right) \subseteq P$.This implies that f(Q) $\subseteq P$. Since f is somewhat PNP open, there exists a S$\in \tau$ such that S $\neq 0_R$ and S $\subseteq f(Q)$. Therefore S $\subseteq f(Q) \subseteq P$ which implies that S $\subseteq A$. Hence PNPInt((P) $\neq 0_R$ which is a contradiction. Hence PNPInt(f^{-1}(P)) = 0_R in (R, τ).

5.6 Theorem

Let (R, τ) and (M, σ) be any two PNP topological spaces Let (R, τ) be a PNP open hereditarily irresolvable space. If the function f: (R, τ) → (M, σ) is somewhat PNP open, somewhat PNP continuous and a bijective function, then (M, σ) is a PNP open hereditarily irresolvable space.

Proof

Let P be a non-zero PNP set in (M, σ) such that PNPInt(P) = 0_R. Now PNPInt(P) = 0_R and f is somewhat PNP open which implies PNPInt(f^{-1}(P)) = 0_R in (R, τ) by theorem 5.5. Since (R, τ) is a PNP open hereditarily irresolvable ,we have Suppose that PNPInt(PNPCl(f^{-1}(P))) = 0_R in (R, τ) by theorem 4.6.Since PNPInt(PNPCl(f^{-1}(P))) = 0_R and f is somewhat PNP continuous by theorem 5.4, we have that PNPInt(PNPCl(f(f^{-1}(P)))) = 0_R. Since f is onto, thus PNPInt(PNPCl(P) = 0_R. Hence, by theorem 4.6, (M, σ) is a PNP open hereditarily irresolvable space.

5. Conclusion

In this paper we have proposed Pentapartitioned neutrosophic pythagorean resolvable and irresolvable spaces and studied some of its properties. Furthermore we also characterized Pentapartitioned Neutrosophic Pythagorean open hereditarily spaces and open functions in Pentapartitioned neutrosophic pythagorean topological spaces. In the future work, we extend the concept to Pentapartitioned Pythagorean almost resolvable and irresolvable spaces.

Funding: "This research received no external funding"

Conflicts of Interest: "The authors declare no conflict of interest."

References

1. K. Atanassov, Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems*.**1986**,volume 20 87-96.

2. Broumi S, Smarandache F (2014) Rough Neutrosophic sets, *Ital J Pure Appl Math* ,***2014***, *volume* 32:493-502

3. M. Caldas, R. Dhavaseelan, M. Ganster and S. Jafari, Neutrosophic Resolvable and irresolvable Spaces, *New trends in Theory and Applications,* volume II .327-335

4. C.L. Chang Fuzzy topological spaces, J. Math. Anal. Appl, **1984**, volume 24, 182-190.

5. Rajashi Chatterjee, P. Majumdar and S. K. Samanta, On some similarity measures and entropy on quadripartitioned single valued neutrosophic sets, *Journal of Intelligent and Fuzzy Systems* ,**2016**, volume 302475-2485.

6. R. Radha, A. Stanis Arul Mary. Pentapartitioned Neutrosophic pythagorean Soft set, *IRJMETS*, **2021 ,** *Volume 3(2)*,905-914.
7. R. Radha, A. Stanis Arul Mary. Pentapartitioned Neutrosophic Pythagorean Set, *IRJASH*, **2021**, *volume 3,* 62-82.
8. R. Radha, A. Stanis Arul Mary. Heptapartitioned neutrosophic sets, *IRJCT, **2021** ,volume 2*,222-230.
9. R. Radha, A. Stanis Arul Mary, F. Smarandache. Quadripartitioned Neutrosophic Pythagorean soft set, *International journal of Neutrosophic Science,* ***2021,*** *volume14(1)*,9-23.
10. R. Radha, A. Stanis Arul Mary, F. Smarandache. Neutrosophic Pythagorean soft set, *Neutrosophic sets and systems,* **2021,***vol 42*,65-78.
11. R. Radha ,A.Stanis Arul Mary, Pentapartitioned neutrosophic pythagorean resolvable and irresolvable spaces(Communicated)
12. Rama Malik, Surapati Pramanik. Pentapartitioned Neutrosophic set and its properties, *Neutrosophic Sets and Systems*, **2020,** *Vol 36*,184-192,2020
13. F.Smarandache, A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic, *American Research Press, Rehoboth*.

14. S. Sharmila, I. Arockiarani, On Intuitionistic Fuzzy almost resolvable and irresolvable spaces, Asian Journal of Applied Sciences,**2015**, *Volume 3, Issue 6*,918-928

15. Wang H, Smarandache F, Zhang YQ, Sunderraman R ,Single valued neutrosophic sets, *Multispace Multistruct ,**2010** ,volume* 4:410-413.

16. L. Zadeh , Fuzzy sets, *Information and Control* **1965**, volume 8, 87-96.

Received: May 7, 2021. Accepted: October 1, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

Wiener index and applications in the Neutrosophic graphs

Masoud Ghods[1,*], Zahra Rostami [2]

[1]Department of Mathematics, Semnan University, Semnan 35131-19111, Iran, mghods@semnan.ac.ir
[2]Department of Mathematics, Semnan University, Semnan 35131-19111, Iran, zahrarostami.98@semnan.ac.ir

* Correspondence: mghods@semnan.ac.ir; Tel.: (09122310288)

Abstract: In this article, we have examined the Wiener index in neutrosophic graphs. Wiener index is one of the most important topological indices. This index is a distance-based index that is calculated based on the geodesic distance between two vertices. Here, after defining the Wiener index in neutrosophic graphs, we calculated this index for some special modes such as the complete neutrosophic graph, cycle, and tree. In the following, by presenting a several theorems, we compared this index with the connectivity index, which is one of the most important degree-based indicators.

Keywords: Wiener index; partial Wiener index; totally Wiener index; neutrosophic graph; neutrosophic tree; strong spanning tree; connectivity index

1. Introduction

The theory of fuzzy sets was first proposed by Zadeh [20] in 1965, and the concept of fuzzy graph was first introduced by Rosenfeld [13] in 1975. Since then, much research has been done on fuzzy graphs, their properties, and applications. One of these problems was the calculation of degree-based topological indices and distance-based indices in fuzzy graphs. These indicators help by providing a numerical value for each graph so that we can have a good criterion for comparing graphs with the same number of vertices.

After that, Atanassov [6] proposed the theory of intuitionistic fuzzy set. Finally, with the generalization of fuzzy theory by Smarandache [15] in 1995, new sets called neutrosophic sets were born. By presenting this theory, researchers tried to introduce other mathematical concepts in this field. Among them was the concept of graphs, which led to the new concept of neutrosophic graphs.

In recent years, many features and applications of neutrosophic graphs have been proposed by theorists in this field. One of them is the problem of the Decision-Making [1], Solving the supply chain problem [2], application in the NeutroHyperAlgebra and AntiHyper Algebra [16], and Energy and Spectrum [7]. One of these topics is the study of topological indices and its applications in neutrosophic graphs. In [8-10], we examined some of these indicators and their applications.

In this paper, we try to define the Wiener index, which is one of the most important topological indices based on distance, in neutrosophic graphs, and then calculate this index for certain conditions. The calculation of this index in neutrosophic graphs is done for the first time in this paper. Finally, we compare the connectivity index, which is one of the most important degree-based indices, with the Wiener index and present the results.

2. Preliminaries

This section, provides some definitions and theorems needed.

Definition 1. [5] Let $G = (N, M)$ be a single-valued Neutrosophic graph, where N is a Neutrosophic set on V and, M is a Neutrosophic set on E, which satisfy the following

$$T_M(u,v) \leq min\big(T_N(u), T_N(v)\big),$$
$$I_M(u,v) \geq max\big(I_N(u), I_N(v)\big),$$
$$F_M(u,v) \geq max\big(F_N(u), F_N(v)\big),$$

where u and v are two vertices of G, and $(u,v) \in E$ is an edge of G.

Definition 2. [5] Let $G = (N, M)$ be a Single-Valued Neutrosophic Graph and P is a path in G. P is a collection of different vertices, $v_0, v_1, v_2, \dots, v_n$ such that $\big(T_M(v_{i-1}, v_i), I_M(v_{i-1}, v_i), F_M(v_{i-1}, v_i)\big) > 0$ for $0 \leq i \leq n$. P is a Neutrosophic cycle if $v_0 = v_n$ and $n \geq 3$.

Definition 3. [5] Suppose $G = (N, M)$ a single-valued Neutrosophic graph. G is a connected Single-Valued Neutrosophic Graph if there exists no isolated vertex in G. ($v \in V_G$ is the isolated vertex, if there exists no incident edge to the vertex v.)

Definition 4. [9] Let $G = (N, M)$ be the connected Neutrosophic Graph. The partial connectivity index of G is defined as

$$PCI_T(G) = \sum_{u,v\,\in N} T_N(u)T_N(v)CONN_{T_G}(u,v),$$
$$PCI_I(G) = \sum_{u,v\,\in N} I_N(u)I_N(v)CONN_{I_G}(u,v),$$
$$PCI_F(G) = \sum_{u,v\,\in N} F_N(u)F_N(v)CONN_{F_G}(u,v),$$

where $CONN_{T_G}(u,v)$ is the strength of truth, $CONN_{I_G}(u,v)$ the strength of indeterminacy and $CONN_{F_G}(u,v)$ the strength of falsity between two vertices u and v. We have

$$CONN_{T_G}(u,v) = \max\{\min T_M(e) \mid e \in P \text{ and } P \text{ is a path between } u \text{ and } v\},$$
$$CONN_{I_G}(u,v) = \min\{\max I_M(e) \mid e \in P \text{ and } P \text{ is a path between } u \text{ and } v\},$$
$$CONN_{F_G}(u,v) = \min\{\max F_M(e) \mid e \in P \text{ and } P \text{ is a path between } u \text{ and } v\}.$$

Also, the totally connectivity index of G is defined as

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6}.$$

Theorem 1. [9] Let $G = (N, M)$ be a complete neutrosophic graph whit $V = \{v_1, v_2, \dots, v_n\}$ such that $t_1 \leq t_2 \leq \cdots \leq t_n$, $i_1 \leq i_2 \leq \cdots \leq i_n$ and $f_1 \geq f_2 \geq \cdots \geq f_n$ where $t_j = T_N(v_j)$, $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$. Then

$$PCI_T(G) = \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k\,, \qquad PCI_I(G) = \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^{n} i_k\,, \qquad PCI_F(G) = \sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^{n} f_k.$$

3. Wiener Index in Neutrosophic Graphs

In this section, which is the main part of the article, we will introduce the Wiener index in neutrosophic graphs. The Wiener index is a distance-based index that is widely used in symmetric graphs.

Like the connectivity index, we divide the Wiener index into a Totally and Partial Wiener index and define it as follows

Definition 5. Let $G = (N, M)$ be the Neutrosophic Graph and $v_1, v_2 \in V$. A strong path P from v_1 to v_2 is called **a neutrosophic geodesic** if there is no strong shorter path between v_1 and v_2.
Note that in the above definition, the shortest strong path must be calculated separately for each of truth (T), indeterminacy (I), and falsity (F) states.

Definition 6. Let $G = (N, M)$ be the Neutrosophic Graph. The **Partial Wiener Index** $(\boldsymbol{PWI})$ of G is defined as

$$PWI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)d_{s_T}(u,v),$$

$$PWI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)d_{s_I}(u,v),$$

$$PWI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)d_{s_F}(u,v),$$

when $d_s(u, v)$ is the minimum, the sum of the weights of the edges in geodesic between u and v.
Also, the **Totally Wiener Index** $(\boldsymbol{TWI})$ of G is defined by

$$TWI(G) = \frac{4 + 2PCWI_T(G) - 2PWI_F(G) - PWI_I(G)}{6}.$$

Example 1. Consider the Neutrosophic Graph $G = (N, M)$ as shown in figure 1, with the vertex set $V = \{a, b, c, d\}$ where $(T_N, I_N, F_N)(a) = (0.4, 0.3, 0.2)$, $(T_N, I_N, F_N)(b) = (0.6, 0.5, 0.2)$, $(T_N, I_N, F_N)(c) = (0.7, 0.2, 0.2)$, and $(T_N, I_N, F_N)(d) = (0.4, 0.2, 0.3)$, whit the edge set $(T_M, I_M, F_M)(a, b) = (0.3, 0.3, 0.3)$, $(T_M, I_M, F_M)(a, c) = (0.4, 0.3, 0.2)$, $(T_M, I_M, F_M)(a, \mathrm{d}) = (0.3, 0.3, 0.2)$, $(T_M, I_M, F_M)(b, \mathrm{d}) = (0.4, 0.4, 0.3)$, $(T_M, I_M, F_M)(c, \mathrm{d}) = (0.4, 0.2, 0.2)$, We have,

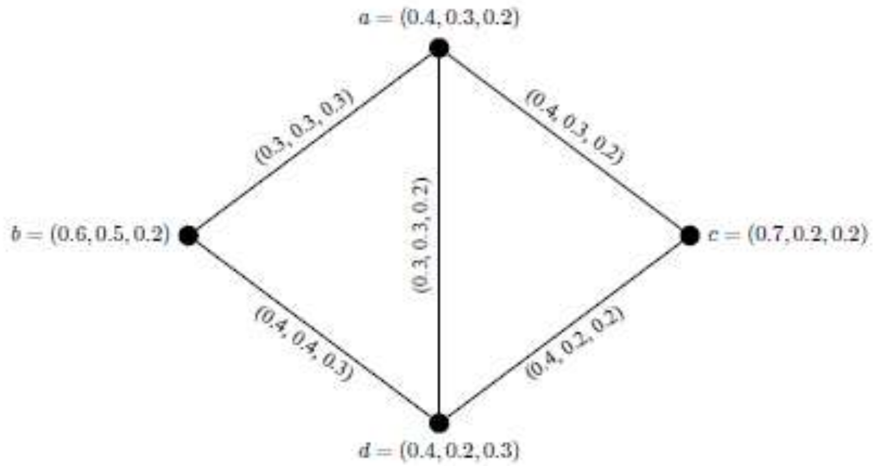


Figure 1. A neutrosophic graph G

Table 1. The sum of the weights of the edges in geodesic between each pair of vertices u and v.

	$d_{s_T}(u,v)$	$d_{s_I}(u,v)$	$d_{s_F}(u,v)$
a,b	$0.4+0.4+0.4=1.2$	0.3	0.3
a,c	0.4	0.3	0.2
a,d	$0.4+0.4=0.8$	0.3	0.2
b,c	$0.4+0.4=0.8$	$0.3+0.3=0.6$	$0.2+0.3=0.5$
b,d	0.4	$0.3+0.3=0.6$	0.3
c,d	0.4	0.2	0.2

$$PWI_T(G)=\sum_{u,v\in N}T_N(u)T_N(v)d_{s_T}(u,v)$$
$$=(0.4)(0.6)(1.2)+(0.4)(0.7)(0.4)+(0.4)(0.4)(0.8)+(0.6)(0.7)(0.8)$$
$$+(0.6)(0.4)(0.4)+(0.7)(0.4)(0.4)=0.288+0.112+0.128+0.336+0.096+0.112$$
$$=1.072,$$

$$PWI_I(G)=\sum_{u,v\in N}I_N(u)I_N(v)d_{s_I}(u,v)$$
$$=(0.3)(0.5)(0.3)+(0.3)(0.2)(0.3)+(0.3)(0.2)(0.3)+(0.5)(0.2)(0.6)$$
$$+(0.5)(0.2)(0.6)+(0.2)(0.2)(0.2)=0.045+0.018+0.018+0.060+0.060+0.008$$
$$=0.209,$$

$$PWI_F(G)=\sum_{u,v\in N}F_N(u)F_N(v)d_{s_F}(u,v)$$
$$=(0.2)(0.2)(0.3)+(0.2)(0.2)(0.2)+(0.2)(0.3)(0.2)+(0.2)(0.2)(0.5)$$
$$+(0.2)(0.3)(0.3)+(0.2)(0.3)(0.2)=0.012+0.008+0.012+0.020+0.018+0.012$$
$$=0.082.$$

$$TWI(G)=\frac{4+2PCWI_T(G)-2PWI_F(G)-PWI_I(G)}{6}=\frac{4+2(1.072)-2(0.209)-(0.082)}{6}=\frac{5.644}{6}$$
$$=0.941.$$

Theorem 2. Let $G=(N,M)$ be a complete neutrosophic graph whit $V=\{v_1,v_2,\dots,v_n\}$ such that $t_1\le t_2\le\cdots\le t_n$, $i_1\le i_2\le\cdots\le i_n$ and $f_1\ge f_2\ge\cdots\ge f_n$ where $t_j=T_N(v_j)$, $i_j=I_N(v_j)$ and $f_j=F_N(v_j)$ for $j=1,2,\dots,n$. Then

$$PWI_T(G)=\sum_{j=1}^{n-1}t_j^2\sum_{k=j+1}^{n}t_k\,,\qquad PWI_I(G)=\sum_{j=1}^{n-1}i_j^2\sum_{k=j+1}^{n}i_k\,,\qquad PWI_F(G)=\sum_{j=1}^{n-1}f_j^2\sum_{k=j+1}^{n}f_k.$$

Proof. Consider neutrosophic graph $G=(N,M)$ with the conditions given in the theorem. According to the definition of the Wiener index

$$PWI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)d_{s_T}(u,v), \qquad (1)$$

Since G is a complete neutrosophic graph, there is a path of length one between the two vertices. We show that the path is geodesic. Let $u = v_1$. Then for any $2 \le i \le n$, we have $t_1 \le t_i$, it is easy to see that

$$d_{s_T}(v_1,\ v_i) = t_1, \qquad 2 \le i \le n,$$

now, we have for v_2,

$$d_{s_T}(v_2,\ v_i) = t_2, \qquad 3 \le i \le n,$$

for v_k,

$$d_{s_T}(v_k,\ v_i) = t_k, \qquad k+1 \le i \le n,$$

and, we have for v_{n-1},

$$d_{s_T}(v_{n-1},\ v_n) = t_{n-1},$$

now, we get by placing the above relation in (1)

$$\begin{aligned} PWI_T(G) = {} & T_N(v_1)T_N(v_2)t_1 + \cdots + T_N(v_1)T_N(v_n)t_1 + T_N(v_2)T_N(v_3)t_2 + \cdots + T_N(v_2)T_N(v_n)t_2 + \cdots \\ & + T_N(v_k)T_N(v_{k+1})t_k + \cdots + T_N(v_k)T_N(v_n)t_k + \cdots + T_N(v_{n-1})T_N(v_n)t_{n-1} \\ & = t_1t_2t_1 + \cdots + t_1t_nt_1 + t_2t_3t_2 + \cdots + t_2t_nt_2 + \cdots + t_kt_{k+1}t_k + \cdots + t_kt_nt_k + \cdots \\ & + t_{n-1}t_nt_{n-1} \\ & = t_1^2(t_2 + \cdots + t_n) + t_2^2(t_3 + \cdots + t_n) + \cdots + t_k^2(t_{k+1} + \cdots + t_n) + \cdots + t_{n-1}^2 t_n \\ & = \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k. \end{aligned}$$

Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved.

□

Corollary 1. Consider the complete neutrosophic graph $G = (N, M)$ with the above theorem conditions, then

$$PWI_T(G) = PCI_T(G),$$
$$PWI_I(G) = PCI_I(G),$$
$$PWI_F(G) = PCI_F(G).$$

Also, $TWI(G) = TCI(G)$.

Proof. According to theorem 1, and the above theorem is clear.

□

Theorem 3. Let $G = (N, M)$ be a neutrosophic graph with $|N^*| = n$, such that G^* is a tree. If for each $uv \in M$, $G - uv$ has two connecting components w_1 and w_2, it has l and k vertices, respectively such that $l + k = n$. Then

$$PWI_T(G) = \sum_{uv \in G} T_M(uv) \sum_{i=1}^{l} T_N(u_i) \sum_{j=1}^{k} T_N(v_j),$$
$$PWI_I(G) = \sum_{uv \in G} I_M(uv) \sum_{i=1}^{l} I_N(u_i) \sum_{j=1}^{k} I_N(v_j),$$

$$PWI_F(G) = \sum_{uv\in G} F_M(uv) \sum_{i=1}^{l} F_N(u_i) \sum_{j=1}^{k} F_N(v_j).$$

Proof. Let $G = (N, M)$ be a neutrosophic graph with $|N^*| = n$, and G^* is a tree. Now suppose we remove the desired edge uv, $uv \in M$, from G. Graph G is divided into two connecting components w_1 and w_2, so that w_1 will contain l vertices and w_2 will contain $k = n - l$ vertices. If $l = 1$ and $k = n - 1$, and $v_1 \in w_1$ then

$$\begin{aligned} PWI_T(G) = &\sum_{u,v\in N} T_N(u)T_N(v)d_{s_T}(u,v) \\ &= T_N(v_1)T_N(v_2)T_M(uv) + T_N(v_1)T_N(v_3)(T_M(uv) + e_1) + \cdots \\ &+ T_N(v_1)T_N(v_n)(T_M(uv) + \cdots + e_m) + \sum_{u,v\in N-v_1} T_N(u)T_N(v)d_{s_T}(u,v). \end{aligned}$$

where $e_i \in M$, and $e_i \neq uv$. Repeat the same process for $\sum_{u,v\in N-v_1} T_N(u)T_N(v)d_{s_T}(u,v)$. We continue this until only one vertex remains in w_2. Then, by factoring and summing the number of vertices of the two components, we reach the desired result. Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved. □

Theorem 4. Let $G = (N, M)$ be a connected neutrosophic graph with the unique strong spanning tree T. then

$$PWI_T(G) = PWI_T(T), \quad PWI_I(G) = PWI_I(T), \quad PWI_F(G) = PWI_F(T).$$

Hence $TWI(G) = TWI(T)$.

Proof. Let G be a connected neutrosophic graph and T is the unique strong spanning tree of G. By definition of strong spanning tree, if u and v are two vertices of G, we have

$$d_{s_T}(u,v)(G) = d_{s_T}(u,v)(T), \quad d_{s_I}(u,v)(G) = d_{s_I}(u,v)(T), \quad d_{s_F}(u,v)(G) = d_{s_F}(u,v)(T).$$

Since, it is clear from the above relation that

$$PWI_T(G) = PWI_T(T), \quad PWI_I(G) = PWI_I(T), \quad PWI_F(G) = PWI_F(T).$$

Therefore $TWI(G) = TWI(T)$.
□

Theorem 5. Let $G = (N, M)$ be a neutrosophic graph with $G^* = C_n$. Let M be a constant function. Then

1. For $n = 2m, m \in \mathbb{N}$

$$PWI_T(G) = \sum_{k=1}^{\frac{n}{2}-1} kt\left(\sum_{j=1}^{n} T_N(u_j)T_N(u_{j+k})\right) + \frac{n}{2}t\sum_{l=1}^{\frac{n}{2}} T_N(u_l)T_N(u_{l+\frac{n}{2}}),$$

$$PWI_I(G) = \sum_{k=1}^{\frac{n}{2}-1} ki\left(\sum_{j=1}^{n} I_N(u_j)I_N(u_{j+k})\right) + \frac{n}{2}i\sum_{l=1}^{\frac{n}{2}} I_N(u_l)I_N(u_{l+\frac{n}{2}}),$$

$$PWI_F(G)=\sum_{k=1}^{\frac{n}{2}-1}kf\left(\sum_{j=1}^{n}F_N(u_j)F_N(u_{j+k})\right)+\frac{n}{2}f\sum_{l=1}^{\frac{n}{2}}F_N(u_l)F_N(u_{l+\frac{n}{2}}),$$

2. For $n=2m+1, m\in\mathbb{N}$

$$PWI_T(G)=\sum_{k=1}^{\frac{n-1}{2}}kt\left(\sum_{j=1}^{n}T_N(u_j)T_N(u_{j+k})\right),$$

$$PWI_I(G)=\sum_{k=1}^{\frac{n-1}{2}}ki\left(\sum_{j=1}^{n}I_N(u_j)I_N(u_{j+k})\right),$$

$$PWI_F(G)=\sum_{k=1}^{\frac{n-1}{2}}kf\left(\sum_{j=1}^{n}F_N(u_j)F_N(u_{j+k})\right).$$

Note that for $j+k>n$, $u_{j+k}=u_d$, this is, $j+k\equiv d$ (mode n).

Also for $G-uv$, we have

$$PWI_T(G-uv)=\sum_{k=1}^{n-1}kt\left(\sum_{j=1}^{n-k}T_N(u_j)T_N(u_{j+k})\right),$$

$$PWI_I(G-uv)=\sum_{k=1}^{n-1}ki\left(\sum_{j=1}^{n-k}I_N(u_j)I_N(u_{j+k})\right),$$

$$PWI_F(G-uv)=\sum_{k=1}^{n-1}kf\left(\sum_{j=1}^{n-k}F_N(u_j)F_N(u_{j+k})\right).$$

Where $M=(t,i,f)$ is a constant function.

Proof. First, we assume that G^* is a cycle of even length, and $M=(t,i,f)$ is a constant function. Hence each edge of G is a neutral edge. Then, the maximum length of a neutrosophic geodesic in G is $\frac{n}{2}$. Now consider a case where the distance between two vertices is less than $\frac{n}{2}$. Suppose the distance between u and v is equal to k, where k is less than $\frac{n}{2}$. In that case, we define the geodesic length between the two vertices u and v as follows

$$P_k=\{(u,v)\in N^*\times N^*, \text{k is equal to the geodetic length between u and v}\},$$

On the other hand, we know that there are $\frac{n}{2}$ pairs of vertices (u,v) such that the geodesic length between them is exactly equal to $\frac{n}{2}(t,i,f)$, for these $\frac{n}{2}$ pairs of vertices, it is sufficient to obtain a product of $T_N(u)$ in $T_N(v)$ [Similarly, $I_N(u)$ in $I_N(v)$, and $F_N(u)$ in $F_N(v)$]. And then sum on u and v. Then we get

$$\frac{n}{2}t\sum_{l=1}^{\frac{n}{2}}T_N(u_l)T_N(u_{l+\frac{n}{2}}), \qquad (1)$$

[Similarly for I and F]. Now back to the state that $1 \leq k < \frac{n}{2}$. For each vertex such as u on the cycle C_n, there is a vertex with distance kt from it. Suppose $k = 1$, so we have

$$T_N(u_1)T_N(u_2) + T_N(u_2)T_N(u_3) + \cdots + T_N(u_j)T_N(u_{j+1}) + \cdots + T_N(u_n)T_N(u_{n+1}),$$

since $n + 1 \equiv 1$ (mode n), hence $T_N(u_n)T_N(u_{n+1}) = T_N(u_n)T_N(u_1)$. Then

for $k = 1$, we have

$$1 \times t \times \sum_{j=1}^{n} T_N(u_j)T_N(u_{j+1}),$$

for $k = 2$,

$$2 \times t \times \sum_{j=1}^{n} T_N(u_j)T_N(u_{j+2}),$$

for $k = m$, $m < \frac{n}{2}$,

$$m \times t \times \sum_{j=1}^{n} T_N(u_j)T_N(u_{j+m}),$$

by continuing this process and summing on k, we get

$$\sum_{k=1}^{\frac{n}{2}-1} kt\left(\sum_{j=1}^{n} T_N(u_j)T_N(u_{j+k})\right), \tag{2}$$

use from (1) and (2),

$$PWI_T(G) = (1) + (2) = \sum_{k=1}^{\frac{n}{2}-1} kt\left(\sum_{j=1}^{n} T_N(u_j)T_N(u_{j+k})\right) + \frac{n}{2}t\sum_{l=1}^{\frac{n}{2}} T_N(u_l)T_N(u_{l+\frac{n}{2}}).$$

To prove that n is odd, note that the maximum distance between the vertices u, and v is $\frac{n-1}{2}$. The continuation of the proof is similar to the case where n is even.
□

Theorem 6. Let $G = (N, M)$ be a neutrosophic tree $|N^*| \geq 3$. Then

$$PCI_T(G) < PWI_T(G), \qquad PCI_I(G) < PWI_I(G), \qquad PCI_F(G) < PWI_F(G).$$

But, $TCI(G)$ need not be less than or equal to $TWI(G)$.

Proof. Let $G = (N, M)$ be a neutrosophic tree and $|N^*| \geq 3$. Since in the neutrosophic tree, there is a unique strong path between vertices u and v, for any u and v. hence this path is the unique strongest path from u to v. then, $d_{s_T}(u, v)$, for each u and v, is equal the sum of the truth-membership values of edges where those edges belong to the strong path from u to v. In other hands, $CONN_{TG}(u, v)$ is truth-membership values of the weakest edge of the $(u - v)-$path. It follows that

$$CONN_{TG}(u, v) \leq d_{s_T}(u, v),$$

In the above relation, equality occurs when uv is a strong edge. Otherwise

$$CONN_{TG}(u,v) < d_{S_T}(u,v),$$

hhen, we have

$$PCI_T(G) < PWI_T(G).$$

Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved.

□

Here we show with an example that $TCI(G)$ dose not always have to be less than $TWI(G)$.

Example 2. Consider the Neutrosophic tree $G = (N, M)$ as shown in figure 2,

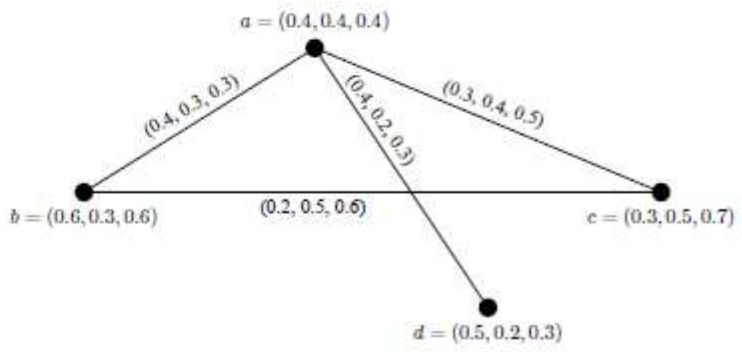


Figure 2. A neutrosophic tree with $V = \{a, b, c, d\}$

Note that here bc is a weak edge.

Table 2. The strength of connectedness and the geodesic between each pair of vertices u and v.

	$CONN_{TG}(u,v)$	$CONN_{IG}(u,v)$	$CONN_{FG}(u,v)$	$d_{S_T}(u,v)$	$d_{S_I}(u,v)$	$d_{S_F}(u,v)$
a,b	0.4	0.3	0.3	0.4	0.3	0.3
a,c	0.3	0.4	0.5	0.3	0.4	0.5
a,d	0.4	0.2	0.3	0.4	0.2	0.3
b,c	0.3	0.4	0.5	0.7	0.7	0.8
b,d	0.4	0.3	0.3	0.8	0.5	0.6
c,d	0.3	0.4	0.5	0.7	0.6	0.8

By direct calculations, we have

$$PCI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)CONN_{T_G}(u,v)$$

$$= 0.4 * 0.6 * 0.4 + 0.4 * 0.3 * 0.3 + 0.4 * 0.5 * 0.4 + 0.6 * 0.3 * 0.3 + 0.6 * 0.5 * 0.4 + 0.3 * 0.5 * 0.3 = 0.096 + 0.036 + 0.080 + 0.054 + 0.120 + 0.045 = 0.431,$$

$$PCI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)CONN_{I_G}(u,v) = 0.036 + 0.080 + 0.016 + 0.060 + 0.018 + 0.040 = 0.25,$$

$$PCI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)CONN_{F_G}(u,v) = 0.072 + 0.14 + 0.036 + 0.21 + 0.054 + 0.105 = 0.617,$$

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} = \frac{3.378}{6} = 0.563.$$

$$PWI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)d_{s_T}(u,v) = 0.096 + 0.036 + 0.08 + 0.126 + 0.24 + 0.105 = 0.683,$$

$$PWI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)d_{s_I}(u,v) = 0.036 + 0.08 + 0.016 + 0.105 + 0.03 + 0.060 = 0.327,$$

$$PWI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)d_{s_F}(u,v) = 0.072 + 0.14 + 0.036 + 0.336 + 0.108 + 0.168 = 0.86,$$

$$TWI(G) = \frac{4 + 2PCWI_T(G) - 2PWI_F(G) - PWI_I(G)}{6} = \frac{3.319}{6} = 0.553.$$

As seen in this example

$$PCI_T(G) = 0.431 < PWI_T(G) = 0.683,$$
$$PCI_I(G) = 0.25 < PWI_I(G) = 0.327,$$
$$PCI_F(G) = 0.617 < PWI_F(G) = 0.86.$$

But, we have $TCI(G) = 0.563 > TWI(G) = 0.553$.

The neutrosophic graph shown in the figure below is also a tree in which $PCI_T(G) < PWI_T(G)$, $PCI_I(G) < PWI_I(G)$, $PCI_F(G) < PWI_F(G)$. And, $TCI(G) < TWI(G)$.

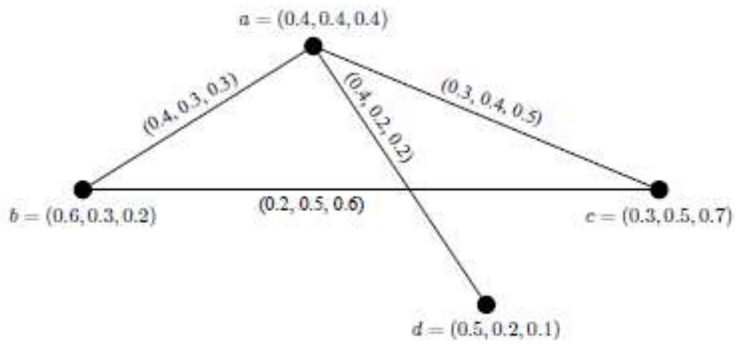


Figure 3. A neutrosophic tree with $V = \{a, b, c, d\}$

Theorem 7. Let $G = (N, M)$ be a neutrosophic tree $|N^*| \geq 3$, With G^* is a star. Let M be a constant function. if v_1 is the center vertex and $v_2, v_3, \dots, v_n$ are the vertices adjacent to vertex v_1, then

$$PWI_T(G) = 2t \sum_{j=1}^{n-1} T_N(v_j) \sum_{k=j+1}^{n} T_N(v_k) - tT_N(v_1) \sum_{j=2}^{n} T(v_j),$$

$$PWI_I(G) = 2i \sum_{j=1}^{n-1} I_N(v_j) \sum_{k=j+1}^{n} I_N(v_k) - iI_N(v_1) \sum_{j=2}^{n} I(v_j),$$

$$PWI_F(G) = 2f \sum_{j=1}^{n-1} F_N(v_j) \sum_{k=j+1}^{n} F_N(v_k) - fF_N(v_1) \sum_{j=2}^{n} F(v_j),$$

where $M = (t, i, f)$.

Proof. Let $G = (N, M)$ be a neutrosophic tree $|N^*| \geq 3$, With G^* is a star. Since $= (t, i, f)$ is a constant function and v_i is the center vertex, for each $v_i,\ 1 < i \leq n$, we have

$$d_{s_T}(v_1, v_i) = t, \qquad d_{s_I}(v_1, v_i) = i, \qquad d_{s_F}(v_1, v_i) = f.$$

Also, for v_i and $v_j,\ i, j \neq 1$, then

$$d_{s_T}(v_j, v_i) = 2t, \qquad d_{s_I}(v_j, v_i) = 2i, \qquad d_{s_F}(v_j, v_i) = 2f.$$

Then

$$\begin{aligned} PWI_T(G) &= \sum_{v_i, v_j \in N} T_N(v_i) T_N(v_j) d_{s_T}(v_i,\ v_j) \\ &= \sum_{v_j \in N} T_N(v_1) T_N(v_j) d_{s_T}(v_1,\ v_j) + \sum_{\substack{v_i, v_j \in N \\ i \neq 1}} T_N(v_i) T_N(v_j) d_{s_T}(v_i,\ v_j) \\ &= tT_N(v_1) \sum_{j=2}^{n} T(v_j) + 2t \sum_{\substack{v_i, v_j \in N \\ i \neq 1}} T_N(v_i) T_N(v_j) d_{s_T}(v_i,\ v_j) \\ &= tT_N(v_1) \sum_{j=2}^{n} T(v_j) + \left[2t \sum_{j=1}^{n-1} T_N(v_j) \sum_{k=j+1}^{n} T_N(v_k) - 2tT_N(v_1) \sum_{j=2}^{n} T(v_j) \right] \\ &= 2t \sum_{j=1}^{n-1} T_N(v_j) \sum_{k=j+1}^{n} T_N(v_k) - tT_N(v_1) \sum_{j=2}^{n} T(v_j). \end{aligned}$$

Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved.

□

4. Applications

One of the most important topics is the use of neutrosophic sets in other sciences and also the use of these assemblies to model various problems. Many applications have been discussed by experts so far. Which can be referred to as application of neutrosophic in graphs [12, 17-19], application in algebraic topics [11, 14], application in intelligent systems and optimization [3, 4].

Here the Wiener index is calculated for a neutrosophic graph associated with a real-time example. You can see this issue and its explanation on the website www.pantechsolutions.net. The neutrosophic graph of this issue is also given in [5]. There, the author examines energy, Laplacian energy, and signless Laplacian energy. We also use the modeling used in [5] here. This neutrosophic graph is intended for four different time periods. According to each time period, we define a neutrosophic graph in the following order:

- G_1 from 16 January 2018 to 15 February 2018 (figure 3);
- G_2 from 16 February 2018 to 15 March 2018 (figure 4);
- G_3 from 16 March 2018 to 15 April 2018 (figure 5);
- G_4 from 16 April 2018 to 15 May 2018 (figure 6);

We now calculate the Wiener index (partial Wiener index and totally Wiener index) for each of the above time periods.

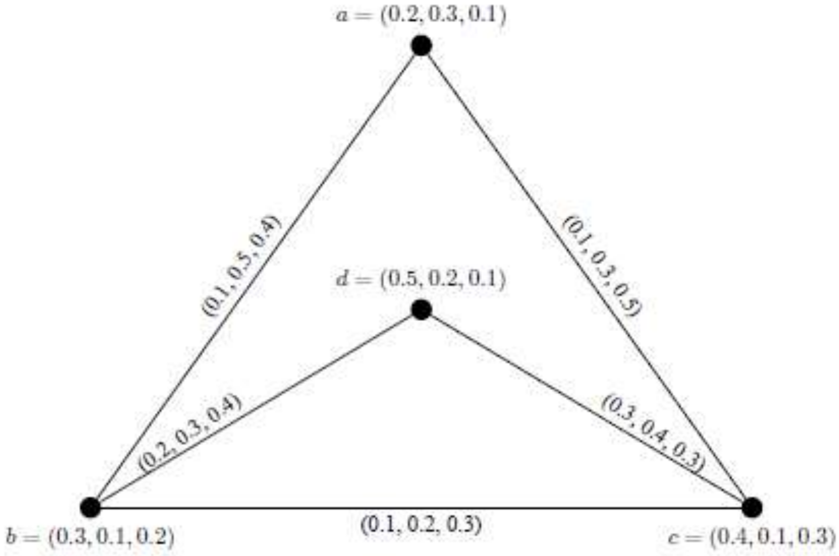


Figure 4. Neutrosophic graph G_1

Table 3. The sum of the weights of the edges in geodesic between each pair of vertices u and v.

	$d_{s_T}(u,v)$	$d_{s_I}(u,v)$	$d_{s_F}(u,v)$
a,b	0.1	$0.3+0.2=0.5$	0.4
a,c	0.1	0.3	$0.4+0.3=0.7$
a,d	$0.1+0.2=0.3$	$0.3+0.4=0.7$	$0.4+0.4=0.8$
b,c	$0.2+0.3=0.5$	0.2	0.3
b,d	0.2	0.3	$0.3+0.3=0.6$
c,d	0.3	$0.2+0.3=0.5$	0.3

$$PWI_T(G_1)=\sum_{u,v\in N}T_N(u)T_N(v)d_{s_T}(u,v)$$
$$=(0.2)(0.3)(0.1)+(0.2)(0.4)(0.1)+(0.2)(0.5)(0.3)+(0.3)(0.4)(0.5)$$
$$+(0.3)(0.5)(0.2)+(0.4)(0.5)(0.3)=0.194,$$

$$PWI_I(G_1)=\sum_{u,v\in N}I_N(u)I_N(v)d_{s_I}(u,v)$$
$$=(0.3)(0.1)(0.5)+(0.3)(0.1)(0.3)+(0.3)(0.2)(0.7)+(0.1)(0.1)(0.2)$$
$$+(0.1)(0.2)(0.3)+(0.1)(0.2)(0.5)=0.084,$$

$$PWI_F(G_1)=\sum_{u,v\in N}F_N(u)F_N(v)d_{s_F}(u,v)$$
$$=(0.1)(0.2)(0.4)+(0.1)(0.3)(0.7)+(0.1)(0.1)(0.8)+(0.2)(0.3)(0.3)$$
$$+(0.2)(0.1)(0.6)+(0.3)(0.1)(0.3)=0.076,$$

$$TWI(G_1)=\frac{4+2PCWI_T(G_1)-2PWI_F(G_1)-PWI_I(G_1)}{6}=\frac{4+2(0.194)-2(0.076)-(0.084)}{6}=\frac{4.152}{6}$$
$$=0.692.$$

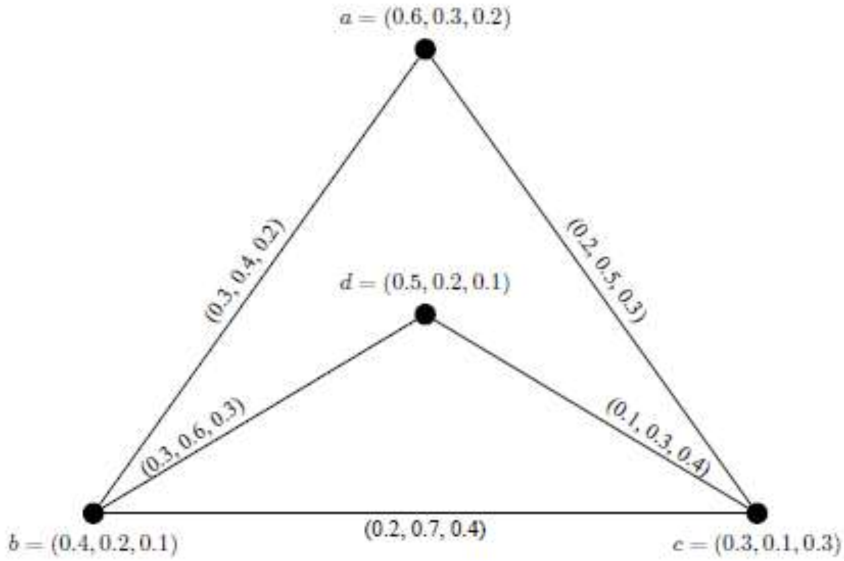


Figure 5. Neutrosophic graph G_2

Table 4. The sum of the weights of the edges in geodesic between each pair of vertices u and v.

	$d_{S_T}(u,v)$	$d_{S_I}(u,v)$	$d_{S_F}(u,v)$
a,b	0.3	0.4	0.2
a,c	0.2	0.5	0.3
a,d	0.3	$0.3+0.5=0.8$	$0.2+0.3=0.5$
b,c	0.2	$0.4+0.5=0.9$	$0.2+0.3=0.5$
b,d	0.3	$0.4+0.5+0.3=1.2$	0.3
c,d	$0.2+0.3=0.5$	0.3	$0.3+0.2+0.3=0.8$

$$PWI_T(G_2) = \sum_{u,v\,\in N} T_N(u)T_N(v)d_{S_T}(u,v) = 0.072 + 0.036 + 0.090 + 0.024 + 0.060 + 0.075 = 0.357,$$

$$PWI_I(G_2) = \sum_{u,v\,\in N} I_N(u)I_N(v)d_{S_I}(u,v) = 0.024 + 0.015 + 0.048 + 0.018 + 0.048 + 0.006 = 0.159,$$

$$PWI_F(G_2) = \sum_{u,v\,\in N} F_N(u)F_N(v)d_{S_F}(u,v) = 0.004 + 0.018 + 0.010 + 0.015 + 0.003 + 0.024 = 0.074,$$

$$TWI(G_2) = \frac{4 + 2PCWI_T(G_2) - 2PWI_F(G_2) - PWI_I(G_2)}{6} = \frac{4 + 2(0.357) - 2(0.074) - (0.159)}{6} = \frac{4.307}{6}$$

$$= 0.718.$$

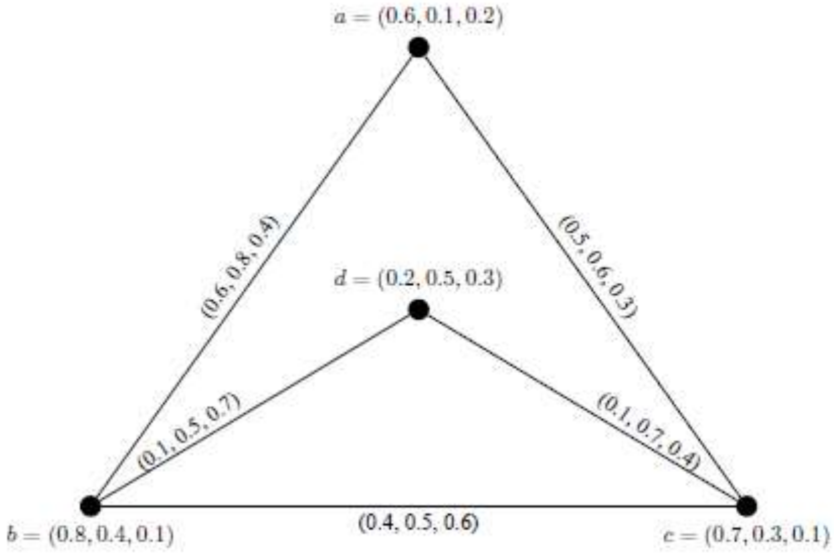


Figure 6. Neutrosophic graph G_3

Table 5. The sum of the weights of the edges in geodesic between each pair of vertices u and v.

	$d_{s_T}(u,v)$	$d_{s_I}(u,v)$	$d_{s_F}(u,v)$
a,b	0.6	$0.5+0.6=1.1$	0.4
a,c	0.5	0.6	0.3
a,d	$0.5+0.1=0.6$	$0.6+0.5+0.5=1.6$	$0.3+0.4=0.7$
b,c	$0.6+0.5=1.1$	0.5	$0.4+0.3=0.7$
b,d	0.1	0.5	$0.4+0.3+0.4=1.1$
c,d	0.1	$0.5+0.5=1$	0.4

$$PWI_T(G_3)=\sum_{u,v\in N} T_N(u)T_N(v)d_{s_T}(u,v)=0.288+0.210+0.072+0.616+0.016+0.014=1.216,$$

$$PWI_I(G_3)=\sum_{u,v\in N} I_N(u)I_N(v)d_{s_I}(u,v)=0.044+0.018+0.080+0.060+0.1+0.15=0.452,$$

$$PWI_F(G_3)=\sum_{u,v\in N} F_N(u)F_N(v)d_{s_F}(u,v)=0.008+0.006+0.042+0.007+0.033+0.012=0.108$$

$$TWI(G_3)=\frac{4+2PCWI_T(G_3)-2PWI_F(G_3)-PWI_I(G_3)}{6}=\frac{4+2(1.216)-2(0.108)-(0.452)}{6}=\frac{5.548}{6}$$
$$=0.925.$$

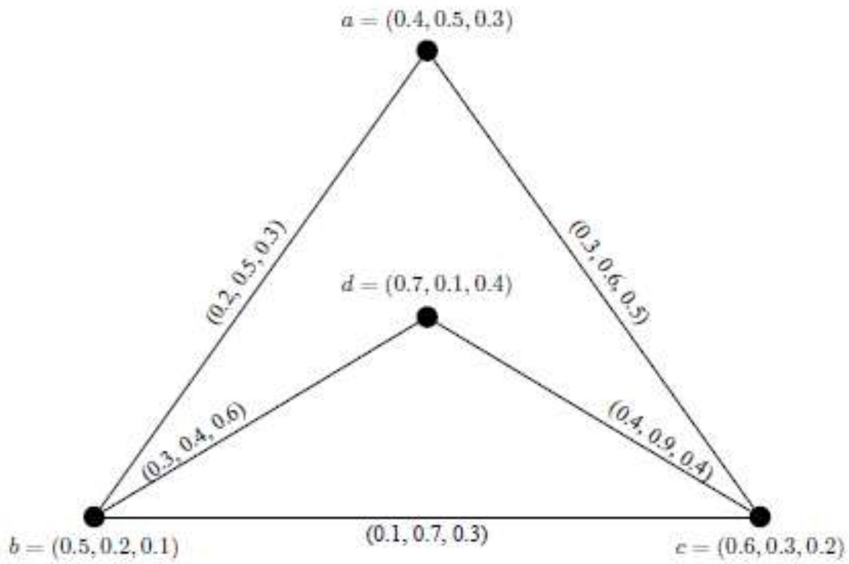


Figure 7. Neutrosophic graph G_4

Table 6. The sum of the weights of the edges in geodesic between each pair of vertices u and v.

	$d_{s_T}(u,v)$	$d_{s_I}(u,v)$	$d_{s_F}(u,v)$
a,b	$0.3+0.4+0.3=1$	0.5	0.3
a,c	0.3	0.6	$0.3+0.3=0.6$
a,d	$0.3+0.4=0.7$	$0.5+0.4=0.9$	$0.5+0.4=0.9$
b,c	$0.3+0.4=0.7$	$0.5+0.6=1.1$	0.3
b,d	0.3	0.4	$0.3+0.4=0.7$
c,d	0.4	$0.6+0.5=1.1$	0.4

$$PWI_T(G_4)=\sum_{u,v\in N} T_N(u)T_N(v)d_{s_T}(u,v)=0.20+0.072+0.196+0.210+0.105+0.168=0.951,$$

$$PWI_I(G_4)=\sum_{u,v\in N} I_N(u)I_N(v)d_{s_I}(u,v)=0.050+0.180+0.045+0.066+0.008+0.033=0.382,$$

$$PWI_F(G_4)=\sum_{u,v\in N} F_N(u)F_N(v)d_{s_F}(u,v)=0.009+0.036+0.108+0.006+0.028+0.032=0.219,$$

$$TWI(G_4)=\frac{4+2PCWI_T(G_4)-2PWI_F(G_4)-PWI_I(G_4)}{6}=\frac{4+2(0.951)-2(0.219)-(0.382)}{6}=\frac{5.082}{6}$$
$$=0.847.$$

Now, using the Wiener index obtained for each of the neutrosophic graphs G_1, G_2, G_3, and G_4, we can compare these four components in the time intervals given in the problem.

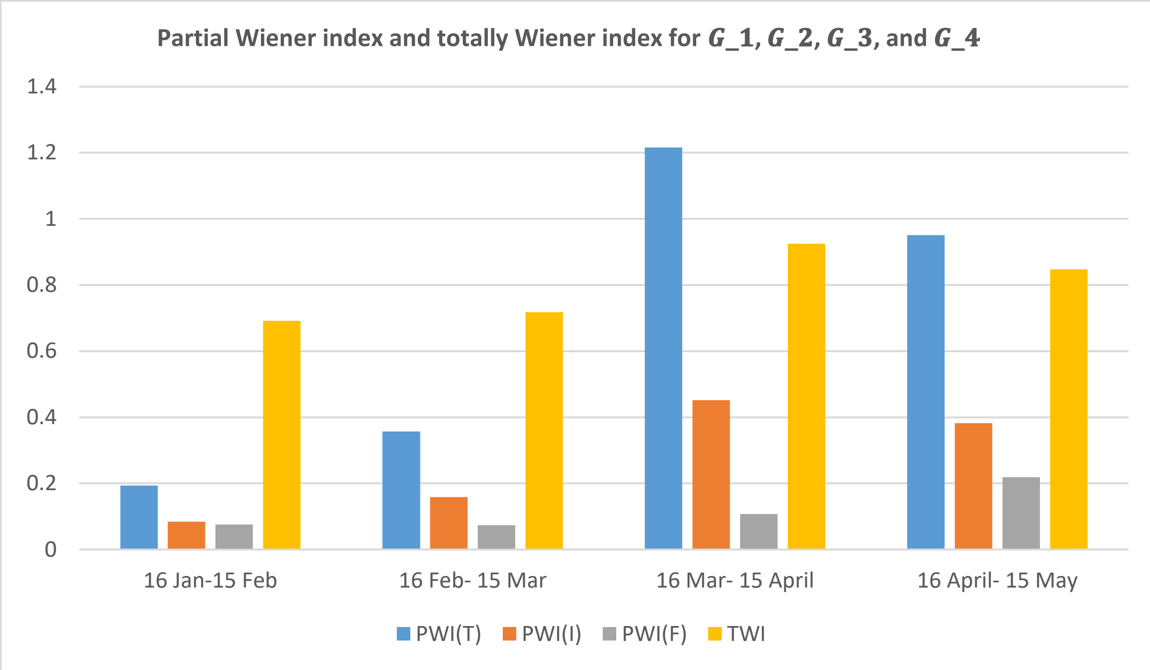


Figure 8. Wiener index comparison chart in G_1, G_2, G_3, and G_4

As shown in Figure 7, they can be easily studied using the Wiener index and assigning a logical value to each of the neutrosophic graphs.

Conclusion

In this article, we examine the Wiener index in neutrosophic graphs. First, this index was defined for this group of graphs and then it was calculated for certain modes of neutrosophic graphs. In the following, we provide an example of the application of this index in real problems. As you can see here, this index, which is one of the most important topological indices based on distance, can be a good criterion for comparing neutrosophic graphs under the same conditions. This index can also be studied and used for bipolar and interval valued neutrosophic graphs.

Funding: "This research received no external funding"

Acknowledgments:
The authors thank the reviewers for their constructive feedback.

Conflicts of Interest: "The authors declare no conflict of interest."

References

1. Abdel-Basset, M., Gamal, A., Son, L. H., & Smarandache, F. 2020. A Bipolar Neutrosophic Multi Criteria Decision Making Framework for Professional Selection. Applied Sciences, 10(4), 1202.
2. Abdel-Basset, M., Mohamed, R., Zaied, A. E. N. H., Gamal, A., & Smarandache, F. 2020. Solving the supply chain problem using the best-worst method based on a novel Plithogenic model. In Optimization Theory Based on Neutrosophic and Plithogenic Sets (pp. 1-19). Academic Press.
3. Abdel-Basset, M.; Mohamed, R.; Elhoseny, M. & Chang, V. Evaluation framework for smart disaster response systems in uncertainty environment, 2020; Mechanical Systems and Signal Processing, 145, 106941.

4. Abdel-Basset, M.; Ali, M.; & Atef, A. Uncertainty assessments of linear time-cost tradeoffs using neutrosophic set, 2020, Computers & Industrial Engineering, 141, 106286.
5. Akram, M. single-Valued Neutrosophic Graphs, 2018; Springer Nature Singapore, Pte Ltd.
6. Atanassov, K. T. Intuitionstic fuzzy sets. 1983. In proceedings of the VII ITKR Session, Sofia, Bulgaria, 79 June 1983, (Deposed in Central Sci.-Techn. Library of Bulg. Acad. Of Sci., 1697/84) (in Bulgaria).
7. Broumi, S.; Talea, M.; Bakali, A.; Singh, P.K.; & Smarandache, F. Energy and Spectrum Analysis of Interval Valued Neutrosophic Graph using MATLAB, 2019, Neutrosophic Sets and Systems, vol. 24, pp. 46-60. DOI: 10.5281/zenodo.2593919.
8. Ghods, M.; & Rostami, Z. Introduction to Topological Indices in Neutrosophic Graphs, 2020; Neutrosophic Sets and Systems, Vol. 35, pp. 68-77. DOI: 10.5281/zenodo.3951641.
9. Ghods, M.; & Rostami, Z. Connectivity index in neutrosophic trees and the algorithm to find its maximum spanning, Neutrosophic Sets and Systems, vol. 36, 2020, pp. 37-49. DOI: 10.5281/zenodo.4065371.
10. Ghods, M.; & Rostami, Z. Introduction Totally and Partial Connectivity Indices in Neutrosophic graphs with Application in Behavioral Sciences, Neutrosophic Sets and Systems, vol. 36, 2020, pp.81-95. DOI: 10.5281/zenodo.4065391.
11. K. Porselvi, B. Elavarasan and F. Smarandache, Neutrosophic N-interior ideals in semigroups, Neutrosophic Sets and Systems, vol. 36, 2020, pp. 70-80. DOI: 10.5281/zenodo.4065385.
12. Nagarajan, D.; Lathamaheswari, M.; Broumi, S.; & Kavikumar, J. Blockchain Single and Interval Valued Neutrosophic Graphs, 2019, Neutrosophic Sets and Systems, vol. 24, pp. 23-35. DOI: 10.5281/zenodo.2593909.
13. Rosenfeld, A. Fuzzy graphs, in: L. A. Zadeh, K. S. Fu, M. Shimura (Eds), Fuzzy Sets and Their Applications, Academic Press, New York, 1975, pp. 77-95.
14. Sowndrara jan, M Jeyaraman and Florentin Smarandache, Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces, Neutrosophic Sets and Systems, vol. 36, 2020, pp. 308-318. DOI: 10.5281/zenodo.4065458.
15. Smarandach, F. A unifying field in logics: Neutrosophic logic. 1995. Rehoboth: American Research Press.
16. Smarandache, F. the neutrosophic triplet (n-ary HyperAlgebra, n-ary NeutroHyperAlgebra, n-ary AntiHyper Algebra). 2019. In Nidus Idearum, Vol. 7, second edition, Pons asbl, Bruxelles, pp. 104-106.
17. Satham Hussain, S.; Jahir Hussain, R.; Jun, Y.B.; & Smarandache, F. Neutrosophic Bipolar Vague Set and its Application to Neutrosophic Bipolar Vague Graphs, 2019, Neutrosophic Sets and Systems, vol. 28, pp. 69-86. DOI: 10.5281/zenodo.3387802
18. Satham Hussain, S.; Jahir Hussain, R.; & Smarandache, F. Domination Number in Neutrosophic Soft Graphs, 2019, Neutrosophic Sets and Systems, vol. 28, pp. 228-244. DOI: 10.5281/zenodo.3382548
19. Satham Hussain, S.; Jahir Hussain, R.; & Smarandache, F. On Neutrosophic Vague Graphs, 2019, Neutrosophic Sets and Systems, vol. 28, pp. 245-258. DOI: 10.5281/zenodo.3382550
20. Zadeh, L.A. Fuzzy sets. Information and Control 1965, 8, 338-353. [CrossRef].

Received: May 18, 2021. Accepted: October 3, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021
University of New Mexico

NEUTROSOPHIC *d*-IDEAL OF NEUTROSOPHIC *d*-ALGEBRA

Suman Das[1,]* and Ali Khalid Hasan[2]

[1]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

[2]Directorate General of Education in Karbala province, Ministry of Education, Iraq.

***Correspondence:** sumandas18843@gmail.com

Abstract: In this article, we introduce the concept of neutrosophic *d*-ideal of neutrosophic *d*-algebra. Also we have studied several properties of them. We also furnish some suitable examples.

Keywords: *Fuzzy Set*; *Intuitionistic Fuzzy Set; Neutrosophic Set*; *d-Algebra*; *d-Ideal*; *d-Sub-Algebra*.

1. Introduction:

The concept of *BCK* algebra and *BCI* algebra are introduced by Imai & Iseki [18]. Thereafter, Negger & Kim [23] introduced the *d*-algebra as a generalization of *BCK* algebra. Negger et al. [22] discussed the ideal theory in *d*-algebra. In the year 1965, Zadeh introduced the idea of fuzzy set [26]. Thereafter, Atanassov introduced the notion of intuitionistic fuzzy set [1], which is the natural generalization of fuzzy set. Later on, Jun et al. [20] applied the notion of intuitionistic fuzzy set on *d*-algebra. Afterwards, the notion of intuitionistic fuzzy *d*-ideal of *d*-algebra was introduced by Hasan [16] in 2017. Thereafter, the concept of intuitionistic fuzzy *d*-filter was introduced by Hasan [17] in 2020. The concept of neutrosophic set was introduced by Smarandache [24]. In this article, we procure the notion of neutrosophic *d*-algebra and neutrosophic *d*-ideal by extending the notion of intuitionistic fuzzy *d*-ideal of *d*-algebra.

Research gap: No investigation on neutrosophic *d*-algebra and neutrosophic *d*-ideal has been reported in the recent literature.

Motivation: To fill the research gap, we introduce the neutrosophic *d*-algebra and neutrosophic *d*-ideal.

The rest of the paper is designed as follows:

In section-2, we recall *d*-algebra, *d*-ideal, fuzzy *d*-algebra, fuzzy *d*-ideal, intuitionistic fuzzy *d*-algebra, intuitionistic fuzzy *d*-ideal. In section-3, we introduce the notion of neutrosophic *d*-algebra, neutrosophic *d*-ideal, and the proofs of some propositions, theorems on neutrosophic *d*-algebra, and neutrosophic *d*-ideal. In section-4, we give the conclusions of work done in this paper.

2. Preliminaries and Some Results:

Definition 2.1.[17] Assume that W be a non-empty set and 0 be a constant. Then, W with a binary operation $*$ is called a d-algebra if it satisfies the following three axioms:

(i) $c*c=0$, $\forall\, c\in W$

(ii) $0*c=0$, $\forall\, c\in W$

(iii) $c*d=0$ and $d*c=0 \Rightarrow c=d$, $\forall\, c, d\in W$.

We will refer to $c*d$ by cd. And $c\leq d$ iff $cd=0$.

Definition 2.2.[17] A d-algebra W is called commutative if $c(cd)=d(dc)$, $\forall c, d\in W$, and $d(dc)$ is denoted by $(c\wedge d)$.

Definition 2.3.[17] A d-algebra W is called bounded if there exist $a\in W$ such that $c\leq a$ for all $c\in W$, i.e. $ca=0$, $\forall c\in W$.

Definition 2.4.[17] Let W be a d-algebra with binary operator $*$ and $A\subseteq W$. Then, A is said to be a d-sub-algebra of W, if $c, d\in A \Rightarrow cd\in A$.

Definition 2.5.[16] Let W be a d-algebra with binary operator $*$ and a constant 0. Then, $D\subseteq W$ is called a d-ideal of W if it satisfies the following:

(i) $a*b\in D$ and $b\in D \Rightarrow a\in D$;

(ii) $a\in D$ and $b\in W \Rightarrow a*b\in D$.

Definition 2.6.[15] Let $Y=\{(c,T_Y(c)):c\in W\}$ be a fuzzy set over a d-algebra W. Then, A is called a fuzzy d-algebra if $T_Y(cd)\geq \min\{T_Y(c), T_Y(d)\}$, for all $c, d\in W$.

Definition 2.7.[15] An fuzzy set $Y=\{(c,T_Y(c),F_Y(c)): c\in W\}$ over a d-algebra W is called the fuzzy d-ideal if it satisfies the following inequalities:

(i) $T_Y(c)\geq \min\{T_Y(cd), T_Y(d)\}$;

(iii) $T_Y(cd)\geq T_Y(c)$, for all $c, d\in W$.

Definition 2.8.[14] Let $Y=\{(c,T_Y(c),F_Y(c)): c\in W\}$ be an intuitionistic fuzzy set over a d-algebra W. Then, A is called an intuitionistic fuzzy d-algebra if it satisfies the followings:

(i) $T_Y(cd)\geq \min\{T_Y(c), T_Y(d)\}$;

(ii) $F_Y(cd)\leq \max\{F_Y(c), F_Y(d)\}$;

where $c, d\in W$.

Proposition 2.1.[14] Every intuitionistic fuzzy d-algebra $Y=\{(c,T_Y(c),F_Y(c)): c\in W\}$ of W satisfies the following inequalities:

(i) $T_Y(0)\geq T_Y(c)$, for all $c\in W$;

(ii) $F_Y(0)\leq F_Y(c)$, for all $c\in W$.

Definition 2.9.[10] An intuitionistic fuzzy set $Y=\{(c,T_Y(c),F_Y(c)): c\in W\}$ over a d-algebra W is called the intuitionistic fuzzy d-ideal if it satisfies the following inequalities:

(i) $T_Y(c)\geq \min\{T_Y(cd), T_Y(d)\}$;

(ii) $F_Y(c)\leq \max\{F_Y(cd), F_Y(d)\}$;

(iii) $T_Y(cd)\geq T_Y(c)$;

(iv) $F_Y(cd)\geq F_Y(c)$; for all $c, d\in Y$.

Proposition 2.2.[10] Let Y={(c,$T_Y(c)$,$F_Y(c)$): $c\in W$} be an intuitionistic fuzzy d-ideal over a d-algebra W. Then, the following inequalities hold:

$T_Y(0)\geq T_Y(c)$, $F_Y(0)\leq F_Y(c)$, for all $c\in W$.

Definition 2.10.[18] A neutrosophic set over a universal set W is defined as follows:

H = {(y, $T_H(y)$, $I_H(y)$, $F_H(y)$): $y\in W$}, where $T_H(y)$, $I_H(y)$ and $F_H(y)$ ($\in]^-0,1^+[$) are the truth, indeterminacy and false membership value of y and $^-0 \leq T_H(y) + I_H(y) + F_H(y) \leq 3^+$.

Definition 2.11.[18] The neutrosophic whole set (1_N) and neutrosophic null set (0_N) over a universal set W is defined as follows:

(i) 1_N={(y,1,0,0): $y\in W$}.

(ii) 0_N ={(y,0,0,1): $y\in W$}.

Definition 2.12.[18] Assume that H={(y, $T_H(y)$, $I_H(y)$, $F_H(y)$):$y\in W$} and K={(y, $T_K(y)$, $I_K(y)$, $F_K(y)$):$y\in W$} are any two neutrosophic sets over X. Then,

(i) $H\cup K$= {(y,$T_H(y)\vee T_K(y)$, $I_H(y)\wedge I_K(y)$, $F_H(y)\wedge F_K(y)$): $y\in W$};

(ii) $H\cap K$= {(y, $T_H(y)\wedge T_K(y)$, $I_H(y)\vee I_K(y)$, $F_H(y)\vee F_K(y)$): $y\in W$};

(iii) H^c= {(y, 1-$T_H(y)$, 1-$I_H(y)$, 1-$F_H(y)$): $y\in W$};

(iv) $H\subseteq K \Leftrightarrow T_H(y)\leq T_K(y)$, $I_H(y) \geq I_K(y)$, $F_H(y) \geq F_K(y)$, for each $y\in W$.

3. Neutrosophic d-Algebra and Neutrosophic d-Ideal:

Definition 3.1. Let Y={(c,$T_Y(c)$,$I_Y(c)$,$F_Y(c)$): $c\in W$} be an neutrosophic set over a d-algebra W. Then, A is called a neutrosophic d-algebra if it satisfies the followings:

(i) $T_Y(c*d)\geq \min\{T_Y(c), T_Y(d)\}$;

(ii) $I_Y(c*d)\leq \max\{I_Y(c), I_Y(d)\}$;

(iii) $F_Y(c*d)\leq \max\{F_Y(c), F_Y(d)\}$;

where c, $d\in W$.

Example 3.1. Take $W = \{0, c, d, w\}$ with the following table

$\star$	0	c	d	w
0	0	0	0	0
c	c	0	0	c
d	d	d	0	0
w	w	w	d	0

Note that if we define

$T_Y(a) = \begin{cases} 0.2 & if a = 0, c \\ 0.02 & if a = d, w \end{cases}$, $I_Y(a) = \begin{cases} 0.09 & if a = 0, c \\ 0.8 & if a = d, w \end{cases}$ and $F_Y(a) = \begin{cases} 0.05 & if a = 0, c \\ 0.7 & if a = d, w \end{cases}$

So we can show easily that $Y = \{(c, T_Y(c), I_Y(c), F_Y(c)): c\in W\}$ is a neutrosophic d-algebra

Proposition 3.1. Every neutrosophic d-algebra Y={(c,$T_Y(c)$,$I_Y(c)$,$F_Y(c)$): $c\in W$} of W satisfies the following inequalities:

(i) $T_Y(0)\geq T_Y(c)$, for all $c\in W$;

(ii) $I_Y(0)\leq I_Y(c)$, for all $c\in W$;

(*iii*) $F_Y(0) \leq F_Y(c)$, for all $c \in W$.

Proof. Assume that $Y=\{(c,T_Y(c),I_Y(c),F_Y(c)): c \in W\}$ be a neutrosophic *d*-algebra of *W*. Let $c \in W$. Then

(*i*) $T_Y(0)= T_Y(c*c) \geq \min\{ T_Y(c), T_Y(c)\}= T_Y(c)$, (using definition 2.1. & 3.1.)

(*ii*) $I_Y(0)= I_Y(c*c) \leq \max\{ I_Y(c), I_Y(c)\}= I_Y(c)$, (using definition 2.1. & 3.1.)

(*iii*) $F_Y(0)= F_Y(c*c) \leq \max\{ F_Y(c), F_Y(c)\}= F_Y(c)$, (using definition 2.1. & 3.1.)

Theorem 3.1. Let $\{Y_i: i \in \Delta\}$ be the family of neutrosophic *d*-algebra of *W*. Then, $\bigcap_{i\in\Delta} Y_i$ is a neutrosophic *d*-algebra of *W*.

Proof. Assume that $\{Y_i: i \in \Delta\}$ be the family of neutrosophic *d*-algebra of *W*. Now, $\bigcap_{i\in\Delta} Y_i=\{(c,\wedge T_{Y_i}(c),\vee I_{Y_i}(c),\vee F_{Y_i}(c)): c \in W\}$. Let $c, d \in W$. Then,

(*i*) $\wedge T_{Y_i}(c*d) \geq \wedge \min\{T_{Y_i}(c), T_{Y_i}(d)\}= \min\{\wedge T_{Y_i}(c),\wedge T_{Y_i}(d)\}$

$\Rightarrow \wedge T_{Y_i}(c*d) \geq \min\{\wedge T_{Y_i}(c),\wedge T_{Y_i}(d)\}$;

(*ii*) $\vee I_{Y_i}(c*d) \leq \vee \max\{I_{Y_i}(c), I_{Y_i}(d)\}= \max\{\vee I_{Y_i}(c),\wedge I_{Y_i}(d)\}$

$\Rightarrow \vee I_{Y_i}(c*d) \leq \max\{\vee I_{Y_i}(c),\wedge I_{Y_i}(d)\}$;

(*iii*) $\vee F_{Y_i}(c*d) \leq \vee \max\{F_{Y_i}(c), F_{Y_i}(d)\}= \max\{\vee F_{Y_i}(c),\wedge F_{Y_i}(d)\}$

$\Rightarrow \vee F_{Y_i}(c*d) \leq \max\{\vee F_{Y_i}(c),\wedge F_{Y_i}(d)\}$;

Therefore, $\bigcap_{i\in\Delta} Y_i$ is also a neutrosophic *d*-algebra of *W*.

Theorem 3.3. If $Y=\{(c,T_Y(c),I_Y(c),F_Y(c)): c \in W\}$ is a neutrosophic *d*-algebra of *W*, then the sets $W_T=\{c \in W: T_Y(c)=T_Y(0)\}$, $W_I=\{c \in W: I_Y(c)=I_Y(0)\}$, and $W_F=\{c \in W: F_Y(c)=F_Y(0)\}$ are *d*-sub-algebras of *W*.

Proof. Assume that $Y=\{(c,T_Y(c),I_Y(c),F_Y(c)): c \in W\}$ be a neutrosophic *d*-algebra of *W*. Given $W_T=\{c \in W: T_Y(c)=T_Y(0)\}$, $W_I=\{c \in W: I_Y(c)=I_Y(0)\}$, and $W_F=\{c \in W: F_Y(c)=F_Y(0)\}$. Let $c, d \in W_T$. Therefore, $T_Y(c)=T_Y(0)$, $T_Y(d)=T_Y(0)$. Now by definition 2.1, $T_Y(c*d) \geq \min\{T_Y(c), T_Y(d)\}= \min\{T_Y(0), T_Y(0)\}=T_Y(0)$, i.e. $T_Y(c*d) \geq T_Y(0)$. Again from proposition 3.1, it is clear that $T_Y(0) \leq T_Y(c*d)$. Therefore $T_Y(c*d)= T_Y(0)$. This implies that $c*d \in W_T$. Hence $c, d \in W_T \Rightarrow c*d \in W_T$. Therefore the set $W_T=\{c \in W: T_Y(c)=T_Y(0)\}$ is a *d*-sub-algebra of *W*.

Similarly we can easily show that $W_I=\{c \in W: I_Y(c)=I_Y(0)\}$ and $W_F=\{c \in W: F_Y(c)=F_Y(0)\}$ are *d*-sub-algebras of *W*.

Definition 3.2. Assume that $Y = \{(c,T_Y(c),I_Y(c),F_Y(c)): c \in W\}$ be a neutrosophic set over *W*. Then, the sets $W(T_Y,\alpha)=\{c \in W: T_Y(c) \geq \alpha\}$, $W(I_Y,\alpha)=\{c \in W: I_Y(c) \leq \alpha\}$, $W(F_Y,\alpha)=\{c \in W: F_Y(c) \leq \alpha\}$ are respectively called *T*-level α-cut, *I*-level α-cut, *F*-level α-cut of *Y*.

Theorem 3.4. Assume that $Y=\{(c,T_Y(c),I_Y(c),F_Y(c)): c \in W\}$ be a neutrosophic *d*-algebra of *W*. Then, for any $\alpha \in [0,1]$, the *T*-level α-cut, *I*-level α-cut, *F*-level α-cut of *Y* are *d*-sub-algebra of *W*.

Proof. Assume that $Y=\{(c,T_Y(c),I_Y(c),F_Y(c)): c \in W\}$ be a neutrosophic *d*-algebra of *W*. Then, *T*-level α-cut of $Y= W(T_Y,\alpha)=\{c \in W: T_Y(c) \geq \alpha\}$, *I*-level α-cut of $Y= W(I_Y,\alpha)=\{c \in W: I_Y(c) \leq \alpha\}$, and *F*-level α-cut of $Y= W(F_Y,\alpha)=\{c \in W: F_Y(c) \leq \alpha\}$.

Let $c, d \in W(T_Y,\alpha)$. Therefore, $T_Y(c) \geq \alpha$, $T_Y(d) \geq \alpha$. Now $T_Y(c*d) \geq \min\{ T_Y(c), T_Y(d)\} \geq \min\{ \alpha, \alpha\} \geq \alpha$. This implies, $c*d \in W(T_Y,\alpha)$. Hence, $c, d \in W(T_Y,\alpha) \Rightarrow c*d \in W(T_Y,\alpha)$. Therefore, $W(T_Y,\alpha)$ i.e. *T*-level α-cut of *Y* is a *d*-sub-algebra of *W*.

Similarly, we can easily show that *I*-level α-cut, *F*-level α-cut of *Y* are *d*-sub-algebra of *W*.

Definition 3.3. An neutrosophic set Y={(c,$T_Y(c)$,$I_Y(c)$,$F_Y(c)$):$c\in W$} over a d-algebra W is called a neutrosophic d-ideal if it satisfies the following inequalities:

(i) $T_Y(c)\geq\min\{T_Y(cd), T_Y(d)\}$;

(ii) $F_Y(c)\leq\max\{F_Y(cd), F_Y(d)\}$;

(iii) $I_Y(c)\leq\max\{I_Y(cd), I_Y(d)\}$;

(iv) $T_Y(cd)\geq T_Y(c)$;

(v) $F_Y(cd)\geq F_Y(c)$;

(vi) $I_Y(cd)\geq I_Y(c)$, for all c, $d\in Y$.

Example 3.2. Take $W = \{0, c, d, w\}$ with the following table

⋆	0	c	d
0	0	0	0
c	d	0	d
d	c	c	0

Note that if we define

$T_Y(a) = \begin{cases} 0.9 & if a = 0 \\ 0.01 & if a = c, d \end{cases}$, $I_Y(a) = \begin{cases} 0.1 & if a = 0 \\ 0.5 & if a = \text{c}, d \end{cases}$ and $F_Y(a) = \begin{cases} 0.2 & if a = 0 \\ 0.6 & if a = c, d \end{cases}$

Then $Y = \{(c, T_Y(c), I_Y(c), F_Y(c)) \colon c\in W\}$ is a neutrosophic d-ideal of d-algebra

Proposition 3.2. If Y={(c,$T_Y(c)$,$I_Y(c)$,$F_Y(c)$): $c\in W$} is a neutrosophic d-ideal of W, then $T_Y(0)\geq T_Y(c)$, $I_Y(0)\leq I_Y(c)$, $F_Y(0)\leq F_Y(c)$, for all $c\in W$.

Proof. Assume that Y={(c,$T_Y(c)$,$I_Y(c)$,$F_Y(c)$): $c\in W$} is a neutrosophic d-ideal of W, and c be any arbitrary element of W. Since $T_Y(c*c)\geq T_Y(c)$, so $T_Y(0)\geq T_Y(c)$. Similarly, since $I_Y(c*c)\leq I_Y(c)$, so $I_Y(0)\leq I_Y(c)$. Again, since $F_Y(c*c)\leq F_Y(c)$, so $F_Y(0)\leq F_Y(c)$.

Theorem 3.6. Assume that Y={(x,$T_Y(x)$,$I_Y(x)$,$F_Y(x)$):$x\in W$} is a neutrosophic d-ideal of W. If $x*y \leq z$, then $T_Y(x)\geq \min\{T_Y(y), T_Y(z)\}$, $I_Y(x)\leq \max\{I_Y(y), I_Y(z)\}$, $F_Y(x)\leq \max\{F_Y(y), F_Y(z)\}$.

Proof. Assume that Y={(x,$T_Y(x)$,$I_Y(x)$,$F_Y(x)$):$x\in W$} be an neutrosophic d-ideal of W. Let x, y, z be any three element of W such that $x*y \leq z$. Then by definition 2.1, $(x*y)*z=0$.

Now, $T_Y(x)\geq \min\{T_Y(x*y), T_Y(y)\}\geq \min\{\min\{T_Y((x*y)*z), T_Y(z)\}, T_Y(y)\}= \min\{\min\{T_Y(0),T_Y(z)\}, T_Y(y)\}\geq \min\{T_Y(z), T_Y(y)\}$. Therefore, $T_Y(x)\geq \min\{T_Y(y), T_Y(z)\}$.

Now, $I_Y(x)\leq \max\{I_Y(x*y), I_Y(y)\}\leq \max\{\max\{I_Y((x*y)*z), I_Y(z)\}, I_Y(y)\}= \max\{\max\{I_Y(0),I_Y(z)\}, I_Y(y)\}\leq \max\{I_Y(z), I_Y(y)\}$. Therefore, $I_Y(x)\leq \max\{I_Y(y), I_Y(z)\}$.

Again, $F_Y(x)\leq \max\{F_Y(x*y), F_Y(y)\}\leq \max\{\max\{F_Y((x*y)*z), F_Y(z)\}, F_Y(y)\}= \max\{\max\{F_Y(0),F_Y(z)\}, F_Y(y)\}\leq \max\{F_Y(z), F_Y(y)\}$. Therefore, $F_Y(x)\leq \max\{F_Y(y), F_Y(z)\}$.

Theorem 3.7. Assume that Y={(c,$T_Y(c)$,$I_Y(c)$,$F_Y(c)$): $c\in W$} is a neutrosophic d-ideal of W. If $x \leq z$, then $T_Y(x)\geq T_Y(z)$, $I_Y(x)\leq I_Y(z)$, $F_Y(x)\leq F_Y(z)$.

Proof. Assume that Y={(c,$T_Y(c)$,$I_Y(c)$,$F_Y(c)$): $c\in W$} is a neutrosophic d-ideal of W. Also let x, z be any two element of W such that $x \leq z$. Then by the definition 2.1, $x*z=0$.

Now, $T_Y(x)\geq \min\{T_Y(x*z),T_Y(z)\}= \min\{T_Y(0),T_Y(z)\}, T_Y(z)\}=T_Y(z)$. Therefore, $T_Y(x)\geq T_Y(z)$.

Now, $I_Y(x)\leq \max\{I_Y(x*z),I_Y(z)\}= \max\{I_Y(0),I_Y(z)\}, I_Y(z)\}=I_Y(z)$. Therefore, $I_Y(x)\leq I_Y(z)$.

Now, $F_Y(x)\leq \max\{F_Y(x*z),F_Y(z)\}= \max\{F_Y(0),F_Y(z)\}, F_Y(z)\}=F_Y(z)$. Therefore, $F_Y(x)\leq F_Y(z)$.

Theorem 3.10. If $\{D_i: i\in\Delta\}$ be the collection of neutrosophic d-ideals of d-algebra W, then $\bigcap_{i\in\Delta} D_i$ is also a neutrosophic d-ideal of d-algebra W.

Proof. Assume that $\{D_i: i\in\Delta\}$ be the collection of neutrosophic d-ideals of d-algebra W. We have $\bigcap_{i\in\Delta} D_i= \{(c,\wedge T_{Y_i}(c),\vee I_{Y_i}(c),\vee F_{Y_i}(c)): c\in W\}$.

Now $\wedge T_{Y_i}(c)\geq \wedge\{\min\{T_{Y_i}(c*d),T_{Y_i}(d)\}\} \geq \min\{\wedge T_{Y_i}(c*d), \wedge T_{Y_i}(d)\}$,

$\vee I_{Y_i}(c)\leq \vee\{\max\{I_{Y_i}(c*d),I_{Y_i}(d)\}\} \leq \max\{\vee I_{Y_i}(c*d), \vee I_{Y_i}(d)\}$,

and $\vee I_{Y_i}(c)\leq \vee\{\max\{I_{Y_i}(c*d),I_{Y_i}(d)\}\} \leq \max\{\vee I_{Y_i}(c*d), \vee I_{Y_i}(d)\}$.

Since $T_{Y_i}(c*d)\geq T_{Y_i}(c)$, $I_{Y_i}(c*d)\leq I_{Y_i}(c)$, $I_{Y_i}(c*d)\leq I_{Y_i}(c)$, for all i, we have $\wedge T_{Y_i}(c*d) \geq \wedge T_{Y_i}(c)$, $\vee I_{Y_i}(c*d) \leq \vee I_{Y_i}(c)$, $\vee I_{Y_i}(c*d) \leq \vee I_{Y_i}(c)$, for all i. Hence $\bigcap_{i\in\Delta} D_i= \{(c,\wedge T_{Y_i}(c),\vee I_{Y_i}(c),\vee F_{Y_i}(c)): c\in W\}$ is a neutrosophic d-ideal of W.

Theorem 3.11. A neutrosophic set $Y=\{(c,T_Y(c),F_Y(c),I_Y(c)): c\in W\}$ is neutrosophic d-ideal of d-algebra W if and only if the corresponding fuzzy set $\{(c,T_Y(c)): c\in W\}$, $\{(c,1\text{-}I_Y(c)): c\in W\}$, $\{(c,1\text{-}F_Y(c)): c\in W\}$ are fuzzy d-ideal of W.

Proof. Assume that $Y=\{(c,T_Y(c),F_Y(c),I_Y(c)): c\in W\}$ be a neutrosophic d-ideal of W. Therefore for all $c,d\in W$, $T_Y(c)\geq \min\{T_Y(cd), T_Y(d)\}$; $T_Y(cd)\geq T_Y(c)$; $I_Y(c)\leq \max\{I_Y(cd), I_Y(d)\}$; $I_Y(cd)\leq I_Y(c)$; $F_Y(c)\leq \max\{F_Y(cd), F_Y(d)\}$; $F_Y(cd)\leq F_Y(c)$.

Since for all c, $d\in W$, $T_Y(c)\geq \min\{T_Y(cd), T_Y(d)\}$; $T_Y(cd)\geq T_Y(c)$, so the fuzzy set $\{(c,T_Y(c)): c\in W\}$ is a fuzzy d-ideal of W.

Now, for all c, $d\in W$,

$I_Y(c) \leq \max\{I_Y(cd), I_Y(d)\} \Rightarrow 1\text{-}I_Y(c)\geq \min\{1\text{-}I_Y(cd), 1\text{-}I_Y(d)\}$;

$I_Y(cd) \leq I_Y(c) \Rightarrow 1\text{-}I_Y(cd) \geq 1\text{-}I_Y(c)$;

Therefore, the fuzzy set $\{(c,1\text{-}I_Y(c)): c\in W\}$ is a fuzzy d-ideal of W.

Again, for all c, $d\in W$,

$F_Y(c)\leq \max\{F_Y(cd), F_Y(d)\} \Rightarrow 1\text{-}F_Y(c)\geq \min\{1\text{-}F_Y(cd), 1\text{-}F_Y(d)\}$;

$F_Y(cd)\leq F_Y(c) \Rightarrow 1\text{-}F_Y(cd)\geq 1\text{-}F_Y(c)$;

Therefore, the fuzzy set $\{(c,1\text{-}F_Y(c)): c\in W\}$ is a fuzzy d-ideal of W.

Hence for an neutrosophic d-ideal $Y=\{(c,T_Y(c),F_Y(c),I_Y(c)): c\in W\}$ of W, the corresponding fuzzy sets $\{(c,T_Y(c)): c\in W\}$, $\{(c,1\text{-}I_Y(c)): c\in W\}$, $\{(c,1\text{-}F_Y(c)): c\in W\}$ are fuzzy d-ideal of W.

Theorem 3.12. If a neutrosophic set $Y=\{(c,T_Y(c),F_Y(c),I_Y(c)): c\in W\}$ is neutrosophic d-ideal of d-algebra W, then the sets $W(T_Y)=\{c\in W: T_Y(c)=T_Y(0)\}$, $W(I_Y)=\{c\in W: I_Y(c)=I_Y(0)\}$, and $W(F_Y)=\{c\in W: F_Y(c)=F_Y(0)\}$ are d-ideal of W.

Proof. Assume that $Y=\{(c,T_Y(c),F_Y(c),I_Y(c)): c\in W\}$ be a neutrosophic d-ideal of a d-algebra W.

Let $a*b\in W(T_Y)$ and $b\in W(T_Y)$. Therefore, $T_Y(a*b)=T_Y(0)$ and $T_Y(b)=T_Y(0)$. Since Y is a neutrosophic d-ideal of a d-algebra W, so $T_Y(a)\geq \min\{T_Y(a*b), T_Y(b)\}= \min\{ T_Y(0), T_Y(0)\}= T_Y(0)$. This implies that $T_Y(a)\geq T_Y(0)$. Again by proposition 3.2, it is clear that $T_Y(0)\geq T_Y(a)$. Hence $T_Y(a)=T_Y(0)$, i.e. $a\in W(T_Y)$. Therefore, $a*b\in W(T_Y)$ and $b\in W(T_Y)\Rightarrow a\in W(T_Y)$.

Again let $a\in W(T_Y)$ and $b\in W$. Therefore, $T_Y(a)=T_Y(0)$. Since Y is a neutrosophic d-ideal of a d-algebra W, so $T_Y(a*b)\geq T_Y(a)=T_Y(0)$. This implies that $T_Y(a*b)\geq T_Y(0)$. From proposition 3.2, it is clear that $T_Y(0)\geq$

$T_Y(a*b)$. Hence $T_Y(a*b)=T_Y(0)$, i.e. $a*b\in W(T_Y)$. Therefore, $a\in W(T_Y)$ and $b\in W \Rightarrow a*b\in W(T_Y)$. Hence the set $W(T_Y)=\{c\in W: T_Y(c)=T_Y(0)\}$ is a d-ideal of W.

Similarly we can show that, the sets $W(I_Y)=\{c\in W: I_Y(c)=I_Y(0)\}$, and $W(F_Y)=\{c\in W: F_Y(c)=F_Y(0)\}$ are d-ideal of W.

5. Conclusions:

In this article, we introduce the notion of neutrosophic d-ideals of d-algebra. Further we have investigated different properties and study some relations on neutrosophic d-algebra. By defining neutrosophic d-algebra, neutrosophic d-ideals, we prove some propositions, theorems on neutrosophic d-algebra and d-ideal.

In the future, we hope that many new notions namely neutrosophic d-filter, neutrosophic d-topology can be introduce based on these notions of neutrosophic d-algebra.

References:

1. Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 35, 87-96.
2. Coker, D. (1997). An introduction to intuitionistic fuzzy topological spaces. *Fuzzy Sets and Systems*, 88, 81-89.
3. Das, S. (2021). Neutrosophic Supra Simply Open Set and Neutrosophic Supra Simply Compact Space. *Neutrosophic Sets and Systems*, 43, 105-113.
4. Das, S., Das, R., Granados, C., & Mukherjee, A. (2021). Pentapartitioned Neutrosophic *Q*-Ideals of *Q*-Algebra. *Neutrosophic Sets and Systems*, 41, 52-63.
5. Das, S., Das, R., & Tripathy, B. C. (2020). Multi criteria group decision making model using single-valued neutrosophic set. *LogForum*, 16(3), 421-429.
6. Das, S., Shil, B., & Tripathy, B. C. (2021). Tangent Similarity Measure Based MADM-Strategy under SVPNS-Environment. *Neutrosophic Sets and Systems*, 43, 93-104.
7. Das, R., Smarandache, F. & Tripathy, B. C. (2020). Neutrosophic fuzzy matrices and some algebraic operation. *Neutrosophic Sets and Systems*, 32, 401-409.
8. Das, S., & Pramanik, S. (2020). Generalized neutrosophic b-open sets in neutrosophic topological space. *Neutrosophic Sets and Systems*, 35, 522-530.
9. Das, S., & Pramanik, S. (2020). Neutrosophic Φ-open sets and neutrosophic Φ-continuous functions. *Neutrosophic Sets and Systems*, 38, 355-367.
10. Das, S., & Pramanik, S. (2020). Neutrosophic simply soft open set in neutrosophic soft topological space. *Neutrosophic Sets and Systems*, 38, 235-243.
11. Das, R. & Tripathy B. C. (2020). Neutrosophic multiset topological space. *Neutrosophic Sets and Systems*, 35, 142-152.
12. Das, S., & Tripathy, B. C. (2020). Pairwise neutrosophic-*b*-open set in neutrosophic bitopological spaces. *Neutrosophic Sets and Systems*, 38, 135-144.

13. Das, S., & Tripathy, B. C. (2021). Neutrosophic simply *b*-open set in neutrosophic topological spaces. *Iraqi Journal of Science*, In Press.
14. Das, S., & Tripathy, B. C. Pentapartitioned Neutrosophic Topological Space. *Neutrosophic Sets and Systems*, In Press.
15. Granados, C., Das, S., & Tripathy, B. C. On I_3-convergence, I^*_3-convergence and I_3-Cauchy sequences in fuzzy normed spaces. *Transactions of A. Razmadze Mathematical Institute*, In press.
16. Hasan, A. K. (2017). On intuitionistic fuzzy *d*-ideal of *d*-algebra. *Journal University of Kerbala*, 15(1), 161-169.
17. Hasan, A. K. (2020). Intuitionistic fuzzy *d*-filter of *d*-algebra. *Journal of mechanics of continua and mathematical sciences*, 15(6), 360-370.
18. Iami, Y., & Iseki, K. (1966). On Axiom System of Propositional Calculi XIV. *Proceedings of the Japan Academy*, 42, 19-20.
19. Iseki, K. (1966). An algebra Relation with Propositional Calculus. *Proceedings of the Japan Academy*, 42, 26-29.
20. Jun, Y. B., Kim, H. S, & Yoo, D. S. (e-2006). Intuitionistic fuzzy *d*-algebra. *Scientiae Mathematicae Japonicae Online*, 1289-1297.
21. Jun, Y. B., Neggers, J., & Kim, H. S. (2000). Fuzzy *d*-ideals of *d*-algebras. *Journal of Fuzzy Mathematics*, 8(1), 123-130.
22. Neggers, J., Jun, Y. B., & Kim, H. S. (1999). On *d*-ideals in *d*-algebras. *Mathematica Slovaca*, 49(3), 243-251.
23. Neggers, J., & Kim, H. S. (1999). On *d*-algebra. *Mathematica Slovaca*, 49(1), 19-26.
24. Smarandache, F. (2005). Neutrosophic set: a generalization of the intuitionistic fuzzy sets. *International Journal of Pure and Applied Mathematics*, 24, 287-297.
25. Tripathy, B. C., & Das, S. (2021). Pairwise Neutrosophic *b*-Continuous Function in Neutrosophic Bitopological Spaces. *Neutrosophic Sets and Systems*, 43, 82-92.
26. Zadeh, L. A. (1965). Fuzzy set, *Information And Control*, 8, 338-353.

Received: May 4, 2021. Accepted: October 1, 2021

NSS

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

Linguistic Neutrosophic Topology

N. Gayathri[1] and M. Helen [2]

[1]Research scholar; Nirmala college for women; Coimbatore ;Tamil Nadu; India

[2]Associate professor; Nirmala college for women;Coimbatore; Tamil Nadu; India

* Correspondence: gayupadmagayu@gmail.com

Abstract. By utilizing linguistic neutrosophic sets and topological spaces, we construct and develop a new notion called "linguistic neutrosophic topological spaces", in this article. Many basic definitions, theorems and properties were defined with suitable examples.

Keywords: Linguistic Neutrosophic topology; Linguistic Neutrosophic open set; Linguistic Neutrosophic closed set; Linguistic Neutrosophic interior; Linguistic Neutrosophic closure; Linguistic Neutrosophic continuous function; Linguistic Neutrosophic neighborhood; Linguistic Neutrosophic derived sets; Linguistic Neutrosophic dense sets;

1. Introduction

Many investigators in business, science, economy and a variety of other branches deal with modeling unknown data on a regular basis. For these ambiguous and uncertainties, traditional techniques are not always successful. Lotfi A. Zadeh [16] instigated the idea of membership or truth value to the elements of collection of well-defined objects called, sets. These systems can handle a variety of inputs, including ambiguous, distorted, or inaccurate data. The idea of fuzzy topology was initially developed by Chang [2] in 1967. Many topological structures and generalizations have developed in time utilizing fuzzy sets. In addition to the degree of truth membership, Atanassov [1] paired non-membership value called false membership, which was the generalization of fuzzy sets, called intuitionistic fuzzy sets. In 1997, intuitionistic fuzzy topology was found by Coker [4]. Along with the two membership values, Smarandache [11] introduced the idea of indeterminacy membership function in 1999. Neutrosophic sets play an important part in many aspects like, decision making, medical diagnosis, etc., Wang and Smarandache [13] introduced the notion of interval valued neutrosophic sets.

Qualitative attributes can be easily expressed in linguistic terms, which was developed by Zadeh [17]. The idea of linguistic variables was applied in decision making by Herrara etc.,al [9] in 2000 and Herrara-Viedma, Vergegay [8] in 1996. Su [12] used linguistic preference information in group decision making. Chen, Liu, etc.,al [3] introduced linguistic intuitionistic fuzzy number(LIFN) in 2015. As LIFN lacks indeterminacy, Ye [15] in 2015, proposed the notion of single valued neutrosophic linguistic numbers and developed an extended TOPSIS model for MAGDM approach utilizing SVNLNs. An extended COPRAS model for MAGDM based on SVN 2-tuple neutrosophic environment was developed by Wei, Wu, etc.,al [14]. Fang, Zebo etc.,al [6] found linguistic neutrosophic numbers in 2017, with a concrete definition.This paper is categorized as follows: Section 2 deals with the basic definitions of LNNs. In Section 3, the idea of linguistic neutrosophic topology is introduced and some properties are discussed. Linguistic neutrosophic derived set is discussed in section 4. In last section, the notion of linguistic neutrosophic continuity and linguistic neutrosophic dense sets are defined and discussed with suitable examples.

2. Preambles

Definition 2.1. [11] Let S be a space of points (objects), with a generic element in x denoted by S. A neutrosophic set A in S is characterized by a truth-membership function T_A, an indeterminacy membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is
$T_A : S \to]0^-, 1^+[, I_A : S \to]0^-, 1^+[, F_A : S \to]0^-, 1^+[$
There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2.2. [11] Let S be a space of points (objects), with a generic element in x denoted by S. A single valued neutrosophic set (SVNS) A in S is characterized by truth-membership function T_A, indeterminacy-membership function I_A and falsity-membership function F_A. For each point S in S, $T_A(x), I_A(x), F_A(x) \in [0, 1]$.
When S is continuous, a SVNS A can be written as $A = \int \langle T(x), I(x), F(x)\rangle / x \in S$.
When S is discrete, a SVNS A can be written as $A = \sum \langle T(x_i), I(x_i), F(x_i)\rangle / x_i \in S$.

Definition 2.3. [6] Let $S = \{s_\theta | \theta = 0, 1, 2, ..., \tau\}$ be a finite and totally ordered discrete term set, where τ is the even value and s_θ represents a possible value for a linguistic variable. For example, when $\tau = 6$, S can be expressed as, $S = \{$very bad, bad, fair, very fair, good, very good$\}$.

Su [12] extended the discrete linguistic term set S into a continuous term set $S = \{s_\theta | \theta \in [0, q]\}$, where, if $s_\theta \in S$, then we call s_θ the original term, otherwise it is called as a virtual term.

Definition 2.4. [6] Let $Q = \{s_0, s_1, s_2, ..., s_t\}$ be a linguistic term set (LTS) with odd cardinality $t+1$ and $\overline{Q} = \{s_h / s_0 \leq s_h \leq s_t, h \in [0, t]\}$. Then, a linguistic single valued neutrosophic set A is defined by,

$A = \{\langle x, s_\theta(x), s_\psi(x), s_\sigma(x)\rangle | x \in S\}$, where $s_\theta(x), s_\psi(x), s_\sigma(x) \in \overline{Q}$ represent the linguistic truth, linguistic indeterminacy and linguistic falsity degrees of S to A, respectively, with condition $0 \leq \theta+\psi+\sigma \leq 3t$. This triplet $(s_\theta, s_\psi, s_\sigma)$ is called a linguistic single valued neutrosophic number.

Definition 2.5. [6] Let $\alpha = (s_\theta, s_\psi, s_\sigma), \alpha_1 = (s_{\theta 1}, s_{\psi_1}, s_{\sigma 1}), \alpha_2 = (s_{\theta 2}, s_{\psi_2}, s_{\sigma 2})$ be three LSVNNs, then

(1) $\alpha^c = (s_\sigma, s_\psi, s_\theta)$;
(2) $\alpha_1 \cup \alpha_2 = (max(\theta_1, \theta_2), min(\psi_1, \psi_2), min(\sigma_1, \sigma_2))$;
(3) $\alpha_1 \cap \alpha_2 = (min(\theta_1, \theta_2), max(\psi_1, \psi_2), max(\sigma_1, \sigma_2))$;
(4) $\alpha_1 = \alpha_2$ iff $\theta_1 = \theta_2, \psi_1 = \psi_2, \sigma_1 = \sigma_2$;

Definition 2.6. Let $\alpha = (l_\theta, l_\psi, l_\sigma)$ be a LSVNN. The set of all labels is, $L = \{l_0, l_1, l_2, ..., l_t\}$. Then the unit linguistic neutrosophic set (1_{LN}) is defined as $1_{LN} = (l_t, l_0, l_0)$, which is the truth membership,and the zero linguistic neutrosophic set (0_{LN}) is defined as $0_{LN} = (l_0, l_t, l_t)$, which is the falsehood membership.

Example 2.7. For the linguistic neutrosophic set, $L = \{$very bad, bad, fair, very fair, good, very good$\}$, the set of all labels be, $L = \{l_0, l_1, l_2, l_3, l_4, l_5\}$.
Then the unit LNs is defined as $1_{LN} = (l_5, l_0, l_0)$, and the zero LNs is defined as $0_{LN} = (l_0, l_5, l_5)$.

3. Linguistic Neutrosophic Topology

In this chapter, we introduce the concepts of linguistic neutrosophic topological spaces.

Definition 3.1. For a linguistic neutrosophic topology π, the collection of linguistic neutrosophic sets should obey,

(1) $0_{LN}, 1_{LN} \in \pi$
(2) $K_1 \bigcap K_2 \in \pi$ for any $K_1, K_2 \in \pi$
(3) $\bigcup K_i \in \pi, \forall \{K_i : i \in J\} \subseteq \pi$

We call, the pair (S_{LN}, π_{LN}), a linguistic neutrosophic topological space.

Remark 3.2. Let (S_{LN}, π_{LN}) be a linguistic neutrosophic topological space (LNTS). Then, $(S_{LN}, \pi_{LN})^c$ is the dual LN topology, whose elements are $K^c{}_{LN}$ for $K_{LN} \in (S_{LN}, \pi_{LN})$. Any open set in π_{LN} is known as linguistic neutrosophic open set(LNOsS). Any closed set in π_{LN} is known as linguistic neutrosophic closed set(LNCS) iff it's complement is linguistic neutrosophic open set.

Example 3.3. Let U_{LN} be the universe of discourse $U_{LN} = \{u, v, w, z\}$ and $S_{LN} = \{u, v\}$ and the linguistic term set be, $L = \{$ very poor, poor, very bad, bad, fair, good, very good$\}$
Then L can be taken as, $L = \{l_0, l_1, l_2, l_3, l_4, l_5, l_6\}$.
Let $K_{LN} = \{\langle u,(l_5, l_3, l_4)\rangle, \langle v,(l_4, l_2, l_3)\rangle\}$,
That is, the element u's degree of appurtenance to the set K_{LN} is good(l_5)
the element u's degree of indeterminate-appurtenance to the set K_{LN} is bad(l_3)
the element u's degree of non-appurtenance to the set K_{LN} is fair(l_4).
And the element v's degree of appurtenance to the set K_{LN} is fair(l_4)
the element v's degree of indeterminate-appurtenance to the set K_{LN} is very bad(l_2)
the element v's degree of non-appurtenance to the set K_{LN} is bad(l_3).
Let, $H_{LN} = \{\langle u,(l_6, l_2, l_2)\rangle, \langle v,(l_6, l_1, l_0)\rangle\}$
That is, the element u's degree of appurtenance to the set H_{LN} is very good(l_6)
the element u's degree of indeterminate-appurtenance to the set H_{LN} is very bad(l_2)
the element u's degree of non-appurtenance to the set H_{LN} is very bad(l_2).
And the element v's degree of appurtenance to the set H_{LN} is very good(l_6)
the element v's degree of indeterminate-appurtenance to the set H_{LN} is poor(l_1)
the element v's degree of non-appurtenance to the set H_{LN} is very poor(l_0).
Similarly, let $M_{LN} = \{\langle u,(l_6, l_3, l_2)\rangle, \langle v,(l_6, l_2, l_0)\rangle\}$
That is, the element u's degree of appurtenance to the set M_{LN} is very good(l_6)
the element u's degree of indeterminate-appurtenance to the set M_{LN} is bad(l_3)
the element u's degree of non-appurtenance to the set M_{LN} is very bad(l_2).
And the element v's degree of appurtenance to the set M_{LN} is very good(l_6)
the element v's degree of indeterminate-appurtenance to the set M_{LN} is very bad(l_2)
the element v's degree of non-appurtenance to the set M_{LN} is very poor(l_0).

Then the collection $\pi_{LN} = \{0_{LN}, K_{LN}, H_{LN}, M_{LN}, K_{LN} \vee H_{LN}, 1_{LN}\}$ forms a LN topology on (S_{LN}, π_{LN}).

Definition 3.4. The linguistic neutrosophic closure and linguistic neutrosophic interior are given by,

(i) $LNint(K_{LN}) = \bigcup\{O_{LN}/O_{LN}$ is a $LNOSinS_{LN}$ where $O_{LN} \subseteq K_{LN}\}$ and it is the largest LN open subset of K_{LN}.

(ii) $LNcl(H_{LN}) = \bigcap\{J_{LN}/J_{LN}$ is a $LNCSinS_{LN}$ where $H_{LN} \subseteq J_{LN}\}$ and it is the smallest LN closed set containing H_{LN}.

Example 3.5. In example 3.3, $LNint(K_{LN}) = N_{LN}$ and $LNcl(K_{LN}) = 1_{LN}$

Theorem 3.6. *Let (S_{LN}, π_{LN}) be a LNTS and $K_{LN}, H_{LN} \in S_{LN}$. Then*

(i) $K_{LN} \in LNcl(K_{LN})$

(ii) K_{LN} *is LN closed if and only if* $K_{LN} = LNcl(K_{LN})$

(iii) $LNcl(\phi_{LN}) = \phi_{LN}$ *and* $LNcl(S_{LN}) = S_{LN}$.

(iv) $K_{LN} \subseteq H_{LN} \Rightarrow LNcl(K_{LN}) \subseteq LNcl(H_{LN})$

(v) $LNcl(K_{LN} \cup H_{LN}) = LNcl(K_{LN}) \cup LNcl(H_{LN})$

(vi) $LNcl(K_{LN} \cap H_{LN}) \subseteq LNcl(K_{LN}) \cap LNcl(H_{LN})$

(vii) $LNcl(LNcl(K_{LN})) = LNcl(K_{LN})$

Proof:

(i) From the definition, $K_{LN} \in LNcl(K_{LN})$

(ii) If K_{LN} is LN closed, then K_{LN} is the smallest LN closed set containing K_{LN}. So, $K_{LN} = LNcl(K_{LN})$.

Conversely, if $K_{LN} = LNcl(K_{LN})$, then K_{LN} is the smallest LN closed set containing K_{LN} and hence K_{LN} is LN closed.

(iii) If K_{LN} is LN closed, then $K_{LN} = LNcl(K_{LN})$. As ϕ_{LN} and S_{LN} are LN closed, $LNcl(\phi_{LN}) = \phi_{LN}$ and $LNcl(S_{LN}) = S_{LN}$.

(iv) When $K_{LN} \subseteq H_{LN}$, since $H_{LN} \subseteq LNcl(H_{LN})$ and $K_{LN} \subseteq LNcl(H_{LN})$. That is, $LNcl(H_{LN})$ is a LN closed set that contains K. But $LNcl(K_{LN})$ is the smallest LN closed set contains K. Thus, $LNcl(K_{LN}) \subseteq LNcl(H_{LN})$

(v) As $K_{LN} \subseteq K_{LN} \cap H_{LN}$ and $H_{LN} \subseteq K_{LN} \cap H_{LN}$, $LNcl(K_{LN}) \subseteq LNcl(K_{LN} \cap H_{LN})$ and $LNcl(H_{LN}) \subseteq LNcl(K_{LN} \cap H_{LN})$. Thus, $LNcl(K_{LN}) \cap LNcl(H_{LN}) \subseteq LNcl(K_{LN} \cap H_{LN})$. Since, $K_{LN} \cup H_{LN} \subseteq LNcl(K_{LN}) \cap LNcl(H_{LN})$, and since $LNcl(K_{LN} \cup H_{LN})$ is the smallest LN closed set containing $K_{LN} \cup H_{LN}, LNcl(K_{LN} \cup H_{LN}) \subseteq LNcl(K_{LN}) \cup LNcl(H_{LN})$.

Thus, $LNcl(K_{LN} \cup H_{LN}) = LNcl(K_{LN}) \cup LNcl(H_{LN})$.

(vi) Since $(K_{LN} \cap H_{LN}) \subseteq K_{LN}$ and $(K_{LN} \cap H_{LN}) \subseteq H_{LN}, LNcl(K_{LN} \cap H_{LN}) \subseteq LNcl(H_{LN}) \subseteq LNcl(H_{LN})$.

(vii) AS $LNcl(K_{LN})$ is a LN closed set, $LNcl(LNcl(K_{LN})) = LNcl(K_{LN})$.

Remark 3.7. If $LNint(K_{LN})$ is $LNcl(K_{LN})$ is a LNCS, then we have,

(i) $LNint(K_{LN}) = K_{LN}$ if and only if K_{LN} is LNOS in (S_{LN}, π_{LN}).

(ii) $LNcl(K_{LN}) = K_{LN}$ if and only if K_{LN} is LNCS in (S_{LN}, π_{LN}).

Theorem 3.8. *Let* (S_{LN}, π_{LN}) *be a LNTS and* $K_{LN} \in S_{LN}$. *Then*

(i) $S - LNint(K_{LN}) = LNint(S_{LN} - K_{LN})$

(ii) $S - LNcl(K_{LN}) = LNcl(S_{LN} - K_{LN})$

Proof: (i): Let $S \in S_{LN} - LNint(K_{LN}) \Rightarrow S \notin LNint(K_{LN})$. Thus, $G \nsubseteq K_{LN} \forall$ LN open set G containing S, (i.e) $C_{LN} \cap (S - K_{LN}) = \neq \phi_{LN}, \forall$ LN open set G. Hence, $S \in LNcl(S_{LN} - K_{LN})$ and $S_{LN} - LNint(K_{LN}) \subseteq LNcl(S_{LN} - K_{LN})$.
Conversely, if $S \in LNcl(S_{LN} - K_{LN})$, then $G_{LN} \cap (S_{LN} - K_{LN}) \neq \phi_{LN}$ for every LN open set containing S, (i.e) $G \nsubseteq A \forall$ LN open set G containing S. That is, $S \notin LNint(A) \Rightarrow S \in S - LNint(A)$. Then, $LNcl(S_{LN} - K_{LN}) \subseteq S_{LN} - LNint(K_{LN})$. Thus, $S_{LN} - LNint(K_{LN}) = LNint(S_{LN} - K_{LN})$
(ii) Proof is similar to (i).

Remark 3.9. On taking complements on both sides of $S_{LN} - LNint(K_{LN}) = LNint(S_{LN} - K_{LN})$ and $S_{LN} - LNcl(K_{LN}) = LNcl(S_{LN} - K_{LN})$, we have,
$LNint(K_{LN}) = S_{LN} - LNcl(S_{LN} - K_{LN})$ and $LNcl(K_{LN}) = S_{LN} - LNint(S_{LN} - K_{LN})$

Theorem 3.10. *Let (S_{LN}, π_{LN}) be a LNTS and $K_{LN}, H_{LN} \in S_{LN}$. Then*

(i) $LNint(K_{LN}) = K_{LN}$ *if and only if* K_{LN} *is LN open.*
(ii) $LNint(\phi_{LN}) = \phi_{LN}$ *and* $LNint(S_{LN}) = S_{LN}$.
(iii) $K_{LN} \subseteq H_{LN} \Rightarrow LNint(K_{LN}) \subseteq LNint(H_{LN})$
(iv) $LNint(K_{LN}) \cup LNint(H_{LN}) \subseteq LNint(K_{LN} \cup H_{LN})$
(iv) $LNint(K_{LN} \cap H_{LN}) = LNint(K_{LN}) \cap LNint(H_{LN})$
(vi) $LNint(LNint(K_{LN})) = LNint(K_{LN})$

Proof: (i): K_{LN} is LN open if and only if $S_{LN} - K_{LN}$ is LN closed, if and only if, $LNcl(S_{LN} - K_{LN}) = S_{LN} - K_{LN}$, if and only if, $S_{LN} - LNcl(K_{LN}) = K_{LN}$ iff $LNint(K_{LN}) = K_{LN}$ bT remark.
(ii): Since ϕ_{LN} and S_{LN} are LN open, $LNint(\phi_{LN}) = \phi_{LN}$ and $LNint(S_{LN}) = S_{LN}$
(iii): $K_{LN} \subseteq H_{LN} \Rightarrow S_{LN} - H_{LN} \subseteq S_{LN} - K_{LN}$. Thus, $LNcl(S_{LN} - H_{LN}) \subseteq LNcl(S_{LN} - K_{LN}), (i.e) S_{LN} - LNcl(S_{LN} - K_{LN}) \subseteq S_{LN} - LNcl(S_{LN} - H_{LN})$. Therefore, $LNint(K_{LN}) \subseteq LNint(H_{LN})$.

Definition 3.11. Let S_{LN} be a non-void set and $K_{LN} = \{\langle S, [T_{K_{LN}}, I_{K_{LN}}, F_{K_{LN}}]\rangle\}$ and $H_{LN} = \{\langle S, [T_{H_{LN}}, I_{H_{LN}}, F_{H_{LN}}]\rangle\}$ are LNSs in LNTS.

(I) $K_{LN} \cup H_{LN}$ can be defined as
  (a) $K_{LN} \cup H_{LN} = \{\langle S, [T_{K_{LN}} \wedge T_{H_{LN}}, I_{K_{LN}} \wedge I_{H_{LN}}, F_{K_{LN}} \vee F_{H_{LN}}]\rangle\}$
(II) $K_{LN} \cap H_{LN}$ can be defined as
  (a) $K_{LN} \cap H_{LN} = \{\langle S, [T_{K_{LN}} \wedge T_{H_{LN}}, I_{K_{LN}} \wedge I_{H_{LN}}, F_{K_{LN}} \vee F_{H_{LN}}]\rangle\}$

(III) The complement of $K_{LN} = \{\langle S, [T_{K_{LN}}, I_{K_{LN}}, F_{K_{LN}}]\rangle\}$ is defined as,

(a) ${K_{LN}}^c = \{\langle S, [1 - F_{K_{LN}}, I_{K_{LN}}, 1 - T_{K_{LN}}]\rangle\}$

(b) $({K_{LN}}^c)^c = K_{LN}$

(c) $(K_{LN} \cap H_{LN})^c = {K_{LN}}^c \cup {H_{LN}}^c$

(d) $(K_{LN} \cup H_{LN})^c = {K_{LN}}^c \cap {H_{LN}}^c$

Theorem 3.12. *Let (S_{LN}, π_{LN}) be a LNTS. $S \in LNcl(K_{LN})$ iff $U_{LN} \cap K_{LN} \neq \phi_{LN}$ for every LN open set U_{LN} containing S, where $K_{LN} \subseteq S_{LN}$.*

Proof:

If U_{LN} is a LN open set and if $S \in LNcl(K_{LN})$, then$S_{LN} - U_{LN}$ is LN closed. If $K_{LN} \cap U_{LN} = \phi_{LN}$, then $K_{LN} \subseteq S_{LN} - U_{LN}$.

That is, $S_{LN} - U_{LN}$ is LN closed set containing K_{LN}. Therefore, $LNcl(K_{LN}) \subseteq S_{LN} - U_{LN}$, which is a contradiction, since $S \in LNcl(K_{LN})$ but $S \notin S_{LN} - U_{LN}$. Hence, $K_{LN} \cap U_{LN} \neq \phi_{LN}$, for every LN open set U_{LN} containing S.

Conversely, if $K_{LN} \cap U_{LN} \neq \phi_{LN}$, for every LN open set U_{LN} containing S and if $S \notin LNcl(K_{LN}), S \in S_{LN} - LNcl(K_{LN})$ which is LN open.

Hence, $(S_{LN} - LNcl(K_{LN})) \cap K_{LN} \neq \phi_{LN}$. But $K_{LN} \subseteq LNcl(K_{LN})$ and hence $S_{LN} - LNcl(K_{LN}) \subseteq S_{LN} - K_{LN}$, that implies $(S_{LN} - LNcl(K_{LN})) \cap K_{LN} \subseteq (S_{LN} - K_{LN}) \cap K_{LN}$.

Thus, $(S_{LN} - K_{LN}) \cap K_{LN} \neq \phi_{LN}$, which is a contradiction. Hence, $S \in LNcl(K_{LN})$.

Definition 3.13. Let (S_{LN}, π_{LN}) be a LNTS and $\pi_{LN} = \{0, S_{LN}\}$. Then, π is called the LN indiscrete topology over S.

Definition 3.14. Let π be the collection of all LN sets that can be defined over S_{LN}. Then, (S_{LN}, π_{LN}) is the called the LN discrete topology over S_{LN}.

Theorem 3.15. *Let $(S_{LN}, {\pi^1}_{LN})$ and $(S_{LN}, {\pi^2}_{LN})$ be two LNTSs, then $(S_{LN}, \pi^1 \cap {\pi^2}_{LN})$ is a LNTS on S_{LN}.*

Proof:

(1) clearly, 0_{LN} and $1_{LN} \in {\pi^1}_{LN} \cap {\pi^2}_{LN}$

(2) Let $F_i \in {\pi^1}_{LN} \cap {\pi^2}_{LN}$. Then, $F_i \in {\pi^1}_{LN}$ and $F_i \in {\pi^2}_{LN} \forall i \in I$.
Therefore, $\cup_{i \in I} F_i \in {\pi^1}_{LN}$ and $\cup_{i \in I} F_i \in {\pi^2}_{LN}$. Thus, $\bigcup_{i \in I} F_i \in {\pi^1}_{LN} \cap {\pi^2}_{LN}$.

(3) Let K_{LN} and $H_{LN} \in {\pi^1}_{LN} \cap {\pi^2}_{LN}$, which implies, $K_{LN}, H_{LN} \in {\pi^1}_{LN}$ and $K_{LN}, H_{LN} \in {\pi^2}_{LN}$. Since, $K_{LN} \cap H_{LN} \in {\pi^1}_{LN}$ and $K_{LN} \cap H_{LN} \in {\pi^2}_{LN}$, $K_{LN} \cap H_{LN} \in {\pi^1}_{LN} \cap {\pi^2}_{LN}$

Thus, $(S_{LN}, {\pi^1}_{LN} \bigcap {\pi^2}_{LN})$ is a LNTS on S_{LN}..

Remark 3.16. Union of two LNTSs may not be a LN topology over S_{LN}..

Example 3.17. Let the universe of discourse be $U = \{a, b, c\}$ and $S = \{a\}$. The set of all linguistic term is, $L = \{$ very salt(l_0), salt(l_1), very sour(l_2), sour(l_3), bitter(l_4), sweet(l_5), very sweet$(l_6)\}$.

And $\pi^1{}_{LN} = \{0_{LN}, 1_{LN}, K_{LN}\}$ where $K_{LN} = \{\langle a,(l_6, l_3, l_3)\rangle\}$, where the element a's degree of appurtenance to the set K_{LN} is very sweet(l_6), the element a's degree of indeterminate-appurtenance to the set K_{LN} is sour(l_3), the element a's degree of non-appurtenance to the set K_{LN} is bitter(l_4).

Let $\pi^2{}_{LN} = \{0_{LN}, 1_{LN}, H_{LN}\}$ where $H_{LN} = \{\langle a,(l_4, l_5, l_2)\rangle\}$, where the element a's degree of appurtenance to the set H_{LN} is bitter(l_4), the element a's degree of indeterminate-appurtenance to the set H_{LN} is sweet(l_5), the element a's degree of non-appurtenance to the set H_{LN} is very sour(l_2).

Let $\pi^1{}_{LN}$ and $\pi^2{}_{LN}$ be two LN topologies on S_{LN}.

Then, $\pi^1{}_{LN} \cup \pi^2{}_{LN} = \{0_{LN}, 1_{LN}, K_{LN}, H_{LN}\} = \{0_{LN}, 1_{LN}, \{\langle a,(l_6, l_3, l_3)\rangle\}, \{\langle a,(l_6, l_5, l_2)\rangle\}\}$.

Now, $K_{LN} \cup H_{LN} = \{\langle a,(l_6, l_5, l_2)\rangle\} \notin \pi^1{}_{LN} \cup \pi^2{}_{LN}$.

$K_{LN} \cap H_{LN} = \{\langle a,(l_4, l_3, l_3)\rangle\} \notin \pi^1{}_{LN} \cup \pi^2{}_{LN}$.

Therefore, union of any two linguistic neutrosophic topologies need not be a linguistic neutrosophic topology.

Definition 3.18. Let (S_{LN}, π_{LN}) be a LNTS and U_{LN} be a LN set over S_{LN}.. Then any point S is a LN interior point of U_{LN}, if there exists a LN open set V_{LN} such that $S \in U_{LN} \subseteq V_{LN}$.

Definition 3.19. Let (S_{LN}, π_{LN}) be a LNTS and U_{LN} be a LN set over S_{LN}.. Then, V_{LN} is called a LN neighborhood if there exists a LN open set V_{LN} such that $S \in U_{LN} \subseteq V_{LN}$.

Theorem 3.20. *Let (S_{LN}, π_{LN}) be a LNTS, then*

(1) *each $s \in S$ has a neighborhood.*

(2) *If U_{LN} and V_{LN} are LN neighborhoods of some $x \in S_{LN}$, then $U_{LN} \cap V_{LN}$ is also a LN neighborhood of s.*

(3) *If U_{LN} is a LN neighborhood of S and $U_{LN} \cap V_{LN}$, then V_{LN} is also a LN neighborhood of $s \in S_{LN}$.*

Proof:

(1) : (2): Let U_{LN} and V_{LN} are LN neighborhoods of some $s \in S$, then there exists $U^1{}_{LN}$ and $V^1{}_{LN} \in \tau$ such that $S \in U^1{}_{LN} \subseteq U_{LN}$ and

$S \in V^1{}_{LN} \subseteq V_{LN}$

Now, $S \in U_{LN}$ and $S \in V_{LN}$ implies that $S \in U^1{}_{LN} \cap V^1{}_{LN}$ and $U^1{}_{LN} \cap V^1{}_{LN} \in \tau$. So w have $S \in U_{1LN} \cap V_{1LN} \subseteq U_{LN} \cap V_{LN}$.

Thus, $U_{LN} \bigcap V_{LN}$ is a LN neighborhood of s.

(3): Let U_{LN} is a LN neighborhood of s and $U_{LN} \bigcap V_{LN}$. By definition, there exists a LN open

set ${U^1}_{LN}$ such that $s \in {U^1}_{LN} \subseteq U_{LN} \subseteq V_{LN}$.
Then, $s \in U_{LN} \subseteq V_{LN}$.
Therefore, V_{LN} is also a LN neighborhood of $s \in S$.

Theorem 3.21. *Let (S_{LN}, π_{LN}) be a LNTS. For any LN open set K_{LN} over S, K_{LN} is a LN neighborhood of each point of $\cap_{i \in I} A_i$.*

Proof:

Let $K_{LN} \in \pi_{LN}$. For any $S \in \cap_{i \in I} K_{LNi}$, we have $S \in A_i \forall i \in I$. Thus, $S \in K_{LN}$ and hence K_{LN} is a LN neighborhood of S.

4. **Linguistic neutrosophic derived sets**

Definition 4.1. Let (S_{LN}, π_{LN}) be a LNTS and $K_{LN} \subseteq S_{LN}$. Let $s \in S_{LN}$. s is called as a LN limit point of K_{LN} if $E_{LN} \cap (K_{LN} - \{s\}) \neq \phi$ for every LN open set E_{LN} containing s. The collection of all LN limit points of K_{LN} is the LN derived set $(LND(K_{LN})) of K_{LN}$.

Theorem 4.2. *$LNcl(K_{LN}) = K_{LN} \cup LND(K_{LN})$ where $K_{LN} \subseteq S_{LN}$*

Proof:

If $s \in K_{LN} \cup LND(K_{LN})$, then $s \in K_{LN}$ or $S \in LND(K_{LN})$. If $s \in K_{LN}$, then $s \in LNcl(K_{LN})$. Therefore, let $s \notin K_{LN}$. That is, $s \in LND(K_{LN})$. Then, $\forall$ LN open set E_{LN} containing s, $E_{LN} \cap (K_{LN} - s) \neq \phi$. Since $s \notin K_{LN}, E_{LN} \cap K_{LN} \notin \phi$. Thus, $s \in LNcl(K_{LN})$. Hence, $K_{LN} \cup LND(K_{LN}) \subseteq LNcl(K_{LN})$. If $s \in LNcl(K_{LN})$ and $s \in K_{LN}$, then $s \in K_{LN} \cup LND(K_{LN})$. If $s \in LNcl(K_{LN})$ but $s \notin K_{LN}$, then $E_{LN} \cap K_{LN} \notin \phi$ for every LN open set E_{LN} containing s and hence $E_{LN} \cap (K_{LN} - s) \notin \phi$. Therefore, $s \in LND(K_{LN})$, (i.e) $s \in K_{LN} \cup LND(K_{LN})$. Thus, $LNcl(K_{LN}) \subseteq K_{LN} \cup LND(K_{LN})$. Therefore, $LNcl(K_{LN}) = K_{LN} \cup LND(K_{LN})$.

Theorem 4.3. *If the derived set of K_{LN} is a subset of K_{LN}, then K_{LN} is LN closed.*

Proof:

K_{LN} is LN closed if and only if $LNcl(K_{LN}) = K_{LN}$, iff $K_{LN} \cup LND(K_{LN}) = K_{LN}$, iff $LND(K_{LN}) \subseteq K_{LN}$.

Theorem 4.4. *If K_{LN} is a singleton subset of S_{LN}, then $LND(K_{LN}) = LNcl(K_{LN}) - K_{LN}$.*

Proof:

If $s \in LND(K_{LN})$, then for every LN open set E_{LN} containing , $E_{LN} \cap (K_{LN} - s) \neq \phi$. Then $s \notin K_{LN}$. Suppose if $s \in K_{LN}$, then $K_{LN} = \{s\}$, and $E_{LN} \cap (K_{LN} - s) = \phi$. It is true that, $LND(K_{LN}) \subseteq LNcl(K_{LN})$. Then, $s \in LNcl(K_{LN})$ but $s \notin K_{LN}$, when $s \in LND(K_{LN})$. Thus, $LND(K_{LN}) \subseteq LNcl(K_{LN}) - K_{LN}$. Thus, $s \in LNcl(K_{LN}) - K_{LN}, s \in LNcl(K_{LN})$

but $s \notin K_{LN}$. Thus, $E_{LN} \cap K_{LN} \neq \phi$ for every LN open set E_{LN} containing s, (i.e) $E_{LN} \cap (K_{LN} - s) \neq \phi$ for every LN open set E_{LN} containing s. Thus, $s \in LND(K_{LN})$. Thus, $LNcl(K_{LN}) - K_{LN} \subseteq LND(K_{LN})$. Hence, $LND(K_{LN}) = LNcl(K_{LN}) - K_{LN}$, if K_{LN} is a singleton set.

Definition 4.5. (1) Linguistic Neutrosophic semi-closed set if $LNint(LNcl(K_{LN})) \subseteq K_{LN}$

(2) Linguistic Neutrosophic semi-open set if $K_{LN} \subseteq LNcl(LNint(K_{LN}))$

(3) Linguistic Neutrosophic semi-pre closed if $LNint(LNcl(LNint(K_{LN}))_{\subseteq} K_{LN}$

(4) Linguistic Neutrosophic semi-pre open if $K_{LN} \subseteq LNcl(LNint(LNcl(K_{LN})))$

(5) Linguistic Neutrosophic pre-closed if $LNcl(LNint(K_{LN}) \subseteq K_{LN}$ -doubt

(6) Linguistic Neutrosophic pre-open if $K_{LN} \subseteq LNint(LNcl(K_{LN}))$

(7) Linguistic Neutrosophic regular closed if $K_{LN} = LNint(LNcl(K_{LN}))$

(8) Linguistic Neutrosophic regular open if $K_{LN} = LNcl(LNint(K_{LN}))$

5. Linguistic Neutrosophic continuity

Definition 5.1. Define the image and pre-image of linguistic neutrosophic sets. Let S_{LN} and T_{LN} be two non-void sets and $f : S_{LN} \to T_{LN}$ be a function, then

(i) If $E_{LN} = \{\langle S, [T_{E_{LN}}(S), I_{E_{LN}}(S), F_{E_{LN}}(S)]\rangle\}$ is a LN set in T_{LN}, then the pre image of E_{LN} under f is denoted by, $f^{-1}(E_{LN})$ is defined by,
$f^{-1}(E_{LN}) = \{\langle S, [f^{-1}(T_{E_{LN}}(S)), f^{-1}(I_{E_{LN}}(S)), f^{-1}(F_{E_{LN}}(S))]\rangle\}$

(ii) If $F_{LN} = \{\langle S, [T_{F_{LN}}(S), I_{F_{LN}}(S), F_{_{LN}}F(S)]\rangle; S \in S_{LN}\}$ is a LN set in S_{LN}, then the image of F_{LN} under f is denoted bT,
$f(F_{LN}) = \{\langle T, [f(T_{F_{LN}}(T)), f(I_{F_{LN}}(T)), f(F_{F_{LN}}(T))]\rangle; T \in T_{LN}\}$

Definition 5.2. A function $f : S_{LN} \to T_{LN}$ is called a linguistic neutrosophic continuous function if the inverse image of every linguistic neutrosophic open set F_{LN} is linguistic neutrosophic open in S_{LN}.

Example 5.3. Let the universe of discourse be $U_{LN} = \{a, b, c, d, x, y, z, w\}$ and $S_1 = \{a, b, c\}$ and $S_2 = \{x, y, z\}$. The set of all linguistic term is, $L = \{$ very salt(l_0), salt(l_1), very sour(l_2), sour(l_3), bitter(l_4), sweet(l_5), very sweet(l_6)$\}$. Define linguistic neutrosophic sets K_{LN} and H_{LN} as $K_{LN} = \{s_1, (a, \langle l_0, l_6, l_0\rangle), (b, \langle l_4, l_0, l_2\rangle), (c, \langle l_2, l_3, l_1\rangle)\}$, where the element a's degree of appurtenance to the set K_{LN} is very sweet(l_0), the element a's degree of indeterminate-appurtenance to the set K_{LN} is very sweet(l_6), the element a's degree of non-appurtenance to the set K_{LN} is very salt(l_0).

Similarly, b's degree of appurtenance to the set K_{LN} is bitter(l_4), b's degree of indeterminate-appurtenance to the set K_{LN} is very salt(l_0), b's degree of non-appurtenance to the set K_{LN} is very sour(l_2).

And, c's degree of appurtenance to the set K_{LN} is very sour(l_2), c's degree of indeterminate-appurtenance to the set K_{LN} is sour(l_3), c's degree of non-appurtenance to the set K_{LN} is very salt(l_1).

Also, let $H_{LN} = \{s_2, (x, \langle l_6, l_0, l_0\rangle), (y, \langle l_0, l_4, l_2\rangle), (z, \langle l_3, l_2, l_1\rangle)\}$. Then, $\pi_{LN} = \{0_{LN}, 1_{LN}, K_{LN}\}$ and $\eta_{LN} = \{0_{LN}, 1_{LN}, H_{LN}\}$ are linguistic neutrosophic topologies on S_1, S_2 respectively. Let $g : (S_1, \pi_{LN}) \to (S_2, \eta_{LN})$ be defined by $g(a) = b, g(b) = a, g(c) = c$. Then, g is linguistic neutrosophic continuous function.

Theorem 5.4. *A function $f : S_{LN} \to T_{LN}$ is linguistic neutrosophic continuous if and only if the pre image of every linguistic neutrosophic closed set in T_{LN} is linguistic neutrosophic closed in S_{LN}.*

Proof:

Let f be a LN continuous function and E_{LN} be a LN closed set in T_{LN}, (i.e) $T_{LN} - E_{LN}$ is LN open in T_{LN}. $f^{-1}(T_{LN} - E_{LN})$ is a LN open set in S_{LN}, as f is LN continuous function. Thus, $S_{LN} - f^{-1}(E_{LN})$ is LN open set in S_{LN}. That is, $f^{-1}(E_{LN})$ is LN closed set in S_{LN}. Conversely, let the inverse image of each LN closed set be LN closed. Let F_{LN} be a LN open set in T_{LN}, (i.e) $T_{LN} - F_{LN}$ is LN closed. Then, $S_{LN} - f^{-1}(F_{LN})$ is LN closed set in S_{LN}, which implies, $f^{-1}(F_{LN})$ is LN open set in S_{LN}. Thus, f is LN continuous function on S_{LN}.

Theorem 5.5. *A function $f : S_{LN} \to T_{LN}$ is LN continuous if and only if $f(LNcl(K_{LN})) \subseteq LNcl(f(K_{LN}))$ for each subset K_{LN} of S_{LN}.*

Proof:

Let f be LN continuous function. If $K_{LN} \subseteq S_{LN}$, then $f(K_{LN}) \subseteq T_{LN}$. As f is LN continuous and $LNcl(f(K_{LN}))$ is LN closed in T_{LN}, $f^{-1}(LNcl(f(K_{LN})))$ is LN closed set in S_{LN}. Since, $f(K_{LN}) \subseteq LNcl(f(K_{LN})), K_{LN} \subseteq f^{-1}(LNcl(f(K_{LN})))$, which implies, $f^{-1}(LNcl(f(K_{LN})))$ is the smallest LN closed set that contains K_{LN}. But, $LNcl(K_{LN})$ is the smallest LN closed set that contains K_{LN}. Hence, $LNcl(K_{LN}) \subseteq f^{-1}(LNcl(f(K_{LN})))$, (i.e) $f(LNcl(K_{LN})) \subseteq LNcl(f(K_{LN}))$. Conversely, let $f(LNcl(H_{LN})) \subseteq LNcl(f(H_{LN}))$. If H_{LN} is LN closed in T_{LN}, $f(LNcl(f^{-1}(H_{LN}))) \subseteq LNcl(H_{LN})$. Thus, $LNcl(f^{-1}(H_{LN})) \subseteq f^{-1}((LNcl(H_{LN}))) = f^{-1}((H_{LN}))$. But, $f^{-1}((H_{LN})) \subseteq LNcl(f^{-1}((H_{LN})))$, that implies, $LNcl(f^{-1}((H_{LN}))) = f^{-1}((H_{LN})) \Rightarrow f^{-1}((H_{LN}))$ is LN closed set in S_{LN} for each LN closed set H_{LN} in T_{LN}. Therefore, f is LN continuous.

Example 5.6. In example(5.3), g is a linguistic neutrosophic continuous function. Let $K_{LN} = \{s_1, \langle a, (l_0, l_6, l_0)\rangle, \langle b, (l_4, l_0, l_2)\rangle, \langle c, (l_2, l_3, l_1)\rangle\} \subseteq (S_1, \pi_{LN})$. Then, $g(LNcl(K_{LN})) =$

$\{s_1, \langle a, (l_6, l_6, l_6)\rangle, \langle b, (l_0, l_4, l_2)\rangle, \langle c, (l_3, l_5, l_4)\rangle\}$.
But, $LNcl(g(K_{LN})) = LNcl(H_{LN}) = B^c \neq \{s_1, \langle a, (l_6, l_6, l_6)\rangle, \langle b, (l_0, l_4, l_2)\rangle, \langle c, (l_3, l_5, l_4)\rangle\}$, even though g is linguistic neutrosophic continuous function. Thus, equality is not necessarily holds when g is linguistic neutrosophic continuous function.

Theorem 5.7. *A function $f : S_{LN} \to T_{LN}$ is LN continuous if and only if $LNcl(f^{-1}(E_{LN})) \subseteq f^{-1}(LNcl(E_{LN}))$ for each subset E_{LN} of T_{LN}.*

Proof:

If f is LN continuous and $E_{LN} \subseteq T_{LN}$, then $LNcl(E_{LN})$ is LN closed in T_{LN} and hence $f^{-1}(LNcl(E_{LN}))$ is LN closed in S_{LN}. Thus, $LNcl(f^{-1}(LNcl(E_{LN}))) = f^{-1}(LNcl(E_{LN}))$. Since, $E_{LN} \subseteq LNcl(E_{LN}), f^{-1}(E_{LN}) \subseteq f^{-1}(LNcl(E_{LN}))$. Therefore, $LNcl(f^{-1}(E_{LN})) \subseteq LNcl(f^{-1}(LNcl(E_{LN}))) = f^{-1}(LNcl(E_{LN}))$, (i.e) $LNcl(f^{-1}(E_{LN})) \subseteq f^{-1}(LNcl(E_{LN}))$. Conversely, let $LNcl(f^{-1}(E_{LN}))f^{-1}(LNcl(E_{LN}))$ for all E_{LN} of T_{LN}. If E_{LN} is LN closed, then $LNcl(E_{LN}) = E_{LN}$. By assumption, $LNcl(f^{-1}(E_{LN})) \subseteq f^{-1}(LNcl(E_{LN}))$. Thus, $LNcl(f^{-1}(E_{LN})) \subseteq f^{-1}(E_{LN})$. But, $f^{-1}(E_{LN}) \subseteq LNcl(f^{-1}(E_{LN}))$. Thus, $LNcl(f^{-1}(E_{LN})) = f^{-1}(E_{LN})$, (i.e) $f^{-1}(E_{LN})$ is LN closed in S_{LN} for every LN closed set E_{LN} in T_{LN}. Hence, f is LN continuous.

Theorem 5.8. *A function $f : S_{LN} \to T_{LN}$ is LN continuous if and only if $f^{-1}(LNint(E_{LN})) \subseteq LNint(f^{-1}(E_{LN}))$ for each subset E_{LN} of T_{LN}.*

Proof:

Let f be LN continuous function and $E \subseteq T_{LN}$. Then, $f^{-1}(LNint(E_{LN}))$ is LN open in S_{LN}. That means, $f^{-1}(LNint(E_{LN})) = LNint(f^{-1}(LNint(E_{LN})))$. As $LNint(E_{LN}) \subseteq E_{LN}$, implies $f^{-1}(LNint(E_{LN})) \subseteq f^{-1}(E_{LN})$. Thus, $LNint(f^{-1}(LNint(E_{LN}))) \subseteq LNint(f^{-1}(E_{LN}))$. Therefore,$f^{-1}(LNint(E_{LN})) \subseteq LNint(f^{-1}(E_{LN}))$. Conversely, let $f^{-1}(LNint(E_{LN})) \subseteq LNint(f^{-1}(E_{LN}))$, for each subset E_{LN} of T_{LN}. If E_{LN} is LN open, then $f^{-1}(E_{LN}) \subseteq LNint(f^{-1}(E_{LN}))$. But, $LNint(f^{-1}(E_{LN})) \subseteq f^{-1}(E_{LN})$. Thus, $f^{-1}(E_{LN}) = LNint(f^{-1}(E_{LN}))$. Hence, f is LN continuous.

Example 5.9. In example(5.3), $H_{LN} = \{s_2, \langle x, (l_6, l_0, l_0)\rangle, \langle y, (l_0, l_4, l_2)\rangle, \langle z, (l_3, l_2, l_1)\rangle\}$. Then, $g^{-1}(LNcl(H_{LN})) = g^{-1}({H_{LN}}^c) = \{s_2, \langle z, (l_0, l_6, l_0)\rangle, \langle y, (l_4, l_4, l_6)\rangle, \langle z, (l_2, l_5, l_3)\rangle\}$. And $LNcl(g^{-1}(H_{LN})) = {K_{LN}}^c$. Thus, $g^{-1}(LNcl(E_{LN})) \subseteq LNcl(g^{-1}(E_{LN}))$. Similarly, $g^{-1}(LNint(E_{LN})) \subseteq LNint(g^{-1}(E_{LN}))$. Even if g is LN continuous, equality does not hold in theorems (5.7) and (5.8).

Definition 5.10. Any subset of a LN topological space (S_{LN}, π_{LN}) is a LN dense set if $LNcl(K_{LN}) = S_{LN})$.

Theorem 5.11. *Let $f : S_{LN} \to T_{LN}$ be an onto function and linguistic neutrosophic continuous function. If U_{LN} is LN dense in S_{LN}, then $f(U_{LN})$ is LN dense in T_{LN}.*

Proof:

As U_{LN} is LN dense in S_{LN}, $f(LNcl(U_{LN})) = f(S_{LN}) = T_{LN}$, since f is onto. Also, $f(LNcl(U_{LN})) \subseteq LNcl(U_{LN})$, as f is LN continuous. Thus, $T_{LN} = LNcl(f(U_{LN}))$. But $LNcl(f(U_{LN})) \subseteq T_{LN}$. Hence,$LNcl(f(U_{LN})) = T_{LN}$, which implies, $f(U_{LN})$ is LN dense set.

Conclusion

We have introduced a new type of topology called linguistic neutrosophic topology and it was established with apt examples. Moreover, the basic properties of linguistic neutrosophic were discussed. In addition to this, the ideas of linguistic neutrosophic continuity and linguistic neutrosophic neighborhood were introduced and established. Linguistic neutrosophic derived sets and linguistic neutrosophic dense sets were talked through.

References

[1] Atanassov, K.T. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 1986; 20, pp. 87–96.

[2] Chang, C.L. Fuzzy topological spaces, J Math.Anal.Appl. 1968; 24, pp. 182–190.

[3] Chen, Z.C.; Liu, P.H.; Pei, Z. An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. Int. J. Comput. Intell. Syst. 2015, 8, 747–760.

[4] Coker, D. An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems, 1997; 88, pp. 81-89.

[5] Fan, Feng, and Hu. (2019). Linguistic Neutrosophic Numbers Einstein Operator and Its Application in Decision Making. Mathematics, 7(5), 389. doi:10.3390/math7050389

[6] Fang, Zebo and Te, Jun. Multiple Attribute Group Decision-Making Method Based on Linguistic Neutrosophic Numbers.Symmetry,9(7), 2017. 111; https://doi.org/10.3390/sTm9070111.

[7] Garg H and Nancy. Linguistic single-valued neutrosophic power aggregation operators and their applications to group decision-making problems. IEEE/CAA J. Autom. Sinica, vol. 7, no. 2, pp. 546–558, Mar. 2020.

[8] Herrera, F.; Herrera-Viedma, E.; Verdegay, L. A model of consensus in group decision making under linguistic assessments. Fuzzy Sets Syst. 1996, 79, 73–87.

[9] Herrera F.; Herrera-Viedma E. Linguistic decision analysis . Steps for solving decision problems under linguistic information. Fuzzy Sets and Systems, 2000, 115(1): 67–82.

[10] Munkres, James R. Topology: a First Course. Englewood Cliffs, N.J.: Prentice-Hall, 1974.

[11] Smarandache F. A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic. American Research Press, Rehoboth, 1999.

[12] Su Z S. Deviation measures of linguistic preference relations in group decision making. Omega, 2005, 33(3):249–254.

[13] Wang H, Smarandache F, Zhang T Q, Sunderraman R. Interval neutrosophic sets and logic: Theory and applications in computing. Hexis, Phoenis, AZ, 2005.

[14] Wei, G., Wu, J., Guo, Y., Wang, J., and Wei, C. (2021). An extended COPRAS model for multiple attribute group decision making based on single-valued neutrosophic 2-tuple linguistic environment. Technological and Economic Development of Economy, 27(2), 353-368. https://doi.org/10.3846/tede.2021.14057.

[15] Ye, J. An extended TOPSIS method for multiple attribute group decision making based on single valued neutrosophic linguistic numbers. J. Intell. Fuzzy Syst. 2015, 28, 247–255.

[16] Zadeh, L.A. Fuzzy Sets. Information and Control, 1965; 8,pp. 338–353.

[17] Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning Part I. Inf. Sci. 1975, 8, 199–249.

Received: May 29, 2021. Accepted: October 1, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

Reverse Subsystems of Interval Neutrosophic Automata

V. Karthikeyan [1*], R. Karuppaiya[2]

[1]Department of Mathematics, Government College of Engineering, Dharmapuri
Tamil Nadu, India ; vkarthikau@gmail.com

[2]Department of Mathematics, Annamalai University, Chidambaram; rajaanju.40@gmail.com

[*]Correspondence: vkarthikau@gmail.com; Tel.: (+91-9489447287)

Abstract. We introduce the concept of reverse subsystem and prove that the necessary and sufficient condition for R_{N_Q} to be a reverse subsystem. Also, we prove that the union and intersection of reverse subsystems of interval neutrosophic automata is reverse subsystem.

Keywords: Interval neutrosophic set, Interval neutrosophic automaton, Subsystem, Reverse Subsystem.

1. Introduction

Fuzzy set theory which is a generalization of conventional set theory was proposed by Lofti A. Zadeh in 1965 with his seminal paper 'Fuzzy Sets' . Fuzzy set provides a simple mathematical tool to represent vagueness, uncertainty and imprecision inherently present in day to day life.

Fuzzy Logic provides a simple way to arrive at a definite conclusion based on vague, ambiguous, imprecise, noisy or missing input information. Since 1965, fuzzy set theory has witnessed enormous development by several researchers. Fuzzy logic based applications range from consumer products and industrial systems to biomedicine, decision analysis, information sciences and control engineering.

Fuzzy automata was introduced by W. G. Wee [17]. Subsequently, number of works have been contributed by many authors for development of generalizations of finite automata. General fuzzy automata was introduced by Doostfatemeh in [3]. It deals the problem of assigning membership values to the active states.

The neutrosophic set is the generalization of classical sets, fuzzy set [18], intuitionstic fuzzy set [1], interval valued intuitionistic fuzzy sets [2], vague set [4] and so on. Florentin Smarandache in 1998 [14] introduced the concept of neutrosophy and neutrosophic set. Single valued and interval valued neutrosophic sets were introuced by Wang $etal.$ in [15,16]. Recently, neutrosophic sets and systems have important applications in various fields especially in multicriteria decision making problems.

Tahir Mahmood et. al in [11, 12] were introduced single valued and interval neutrosophic finite automata. Consequently, J. Kavikumar et.al were introduced neutrosophic general finite automata and composite neutrosophic finite automata [9, 10].

Subsystems of finite fuzzy state machines was discussed in [13]. Later, Retrievability, subsystems, strong subsystems, and characterizations of submachines of Interval neutrosophic automata were discussed by V. Karthikeyan in [5–8]. In this paper, we introduce reverse subsystem (R.S) of interval neutrosophic automata and discuss their properties. We prove that the necessary and sufficient condition for R_{N_Q} to be a reverse subsystem, union and intersection of reverse subsystems of interval neutrosophic automata is reverse subsystems.

2. Preliminaries

Definition 2.1. [14] Let U be the universe of discourse. A neutrosophic set (NS) N in U is defined by a truth membership T_N, indeterminacy membership I_N and a falsity membership F_N, where T_N, I_N, and F_N are real standard or non-standard subsets of $]0^-, 1^+[$. That is
$N = \{\langle x, (T_N(x), I_N(x), F_N(x))\rangle, x \in U,\ T_N, I_N, F_N \in\ ]0^-, 1^+[\ \}$ and
$0^- \leq sup\ T_N(x) + sup\ I_N(x) + sup\ F_N(x) \leq 3^+$. We use the interval $[0, 1]$ instead of $]0^-, 1^+[$.

Definition 2.2. [16] Interval neutrosophic set (INS for short) is of the form $N = \{\langle \alpha_N(x), \beta_N(x), \gamma_N(x)\rangle \,|x \in U\}$
$= \{\langle x, [\inf\ \alpha_N(x), \sup\ \alpha_N(x)], [\inf\ \beta_N(x), \sup\ \beta_N(x)], [\inf\ \gamma_N(x), \sup\gamma_N(x)]\rangle\}$,
$x \in U$, $\alpha_N(x),\ \beta_N(x),\ \gamma_N(x) \subseteq [0, 1]$ and
$0 \leq \sup\ \alpha_N(x) +\ \sup\ \beta_N(x) +\ \sup\ \gamma_N(x) \leq\ 3$.

Definition 2.3. [16] An INS N is empty if $\inf\ \alpha_N(x) = \sup\ \alpha_N(x) = 0,\ \inf\ \beta_N(x) = \sup\ \beta_N(x) = 1,\ \inf\ \gamma_N(x) = \sup\ \gamma_N(x) = 1$ forall $x\ \in\ U$.

Definition 2.4. [11] Interval neutrosophic automaton $M = (Q,\ \Sigma,\ N)$ $(INAforshort)$, where Q and Σ are non-empty finite sets called the set of states and input symbols respectively, and $N = \{\langle \alpha_N(x), \beta_N(x), \gamma_N(x)\rangle\}$ is an INS in $Q \times \Sigma \times Q$.

The set of all words of finite length of Σ is denoted by Σ^*. The empty word is denoted by ϵ, and the length of each $x \in \Sigma^*$ is denoted by $|x|$.

Definition 2.5. [11] Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton and extended interval neutrosophic set is defined as $N^* = \{\langle \alpha_{N^*}(x), \beta_{N^*}(x), \gamma_{N^*}(x)\rangle\}$ in $Q \times \Sigma^* \times Q$ by

$$\alpha_{N^*}(q_i,\ \epsilon,\ q_j) = \begin{cases} [1,1] & \text{if } q_i = q_j \\ [0,0] & \text{if } q_i \neq q_j \end{cases}$$

$$\beta_{N^*}(q_i,\ \epsilon,\ q_j) = \begin{cases} [0,0] & \text{if } q_i = q_j \\ [1,1] & \text{if } q_i \neq q_j \end{cases}$$

$$\gamma_{N^*}(q_i,\ \epsilon,\ q_j) = \begin{cases} [0,0] & \text{if } q_i = q_j \\ [1,1] & \text{if } q_i \neq q_j \end{cases}$$

$\alpha_{N^*}(q_i, w, q_j) = \alpha_{N^*}(q_i, xy, q_j) = \vee_{q_r \in Q}[\alpha_{N^*}(q_i, x, q_r) \wedge \alpha_{N^*}(q_r, y, q_j)]$,
$\beta_{N^*}(q_i, w, q_j) = \beta_{N^*}(q_i, xy, q_j) = \wedge_{q_r \in Q}[\beta_{N^*}(q_i, x, q_r) \vee \beta_{N^*}(q_r, y, q_j)]$,
$\gamma_{N^*}(q_i, w, q_j) = \gamma_{N^*}(q_i, xy, q_j) = \wedge_{q_r \in Q}[\gamma_{N^*}(q_i, x, q_r) \vee \gamma_{N^*}(q_r, y, q_j)], \forall q_i, q_j \in Q$,
$w = xy, x \in \Sigma^*$ and $y \in \Sigma$.

3. Reverse Subsystems of Interval Neutrosophic Automata

Definition 3.1. Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton and R_{N_Q} be an interval neutrosophic set of Q. Let $q_i \in Q$, and R_{N_Q} is defined as $R_{N_Q} = \left\{\left\langle \alpha_{R_{N_Q}}(q_i),\ \beta_{R_{N_Q}}(q_i),\ \gamma_{R_{N_Q}}(q_i)\right\rangle\right\}$
$= \left\{\left\langle q_i, [\inf\alpha_{R_{N_Q}}(q_i), \sup\alpha_{R_{N_Q}}(q_i)], [\inf\beta_{R_{N_Q}}(q_i), \sup\beta_{R_{N_Q}}(q_i)], [\inf\gamma_{R_{N_Q}}(q_i), \sup\gamma_{R_{N_Q}}(q_i)]\right\rangle\right\}$.
Here, $\alpha_{R_{N_Q}}(q_i),\ \beta_{R_{N_Q}}(q_i), \gamma_{R_{N_Q}}(q_i) \subseteq [0,1]$.
Then (Q, R_{N_Q}, Σ, N) is said to be an reverse subsystem of M if $\forall q_i, q_j \in Q$ and $x \in \Sigma$ such that $\alpha_{R_{N_Q}}(q_j) \leq \vee_{q_i \in Q}\{\alpha_{R_{N_Q}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,
$\beta_{R_{N_Q}}(q_j) \geq \wedge_{q_i \in Q}\{\beta_{R_{N_Q}}(q_i) \vee \beta_N(q_i, x, q_j)\}$ and
$\gamma_{R_{N_Q}}(q_j) \geq \wedge_{q_i \in Q}\{\gamma_{R_{N_Q}}(q_i) \vee \gamma_N(q_i, x, q_j)\}$.
In this case, the reverse subsystem (Q, R_{N_Q}, Σ, N) of M is denoted by R_{N_Q}.

Example 3.2. Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton, where $Q = \{q_1, q_2, q_3, q_4, q_5\},\ \Sigma = \{x\}$, and $N,\ N_Q(q_i),\ i = 1, 2, 3, 4, 5$ are defined as below.

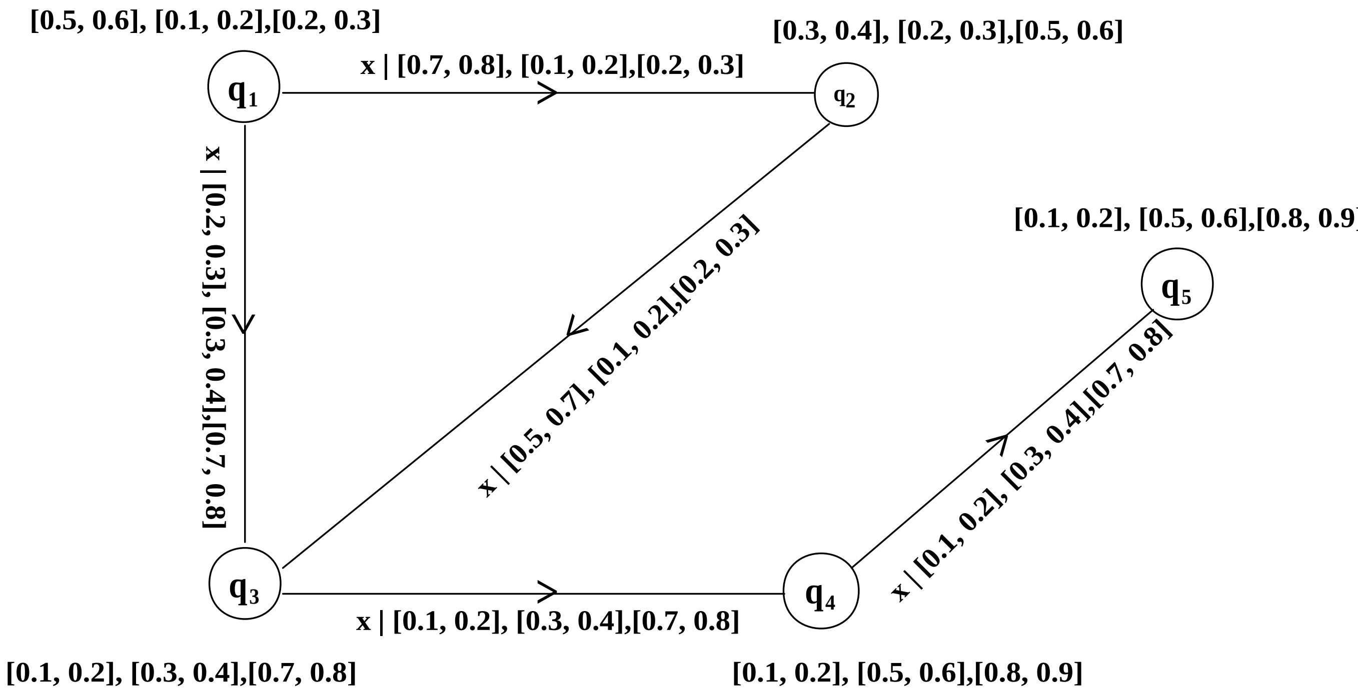


Fig- 3.1

In this case the above interval neutrosophic automaton M is said to be reverse subsystem.

Theorem 3.3. Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton and $R_{N_Q} = \left\{\left\langle \alpha_{R_{N_Q}}, \beta_{R_{N_Q}}, \gamma_{R_{N_Q}} \right\rangle\right\}$ be an interval neutrosophic subset in Q. Then R_{N_Q} is an reverse subsystem of M if and only if $\forall q_i, q_j \in Q, \forall x \in \Sigma^*$,

$\alpha_{R_{N_Q}}(q_j) \leq \vee_{q_i \in Q}\{\alpha_{R_{N_Q}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,

$\beta_{R_{N_Q}}(q_j) \geq \wedge_{q_i \in Q}\{\beta_{R_{N_Q}}(q_i) \vee \beta_N(q_i, x, q_j)\}$ and

$\gamma_{R_{N_Q}}(q_j) \leq \wedge_{q_i \in Q}\{\gamma_{R_{N_Q}}(q_i) \vee \gamma_N(q_i, x, q_j)\}$.

Proof. Suppose $R_{N_Q}$is an reverse subsystem of M. Let $q_i, q_j \in Q$ and $x \in \Sigma^*$. We prove this by induction on $|x| = n$. If $n = 0$, then $x = \epsilon$. Now if $q_i = q_j$, then

$\alpha_{R_{N_Q}}(q_j) \wedge \alpha_{N*}(q_i, \epsilon, q_j) = \alpha_{R_{N_Q}}(q_j)$, $\beta_{R_{N_Q}}(q_j) \vee \beta_{N*}(q_i, \epsilon, q_j) = \beta_{R_{N_Q}}(q_j)$, and

$\gamma_{R_{N_Q}}(q_j) \vee \gamma_{N*}(q_i, \epsilon, q_j) = \gamma_{R_{N_Q}}(q_j)$.

Now if $q_i \neq q_j$, then

$\alpha_{R_{N_Q}}(q_i) \wedge \alpha_{N^*}(q_i, \epsilon, q_j) \geq \alpha_{R_{N_Q}}(q_j)$, $\beta_{R_{N_Q}}(q_i) \vee \beta_{N^*}(q_i, \epsilon, q_j) \leq \beta_{R_{N_Q}}(q_j)$, and

$\gamma_{R_{N_Q}}(q_i) \vee \gamma_{N^*}(q_i, \epsilon, q_j) \leq \gamma_{R_{N_Q}}(q_j)$.

Therefore, the statement is true for $n = 0$.

Assume the statement is true for all $y \in \Sigma^*$ such that $|y| = n - 1, n > 0$.

Let $x = ya, |y| = n - 1, y \in \Sigma^*, a \in \Sigma$. Then

$\vee_{q_i \in Q}\{\alpha_{R_{N_Q}}(q_i) \wedge \alpha_N(q_i, x, q_j)\} = \vee_{q_i \in Q}\{\alpha_{R_{N_Q}}(q_i) \wedge \alpha_{N^*}(q_i, ya, q_j)\}$

$= \vee_{q_i \in Q}\{\alpha_{R_{N_Q}}(q_i) \wedge \{\vee_{q_k \in Q}\ \{\alpha_{N^*}(q_i, y, q_k) \wedge \alpha_N(q_k, a, q_j)\}\}\}$

$= \vee_{q_i \in Q}\{\vee_{q_k \in Q}\left\{\alpha_{R_{N_Q}}(q_i) \wedge \alpha_{N^*}(q_i, y, q_k) \wedge \alpha_N(q_k, a, q_j)\right\}\}$
$\geq \vee_{q_k \in Q}\left\{\alpha_{R_{N_Q}}(q_k) \wedge \alpha_N(q_k, a, q_j)\right\}$
$\geq \alpha_{R_{N_Q}}(q_j)$.
$\vee_{q_i \in Q}\{\alpha_{R_{N_Q}}(q_i) \wedge \alpha_N(q_i, x, q_j)\} \geq \alpha_{R_{N_Q}}(q_j)$.
Thus, $\alpha_{R_{N_Q}}(q_j) \leq \vee_{q_i \in Q}\{\alpha_{R_{N_Q}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}$
$\wedge_{q_i \in Q}\{\beta_{R_{N_Q}}(q_i) \vee \beta_N(q_i, x, q_j)\} = \wedge_{q_i \in Q}\{\beta_{R_{N_Q}}(q_i) \vee \beta_N(q_i, ya, q_j)\}$
$= \wedge_{q_i \in Q}\{\beta_{R_{N_Q}}(q_i) \vee \{\wedge_{q_k \in Q}\{\beta_{N^*}(q_i, y, q_k) \vee \beta_N(q_k, a, q_j)\}\}\}$
$= \wedge_{q_i \in Q}\{\wedge_{q_k \in Q}\left\{\beta_{R_{N_Q}}(q_i) \vee \beta_{N^*}(q_i, y, q_k) \vee \beta_N(q_k, a, q_j)\right\}\}$
$\leq \wedge_{q_k \in Q}\left\{\beta_{R_{N_Q}}(q_k) \vee \beta_N(q_k, a, q_j)\right\}$
$\leq \beta_{R_{N_Q}}(q_j)$.
$\wedge_{q_i \in Q}\{\beta_{R_{N_Q}}(q_i) \vee \beta_N(q_i, x, q_j)\} \leq \beta_{R_{N_Q}}(q_j)$
Thus,$\beta_{R_{N_Q}}(q_j) \geq \wedge_{q_i \in Q}\{\beta_{R_{N_Q}}(q_i) \vee \beta_N(q_i, x, q_j)\}$ and
$\wedge_{q_i \in Q}\{\gamma_{R_{N_Q}}(q_i) \vee \gamma_N(q_i, x, q_j)\} = \wedge_{q_i \in Q}\{\gamma_{R_{N_Q}}(q_i) \vee \gamma_N(q_i, ya, q_j)\}$
$= \wedge_{q_i \in Q}\{\gamma_{R_{N_Q}}(q_i) \vee \{\wedge_{q_k \in Q}\{\gamma_{N^*}(q_i, y, q_k) \vee \gamma_N(q_k, a, q_j)\}\}\}$
$= \wedge_{q_i \in Q}\{\wedge_{q_k \in Q}\left\{\gamma_{R_{N_Q}}(q_i) \vee \gamma_{N^*}(q_i, y, q_k) \vee \gamma_N(q_k, a, q_j)\right\}\}$
$\leq \wedge_{q_k \in Q}\left\{\gamma_{R_{N_Q}}(q_k) \vee \gamma_N(q_k, a, q_j)\right\}$
$\leq \gamma_{R_{N_Q}}(q_j)$.
$\wedge_{q_i \in Q}\{\gamma_{R_{N_Q}}(q_i) \vee \gamma_N(q_i, x, q_j)\} \leq \gamma_{R_{N_Q}}(q_j)$.
Thus, $\gamma_{R_{N_Q}}(q_j) \geq \wedge_{q_i \in Q}\{\gamma_{R_{N_Q}}(q_i) \vee \gamma_N(q_i, x, q_j)\}$.
The converse part is obvious.

Theorem 3.4. Let$M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton. Let $R_{N_{Q_1}}$, and $R_{N_{Q_2}}$ be reverse subsystems of M. Then $R_{N_{Q_1}} \vee R_{N_{Q_2}}$ is an reverse subsystem of M.

Proof. Since $R_{N_{Q_1}}$ and $R_{N_{Q_2}}$ are reverse subsystem of an interval neutrosophic automaton M. Then $\forall\ q_i, q_j\ \in\ Q$ and $x\ \in\ \Sigma$ such that
$\alpha_{R_{N_{Q_1}}}(q_j) \leq \vee_{q_i \in Q}\{\alpha_{R_{N_{Q_1}}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,
$\beta_{R_{N_{Q_1}}}(q_j) \geq \wedge_{q_i \in Q}\{\beta_{R_{N_{Q_1}}}(q_i) \vee \beta_N(q_i, x, q_j)\}$,
$\gamma_{R_{N_{Q_1}}}(q_j) \geq \wedge_{q_i \in Q}\{\gamma_{R_{N_{Q_1}}}(q_i) \vee \gamma_N(q_i, x, q_j)\}$ and
$\alpha_{R_{N_{Q_2}}}(q_j) \leq \vee_{q_i \in Q}\{\alpha_{R_{N_{Q_2}}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,
$\beta_{R_{N_{Q_2}}}(q_j) \leq \wedge_{q_i \in Q}\{\beta_{R_{N_{Q_2}}}(q_i) \vee \beta_N(q_i, x, q_j)\}$,
$\gamma_{R_{N_{Q_2}}}(q_j) \leq \wedge_{q_i \in Q}\{\gamma_{R_{N_{Q_2}}}(q_i) \vee \gamma_N(q_i, x, q_j)\}$.
Now to prove $R_{N_{Q_1}} \vee R_{N_{Q_2}}$ is reverse subsystem of interval neutrosophic automaton M, it is enough to prove that
$(\alpha_{R_{N_{Q_1}}} \vee \alpha_{R_{N_{Q_2}}})(q_j) \leq \vee_{q_i \in Q}\{(\alpha_{R_{N_{Q_1}}} \vee \alpha_{R_{N_{Q_2}}})(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,
$(\beta_{R_{N_{Q_1}}} \vee \beta_{R_{N_{Q_2}}})(q_j) \geq \wedge_{q_i \in Q}\{(\beta_{R_{N_{Q_1}}} \vee \beta_{R_{N_{Q_2}}})(q_i) \vee \beta_N(q_i, x, q_j)\}$, and
$(\gamma_{R_{N_{Q_1}}} \vee \gamma_{R_{N_{Q_2}}})(q_j) \geq \wedge_{q_i \in Q}\{(\gamma_{R_{N_{Q_1}}} \vee \gamma_{R_{N_{Q_2}}})(q_i) \vee \gamma_N(q_i, x, q_j)\}$.

Now, $(\alpha_{R_{N_{Q_1}}} \vee \alpha_{R_{N_{Q_2}}})(q_j) = (\alpha_{R_{N_{Q_1}}}(q_j) \vee \alpha_{R_{N_{Q_2}}}(q_j))$
$\leq \{\vee_{q_i \in Q}\{\alpha_{R_{N_{Q_1}}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}\} \vee \{\vee_{q_i \in Q}\{\alpha_{R_{N_{Q_2}}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}\}$
$= \{\vee_{q_i \in Q}\{\alpha_{R_{N_{Q_1}}}(q_i) \vee \alpha_{R_{N_{Q_2}}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}\}$
$= \vee_{q_i \in Q}\{(\alpha_{R_{N_{Q_1}}} \vee \alpha_{R_{N_{Q_2}}})(q_i) \wedge \alpha_N(q_i, x, q_j)\}$
Thus, $(\alpha_{R_{N_{Q_1}}} \vee \alpha_{R_{N_{Q_2}}})(q_j) \leq \vee_{q_i \in Q}\{(\alpha_{R_{N_{Q_1}}} \vee \alpha_{R_{N_{Q_2}}})(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,—(1)
$(\beta_{R_{N_{Q_1}}} \vee \beta_{R_{N_{Q_2}}})(q_j) = (\beta_{R_{N_{Q_1}}}(q_j) \vee \beta_{R_{N_{Q_2}}}(q_j))$
$\geq \{\wedge_{q_i \in Q}\{\beta_{R_{N_{Q_1}}}(q_i) \vee \beta_N(q_i, x, q_j)\}\} \vee \{\wedge_{q_i \in Q}\{\beta_{R_{N_{Q_2}}}(q_i) \vee \beta_N(q_i, x, q_j)\}\}$
$= \{\wedge_{q_i \in Q}\{\beta_{R_{N_{Q_1}}}(q_i) \vee \beta_{R_{N_{Q_2}}}(q_i) \vee \beta_N(q_i, x, q_j)\}\}$
$= \wedge_{q_i \in Q}\{(\beta_{R_{N_{Q_1}}} \vee \beta_{R_{N_{Q_2}}})(q_i) \vee \beta_N(q_i, x, q_j)\}$,
Thus, $(\beta_{R_{N_{Q_1}}} \vee \beta_{R_{N_{Q_2}}})(q_j) \geq \wedge_{q_i \in Q}\{(\beta_{R_{N_{Q_1}}} \vee \beta_{R_{N_{Q_2}}})(q_i) \vee \beta_N(q_i, x, q_j)\}$,—(2) and
$(\gamma_{R_{N_{Q_1}}} \vee \gamma_{R_{N_{Q_2}}})(q_j) = (\gamma_{R_{N_{Q_1}}}(q_j) \vee \gamma_{R_{N_{Q_2}}}(q_j))$
$\geq \{\wedge_{q_i \in Q}\{\gamma_{R_{N_{Q_1}}}(q_i) \vee \gamma_N(q_i, x, q_j)\}\} \vee \{\wedge_{q_i \in Q}\{\gamma_{R_{N_{Q_2}}}(q_i) \vee \gamma_N(q_i, x, q_j)\}\}$
$= \{\wedge_{q_i \in Q}\{\gamma_{R_{N_{Q_1}}}(q_i) \vee \gamma_{R_{N_{Q_2}}}(q_i) \vee \gamma_N(q_i, x, q_j)\}\}$
$= \wedge_{q_i \in Q}\{(\gamma_{R_{N_{Q_1}}} \vee \gamma_{R_{N_{Q_2}}})(q_i) \vee \gamma_N(q_i, x, q_j)\}$.
Thus, $(\gamma_{R_{N_{Q_1}}} \vee \gamma_{R_{N_{Q_2}}})(q_j) \geq \wedge_{q_i \in Q}\{(\gamma_{R_{N_{Q_1}}} \vee \gamma_{R_{N_{Q_2}}})(q_i) \vee \gamma_N(q_i, x, q_j)\}$.—(3)
Hence from (1), (2), and (3),$R_{N_{Q_1}} \vee R_{N_{Q_2}}$ is reverse subsystem of interval neutrosophic automaton M.

Theorem 3.5. Let$M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton. and $R_{N_{Q_1}}$, and $R_{N_{Q_2}}$ be reverse subsystems of M. Then $R_{N_{Q_1}} \wedge R_{N_{Q_2}}$ is reverse subsystem of M.

Proof:

Since $R_{N_{Q_1}}$ and $R_{N_{Q_2}}$ are reverse subsystem of interval neutrosophic automaton M. Then $\forall\ q_i, q_j\ \in\ Q$ and $x\ \in\ \Sigma$ such that
$\alpha_{R_{N_{Q_1}}}(q_j) \leq \vee_{q_i \in Q}\{\alpha_{R_{N_{Q_1}}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,
$\beta_{R_{N_{Q_1}}}(q_j) \geq \wedge_{q_i \in Q}\{\beta_{R_{N_{Q_1}}}(q_i) \vee \beta_N(q_i, x, q_j)\}$,
$\gamma_{R_{N_{Q_1}}}(q_j) \geq \wedge_{q_i \in Q}\{\gamma_{R_{N_{Q_1}}}(q_i) \vee \gamma_N(q_i, x, q_j)\}$ and
$\alpha_{R_{N_{Q_2}}}(q_j) \leq \vee_{q_i \in Q}\{\alpha_{R_{N_{Q_2}}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,
$\beta_{R_{N_{Q_2}}}(q_j) \leq \wedge_{q_i \in Q}\{\beta_{R_{N_{Q_2}}}(q_i) \vee \beta_N(q_i, x, q_j)\}$,
$\gamma_{R_{N_{Q_2}}}(q_j) \leq \wedge_{q_i \in Q}\{\gamma_{R_{N_{Q_2}}}(q_i) \vee \gamma_N(q_i, x, q_j)\}$.
Now we have to prove $R_{N_{Q_1}} \wedge R_{N_{Q_2}}$ is a reverse subsystem of M.
It is enough to prove that
$(\alpha_{R_{N_{Q_1}}} \wedge \alpha_{R_{N_{Q_2}}})(q_j) \leq \vee_{q_i \in Q}\{(\alpha_{R_{N_{Q_1}}} \wedge \alpha_{R_{N_{Q_2}}})(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,
$(\beta_{R_{N_{Q_1}}} \wedge \beta_{R_{N_{Q_2}}})(q_j) \geq \wedge_{q_i \in Q}\{(\beta_{R_{N_{Q_1}}} \wedge \beta_{R_{N_{Q_2}}})(q_i) \vee \beta_N(q_i, x, q_j)\}$, and
$(\gamma_{R_{N_{Q_1}}} \wedge \gamma_{R_{N_{Q_2}}})(q_j) \geq \wedge_{q_i \in Q}\{(\gamma_{R_{N_{Q_1}}} \wedge \gamma_{R_{N_{Q_2}}})(q_i) \vee \gamma_N(q_i, x, q_j)\}$.
Now, $(\alpha_{R_{N_{Q_1}}} \wedge \alpha_{R_{N_{Q_2}}})(q_j) = (\alpha_{R_{N_{Q_1}}}(q_j) \wedge \alpha_{R_{N_{Q_2}}}(q_j))$
$\leq \{\vee_{q_i \in Q}\{\alpha_{R_{N_{Q_1}}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}\} \wedge \{\vee_{q_i \in Q}\{\alpha_{R_{N_{Q_2}}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}\}$

$= \{\vee_{q_i \in Q}\{\alpha_{R_{N_{Q_1}}}(q_i) \wedge \alpha_{R_{N_{Q_2}}}(q_i) \wedge \alpha_N(q_i, x, q_j)\}\}$
$= \vee_{q_i \in Q}\{(\alpha_{R_{N_{Q_1}}} \wedge \alpha_{R_{N_{Q_2}}})(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,
Thus, $(\alpha_{R_{N_{Q_1}}} \wedge \alpha_{R_{N_{Q_2}}})(q_j) \leq \vee_{q_i \in Q}\{(\alpha_{R_{N_{Q_1}}} \wedge \alpha_{R_{N_{Q_2}}})(q_i) \wedge \alpha_N(q_i, x, q_j)\}$,—(4) $(\beta_{R_{N_{Q_1}}} \wedge \beta_{R_{N_{Q_2}}})(q_j) = (\beta_{R_{N_{Q_1}}}(q_j) \wedge \beta_{R_{N_{Q_2}}}(q_j))$
$\geq \{\wedge_{q_i \in Q}\{\beta_{R_{N_{Q_1}}}(q_i) \vee \beta_N(q_i, x, q_j)\}\} \wedge \{\wedge_{q_i \in Q}\{\beta_{R_{N_{Q_2}}}(q_i) \vee \beta_N(q_i, x, q_j)\}\}$
$= \{\wedge_{q_i \in Q}\{\beta_{R_{N_{Q_1}}}(q_i) \wedge \beta_{R_{N_{Q_2}}}(q_i) \vee \beta_N(q_i, x, q_j)\}\}$
$= \wedge_{q_i \in Q}\{(\beta_{R_{N_{Q_1}}} \wedge \beta_{R_{N_{Q_2}}})(q_i) \vee \beta_N(q_i, x, q_j)\}$, Thus, $(\beta_{R_{N_{Q_1}}} \wedge \beta_{R_{N_{Q_2}}})(q_j) \geq \wedge_{q_i \in Q}\{(\beta_{R_{N_{Q_1}}} \wedge \beta_{R_{N_{Q_2}}})(q_i) \vee \beta_N(q_i, x, q_j)\}$,—(5) and
$(\gamma_{R_{N_{Q_1}}} \wedge \gamma_{R_{N_{Q_2}}})(q_j) = (\gamma_{R_{N_{Q_1}}}(q_j) \wedge \gamma_{R_{N_{Q_2}}}(q_j))$
$\geq \{\wedge_{q_i \in Q}\{\gamma_{R_{N_{Q_1}}}(q_i) \vee \gamma_N(q_i, x, q_j)\}\} \wedge \{\wedge_{q_i \in Q}\{\gamma_{R_{N_{Q_2}}}(q_i) \vee \gamma_N(q_i, x, q_j)\}\}$
$= \{\wedge_{q_i \in Q}\{\gamma_{R_{N_{Q_1}}}(q_i) \wedge \gamma_{R_{N_{Q_2}}}(q_i) \vee \gamma_N(q_i, x, q_j)\}\}$
$= \wedge_{q_i \in Q}\{(\gamma_{R_{N_{Q_1}}} \wedge \gamma_{R_{N_{Q_2}}})(q_i) \vee \gamma_N(q_i, x, q_j)\}$
Thus, $(\gamma_{R_{N_{Q_1}}} \wedge \gamma_{R_{N_{Q_2}}})(q_j) \geq \wedge_{q_i \in Q}\{(\gamma_{R_{N_{Q_1}}} \wedge \gamma_{R_{N_{Q_2}}})(q_i) \vee \gamma_N(q_i, x, q_j)\}$.—(6)
Thus, From (4), (5) and (6) $R_{N_{Q_1}} \wedge R_{N_{Q_2}}$ is reverse subsystem of interval neutrosophic automaton M.

4. Conclusions

In this paper, we introduce reverse subsystem of interval neutrosophic automata with example. Also, we establish necessary and sufficient condition for R_{N_Q} to be a reverse subsystem in interval neutrosophic automaton. Finally, we prove that the union and intersection of reverse subsystems of interval neutrosophic automaton is reverse subsystem of an interval neutrosophic automaton.

Conflicts of Interest: "The authors declare no conflict of interest."

References

1. K. Atanassov, *Intuitionistic fuzzy sets,* Fuzzy Sets and Systems, 20, 1986, 87-96.
2. K. Atanassov, *Interval valued intuitionistic fuzzy sets,* Fuzzy Sets and Systems, 31(3), (1989), 343-349.
3. M. Doostfatemeh and S. C. Kremer. *New directions in fuzzy automata,* International Journal of Approximate Reasoning, 38 (2005), 175-214.
4. W. L. Gau, D. J. Buehrer *Vague sets,* IEEE Trans. Syst. Man Cybern. 23(2), (1993), 610-614.
5. V. Karthikeyan, and R. Karuppaiya *Retrievability in Interval Neutrosophic Automata,* Advances in Mathematics: Scientific Journal 9(4), (2020), 1637-1644.
6. V. Karthikeyan, and R. Karuppaiya *Strong Subsystems of Interval Neutrosophic Automata,* Advances in Mathematics: Scientific Journal 9(4), (2020), 1645-1651.
7. V. Karthikeyan, and R. Karuppaiya *Subsystems of Interval Neutrosophic Automata,* Advances in Mathematics: Scientific Journal 9(4), (2020), 1653-1659.
8. V. Karthikeyan, and R. Karuppaiya *Characterizations of Submachine of Interval Neutrosophic Automata,* Advances in Mathematics: Scientific Journal 9(4), (2020), 2273-2277.

9. J. Kavikumar, D. Nagarajan, Said Broumi, F. Smarandache, M. Lathamaheswari, and Nur Ain Ebas *Neutrosophic General Finite Automata,* Neutrosophic Sets and Systems 27, (2019), 17-34.
10. J. Kavikumar, D. Nagarajan, S.P. Tiwari, Said Broumi and F. Smarandache *Composite Neutrosophic Finite Automata,* Neutrosophic Sets and Systems 36, (2020), 282-291.
11. T. Mahmood, and Q. Khan *Interval neutrosophic finite switchboard state machine,* Afr. Mat. 27, (2016),1361-1376.
12. T. Mahmood, Q. Khan, K. Ullah, and N. Jan. *Single valued neutrosophic finite state machine and switchboard state machine,* New Trends in Neutrosophic Theory and Applications, II, (2018), 384-402.
13. D. S. Malik, J. N. Mordeson and M. K. Sen *On subsystems of fuzzy finite state machines* Fuzzy Sets and Systems, 68, 1994, 83-92.
14. F. Smarandache, *A Unifying Field in Logics, Neutrosophy: Neutrosophic Probability, set and Logic, Rehoboth: American Research Press,* (1998).
15. H. Wang, F. Smarandache, Y.Q. Zhang, and R. Sunderraman *Single-valued neutrosophic sets* , Multispace and Multistructure, 4, (2010), Hexis, Phoenix, AZ.
16. H. Wang, F. Smarandache, Y.Q. Zhang, and R. Sunderraman *Interval Neutrosophic Sets and Logic:*, Theory and Applications in Computing, 5, (2005), Hexis, Phoenix, AZ.
17. W. G. Wee, *On generalizations of adaptive algoriths and application of the fuzzy sets concepts to pattern classification* Ph.D. Thesis, Purdue University, 1967.
18. L. A. Zadeh, *Fuzzy sets,* Information and Control, 8(3), 1965,338-353.

Received: May 21, 2021. Accepted: October 2, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

Continuous and bounded operators on neutrosophic normed spaces

Saleh Omran[1], and A. Elrawy[1,*]

[1]Department of Mathematics, Faculty of Science, SVU, Qena 83523, Egypt

*Correspondence: amr.elsaman@sci.svu.edu.eg

Abstract. In this paper, we define the concept of continuous, sequentially continuous, and strongly continuous mappings neutrosophic normed spaces. Also, we have some important relationships between continuous, sequentially continuous, strongly continuous relationships mappings. Furthermore, the concept of neutrosophic Lipschitzian mapping is introduced and a neutrosophic version of Banach's contraction principle is achieved. Finally, the definition of neutrosophic bounded and weakly bounded linear operators are discussed and studied.

Keywords: Neutrosophic sets, neutrosophic normed spaces, continuous mappings, bounded linear operators, Banach spaces.

Mathematics Subject Classication: 46S40, 11B39, 03E72, 40G15.

1. Introduction

The concept of neutrosophic set, as a generalization of fuzzy set [18] and intuitionistic fuzzy set [5] was introduced by smarandache [16, 17]. Since 2005, the notion of the neutrosophic set received by attantion and have many applications [1–3]. The concept of neutrosophic normed space is a natural generalization of fuzzy normed space and intuitionistic fuzzy normed space. However, many different types of fuzzy normed spaces were introduced in [10,11,13]. In [6] Bag and Samanta introduced a new concept of fuzzy norm its more natural to the usual norm, they studied the properties of bounded sets and compact set in finite dimensional fuzzy normed linear spaces. Also, in [7] Bag and Samanta introduced types of continuous and bounded of linear operators. In [4] Abdulgawad et al present the notion of fuzzy strongly continuous, sequentially continuous, and continuous mappings. As well as they discussed the bounded and isometry of the fuzzy linear operator between fuzzy normed.

Recently, the concept of neutrosophic normed space, as a generalization of fuzzy normed spaces and the intuitionistic fuzzy normed space was introduced in [9], they studied the properties of convergence, completeness of such spaces.

In this paper, we extend the definitions of continuous and bounded operators in neutrosophic normed spaces. Moreover, we establish the main properties of bounded linear operators and continuous linear operators. We obtain a generalized version of boundedness and continuity of intuitionistic fuzzy norms, while will play an important role in study neutrosophic analysis. Furthermore, we introduce the notion of neutrosophic Lipsechitzian mapping and neutrosophic Banach space.

The paper is divided into the following sections:

Section 2 includes some basic results. In section 3, we introduce and study some types of continuous linear operators in neutrosophic normed spaces and neutrosophic Lipschitzian mapping. In section 4, we define and study some types of bounded and isometry linear operators in neutrosophic normed spaces. In section 5, we draw some conclusions.

2. Basic concepts

In this section, we remember the basic concepts and results that are required for the present work.

Definition 2.1. [12] A continuous t-norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ with the following axioms:

- (i) $*$ is commutative and associative.
- (ii) $*$ is continuous.
- (iii) $\ell * 1 = \ell, \, , \forall \, \ell \in [0,1]$.
- (iv) $x * y \leq u * v$, $y \leq v$, $x \leq u$ and $x, y, u, v \in [0,1]$.

Definition 2.2. [14,15] A continuous t-co-norm is a binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ with the following axioms:

- (i) $\diamond$ is commutative and associative.
- (ii) $\diamond$ is continuous.
- (iii) $\ell \diamond 0 = \ell, \, \forall \, \ell \in [0,1]$.
- (iv) $x \diamond y \leq u \diamond v$, $y \leq v$, $x \leq u$ and $x, y, u, v \in [0,1]$.

Definition 2.3. [17] Let N be the universe set. A neutrosophic set $\mathcal{N}$ on N (NS $\mathcal{N}$) is defined as:

$$\mathcal{N} = \{< a, \rho(a), \xi(a), \eta(a) > | a \in N\} .$$

where $\rho, \xi, \eta : N \to [0,1]$.

Definition 2.4. [15] Let U be a linear space over $\mathbb{R}$ and $*, \diamond$ be a continuous t-norm, a continuous t-co-norm, respectively, then a neutrosophic subset $\mathcal{N} :< \rho, \xi, \eta >$ on $V \times \mathbb{R}$ be a neutrosophic norm on U if for $a, b \times U$ and $c,\ t,\ s \in \mathbb{R}$, if the following conditions hold.

(1) $0 \leq \rho(a,t), \xi(a,t), \eta(a,t) \leq 1$.
(2) $0 \leq \rho(a,t) + \xi(a,t) + \eta(a,t) \leq 3$.
(3) $\rho(a,t) = 0$ with $t \leq 0$.
(4) $\rho(a,t) = 1$ with $t > 0$ iff $x = 0$.
(5) $\rho(ca,t) = \rho(x, \dfrac{t}{\mid c \mid})\ \forall c \neq 0,\ t > 0$.
(6) $\rho(a,s) * \rho(b,t) \leq \rho(a+b, s+t)\ \forall s,t \in \mathbb{R}$.
(7) $\rho(a,.)$ is continuous non-decreasing function for $t > 0$, $\lim_{t\to\infty} \rho(a,t) = 1$.
(8) $\xi(a,t) = 1$ with $t \leq 0$.
(9) $\xi(a,t) = 0$ with $t > 0$ iff $x = 0$.
(10) $\xi(ca,t) = \xi(x, \dfrac{t}{\mid c \mid})\ \forall c \neq 0,\ t > 0$.
(11) $\xi(a,s) \diamond \xi(b,t) \geq \xi(a+b, s+t)$.
(12) $\xi(a,.)$ is continuous non-increasing function for $t > 0$, $\lim_{t\to\infty} \xi(a,t) = 0$.
(13) $\eta(a,t) = 1$ with $t \leq 0$.
(14) $\eta(a,t) = 0$ and $t > 0$ if and only if $x = 0$.
(15) $\eta(ca,t) = \xi(a, \dfrac{t}{\mid c \mid})\ \forall c \neq 0,\ t > 0$.
(16) $\eta(a,s) \diamond \eta(b,t) \geq \eta(a+b, s+t)$.
(17) $\eta(a,.)$ is continuous non-increasing function for $t > 0$, $\lim_{t\to\infty} \eta(a,t) = 0$.

Further $(V, \mathcal{N}, *, \diamond)$ is neutrosophic normed linear space (N$\mathcal{N}$LS).

Definition 2.5. [14,15] Let (a_n) be a sequence of points in an N$\mathcal{N}$LS $(U, \mathcal{N}, *, \diamond)$, then the sequence converges to a point $a \in U$ if and only if for given $0 < e < 1,\ t > 0\ \exists\ n_0 \in N$ such that,

$$\rho(a_n - a, t) > 1 - e,\ \xi(a_n - a, t) < e,\ \eta(a_n - a, t) < e\ \forall\ n \geq n_0.$$

$$\lim_{n\to\infty} \rho(a_n - a, t) = 1,\ \lim_{n\to\infty} \xi(a_n - a, t) = 0,\ \lim_{n\to\infty} \eta(a_n - a, t) = 0.$$

Then the sequence (a_n) is called a convergent sequence in the N$\mathcal{N}$LS $(U, \mathcal{N}, *, \diamond)$.

Definition 2.6. [15] Let (a_n) be a sequence in an N$\mathcal{N}$LS $(U, \mathcal{N}, *, \diamond)$, is said to be bounded for $0 < e < 1,\ t > 0$ if the following hold,

$$\rho(a_n, t) > 1 - e,\ \xi(a_n, t) < e,\ \eta(a_n, t) < e\ \forall n \in N.$$

Definition 2.7. [15] A sequence (a_n) of points in an N$\mathcal{N}$LS $(U, \mathcal{N}, *, \diamond)$, is called a Cauchy sequence if for given $0 < e < 1,\ t > 0\ \exists\ n_0 \in N$ such that,

$$\rho(a_n - a_m, t) > 1 - e,\ \xi(a_n - a_m, t) < e,\ \eta(a_n - a_m, t) < e\ \forall\ n, m \geq n_0.$$

$$\lim_{n,m\to\infty}\rho(a_n-a_m,t)=1,\ \lim_{n,m\to\infty}\xi(a_n-a_m,t)=0,\ \lim_{n,m\to\infty}\eta(a_n-a_m,t)=0.$$

3. Continuous mappings

In this section, we introduce the concept of continuous, sequentially continuous, and strongly continuous mappings neutrosophic normed spaces. Also, we study the relationships between continuous, sequentially continuous, strongly continuous mappings. Moreover, this study is enhanced with an application

Definition 3.1. Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. The mapping $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ is said to be continuous at $x_0 \in U$ if for all $x \in U$, for each $0 < \epsilon < 1$ and $t > 0$, there exists $0 < \delta < 1$ and $s > 0$, such that

$$\begin{aligned}&\rho_V(\mathcal{T}(x)-\mathcal{T}(x_0),t) > (1-\epsilon),\\ &\xi_V(\mathcal{T}(x)-\mathcal{T}(x_0),t) < \epsilon,\\ &\eta_V(\mathcal{T}(x)-\mathcal{T}(x_0),t) < \epsilon,\end{aligned}$$

whenever

$$\begin{aligned}&\rho_U(x-x_0,s) > (1-\delta),\\ &\xi_U(x-x_0,s) < \delta,\\ &\eta_U(x-x_0,s) < \delta,\end{aligned}$$

respectively. In other words:

$$\begin{aligned}\rho_U(x-x_0,s) > (1-\delta) &\Rightarrow \rho_V(\mathcal{T}(x)-\mathcal{T}(x_0),t) > (1-\epsilon),\\ \xi_U(x-x_0,s) < \delta &\Rightarrow \xi_V(\mathcal{T}(x)-\mathcal{T}(x_0),t) < \epsilon, \qquad (1)\\ \eta_U(x-x_0,s) < \delta &\Rightarrow \eta_V(\mathcal{T}(x)-\mathcal{T}(x_0),t) < \epsilon,\end{aligned}$$

$\mathcal{T}$ is continuous on U if it is continuous at every point in U.

Definition 3.2. Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. The mapping $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ is called sequentially continuous at $x_0 \in U$, any sequence (x_n) in U satisfying $x_n \to x_0$ leads to $\mathcal{T}(x_n) \to \mathcal{T}(x_0)$. In other words:

$$\begin{aligned}\lim_{n\to\infty}\rho_U(x_n-x_0,t)=1 &\Rightarrow \lim_{n\to\infty}\rho_V(\mathcal{T}(x_n)-\mathcal{T}(x_0),t)=1,\\ \lim_{n\to\infty}\xi_U(x_n-x_0,t)=0 &\Rightarrow \lim_{n\to\infty}\xi_V(\mathcal{T}(x_n)-\mathcal{T}(x_0),t)=0, \qquad (2)\\ \lim_{n\to\infty}\eta_U(x_n-x_0,t)=0 &\Rightarrow \lim_{n\to\infty}\eta_V(\mathcal{T}(x_n)-\mathcal{T}(x_0),t)=0,\end{aligned}$$

where $t > 0$. We call $\mathcal{T}$ is sequentially continuous on U when $\mathcal{T}$ is sequentially continuous at each point of U

Definition 3.3. Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. The mapping $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ is called strongly continuous at $x_0 \in U$ if for each $t > 0$. $\exists\ s > 0$ such that $\forall\ x \in U$,

$$\begin{aligned} \rho_U(x - x_0, s) &\leq \rho_V(\mathcal{T}(x) - \mathcal{T}(x_0), t), \\ \xi_U(x - x_0, s) &\geq \xi_V(\mathcal{T}(x) - \mathcal{T}(x_0), t), \\ \eta_U(x - x_0, s) &\geq \eta_V(\mathcal{T}(x) - \mathcal{T}(x_0), t), \end{aligned} \tag{3}$$

we say $\mathcal{T}$ is strongly continuous on U when it is strongly continuous at every point in U.

Theorem 3.4. *Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. The mapping $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ be continuous at $x_0 \in U$ if and only if $\mathcal{T}$ is sequentially continuous at $x_0 \in U$.*

Proof. Assume that $\mathcal{T}$ is continuous at $x_0 \in U$, $(x_n) \subset U$ if for all $x \in U$, for each $0 < \epsilon < 1$ and $t > 0\ \exists\ 0 < \delta < 1$ and $s > 0$, such that

$$\begin{aligned} \rho_U(x - x_0, s) > (1 - \delta) &\Rightarrow \rho_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) > (1 - \epsilon), \\ \xi_U(x - x_0, s) < \delta &\Rightarrow \xi_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) < \epsilon, \\ \eta_U(x - x_0, s) < \delta &\Rightarrow \eta_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) < \epsilon, \end{aligned}$$

Since $x_n \to x_0$, then there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\rho_U(x_n - x_0, s) > (1 - \delta), \\ &\xi_U(x_n - x_0, s) < \delta, \\ &\eta_U(x_n - x_0, s) < \delta. \end{aligned}$$

□

Hence

$$\begin{aligned} &\rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) > (1 - \epsilon), \\ &\xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) < \epsilon, \\ &\eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) < \epsilon, \end{aligned}$$

as $0 < \epsilon < 1$ arbitrary; so $\mathcal{T}(x_n) \to \mathcal{T}(x_0)$. Thus, $\mathcal{T}$ is sequentially continuous at $x_0 \in U$.

Another direction, we suppose that $\mathcal{T}$ is sequentially continuous at $x_0 \in U$ and $\mathcal{T}$ is not continuous at x_0. Then there exists $0 < \epsilon < 1$ and $t > 0$, such that for any $0 < \delta < 1$ and $s > 0$, there exists $x \in U$, such that

$$\begin{aligned} \rho_U(x - x_0, s) > (1 - \delta) &\text{ but } \rho_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) \leq (1 - \epsilon), \\ \xi_U(x - x_0, s) < \delta &\text{ but } \xi_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) \geq \epsilon, \\ \eta_U(x - x_0, s) < \delta &\text{ but } \eta_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) \geq \epsilon. \end{aligned} \tag{4}$$

So, for $\delta = 1 - \dfrac{1}{n+1}$, $s = \dfrac{1}{n+1}$, $n \in \mathbb{N}$ $\exists$ x_n such that

$$\rho_U(x_n - x_0, \frac{1}{n+1}) > (\frac{1}{n+1}) \text{ but } \rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \leq (1-\epsilon),$$
$$\xi_U(x_n - x_0, \frac{1}{n+1}) < 1 - \frac{1}{n+1} \text{ but } \xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \geq \epsilon,$$
$$\eta_U(x_n - x_0, \frac{1}{n+1}) < 1 - \frac{1}{n+1} \text{ but } \eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \geq \epsilon.$$

Taking $s > 0$, there exists n_0, such that $\dfrac{1}{n+1} < s$ for all $n \geq n_0$ then

$$\rho_U(x_n - x_0, s) > (\frac{1}{n+1}),$$
$$\xi_U(x_n - x_0, s) < 1 - \frac{1}{n+1},$$
$$\eta_U(x_n - x_0, s) < 1 - \frac{1}{n+1},$$

hence

$$\lim_{n\to\infty} \rho_U(x_n - x_0, s) = 1,$$
$$\lim_{n\to\infty} \xi_U(x_n - x_0, s) = 0,$$
$$\lim_{n\to\infty} \eta_U(x_n - x_0, s) = 0,$$

this lead to $x_n \to x_0$. However by (4),

$$\rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \leq (1-\epsilon),$$
$$\xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \geq \epsilon,$$
$$\eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \geq \epsilon.$$

Thus, $\mathcal{T}(x_n)$ does not converges to $\mathcal{T}(x_0)$ but $x_n \to x_0$, which gives contradiction. Therefor, the mapping $\mathcal{T}$ is continuous at $x_0 \in U$.

Theorem 3.5. *Let $(U, \mathcal{N}_U, *, \diamond)$, $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces and $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$. If $\mathcal{T}$ is a strongly continuous, then $\mathcal{T}$ is sequentially continuous at $x_0 \in U$.*

Proof. Suppose that $\mathcal{T}$ is strongly continuous at x_0, then for each $t > 0$, there exists $s > 0$ such that for all $x \in U$ sequence (x_n) in U satisfying (3). Suppose that (x_n) is a sequence such that $x_n \to x_0$. If we put $x = x_n$ in (3), then we have

$$\rho_U(x_n - x_0, s) \leq \rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t),$$
$$\xi_U(x_n - x_0, s) \geq \xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t),$$
$$\eta_U(x_n - x_0, s) \geq \eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t).$$

This implies that

$$\lim_{n\to\infty} \rho_U(x_n - x_0, s) \leq \lim_{n\to\infty} \rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t),$$
$$\lim_{n\to\infty} \xi_U(x_n - x_0, s) \geq \lim_{n\to\infty} \xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t),$$
$$\lim_{n\to\infty} \eta_U(x_n - x_0, s) \geq \lim_{n\to\infty} \eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t).$$

Therefore,

$$\lim_{n\to\infty} \rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) = 1,$$
$$\lim_{n\to\infty} \xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) = 0,$$
$$\lim_{n\to\infty} \eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) = 0.$$

Since $t > 0$ is arbitrary, we obtain that $\mathcal{T}(x_n) \to \mathcal{T}(x_0)$. Thus, $\mathcal{T}$ is sequentially continuous. □

Remark 3.6. The converse of the above Theorem 3.5 is not true, i.e., the sequentially continuity does not imply the strongly continuity.

Now, we give an example that illustrates the above remark.

Example 3.7. Let $(U = \mathbb{R}, \| x \|)$ be a normed linear space, where $\| x \| = | x | \; \forall x \in U$, and $a * b = min\,\{a, b\}\,,\; a \diamond b = max\,\{a, b\}\,\forall a, b \in [0, 1]$. Define $\rho_1, \rho_2, \xi_1, \xi_2, \eta_1, \eta_2 : U \times \mathbb{R}^+ \to [0, 1]$ by

$$\rho_1(x, t) = \frac{t}{t + | x |}, \qquad \rho_2(x, t) = \frac{t}{t + c\,| x |},\; c > 0,$$

$$\xi_1(x, t) = \frac{| x |}{t + | x |}, \qquad \xi_2(x, t) = \frac{c\,| x |}{t + c\,| x |},\; c > 0,$$

$$\eta_1(x, t) = \frac{| x |}{t}, \qquad \eta_2(x, t) = \frac{c\,| x |}{t},\; c > 0.$$

It is easy to see that $(U, \mathcal{N}_1, *, \diamond)$ and $(U, \mathcal{N}_2, *, \diamond)$ are N$\mathcal{N}$LS. Let us now define, $f : (U, \mathcal{N}_1, *, \diamond) \to (U, \mathcal{N}_2, *, \diamond)$, $f(x) = \dfrac{x^4}{1 + x^2}$ for all $x \in U$. Let $x_0 \in U$ and (x_n) be a sequence in U such that $x_n \to x_0$ in $(U, \mathcal{N}_1, *, \diamond)$, that is, for all $t > 0$,

$$\lim_{n\to\infty} \rho_1(x_n - x_0, t) = \lim_{n\to\infty} \frac{t}{t + | x_n - x_0 |} = 1,$$
$$\lim_{n\to\infty} \xi_1(x_n - x_0, t) = \lim_{n\to\infty} \frac{c\,| x_n - x_0 |}{t + | x_n - x_0 |} = 0,$$
$$\lim_{n\to\infty} \eta_1(x_n - x_0, t) = \lim_{n\to\infty} \frac{| x_n - x_0 |}{t} = 0.$$

In other hand,

$$\begin{aligned}\rho_2(f(x_n)-f(x_0),t) &= \frac{t}{t+c\mid f(x_n)-f(x_0)\mid}\\ &= \frac{t}{t+c\mid \dfrac{x_n^4}{1+x_n^2}-\dfrac{x_0^4}{1+x_0^2}\mid}\\ &= \frac{t(1+x_n^2)(1+x_0^2)}{t(1+x_n^2)(1+x_0^2)+c\mid x_n^4(1+x_0^2)-x_0^4(1+x_n^2)\mid}\\ &= \frac{t(1+x_n^2)(1+x_0^2)}{t(1+x_n^2)(1+x_0^2)+c\mid (x_n^2+x_0^2)(x_n^2-x_0^2+x_n^2x_0^2(x_n^2-x_0^2))\mid}\\ &= \frac{t(1+x_n^2)(1+x_0^2)}{t(1+x_n^2)(1+x_0^2)+c\mid (x_n-x_0)(x_n+x_0)(x_n^2+x_0^2+x_n^2x_0^2)\mid}.\end{aligned}$$

So

$$\lim_{n\to\infty}\rho_2(x_n-x_0,t)=1.$$

$$\begin{aligned}\xi_2(f(x_n)-f(x_0),t) &= \frac{c\mid f(x_n)-f(x_0)\mid}{t+c\mid f(x_n)-f(x_0)\mid}\\ &= \frac{c\mid (x_n-x_0)(x_n+x_0)(x_n^2+x_0^2+x_n^2x_0^2)\mid}{t(1+x_n^2)(1+x_0^2)+c\mid (x_n-x_0)(x_n+x_0)(x_n^2+x_0^2+x_n^2x_0^2)\mid}.\end{aligned}$$

So

$$\lim_{n\to\infty}\xi_2(x_n-x_0,t)=0.$$

Finally,

$$\begin{aligned}\eta_2(f(x_n)-f(x_0),t) &= \frac{c\mid f(x_n)-f(x_0)\mid}{t}\\ &= \frac{c\mid (x_n-x_0)(x_n+x_0)(x_n^2+x_0^2+x_n^2x_0^2)\mid}{t\mid (1+x_n^2)(1+x_0^2)\mid},\end{aligned}$$

and this lead to

$$\lim_{n\to\infty}\eta_2(x_n-x_0,t)=0.$$

Thus, we see that f is sequentially continuous on U.

Now, we will explain that f is not strongly continuous by a contradiction. Let f be strongly continuous, then it holds that for all $x_0\in U$ and for each $t>0$ there exist $s>0$ such that for all $x_0\in U$,

$$\rho_1(x-x_0,s)\le \rho_2(f(x)-f(x_0),t),$$

$$\xi_1(x-x_0,s)\ge \xi_2(f(x)-f(x_0),t),$$

$$\eta_1(x-x_0,s)\ge \eta_2(f(x)-f(x_0),t).$$

Firstly, from the calculation of example [7,8] and

$$\frac{c \mid (x-x_0)(x+x_0)(x^2+x_0^2+x^2x_0^2) \mid}{t \mid (1+x^2)(1+x_0^2) \mid} \leq \frac{\mid x-x_0 \mid}{s}$$

$$\frac{t \mid (1+x^2)(1+x_0^2) \mid}{\mid (x+x_0)(x^2+x_0^2+x^2x_0^2) \mid} \geq \frac{c}{t}s.$$

Then it holds that

$$Inf_{x\in U}\left\{\frac{t \mid (1+x^2)(1+x_0^2) \mid}{\mid (x+x_0)(x^2+x_0^2+x^2x_0^2) \mid}\right\} \geq \frac{c}{t}s.$$

Thus, $\frac{c}{t}s = 0$. Since $k, t > 0$ then it holds that $s = 0$. This gives a contradiction with the fact that $s > 0$. So f is not strongly continuous.

3.1. *Application*

Definition 3.8. A mapping $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ is said to be neutrosophic Lipschitzian on U if $\exists\ c > 0$ such that

$$\rho_V(\mathcal{T}(x) - \mathcal{T}(y), t) \geq \rho_U(x - y, \frac{t}{c}),$$

$$\xi_V(\mathcal{T}(x) - \mathcal{T}(y), t) \leq \xi_U(x - y, \frac{t}{c}),$$

$$\eta_V(\mathcal{T}(x) - \mathcal{T}(y), t) \leq \eta_U(x - y, \frac{t}{c}),$$

$\forall t > 0,\ \forall x,\ y \in U$. If $c < 1$, we say that $\mathcal{T}$ is a neutrosophic contraction.

Remark 3.9. If $\mathcal{T}$ is a neutrosophic Lipschitzian mapping, then $\mathcal{T}$ is a neutrosophic continuous.

Definition 3.10. A neutrosophic Banach space is a complete neutrosophic normed linear space.

Theorem 3.11. *Let $(U, \mathcal{N}_U, *, \diamond)$ be a neutrosophic Banach space and $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (U, \mathcal{N}_U, *, \diamond)$ be a neutrosophic contraction, then $\mathcal{T}$ has a unique fixed point.*

Proof. Let x be arbitary point in U, then $\{\mathcal{T}^n(x)\}$ is a Cauchy sequence. In fact, for $t > 0$ and $m \in \mathbb{N} - \{0\}$, we get

$$\rho(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) \geq \rho(\mathcal{T}^{n+m-1}(x) - \mathcal{T}^{n-1}(x), \frac{t}{c}) \geq ... \geq \rho(\mathcal{T}^m(x) - x, \frac{t}{c^n}),$$

$$\xi(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) \leq \xi(\mathcal{T}^{n+m-1}(x) - \mathcal{T}^{n-1}(x), \frac{t}{c}) \leq ... \leq \xi(\mathcal{T}^m(x) - x, \frac{t}{c^n}),$$

$$\eta(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) \leq \eta(\mathcal{T}^{n+m-1}(x) - \mathcal{T}^{n-1}(x), \frac{t}{c}) \leq ... \leq \eta(\mathcal{T}^m(x) - x, \frac{t}{c^n}).$$

As $0 < c < 1$, we have that $\lim_{n\to\infty} \dfrac{t}{c^n} = \infty$. So

$$\lim_{n\to\infty} \rho(\mathcal{T}^m(x) - x, \frac{t}{c^n}) = 1,$$
$$\lim_{n\to\infty} \xi(\mathcal{T}^m(x) - x, \frac{t}{c^n}) = 0,$$
$$\lim_{n\to\infty} \eta(\mathcal{T}^m(x) - x, \frac{t}{c^n}) = 0.$$

Thus,

$$\lim_{n\to\infty} \rho(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) = 1,$$
$$\lim_{n\to\infty} \xi(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) = 0,$$
$$\lim_{n\to\infty} \eta(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) = 0.$$

Since U is complete, we have that $\{\mathcal{T}^n(x)\}$ is a convergent sequence. So there exists $u \in U$ such that $\lim_{n\to\infty} \mathcal{T}^n(x) = u$. We find that

$$u = \lim_{n\to\infty} \mathcal{T}^{n+1}(x) = \lim_{n\to\infty} \mathcal{T}(\mathcal{T}^n(x)) = \mathcal{T}(u).$$

Now, we exhibit the uniqueness. Assume that $\exists u, v \in U$ with $u \neq v$ and $u = \mathcal{T}(u)$, $v = \mathcal{T}(v)$. As $u \neq v$, $\exists s > 0$ such that

$$\rho(u - v, s) = a < 1,$$
$$\xi(u - v, s) = b > 0,$$
$$\eta(u - v, s) = c > 0,$$

then, for all $n \in \mathbb{N}^*$ we obtain

$$a = \rho(v - u, s) = \rho(\mathcal{T}^n(v) - \mathcal{T}^n(u), s) \geq \rho(v - u, \frac{s}{c^n}) \to 1,$$
$$b = \xi(v - u, s) = \xi(\mathcal{T}^n(v) - \mathcal{T}^n(u), s) \leq \xi(v - u, \frac{s}{c^n}) \to 0,$$
$$c = \eta(v - u, s) = \eta(\mathcal{T}^n(v) - \mathcal{T}^n(u), s) \leq \eta(v - u, \frac{s}{c^n}) \to 0,$$

thus, $a = 1, b = 0, c = 0$, which gives contradiction, hence the claims of theorem. □

4. **Neutrosophic bounded**

In this section, we introduce the concept of boundedness and isometry of mappings neutrosophic linear operators between neutrosophic normed spaces. Also, we study the relationships between bounded and weakly bounded linear operators.

Definition 4.1. Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. A mapping $\mathcal{T} : U \to V$ is called neutrosophic isometry if for each $x \in U$, $t > 0$ such that for all $x \in D$,

$$\begin{aligned} \rho_V(\mathcal{T}(x), t) &= \rho_U(x, t), \\ \xi_V(\mathcal{T}(x), t) &= \xi_U(x, t), \\ \eta_V(\mathcal{T}(x), t) &= \eta_U(x, t). \end{aligned} \tag{5}$$

Definition 4.2. Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces and $\mathcal{T} : U \to V$ be a linear operator. The operator $\mathcal{T}$ is called neutrosophic bounded if there exist a constant $k \in \mathbb{R} - \{0\}$ such that for each $x \in U$ and $t > 0$,

$$\begin{aligned} \rho_V(\mathcal{T}(x), t) &\geq \rho_U(kx, t), \\ \xi_V(\mathcal{T}(x), t) &\leq \xi_U(kx, t), \\ \eta_V(\mathcal{T}(x), t) &\leq \eta_U(kx, t). \end{aligned} \tag{6}$$

Definition 4.3. Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces and $\mathcal{T} : U \to V$ be a linear operator. The operator $\mathcal{T}$ is called weakly neutrosophic bounded if for all $0 < r < 1$ there exist a constant $k \in \mathbb{R} - \{0\}$ such that for each $x \in U$ and $t > 0$,

$$\begin{aligned} \rho_U(kx, t) \geq 1 - r &\Rightarrow \rho_V(\mathcal{T}(x), t) \geq 1 - r, \\ \xi_U(kx, t) \leq r &\Rightarrow \xi_V(\mathcal{T}(x), t) \leq r, \\ \eta_U(kx, t) \leq r &\Rightarrow \eta_V(\mathcal{T}(x), t) \leq r. \end{aligned} \tag{7}$$

Theorem 4.4. *Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. The linear operator $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ be neutrosophic bounded if $\mathcal{T}$ is weakly neutrosophic bounded.*

Proof. Suppose that $\mathcal{T}$ is a neutrosophic bounded operator. Then there exist a constant $k \in \mathbb{R} - \{0\}$ such that for each $x \in U$, $t > 0$, and satisfied (6). Using the fact that $\rho_U(kx, t)$, $\xi_U(kx, t)$, $\eta_U(kx, t) \in [0, 1]$, we obtain that for any $0 < r < 1$ there exist a k_r depends on k such that

$$\begin{aligned} \rho_U(kx, t) \geq \rho_U(k_r x, t) &\geq 1 - r, \\ \rho_U(kx, t) \leq \rho_U(k_r x, t) &\leq r, \\ \rho_U(kx, t) \leq \rho_U(k_r x, t) &\leq r. \end{aligned}$$

Since (6) it holds that

$$\begin{aligned} \rho_V(\mathcal{T}(x), t) &\geq 1 - r, \\ \xi_V(\mathcal{T}(x), t) &\leq r, \\ \eta_V(\mathcal{T}(x), t) &\leq r. \end{aligned}$$

Thus, T is weakly neutrosophic bounded □

Theorem 4.5. *Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. The linear operator $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ is continuous iff it is neutrosophic bounded.*

Proof. The first direction, let $\mathcal{T}$ be continuous on $(U, \mathcal{N}_U, *, \diamond)$, then it is continuous at $0 \in U$. Thus, for all $x \in U$, for each $0 < \epsilon < 1$ and $t > 0$, there exists $0 < \delta < 1$ and $s > 0$, such that if

$$\rho_U(x - 0, s) > (1 - \delta) \Rightarrow \rho_V(\mathcal{T}(x) - \mathcal{T}(0), t) > (1 - \epsilon),$$
$$\xi_U(x - 0, s) < \delta \Rightarrow \xi_V(\mathcal{T}(x) - \mathcal{T}(0), t) < \epsilon,$$
$$\eta_U(x - 0, s) < \delta \Rightarrow \eta_V(\mathcal{T}(x) - \mathcal{T}(0), t) < \epsilon.$$

Now, any way there exists $0 < \delta < 1$ such that

$$\rho_U(kx, t) > (1 - \delta),$$
$$\xi_U(kx, t) < \delta,$$
$$\eta_U(kx, t) < \delta.$$

So

$$\rho_U(x, \frac{t}{\mid k \mid}) = \rho_U(x, t) > (1 - \delta),$$
$$\xi_U(x, \frac{t}{\mid k \mid}) = \xi_U(kx, t) < \delta,$$
$$\eta_U(x, \frac{t}{\mid k \mid}) = \eta_U(kx, t) < \delta.$$

By putting $s = \frac{t}{\mid k \mid}$ we obtain that

$$\rho_U(x, s) > (1 - \delta) \Rightarrow \rho_V(\mathcal{T}(x), t) > (1 - \epsilon),$$
$$\xi_U(x, s) < \delta \Rightarrow \xi_V(\mathcal{T}(x), t) < \epsilon,$$
$$\eta_U(x, s) < \delta \Rightarrow \eta_V(\mathcal{T}(x), t) < \epsilon.$$

Hence

$$\rho_V(\mathcal{T}(x), t) \geq \rho_U(kx, t),$$
$$\xi_V(\mathcal{T}(x), t) \leq \xi_U(kx, t),$$
$$\eta_V(\mathcal{T}(x), t) \leq \eta_U(kx, t).$$

Therefore, $\mathcal{T}$ is neutrosophic bounded.

For the other direction, suppose that $\mathcal{T}$ is neutrosophic bounded, then there exist a constant

$k \in \mathbb{R} - \{0\}$ such that for each $x \in U$, $t > 0$, and satisfied (6). We have

$$\begin{aligned} \rho_V(\mathcal{T}(x), t) &\geq \rho_U(kx, t) = \rho_U(x, \frac{t}{\mid k \mid}) = \rho_U(x, s), \\ \xi_V(\mathcal{T}(x), t) &\leq \xi(kx, t) = \xi_U(x, \frac{t}{\mid k \mid}) = \xi_U(x, s), \\ \eta_V(\mathcal{T}(x), t) &\leq \eta_U(kx, t) = \eta_U(x, \frac{t}{\mid k \mid}) = \eta_U(x, s). \end{aligned} \tag{8}$$

Let $x_0 \in U$, $0 < \epsilon < 1,\ t > 0$, put $\delta = \epsilon$ and $s = \frac{t}{\mid k \mid} > 0$. Suppose that

$$\begin{aligned} &\rho_U(x - x_0) \geq (1 - \delta), \\ &\xi_U(x - x_0) \leq \delta, \\ &\eta_U(x - x_0) \leq \delta. \end{aligned}$$

Since (8) it holds that

$$\begin{aligned} &\rho_V(\mathcal{T}(x) - \mathcal{T}(x_0)) > (1 - \delta), \\ &\xi_V(\mathcal{T}(x) - \mathcal{T}(x_0)) < \delta, \\ &\eta_V(\mathcal{T}(x) - \mathcal{T}(x_0)) < \delta. \end{aligned}$$

Thus, $\mathcal{T}$ is continuous. □

5. **Conclusions**

In this paper, we have extended the definitions of continuous and bounded operators in neutrosophic normed spaces. Also, we have introduced a type of continuous and bounded operators in neutrosophic normed spaces. Moreover, we have studied some interesting relationships. These are illustrated by examples that are appropriate.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] Abdel-Basset, M., Mohamed, R., Zaied, A. E. N. H., & Smarandache, F. (2019). A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. Symmetry, 11(7), 903.

[2] Abdel-Basset, M., Atef, A., & Smarandache, F. (2019). A hybrid Neutrosophic multiple criteria group decision making approach for project selection. Cognitive Systems Research, 57, 216-227.

[3] Abdel-Basset, M., Gamal, A., Chakrabortty, R. K., & Ryan, M. (2021). Development of a hybrid multi-criteria decision-making approach for sustainability evaluation of bioenergy production technologies: A case study. Journal of Cleaner Production, 290, 125805.

[4] Al-Qubati, A. A., & Zedam, L. (2015). Properties of continuous maps and bounded linear operators in fuzzy normed spaces. Journal of Intelligent & Fuzzy Systems, 29(1), 127-133.

[5] Atanassov, K. (1986). Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20, 87 - 96.

[6] Bag, T., & Samanta, S. K. (2003). Finite dimensional fuzzy normed linear spaces. Journal of Fuzzy Mathematics, 11(3), 687-706.

[7] Bag, T., & Samanta, S. K. (2005). Fuzzy bounded linear operators. Fuzzy sets and Systems, 151(3), 513-547.

[8] Bag, T., & Samanta, S. K. (2013). Finite dimensional intuitionistic fuzzy normed linear spaces. Ann. Fuzzy Math. Inform, 6(2), 45-57.

[9] Bera, T., & Mahapatra, N. K. (2018). Neutrosophic soft normed linear spaces. Neutrosophic Sets and Systems. 23, 52-71.

[10] Cheng, S. C., & Mordeson, J. N. (1992). Fuzzy linear operators and fuzzy normed linear spaces. In First International Conference on Fuzzy Theory and Technology Proceedings, Abstracts and Summaries (pp. 193-197).

[11] Felbin, C. (1992). Finite dimensional fuzzy normed linear space. Fuzzy sets and Systems, 48(2), 239-248.

[12] Issac, P., & Maya, K. (2012). On the Intuitionistic fuzzy normed linear space (R^n, A), Inter. J. Fuzzy Math. And Systems, 2(2), 95-110.

[13] Katsaras, A. K. (1984). Fuzzy topological vector spaces II. Fuzzy sets and systems, 12(2), 143-154.

[14] Kirici, M., & imek, N. (2020). Neutrosophic normed spaces and statistical convergence. The Journal of Analysis, 28(4), 1059-1073.

[15] Muralikrishna, P., & Kumar, D. S. (2019). Neutrosophic approach on normed linear space. Neutrosophic Sets and Systems, 30, 255-240.

[16] Smarandache, F. (1998). Neutrosophy, Neutrosophic Probability, Set and logic, Amer.Res. Press, Rehoboth, USA, 105, http://fs.gallup.unm.edu/eBOOK- neutrosophics 4.pdf (fourth version).

[17] Smarandache, F. (2005). Neutrosophic set-a generalization of the intuitionistic fuzzy set. International journal of pure and applied mathematics, 24(3), 287.

[18] Zadeh, L. A. (1965). Information and control. Fuzzy sets, 8(3), 338-353.

Received: May 14, 2021. Accepted: October 3, 2021

NSS

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

A neutrosophic folding on a neutrosophic fundamental group

M. Abu-Saleem[1,*]

[1]Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 19117, Jordan;m_abusaleem@bau.edu.jo

[*]Correspondence: m_abusaleem@bau.edu.jo

Abstract. In this paper, we create a new type of fundamental group called the neutrosophic fundamental group. We obtain some kinds of conditional foldings that are confined to the elements of the neutrosophic fundamental groups. Also, we deduce the limit foldings of a neutrosophic fundamental group. We present the variant and invariant of the neutrosophic fundamental group under the folding of the neutrosophic manifold into itself. We show that the neutrosophic fundamental group at the ending limits of neutrosophic foldings on the n-dimensional neutrosophic manifold into itself is the neutrosophic identity group.

Keywords: Manifold; Neutrosophic folding; Neutrosophic fundamental group.

---

1. **Introduction**

In daily natural life, there are many uncertainties. However, standard mathematical logic is inadequate to account for these uncertainties to describe these uncertainties mathematically and to employ them in practice. The theory of the fuzzy set has occupied just about all areas of mathematics waz introduced by Zadeh [1]. The concept of "intuitionistic fuzzy set" was first introduced by Krassimir Atanassov [2]. A neutrosophic controller has been applied to many industrial applications, a neutrosophic controller uses scaling functions of physical variables to cope with uncertainly in process dynamics or the control environment [3]. Robertson proposed the folding of a manifold [4]. Many kinds of foldings and retractions were discussed in [5–9]. The fundamental group of quotient spaces was studied in [10]. Different groups are very significant in algebraic structures since they perform the role of a fundamental in almost all algebraic structures theories. Groups are as well important in plentiful other areas such as combinatorics, biology, physics, chemistry, etc., in order to study the symmetries and other performance among their components. For a continuous map $F : (W,\ w) \rightarrow (V,\ v)$ and $\widetilde{F} : \pi_1(W,\ w) \rightarrow \pi_1(V,\ v)$ is an induced map gained by using fundamental group functor [11].

One of the standard problems in the transformation of the fundamental groups has been to study the properties of a manifold and to illustrate them whenever possible.
section Preliminaries

Some essential concepts related to single-valued neutrosophic sets and the fundamental groups are shown in this section.

Definition 1.1. Let W be a topological space. Then the homotopy classes of loops at a given point $\mathrm{w}_\circ$ with an operation $[\alpha][\beta] = [\alpha \cdot \beta]$ is called the fundamental group and denoted by $\pi_1(\mathrm{W},w_\circ)$ [12].

Definition 1.2. Given spaces V and W with chosen points $\mathrm{v}_\circ \in V$ and $w_\circ \in W$, then the wedge sum $V \vee W$ is the quotient of the disjoint union $V \cup W$obtained by identifying $\mathrm{v}_\circ$ and $w_\circ$ to a single point [13].

Definition 1.3. Let $\check{\mathrm{W}}$ be a space of objects, a neutrosophic set $\check{\mathrm{B}}$ in $\check{\mathrm{W}}$ is branded by three functions called truth membership function $\lambda_{\check{\mathrm{B}}}(\mathrm{w})$, indeterminacy membership function $\xi_{\check{\mathrm{B}}}(\mathrm{w})$, and falsity membership function $\sigma_{\check{\mathrm{B}}}(\mathrm{y})$, for which $\lambda_{\check{\mathrm{B}}},\ \xi_{\check{\mathrm{B}}},\ \sigma_{\check{\mathrm{B}}} : \check{\mathrm{W}} \to]\ ^{-}0,\ 1^{+}[$ and $^{-}0 \leq \lambda_{\check{\mathrm{B}}}(\mathrm{w}) + \xi_{\check{\mathrm{B}}}(\mathrm{w}) + \sigma_{\check{\mathrm{B}}}(\mathrm{w}) \leq 3^{+}$. But in real-life application in scientific and engineering problems it is hard to utilize neutrosophic set on a value of $]\ ^{-}0,\ 1^{+}[$ [3, 14, 15]. We also note that in [16] for (SVNS) all values are taken as the subsets of $[0,\ 1]$. We'll utilize the symbol for convenience's sake $\langle\ \lambda_{\check{\mathrm{B}}},\ \xi_{\check{\mathrm{B}}}, \sigma_{\check{\mathrm{B}}}\ \rangle$ for the neutrosophic set $\check{\mathrm{B}} = \left\{\langle \mathrm{w}, \lambda_{\check{\mathrm{B}}}(\mathrm{w}),\ \xi_{\check{\mathrm{B}}}(\mathrm{w}),\ \sigma_{\check{\mathrm{B}}}(\mathrm{w})\ \rangle : \mathrm{w}\epsilon\check{\mathrm{W}}\right\}$, $\mathcal{F}_{\check{\mathrm{B}}} = \langle \lambda_{\check{\mathrm{B}}},\ \xi_{\check{\mathrm{B}}},\ \sigma_{\check{\mathrm{B}}}\rangle$ and $\mathcal{F}_{\check{\mathrm{B}}}(\mathrm{w}) = \langle \lambda_{\check{\mathrm{B}}}(\mathrm{w}),\ \xi_{\check{\mathrm{B}}}(\mathrm{w}),\ \sigma_{\check{\mathrm{B}}}(\mathrm{w})\rangle$, for which $\lambda_{\check{\mathrm{B}}}(\mathrm{w}),\ \xi_{\check{\mathrm{B}}}(\mathrm{w}),\ \sigma_{\check{\mathrm{B}}}(\mathrm{w}) \in [0,\ 1]$ for all $\mathrm{w} \in \check{\mathrm{W}}$. For a neutrosophic point and simlicity we use $(w,\ \mathcal{F}(\mathrm{w}))$ for $(w,\ \mathcal{F}_{\check{\mathrm{B}}}(\mathrm{w}))$.

2. Main Results

Intending to our study we will create the following definitions.

Definition 2.1. A neutrosophic path in a topological space $\tilde{\mathrm{W}}$ from $(\mathrm{w}_0, \mathcal{F}(\mathrm{w}_0))$ to $(\mathrm{w}_1,\ \mathcal{F}(\mathrm{w}_1))$ is a neutrosophic continuous map $\breve{\eta} : \widetilde{[0,\ 1]} \longrightarrow \tilde{\mathrm{W}}$ in which $\breve{\eta}(0,\ \mathcal{F}(0)) = (\mathrm{w}_0,\ \mathcal{F}(\mathrm{w}_0))$ and $\breve{\eta}(1,\ \mathcal{F}(1)) = (\mathrm{w}_1,\ \mathcal{F}(\mathrm{w}_1))$.

Definition 2.2. A space $\tilde{\mathrm{W}}$ is called neutrosophic arcwise connected if for any two points $(\mathrm{w}_0,\ \mathcal{F}(\mathrm{w}_0))$ and $(\mathrm{w}_1,\ \mathcal{F}(\mathrm{w}_1))$ in $\tilde{\mathrm{W}}$, there exists a neutrosophic path with begin $(\mathrm{w}_0,\ \mathcal{F}(\mathrm{w}_0))$ and end $(\mathrm{w}_1,\ \mathcal{F}(\mathrm{w}_1))$.

Definition 2.3. Two neutrosophic continuous maps $\breve{\eta}$, $\breve{\xi}$: $\tilde{\mathrm{V}} \longrightarrow \tilde{\mathrm{W}}$ are neutrosophy homotopic written ($\breve{\eta} \cong \breve{\xi}$) if there exists a neutrosophic continuous map $\breve{\phi} : \tilde{\mathrm{V}} \times \widetilde{[0,\ 1]} \to \tilde{\mathrm{W}}$, for $(\mathrm{v},\ \mathcal{F}(\mathrm{v})) \in \tilde{\mathrm{V}}$,

$\check{\phi}((\mathrm{v},\, \mathit{Ŧ}(\mathrm{v}))\,,\, (0,\, \mathit{Ŧ}(0))) = \check{\eta}(\mathrm{v}, \mathit{Ŧ}(\mathrm{v}))$,
$\check{\phi}((\mathrm{v},\, \mathit{Ŧ}(\mathrm{v}))\,,\, (1,\, \mathit{Ŧ}(1))) = \check{\xi}(\mathrm{v},\, \mathit{Ŧ}(\mathrm{v}))$.

Definition 2.4. A neutrosophic path $\check{\mathrm{W}}$ is called a neutrosophic loop if $\check{\eta}(0,\ \mathit{Ŧ}(0)) = \check{\eta}(1\,,\ \mathit{Ŧ}(1))$.

Definition 2.5. Let $\check{\eta}$ is a path from $(\mathrm{w}_0,\ \mathit{Ŧ}(\mathrm{w}_0))$ to $(\mathrm{w}_1,\ \mathit{Ŧ}(\mathrm{w}_1))$ and let $\check{\xi}$ is a path from $(\mathrm{w}_1,\ \mathit{Ŧ}(\mathrm{w}_1))$to $(\mathrm{w}_2,\ \mathit{Ŧ}(\mathrm{w}_2))$, then

$$\check{\eta}\cdot\check{\xi}(t,\ \mathit{Ŧ}(t)) = \begin{cases} \check{\eta}(2t,\ \mathit{Ŧ}(2t)) & (0,\ \mathit{Ŧ}(0)) \le (t,\ \mathit{Ŧ}(t)) \le (\frac{1}{2},\ \mathit{Ŧ}(\frac{1}{2})) \\ \check{\xi}(2t-1),\ \mathit{Ŧ}(2t-1)) & (\frac{1}{2},\ \mathit{Ŧ}(\frac{1}{2})) \le (t,\ \mathit{Ŧ}(t)) \le (1,\ \mathit{Ŧ}(1)\,) \end{cases}$$

Definition 2.6. The neutrosophic fundamental group in neutrosophic space $\check{\mathrm{W}}$ at the neutrosophic base point $\check{\mathrm{b}}$ is the set of neutrosophic homotopy classes of neutrosophic loops with the product operation $[\ \check{\eta}][\check{\xi}] = [\ \check{\eta}\cdot\check{\xi}]$ and denoted as $\overset{\mathrm{ne}}{\pi}(\check{\mathrm{W}},\ \check{\mathrm{b}})$.

Definition 2.7. The neutrosophic group $\tilde{G}$ is a group with neutrosophic elements, i.e. the neutrosophic elements $\tilde{\mathrm{g}}$ can be represented as $\tilde{\mathrm{g}}=(g,\ \Upsilon(g))$.

Definition 2.8. Let $\tilde{N}_1$ and $\tilde{N}_2$ be two neutrosophic manifolds of dimension n_1 and n_2 respectively. A neutrosophic map $\Im : \tilde{N}_1 \to \tilde{N}_2$ is called a neutrosophic topological folding iff $\Im\circ\tilde{\delta} : \widetilde{[0,\ 1]} \to \tilde{N}_2$ is an induced piecewise neutrosophic geodesic that does not preserve length as $\tilde{\delta}$, whenever $\tilde{\delta} : \widetilde{[0,\ 1]} \to \tilde{N}_1$ is a piecewise geodesic neutrosophic path **.** For simplicity, we denote the neutrosophic topological folding by neutrosophic folding.

Example 2.9. Let $\tilde{\mathfrak{S}}^n$ be a neutrosophic sphere of dimension n. Then, $\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^1,\ \breve{\mathrm{a}}) \approx \tilde{Z}$**,** $\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^n,\ \breve{\mathrm{a}}) = \check{0}$ (neutrosophic identity group) for $n \ge 2$. Also, $\overset{\mathrm{ne}}{\pi}(\tilde{R}^n,\ \breve{\mathrm{a}}) = \check{0}$ for $n \ge 1$.

Lemma 2.10. *Two types of neutrosophic foldings* $\Im_{\mathrm{j}}$:$\tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$, $(\mathrm{j} = 1,\ 2)$ *without singularities induce neutrosophic foldings*
$\widehat{\Im}_{\mathrm{j}}$: $\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}_1^1\,) \to \overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}_2^1)$ *such that* $\widehat{\Im_{\mathrm{j}}}\left(\overset{\mathrm{ne}}{\pi}(\ \tilde{\mathfrak{S}}_1^1\,)\right) = \overset{\mathrm{ne}}{\pi}\left(\Im_{\mathrm{j}}\left(\tilde{\mathfrak{S}}_1^1\right)\right)$.

Proof. Let $\Im_1$:$\tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$ be a neutrosophic folding such that $\Im_1\left(e^{i\theta}, \mathit{Ŧ}\left(e^{i\theta}\right)\right) = (re^{i\theta},\ \mathit{Ŧ}(re^{i\theta}))$, $r > 0$, $\theta \in [0,\ 2\pi)$ then we obtain an induced neutrosophic folding $\widehat{\Im}_1$:$\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}_1^1\,) \to \overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}_2^1)$ such that $\widehat{\Im}_1\,[\alpha,\ \mathit{Ŧ}(\alpha)] = [r\alpha,\ \mathit{Ŧ}(r\alpha)]$, where $\alpha = e^{i(2m\pi\theta)}$, $m \in Z$, and so $\widehat{\Im}_1\left(\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}_1^1\,)\right) = \overset{\mathrm{ne}}{\pi}\left(\Im_1\left(\tilde{\mathfrak{S}}_1^1\right)\right)$. Also, let
$\Im_2$:$\tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$ be a neutrosophic folding such that
$\Im_2\left(e^{i\theta},\ \mathit{Ŧ}\left(e^{i\theta}\right)\right) = (re^{i\varphi},\ \mathit{Ŧ}\left(re^{i\varphi}\right))$, $\theta,\ \varphi \in [0,\ 2\pi)\ 0 \le \theta < 2\pi,\ \theta - \varphi \in [0,\ 2\pi)$, then we get an induced neutrosophic folding $\widehat{\Im}_2$:$\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}_1^1\,) \to \overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}_2^1)$ such that $\widehat{\Im}\,([\alpha, \mathit{Ŧ}(\alpha)]) = [\beta, \mathit{Ŧ}(\beta)]$, where $\alpha = e^{i(2m\pi\theta)}$, $\beta = e^{i(2m\pi\varphi)}$, $m \in Z$, and so $\widehat{\Im}\left(\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}_1^1\,)\right) = \overset{\mathrm{ne}}{\pi}\left(\Im_2\left(\tilde{\mathfrak{S}}_1^1\right)\right)$. □

Now we use some interesting transformations on a neutrosophic manifold to describe the structure of a neutrosophic fundamental group. However, in the following theorem, we will describe many types of neutrosophic foldings on the neutrosophic sphere of dimension 1 and neutrosophic torus.

Theorem 2.11. *There are different types of neutrosophic foldings* $\Im : \tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$ *which induced neutrosophic foldings* $\widehat{\Im}:\overset{\text{ne}}{\pi}(\tilde{\mathfrak{S}}_1^1) \to \overset{\text{ne}}{\pi}(\tilde{\mathfrak{S}}_2^1)$ *such that* $\widehat{\Im}(\overset{\text{ne}}{\pi}(\tilde{\mathfrak{S}}_1^1))$ *is either isomorphic to* $\check{0}$, $\tilde{Z}$ *or* $\tilde{G}$, *where* $\tilde{G} = \{(n,\ 0,\ 0,\ 0) : n \in Z\}$.

Proof. (i) If $\Im$: $\tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$ is a neutrosophic folding by a cut as in Fig.(1.a), then clearly $\widehat{\Im}_{\text{j}}\left(\overset{\text{ne}}{\pi}(\tilde{\mathfrak{S}}_1^1)\right) = \overset{\text{ne}}{\pi}\left(\Im\left(\tilde{\mathfrak{S}}_1^1\right)\right) = \tilde{0}$.

(ii) If $\Im$: $\tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$ is a neutrosophic folding without singularity of on $\tilde{\mathfrak{S}}_1^1$, then $\Im(\tilde{\mathrm{S}}^1)$ is a neutrosophic manifold which is homeomorphic to $\tilde{\mathfrak{S}}_1^1$ as in Fig.(1.b) , and so $\widehat{\Im}_{\text{j}}\left(\overset{\text{ne}}{\pi}(\tilde{\mathfrak{S}}_1^1)\right) = \overset{\text{ne}}{\pi}\left(\Im\left(\tilde{\mathfrak{S}}_1^1\right)\right) \approx \tilde{Z}$.

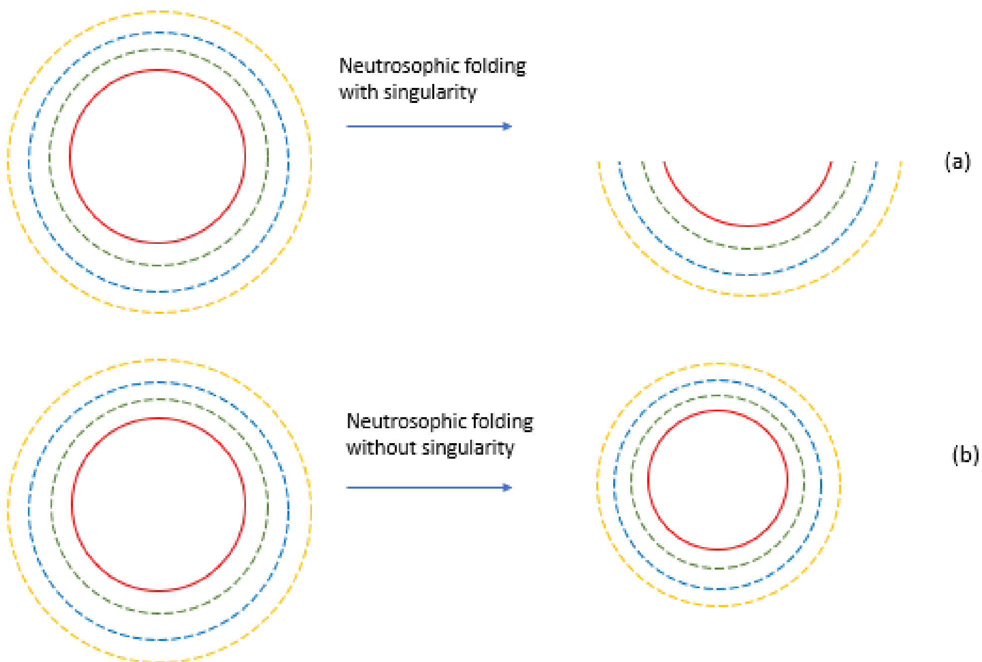


FIGURE 1

(iii) If $\Im$: $\tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$ is a neutrosophic folding such that $\Im$(geometry) = geometry and neutrosophic folding by a cut to all other neutrosophy as in Fig.(2) Then there is an induced neutrosophic folding $\Im$: $\tilde{\pi}_1(\tilde{\mathfrak{S}}_1^1) \to \overset{\text{ne}}{\pi}(\tilde{\mathfrak{S}}_2^1)$ for which $\widehat{\Im}_{\text{j}}\left(\overset{\text{ne}}{\pi}(\tilde{\mathfrak{S}}_1^1)\right) = \overset{\text{ne}}{\pi}\left(\Im\left(\tilde{\mathfrak{S}}_1^1\right)\right) \approx \tilde{G}$, where $\tilde{G} = \{(n,\ 0,\ 0,\ 0) : n \in Z\}$.

□

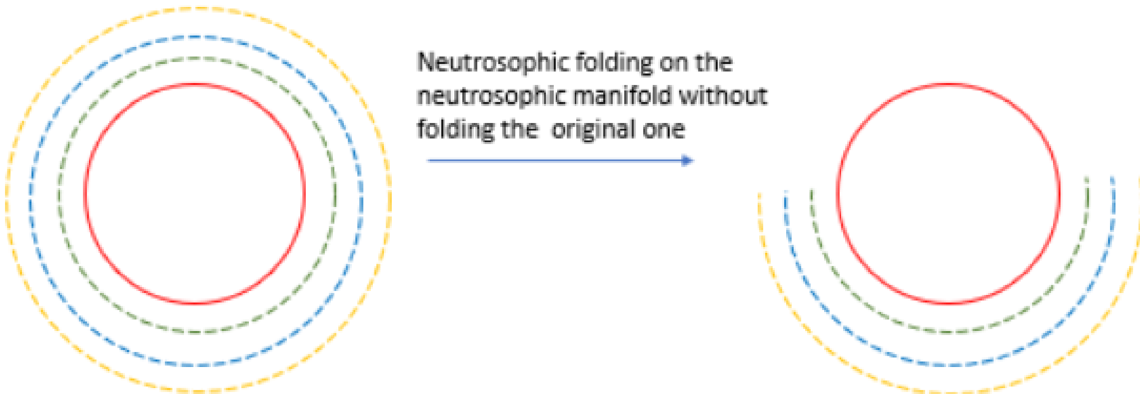


FIGURE 2

Theorem 2.12. *There are different types of neutrosophic foldings* $\Im : \tilde{T}_1^1 \rightarrow \tilde{T}_2^1$ *which induce neutrosophic foldings* $\widehat{\Im}_j:\tilde{\pi}_1(\tilde{T}_1^1)\rightarrow\tilde{\pi}_1(\tilde{T}_2^1)$ *such that* $\widehat{\Im}_j\left(\tilde{\pi}_1(\tilde{T}_1^1)\right) \approx G_1 \times G_2$ *where* G_1 *and* G_2 *are either* $\tilde{0}$, $\tilde{Z}$ *or* $\{(n,\ 0,\ 0,\ 0) : n \in Z\ \}$.

Proof. The proof of this theorem is similar to the proof of theorem 2.11. □

Theorem 2.13. *Let* $\tilde{\mathcal{M}}$ *be the neutrosophic annulus and let* $\tilde{E}^2$ *denote the closed neutrosophic unit ball in* $\tilde{R}^2$. *Then there is a sequence of neutrosophic foldings* $\Im_m : \tilde{\mathcal{M}} \longrightarrow \tilde{E}^2, m = 1, 2, .., k$ *for which* $\overset{ne}{\pi}(\lim_{k\to\infty}(\Im_k(\tilde{\mathcal{M}}))) = \check{0}$.

Proof. We can define a sequence of neutrosophic foldings as follows:

$$\Im_1 : \tilde{\mathcal{M}}_1 \longrightarrow \tilde{\mathcal{M}}_2, \quad \tilde{\mathcal{M}}_1 \subseteq \tilde{\mathcal{M}}_2 \subseteq \tilde{E}$$
$$\Im_2 : \tilde{\mathcal{M}}_2 \longrightarrow \tilde{\mathcal{M}}_3, \quad \tilde{\mathcal{M}}_2 \subseteq \tilde{\mathcal{M}}_3 \subseteq \tilde{E}$$
$$\vdots \qquad\qquad \vdots$$
$$\Im_k : \tilde{\mathcal{M}}_{k-1} \subseteq \tilde{\mathcal{M}}_k \quad \tilde{\mathcal{M}}_{k-1} \subseteq \tilde{\mathcal{M}}_k \subseteq \tilde{E}^2$$

and so $\lim_{k\to\infty}(\Im_k(\tilde{\mathcal{M}})) = \tilde{0}$ as in Fig.(3). Hence, $\overset{ne}{\pi}(\lim_{k\to\infty}(\Im_k(\tilde{\mathcal{M}}))) = \overset{ne}{\pi}(\tilde{E}^2)$, thus $\overset{ne}{\pi}(\lim_{k\to\infty}(\Im_k(\tilde{\mathcal{M}}))) = \check{0}$. □

Lemma 2.14. *The neutrosophic fundamental group of the limit neutrosophic foldings of a neutrosophic manifold which is homeomorphic to* $\tilde{\mathfrak{S}}^n, n \geq 2$ *is the neutrosophic identity group.*

Proof. The proof follows explicitly from the concept of a neutrosophic folding. □

Theorem 2.15. *Let* $\tilde{D}_n$ *be the disjoint union of neutrosophic* n *discs on the neutrosophic sphere* $\tilde{\mathfrak{S}}^2$. *Then there is a sequence of neutrosophic foldings*

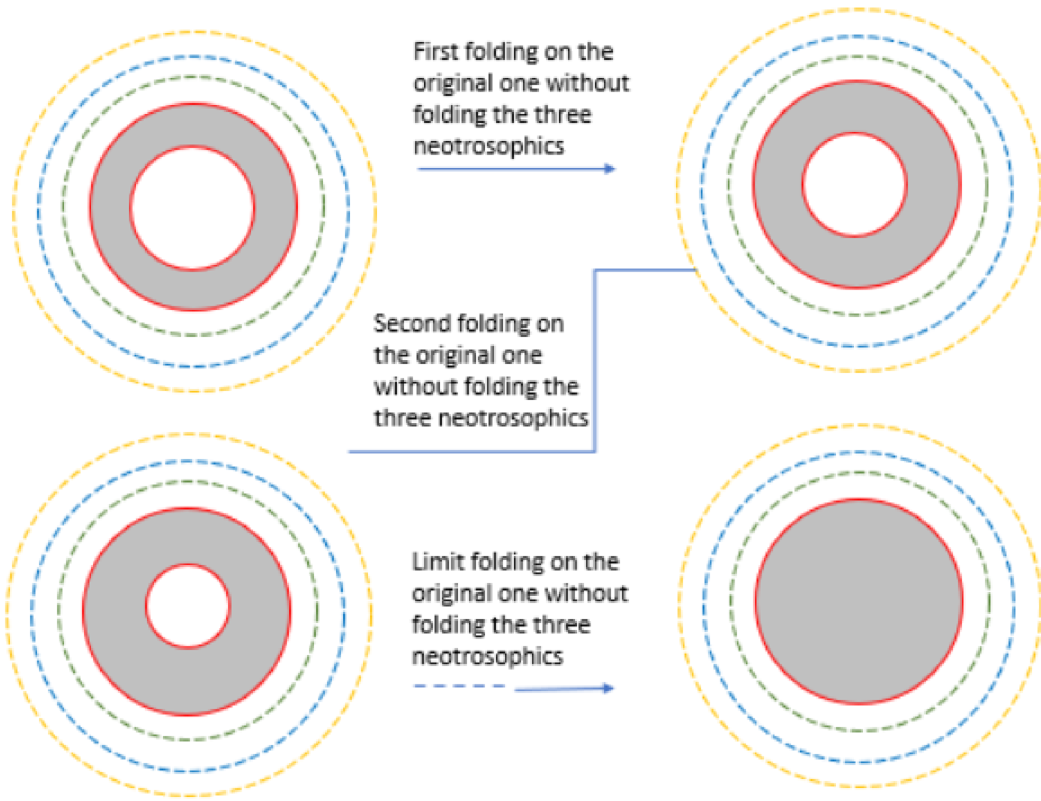


FIGURE 3

$\Im_m\colon \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_{m-1} \to \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_m : m = 1, 2, \ldots k$ *in which*
$\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{k}\to\infty} \Im_{\mathrm{k}}(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}})) = \breve{0}$.

Proof. Let $\tilde{\mathrm{D}}_{\mathrm{n}}$ be the disjoint union of neutrosophic n discs on the neutrosophic sphere $\tilde{\mathfrak{S}}^2$. Then, we can define a sequence of neutrosophic foldings as:

$$\begin{array}{llll}
\Im_1\colon \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_0 & \to & \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_1 \subseteq & \tilde{\mathfrak{S}}^2 \\
\Im_2\colon \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_1 & \to & \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_2 \subseteq & \tilde{\mathfrak{S}}^2 \\
\vdots & \vdots & & \vdots \\
\Im_k\colon \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_{k-1} & \to & \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_k \subseteq & \tilde{\mathfrak{S}}^2,
\end{array}$$

for which $\lim_{\mathrm{k}\to\infty} \Im_{\mathrm{k}} \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_{k-1} = \tilde{\mathfrak{S}}^2$ as in Fig.(4) for $n = 2$. Hence,
$\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{k}\to\infty} \Im_{\mathrm{k}} \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_{k-1}) = \overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^2)$. Therefore, $\overset{\mathrm{ne}}{\pi}(\Im_{\mathrm{k}} \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_{k-1}) = \tilde{0}$. □

Theorem 2.16. *There is a kind of a neutrosophic folding on* $\tilde{\mathfrak{S}}^2$ *for which* $\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{m}\to\infty} \Im_{\mathrm{k}}(\tilde{\mathfrak{S}}^2))$ *is either a free neutrosophic group of rank n or a neutrosophic identity group.*

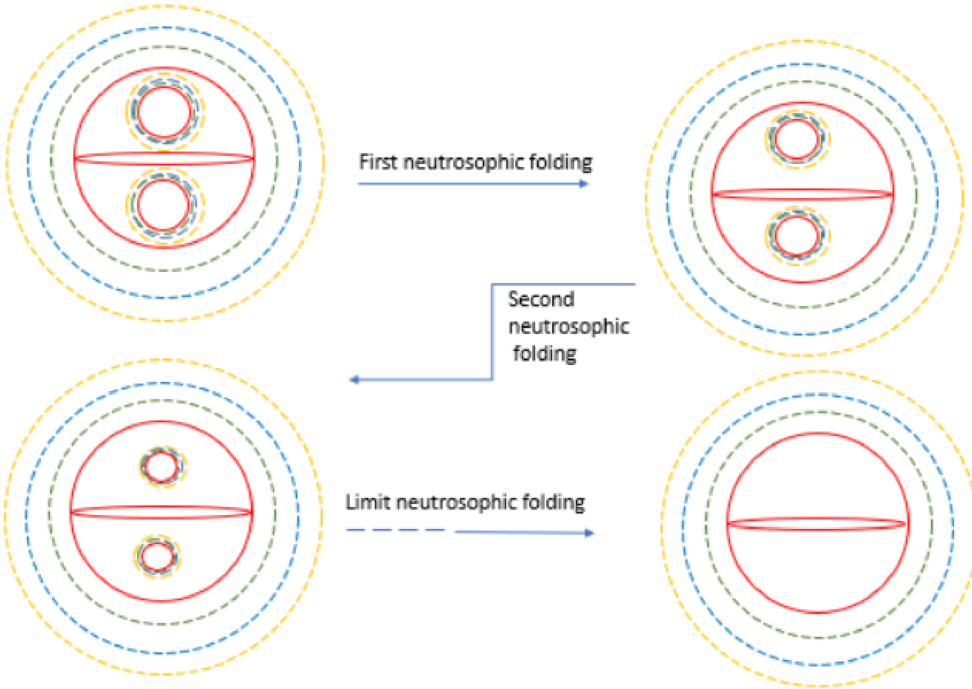


FIGURE 4

Proof. Consider the following chain of neutrosophic folding on $\tilde{\mathfrak{S}}^2$

$\Im_1 : \tilde{\mathfrak{S}}^2 \to \tilde{\mathfrak{S}}^2_1$, radius $(\tilde{\mathfrak{S}}^2) <$ radius $(\tilde{\mathfrak{S}}^2_1)$,

$\Im_2 : \tilde{\mathfrak{S}}^2_1 \to \tilde{\mathfrak{S}}^2_2$, radius $(\tilde{\mathfrak{S}}^2_1) <$ radius $(\tilde{\mathfrak{S}}^2_2)$,

$\vdots$ $\vdots$

$\Im_k : \tilde{\mathfrak{S}}^2_{k-1} \to \tilde{\mathfrak{S}}^2_k$, radius $(\tilde{\mathfrak{S}}^2_{k-1}) <$ radius $(\tilde{\mathfrak{S}}^2_k)$,

then we have two cases:

Case (1) If $\lim_{k\to\infty} \Im_k(\tilde{\mathfrak{S}}^2) = \tilde{\mathfrak{S}}^2 - \tilde{D}_n$, for some n as in Fig. (5) for n = 2 then $\overset{ne}{\pi}(\lim_{k\to\infty} \Im_k(\tilde{\mathfrak{S}}^2)) = \overset{ne}{\pi}(\lim_{k\to\infty}(\Im_k(\tilde{\mathfrak{S}}^2 - \tilde{D}_n)))$. Since $\overset{n}{\underset{j=1}{\vee}} \tilde{\mathfrak{S}}_j$ is a neutrosophic deformation retract of $\tilde{\mathfrak{S}}^2 - \tilde{D}_n$, it follows that $\overset{ne}{\pi}(\tilde{\mathfrak{S}}^2 - \tilde{D}_n)$ is a free neutrosophic group of rank n. Thus, $\overset{ne}{\pi}(\lim_{m\to\infty} \Im_k(\tilde{\mathfrak{S}}^2))$ is a free neutrosophic group of rank n.

Case(2) If $\overset{ne}{\pi}(\lim_{k\to\infty} \Im_k(\tilde{\mathfrak{S}}^2))$ is a neutrosophic sphere of radius ∞, then $\overset{ne}{\pi}(\lim_{k\to\infty} \Im_k(\tilde{\mathfrak{S}}^2) = \tilde{0}$.

Hence, the proof is complete. □

Remark 2.17. Let $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_1$ be two neutrosophic manifolds of the same dimension and let $\Im : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}_1$ be any neutrosophic folding of $\tilde{\mathcal{M}}$ into $\tilde{\mathcal{M}}_1$. Then, $\overset{ne}{\pi}(\lim_{k\to\infty} \Im_k(\tilde{\mathcal{M}}_{k-1}))$ and $\overset{ne}{\pi}(\Im(\tilde{\mathcal{M}}))$ needed not to be equal.

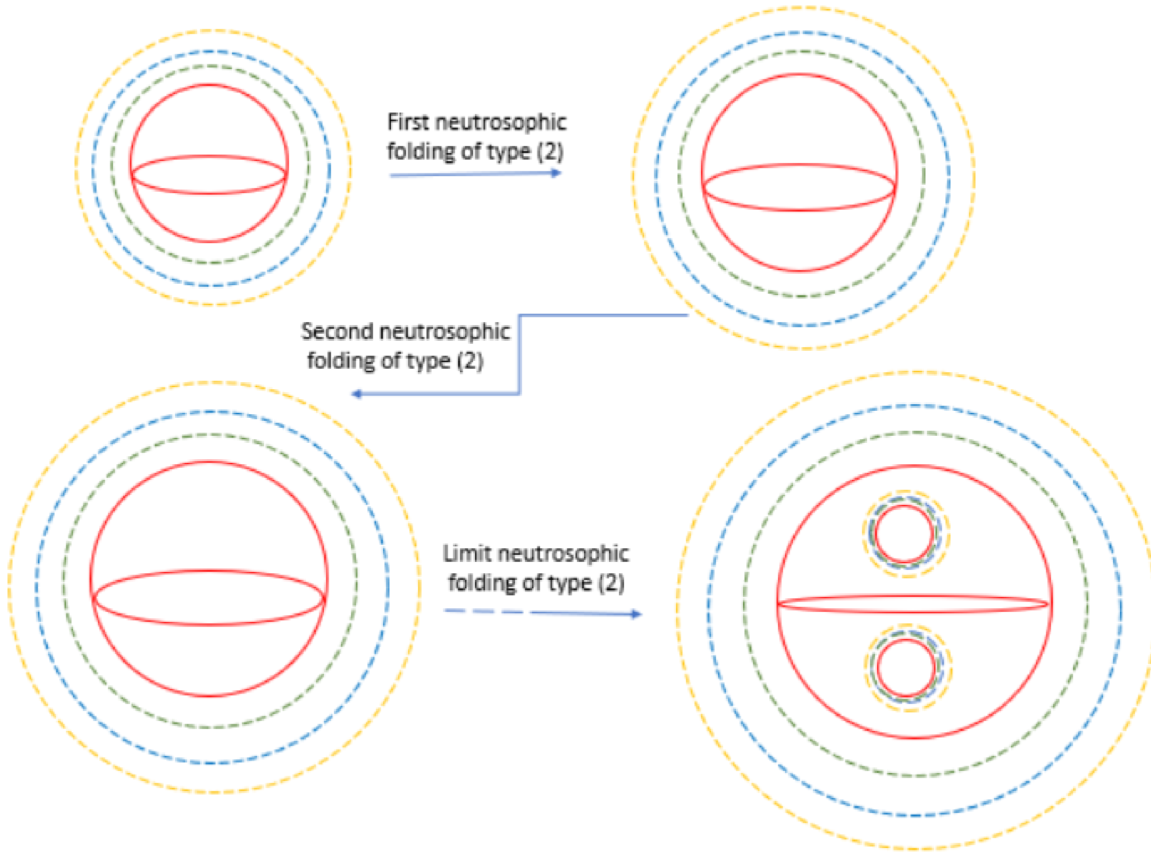


FIGURE 5

Proof. We will show this result by considering the following counter-example, Let $\tilde{\mathcal{M}} = \tilde{\mathfrak{S}}^1$, then we have a chain of neutrosophic folding as in Fig.(6), we have $\lim_{k\to\infty} \Im_k(\tilde{\mathfrak{S}}^1_{k-1}) \approx \tilde{L}$ (neutrosophic line), and so $\overset{ne}{\pi}(\lim_{k\to\infty} \Im_k(\tilde{\mathfrak{S}}^1_{k-1})) = \overset{ne}{\pi}(\tilde{L}) = \check{0}$ but
$\overset{ne}{\pi}(\Im(\tilde{\mathfrak{S}}^1)) = \overset{ne}{\pi}(\tilde{\mathfrak{S}}^1) \approx \tilde{Z}$. Hence, $\overset{ne}{\pi}(\lim_{k\to\infty} \Im_k(\tilde{\mathcal{M}}_{k-1})) \not\approx \overset{ne}{\pi}(\Im(\tilde{\mathcal{M}}))$. □

Theorem 2.18. *The neutrosophic fundamental group at end limits of neutrosophic foldings of an n-dimensional neutrosophic manifold $\tilde{\mathcal{M}}^n$ into itself is the neutrosophic identity group.*

Proof. Let $\Im_i$ be a neutrosophic folding of an n-dimensional neutrosophic manifold $\tilde{\mathcal{M}}^n$. Then we, get the following chains,

$$\begin{array}{ccccccccc}
\tilde{\mathcal{M}}^n & \xrightarrow{\Im_1^1} & \tilde{\mathcal{M}}_1^n & \xrightarrow{\Im_2^1} & \tilde{\mathcal{M}}_2^n & \rightarrow & \cdots \tilde{\mathcal{M}}_{k-1}^n & \cdots \xrightarrow{\lim_{k\to\infty} \Im_k^1} & \tilde{\mathcal{M}}^{n-1} \\
\tilde{\mathcal{M}}^{n-1} & \xrightarrow{\Im_1^2} & \tilde{\mathcal{M}}_1^{n-1} & \xrightarrow{\Im_2^2} & \tilde{\mathcal{M}}_2^{n-1} & \rightarrow & \cdots \tilde{\mathcal{M}}_{k-1}^{n-1} & \cdots \xrightarrow{\lim_{k\to\infty} \Im_k^2} & \tilde{\mathcal{M}}^{n-2} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\tilde{\mathcal{M}}^1 & \xrightarrow{\Im_1^n} & \tilde{\mathcal{M}}_1^1 & \xrightarrow{\Im_2^n} & \tilde{\mathcal{M}}_2^1 & & \cdots \tilde{\mathcal{M}}_{k-1}^1 & \cdots \xrightarrow{\lim_{k\to\infty} \Im_k^n} & \tilde{\mathcal{M}}^0.
\end{array}$$

As a result, of the last sequence, the ending limits of neutrosophic foldings is a zero-dimensional

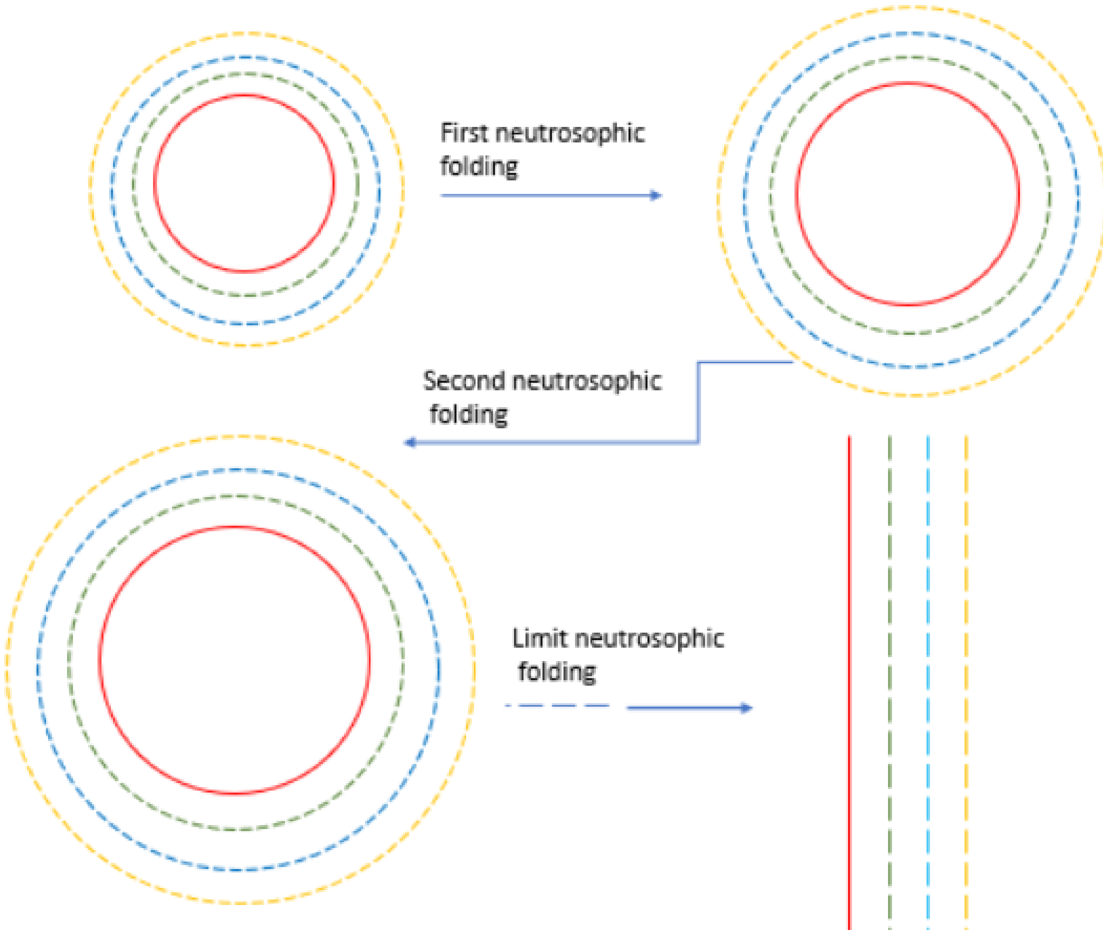


FIGURE 6

neutrosophic manifold, it is a neutrosophic point and the neutrosophic the fundamental group of a neutrosophic point is the neutrosophic identity group. □

3. conclusions

As a result, the neutrosophic fundamental group and foldings map impact on a neutrosophic fundamental group is introduced. We use the transformation to describe the elements of a neutrosophic fundamental group. Under the folding map, many kinds of the isomorphic neutrosophic group are obtained.

References

[1] Zadeh, L. A. Fuzzy sets. Information and Control, 1965; Volume 8, pp. 338-353.
[2] Atanassov, K. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 1986; Volume 20, pp. 87-96.
[3] Smarandache, F.; Pramanik, S. (Editors), New Trends in Neutrosophic Theory and Applications, Volume II. Brussels: Pons Editions, 2018.

[4] Robertson, S.A. Isometric folding of Riemannian manifolds, Proc. CityplaceRoy. Soc Edinburgh, 1977; Volume 77, pp. 275-289.

[5] Abu-Saleem, M. Folding on manifolds and their fundamental group, Journal of Taibah University for Science, 2018; Volume 13 (1) pp.16-19, https://doi, 10.1080/16583655.2018.1516396.

[6] Abu-Saleem, M. Retractions and homomorphisms on some operations of graphs, Journal of Mathematics, 2018; Volume 2018, pp. 1-4, https://doi.org/10.1155/2018/7328065

[7] Abu-Saleem, M. A neutrosophic folding and retraction on a single-valued neutrosophic graph , Journal of Intelligent and Fuzzy Systems, 2021; Volume 40 (3), pp. 5207-5213, https://doi: 10.3233/JIFS-201957

[8] Abu-Saleem, M. On the dynamical hyperbolic 3-spaces and their deformation retracts, Proceedings of the Jangjeon Mathematical society, 2012; Volume 15(2), pp. 189-193

[9] El-Ghoul, M.; El-Ahmady, A.E.; Rafat, H.; Abu-Saleem, M. The fundamental group of the connected sum of manifolds and their foldings, Journal of the Chungcheong Mathematical Society, 2005; Volume 18 (2), pp. 161-173.

[10] Montagantirud, P. Density of the topological fundamental groups of quotient spaces, Topology and its Applications, 2020; Volume 284(1), https://doi.org/10.1016/j.topol.2020.107358.

[11] Smale, S. A Vietoris mapping theorem for homotopy, Proceedings of the American Mathematical Society, 1957; Volume 8(3), https://doi.org/10.1090/S0002-9939-1957-0087106-9

[12] Massey, W. S. Algebraic topology: An introduction, Harcourt Brace and world, New York, 1967.

[13] Hatcher, A. Algebraic Topology, Cambridge University Press, 2002.

[14] Salama, A. A; Alblowi, S. A. Neutrosophic Set and Neutrosophic Topological Spaces, IOSR, Journal of Mathematics, 2012; Volume 3 (4), pp. 31-35.

[15] Salama, A. A.; Samarandache, F. Neutrosophic Set Theory, The Educational Publisher 415 Columbus, Ohio 2015.

[16] Smarandache, F. A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability and statistics: American Research Press, 2005.

Received: May 12, 2021. Accepted: October 4, 2021

NSS

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

Decision-Making Application Based on Aggregations of Complex Fuzzy Hypersoft Set and Development of Interval-Valued Complex Fuzzy Hypersoft Set

Atiqe Ur Rahman[1,*], Muhammad Saeed[2], Arooj Khalid[3], Muhammad Rayees Ahmad[4] and Sehrish Ayaz[5]

[1,2,3,4,5] Department of Mathematics, University of Management and Techology, Lahore, Pakistan ; aurkhb@gmail.com; muhammad.saeed@umt.edu.pk; ; f2019109013@umt.edu.pk; rayeesmalik.ravian@gmail.com; f2019349001@umt.edu.pk

*Correspondence: aurkhb@gmail.com

Abstract. Hypersoft set, an extension of soft set, deals with disjoint attribute-valued sets corresponding to distinct attributes. In this study, the innovation of complex fuzzy hypersoft set (CFH-set) is conferred, which can tackle with uncertainties and vagueness that lie in the data by taking into account the amplitude and phase terms of the complex numbers at the same time. This model establishes a gluing framework of the fuzzy set and hypersoft set characterized in the complex plane. This structure is more flexible and useful as it consents a broad range of values for membership function by expanding them to the unit circle in a complex plane through the characterization of the fuzzy hypersoft set to consider the periodic nature of the information and the attributes can further be classified into attribute-values sets for vivid understanding. With the characterization of its some fundamental properties and operations, aggregations of complex fuzzy hypersoft set: matrix, cardinal set, cardinal matrix of cardinal set, aggregation operator/set and matrix of aggregation set, are conceptualized along with application in decision-making. Moreover, complex interval-valued fuzzy hypersoft set is developed and some of its fundamentals i.e. subset, equal sets, null set, absolute set etc. and theoretic operations i.e. compliment, union, intersection etc. are investigated.

Keywords: Complex fuzzy sets (CF-Sets), soft set, hypersoft set and complex fuzzy hypersoft set.

---

1. Introduction

The concept of complex fuzzy set theory (CFS-Theory) [1] is an extension of fuzzy set theory (FS-Theory) [2], which uses complex-valued state for the membership of its elements. FS-Theory and CFS-Theory have some kind of complexities which restrain them to solve problem involving uncertainty professionally. The reason for these hurdles is, possibly, the inadequacy of the parametrization tool. It demands a mathematical tool free of all such impediments to tackle such issues. This scantiness is resolved with the development of soft set theory (SS-Theory) [7] which is a new parameterized family of subsets of the universe of discourse. The researchers [8]- [17] studied and investigated some elementary properties, operations, laws and hybrids of SS-Theory with applications in decision making. The gluing concept of NS-Theory and SS-Theory, is studied in [18] to make the NS-Theory adequate with parameterized tool. In many real life situations, distinct attributes are further partitioned in disjoint attribute-valued sets but existing SS-Theory is insufficient for dealing with such kind of attribute-valued sets. Hypersoft set theory (HS-Theory) [19] is developed to make the SST in line with attribute-valued sets to tackle real life scenarios. HS-Theory is an extension of SS-Theory as it transforms the single argument function into a multi-argument function. Certain elementary properties, aggregation operations, laws, relations and functions of HS-Theory, are investigated by [20]- [22] for proper understanding and further utilization in different fields. The applications of HS-Theory in decision making is studied by [23]- [27] and the intermingling study of HS-Theory with complex sets, convex and concave sets is studied by [28, 29]. Deli [30] characterized hybrid set structures under uncertainly parameterized hypersoft sets with theory and applications. Gayen et al. [31] analyzed some essential aspects of plithogenic hypersoft algebraic structures. They also investigated the notions and basic properties of plithogenic hypersoft subgroups ie plithogenic fuzzy hypersoft subgroup, plithogenic intuitionistic fuzzy hypersoft subgroup, plithogenic neutrosophic hypersoft subgroup. Saeed et al. [32, 33] discussed decision making techniques for neutrosophic hypersoft mapping and complex multi-fuzzy hypersoft set. Rahman et al. [34–36] studied decision making applications based on neutrosophic parameterized hypersoft Set, fuzzy parameterized hypersoft set and rough hypersoft set. Ihsan et al. [37] investigated hypersoft expert set with application in decision making for the best selection of product.

1.1. *Motivation*

In order to address the limitation of fuzzy soft set for dealing with periodic nature of data, Thirunavukarasu et al. [38] developed the theory of complex fuzzy soft set and discussed its some fundamentals along with applications. Kumar et al. [39] extended the work of Thirunavukarasu et al. to complex intuitionistic fuzzy soft sets and calculated its distance measures and entropies. Selvachandran et al. [40] investigated interval-valued complex fuzzy soft set with application. Abd et al. [41] discussed the fundamentals, properties and application of complex generalised fuzzy soft sets. These existing models employed single set of attributes for dealing uncertainties under fuzzy set-like environments but there are many situations when each attribute is required to be further partitioned into its attribute-valued set.

These existing structures has limitation regarding the consideration of such attribute-valued sets. Inspiring from the above literature, the decision system of complex fuzzy hypersoft set is developed with the help of the characterization of its aggregation operations and fundamental theory of interval-valued complex fuzzy hypersoft set is investigated. The proposed structure complex fuzzy hypersoft set (CFH-set) and interval-valued complex hypersoft set (IV-CFHS) are more flexible and useful as they

(i) generalize the existing structures of complex fuzzy soft set.
(ii) permit a broad range of values for membership function by expanding them to the unit circle in a complex plane.
(iii) consider the periodic nature of the information through the phase-terms.
(iv) classify distinct attributes into corresponding attribute-values sets for vivid understanding.

1.2. *Organization of Paper*

The rest of the paper is organized as: section 2 reviews the notions of fuzzy set, soft set, complex fuzzy set and relevant definitions used in the proposed work. Section 3, presents the decision system of complex fuzzy hypersoft set based on its some decisive aggregation operations along with application in decision-making. Section 4, investigates the fundamental theory of interval-valued complex fuzzy hypersoft set. Lastly, paper is summarized with future directions.

2. Preliminaries

Here some existing fundamental concepts regarding fuzzy set, fuzzy soft set and fuzzy hypersoft set are presented along with their structures with complex fuzzy set from literature. Throughout the paper, $\mathbb{U}$, $P(\mathbb{U})$, $F(\mathbb{U})$, $C(\mathbb{U})$ and $C_h(\mathbb{U})$ will present universe of discourse, power set of $\mathbb{U}$, collection of fuzzy sets, collection of complex fuzzy sets on soft sets and collection of complex fuzzy sets on hypersoft sets respectively.

Definition 2.1. [2]
Suppose a universal set $\mathbb{U}$ and a *fuzzy set* $X \subseteq \mathbb{U}$. The set X will be written as $X = \{(x, \alpha_X(x)) | x \in \mathbb{U}\}$ such that

$$\alpha_X : \mathbb{U} \to [0, 1]$$

where $\alpha_X(x)$ describes the membership percentage of $x \in X$.

Definition 2.2. [1]
A *complex fuzzy set* $\mathbb{C}_f$ is of the form

$$\mathbb{C}_f = \left\{(\epsilon, \mu_{\mathbb{C}_f}(\epsilon)) : \epsilon \in \mathbb{U}\right\} = \left\{(\epsilon, r_{\mathbb{C}_f}(\epsilon) e^{i\omega_{\mathbb{C}_f}(\epsilon)}) : \epsilon \in \mathbb{U}\right\}.$$

where $\mu_{\mathbb{C}_f}(\epsilon)$ is a membership function of $\mathbb{C}_f$ with $r_{\mathbb{C}_f}(\epsilon) \in [0,1]$ and $\omega_{\mathbb{C}_f}(\epsilon) \in (0,2\pi]$ as amplitude and phase terms respectively and $i=\sqrt{-1}$.

Buckley [3] and Zhang et al. [4] presented fuzzy complex number in different way. However, according to [5]- [6], both amplitude and phase terms are captured by fuzzy sets.

Definition 2.3. [7]
A *soft set* $\mathfrak{S}$ over $\mathbb{U}$, is defined as

$$\mathfrak{S} = \{(\epsilon, f_{\mathfrak{S}}(\epsilon)) : \epsilon \in E_1\}$$

where $f_{\mathfrak{S}} : E_1 \rightarrow P(\mathbb{U})$. and $E_1 \subseteq E$ (set of parameters).

Definition 2.4. [9]
A *fuzzy soft set* (FS-set) Γ_{E_1} on $\mathbb{U}$, is defined as

$$\Gamma_{E_1} = \{(\epsilon, \gamma_{E_1}(\epsilon)) : \epsilon \in E_1, \gamma_{E_1}(\epsilon) \in F(\mathbb{U})\}$$

where $\gamma_{E_1} : E_1 \rightarrow F(\mathbb{U})$ such that $\gamma_{E_1}(\epsilon) = \emptyset$ if $\epsilon \notin E_1$, and for all $\epsilon \in E_1$,

$$\gamma_{E_1}(\epsilon) = \left\{\mu_{\gamma_{E_1}(\epsilon)}(v)/v : v \in \mathbb{U}, \mu_{\gamma_{E_1}(\epsilon)}(v) \in [0,1]\right\}$$

is a fuzzy set over $\mathbb{U}$. Also γ_{E_1} is the approximate function of Γ_{E_1} and the value $\gamma_A(x)$ is a fuzzy set called ϵ-element of FS-set. Note that if $\gamma_{E_1}(\epsilon) = \emptyset$, then $(\epsilon, \gamma_{E_1}(\epsilon)) \notin \Gamma_{E_1}$.

Definition 2.5. [38]
A *complex fuzzy soft set* (CFS-set) χ_{E_1} over $\mathbb{U}$, is defined as

$$\chi_{E_1} = \{(\epsilon, \psi_{E_1}(\epsilon)) : \epsilon \in E_1, \psi_{E_1}(\epsilon) \in C(\mathbb{U})\}.$$

where $\psi_{E_1} : E_1 \rightarrow C(\mathbb{U})$ such that $\psi_{E_1}(\epsilon) = \emptyset$ if $\epsilon \notin E_1$ and it is complex fuzzy approximate function of CFS-set χ_{E_1} and its value $\psi_{E_1}(\epsilon)$ is called ϵ-member of CFS-set χ_{E_1} for all $\epsilon \in E_1$. Operations of CF-sets and CFS-sets were defined in [1] and [38] respectively.

Definition 2.6. [19]
The pair (H,G) is called a *hypersoft set* over $\mathbb{U}$, where G is the cartesian product of n disjoint sets $H_1, H_2, H_3, ..., H_n$ having attribute values of n distinct attributes $h_1, h_2, h_3, ..., h_n$ respectively and $H : G \rightarrow P(\mathbb{U})$.

Definition 2.7. [19]
A hypersoft set over a fuzzy universe of discourse is called *fuzzy hypersoft set.*

For more definitions and operations of hypersoft set, see [20]- [22]

2.1. *Complex Fuzzy Hypersoft Set*

The following subsections 2.1 and 2.2 are reviewed from [28].

Definition 2.8. Let $A_1, A_2, A_3, ..., A_n$ are disjoint sets having attribute values of n distinct attributes $a_1, a_2, a_3, ..., a_n$ respectively for $n \geq 1, G = A_1 \times A_2 \times A_3 \times ..... \times A_n$ and $\psi(\underline{x})$ be a CF-set over $\mathbb{U}$ for all $\underline{\epsilon} = (d_1, d_2, d_3, ..., d_n) \in G$. Then, *complex fuzzy hypersoft set* (CFH-set) χ_G over $\mathbb{U}$ is defined as

$$\chi_G = \{(\underline{\epsilon}, \psi(\underline{\epsilon})) : \underline{\epsilon} \in G, \psi(\underline{\epsilon}) \in C(\mathbb{U})\}$$

where

$$\psi : G \rightarrow C(\mathbb{U}), \qquad \psi(\underline{\epsilon}) = \emptyset \ if \ \underline{\epsilon} \notin G.$$

is a CF-approximate function of χ_G and its value $\psi(\underline{\epsilon})$ is called $\underline{\epsilon}$-member of CFH-set $\forall \underline{\epsilon} \in G$.

Example 2.9. Suppose a Department Promotion Committee (DPC) wants to observe(evaluate) the characteristics of some teachers by some defined indicators for departmental promotion. For this purpose, consider a set of teachers as a universe of discourse $\mathbb{U} = \{t_1, t_2, t_3, t_4\}$. The attributes of the teachers under consideration are the set $E = \{A_1, A_2, A_3\}$, where

A_1 = Total experience in years = $\{3, < 10\} = \{e_{11}, e_{12}\}$

A_2 = Total no. of publications= $\{10, 10 <\} = \{e_{21}, e_{22}\}$

A_3 = Performance Evaluation Report (PER) remarks = $\{eligible, not\, eligible\} = \{e_{31}, e_{32}\}$

and

$$G = A_1 \times A_2 \times A_3 = \left\{ \begin{array}{l} (e_{11}, e_{21}, e_{31}), (e_{11}, e_{21}, e_{32}), (e_{11}, e_{22}, e_{31}), \\ (e_{11}, e_{22}, e_{32}), (e_{12}, e_{21}, e_{31}), (e_{12}, e_{21}, e_{32}), \\ (e_{12}, e_{22}, e_{31}), (e_{12}, e_{22}, e_{32}) \end{array} \right\} = \{e_1, e_2, e_3, ..., e_8\}$$

Complex fuzzy set $\psi_G(e_1), \psi_G(e_2), ..., \psi_G(e_8)$ are defined as,

$$\psi_G(e_1) = \left\{ \frac{0.4e^{i0.5\pi}}{t_1}, \frac{0.8e^{i0.6\pi}}{t_2}, \frac{0.8e^{i0.8\pi}}{t_3}, \frac{1.0e^{i0.75\pi}}{t_4} \right\},$$

$$\psi_G(e_2) = \left\{ \frac{0.6e^{i0.7\pi}}{t_1}, \frac{0.9e^{i0.9\pi}}{t_2}, \frac{0.7e^{i0.9\pi}}{t_3}, \frac{0.75e^{i0.95\pi}}{t_4} \right\},$$

$$\psi_G(e_3) = \left\{ \frac{0.5e^{i0.6\pi}}{t_1}, \frac{0.8e^{i0.9\pi}}{t_2}, \frac{0.6e^{i0.9\pi}}{t_3}, \frac{0.65e^{i0.95\pi}}{t_4} \right\},$$

$$\psi_G(e_4) = \left\{ \frac{0.3e^{i0.7\pi}}{t_1}, \frac{0.7e^{i0.9\pi}}{t_2}, \frac{0.5e^{i0.9\pi}}{t_3}, \frac{0.75e^{i0.65\pi}}{t_4} \right\},$$

$$\psi_G(e_5) = \left\{ \frac{0.2e^{i0.5\pi}}{t_1}, \frac{0.3e^{i0.8\pi}}{t_2}, \frac{0.8e^{i0.7\pi}}{t_3}, \frac{0.45e^{i0.65\pi}}{t_4} \right\},$$

$$\psi_G(e_6) = \left\{ \frac{0.5e^{i0.9\pi}}{t_1}, \frac{0.3e^{i0.9\pi}}{t_2}, \frac{0.7e^{i0.8\pi}}{t_3}, \frac{0.85e^{i0.95\pi}}{t_4} \right\},$$

$$\psi_G(e_7) = \left\{ \frac{0.6e^{i0.9\pi}}{t_1}, \frac{0.9e^{i0.6\pi}}{t_2}, \frac{0.5e^{i0.6\pi}}{t_3}, \frac{0.85e^{i0.75\pi}}{t_4} \right\},$$

and

$$\psi_G(e_8)=\left\{\frac{0.8e^{i0.9\pi}}{t_1},\frac{0.8e^{i0.8\pi}}{t_2},\frac{0.6e^{i0.8\pi}}{t_3},\frac{0.65e^{i0.85\pi}}{t_4}\right\}$$

then CFH-set χ_G is written by,

$$\chi_G=\left\{\begin{array}{l}(e_1,\frac{0.4e^{i0.5\pi}}{t_1},\frac{0.8e^{i0.6\pi}}{t_2},\frac{0.8e^{i0.8\pi}}{t_3},\frac{1.0e^{i0.75\pi}}{t_4}),(e_2,\frac{0.6e^{i0.7\pi}}{t_1},\frac{0.9e^{i0.9\pi}}{t_2},\frac{0.7e^{i0.9\pi}}{t_3},\frac{0.75e^{i0.95\pi}}{t_4}),\\(e_3,\frac{0.5e^{i0.6\pi}}{t_1},\frac{0.8e^{i0.9\pi}}{t_2},\frac{0.6e^{i0.9\pi}}{t_3},\frac{0.65e^{i0.95\pi}}{t_4}),(e_4,\frac{0.3e^{i0.7\pi}}{t_1},\frac{0.7e^{i0.9\pi}}{t_2},\frac{0.5e^{i0.9\pi}}{t_3},\frac{0.75e^{i0.65\pi}}{t_4}),\\(e_5,\frac{0.2e^{i0.5\pi}}{t_1},\frac{0.3e^{i0.8\pi}}{t_2},\frac{0.8e^{i0.7\pi}}{t_3},\frac{0.45e^{i0.65\pi}}{t_4}),(e_6,\frac{0.5e^{i0.9\pi}}{t_1},\frac{0.3e^{i0.9\pi}}{t_2},\frac{0.7e^{i0.8\pi}}{t_3},\frac{0.85e^{i0.95\pi}}{t_4}),\\(e_7,\frac{0.6e^{i0.9\pi}}{t_1},\frac{0.9e^{i0.6\pi}}{t_2},\frac{0.5e^{i0.6\pi}}{t_3},\frac{0.85e^{i0.75\pi}}{t_4}),(e_8,\frac{0.8e^{i0.9\pi}}{t_1},\frac{0.8e^{i0.8\pi}}{t_2},\frac{0.6e^{i0.8\pi}}{t_3},\frac{0.65e^{i0.85\pi}}{t_4})\end{array}\right\}$$

Definition 2.10. Let $\chi_{G_1}=(\psi_1,G_1)$ and $\chi_{G_2}=(\psi_2,G_2)$ be two CFH-sets over the same $\mathbb{U}$. The set $\chi_{G_1}=(\psi_1,G_1)$ is said to be the *subset* of $\chi_{G_2}=(\psi_2,G_2)$, if

i. $G_1\subseteq G_2$

ii. $\forall\ \underline{x}\in G_1,\psi_1(\underline{x})\subseteq\psi_2(\underline{x})$ i.e. $r_{G_1}(\underline{x})\le r_{G_2}(\underline{x})$ and $\omega_{G_1}(\underline{x})\le\omega_{G_2}(\underline{x})$, where $r_{G_1}(\underline{x})$ and $\omega_{G_1}(\underline{x})$ are amplitude and phase terms of $\psi_1(\underline{x})$, whereas $r_{G_2}(\underline{x})$ and $\omega_{G_2}(\underline{x})$ are amplitude and phase terms of $\psi_2(\underline{x})$.

Definition 2.11. Two CFH-sets $\chi_{G_1}=(\psi_1,G_1)$ and $\chi_{G_2}=(\psi_2,G_2)$ over the same $\mathbb{U}$, are said to be *equal* if

i. $(\psi_1,G_1)\subseteq(\psi_2,G_2)$

ii. $(\psi_2,G_2)\subseteq(\psi_1,G_1)$.

Definition 2.12. Let (ψ,G) be a CFH-set over $\mathbb{U}$.Then

i. (ψ,G) is called a *null CFH-set*, denoted by $(\psi,G)_\Phi$ if for all $\underline{x}\in G$, the amplitude and phase terms of the membership function are given by $r_G(\underline{x})=0$ and $\omega_G(\underline{x})=0\pi$ respectively.

ii. (ψ,G) is called a *absolute CFH-set*, denoted by $(\psi,G)_\Delta$ if for all $\underline{x}\in G$, the amplitude and phase terms of the membership function are given by $r_G(\underline{x})=1$ and $\omega_G(\underline{x})=2\pi$ respectively.

Definition 2.13. Let (ψ_1,G_1) and (ψ_2,G_2) are two CFH-sets over the same universe $\mathbb{U}$.Then

i. A CFH-set (ψ_1,G_1) is called a *homogeneous CFH-set*, denoted by $(\psi_1,G_1)_{Hom}$ if and only if $\psi_1(\underline{x})$ is a homogeneous CF-set for all $\underline{x}\in G_1$.

ii. A CFH-set (ψ_1,G_1) is called a *completely homogeneous CFH-set*, denoted by $(\psi_1,G_1)_{CHom}$ if and only if $\psi_1(\underline{x})$ is a homogeneous with $\psi_1(\underline{y})$ for all $\underline{x},\underline{y}\in G_1$.

iii. A CFH-set (ψ_1,G_1) is said to be a completely homogeneous CFH-set with (ψ_2,G_2) if and only if $\psi_1(\underline{x})$ is a homogeneous with $\psi_2(\underline{x})$ for all $\underline{x}\in G_1\cap G_2$.

2.2. *Set Theoretic Operations and Laws on CFH-Sets*

Here some basic set theoretic operations (i.e.complement, union and intersection) and laws (commutative laws, associative laws etc.) are discussed on CFH-sets.

Definition 2.14. The *complement* of CFH-set (ψ, G), denoted by $(\psi, G)^c$ is defined as

$$(\psi, G)^c = \{(\underline{x}, \psi^c(\underline{x})) : \underline{x} \in G, \psi^c(\underline{x}) \in C(\mathbb{U})\}$$

such that the amplitude and phase terms of the membership function $\psi^c(\underline{x})$ are given by $r^c_G(\underline{x}) = 1 - r_G(\underline{x})$ and $\omega^c_G(\underline{x}) = 2\pi - \omega_G(\underline{x})$ respectively.

Proposition 2.15. *Let* (ψ, G) *be a CFH-set over* $\mathbb{U}$*. Then* $((\psi, G)^c)^c = (\psi, G)$.

Proof. Since $\psi(\underline{x}) \in C(\mathbb{U})$, therefore (ψ, G) can be written in terms of its amplitude and phase terms as

$$(\psi, G) = \left\{ \left(\underline{x}, r_G(\underline{x}) e^{i\omega_G(\underline{x})} \right) : \underline{x} \in G \right\} \tag{1}$$

Now

$$\psi^c(\underline{x}) = \left\{ \left(\underline{x}, r^c_G(\underline{x}) e^{i\omega^c_G(\underline{x})} \right) : \underline{x} \in G \right\}$$

$$\psi^c(\underline{x}) = \left\{ \left(\underline{x}, (1 - r_G(\underline{x})) e^{i(2\pi - \omega_G(\underline{x}))} \right) : \underline{x} \in G \right\}$$

$$((\psi, G)^c)^c = \left\{ \left(\underline{x}, (1 - r_G(\underline{x}))^c e^{i(2\pi - \omega_G(\underline{x})^c)} \right) : \underline{x} \in G \right\}$$

$$((\psi, G)^c)^c = \left\{ \left(\underline{x}, (1 - (1 - r_G(\underline{x}))) e^{i(2\pi - (2\pi - \omega_G(\underline{x})))} \right) : \underline{x} \in G \right\}$$

$$((\psi, G)^c)^c = \left\{ \left(\underline{x}, r_G(\underline{x}) e^{i\omega_G(\underline{x})} \right) : \underline{x} \in G \right\} \tag{2}$$

from equations (1) and (2), we have $((\psi, G)^c)^c = (\psi, G)$. □

Proposition 2.16. *Let* (ψ, G) *be a CFH-set over* $\mathbb{U}$*. Then*

i. $((\psi, G)_\Phi)^c = (\psi, G)_\Delta$
ii. $((\psi, G)_\Delta)^c = (\psi, G)_\Phi$

Definition 2.17. The *intersection* of two CFH-sets (ψ_1, G_1) and (ψ_2, G_2) over the same universe $\mathbb{U}$, denoted by $(\psi_1, G_1) \cap (\psi_2, G_2)$, is the CFH-set (ψ_3, G_3), where $G_3 = G_1 \cap G_2$, and $\psi_3(\underline{x}) = \psi_1(\underline{x}) \cap \psi_2(\underline{x})$ for all $\underline{x} \in G_3$.

Definition 2.18. The *difference* between two CFH-sets (ψ_1, G_1) and (ψ_2, G_2) is defined as

$$(\psi_1, G_1) \setminus (\psi_2, G_2) = (\psi_1, G_1) \cap (\psi_2, G_2)^c$$

Definition 2.19. The *union* of two CFH-sets (ψ_1, G_1) and (ψ_2, G_2) over the same universe $\mathbb{U}$, denoted by $(\psi_1, G_1) \cup (\psi_2, G_2)$, is the CFH-set (ψ_3, G_3), where $G_3 = G_1 \cup G_2$, and for all $\underline{x} \in G_3$,

$$\psi_3(\underline{x}) = \begin{cases} \psi_1(\underline{x}) & , if\ \underline{x} \in G_1 \backslash G_2 \\ \psi_2(\underline{x}) & , if\ \underline{x} \in G_2 \backslash G_1 \\ \psi_1(\underline{x}) \cup \psi_2(\underline{x}) & , if\ \underline{x} \in G_1 \cap G_2 \end{cases}$$

Proposition 2.20. *Let (ψ, G) be a CFH-set over $\mathbb{U}$.Then the following results hold true:*

i. $(\psi, G) \cup (\psi, G)_\Phi = (\psi, G)$
ii. $(\psi, G) \cup (\psi, G)_\Delta = (\psi, G)_\Delta$
iii. $(\psi, G) \cap (\psi, G)_\Phi = (\psi, G)_\Phi$
iv. $(\psi, G) \cap (\psi, G)_\Delta = (\psi, G)$
v. $(\psi, G)_\Phi \cup (\psi, G)_\Delta = (\psi, G)_\Delta$
vi. $(\psi, G)_\Phi \cap (\psi, G)_\Delta = (\psi, G)_\Phi$

Proposition 2.21. *Let (ψ_1, G_1), (ψ_2, G_2) and (ψ_3, G_3) are three CFH-sets over the same universe $\mathbb{U}$. Then the following commutative and associative laws hold true:*

i. $(\psi_1, G_1) \cap (\psi_2, G_2) = (\psi_2, G_2) \cap (\psi_1, G_1)$
ii. $(\psi_1, G_1) \cup (\psi_2, G_2) = (\psi_2, G_2) \cup (\psi_1, G_1)$
iii. $(\psi_1, G_1) \cap ((\psi_2, G_2) \cap (\psi_3, G_3)) = ((\psi_1, G_1) \cap (\psi_2, G_2)) \cap (\psi_3, G_3)$
iv. $(\psi_1, G_1) \cup ((\psi_2, G_2) \cup (\psi_3, G_3)) = ((\psi_1, G_1) \cup (\psi_2, G_2)) \cup (\psi_3, G_3)$

Proposition 2.22. *Let (ψ_1, G_1) and (ψ_2, G_2) are two CFH-sets over the same universe $\mathbb{U}$. Then the following De Morganss laws hold true:*

i. $((\psi_1, G_1) \cap (\psi_2, G_2))^c = (\psi_1, G_1)^c \cup (\psi_2, G_2)^c$
ii. $(\psi_1, G_1) \cup (\psi_2, G_2))^c = (\psi_1, G_1)^c \cap (\psi_2, G_2)^c$

3. Aggregation of Complex Fuzzy Hypersoft Set

In this section, we define an aggregation operator on complex fuzzy hypersoft set that produces an aggregate fuzzy set from a complex fuzzy hypersoft set and its cardinal set. The approximate functions of a complex fuzzy hypersoft set are fuzzy. Here G, E, χ_G and $C_H(\mathbb{U})$ will be in accordance with definition (2.8).

Definition 3.1. Let $\chi_G \in C_H(\mathbb{U})$. Assume that $\mathbb{U} = \{u_1, u_2, ..., u_m\}$ and $E = \{A_1, A_2, ..., A_n\}$ with
$A_1 = \{e_{11}, e_{12}, ..., e_{1n}\}, A_2 = \{e_{21}, e_{22}, ..., e_{2n}\}, ..., A_n = \{e_{n1}, e_{n2}, ..., e_{nn}\}$
and $G = A_1 \times A_2 \times ..... \times A_n = \{x_1, x_2, ..., x_n, ..., x_{n^n} = x_r\}$, each x_i is n-tuple element of G and $|G| = r = n^n$ then the χ_G can be presented as

χ_G	x_1	x_2	...	x_r
u_1	$\mu_{\psi_G(x_1)}(u_1)$	$\mu_{\psi_G(x_2)}(u_1)$	$\cdots$	$\mu_{\psi_G(x_n)}(u_1)$
u_2	$\mu_{\psi_G(x_1)}(u_2)$	$\mu_{\psi_G(x_2)}(u_2)$	$\cdots$	$\mu_{\psi_G(x_n)}(u_2)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$\mu_{\psi_G(x_1)}(u_m)$	$\mu_{\psi_G(x_2)}(u_m)$	$\cdots$	$\mu_{\psi_G(x_n)}(u_m)$

Where $\mu_{\psi_G(x)}$ is the membership function of ψ_G. If $a_{ij} = \mu_{\psi_G(x_j)}(u_i)$, for $i = \mathbb{N}_1^m$ and $j = \mathbb{N}_1^r$ then CFH-set χ_G is uniquely characterized by a matrix,

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called an $m \times r$ CFH-set matrix..

Definition 3.2. Let $\chi_G \in C_H(\mathbb{U})$. Then, the *cardinal set* of χ_G is defined as

$$\|\chi_G\| = \left\{\mu_{\|\chi_G\|}(\underline{x})/\underline{x} : \underline{x} \in G\right\},$$

where $\mu_{\|\chi_G\|} : G \to [0,1]$ is a membership function of $\|\chi_G\|$ with $\mu_{Card(\chi_G)(\underline{x})} = \frac{|\psi_G(\underline{x})|}{|U|}$.
Note that $\|C_H(\mathbb{U})\|$ is the collection of all cardinal sets of CFH-sets and $\|C_H(\mathbb{U})\| \subseteq F(G)$.

Definition 3.3. Let $\chi_G \in C_H(\mathbb{U})$ and $\|\chi_G\| \in \|C_H(\mathbb{U})\|$. Consider E as in definition (4.1) then $\|\chi_G\|$ can be presented as

G	x_1	x_2	$\cdots$	x_r
$\mu_{\|\chi_G\|}$	$\mu_{\|\chi_G\|}(x_1)$	$\mu_{\|\chi_G\|}(x_2)$	$\cdots$	$\mu_{\|\chi_G\|}(x_r)$

If $a_{1j} = \mu_{\|\chi_G\|(x_j)}$, for $j = \mathbb{N}_1^r$ then the cardinal set $\|\chi_G\|$ is represented by a matrix,

$$[a_{ij}]_{1\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

and is called *cardinal matrix* of $\|\chi_G\|$.

Definition 3.4. Let $\chi_G \in C_H(\mathbb{U})$ and $\|\chi_G\| \in \|C_H(\mathbb{U})\|$. Then *CFH-aggregation operator* is defined as

$$\overbrace{\chi_G} = A_{CFH}\left(\|\chi_G\|, \chi_G\right)$$

where

$$A_{CFH} : \|C_H(\mathbb{U})\| \times C_H(\mathbb{U}) \to F(U).$$

$\overbrace{\chi_G}$ is called the aggregate fuzzy set of CFH-set χ_G.
Its membership function is given as

$$\mu_{\overbrace{\chi_G}} : U \to [0,1]$$

with

$$\mu_{\overbrace{\chi_G}}(u) = \frac{1}{|G|} \sum_{\underline{x} \in G} \mu_{Card(\chi_G)}(\underline{x}) \mu_{Card(\psi_G)}(u).$$

Definition 3.5. Let $\chi_G \in C_H(\mathbb{U})$ and $\overbrace{\chi_G}$ be its aggregate fuzzy set. Assume that $\mathbb{U} = \{u_1, u_2, ..., u_m\}$, then $\overbrace{\chi_G}$ can be presented as

χ_G	$\mu_{\overbrace{\chi_G}}$
u_1	$\mu_{\overbrace{\chi_G}}(u_1)$
u_2	$\mu_{\overbrace{\chi_G}}(u_2)$
$\vdots$	$\vdots$
u_m	$\mu_{\overbrace{\chi_G}}(u_m)$

If $a_{i1} = \mu_{\overbrace{\chi_G}}(u_i)$ for $i = \mathbb{N}_1^m$ then $\overbrace{\chi_G}$ is represented by the matrix,

$$[a_{i1}]_{m\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

which is called *aggregate matrix* of $\overbrace{\chi_G}$ over $\mathbb{U}$.

3.1. *Applications of Complex Fuzzy Hypersoft Set*

In this section, an algorithm is presented to solve the problems in decision making by having under consideration the concept of aggregations defined in previous section. An example is demonstrated to explain the proposed algorithm.
It is necessary to determine an aggregate fuzzy set of CFH-set for choosing the best option (parameter) from the given set (set of choices/alternatives). Following algorithm may help in making appropriate decision.

Step 1: Determine a CFH-set χ_G over $\mathbb{U}$,

Step 2: Determine $\parallel \chi_G \parallel$ for amplitude term and phase term separately,

Step 3: Find $\overbrace{\chi_G}$ for amplitude term and phase term separately,

Step 4: Find the best option by max modulus of $\mu_{\overbrace{\chi_G}}(u)$

Example 3.6. Suppose a business man wants to buy a share from share market. There are four same kind of share which form the set , $\mathbb{U} = \{s_1, s_2, s_3, s_4\}$. The expert committee consider a set of attributes , $E = \{e_1, e_2, e_3\}$. For i = 1, 2, 3, 4, the attributes e_i stand for current trend of company performance, particular companys stock price for last one year, and Home country inflation rate, respectively. Corresponding to each attribute, the sets of attribute values are: $A_1 = \{e_{11}, e_{12}\}$; $A_2 = \{e_{21}\}$ and $A_3 = \{e_{31}, e_{32}\}$. Then the set $G = A_1 \times A_2 \times A_3 = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ where each ϵ_i is a 3-tuple. Complex fuzzy sets $\psi_G(\epsilon_1), \psi_G(\epsilon_2), \psi_G(\epsilon_3), \psi_G(\epsilon_4)$ are defined as,

$$\psi_G(\epsilon_1) = \left\{\frac{0.4e^{i0.5\pi}}{s_1}, \frac{0.8e^{i0.6\pi}}{s_2}, \frac{0.8e^{i0.8\pi}}{s_3}, \frac{1.0e^{i0.75\pi}}{s_4}\right\},$$

$$\psi_G(\epsilon_2) = \left\{\frac{0.3e^{i0.7\pi}}{s_1}, \frac{0.6e^{i0.8\pi}}{s_2}, \frac{0.5e^{i0.2\pi}}{s_3}, \frac{1.0e^{i0.85\pi}}{s_4}\right\},$$

$$\psi_G(\epsilon_3) = \left\{\frac{0.6e^{i0.7\pi}}{s_1}, \frac{0.9e^{i0.9\pi}}{s_2}, \frac{0.7e^{i0.95\pi}}{s_3}, \frac{0.75e^{i0.95\pi}}{s_4}\right\},$$

and

$$\psi_G(\epsilon_4) = \left\{\frac{0.5e^{i0.6\pi}}{s_1}, \frac{0.7e^{i0.8\pi}}{s_2}, \frac{0.6e^{i0.85\pi}}{s_3}, \frac{0.75e^{i0.85\pi}}{s_4}\right\},$$

Step 1: CFH-set χ_G is written as,

$$\chi_G = \left\{ \begin{array}{l} \left(\epsilon_1, \frac{0.4e^{i0.5\pi}}{s_1}, \frac{0.8e^{i0.6\pi}}{s_2}, \frac{0.8e^{i0.8\pi}}{s_3}, \frac{1.0e^{i0.75\pi}}{s_4}\right), \left(\epsilon_2, \frac{0.3e^{i0.7\pi}}{s_1}, \frac{0.6e^{i0.8\pi}}{s_2}, \frac{0.5e^{i0.2\pi}}{s_3}, \frac{1.0e^{i0.85\pi}}{s_4}\right), \\ \left(\epsilon_3, \frac{0.6e^{i0.7\pi}}{s_1}, \frac{0.9e^{i0.9\pi}}{s_2}, \frac{0.7e^{i0.95\pi}}{s_3}, \frac{0.75e^{i0.95\pi}}{s_4}\right), \left(\epsilon_4, \frac{0.5e^{i0.6\pi}}{s_1}, \frac{0.7e^{i0.8\pi}}{s_2}, \frac{0.6e^{i0.85\pi}}{s_3}, \frac{0.75e^{i0.85\pi}}{s_4}\right) \end{array} \right\}$$

Step 2: The cardinal is computed as,

$$\parallel \chi_G \parallel (Amplitude\ Term) = \{0.75/\epsilon_1, 0.6/\epsilon_2, 0.74/\epsilon_3, 0.64/\epsilon_4\}$$

$$\parallel \chi_G \parallel (Phase\ Term) = \{0.66/\epsilon_1, 0.64/\epsilon_2, 0.87/\epsilon_3, 0.78/\epsilon_4\}$$

Step 3: The set $\widehat{\chi_G}$ can be determined as,

$$\widehat{\chi_G}(Apmlitude\ Term) = \frac{1}{4}\begin{bmatrix} 0.4 & 0.3 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.9 & 0.7 \\ 0.8 & 0.5 & 0.7 & 0.6 \\ 1.0 & 1.0 & 0.75 & 0.75 \end{bmatrix}\begin{bmatrix} 0.75 \\ 0.6 \\ 0.74 \\ 0.64 \end{bmatrix} = \begin{bmatrix} 0.3110 \\ 0.5185 \\ 0.4505 \\ 0.5963 \end{bmatrix}$$

$$\widehat{\chi_G}(Phase\ Term) = \frac{1}{4}\begin{bmatrix} 0.5 & 0.7 & 0.7 & 0.6 \\ 0.6 & 0.8 & 0.9 & 0.8 \\ 0.8 & 0.2 & 0.95 & 0.85 \\ 0.75 & 0.85 & 0.95 & 0.85 \end{bmatrix}\begin{bmatrix} 0.66 \\ 0.64 \\ 0.87 \\ 0.78 \end{bmatrix} = \begin{bmatrix} 0.4638 \\ 0.5788 \\ 0.5364 \\ 0.6321 \end{bmatrix}$$

$$\widehat{\chi_G} = \left\{0.3110e^{i0.4638\pi}/s_1, 0.5185e^{i0.5788\pi}/s_2, 0.4505e^{i0.5364\pi}/s_3, 0.5963e^{i0.6321\pi}/s_4\right\}$$

Consider the modulus value of $Max(\mu_{\widehat{\chi_G}}) = \{0.31098/s_1, 0.5185/s_2, 0.4504/s_3, 0.5963/s_4\} = 0.5963/s_4$ This means that the 4th share s_4 may be recommended for suitable investment.

4. Interval-Valued Complex Fuzzy Hypersoft Set(IV-CFHS)

In this section, the basic theory of interval-valued complex fuzzy hypersoft set is developed.

Definition 4.1. Let $W_1, W_2, W_3, ..., W_n$ are disjoint sets having attribute values of n distinct attributes $w_1, w_2, w_3, ..., w_n$ respectively for $n \geq 1, W = W_1 \times W_2 \times W_3 \times ..... \times W_n$ and $\Psi(\underline{\omega})$ be a IV-CFS over $\mathbb{U}$ for all $\underline{\omega} = (b_1, b_2, b_3, ..., b_n) \in W$. Then, *interval-valued complex fuzzy hypersoft set* (IV-CFHS) $\Omega_W = (\Psi, W)$ over $\mathbb{U}$ is defined as

$$\Omega_W = \{(\underline{\omega}, \Psi(\underline{\omega})) : \underline{\omega} \in W, \Psi(\underline{\omega}) \in C_{IV}(\mathbb{U})\}$$

where

$$\Psi : W \to C_{IV}(\mathbb{U}), \qquad \Psi(\underline{\omega}) = \emptyset \; if \; \underline{\omega} \notin W.$$

is a IV-CF approximate function of Ω_W and $\Psi(\underline{\omega}) = (\overleftarrow{\Psi}(\underline{\omega}), \overrightarrow{\Psi}(\underline{\omega}))$. $\overleftarrow{\Psi}(\underline{\omega}) = \overleftarrow{r} e^{i\overleftarrow{\theta}}$ and $\overrightarrow{\Psi}(\underline{\omega}) = \overrightarrow{r} e^{i\overrightarrow{\theta}}$ are lower and upper bounds of the membership function of Ω_W respectively and its value $\Psi(\underline{\omega})$ is called $\underline{\omega}$-member of IV-CFHS $\forall \underline{\omega} \in W$.

Example 4.2. Considering example 2.9 with $W = \{e_1, e_2, e_3, ..., e_8\}$, IV-Complex fuzzy sets $\Psi_W(e_1), \Psi_W(e_2), ..., \Psi_W(e_8)$ are defined as,

$$\Psi_W(e_1) = \left\{ \frac{[0.4, 0.5]e^{i[0.5,06]\pi}}{t_1}, \frac{[0.7, 0.8]e^{i[0.5,0.6]\pi}}{t_2}, \frac{[0.6, 0.7]e^{i[0.7,0.8]\pi}}{t_3}, \frac{[0.3, 0.4]e^{i[0.65,0.75]\pi}}{t_4} \right\},$$

$$\Psi_W(e_2) = \left\{ \frac{[0.5, 0.6]e^{i[0.6,0.7]\pi}}{t_1}, \frac{[0.8, 0.9]e^{i[0.8,0.9]\pi}}{t_2}, \frac{[0.6, 0.7]e^{i[0.8,0.9]\pi}}{t_3}, \frac{[0.65, 0.75]e^{i[0.85,0.95]\pi}}{t_4} \right\},$$

$$\Psi_W(e_3) = \left\{ \frac{[0.4, 0.5]e^{i[0.5,0.6]\pi}}{t_1}, \frac{[0.7, 0.8]e^{i[0.8,0.9]\pi}}{t_2}, \frac{[0.5, 0.6]e^{i[0.8,0.9]\pi}}{t_3}, \frac{[0.55, 0.65]e^{i[0.85,0.95]\pi}}{t_4} \right\},$$

$$\Psi_W(e_4) = \left\{ \frac{[0.2, 0.3]e^{i[0.6,0.7]\pi}}{t_1}, \frac{[0.6, 0.7]e^{i[0.8,0.9]\pi}}{t_2}, \frac{[0.4, 0.5]e^{i[0.8,0.9]\pi}}{t_3}, \frac{[0.65, 0.75]e^{i[0.55,0.65]\pi}}{t_4} \right\},$$

$$\Psi_W(e_5) = \left\{ \frac{[0.1, 0.2]e^{i[0.4,0.5]\pi}}{t_1}, \frac{[0.2, 0.3]e^{i[0.7,0.8]\pi}}{t_2}, \frac{[0.7, 0.8]e^{i[0.6,0.7]\pi}}{t_3}, \frac{[0.35, 0.45]e^{i[0.55,0.65]\pi}}{t_4} \right\},$$

$$\Psi_W(e_6) = \left\{ \frac{[0.4, 0.5]e^{i[0.8,0.9]\pi}}{t_1}, \frac{[0.2, 0.3]e^{i[0.8,0.9]\pi}}{t_2}, \frac{[0.6, 0.7]e^{i[0.7,0.8]\pi}}{t_3}, \frac{[0.75, 0.85]e^{i[0.85,0.95]\pi}}{t_4} \right\},$$

$$\Psi_W(e_7) = \left\{ \frac{[0.5, 0.6]e^{i[0.8,0.9]\pi}}{t_1}, \frac{[0.8, 0.9]e^{i[0.5,0.6]\pi}}{t_2}, \frac{[0.4, 0.5]e^{i[0.5,0.6]\pi}}{t_3}, \frac{[0.75, 0.85]e^{i[0.65,0.75]\pi}}{t_4} \right\},$$

and

$$\Psi_W(e_8) = \left\{ \frac{[0.7, 0.8]e^{i[0.8,0.9]\pi}}{t_1}, \frac{[0.7, 0.8]e^{i[0.7,0.8]\pi}}{t_2}, \frac{[0.5, 0.6]e^{i[0.7,0.8]\pi}}{t_3}, \frac{[0.55, 0.65]e^{i[0.75,0.85]\pi}}{t_4} \right\}$$

then IV-CFHS Ω_W is written by,

$$\Omega_W = \left\{ \begin{array}{l} (e_1, \frac{[0.4,0.5]e^{i[0.5,06]\pi}}{t_1}, \frac{[0.7,0.8]e^{i[0.5,0.6]\pi}}{t_2}, \frac{[0.6,0.7]e^{i[0.7,0.8]\pi}}{t_3}, \frac{[0.3,0.4]e^{i[0.65,0.75]\pi}}{t_4}), \\ (e_2, \frac{[0.5,0.6]e^{i[0.6,0.7]\pi}}{t_1}, \frac{[0.8,0.9]e^{i[0.8,0.9]\pi}}{t_2}, \frac{[0.6,0.7]e^{i[0.8,0.9]\pi}}{t_3}, \frac{[0.65,0.75]e^{i[0.85,0.95]\pi}}{t_4}), \\ (e_3, \frac{[0.4,0.5]e^{i[0.5,0.6]\pi}}{t_1}, \frac{[0.7,0.8]e^{i[0.8,0.9]\pi}}{t_2}, \frac{[0.5,0.6]e^{i[0.8,0.9]\pi}}{t_3}, \frac{[0.55,0.65]e^{i[0.85,0.95]\pi}}{t_4}), \\ (e_4, \frac{[0.2,0.3]e^{i[0.6,0.7]\pi}}{t_1}, \frac{[0.6,0.7]e^{i[0.8,0.9]\pi}}{t_2}, \frac{[0.4,0.5]e^{i[0.8,0.9]\pi}}{t_3}, \frac{[0.65,0.75]e^{i[0.55,0.65]\pi}}{t_4}), \\ (e_5, \frac{[0.1,0.2]e^{i[0.4,0.5]\pi}}{t_1}, \frac{[0.2,0.3]e^{i[0.7,0.8]\pi}}{t_2}, \frac{[0.7,0.8]e^{i[0.6,0.7]\pi}}{t_3}, \frac{[0.35,0.45]e^{i[0.55,0.65]\pi}}{t_4}), \\ (e_6, \frac{[0.4,0.5]e^{i[0.8,0.9]\pi}}{t_1}, \frac{[0.2,0.3]e^{i[0.8,0.9]\pi}}{t_2}, \frac{[0.6,0.7]e^{i[0.7,0.8]\pi}}{t_3}, \frac{[0.75,0.85]e^{i[0.85,0.95]\pi}}{t_4}), \\ (e_7, \frac{[0.5,0.6]e^{i[0.8,0.9]\pi}}{t_1}, \frac{[0.8,0.9]e^{i[0.5,0.6]\pi}}{t_2}, \frac{[0.4,0.5]e^{i[0.5,0.6]\pi}}{t_3}, \frac{[0.75,0.85]e^{i[0.65,0.75]\pi}}{t_4}), \\ (e_8, \frac{[0.7,0.8]e^{i[0.8,0.9]\pi}}{t_1}, \frac{[0.7,0.8]e^{i[0.7,0.8]\pi}}{t_2}, \frac{[0.5,0.6]e^{i[0.7,0.8]\pi}}{t_3}, \frac{[0.55,0.65]e^{i[0.75,0.85]\pi}}{t_4}) \end{array} \right\}$$

Definition 4.3. Let $\Omega_{W_1} = (\Psi_1, W_1)$ and $\Omega_{W_2} = (\Psi_2, W_2)$ be two IV-CFHS over the same $\mathbb{U}$. The set $\Omega_{W_1} = (\Psi_1, W_1)$ is said to be the *subset* of $\Omega_{W_2} = (\Psi_2, W_2)$, if

i. $W_1 \subseteq W_2$

ii. $\forall\, \underline{x} \in W_1, \Psi_1(\underline{x}) \subseteq \Psi_2(\underline{x})$ implies $\overleftarrow{\Psi}_1(\underline{x}) \subseteq \overleftarrow{\Psi}_2(\underline{x}), \overrightarrow{\Psi}_1(\underline{x}) \subseteq \overrightarrow{\Psi}_2(\underline{x})$ i.e. $\overleftarrow{r}_{W_1}(\underline{x}) \leq \overleftarrow{r}_{W_2}(\underline{x}), \overrightarrow{r}_{W_1}(\underline{x}) \leq \overrightarrow{r}_{W_2}(\underline{x}), \overleftarrow{\theta}_{W_1}(\underline{x}) \leq \overleftarrow{\theta}_{W_2}(\underline{x})$ and $\overrightarrow{\theta}_{W_1}(\underline{x}) \leq \overrightarrow{\theta}_{W_2}(\underline{x})$, where

$\overleftarrow{r}_{W_1}(\underline{x})$ and $\overleftarrow{\theta}_{W_1}(\underline{x})$ are amplitude and phase terms of $\overleftarrow{\Psi}_1(\underline{x})$,
$\overrightarrow{r}_{W_1}(\underline{x})$ and $\overrightarrow{\theta}_{W_1}(\underline{x})$ are amplitude and phase terms of $\overrightarrow{\Psi}_1(\underline{x})$,
$\overleftarrow{r}_{W_2}(\underline{x})$ and $\overleftarrow{\theta}_{W_2}(\underline{x})$ are amplitude and phase terms of $\overleftarrow{\Psi}_2(\underline{x})$, and
$\overrightarrow{r}_{W_2}(\underline{x})$ and $\overrightarrow{\theta}_{W_2}(\underline{x})$ are amplitude and phase terms of $\overrightarrow{\Psi}_2(\underline{x})$.

Definition 4.4. Two IV-CFHS $\Omega_{W_1} = (\Psi_1, W_1)$ and $\Omega_{W_2} = (\Psi_2, W_2)$ over the same $\mathbb{U}$, are said to be *equal* if

i. $(\Psi_1, W_1) \subseteq (\Psi_2, W_2)$

ii. $(\Psi_2, W_2) \subseteq (\Psi_1, W_1)$.

Definition 4.5. Let (Ψ, W) be a IV-CFHS over $\mathbb{U}$.Then

i. (Ψ, W) is called a *null IV-CFHS*, denoted by $(\Psi, W)_\Phi$ if for all $\underline{x} \in W$, the amplitude and phase terms of the membership function are given by $\overleftarrow{r}_W(\underline{x}) = \overrightarrow{r}_W(\underline{x}) = 0$ and $\overleftarrow{\theta}_W(\underline{x}) = \overrightarrow{\theta}_W(\underline{x}) = 0\pi$ respectively.

ii. (Ψ, W) is called a *absolute IV-CFHS*, denoted by $(\Psi, W)_\Delta$ if for all $\underline{x} \in W$, the amplitude and phase terms of the membership function are given by $\overleftarrow{r}_W(\underline{x}) = \overrightarrow{r}_W(\underline{x}) = 1$ and $\overleftarrow{\theta}_W(\underline{x}) = \overrightarrow{\theta}_W(\underline{x}) = 2\pi$ respectively.

Definition 4.6. Let (Ψ_1, W_1) and (Ψ_2, W_2) are two CFH-sets over the same universe $\mathbb{U}$.Then

i. A IV-CFHS (Ψ_1, W_1) is called a *homogeneous IV-CFHS*, denoted by $(\Psi_1, W_1)_{Hom}$ if and only if $\Psi_1(\underline{x})$ is a homogeneous CF-set for all $\underline{x} \in W_1$.

ii. A IV-CFHS (Ψ_1, W_1) is called a *completely homogeneous IV-CFHS*, denoted by $(\Psi_1, W_1)_{CHom}$ if and only if $\Psi_1(\underline{x})$ is a homogeneous with $\Psi_1(\underline{y})$ for all $\underline{x}, \underline{y} \in W_1$.

iii. A IV-CFHS (Ψ_1, W_1) is said to be a completely homogeneous IV-CFHS with (Ψ_2, W_2) if and only if $\Psi_1(\underline{x})$ is a homogeneous with $\Psi_2(\underline{x})$ for all $\underline{x} \in W_1 \prod W_2$.

4.1. *Set Theoretic Operations and Laws on IV-CFHS*

Here some basic set theoretic operations (i.e.complement, union and intersection) and laws (commutative laws, associative laws etc.) are discussed on IV-CFHS.

Definition 4.7. The *complement* of IV-CFHS (Ψ, W), denoted by $(\Psi, W)^c$ is defined as

$$(\Psi, W)^c = \{(\underline{x}, (\Psi(\underline{x}))^c) : \underline{x} \in W, (\Psi(\underline{x}))^c \in C_{IV}(\mathbb{U})\}$$

such that the amplitude and phase terms of the membership function $(\Psi(\underline{x}))^c$ are given by
$(\overleftarrow{r}_W(\underline{x}))^c = 1 - \overleftarrow{r}_W(\underline{x})$
$(\overrightarrow{r}_W(\underline{x}))^c = 1 - \overrightarrow{r}_W(\underline{x})$
and
$(\overleftarrow{\theta}_W(\underline{x}))^c = 2\pi - \overleftarrow{\theta}_W(\underline{x})$,
$(\overrightarrow{\theta}_W(\underline{x}))^c = 2\pi - \overrightarrow{\theta}_W(\underline{x})$ respectively.

Proposition 4.8. *Let (Ψ, W) be a IV-CFHS over $\mathbb{U}$. Then $((\Psi, W)^c)^c = (\Psi, W)$.*

Proof. Since $\Psi(\underline{x}) \in C_{IV}(\mathbb{U})$, therefore (Ψ, W) can be written in terms of its amplitude and phase terms as

$$(\Psi, W) = \left\{\left(\underline{x}, \left(\overleftarrow{r}_W(\underline{x})e^{i\overleftarrow{\theta}_W(\underline{x})}, \overrightarrow{r}_W(\underline{x})e^{i\overrightarrow{\theta}_G(\underline{x})}\right)\right) : \underline{x} \in W\right\} \tag{3}$$

Now

$$(\Psi, W)^c(\underline{x}) = \left\{\left(\underline{x}, \left((\overleftarrow{r}_W(\underline{x}))^c e^{i(\overleftarrow{\theta}_W(\underline{x}))^c}, (\overrightarrow{r}_W(\underline{x}))^c e^{i(\overrightarrow{\theta}_G(\underline{x}))^c}\right)\right) : \underline{x} \in W\right\}$$

$$(\Psi, W)^c(\underline{x}) = \left\{\left(\underline{x}, \left((1 - \overleftarrow{r}_W(\underline{x}))e^{i(2\pi - \overleftarrow{\theta}_W(\underline{x}))}, (1 - \overrightarrow{r}_W(\underline{x}))e^{i(2\pi - \overrightarrow{\theta}_G(\underline{x}))}\right)\right) : \underline{x} \in W\right\}$$

$$((\psi, G)^c)^c = \left\{\left(\underline{x}, \left((1 - \overleftarrow{r}_W(\underline{x}))^c e^{i(2\pi - \overleftarrow{\theta}_W(\underline{x}))^c}, (1 - \overrightarrow{r}_W(\underline{x}))^c e^{i(2\pi - \overrightarrow{\theta}_G(\underline{x}))^c}\right)\right) : \underline{x} \in W\right\}$$

$$((\Psi, W)^c)^c = \left\{\left(\underline{x}, \left((1 - (1 - \overleftarrow{r}_W(\underline{x})))e^{i(2\pi - (2\pi - \overleftarrow{\theta}_W(\underline{x})))}, (1 - (1 - \overrightarrow{r}_W(\underline{x})))e^{i(2\pi - (2\pi - \overrightarrow{\theta}_G(\underline{x})))}\right)\right) : \underline{x} \in W\right\}$$

$$((\Psi, W)^c)^c = \left\{\left(\underline{x}, \left(\overleftarrow{r}_W(\underline{x})e^{i\overleftarrow{\theta}_W(\underline{x})}, \overrightarrow{r}_W(\underline{x})e^{i\overrightarrow{\theta}_G(\underline{x})}\right)\right) : \underline{x} \in W\right\} \tag{4}$$

from equations (3) and (4), we have $((\Psi, W)^c)^c = (\Psi, W)$. □

Proposition 4.9. *Let (Ψ, W) be a IV-CFHS over $\mathbb{U}$. Then*

i. $((\Psi, W)_\Phi)^c = (\Psi, W)_\Delta$
ii. $((\Psi, W)_\Delta)^c = (\Psi, W)_\Phi$

Definition 4.10. The *intersection* of two IV-CFHS (Ψ_1, W_1) and (Ψ_2, W_2) over the same universe $\mathbb{U}$, denoted by $(\Psi_1, W_1)\prod(\Psi_2, W_2)$, is the IV-CFHS (Ψ_3, W_3), where $W_3 = W_1 \coprod W_2$, and for all $\underline{x} \in W_3$,

$$\overleftarrow{\Psi}_3(\underline{x}) = \begin{cases} \overleftarrow{r}_{W_1}(\underline{x})e^{i\overleftarrow{\theta}_{W_1}(\underline{x})} & , if\ \underline{x} \in W_1 \backslash W_2 \\ \overleftarrow{r}_{W_2}(\underline{x})e^{i\overleftarrow{\theta}_{W_2}(\underline{x})} & , if\ \underline{x} \in W_2 \backslash W_1 \\ min[\overleftarrow{r}_{W_1}(\underline{x}), \overleftarrow{r}_{W_2}(\underline{x})]e^{i\,min[\overleftarrow{\theta}_{W_1}(\underline{x}), \overleftarrow{\theta}_{W_2}(\underline{x})]} & , if\ \underline{x} \in W_1 \prod W_2 \end{cases}$$

and

$$\overrightarrow{\Psi}_3(\underline{x}) = \begin{cases} \overrightarrow{r}_{W_1}(\underline{x})e^{i\overrightarrow{\theta}_{W_1}(\underline{x})} & , if\ \underline{x} \in W_1 \backslash W_2 \\ \overrightarrow{r}_{W_2}(\underline{x})e^{i\overrightarrow{\theta}_{W_2}(\underline{x})} & , if\ \underline{x} \in W_2 \backslash W_1 \\ min[\overrightarrow{r}_{W_1}(\underline{x}), \overrightarrow{r}_{W_2}(\underline{x})]e^{i\,min[\overrightarrow{\theta}_{W_1}(\underline{x}), \overrightarrow{\theta}_{W_2}(\underline{x})]} & , if\ \underline{x} \in W_1 \prod W_2 \end{cases}$$

Definition 4.11. The *difference* between two IV-CFHS (Ψ_1, W_1) and (Ψ_2, W_2) is defined as

$$(\Psi_1, W_1) \setminus (\Psi_2, W_2) = (\Psi_1, W_1) \prod (\Psi_2, W_2)^c$$

Definition 4.12. The *union* of two IV-CFHS (Ψ_1, W_1) and (Ψ_2, W_2) over the same universe $\mathbb{U}$, denoted by $(\Psi_1, W_1)\coprod(\Psi_2, W_2)$, is the IV-CFHS (Ψ_3, W_3), where $W_3 = W_1 \coprod W_2$, and for all $\underline{x} \in W_3$,

$$\overleftarrow{\Psi}_3(\underline{x}) = \begin{cases} \overleftarrow{r}_{W_1}(\underline{x})e^{i\overleftarrow{\theta}_{W_1}(\underline{x})} & , if\ \underline{x} \in W_1 \backslash W_2 \\ \overleftarrow{r}_{W_2}(\underline{x})e^{i\overleftarrow{\theta}_{W_2}(\underline{x})} & , if\ \underline{x} \in W_2 \backslash W_1 \\ max[\overleftarrow{r}_{W_1}(\underline{x}), \overleftarrow{r}_{W_2}(\underline{x})]e^{i\,max[\overleftarrow{\theta}_{W_1}(\underline{x}), \overleftarrow{\theta}_{W_2}(\underline{x})]} & , if\ \underline{x} \in W_1 \prod W_2 \end{cases}$$

and

$$\overrightarrow{\Psi}_3(\underline{x}) = \begin{cases} \overrightarrow{r}_{W_1}(\underline{x})e^{i\overrightarrow{\theta}_{W_1}(\underline{x})} & , if\ \underline{x} \in W_1 \backslash W_2 \\ \overrightarrow{r}_{W_2}(\underline{x})e^{i\overrightarrow{\theta}_{W_2}(\underline{x})} & , if\ \underline{x} \in W_2 \backslash W_1 \\ max[\overrightarrow{r}_{W_1}(\underline{x}), \overrightarrow{r}_{W_2}(\underline{x})]e^{i\,max[\overrightarrow{\theta}_{W_1}(\underline{x}), \overrightarrow{\theta}_{W_2}(\underline{x})]} & , if\ \underline{x} \in W_1 \prod W_2 \end{cases}$$

Proposition 4.13. *Let (Ψ, W) be a IV-CFHS over $\mathbb{U}$. Then the following results hold true:*

i. $(\Psi, W)\coprod(\Psi, W)_\Phi = (\Psi, W)$
ii. $(\Psi, W)\coprod(\Psi, W)_\Delta = (\Psi, W)_\Delta$
iii. $(\Psi, W)\prod(\Psi, W)_\Phi = (\Psi, W)_\Phi$
iv. $(\Psi, W)\prod(\Psi, W)_\Delta = (\Psi, W)$
v. $(\Psi, W)_\Phi\coprod(\Psi, W)_\Delta = (\Psi, W)_\Delta$
vi. $(\Psi, W)_\Phi\prod(\Psi, W)_\Delta = (\Psi, W)_\Phi$

Proposition 4.14. *Let (Ψ_1, W_1), (Ψ_2, W_2) and (Ψ_3, W_3) are three CFH-sets over the same universe $\mathbb{U}$. Then the following commutative and associative laws hold true:*

i. $(\Psi_1, W_1)\prod(\Psi_2, W_2) = (\Psi_2, W_2)\prod(\Psi_1, W_1)$
ii. $(\Psi_1, W_1)\coprod(\Psi_2, W_2) = (\Psi_2, W_2)\coprod(\Psi_1, W_1)$
iii. $(\Psi_1, W_1)\prod((\Psi_2, W_2)\prod(\Psi_3, W_3)) = ((\Psi_1, W_1)\prod(\Psi_2, W_2))\prod(\Psi_3, W_3)$

iv. $(\Psi_1, W_1) \coprod ((\Psi_2, W_2) \coprod (\Psi_3, W_3)) = ((\Psi_1, W_1) \coprod (\Psi_2, W_2)) \coprod (\Psi_3, W_3)$

Proposition 4.15. *Let* (Ψ_1, W_1) *and* (Ψ_2, W_2) *are two CFH-sets over the same universe* $\mathbb{U}$. *Then the following De Morganss laws hold true:*

i. $((\Psi_1, W_1) \prod (\Psi_2, W_2))^c = (\Psi_1, W_1)^c \coprod (\Psi_2, W_2)^c$
ii. $(\Psi_1, W_1) \coprod (\Psi_2, W_2))^c = (\Psi_1, W_1)^c \prod (\Psi_2, W_2)^c$

Conclusion

In this work, the complex fuzzy hypersoft sets (CFH-sets) are developed along with some fundamentals, theoretic set operations and aggregations. Also a method is proposed to solve decision making problems and demonstrated with a commerce-based application. Moreover, the rudiments of interval-valued fuzzy hypersoft set (IV-CFHS) are characterized with suitable examples. CFH-sets and IV-CFHS generalize the existing structures of complex fuzzy soft set, permit a broad range of values for membership function by expanding them to the unit circle in a complex plane, consider the periodic nature of the information through the phase-terms and classify distinct attributes into corresponding attribute-values sets for vivid understanding. Further work may include:

(i) the extension of proposed work to the development of:
   - complex intuitionistic fuzzy hypersoft set,
   - complex neutrosophic hypersoft set,
   - interval-valued complex intuitionistic fuzzy hypersoft set,
   - interval-valued complex neutrosophic hypersoft set,

(ii) the application of proposed work in multi-criteria decision-making,
(iii) the determination of similarity measures and entropies for proposed structures,
(iv) the parameterization of proposed structures with fuzzy, intuitionistic fuzzy and neutrosophic settings,
(v) the characterization of proposed structures under multi-decisive environment,
(vi) the introduction of refinement in the proposed structures for sub-membership grades.

References

[1] Ramot, D.; Milo, R.; Friedman, M.; Kandel, A. Complex fuzzy sets. *IEEE Transactions on Fuzzy Systems* **2002**, *10*(2), 171-186. http://doi.org/10.1109/91.995119

[2] Zadeh, L. Fuzzy sets. *Information and control* **1965**, *8*(3), 338-353. http://doi.org/10.1016/S0019-9958(65)90241-X

[3] Buckley. J.J., Fuzzy complex numbers. *Fuzzy Sets and Systems* **1989**, *33*(3), 333-345. https://doi.org/10.1016/0165-0114(89)90122-X

[4] Zhang, G.; Dillon, T. S.; Cai, K. Y.; Ma, J.; Lu, J. Operation properties and δ-equalities of complex fuzzy sets. *International journal of approximate reasoning* **2009**, *50*(8), 1227-1249. https://doi.org/10.1016/j.ijar.2009.05.010

[5] Ramot, D.; Milo, R.; Friedman, M.; Kandel, A. Complex fuzzy sets. *IEEE Transactions on Fuzzy Systems* **2002**, *10*(2), 171-186. https://doi.org/10.1109/91.995119

[6] Ramot, D.; Friedman, M.; Langholz, G.; Kandel, A. Complex fuzzy logic. *IEEE Transactions on Fuzzy Systems* **2003**, *11*(4), 450-461. https://doi.org/10.1109/TFUZZ.2003.814832

[7] Molodtsov, D. Soft Set Theory - First Results. *Computers and Mathematics with Applications* **1999**, *37*, 19-31. http://doi.org/10.1016/S0898-1221(99)00056-5

[8] Maji. P.K.; Biswas R.; Roy, A. R. Soft Set Theory. *Computers and Mathematics with Applications* **2003**, *45*, 555-562. http://doi.org/10.1016/S0898-1221(03)00016-6

[9] Maji, P. K.; Biswas, R.; Roy, A. R. Fuzzy soft sets. *Journal of Fuzzy Mathematics* **2001**, *9*(3), 589-602.

[10] Pei, D.; Miao, D. From soft set to information system. *In international conference of granular computing IEEE* **2005**, *2* , 617-621. http://doi.org/10.1109/GRC.2005.1547365

[11] Ali, M. I.; Feng, F.; Liu, X.; Min, W. K.; Sabir, M. On some new operations in soft set theory. *Computers and Mathematics with Applications 2009*, *57* , 1547-1553. http://doi.org/10.1016/j.camwa.2008.11.009

[12] Babitha, K. V.; Sunil, J. J. Soft set relations and functions. *Computers and Mathematics with Applications* **2010**, *60*, 1840-1849. http://doi.org/10.1016/j.camwa.2010.07.014

[13] Babitha, K. V.; Sunil, J. J. Transitive closure and ordering in soft set. *Computers and Mathematics with Applications* **2010**, *61* 2235-2239. http://doi.org/10.1016/j.camwa.2011.07.010

[14] Sezgin, A.; Atagün A. O. On operations of soft sets. *Computers and Mathematics with Applications* **2011**, *61*(5), 1457-1467. http://doi.org/10.1016/j.camwa.2011.01.018

[15] Ge, X.; Yang, S. Investigations on some operations of soft sets. *World Academy of Science Engineering and Technology* **2011**, *75*, 1113-1116.

[16] Li, F. Notes on soft set operations. *ARPN Journal of systems and softwares* **2011**, 205-208.

[17] Maji, P. K.; Biswas, R.; Roy, A. R. Intuitionistic fuzzy soft sets. *The Journal of Fuzzy Mathematics* **2001**, *9*(3), 677-692.

[18] Maji, P. K. Neutrosophic Soft Set. *Annals of Fuzzy Mathematics and Informatics* **2013**, *5*(1), 2287-623.

[19] Smarandache, F. Extension of Soft Set of Hypersoft Set, and then to Plithogenic Hypersoft Set. *Neutrosophic Sets and Systems* **2018**, *22* , 168-170. http://doi.org/10.5281/zenodo.2838716

[20] Saeed, M.; Ahsan, M.; Siddique, M.k.; Ahmad, M.R. A Study of the Fundamentals of Hypersoft Set Theory. *International Journal of Scientific and Engineering Research* **2020**, *11*(1), 320-329.

[21] Saeed, M.; Rahman, A. U.; Ahsan, M.; Smarandache, F. An Inclusive Study on Fundamentals of Hypersoft Set, In Theory and Application of Hypersoft Set, Pons Publishing House, Brussel, 2021, 1-23.

[22] Abbas, F.; Murtaza, G.; Smarandache, F. (2020). Basic operations on hypersoft sets and hypersoft points. *Neutrosophic Sets and Systems* **2020**, *35*, 407-421. http://doi.org/10.5281/zenodo.3951694

[23] Saqlain, M.; Jafar, N.; Moin, S.; Saeed, M.; Broumi, S. Single and Multi-valued Neutrosophic Hypersoft Set and Tangent Similarity Measure of Single valued Neutrosophic Hypersoft Sets. *Neutrosophic Sets and Systems* **2020**, *32*, 317-329. http://doi.org/10.5281/zenodo.3723165

[24] Saqlain, M.; Moin, S.; Jafar, N.; Saeed, M.; Smarandache, F. Aggregate Operators of Neutrosophic Hypersoft Sets. *Neutrosophic Sets and Systems* **2020**, 32, 294-306. http://doi.org/10.5281/zenodo.3723155

[25] Saqlain, M.; Saeed, M.; Ahmad, M.R.; Smarandache, F. Generalization of TOPSIS for Neutrosophic Hypersoft Sets using Accuracy Function and its Application. *Neutrosophic Sets and Systems* **2020**, *27*, 131-137. http://doi.org/10.5281/zenodo.3275533

[26] Martin, N.; Smarandache, F. Concentric Plithogenic Hypergraph based on Plithogenic Hypersoft Sets A Novel Outlook. *Neutrosophic Sets and Systems* **2020**, *33*, 78-91. http://doi.org/10.5281/zenodo.3782824

[27] Rahman, A. U.; Saeed, M.; Alodhaibi, S.S.; Abd, H. Decision Making Algorithmic Approaches Based on Parameterization of Neutrosophic Set under Hypersoft Set Environment with Fuzzy, Intuitionistic Fuzzy

and Neutrosophic Settings. *CMES-Computer Modeling in Engineering & Sciences* **2021**, *128*(2), 743-777. http://doi.org/10.32604/cmes.2021.016736

[28] Rahman, A. U.; Saeed, M.; Smarandache, F.; Ahmad, M. R. Development of Hybrids of Hypersoft Set with Complex Fuzzy Set, Complex Intuitionistic Fuzzy set and Complex Neutrosophic Set. *Neutrosophic Sets and Systems* **2020**, 38, 335-354. http://doi.org/10.5281/zenodo.4300520

[29] Rahman, A. U.; Saeed, M.; Smarandache, F. Convex and Concave Hypersoft Sets with Some Properties. *Neutrosophic Sets and Systems* **2020**, 38, 497-508. http://doi.org/10.5281/zenodo.4300580

[30] Deli, I. Hybrid set structures under uncertainly parameterized hypersoft sets, Theory and applications. In Theory and Application of Hypersoft Set, Pons Publishing House, Brussel, 2021, 24-49.

[31] Gayen, S.; Smarandache, F.; Jha, S.; Singh, M. K.; Broumi, S.; Kumar, R. Introduction to Plithogenic Hypersoft Subgroup. *Neutrosophic Sets and Systems 2020*, *33*, 14. http://doi.org/10.5281/zenodo.3782897

[32] Saeed, M.; Ahsan, M.; Saeed, M. H.; Mehmood, A.; Abdeljawad, T. An Application of Neutrosophic Hypersoft Mapping to Diagnose Hepatitis and Propose Appropriate Treatment. *IEEE Access* **2021**, *9*, 70455-70471. http://doi.org/10.1109/ACCESS.2021.3077867

[33] Saeed, M.; Ahsan, M.; Abdeljawad, T. A Development of Complex Multi-Fuzzy Hypersoft Set With Application in MCDM Based on Entropy and Similarity Measure. *IEEE Access* **2021**, *9*, 60026-60042. 10.1109/ACCESS.2021.3073206

[34] Rahman, A.U.; Saeed, M.; Dhital, A. Decision Making Application Based on Neutrosophic Parameterized Hypersoft Set Theory. *Neutrosophic Sets and Systems* **2021**, *41*, 1-14. http://doi.org/10.5281/zenodo.4625665

[35] Rahman, A.U.; Saeed, M.; Zahid, S. Application in Decision Making Based on Fuzzy Parameterized Hypersoft Set Theory. *Asia Mathematika 2021*, *5*(1), 19-27. http://doi.org/10.5281/zenodo.4721481

[36] Rahman, A.U.; Hafeez, A.; Saeed, M.; Ahmad, M.R.; Farwa, U. Development of Rough Hypersoft Set with Application in Decision Making for the Best Choice of Chemical Material, In Theory and Application of Hypersoft Set, Pons Publication House, Brussel, 2021, 192-202. http://doi.org/10.5281/zenodo.4743367

[37] Ihsan,M.; Rahman, A.U.; Saeed, M. Hypersoft Expert Set With Application in Decision Making for Recruitment Process. *Neutrosophic Sets and Systems* **2021**, *42*, 191-207. http://doi.org/10.5281/zenodo.4711524

[38] Thirunavukarasu, P,; Suresh, R.; Ashokkumar, V. Theory of Complex Fuzzy Soft Set and its Applications. *International Journal for Innovative Research in Science and Technology* **2013**, *3*, 13-18.

[39] Kumar, T.; Bajaj, R. K. On Complex Intuitionistic Fuzzy Soft Sets with Distance Measures and Entropies. *Journal of Mathematics* **2014**, *2014*(3), 1-12. http://doi.org/10.1155/2014/972198

[40] Selvachandran, G.; Singh, P. K. Interval-valued complex fuzzy soft set and its application. *International Journal for Uncertainty Quantification* **2018**, *8*(2), 101-117. http://doi.org/10.1615/Int.J.UncertaintyQuantification.2018020362

[41] Abd Ulzeez M. J. S. Alkouri. Complex Generalised Fuzzy Soft Sets and its Application. *WSEAS Transactions on Mathematics* **2020**, *19*, 323-333. http://doi.org/10.37394/23206.2020.19.32

Received: May 5, 2021. Accepted: October 5, 2021

NSS

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

Fuzzy Hypersoft Expert Set with Application in Decision Making for the Best Selection of Product

Muhammad Ihsan[1,*], Atiqe Ur Rahman[1], Muhammad Saeed[2]

[1,1,*] University of Management and Technology, Lahore, Pakistan. 1;mihkhb@gmail.com, aurkhb@gmail.com

[2] University of Management and Technology, Lahore, Pakistan. 2; muhammad.saeed@umt.edu.pk

Correspondence: mihkhb@gmail.com

*

Abstract. Numerous researchers have made a few models dependent on soft set, to tackle issues in decision making and clinical analysis, yet a large portion of these models manage one expert. This causes an issue with the clients, particularly with the individuals who use polls in their work and studies. Accordingly we present another model i.e. fuzzy hypersoft expert set which not just addresses this constraint of fuzzy soft-like models by accentuating the assessment, all things considered, yet additionally settle the deficiency of soft set for disjoint attribute-valued sets comparing to distinct attributes. In this study, the existing concept of fuzzy soft expert set is generalized to fuzzy hypersoft expert set which is more flexible and useful. Some fundamental properties (i.e. subset, not set and equal set), results (i.e. commutative, associative, distributive and D Morgan Laws) and set-theoretic operations (i.e. complement, union, intersection AND, and OR) are discussed. An algorithm is proposed to solve decision-making problems and is applied to select the best product.

Keywords: Soft Set; Fuzzy Soft Set; Fuzzy Soft Expert Set; Hypersoft Set; Fuzzy Hypersoft Expert Set.

---

1. Introduction

Zadeh [1] initiated fuzzy set theory as a basic model to tackle uncertainties in the data. Molodtsov [2] presented soft set theory that is supposed to be a new parameterized class of subsets of the universe of discourse, which addresses the inadequacy of fuzzy set-like structures for parameterization tools. It has helped the researcher (experts) to resolve efficiently the decision-making problems involving vagueness and uncertainty. The researchers [3–15] studied and broadened the concept of soft set and applied to different fields. The gluing concept of soft set with expert system initiated by Alkhazaleh et al. [16] to emphasize the due status of the opinions of all experts regarding taking any decision in decision-making system. Al-Quran et al. [17] proposed neutrosophic vague soft expert set theory, Alkhazaleh et al. [18] characterized fuzzy soft expert set. and its application. Bashir et al. [19, 20] presented possibility fuzzy soft expert set and fuzzy parameterized soft expert set. Sahin et al. [21] investigated neutrosophic soft expert sets. Alhazaymeh et al. [22,23] studied mapping on generalized vague soft expert set and generalized vague soft expert set. Alhazaymeh et al. [24] explained the application of generalized vague soft expert set in decision making. Hassan et al. [25] reviewed Q-neutrosophic soft expert set and its application in decision making. Ulucay et al. [26] studied generalized neutrosophic soft expert set for multiple-criteria decision-making. Al-Qudah et al. [27] explained bipolar fuzzy soft expert set and its application in decision making. Al-Qudah et al. [28] investigated complex multi-fuzzy soft expert set and its application. Al-Quran et al. [29] presented the com-plex neutrosophic soft expert set and its application in decision making. Pramanik et al. [30] studied the topsis for single valued neutrosophic soft expert set based multi-attribute decision making problems. Abu Qamar et al. [31] investigated the generalized Q-neutrosophic soft ex-pert set for decision under uncertainty. Adam et al. [32] characterized the multi Q-fuzzy soft expert set and its application. Ulucay et al. [33] presented the time-neutrosophic soft expert sets and its decision making problem. Al-Quran et al. [34] studied fuzzy parameterised single valued neutrosophic soft expert set theory and its application in decision making. Hazaymeh et al. [35] researched generalized fuzzy soft expert set.

There are many real life scenarios when we are to deal with disjoint attribute-valued set for distinct attributes. In 2018, Smarandache [36] addressed this inadequacy of soft with the development of new structure hypersoft set by replacing single attribute-valued function to multi-attribute valued function. In 2020, Saeed et al. [37, 38] extended the concept and discussed the fundamentals of hypersoft set such as hypersoft subset, complement, not hypersoft set, aggregation operators along with hypersoft set relation, sub relation,complement relation, function, matrices and operations on hypersoft matrices. In the same year, Mujahid et al. [39] discussed hypersoft points in different fuzzy-like environments. In 2020, Rahman et al. [40] defined complex hypersoft set and developed the hybrids of hypersoft set with complex fuzzy set, complex intuitionistic fuzzy set and complex neutrosophic set respectively. They also discussed their fundamentals i.e. subset, equal sets, null set, absolute set etc. and theoretic operations i.e. complement, union, intersection etc. In 2020, Rahman et al. [41] conceptualized convexity cum concavity on hypersoft set and presented its pictorial versions with illustrative examples. Recently the researchers [42–49] investigated on the theory of hypersoft set and developed certain its hybrids with discussion and applications in decision making.

Dealing with disjoint attribute-valued sets is of great importance and it is vital for sensible decisions in decision-making techniques. Results will be varied and be considered inclined and odd on ignoring such kind of sets. Therefore, it is the need of the literature to adequate the exiting literature of soft and expert set for multi-attribute function. Having motivation from [10]- [19], new notions of fuzzy hypersoft expert set are developed and an application is discussed in decision making through a proposed method. The pattern of rest of the paper is: section 2 reviews the notions of soft sets, fuzzy soft set, fuzzy soft expert set, hypersoft set and relevant definitions used in the proposed work. Section 3, presents notions of fuzzy hypersoft expert set with properties. Section 4, demonstrates an application of this concept in a decision-making problem. Section 5, concludes the paper.

1.1. *Motivation*

The novelty of fuzzy hypersoft expert set (FHSE-set) is as:

- It is the extension of soft set, fuzzy soft set, soft expert set and fuzzy soft expert set,
- It tackles all the hindrances of soft set, fuzzy soft set, soft expert set and fuzzy soft expert set for dealing with further partitions of attributes into attribute-valued sets,
- It facilitates the decision-makers to have decisions for uncertain scenarios without encountering with any inclined situation.

2. **Preliminaries**

In this section, some basic definitions and terms regarding the main study, are presented from the literature.

Definition 2.1. [1]
Let $P(\Omega)$ denote power set of Ω(universe of discourse) and F be a collection of parameters defining Ω. A *soft set* M is defined by mapping

$$\Psi : F \to P(\Omega)$$

Definition 2.2. [3] Suppose Ω be a set of universe, while F is a set of parameters. Here I^{Ω} represents the power set of all fuzzy subsets of Ω. Let $C \subseteq F$. A pair (R, F) is called a fuzzy soft set with R is a mapping given by

$$R : C \to I^{\Omega}$$

Definition 2.3. [4]
The union of two soft sets (Ψ_1, A_1) and (Ψ_2, A_2) over Ω is the soft set (Ψ_3, A_3) ; $A_3 \doteq A_1 \cup A_2$, and $\forall\ \xi \in A_3$,

$$\Psi_3(\xi) = \begin{cases} \Psi_1(\xi) & ; \xi \in A_1 - A_2 \\ \Psi_2(\xi) & ; \xi \in A_2 - A_1 \\ \Psi_1(\xi) \cup\ \Psi_2(\xi) & ; \xi \in A_1 \cap A_2 \end{cases}$$

Definition 2.4. [15]
The extended intersection of two soft sets (Ψ_1, A_1) and (Ψ_2, A_2) with Ω is the soft set (Ψ_3, A_3) while $A_3 \doteq A_1 \cup A_2,\ ;\ \xi \in A_3,$

$$\Psi_3(\xi) = \begin{cases} \Psi_1(\xi) & ; \xi \in A_1 - A_2 \\ \Psi_2(\xi) & ; \xi \in A_2 - A_1 \\ \Psi_1(\xi) \cup \Psi_2(\xi) & ; \xi \in A_1 \cap A_2 \end{cases}$$

Definition 2.5. [16]
Assume that Y be a set of specialists (operators) and $\ddot{O}$ be a set of conclusions, $T = F \times Y \times \ddot{O}$ with $S \subseteq T$ where Ω denotes the universe , F a set of parameters. A pair(Φ, S) is known as a *soft expert set* over Ω, where H is a mapping given by

$$\Phi : S \to P(\Omega)$$

Definition 2.6. [16]
A $(\Phi_1, S) \subseteq (\Phi_2, P)$ over Ω, if
(i) $S \subseteq P$,
(ii) $\forall\ \alpha \in S, \Phi_1(\alpha) \subseteq \Phi_2(\alpha)$.
While (Φ_2, P) is known as a *soft expert superset* of (Φ_1, S).

Definition 2.7. [18] A pair (H, C) is called a fuzzy soft expert set over Ω where F is a mapping given by

$$H : C \to I^{\Omega}$$

where I^{Ω} the set of all fuzzy subsets of Ω.

Definition 2.8. [36]
Let $h_1, h_2, h_3, ..., h_m$, for $m \geq 1$, be m distinct attributes, whose corresponding attribute values are respectively the sets $H_1, H_2, H_3, ..., H_m$, with $H_i \cap H_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., m\}$. Then the pair (Ψ, G), where $G = H_1 \times H_2 \times H_3 \times ..... \times H_m$ and $\Psi : G \to P(\Omega)$ is called a *hypersoft Set* over Ω.

Definition 2.9. [38]
Let $\Gamma_1, \Gamma_2, \Gamma_3, ...\Gamma_m$ be disjoint attribute-valued sets for m distinct attributes . A pair (Φ, Γ) is called fuzzy hypersoft set over Ω with Γ is the cartesian product of $\Gamma_i, i = 1, 2, ..., m$, and $\Phi : \Gamma \to P(\Omega)$. In general, $\Phi(\alpha) = \{(x, \Phi(\alpha))(x)/x \in \Omega\}; \alpha \in \Gamma$. Here $P(\Omega)$ is the collection of all fuzzy sets.

3. Fuzzy Hypersoft Expert set (FHSE-Set)

Definition 3.1. Fuzzy Hypersoft Expert set (FHSE-Set)
A pair$(\xi, \mathbb{S})$ is known as a *fuzzy hypersoft expert set* over $\coprod$, where

$$\xi : \mathbb{S} \to I^{\coprod}$$

where

- $I^{\coprod}$ is collection of all fuzzy subsets of $\coprod$
- $\mathbb{S} \subseteq \mathcal{H} = \mathcal{G} \times \mathcal{D} \times \mathbb{C}$
- $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \times .... \times \mathcal{G}_p$ where $\mathcal{G}_i$ are disjoint attributive-valued sets corresponding to distinct attributes $g_i, i = 1, 2, 3, ..., p$
- $\mathcal{D}$ be a set of specialists (operators)
- $\mathbb{C}$ be a set of conclusions.

For simplicity, $\mathbb{C} = \{0 = disagree, 1 = agree\}$.

Example 3.2. Suppose that a multi-national company aims to proceed the valuation of certain specialists about its certain products. Let $\coprod = \{m_1, m_2, m_3, m_4\}$ be a set of products and
$\mathcal{G}_1 = \{q_{11}, q_{12}\}$
$\mathcal{G}_2 = \{q_{21}, q_{22}\}$
$\mathcal{G}_3 = \{q_{31}, q_{32}\}$
be disjoint attributive sets for distinct attributes q_1= simple to utilize, q_2= nature, q_3= modest.
Now
$\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3$

$$\mathcal{G} = \left\{ \begin{array}{l} \mu_1 = (q_{11}, q_{21}, q_{31}), \mu_2 = (q_{11}, q_{21}, q_{32}), \mu_3 = (q_{11}, q_{22}, q_{31}), \mu_4 = (q_{11}, q_{22}, q_{32}), \\ \mu_5 = (q_{12}, q_{21}, q_{31}), \mu_6 = (q_{12}, q_{21}, q_{32}), \mu_7 = (q_{12}, q_{22}, q_{31}), \mu_8 = (q_{12}, q_{22}, q_{32}) \end{array} \right\}$$

Now $\mathcal{H} = \mathcal{G} \times \mathcal{D} \times \mathbb{C}$

$$\mathcal{H} = \left\{ \begin{array}{l} (\mu_1, s, 0), (\mu_1, s, 1), (\mu_1, t, 0), (\mu_1, t, 1), (\mu_1, u, 0), (\mu_1, u, 1), \\ (\mu_2, s, 0), (\mu_2, s, 1), (\mu_2, t, 0), (\mu_2, t, 1), (\mu_2, u, 0), (\mu_2, u, 1), \\ (\mu_3, s, 0), (\mu_3, s, 1), (\mu_3, t, 0), (\mu_3, t, 1), (\mu_3, u, 0), (\mu_3, u, 1), \\ (\mu_4, s, 0), (\mu_4, s, 1), (\mu_4, t, 0), (\mu_4, t, 1), (\mu_4, u, 0), (\mu_4, u, 1), \\ (\mu_5, s, 0), (\mu_5, s, 1), (\mu_5, t, 0), (\mu_5, t, 1), (\mu_5, u, 0), (\mu_5, u, 1), \\ (\mu_6, s, 0), (\mu_6, s, 1), (\mu_6, t, 0), (\mu_6, t, 1), (\mu_6, u, 0), (\mu_6, u, 1), \\ (\mu_7, s, 0), (\mu_7, s, 1), (\mu_7, t, 0), (\mu_7, t, 1), (\mu_7, u, 0), (\mu_7, u, 1), \\ (\mu_8, s, 0), (\mu_8, s, 1), (\mu_8, t, 0), (\mu_8, t, 1), (\mu_8, u, 0), (\mu_8, u, 1) \end{array} \right\}$$

let

$$\mathbb{S} = \left\{ \begin{array}{l} (\mu_1, s, 0), (\mu_1, s, 1), (\mu_1, t, 0), (\mu_1, t, 1), (\mu_1, u, 0), (\mu_1, u, 1), \\ (\mu_2, s, 0), (\mu_2, s, 1), (\mu_2, t, 0), (\mu_2, t, 1), (\mu_2, u, 0), (\mu_2, u, 1) \\ (\mu_3, s, 0), (\mu_3, s, 1), (\mu_3, t, 0), (\mu_3, t, 1), (\mu_3, u, 0), (\mu_3, u, 1), \end{array} \right\}$$

be a subset of $\mathcal{H}$ and $\mathcal{D} = \{s, t, u,\}$ be a set of specialists.

Following survey depicts choices of three specialists:

$\xi_1 = \xi(\mu_1, s, 1) = \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}$, $\xi_2 = \xi(\mu_1, t, 1) = \left\{\frac{m_1}{0.4}, \frac{m_2}{0.8}, \frac{m_3}{0.4}, \frac{m_4}{0.2}\right\}$,
$\xi_3 = \xi(\mu_1, u, 1) = \left\{\frac{m_1}{0.7}, \frac{m_2}{0.5}, \frac{m_3}{0.6}, \frac{m_4}{0.3}\right\}$, $\xi_4 = \xi(\mu_2, s, 1) = \left\{\frac{m_1}{0.9}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.3}\right\}$,
$\xi_5 = \xi(\mu_2, t, 1) = \left\{\frac{m_1}{0.4}, \frac{m_2}{0.8}, \frac{m_3}{0.3}, \frac{m_4}{0.2}\right\}$, $\xi_6 = \xi(\mu_2, u, 1) = \left\{\frac{m_1}{0.5}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.8}\right\}$,
$\xi_7 = \xi(\mu_3, s, 1) = \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.4}, \frac{m_4}{0.5}\right\}$, $\xi_8 = \xi(\mu_3, t, 1) = \left\{\frac{m_1}{0.4}, \frac{m_2}{0.6}, \frac{m_3}{0.7}, \frac{m_4}{0.9}\right\}$,
$\xi_9 = \xi(\mu_3, u, 1) = \left\{\frac{m_1}{0.7}, \frac{m_2}{0.3}, \frac{m_3}{0.5}, \frac{m_4}{0.2}\right\}$, $\xi_{10} = \xi(\mu_1, s, 0) = \left\{\frac{m_1}{0.3}, \frac{m_2}{0.2}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}$,
$\xi_{11} = \xi(\mu_1, t, 0) = \left\{\frac{m_1}{0.1}, \frac{m_2}{0.9}, \frac{m_3}{0.6}, \frac{m_4}{0.2}\right\}$, $\xi_{12} = \xi(\mu_1, u, 0) = \left\{\frac{m_1}{0.2}, \frac{m_2}{0.1}, \frac{m_3}{0.3}, \frac{m_4}{0.5}\right\}$,
$\xi_{13} = \xi(\mu_2, s, 0) = \left\{\frac{m_1}{0.8}, \frac{m_2}{0.3}, \frac{m_3}{0.5}, \frac{m_4}{0.7}\right\}$, $\xi_{14} = \xi(\mu_2, t, 0) = \left\{\frac{m_1}{0.7}, \frac{m_2}{0.2}, \frac{m_3}{0.9}, \frac{m_4}{0.4}\right\}$,
$\xi_{15} = \xi(\mu_2, u, 0) = \left\{\frac{m_1}{0.6}, \frac{m_2}{0.7}, \frac{m_3}{0.3}, \frac{m_4}{0.2}\right\}$, $\xi_{16} = \xi(\mu_3, s, 0) = \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}$,
$\xi_{17} = \xi(\mu_3, t, 0) = \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.8}, \frac{m_4}{0.3}\right\}$, $\xi_{18} = \xi(\mu_3, u, 0) = \left\{\frac{m_1}{0.5}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.1}\right\}$,

The fuzzy soft expert set can be described as

$$(\xi, \mathbb{S}) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}\right), \left((\mu_1, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.8}, \frac{m_3}{0.4}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_1, u, 1), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.5}, \frac{m_3}{0.6}, \frac{m_4}{0.3}\right\}\right), \left((\mu_2, s, 1), \left\{\frac{m_1}{0.9}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.3}\right\}\right), \\ \left(\mu_2, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.8}, \frac{m_3}{0.3}, \frac{m_4}{0.2}\right\}\right), \left((\mu_2, u, 1), \left\{\frac{m_1}{0.5}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.8}\right\}\right), \\ \left((\mu_3, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.4}, \frac{m_4}{0.5}\right\}\right), \left((\mu_3, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.6}, \frac{m_3}{0.7}, \frac{m_4}{0.9}\right\}\right), \\ \left((\mu_3, u, 1), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.3}, \frac{m_3}{0.5}, \frac{m_4}{0.2}\right\}\right), \left((\mu_1, s, 0), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.2}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \\ \left((\mu_1, t, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.9}, \frac{m_3}{0.6}, \frac{m_4}{0.2}\right\}\right), \left((\mu_1, u, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.1}, \frac{m_3}{0.3}, \frac{m_4}{0.5}\right\}\right), \\ \left((\mu_2, s, 0), \left\{\frac{m_1}{0.8}, \frac{m_2}{0.3}, \frac{m_3}{0.5}, \frac{m_4}{0.7}\right\}\right), \left((\mu_2, t, 0), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.2}, \frac{m_3}{0.9}, \frac{m_4}{0.4}\right\}\right), \\ \left((\mu_2, u, 0), \left\{\frac{m_1}{0.6}, \frac{m_2}{0.7}, \frac{m_3}{0.3}, \frac{m_4}{0.2}\right\}\right), \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right), \\ \left((\mu_3, t, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.8}, \frac{m_4}{0.3}\right\}\right), \left((\mu_3, u, 0), \left\{\frac{m_1}{0.5}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.1}\right\}\right), \end{array} \right\}$$

Definition 3.3. Fuzzy Hypersoft Expert subset

A hypersoft expert set $(\xi_1, \mathbb{S})$ is said to be fuzzy hypersoft expert subset of (ξ_2, R) over $\coprod$, if

(i) $\mathbb{S} \subseteq R$,

(ii) $\forall\ \alpha \in \mathbb{S}, \xi_1(\alpha) \subseteq \xi_2(\alpha)$.

and denoted by $(\xi_1, \mathbb{S}) \subseteq (\xi_2, R)$. Similarly (ξ_2, R) is said to be *fuzzy hypersoft expert superset* of $(\xi_1, \mathbb{S})$.

Example 3.4. Considering Example 3.2, Suppose

$$A_1 = \Big\{ (\mu_1, s, 1), (\mu_3, s, 0), (\mu_1, t, 1), (\mu_3, t, 1), (\mu_3, t, 0), (\mu_1, u, 0), (\mu_3, u, 1) \Big\}$$

$$A_2 = \Big\{ (\mu_1, s, 1), (\mu_3, s, 0), (\mu_3, s, 1), (\mu_1, t, 1), (\mu_3, t, 1), (\mu_1, t, 0), (\mu_3, t, 0), (\mu_1, u, 0), (\mu_3, u, 1), (\mu_1, u, 1) \Big\}$$

It is clear that $A_1 \subset A_2$.Suppose (ξ_1, A_1) and (ξ_2, A_2) be defined as following

$$(\xi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.6}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \left((\mu_1, t, 1), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.6}, \frac{m_3}{0.2}, \frac{m_4}{0.1}\right\}\right), \\ \left((\mu_3, t, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.5}, \frac{m_3}{0.6}, \frac{m_4}{0.8}\right\}\right), \left((\mu_3, u, 1), \left\{\frac{m_1}{0.6}, \frac{m_2}{0.2}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \\ \left((\mu_1, u, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.1}, \frac{m_3}{0.2}, \frac{m_4}{0.4}\right\}\right), \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.7}\right\}\right), \\ \left((\mu_3, t, 0) \left\{\frac{m_1}{0.1}, \frac{m_2}{0.8}, \frac{m_3}{0.7}, \frac{m_4}{0.2}\right\}\right) \end{array} \right\}$$

$$(\xi_2, \mathrm{A}_2) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}\right), \left((\mu_1, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.8}, \frac{m_3}{0.4}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_3, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.4}, \frac{m_4}{0.5}\right\}\right), \left((\mu_3, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.6}, \frac{m_3}{0.7}, \frac{m_4}{0.9}\right\}\right), \\ \left((\mu_1, u, 1), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.5}, \frac{m_3}{0.6}, \frac{m_4}{0.3}\right\}\right), \left((\mu_3, u, 1), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.3}, \frac{m_3}{0.5}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_1, u, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.1}, \frac{m_3}{0.3}, \frac{m_4}{0.5}\right\}\right), \left((\mu_1, t, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.9}, \frac{m_3}{0.6}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right), \left((\mu_3, t, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.8}, \frac{m_4}{0.3}\right\}\right) \end{array} \right\}$$

which implies that $(\xi_1, A_1) \subseteq (\xi_2, A_2)$.

Definition 3.5. Two fuzzy hypersoft expert sets (ξ_1, A_1) and (ξ_2, A_2) over $\coprod$ are said to be equal if (ξ_1, A_1) is a hypersoft expert subset of (ξ_2, A_2) and (ξ_2, A_2) is a fuzzy hypersoft expert subset of (ξ_1, A_1).

Definition 3.6. Let $\mathcal{G}$ be a set as defined in definition 3.1 and $\mathcal{D}$, a set of experts. The NOT set of $\mathcal{H} = \mathcal{G} \times \mathcal{D} \times \mathbb{C}$ denoted by $\sim \mathcal{H}$, is defined by $\sim \mathcal{T} = \{(\sim g_i, d_j, c_k) \forall i, j, k\}$ where $\sim g_i$ is not g_i.

Definition 3.7. The complement of a fuzzy hypersoft expert set $(\xi, \mathbb{S})$, denoted by $(\xi, \mathbb{S})^c$, is defined by $(\xi, \mathbb{S})^c = (\xi^c, \sim \mathbb{S})$ while $\xi^c : \sim \mathbb{S} \to P(\coprod)$ is a mapping given by $\xi^c(\beta) = \coprod - \xi(\sim \beta)$, where $\beta \in \sim \mathbb{S}$.

Example 3.8. Taking complement of fuzzy hypersoft expert set determined in 3.2, we have

$$(\xi, \mathbb{S})^{\mathrm{c}} = \left\{ \begin{array}{l} \left((\sim \mu_1, s, 1), \left\{\frac{m_1}{0.8}, \frac{m_2}{0.3}, \frac{m_3}{0.5}, \frac{m_4}{0.9}\right\}\right), \left((\sim \mu_1, t, 1), \left\{\frac{m_1}{0.6}, \frac{m_2}{0.2}, \frac{m_3}{0.6}, \frac{m_4}{0.8}\right\}\right), \\ \left((\sim \mu_1, u, 1), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.5}, \frac{m_3}{0.4}, \frac{m_4}{0.7}\right\}\right), \left((\sim \mu_3, s, 1), \left\{\frac{m_1}{0.8}, \frac{m_2}{0.1}, \frac{m_3}{0.6}, \frac{m_4}{0.5}\right\}\right), \\ \left((\sim \mu_3, t, 1), \left\{\frac{m_1}{0.6}, \frac{m_2}{0.4}, \frac{m_3}{0.3}, \frac{m_4}{0.1}\right\}\right), \left((\sim \mu_3, u, 1), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.8}\right\}\right), \\ \left((\sim \mu_2, s, 1), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.6}, \frac{m_3}{0.3}, \frac{m_4}{0.7}\right\}\right), \left((\sim \mu_2, t, 1), \left\{\frac{m_1}{0.6}, \frac{m_2}{0.2}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right), \\ \left((\sim \mu_2, u, 1), \left\{\frac{m_1}{0.5}, \frac{m_2}{0.7}, \frac{m_3}{0.4}, \frac{m_4}{0.2}\right\}\right), \left((\sim \mu_1, s, 0), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.9}\right\}\right), \\ \left((\sim \mu_1, t, 0), \left\{\frac{m_1}{0.9}, \frac{m_2}{0.1}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \left((\sim \mu_1, u, 0), \left\{\frac{m_1}{0.8}, \frac{m_2}{0.9}, \frac{m_3}{0.7}, \frac{m_4}{0.5}\right\}\right), \\ \left((\sim \mu_3, s, 0), \left\{\frac{m_1}{0.9}, \frac{m_2}{0.6}, \frac{m_3}{0.3}, \frac{m_4}{0.2}\right\}\right), \left((\sim \mu_3, t, 0), \left\{\frac{m_1}{0.8}, \frac{m_2}{0.1}, \frac{m_3}{0.2}, \frac{m_4}{0.7}\right\}\right), \\ \left((\sim (\mu_3, u, 0), \left\{\frac{m_1}{0.5}, \frac{m_2}{0.7}, \frac{m_3}{0.4}, \frac{m_4}{0.9}\right\}\right) \left((\sim \mu_2, s, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.3}\right\}\right), \\ \left((\sim \mu_2, t, 0), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.8}, \frac{m_3}{0.1}, \frac{m_4}{0.4}\right\},\right), \left((\sim \mu_2, u, 0), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.3}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right) \end{array} \right\}$$

Definition 3.9. An agree-fuzzy hypersoft expert set $(\xi, \mathbb{S})_{ag}$ over $\coprod$, is a fuzzy hypersoft expert subset of $(\xi, \mathbb{S})$ and is characterized as

$(\xi, \mathbb{S})_{ag} = \{\xi_{ag}(\beta) : \beta \in G \times D \times \{1\}\}$.

Example 3.10. Finding agree-fuzzy hypersoft expert set determined in 3.2, we get

$$(\xi,\mathbb{S}) = \left\{ \begin{array}{l} \left((\mu_1,s,1),\left\{\frac{m_1}{0.2},\frac{m_2}{0.7},\frac{m_3}{0.5},\frac{m_4}{0.1}\right\}\right),\left((\mu_1,t,1),\left\{\frac{m_1}{0.4},\frac{m_2}{0.8},\frac{m_3}{0.4},\frac{m_4}{0.2}\right\}\right), \\ \left((\mu_1,u,1),\left\{\frac{m_1}{0.7},\frac{m_2}{0.5},\frac{m_3}{0.6},\frac{m_4}{0.3}\right\}\right),\left((\mu_3,s,1),\left\{\frac{m_1}{0.2},\frac{m_2}{0.9},\frac{m_3}{0.4},\frac{m_4}{0.5}\right\}\right), \\ \left((\mu_3,t,1),\left\{\frac{m_1}{0.4},\frac{m_2}{0.6},\frac{m_3}{0.7},\frac{m_4}{0.9}\right\}\right),\left((\mu_3,u,1),\left\{\frac{m_1}{0.7},\frac{m_2}{0.3},\frac{m_3}{0.5},\frac{m_4}{0.2}\right\}\right), \\ \left((\mu_2,s,1),\left\{\frac{m_1}{0.9},\frac{m_2}{0.4},\frac{m_3}{0.7},\frac{m_4}{0.3}\right\}\right),\left((\mu_2,u,1),\left\{\frac{m_1}{0.5},\frac{m_2}{0.3},\frac{m_3}{0.6},\frac{m_4}{0.8}\right\}\right) \end{array} \right\}$$

Definition 3.11. A disagree- fuzzy hypersoft expert set $(\xi,\mathbb{S})_{dag}$ over $\coprod$, is a fuzzy hypersoft expert subset of $(\xi,\mathbb{S})$ and is characterized as
$(\xi,\mathbb{S})_{dag} = \{\xi_{dag}(\beta) : \beta \in \mathbb{G}\times\mathcal{D}\times\{0\}\}$.

Example 3.12. Getting disagree-fuzzy hypersoft expert set determined in 3.2,

$$(\xi,\mathbb{S}) = \left\{ \begin{array}{l} \left((\mu_1,s,0),\left\{\frac{m_1}{0.3},\frac{m_2}{0.2},\frac{m_3}{0.4},\frac{m_4}{0.1}\right\}\right),\left((\mu_1,t,0),\left\{\frac{m_1}{0.1},\frac{m_2}{0.9},\frac{m_3}{0.6},\frac{m_4}{0.2}\right\}\right), \\ \left((\mu_1,u,0),\left\{\frac{m_1}{0.2},\frac{m_2}{0.1},\frac{m_3}{0.3},\frac{m_4}{0.5}\right\}\right),\left((\mu_3,s,0),\left\{\frac{m_1}{0.1},\frac{m_2}{0.4},\frac{m_3}{0.7},\frac{m_4}{0.8}\right\}\right), \\ \left((\mu_3,t,0),\left\{\frac{m_1}{0.2},\frac{m_2}{0.9},\frac{m_3}{0.8},\frac{m_4}{0.3}\right\}\right),\left((\mu_3,u,0),\left\{\frac{m_1}{0.5},\frac{m_2}{0.3},\frac{m_3}{0.6},\frac{m_4}{0.1}\right\}\right) \\ \left((\mu_2,s,0),\left\{\frac{m_1}{0.8},\frac{m_2}{0.3},\frac{m_3}{0.5},\frac{m_4}{0.7}\right\}\right),\left((\mu_2,t,0),\left\{\frac{m_1}{0.7},\frac{m_2}{0.2},\frac{m_3}{0.9},\frac{m_4}{0.4}\right\}\right), \\ \left((\mu_2,u,0),\left\{\frac{m_1}{0.6},\frac{m_2}{0.7},\frac{m_3}{0.3},\frac{m_4}{0.2}\right\}\right) \end{array} \right\}$$

Proposition 3.13. *If $(\xi,\mathbb{S})$ is a fuzzy hypersoft expert set over $\coprod$, then*

(1). $((\xi,\mathbb{S})^c)^c = (\xi,\mathbb{S})$
(2). $(\xi,\mathbb{S})^c_{ag} = (\xi,\mathbb{S})_{dag}$
(3). $(\xi,\mathbb{S})^c_{dag} = (\xi,\mathbb{S})_{ag}$

Definition 3.14. The union of $(\xi_1,\mathbb{S})$ and (ξ_2,R) over $\coprod$ is (ξ_3,L) with $L=\mathbb{S}\cup R$, defined as

$$\xi_3(\beta) = \left\{ \begin{array}{cl} \xi_1(\beta) & ;\ \beta\in\mathbb{S}-R \\ \xi_2(\beta) & ;\ \beta\in\mathbb{R}-S \\ \xi_1(\beta)\cup\xi_2(\beta) & ;\ \beta\in\mathbb{S}\cap R \end{array} \right.$$

Example 3.15. Taking into consideration the concept of example 3.2, consider the following two sets

$$A_1 = \Big\{ (\mu_1,s,1),(\mu_3,s,0),(\mu_1,t,1),(\mu_3,t,1),(\mu_3,t,0),(\mu_1,u,0),(\mu_3,u,1)\Big\}$$

$$A_2 = \Big\{ (\mu_1,s,1),(\mu_3,s,0),(\mu_3,s,1),(\mu_1,t,1),(\mu_3,t,1),(\mu_1,u,1),(\mu_3,t,0),(\mu_1,u,0),(\mu_3,u,1),(\mu_1,t,0)\Big\}$$

Suppose (ξ_1,A_1) and (ξ_2,A_2) over $\coprod$ are two fuzzy hypersoft expert sets such that

$$(\xi_1,\mathrm{A}_1) = \left\{ \begin{array}{l} \left((\mu_1,s,1),\left\{\frac{m_1}{0.1},\frac{m_2}{0.6},\frac{m_3}{0.4},\frac{m_4}{0.1}\right\}\right),\left((\mu_1,t,1),\left\{\frac{m_1}{0.3},\frac{m_2}{0.6},\frac{m_3}{0.2},\frac{m_4}{0.1}\right\}\right), \\ \left((\mu_3,t,1),\left\{\frac{m_1}{0.2},\frac{m_2}{0.5},\frac{m_3}{0.6},\frac{m_4}{0.8}\right\}\right),\left((\mu_3,u,1),\left\{\frac{m_1}{0.6},\frac{m_2}{0.2},\frac{m_3}{0.4},\frac{m_4}{0.1}\right\}\right), \\ \left((\mu_1,u,0),\left\{\frac{m_1}{0.1},\frac{m_2}{0.1},\frac{m_3}{0.2},\frac{m_4}{0.4}\right\}\right),\left((\mu_3,s,0),\left\{\frac{m_1}{0.1},\frac{m_2}{0.3},\frac{m_3}{0.6},\frac{m_4}{0.7}\right\}\right), \\ \left((\mu_3,t,0)\left\{\frac{m_1}{0.1},\frac{m_2}{0.8},\frac{m_3}{0.7},\frac{m_4}{0.2}\right\}\right) \end{array} \right\}$$

$$(\xi_2, \mathrm{A}_2) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}\right), \left((\mu_1, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.8}, \frac{m_3}{0.4}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_3, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.4}, \frac{m_4}{0.5}\right\}\right), \left((\mu_3, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.6}, \frac{m_3}{0.7}, \frac{m_4}{0.9}\right\}\right), \\ \left((\mu_1, u, 1), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.5}, \frac{m_3}{0.6}, \frac{m_4}{0.3}\right\}\right), \left((\mu_3, u, 1), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.3}, \frac{m_3}{0.5}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_1, u, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.1}, \frac{m_3}{0.3}, \frac{m_4}{0.5}\right\}\right), \left((\mu_1, t, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.9}, \frac{m_3}{0.6}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right), \left((\mu_3, t, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.8}, \frac{m_4}{0.3}\right\}\right) \end{array} \right\}$$

Then $(\xi_1, A_1) \cup (\xi_2, A_2) = (\xi_3, A_3)$

$$(\xi_2, \mathrm{A}_2) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}\right), \left((\mu_1, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.8}, \frac{m_3}{0.4}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_3, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.4}, \frac{m_4}{0.5}\right\}\right), \left((\mu_3, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.6}, \frac{m_3}{0.7}, \frac{m_4}{0.9}\right\}\right), \\ \left((\mu_1, u, 1), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.5}, \frac{m_3}{0.6}, \frac{m_4}{0.3}\right\}\right), \left((\mu_3, u, 1), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.3}, \frac{m_3}{0.5}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_1, u, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.1}, \frac{m_3}{0.3}, \frac{m_4}{0.5}\right\}\right), \left((\mu_1, t, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.9}, \frac{m_3}{0.6}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right), \left((\mu_3, t, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.8}, \frac{m_4}{0.3}\right\}\right) \end{array} \right\}$$

Proposition 3.16. *If (ξ_1, A_1),(ξ_2, A_2) and (ξ_3, A_3) are three fuzzy hypersoft expert sets over $\coprod$, then*

(1). $(\xi_1, A_1) \cup (\xi_2, A_2) = (\xi_2, A_2) \cup (\xi_1, A_1)$

(2). $((\xi_1, A_1) \cup (\xi_2, A_2)) \cup (\xi_3, A_3) = (\xi_1, A_1) \cup ((\xi_2, A_2) \cup (\xi_3, A_3))$

Definition 3.17. The intersection of $(\xi_1, \mathbb{S})$ and (ξ_2, R) over $\coprod$ is (ξ_3, L) with $L = \mathbb{S} \cap R$, defined as

$$\xi_3(\beta) = \left\{ \begin{array}{cl} \xi_1(\beta) & ; \beta \in \mathbb{S} - R \\ \xi_2(\beta) & ; \beta \in R - S \\ \xi_1(\beta) \cap \xi_2(\beta) & ; \beta \in \mathbb{S} \cap R \end{array} \right.$$

Example 3.18. Taking into consideration the concept of example 3.2, consider the following two sets

$$A_1 = \left\{ (\mu_1, s, 1), (\mu_3, s, 0), (\mu_1, t, 1), (\mu_3, t, 1), (\mu_3, t, 0), (\mu_1, u, 0), (\mu_3, u, 1) \right\}$$

$$A_2 = \left\{ (\mu_1, s, 1), (\mu_3, s, 0), (\mu_3, s, 1), (\mu_1, t, 1), (\mu_3, t, 1), (\mu_1, t, 0), (\mu_3, t, 0), (\mu_1, u, 0), (\mu_3, u, 1), , (\mu_1, u, 1) \right\}$$

Suppose (ξ_1, A_1) and (ξ_2, A_2) over $\coprod$ are two fuzzy hypersoft expert sets such that

$$(\xi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.6}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \left((\mu_1, t, 1), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.6}, \frac{m_3}{0.2}, \frac{m_4}{0.1}\right\}\right), \\ \left((\mu_3, t, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.5}, \frac{m_3}{0.6}, \frac{m_4}{0.8}\right\}\right), \left((\mu_3, u, 1), \left\{\frac{m_1}{0.6}, \frac{m_2}{0.2}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \\ \left((\mu_1, u, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.1}, \frac{m_3}{0.2}, \frac{m_4}{0.4}\right\}\right), \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.7}\right\}\right), \\ \left((\mu_3, t, 0) \left\{\frac{m_1}{0.1}, \frac{m_2}{0.8}, \frac{m_3}{0.7}, \frac{m_4}{0.2}\right\}\right) \end{array} \right\}$$

$$(\xi_2, \mathrm{A}_2) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}\right), \left((\mu_1, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.8}, \frac{m_3}{0.4}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_3, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.4}, \frac{m_4}{0.5}\right\}\right), \left((\mu_3, t, 1), \left\{\frac{m_1}{0.4}, \frac{m_2}{0.6}, \frac{m_3}{0.7}, \frac{m_4}{0.9}\right\}\right), \\ \left((\mu_1, u, 1), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.5}, \frac{m_3}{0.6}, \frac{m_4}{0.3}\right\}\right), \left((\mu_3, u, 1), \left\{\frac{m_1}{0.7}, \frac{m_2}{0.3}, \frac{m_3}{0.5}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_1, u, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.1}, \frac{m_3}{0.3}, \frac{m_4}{0.5}\right\}\right), \left((\mu_1, t, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.9}, \frac{m_3}{0.6}, \frac{m_4}{0.2}\right\}\right), \\ \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right), \left((\mu_3, t, 0), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.9}, \frac{m_3}{0.8}, \frac{m_4}{0.3}\right\}\right) \end{array} \right\}$$

Then $(\xi_1, A_1) \cap (\xi_2, A_2) = (\xi_3, A_3)$

$$(\xi_3, \mathrm{A}_3) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.6}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \left((\mu_1, t, 1), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.6}, \frac{m_3}{0.2}, \frac{m_4}{0.1}\right\}\right), \\ \left((\mu_3, t, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.5}, \frac{m_3}{0.4}, \frac{m_4}{0.5}\right\}\right), \left((\mu_3, u, 1), \left\{\frac{m_1}{0.6}, \frac{m_2}{0.2}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \\ \left((\mu_1, u, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.1}, \frac{m_3}{0.2}, \frac{m_4}{0.4}\right\}\right), \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.7}\right\}\right), \\ \left((\mu_3, t, 0) \left\{\frac{m_1}{0.1}, \frac{m_2}{0.8}, \frac{m_3}{0.7}, \frac{m_4}{0.2}\right\}\right) \end{array} \right\}$$

Proposition 3.19. *If* (ξ_1, A_1),(ξ_2, A_2) *and* (ξ_3, A_3) *are three fuzzy hypersoft expert sets over* $\coprod$*, then*

(1). $(\xi_1, A_1) \cap (\xi_2, A_2) = (\xi_2, A_2) \cap (\xi_1, A_1)$

(2). $((\xi_1, A_1) \cap (\xi_2, A_2)) \cap (\xi_3, A_3) = (\xi_1, A_1) \cap ((\xi_2, A_2) \cap (\xi_3, A_3))$

Proposition 3.20. *If* (ξ_1, A_1),(ξ_2, A_2) *and* (ξ_3, A_3) *are three fuzzy hypersoft expert sets over* $\coprod$*, then*

(1). $(\xi_1, A_1) \cup ((\xi_2, A_2) \cap (\xi_3, A_3)) = ((\xi_1, A_1) \cup ((\xi_2, A_2)) \cap ((\xi_1, A_1) \cup (\xi_3, A_3))$

(2). $(\xi_1, A_1) \cap ((\xi_2, A_2) \cup (\xi_3, A_3)) = ((\xi_1, A_1) \cap ((\xi_2, A_2)) \cup ((\xi_1, A_1) \cap (\xi_3, A_3))$

Definition 3.21. If (ξ_1, A_1) and (ξ_2, A_2) are two fuzzy hypersoft expert sets over $\coprod$ then (ξ_1, A_1) AND (ξ_2, A_2) denoted by $(\xi_1, A_1) \wedge (\xi_2, A_2)$ is defined by
$(\xi_1, A_1) \wedge (\xi_2, A_2) = (\xi_3, A_1 \times A_2)$,
while $\xi_3(\beta, \gamma) = \xi_1(\beta) \cap \xi_2(\gamma), \forall (\beta, \gamma) \in A_1 \times A_2$.

Example 3.22. Taking into consideration the concept of example 3.2, let two sets

$$A_1 = \left\{ (\mu_1, s, 1), (\mu_1, t, 1), (\mu_3, s, 0) \right\}$$

$$A_2 = \left\{ (\mu_1, s, 1), (\mu_3, s, 0) \right\}$$

Suppose (ξ_1, A_1) and (ξ_2, A_2) over $\coprod$ are two fuzzy hypersoft expert sets such that

$$(\xi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.6}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \left((\mu_1, t, 1), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.6}, \frac{m_3}{0.2}, \frac{m_4}{0.1}\right\}\right), \\ , \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.7}\right\}\right), \end{array} \right\}$$

$$(\xi_2, \mathrm{A}_2) = \left\{ \left((\mu_1, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}\right), \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right), \right\}$$

Then $(\xi_3, A_3) \wedge (\xi_2, A_2) = (\xi_3, A_1 \times A_2)$,

$$(\xi_3, \mathrm{A}_1 \times \mathrm{A}_2) = \left\{ \begin{array}{l} \left(((\mu_1, s, 1), (\mu_1, s, 1)), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}\right), \\ \left(((\mu_1, t, 1), (\mu_1, s, 1)), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}\right), \\ \left(((\mu_1, t, 1), (\mu_3, s, 0)), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.6}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right), \\ \left(((\mu_1, s, 1), (\mu_3, s, 0)), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.6}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right), \\ \left(((\mu_3, s, 0), (\mu_1, s, 1)), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.6}, \frac{m_4}{0.7}\right\}\right), \\ \left(((\mu_3, s, 0), (\mu_3, s, 0)), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right) \end{array} \right\}$$

Definition 3.23. If (ξ_1, A_1) and (ξ_2, A_2) are two fuzzy hypersoft expert sets over $\coprod$ then (ξ_1, A_1) OR (ξ_2, A_2) denoted by $(\xi_1, A_1) \vee (\xi_2, A_2)$ is defined by
$(\xi_1, A_1) \vee (\xi_2, A_2) = (\xi_3, A_1 \times A_2)$,
while $\xi_3(\beta, \gamma) = \xi_1(\beta) \cup \xi_2(\gamma), \forall (\beta, \gamma) \in A_1 \times A_2$.

Example 3.24. Taking into consideration the concept of example 3.2, suppose the following sets

$$A_1 = \Big\{ (\mu_1, s, 1), (\mu_1, t, 1), (\mu_3, s, 0) \Big\}$$
$$A_2 = \Big\{ (\mu_1, s, 1), (\mu_3, s, 0) \Big\}$$

Suppose (ξ_1, A_1) and (ξ_2, A_2) over $\coprod$ are two fuzzy hypersoft expert sets such that

$$(\xi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.6}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \left((\mu_1, t, 1), \left\{\frac{m_1}{0.3}, \frac{m_2}{0.6}, \frac{m_3}{0.2}, \frac{m_4}{0.1}\right\}\right), \\ , \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.7}\right\}\right), \end{array} \right\}$$

$$(\xi_2, \mathrm{A}_2) = \left\{ \left((\mu_1, s, 1), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.7}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}\right), \left((\mu_3, s, 0), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.7}, \frac{m_4}{0.8}\right\}\right), \right\}$$

Then $(\xi_3, A_3) \wedge (\xi_2, A_2) = (\xi_3, A_1 \times A_2)$,

$$(\xi_3, \mathrm{A}_1 \times \mathrm{A}_2) = \left\{ \begin{array}{l} \left(((\mu_1, s, 1), (\mu_1, s, 1)), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.6}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \\ \left(((\mu_1, t, 1), (\mu_1, s, 1)), \left\{\frac{m_1}{0.2}, \frac{m_2}{0.6}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \\ \left(((\mu_1, t, 1), (\mu_3, s, 0)), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.2}, \frac{m_4}{0.1}\right\}\right), \\ \left(((\mu_1, s, 1), (\mu_3, s, 0)), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.4}, \frac{m_3}{0.4}, \frac{m_4}{0.1}\right\}\right), \\ \left(((\mu_3, s, 0), (\mu_1, s, 1)), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.3}, \frac{m_3}{0.5}, \frac{m_4}{0.1}\right\}\right), \\ \left(((\mu_3, s, 0), (\mu_3, s, 0)), \left\{\frac{m_1}{0.1}, \frac{m_2}{0.3}, \frac{m_3}{0.6}, \frac{m_4}{0.7}\right\}\right) \end{array} \right\}$$

Proposition 3.25. *If* (ξ_1, A_1),(ξ_2, A_2) *and* (ξ_3, A_3) *are three fuzzy hypersoft expert sets over* $\coprod$, *then*

(1). $((\xi_1, A_1) \wedge (\xi_2, A_2))^c = ((\xi_1, A_1))^c \vee ((\xi_2, A_2))^c$

(2). $((\xi_1, A_1) \vee (\xi_2, A_2))^c = ((\xi_1, A_1))^c \wedge ((\xi_2, A_2))^c$

Proposition 3.26. *If* (ξ_1, A_1),(ξ_2, A_2) *and* (ξ_3, A_3) *are three fuzzy hypersoft expert sets over* $\coprod$, *then*

(1). $((\xi_1, A_1) \wedge (\xi_2, A_2)) \wedge (\xi_3, A_3) = (\xi_1, A_1) \wedge ((\xi_2, A_2) \wedge (\xi_3, A_3))$

(2). $((\xi_1, A_1) \vee (\xi_2, A_2)) \vee (\xi_3, A_3) = (\xi_1, A_1) \vee ((\xi_2, A_2) \vee (\xi_3, A_3))$

(3). $(\xi_1, A_1) \vee ((\xi_2, A_2) \wedge (\xi_3, A_3) = ((\xi_1, A_1) \vee ((\xi_2, A_2)) \wedge ((\xi_1, A_1) \vee (\xi_3, A_3))$

(4). $(\xi_1, A_1) \wedge ((\xi_2, A_2) \vee (\xi_3, A_3)) = ((\xi_1, A_1) \wedge ((\xi_2, A_2)) \vee ((\xi_1, A_1) \wedge (\xi_3, A_3))$

4. An Applications to Fuzzy Hypersoft expert set

In this section, an application of fuzzy hypersoft expert set theory in a decision making problem, is presented.

Statement of the problem

Mr. John wants to purchase a mobile from a mobile market for his personal use. He takes help from his some friends (Stephen, Thomas and Umar) who have expertise in mobile purchase.

Proposed Algorithm

The following algorithm is adopted for this selection (purchase).

(1). Construct fuzzy hypersoft soft expert set (ξ, K),

(2). Determine an Agree-fuzzy hypersoft expert set and a Disagree-fuzzy hypersoft expert set,

(3). Compute $d_i = \sum_i c_{ij}$ for Agree-fuzzy hypersoft expert set,

(4). Determine $f_i = \sum_i c_{ij}$ for Disagree-fuzzy hypersoft expert set,

(5). Determine $g_j = d_j - f_j$ for Agree-fuzzy hypersoft expert set,

(6). Compute n, for which $p_n = \max p_j$ for Agree-fuzzy hypersoft expert set,

Step-1

Let eight categories of mobile are there which form the universe of discourse $\Omega = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$ and X $= \{E_1 = Stephen, E_2 = Thomas, E_3 = Umar\}$ be a set of experts for this purchase. The following are the attribute-valued sets for prescribed attributes:

$L_1 = Brand = \{l_1, l_2\}$

$L_2 = Price = \{l_3, l_4\}$

$L_3 = Colour = \{l_5, l_6\}$

$L_4 = Memory = \{l_7, l_8\}$

$L_5 = Resolution = \{l_9, l_{10}\}$

and then

$L = L_1 \times L_2 \times L_3 \times L_4 \times L_5$

$$L = \left\{ \begin{array}{l} (l_1, l_3, l_5, l_7, l_9), (l_1, l_3, l_5, l_7, l_{10}), (l_1, l_3, l_5, l_8, l_9), (l_1, l_3, l_5, l_8, l_{10}), (l_1, l_3, l_6, l_7, l_9), \\ (l_1, l_3, l_6, l_7, l_{10}), (l_1, l_3, l_6, l_8, l_9), (l_1, l_3, l_6, l_8, l_{10}), (l_1, l_4, l_5, l_7, l_9), (l_1, l_4, l_5, l_7, l_{10}), \\ (l_1, l_4, l_5, l_8, l_9), (l_1, l_4, l_5, l_8, l_{10}), (l_1, l_4, l_6, l_7, l_9), (l_1, l_4, l_6, l_7, l_{10}), (l_1, l_4, l_6, l_8, l_9), \\ (l_1, l_4, l_6, l_8, l_{10}), (l_2, l_3, l_5, l_7, l_9), (l_2, l_3, l_5, l_7, l_{10}), (l_2, l_3, l_5, l_8, l_9), (l_2, l_3, l_5, l_8, l_{10}), \\ (l_2, l_3, l_6, l_7, l_9), (l_2, l_3, l_6, l_7, l_{10}), (l_2, l_3, l_6, l_8, l_9), (l_2, l_3, l_6, l_8, l_{10}), (l_2, l_4, l_5, l_7, l_9), \\ (l_2, l_4, l_5, l_7, l_{10}), (l_2, l_4, l_5, l_8, l_9), (l_2, l_4, l_5, l_8, l_{10}), (l_2, l_4, l_6, l_7, l_9), (l_2, l_4, l_6, l_7, l_{10}), \\ (l_2, l_4, l_6, l_8, l_9), (l_2, l_4, l_6, l_8, l_{10}) \end{array} \right\}$$

and now take $K \subseteq H$ as

$$K = \left\{ \begin{array}{l} k_1 = (l_1, l_3, l_5, l_7, l_9), k_2 = (l_1, l_3, l_6, l_7, l_{10}), k_3 = (l_1, l_4, l_6, l_8, l_9), \\ k_4 = (l_2, l_3, l_6, l_8, l_9), k_5 = (l_2, l_4, l_6, l_7, l_{10}) \end{array} \right\}$$

and

$$(\xi, \mathrm{K}) = \left\{ \begin{array}{l} \left((k_1,E_1,1),\left\{\frac{c_1}{0.3},\frac{c_2}{0.8},\frac{c_4}{0.5},\frac{c_7}{0.2},\frac{c_8}{0.5}\right\}\right),\left((k_1,E_2,1),\left\{\frac{c_1}{0.8},\frac{c_4}{0.7},\frac{c_5}{0.8},\frac{c_8}{0.2}\right\}\right), \\ \left((k_1,E_3,1),\left\{\frac{c_1}{0.7},\frac{c_3}{0.8},\frac{c_4}{0.6},\frac{c_5}{0.1},\frac{c_6}{0.5},\frac{c_7}{0.3},\frac{c_8}{0.2}\right\}\right),\left((k_2,E_1,1),\left\{\frac{c_1}{0.3},\frac{c_3}{0.6},\frac{c_5}{0.2},\frac{c_8}{0.1}\right\}\right), \\ \left((k_2,E_2,1),\left\{\frac{c_1}{0.3},\frac{c_3}{0.8},\frac{c_4}{0.6},\frac{c_5}{0.1},\frac{c_6}{0.9},\frac{c_8}{0.1}\right\}\right),\left((k_2,E_3,1),\left\{\frac{c_1}{0.2},\frac{c_2}{0.5},\frac{c_4}{0.1},\frac{c_5}{0.1},\frac{c_6}{0.7},\frac{c_8}{0.9}\right\}\right), \\ \left((k_3,E_1,1),\left\{\frac{c_1}{0.2},\frac{c_4}{0.8},\frac{c_5}{0.1},\frac{c_7}{0.9}\right\}\right),\left((k_3,E_2,1),\left\{\frac{c_1}{0.3},\frac{c_2}{0.7},\frac{c_5}{0.8},\frac{c_8}{0.1}\right\}\right), \\ \left((k_3,E_3,1),\left\{\frac{c_1}{0.5},\frac{c_3}{0.7},\frac{c_5}{0.7},\frac{c_8}{0.1}\right\}\right),\left((k_4,E_1,1),\left\{\frac{c_1}{0.4},\frac{c_7}{0.7},\frac{c_8}{0.8}\right\}\right), \\ \left((k_4,E_2,1),\left\{\frac{c_1}{0.5},\frac{c_4}{0.8},\frac{c_5}{0.9},\frac{c_8}{0.1}\right\}\right),\left((k_4,E_3,1),\left\{\frac{c_1}{0.3},\frac{c_6}{0.7},\frac{c_7}{0.9},\frac{c_8}{0.7}\right\}\right), \\ \left((k_5,E_1,1),\left\{\frac{c_1}{0.2},\frac{c_3}{0.7},\frac{c_4}{0.3},\frac{c_5}{0.6},\frac{c_7}{0.1},\frac{c_8}{0.8}\right\}\right),\left((k_5,E_2,1),\left\{\frac{c_1}{0.1},\frac{c_4}{0.8},\frac{c_5}{0.8},\frac{c_6}{0.2},\frac{c_8}{0.3}\right\}\right), \\ \left((k_5,E_3,1),\left\{\frac{c_1}{0.2},\frac{c_3}{0.7},\frac{c_4}{0.3},\frac{c_5}{0.9},\frac{c_6}{0.4},\frac{c_7}{0.1},\frac{c_8}{0.8}\right\}\right),\left((k_1,E_1,0),\left\{\frac{c_1}{0.3},\frac{c_6}{0.7},\frac{c_7}{0.4},\frac{c_8}{0.7}\right\}\right), \\ \left((k_1,E_2,0),\left\{\frac{c_2}{0.4},\frac{c_3}{0.8},\frac{c_6}{0.7},\frac{c_7}{0.1},\frac{c_8}{0.8}\right\}\right),\left((k_1,E_3,0),\left\{\frac{c_1}{0.2},\frac{c_5}{0.3}\right\}\right), \\ \left((k_2,E_1,0),\left\{\frac{c_1}{0.1},\frac{c_2}{0.7},\frac{c_4}{0.3},\frac{c_5}{0.4},\frac{c_6}{0.5}\right\}\right),\left((k_2,E_2,0),\left\{\frac{c_1}{0.3},\frac{c_7}{0.2}\right\}\right), \\ \left((k_2,E_3,0),\left\{\frac{c_1}{0.4},\frac{c_3}{0.5},\frac{c_4}{0.1},\frac{c_5}{0.8},\frac{c_6}{0.9},\frac{c_7}{0.2}\right\}\right),\left((k_3,E_1,0),\left\{\frac{c_1}{0.6},\frac{c_2}{0.7},\frac{c_6}{0.8},\frac{c_8}{0.9}\right\}\right), \\ \left((k_3,E_2,0),\left\{\frac{c_3}{0.2},\frac{c_4}{0.4},\frac{c_6}{0.5},\frac{c_7}{0.6}\right\}\right),\left((k_3,E_3,0),\left\{\frac{c_1}{0.2},\frac{c_3}{0.7},\frac{c_4}{0.8},\frac{c_5}{0.1},\frac{c_7}{0.9}\right\}\right), \\ \left((k_4,E_1,0),\left\{\frac{c_1}{0.2},\frac{c_2}{0.6},\frac{c_3}{0.7},\frac{c_4}{0.9},\frac{c_5}{0.1},\frac{c_7}{0.4}\right\}\right),\left((k_4,E_2,0),\left\{\frac{c_2}{0.3},\frac{c_3}{0.7},\frac{c_6}{0.5},\frac{c_7}{0.1}\right\}\right), \\ \left((k_4,E_3,0),\left\{\frac{c_1}{0.4},\frac{c_3}{0.7},\frac{c_4}{0.8},\frac{c_5}{0.1}\right\}\right),\left((k_5,E_1,0),\left\{\frac{c_1}{0.6},\frac{c_6}{0.7}\right\}\right), \\ \left((k_5,E_2,0),\left\{\frac{c_1}{0.7},\frac{c_2}{0.5},\frac{c_6}{0.1},\frac{c_7}{0.2}\right\}\right),\left((k_5,E_3,0),\left\{\frac{c_1}{0.2},\frac{c_4}{0.3},\frac{c_6}{0.1}\right\}\right), \end{array} \right\}$$

is a fuzzy hypersoft expert set.

Step-2

Table 1 presents an Agree-fuzzy hypersoft expert set and table 2 presents a Disagree-fuzzy hypersoft expert set respectively, such that if $c_i \in \xi_1(\beta)$ then $c_{ij} \in [0,1]$ otherwise $c_{ij} = \times = 0$, and if

$$c_i \in \xi_0(\beta)$$

then $c_{ij} \in [0,1]$ otherwise $c_{ij} = \times = 0$ where c_{ij} are the entries in tables 1 and 2.

Step-(3-6)

Table 3 presents $d_i = \sum_i c_{ij}$ for Agree-fuzzy hypersoft expert set, $f_i = \sum_i c_{ij}$ for Disagree-fuzzy hypersoft expert set, $g_j = d_j - f_j$ for Agree-fuzzy hypersoft expert set, and n, for which $p_n =$ max p_j for Agree-fuzzy hypersoft expert set.

Decision

As g_5 is maximum, so category c_5 is preferred to be best for purchase.

5. Conclusions

Insufficiency of soft set, fuzzy soft set, soft expert set and fuzzy soft expert set for multi-attribute function (attribute-valued sets) is addressed with the development and characterization of novel hybrid structure i.e. fuzzy hypersoft expert set, in this study. Moreover

(1) The fundamentals of fuzzy hypersoft expert set (FHSE-Set) are established and the basic properties of FHSE-Set like subset, superset, equal sets, not set, agree FHSE-Set and disagree FHSE-Set are described with examples.

Table 1. Agree-fuzzy hypersoft expert set

C	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
(k_1, E_1)	0.3	0.8	×	0.5	×	×	0.2	0.5
(k_1, E_2)	0.8	×	×	0.7	0.8	×	×	0.2
(k_1, E_3)	0.7	×	0.8	0.6	0.1	0.5	0.3	0.2
(k_2, E_1)	0.3	×	0.6	×	0.2	×	×	0.1
(k_2, E_2)	0.3	×	0.8	0.6	0.1	0.9	×	0.1
(k_2, E_3)	0.2	0.5	×	0.1	0.1	0.7	×	0.9
(k_3, E_1)	0.2	×	×	0.8	0.1	×	0.9	×
(k_3, E_2)	0.3	0.7	×	×	0.8	×	×	0.1
(k_3, E_3)	0.5	×	0.7	×	0.7	×	×	0.1
(k_4, E_1)	0.4	×	×	×	×	×	0.7	0.8
(k_4, E_2)	0.5	×	×	0.8	0.9	×	×	0.1
(k_4, E_3)	0.3	×	×	×	×	0.7	0.9	0.7
(k_5, E_1)	0.2	×	0.7	0.3	0.6	×	0.1	0.8
(k_5, E_2)	0.1	×	×	0.8	0.8	0.2	×	0.3
(k_5, E_3)	0.2	×	0.7	0.3	0.9	0.4	0.1	0.8
$d_j = \sum_i c_{ij}$	5.3	2.0	4.3	5.5	5.9	3.4	3.2	5.7
	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8

Table 2. Disagree-fuzzy hypersoft expert set

C	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
(k_1, E_1)	0.3	×	×	×	×	0.7	0.4	0.7
(k_1, E_2)	×	0.4	0.8	×	×	0.7	0.1	0.8
(k_1, E_3)	0.2	×	×	×	0.3	×	×	×
(k_2, E_1)	0.1	0.7	×	0.3	0.4	0.5	×	×
(k_2, E_2)	0.3	×	×	×	×	×	0.2	×
(k_2, E_3)	0.4	×	0.5	0.1	0.8	0.9	0.2	×
(k_3, E_1)	0.6	0.7	×	×	×	0.8	×	0.9
(k_3, E_2)	×	×	0.2	0.4	×	0.5	0.6	×
(k_3, E_3)	0.2	×	0.7	0.8	0.1	×	0.9	×
(k_4, E_1)	0.2	0.6	0.7	0.9	0.1	×	0.4	×
(k_4, E_2)	×	0.3	0.7	×	×	0.5	0.1	×
(k_4, E_3)	0.4	×	0.7	0.8	0.1	×	×	×
(k_5, E_1)	0.6	×	×	×	×	0.7	×	×
(k_5, E_2)	0.7	0.5	×	×	×	0.1	0.2	×
(k_5, E_3)	0.2	×	×	0.3	×	0.1	×	×
$f_i = \sum_i c_{ij}$	4.2	3.2	4.3	3.6	1.8	5.5	3.1	2.4
	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8

Table 3. Optimal

$d_i = \sum_i c_{ij}$	$f_i = \sum_i c_{ij}$	$g_j = d_j - f_j$
$d_1 = 5.3$	$f_1 = 4.2$	$g_1 = 1.1$
$d_2 = 2.0$	$f_2 = 3.2$	$g_2 = -1.0$
$d_3 = 4.3$	$f_3 = 4.3$	$g_3 = 0.0$
$d_4 = 5.5$	$f_4 = 3.6$	$g_4 = 1.9$
$d_5 = 5.9$	$f_5 = 1.8$	$g_5 = 4.1$
$d_6 = 3.4$	$f_6 = 5.5$	$g_6 = -2.1$
$d_7 = 3.2$	$f_7 = 3.1$	$g_7 = 1.0$
$d_8 = 5.7$	$f_8 = 2.4$	$g_8 = 3.3$

(2) The essential set-theoretic operations on FHSE-Set like complement, union, intersection, OR and AND operations are established and some laws such as commutative, associative and De Morgan are presented with suitable examples.

(3) A decision-making application regarding the best selection of a certain product is presented with the help of proposed algorithm.

(4) A daily life based example is discussed for the understanding of decision making process.

(5) Future work may include the extension of the presented work for other hypersoft-like hybrids i.e. intuitionistic fuzzy set, interval-valued fuzzy set, pythagorean fuzzy set, neutrosophic set etc.

Conflicts of Interest:

The authors declare no conflict of interest.

References

1. Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*, 338-353.
2. Molodtsov, D. (1999). Soft set theoryfirst results. *Computers and Mathematics with Applications*, *37*(4-5), 19-31.
3. Maji, P. K.; Biswas, R., & Roy, A. R. (2001). Fuzzy soft sets. *Journal of Fuzzy Mathematics*, *9*(3), 589-602.
4. Maji, P. K., Roy, A. R., & Biswas, R. (2003). Soft set theory. *Computers and Mathematics with Applications*, *45*(4-5), 555-562.
5. Maji, P. K., Roy, A. R., & Biswas, R. (2002). An application of soft sets in a decision making problem. *Computers and Mathematics with Applications*, *44*(8-9), 1077-1083.
6. Alkhazaleh, S., A. R. Salleh, & Hassan, N. (2011). Soft multisets theory. *Applied Mathematical Sciences*, *5*(72), 3561-3573.
7. Alkhazaleh, S., A. R. Salleh, & Hassan, N. (2011). Possibility fuzzy soft set. *Advances in Decision Sciences*, *2011*.
8. Alkhazaleh, S., A. R. Salleh, & Hassan, N. (2011). Fuzzy parameterized interval-valued fuzzy soft set. *Applied Mathematical Sciences*, *5*(67), 3335-3346.

9. Abdel-Basset, M., Saleh, M., Gamal, A., & Smarandache, F. (2019). An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. *Applied Soft Computing*, *77*, 438-452.
10. Abdel-Baset, M., Chang, V., Gamal, A., & Smarandache, F. (2019). An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field. *Computers in Industry*, *106*, 94-110.
11. Abdel-Baset, M., Chang, V., & Gamal, A. (2019). Evaluation of the green supply chain management practices: A novel neutrosophic approach. *Computers in Industry*, *108*, 210-220.
12. Abdel-Basset, M., Gamal, A., Manogaran, G., & Long, H. V. (2019). A novel group decision making model based on neutrosophic sets for heart disease diagnosis. *Multimedia Tools and Applications*, 1-26.
13. Abdel-Basset, Mohamed, Gamal, A., Son, L.H., & Smarandache, F. (2019). A bipolar neutrosophic multi criteria decision making framework for professional selection. *Applied Sciences*, *10*(4), 1202.
14. Abdel-Basset, M., Manogaran, G., Gamal, A., & Chang, V. (2019). A novel intelligent medical decision support model based on soft computing and IoT. *IEEE Internet of Things Journal*, *7*(5), 4160-4170.
15. Ali, M. I., Feng, F., Liu, X., Min, W. K., & Shabir, M. (2009). On some new operations in soft set theory, *Computers and Mathematics with Applications*, *57*(9), 1547-1553.
16. Alkhazaleh, S., & Salleh, A. R. (2011). Soft Expert Sets. *Advances in Decision Sciences*, *2011*, 757868-1.
17. Al-Quran, A., & Hassan, N. (2016). Neutrosophic vague soft expert set theory. *Journal of Intelligent and Fuzzy Systems*, *30*(6), 3691-3702.
18. Alkhazaleh, S., & Salleh, A. R. (2014). Fuzzy soft expert set and its application. *Applied Mathematics*, *2014*.
19. Bashir, M., & Salleh, A. R. (2012). Possibility fuzzy soft expert set. *Open Journal of Applied Sciences*, *12*, 208-211.
20. Bashir, M., & Salleh, A. R. (2012). Fuzzy parameterized soft expert set. In Abstract and Applied Analysis, Hindawi.
21. Sahin, M., Alkhazaleh, S., & Ulucay, V. (2015). Neutrosophic soft expert sets. *Applied Mathematics*, *6*(1), 116.
22. Alhazaymeh, K., & Hassan, N. (2014). Mapping on generalized vague soft expert set. *International Journal of Pure and Applied Mathematics*, *93*(3), 369-376.
23. Alhazaymeh, K., & Hassan, N. (2014). Generalized vague soft expert set. *International Journal of Pure and Applied Mathematics*, *93*(3), 351-360.
24. Alhazaymeh, K., & Hassan, N. (2014). Application of generalized vague soft expert set in decision making. *International Journal of Pure and Applied Mathematics*, 93(3), 361-367.
25. Hassan, N., Ulucay, V., & Sahin, M. (2018). Q-neutrosophic soft expert set and its application in decision making. *International Journal of Fuzzy System Applications (IJFSA)*, *7*(4), 37-61.
26. Ulucay, V., Sahin, M. , & Hassan, N. (2018). Generalized neutrosophic soft expert set for multiple-criteria decision-making. *Symmetry*, *10*(10), 437.
27. Al-Qudah, Y., & Hassan, N. (2017). Bipolar fuzzy soft expert set and its application in decision making. *International Journal of Applied Decision Sciences*, *10*(2), 175-191.
28. Al-Qudah, Y., & Hassan, N. (2017). Complex multi-fuzzy soft expert set and its application. *International Journal of Mathematics and Computer Science*, *14*(1), 149-176.
29. Al-Quran, A., & Hassan, N. (2018). The complex neutrosophic soft expert set and its application in decision making. *Journal of Intelligent and Fuzzy Systems*, 34(1), 569-582.
30. Pramanik, S., Dey, P. P., & Giri, B. C. (2015). TOPSIS for single valued neutrosophic soft expert set based multi-attribute decision making problems. *Neutrosophic Sets and Systems*, *10*, 88-95.

31. Abu Qamar, M., & Hassan, N. (2018). Generalized Q-neutrosophic soft expert set for decision under uncertainty. *Symmetry*, *10*(11), 621.
32. Adam, F., & Hassan, N. (2016). Multi Q-fuzzy soft expert set and its application. *Journal of Intelligent and Fuzzy Systems*, *30*(2), 943-950.
33. Ulucay, V., Sahin, M., & Olgun, N. (2018). Time-Neutrosophic Soft Expert Sets and Its Decision Making Problem. *MATEMATIKA: Malaysian Journal of Industrial and Applied Mathematics*, 246-260.
34. Al-Quran, A., & Hassan, N. (2016). Fuzzy parameterised single valued neutrosophic soft expert set theory and its application in decision making. *International Journal of Applied Decision Science*, *9*(2), 212-227.
35. Hazaymeh, A. A., Abdullah, I. B., Balkhi, Z. T., & Ibrahim, R. I. (2012). Generalized fuzzy soft expert set. *Journal of Applied Mathematics*, *2012*.
36. Smarandache, F., (2018). Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set. *Neutrosophic Sets and Systems*, *22*, 168-170.
37. Saeed, M.; Ahsan, M.; Siddique, M. K., & Ahmad, M. R. (2020). A Study of The Fundamentals of Hypersoft Set Theory. *International Journal of Scientific and Engineering Research*, *2020*, 11.
38. Saeed, M., Rahman, A. U., Ahsan, M., & Smarandache, F. (2021). An Inclusive Study on Fundamentals of Hypersoft Set. In Theory and Application of Hypersoft Set, Pons Publishing House, Brussels.
39. Abbas, M.; Murtaza, G., & Smarandache, F. (2020). Basic operations on hypersoft sets and hypersoft point. *Neutrosophic Sets and Systems*, *35*, 407-421.
40. Rahman, A.U., Saeed, M., Smarandache, F., & Ahmad, M.R. (2020). Development of Hybrids of Hypersoft Set with Complex Fuzzy Set, Complex Intuitionistic Fuzzy set and Complex Neutrosophic Set. *Neutrosophic Sets and Systems*, *38*, 335-354. http://doi.org/10.5281/zenodo.4300520
41. Rahman, A.U., Saeed, M., & Smarandache, F. (2020). Convex and Concave Hypersoft Sets with Some Properties. *Neutrosophic Sets and Systems*, 38, 497-508. http://doi.org/10.5281/zenodo.4300580
42. Rahman, A. U., Saeed, M., & Zahid, S. (2021). Application in Decision Making Based on Fuzzy Parameterized Hypersoft Set Theory. *Asia Mathematika*, *5*(1) 19-27. http://doi.org/10.5281/zenodo.4721481.
43. Rahman, A. U., Saeed, M., & Dhital, A. (2021). Decision Making Application Based on Neutrosophic Parameterized Hypersoft Set Theory. *Neutrosophic Sets and Systems*, *41*, 1-14. http://doi.org/10.5281/zenodo.4625665.
44. Rahman, A. U., Hafeez, A., Saeed, M., Ahmad, M.R., & Farwa,U. (2021). Development of Rough Hypersoft Set with Application in Decision Making for the Best Choice of Chemical Material, In Theory and Application of Hypersoft Set, Pons Publication House, Brussel, 192-202. http://doi.org/10.5281/zenodo.4743367
45. Saeed, M., Ahsan, M., & Rahman, A. U. (2021). A novel approach to mappings on hypersoft classes with application, In Theory and Application of Hypersoft Set, Pons Publication House, Brussel, 175-191. http://doi.org/10.5281/zenodo.4743384
46. Saeed, M., Siddique, M.K., Ahsan, M., Ahmad, M.R., & Rahman, A. U. (2021). A Novel Approach to the Rudiments of Hypersoft Graphs, In Theory and Application of Hypersoft Set, Pons Publication House, Brussel, 203-214. http://doi.org/10.5281/zenodo.4736620
47. Ihsan, M., Rahman, A. U., & Saeed, M.(2021). Hypersoft Expert Set With Application in Decision Making for Recruitment Process. *Neutrosophic Sets and Systems*, *42*, 191-207. http://doi.org/10.5281/zenodo.4711524
48. Rahman, A. U., Saeed, M., Alodhaibi, S.S., & Abd, H.(2021). Decision Making Algorithmic Approaches Based on Parameterization of Neutrosophic Set under Hypersoft Set Environment with Fuzzy, Intuitionistic Fuzzy and Neutrosophic Settings. *CMES-Computer Modeling in Engineering & Sciences*, *128*(2), 743-777. http://doi.org/10.32604/cmes.2021.016736

49. Saeed, M., Ahsan, M., Rahman, A. U., Saeed, M.H., & Mehmood, A. (2021). An application of neutrosophic hypersoft mapping to diagnose brain tumor and propose appropriate treatment. *Journal of Intelligent & Fuzzy Systems*(In press), 1-23.

Received: May 24, 2021. Accepted: October 1, 2021

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

A Novel Approach to Neutrosophic Hypersoft Graphs with Properties

Muhammad Saeed[1], Atiqe Ur Rahman[2,*], Muhammad Arshad[2],and Alok Dhital[3]

[1,2]Department of Mathematics, University of Management and Technology, Lahore, Pakistan; muhammad.saeed@umt.edu.pk, maakhb84@gmail.com, aurkhb@gmail.com

[3] Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA.
adhital@unm.edu

*Correspondence: aurkhb@gmail.com;

Abstract. Neutrosophic hypersoft set is the combination of neutrosophic set and hypersoft set. It resolves the limitations of intuitionistic fuzzy sets and soft sets for the consideration of the degree of indeterminacy and multi-argument approximate function respectively. In this research article, a novel framework i.e. neutrosophic hypersoft graph, is formulated for handling neutrosophic hypersoft information by combining the concept of neutrosophic hypersoft sets with graph theory. Firstly, some of essential and fundamental notions of neutrosophic hypersoft graph are characterized with the help of numerical examples and graphical representation. Secondly, some set theoretic operations i.e. union, intersection and complement, are investigated with illustrative examples and pictorial depiction.

Keywords: Neutrosophic Set; Soft set; Hypersoft set; Neutrosophic soft graph; Neutrosophic hypersoft set; Neutrosophic hypersoft graph.

---

1. Introduction

In different mathematical disciplines, fuzzy sets theory (FS-Theory) [1] and intuitionistic fuzzy set theory (IFS-Theory) [2] are considered apt mathematical modes to tackle several intricate problems involving various uncertainties. The former emphasizes on a certain object's degree of true belongingness from the initial sample space, while the latter emphasizes degree of true membership and degree of non-membership with the state of their interdependence. These theories portray some kind of inadequacy in terms of providing due status to a degree of indeterminacy. The implementation of neutrosophic set theory (NS-Theory) [3, 4] overcomes this impediment by taking into account not only the proper status of degree of indeterminacy but also the state of dependence. This theory is more adaptable and suitable for dealing with inconsistent data. Wang et el [5] conceptualized single-valued neutrosophic set in which truth membership degree, indeterminacy degree and falsity degree are restricted within unit closed interval. Many researchers [6]- [14] have been drawn to NS-Theory for further application in statistics, topological spaces, and the construction of some neutrosophic-like blended structures with other existing models for useful applications in decision making. Edalatpanah [15] studied a system of neutrosophic linear equations (SNLE) based on the embedding approach. He used (α, β, γ)-cut for transformation of SNLE into a crisp linear system. Kumar et al. [16] exhibited a novel linear programming approach for finding the neutrosophic shortest path problem (NSSPP) considering Gaussian valued neutrosophic number.

FS-Theory, IFS-Theory and NS-Theory have some kind of complexities which restrain them to solve problem involving uncertainty professionally. The reason for these hurdles is, possibly, the inadequacy of the parametrization tool. It demands a mathematical tool free of all such impediments to tackle such issues. This scantiness is resolved with the development of soft set theory (SS-Theory) [17] which is a new parameterized family of subsets of the universe of discourse. The researchers [18]- [27] studied and investigated some elementary properties, operations, laws and hybrids of SS-Theory with applications in decision making. The gluing concept of NS-Theory and SS-Theory, is studied in [28] to make the NS-Theory adequate with parameterized tool. In many real life situations, distinct attributes are further partitioned in disjoint attribute-valued sets but existing SS-Theory is insufficient for dealing with such kind of attribute-valued sets. Hypersoft set theory (HS-Theory) [29] is developed to make the SST in line with attribute-valued sets to tackle real life scenarios. HS-Theory is an extension of SS-Theory as it transforms the single argument function into a multi-argument function. Certain elementary properties, aggregation operations, laws, relations and functions of HS-Theory, are investigated by [30]- [32] for proper understanding and further utilization in different fields. The applications of HS-Theory in decision making is studied by [33]- [37] and the intermingling study of HS-Theory with complex sets, convex and concave sets is studied by [38,39]. Deli [40] characterized hybrid set structures under uncertainly parameterized hypersoft sets with theory and applications. Gayen et al. [41] analyzed some essential aspects of plithogenic hypersoft algebraic structures. They also investigated the notions and basic properties of plithogenic hypersoft subgroups ie plithogenic fuzzy hypersoft subgroup, plithogenic intuitionistic fuzzy hypersoft subgroup, plithogenic neutrosophic hypersoft subgroup. Saeed et al. [42,43] discussed decision making techniques for neutrosophic hypersoft mapping and complex multi-fuzzy hypersoft set. Rahman et al. [44–46] studied decision making applications based on neutrosophic parameterized hypersoft Set, fuzzy parameterized hypersoft set and rough hypersoft set. Ihsan et al. [47] investigated hypersoft expert set with application in decision making for the best

selection of product. The gluing concept of graph theory with uncertain environments like fuzzy, intuitionistic fuzzy, neutrosophic, fuzzy soft, intuitionistic fuzzy soft and neutrosophic soft set, is discussed and characterized by the authors [48]- [54]. Inspiring from above literature in general, and from [55], [56] in specific, new notions of neutrosophic hypersoft graph are conceptualized along with some elementary types, essential properties, aggregation operations and generalized typical results. The rest of the paper is organized as:
In section 2, some basic definitions and terminologies are presented. In section 3, the elementary notions of neutrosophic hypersoft graphs are discussed with properties and results. In section 4, some set theoretic operations of neutrosophic hypersoft graphs are presented with examples. In section 5, paper is summarized with future directions.

2. **Preliminaries**

Here some essential terms and definitions are recalled from existing literature.

Definition 2.1. [3]
A neutrosophic set $\mathcal{K}$ defined as $\mathcal{K} = \{(k, <\mathcal{M}_K(k), \mathcal{I}_K(k), \mathcal{N}_K(k)>)|k \in \mathcal{Z}\}$ such that $\mathcal{M}_K(k) : \mathcal{Z} \rightarrow ^-]0,1[^+$, $\mathcal{I}_K(k) : \mathcal{Z} \rightarrow ^-]0,1[^+$ and $\mathcal{N}_K(k) : \mathcal{Z} \rightarrow ^-]0,1[^+$ where $\mathcal{M}_K(k)$ stands for membership, $\mathcal{N}_K(k)$ stands for non-membership and $\mathcal{I}_K(k)$ stands for indeterminacy under condition $^-0 \leq \mathcal{M}_K(k) + \mathcal{I}_K(k) + \mathcal{N}_K(k) \leq 3^+$.

Definition 2.2. [17]
A pair $(\Psi_M, \mathcal{W})$ is said to be soft set over $\mathcal{Z}$ (universe of discourse), where $\Psi_M : \mathcal{W} \rightarrow \mathcal{P}(\mathcal{Z})$ and $\mathcal{W}$ is a subset of set of attributes $\mathcal{X}$.

For more detail on soft set, see [18, 19].

Definition 2.3. [29]
A pair $(\xi_H, \mathcal{R})$ is said to be hypersoft set over $\mathcal{Z}$, where $\xi_H : \mathcal{R} \rightarrow \mathcal{P}(\mathcal{Z})$ and $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3 \times ... \times \mathcal{R}_n$, $\mathcal{R}_i$ are disjoint attribute-valued sets corresponding to distinct attributes r_i respectively for $1 \leq i \leq n$.

Definition 2.4. [29]
A pair $(\zeta_N, \mathcal{U})$ is said to be neutrosophic hypersoft set over $\mathcal{Z}$ if $\zeta_N : \mathcal{U} \rightarrow \mathcal{P}(\mathcal{Z})$, where $\mathcal{P}(\mathcal{Z})$ is a collection of all neutrosophic subsets and $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3 \times ... \times \mathcal{U}_n$, $\mathcal{U}_i$ are disjoint attribute-valued sets corresponding to distinct attributes u_i respectively for $1 \leq i \leq n$.

For more definitions and operations of hypersoft set, see [30–32].

Definition 2.5. [56]
Let $\mathcal{Q}$ and $\mathfrak{G}^\star = (\mathcal{V}, \mathcal{E})$ be a set of parameters and a simple graph respectively with $\mathcal{V}$ as set

of vertices and $\mathcal{E}$ as set of edges. Let $\mathcal{N}(\mathcal{V})$ be the set of all neutrosophic set in $\mathcal{V}$. By a neutrosophic soft graph (NS-Graph), we mean a 4-tuple $\mathfrak{G} = (\mathfrak{G}^{\star}, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ where $\mathbb{F} : \mathcal{Q} \to \mathcal{N}(\mathcal{V}), \mathbb{G} : \mathcal{Q} \to \mathcal{N}(\mathcal{V} \times \mathcal{V})$ given by

$$\mathbb{F}(\theta) = \mathbb{F}_{\theta} = \{\langle \nu, \mathcal{T}_{\mathbb{F}_{\theta}}(\nu), \mathcal{I}_{\mathbb{F}_{\theta}}(\nu), \mathcal{F}_{\mathbb{F}_{\theta}}(\nu)\rangle, \nu \in \mathcal{V}\}$$

and

$$\mathbb{G}(\theta) = \mathbb{G}_{\theta} = \{\langle (\nu,\mu), \mathcal{T}_{\mathbb{F}_{\theta}}(\nu,\mu), \mathcal{I}_{\mathbb{F}_{\theta}}(\nu,\mu), \mathcal{F}_{\mathbb{F}_{\theta}}(\nu,\mu)\rangle, (\nu,\mu) \in \mathcal{V} \times \mathcal{V}\}$$

are neutrosophic sets over $\mathcal{V}$ and $\mathcal{V} \times \mathcal{V}$ respectively with

$$\mathcal{T}_{\mathbb{F}_{\theta}}(\nu,\mu) \leq \min\{\mathcal{T}_{\mathbb{F}_{\theta}}(\nu), \mathcal{T}_{\mathbb{F}_{\theta}}(\mu)\}$$

$$\mathcal{I}_{\mathbb{F}_{\theta}}(\nu,\mu) \leq \min\{\mathcal{I}_{\mathbb{F}_{\theta}}(\nu), \mathcal{I}_{\mathbb{F}_{\theta}}(\mu)\}$$

$$\mathcal{F}_{\mathbb{F}_{\theta}}(\nu,\mu) \geq \max\{\mathcal{F}_{\mathbb{F}_{\theta}}(\nu), \mathcal{F}_{\mathbb{F}_{\theta}}(\mu)\}$$

for all $(\nu,\mu) \in (\mathcal{V} \times \mathcal{V})$ and $\theta \in \mathcal{Q}$.

3. Neutrosophic Hypersoft Graphs

In this section, notions of neutrosophic hypersoft graph are characterized with some properties and examples.

Definition 3.1. Let $\mathfrak{G}^{\star} = (\mathcal{V}, \mathcal{E})$ be a simple graph with $\mathcal{V}$ as set of vertices and $\mathcal{E}$ as set of edges and $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, ..., \mathcal{Q}_n$ are disjoint attribute-valued sets corresponding to distinct attributes $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ with $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_3 \times ... \times \mathcal{Q}_n$. Let $\mathcal{N}(\mathcal{V})$ be the set of all neutrosophic set in $\mathcal{V}$. By a neutrosophic hypersoft graph (NHS-Graph), we mean a 4-tuple $\mathfrak{G} = (\mathfrak{G}^{\star}, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ where $\mathbb{F} : \mathcal{Q} \to \mathcal{N}(\mathcal{V}), \mathbb{G} : \mathcal{Q} \to \mathcal{N}(\mathcal{V} \times \mathcal{V})$ given by

$$\mathbb{F}(\theta) = \mathbb{F}_{\theta} = \{\langle \nu, \mathcal{T}_{\mathbb{F}_{\theta}}(\nu), \mathcal{I}_{\mathbb{F}_{\theta}}(\nu), \mathcal{F}_{\mathbb{F}_{\theta}}(\nu)\rangle, \nu \in \mathcal{V}\}$$

and

$$\mathbb{G}(\theta) = \mathbb{G}_{\theta} = \{\langle (\nu,\mu), \mathcal{T}_{\mathbb{F}_{\theta}}(\nu,\mu), \mathcal{I}_{\mathbb{F}_{\theta}}(\nu,\mu), \mathcal{F}_{\mathbb{F}_{\theta}}(\nu,\mu)\rangle, (\nu,\mu) \in \mathcal{V} \times \mathcal{V}\}$$

are neutrosophic sets over $\mathcal{V}$ and $\mathcal{V} \times \mathcal{V}$ with

$$\mathcal{T}_{\mathbb{F}_{\theta}}(\nu,\mu) \leq \min\{\mathcal{T}_{\mathbb{F}_{\theta}}(\nu), \mathcal{T}_{\mathbb{F}_{\theta}}(\mu)\}$$

$$\mathcal{I}_{\mathbb{F}_{\theta}}(\nu,\mu) \geq \min\{\mathcal{I}_{\mathbb{F}_{\theta}}(\nu), \mathcal{I}_{\mathbb{F}_{\theta}}(\mu)\}$$

$$\mathcal{F}_{\mathbb{F}_{\theta}}(\nu,\mu) \geq \max\{\mathcal{F}_{\mathbb{F}_{\theta}}(\nu), \mathcal{F}_{\mathbb{F}_{\theta}}(\mu)\}$$

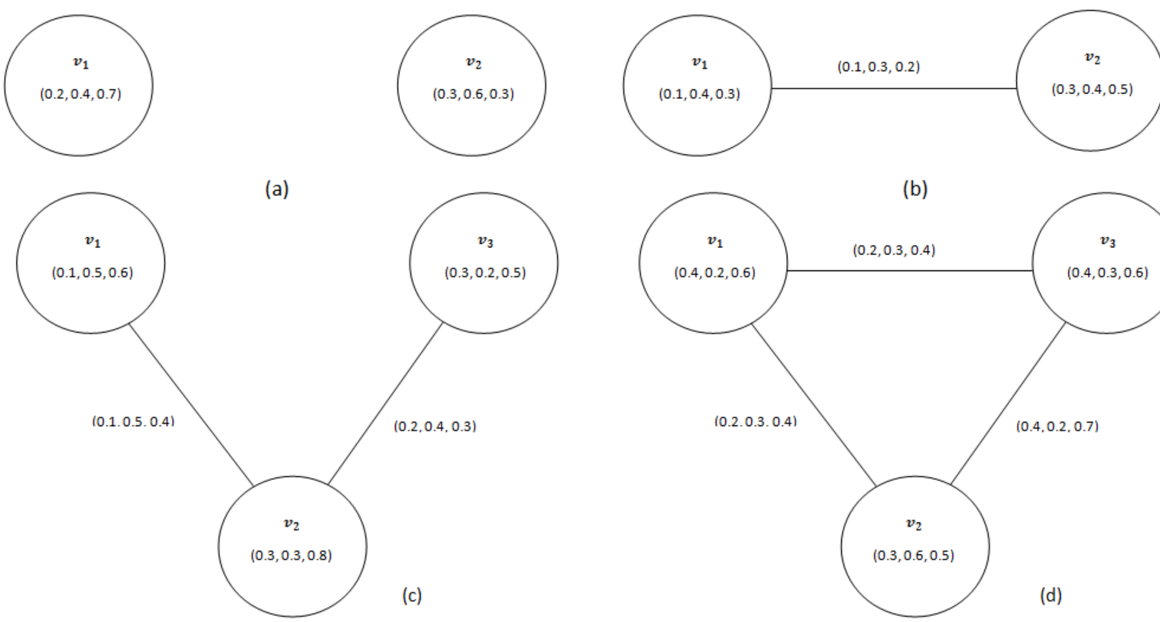
for all $(\nu,\mu) \in (\mathcal{V} \times \mathcal{V})$ and $\theta \in \mathcal{Q}$.

Note: The collection of all neutrosophic hypersoft graphs is denoted by Ω_{NHSG}.

TABLE 1. Tabular Representation of NHS-Graph $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$

$\mathbb{F}$	ν_1	ν_2	ν_3
θ_1	$(0.2, 0.4, 0.7)$	$(0.3, 0.6, 0.3)$	$(0, 0, 1)$
θ_2	$(0.1, 0.4, 0.3)$	$(0.3, 0.4, 0.5)$	$(0, 0, 1)$
θ_3	$(0.1, 0.5, 0.6)$	$(0.3, 0.3, 0.8)$	$(0.3, 0.2, 0.5)$
θ_4	$(0.4, 0.2, 0.6)$	$(0.3, 0.6, 0.5)$	$(0.4, 0.3, 0.6)$
$\mathbb{G}$	(ν_1, ν_2)	(ν_2, ν_3)	(ν_1, ν_3)
θ_1	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$
θ_2	$(0.1, 0.3, 0.2)$	$(0, 0, 1)$	$(0, 0, 1)$
θ_3	$(0.1, 0.5, 0.4)$	$(0.2, 0.4, 0.3)$	$(0, 0, 1)$
θ_4	$(0.2, 0.3, 0.4)$	$(0.2, 0.5, 0.3)$	$(0.4, 0.2, 0.7)$

FIGURE 1. Graphical Representation of TABLE 1 with (a) $\mathcal{N}(\theta_1)$, (b) $\mathcal{N}(\theta_2)$, (c) $\mathcal{N}(\theta_3)$ and (d) $\mathcal{N}(\theta_4)$



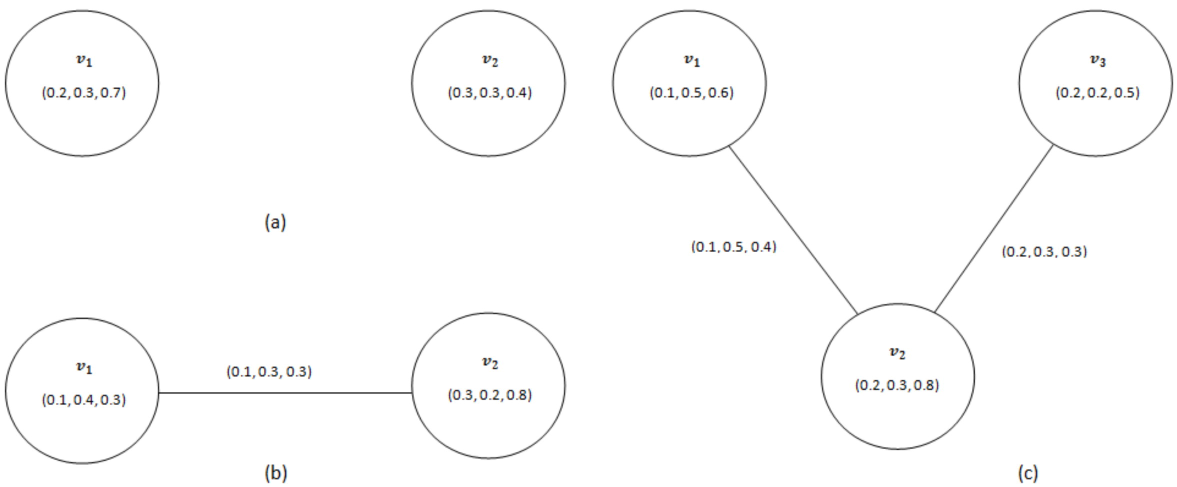
Example 3.2. Let $\mathfrak{G}^\star = (\mathcal{V}, \mathcal{E})$ be a simple graph with $\mathcal{V} = \{\nu_1, \nu_2, \nu_3\}$ and $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ are disjoint attribute-valued sets corresponding to distinct attributes $\alpha_1, \alpha_2, \alpha_3$ where $\mathcal{Q}_1 = \{\alpha_{11}, \alpha_{12}\}$, $\mathcal{Q}_2 = \{\alpha_{21}, \alpha_{22}\}$ and $\mathcal{Q}_3 = \{\alpha_{31}\}$. $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_3 = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ where each θ_i is a 3-tuple element of $\mathcal{Q}$ and $\mathcal{T}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{I}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{F}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 1$ for all $(\nu_i, \nu_j) \in \mathcal{V} \times \mathcal{V} \setminus \{(\nu_1, \nu_2), (\nu_2, \nu_3), (\nu_1, \nu_3)\}$. The tabular and graphical representation of NHS-Graph $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ are given in TABLE 1 and FIGURE 1 respectively.

Definition 3.3. A neutrosophic hypersoft graph $\mathfrak{G} = \left(\mathfrak{G}^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ is called a neutrosophic hypersoft subgraph of $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{A}, \mathbb{F}, \mathbb{G})$ if

TABLE 2. Tabular Representation of NHS-subgraph $\mathfrak{G} = \left(\mathfrak{G}^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$

$\mathbb{F}$	ν_1	ν_2	ν_3
θ_1	$(0.2, 0.3, 0.7)$	$(0.3, 0.3, 0.4)$	$(0, 0, 1)$
θ_2	$(0.1, 0.4, 0.3)$	$(0.3, 0.2, 0.8)$	$(0, 0, 1)$
θ_3	$(0.1, 0.5, 0.6)$	$(0.2, 0.3, 0.8)$	$(0.2, 0.2, 0.5)$
$\mathbb{G}$	(ν_1, ν_2)	(ν_2, ν_3)	(ν_1, ν_3)
θ_1	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$
θ_2	$(0.1, 0.3, 0.3)$	$(0, 0, 1)$	$(0, 0, 1)$
θ_3	$(0.1, 0.5, 0.4)$	$(0.2, 0.3, 0.3)$	$(0, 0, 1)$

FIGURE 2. Graphical Representation of TABLE 2 with (a) $\mathcal{N}(\theta_1)$, (b) $\mathcal{N}(\theta_2)$ and (c) $\mathcal{N}(\theta_3)$



(1) $\mathcal{Q}^1 \subseteq \mathcal{Q}$

(2) $\mathbb{F}^1_\theta \subseteq f$ which means $\mathcal{T}_{\mathbb{F}^1_\theta}(\nu) \leq \mathcal{T}_{\mathbb{F}_\theta}(\nu), \mathcal{I}_{\mathbb{F}^1_\theta}(\nu) \leq \mathcal{I}_{\mathbb{F}_\theta}(\nu), \mathcal{F}_{\mathbb{F}^1_\theta}(\nu) \geq \mathcal{F}_{\mathbb{F}_\theta}(\nu)$.

(3) $\mathbb{G}^1_\theta \subseteq g$ which means $\mathcal{T}_{\mathbb{G}^1_\theta}(\nu) \leq \mathcal{T}_{\mathbb{G}_\theta}(\nu), \mathcal{I}_{\mathbb{G}^1_\theta}(\nu) \leq \mathcal{I}_{\mathbb{G}_\theta}(\nu), \mathcal{F}_{\mathbb{G}^1_\theta}(\nu) \geq \mathcal{F}_{\mathbb{G}_\theta}(\nu)$.

for all $\theta \in \mathcal{Q}^1$ and $\mathcal{Q}^1 = \mathcal{Q}_1 \times \mathcal{Q}_2 \times ... \times \mathcal{Q}_n$ where $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_n$ are disjoint attribute-valued sets corresponding to distinct attributes $\alpha_1, \alpha_2, ..., \alpha_n$ respectively.

Example 3.4. Let $\mathfrak{G}^\star = (\mathcal{V}, E)$ be a simple graph with $\mathcal{V} = \{\nu_1, \nu_2, \nu_3\}$ and $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ are disjoint attribute-valued sets corresponding to disjoint attributes $\alpha_1, \alpha_2, \alpha_3$ where $\mathcal{Q}_1 = \{\alpha_{11}, \alpha_{12}\}$, $\mathcal{Q}_2 = \{\alpha_{21}\}$ and $\mathcal{Q}_3 = \{\alpha_{31}\}$. $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_3 = \{\theta_1, \theta_2, \theta_3\}$ where each θ_i is a 3-tuple element of $\mathcal{Q}$. The tabular and graphical representation of NHS-subgraph $\mathfrak{G} = \left(\mathfrak{G}^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ are given in TABLE 2 and FIGURE 2 respectively. In this graph, $\mathcal{T}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{I}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{F}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 1$ for all $(\nu_i, \nu_j) \in \mathcal{V} \times \mathcal{V} \setminus \{(\nu_1, \nu_2), (\nu_2, \nu_3), (\nu_1, \nu_3)\}$ and for all $\theta \in \mathcal{Q}$.

Definition 3.5. A neutrosophic hypersoft subgraph $\mathfrak{G} = \left(\mathfrak{G}^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ is called a neutrosophic hypersoft spanning subgraph of $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ if $\mathbb{F}^1_\theta(\nu) = \mathbb{F}(\nu)$ for all $\nu \in \mathcal{V}, e \in \mathcal{Q}$ where $\mathcal{Q}^1 = \mathcal{Q}_1 \times \mathcal{Q}_2 \times ... \times \mathcal{Q}_n$ and $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_n$ are disjoint attribute-valued sets corresponding to disjoint attributes $\alpha_1, \alpha_2, ..., \alpha_n$ respectively.

Definition 3.6. A strong neutrosophic hypersoft subgraph $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ is a neutrosophic hypersoft subgraph with condition that $\mathbb{G}_\theta(\nu, \mu) = \mathbb{F}_\theta(\nu) \cap \mathbb{F}_\theta(\mu)$ for $x, y \in \mathcal{V}$ and $e \in \mathcal{Q}$ such that $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2 ... \times \mathcal{Q}_n$ and $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_n$ are disjoint attribute-valued sets corresponding to disjoint attributes $\alpha_1, \alpha_2, ..., \alpha_n$ respectively.

4. Set Theoretic Operations of NHS-Graphs

In this section, some theoretic operations (i.e. union, intersection and complement) of neutrosophic hypersoft graph (NHS-Graphs) are investigated with suitable examples and results.

Definition 4.1. The union of two NHS-Graphs $\mathfrak{G}_1 = \left(\mathfrak{G}^\star_1, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$, $\mathfrak{G}_2 = \left(\mathfrak{G}^\star_2, \mathcal{Q}^2, \mathbb{F}^2, \mathbb{G}^2\right)$, denoted by $\mathfrak{G}_1 \cup \mathfrak{G}_2$, is a NHS-Graph $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ such that $\mathcal{Q} = \mathcal{Q}^1 \cup \mathcal{Q}^2$. In this graph, the neutrosophic components for $\mathbb{F}$ are given as follows:

$$\mathcal{T}_{\mathbb{F}_\theta}(\nu) = \begin{cases} \mathcal{T}_{\mathbb{F}^1_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 - \mathcal{V}_2\, or \\ if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 \cap \mathcal{V}_2\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 - \mathcal{V}_2 \end{cases} \\ \mathcal{T}_{\mathbb{F}^2_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, \nu \in \mathcal{V}_2 - \mathcal{V}_1\, or \\ if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, \nu \in \mathcal{V}_2 \cap \mathcal{V}_1\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_2 - \mathcal{V}_1 \end{cases} \\ \max\left\{\mathcal{T}_{\mathbb{F}^1_\theta}(\nu), \mathcal{T}_{\mathbb{F}^2_\theta}(\nu)\right\} \{if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 \cap \mathcal{V}_2 \\ 0,\, otherwise \end{cases}$$

and

$$\mathcal{I}_{\mathbb{F}_\theta}(\nu) = \begin{cases} \mathcal{I}_{\mathbb{F}^1_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 - \mathcal{V}_2\, or \\ if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 \cap \mathcal{V}_2\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 - \mathcal{V}_2 \end{cases} \\ \mathcal{I}_{\mathbb{F}^2_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, \nu \in \mathcal{V}_2 - \mathcal{V}_1\, or \\ if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, \nu \in \mathcal{V}_2 \cap \mathcal{V}_1\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_2 - \mathcal{V}_1 \end{cases} \\ \max\left\{\mathcal{I}_{\mathbb{F}^1_\theta}(\nu), \mathcal{I}_{\mathbb{F}^2_\theta}(\nu)\right\} \{if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 \cap \mathcal{V}_2 \\ 0,\, otherwise \end{cases}$$

and

$$\mathcal{F}_{\mathbb{F}_\theta}(\nu) = \begin{cases} \mathcal{F}_{\mathbb{F}^1_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 - \mathcal{V}_2\, or \\ if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 \cap \mathcal{V}_2\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 - \mathcal{V}_2 \end{cases} \\ \mathcal{F}_{\mathbb{F}^2_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, \nu \in \mathcal{V}_2 - \mathcal{V}_1\, or \\ if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, \nu \in \mathcal{V}_2 \cap \mathcal{V}_1\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_2 - \mathcal{V}_1 \end{cases} \\ \min\left\{\mathcal{F}_{\mathbb{F}^1_\theta}(\nu), \mathcal{F}_{\mathbb{F}^2_\theta}(\nu)\right\} \left\{ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, \nu \in \mathcal{V}_1 \cap \mathcal{V}_2 \right. \\ 0,\, otherwise \end{cases}.$$

Also the neutrosophic components for $\mathbb{G}$ are given as follows:

$$\mathcal{T}_{\mathbb{G}_\theta}(\nu) = \begin{cases} \mathcal{T}_{\mathbb{G}^1_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) - (\mathcal{V}_2 \times \mathcal{V}_2)\, or \\ if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) \cap (\mathcal{V}_2 \times \mathcal{V}_2)\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) - (\mathcal{V}_2 \times \mathcal{V}_2) \end{cases} \\ \mathcal{T}_{\mathbb{G}^2_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, (\nu,\mu) \in (\mathcal{V}_2 \times \mathcal{V}_2) - (\mathcal{V}_1 \times \mathcal{V}_1)\, or \\ if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, (\nu,\mu) \in (\mathcal{V}_2 \times \mathcal{V}_2) \cap (\mathcal{V}_1 \times \mathcal{V}_1)\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_2 \times \mathcal{V}_2) - (\mathcal{V}_1 \times \mathcal{V}_1) \end{cases} \\ \max\left\{\mathcal{T}_{\mathbb{G}^1_\theta}(\nu), \mathcal{T}_{\mathbb{G}^2_\theta}(\nu)\right\} \left\{ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) \cap (\mathcal{V}_2 \times \mathcal{V}_2) \right. \\ 0,\, otherwise \end{cases}$$

and

$$\mathcal{I}_{\mathbb{G}_\theta}(\nu) = \begin{cases} \mathcal{I}_{\mathbb{G}^1_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) - (\mathcal{V}_2 \times \mathcal{V}_2)\, or \\ if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) \cap (\mathcal{V}_2 \times \mathcal{V}_2)\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) - (\mathcal{V}_2 \times \mathcal{V}_2) \end{cases} \\ \mathcal{I}_{\mathbb{G}^2_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, (\nu,\mu) \in (\mathcal{V}_2 \times \mathcal{V}_2) - (\mathcal{V}_1 \times \mathcal{V}_1)\, or \\ if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, (\nu,\mu) \in (\mathcal{V}_2 \times \mathcal{V}_2) \cap (\mathcal{V}_1 \times \mathcal{V}_1)\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_2 \times \mathcal{V}_2) - (\mathcal{V}_1 \times \mathcal{V}_1) \end{cases} \\ \max\left\{\mathcal{I}_{\mathbb{G}^1_\theta}(\nu), \mathcal{I}_{\mathbb{G}^2_\theta}(\nu)\right\} \left\{ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) \cap (\mathcal{V}_2 \times \mathcal{V}_2) \right. \\ 0,\, otherwise \end{cases}$$

and

$$\mathcal{F}_{\mathbb{G}_\theta}(\nu) = \begin{cases} \mathcal{F}_{\mathbb{G}^1_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) - (\mathcal{V}_2 \times \mathcal{V}_2)\, or \\ if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) \cap (\mathcal{V}_2 \times \mathcal{V}_2)\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) - (\mathcal{V}_2 \times \mathcal{V}_2) \end{cases} \\ \mathcal{F}_{\mathbb{G}^2_\theta}(\nu) \begin{cases} if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, (\nu,\mu) \in (\mathcal{V}_2 \times \mathcal{V}_2) - (\mathcal{V}_1 \times \mathcal{V}_1)\, or \\ if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1\, and\, (\nu,\mu) \in (\mathcal{V}_2 \times \mathcal{V}_2) \cap (\mathcal{V}_1 \times \mathcal{V}_1)\, or \\ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_2 \times \mathcal{V}_2) - (\mathcal{V}_1 \times \mathcal{V}_1) \end{cases} \\ \min\left\{\mathcal{F}_{\mathbb{G}^1_\theta}(\nu), \mathcal{F}_{\mathbb{G}^2_\theta}(\nu)\right\} \left\{ if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2\, and\, (\nu,\mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) \cap (\mathcal{V}_2 \times \mathcal{V}_2) \right. \\ 0,\, otherwise \end{cases}.$$

Theorem 4.2. *If $\mathfrak{G}_1, \mathfrak{G}_2 \in \Omega_{NHSG}$ then $\mathfrak{G}_1 \cup \mathfrak{G}_2 \in \Omega_{NHSG}$.*

Proof. Consider two NHS-Graphs $\mathfrak{G}_1 = \left(\mathfrak{G}_1^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ and $\mathfrak{G}_2 = \left(\mathfrak{G}_2^\star, \mathcal{Q}^2, \mathbb{F}^2, \mathbb{G}^2\right)$ as defined in 3.1. Let $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ be the union of NHS-Graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ where $\mathcal{Q} = \mathcal{Q}^1 \cup \mathcal{Q}^2$. Now let $\theta \in \mathcal{Q}^1 - \mathcal{Q}^2$ and $(\nu, \mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) - (\mathcal{V}_2 \times \mathcal{V}_2)$, then

$$\begin{aligned}\mathcal{T}_{\mathbb{G}_\theta}(\nu,\mu) = \mathcal{T}_{\mathbb{G}^1_\theta}(\nu,\mu) &\leq \min\left\{\mathcal{T}_{\mathbb{F}^1_\theta}(\nu), \mathcal{T}_{\mathbb{F}^1_\theta}(\mu)\right\} \\ &= \min\left\{\mathcal{T}_{\mathbb{F}_\theta}(\nu), \mathcal{T}_{\mathbb{F}_\theta}(\mu)\right\}\end{aligned}$$

so

$$\mathcal{T}_{\mathbb{G}_\theta}(\nu,\mu) \leq \min\left\{\mathcal{T}_{\mathbb{F}_\theta}(\nu), \mathcal{T}_{\mathbb{F}_\theta}(\mu)\right\}.$$

Also

$$\begin{aligned}\mathcal{I}_{\mathbb{G}_\theta}(\nu,\mu) = \mathcal{I}_{\mathbb{G}^1_\theta}(\nu,\mu) &\leq \min\left\{\mathcal{I}_{\mathbb{F}^1_\theta}(\nu), \mathcal{I}_{\mathbb{F}^1_\theta}(\mu)\right\} \\ &= \min\left\{\mathcal{I}_{\mathbb{F}_\theta}(\nu), \mathcal{I}_{\mathbb{F}_\theta}(\mu)\right\}\end{aligned}$$

so

$$\mathcal{I}_{\mathbb{G}_\theta}(\nu,\mu) \leq \min\left\{\mathcal{I}_{\mathbb{F}_\theta}(\nu), \mathcal{I}_{\mathbb{F}_\theta}(\mu)\right\}.$$

Now

$$\begin{aligned}\mathcal{F}_{\mathbb{G}_\theta}(\nu,\mu) = \mathcal{F}_{\mathbb{G}^1_\theta}(\nu,\mu) &\geq \max\left\{\mathcal{F}_{\mathbb{F}^1_\theta}(\nu), \mathcal{F}_{\mathbb{F}^1_\theta}(\mu)\right\} \\ &= \max\left\{\mathcal{F}_{\mathbb{F}_\theta}(\nu), \mathcal{F}_{\mathbb{F}_\theta}(\mu)\right\}\end{aligned}$$

so

$$\mathcal{F}_{\mathbb{G}_\theta}(\nu,\mu) \geq \max\left\{\mathcal{F}_{\mathbb{F}_\theta}(\nu), \mathcal{F}_{\mathbb{F}_\theta}(\mu)\right\}.$$

Similar results are obtained when $\theta \in \mathcal{Q}^1 - \mathcal{Q}^2$ and $(\nu, \mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) \cap (\mathcal{V}_2 \times \mathcal{V}_2)$ or $\theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2$ and $(\nu, \mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) - (\mathcal{V}_2 \times \mathcal{V}_2)$.
Now if $\theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2$ and $(\nu, \mu) \in (\mathcal{V}_1 \times \mathcal{V}_1) \cap (\mathcal{V}_2 \times \mathcal{V}_2)$ then

$$\begin{aligned}\mathcal{T}_{\mathbb{G}_\theta}(\nu,\mu) &= \max\left\{\mathcal{T}_{\mathbb{G}^1_\theta}(\nu,\mu), \mathcal{T}_{\mathbb{G}^2_\theta}(\nu,\mu)\right\} \\ &\leq \max\left\{\min\left\{\mathcal{T}_{\mathbb{F}^1_\theta}(\nu), \mathcal{T}_{\mathbb{F}^1_\theta}(\mu)\right\}, \min\left\{\mathcal{T}_{\mathbb{F}^2_\theta}(\nu), \mathcal{T}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\} \\ &\leq \min\left\{\max\left\{\mathcal{T}_{\mathbb{F}^1_\theta}(\nu), \mathcal{T}_{\mathbb{F}^1_\theta}(\mu)\right\}, \max\left\{\mathcal{T}_{\mathbb{F}^2_\theta}(\nu), \mathcal{T}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\} \\ &= \min\left\{\mathcal{T}_{\mathbb{F}_\theta}(\nu), \mathcal{T}_{\mathbb{F}_\theta}(\mu)\right\}.\end{aligned}$$

Also

$$\begin{aligned}\mathcal{I}_{\mathbb{G}_\theta}(\nu,\mu) &= \max\left\{\mathcal{I}_{\mathbb{G}^1_\theta}(\nu,\mu), \mathcal{I}_{\mathbb{G}^2_\theta}(\nu,\mu)\right\} \\ &\leq \max\left\{\min\left\{\mathcal{I}_{\mathbb{F}^1_\theta}(\nu), \mathcal{I}_{\mathbb{F}^1_\theta}(\mu)\right\}, \min\left\{\mathcal{I}_{\mathbb{F}^2_\theta}(\nu), \mathcal{I}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\} \\ &\leq \min\left\{\max\left\{\mathcal{I}_{\mathbb{F}^1_\theta}(\nu), \mathcal{I}_{\mathbb{F}^1_\theta}(\mu)\right\}, \max\left\{\mathcal{I}_{\mathbb{F}^2_\theta}(\nu), \mathcal{I}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\} \\ &= \min\left\{\mathcal{I}_{\mathbb{F}_\theta}(\nu), \mathcal{I}_{\mathbb{F}_\theta}(\mu)\right\}.\end{aligned}$$

In the same way

$$\mathcal{F}_{\mathbb{G}_\theta}(\nu,\mu) = \min\left\{\mathcal{F}_{\mathbb{G}^1_\theta}(\nu,\mu), \mathcal{F}_{\mathbb{G}^2_\theta}(\nu,\mu)\right\}$$

TABLE 3. Tabular Representation of NHS-Graph $\mathfrak{G}_1 = \left(\mathfrak{G}_1^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ according to Example 4.3

$\mathbb{F}$	ν_1	ν_2	ν_3
θ_1	$(0.2, 0.3, 0.4)$	$(0.3, 0.6, 0.8)$	$(0.3, 0.4, 0.5)$
θ_2	$(0.2, 0.4, 0.8)$	$(0.2, 0.3, 0.4)$	$(0.5, 0.7, 0.8)$
θ_3	$(0.6, 0.7, 0.8)$	$(0.4, 0.5, 0.7)$	$(0.7, 0.9, 0.9)$
$\mathbb{G}$	(ν_1, ν_2)	(ν_2, ν_3)	(ν_1, ν_3)
θ_1	$(0.2, 0.3, 0.6)$	$(0.2, 0.4, 0.9)$	$(0.2, 0.3, 0.8)$
θ_2	$(0.2, 0.3, 0.9)$	$(0.2, 0.2, 0.9)$	$(0.2, 0.3, 0.8)$
θ_3	$(0, 0, 1)$	$(0.3, 0.4, 0.9)$	$(0.2, 0.4, 0.9)$

$$\geq \min\left\{\max\left\{\mathcal{F}_{\mathbb{F}^1_\theta}(\nu), \mathcal{F}_{\mathbb{F}^1_\theta}(\mu)\right\}, \max\left\{\mathcal{F}_{\mathbb{F}^2_\theta}(\nu), \mathcal{F}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\}$$
$$\geq \max\left\{\min\left\{\mathcal{F}_{\mathbb{F}^1_\theta}(\nu), \mathcal{F}_{\mathbb{F}^1_\theta}(\mu)\right\}, \min\left\{\mathcal{F}_{\mathbb{F}^2_\theta}(\nu), \mathcal{F}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\}$$
$$= \max\left\{\mathcal{F}_{\mathbb{F}_\theta}(\nu), \mathcal{F}_{\mathbb{F}_\theta}(\mu)\right\}.$$

Hence the union $\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2$ is NHS-Graphs. □

Example 4.3. Let $\mathfrak{G}_1 = \left(\mathfrak{G}_1^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ be a neutrosophic hypersoft graph where $\mathfrak{G}_1^\star = (\mathcal{V}_1, \mathcal{E}_1)$ with $\mathcal{V}_1 = \{\nu_1, \nu_2, \nu_3\}$ and $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ are disjoint attribute-valued sets corresponding to distinct attributes $\alpha_1, \alpha_2, \alpha_3$ where $\mathcal{Q}_1 = \{\alpha_{11}\}$, $\mathcal{Q}_2 = \{\alpha_{21}\}$ and $\mathcal{Q}_3 = \{\alpha_{31}, \alpha_{32}, \alpha_{33}\}$. $\mathcal{Q}^1 = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_3 = \{\theta_1, \theta_2, \theta_3\}$ where each θ_i is a 3-tuple element of $\mathcal{Q}^1$ and $\mathcal{T}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{I}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{F}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 1$ for all $(\nu_i, \nu_j) \in \mathcal{V}_1 \times \mathcal{V}_1 \setminus \{(\nu_1, \nu_2), (\nu_2, \nu_3), (\nu_1, \nu_3)\}$. Its tabular representation is given in TABLE 3. Also let $\mathfrak{G}_2 = \left(\mathfrak{G}_2^\star, \mathcal{Q}^2, \mathbb{F}^2, \mathbb{G}^2\right)$ be a neutrosophic hypersoft graph where $\mathfrak{G}_2^\star = (\mathcal{V}_2, \mathcal{E}_2)$ with $\mathcal{V}_2 = \{\nu_3, \nu_4, \nu_5\}$ and $\mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5$ are disjoint attribute-valued sets corresponding to distinct attributes $\alpha_3, \alpha_4, \alpha_5$ where $\mathcal{Q}_3 = \{\alpha_{31}, \alpha_{32}\}$, $\mathcal{Q}_4 = \{\alpha_{41}\}$, $\mathcal{Q}_5 = \{\alpha_{51}\}$. $\mathcal{Q}^2 = \mathcal{Q}_3 \times \mathcal{Q}_4 \times \mathcal{Q}_5 = \{\theta_2, \theta_4\}$ where each θ_i is a 3-tuple element of $\mathcal{Q}^2$ and $\mathcal{T}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{I}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{F}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 1$ for all $(\nu_i, \nu_j) \in \mathcal{V}_2 \times \mathcal{V}_2 \setminus \{(\nu_3, \nu_4), (\nu_4, \nu_5), (\nu_3, \nu_5)\}$. Its tabular representation is given in TABLE 4.

Now Let $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ be the union of two neutrosophic hypersoft graphs $\mathfrak{G}_1 = \left(\mathfrak{G}_1^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ and $\mathfrak{G}_2 = \left(\mathfrak{G}_2^\star, \mathcal{Q}^2, \mathbb{F}^2, \mathbb{G}^2\right)$ where $\mathcal{Q} = \mathcal{Q}^1 \cup \mathcal{Q}^2$ and $\mathcal{T}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{I}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{F}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 1$ for all $(\nu_i, \nu_j) \in \mathcal{V} \times \mathcal{V} \setminus \{(\nu_1, \nu_2), (\nu_1, \nu_3), (\nu_2, \nu_3), (\nu_3, \nu_4), (\nu_3, \nu_5), (\nu_4, \nu_5)\}$. Its (union of these two graphs) tabular representation is given in TABLE 5.

Definition 4.4. The intersection of two NHS-Graphs $\mathfrak{G}_1 = \left(\mathfrak{G}_1^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$, $\mathfrak{G}_2 = \left(\mathfrak{G}_2^\star, \mathcal{Q}^2, \mathbb{F}^2, \mathbb{G}^2\right)$, denoted by $\mathfrak{G}_1 \cap \mathfrak{G}_2$, is a NHS-Graph $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ such that $\mathcal{Q} =$

FIGURE 3. Graphical Representation of TABLE 3 with (a) $\mathcal{N}(\theta_1)$, (b) $\mathcal{N}(\theta_2)$ and (c) $\mathcal{N}(\theta_3)$

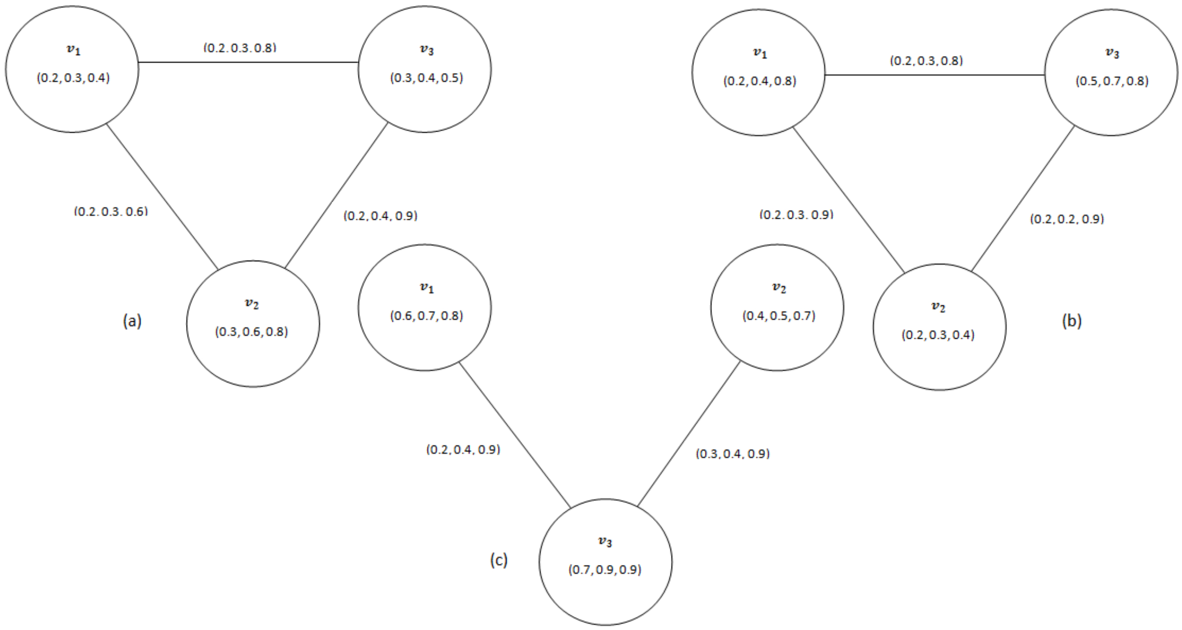


TABLE 4. Tabular Representation of NHS-Graph $\mathfrak{G}_2 = \left(\mathfrak{G}_2^\star, \mathcal{Q}^2, \mathbb{F}^2, \mathbb{G}^2\right)$ according to Example 4.3

$\mathbb{F}$	ν_3	ν_4	ν_5
θ_2	$(0.3, 0.4, 0.5)$	$(0.2, 0.3, 0.5)$	$(0.5, 0.7, 0.8)$
θ_4	$(0.6, 0.8, 0.9)$	$(0.4, 0.7, 0.9)$	$(0.4, 0.5, 0.6)$
$\mathbb{G}$	(ν_3, ν_4)	(ν_4, ν_5)	(ν_3, ν_5)
θ_2	$(0.2, 0.3, 0.9)$	$(0.3, 0.4, 0.9)$	$(0, 0, 1)$
θ_4	$(0.2, 0.2, 0.9)$	$(0.3, 0.3, 0.9)$	$(0.3, 0.4, 0.9)$

TABLE 5. Tabular Representation of $\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2$

$\mathbb{F}$	ν_1	ν_2	ν_3	ν_4	ν_5	
θ_1	$(0.2, 0.3, 0.4)$	$(0.3, 0.4, 0.5)$	$(0.3, 0.6, 0.8)$	$(0, 0, 1)$	$(0, 0, 1)$	
θ_2	$(0.2, 0.4, 0.8)$	$(0.2, 0.3, 0.4)$	$(0.3, 0.5, 0.5)$	$(0.2, 0.3, 0.4)$	$(0.5, 0.7, 0.8)$	
θ_3	$(0.6, 0.7, 0.8)$	$(0.4, 0.5, 0.7)$	$(0.7, 0.9, 0.9)$	$(0, 0, 1)$	$(0, 0, 1)$	
θ_4	$(0, 0, 1)$	$(0, 0, 1)$	$(0.6, 0.8, 0.9)$	$(0.4, 0.7, 0.9)$	$(0.4, 0.5, 0.6)$	
$\mathbb{G}$	(ν_1, ν_2)	(ν_1, ν_3)	(ν_2, ν_3)	(ν_3, ν_4)	(ν_3, ν_5)	(ν_4, ν_5)
θ_1	$(0.2, 0.3, 0.8)$	$(0.2, 0.3, 0.9)$	$(0.2, 0.4, 0.9)$	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$
θ_2	$(0.2, 0.3, 0.8)$	$(0.2, 0.3, 0.9)$	$(0.2, 0.2, 0.9)$	$(0.2, 0.3, 0.9)$	$(0.3, 0.4, 0.9)$	$(0, 0, 1)$
θ_3	$(0.2, 0.4, 0.9)$	$(0, 0, 1)$	$(0.3, 0.4, 0.9)$	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$
θ_4	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$	$(0.2, 0.2, 0.9)$	$(0.3, 0.3, 0.9)$	$(0.3, 0.4, 0.9)$

FIGURE 4. Graphical Representation of TABLE 4 with (a) $\mathcal{N}(\theta_2)$ and (b) $\mathcal{N}(\theta_4)$

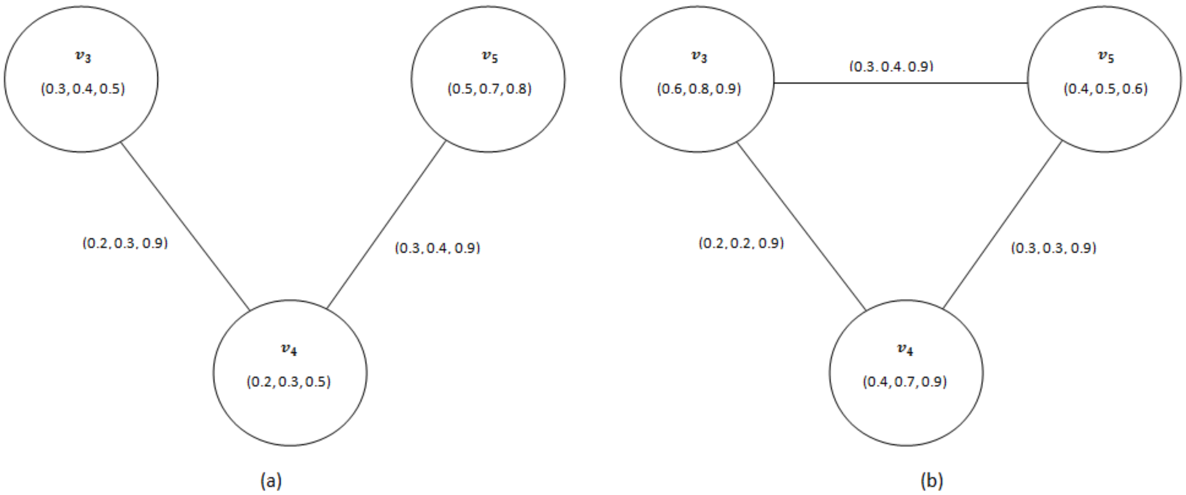
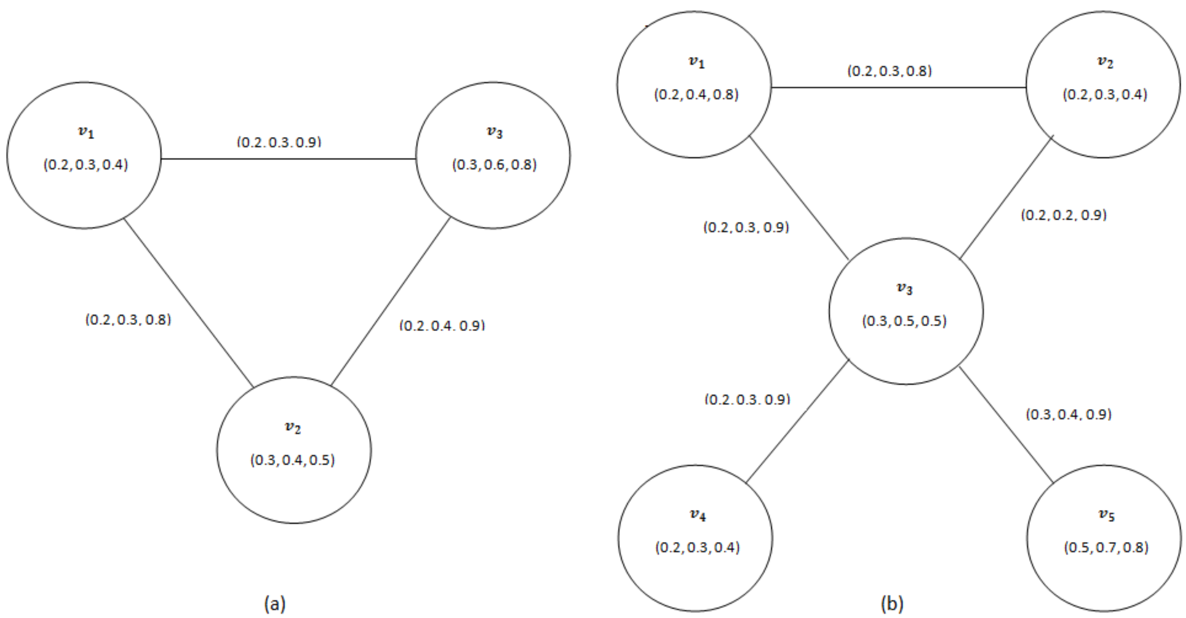


FIGURE 5. Graphical Representation of TABLE 5 with (a) $\mathcal{N}(\theta_1)$ and (b) $\mathcal{N}(\theta_2)$



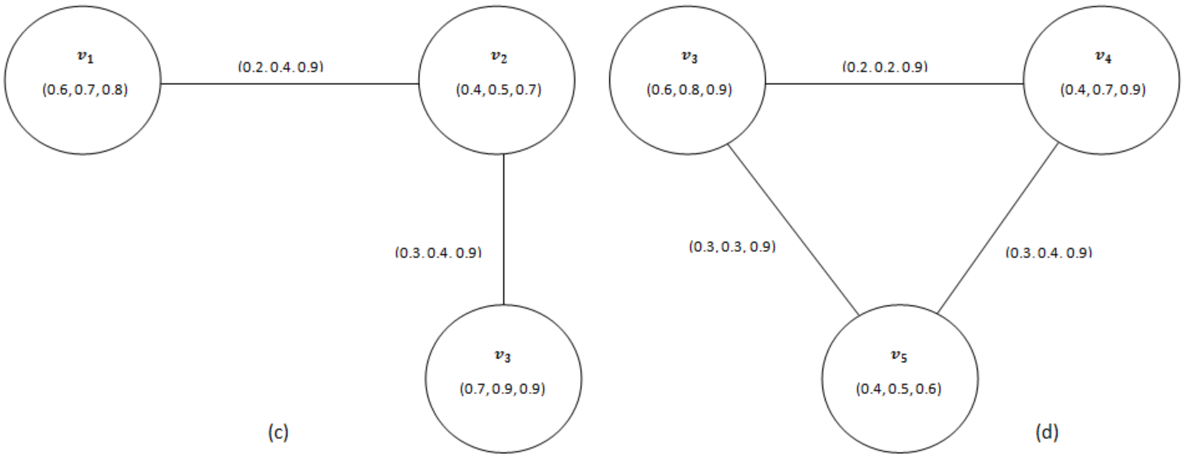
$\mathcal{Q}^1 \cap \mathcal{Q}^2, \mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$. In this graph, the neutrosophic components for $\mathbb{F}$ are given as follows:

$$\mathcal{T}_{\mathbb{F}_\theta} = \begin{cases} \mathcal{T}^1_{\mathbb{F}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2 \\ \mathcal{T}^2_{\mathbb{F}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1 \\ \min\left\{\mathcal{T}^1_{\mathbb{F}_\theta}(\nu), \mathcal{T}^2_{\mathbb{F}_\theta}(\nu)\right\}\, if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \end{cases},$$

and

$$\mathcal{I}_{\mathbb{F}_\theta} = \begin{cases} \mathcal{I}^1_{\mathbb{F}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2 \\ \mathcal{I}^2_{\mathbb{F}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1 \\ \min\left\{\mathcal{I}^1_{\mathbb{F}_\theta}(\nu), \mathcal{I}^2_{\mathbb{F}_\theta}(\nu)\right\}\, if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \end{cases},$$

FIGURE 6. Graphical Representation of TABLE 5 with (c) $\mathcal{N}(\theta_3)$ and (d) $\mathcal{N}(\theta_4)$



and

$$\mathcal{F}_{\mathbb{F}_\theta} = \begin{cases} \mathcal{F}^1_{\mathbb{F}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2 \\ \mathcal{F}^2_{\mathbb{F}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1 \\ \max\left\{\mathcal{F}^1_{\mathbb{F}_\theta}(\nu), \mathcal{F}^2_{\mathbb{F}_\theta}(\nu)\right\}\, if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \end{cases}.$$

The neutrosophic components for $\mathbb{G}$ are given as follows:

$$\mathcal{T}_{\mathbb{G}_\theta} = \begin{cases} \mathcal{T}^1_{\mathbb{G}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2 \\ \mathcal{T}^2_{\mathbb{G}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1 \\ \min\left\{\mathcal{T}^1_{\mathbb{G}_\theta}(\nu), \mathcal{T}^2_{\mathbb{G}_\theta}(\nu)\right\}\, if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \end{cases},$$

and

$$\mathcal{I}_{\mathbb{G}_\theta} = \begin{cases} \mathcal{I}^1_{\mathbb{G}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2 \\ \mathcal{I}^2_{\mathbb{G}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1 \\ \min\left\{\mathcal{I}^1_{\mathbb{G}_\theta}(\nu), \mathcal{I}^2_{\mathbb{G}_\theta}(\nu)\right\}\, if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \end{cases},$$

and

$$\mathcal{F}_{\mathbb{G}_\theta} = \begin{cases} \mathcal{F}^1_{\mathbb{G}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^1 - \mathcal{Q}^2 \\ \mathcal{F}^2_{\mathbb{G}_\theta}(\nu)\, if\, \theta \in \mathcal{Q}^2 - \mathcal{Q}^1 \\ \max\left\{\mathcal{F}^1_{\mathbb{G}_\theta}(\nu), \mathcal{F}^2_{\mathbb{G}_\theta}(\nu)\right\}\, if\, \theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \end{cases}.$$

Theorem 4.5. *If $\mathfrak{G}_1, \mathfrak{G}_2 \in \Omega_{NHSG}$ then $\mathfrak{G}_1 \cap \mathfrak{G}_2 \in \Omega_{NHSG}$.*

Proof. Consider two NHS-Graphs $\mathfrak{G}_1 = \left(\mathfrak{G}^\star_1, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ and $\mathfrak{G}_2 = \left(\mathfrak{G}^\star_2, \mathcal{Q}^2, \mathbb{F}^2, \mathbb{G}^2\right)$ as defined in 3.1. Let $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ be the intersection of NHS-Graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ where $\mathcal{Q} = \mathcal{Q}^1 \cup \mathcal{Q}^2$ and $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$. Let $\theta \in \mathcal{Q}^1 - \mathcal{Q}^2$ then

$$\mathcal{T}_{\mathbb{G}_\theta}(\nu, \mu) = \mathcal{T}_{\mathbb{G}^1_\theta}(\nu, \mu)$$

$$\leq \min\left\{\mathcal{T}_{\mathbb{F}^1_\theta}(\nu), \mathcal{T}_{\mathbb{F}^1_\theta}(\mu)\right\}$$

$$= \min\left\{\mathcal{T}_{\mathbb{F}_\theta}(\nu), \mathcal{T}_{\mathbb{F}_\theta}(\mu)\right\}$$

so

$$\mathcal{T}_{\mathbb{G}_\theta}(\nu,\mu) \leq \min\{\mathcal{T}_{\mathbb{F}_\theta}(\nu), \mathcal{T}_{\mathbb{F}_\theta}(\mu)\}$$

Also

$$\mathcal{I}_{\mathbb{G}_\theta}(\nu,\mu) = \mathcal{I}_{\mathbb{G}^1_\theta}(\nu,\mu)$$
$$\leq \min\left\{\mathcal{I}_{\mathbb{F}^1_\theta}(\nu), \mathcal{I}_{\mathbb{F}^1_\theta}(\mu)\right\}$$
$$= \min\{\mathcal{I}_{\mathbb{F}_\theta}(\nu), \mathcal{I}_{\mathbb{F}_\theta}(\mu)\}$$

so

$$\mathcal{I}_{\mathbb{G}_\theta}(\nu,\mu) \leq \min\{\mathcal{I}_{\mathbb{F}_\theta}(\nu), \mathcal{I}_{\mathbb{F}_\theta}(\mu)\}$$

Now

$$\mathcal{F}_{\mathbb{G}_\theta}(\nu,\mu) = \mathcal{F}_{\mathbb{G}^1_\theta}(\nu,\mu)$$
$$\geq \max\left\{\mathcal{F}_{\mathbb{F}^1_\theta}(\nu), \mathcal{F}_{\mathbb{F}^1_\theta}(\mu)\right\}$$
$$= \max\{\mathcal{F}_{\mathbb{F}_\theta}(\nu), \mathcal{F}_{\mathbb{F}_\theta}(\mu)\}$$

so

$$\mathcal{F}_{\mathbb{G}_\theta}(\nu,\mu) \geq \max\{\mathcal{F}_{\mathbb{F}_\theta}(\nu), \mathcal{F}_{\mathbb{F}_\theta}(\mu)\}$$

Similar results are obtained when $\theta \in \mathcal{Q}^2 - \mathcal{Q}^1$

Now if $\theta \in \mathcal{Q}^1 \cap \mathcal{Q}^2$ then

$$\mathcal{T}_{\mathbb{G}_\theta}(\nu,\mu) = \min\left\{\mathcal{T}_{\mathbb{G}^1_\theta}(\nu,\mu), \mathcal{T}_{\mathbb{G}^2_\theta}(\nu,\mu)\right\}$$
$$\leq \min\left\{\min\left\{\mathcal{T}_{\mathbb{F}^1_\theta}(\nu), \mathcal{T}_{\mathbb{F}^1_\theta}(\mu)\right\}, \min\left\{\mathcal{T}_{\mathbb{F}^2_\theta}(\nu), \mathcal{T}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\}$$
$$\leq \min\left\{\min\left\{\mathcal{T}_{\mathbb{F}^1_\theta}(\nu), \mathcal{T}_{\mathbb{F}^2_\theta}(\mu)\right\}, \min\left\{\mathcal{T}_{\mathbb{F}^1_\theta}(\nu), \mathcal{T}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\}$$
$$= \min\{\mathcal{T}_{\mathbb{F}_\theta}(\nu), \mathcal{T}_{\mathbb{F}_\theta}(\mu)\}$$

Also

$$\mathcal{I}_{\mathbb{G}_\theta}(\nu,\mu) = \min\left\{\mathcal{I}_{\mathbb{G}^1_\theta}(\nu,\mu), \mathcal{I}_{\mathbb{G}^2_\theta}(\nu,\mu)\right\}$$
$$\leq \min\left\{\min\left\{\mathcal{I}_{\mathbb{F}^1_\theta}(\nu), \mathcal{I}_{\mathbb{F}^1_\theta}(\mu)\right\}, \min\left\{\mathcal{I}_{\mathbb{F}^2_\theta}(\nu), \mathcal{I}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\}$$
$$\leq \min\left\{\min\left\{\mathcal{I}_{\mathbb{F}^1_\theta}(\nu), \mathcal{I}_{\mathbb{F}^2_\theta}(\mu)\right\}, \min\left\{\mathcal{I}_{\mathbb{F}^1_\theta}(\nu), \mathcal{I}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\}$$
$$= \min\{\mathcal{I}_{\mathbb{F}_\theta}(\nu), \mathcal{I}_{\mathbb{F}_\theta}(\mu)\}$$

In the same way

$$\mathcal{F}_{\mathbb{G}_\theta}(\nu,\mu) = \max\left\{\mathcal{F}_{\mathbb{G}^1_\theta}(\nu,\mu), \mathcal{F}_{\mathbb{G}^2_\theta}(\nu,\mu)\right\}$$
$$\geq \max\left\{\max\left\{\mathcal{F}_{\mathbb{F}^1_\theta}(\nu), \mathcal{F}_{\mathbb{F}^1_\theta}(\mu)\right\}, \max\left\{\mathcal{F}_{\mathbb{F}^2_\theta}(\nu), \mathcal{F}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\}$$
$$\geq \max\left\{\max\left\{\mathcal{F}_{\mathbb{F}^1_\theta}(\nu), \mathcal{F}_{\mathbb{F}^2_\theta}(\mu)\right\}, \max\left\{\mathcal{F}_{\mathbb{F}^1_\theta}(\nu), \mathcal{F}_{\mathbb{F}^2_\theta}(\mu)\right\}\right\}$$
$$= \max\{\mathcal{F}_{\mathbb{F}_\theta}(\nu), \mathcal{F}_{\mathbb{F}_\theta}(\mu)\}$$

Hence the intersection $\mathfrak{G} = \mathfrak{G}_1 \cap \mathfrak{G}_2$ is NHS-Graphs. □

Table 6. Tabular Representation of NHS-Graph $\mathfrak{G}_1 = \left(\mathfrak{G}_1^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ according to Example 4.6

$\mathbb{F}$	ν_1	ν_2	ν_3
θ_1	$(0.2, 0.3, 0.4)$	$(0.3, 0.5, 0.6)$	$(0.2, 0.6, 0.8)$
θ_2	$(0.3, 0.4, 0.8)$	$(0.5, 0.7, 0.8)$	$(0.4, 0.5, 0.7)$
$\mathbb{G}$	(ν_1, ν_2)	(ν_2, ν_3)	(ν_1, ν_3)
θ_1	$(0.2, 0.2, 0.7)$	$(0.2, 0.4, 0.9)$	$(0.2, 0.2, 0.9)$
θ_2	$(0.3, 0.4, 0.8)$	$(0.4, 0.5, 0.9)$	$(0.3, 0.4, 0.8)$

Table 7. Tabular Representation of NHS-Graph $\mathfrak{G}_2 = \left(\mathfrak{G}_2^\star, \mathcal{Q}^2, \mathbb{F}^2, \mathbb{G}^2\right)$ according to Example 4.6

$\mathbb{F}$	ν_2	ν_3	ν_4
θ_2	$(0.4, 0.6, 0.7)$	$(0.5, 0.6, 0.9)$	$(0.3, 0.5, 0.7)$
θ_3	$(0.3, 0.5, 0.6)$	$(0.2, 0.6, 0.8)$	$(0.2, 0.3, 0.7)$
$\mathbb{G}$	(ν_2, ν_3)	(ν_3, ν_4)	(ν_2, ν_4)
θ_2	$(0.2, 0.2, 0.7)$	$(0.2, 0.4, 0.9)$	$(0.2, 0.2, 0.9)$
θ_3	$(0.3, 0.4, 0.8)$	$(0.4, 0.5, 0.9)$	$(0.3, 0.4, 0.8)$

Example 4.6. Let $\mathfrak{G}_1 = \left(\mathfrak{G}_1^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ be a neutrosophic hypersoft graph where $\mathfrak{G}_1^\star = (\mathcal{V}_1, \mathcal{E}_1)$ with $\mathcal{V}_1 = \{\nu_1, \nu_2, \nu_3\}$ and $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ are disjoint attribute-valued sets corresponding to distinct attributes $\alpha_1, \alpha_2, \alpha_3$ where $\mathcal{Q}_1 = \{\alpha_{11}\}$, $\mathcal{Q}_2 = \{\alpha_{21}\}$ and $\mathcal{Q}_3 = \{\alpha_{31}, \alpha_{32}\}$. $\mathcal{Q}^1 = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_3 = \{\theta_1, \theta_2\}$ where each θ_i is a 3-tuple element of $\mathcal{Q}^1$ and $\mathcal{T}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{I}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{F}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 1$ for all $(\nu_i, \nu_j) \in \mathcal{V}_1 \times \mathcal{V}_1 \backslash \{(\nu_1, \nu_2), (\nu_2, \nu_3), (\nu_1, \nu_3)\}$. Its tabular and graphical representation are given in TABLE 6 and FIGURE 7 respectively. Also let $\mathfrak{G}_2 = \left(\mathfrak{G}_2^\star, \mathcal{Q}^2, \mathbb{F}^2, \mathbb{G}^2\right)$ be a neutrosophic hypersoft graph where $\mathfrak{G}_2^\star = (\mathcal{V}_2, \mathcal{E}_2)$ with $\mathcal{V}_2 = \{\nu_2, \nu_3, \nu_4\}$ and $\mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4$ are disjoint attribute-valued sets corresponding to distinct attributes $\alpha_2, \alpha_3, \alpha_4$ where $\mathcal{Q}_2 = \{\alpha_{21}\}$, $\mathcal{Q}_3 = \{\alpha_{31}, \alpha_{32}\}$, $\mathcal{Q}_4 = \{\alpha_{41}\}$. $\mathcal{Q}^2 = \mathcal{Q}_2 \times \mathcal{Q}_3 \times \mathcal{Q}_4 = \{\theta_2, \theta_3\}$ where each θ_i is a 3-tuple element of $\mathcal{Q}^2$ and $\mathcal{T}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{I}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 0, \mathcal{F}_{\mathbb{G}_\theta}(\nu_i, \nu_j) = 1$ for all $(\nu_i, \nu_j) \in \mathcal{V}_2 \times \mathcal{V}_2 \backslash \{(\nu_2, \nu_3), (\nu_3, \nu_4), (\nu_2, \nu_4)\}$. Its tabular and graphical representation are given in TABLE 7 and FIGURE 8 respectively.
Now Let $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ be the intersection of two neutrosophic hypersoft graphs $\mathfrak{G}_1 = \left(\mathfrak{G}_1^\star, \mathcal{Q}^1, \mathbb{F}^1, \mathbb{G}^1\right)$ and $\mathfrak{G}_2 = \left(\mathfrak{G}_2^\star, \mathcal{Q}^2, \mathbb{F}^2, \mathbb{G}^2\right)$ where $\mathcal{Q} = \mathcal{Q}^1 \cap \mathcal{Q}^2$.Its (intersection of these two NHS-graphs) tabular and graphical representation are given in TABLE 8 and FIGURE 9 respectively.

Definition 4.7. The compliment $\overline{\mathfrak{G}} = \left(\overline{\mathfrak{G}^\star}, \overline{\mathcal{Q}}, \overline{\mathbb{F}}, \overline{\mathbb{G}}\right)$ of strong neutrosophic hypersoft subgraph $\mathfrak{G} = (\mathfrak{G}^\star, \mathcal{Q}, \mathbb{F}, \mathbb{G})$ with $\mathbb{G}_\theta(\nu, \mu) = \mathbb{F}_\theta(\nu) \cap \mathbb{F}_\theta(\mu)$ where

FIGURE 7. Graphical Representation of TABLE 6with (a) $\mathcal{N}(\theta_1)$ and (b) $\mathcal{N}(\theta_2)$

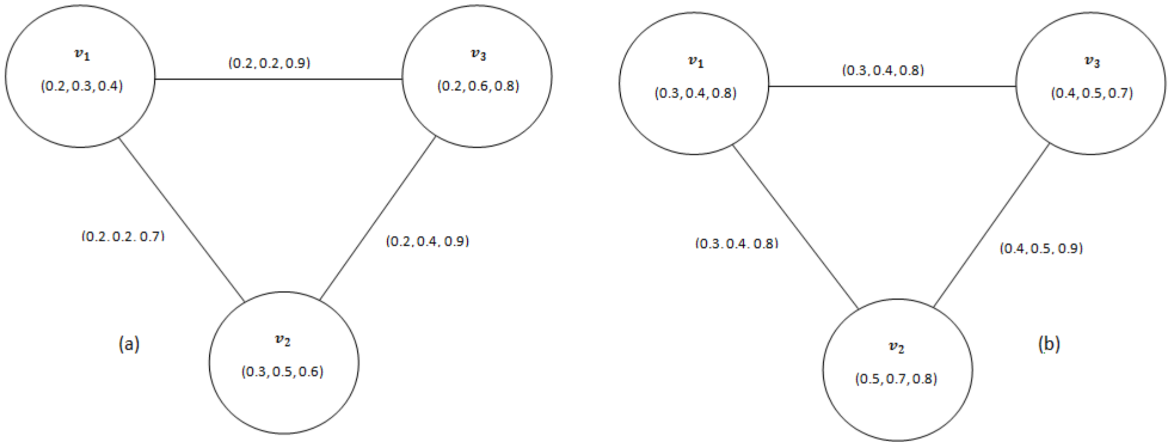


FIGURE 8. Graphical Representation of TABLE 7with (a) $\mathcal{N}(\theta_2)$ and (b) $\mathcal{N}(\theta_3)$

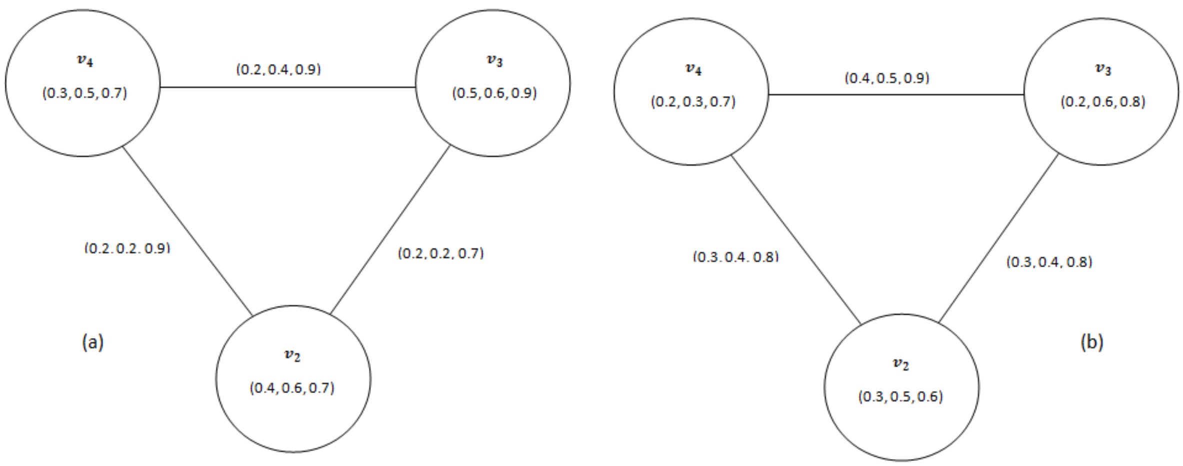


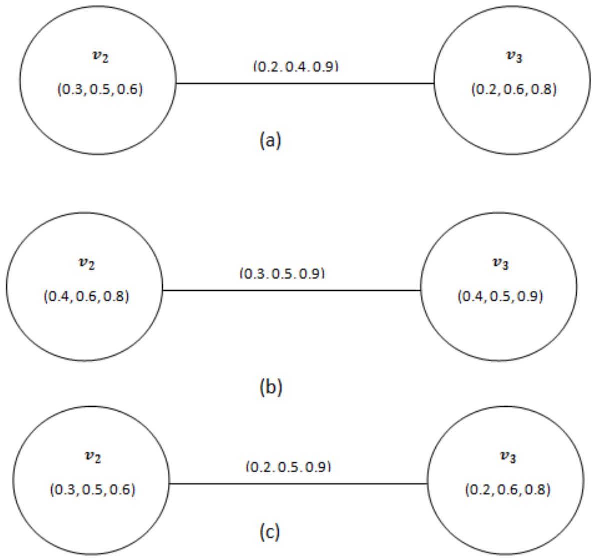
TABLE 8. Tabular Representation of NHS-Graph $\mathfrak{G} = \mathfrak{G}_1 \cap \mathfrak{G}_2$

$\mathbb{F}$	ν_2	ν_3
θ_1	$(0.3, 0.5, 0.6)$	$(0.2, 0.6, 0.8)$
θ_2	$(0.4, 0.6, 0.8)$	$(0.4, 0.5, 0.9)$
θ_3	$(0.3, 0.5, 0.6)$	$(0.2, 0.6, 0.8)$
$\mathbb{G}$	(ν_2, ν_3)	
θ_1	$(0.2, 0.4, 0.9)$	
θ_2	$(0.3, 0.5, 0.9)$	
θ_3	$(0.2, 0.5, 0.9)$	

(1) $\overline{\mathcal{Q}} = \mathcal{Q}$

(2) $\overline{\mathcal{T}_{\mathbb{F}_\theta}(\nu)} = \mathcal{T}_{\mathbb{F}_\theta}(\nu), \overline{\mathcal{I}_{\mathbb{F}_\theta}(\nu)} = \mathcal{I}_{\mathbb{F}_\theta}(\nu), \overline{\mathcal{F}_{\mathbb{F}_\theta}(\nu)} = \mathcal{F}_{\mathbb{F}_\theta}(\nu)$ for all $\nu \in \mathcal{V}$

Figure 9. Graphical Representation of TABLE 8 with (a) $\mathcal{N}(\theta_1)$, (b) $\mathcal{N}(\theta_2)$ and (c) $\mathcal{N}(\theta_3)$



(3) $$\overline{\mathcal{T}_{\mathbb{F}_\theta}(\nu,\mu)} = \begin{cases} \min\{\mathcal{T}_{\mathbb{F}_\theta}(\nu), \mathcal{T}_{\mathbb{F}_\theta}(\mu)\} & if\ \mathcal{T}_{\mathbb{G}_\theta}(\nu,\mu) = 0 \\ 0 & otherwise \end{cases}$$

$$\overline{\mathcal{I}_{\mathbb{F}_\theta}(\nu,\mu)} = \begin{cases} \min\{\mathcal{I}_{\mathbb{F}_\theta}(\nu), \mathcal{I}_{\mathbb{F}_\theta}(\mu)\} & if\ \mathcal{I}_{\mathbb{G}_\theta}(\nu,\mu) = 0 \\ 0 & otherwise \end{cases}$$

$$\overline{\mathcal{F}_{\mathbb{F}_\theta}(\nu,\mu)} = \begin{cases} \max\{\mathcal{F}_{\mathbb{F}_\theta}(\nu), \mathcal{F}_{\mathbb{F}_\theta}(\mu)\} & if\ \mathcal{F}_{\mathbb{G}_\theta}(\nu,\mu) = 0 \\ 0 & otherwise \end{cases}.$$

5. Conclusions

In this study, a gluing concept of neutrosophic hypersoft set and graph theory is characterized. Some of elementary properties, types, operations and results are generalized under neutrosophic hypersoft set environment. Future work may include the extension of this study for the following structures and fields:

- Interval valued neutrosophic hypersoft set
- Neutrosophic parameterized hypersoft set
- m-polar neutrosophic hypersoft set
- Decision making problems
- New kinds of graphs
- Energies of graph

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L. Fuzzy sets. *Information and control* **1965**, *8*(3), 338-353. http://doi.org/10.1016/S0019-9958(65)90241-X
2. Atanassov, K. Intuitionistic Fuzzy Sets. *Fuzzy Sets and Systems* **1986**, *20*, 87-96. http://doi.org/10.1016/S0165-0114(86)80034-3
3. Smarandache, F. *Neutrosophy, neutrosophic probability, set, and logic*, analytic synthesis and synthetic analysis. Rehoboth, American Research Press 1998.
4. Smarandache, F. Neutrosophic set- a generalization of intuitionistic fuzzy set. *Granular Computing, IEEE, International Conference* **2006**, 38-42. http://doi.org/10.1109/GRC.2006.1635754
5. Wang, H.; Smarandache, F.; Zhang, Y.; Sunderraman, R. Single valued neutrosophic sets. *Multissp. Multistruct.* **2010**, *4*, 410-413
6. Smarandache, F. n-Valued refined neutrosophic logic and its applications in Physics. *Progress in Physics* **2013**, *4* , 143-146. http://doi.org/10.5281/zenodo.49149
7. Smarandache, F. Operators on single valued neutrosophic sets, neutrosophic undersets, and neutrosophic offsets. *Journal of Mathematics and Informatics* **2016**, *5* , 63-67. http://doi.org/10.5281/zenodo.57412
8. Smarandache, F. Neutrosophic overset, neutrosophic underset, neutrosophic offset, similarly for neutrosophicover-/under-/offlogic, probability, and statistic. Pons Editions, Brussels, 2016. http://doi.org/10.5281/zenodo.57410
9. Pramanik, S.; Dey, P. P.; Smarandache, F. Correlation coefficient measures of interval bipolar neutrosophic sets for solving multi-attribute decision making problems. *Neutrosophic Sets and Systems* **2018**, *19* , 70-79. http://doi.org/10.5281/zenodo.1235151.
10. Broumi, S.; Smarandache, F. Intuitionistic Neutrosophic Soft Set. *Journal of Information and Computing Science* **2013**, *8*(2) , 130-140. http://doi.org/10.5281/zenodo.2861554.
11. Broumi, S. Generalized Neutrosophic Soft Set. *International Journal of Computer Science, Engineering and Information Technology* **2013**, *3*(2) , 17-30. http://doi.org/10.5121/ijcseit.2013.3202.
12. Broumi, S.; Deli, I.; Smarandache, F. Relations on Interval Valued Neutrosophic Soft Sets. *Journal of New Results in Science* **2014**, *5* , 1-20. http://doi.org/10.5281/zenodo.30306.
13. Deli, I. Interval-valued neutrosophic soft sets and its decision making. *International Journal of Machine Learning and Cybernetics* **2017**, *8*(2) , 665-676. http://doi.org/10.1007/s13042-015-0461-3.
14. Kharal, A. A Neutrosophic Multicriteria Decision Making Method. *New Mathematics and Natural Computation* **2014**, *10*(2), 143-162. http://doi.org/10.1142/S1793005714500070
15. Edalatpanah, S. A. Systems of Neutrosophic Linear Equations. *Neutrosophic Sets and Systems* **2020**, *33*, 92-104. http://doi.org/10.5281/zenodo.3782826.
16. Kumar, R.; Edalatpanah, S. A.; Jha, S.,; Singh, R. A novel approach to solve gaussian valued neutrosophic shortest path problems. *International Journal of Engineering and Advanced Technology* **2019**, *8*(3), 347-353. http://doi.org/10.35940/ijeat.E1049.0585C19.
17. Molodtsov, D. Soft Set Theory - First Results. *Computers and Mathematics with Applications* **1999**, *37*, 19-31. http://doi.org/10.1016/S0898-1221(99)00056-5
18. Maji. P.K.; Biswas R.; Roy, A. R. Soft Set Theory. *Computers and Mathematics with Applications* **2003**, *45*, 555-562. http://doi.org/10.1016/S0898-1221(03)00016-6
19. Maji, P. K.; Biswas, R.; Roy, A. R. Fuzzy soft sets. *Journal of Fuzzy Mathematics* **2001**, *9*(3), 589-602.
20. Pei, D.; Miao, D. From soft set to information system. *In international conference of granular computing IEEE* **2005**, *2* , 617-621. http://doi.org/10.1109/GRC.2005.1547365
21. Ali, M. I.; Feng, F.; Liu, X.; Min, W. K.; Sabir, M. On some new operations in soft set theory. *Computers and Mathematics with Applications 2009*, *57* , 1547-1553. http://doi.org/10.1016/j.camwa.2008.11.009

22. Babitha, K. V.; Sunil, J. J. Soft set relations and functions. *Computers and Mathematics with Applications* **2010**, *60*, 1840-1849. http://doi.org/10.1016/j.camwa.2010.07.014
23. Babitha, K. V.; Sunil, J. J. Transitive closure and ordering in soft set. *Computers and Mathematics with Applications* **2010**, *61* 2235-2239. http://doi.org/10.1016/j.camwa.2011.07.010
24. Sezgin, A.; Atagün A. O. On operations of soft sets. *Computers and Mathematics with Applications* **2011**, *61*(5), 1457-1467. http://doi.org/10.1016/j.camwa.2011.01.018
25. Ge, X.; Yang, S. Investigations on some operations of soft sets. *World Academy of Science Engineering and Technology* **2011**, *75*, 1113-1116.
26. Li, F. Notes on soft set operations. *ARPN Journal of systems and softwares* **2011**, 205-208.
27. Maji, P. K.; Biswas, R.; Roy, A. R. Intuitionistic fuzzy soft sets. *The Journal of Fuzzy Mathematics* **2001**, *9*(3), 677-692.
28. Maji, P. K. Neutrosophic Soft Set. *Annals of Fuzzy Mathematics and Informatics* **2013**, *5*(1), 2287-623.
29. Smarandache, F. Extension of Soft Set of Hypersoft Set, and then to Plithogenic Hypersoft Set. *Neutrosophic Sets and Systems* **2018**, *22* , 168-170. http://doi.org/10.5281/zenodo.2838716
30. Saeed, M.; Ahsan, M.; Siddique, M.k.; Ahmad, M.R. A Study of the Fundamentals of Hypersoft Set Theory. *International Journal of Scientific and Engineering Research* **2020**, *11*(1), 320-329.
31. Saeed, M.; Rahman, A. U.; Ahsan, M.; Smarandache, F. An Inclusive Study on Fundamentals of Hypersoft Set, In Theory and Application of Hypersoft Set, Pons Publishing House, Brussel, 2021, 1-23.
32. Abbas, F.; Murtaza, G.; Smarandache, F. (2020). Basic operations on hypersoft sets and hypersoft points. *Neutrosophic Sets and Systems* **2020**, *35*, 407-421. http://doi.org/10.5281/zenodo.3951694
33. Saqlain, M.; Jafar, N.; Moin, S.; Saeed, M.; Broumi, S. Single and Multi-valued Neutrosophic Hypersoft Set and Tangent Similarity Measure of Single valued Neutrosophic Hypersoft Sets. *Neutrosophic Sets and Systems* **2020**, *32*, 317-329. http://doi.org/10.5281/zenodo.3723165
34. Saqlain, M.; Moin, S.; Jafar, N.; Saeed, M.; Smarandache, F. Aggregate Operators of Neutrosophic Hypersoft Sets. *Neutrosophic Sets and Systems* **2020**, 32, 294-306. http://doi.org/10.5281/zenodo.3723155
35. Saqlain, M.; Saeed, M.; Ahmad, M.R.; Smarandache, F. Generalization of TOPSIS for Neutrosophic Hypersoft Sets using Accuracy Function and its Application. *Neutrosophic Sets and Systems* **2020**, *27*, 131-137. http://doi.org/10.5281/zenodo.3275533
36. Martin, N.; Smarandache, F. Concentric Plithogenic Hypergraph based on Plithogenic Hypersoft Sets A Novel Outlook. *Neutrosophic Sets and Systems* **2020**, *33*, 78-91. http://doi.org/10.5281/zenodo.3782824
37. Rahman, A. U.; Saeed, M.; Alodhaibi, S.S.; Abd, H. Decision Making Algorithmic Approaches Based on Parameterization of Neutrosophic Set under Hypersoft Set Environment with Fuzzy, Intuitionistic Fuzzy and Neutrosophic Settings. *CMES-Computer Modeling in Engineering* & *Sciences* **2021**, *128*(2), 743-777. http://doi.org/10.32604/cmes.2021.016736
38. Rahman, A. U.; Saeed, M.; Smarandache, F.; Ahmad, M. R. Development of Hybrids of Hypersoft Set with Complex Fuzzy Set, Complex Intuitionistic Fuzzy set and Complex Neutrosophic Set. *Neutrosophic Sets and Systems* **2020**, 38, 335-354. http://doi.org/10.5281/zenodo.4300520
39. Rahman, A. U.; Saeed, M.; Smarandache, F. Convex and Concave Hypersoft Sets with Some Properties. *Neutrosophic Sets and Systems* **2020**, 38, 497-508. http://doi.org/10.5281/zenodo.4300580
40. Deli, I. Hybrid set structures under uncertainly parameterized hypersoft sets, Theory and applications. In Theory and Application of Hypersoft Set, Pons Publishing House, Brussel, 2021, 24-49.
41. Gayen, S.; Smarandache, F.; Jha, S.; Singh, M. K.; Broumi, S.; Kumar, R. Introduction to Plithogenic Hypersoft Subgroup. *Neutrosophic Sets and Systems 2020*, *33*, 14. http://doi.org/10.5281/zenodo.3782897
42. Saeed, M.; Ahsan, M.; Saeed, M. H.; Mehmood, A.; Abdeljawad, T. An Application of Neutrosophic Hypersoft Mapping to Diagnose Hepatitis and Propose Appropriate Treatment. *IEEE Access* **2021**, *9*, 70455-70471. http://doi.org/10.1109/ACCESS.2021.3077867

43. Saeed, M.; Ahsan, M.; Abdeljawad, T. A Development of Complex Multi-Fuzzy Hypersoft Set With Application in MCDM Based on Entropy and Similarity Measure. *IEEE Access* **2021**, *9*, 60026-60042. 10.1109/ACCESS.2021.3073206
44. Rahman, A.U.; Saeed, M.; Dhital, A. Decision Making Application Based on Neutrosophic Parameterized Hypersoft Set Theory. *Neutrosophic Sets and Systems* **2021**, *41*, 1-14. http://doi.org/10.5281/zenodo.4625665
45. Rahman, A.U.; Saeed, M.; Zahid, S. Application in Decision Making Based on Fuzzy Parameterized Hypersoft Set Theory. *Asia Mathematika 2021*, *5*(1), 19-27. http://doi.org/10.5281/zenodo.4721481
46. Rahman, A.U.; Hafeez, A.; Saeed, M.; Ahmad, M.R.; Farwa, U. Development of Rough Hypersoft Set with Application in Decision Making for the Best Choice of Chemical Material, In Theory and Application of Hypersoft Set, Pons Publication House, Brussel, 2021, 192-202. http://doi.org/10.5281/zenodo.4743367
47. Ihsan,M.; Rahman, A.U.; Saeed, M. Hypersoft Expert Set With Application in Decision Making for Recruitment Process. *Neutrosophic Sets and Systems* **2021**, *42*, 191-207. http://doi.org/10.5281/zenodo.4711524
48. Thumbakara, R. K.; George, B. Soft graphs. *General Mathematics Notes* **2014**, *21*(2), 75-86.
49. Mohinta, S.; Samanta, T. K. An introduction to fuzzy soft graph. *Mathematica Moravica* **2015**, *19*(2), 35-48. https://doi.org/10.5937/matmor1502035m
50. Shahzadi, S.; Akram, M. Intuitionistic fuzzy soft graphs with applications. *Journal of Applied Mathematics and Computing* **2017**, *55*(1), 369-392. https://doi.org/10.1007/s12190-016-1041-8
51. Shyla, A. M.; Varkey, T. M. Intuitionistic Fuzzy soft graph. *International Journal of Fuzzy Mathematical Archive* **2016**, *11*(2), 63-77.
52. Akram, M.; Nawaz, S. On fuzzy soft graphs. *Italian journal of pure and applied mathematics* **2015**, *34*, 497-514.
53. Al-Masarwah, A.; Qamar, M. A. Some new concepts of fuzzy soft graphs. *Fuzzy Information and Engineering* 2016, *8*(4), 427-438. https://doi.org/10.1016/j.fiae.2017.01.003
54. Akram, M.; Malik, H. M.; Shahzadi, S.; Smarandache, F. Neutrosophic soft rough graphs with application. *Axioms* **2018**, *7*(1), 14. https://doi.org/10.3390/axioms7010014
55. Akram, M.; Shahzadi, S. Neutrosophic soft graphs with application. *Journal of Intelligent and Fuzzy Systems* **2017**, *32*(1), 841-858. https://doi.org/10.3233/jifs-16090
56. Shah, N.; Hussain, A. Neutrosophic soft graphs. *Neutrosophic Sets and Systems* **2016**, *11*, 31-44. https://doi.org/10.5281/zenodo.571574

Received: May 30, 2021. Accepted: October 5, 2021

NSS

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

Neutrosophic Quadratic Residues and Non-Residues

Chalapathi Tekuri [1,*], Sajana Shaik [2] and Smarandache Florentin [3]

[1] Department of Mathematics, Sree Vidyanikethan Eng. College, Tirupati, A.P., India; chalapathi.tekuri@gmail.come
[2] Department of Mathematics, P.R. Govt. College (A), Kakinada, A.P., India; ssajana.maths@gmail.com
[3] Mathematics & Science Division, University of New Mexico, Gallup Campus, USA; smarand@unm.edu

* Correspondence: chalapathi.tekuri@gmail.come

Abstract: In this paper, we present the Neutrosophic quadratic residues and nonresidues with their basic interpretation as graphs in an algebraic manner and analog to the algebraic graphs. We establish the Neutrosophic, number-theoretic, and graph-theoretic properties of the set of Neutrosophic quadratic residues and nonresidues, many of which mirror those of the classical quadratic residues and nonresidues of modulo an odd prime. These properties, especially the algebraic ones, are connected to algebraic graphs, and thus we conclude the paper by studying the structural properties of Neutrosophic quadratic residue and quadratic nonresidue graphs.

Keywords: Quadratic residues; Quadratic nonresidues; Neutrosophic quadratic residues; Neutrosophic quadratic nonresidues; Neutrosophic quadratic residue graph; Neutrosophic quadratic nonresidue graph.

1. Introduction

For any positive integer $n \geq 1$, the set $Z_n = \{0, 1, 2, \ldots, n-1\}$ is a ring under the usual addition and multiplication modulo n. Moreover, for any prime p, the ring Z_p is a field of order p and hence $Z_p^* = \{1, 2, 3, \ldots, p-1\}$ is a group under multiplication modulo p, see [1-2].

For $a \in Z_p^*$, a is a quadratic residue modulo p if and only if $a = x^2$ for some $x \in Z_p^*$. Now suppose Q_p denote the set of all quadratic residues modulo p. Then Q_p is a nonempty subset of Z_p^*, given by $Q_p = \{x^2 \in Z_p^* : x = 1, 2, \ldots, \frac{p-1}{2}\}$. It is clear that for any $a, b \in Q_p$, there exists x and y in Z_p^* such that $a^{-1}b = (x^{-1}y)^2 \in Q_p$. Therefore, Q_p is a subgroup of Z_p^* and also the index $[Z_p^* : Q_p] = 2$. This implies that $xy \in Q_p$ if and only if x and y are both in Q_p or neither of them is in Q_p. This specifies that an element in Z_p^* as a residue or nonresidue according to whether or not it is a quadratic residue modulo p. In particular, the set of all quadratic nonresidues modulo p in Z_p^* is denoted by $\overline{Q_p}$. Hence $|Q_p| = |\overline{Q_p}| = \frac{p-1}{2}$. So, Q_p is the normal subgroup and $\overline{Q_p}$ is the only nonempty subset of Z_p^* whose orders are equal. For more information about Q_p and $\overline{Q_p}$, reader refer [3].

Much of the specific power and utility of modern mathematics arises from its abstraction of important features similar to various circumstances and illustrations. But many sets and systems we encounter have a usual addition and multiplication defined on their elements. These operations often satisfy a few common properties that we want to isolate and study. Besides the obvious illustrations in different number systems and algebraic systems, we can operate polynomials, functions, matrices, etc. Studying the algebraic structure of groups, rings, and fields based on number theoretic and combinatorial properties has caught the interest of many researchers order the last decades. Recently, algebraic systems associated with neutrosophic elements and sets [4] seem to be more interesting and active area compare to those associated with classical algebraic structures. For instance, the Neutrosophic set $N(Z_p^*, I)$ is generated by the multiplicative group Z_p^* and the neutrosophic unit element I ($I^2 = I$ and I^{-1} does not exist), that is, $N(Z_p^*, I)$ or equivalently $N(Z_p^*) = < Z_p^*, I >= Z_p^* \cup Z_p^* I$, where p is prime. This is a Neutrosophic group [5] concerning Neutrosophic multiplication $(aI)(bI) = abI$ for every $aI, bI \in N(Z_p^*, I)$.

The concept of the Neutrosophic graph of Neutrosophic structures was first introduced by Vasanth Kandasamy and Smarandache [6], but this work was mostly concerned with the basic properties of Neutrosophic algebraic structures. Recently, the authors Chalapathi and Kiran studied the Neutrosophic graphs [5] of finite groups. The Neutrosophic graph of a finite group G, which is denoted by $Ne(G, I)$, is an undirected simple graph whose vertices are elements of the neutrosophic group $N(G)$ with two distinct vertices x and y which are adjacent if and only if either $xy = x$ or $xy = y$.

In 1879, author Cayley considered the Cayley graph for finite groups. After that, a lot of research has been done on various families of Cayley graphs. For instance, Quadratic residue Cayley graphs [7], Quartic residue Cayley graphs [8]. Many researchers exist in the literature on Cayley graphs quadratic residues on odd prime and prime power modules. The authors studied quadratic residues modulo 2^n Cayley graphs in [9]. In this paper, we will focus on Neutrosophic quadratic residues and their corresponding algebraic graphs, which are not Cayley graphs.

2. Neutrosophic Quadratic and Non Quadratic Residues

In this section, for convenience and also for later use, we define some definitions and notations concerning integers modulo an odd prime p , and Neutrosophic quadratic and nonquadratic residue modulo p.

First, we recall some results about neutrosophic groups from [5].

Theorem 2.1:

1. $Z_p^* I = \{aI : a \in Z_p^*\}$
2. $N(Z_p^*) = Z_p^* \cup Z_p^* I$, where $Z_p^* \cap Z_p^* I = \emptyset$

Theorem 2.2: Let Z_p^* be a finite group with respect to multiplication modulo n. Then

1. $|Z_p^*| = p - 1$ and $|Z_p^* I| = p - 1$
2. $|N(Z_p^*)| = 2(p - 1)$

Let $aI \in N(Q_p)$. Then aI is a neutrosophic quadratic residue modulo p if and only if $aI = (xI)^2$ for some $xI \in Z_p^* I$. Now suppose $N(Q_p)$ denote the set of all neutrosophic quadratic residues modulo p . Then $Q_p I$ is a nonempty subset of $N(Z_p^*)$ given by $Q_p I = \{(xI)^2 \in N(Z_p^*) : x = 1,2, \dots, \frac{p-1}{2}\}$.

Further, if for any $aI, bI \in Q_p I$, then $aI = (xI)^2$ and $bI = (yI)^2$ for some $xI, yI \in Z_p^* I$, so $(aI)(bI) = (xyI)^2 \in Q_p I$

Hence $Q_p I$ is a neutrosophic subgroup of $N(Z_p^*) = Z_p^* \cup Z_p^* I$ with neutrosophic index, by the Theorem 2.1.

$$[N(Z_p^*) : Q_p I] = \frac{|N(Z_p^*)|}{|Q_p I|} = \frac{2(p-1)}{\left(\frac{p-1}{2}\right)} = 4.$$

Similarly, the set of all neutrosophic quadratic non-residues modulo p in $Z_p^* I$ is denoted by $\overline{Q_p I}$ with $|Q_p I| = |\overline{Q_p I}| = \frac{p-1}{2}$.

Example 2.3: The following shortlist shows that the Neutrosophic quadratic and nonquadratic residues modulo 3, $5, 7$, respectively.

$N(Q_3, I) = \{1, I\}$, $N(\overline{Q_3}, I) = \{2, 2I\}$,

$N(Q_5, I) = \{1, 4, I, 4I\}$, $N(\overline{Q_5}, I) = \{2, 3, 2I, 3I\}$,

$N(Q_7, I) = \{1, 3, 4, 5, 9, I, 3I, 4I, 5I, 9I\}$, $N(\overline{Q_7}, I) = \{2, 6, 7, 8, 10, 2I, 6I, 7I, 8I, 10I\}$.

From the above example, we observe the following:

$N(Q_p, I) = Q_p \cup Q_p I$ and $N(\overline{Q_p}, I) = \overline{Q_p} \cup \overline{Q_p I}$. In particular,

$|N(Q_p, I)| = |Q_p| + |Q_p I| = \frac{p-1}{2} + \frac{p-1}{2} = p - 1$ and

$|N(\overline{Q_p}, I)| = |\overline{Q_p}| + |\overline{Q_p I}| = \frac{p-1}{2} + \frac{p-1}{2} = p - 1$.

Theorem 2.4: Given $p > 2$, $N(W_p^*, I) = W_p^* \cup W_p^* I$, is the neutrosophic prime subgroup of $N(Z_p^*, I)$, where $W_p^* = \{1, p-1\}$.

Proof: It is clear from the well-known result that W_p^* is a subgroup of the group Z_p^*, because $(p-1)^2 \equiv 1 \ (mod\ p)$.

Theorem 2.5: Fundamental Theorem of Neutrosophic Quadratic Residues Modulo p

For each $p > 2$, we have the neutrosophic quotient group $\frac{N(Z_p^*, I)}{N(W_p^*, I)}$ is isomorphic to the neutrosophic group $N(Q_p, I)$.

Proof: For any $p > 2$, we have $(p-1)^2 \equiv 1 \ (mod\ p)$ and $((p-1)I)^2 \equiv I \ (mod\ p)$. Therefore, $N(W_p^*, I) = \{1, p-1, I, (p-1)I\}$ is a neutrosophic subgroup of $N(Z_p^*, I)$. So, there exists a Neutrosophic quotient group $\frac{N(Z_p^*, I)}{N(W_p^*, I)}$. Now we claim that $\frac{N(Z_p^*, I)}{N(W_p^*, I)} \cong N(Q_p, I)$. For this, we define a map $\Psi: N(Z_p^*, I) \to N(Q_p, I)$ by the relation

$$\Psi(x) = \begin{cases} x^2, & if\ x \in Z_p^* \\ (xI)^2, & if\ xI \in Z_p^* I \end{cases}$$

Clearly, Ψ is a well-defined group and Neutrosophic group homomorphism, because $(ab)^2 = a^2 b^2, \forall\ a, b \in Z_p^*$ and $((aI)(bI))^2 = (aI)^2 (bI)^2, \forall\ aI, bI \in Z_p^* I$.

Now to find a kernel of Ψ. If $x \in Z_p^*$, then by the definition of kernel of group (classical) homomorphism,

$$K = \{x \in Z_p^* : x^2 = 1\}$$

$$= \{1, -1\}$$

$$= \{1, p-1\}.$$

Similarly, if $xI \in Z_p^* I$, then by the definition of a kernel of a Neutrosophic group homomorphism,

$$K' = \{xI \in Z_p^* I : (xI)^2 = I\}$$

$$= \{I, -I\}$$

$$= \{I, (p-1)I\}.$$

Hence, $Ker\ \Psi = K \cup K'$

$$= \{1, p-1, I, (p-1)I\}$$

$$= N(W_p^*, I).$$

Finally, to find image of Ψ.

$$Im\,(\Psi) = \{\Psi(x) \in N(Z_p^*, I) : x \in N(Z_p^*, I)\}$$

$$= \{x^2 \in Z_p^* : x \in Z_p^*\} \cup \{(xI)^2 \in Z_p^* I : xI \in Z_p^* I\}$$

$$= Q_p \cup Q_p I$$

$$= N(Q_p, I).$$

By the fundamental theorem of a Neutrosophic group homomorphism, $\frac{N(Z_p^*, I)}{Ker\ \Psi} \cong Im(\Psi)$. This shows that $\frac{N(Z_p^*, I)}{N(W_p^*, I)} \cong N(Q_p, I)$.

Remark 2.6: $x \in N(Z_p^*, I)$ is a Neutrosophic quadratic residue if and only if $x \in Im\,(\Psi)$, otherwise, it is called neutrosophic quadratic residue modulo p.

Example 2.7: For the prime $p = 5$, we have $Z_5^* = \{1, 2, 3, 4\}$, $N(Z_5^*, I) = \{1, 2, 3, 4, I, 2I, 3I, 4I\}$, $W_5^* = \{1, 4\}$, $N(W_5^*, I) = \{1, 4, I, 4I\}$, $\frac{N(Z_5^*, I)}{N(W_5^*, I)} = \{N(W_5^*, I), 2N(W_5^*, I), 3N(W_5^*, I), 4N(W_5^*, I), I\, N(W_5^*, I), 2I\, N(W_5^*, I), 3I\, N(W_5^*, I), 4I\, N(W_5^*, I)\}$.

Theorem 2.8: The neutrosophic product of two neutrosophic quadratic residues is again a neutrosophic a quadratic residue modulo p. Similarly, the neutrosophic product of two Neutrosophic quadratic nonresidues is a Neutrosophic quadratic residue modulo p.

Proof: Since $N(Q_p, I)$ is a Neutrosophic normal subgroup of the Neutrosophic group $N(Z_p^*, I)$ whose index is 4. So there exists a Neutrosophic quotient group $\frac{N(Z_p^*, I)}{N(Q_p, I)}$ such that $\left|\frac{N(Z_p^*, I)}{N(Q_p, I)}\right| = 4$, that is $\frac{N(Z_p^*, I)}{N(Q_p, I)} = \{x\, N(Q_p, I) : x \in N(Z_p^*, I)\}$.

Let $x \in Z_p^*$ such that $x \in Q_p$. Then $(x\, Q_p)^2 = {Q_p}^2 = Q_p$, since $hH = Hh = H$. Let $a \in Z_p^*$ such that $a \notin Q_p$. Then $(a\, Q_p)^2 \neq Q_p$.

Let $x \in Z_p^* I$ such that $x \in Q_p\, I$. Then $(x\, Q_p I)^2 = (Q_p I)^2 = {Q_p}^2\, I^2 = Q_p I$. Let $a \in Z_p^* I$ such that $a \notin Q_p I$. Then $(a\, Q_p I)^2 \neq Q_p I$.

Because $N(Z_p^*, I) = Z_p^* \cup Z_p^* I$ and $N(Q_p, I) = Q_p \cup Q_p I$, we know that the neutrosophic quotient group defined as $\frac{N(Z_p^*, I)}{N(Q_p, I)} = \{Q_p, aQ_p, IQ_p, a\, IQ_p\}$.

(1) If $x, y \in Q_p I$, then

$$
\begin{aligned}
xy\, Q_p I &= (x\, Q_p I)(y\, Q_p I) \\
&= (Q_p I)(Q_p I) \\
&= (Q_p I)^2 \\
&= {Q_p}^2 I^2
\end{aligned}
$$

$= Q_p I$, since ${Q_p}^2 = Q_p$,

and thus $xy \in Q_p I$.

(2) If $x, y \notin Q_p I$, then $x,\ y \in \overline{Q_p}$. So there exists $\bar{a}, \bar{b} \in \overline{Q_p}$ such that $x = \bar{a}\, I$ and $y = \bar{b}\, I$. Then

$$xy\, Q_p I = (\bar{a}\, I)(\bar{b}\, I)\ Q_p$$

$$= (\bar{a}\, \bar{b}) I\, Q_p$$

$$= I\left((\bar{a}\, \bar{b})\ Q_p\right)$$

$= I\, Q_p$, since $\bar{a}, \bar{b} \in \overline{Q_p} \Rightarrow \bar{a}\bar{b} \in Q_p$ and $\bar{a}\bar{b}Q_p = Q_p$.

Hence $xy \in Q_p I$.

(3) If $x \in Q_p I$ and $y \notin Q_p I$, then

$$xy\, Q_p I = (x\, Q_p I)(y\, Q_p I)$$

$= (Q_p I)(y\, Q_p I)$, since $x \in Q_p I \Leftrightarrow x\, Q_p I = Q_p I$

$$= y(Q_p I)^2$$

$$= y\, {Q_p}^2 I^2$$

$$= y\, Q_p I$$

$\neq Q_p I$, since $y \notin Q_p I$ iff $y\, Q_p I \neq Q_p I$.

Hence $xy \notin Q_p I$. This proves the theorem.

Now, let us start with simple undirected graphs of neutrosophic quadratic residue and Neutrosophic quadratic Nonresidue graphs of the Neutrosophic graph $N(Z_p^*,\ I)$ whose vertices are members in the Neutrosophic graph $N(Z_p^*,\ I)$ where p is an odd prime.

3. Neutrosophic Quadratic Residue Graphs

Structurally, many real-world concepts, aspects, and situations can be described by using and applying diagrams of a set of vertices with edges joining pairs of these vertices. So, a mathematical abstraction of this type of diagram gives rise to the concept of a graph. A graph G and is denoted by

$G=(V,\ E)$, where $V=V(G)$ and $E=E(G)$ vertex and edge sets of G, respectively. A graph G is said to be connected if there is at least one path between every two vertices in G and disconnected if G has at least one pair of vertices between which there is no path. Every graph G consists of one or more connected graphs as subgraphs, and each such connected subgraph of G is called a component of G , and each component of G is denoted by $Comp(G)$. It is clear that every connected graph contains only one component and every disconnected graph of more than one vertex contains two or more components. Now a graph G is said to be complete if every vertex in G is connected to another vertex in G.

A complete graph of order n is denoted by K_n and it has exactly $\frac{n(n-1)}{2}=n_{C_2}$ edges, and it is called the size of K_n. If u is a vertex of G, then the number of edges incident on a vertex u is called the degree of u and it is denoted by $\deg(u)$. In particular, if $\deg(u)=k$ for every vertex u in G, then G is called a k −regular graph. A graph G is said to be bipartite if its vertex set V can be partitioned into two non-empty disjoint subsets V_1 and V_2 such that each edge of G connects a vertex of V_1 to a vertex of V_2, and the pair $(V_1,\ V_2)$ is called bipartite of G. Similarly, G is called a complete bipartite graph if each vertex of V_1 is adjacent to each vertex of V_2. Now, consider two graphs $G=(V,\ E)$ and $G^{'}=(V^{'},\ E^{'})$, then G and $G^{'}$ are isomorphic to each other and it is denoted by $G\cong G^{'}$ if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved, see [10].

Definition 3.1: An undirected simple graph $G(Z_p^*,Q_p,I)$ is called a Neutrosophic quadratic residue graph of the Neutrosophic group $N(Z_p^*,I)$ whose vertex set is $N(Z_p^*,I)$ and two distinct vertices x and y are adjacent in $G(Z_p^*,Q_p,I)$ if and only if $xy\in N(Q_p,I)$.

Before studying the properties of neutrosophic quadratic residue graphs, we give two examples to illustrate their usefulness.

Example 3.2: Since $N(Z_5^*, I) = \{1,\ 2,\ 3,\ 4,\ I,\ 2I,\ 3I,\ 4I\}$ is the vertex set of the graph $G(Z_5^*,\ Q_5,\ I)$, where $N(Q_5,\ I) = \{1, 4,\ I,\ 4I\}$.

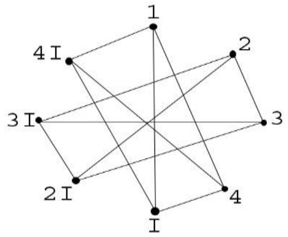


Figure 1. Neutrosophic Quadratic Residue Graph $G(Z_5^*,\ Q_5,\ I)$ of modulo 5.

Example 3.3: For $p = 7$, we have $N(Z_7^*, I) = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ I,\ 2I,\ 3I,\ 4I,\ 5I,\ 6I\}$ and $N(Q_7,\ I) = \{1,\ 2,\ 4,\ I,\ 2I,\ 4I\}$. Then the graph $G(Z_7^*,\ Q_7,\ I)$ is represented as follows.

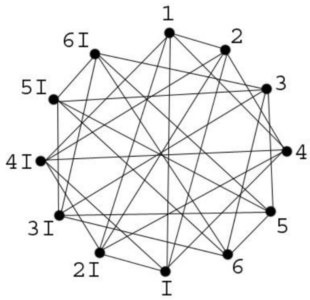


Figure 2. Neutrosophic Quadratic Residue Graph $G(Z_7^*,\ Q_7,\ I)$ of modulo 7.

In this section, the basic properties of $G(Z_p^*,\ Q_p,\ I)$ being studied. We begin with the disconnectedness of the graph $G(Z_p^*,\ Q_p,\ I)$.

Theorem 3.4: For $p > 2$, the graph $G(Z_p^*,\ Q_p,\ I)$ is disconnected. In particular, graph $G(Z_p^*,\ Q_p,\ I)$ is the disjoint union of two complete components.

Proof: Let $p > 2$ be an odd prime. Then the vertex set of neutrosophic quadratic residue graph $G(Z_p^*,\ Q_p,\ I)$ is $N(Z_p^*,\ I)$. But

$$N(Z_p^*,\ I) = N(Q_p,\ I) \cup N(\overline{Q_p},\ I)$$

$$= (Q_p \cup Q_p I) \cup (\overline{Q_p} \cup \overline{Q_p} I)$$

where $(Q_p \cup Q_p I) \cap (\overline{Q_p} \cup \overline{Q_p} I) = \emptyset$. This gives us that the vertex set $N(Z_p^*,\ I)$ is partitioned into two disjoint unions of $(Q_p \cup Q_p I)$ and $(\overline{Q_p} \cup \overline{Q_p} I)$. So, because of Theorem 2.8, we clear that $G(Z_p^*, Q_p, I)$ is disconnected. Now consider the following three cases.

Case 1: Suppose $x, y \in N(Q_p, I)$. Then obviously $xy \in N(Q_p, I)$. This implies that there exists an edge between any two vertices x and y in the graph $G(Z_p^*, Q_p, I)$. Thus, $G(Z_p^*, Q_p, I)$ has a complete subgraph, say $Comp(Z_p^*, Q_p, I)$ whose vertex set is $N(Q_p, I)$.

Case 2: Suppose $x, y \in N(\overline{Q_p}, I)$. Then again by Theorem 2.8, $xy \in N(\overline{Q_p}, I)$. So, in this case also there exists an edge between every two vertices x and y in the graph $G(Z_p^*, Q_p, I)$. Thus, the graph, $G(Z_p^*, Q_p, I)$ has another complete subgraph, say $Comp(Z_p^*, \overline{Q_p}, I)$ whose vertex set is $N(\overline{Q_p}, I)$.

Case 3: Suppose $x \in N(Q_p, I)$ and $y \in N(\overline{Q_p}, I)$. Then $xy \notin N(Q_p, I)$. It gives us that there is no edge between x and y when $x \in N(Q_p, I)$ and $y \in N(\overline{Q_p}, I)$.

From the above three cases, we conclude that $Comp(Z_p^*, Q_p, I)$ and $Comp(Z_p^*, \overline{Q_p}, I)$ are two disjoint complete components of the graph $G(Z_p^*, Q_p, I)$ such that

$$G(Z_p^*, Q_p, I) = Comp(Z_p^*, Q_p, I) \cup Comp(Z_p^*, \overline{Q_p}, I).$$

Example 3.5: Two components of the graph $G(Z_5^*, Q_5, I)$ as shown below.

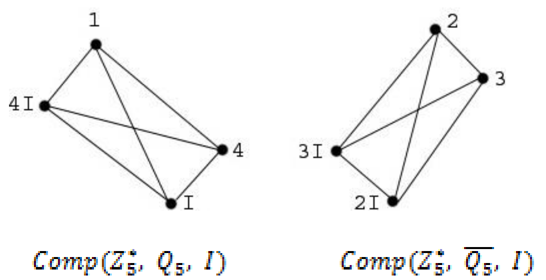


Figure 3. Components of the graph $G(Z_5^*, Q_5, I)$.

For each odd prime p, the structure of $G(Z_p^*, Q_p, I)$ is easy to describe, because it contains the following properties:

1. $G(Z_p^*, Q_p, I)$ contains two disjoint connected components for each $p > 2$.

2. Each component of $G(Z_p^*, Q_p, I)$ contains even and odd cycles for $p \geq 5$.

3. Each component of $G(Z_p^*, Q_p, I)$ is not a bipartite graph for $p \geq 3$.

The next result gives useful and important properties of the components of the graph $G(Z_p^*, Q_p, I)$ when $p > 2$.

Theorem 3.6: For each prime $p > 2$, $Comp\ (Z_p^*, Q_p, I) \cong Comp\ (Z_p^*, \overline{Q_p}, I)$.

Proof: For each prime $p > 2$, the Neutrosophic quadratic residue and non-residue sets of $N(Z_p^*, I)$ are given by $N(Q_p, I) = Q_p \cup Q_p I$ and $N(\overline{Q_p}, I) = \overline{Q_p} \cup \overline{Q_p} I$.

These are the vertex sets of the components $Comp\ (Z_p^*, Q_p, I)$ and $Comp\ (Z_p^*, \overline{Q_p}, I)$, respectively. Also, we have $|N(Q_p, I)| = \frac{p-1}{2} + \frac{p-1}{2} = p - 1 = |N(\overline{Q_p}, I)|$. Now to prove that $Comp\ (Z_p^*, Q_p, I)$ and $Comp\ (Z_p^*, \overline{Q_p}, I)$ are isomorphic as groups. For this, we define a function $f: N(Q_p, I) \to N(\overline{Q_p}, I)$ by the relation $f(u) = v$ for every $u \in N(Q_p, I)$ and $v \in N(\overline{Q_p}, I)$. Because of $|N(Q_p, I)| = p - 1$ and $|N(\overline{Q_p}, I)| = p - 1$, the map f is a one-to-one correspondence.

Now, suppose $\bar{e}$ be an edge with end vertices v and v' in the component $Comp\ (Z_p^*, \overline{Q_p}, I)$. Then $\bar{e} = (v, v') \Leftrightarrow \bar{e} = (f(u), f(u'))$

$$\Leftrightarrow \bar{e} = f(u, u')$$

$$\Leftrightarrow \bar{e} = f(e),$$

where $e = (u, u')$ be an edge with end vertices u and u' in $Comp\ (Z_p^*, Q_p, I)$. This shows that there is a one-to-one correspondence between their vertices and their edges such that the incidence relationship is preserved. Hence, $Comp\ (Z_p^*, Q_p, I) \cong Comp\ (Z_p^*, \overline{Q_p}, I)$.

The following example illustrates the procedure of the above theorem 3.6 clearly.

Example 3.7: Since $N(Q_5, I) = \{1, 4, I, 4I\}$ and $N(\overline{Q_5}, I) = \{2, 3, 2I, 3I\}$. Using the map $f: N(Q_5, I) \to N(\overline{Q_5}, I)$ as above, write the equations $f(1) = 2$, $f(4) = 3$, $f(I) = f(2I)$ and

$f(4I) = f(3I)$. These equations show that f is a one-to-one correspondence between the graph components $Comp\ (Z_5^*,\ Q_5,\ I)$ and $Comp\ (Z_5^*,\ \overline{Q_5},\ I)$, and thus which are isomorphic as graphs.

This special case of the above theorem when $p > 2$ occurs frequently and so we isolate it as a corollary.

Corollary 3.8: Each component of the neutrosophic quadratic residue graph is isomorphic to the complete graph K_{p-1}.

Proof: Due to Theorem 3.6, the only possibility of the graph $Comp\ (Z_p^*,\ Q_p,\ I)$ is $Comp\ (Z_p^*,\ Q_p,\ I) \cong Comp\ (Z_p^*,\ \overline{Q_p},\ I)$. Therefore, the order and size of each component are $p-1$ and $\binom{p-1}{2}$, respectively, and thus each component of the graph $G(Z_p^*,\ Q_p,\ I)$ is isomorphic to the complete graph K_{p-1}.

Example 3.9: $Comp\ (Z_5^*,\ Q_5,\ I) \cong K_4$ and $Comp\ (Z_7^*,\ Q_7,\ I) \cong K_6$.

The integer p is prime if and only if $p = 2$ or $p \equiv 3(mod\ 4)$ $p \equiv 1(mod\ 4)$. But, this paper p will denote odd prime integer such that either $p \equiv 1(mod\ 4)$ or $p \equiv 3(mod\ 4)$. These prime integers are weapons for verifying two components of the graph $G(Z_p^*,\ Q_p,\ I)$ are Eulerian or not. It is now the time for determining the cases in which the components of the graph $G(Z_p^*,\ Q_p,\ I)$ are Eulerian, but first, we recall the following well-known result.

Theorem 3.10 [10]: A connected graph G is Eulerian if and only if the degree of each vertex of G is even.

For $p \equiv 1\ (mod\ 4)$ or $p \equiv 3\ (mod\ 4)$, the following theorems show that $G(Z_p^*,\ Q_p,\ I)$ could not be Eulerian.

Theorem 3.11: If $p \equiv 1\ (mod\ 4)$ or $p \equiv 3\ (mod\ 4)$, then each component of $G(Z_p^*,\ Q_p,\ I)$ is not Eulerian.

Proof: Suppose on contrary that each component of $G(Z_p^*,\ Q_p,\ I)$ is Eulerian, which implies that the degree of each vertex is even. By Theorem 3.6, it is clear that

$$Comp\ (Z_p^*,\ Q_p,\ I) \cong Comp\ (Z_p^*,\ \overline{Q_p},\ I) \cong K_{p-1}.$$

So, for every vertex x in $G(Z_p^*, Q_p, I)$, we have

$$\deg(x) = (p-1) - 1 = p-2.$$

$\deg(x) = (4q+1) - 2 = 4q-1$, which is odd. Similarly, we can show that $\deg(x) = (4q+3) - 2 = 4q+1$, which is also odd. Hence, we found that the degree of each vertex in the graph $G(Z_p^*, Q_p, I)$ can not be even. This contraposition shows that each component of $G(Z_p^*, Q_p, I)$ is never Eulerian when $p \equiv 1 (mod\ 4)$ or $p \equiv 3 (mod\ 4)$.

4. Neutrosophic Quadratic Nonresidue Graphs

In this section, we establish a complement graph of the neutrosophic quadratic residue graph $G(Z_p^*, Q_p, I)$, which is denoted by $\bar{G}(Z_p^*, \overline{Q_p}, I)$ and it is called a Neutrosophic quadratic nonresidue graph whose vertex set is the Neutrosophic group $N(Z_p^*, I)$ and edge set is $E(\bar{G}(Z_p^*, \overline{Q_p}, I)) = \{(x, y) : x, y \in N(Z_p^*, I)\ and\ xy \in N(\overline{Q_p}, I)\}$.

Example 4.1: Since $N(Z_3^*, I) = \{1, 2, I, 2I\}$ and $N(\overline{Q_3}, I) = \{2, 2I\}$. The Neutrosophic quadratic nonresidue graph of $N(Z_3^*, I)$ is shown below.

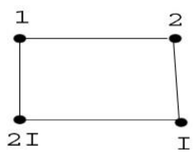


Figure 4. The graph $\bar{G}(Z_3^*, \overline{Q_3}, I)$.

Now several interesting properties of these graphs on Neutrosophic quadratic nonresidues of modulo p have been obtained.

We begin with the basic properties of $\bar{G}(Z_p^*, \overline{Q_p}, I)$.

Theorem 4.2: The Neutrosophic quadratic nonresidue graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is connected.

Proof: By the Theorem 2.8, $xy \in N(\overline{Q_p}, I)$ whenever $x \in N(Q_p, I)$ and $x \in N(\overline{Q_p}, I)$. This relates, for each $1 \le i \le \frac{p-1}{2}$, we have

$Q_p = \left\{x_1, x_2, \ldots, x_{\frac{p-1}{2}}\right\}$,

$Q_p I = \left\{x_1 I, x_2 I, \ldots, x_{\frac{p-1}{2}} I\right\}$,

$\overline{Q_p} = \left\{y_1, y_2, \ldots, y_{\frac{p-1}{2}}\right\}$ and

$\overline{Q_p} I = \left\{y_1 I, y_2 I, \ldots, y_{\frac{p-1}{2}} I\right\}$.

These sets determine the elements

$x_1 y_1, x_1 y_2, \ldots, x_1 y_i$;

$x_2 y_1, x_2 y_2, \ldots, x_2 y_i$;

… … …

$x_i y_1, x_i y_2, \ldots, x_i y_i$;

… … …

$(x_1 I)(y_1 I), (x_1 I)(y_2 I), \ldots, (x_1 I)(y_i I)$;

$(x_2 I)(y_1 I), (x_2 I)(y_2 I), \ldots, (x_2 I)(y_i I)$;

… … …

$(x_i I)(y_1 I), (x_i I)(y_2 I), \ldots, (x_i I)(y_i I)$;

… … …

are elements in $N(\overline{Q_p}, I)$ and which are the edges in the graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$. Consequently, there is a path between any two distinct vertices in $\bar{G}(Z_p^*, \overline{Q_p}, I)$ and hence $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is connected.

Theorem 4.3: The Neutrosophic quadratic nonresidue graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is $(p-1)$- regular.

Proof: If x is any vertex of the Neutrosophic quadratic nonresidue graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$, then x must be an element of the Neutrosophic group $N(Z_p^*, I)$. So there exist Neutrosophic quadratic residues $N(Q_p, I)$ and nonresidues $N(\overline{Q_p}, I)$ such that

$$N(Z_p^*, I) = N(Q_p, I) \cup N(\overline{Q_p}, I).$$

This partition of the vertex set of the graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ implies that either $x \in N(Q_p, I)$ or $x \in N(\overline{Q_p}, I)$.

Now $x \in N(Q_p, I)$, and if $N(\overline{Q_p}, I) = \left\{y_1, y_2, \ldots, y_{\frac{p-1}{2}}, y_1 I, y_2 I, \ldots, y_{\frac{p-1}{2}} I\right\}$ then by Theorem 2.8 $xN(\overline{Q_p}, I) = \left\{xy_1, xy_2, \ldots, xy_{\frac{p-1}{2}}, xy_1 I, xy_2 I, \ldots, xy_{\frac{p-1}{2}} I\right\} = N(\overline{Q_p}, I)$.

It gives that the vertex x is adjacent to every element in $N(\overline{Q_p}, I)$. This means that

$$\begin{aligned}\deg(x) &= |N(\overline{Q_p}, I)| \\ &= |\overline{Q_p} \cup \overline{Q_p} I| \\ &= |\overline{Q_p}| + |\overline{Q_p} I| \\ &= \frac{p-1}{2} + \frac{p-1}{2} \\ &= p-1.\end{aligned}$$

Next $x \in N(\overline{Q_p}, I)$ and if $N(Q_p, I) = \left\{x_1, x_2, \ldots, x_{\frac{p-1}{2}}, x_1 I, x_2 I, \ldots, x_{\frac{p-1}{2}} I\right\}$. Then, again by the Theorem 2.8,

$$x\, N(Q_p, I) = N(\overline{Q_p}, I).$$

It yields that $\deg(x) = p-1$, proving that the Neutrosophic Quadratic nonresidue Graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is $(p-1)-$ regular.

Finally looking at another basic property of the Neutrosophic quadratic nonresidue graph, we state the following fundamental theorem of graph theory.

Theorem 4.4 [10]: If G is a simple undirected graph of the size $|E|$. Then $\sum_{x \in V(G)} \deg(x) = 2|E|$.

Theorem 4.5: The size of the graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is $(p-1)^2$.

Proof: By the Theorem 4.3 and theorem 2.5, the size of the graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is denoted by $|E(\bar{G})|$ and defined as

$$|E(\bar{G})| = \frac{1}{2} \sum_{x \in N(Z_p^*, I)} \deg(x)$$

$$= \frac{1}{2}\sum_{x\in N(Z_p^*,\ I)}(p-1)$$

$$= \frac{1}{2}(p-1)\left|N(Z_p^*,\ I)\right|$$

$$= \frac{1}{2}(p-1)(2p-2)$$

$$= (p-1)^2.$$

5. Conclusions

In this paper, we have studied two Neutrosophic graphical representations for determining the Neutrosophic Quadratic residues and nonresidues of the Neutrosophic group of modulo prime by using Neutrosophic algebraic theory, number theory, and classical algebraic theory. In addition to these, the Neutrosophic algebraic system can find Neutrosophic properties of Quadratic residues and nonresidues. Also, this algebraic-based application produces the complement neutrosophic graphs of each disjoint union of Neutrosophic Quadratic residue and nonresidue sets.

References

1. Judson, T. W., Abstract Algebra: Theory and Applications, 2020 Edition, *Orthogonal Publishing L3C*, Michigan, 2000.
2. Chalapathi, T.; Sajana, S.; Bharathi, D., Classical Pairs in Z_n, *Notes on Number Theory and Discrete Mathematics*, 2020, *Vol. 26*, No. 1, pp. 59 - 69.
3. Apostol, T.M., Introduction to Analytic Number Theory, *Springer International Student Edition*, 1989.
4. Kandasamy, W. B. V.; Smarandache, F., Basic Neutrosophic Algebraic Structures and Their Application to Fuzzy and Neutrosophic Models, *Hexis, USA*, 2004.
5. Chalapathi, T.; VMSS Kiran Kumar, R., Neutrosophic Graphs of Finite Groups, Neutrosophic Sets and Systems, *University of New Mexico*, USA, 2017, *Vol.15.*
6. Smarandache, F.; Ali, M., Neutrosophic Sets and Systems, *University of New Mexico*, USA, 2017, *Vol.15.*
7. Maheswari, B.; Madhavi, L., Enumeration of Triangles and Hamilton Cycles in Quadratic Residue Cayley Graphs, *Chamchuri Journal of Mathematics*, 2009, *Vol. 1,* No 1, pp. 95 – 103.
8. Budden, M., et al., Enumeration of Triangles in Quartic Residue Graphs, *Integers 12*, 2012, pp. 57 – 73.
9. Giudici, R. E.; Olivieri, A. A., Quadratic modulo 2^n Cayley graphs, *Discrete Mathematics*, 2000, *Vol. 215*, No. 1, pp. 73 - 79.
10. Bondy, J. A.; Murty, U. S. R., Graph Theory with Applications, *Macmillan Press Ltd.*, Great Britain, 1976.

Received: May 26, 2021. Accepted: October 5, 2021

NSS

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

A Note on Neutrosophic Almost Bitopological Group

Bhimraj Basumatary[1,*] and Nijwm Wary[2]

[1]Department of Mathematical Sciences, Bodoland University, Kokrajhar; brbasumatary14@gmail.com

[2]Department of Mathematical Sciences, Bodoland University, Kokrajhar; nijwmwr0@gmail.com

[*]Correspondence: brbasumatary14@gmail.com

Abstract. Because the concept of an almost topological group is relatively new, so, in this paper, we introduce the notion of the neutrosophic almost bitopological group. In this work, we define the definition of the neutrosophic almost continuous mapping and then we define the neutrosophic almost bitopological group. Also, we investigate some properties of the neutrosophic almost bitopological group.

Keywords: Neutrosophic Set; Neutrosophic Almost Continuous Mapping; Neutrosophic Bitopological Group; Neutrosophic Almost Bitopological Group.

A list of Abbreviations

NS - neutrosophic set.

NG - neutrosophic group.

NT - neutrosophic topology.

NTS - neutrosophic topological space.

NOS - neutrosophic open set.

NCoS - neutrosophic closed set.

NBTS - neutrosophic bitopological space.

NTG - neutrosophic topological group.

NSOS - neutrosophic semi open set.

NSCoS - neutrosophic semi closed set.

NROS - neutrosophic regularly open set.

NRCS - neutrosophic regularly closed set.

NABTG - neutrosophic almost bitopological group.

1. Introduction

The Fuzzy set (FS) concept was first introduced by Zadeh [29] in 1965. The concept of membership function and explained the idea of uncertainty defined with the help of FS. Atanassov [8] generalized the concept of fuzzy set theory (FST) and introduced the degree of non-membership and proposed intuitionistic fuzzy set theory (IFST). Azad [9] discussed the Fuzzy Semi-continuity (FSC), Fuzzy Almost Continuity (FAC), and Fuzzy Weakly Continuity (FWC). Chang [11] defined the concept of the fuzzy topology space (FTS) and Coker [12] introduced the Intuitionistic fuzzy topological space (IFTS). Kandil [15] and Kelly [16] discussed the fuzzy bitopological spaces and bitopological space. Rosenfeld [20] introduced the fuzzy groups and Foster [13] defined the fuzzy topological groups.

F. Smarandache [25, 26] was introduced as an independent component of the degree of uncertainty and discovered the neutrosophic set (NS). After the discovery of NS, many researchers have developed the neutrosophic set theory for various branches of Science and Technology. NS is used as an independent measure of uncertainty Membership and Non-Membership Function. FS is used to control uncertainty by using the membership function only. While NS is used to control uncertainty by using the truth membership function, indeterminacy membership function, and falsity membership function. Salama and Alblowi [21] introduced the concept of neutrosophic topological space (NTS). Salama et. al [22] studied closed sets and neutrosophic continuous functions. Imran et. al [14] discussed some types of neutrosophic topological groups in relation to neutrosophic alpha open sets. Abdel-Basset et. al [1] have applied neutrosophic set theory (NST) as a tool on group discussion making framework. Abdel-Basset et al [2] done the work in solving chain problems using a base-worst method based on a novel plithogenic model. Sumathi et. al [27, 28] studied the Fuzzy Neutrosophic Groups (FNG) and Topological Group Structure of Neutrosophic set. Mwchahary et. al [17] did their work in neutrosophic bitopological space. Abdel-Basset et. al [3] developed supplier selection with group decision-making under the type-2 neutrosophic number of TOPSIS technology. Abdel-Basset et. al [4, 5] studied the chain management practices of evaluation of the green supply and defined for achieving sustainable supplier selection of VIKOR method. Also, Abdel-Basset et. al [6, 7] developed hybrid multi-criteria decision-making for the sustainability assessment of bioenergy production technologies and employed an evaluation approach for sustainable renewable energy systems under uncertain environments.

In this paper, we try to study the neutrosophic almost bitopological group (NABTG) and some of their properties by using the definition of neutrosophic almost continuous mapping (NACM).

2. Preliminaries

2.1. Definition:[28]

A NS A on a universe of discourse X can be expressed as $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \Gamma_A(x)\rangle : x \in X\}$, where $\mathcal{T}, \mathcal{I}, \Gamma : X \to]^-0, 1^+[$. Note that $0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \Gamma_A(x) \leq 3$.

2.2. Definition:[28]

The complement of NS A is expressed as $A^c(x) = \{\langle x, \mathcal{T}_{A^c}(x) = \Gamma_A(x), \mathcal{I}_{A^c}(x) = 1 - \mathcal{I}_A(x), \Gamma_{A^c}(x) = \mathcal{T}_A(x)\rangle : x \in X\}$.

2.3. Definition:[28]

Let X be non-empty set and $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \Gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mathcal{T}_B(x), \mathcal{I}_B(x), \Gamma_B(x)\rangle : x \in X\}$, are NSs. Then

(i) $A \Cap B = \{\langle x, min(\mathcal{T}_A(x), \mathcal{T}_B(x)), min(\mathcal{I}_A(x), \mathcal{I}_B(x)), max(\Gamma_A(x), \Gamma_B(x)), \rangle : x \in X\}$
(ii) $A \Cup B = \{\langle x, max(\mathcal{T}_A(x), \mathcal{T}_B(x)), max(\mathcal{I}_A(x), \mathcal{I}_B(x)), min(\Gamma_A(x), \Gamma_B(x)), \rangle : x \in X\}$
(iii) $A \leq B$ if for each $x \in X, \mathcal{T}_A(x) \leq \mathcal{T}_B(x), \mathcal{I}_A(x) \leq \mathcal{I}_B(x), \Gamma_A(x) \geq \Gamma_B(x)$.

2.4. Definition:[28]

Let $(X, *)$ be a group and let A be a NG in X. Then A is said to be a NG in X if it satisfies the following conditions:

(i) $\mathcal{T}_A(xy) \geq \mathcal{T}_A(x) \Cap \mathcal{T}_A(y)$, $\mathcal{I}_A(xy) \geq \mathcal{I}_A(x) \Cap \mathcal{I}_A(y)$ and $\Gamma_A(xy) \leq \Gamma_A(x) \Cup \Gamma_A(y)$,
(ii) $\mathcal{T}_A(x^{-1}) \geq \mathcal{T}_A(x)$, $\mathcal{I}_A(x^{-1}) \geq \mathcal{I}_A(x)$, and $\Gamma_A(x^{-1}) \leq \Gamma_A(x)$.

2.5. Definition:[21]

Let X be a group and let $\mathbb{G}$ be NG in X and e be the identity of X. We define the NS $\mathbb{G}_e$ by

$$\mathbb{G}_e = \{x \in X : \mathcal{T}_{\mathbb{G}}(x) = \mathcal{T}_{\mathbb{G}}(e), \mathcal{I}_{\mathcal{G}}(x) = \mathcal{I}_{\mathcal{G}}(e), \Gamma_{\mathbb{G}}(x) = \Gamma_{\mathbb{G}}(e)\}.$$

We note for a NG $\mathbb{G}$ in a group X, for every $x \in X : \mathcal{T}_{\mathbb{G}}(x^{-1}) = \mathcal{T}_{\mathbb{G}}(x)$, $\mathcal{I}_{\mathbb{G}}(x^{-1}) = \mathcal{I}_{\mathbb{G}}(x)$ and $\Gamma_{\mathbb{G}}(x^{-1}) = \Gamma_{\mathbb{G}}(x)$. Also for the identity e of the group $X : \mathcal{T}_{\mathbb{G}}(e) \geq \mathcal{T}_{\mathbb{G}}(x), \mathcal{I}_{\mathbb{G}}(e) \geq \mathcal{I}_{\mathbb{G}}(x)$, and $\Gamma_{\mathbb{G}}(e) \leq \Gamma_{\mathbb{G}}(x)$.

2.6. Definition:[21]

Let X be a non-empty set and a NT on X is a family $\daleth_X$ of neutrosophic subsets of X satisfying the following axioms:

(i) $0_N, 1_N \in \daleth_X$
(ii) $G_1 \Cap G_2 \in \daleth_X$ for any $G_1, G_2 \in \daleth_X$

(iii) $\Cup G_i \in \daleth_X \ \forall \ \{G_i : i \in J\} \subseteq \daleth_X$

In this case, the pair $(X, \daleth_X)$ is called a neutrosophic topological space (NTS) and any NS in $\daleth_X$ is known as neuterosophic open set (NOS). The elements of $\daleth_X$ are called NOSs, a NS F is neutrosophic closed set (NCoS) if and only if it F^C is NOS.

2.7. **Definition:[8]**

It is known that $f : (X, \daleth_X) \to (Y, \daleth_Y)$ is neutrosophic continuous if the preimage of each neutrosophic open set in Y is neutrosophic open set in X.

2.8. **Definition:[17]**

Let $(X, \daleth_i^X)$ and $(X, \daleth_j^X)$ be the two neutrosophic topologies on X, then $(X, \daleth_i^X, \daleth_j^X)$ is said to be a neutrosophic bitopological space (NBTS). Throughout in this paper the indices i, j take the value $\in \{i, j\}$ and $i \neq j$.

2.9. **Definition:[17]**

Let $(X, \daleth_i^X, \daleth_j^X)$ be a NBTS. Then a set for $A = \big\{\langle x, \alpha_{ij}, \beta_{ij}, \gamma_{ij}\rangle : x \in X\big\}$, neutrosophic $(\daleth_i^X, \daleth_j^X)N\vdash$ interior of A is the union of all $(\daleth_i^X, \daleth_j^X)N\vdash$ open sets of X contained in A and can be defined as follows:
$(\daleth_i^{\mathcal{G}}, \daleth_j^{\mathcal{G}})N \vdash Int(A) = \big\{\langle x, \Cup_{\daleth_i^X} \Cup_{\daleth_j^X} \alpha_{ij}, \Cap_{\daleth_i^X} \Cap_{\daleth_j^X} \beta_{ij}, \Cap_{\daleth_i^X} \Cap_{\daleth_j^X} \gamma_{ij}\rangle : x \in X\big\}$

2.10. **Definition:[17]**

Let $(X, \daleth_i^X, \daleth_j^X)$ be a NBTS. Then a set for $A = \big\{\langle x, \alpha_{ij}, \beta_{ij}, \gamma_{ij}\rangle : x \in X\big\}$, neutrosophic $(\daleth_i^X, \daleth_j^X)N\vdash$ closure of A is the intersection of all $(\daleth_i^X, \daleth_j^X)N\vdash$ closed sets of X contained in A and can be defined as follows:
$(\daleth_i^{\mathcal{G}}, \daleth_j^{\mathcal{G}})N \vdash Cl(A) = \big\{\langle x, \Cap_{\daleth_i^X} \Cap_{\daleth_j^X} \alpha_{ij}, \Cup_{\daleth_i^X} \Cup_{\daleth_j^X} \beta_{ij}, \Cup_{\daleth_i^X} \Cup_{\daleth_j^X} \gamma_{ij}\rangle : x \in X\big\}$

2.11. **Definition:[10]**

Let X be a group and $\mathbb{G}$ be a NG on X. Let $\daleth_X$ be a NT on $\mathbb{G}$ then $(\mathbb{G}, \daleth_X)$ is called a neutrosophic topological group (NTG) if the following conditions are satisfied:

(i) The mapping $\lambda : (\mathbb{G}, \daleth_X) \times (\mathbb{G}, \daleth_X) \to (\mathbb{G}, \daleth_X)$ defined by $\lambda(g_1, g_2) = g_1 g_2$, for all $g_1, g_2 \in X$, is relatively neutrosophic continuous.

(ii) The mapping $\mu : (\mathbb{G}, \daleth_X) \to (\mathbb{G}, \daleth_X)$ defined by $\mu(g) = g^{-1}$, for all $g \in X$, is relatively neutrosophic continuous.

3. Main Works:

3.1. Definition:

Let $\mathscr{A}$ be a NS of a NBTS $(X, \daleth_i^X, \daleth_j^X)$, then $\mathscr{A}$ is called a neutrosophic semi-open set (NSOS) of X if there exists a $\mathscr{B} \in (X, \daleth_i^X, \daleth_j^X)$ such that $\mathscr{A} \subseteq (\daleth_i^{\mathscr{G}}, \daleth_j^{\mathscr{G}})N \vdash Cl(\mathscr{B})$, for each $i = j = 1, 2$.

3.2. Definition:

Let $\mathscr{A}$ be a NS of a NBTS $(X, \daleth_i^X, \daleth_j^X)$, then $\mathscr{A}$ is called a neutrosophic semi-closed set (NSCoS) of X if there exists a $\mathscr{B}^c \in (X, \daleth_i^X, \daleth_j^X)$ such that $(\daleth_i^{\mathscr{G}}, \daleth_j^{\mathscr{G}})N \vdash Int(\mathscr{B}) \subseteq \mathscr{A}$, for each $i = j = 1, 2$.

3.3. Definition:

A NS $\mathscr{A}$ of a NBTS $(X, \daleth_i^X, \daleth_j^X)$ is said to be a neutrosophic regularly open set (NROS), if $(\daleth_i^{\mathscr{G}}, \daleth_j^{\mathscr{G}})N \vdash Int\Big((\daleth_i^X, \daleth_j^X)N \vdash Cl(\mathscr{A})\Big) = \mathscr{A}$, for each $i = j = 1, 2$.

3.4. Definition:

A NS $\mathscr{A}$ of a NBTS $(X, \daleth_i^X, \daleth_j^X)$ is said to be a neutrosophic regularly closed set (NRCS), if $(\daleth_i^{\mathscr{G}}, \daleth_j^{\mathscr{G}})N \vdash Cl\Big((\daleth_i^X, \daleth_j^X)N \vdash Int(\mathscr{A})\Big) = \mathscr{A}$, for each $i = j = 1, 2$.

3.5. Definition:

A mapping $\phi : (X, \daleth_i^X, \daleth_j^X) \rightarrow (Y, \daleth_i^Y, \daleth_j^Y)$ is said to be a neutrosophic almost continuous mapping (NACM), if $\phi^{-1}(\mathscr{A}) \in (X, \daleth_i^X, \daleth_j^X)$ for each neutrosophic regularly closed set $\mathscr{A}$ of $(Y, \daleth_i^Y, \daleth_j^Y); i = j = 1, 2$.

Example 3.1. Let $X = \{x, y\}$ and $Y = \{p, q\}$
$P = \{\langle \frac{0.6,0.4,0.4}{a} \rangle, \langle \frac{0.7,0.2,0.3}{b} \rangle\}$, $Q = \{\langle \frac{0.2,0.3,0.7}{a} \rangle, \langle \frac{0.2,0.1,0.7}{b} \rangle\}$, $R = \{\langle \frac{0.6,0.3,0.3}{a} \rangle, \langle \frac{0.7,0.1,0.2}{b} \rangle\}$,
$S = \{\langle \frac{0.1,0.8,0.7}{a} \rangle, \langle \frac{0.1,0.8,0.7}{b} \rangle\}$.
Then $\daleth_{N_1} = \{0_{N_X}, 1_{N_X}, P\}$, $\daleth_{N_2} = \{0_{N_X}, 1_{N_X}, Q\}$, $\daleth_{N_3} = \{0_{N_X}, 1_{N_X}, R\}$, $\daleth_{N_4} = \{0_{N_X}, 1_{N_X}, S\}$ are neutrosophic topological spaces. Then
$(\daleth_{N_1}, \daleth_{N_2}) \vdash$ open sets $= \{0_{N_X}, 1_{N_X}, P, Q\}$, $(\daleth_{N_1}, \daleth_{N_2}) \vdash$ closed sets $= \{0_{N_X}, 1_{N_X}, P^C, Q^C\}$,
$(\daleth_{N_3}, \daleth_{N_4}) \vdash$ open sets $= \{0_{N_X}, 1_{N_X}, R, S\}$, $(\daleth_{N_1}, \daleth_{N_2}) \vdash$ closed sets $= \{0_{N_X}, 1_{N_X}, R^C, S^C\}$.
Let $f : (X, \daleth_{N_1}, \daleth_{N_2}) \rightarrow (Y, \daleth_{N_1}, \daleth_{N_2})$ define by $f(x) = p$ and $f(y) = q$. Then we have $(\daleth_{N_3}, \daleth_{N_4}) \vdash Cl(R^C)$ is closed set.
Now, R^C is $(\daleth_{N_3}, \daleth_{N_4}) \vdash$ closed set in Y and $f^{-1}(R^C) \subseteq P$, where P is $(\daleth_{N_1}, \daleth_{N_2}) \vdash$ open set in X. Also, $Cl(f^{-1}(R^C)) \subseteq P$.
Therefore, f is neutrosophic almost continuous mapping.

3.6. **Definition:**

Let $\mathbb{G}$ be a NG on a group X. Let $\daleth_i^{\mathbb{G}}$ be a NT on $\mathbb{G}$ for each $i = 1, 2$; then $(\mathbb{G}, \daleth_1^{\mathbb{G}}, \daleth_2^{\mathbb{G}})$ is said to be a neutrosophic almost bitopological group (NABTG) if the following conditions are satisfied:

(i) A mapping $\lambda : (\mathbb{G}, \daleth_i^{\mathbb{G}}) \times (\mathbb{G}, \daleth_i^{\mathbb{G}}) \to (\mathbb{G}, \daleth_i^{\mathbb{G}})$ defined by $\lambda(x, y) = xy$, for all $x, y \in X$, is neutrosophic almost i-continuous, for each $i = 1, 2$.

(ii) A mapping $\mu : (\mathbb{G}, \daleth_i^{\mathbb{G}}) \to (\mathbb{G}, \daleth_i^{\mathbb{G}})$ defined by $\mu(x) = x^{-1}$, for all $x \in X$, is neutrosophic almost i-continuous, for each $i = 1, 2$.

Example 3.2. Let $\mathbb{G} = \{a, e\}$, where e is the identity element of $\mathbb{G}$. Then $\mathbb{G}$ is a group. Let $P = \{\langle \frac{0.6,0.4,0.4}{a} \rangle, \langle \frac{0.7,0.2,0.3}{b} \rangle\}$, $Q = \{\langle \frac{0.2,0.3,0.7}{a} \rangle, \langle \frac{0.2,0.1,0.7}{b} \rangle\}$, $R = \{\langle \frac{0.6,0.3,0.3}{a} \rangle, \langle \frac{0.7,0.1,0.2}{b} \rangle\}$, $S = \{\langle \frac{0.1,0.8,0.7}{a} \rangle, \langle \frac{0.1,0.8,0.7}{b} \rangle\}$.
Then, it is clear that the mapping $\mu : (a, b) \to ab$ of $(\mathbb{G}, \daleth_1^{\mathbb{G}}, \daleth_2^{\mathbb{G}}) \times (\mathbb{G}, \daleth_1^{\mathbb{G}}, \daleth_2^{\mathbb{G}})$ in to $(\mathbb{G}, \daleth_1^{\mathbb{G}}, \daleth_2^{\mathbb{G}})$ is neutrosophic almost continuous and $\lambda : a \to a^{-1}$ of $(\mathbb{G}, \daleth_1^{\mathbb{G}}, \daleth_2^{\mathbb{G}})$ in to $(\mathbb{G}, \daleth_1^{\mathbb{G}}, \daleth_2^{\mathbb{G}})$ is neutrosophic almost continuous.
Hence, $(\mathbb{G}, \daleth_1^{\mathbb{G}}, \daleth_2^{\mathbb{G}})$ is neutrosophic almost topological group.

Remark 3.1. $(\mathbb{G}, \daleth_i^{\mathbb{G}})$ is a NABTG for each $i = 1, 2$; if following conditions hold good:

(i) for $g_1, g_2 \in \mathbb{G}$ and for every $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS $\mathcal{U}$ containing g_1, g_2 in $\mathbb{G}$, $\exists\ \daleth_i^{\mathbb{G}}-$ neutrosophic open, $i = 1, 2$ nbds $\mathcal{P}$ and $\mathcal{Q}$ of g_1 and g_2 respectively in $\mathbb{G}$ so that $\mathcal{P} * \mathcal{Q} \subseteq \mathcal{U}$ and

(ii) for $g \in \mathbb{G}$ and for every $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS $\mathcal{Q}$ in $\mathbb{G}$ containing g^{-1}, $\exists\ \daleth_i^{\mathbb{G}}-$ neutrosophic open, $i = 1, 2$ nbd $\mathcal{P}$ of g in $\mathbb{G}$ such that $\mathcal{P}^{-1} \subseteq \mathcal{Q}$.

Theorem 3.1. *Let $(\mathbb{G}, \daleth_i^{\mathbb{G}})$, $i = 1, 2$; be a NABTG and let $a \in \mathbb{G}$ be any element of $\mathbb{G}$. Then*

(i) $\pi_a : (\mathbb{G}, \daleth_i^{\mathbb{G}}) \to (\mathbb{G}, \daleth_i^{\mathbb{G}})$: $\pi_a(x) = ax$, for all $x \in \mathbb{G}$, is neutrosophic almost i-continuous, for each $i = 1, 2$.

(ii) $\sigma_a : (\mathbb{G}, \daleth_i^{\mathbb{G}}) \to (\mathbb{G}, \daleth_i^{\mathbb{G}})$: $\sigma_a(x) = xa$, for all $x \in \mathbb{G}$, is neutrosophic almost i-continuous, for each $i = 1, 2$.

Proof.

(i) Let $p \in \mathbb{G}$ and let $\mathcal{W}$ be a $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS, $i = j = 1, 2$; containing ap in $\mathbb{G}$. By Definition 3.6, $\exists\ \daleth_i^{\mathbb{G}}-$ neutrosophic open, $i = 1, 2$ neighborhoods $\mathcal{U}, \mathcal{V}$ of ap in $\mathbb{G}$ so that $\mathcal{U}\mathcal{V} \subseteq \mathcal{W}$. Especially, $a\mathcal{V} \subseteq \mathcal{W}$ that is $\pi_a(\mathcal{V}) \subseteq \mathcal{W}$. This shows that π_a is NACM at p and therefore π_a is NACM.

(ii) Suppose $p \in \mathbb{G}$ and $\mathcal{W} \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS$(\mathbb{G})$, containing pa. Then $\exists\ \daleth_i^{\mathbb{G}}-$ NOSs, $i = 1, 2$; $p \in \mathcal{U}$ and $a \in \mathcal{V}$ in $\mathbb{G}$ so that $\mathcal{U}\mathcal{V} \subseteq \mathcal{W}$. This shows $\mathcal{U}a \subseteq \mathcal{W}$, i.e., $\sigma_a(\mathcal{U}) \subseteq \mathcal{W}$. This implies σ_a is NACM at p. As arbitrary element p is in $\mathbb{G}$, σ_a is NACM.

Theorem 3.2. *Let $\mathbb{G}$ be any $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS, $i = j = 1, 2$; in a NABTG $(\mathbb{G}, \daleth_i^{\mathbb{G}})$. The following conditions hold good:*

(1) $a\mathcal{U} \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS$(\mathbb{G})$, for all $a \in \mathbb{G}$.

(2) $\mathcal{U}a \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS$(\mathbb{G})$, for all $a \in \mathbb{G}$.

(3) $\mathcal{U}^{-1} \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS$(\mathbb{G})$.

Proof.

(1) First, we have to prove that $a\mathcal{U} \in \daleth_i^{\mathbb{G}}, i = 1, 2$. Let $p \in a\mathcal{U}$. Then from Definition 3.6 of NABTGs, $\exists\ \daleth_i^{\mathbb{G}}-$ neutrosophic open neighborhoods, $i = 1, 2$; $a^{-1} \in \mathcal{W}_1$ and $p \in \mathcal{W}_2$ in $\mathbb{G}$ so that $\mathcal{W}_1\mathcal{W}_2 \subseteq \mathcal{U}$. Especially, $a^{-1}\mathcal{W}_2 \subseteq \mathcal{U}$. i.e., equivalently $\mathcal{W}_2 \subseteq a\mathcal{U}$. This indicates that $p \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(a\mathcal{U})$ and thus, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(a\mathcal{U}) = a\mathcal{U}$. i.e., $a\mathcal{U} \in \daleth_i^{\mathbb{G}}, i = 1, 2$. Consequently, $a\mathcal{U} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U})\Big)$.

Now, we have to prove that $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U})\Big) \subseteq a\mathcal{U}$. Since $\mathcal{U}$ is $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$; $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}) \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS$(\mathbb{G})$. From theorem 3.1, $\pi_{a^{-1}} : (\mathbb{G}, \daleth_i^{\mathbb{G}}) \to (\mathbb{G}, \daleth_i^{\mathbb{G}})$ is NACM, $i = 1, 2$ and therefore, $a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i = 1, 2$. Thus, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U})\Big) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U}) \subseteq a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})$. i.e., $a^{-1}(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U})\Big) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})$. Since $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U})\Big)$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS, $i = j = 1, 2$; it follows that $a^{-1}(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U})\Big) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})\Big) = \mathcal{U}$. i.e., $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U})\Big) \subseteq a\mathcal{U}$. Thus, $a\mathcal{U} = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U})\Big)$. This shows that $a\mathcal{U} \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS$(\mathbb{G})$.

(2) Following the same steps as in part (1) above, then we can prove $\mathcal{U}a \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS$(\mathbb{G})$, for all $a \in \mathbb{G}$.

(3) Let $x \in \mathcal{U}^{-1}$, then $\exists\ \daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$; $x \in \mathcal{W}$ in $\mathbb{G}$ so that $\mathcal{W}^{-1} \subseteq \mathcal{U}$ $\Rightarrow \mathcal{W} \subseteq \mathcal{U}^{-1}$. Therefore $\mathcal{U}^{-1}$ has interior point x. Thus, $\mathcal{U}^{-1}$ is $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$ i.e., $\mathcal{U}^{-1} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}^{-1})\Big)$. Now we have to prove that $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}^{-1})\Big) \subseteq \mathcal{U}^{-1}$. Since $\mathcal{U}$ is $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$; $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$ and hence $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})^{-1}$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i = 1, 2$ in $\mathbb{G}$. Therefore, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}^{-1})\Big) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}^{-1}) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})^{-1}$ $\Rightarrow (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}^{-1})\Big) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})\Big)^{-1} = \mathcal{U}^{-1}$. Thus, $\mathcal{U}^{-1} = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}^{-1})\Big)$. This shows that $\mathcal{U}^{-1} \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS$(\mathbb{G})$.

Corollary 3.1. *Let $\mathcal{Q}$ be any $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$; in a NABTG. Then*

(1) $a\mathcal{Q} \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ *NRCS*$(\mathbb{G})$, *for all* $a \in \mathbb{G}$.

(2) $\mathcal{Q}^{-1} \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ *NRCS*$(\mathbb{G})$.

Theorem 3.3. *Let* $\mathcal{U}$ *be any* $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ *NROS,* $i = j = 1, 2$*; in a NABTG. Then*

(1) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}a) = a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})$, *for each* $a \in \mathbb{G}$.

(2) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U}) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})a$, *for each* $a \in \mathbb{G}$.

(3) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}^{-1}) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})^{-1}$.

Proof.

(1) Taking $p \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}a)$ and consider $q = pa^{-1}$. Let $q \in \mathcal{W}$ be $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$ in $\mathbb{G}$. Then $\exists$ $\daleth_i^{\mathbb{G}}-$ NOSs, $i = 1, 2$; $a^{-1} \in \mathcal{V}_1$ and $p \in \mathcal{V}_2$ in $\mathbb{G}$, so that $\mathcal{V}_1\mathcal{V}_2 \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{W})\Big)$. By assumption, there is $g \in \mathcal{U}a \Cap \mathcal{V}_2 \Rightarrow ga^{-1} \in \mathcal{U} \Cap \mathcal{V}_1\mathcal{V}_2 \subseteq \mathcal{U} \Cap (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{W})\Big) \Rightarrow \mathcal{U} \Cap (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{W})\Big) \neq 0_N \Rightarrow \mathcal{U} \Cap \Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{W})\Big) \neq 0_N$.

Since $\mathcal{U}$ is $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$; $\mathcal{U} \Cap \mathcal{W} \neq 0_N$. i.e., $p \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})a$.

Conversely, let $q \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})a$. Then $q = pa$ for some $p \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})$. To prove $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})a \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}a)$.

Let $pa \in \mathcal{W}$ be an $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$ in $\mathbb{G}$. Then $\exists$ $\daleth_i^{\mathbb{G}}-$ NOSs, $i = 1, 2$; $a \in \mathcal{V}_1$ in $\mathbb{G}$ and $p \in \mathcal{V}_2$ in $\mathbb{G}$, so that $\mathcal{V}_1\mathcal{V}_2 \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{W})\Big)$. Since $p \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})$, $\mathcal{U} \Cap \mathcal{V}_2 \neq 0_N$. There is $g \in \mathcal{U} \Cap \mathcal{V}_2$. This gives $ag \in (\mathcal{U}a) \Cap (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{W})\Big) \Rightarrow (\mathcal{U}a) \Cap \Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{W})\Big) \neq 0_N$. From Theorem 3.2, $\mathcal{U}a$ is $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$ and thus $(\mathcal{U}a) \Cap \mathcal{W} \neq 0_N$, so $q \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}a)$.

Therefore, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}a) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})a$.

(2) Following the same steps as in part (1) above, then we can prove $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{U}) = a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})$.

(3) Since $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$; $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})^{-1}$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i = 1, 2$ in $\mathbb{G}$. So, $\mathcal{U}^{-1} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})^{-1}$ this implies $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}^{-1}) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})^{-1}$. Next, $q \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})^{-1}$. Then $q = p^{-1}$ for some $p \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})$. Let $q \in \mathcal{V}_2$ be any $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$ in $\mathbb{G}$. Then $\exists$ $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$; $\mathcal{V}_1$ in $\mathbb{G}$ so that $p \in \mathcal{V}_1$ with $\mathcal{V}_1^{-1} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\big\{(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{V}_2)\big\}$. Also, there is $g \in \mathcal{U} \Cap \mathcal{V}_1$ which implies $g^{-1} \in \mathcal{U}^{-1} \Cap (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\big\{(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{V}_2)\big\}$. i.e., $\in \mathcal{U}^{-1} \Cap (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\big\{(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{V}_2)\big\} \neq 0_N \Rightarrow \mathcal{U}^{-1} \Cap (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{V}_2) \neq 0_N \Rightarrow \mathcal{U}^{-1} \Cap \mathcal{V}_2 \neq 0_N$, since $\mathcal{U}^{-1}$ is $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$. Therefore, $q \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}^{-1})$.

Hence $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U}^{-1}) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{U})^{-1}$.

Theorem 3.4. *Let $\mathcal{Q}$ be $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ neutrosophic regularly closed, $i = j = 1, 2$ subset in a NABTG $\mathbb{G}$. Then the following statements are satisfied:*

(1) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(a\mathcal{Q}) = a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})$, *for all* $a \in \mathbb{G}$.

(2) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q}a) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})a$, *for all* $a \in \mathbb{G}$.

(3) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q}^{-1}) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})^{-1}$.

Proof.

(1) Since $\mathcal{Q}$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS, and $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS, $i = j = 1, 2$ in $\mathbb{G}$. Consequently, $a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q}) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(a\mathcal{Q})$.

Conversely, let q be an arbitrary element of $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(a\mathcal{Q})$. Assume that $q = ap$ for some $p \in \mathcal{Q}$. By assumption, this shows $a\mathcal{Q}$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i = 1, 2$; and that is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(a\mathcal{Q})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS, $i = j = 1, 2$ in $\mathbb{G}$. Suppose $a \in \mathcal{U}$ and $p \in \mathcal{V}$ be $\daleth_i^{\mathbb{G}}-$ NOSs, $i = 1, 2$ in $\mathbb{G}$, so that $\mathcal{U}\mathcal{V} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(a\mathcal{Q})$. Then $a\mathcal{V} \subseteq a\mathcal{Q}$, which it follows that $a\mathcal{V} \subseteq a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})$. Thus, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(a\mathcal{Q}) \subseteq a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})$. Hence the statement follows.

(2) Following the same steps as in part (1) above, then we can prove $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q}a) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})a$.

(3) Since $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS, $i = j = 1, 2$; so, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})^{-1}$ is $\daleth_i^{\mathbb{G}}-$ NOSs, $i = 1, 2$ in $\mathbb{G}$. Therefore, $\mathcal{Q}^{-1} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})^{-1}$ implies that $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q}^{-1}) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})^{-1}$. Next, let q be an arbitrary element of $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})^{-1}$. Then $q = p^{-1}$ for some $p \in (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})$. Let $q \in \mathcal{V}$ be $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$ in $\mathbb{G}$. Then $\exists$ $\daleth_i^{\mathbb{G}}-$ NOS, $i = 1, 2$; $\mathcal{U}$ is in $\mathbb{G}$ so that $p \in \mathcal{U}$ with $\mathcal{U}^{-1} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{V})\Big)$. Also, there is $g \in \mathcal{Q} \Cap \mathcal{U}$ which implies $g^{-1} \in \mathcal{Q}^{-1} \Cap (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{V})\Big)$. That is $\mathcal{Q}^{-1} \Cap (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{V})\Big) \neq 0_N \Rightarrow \mathcal{Q}^{-1} \Cap (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{V}) \neq 0_N \Rightarrow \mathcal{Q}^{-1} \Cap \mathcal{V} \neq 0_N$, since $\mathcal{Q}^{-1}$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i = 1, 2$. Hence $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q}^{-1}) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})^{-1}$.

Theorem 3.5. *Let $\mathcal{A}$ be any $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NSOS, $i = 1, 2$ in a NABTG $\mathbb{G}$. Then*

(1) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathcal{A}) \subseteq a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{A})$, *for all* $a \in \mathbb{G}$.

(2) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{A}a) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{A})a$, *for all* $a \in \mathbb{G}$.

(3) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{A}^{-1}) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{A})^{-1}$.

Proof.

(1) As $\mathcal{A}$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NSOS, and $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathcal{A})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$. From Theorem 3.1, $\pi_{a^{-1}} : (\mathbb{G}, \daleth_i^{\mathbb{G}}) \to (\mathbb{G}, \daleth_i^{\mathbb{G}})$ is NACM, for each $i = 1, 2$.

So, $a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i = 1, 2$. Hence $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathscr{A}) \subseteq a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$.

(2) As $\mathscr{A}$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NSOS, and $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$. From Theorem 3.1, $\sigma_{a^{-1}} : (\mathbb{G}, \daleth_i^{\mathbb{G}}) \to (\mathbb{G}, \daleth_i^{\mathbb{G}})$ is NACM, for each $i = 1, 2$. So, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})a$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i = 1, 2$. Thus, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}a) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})a$.

(3) Since $\mathscr{A}$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NSOS, $i = j = 1, 2$; so, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$ and hence $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})^{-1}$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i = 1, 2$. Conserquently, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}^{-1}) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})^{-1}$.

Theorem 3.6. *Let $\mathscr{A}$ be both $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ neutrosophic semi open and $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ neutrosophic semi closed subset of a NABTG, $i = j = 1, 2$. Then the following statements hold:*

(1) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathscr{A}) = a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$, *for all* $a \in \mathbb{G}$.

(2) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}a) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})a$, *for all* $a \in \mathbb{G}$.

(3) $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}^{-1}) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})^{-1}$.

Proof.

(1) Since $\mathscr{A}$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NSOS, and $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$; from which it follows that $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathscr{A}) \subseteq a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$. Further, $\daleth_i^{\mathbb{G}}-$ neutrosophic semi-openness of $\mathscr{A}$, $i = 1, 2$ implies $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathscr{A})\Big) \Rightarrow a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}) = a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathscr{A})\Big)$. As $\mathscr{A}$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NSCoS, and $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathscr{A})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS in $\mathbb{G}$, $i = j = 1, 2$. From Theorem 3.5, $a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}) = a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathscr{A})\Big) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big(a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathscr{A})\Big) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathscr{A})$. Hence $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(a\mathscr{A}) = a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$.

(2) Following the same steps as in part (1) above, then we can prove $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}a) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})a$.

(3) By assumption, this shows $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$ and therefore $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})^{-1}$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i = 1, 2$. Consequently, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}^{-1}) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})^{-1}$. Next, as $\mathscr{A}$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NSOS, $i = j = 1, 2$; $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathscr{A})\Big) \Rightarrow (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})^{-1} = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathscr{A})\Big)^{-1}$. Also, as $\mathscr{A}$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NSCoS, and $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathscr{A})$ is $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS, $i = j = 1, 2$. From Theorem 3.3, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})^{-1} = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathscr{A})^{-1}\Big) \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}^{-1})$. This shows that $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A}^{-1}) = (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})^{-1}$.

Theorem 3.7. *From Theorem 3.6, the following statements hold:*

(1) $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(a\mathscr{A}) = a(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})$, *for all* $a \in \mathbb{G}$.

(2) $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}a) = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})a$, *for all* $a \in \mathbb{G}$.

(3) $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}^{-1}) = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})^{-1}$.

Proof.

(1) As $\mathscr{A}$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NSCoS, and $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NROS, $i = j = 1, 2$. From Theorem 3.1, $\pi_{a^{-1}} : (\mathbb{G}, \urcorner_i^{\mathbb{G}}) \to (\mathbb{G}, \urcorner_i^{\mathbb{G}})$ is NACM, for each $i = 1, 2$. So, $\pi_{a^{-1}}^{-1}\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})\Big) = a(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})$ is $\urcorner_i^{\mathbb{G}}-$ NOS, $i = 1, 2$. Thus, $a(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}) \subseteq (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(a\mathscr{A})$. Next, by hypothesis, it follows $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}) = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\Big) \Rightarrow a(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}) = a(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\Big)$. As $\mathscr{A}$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NSOS, and $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$. From Theorem 3.4, $a(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\Big) = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big(a(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\Big)$ $\supseteq (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(a\mathscr{A})$. That is, $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(a\mathscr{A}) \subseteq a(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})$. Therefore, $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(a\mathscr{A}) = a(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})$. Hence proved.

(2) As $\mathscr{A}$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NSCoS, and $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NROS, $i = j = 1, 2$. From Theorem 3.1, $\sigma_{a^{-1}} : (\mathbb{G}, \urcorner_i^{\mathbb{G}}) \to (\mathbb{G}, \urcorner_i^{\mathbb{G}})$ is NACM, for each $i = 1, 2$. So, $\sigma_{a^{-1}}^{-1}\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})\Big) = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})a$ is $\urcorner_i^{\mathbb{G}}-$ NOS, $i = 1, 2$. Thus, $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})a \subseteq (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}a)$. Next, by hypothesis, it shows that $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}) = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\Big) \Rightarrow (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})a = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\Big)a$. Since $\mathscr{A}$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NSOS, and $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$. From Theorem 3.4, $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\Big)a = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})a\Big)$ $\supseteq (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}a)$. That is, $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}a) \subseteq (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})a$. Therefore, $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}a) = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})a$. Hence proved.

(3) From hypothesis, this shows that $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NROS, $i = j = 1, 2$ and therefore $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})^{-1}$ is $\urcorner_i^{\mathbb{G}}-$ NOS, $i = 1, 2$. Consequently, $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}^{-1}) \subseteq (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})^{-1}$. Next, as $\mathscr{A}$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NSCoS, $i = j = 1, 2$; $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}) = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\Big)$ $\Rightarrow (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})^{-1} = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\Big)^{-1}$. Also, as $\mathscr{A}$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NSOS, and $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})$ is $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})-$ NRCS, $i = j = 1, 2$. From Theorem 3.4, $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})^{-1} = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int\Big((\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})^{-1}\Big)$. $\subseteq (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}^{-1})$. This proves that $(\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A}^{-1}) = (\urcorner_i^{\mathbb{G}}, \urcorner_j^{\mathbb{G}})N \vdash Int(\mathscr{A})^{-1}$.

Theorem 3.8. *Let $\mathscr{A}$ be $\daleth_i^{\mathbb{G}}-$ NOS in a NABTG $\mathbb{G}, i=1,2$. Then $a\mathscr{A} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big(a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\big)\Big)$, for each $a \in \mathbb{G}$.*

Proof.

Since $\mathscr{A}$ is $\daleth_i^{\mathbb{G}}-$ NOS, $i=1,2$; so $\mathscr{A} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\big) \Rightarrow a\mathscr{A} \subseteq a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\big)$. From Theorem 3.2, $a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\big)$ is $\daleth_i^{\mathbb{G}}-$ NOS, $i=1,2$; $\big($in fact,$(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NROS$, i=j=1,2\big)$.
Hence, $a\mathscr{A} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\Big(a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl(\mathscr{A})\big)\Big)$.

Theorem 3.9. *Let $\mathcal{Q}$ be any $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ Neutrosophic closed subset in a NABTG $\mathbb{G}, i=j=1,2$. Then $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big(a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})\big)\Big) \subseteq a\mathcal{Q}$, for each $a \in \mathbb{G}$.*

Proof.

Since $\mathcal{Q}$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i=1,2$; so $\mathcal{Q} \supseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})\big) \Rightarrow a\mathcal{Q} \supseteq a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})\big)$. From Theorem 3.2, $a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})\big)$ is $\daleth_i^{\mathbb{G}}-$ NCoS, $i=1,2$; $\big($in fact,$(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})-$ NRCS$, i=j=1,2\big)$. Therefore, $a\mathcal{Q} \subseteq (\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big(a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})\big)\Big)$.
Hence, $(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\Big(a(\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Cl\big((\daleth_i^{\mathbb{G}}, \daleth_j^{\mathbb{G}})N \vdash Int(\mathcal{Q})\big)\Big) \subseteq a\mathcal{Q}$.

4. Conclusion:

In this paper, we have studied the neutrosophic almost bitopological group. We defined the definition of the neutrosophic regularly open (closed) set and proved some of their properties. After defining the definition of the neutrosophic regularly open set we have defined the definition of the neutrosophic almost continuous mapping with an example and we proved some properties of the neutrosophic almost continuous mapping. Finally, by using the definition of the neutrosophic almost continuous mapping, we defined the neutrosophic almost bitopological group and cited an example. Also, we have proved some of their properties. In future, we try to study the Neutrosophic Almost Ideal Bitopological Group. We hope that this work shall bring some new ideas in the development of the neutrosophic almost bitopological group.

References

1. Abdel-Baset, M.; Mohamed, R.; Zaied, A. E. N. H. and Smarandache, F. (2019). A Hybride plitogenic decision making approach with quality function deployment for selecting supply chain sustainability metrics, Symmetry, 11(7), 903.
2. Abdel-Baset,M.; Manogaran, G.; Gamal, A. and Smarandache, F. (2019). A Group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection , Journal of Medical System, 43(2), 38.

3. Abdel-Basset, M.; Saleh, M.; Gamal, A.; Smarandache, F. (2019). An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Applied Soft Computing 77: 438-452.
4. Abdel-Baset, M.; Chang, V.; Gamal, A. (2019). Evaluation of the green supply chain management practices: A novel neutrosophic approach. Comput. Ind. 108, 210–220. https://doi.org/10.1016/j.compind.2019.02.013
5. Abdel-Baset, M.; Chang, V.; Gamal, A.; Smarandache, F. (2019). An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field. Comput. Ind. 106, 94–110. https://doi.org/10.1016/j.compind.2018.12.017
6. Abdel-Basset, M.; Gamal, A.; Chakrabortty, R. K. and Ryan, M. (2021). Development of a Hybrid Multi-Criteria Decision-making Approach for Sustainability Evaluation of Bioenergy Production Technologies: A Case Study. Journal of Cleaner Production, 125805.
7. Abdel-Basset, M.; Gamal, A.; Chakrabortty, R. K. and Ryan, M. J. (2021). Evaluation Approach for Sustainable Renewable Energy Systems Under Uncertain Environment: A Case study. Renewable Energy.
8. Atanassov, K. T. (1986). Intuitionistic fuzzy sets. Fuzzy sets and systems, 20, 87-96.
9. Azad, K.K. (1981). On Fuzzy Semi-continuity, Fuzzy Almost Continuity and Fuzzy Weakly Continuity. Journal of Mathematical Analysis and Applications, 82:14-32.
10. Basumatary, B.; Wary, N. (2020). A Study on Neutrosophic Bitopological Group. Neutrosophic Sets and Systems, 38, 164-178.
11. Chang, C. L. (1968). Fuzzy Topological Space. Journal of Mathematical Analysis and Application, 24, 182-190.
12. Coker, D. (1997). An introduction to intuitionistic fuzzy topological spaces. Fuzzy Sets and Systems, 88, 81-89.
13. Foster, D. H. (1979). Fuzzy topological groups. J. Math. Anal. Appl., 67, 549-564.
14. Imran, Q. H.; Al-Obaidi, A. H. M. ; Smarandache, F. (2020). On Some Types of Neutrosophic Topological Groups with Respect to Neutrosophic Alpha Open Sets. Neutrosophic Sets and Systems, 32, 425-434.
15. Kandil, A.; Nouth, A. A.; El-Sheikh, S. A. (1995). On fuzzy bitopological spaces. Fuzzy sets and system, 74, 353-363.
16. Kelly, J. C. (1963). Bitopological spaces. Proceedings of the London Mathematical Society, 3(1), 71-89.
17. Mwchahary, D. D.; Basumatary, B. (2020). A note on Neutrosophic Bitopological Space. Neutrosophic Sets and Systems, 33, 134-144.
18. Noiri, T. (1973). On Semi-Continuous Mapping. Atti Accud. Naz Lincei Rend, Cl. Sci. Fis. Mat. Nutur, 54(8):132-136.
19. Ozturk, T. Y.; Ozkan, A. (2019). Neutrosophic Bitopological Spaces. Neutrosophic Sets and Systems, Vol. 30, 88-97.
20. Rosenfeld, A. (1971). Fuzzy groups. J. Math. Anal. Appl., 35, 512-517.
21. Salama, A. A.; Alblowi, S. A. (2012). Neutrosophic Set and Neutrosophic Topological Spaces. IOSR, Journal of Mathematics, 3(4), 31-35.
22. Salama, A. A.; Smarandache, F.; Kroumov, V. (2014). Closed sets and Neutrosophic Continuous Functions. Neutrosophic Sets and Systems, 4, 4-8.
23. Sherwood, H. (1983). Products of fuzzy subgroups. Fuzzy sets and systems, 11, 79-89.
24. Singal, M. K.; Singal, A. R. (1968). Almost Continuous Mappings. Yokohama Math. J., 16:63-73.
25. Smarandache, F.(2005). Neutrosophic set - a generalization of the intuitionistic fuzzy set. International Journal of Pure and Applied Mathematics, 24(3), 287–297.
26. Smarandache, F. (2002). Neutrosophy and neutrosophic logic, first international conference on neutrosophy, neutrosophic logic, set, probability, and statistics, University of New Mexico, Gallup, NM 87301, USA.

27. Sumathi, I. R.; Arockiarani, I. (2015). Fuzzy Neutrosophic Groups. Advanced in Fuzzy Mathematics, 10 (2), 117-122.
28. Sumathi, I. R.; Arockiarani, I. (2016). Topological Group Structure of Neutrosophic set. Journal of Advanced Studies in Topology, 7(1), 12-20.
29. Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8, 338-353.

Received: May 19, 2021. Accepted: October 2, 2021

NSS

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

A Model Describing the Neutrosophic Differential Equation and Its Application on Mine Safety

Debapriya Mondal[1], Suklal Tudu[2], Gopal Chandra Roy [3] and Tapan Kumar Roy[4]

[1] Department of Mathematics,IIEST,Shibpur,Howrah-03,West Bengal,India.; dmbiltu@gamil.com

[2]Department of Mathematics,IIEST,Shibpur,Howrah-03,West Bengal,India.; suklaltudu5@gamil.com

[3]Department of Mining Engineering,IIEST,Shibpur,Howrah-03,West Bengal,India.; gcr@mining.iiests.ac.in

[4]Department of Mathematics,IIEST,Shibpur,Howrah-03,West Bengal,India.; roy_t_k@yahoo.co.in

Abstract. In the theory of uncertainty and approximation neutrosophy plays a significant role. Neutrosophy is tool emerged on standard or non-standard to measured the mathematical model of uncertainty, vagueness, ambiguity etc. In light of these major issues, the paper outlines of Neutrosophic Set, Single Valued Neutrosophic Set, Triangular Single Valued Neutrosophic Number and Trapezoidal Single Valued Neutrosophic Number. It also propose Neutrosophic Differential Equation and shown its solution in different conditions. Thereafter mining safety model via Single Valued neutrosophic number be epitomized. At last a mathematical experiment is done to exhibit its reality and use fullness of this Number.

Keywords: neutrosophic set(NS); single valued neutrosophic set(SVNS); triangular single valued neutrosophic number(TSVNNs); trapezoidal single valued neutrosophic number(TrSVNNs); neutrosophic differential equation(NDE); mining safety model

---

1. Introduction

NS highlights the origin and nature of neutralise in different fields which is the generalization of classical set, fuzzy set(FS), intuitionistic fuzzy set(IFS) etc. Gradually varying value is used in FS theory rather than precise or sharp value. In 1965 [1], a famous paper was published by Prof. L.A. Zadeh as "Fuzzy sets" in "Information and Control" that provided some new mathematical tool which enable us to described and handle dubious or unclear notions. FS theory, only shows membership degree and do not provide any idea about non-membership degree. In reality, this linguistic statement don't fulfill the logical statement. When choosing the membership degree there may exist some types of doubtfulness or absence of information are present while defining the membership. Due to this doubtfulness, an idea of IFS as generalization of FS was introduced by Atanassov in 1983 [2]. IFS consider both membership and non-membership function. IFS only pick up incomplete information. In 2003 [3], A new concept, say, NS was innovated by Smarandache. It deals with the study of origin, nature and scope of neutralise, as well as their interaction with different idealism spectra. NS is the generalization of CS, FS, IFS and so on. A NS can be distinguish by a truth membership function 'μ_T', an indeterminacy membership function 'ν_I', and a falsity membership function 'σ_F'. In NS: μ_T, ν_I and σ_F are not dependent, which is useful in situations such as information fusion. In NS μ_T, ν_I, σ_F being the real standard or non standard subset of $^{-}]0,1[^{+}$,moreover in SNVS, μ_T, ν_I, σ_F be the subset of [0,1]. From philosophical point of view, NS generalised the above mentioned sets but from scientific or engineering point of view, need to be defined. Else it's difficult to apply in many real application.

It is much noticed that when modeling some problems related to physical science and engineering, where the parameters are unknown but performed in an interval. Before, the application of interval arithmetic managed such circumstances, where mathematical calculation is done on intervals to get the estimate of target quantities in respective intervals. Fuzzy arithmetic is the generalization of the intervals arithmetic. As the principle definition of FS which approve gradation of membership for an element of the Universal set. So the situation of the modeling based on fuzzy arithmetic is awaited to publish more realistically. There are several types of fuzzy number are exist. These are applied in Decision-making problem and so on [4]. But it is not efficient for any application where the knowledge about membership degree is lacking. Latter generalization it to intuitionistic fuzzy number [5] were developed. In these paper we define several types of neutrosophic numbers and their cuts.

In the field of science & engineering, differentiation takes on an evidential role. Many problems stand up with uncertain or imprecise parameters. Due to this naiveness, we bear upon the differential equation with imprecise parameters. Fuzzy differential equation [6] has been proposed to model this uncertainty. However, it consider only membership value. Later, intuitionistic fuzzy differential [7] equation was founded with degree of membership and non-membership function. However, the term indeterminacy is absent in the above logic's. Hence, neutrosophic differential eqaution(NDE) [8–10] was developed to model indeterminacy. In this paper, a mining safety model describe [11], this model consist of three differential equations, those differential equations describe via Single Valued Neutrosophic Number(SVNNs). The solution of the equation is describe later.

In reality, the collected data, in many situations, it was observed that is insufficient and transmit some misinformation. As a result, the solution obtained from these data suffers with insufficiency and inconsistency. In these situations, the neutrosophic sets offer better result.

We have designed the paper in the following way: Section-2 gives some preliminaries concept and definition. Section-3 contains definition of NDE. Section-4 contains solution of NDE with numerical example. Section-5 contains Mining Safety model. Section-6 contains Mining safety model formulation. Section-7 described solution mode of the model. Section-8 contains numerical experiment and consequently, conclusions are discussed in Section-9. The references are shown in Section-10.

2. **Preliminaries**

2.1. ***Definition of NS*** *[12]*

Let $\mathfrak{U}$ be a Universal set. A NS $\tilde{\mathcal{A}}^{NS}$ of $\mathfrak{U}$ be defined by $\tilde{\mathcal{A}}^{NS} = \langle(\mathfrak{u}; \mu_T(\mathfrak{u}), \nu_I(\mathfrak{u}), \sigma_F(\mathfrak{u})) : \mathfrak{u} \in \mathfrak{U}\rangle$ where $\mu_T(\mathfrak{u}), \nu_I(\mathfrak{u}), \sigma_F(\mathfrak{u})$ be outlined as the truth membership, indeterminacy membership, falsity membership grade of $\mathfrak{u}$ in $\tilde{\mathcal{A}}^{NS}$ which are real standard or non-standard subsets of $^{-}]0, 1[^{+}$ & $\mu_T(\mathfrak{u}) + \nu_I(\mathfrak{u}) + \sigma_F(\mathfrak{u}) \leqslant 3^{+}$.

2.2. ***Definition of SVNS*** *[12]*

Let $\mathfrak{U}$ be a Universal set. A SVNS $\tilde{\mathcal{A}}^{Ne}$ of $\mathfrak{U}$ be defined by $\tilde{\mathcal{A}}^{Ne} = \langle(\mathfrak{u}; \mu_T(\mathfrak{u}), \nu_I(\mathfrak{u}), \sigma_F(\mathfrak{u})) : \mathfrak{u} \in \mathfrak{U}\rangle$ where $\mu_T(\mathfrak{u}), \nu_I(\mathfrak{u}), \sigma_F(\mathfrak{u})$ be outlined as the truth membership, indeterminacy membership, falsity membership grade of $\mathfrak{u}$ in $\tilde{\mathcal{A}}^{Ne}$ which are subset of $[0, 1]$ & $\mu_T(\mathfrak{u})+\nu_I(\mathfrak{u})+\sigma_F(\mathfrak{u}) \leqslant 3$.

2.3. ***Definition of TSVNNs*** *[8]*

A TSVNNs is denoted by $\tilde{\mathcal{A}}^{Ne} = \langle \mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{a}'_3; w_\mu, w_\nu, w_\sigma\rangle$ whose truth, indeterminacy and falsity membership functions are defined by

$$\mu_T(\mathfrak{u}) = \begin{cases} (\frac{\mathfrak{u}-\mathfrak{a}'_1}{\mathfrak{a}'_2-\mathfrak{a}'_1})w_\mu & \text{when } \mathfrak{a}'_1 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_2 \\ w_\mu & \text{when } \mathfrak{u} = \mathfrak{a}'_2 \\ (\frac{\mathfrak{a}'_3-\mathfrak{u}}{\mathfrak{a}'_3-\mathfrak{a}'_2})w_\mu & \text{when } a'_2 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_3 \\ 0 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}'_1 \text{ or } \mathfrak{u} \geqslant \mathfrak{a}'_3 \end{cases}$$

$$\nu_I(\mathfrak{u}) = \begin{cases} \frac{(\mathfrak{a}'_2-\mathfrak{u})+(\mathfrak{u}-\mathfrak{a}'_1)w_\nu}{\mathfrak{a}'_2-\mathfrak{a}'_1} & \text{when } \mathfrak{a}'_1 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_2 \\ w_\nu & \text{when } \mathfrak{u} = \mathfrak{a}'_2 \\ \frac{(\mathfrak{u}-\mathfrak{a}'_2)+(\mathfrak{a}'_3-\mathfrak{u})w_\nu}{\mathfrak{a}'_3-\mathfrak{a}'_2} & \text{when } \mathfrak{a}'_2 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_3 \\ 1 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}'_1 \text{ or } \mathfrak{u} \geqslant \mathfrak{a}'_3 \end{cases}$$

$$\sigma_F(\mathfrak{u}) = \begin{cases} \frac{(\mathfrak{a}'_2-\mathfrak{u})+(\mathfrak{u}-\mathfrak{a}'_1)w_\sigma}{\mathfrak{a}'_2-\mathfrak{a}'_1} & \text{when } \mathfrak{a}'_1 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_2 \\ w_\sigma & \text{when } \mathfrak{u} = \mathfrak{a}'_2 \\ \frac{(\mathfrak{u}-\mathfrak{a}'_2)+(\mathfrak{a}'_3-\mathfrak{u})w_\sigma}{\mathfrak{a}'_3-\mathfrak{a}'_2} & \text{when } \mathfrak{a}'_2 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_3 \\ 1 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}'_1 \text{ or } \mathfrak{u} \geqslant \mathfrak{a}'_3 \end{cases}$$

where $\mu_T(\mathfrak{u}) + \nu_I(\mathfrak{u}) + \sigma_F(\mathfrak{u}) \leqslant 3$ & $w_\mu \in (0,1]$, $w_\nu, w_\sigma \in [0,1)$.

2.4. ***Definition of TrSVNNs*** *[9]*

A TrSVNNs is denoted by $\tilde{\mathcal{A}}^{Ne} = \langle \mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{a}'_3, \mathfrak{a}'_4; w_\mu, w_\nu, w_\sigma \rangle$ whose truth, indeterminacy and falsity membership functions are defined by

$$\mu_T(\mathfrak{u}) = \begin{cases} (\frac{\mathfrak{u}-\mathfrak{a}'_1}{\mathfrak{a}'_2-\mathfrak{a}'_1})w_\mu & \text{when } \mathfrak{a}'_1 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_2 \\ w_\mu & \text{when } \mathfrak{a}'_2 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_3 \\ (\frac{\mathfrak{a}'_4-\mathfrak{u}}{\mathfrak{a}'_4-\mathfrak{a}'_3})w_\mu & \text{when } a'_3 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_4 \\ 0 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}'_1 \text{ or } \mathfrak{u} \geqslant \mathfrak{a}'_4 \end{cases}$$

$$\nu_I(\mathfrak{u}) = \begin{cases} \frac{(\mathfrak{a}'_2-\mathfrak{u})+(\mathfrak{u}-\mathfrak{a}'_1)w_\nu}{\mathfrak{a}'_2-\mathfrak{a}'_1} & \text{when } \mathfrak{a}'_1 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_2 \\ w_\nu & \text{when } \mathfrak{a}'_2 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_3 \\ \frac{(\mathfrak{u}-\mathfrak{a}'_3)+(\mathfrak{a}'_4-\mathfrak{u})w_\nu}{\mathfrak{a}'_4-\mathfrak{a}'_3} & \text{when } \mathfrak{a}'_3 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_4 \\ 1 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}'_1 \text{ or } \mathfrak{u} \geqslant \mathfrak{a}'_4 \end{cases}$$

$$\sigma_F(\mathfrak{u}) = \begin{cases} \frac{(\mathfrak{a}'_2-\mathfrak{u})+(\mathfrak{u}-\mathfrak{a}'_1)w_\sigma}{\mathfrak{a}'_2-\mathfrak{a}'_1} & \text{when } \mathfrak{a}'_1 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_2 \\ w_\sigma & \text{when } \mathfrak{a}'_2 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_3 \\ \frac{(\mathfrak{u}-\mathfrak{a}'_3)+(\mathfrak{a}'_4-\mathfrak{u})w_\sigma}{\mathfrak{a}'_4-\mathfrak{a}'_3} & \text{when } \mathfrak{a}'_3 \leqslant \mathfrak{u} \leqslant \mathfrak{a}'_4 \\ 1 & \text{when } \mathfrak{u} \leqslant \mathfrak{a}'_1 \text{ or } \mathfrak{u} \geqslant \mathfrak{a}'_4 \end{cases}$$

where $\mu_T(\mathfrak{u}) + \nu_I(\mathfrak{u}) + \sigma_F(\mathfrak{u}) \leqslant 3$ & $w_\mu \in (0,1]$, $w_\nu, w_\sigma \in [0,1)$.

2.5. ***Cut Set*** *[8]*

Let $\tilde{\mathcal{A}}^{Ne}$ be any SVNS, then (r,β,γ)-cut of SVNS is denoted by $\tilde{\mathcal{A}}^{Ne}(r,\beta,\gamma)$and it is defined by $\tilde{\mathcal{A}}^{Ne}(r,\beta,\gamma)$=$\langle \mathfrak{u} \in \mathfrak{U} : \mu_T(\mathfrak{u}) \geqslant r, \nu_I(\mathfrak{u}) \leqslant \beta, \sigma_F(\mathfrak{u}) \leqslant \gamma; 0 < r \leqslant 1, 0 \leqslant \beta < 1, 0 \leqslant \gamma < 1 \rangle$.

2.6. ***Operation Using SVNNs:*** *[13]*

Consider two TSVNNs, $\tilde{\mathcal{A}}^{Ne}$=$\langle \mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{a}'_3; w_\mu, w_\nu, w_\sigma \rangle$; $\tilde{\mathcal{B}}^{Ne}$=$\langle \mathfrak{b}'_1, \mathfrak{b}'_2, \mathfrak{b}'_3; u_\mu, u_\nu, u_\sigma \rangle$, the following operation are:

• **Addition:**

$\tilde{\mathcal{A}}^{Ne}+\tilde{\mathcal{B}}^{Ne} = \langle[(\mathfrak{a}_1'+\mathfrak{b}_1', \mathfrak{a}_2'+\mathfrak{b}_2', \mathfrak{a}_3'+\mathfrak{b}_3'); w_\mu \wedge u_\mu, w_\nu \vee u_\nu, w_\sigma \vee u_\sigma]\rangle$

• **Substraction:**

$\tilde{\mathcal{A}}^{Ne}\text{-}\tilde{\mathcal{B}}^{Ne} = \langle[(\mathfrak{a}_1'-\mathfrak{b}_3', \mathfrak{a}_2'-\mathfrak{b}_2', \mathfrak{a}_3'-\mathfrak{b}_1'); w_\mu \wedge u_\mu, w_\nu \vee u_\nu, w_\sigma \vee u_\sigma]\rangle$

• **Multiplication:**

$\tilde{\mathcal{A}}^{Ne}.\tilde{\mathcal{B}}^{Ne} = \langle[(\mathfrak{a}_1'\mathfrak{b}_1', \mathfrak{a}_2'\mathfrak{b}_2', \mathfrak{a}_3'\mathfrak{b}_3'); w_\mu \wedge u_\mu, w_\nu \vee u_\nu, w_\sigma \vee u_\sigma]\rangle$

• **Division:**

$\frac{\tilde{\mathcal{A}}^{Ne}}{\tilde{\mathcal{B}}^{Ne}}=\langle[(\frac{\mathfrak{a}_1'}{\mathfrak{b}_3'}, \frac{\mathfrak{a}_2'}{\mathfrak{b}_2'}, \frac{\mathfrak{a}_3'}{\mathfrak{b}_1'}); w_\mu \wedge u_\mu, w_\nu \vee u_\nu, w_\sigma \vee u_\sigma]\rangle$

Where $\wedge = Min, \vee = Max$

3. Definition of NDE: [8]

Consider an Ordinary differential equation $\frac{d\mathbb{Y}}{d\mathfrak{t}} = \mathcal{K}\mathbb{Y}$, $\mathfrak{t} \in [0,\infty)$ with initial condition(IC) $\mathbb{Y}(\mathfrak{t}_0) = \mathbb{Y}_0$. The above ODE is called NDE if any one of the following three cases hold:

(i) $\tilde{\mathcal{K}}^{Ne}$ is SVNNs & $\mathbb{Y}_0$ is Crisp number.

(ii) $\mathcal{K}$ is Crisp number & $\tilde{\mathbb{Y}}_0^{Ne}$ is SVNNs.

(iii) Both $\tilde{\mathcal{K}}^{Ne}$ & $\tilde{\mathbb{Y}}_0^{Ne}$ are SVNNs.

Let the classical solution [14] be $\tilde{\mathbb{Y}}^{Ne}(\mathfrak{t})$ and its Cut be $\mathbb{Y}(\mathfrak{t},r,\beta,\gamma)=\langle[\mathbb{Y}_1(\mathfrak{t},r), \mathbb{Y}_2(\mathfrak{t},r)], [\mathbb{Y}_1'(\mathfrak{t},\beta), \mathbb{Y}_2'(\mathfrak{t},\beta)], [\mathbb{Y}_1''(\mathfrak{t},\gamma), \mathbb{Y}_2''(\mathfrak{t},\gamma)]\rangle$.

The solution is strong if

(i) $\frac{d\mathbb{Y}_1(\mathfrak{t},r)}{dr} > 0$, $\frac{d\mathbb{Y}_2(\mathfrak{t},r)}{dr} < 0 \;\forall\; r \in (0,1]$, $\mathbb{Y}_1(\mathfrak{t},1) \leqslant \mathbb{Y}_2(\mathfrak{t},1)$

(ii) $\frac{d\mathbb{Y}_1'(\mathfrak{t},\beta)}{d\beta} < 0$, $\frac{d\mathbb{Y}_2'(\mathfrak{t},\beta)}{d\beta} > 0 \;\forall\; \beta \in [0,1)$, $\mathbb{Y}_1'(\mathfrak{t},0) \leqslant \mathbb{Y}_2'(\mathfrak{t},0)$

(iii) $\frac{d\mathbb{Y}_1''(\mathfrak{t},\gamma)}{d\gamma} < 0$, $\frac{d\mathbb{Y}_2''(\mathfrak{t},\gamma)}{d\gamma} > 0 \;\forall\; \gamma \in [0,1)$, $\mathbb{Y}_1''(\mathfrak{t},0) \leqslant \mathbb{Y}_2''(\mathfrak{t},0)$

Otherwise the solution is weak solution.

4. Solution of NDE

(i) $\tilde{\mathcal{K}}^{Ne}$ is SVNNs & $\mathbb{Y}_0$ is Crisp number.

Case 1 When Sign of $\tilde{\mathcal{K}}^{Ne}$ is positive.

Therefore required solutions are

$\mathbb{Y}_1(\mathfrak{t},r) = \mathbb{Y}_0 e^{\mathbb{K}_1(r)(\mathfrak{t}-\mathfrak{t}_0)}$; $\mathbb{Y}_2(\mathfrak{t},r) = \mathbb{Y}_0 e^{\mathbb{K}_2(r)(\mathfrak{t}-\mathfrak{t}_0)}$

$\mathbb{Y}_1'(\mathfrak{t},\beta) = \mathbb{Y}_0 e^{\mathbb{K}_1'(\beta)(\mathfrak{t}-\mathfrak{t}_0)}$; $\mathbb{Y}_2'(\mathfrak{t},\beta) = \mathbb{Y}_0 e^{\mathbb{K}_2'(\beta)(\mathfrak{t}-\mathfrak{t}_0)}$

$\mathbb{Y}_1''(\mathfrak{t},\gamma) = \mathbb{Y}_0 e^{\mathbb{K}_1''(\gamma)(\mathfrak{t}-\mathfrak{t}_0)}$; $\mathbb{Y}_2''(\mathfrak{t},\gamma) = \mathbb{Y}_0 e^{\mathbb{K}_2''(\gamma)(\mathfrak{t}-\mathfrak{t}_0)}$

Case 2 When Sign of $\tilde{\mathcal{K}}^{Ne}$ is negative.

Therefore required solutions are

$$\mathbb{Y}_1(\mathfrak{t},r)=\frac{\mathbb{Y}_0}{2}[(1+\sqrt{\frac{\mathbb{K}_2(r)}{\mathbb{K}_1(r)}})e^{-\sqrt{\mathbb{K}_1(r)\mathbb{K}_2(r)}(\mathfrak{t}-\mathfrak{t}_\circ)}+(1-\sqrt{\frac{\mathbb{K}_2(r)}{\mathbb{K}_1(r)}})e^{\sqrt{\mathbb{K}_1(r)\mathbb{K}_2(r)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

$$\mathbb{Y}_2(\mathfrak{t},r)=\frac{\mathbb{Y}_0}{2}[(\sqrt{\frac{\mathbb{K}_1(r)}{\mathbb{K}_2(r)}}+1)e^{-\sqrt{\mathbb{K}_1(r)\mathbb{K}_2(r)}(\mathfrak{t}-\mathfrak{t}_\circ)}-(\sqrt{\frac{\mathbb{K}_1(r)}{\mathbb{K}_2(r)}}-1)e^{\sqrt{\mathbb{K}_1(r)\mathbb{K}_2(r)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

$$\mathbb{Y}'_1(\mathfrak{t},\beta)=\frac{\mathbb{Y}_0}{2}[(1+\sqrt{\frac{\mathbb{K}'_2(\beta)}{\mathbb{K}'_1(\beta)}})e^{-\sqrt{\mathbb{K}'_1(\beta)\mathbb{K}'_2(\beta)}(\mathfrak{t}-\mathfrak{t}_\circ)}+(1-\sqrt{\frac{\mathbb{K}'_2(\beta)}{\mathbb{K}'_1(\beta)}})e^{\sqrt{\mathbb{K}'_1(\beta)\mathbb{K}'_2(\beta)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

$$\mathbb{Y}'_2(\mathfrak{t},\beta)=\frac{\mathbb{Y}_0}{2}[(\sqrt{\frac{\mathbb{K}'_1(\beta)}{\mathbb{K}'_2(\beta)}}+1)e^{-\sqrt{\mathbb{K}'_1(\beta)\mathbb{K}'_2(\beta)}(\mathfrak{t}-\mathfrak{t}_\circ)}-(\sqrt{\frac{\mathbb{K}'_1(\beta)}{\mathbb{K}'_2(\beta)}}-1)e^{\sqrt{\mathbb{K}'_1(\beta)\mathbb{K}'_2(\beta)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

$$\mathbb{Y}''_1(\mathfrak{t},\gamma)=\frac{\mathbb{Y}_0}{2}[(1+\sqrt{\frac{\mathbb{K}''_2(\gamma)}{\mathbb{K}''_1(\gamma)}})e^{-\sqrt{\mathbb{K}''_1(\gamma)\mathbb{K}''_2(\gamma)}(\mathfrak{t}-\mathfrak{t}_\circ)}+(1-\sqrt{\frac{\mathbb{K}''_2(\gamma)}{\mathbb{K}''_1(\gamma)}})e^{\sqrt{\mathbb{K}''_1(\gamma)\mathbb{K}''_2(\gamma)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

$$\mathbb{Y}''_2(\mathfrak{t},\gamma)=\frac{\mathbb{Y}_0}{2}[(\sqrt{\frac{\mathbb{K}''_1(\gamma)}{\mathbb{K}''_2(\gamma)}}+1)e^{-\sqrt{\mathbb{K}''_1(\gamma)\mathbb{K}''_2(\gamma)}(\mathfrak{t}-\mathfrak{t}_\circ)}-(\sqrt{\frac{\mathbb{K}''_1(\gamma)}{\mathbb{K}''_2(\gamma)}}-1)e^{\sqrt{\mathbb{K}''_1(\gamma)\mathbb{K}''_2(\gamma)}(\mathfrak{t}-\mathfrak{t}_\circ)}]$$

Where $\langle[\mathbb{K}_1(r),\mathbb{K}_2(r)],[\mathbb{K}'_1(\beta),\mathbb{K}'_2(\beta)],[\mathbb{K}''_1(\gamma),\mathbb{K}''_2(\gamma)]\rangle$ is the cut set of $\tilde{\mathcal{K}}^{Ne}$. Solutions are strong or week if it satisfies the condition of NDE.

Similarly, we can get the solution of other two cases.

Numerical Example: Let us consider NDE $\frac{d\mathbb{Y}}{d\mathfrak{t}}=\mathcal{K}\mathbb{Y}$, with IC $\tilde{\mathbb{Y}}^{Ne}(0)=\langle 3,4,5;0.8,0.2,0.3\rangle$, $\mathcal{K}=\frac{1}{3}$.

Solution: Required (r,β,γ)-cut solution at $\mathfrak{t}=2$ we get $\mathbb{Y}_1(\mathfrak{t},r)=[3+1.25r]e^{\frac{2}{3}}$; $\mathbb{Y}_2(\mathfrak{t},r)=[5-1.25r]e^{\frac{2}{3}}$; $\mathbb{Y}'_1(\mathfrak{t},\beta)=[\frac{3.4-\beta}{0.8}]e^{\frac{2}{3}}$; $\mathbb{Y}'_2(\mathfrak{t},\beta)=[\frac{3+\beta}{0.8}]e^{\frac{2}{3}}$; $\mathbb{Y}''_1(\mathfrak{t},\gamma)=[\frac{3.1-\gamma}{0.7}]e^{\frac{2}{3}}$; $\mathbb{Y}''_2(\mathfrak{t},\gamma)=[\frac{2.5+\gamma}{0.7}]e^{\frac{2}{3}}$.

When we take $\mathfrak{t}=2$ and for different values of r,β,γ the solution is given in Table 1. The graphical interpretation of the table is also shown in the form of membership function in the Figure. 1.

5. **Mining Safety Model**

The miming industry has played an important role in development in the human civilization. Extraction of minerals from the underground system of work has involved a considerable amount of risks like roof fall over the workplace, inundation of the workplace due to the influx of water from the old working, explosion, influx of poisonous gases in the workplace, etc. Similarly, the opencast system of work has involved chances of runway of dumpers, sliding of benches in the workplace, striking by the fly rocks blasting, etc. These phenomenon's not only

r, β, γ	$\mathbb{Y}_1(t, r)$	$\mathbb{Y}_2(t, r)$	$\mathbb{Y}'_1(t, \beta)$	$\mathbb{Y}'_2(t, \beta)$	$\mathbb{Y}''_1(t, \gamma)$	$\mathbb{Y}''_2(t, \gamma)$
0	5.8432	9.7387	8.2778	7.3040	8.6257	6.9561
0.1	6.0866	9.4952	8.0344	7.5474	8.3474	7.2344
0.2	6.3301	9.2517	7.7909	7.7909	8.0692	7.5127
0.3	6.5736	9.0083	7.5475	8.0344	7.7909	7.7909
0.4	6.8171	8.7648	7.3040	8.2779	7.5127	8.0692
0.5	7.0605	8.5213	7.0605	8.5213	7.2344	8.3474
0.6	7.3040	8.2779	6.8171	8.7648	6.9562	8.6257
0.7	7.5475	8.0344	6.5736	9.0082	6.6779	8.9039
0.8	7.7909	7.7909	6.3301	9.2517	6.3997	9.1822
0.9	8.0344	7.5475	6.0867	9.4952	6.1214	9.4604
1.0	8.2779	7.3040	5.8432	9.7387	5.8432	9.7387

TABLE 1. Solution for $t = 2$

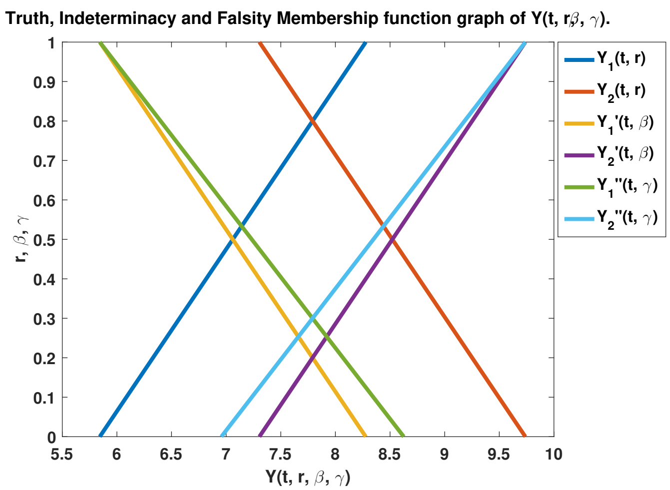


FIGURE 1. Membership Function Graph (at t=2).

causes injury to the workmen, sometimes lead to fatal. Improper used and malfunctioning mining equipment or system also results an accident.

The system fails safely is denoted by λ_1 and system fails unsafely is denoted by λ_2 for the mining safety model used here. Either λ_1 or λ_2 or both λ_1 and λ_2 are imprecise in nature. Our main interest in this paper are given below:

• Formulate Mining Safety model.

• Observe solution of the model in Crisp environment.

• Observe solution of the mining model in three ways:

(i) when λ_1 is SVNNs and λ_2 is crisp number.

(ii) when λ_1 is crisp number and λ_2 is SVNNs.

(iii) both λ_1 and λ_2 are SVNNs.

• Observe cut value in table form of the solution of the mining model in each of the cases mention above and show its graphical representation.

5.1. ***Acceptation***

(I) All events are not dependent to one another.

(II) The probability of progression from one condition to another is $\Psi\delta\mathfrak{t}$; $\delta\mathfrak{t}$ indicates finite time interval, Ψ indicate the progression rate from one condition to another.

(III) $(\Psi\delta\mathfrak{t})(\Psi\delta\mathfrak{t}) \to 0$.

(IV) $\mathcal{P}\{\eta(\delta\mathfrak{t}) \geqslant 2\} = o(\delta\mathfrak{t})$, where $\eta(\delta\mathfrak{t})$ be the number of event that occur in $\delta\mathfrak{t}$.

(V) $\mathcal{P}\{\eta(\delta\mathfrak{t}) = 1\} = \Psi\delta\mathfrak{t} + o(\delta\mathfrak{t})$, where $\Psi > 0$.

(VI) $\lim_{\delta\mathfrak{t}\to 0} \frac{o(\delta\mathfrak{t})}{\delta\mathfrak{t}} = 0$.

5.2. ***Input data***

$\mathfrak{t}$= time.

λ_1= mining system safe failure rate.

λ_2= mining system unsafe failure rate.

5.3. ***Output data***

$\mathcal{P}_0(\mathfrak{t})$= Probability of Mining system operating normally.

$\mathcal{P}_1(\mathfrak{t})$= Probability of Mining system failed safely.

$\mathcal{P}_2(\mathfrak{t})$= Probability of Mining system failed unsafely.

5.4. *Modulator*

$\mathfrak{t}$ time.

$\delta\mathfrak{t}$ finite time intervall.

$\mathcal{P}_0(\mathfrak{t}+\delta\mathfrak{t})$ operating probability in state 0 at time $\mathfrak{t}+\delta\mathfrak{t}$.

$\mathcal{P}_1(\mathfrak{t}+\delta\mathfrak{t})$ safe fail probability in state 1 at time $\mathfrak{t}+\delta\mathfrak{t}$.

$\mathcal{P}_2(\mathfrak{t}+\delta\mathfrak{t})$ unsafe fail probability in state 2 at time $\mathfrak{t}+\delta\mathfrak{t}$.

j=0 state operating normal.

j=1 state fail safe.

j=2 state fail unsafe.

$\mathcal{P}_j(\mathfrak{t})$ probability in state j at time $\mathfrak{t}$.

$\lambda_1\delta\mathfrak{t}$ safe fail probability in finite time interval $\delta\mathfrak{t}$

$\lambda_2\delta\mathfrak{t}$ unsafe fail probability in $\delta\mathfrak{t}$

(1 - $\lambda_1\delta\mathfrak{t}$) no safe fail probability in $\delta\mathfrak{t}$

(1 - $\lambda_1\delta\mathfrak{t}$) no unsafe fail probability in $\delta\mathfrak{t}$

6. Model Formulation

Consider a mining system, the state space diagram is shown in Figure-2.

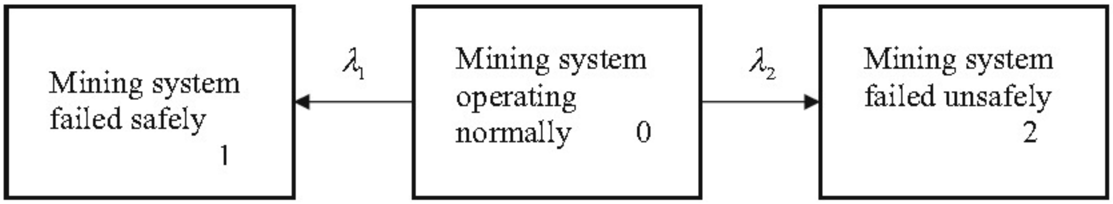


FIGURE 2. Mining system state space diagram

From Fig.2,we get following three equations

$$\mathcal{P}_0(\mathfrak{t}+\delta\mathfrak{t}) = \mathcal{P}_0(\mathfrak{t})(1-\lambda_1\delta\mathfrak{t})(1-\lambda_2\delta\mathfrak{t}) \tag{1}$$

$$\mathcal{P}_1(\mathfrak{t}+\delta\mathfrak{t}) = \mathcal{P}_1(\mathfrak{t})(1-o(\delta\mathfrak{t})) + \mathcal{P}_0(\mathfrak{t})\lambda_1\delta\mathfrak{t} \tag{2}$$

$$\mathcal{P}_2(\mathfrak{t}+\delta\mathfrak{t}) = \mathcal{P}_2(\mathfrak{t})(1-o(\delta\mathfrak{t})) + \mathcal{P}_0(\mathfrak{t})\lambda_2\delta\mathfrak{t} \tag{3}$$

From (1), (2), (3) we get

$$\therefore \frac{d\mathcal{P}_0(\mathfrak{t})}{d\mathfrak{t}} = -(\lambda_1+\lambda_2)\mathcal{P}_0(\mathfrak{t}) \tag{4}$$

$$\frac{d\mathcal{P}_1(\mathfrak{t})}{d\mathfrak{t}} = \lambda_1\mathcal{P}_0(\mathfrak{t}) \tag{5}$$

$$\frac{d\mathcal{P}_2(\mathfrak{t})}{d\mathfrak{t}} = \lambda_2\mathcal{P}_0(\mathfrak{t}) \tag{6}$$

with IC: $\mathcal{P}_j(0) = 1$ for j=0 & $\mathcal{P}_j(0) = 0$ for j=1,2.

7. Solution mode

7.1. *Crisp Solution:*

Input data: Both λ_1 and λ_2 are Crisp number..

Output data: We get the values of $\mathcal{P}_0(\mathfrak{t})$, $\mathcal{P}_1(\mathfrak{t})$, $\mathcal{P}_2(\mathfrak{t})$.

7.2. *Neutrosophic Solution:*

Input data: Three cases arise

Case-1: $\tilde{\lambda_1}^{Ne}$=$\langle \mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{a}'_3; w_\mu, w_\nu, w_\sigma\rangle$ & λ_2 is Crisp number.

Case-2: λ_1 is Crisp number & $\tilde{\lambda_2}^{Ne}$=$\langle \mathfrak{b}'_1, \mathfrak{b}'_2, \mathfrak{b}'_3; u_\mu, u_\nu, u_\sigma\rangle$

Case-3: $\tilde{\lambda_1}^{Ne}$=$\langle \mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{a}'_3; w_\mu, w_\nu, w_\sigma\rangle$ & $\tilde{\lambda_2}^{Ne}$=$\langle \mathfrak{b}'_1, \mathfrak{b}'_2, \mathfrak{b}'_3; u_\mu, u_\nu, u_\sigma\rangle$

Output data:

Let, $\tilde{\mathcal{P}_0(t)}^{Ne}$,$\tilde{\mathcal{P}_1(t)}^{Ne}$,$\tilde{\mathcal{P}_2(t)}^{Ne}$ be the solution of the modified model with Cut

$\mathcal{P}_0(t,r,\beta,\gamma)$=$\langle[\mathcal{P}_{01}(t,r),\mathcal{P}_{02}(t,r)],[\mathcal{P}'_{01}(t,\beta),\mathcal{P}'_{02}(t,\beta)],[\mathcal{P}''_{01}(t,\gamma),\mathcal{P}''_{02}(t,\gamma)]\rangle$

$\mathcal{P}_1(t,r,\beta,\gamma)$=$\langle[\mathcal{P}_{11}(t,r),\mathcal{P}_{12}(t,r)],[\mathcal{P}'_{11}(t,\beta),\mathcal{P}'_{12}(t,\beta)],[\mathcal{P}''_{11}(t,\gamma),\mathcal{P}''_{12}(t,\gamma)]\rangle$

$\mathcal{P}_2(t,r,\beta,\gamma)$=$\langle[\mathcal{P}_{21}(t,r),\mathcal{P}_{22}(t,r)],[\mathcal{P}'_{21}(t,\beta),\mathcal{P}'_{22}(t,\beta)],[\mathcal{P}''_{21}(t,\gamma),\mathcal{P}''_{22}(t,\gamma)]\rangle$

Solution is strong or weak if it satisfies the condition of NDE.

8. Numerical Experiment

8.1. *Crisp Solution*

Input data: $\lambda_1 = 0.009; \lambda_2 = 0.001;$ t=20-h.

Output: $\mathcal{P}_2(20)$=0.018127

8.2. *NS Solution*

Case: 1

Input data: $\tilde{\lambda_1}^{Ne}$=$\langle 0.007, 0.009, 0.011; 0.5, 0.3, 0.2\rangle; \lambda_2 = 0.001;$ t=20-h.

Output: When we take the value t=20-h the output of λ_1^{Ne} is TSVNNs & λ_2 is crisp number are shown in Table-2 and the corresponding membership function shown in Figure-3.

Case: 2

Input data: λ_1=0.009; $\tilde{\lambda_2}^{Ne} = \langle 0.0007, 0.001, 0.0013; 0.7, 0.5, 0.4\rangle$; t=20-h.

Output: When we take the value t=20-h the output of λ_1 is Crisp number and $\tilde{\lambda_2}^{Ne}$ is TSVNNs are shown in Table-3 and the corresponding membership function shown in Figure-4.

Case: 3

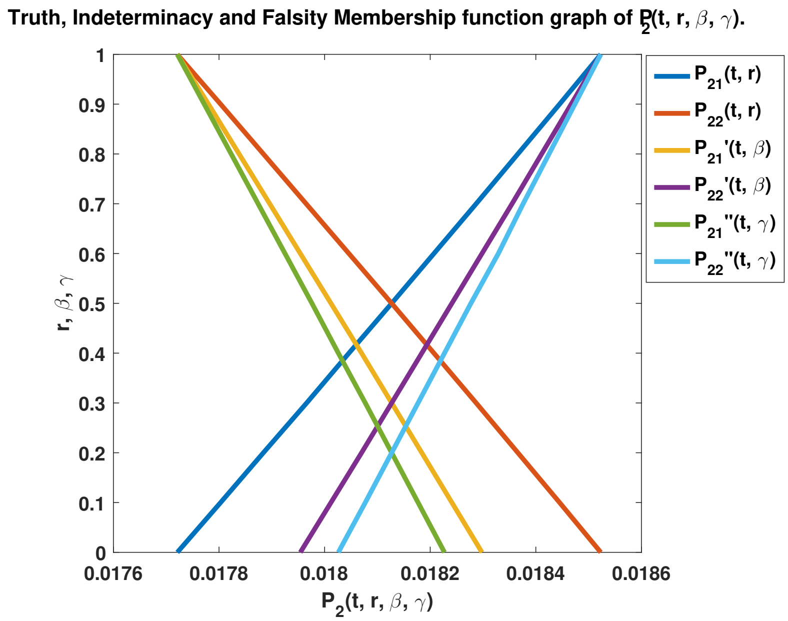


FIGURE 3. Membership Function Graph (at t=20).

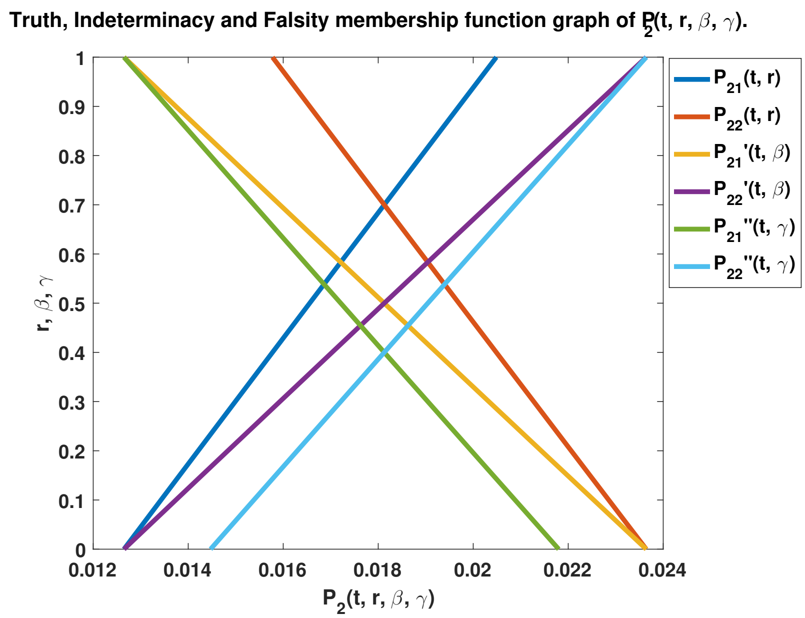


FIGURE 4. Membership Function Graph (at t=20).

r,β,γ	$\mathcal{P}_{21}(\mathfrak{t},r)$	$\mathcal{P}_{22}(\mathfrak{t},r)$	$\mathcal{P}'_{21}(\mathfrak{t},\beta)$	$\mathcal{P}'_{22}(\mathfrak{t},\beta)$	$\mathcal{P}''_{21}(\mathfrak{t},\gamma)$	$\mathcal{P}''_{22}(\mathfrak{t},\gamma)$
0	0.017721	0.018523	0.018298	0.017954	0.018227	0.018026
0.1	0.017803	0.018445	0.018241	0.018012	0.018177	0.018077
0.2	0.017884	0.018366	0.018184	0.018070	0.018127	0.018127
0.3	0.017966	0.018287	0.018127	0.018127	0.018077	0.018177
0.4	0.018046	0.018207	0.018070	0.018184	0.018026	0.018227
0.5	0.018127	0.018127	0.018012	0.018241	0.017976	0.018277
0.6	0.018207	0.018046	0.017954	0.018298	0.017925	0.018329
0.7	0.018287	0.017965	0.017896	0.018355	0.017874	0.018376
0.8	0.018366	0.017884	0.017838	0.018411	0.017823	0.018425
0.9	0.018445	0.017803	0.017780	0.018467	0.017772	0.018474
1.0	0.018523	0.017721	0.017721	0.018523	0.017721	0.018523

TABLE 2. $\tilde{\lambda_1}^{Ne}$ is TSVNNs & λ_2 is Crisp number.

r,β,γ	$\mathcal{P}_{21}(\mathfrak{t},r)$	$\mathcal{P}_{22}(\mathfrak{t},r)$	$\mathcal{P}'_{21}(\mathfrak{t},\beta)$	$\mathcal{P}'_{22}(\mathfrak{t},\beta)$	$\mathcal{P}''_{21}(\mathfrak{t},\gamma)$	$\mathcal{P}''_{22}(\mathfrak{t},\gamma)$
0	0.012647	0.023643	0.023643	0.012647	0.021800	0.014470
0.1	0.013427	0.022853	0.022537	0.013740	0.020881	0.015382
0.2	0.014209	0.022063	0.021432	0.014834	0.019962	0.016296
0.3	0.014991	0.021275	0.020329	0.015930	0.019044	0.017211
0.4	0.015774	0.020487	0.019227	0.017028	0.018127	0.018127
0.5	0.016557	0.019699	0.018127	0.018127	0.017211	0.019044
0.6	0.017342	0.018913	0.017028	0.019227	0.016296	0.019962
0.7	0.018127	0.018127	0.015930	0.020329	0.015382	0.020881
0.8	0.018913	0.017342	0.014834	0.021432	0.014470	0.021800
0.9	0.019699	0.016557	0.013740	0.022537	0.013558	0.022721
1.0	0.020487	0.015774	0.012647	0.023643	0.012647	0.023643

TABLE 3. λ_1 is Crisp number & $\tilde{\lambda_2}^{Ne}$ is TSVNNs

Input data:
$\tilde{\lambda_1}^{Ne}$=$\langle 0.007, 0.009, 0.011; 0.5, 0.3, 0.2\rangle$; $\tilde{\lambda_2}^{Ne} = \langle 0.0007, 0.001, 0.0013; 0.7, 0.5, 0.4\rangle$; $\mathfrak{t}$=20-h.
Output: When we take the value $\mathfrak{t}$=20-h the output of $\tilde{\lambda_1}^{Ne}$ & $\tilde{\lambda_2}^{Ne}$ are TSVNNs are shown in Table-4 and the corresponding membership function shown in Figure-5.

From the table values and graph, we see that
$\mathcal{P}_1(\mathfrak{t},r)$ is increasing function and
$\mathcal{P}_2(\mathfrak{t},r)$ is decreasing function, whereas

r,β,γ	$\mathcal{P}_{21}(t,r)$	$\mathcal{P}_{22}(t,r)$	$\mathcal{P}'_{21}(t,\beta)$	$\mathcal{P}'_{22}(t,\beta)$	$\mathcal{P}''_{21}(t,\gamma)$	$\mathcal{P}''_{22}(t,\gamma)$
0	0.012361	0.024156	0.024156	0.012361	0.022118	0.014253
0.1	0.013188	0.023247	0.022930	0.013493	0.021109	0.015210
0.2	0.014023	0.022345	0.021714	0.014635	0.020108	0.016175
0.3	0.014866	0.021449	0.020508	0.015788	0.019165	0.017147
0.4	0.015715	0.020561	0.019312	0.016952	0.018127	0.018127
0.5	0.016573	0.019681	0.018127	0.018127	0.017147	0.019114
0.6	0.017438	0.018807	0.016952	0.019312	0.016175	0.020108
0.7	0.018310	0.017941	0.015788	0.020508	0.015210	0.021109
0.8	0.019190	0.017083	0.014635	0.021714	0.014253	0.022118
0.9	0.020077	0.016232	0.013493	0.022930	0.013303	0.023134
1.0	0.020971	0.015388	0.012361	0.024156	0.012361	0.024156

TABLE 4. Both $\tilde{\lambda_1}^{Ne}$ & $\tilde{\lambda_2}^{Ne}$ are TSVNNs

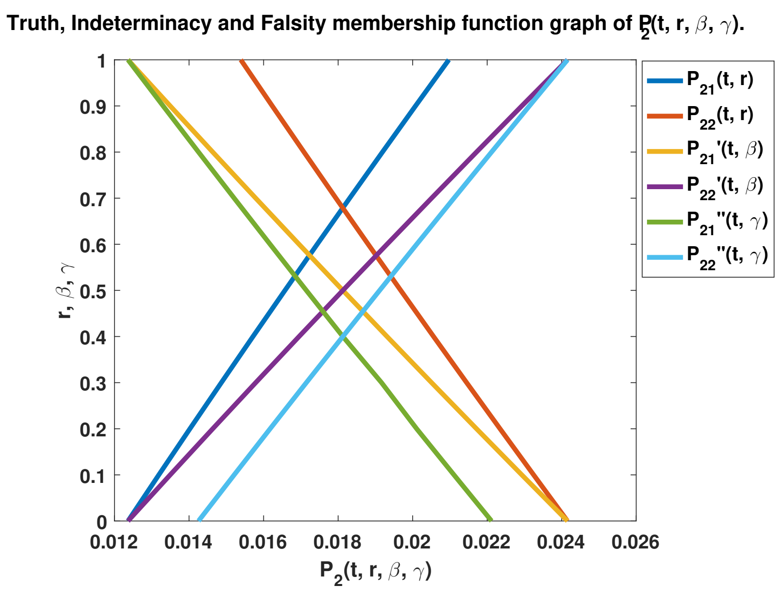


FIGURE 5. Membership Function Graph (at t=20).

$\mathcal{P}'_1(t,\beta)$, $\mathcal{P}''_1(t,\gamma)$ are decreasing functions and
$\mathcal{P}'_2(t,\beta)$, $\mathcal{P}''_2(t,\gamma)$ are increasing functions. Hence, the solution is strong solution.

9. Conclusion

• NS is a hot research topic and can be applied for solving the mathematical model of uncertainty, vagueness, ambiguity, etc.

• The mining safety model described in this paper with two parameters which satisfies the condition of NDE has got strong solutions.

• The solutions of the three differential equations of the mining safety model have been described via TSVNNs.

• The paper has also proposed numerical experiment and graphical representation of truth, indeterminacy and falsity membership function.

This will promote the future study of trapezoidal single valued neutrosophic numbers.

References

[1] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[2] KT Atanassov. Intuitionistic fuzzy sets vii itkr's session. *Sofia, June*, 1:983, 1983.

[3] Florentin Smarandache. *A unifying field in logics: Neutrosophic logic. neutrosophy, neutrosophic set, neutrosophic probability: Neutrosophic logic: neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2003.

[4] Suresh Chandra C. R. Bector. *Fuzzy Mathematical Programming and Fuzzy Matrix Games*. Studies in Fuzziness and Soft Computing. Springer, 1st edition. edition, 2010.

[5] Tamalika Chaira. *Fuzzy Set and its Extension. The Intuitionistic Fuzzy Set*. Wiley, 2019.

[6] Sankar Prasad Mondal, Sanhita Banerjee, and Tapan Kumar Roy. First order linear homogeneous ordinary differential equation in fuzzy environment. *International journal of pure and applied sciences and technology*, 14(1):16, 2013.

[7] Sankar Prasad Mondal and Tapan Kumar Roy. First order homogeneous ordinary differential equation with initial value as triangular intuitionistic fuzzy number. *Journal of Uncertainty in Mathematics Science*, 2014:1–17, 2014.

[8] IR Sumathi and V Mohana Priya. *A new perspective on neutrosophic differential equation*. Infinite Study, 2018.

[9] IR Sumathi and C Antony Crispin Sweety. New approach on differential equation via trapezoidal neutrosophic number. *Complex & Intelligent Systems*, 5(4):417–424, 2019.

[10] Nguyen Thi Kim Son, Nguyen Phuong Dong, Hoang Viet Long, Alireza Khastan, et al. Linear quadratic regulator problem governed by granular neutrosophic fractional differential equations. *ISA transactions*, 97:296–316, 2020.

[11] B.S. Dhillon (auth.). *Mine Safety: A Modern Approach*. Springer Series in Reliability Engineering. Springer-Verlag London, 1 edition, 2010.

[12] Wasim Akram Mandal Sahidul Islam. *Fuzzy Geometric Programming Techniques and Applications*. Springer, 2019.

[13] Nguyen Thi Kim Son, Nguyen Phuong Dong, Hoang Viet Long, et al. Towards granular calculus of single-valued neutrosophic functions under granular computing. *Multimedia Tools and Applications*, pages 1–37, 2019.

[14] James J Buckley, Esfandiar Eslami, and Thomas Feuring. *Fuzzy mathematics in economics and engineering*, volume 91. Physica, 2013.

Received: May 7, 2021. Accepted: October 5, 2021

NSS

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

Relation of Quasi-coincidence for Neutrosophic Sets

Gautam Chandra Ray[1,*] and Sudeep Dey[2]

[1]Department of Mathematics, Central Institute of Technology, Kokrajhar ; gautomofcit@gmail.com

[2]Department of Mathematics, Science College, Kokrajhar ; sudeep.dey.1976@gmail.com

*Correspondence: gautomofcit@gmail.com

Abstract. We define the relation of quasi-coincidence between a neutrosophic point and a neutrosophic set as well as between two neutrosophic sets and investigate some properties based on that. We define the quasi-neighbourhood of a neutrosophic point and examine some properties. We also study the characterization of neutrosophic topological space in terms of quasi-neighbourhoods.

Keywords: Neutrosophic set ; Neutrosophic point ; Quasi-coincidence ; Quasi-neighbourhood.

---

1. **Introduction**

The notion of Fuzzy set was brought to light by Zadeh [38] in 1965 and Intuitionistic fuzzy set, a generalized version of fuzzy set, was introduced by Atanassov [1] in 1986. After a decade, a new branch of philosophy recognised as Neutrosophy was developed and studied by Florentin Smarandache [25–27]. Smarandache [27] proved that neutrosophic set was a generalization of intuitionistic fuzzy set. Like intuitionistic fuzzy set, an element in a neutrosophic set has the degree of membership and the degree of non-membership but it has another grade of membership known as the degree of indeterminacy and one very important point about neutrosophic set is that all the three neutrosophic components are independent of one another.

After Smarandache had brought the thought of neutrosophy, it was studied and taken ahead by many researchers [11, 31, 32, 35]. In the year 2002, Smarandache [26] added the thinking of neutrosophic topology on the non-standard interval and thereafter Lupiáñez [16–19] studied and investigated many properties of neutrosophic topological space. The author [17] also studied the relation between interval neutrosophic sets and topology. Salma et.al. [28–30] studied neutrosophic topological space, generalised neutrosophic topological space and neutrosophic continuous functions. In the year 2016, Karatas and Kuru [15] redefined the single valued neutrosophic set operations and introduced a new neutrosophic topology and then investigated some important properties of general topology on the redefined neutrosophic topological space. Later, various aspects of neutrosophic topology were developed by many researchers [2, 12, 14, 33].

Neutrosophy, due to the fact of its of flexibility and effectiveness, is attracting the researchers throughout the world and is very useful not only in the developement of science and technology but also in various other fields. For instance, Abdel-Basset et.al. [3–6] studied the applications of neutrosophic theory in a number of scientific fields. Pramanik and Roy [24] in 2014 studied on the conflict between India and Pakistan over Jammu-Kashmir through neutrosophic game theory. Works on medical diagnosis [7, 36], decision making problem [8, 37], image processing [10, 13], social issues [20, 23], educational problems [21, 22] were also done under neutrosophic environment.

In the year 1995 Coker and Demirci [9] introduced the idea of intuitionistic fuzzy points and their quasi-coincident relation. Very recently Ray and Dey [34] introduced the idea of neutrosophic point on single-valued neutrosophic sets and studied various properties. But the relation of quasi-coincidence in case of neutrosophic points or neutrosophic sets has not been studied so far. In this article, we define the relation of quasi-coincidence between a neutrosophic point and a neutrosophic set as well as between two neutrosophic sets and examine some properties based on the relation of quasi-coincidence. We then define neutrosophic quasi-neighbourhood of a neutrosophic point and investigate some properties. Lastly we study the characterization of neutrosophic topological space in terms of neutrosophic quasi-neighbourhoods.

2. Preliminaries

In this section we discuss some concepts related with neutrosophic sets.

2.1. Definition: [35]

Let X be the universe of discourse. A single valued neutrosophic set A over X is defined as $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x)\rangle : x \in X\}$, where $\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A$ are functions from X to $[0, 1]$ and $0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \mathcal{F}_A(x) \leq 3$.

The set of all single valued neutrosophic sets over X is denoted by $\mathcal{N}(X)$.

Throughout this article, a single valued neutrosophic set will simply be called a neutrosophic set (NS, for short).

2.2. Definition: [15]

Let $A, B \in \mathcal{N}(X)$. Then

(i) (Inclusion): If $\mathcal{T}_A(x) \leq \mathcal{T}_B(x), \mathcal{I}_A(x) \geq \mathcal{I}_B(x), \mathcal{F}_A(x) \geq \mathcal{F}_B(x)$ for all $x \in X$ then A is said to be a neutrosophic subset of B and which is denoted by $A \subseteq B$.

(ii) (Equality): If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(iii) (Intersection): The intersection of A and B, denoted by $A \cap B$, is defined as $A \cap B = \{\langle x, \mathcal{T}_A(x) \wedge \mathcal{T}_B(x), \mathcal{I}_A(x) \vee \mathcal{I}_B(x), \mathcal{F}_A(x) \vee \mathcal{F}_B(x)\rangle : x \in X\}$.

(iv) (Union): The union of A and B, denoted by $A \cup B$, is defined as $A \cup B = \{\langle x, \mathcal{T}_A(x) \vee \mathcal{T}_B(x), \mathcal{I}_A(x) \wedge \mathcal{I}_B(x), \mathcal{F}_A(x) \wedge \mathcal{F}_B(x)\rangle : x \in X\}$.

(v) (Complement): The complement of the NS A, denoted by A^c, is defined as $A^c = \{\langle x, \mathcal{F}_A(x), 1 - \mathcal{I}_A(x), \mathcal{T}_A(x)\rangle : x \in X\}$

(vi) (Universal Set): If $\mathcal{T}_A(x) = 1, \mathcal{I}_A(x) = 0, \mathcal{F}_A(x) = 0$ for all $x \in X$ then A is said to be neutrosophic universal set and which is denoted by $\tilde{X}$.

(vii) (Empty Set): If $\mathcal{T}_A(x) = 0, \mathcal{I}_A(x) = 1, \mathcal{F}_A(x) = 1$ for all $x \in X$ then A is said to be neutrosophic empty set and which is denoted by $\tilde{\emptyset}$.

2.3. **Definition: [29]**

Let $\{A_i : i \in \triangle\} \subseteq \mathcal{N}(X)$, where $\triangle$ is an index set. Then

(i) $\cup_{i\in\triangle} A_i = \{\langle x, \vee_{i\in\triangle}\mathcal{T}_{A_i}(x), \wedge_{i\in\triangle}\mathcal{I}_{A_i}(x), \wedge_{i\in\triangle}\mathcal{F}_{A_i}(x)\rangle : x \in X\}$.

(ii) $\cap_{i\in\triangle} A_i = \{\langle x, \wedge_{i\in\triangle}\mathcal{T}_{A_i}(x), \vee_{i\in\triangle}\mathcal{I}_{A_i}(x), \vee_{i\in\triangle}\mathcal{F}_{A_i}(x)\rangle : x \in X\}$.

2.4. **Neutrosophic topological space:**

2.4.1. **Definition: [15]**

Let $\tau \subseteq \mathcal{N}(X)$. Then τ is called a neutrosophic topology on X if

(i) $\tilde{\emptyset}$ and $\tilde{X}$ belong to τ.

(ii) The union of any number of neutrosophic sets in τ belongs to τ.

(iii) The intersection of any two neutrosophic sets in τ belongs to τ.

If τ is a neutrosophic topology on X then the pair (X, τ) is called a neutrosophic topological space (NTS, for short) over X. The members of τ are called neutrosophic open sets in X. If for a neutrosophic set A, $A^c \in \tau$ then A is said to be a neutrosophic closed set in X.

2.4.2. **Theorem: [15]**

Let (X, τ) be a neutrosophic topological space over X. Then

(i) $\tilde{\emptyset}$ and $\tilde{X}$ are neutrosophic closed sets over X.

(ii) The intersection of any number of neutrosophic closed sets is a neutrosophic closed set over X.

(iii) The union of any two neutrosophic closed sets is a neutrosophic closed set over X.

2.5. **Definition: [34]**

Let $\mathcal{N}(X)$ be the set of all neutrosophic sets over X. A NS $P = \{\langle x, \mathcal{T}_P(x), \mathcal{I}_P(x), \mathcal{F}_P(x)\rangle : x \in X\}$ is called a neutrosophic point (NP, for short) iff for any element $y \in X$, $\mathcal{T}_P(y) = \alpha, \mathcal{I}_P(y) = \beta, \mathcal{F}_P(y) = \gamma$ for $y = x$ and $\mathcal{T}_P(y) = 0, \mathcal{I}_P(y) = 1, \mathcal{F}_P(y) = 1$ for $y \neq x$, where $0 < \alpha \leq 1, 0 \leq \beta < 1, 0 \leq \gamma < 1$.

A neutrosophic point $P = \{\langle x, \mathcal{T}_P(x), \mathcal{I}_P(x), \mathcal{F}_P(x)\rangle : x \in X\}$ will be denoted by $P^x_{\alpha,\beta,\gamma}$ or $P < x, \alpha, \beta, \gamma >$ or simply by $x_{\alpha,\beta,\gamma}$. For the NP $x_{\alpha,\beta,\gamma}$, x will be called its support.

The complement of the NP $P^x_{\alpha,\beta,\gamma}$ will be denoted by $(P^x_{\alpha,\beta,\gamma})^c$ or by $x^c_{\alpha,\beta,\gamma}$.

2.6. **Definition: [34]**

Let A be a neutrosophic set over X. Also let $x_{\alpha,\beta,\gamma}$ and $y_{\alpha^/,\beta^/,\gamma^/}$ be two neutrosophic points in X. Then

(i) $x_{\alpha,\beta,\gamma}$ is said to be contained in A, denoted by $x_{\alpha,\beta,\gamma} \subseteq A$, iff $\alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x)$.

(ii) $x_{\alpha,\beta,\gamma}$ is said to belong to A, denoted by $x_{\alpha,\beta,\gamma} \in A$, iff $\alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x)$.

(iii) $x_{\alpha,\beta,\gamma}$ is said to be contained in $y_{\alpha^/,\beta^/,\gamma^/}$, denoted by $x_{\alpha,\beta,\gamma} \subseteq y_{\alpha^/,\beta^/,\gamma^/}$, iff $x = y$ and $\alpha \leq \alpha^/, \beta \geq \beta^/, \gamma \geq \gamma^/$.

(iv) $x_{\alpha,\beta,\gamma}$ is said to belong to $y_{\alpha^/,\beta^/,\gamma^/}$, denoted by $x_{\alpha,\beta,\gamma} \in y_{\alpha^/,\beta^/,\gamma^/}$, iff $x = y$ and $\alpha \leq \alpha^/, \beta \geq \beta^/, \gamma \geq \gamma^/$.

2.7. **Proposition: [34]**

Let $\{A_i : i \in \triangle\} \subseteq \mathcal{N}(X)$, where $\triangle$ is an index set. Let $x_{\alpha,\beta,\gamma}$ and $y_{\alpha^/,\beta^/,\gamma^/}$ be any two neutrosophic points over X. Then the following hold good.

(i) $x_{\alpha,\beta,\gamma} \in \bigcap\{A_i : i \in \triangle\} \Longleftrightarrow x_{\alpha,\beta,\gamma} \in A_i \, \forall \, i \in \triangle$.

(ii) If $x_{\alpha,\beta,\gamma} \in A_i$ for some $i \in \triangle$ then $x_{\alpha,\beta,\gamma} \in \bigcup\{A_i : i \in \triangle\}$.

(iii) If $x_{\alpha,\beta,\gamma} \in \bigcup\{A_i : i \in \triangle\}$ then there exists a NS $A(x_{\alpha,\beta,\gamma})$ such that $x_{\alpha,\beta,\gamma} \in A(x_{\alpha,\beta,\gamma}) \subseteq \bigcup\{A_i : i \in \triangle\}$.

For other definitions and results concerning neutrosophic points used in this article, please see [34]

3. Main Results

3.1. Definition:

A NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$ is said to be quasi-coincident with a NS $A \in \mathcal{N}(X)$ or $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$ quasi-coincides with a NS $A \in \mathcal{N}(X)$, denoted by $x_{\alpha,\beta,\gamma} qA$, iff $\alpha > \mathcal{T}_{A^c}(x)$ or $\beta < \mathcal{I}_{A^c}(x)$ or $\gamma < \mathcal{F}_{A^c}(x)$, i.e., $\alpha > \mathcal{F}_A(x)$ or $\beta < 1 - \mathcal{I}_A(x)$ or $\gamma < \mathcal{T}_A(x)$.

A NS A is said to be quasi-coincident with a NS B at $x \in X$ or A quasi-coincides with B at $x \in X$, denoted by AqB at x, iff $\mathcal{T}_A(x) > \mathcal{T}_{B^c}(x)$ or $\mathcal{I}_A(x) < \mathcal{I}_{B^c}(x)$ or $\mathcal{F}_A(x) < \mathcal{F}_{B^c}(x)$. We say A quasi-coincides with B or A is quasi-coincident with B, denoted by AqB, iff A quasi-coincides with B at some point $x \in X$. Thus A quasi-coincides with B or A is quasi-coincident with B iff there exists an element $x \in X$ such that $\mathcal{T}_A(x) > \mathcal{T}_{B^c}(x)$ or $\mathcal{I}_A(x) < \mathcal{I}_{B^c}(x)$ or $\mathcal{F}_A(x) < \mathcal{F}_{B^c}(x)$, i.e., $\mathcal{T}_A(x) > \mathcal{F}_B(x)$ or $\mathcal{I}_A(x) < 1 - \mathcal{I}_B(x)$ or $\mathcal{F}_A(x) < \mathcal{T}_B(x)$.

If the NP $x_{\alpha,\beta,\gamma}$ is not quasi-coincident with a NS A, we shall denote it by $x_{\alpha,\beta,\gamma}\hat{q}A$. Similarly if the NS A is not quasi-coincident with the NS B, we shall denote it by $A\hat{q}B$.

The set of all the points in X, at which AqB, will be denoted by $A\Omega B$, i.e., $A\Omega B = \{x \in X : AqB \text{ at } x\}$.

Before proceeding to the results connected to quasi-coincident relation we first prove a simple result on neutrosophic sets.

3.2. Proposition:

Let $A, B \in \mathcal{N}(X)$. Then $A \subseteq B \Leftrightarrow B^c \subseteq A^c$.

Proof:

$$
\begin{aligned}
& A \subseteq B \\
\Leftrightarrow & \mathcal{T}_A(x) \le \mathcal{T}_B(x), \mathcal{I}_A(x) \ge \mathcal{I}_B(x), \mathcal{F}_A(x) \ge \mathcal{F}_B(x) \text{ for all } x \in X \\
\Leftrightarrow & \mathcal{F}_B(x) \le \mathcal{F}_A(x), 1 - \mathcal{I}_A(x) \le 1 - \mathcal{I}_B(x), \mathcal{T}_B(x) \ge \mathcal{T}_A(x) \text{ for all } x \in X \\
\Leftrightarrow & \mathcal{T}_{B^c}(x) \le \mathcal{T}_{A^c}(x), \mathcal{I}_{B^c}(x) \ge \mathcal{I}_{A^c}(x), \mathcal{F}_{B^c}(x) \ge \mathcal{F}_{A^c}(x) \text{ for all } x \in X \\
\Leftrightarrow & B^c \subseteq A^c
\end{aligned}
$$

3.3. Proposition:

Let A, B, C be three neutrosophic sets and $x_{\alpha,\beta,\gamma}$ be a neutrosophic point in X. Then

(i) $x_{\alpha,\beta,\gamma}\hat{q}\tilde{\emptyset}$.

(ii) $x_{\alpha,\beta,\gamma}q\tilde{X}$.

(iii) $x_{\alpha,\beta,\gamma} \in A \Leftrightarrow x_{\alpha,\beta,\gamma}\hat{q}A^c$.

(iv) $x_{\alpha,\beta,\gamma}qA \Leftrightarrow x_{\alpha,\beta,\gamma} \notin A^c$.

(v) $A \subseteq B \Leftrightarrow A\hat{q}B^c$.

(vi) $AqB \Leftrightarrow A \nsubseteq B^c$

(vii) $x_{\alpha,\beta,\gamma}qA$ and $A \subseteq B$ then $x_{\alpha,\beta,\gamma}qB$.

(viii) CqA and $A \subseteq B$ then CqB.

(ix) AqB at $x \Leftrightarrow BqA$ at x.

(x) $AqB \Leftrightarrow BqA$.

Proofs:

(i) Very obvious.

(ii) Very obvious.

(iii)

$$\begin{aligned}
&x_{\alpha,\beta,\gamma} \in A \\
\Leftrightarrow&\alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x) \\
\Leftrightarrow&\alpha \ngtr \mathcal{T}_A(x), \beta \nless \mathcal{I}_A(x), \gamma \nless \mathcal{F}_A(x) \\
\Leftrightarrow&\alpha \ngtr \mathcal{T}_{(A^c)^c}(x), \beta \nless \mathcal{I}_{(A^c)^c}(x), \gamma \nless \mathcal{F}_{(A^c)^c}(x) \\
\Leftrightarrow&x_{\alpha,\beta,\gamma}\hat{q}A^c
\end{aligned}$$

(iv)

$$\begin{aligned}
&x_{\alpha,\beta,\gamma}qA \\
\Leftrightarrow&\alpha > \mathcal{T}_{A^c}(x) \text{ or } \beta < \mathcal{I}_{A^c}(x) \text{ or } \gamma < \mathcal{F}_{A^c}(x) \\
\Leftrightarrow&\alpha \nleq \mathcal{T}_{A^c}(x) \text{ or } \beta \ngeq \mathcal{I}_{A^c}(x) \text{ or } \gamma \ngeq \mathcal{F}_{A^c}(x) \\
\Leftrightarrow&x_{\alpha,\beta,\gamma} \notin A^c
\end{aligned}$$

(v)

$$\begin{aligned}
&A \subseteq B \\
\Leftrightarrow&\mathcal{T}_A(x) \leq \mathcal{T}_B(x), \mathcal{I}_A(x) \geq \mathcal{I}_B(x), \mathcal{F}_A(x) \geq \mathcal{F}_B(x) \,\forall\, x \in X \\
\Leftrightarrow&\mathcal{T}_A(x) \ngtr \mathcal{T}_B(x), \mathcal{I}_A(x) \nless \mathcal{I}_B(x), \mathcal{F}_A(x) \nless \mathcal{F}_B(x) \,\forall\, x \in X \\
\Leftrightarrow&\mathcal{T}_A(x) \ngtr \mathcal{T}_{(B^c)^c}(x), \mathcal{I}_A(x) \nless \mathcal{I}_{(B^c)^c}(x), \mathcal{F}_A(x) \nless \mathcal{F}_{(B^c)^c}(x) \,\forall\, x \in X \\
\Leftrightarrow&A\hat{q}B^c
\end{aligned}$$

(vi)

$$\begin{aligned}
&AqB\\
\Leftrightarrow&\mathcal{T}_A(x) > \mathcal{T}_{B^c}(x)\,\text{or}\,\mathcal{I}_A(x) < \mathcal{I}_{B^c}(x)\,\text{or}\,\mathcal{F}_A(x) < \mathcal{F}_{B^c}(x)\,\text{for some}\,x \in X\\
\Leftrightarrow&\mathcal{T}_A(x) \nleq \mathcal{T}_{B^c}(x)\,\text{or}\,\mathcal{I}_A(x) \ngeq \mathcal{I}_{B^c}(x)\,\text{or}\,\mathcal{F}_A(x) \ngeq \mathcal{F}_{B^c}(x)\,\text{for some}\,x \in X\\
\Leftrightarrow&A \nsubseteq B^c
\end{aligned}$$

(vii) Since $x_{\alpha,\beta,\gamma}qA$, so $\alpha > \mathcal{T}_{A^c}(x)\,\text{or}\,\beta < \mathcal{I}_{A^c}(x)\,\text{or}\,\gamma < \mathcal{F}_{A^c}(x)$. Now

$$\begin{aligned}
&A \subseteq B\\
\Rightarrow&B^c \subseteq A^c\\
\Rightarrow&\mathcal{T}_{B^c}(x) \leq \mathcal{T}_{A^c}(x), \mathcal{I}_{B^c}(x) \geq \mathcal{I}_{A^c}(x), \mathcal{F}_{B^c}(x) \geq \mathcal{F}_{A^c}(x) \text{ for all } x \in X\\
\Rightarrow&\mathcal{T}_{A^c}(x) \geq \mathcal{T}_{B^c}(x), \mathcal{I}_{A^c}(x) \leq \mathcal{I}_{B^c}(x), \mathcal{F}_{A^c}(x) \leq \mathcal{F}_{B^c}(x) \text{ for all } x \in X\\
\Rightarrow&\alpha > \mathcal{T}_{B^c}(x) \text{ or } \beta < \mathcal{I}_{B^c}(x) \text{ or } \gamma < \mathcal{F}_{B^c}(x)\\
\Rightarrow&x_{\alpha,\beta,\gamma}qB
\end{aligned}$$

(viii) $CqA \Rightarrow C \nsubseteq A^c \Rightarrow C \nsubseteq B^c$ [$\because A \subseteq B \Rightarrow B^c \subseteq A^c$]$\Rightarrow CqB$.

(ix)

$$\begin{aligned}
&AqB \text{ at } x\\
\Leftrightarrow&\mathcal{T}_A(x) > \mathcal{T}_{B^c}(x) \text{ or } \mathcal{I}_A(x) < \mathcal{I}_{B^c}(x) \text{ or } \mathcal{F}_A(x) < \mathcal{F}_{B^c}(x)\\
\Leftrightarrow&\mathcal{T}_A(x) > \mathcal{F}_B(x) \text{ or } \mathcal{I}_A(x) < 1 - \mathcal{I}_B(x) \text{ or } \mathcal{F}_A(x) < \mathcal{T}_B(x)\\
\Leftrightarrow&\mathcal{T}_B(x) > \mathcal{F}_A(x) \text{ or } \mathcal{I}_B(x) < 1 - \mathcal{I}_A(x) \text{ or } \mathcal{F}_B(x) < \mathcal{T}_A(x)\\
\Leftrightarrow&\mathcal{T}_B(x) > \mathcal{T}_{A^c}(x) \text{ or } \mathcal{I}_B(x) < \mathcal{I}_{A^c}(x) \text{ or } \mathcal{F}_B(x) < \mathcal{F}_{A^c}(x)\\
\Leftrightarrow&BqA \text{ at } x
\end{aligned}$$

(x) Obvious from (ix).

3.4. **Proposition:**

Let $x_{\alpha,\beta,\gamma}$ be a NP in X, $A \in \mathcal{N}(X)$ and $\{A_i : i \in \triangle\} \subseteq \mathcal{N}(X)$, $\triangle$ is an index set. Then

(i) $x_{\alpha,\beta,\gamma}q \cup_{i\in\triangle} A_i \Leftrightarrow x_{\alpha,\beta,\gamma}qA_j$ for some $j \in\triangle$.

(ii) $Aq \cup_{i\in\triangle} A_i \Leftrightarrow AqA_j$ for some $j \in\triangle$.

(iii) $x_{\alpha,\beta,\gamma}q \cap_{i\in\triangle} A_i \Rightarrow x_{\alpha,\beta,\gamma}qA_i$ for all $i \in\triangle$. Converse is not true.

(iv) $Aq \cap_{i\in\triangle} A_i \Rightarrow AqA_i$ for all $i \in\triangle$. Converse is not true.

Proofs: (i)

$$x_{\alpha,\beta,\gamma} q \cup_{i\in\triangle} A_i$$
$$\Leftrightarrow x_{\alpha,\beta,\gamma} \notin (\cup_{i\in\triangle} A_i)^c$$
$$\Leftrightarrow x_{\alpha,\beta,\gamma} \notin \cap_{i\in\triangle} A_i^c$$
$$\Leftrightarrow x_{\alpha,\beta,\gamma} \notin A_j^c \text{ for some } j \in\triangle$$
$$\Leftrightarrow x_{\alpha,\beta,\gamma} q A_j \text{ for some } j \in\triangle$$

(ii)

$$A q \cup_{i\in\triangle} A_i$$
$$\Leftrightarrow A \nsubseteq (\cup_{i\in\triangle} A_i)^c$$
$$\Leftrightarrow A \nsubseteq \cap_{i\in\triangle} A_i^c$$
$$\Leftrightarrow A \nsubseteq A_j^c \text{ for some } j \in\triangle$$
$$\Leftrightarrow A q A_j \text{ for some } j \in\triangle$$

(iii)

$$x_{\alpha,\beta,\gamma} q \cap_{i\in\triangle} A_i$$
$$\Rightarrow x_{\alpha,\beta,\gamma} \notin (\cap_{i\in\triangle} A_i)^c$$
$$\Rightarrow x_{\alpha,\beta,\gamma} \notin \cup_{i\in\triangle} A_i^c$$
$$\Rightarrow x_{\alpha,\beta,\gamma} \notin A_i^c \text{ for all } i \in\triangle$$
$$\Rightarrow x_{\alpha,\beta,\gamma} q A_i \text{ for all } i \in\triangle$$

Converse is not true. We establish it by the following counter example.

Let $X = \{x, y\}$. Also let $A = \{\langle x, 0.3, 0.6, 0.2\rangle, \langle y, 0.6, 0.7, 0.7\rangle\}, B = \{\langle x, 0.3, 0.5, 0.6\rangle, \langle y, 0.3, 0.8, 0.7\rangle\}$ and $C = \{\langle x, 0.4, 0.5, 0.7\rangle, \langle y, 0.6, 0.1, 0.7\rangle\}$ be three neutrosophic sets over X. Then $A \cap B \cap C = \{\langle x, 0.3, 0.6, 0.7\rangle, \langle y, 0.3, 0.8, 0.7\rangle\}$ Let us consider the neutrosophic point $x_{0.3,0.4,0.8}$. Clearly $x_{0.3,0.4,0.8} qA$, $x_{0.3,0.4,0.8} qB$ and $x_{0.3,0.4,0.8} qC$ but $x_{0.3,0.4,0.8}$ is not quasi-coincident with $A \cap B \cap C$.

(iv)

$$A q \cap_{i\in\triangle} A_i$$
$$\Rightarrow A \nsubseteq (\cap_{i\in\triangle} A_i)^c$$
$$\Rightarrow A \nsubseteq \cup_{i\in\triangle} A_i^c$$
$$\Rightarrow A \nsubseteq A_i^c \text{ for all } i \in\triangle$$
$$\Rightarrow A q A_i \text{ for all } i \in\triangle$$

Converse is not true. We establish it by the following counter example.

Let $X = \{x, y\}$. Also let $A = \{\langle x, 0.3, 0.6, 0.2\rangle, \langle y, 0.6, 0.7, 0.7\rangle\}, B = \{\langle x, 0.3, 0.5, 0.6\rangle, \langle y, 0.3, 0.8, 0.7\rangle\}$ and $C = \{\langle x, 0.4, 0.5, 0.7\rangle, \langle y, 0.6, 0.1, 0.7\rangle\}$ be three neutrosophic sets over X. Then $A \cap B \cap C = \{\langle x, 0.3, 0.6, 0.7\rangle, \langle y, 0.3, 0.8, 0.7\rangle\}$ Let us consider the neutrosophic set $D = \{\langle x, 0.3, 0.4, 0.8\rangle, \langle y, 0.5, 0.7, 0.7\rangle\}$. Clearly DqA , DqB and DqC but D is not quasi-coincident with $A \cap B \cap C$.

3.5. **Proposition:**

(i) $A\Omega B = B\Omega A$.
(ii) $AqB \Leftrightarrow A\Omega B \neq \emptyset$.
(iii) $A \subseteq B \Rightarrow A\Omega C \subseteq B\Omega C$.
(iv) $A\Omega(\cup_{i\in\triangle} A_i) = \cup_{i\in\triangle}(A\Omega A_i)$.
(v) $A\Omega(\cap_{i\in\triangle} A_i) \subseteq \cap_{i\in\triangle}(A\Omega A_i)$. Converse is not true.

Proofs:

(i) $A\Omega B = \{x \in X : AqB \,\text{at}\, x\} = \{x \in X : BqA \,\text{at}\, x\} = B\Omega A$.

(ii)$AqB \Leftrightarrow AqB$ at some $x \in X \Leftrightarrow x \in A\Omega B$. Therefore $AqB \Leftrightarrow A\Omega B \neq \emptyset$.

(iii)$A \subseteq B \Rightarrow \mathcal{T}_A(x) \leq \mathcal{T}_B(x), \mathcal{I}_A(x) \geq \mathcal{I}_B(x), \mathcal{F}_A(x) \geq \mathcal{F}_B(x)$ for all $x \in X$. Now

$$\begin{aligned}
& x \in A\Omega C \\
\Rightarrow & AqC \text{ at } x \in X \\
\Rightarrow & \mathcal{T}_A(x) > \mathcal{T}_{C^c}(x) \text{ or } \mathcal{I}_A(x) < \mathcal{I}_{C^c}(x) \text{ or } \mathcal{F}_A(x) < \mathcal{F}_{C^c}(x) \\
\Rightarrow & \mathcal{T}_B(x) > \mathcal{T}_{C^c}(x) \text{ or } \mathcal{I}_B(x) < \mathcal{I}_{C^c}(x) \text{ or } \mathcal{F}_B(x) < \mathcal{F}_{C^c}(x) \\
\Rightarrow & BqC \text{ at } x \in X \\
\Rightarrow & x \in B\Omega C \\
& \therefore A\Omega C \subseteq B\Omega C.
\end{aligned}$$

(iv)

$$\begin{aligned}
& x \in A\Omega(\cup_{i\in\triangle} A_i) \\
\Rightarrow & Aq(\cup_{i\in\triangle} A_i) \text{ at } x \in X \\
\Rightarrow & \exists\ j \in\triangle \text{ such that } AqA_j \text{ at } x \in X \\
\Rightarrow & \exists\ j \in\triangle \text{ such that } x \in A\Omega A_j \\
\Rightarrow & x \in \cup_{i\in\triangle}(A\Omega A_i) \\
& \therefore A\Omega(\cup_{i\in\triangle} A_i) \subseteq \cup_{i\in\triangle}(A\Omega A_i).
\end{aligned}$$

Again

$$x \in \cup_{i\in\triangle}(A\Omega A_i)$$
$$\Rightarrow \bigvee_{i\in\triangle}(AqA_i \text{ at } x \in X)$$
$$\Rightarrow \bigvee_{i\in\triangle}(A_iqA \text{ at } x \in X)$$
$$\Rightarrow \bigvee_{i\in\triangle}[\mathcal{T}_{A_i}(x) > \mathcal{T}_{A^c}(x) \text{ or } \mathcal{I}_{A_i}(x) < \mathcal{I}_{A^c}(x) \text{ or } \mathcal{F}_{A_i}(x) < \mathcal{F}_{A^c}(x)]$$
$$\Rightarrow \sup_{i\in\triangle}\mathcal{T}_{A_i}(x) > \mathcal{T}_{A^c}(x) \text{ or } \inf_{i\in\triangle}\mathcal{I}_{A_i}(x) < \mathcal{I}_{A^c}(x) \text{ or } \inf_{i\in\triangle}\mathcal{F}_{A_i}(x) < \mathcal{F}_{A^c}(x)$$
$$\Rightarrow \mathcal{T}_{\cup A_i}(x) > \mathcal{T}_{A^c}(x) \text{ or } \mathcal{I}_{\cup A_i}(x) < \mathcal{I}_{A^c}(x) \text{ or } \mathcal{F}_{\cup A_i}(x) < \mathcal{F}_{A^c}(x)$$
$$\Rightarrow (\cup_{i\in\triangle}A_i)qA \text{ at } x \in X$$
$$\Rightarrow Aq(\cup_{i\in\triangle}A_i) \text{ at } x \in X$$
$$\Rightarrow x \in A\Omega(\cup_{i\in\triangle}A_i)$$
$$\therefore \cup_{i\in\triangle}(A\Omega A_i) \subseteq A\Omega(\cup_{i\in\triangle}A_i)$$

Hence $A\Omega(\cup_{i\in\triangle}A_i) = \cup_{i\in\triangle}(A\Omega A_i)$.

(v)

$$x \in A\Omega(\cap_{i\in\triangle}A_i)$$
$$\Rightarrow Aq(\cap_{i\in\triangle}A_i) \text{ at } x \in X$$
$$\Rightarrow AqA_i \text{ at } x \in X \text{for all} i \in\triangle$$
$$\Rightarrow x \in A\Omega A_i \text{for all} i \in\triangle$$
$$\Rightarrow x \in \cap_{i\in\triangle}(A\Omega A_i)$$
$$\therefore A\Omega(\cap_{i\in\triangle}A_i) \subseteq \cap_{i\in\triangle}(A\Omega A_i).$$

Converse is not true We establish it by the following counter example.

Let $X = \{x, y\}$. Also let $A = \{\langle x, 0.3, 0.6, 0.2\rangle, \langle y, 0.6, 0.7, 0.7\rangle\}, B = \{\langle x, 0.3, 0.5, 0.6\rangle, \langle y, 0.3, 0.8, 0.7\rangle\}$ and $C = \{\langle x, 0.4, 0.5, 0.7\rangle, \langle y, 0.6, 0.1, 0.7\rangle\}$ be three neutrosophic sets over X. Then $A \cap B \cap C = \{\langle x, 0.3, 0.6, 0.7\rangle, \langle y, 0.3, 0.8, 0.7\rangle\}$ Let us consider the neutrosophic set $D = \{\langle x, 0.3, 0.4, 0.8\rangle, \langle y, 0.5, 0.7, 0.7\rangle\}$. Clearly $D\Omega A = \{x\}$, $D\Omega B = \{x\}$, $D\Omega C = \{x, y\}$ and $D\Omega(A \cap B \cap C) = \emptyset$. Therefore $(D\Omega A) \cap (D\Omega B) \cap (D\Omega C) = \{x\} \nsubseteq D\Omega(A \cap B \cap C)$.

3.6. **Definition:**

Let (X, τ) be a NTS. A neutrosophic set A is called a neutrosophic quasi-neighbourhood or simply Q-neighbourhood (Q-nhbd, for short) of a neutrosophic point $x_{\alpha,\beta,\gamma}$ iff there exists a NS $B \in \tau$ such that $x_{\alpha,\beta,\gamma}qB \subseteq A$.

The family consisting of all the Q-neighbourhoods of the NP $x_{\alpha,\beta,\gamma}$ is called the system of Q-neighbourhoods or Q-neighbourhood system of $x_{\alpha,\beta,\gamma}$. This family is denoted by $\mathbf{N_Q}(x_{\alpha,\beta,\gamma})$.

3.7. **Proposition:**

Every neutrosophic open set A in a NTS (X,τ) is a Q-nhbd of every NP quasi-coincident with A.

Proofs: Obvious because for every NP $x_{\alpha,\beta,\gamma}qA$, we have $x_{\alpha,\beta,\gamma}qA \subseteq A$.

3.8. **Properties of Neutrosophic Q-neighbourhoods :**

Let $\mathbf{N_Q}(x_{\alpha,\beta,\gamma})$ be the collection of all Q-neighbourhoods of the NP $x_{\alpha,\beta,\gamma}$ in a NTS (X,τ). Then

N1) $\mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \neq \emptyset$ for every NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$.

N2) $P \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \Rightarrow x_{\alpha,\beta,\gamma}qP$.

N3) $P \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}), P \subseteq Q \Rightarrow Q \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$.

N4) $P \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \Rightarrow$ there exists a $Q \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$ such that $Q \subseteq P$ and $Q \in \mathbf{N_Q}(y_{\alpha/,\beta/,\gamma/})$ for every NP $y_{\alpha/,\beta/,\gamma/}$ quasi-coincident with Q.

Proofs:

N1) Obviously $\tilde{X}$ is a Q-nhbd of every NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$. Thus there exists at least one Q-nhbd for every NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$. Therefore $\mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \neq \emptyset$ for every NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$.

N2) $P \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \Rightarrow P$ is a Q-nhbd of $x_{\alpha,\beta,\gamma} \Rightarrow \exists$ a $S \in \tau$ such that $x_{\alpha,\beta,\gamma}qS \subseteq P$. Therefore $x_{\alpha,\beta,\gamma}qP$.

N3) $P \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \Rightarrow P$ is a Q-nhbd of $x_{\alpha,\beta,\gamma} \Rightarrow \exists$ an open set G such that $x_{\alpha,\beta,\gamma}qG \subseteq P \Rightarrow \exists$ an open set G such that $x_{\alpha,\beta,\gamma}qG \subseteq Q \Rightarrow Q$ is a Q-nhbd of $x_{\alpha,\beta,\gamma} \Rightarrow Q \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$

N4) Since $P \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$, so there exists a τ-open set Q such that $x_{\alpha,\beta,\gamma}qQ \subseteq P$. Since Q is an open set, so $Q \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$. Thus $Q \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$ and $Q \subseteq P$.

Again since Q is an open set, so Q is a Q-nhbd of every NP quasi-coincident with Q. Therefore $Q \in \mathbf{N_Q}(y_{\alpha/,\beta/,\gamma/})$ for every NP $y_{\alpha/,\beta/,\gamma/}$ quasi-coincident with Q .

Hence proved.

3.9. **Characterization of NTS in terms of Neutrosophic Q-neighbourhoods:**

Let X be a non-empty set and let $x \in X$. Let $\mathbf{N_Q}(x_{\alpha,\beta,\gamma})$ be a family of all neutrosophic sets over X satisfying the following conditions :

N1) $P \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \Rightarrow x_{\alpha,\beta,\gamma}qP$.

N2) $P, Q \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \Rightarrow P \cap Q \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$.

N3) $P \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}), P \subseteq Q \Rightarrow Q \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$.

Then there exists a neutrosophic topology τ on X. If, in addition to that, the following condition (N4) is also satisfied then $\mathbf{N_Q}(x_{\alpha,\beta,\gamma})$ is exactly the Q-neighbourhood system of $x_{\alpha,\beta,\gamma}$ in the NTS (X,τ).

N4) $P \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \Rightarrow$ there exists a $Q \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$ such that $Q \subseteq P$ and $Q \in \mathbf{N_Q}(y_{\alpha',\beta',\gamma'})$ for every NP $y_{\alpha',\beta',\gamma'}$ quasi-coincident with Q.

Proof: We define τ as follows :

A NS $G \in \tau$ iff $G \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$ whenever $x_{\alpha,\beta,\gamma} qG$.

We claim that τ is a neutrosophic topology on X.

T1) $\tilde{\emptyset} \in \tau$ as no NP is quasi-coincident with $\tilde{\emptyset}$. By (N3), $\tilde{X} \in \tau$. Thus $\tilde{\emptyset}, \tilde{X} \in \tau$.

T2) Suppose $G_1, G_2 \in \tau$ and $x_{\alpha,\beta,\gamma} q(G_1 \cap G_2)$. Since $x_{\alpha,\beta,\gamma} q(G_1 \cap G_2)$, so $x_{\alpha,\beta,\gamma} qG_1$ and $x_{\alpha,\beta,\gamma} qG_2$. Therefore $G_1, G_2 \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$ and so, by (N2), $G_1 \cap G_2 \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$.

T3) Suppose $\{G_i : i \in\triangle\} \subseteq \tau$ and $x_{\alpha,\beta,\gamma} q(\cup_{i\in\triangle} G_i)$. We show that $\cup\{G_i : i \in\triangle\} \in \tau$. Now $x_{\alpha,\beta,\gamma} q(\cup_{i\in\triangle} G_i) \Rightarrow \exists$ a $j \in\triangle$ such that $x_{\alpha,\beta,\gamma} qG_j \Rightarrow \exists$ a $j \in\triangle$ such that $G_j \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \Rightarrow \cup\{G_i : i \in\triangle\} \in \mathbf{N}(x_{\alpha,\beta,\gamma})$[by (N3)] $\Rightarrow \cup\{G_i : i \in\triangle\} \in \tau$.

Therefore τ is a neutrosophic topology on X.

Let the condition (N4) be satisfied. Suppose that $\mathbf{N^*_Q}(x_{\alpha,\beta,\gamma})$, is the family of all Q-neighbourhoods of the NP $x_{\alpha,\beta,\gamma}$ in (X,τ). We show that $\mathbf{N^*_Q}(x_{\alpha,\beta,\gamma}) = \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$.

Let $N \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$. Then by (N4) there exists a $M \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$ such that $M \subseteq N$ and $M \in \mathbf{N_Q}(y_{\alpha',\beta',\gamma'})$ for every NP $y_{\alpha',\beta',\gamma'}$ quasi-coincident with M. Now $M \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \Rightarrow x_{\alpha,\beta,\gamma} qM$ [by (N1)]. Therefore $M \in \tau$. Thus M is a τ-open set such that $x_{\alpha,\beta,\gamma} qM \subseteq N$. Therefore $N \in \mathbf{N^*_Q}(x_{\alpha,\beta,\gamma})$ and so $\mathbf{N_Q}(x_{\alpha,\beta,\gamma}) \subseteq \mathbf{N^*_Q}(x_{\alpha,\beta,\gamma})$. Conversely let $N \in \mathbf{N^*_Q}(x_{\alpha,\beta,\gamma})$ so that N is a Q-nhbd of $x_{\alpha,\beta,\gamma}$. Then there exists a τ-open set G such that $x_{\alpha,\beta,\gamma} qG \subseteq N$. Therefore $G \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$. But $G \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$ and $G \subseteq N$ together imply by (N3) that $N \in \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$. Therefore $\mathbf{N^*_Q}(x_{\alpha,\beta,\gamma}) \subseteq \mathbf{N_Q}(x_{\alpha,\beta,\gamma})$.Thus $\mathbf{N_Q}(x_{\alpha,\beta,\gamma}) = \mathbf{N^*_Q}(x_{\alpha,\beta,\gamma})$.

Hence proved.

4. **Conclusion**

In this article we have introduced the notion of quasi-coincident relation and established some vital properties based on that. We have also defined the quasi-neighbourhood of a neutrosophic point and studied some properties. At last we have thrown light on the characterization of neutrosophic topological space through the the quasi-neighbourhoods of the neutrosophic points. Hope that the findings in this article will assist the research fraternity to move forward for the development of different aspects of neutrosophic topology.

5. **Conflict of Interest**

We certify that there is no actual or potential conflict of interest in relation to this article.

References

1. Atanassov, K. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20, 87-96.
2. Arar, M. (2020). About Neutrosophic Countably Compactness. Neutrosophic Sets and Systems, 36(1), 246-255.
3. Abdel-Basset, M. ; Gamal, A. ; Chakrabortty, R.K. ; Ryan, M.J. (2021). A new hybrid multi-criteria decision-making approach for location selection of sustainable offshore wind energy stations : A case study. Journal of Cleaner Production, 280, DOI : 10.1016/j.jclepro.2020.124462
4. Abdel-Basset, M. ; Manogaran, G. ; Mohamed, M. (2019). A neutrosophic theory based security approach for fog and mobile-edge computing. Computer Networks, 157, 122-132.
5. Abdel-Basset, M. ; Gamal, A. ; Chakrabortty, R.K. ; Ryan, M.J. (2020). Evaluation of sustainable hydrogen production options using an advanced hybrid MCDM approach : A case study. International Journal of Hydrogen Energy, DOI : 10.1016/j.ijhydene.2020.10.232.
6. Abdel-Basset, M. ; Mohamed, R. ; Smarandache, F. ; Elhoseny, M. (2021). A New Decision-Making Model Based on Plithogenic Set for Supplier Selection. Computers, Materials & Continua, 66(3), 2751-2769.
7. Abdel-Basset, M. ; Gamal, A. ; Manogaran, G. ; Long, H.V. (2020). A novel group decision making model based on neutrosophic sets for heart disease diagnosis. Multimedia Tools and Applications, 79, 9977-10002.
8. Biswas, P. ; Pramanik, S. ; Giri, B.C. (2014). A new methodology for neutrosophic multi-attribute decision making with unknown weight information. Neutrosophic Sets and Systems, 3, 42-52.
9. Coker, D. C. ; Demirci, M. (1995). On intuitionistic fuzzy points. Notes IFS, 1(2), 79-84.
10. Cheng, H.D. ; Guo, Y. (2008). A new neutrosophic approach to image thresholding. New Mathematics and Natural Computation, 4(3), 291-308.
11. Broumi, S. ; Smarandache, F. (2014). On Neutrosophic Implications. Neutrosophic Sets and Systems, 2, 9-17.
12. Das, S. ; Pramanik, S. (2020). Generalized neutrosophic b-open sets in neutrosophic topological space. Neutrosophic Sets and Systems, 38, 235-243.
13. Guo, Y. ; Cheng, H.D. (2009). New neutrosophic approach to image segmentation. Pattern Recognition, 42, 587-595.
14. Ishwarya, P. ; Bageerathi, K. (2016). On Neutrosophic Semi-Open sets in Neutrosophic Topological Spaces. International Journal of Math. Trends and Tech., 37(3), 214-223.
15. Karatas, S. ; Kuru, C. (2016). Neutrosophic Topology. Neutrosophic Sets and Systems, 13(1), 90-95.
16. Lupiáñez, F.G. (2008). On neutrosophic topology. The International Journal of Systems and Cybernetics, 37(6), 797-800.
17. Lupiáñez, F.G. (2009). Interval neutrosophic sets and topology. The International Journal of Systems and Cybernetics, 38(3/4), 621-624.
18. Lupiáñez, F.G. (2009). On various neutrosophic topologies. The International Journal of Systems and Cybernetics, 38(6), 1009-1013.
19. Lupiáñez, F.G. (2010). On neutrosophic paraconsistent topology. The International Journal of Systems and Cybernetics, 39(4), 598-601.
20. Mondal, K. ; Pramanik, S. (2014). A Study on Problems of Hijras in West Bengal Based on Neutrosophic Cognitive Maps. Neutrosophic Sets and Systems, 5, 21-26.
21. Mondal, K. ; Pramanik, S. (2014). Multi-criteria group decision making approach for teacher recruitment in higher education under simplified neutrosophic environment. Neutrosophic Sets and Systems, 6, 28-34.
22. Mondal, K. ; Pramanik, S. (2015). Neutrosophic decision making model of school choice. Neutrosophic Sets and Systems, 7, 62-68.

23. Pramanik, S. ; Chackrabarti, S.N. (2013). A study on problems of construction workers in West Bengal based on neutrosophic cognitive maps. International Journal of Innovative Research in Science, Engineering and Technology, 2(11), 6387-6394.
24. Pramanik, S. ; Roy, T. (2014). Neutrosophic Game Theoretic Approach to Indo-Pak Conflict over Jammu-Kashmir. Neutrosophic Sets and Systems, 2(1), 82-101.
25. Smarandache, F. (1999). A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. American Research Press, Rehoboth, NM.
26. Smarandache, F. (2002). Neutrosophy and neutrosophic logic. First international conference on neutrosophy, neutrosophic logic, set, probability, and statistics, University of New Mexico, Gallup, NM 87301, USA .
27. Smarandache,F. (2005). Neutrosophic set - a generalization of the intuitionistic fuzzy set. International Journal of Pure and Applied Mathematics, 24(3), 287-297.
28. Salama, A.A. ; Alblowi, S. (2012). Generalized neutrosophic set and generalized neutrosophic topological spaces. Computer Science and Engineering, 2(7), 129-132.
29. Salama, A.A. ; Alblowi, S. (2012). Neutrosophic set and Neutrosophic Topological Spaces. IOSR Journal of Mathematics, 3(4), 31-35.
30. Salama, A. A. ; Smarandache, F. ; Kroumov, V. (2014). Closed sets and Neutrosophic Continuous Functions. Neutrosophic Sets and Systems, 4, 4-8.
31. Salma, A.A. ; Smarandache, F. (2015). Neutrosophic Set Theory. The Educational Publisher 415 Columbus, Ohio.
32. Alblowi, S.A. ; Salma, A.A. ; Eisa, M. (2014). New concepts of neutrosophic sets. Int.J. of Math and Comp. Appl. Research, 4(1), 59-66.
33. Saber, Y. ; Alsharari, F. ; Smarandache, F. ; Abdel-Sattar, M. (2020). Connectedness and Stratification of Single-Valued Neutrosophic Topological Spaces. Symmetry, 12, 1464, DOI- 10.3390/sym12091464.
34. Ray, G.C. ; Dey, S. (2021). Neutrosophic point and its neighbourhood structure. Neutrosophic Sets and Systems, 43, 156-168.
35. Wang, H. ; Smarandache, F. ; Zhang, Y.Q. ; Sunderraman, R. (2010). Single valued neutrosophic sets. Multispace Multistruct, 4, 410-413.
36. Ye, S. ; Fu, J. ; Ye, J. (2014). Medical diagnosis using distance based similarity measures of single valued neutrosophic multisets. Neutrosophic Sets and Systems, 7, 47-52.
37. Ye, J. (2014). Single valued neutrosophic cross entropy for multicriteria decision making problems. Applied Mathematical Modeling, 38, 1170-1175.
38. Zadeh, L.A. (1965). Fuzzy sets. Inform. and Control, 8, 338-353.

Received: May 24, 2021. Accepted: October 4, 2021

NSS

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

LR-Type Fully Single-Valued Neutrosophic Linear Programming Problems

Jamil Ahmed

Department of Mathematics, University of Gujrat, Gujrat, Pakistan.

E-mail: jamilsial47401@gmail.com

Abstract: A single-valued neutrosophic set, an generalization of intuitionistic fuzzy set, is a powerful model to deal with uncertainty. In this study we present a method to solve LR-type single-valued neutrosophic linear programming problems by using unrestricted LR-type single-valued neutrosophic numbers. We propose the ranking function to transform LR-type single-valued neutrosophic problems into crisp problems. The arithmetic operations for unrestricted LR-type single-valued neutrosophic numbers are introduced. We propose a method to solve the fully single-valued neutrosophic linear programming problems with equality constraints having LR-type single-valued neutrosophic numbers as right hand sides, parameters and variables. We describe our proposed method by solving real life examples.

Keywords: LR-type single-valued neutrosophic numbers, Ranking function, Arithmetic operations.

1 Introduction

First time in the history, Zadeh [53] introduced the theory of fuzzy sets to handel vagueness. Fuzzy logic and fuzzy sets have been applied to many real life applications. Atanassove [8] gave the concept of intuitionistic fuzzy sets which is an extension of fuzzy set. In intuitionistic fuzzy set, we deal with non-membership function as well as membership function. Intuitionistic fuzzy sets are fail to deal with complete information. Intuitionistic fuzzy sets are not able to handle inconsistent information and indeterminate information which exist commonly in the belief system. Smarandache [46] introduced the concept of neutrosophic set theory. Neutrosophic set is an extension of Intuitionistic fuzzy set, there are three independent membership functions namely truth membership, falsity membership and hesitancy membership function to deal with vague information.

Linear programming is a quantitative tool to allocate optimal allocation available sources between competing procedures. It is among the popular techniques applied to several areas like marketing, production, advertising, finance and distribution and so forth. Many problems of science and engineering are modeled in such a way that information about the situation is vague, imprecise or incomplete. Many scientists have been worked on linear programming (LP) and fuzzy linear programming (FLP). First time Bellman and Zadeh [10] introduced the idea of decision making in fuzzy environment. By using multi-objective function Zimmerman [54] gave a technique to solve LP problem. Behera et al. [9] proposed two new methods to solve FLP problems. They solved two types of problems with two different methods. Kaur and Kumar [24] gave an introduction to fuzzy linear programming problems. Kumar et al. [28] presented a method to solve fully fuzzy linear programming (FFLP) problems. Kaur and Kumar [25] presented a method to find exact fuzzy optimal solution of FFLP problems by using unrestricted fuzzy variables. Najafi and Edalatpanah [38] proposed a better technique to solve FFLP problem than Kumar et al. [28]. Kaur and Kumar [26] proposed Mehar's method for solving FFLP problems with LR fuzzy parameters. Najafi et al. [39] solved a nonlinear model for FFLP by using unrestricted fuzzy numbers. Based on multi objective LP problems and lexicographic method Das et al. [17] proposed a new technique to solve FFLP problem with trapezoidal fuzzy numbers. Allahviranloo et al. [6] solved FFLP problem by using a kind of defuzzification approach. Lotfi et al. [29] considered FFLP problems in which all parameters and variable are

triangular fuzzy numbers. They used the concept of the symmetric triangular fuzzy number and popularized a method to defuzzify a general fuzzy quantity. For solving FFLP problems with inequality constraints Prez-Caedo et al. [43] suggested a revised version of a lexicographical-based method.
Later on, in 1983 Atanassove [8] introduced the concept of intuitionistic fuzzy set which is an extension of fuzzy set. In intuitionistic fuzzy set there is a non-membership function along with membership function. Many researchers have worked at certain techniques to solve LP problems in an intuitionistic fuzzy environment by using intuitionistic fuzzy numbers (IFNs) or LR-type IFNs. In an intuitionistic fuzzy environment Angelov [7] has introduced a new technique to the optimization problem. Singh and Yadav [44] introduced the product of LR-type IFNs and solved LR-type intuitionistic fuzzy linear programming (IFLP) problems. Abhishekh and Nishad [2] proposed a new ranking function to obtain an optimal solution of fully LR-intuitionistic fuzzy transportation problem by using LR-type IFNs. Dubey and Mehra [18] solved LP problems with triangular intuitionistic fuzzy numbers (TIFNs). Nagoorgani and Ponnalagu [36] introduced division of TIFN by using accuracy function, score function, α-cut and β-cut. Edalatpanah [19] designed a model of data envelopment analysis with TIFNs and established a strategy to solve it. Kabiraj et al. [23] solved IFLP problems by using a method based on a method suggested by Zimmermann [54]. Malathi and Umadevi [30] IFLP problems in an intuitionistic fuzzy environment. Prez-Caedo and Concepcin-Morales [42] proposed a method to solve LR-type fully intuitionistic fuzzy linear programming (FIFLP) having inequality constraints in which variables and constrains are unrestricted LR-type IFNs. Pythagorean fuzzy linear programming is an extension of intuitionistic fuzzy linear programming. Akram et al. [4, 5] proposed a method to solve pythagorean fuzzy linear programming problems by using pythagorean fuzzy numbers and LR-type pythagorean fuzzy numbers.
Neutrosophic set is an extension of intuitionistic fuzzy set. In neutrosophic set there are three independent membership functions namely truth membership, falsity membership and hesitancy membership function. Smarandache [46] introduced the concept of neutrosophic set theory. Abdel-Basset et al. [1] suggested a technique to solve the fully neutrosophic linear programming (FNLP) problems. Bera and Mahapatra [12] developed the Big-M simplex method to solve neutrosophic linear programming (NLP) problem. Das and Chakraborty [15] considered a pentagonal NLP problem to solve it. Das and Dash [16] solved NLP problems with mixed constraints. Edalatpanah [20] presented a direct algorithm to solve the linear programming problems. Khalifa et al. [27] solved NLP problem with single-valued trapezoidal neutrosophic numbers. Recently, Ahmad et al. [3] have presented a new method to solve LPP using bipolar single-valued neutrosophic sets.
The main contribution of this article is as follows.

1. We present the concept of LR-type SNN and arithmetic operations of LR-type SNNs by using α-cut, β-cut and γ-cut.
2. We propose the idea of ranking function for LR-type SNNs.
3. We promote a technique to solve FSNLPP with equality constraints in which all the parameters and variables are unrestricted LR-type SNNs.
4. We apply proposed method for solving real life problems.

This paper is arranged as follows: In Section 2, basic preliminaries and arithmetic operations are discussed. In Sections 3, methodology for solving problems are explained. In Section 4, numerical problems are solved. In Section 5, conclusion is given.
For more information, the readers are referred to [11, 13, 14, 21, 22, 31, 32, 33, 34, 35, 40, 41, 45, 46, 47, 48, 49, 50, 51, 52].

2 Preliminaries

Definition 1. [46] Let X be a nonempty set. A SNS $\tilde{B}$ in X is an object having the form

$$\tilde{B} = \{x, T_{\tilde{B}(x)}, I_{\tilde{B}(x)}, F_{\tilde{B}(x)} : x \in X\},$$

where the truth membership function $T_{\tilde{B}(x)} : X \to [0,1]$, indeterminacy membership function $I_{\tilde{B}(x)} : X \to [0,1]$ and the falsity membership function $F_{\tilde{B}(x)} : X \to [0,1]$.

Definition 2. [11] Let $\tilde{B}$ be a SNS in X, then its α-cut, β-cut and γ-cut are defined as $\tilde{B}^{\alpha} = \{x \in X : T(x) \geq \alpha\}$, $\tilde{B}^{\beta} = \{x \in X : I(x) \leq \beta\}$ and $\tilde{B}^{\gamma} = \{x \in X : F(x) \leq \gamma\}$ with $\alpha, \beta, \gamma \in [0,1]$.

Definition 3. *A SNN $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ is defined as an LR-type SNN, if its truth membership $(T_{\tilde{B}}(x))$, Indeterminacy membership $(I_{\tilde{B}}(x))$ and falsity membership $(F_{\tilde{B}}(x))$ functions are defined as:*

$$T_{\tilde{B}}(x) = \begin{cases} L(\frac{b-x}{l}), & x \leq b, l > 0, \\ R(\frac{x-b}{r}), & x \geq b, r > 0, \end{cases}$$

$$I_{\tilde{B}}(x) = \begin{cases} L'(\frac{b-x}{l'}), & x \leq b, l' > 0, \\ R'(\frac{x-b}{r'}), & x \geq b, r' > 0, \end{cases}$$

and

$$F_{\tilde{B}}(x) = \begin{cases} L''(\frac{b-x}{l''}), & x \leq b, l'' > 0, \\ R''(\frac{x-b}{r''}), & x \geq b, r'' > 0, \end{cases}$$

where $l \leq l' \leq l'', r \leq r' \leq r''$, L and R are continues, non-increasing functions on $[0, \infty)$ and $L', R', L'' and R''$ are continuous and non-decreasing functions on $[0, \infty)$ such that

1. $L(0) = R(0) = \chi$,
2. $lim_{x\to\infty} R(x) = lim_{x\to\infty} L(x) = 0$,
3. $L'(0) = R'(0) = \eta$,
4. $lim_{x\to\infty} R'(x) = lim_{x\to\infty} L'(x) = 1$,
5. $L''(0) = R''(0) = \zeta$,
6. $lim_{x\to\infty} R''(x) = lim_{x\to\infty} L''(x) = 1$,

b is called the mean value of $\tilde{B}$, l and r are the left and right spreads of $(T_{\tilde{B}}(x))$, l' and r' are the left and right spreads of $(I_{\tilde{B}}(x))$ and l'' and r'' are the left and right spreads of $(F_{\tilde{B}}(x))$, respectively.

Remark

If we set $L(x) = R(x) = max\{0, \chi - x\}, L'(x) = R'(x) = min\{1, \eta + x\}$ and $L''(x) = R''(x) = min\{1, \zeta + x\}$ then $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ becomes LR-type triangular single-valued neutrosophic number.

$$L(x) = R(x) = \begin{cases} \chi - x, & 0 \leq x \leq \chi, \\ 0, & otherwise, \end{cases}$$

$$L'(x) = R'(x) = \begin{cases} \eta + x, & \eta \leq x \leq 1, \\ 1, & otherwise, \end{cases}$$

$$L''(x) = R''(x) = \begin{cases} \zeta + x, & \zeta \leq x \leq 1, \\ 1, & otherwise, \end{cases}$$

$\chi, \eta, \zeta \in [0,1]$.

Definition 4. Based on [44], An LR-type SNN $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ is non-negative, if $b - l'' \geq 0$ and denoted as $\tilde{B} \geq 0$.

Definition 5. Based on [44], An LR-type SNN $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ is non-positive, if $b + r'' < 0$.

Definition 6. Based on [44], An LR-type SNN $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ is unrestricted, if b is any real number.

Theorem 7. *Let $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ be an LR-type SNN, then its α-cut, β-cut and γ-cut are $\tilde{B}^{\alpha} = [b - lL^{-1}(\alpha), b + rR^{-1}(\alpha)]$, $\tilde{B}^{\beta} = [b - l'L^{'-1}(\beta), b + r'R^{'-1}(\beta)]$ and $\tilde{B}^{\gamma} = [b - l''L^{''-1}(\gamma), b + r''R^{''-1}(\gamma)]$, with $\alpha, \beta, \gamma \in [0, 1]$.*
Proof. By using the Definition 2, the theorem can be proved easily.

Definition 8. *Let $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ be an LR-type SNN, then ranking of $\tilde{B}$, denoted $\Re(\tilde{B})$, can be defined as*
$\Re(\tilde{B}) = \frac{1}{6}[(\int_0^{\chi} b - lL^{-1}(\alpha))d\alpha + (\int_0^{\chi} b + rR^{-1}(\alpha))d\alpha + (\int_{\eta}^{1} b - l'L^{'-1}(\beta))d\beta + (\int_{\eta}^{1} b + r'R^{'-1}(\beta))d\beta + (\int_{\zeta}^{1} b - l''L^{''-1}(\gamma))d\gamma + (\int_{\zeta}^{1} b + r''R^{''-1}(\gamma))d\gamma]$.
Let $\tilde{B}_1$ and $\tilde{B}_2$ be two LR-type SNNs,

- $\tilde{B}_1 \prec \tilde{B}_2$ *if* $\Re(\tilde{B}_1) < \Re(\tilde{B}_1)$,
- $\tilde{B}_1 \succ \tilde{B}_2$ *if* $\Re(\tilde{B}_1) > \Re(\tilde{B}_1)$,
- $\tilde{B}_1 \approx \tilde{B}_2$ *if* $\Re(\tilde{B}_1) = \Re(\tilde{B}_1)$.

2.1 Arithmetic Operations

Theorem 9. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be two LR-type SNNs, then $\tilde{B}_1 \oplus \tilde{B}_2 = ([b_1 + b_2; l_1 + l_2, r_1 + r_2; l'_1 + l'_2, r'_1 + r'_2; l''_1 + l''_2, r''_1 + r''_2]; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)_{LR}$*
proof. Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be two LR-type SNNs, then their α-cut, β-cut and γ-cut are given as;
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l'_1L^{'-1}(\beta), b_1 + r'_1R^{'-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l'_2L^{'-1}(\beta), b_2 + r'_2R^{'-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l''_1L^{''-1}(\gamma), b_1 + r''_1R^{''-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l''_2L^{''-1}(\gamma), b_2 + r''_2R^{''-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha} + \tilde{B}_2^{\alpha} = [b_1 - l_1L^{-1}(\alpha) + b_2 - l_2L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha) + b_2 + r_2R^{-1}(\alpha)]. \tag{1}$$

By taking $\alpha = \chi$ in equation (1), we have

$$(\tilde{B}_1 + \tilde{B}_2)^{\alpha=\chi} = b_1 + b_2. \tag{2}$$

By taking $\alpha = 0$ in equation (1), we have

$$(\tilde{B}_1 + \tilde{B}_2)^{\alpha=0} = [b_1 + b_2 - l_1 - l_2, b_1 + b_2 + r_1 + r_2]. \tag{3}$$

Now

$$\tilde{B}_1^{\beta} + \tilde{B}_2^{\beta} = [b_1 - l'_1L^{'-1}(\beta) + b_2 - l'_2L^{'-1}(\beta), b_1 + r'_1R^{'-1}(\beta) + b_2 + r'_2R^{'-1}(\beta)]. \tag{4}$$

By taking $\beta = \eta$ in equation (4), we have

$$(\tilde{B}_1 + \tilde{B}_2)^{\beta=\eta} = b_1 + b_2. \tag{5}$$

By taking $\beta = 1$ in equation (4), we have

$$(\tilde{B}_1 + \tilde{B}_2)^{\beta=1} = [b_1 + b_2 - l_1' - l_2', b_1 + b_2 + r_1' + r_2']. \tag{6}$$

Further,

$$\tilde{B}_1^{\gamma} + \tilde{B}_2^{\gamma} = [b_1 - l_1'' L^{''-1}(\gamma) + b_2 - l_2'' L^{''-1}(\gamma), b_1 + r_1'' R^{''-1}(\gamma) + b_2 + r_2'' R^{''-1}(\gamma)]. \tag{7}$$

By taking $\gamma = \zeta$ in equation (7), we have

$$(\tilde{B}_1 + \tilde{B}_2)^{\gamma=\zeta} = b_1 + b_2. \tag{8}$$

By taking $\gamma = 1$ in equation (7), we have

$$(\tilde{B}_1 + \tilde{B}_2)^{\gamma=1} = [b_1 + b_2 - l_1'' - l_2'', b_1 + b_2 + r_1'' + r_2'']. \tag{9}$$

By combining the equations (2),(3),(5),(6),(8), and (9), the result follows.

Theorem 10. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be two LR-type SNNs, then $\tilde{B}_1 \ominus \tilde{B}_2 = ([b_1 - b_2; l_1 - r_2, r_1 - l_2; l_1' - r_2', r_1' - l_2'; l_1'' - r_2'', r_1'' - l_2'']; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)_{LR}$*
proof. Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be two LR-type SNNs, then their α-cut, β-cut and γ-cut are given as;
$\tilde{B}_1^{\alpha} = [b_1 - l_1 L^{-1}(\alpha), b_1 + r_1 R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2 L^{-1}(\alpha), b_2 + r_2 R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l_1' L^{'-1}(\beta), b_1 + r_1' R^{'-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l_2' L^{'-1}(\beta), b_2 + r_2' R^{'-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l_1'' L^{''-1}(\gamma), b_1 + r_1'' R^{''-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l_2'' L^{''-1}(\gamma), b_2 + r_2'' R^{''-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha} - \tilde{B}_2^{\alpha} = [b_1 - l_1 L^{-1}(\alpha) - b_2 - r_2 R^{-1}(\alpha), b_1 + r_1 R^{-1}(\alpha) - b_2 + l_2 L^{-1}(\alpha)]. \tag{10}$$

By taking $\alpha = \chi$ in equation (10), we have

$$(\tilde{B}_1 - \tilde{B}_2)^{\alpha=\chi} = b_1 - b_2. \tag{11}$$

By taking $\alpha = 0$ in equation (10), we have

$$(\tilde{B}_1 - \tilde{B}_2)^{\alpha=0} = [b_1 - b_2 - l_1 - r_2, b_1 - b_2 + r_1 + l_2]. \tag{12}$$

Now

$$\tilde{B}_1^{\beta} - \tilde{B}_2^{\beta} = [b_1 - l_1' L^{'-1}(\beta) - b_2 - r_2' R^{'-1}(\beta), b_1 + r_1' R^{'-1}(\beta) - b_2 + l_2' L^{'-1}(\beta)]. \tag{13}$$

By taking $\beta = \eta$ in equation (13), we have

$$(\tilde{B}_1 - \tilde{B}_2)^{\beta=\eta} = b_1 - b_2. \tag{14}$$

By taking $\beta = 1$ in equation (13), we have

$$(\tilde{B}_1 - \tilde{B}_2)^{\beta=1} = [b_1 - b_2 - l_1' - r_2', b_1 - b_2 + r_1' + l_2']. \tag{15}$$

Further,

$$\tilde{B}_1^{\gamma} - \tilde{B}_2^{\gamma} = [b_1 - l_1'' L^{''-1}(\gamma) - b_2 - r_2'' L^{''-1}(\gamma), b_1 + r_1'' R^{''-1}(\gamma) - b_2 + l_2'' R^{''-1}(\gamma)]. \tag{16}$$

By taking $\gamma = \zeta$ *in equation (16), we have*

$$(\tilde{B}_1 - \tilde{B}_2)^{\gamma=\zeta} = b_1 - b_2. \tag{17}$$

By taking $\gamma = 1$ *in equation (16), we have*

$$(\tilde{B}_1 - \tilde{B}_2)^{\gamma=1} = [b_1 - b_2 - l''_1 - r''_2, b_1 - b_2 + r''_1 + l''_2]. \tag{18}$$

By combining the equations (11),(12),(14),(15),(17), and (18), the result follows.

Theorem 11. *Let* $\tilde{B} = ([b; l, r; l', r'1; l'', r'']; \chi, \eta, \zeta)_{LR}$ *be an* LR*-type SNN and c be any arbitrary real number, then*

$$c\tilde{B} = \begin{cases} (cb; cl, cr; cl', cr'; cl'', cr''), & c \geq 0, \\ (cb; -cr, -cl; -cr', -cl'; -cr'', -cl''), & c < 0. \end{cases}$$

Proof. Let $\tilde{B} = ([b; l, r; l', r'1; l'', r'']; \chi, \eta, \zeta)_{LR}$ *be an* LR*-type SNN and c be any arbitrary real number, then* $\tilde{B}^{\alpha} = [b - lL^{-1}(\alpha), b + rR^{-1}(\alpha)]$, $\tilde{B}^{\beta} = [b - l'L'^{-1}(\beta), b + r'R'^{-1}(\beta)]$, $\tilde{B}^{\gamma} = [b - l''L''^{-1}(\gamma), b + r''R''^{-1}(\gamma)]$.
Now, if $c \geq 0$, *then*

$$c\tilde{B}^{\alpha} = [cb - clL^{-1}(\alpha), cb + crR^{-1}(\alpha)]. \tag{19}$$

By taking $\alpha = \chi$ *in equation (19), we have*

$$c\tilde{B}^{\alpha=\chi} = cb. \tag{20}$$

By taking $\alpha = 0$ *in equation (19), we have*

$$c\tilde{B}^{\alpha=0} = [cb - cl, cb + cr]. \tag{21}$$

Also,

$$c\tilde{B}^{\beta} = [cb - cl'L'^{-1}(\beta), cb + cr'R'^{-1}(\beta)]. \tag{22}$$

By taking $\beta = \eta$ *in equation (22), we have*

$$c\tilde{B}^{\beta=\eta} = cb. \tag{23}$$

By taking $\beta = 1$ *in equation (22), we have*

$$c\tilde{B}^{\beta=1} = [cb - cl', cb + cr']. \tag{24}$$

Further,

$$c\tilde{B}^{\gamma} = [cb - cl''L''^{-1}(\gamma), cb + cr''R''^{-1}(\gamma)]. \tag{25}$$

By taking $\gamma = \zeta$ *in equation (25), we have*

$$c\tilde{B}^{\gamma=\zeta} = cb. \tag{26}$$

By taking $\gamma = 1$ *in equation (25), we have*

$$c\tilde{B}^{\gamma=1} = [cb - cl'', cb + cr'']. \tag{27}$$

By combining the equations (20),(21),(23),(24),(26), and (27), the case $c \geq 0$ follows.
If c¡0, then

$$c\tilde{B}^{\alpha} = [cb + crR^{-1}(\alpha), cb - clL^{-1}(\alpha)]. \tag{28}$$

By taking $\alpha = \chi$ in equation (28), we have

$$c\tilde{B}^{\alpha=\chi} = cb. \tag{29}$$

By taking $\alpha = 0$ in equation (28), we have

$$c\tilde{B}^{\alpha=0} = [cb + cr, cb - cl]. \tag{30}$$

Also,

$$c\tilde{B}^{\beta} = [cb + cr'R'^{-1}(\beta), cb - cl'L'^{-1}(\beta)]. \tag{31}$$

By taking $\beta = \eta$ in equation (31), we have

$$c\tilde{B}^{\beta=\eta} = cb. \tag{32}$$

By taking $\beta = 1$ in equation (31), we have

$$c\tilde{B}^{\beta=1} = [cb + cr', cb - cl']. \tag{33}$$

Further,

$$c\tilde{B}^{\gamma} = [cb + cr''R''^{-1}(\gamma), cb - cl''L''^{-1}(\gamma)]. \tag{34}$$

By taking $\gamma = \zeta$ in equation (34), we have

$$c\tilde{B}^{\gamma=\zeta} = cb. \tag{35}$$

By taking $\gamma = 1$ in equation (34), we have

$$c\tilde{B}^{\gamma=1} = [cb + cr'', cb - cl'']. \tag{36}$$

On combining the equations (29),(30),(32),(33),(35), and (36), the case c¡0 follows. thus proof completed.

Theorem 12. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be two non-negative LR-type SNNs, then*
$\tilde{B}_1 \otimes \tilde{B}_2 = ([b_1b_2; b_1l_2 + b_2l_1 - l_1l_2, b_1r_2 + b_2r_1 + r_1r_2; b_1l'_2 + b_2l'_1 - l'_1l'_2, b_1r'_2 + b_2r'_1 + r'_1r'_2; b_1l''_2 + b_2l''_1 - l''_1l''_2, b_1r''_2 + b_2r''_1 + r''_1r''_2]; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)$.
Proof. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be two non-negative LR-type SNNs, their $\alpha-$cut, $\beta-$cut and $\gamma-$cut are given as:*
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l'_1L'^{-1}(\beta), b_1 + r'_1R'^{-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l'_2L'^{-1}(\beta), b_2 + r'_2R'^{-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l''_1L''^{-1}(\gamma), b_1 + r''_1R''^{-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l''_2L''^{-1}(\gamma), b_2 + r''_2R''^{-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [(b_1 - l_1L^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha)), (b_1 + r_1R^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha))]. \tag{37}$$

By taking $\alpha = \chi$ in equation (37), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=\chi} = b_1 b_2. \tag{38}$$

By taking $\alpha = 0$ in equation (37), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=0} = [b_1b_2 - b_1l_2 - b_2l_1 + l_1l_2, b_1b_2 + b_1r_2 + b_2r_1 + r_1r_2]. \tag{39}$$

Now

$$\tilde{B}_1^{\beta}\tilde{B}_2^{\beta} = [(b_1 - l_1'L^{'-1}(\beta))(b_2 - l_2'L^{'-1}(\beta)), (b_1 + r_1'R^{'-1}(\beta))(b_2 + r_2'R^{'-1}(\beta))]. \tag{40}$$

By taking $\beta = \eta$ in equation (40), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=\eta} = b_1 b_2. \tag{41}$$

By taking $\beta = 1$ in equation (40), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=1} = [b_1b_2 - b_1l_2' - b_2l_1' + l_1'l_2', b_1b_2 + b_1r_2' + b_2r_1' + r_1'r_2']. \tag{42}$$

Also

$$\tilde{B}_1^{\gamma}\tilde{B}_2^{\gamma} = [(b_1 - l_1''L^{''-1}(\gamma))(b_2 - l_2''L^{''-1}(\gamma)), (b_1 + r_1''R^{''-1}(\gamma))(b_2 + r_2''R^{''-1}(\gamma))]. \tag{43}$$

By taking $\gamma = \zeta$ in equation (43), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=\zeta} = b_1 b_2. \tag{44}$$

By taking $\gamma = 1$ in equation (43), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=1} = [b_1b_2 - b_1l_2'' - b_2l_1'' + l_1''l_2'', b_1b_2 + b_1r_2'' + b_2r_1'' + r_1''r_2'']. \tag{45}$$

By combining the equations (38),(39),(41),(42),(44), and (45), the result follows.

Theorem 13. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ be non-negative LR-type SNN, and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be non-positive LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b_1b_2; b_1l_2 - b_2r_1 + l_2r_1, b_1r_2 - b_2l_1 - l_1r_2; b_1l_2' - b_2r_1' + l_2'r_1', b_1r_2' - b_2l_1' - l_1'r_2'; b_1l_2'' - b_2r_1'' + l_2''r_1'', b_1r_2'' - b_2l_1'' - l_1''r_2'']; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)_{LR}$.*

Proof. Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ be non-negative LR-type SNN, and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be non-positive LR-type SNN, their $\alpha-$cut, $\beta-$cut and $\gamma-$cut are given as:
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l_1'L^{'-1}(\beta), b_1 + r_1'R^{'-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l_2'L^{'-1}(\beta), b_2 + r_2'R^{'-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l_1''L^{''-1}(\gamma), b_1 + r_1''R^{''-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l_2''L^{''-1}(\gamma), b_2 + r_2''R^{''-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [(b_1 + r_1R^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha)), (b_1 - l_1L^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha))]. \tag{46}$$

By taking $\alpha = \chi$ in equation (46), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=\chi} = b_1 b_2. \tag{47}$$

By taking $\alpha = 0$ in equation (46), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=0} = [b_1b_2 - b_1l_2 + b_2r_1 - l_2r_1, b_1b_2 + b_1r_2 - b_2l_1 - l_1r_2]. \tag{48}$$

Now

$$\tilde{B}_1^{\beta}\tilde{B}_2^{\beta} = [(b_1 + r_1'R'^{-1}(\beta))(b_2 - l_2'L'^{-1}(\beta)), (b_1 - l_1'L'^{-1}(\beta))(b_2 + r_2'R'^{-1}(\beta))]. \tag{49}$$

By taking $\beta = \eta$ in equation (49), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=\eta} = b_1b_2. \tag{50}$$

By taking $\beta = 1$ in equation (49), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=1} = [b_1b_2 - b_1l_2' + b_2r_1' - l_2'r_1', b_1b_2 + b_1r_2' - b_2l_1' - l_1'r_2']. \tag{51}$$

Also

$$\tilde{B}_1^{\gamma}\tilde{B}_2^{\gamma} = [(b_1 + r_1''R''^{-1}(\gamma))(b_2 - l_2''L''^{-1}(\gamma)), (b_1 - l_1''L''^{-1}(\gamma))(b_2 + r_2''R''^{-1}(\gamma))]. \tag{52}$$

By taking $\gamma = \zeta$ in equation (52), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=\zeta} = b_1b_2. \tag{53}$$

By taking $\gamma = 1$ in equation (52), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=1} = [b_1b_2 - b_1l_2'' + b_2r_1'' - l_2''r_1'', b_1b_2 + b_1r_2'' - b_2l_1'' - l_1''r_2'']. \tag{54}$$

By combining the equations (47),(48),(50),(51),(53), and (54), the result follows.

Theorem 14. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ be non-positive LR-type SNN, and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be non-negative LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b_1b_2; b_2l_1 - b_1r_2 + l_1r_2, -b_1l_2 + b_2r_1 - l_2r_1; b_2l_1' - b_1r_2' + l_1'r_2', -b_1l_2' + b_2r_1' - l_2'r_1'; b_2l_1'' - b_1r_2'' + l_1''r_2'', -b_1l_2'' + b_2r_1'' - l_2''r_1'']; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)_{LR}$.*
Proof. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ be non-positive LR-type SNN, and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be non-negative LR-type SNN, their $\alpha-$cut, $\beta-$cut and $\gamma-$cut are given as:*
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l_1'L'^{-1}(\beta), b_1 + r_1'R'^{-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l_2'L'^{-1}(\beta), b_2 + r_2'R'^{-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l_1''L''^{-1}(\gamma), b_1 + r_1''R''^{-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l_2''L''^{-1}(\gamma), b_2 + r_2''R''^{-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [(b_1 - l_1L^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha)), (b_1 + r_1R^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha))]. \tag{55}$$

By taking $\alpha = \chi$ in equation (55), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=\chi} = b_1b_2. \tag{56}$$

By taking $\alpha = 0$ in equation (55), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=0} = [b_1b_2 - b_2l_1 + b_1r_2 - l_1r_2, b_1b_2 - b_1l_2 + b_2r_1 - l_2r_1]. \tag{57}$$

Now

$$\tilde{B}_1^{\beta}\tilde{B}_2^{\beta} = [(b_1 - l_1'L'^{-1}(\beta))(b_2 + r_2'R'^{-1}(\beta)), (b_1 + r_1'R'^{-1}(\beta))(b_2 - l_2'L'^{-1}(\beta))]. \tag{58}$$

By taking $\beta = \eta$ in equation (58), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=\eta} = b_1b_2. \tag{59}$$

By taking $\beta = 1$ *in equation (58), we have*

$$(\tilde{B}_1\tilde{B}_2)^{\beta=1} = [b_1b_2 - b_2l'_1 + b_1r'_2 - l'_1r'_2, b_1b_2 - b_1l'_2 + b_2r'_1 - l'_2r'_1]. \tag{60}$$

Also

$$\tilde{B}_1^{\gamma}\tilde{B}_2^{\gamma} = [(b_1 - l''_1L''^{-1}(\gamma))(b_2 + r''_2R''^{-1}(\gamma)), (b_1 + r''_1R''^{-1}(\gamma))(b_2 - l''_2L''^{-1}(\gamma))]. \tag{61}$$

By taking $\gamma = \zeta$ *in equation (61), we have*

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=\zeta} = b_1b_2. \tag{62}$$

By taking $\gamma = 1$ *in equation (61), we have*

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=1} = [b_1b_2 - b_2l''_1 + b_1r''_2 - l''_1r''_2, b_1b_2 - b_1l''_2 + b_2r''_1 - l''_2r''_1]. \tag{63}$$

By combining the equations (56),(57),(59),(60),(62), and (63), the result follows.

Theorem 15. *Let* $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ *and* $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *be two non-positive LR-type SNNs, then* $\tilde{B}_1 \otimes \tilde{B}_2 = ([b_1b_2; -b_1r_2 - b_2r_1 - r_1r_2, -b_1l_2 - b_2l_1 + l_1l_2; -b_1r'_2 - b_2r'_1 - r'_1r'_2, -b_1l'_2 - b_2l'_1 + l'_1l'_2; -b_1r''_2 - b_2r''_1 - r''_1r''_2, -b_1l''_2 - b_2l''_1 + l''_1l''_2]; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)$.

Proof. *Let* $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ *and* $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *be two non-negative LR-type SNNs, their* $\alpha-$*cut,* $\beta-$*cut and* $\gamma-$*cut are given as:*
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l'_1L'^{-1}(\beta), b_1 + r'_1R'^{-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l'_2L'^{-1}(\beta), b_2 + r'_2R'^{-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l''_1L''^{-1}(\gamma), b_1 + r''_1R''^{-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l''_2L''^{-1}(\gamma), b_2 + r''_2R''^{-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [(b_1 + r_1R^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha)), (b_1 - l_1L^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha))]. \tag{64}$$

By taking $\alpha = \chi$ *in equation (64), we have*

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=\chi} = b_1b_2. \tag{65}$$

By taking $\alpha = 0$ *in equation (64), we have*

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=0} = [b_1b_2 + b_1r_2 + b_2r_1 + r_1r_2, b_1b_2 - b_1l_2 - b_2l_1 + l_1l_2]. \tag{66}$$

Now

$$\tilde{B}_1^{\beta}\tilde{B}_2^{\beta} = [(b_1 + r'_1R'^{-1}(\beta))(b_2 + r'_2R'^{-1}(\beta)), (b_1 - l'_1L'^{-1}(\beta))(b_2 - l'_2L'^{-1}(\beta))]. \tag{67}$$

By taking $\beta = \eta$ *in equation (67), we have*

$$(\tilde{B}_1\tilde{B}_2)^{\beta=\eta} = b_1b_2. \tag{68}$$

By taking $\beta = 1$ *in equation (67), we have*

$$(\tilde{B}_1\tilde{B}_2)^{\beta=1} = [b_1b_2 + b_1r'_2 + b_2r'_1 + r'_1r'_2, b_1b_2 - b_1l'_2 - b_2l'_1 + l'_1l'_2]. \tag{69}$$

Also

$$\tilde{B}_1^{\gamma}\tilde{B}_2^{\gamma} = [(b_1 + r''_1R''^{-1}(\gamma))(b_2 + r''_2R''^{-1}(\gamma)), (b_1 - l''_1L''^{-1}(\gamma))(b_2 - l''_2L''^{-1}(\gamma))]. \tag{70}$$

By taking $\gamma = \zeta$ in equation (70), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=\zeta} = b_1b_2. \tag{71}$$

By taking $\gamma = 1$ in equation (70), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=1} = [b_1b_2 + b_1r_2'' + b_2r_1'' + r_1''r_2'', b_1b_2 - b_1l_2'' - b_2l_1'' + l_1''l_2'']. \tag{72}$$

By combining the equations (65),(66),(68),(69),(71), and (72), the result follows.

Theorem 16. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN in which $b_1 - l_1'' < 0$, $b_1 - l_1' \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, where $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2, b_1b_2 - l_2b_1 + r_1b_2 - l_2r_1\}, r = max\{b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2, b_1b_2 + r_2b_1 - l_1b_2 - l_1r_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l_2'b_1 - l_1'b_2 + l_1'l_2', b_1b_2 - l_2'b_1 + r_1'b_2 - l_2'r_1'\}, r' = max\{b_1b_2 + r_2'b_1 + r_1'b_2 + r_1'r_2', b_1b_2 + r_2'b_1 - l_1'b_2 - l_1'r_2'\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l_1''b_2 + r_2''b_1 - l_1''r_2'', b_1b_2 + r_1''b_2 - l_2''b_1 - l_2''r_1''\}$ and $r'' = max\{b_1b_2 - l_1''b_2 - l_2''b_1 + l_1''l_2'', b_1b_2 + r_1''b_2 + r_2''b_1 + r_1''r_2''\} - b_1b_2$.*
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.
Proof. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN such that $b_1 - l_1'' < 0$, $b_1 - l_1' \geq 0$, $b_1 - l_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, and their $\alpha-$cut, $\beta-$cut and $\gamma-$cut are given as:*
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l_1'L'^{-1}(\beta), b_1 + r_1'R'^{-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l_2'L'^{-1}(\beta), b_2 + r_2'R'^{-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l_1''L''^{-1}(\gamma), b_1 + r_1''R''^{-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l_2''L''^{-1}(\gamma), b_2 + r_2''R''^{-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [min\{(b_1 - l_1L^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha)), (b_1 + r_1R^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha))\},$$

$$max\{(b_1 + r_1R^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha)), (b_1 - l_1L^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha))\}]. \tag{73}$$

By taking $\alpha = \chi$ in equation (73), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=\chi} = b_1b_2. \tag{74}$$

By taking $\alpha = 0$ in equation (73), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=0} = [min\{b_1b_2 - b_1l_2 - b_2l_1 + l_1l_2, b_1b_2 - b_1l_2 + b_2r_1 - l_2r_1\},$$

$$max\{b_1b_2 + b_1r_2 + b_2r_1 + r_1r_2, b_1b_2 + b_1r_2 - b_2l_1 - l_1r_2\}]. \tag{75}$$

Now

$$\tilde{B}_1^{\beta}\tilde{B}_2^{\beta} = [min\{(b_1 - l_1'L'^{-1}(\beta))(b_2 - l_2'L'^{-1}(\beta)), (b_1 + r_1'R'^{-1}(\beta))(b_2 + r_2'R'^{-1}(\beta))\},$$

$$max\{(b_1 + r_1'R'^{-1}(\beta))(b_2 + r_2'R'^{-1}(\beta)), \{(b_1 - l_1'L'^{-1}(\beta))(b_2 - l_2'L'^{-1}(\beta))\}]. \tag{76}$$

By taking $\beta = \eta$ in equation (76), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=\eta} = b_1b_2. \tag{77}$$

By taking $\beta = 1$ in equation (76), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=1} = [min\{b_1b_2 - b_1l'_2 - b_2l'_1 + l'_1l'_2, b_1b_2 - b_1l'_2 + b_2r'_1 - l'_2r'_1\},$$

$$max\{b_1b_2 + b_1r'_2 + b_2r'_1 + r'_1r'_2, b_1b_2 + b_1r'_2 - b_2l'_1 - l'_1r'_2\}]. \quad (78)$$

Also

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [min\{(b_1 - l''_1L''^{-1}(\gamma))(b_2 + r''_2R''^{-1}(\gamma)), (b_1 + r''_1R''^{-1}(\gamma))(b_2 - l''_2L''^{-1}(\gamma))\},$$

$$max\{\{(b_1 - l''_1L''^{-1}(\gamma))(b_2 - l''_2L''^{-1}(\gamma)), (b_1 + r''_1R''^{-1}(\gamma))(b_2 + r''_2R''^{-1}(\gamma))\}]. \quad (79)$$

By taking $\gamma = \zeta$ in equation (79), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=\zeta} = b_1b_2. \quad (80)$$

By taking $\gamma = 1$ in equation (79), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=1} = [min\{b_1b_2 + b_1r''_2 - b_2l''_1 - l''_1r''_2, b_1b_2 - b_1l''_2 + b_2r''_1 - l''_2r''_1\},$$

$$max\{b_1b_2 - b_1l''_2 - b_2l''_1 + l''_1l''_2, b_1b_2 + b_1r''_2 + b_2r''_1 + r''_1r''_2\}]. \quad (81)$$

By combining the equations (74),(75),(77),(78),(80), and (81), the result follows.
By using similar method as used in the above theorem the following theorems can be proved easily.

Theorem 17. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN in which $b_1 - l'_1 < 0$, $b_1 - l_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, where $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2, b_1b_2 - l_2b_1 + r_1b_2 - l_2r_1\}, r = max\{b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2, b_1b_2 + r_2b_1 - l_1b_2 - l_1r_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1\}, r' = max\{b_1b_2 - l'_1b_2 - l'_2b_1 + l'_1l'_2, b_1b_2 + r'_1b_2 + r'_2b_1 + r'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ and $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2$.*
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.

Theorem 18. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN in which $b_1 - l_1 < 0$, $b_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, where $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_1b_2 + r_2b_1 - l_1r_2, b_1b_2 + r_1b_2 - l_2b_1 - l_2r_1\}, r = max\{b_1b_2 - l_1b_2 - l_2b_1 + l_1l_2, b_1b_2 + r_1b_2 + r_2b_1 + r_1r_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1\}, r' = max\{b_1b_2 - l'_1b_2 - l'_2b_1 + l'_1l'_2, b_1b_2 + r'_1b_2 + r'_2b_1 + r'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ and $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2$.*
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.

Theorem 19. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN in which $b_1 < 0$, $b_1 + r_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, where $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_1b_2 + r_2b_1 - l_1r_2, b_1b_2 + r_1b_2 - l_2b_1 - l_2r_1\}, r = max\{b_1b_2 - l_1b_2 - l_2b_1 + l_1l_2, b_1b_2 + r_1b_2 + r_2b_1 + r_1r_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1\}, r' = max\{b_1b_2 - l'_1b_2 - l'_2b_1 + l'_1l'_2, b_1b_2 + r'_1b_2 + r'_2b_1 + r'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ and $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2$.*
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.

Theorem 20. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN in which $b_1 + r_1 < 0$, $b_1 + r'_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$,*

where $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_1b_2 + r_2b_1 - l_1r_2, b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2\}, r = max\{b_1b_2 + r_1b_2 - l_2b_1 - l_2r_1, b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1\}, r' = max\{b_1b_2 - l'_1b_2 - l'_2b_1 + l'_1l'_2, b_1b_2 + r'_1b_2 + r'_2b_1 + r'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ *and* $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2$.
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.

Theorem 21. *Let* $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ *be an* LR*-type SNN in which* $b_1 + r'_1 < 0$, $b_1 + r''_1 \geq 0$, *and* $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *be an unrestricted* LR*-type SNN, then* $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, *where* $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_1b_2 + r_2b_1 - l_1r_2, b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2\}, r = max\{b_1b_2 + r_1b_2 - l_2b_1 - l_2r_1, b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_2b_1 + r'_1b_2 + r'_1r'_2\}, r' = max\{b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1, b_1b_2 - l_2b_1 - l'_1b_2 + l'_1l'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ *and* $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2$.
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.

Theorem 22. *Let* $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ *be an* LR*-type SNN in which* $b_1 + r''_1 < 0$, *and* $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *be an unrestricted* LR*-type SNN, then* $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, *where* $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_1b_2 + r_2b_1 - l_1r_2, b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2\}, r = max\{b_1b_2 + r_1b_2 - l_2b_1 - l_2r_1, b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_2b_1 + r'_1b_2 + r'_1r'_2\}, r' = max\{b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1, b_1b_2 - l_2b_1 - l'_1b_2 + l'_1l'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_2b_1 + r''_1b_2 + r''_1r''_2\}$ *and* $r'' = max\{b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1, b_1b_2 - l_2b_1 - l''_1b_2 + l''_1l''_2\} - b_1b_2$.
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.

Theorem 23. *Let* $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ *be an* LR*-type SNN in which* $b_1 - l''_1 \geq 0$, *and* $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *be an unrestricted* LR*-type SNN, then* $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, *where* $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2, b_1b_2 - l_2b_1 + r_1b_2 - l_2r_1\}, r = max\{b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2, b_1b_2 + r_2b_1 - l_1b_2 - l_1r_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_2b_1 - l'_1b_2 + l'_1l'_2, b_1b_2 - l'_2b_1 + r'_1b_2 - l'_2r'_1\}, r' = max\{b_1b_2 + r'_2b_1 + r'_1b_2 + r'_1r'_2, b_1b_2 + r'_2b_1 - l'_1b_2 - l'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_2b_1 - l''_1b_2 + l''_1l''_2, b_1b_2 - l''_2b_1 + r''_1b_2 - l''_2r''_1\}$ *and* $r'' = max\{b_1b_2 + r''_2b_1 + r''_1b_2 + r''_1r''_2, b_1b_2 + r''_2b_1 - l''_1b_2 - l''_1r''_2\} - b_1b_2$.
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.

3 Methodology

In this section, a new method is presented to find the single-valued neutrosophic optimal solution of FSNLP problems with equality constraints, in which all the parameters are represented by LR-type SNNs.

$$\text{Maximize/ Minimize } \sum_{j=1}^{n} C_j \otimes X_j; \tag{82}$$

subject to

$$\sum_{j=1}^{n} A_{ij} \otimes X_j = B_i, \forall i = 1, 2, 3, \cdots, m.$$

where C_j, A_{ij}, B_i and X_j are LR-type SNNs.
Step 1. Assuming $C_j = ([c_j; p_j, q_j; p'_j, q'_j; p''_j, q''_j]; \chi_j, \eta_j, \zeta_j)_{LR}, X_j = ([x_j; y_j, z_j; y'_j, z'_j; y''_j, z''_j]; \phi_j, \theta_j, \kappa_j)_{LR}, A_{ij} = ([a_{ij}; l_j, r_j; l'_j, r'_j; l''_j, r''_j]; \xi_{ij}, \psi_{ij}, \mathit{\Gamma}_{ij})_{LR}$, and $B_i = ([b_i; s_j, t_j; s'_j, t'_j; s''_j, t''_j]; \epsilon_j, \varepsilon_j, \phi_j)$, the FSNLP problem can be trans-

formed as follows;

$$\text{Maximize/ Minimize}\left(\sum_{j=1}^{n}([c_j;p_j,q_j;p'_j,q'_j;p''_j,q''_j];\chi_j,\eta_j,\zeta_j)_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}\right); \quad (83)$$

subject to

$$\sum_{j=1}^{n}([a_{ij};l_j,r_j;l'_j,r'_j;l''_j,r''_j];\xi_{ij},\psi_{ij},\Gamma_{ij})_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}$$
$$=([b_i;s_j,t_j;s'_j,t'_j;s''_j,t''_j];\epsilon_j,\varepsilon_j,\phi_j)_{LR},\forall i=1,2,3,\cdots,m.$$

where $([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}$ are LR-type SNN, $\forall j=1,2,3\cdots,n$.
Step 2. Using product of LR-type SNNs defined in Section (2.1) and assuming

$$([a_{ij};l_j,r_j;l'_j,r'_j;l''_j,r''_j];\xi_{ij},\psi_{ij},\Gamma_{ij})_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}$$
$$=([a^*_{ij};l^*_j,r^*_j;l^{*'}_j,r^{*'}_j;l^{*''}_j,r^{*''}_j];\xi^*_{ij},\psi^*_{ij},\Gamma^*_{ij})_{LR}.$$
$$Here$$
$$\xi_{ij}\wedge\phi_j=\xi^*_{ij},\psi_{ij}\vee\theta_j=\psi^*_{ij},\Gamma_{ij}\vee\kappa_j=\Gamma^*_{ij}.$$

The FSNLP problem (83) can be transformed as follows;

$$\text{Maximize/ Minimize}\left(\sum_{j=1}^{n}([c_j;p_j,q_j;p'_j,q'_j;p''_j,q''_j];\chi_j,\eta_j,\zeta_j)_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}\right); \quad (84)$$

subject to

$$([a^*_{ij};l^*_j,r^*_j;l^{*'}_j,r^{*'}_j;l^{*''}_j,r^{*''}_j];\xi^*_{ij},\psi^*_{ij},\Gamma^*_{ij})_{LR}=([b_i;s_j,t_j;s'_j,t'_j;s''_j,t''_j];\epsilon_j,\varepsilon_j,\phi_j)_{LR},\forall i=1,2,3,\cdots,m.$$

where $([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}$ are LR-type SNN, $\forall j=1,2,3\cdots,n$.
Step 3. Using arithmetic operations defined in Section (2.1), above problem becomes:

$$\text{Maximize/ Minimize}\left(\sum_{j=1}^{n}([c_j;p_j,q_j;p'_j,q'_j;p''_j,q''_j];\chi_j,\eta_j,\zeta_j)_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}\right); \quad (85)$$

subject to

$$\sum_{j=1}^{n}a^*_{ij}=b_i,\sum_{j=1}^{n}l^*_j=s_j,\sum_{j=1}^{n}r^*_j=t_j,\sum_{j=1}^{n}l^{*'}_j=s'_j,\sum_{j=1}^{n}r^{*'}_j=t'_j,\sum_{j=1}^{n}l^{*''}_j=s''_j,\sum_{j=1}^{n}r^{*''}_j=t''_j,$$
$$\wedge[\xi^*_{ij}]=\epsilon_j,\vee[\psi^*_{ij}]=\varepsilon_j,\vee[\Gamma^*_{ij}]=\phi_j,$$

$y_j\geq 0, z_j\geq 0,\quad y'_j-y_j\geq 0,\quad z'_j-z_j\geq 0,\quad y''_j-y'_j\geq 0,\quad z''_j-z'_j\geq 0,\forall j=1,2,3,\cdots,n.$
and $\phi_j,\theta_j,\kappa_j\in[0,1]$.
Step 4. Now we have to find LR-type single-valued neutrosophic feasible solution $X^k=([x^k_j;y^k_j,z^k_j;y^{k'}_j,z^{k'}_j;y^{k''}_j,z^{k''}_j];\phi_j,\theta_j,\kappa_j)_{LR}$. By applying ranking, the FSNLP problem can be solved

$$\text{Maximize/ Minimize}\ \Re\left(\sum_{j=1}^{n}([c_j;p_j,q_j;p'_j,q'_j;p''_j,q''_j];\chi_j,\eta_j,\zeta_j)_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}\right); \quad (86)$$

subject to

$$\sum_{j=1}^{n} a_{ij}^{*} = b_i, \sum_{j=1}^{n} l_j^{*} = s_j, \sum_{j=1}^{n} r_j^{*} = t_j, \sum_{j=1}^{n} l_j^{*'} = s'_j, \sum_{j=1}^{n} r_j^{*'} = t'_j, \sum_{j=1}^{n} l_j^{*''} = s''_j, \sum_{j=1}^{n} r_j^{*''} = t''_j,$$
$$\wedge[\xi_{ij}^{*}] = \epsilon_j, \vee[\psi_{ij}^{*}] = \varepsilon_j, \vee[\Gamma_{ij}^{*}] = \phi_j,$$

$y_j \geq 0, z_j \geq 0, \quad y'_j - y_j \geq 0, \quad z'_j - z_j \geq 0, \quad y''_j - y'_j \geq 0, \quad z''_j - z'_j \geq 0, \forall j = 1, 2, 3, \cdots, n.$
and $\phi_j, \theta_j, \kappa_j \in [0, 1]$.
Step 5. Assuming $([c_j; p_j, q_j; p'_j, q'_j; p''_j, q''_j]; \chi_j, \eta_j, \zeta_j)_{LR} \otimes ([x_j; y_j, z_j; y'_j, z'_j; y''_j, z''_j]; \phi_j, \theta_j, \kappa_j)_{LR}$ $= ([x_j^s; y_j^s, z_j^s; ys'_j, zs'_j; ys''_j, zs''_j]; \phi_j^s, \theta_j^s, \kappa_j^s)_{LR}$ the problem (86) can be written as:

$$\text{Maximize/Minimize } \Re\left(\sum_{j=1}^{n}([x_j^s; y_j^s, z_j^s; ys'_j, zs'_j; ys''_j, zs''_j]; \phi_j^s, \theta_j^s, \kappa_j^s)_{LR} \right); \tag{87}$$

subject to

$$\sum_{j=1}^{n} a_{ij}^{*} = b_i, \sum_{j=1}^{n} l_j^{*} = s_j, \sum_{j=1}^{n} r_j^{*} = t_j, \sum_{j=1}^{n} l_j^{*'} = s'_j, \sum_{j=1}^{n} r_j^{*'} = t'_j, \sum_{j=1}^{n} l_j^{*''} = s''_j, \sum_{j=1}^{n} r_j^{*''} = t''_j,$$
$$\wedge[\xi_{ij}^{*}] = \epsilon_j, \vee[\psi_{ij}^{*}] = \varepsilon_j, \vee[\Gamma_{ij}^{*}] = \phi_j,$$

$y_j \geq 0, z_j \geq 0, \quad y'_j - y_j \geq 0, \quad z'_j - z_j \geq 0, \quad y''_j - y'_j \geq 0, \quad z''_j - z'_j \geq 0, \forall j = 1, 2, 3, \cdots, n.$
and $\phi_j, \theta_j, \kappa_j \in [0, 1]$.
Step 6. As ranking function is linear thus the problem (87) can be written as:

$$\text{Maximize/Minimize } \left(\sum_{j=1}^{n} \Re([x_j^s; y_j^s, z_j^s; ys'_j, zs'_j; ys''_j, zs''_j]; \phi_j^s, \theta_j^s, \kappa_j^s)_{LR} \right); \tag{88}$$

subject to

$$\begin{aligned}&\sum_{j=1}^{n} a_{ij}^{*} = b_i, \sum_{j=1}^{n} l_j^{*} = s_j, \sum_{j=1}^{n} r_j^{*} = t_j, \sum_{j=1}^{n} l_j^{*'} = s'_j, \sum_{j=1}^{n} r_j^{*'} = t'_j, \sum_{j=1}^{n} l_j^{*''} = s''_j, \sum_{j=1}^{n} r_j^{*''} = t''_j,\\ &\wedge[\xi_{ij}^{*}] = \epsilon_j, \vee[\psi_{ij}^{*}] = \varepsilon_j, \vee[\Gamma_{ij}^{*}] = \phi_j,\end{aligned} \tag{89}$$

$y_j \geq 0, z_j \geq 0, \quad y'_j - y_j \geq 0, \quad z'_j - z_j \geq 0, \quad y''_j - y'_j \geq 0, \quad z''_j - z'_j \geq 0, \forall j = 1, 2, 3, \cdots, n.$
and $\phi_j, \theta_j, \kappa_j \in [0, 1]$.
Step 7. Using the definition of ranking function defined in Section (2.1), the problem can be converted into:

$$\text{Maximize/Minimize} \begin{pmatrix} \sum_{j=1}^{n}[\frac{1}{6}\{(\int_0^{\chi} x_j^s - y_j^s L^{-1}(\alpha)d\alpha) + (\int_0^{\chi} x_j^s - z_j^s R^{-1}(\alpha)d\alpha) + \\ (\int_{\eta}^{1} x_j^s - y_j^{s'} L'^{-1}(\beta)d\beta) + (\int_{\eta}^{1} x_j^s - z_j^{s'} R'^{-1}(\beta)d\beta) \\ +(\int_{\zeta}^{1} x_j^s - y_j^{s''} L''^{-1}(\gamma)d\gamma) + (\int_{\zeta}^{1} x_j^s - z_j^{s''} R''^{-1}(\gamma)d\gamma)\}] \end{pmatrix}; \tag{90}$$

subject to

$$\sum_{j=1}^{n} a_{ij}^{*} = b_i, \sum_{j=1}^{n} l_j^{*} = s_j, \sum_{j=1}^{n} r_j^{*} = t_j, \sum_{j=1}^{n} l_j^{*'} = s'_j, \sum_{j=1}^{n} r_j^{*'} = t'_j, \sum_{j=1}^{n} l_j^{*''} = s''_j, \sum_{j=1}^{n} r_j^{*''} = t''_j,$$
$$\wedge[\xi_{ij}^{*}] = \epsilon_j, \vee[\psi_{ij}^{*}] = \varepsilon_j, \vee[\Gamma_{ij}^{*}] = \phi_j,$$

$y_j \geq 0, z_j \geq 0, \quad y'_j - y_j \geq 0, \quad z'_j - z_j \geq 0, \quad y''_j - y'_j \geq 0, \quad z''_j - z'_j \geq 0, \forall j = 1, 2, 3, \cdots, n.$
and $\phi_j, \theta_j, \kappa_j \in [0, 1]$.
Step 8 Solve the crisp linear programming problem (90) by proposed method to find the optimal solution $x_j, y_j, z_j, y'_j, z'_j, y''_j, z''_j, \chi_j, \eta_j, \zeta_j$. **Step 9** Find the LR-type single-valued neutrosophic optimal solution X_j of the FSNLP problem by substituting the values of $x_j, y_j, z_j, y'_j, z'_j, y''_j, z''_j, \chi_j, \eta_j$ and ζ_j in $X_j = ([x_j; y_j, z_j; y'_j, z'_j; y''_j, z''_j]; \chi_j, \eta_j, \zeta_j)_{LR}$.
Step 10 Find the LR-type single-valued neutrosophic optimal solution of the FSNLP problem (82) by substituting the

values of X_j in $\sum_{j=1}^{n} C_j \otimes X_j$.

Theorem 24. *The solution of FSNLP problem with LR-type SNNs*

$$\textit{Maximize/ Minimize} \sum_{j=1}^{n} C_j \otimes X_j \quad \textit{subject to} \quad \sum_{j=1}^{n} A_{ij} \otimes X_j = B_i, \forall i = 1, 2, 3, \cdots, m. \tag{91}$$

where C_j, A_{ij}, B_i and X_j are LR-type SNNs, exists when the solution of the associated crisp LPP

$$\textit{Maximize/ Minimize}\ \Re\left(\sum_{j=1}^{n}([x_j^s; y_j^s, z_j^s; ys'_j, zs'_j; ys''_j, zs''_j]; \phi_j^s, \theta_j^s, \kappa_j^s)_{LR} \right);$$

subject to

$$\sum_{j=1}^{n} a_{ij}^* = b_i, \sum_{j=1}^{n} l_j^* = s_j, \sum_{j=1}^{n} r_j^* = t_j, \sum_{j=1}^{n} l_j^{*'} = s'_j, \sum_{j=1}^{n} r_j^{*'} = t'_j, \sum_{j=1}^{n} l_j^{*''} = s''_j, \sum_{j=1}^{n} r_j^{*''} = t''_j,$$
$$\wedge[\xi_{ij}^*] = \epsilon_j, \vee[\psi_{ij}^*] = \varepsilon_j, \vee[\Gamma_{ij}^*] = \phi_j,$$

$y_j \geq 0, z_j \geq 0, \quad y'_j - y_j \geq 0, \quad z'_j - z_j \geq 0, \quad y''_j - y'_j \geq 0, \quad z''_j - z'_j \geq 0, \forall j = 1, 2, 3, \cdots, n.$
and $\phi_j, \theta_j, \kappa_j \in [0, 1]$, exists. Otherwise, there is no guarantee that the LR-type single-valued neutrosophic optimal solution exists.

Proof. Straightforward. □

4 Numerical Examples

Example 1. A Company Manufacturing Problem. A company manufactures two types of face mask: cotton face mask and wool face mask. Each face mask has to pass through two different machines: M_1 and M_2. M_1 machine can work for $([50; 26, 62; 38, 101; 49.2, 192]; 0.7, 0.6, 0.5)_{LR}$ minutes per week and M_2 machine can work for $([68; 41, 94; 54, 133; 67.8, 262]; 0.6, 0.5, 0.3)_{LR}$ minutes per week. Ten hundred cotton face masks required $([5; 2, 3; 3, 5; 4.7, 6]; 0.9, 0.6, 0.3)_{LR}$ minutes on M_1 machine and $([6; 1, 2; 2, 3; 5.5, 5]; 0.8, 0.3, 0.2)_{LR}$ minutes on M_2 machine. Ten hundred wool masks required $([7; 3, 4; 4, 5; 6.9, 7]; 0.6, 0.1, 0.2)_{LR}$ minutes and $([8; 3, 4; 4, 5; 7.9, 8]; 0.7, 0.5, 0.2)_{LR}$ minutes on M_1 and M_2, respectively. The profit is Rs. $([12; 5, 6; 7, 8; 10, 11]; 0.6, 0.2, 0.3)_{LR}$ per thousand for cotton face masks and Rs. $([14; 4, 7; 6, 9; 8, 13]; 0.8, 0.5, 0.4)_{LR}$ per thousand for wool face masks. The company wants to maximize the profit.
We apply the method discussed in Section (3).

$$Maximize([12; 5, 6; 7, 8; 10, 11]; 0.6, 0.2, 0.3)_{LR} \otimes X_1 \oplus ([14; 4, 7; 6, 9; 8, 13]; 0.8, 0.5, 0.4)_{LR} \otimes X_2$$

subject to

$$\begin{aligned}
&([5; 2, 3; 3, 5; 4.7, 6]; 0.9, 0.6, 0.3)_{LR} \otimes X_1 \oplus ([6; 1, 2; 2, 3; 5.5, 5]; 0.8, 0.3, 0.2)_{LR} \otimes X_2 \\
&= ([50; 26, 62; 38, 101; 49.2, 192]; 0.7, 0.6, 0.5)_{LR} \\
&([7; 3, 4; 4, 5; 6.9, 7]; 0.6, 0.1, 0.2)_{LR} \otimes X_1 \oplus ([8; 3, 4; 4, 5; 7.9, 8]; 0.7, 0.5, 0.2)_{LR} \otimes X_2 \\
&= ([68; 41, 94; 54, 133; 67.8, 262]; 0.6, 0.5, 0.3)_{LR}
\end{aligned}$$

where $([x_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)$ and $([x_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ are LR-type SNNs.
$\chi_1, \eta_1, \zeta_1, \chi_2, \eta_2, \zeta_2 \in [0, 1]$.
Step 1: Let $X_1 = ([x_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ and $X_2 = ([x_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$, then prob-

lems can be written as

$$Maximize([12;5,6;7,8;10,11];0.6,0.2,0.3)_{LR}\otimes([x_1;l_1,r_1;l'_1,r'_1;l''_1,r''_1];\chi_1,\eta_1,\zeta_1)_{LR}\oplus$$
$$([14;4,7;6,9;8,13];0.8,0.5,0.4)_{LR}\otimes([x_2;l_2,r_2;l'_2,r'_2;l''_2,r''_2];\chi_2,\eta_2,\zeta_2)_{LR}$$

subject to

$$([5;2,3;3,5;4.7,6];0.9,0.6,0.3)_{LR}\otimes([x_1;l_1,r_1;l'_1,r'_1;l''_1,r''_1];\chi_1,\eta_1,\zeta_1)_{LR}\oplus([6;1,2;2,3;5.5,5];0.8,0.3,0.2)_{LR}$$
$$\otimes([x_2;l_2,r_2;l'_2,r'_2;l''_2,r''_2];\chi_2,\eta_2,\zeta_2)_{LR}=([50;26,62;38,101;49.2,192];0.7,0.6,0.5)_{LR}$$
$$([7;3,4;4,5;6.9,7];0.6,0.1,0.2)_{LR}\otimes([x_1;l_1,r_1;l'_1,r'_1;l''_1,r''_1];\chi_1,\eta_1,\zeta_1)_{LR}\oplus([8;3,4;4,5;7.9,8];0.7,0.5,0.2)_{LR}$$
$$\otimes([x_2;l_2,r_2;l'_2,r'_2;l''_2,r''_2];\chi_2,\eta_2,\zeta_2)_{LR}=([68;41,94;54,133;67.8,262];0.6,0.5,0.3)_{LR}$$

where $([x_1;l_1,r_1;l'_1,r'_1;l''_1,r''_1];\chi_1,\eta_1,\zeta_1)$ and $([x_2;l_2,r_2;l'_2,r'_2;l''_2,r''_2];\chi_2,\eta_2,\zeta_2)_{LR}$ are LR-type SNNs. $\chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1]$.

Step 2: Using product defined in Section (2.1), the FSNLP, problem obtained in step 1, can be written as:

$$\text{Maximize}([12x_1;12x_1-min\{7x_1-7l_1,18x_1-18l_1\},max\{18x_1+18r_1,7x_1+7r_1\}-12x_1;12x_1$$
$$-min\{5x_1-5l'_1,20x_1-20l'_1\},max\{20x_1+20r'_1,5x_1+5r'_1\}-12x_1;12x_1-min$$
$$\{2x_1-2l''_1,23x_1-23l''_1\},max\{23x_1+23r''_1,2x_1+2r''_1\}-12x_1];0.6\wedge\chi_1,0.2\vee\eta_1,0.3\vee\zeta_1)_{LR}$$
$$\oplus([14x_2;14x_2-min\{10x_2-10l_2,21x_2-21l_2\},max\{21x_2+21r_2,10x_2+10r_2\}-14x_2;14x_2$$
$$-min\{8x_2-8l'_2,23x_2-23l'_2\},max\{23x_2+23r'_2,8x_2+8r'_2\}-14x_2;14x_2-min$$
$$\{6x_2-6l''_2,27x_2-27l''_2\},max\{27x_2+27r''_2,6x_2+6r''_2\}-14x_2];0.8\wedge\chi_2,0.5\vee\eta_2,0.4\vee\zeta_2)_{LR}$$

subject to

$$([5x_1;5x_1-min\{3x_1-3l_1,8x_1-8l_1\},max\{8x_1+8r_1,3x_1+3r_1\}-5x_1;5x_1$$
$$-min\{2x_1-2l'_1,10x_1-10l'_1\},max\{10x_1+10r'_1,2x_1+2r'_1\}-5x_1;5x_1-min$$
$$\{0.3x_1-0.3l''_1,11x_1-11l''_1\},max\{11x_1+11r''_1,x_1+r''_1\}-5x_1];0.9\wedge\chi_1,0.6\vee\eta_1,0.3\vee\zeta_1)_{LR}$$
$$\oplus([6x_2;6x_2-min\{5x_2-5l_2,8x_2-8l_2\},max\{8x_2+8r_2,5x_2+5r_2\}-6x_2;6x_2$$
$$-min\{4x_2-4l'_2,9x_2-9l'_2\},max\{9x_2+9r'_2,4x_2+4r'_2\}-6x_2;6x_2-min$$
$$\{0.5x_2-0.5l''_2,11x_2-11l''_2\},max\{11x_2+11r''_2,2x_2+2r''_2\}-6x_2];0.8\wedge\chi_2,0.3\vee\eta_2,0.2\vee\zeta_2)_{LR}$$
$$=([50;26,62;38,101;49.2,192];0.7,0.6,0.5)_{LR}$$
$$([7x_1;7x_1-min\{4x_1-4l_1,11x_1-11l_1\},max\{11x_1+11r_1,4x_1+4r_1\}-7x_1;7x_1$$
$$-min\{3x_1-3l'_1,12x_1-12l'_1\},max\{12x_1+12r'_1,3x_1+3r'_1\}-7x_1;7x_1-min$$
$$\{0.1x_1-0.1l''_1,14x_1-14l''_1\},max\{14x_1+14r''_1,x_1+r''_1\}-7x_1];0.6\wedge\chi_1,0.1\vee\eta_1,0.2\vee\zeta_1)_{LR}$$
$$\oplus([8x_2;8x_2-min\{5x_2-5l_2,12x_2-12l_2\},max\{12x_2+12r_2,5x_2+5r_2\}-8x_2;8x_2$$
$$-min\{4x_2-4l'_2,13x_2-13l'_2\},max\{13x_2+13r'_2,4x_2+4r'_2\}-8x_2;8x_2-min$$
$$\{0.1x_2-0.1l''_2,16x_2-16l''_2\},max\{16x_2+16r''_2,2x_2+2r''_2\}-8x_2];0.7\wedge\chi_2,0.5\vee\eta_2,0.2\vee\zeta_2)_{LR}$$
$$=([68;41,94;54,133;67.8,262];0.6,0.5,0.3)_{LR}$$

where $([x_1;l_1,r_1;l'_1,r'_1;l''_1,r''_1];\chi_1,\eta_1,\zeta_1)$ and $([x_2;l_2,r_2;l'_2,r'_2;l''_2,r''_2];\chi_2,\eta_2,\zeta_2)_{LR}$ are LR-type SNNs. $\chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1]$.

Step 3: Using the arithmetic operations which are defined in Section (2.1), the FSNLP, problem obtained in step 2, can be rewritten as:

$$\begin{aligned}\text{Maximize}([&12x_1+14x_2; 12x_1-min\{7x_1-7l_1, 18x_1-18l_1\}+14x_2-min\{10x_2-10l_2, 21x_2-21l_2\},\\&+max\{18x_1+18r_1, 7x_1+7r_1\}-12x_1+max\{21x_2+21r_2, 10x_2+10r_2\}-14x_2; 12x_1\\&-min\{5x_1-5l_1', 20x_1-20l_1'\}+14x_2-min\{8x_2-8l_2', 23x_2-23l_2'\}, max\{20x_1+20r_1',\\&5x_1+5r_1'\}-12x_1+max\{23x_2+23r_2', 8x_2+8r_2'\}-14x_2; 12x_1-min\{2x_1-2l_1'', 23x_1-23l_1''\}\\&+14x_2-min\{6x_2-6l_2'', 27x_2-27l_2''\}, max\{23x_1+23r_1'', 2x_1+2r_1''\}-12x_1+max\{27x_2+27r_2''\\&, 6x_2+6r_2''\}-14x_2]; \wedge[(0.6\wedge\chi_1)\wedge(0.8\wedge\chi_2)], \vee[(0.2\vee\eta_1)\vee(0.5\vee\eta_2)], \vee[(0.3\vee\zeta_1)\vee(0.4\vee\zeta_2)]_{LR}\end{aligned}$$

subject to

$$\begin{aligned}5x_1+6x_2=50, 7x_1+8x_2=68,\\5x_1-min\{3x_1-3l_1, 8x_1-8l_1\}+6x_2-min\{5x_2-5l_2, 8x_2-8l_2\}=26,\\max\{8x_1+8r_1, 3x_1+3r_1\}-5x_1+max\{8x_2+8r_2, 5x_2+5r_2\}-6x_2=62,\\5x_1-min\{2x_1-2l_1', 10x_1-10l_1'\}+6x_2-min\{4x_2-4l_2', 9x_2-9l_2'\}=38,\\max\{10x_1+10r_1', 2x_1+2r_1'\}-5x_1+max\{9x_2+9r_2', 4x_2+4r_2'\}-6x_2=101,\\5x_1-min\{0.3x_1-0.3l_1'', 11x_1-11l_1''\}+6x_2-min\{0.5x_2-0.5l_2'', 11x_2-11l_2''\}=49.2,\\max\{11x_1+11r_1'', x_1+r_1''\}-5x_1+max\{11x_2+11r_2'', 2x_2+2r_2''\}-6x_2=192,\\\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7, \vee[(0.6\vee\eta_1)\vee(0.3\vee\eta_2)]=0.6, \vee[(0.3\vee\zeta_1)\vee(0.2\vee\zeta_2)]=0.5,\\7x_1-min\{4x_1-4l_1, 11x_1-11l_1\}+8x_2-min\{5x_2-5l_2, 12x_2-12l_2\}=41,\\max\{11x_1+11r_1, 4x_1+4r_1\}-7x_1+max\{12x_2+12r_2, 5x_2+5r_2\}-8x_2=94,\\7x_1-min\{3x_1-3l_1', 12x_1-12l_1'\}+8x_2-min\{4x_2-4l_2', 13x_2-13l_2'\}=54,\\max\{12x_1+12r_1', 3x_1+3r_1'\}-7x_1+max\{13x_2+13r_2', 4x_2+4r_2'\}-8x_2=133,\\7x_1-min\{0.1x_1-0.1l_1'', 14x_1-14l_1''\}+8x_2-min\{0.1x_2-0.1l_2'', 16x_2-16l_2''\}=67.8,\\max\{14x_1+14r_1'', x_1+r_1''\}-7x_1+max\{16x_2+16r_2'', 2x_2+2r_2''\}-8x_2=262,\\\wedge[(0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6, \vee[(0.1\vee\eta_1)\vee(0.5\vee\eta_2)]=0.5, \vee[(0.2\vee\zeta_1)\vee(0.2\vee\zeta_2)]=0.3,\\l_1\geq 0, r_1\geq 0, l_1'-l_1\geq 0, r_1'-r_1\geq 0, l_1''-l_1'\geq 0, r_1''-r_1'\geq 0, l_2\geq 0, r_2\geq 0, l_2'-l_2\geq 0, r_2'-r_2\geq 0,\\l_2''-l_2'\geq 0, r_2''-r_2'\geq 0, \chi_1, \eta_1, \zeta_1, \chi_2, \eta_2, \zeta_2\in[0,1].\end{aligned}$$

Step 4: Using the ranking function which are defined in Section (3), the FSNLP, problem obtained in step 3, can be rewritten as:

$$\begin{aligned}\text{Maximize}\Re([&12x_1+14x_2; 12x_1-min\{7x_1-7l_1, 18x_1-18l_1\}+14x_2-min\{10x_2-10l_2, 21x_2-21l_2\},\\&+max\{18x_1+18r_1, 7x_1+7r_1\}-12x_1+max\{21x_2+21r_2, 10x_2+10r_2\}-14x_2; 12x_1\\&-min\{5x_1-5l_1', 20x_1-20l_1'\}+14x_2-min\{8x_2-8l_2', 23x_2-23l_2'\}, max\{20x_1+20r_1',\\&5x_1+5r_1'\}-12x_1+max\{23x_2+23r_2', 8x_2+8r_2'\}-14x_2; 12x_1-min\{2x_1-2l_1'', 23x_1-23l_1''\}\\&+14x_2-min\{6x_2-6l_2'', 27x_2-27l_2''\}, max\{23x_1+23r_1'', 2x_1+2r_1''\}-12x_1+max\{27x_2+27r_2''\\&, 6x_2+6r_2''\}-14x_2]; \wedge[(0.6\wedge\chi_1)\wedge(0.8\wedge\chi_2)], \vee[(0.2\vee\eta_1)\vee(0.5\vee\eta_2)], \vee[(0.3\vee\zeta_1)\vee(0.4\vee\zeta_2)])_{LR}\end{aligned}$$

subject to

$$5x_1+6x_2=50, 7x_1+8x_2=68,$$
$$5x_1-min\{3x_1-3l_1, 8x_1-8l_1\}+6x_2-min\{5x_2-5l_2, 8x_2-8l_2\}=26,$$
$$max\{8x_1+8r_1, 3x_1+3r_1\}-5x_1+max\{8x_2+8r_2, 5x_2+5r_2\}-6x_2=62,$$
$$5x_1-min\{2x_1-2l_1', 10x_1-10l_1'\}+6x_2-min\{4x_2-4l_2', 9x_2-9l_2'\}=38,$$
$$max\{10x_1+10r_1', 2x_1+2r_1'\}-5x_1+max\{9x_2+9r_2', 4x_2+4r_2'\}-6x_2=101,$$
$$5x_1-min\{0.3x_1-0.3l_1'', 11x_1-11l_1''\}+6x_2-min\{0.5x_2-0.5l_2'', 11x_2-11l_2''\}=49.2,$$
$$max\{11x_1+11r_1'', x_1+r_1''\}-5x_1+max\{11x_2+11r_2'', 2x_2+2r_2''\}-6x_2=192,$$
$$\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7, \vee[(0.6\vee\eta_1)\vee(0.3\vee\eta_2)]=0.6, \vee[(0.3\vee\zeta_1)\vee(0.2\vee\zeta_2)]=0.5,$$
$$7x_1-min\{4x_1-4l_1, 11x_1-11l_1\}+8x_2-min\{5x_2-5l_2, 12x_2-12l_2\}=41,$$
$$max\{11x_1+11r_1, 4x_1+4r_1\}-7x_1+max\{12x_2+12r_2, 5x_2+5r_2\}-8x_2=94,$$
$$7x_1-min\{3x_1-3l_1', 12x_1-12l_1'\}+8x_2-min\{4x_2-4l_2', 13x_2-13l_2'\}=54,$$
$$max\{12x_1+12r_1', 3x_1+3r_1'\}-7x_1+max\{13x_2+13r_2', 4x_2+4r_2'\}-8x_2=133,$$
$$7x_1-min\{0.1x_1-0.1l_1'', 14x_1-14l_1''\}+8x_2-min\{0.1x_2-0.1l_2'', 16x_2-16l_2''\}=67.8,$$
$$max\{14x_1+14r_1'', x_1+r_1''\}-7x_1+max\{16x_2+16r_2'', 2x_2+2r_2''\}-8x_2=262,$$
$$\wedge[(0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6, \vee[(0.1\vee\eta_1)\vee(0.5\vee\eta_2)]=0.5, \vee[(0.2\vee\zeta_1)\vee(0.2\vee\zeta_2)]=0.3,$$
$$l_1\geq0, r_1\geq0, l_1'-l_1\geq0, r_1'-r_1\geq0, l_1''-l_1'\geq0, r_1''-r_1'\geq0, l_2\geq0, r_2\geq0, l_2'-l_2\geq0, r_2'-r_2\geq0,$$
$$l_2''-l_2'\geq0, r_2''-r_2'\geq0, \chi_1, \eta_1, \zeta_1, \chi_2, \eta_2, \zeta_2\in[0,1].$$

Step 5: Using $\min\{a,b\}=\frac{a+b}{2}-|\frac{a-b}{2}|$, $\max\{a,b\}=\frac{a+b}{2}+|\frac{a-b}{2}|$, the FSNLP, problem obtained in step 4, can be rewritten as:

$$\begin{aligned}
\text{Maximize}([&\frac{(96+48\chi-48\eta-48\zeta+\chi^2+(\eta-1)^2+(\zeta-1)^2)}{12}x_1+\\
&\frac{(112+56\chi-56\eta-56\zeta+3\chi^2+3(\eta-1)^2+5(\zeta-1)^2)}{12}x_2-\frac{25}{24}\chi^2l_1-\frac{11}{24}\chi^2|x_1-l_1|-\frac{31}{24}\chi^2l_2\\
&-\frac{11}{24}\chi^2|x_2-l_2|+\frac{25}{24}\chi^2r_1+\frac{11}{24}\chi^2|x_1+r_1|+\frac{31}{24}\chi^2r_2+\frac{11}{24}\chi^2|x_2+r_2|-\frac{25}{24}(\eta-1)^2l_1'\\
&-\frac{15}{24}(\eta-1)^2|x_1-l_1'|-\frac{31}{24}(\eta-1)^2l_2'-\frac{15}{24}(\eta-1)^2|x_2-l_2'|+\frac{25}{24}(\eta-1)^2r_1'+\frac{15}{24}(\eta-1)^2|x_1+r_1'|\\
&+\frac{31}{24}(\eta-1)^2r_2'+\frac{15}{24}(\eta-1)^2|x_2+r_2'|-\frac{25}{24}(\zeta-1)^2l_1''-\frac{21}{24}(\zeta-1)^2|x_1-l_1''|-\frac{33}{24}(\zeta-1)^2l_2''\\
&-\frac{21}{24}(\zeta-1)^2|x_2-l_2''|+\frac{25}{24}(\zeta-1)^2r_1''+\frac{21}{24}(\zeta-1)^2|x_1+r_1''|+\frac{33}{24}(\zeta-1)^2r_2''+\frac{21}{24}(\zeta-1)^2|x_2+r_2''|])
\end{aligned}$$

subject to

$$\begin{aligned}
5x_1+6x_2=50, 7x_1+8x_2=68,\\
-\frac{1}{2}x_1+\frac{11}{2}l_1+\frac{5}{2}|x_1-l_1|-\frac{x_2}{2}+\frac{13}{2}l_2+\frac{3}{2}|x_2-l_2|=26,\\
\frac{1}{2}x_1+\frac{11}{2}r_1+\frac{5}{2}|x_1+r_1|+\frac{x_2}{2}+\frac{13}{2}r_2+\frac{3}{2}|x_2+r_2|=62,\\
-x_1+6l_1'+4|x_1-l_1'|-\frac{x_2}{2}+\frac{13}{2}l_2'+\frac{5}{2}|x_2-l_2'|=38,\\
x_1+6r_1'+4|x_1+r_1'|+\frac{x_2}{2}+\frac{13}{2}r_2'+\frac{5}{2}|x_2+r_2'|=101,\\
-\frac{1.3}{2}x_1+\frac{11.3}{2}l_1''+\frac{10.7}{2}|x_1-l_1''|+\frac{0.5}{2}x_2+\frac{11.5}{2}l_2''+\frac{10.5}{2}|x_2-l_2''|=49.2,\\
x_1+6r_1''+5|x_1+r_1''|+\frac{x_2}{2}+\frac{13}{2}r_2''+\frac{9}{2}|x_2+r_2''|=192,\\
\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7, \vee[(0.6\vee\eta_1)\vee(0.3\vee\eta_2)]=0.6, \vee[(0.3\vee\zeta_1)\vee(0.2\vee\zeta_2)]=0.5,\\
-\frac{1}{2}x_1+\frac{15}{2}l_1+\frac{7}{2}|x_1-l_1|-\frac{x_2}{2}+\frac{17}{2}l_2+\frac{7}{2}|x_2-l_2|=41,\\
\frac{1}{2}x_1+\frac{15}{2}r_1+\frac{7}{2}|x_1+r_1|+\frac{x_2}{2}+\frac{17}{2}r_2+\frac{7}{2}|x_2+r_2|=94,\\
-\frac{1}{2}x_1+\frac{15}{2}l_1'+\frac{9}{2}|x_1-l_1'|-\frac{x_2}{2}+\frac{17}{2}l_2'+\frac{9}{2}|x_2-l_2'|=54,\\
\frac{1}{2}x_1+\frac{15}{2}r_1'+\frac{9}{2}|x_1+r_1'|+\frac{x_2}{2}+\frac{17}{2}r_2'+\frac{9}{2}|x_2+r_2'|=133,\\
-\frac{0.1}{2}x_1+\frac{14.1}{2}l_1''+\frac{13.9}{2}|x_1-l_1''|-\frac{0.1}{2}x_2+\frac{16.1}{2}l_2''+\frac{15.9}{2}|x_2-l_2''|=67.8,\\
\frac{1}{2}x_1+\frac{15}{2}r_1''+\frac{13}{2}|x_1+r_1''|+x_2+9r_2''+7|x_2+r_2''|=262,\\
\wedge[(0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6, \vee[(0.1\vee\eta_1)\vee(0.5\vee\eta_2)]=0.5, \vee[(0.2\vee\zeta_1)\vee(0.2\vee\zeta_2)]=0.3,\\
l_1\geq 0, r_1\geq 0, l_1'-l_1\geq 0, r_1'-r_1\geq 0, l_1''-l_1'\geq 0, r_1''-r_1'\geq 0, l_2\geq 0, r_2\geq 0, l_2'-l_2\geq 0, r_2'-r_2\geq 0,\\
l_2''-l_2'\geq 0, r_2''-r_2'\geq 0, \chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1].
\end{aligned}$$

Step 6: By solving the crisp mathematical problem obtained in step 5, we get the optimal solution $x_1=4, l_1=1, r_1=2, l_1'=2, r_1'=3, l_1''=3, r_1''=7, x_2=5, l_2=2, r_2=3, l_2'=3, r_2'=4, l_2''=4, r_2''=6, \chi_1=0.7, \eta_1=0.5, \zeta_1=0.4, \chi_2=0.9, \eta_2=0.4, \zeta_2=0.5$.
Step 7: Substituting the values of $x_1, l_1, r_1, l_1', r_1', l_1'', r_1'', x_2, l_2, r_2, l_2', r_2', l_2'', r_2'', \chi_1, \eta_1, \zeta_1, \chi_2, \eta_2$ and ζ_2 in $X_1=([x_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ and $X_2=([x_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ the exact LR-type single-valued neutrosophic optimal solution is $X_1=([4;1,2;2,3;3,7];0.7,0.5,0.4)_{LR}, X_2=([5;2,3;3,4;4,6];0.9,0.4,0.5)_{LR}$.
Step 8: By substituting the values of X_1 and X_2, obtained in Step 7, into the objective function, the LR-type single-valued neutrosophic optimal value is $([118;67,158;92,229;110,432];0.6,0.5,0.5)_{LR}$.

Example 2.

$$Minimize([10;3,5;4,6;7,8];0.8,0.4,0.5)_{LR}\otimes X_1\oplus([16;4,6;8,10;12,14];0.7,0.3,0.2)_{LR}\otimes X_2$$

subject to

$$([7;3,4;4,6;5,7];0.6,0.5,0.4)_{LR}\otimes X_1\oplus([9;4,5;6,7;8,9];0.7,0.1,0.3)_{LR}\otimes X_2$$
$$=([87;56,149;75,216;84,297];0.6,0.5,0.4)_{LR}$$
$$([10;4,6;8,9;9,10];0.9,0.2,0.1)_{LR}\otimes X_1\oplus([11;4,6;6,8;9,10];0.8,0.3,0.4)_{LR}\otimes X_2$$
$$=([115;70,198;101,284;112,377];0.7,0.5,0.4)_{LR}$$

Step 1: *Let* $X_1=([x_1;l_1,r_1;l_1',r_1';l_1'',r_1''];\chi_1,\eta_1,\zeta_1)_{LR}$ *and* $X_2=([x_2;l_2,r_2;l_2',r_2';l_2'',r_2''];\chi_2,\eta_2,\zeta_2)_{LR}$*, then problems can be written as*

$$Minimize([10;3,5;4,6;7,8];0.8,0.4,0.5)_{LR}\otimes([x_1;l_1,r_1;l_1',r_1';l_1'',r_1''];\chi_1,\eta_1,\zeta_1)_{LR}\oplus$$
$$([16;4,6;8,10;12,14];0.7,0.3,0.2)_{LR}\otimes([x_2;l_2,r_2;l_2',r_2';l_2'',r_2''];\chi_2,\eta_2,\zeta_2)_{LR}$$

subject to

$$([7;3,4;4,6;5,7];0.6,0.5,0.4)_{LR}\otimes([x_1;l_1,r_1;l_1',r_1';l_1'',r_1''];\chi_1,\eta_1,\zeta_1)_{LR}\oplus([9;4,5;6,7;8,9];0.7,0.1,0.3)_{LR}\otimes$$
$$([x_2;l_2,r_2;l_2',r_2';l_2'',r_2''];\chi_2,\eta_2,\zeta_2)_{LR}=([87;56,149;75,216;84,297];0.6,0.5,0.4)_{LR}$$
$$([10;4,6;8,9;9,10];0.9,0.2,0.1)_{LR}\otimes([x_1;l_1,r_1;l_1',r_1';l_1'',r_1''];\chi_1,\eta_1,\zeta_1)_{LR}\oplus([11;4,6;6,8;9,10];0.8,0.3,0.4)_{LR}\otimes$$
$$([x_2;l_2,r_2;l_2',r_2';l_2'',r_2''];\chi_2,\eta_2,\zeta_2)_{LR}=([115;70,198;101,284;112,377];0.7,0.5,0.4)_{LR}$$

where $([x_1;l_1,r_1;l_1',r_1';l_1'',r_1''];\chi_1,\eta_1,\zeta_1)$ *and* $([x_2;l_2,r_2;l_2',r_2';l_2'',r_2''];\chi_2,\eta_2,\zeta_2)_{LR}$ *are* LR*-type SNNs.*
$\chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1]$.
Step 2: *Using product defined in Section (2.1), the FSNLP, problem obtained in step 1, can be written as:*

$$\textit{Minimize}([10x_1;10x_1-min\{7x_1-7l_1,15x_1-15l_1\},max\{15x_1+15r_1,7x_1+7r_1\}-10x_1;10x_1$$
$$-min\{6x_1-6l_1',16x_1-16l_1'\},max\{16x_1+16r_1',6x_1+6r_1'\}-10x_1;10x_1-min$$
$$\{3x_1-3l_1'',18x_1-18l_1''\},max\{18x_1+18r_1'',3x_1+3r_1''\}-10x_1];0.8\wedge\chi_1,0.4\vee\eta_1,0.5\vee\zeta_1)_{LR}$$
$$\oplus([16x_2;16x_2-min\{12x_2-12l_2,22x_2-22l_2\},max\{22x_2+22r_2,12x_2+12r_2\}-16x_2;16x_2$$
$$-min\{8x_2-8l_2',26x_2-26l_2'\},max\{26x_2+26r_2',8x_2+8r_2'\}-16x_2;16x_2-min$$
$$\{4x_2-4l_2'',30x_2-30l_2''\},max\{30x_2+30r_2'',4x_2+4r_2''\}-16x_2];0.7\wedge\chi_2,0.3\vee\eta_2,0.2\vee\zeta_2)_{LR}$$

subject to

$$([7x_1; 7x_1 - min\{4x_1 - 4l_1, 11x_1 - 11l_1\}, max\{11x_1 + 11r_1, 4x_1 + 4r_1\} - 7x_1; 7x_1 - min\{3x_1 - 3l'_1, 13x_1 - 13l'_1\}, max\{13x_1 + 13r'_1, 3x_1 + 3r'_1\} - 7x_1; 7x_1 - min \{2x_1 - 2l''_1, 14x_1 - 14l''_1\}, max\{14x_1 + 14r''_1, 2x_1 + 2r''_1\} - 7x_1]; 0.6 \wedge \chi_1, 0.5 \vee \eta_1, 0.4 \vee \zeta_1)_{LR} \oplus ([9x_2; 9x_2 - min\{5x_2 - 5l_2, 14x_2 - 14l_2\}, max\{14x_2 + 14r_2, 5x_2 + 5r_2\} - 9x_2; 9x_2 - min\{3x_2 - 3l'_2, 16x_2 - 16l'_2\}, max\{16x_2 + 16r'_2, 3x_2 + 3r'_2\} - 9x_2; 9x_2 - min \{x_2 - l''_2, 18x_2 - 18l''_2\}, max\{18x_2 + 18r''_2, x_2 + r''_2\} - 9x_2]; 0.7 \wedge \chi_2, 0.1 \vee \eta_2, 0.3 \vee \zeta_2)_{LR} = ([87; 56, 149; 75, 216; 84, 297]; 0.6, 0.5, 0.4)_{LR}$$

$$([10x_1; 10x_1 - min\{6x_1 - 6l_1, 16x_1 - 16l_1\}, max\{16x_1 + 16r_1, 6x_1 + 6r_1\} - 10x_1; 10x_1 - min\{2x_1 - 2l'_1, 19x_1 - 19l'_1\}, max\{19x_1 + 19r'_1, 2x_1 + 2r'_1\} - 10x_1; 10x_1 - min \{x_1 - l''_1, 20x_1 - 20l''_1\}, max\{20x_1 + 20r''_1, x_1 + r''_1\} - 10x_1]; 0.9 \wedge \chi_1, 0.2 \vee \eta_1, 0.1 \vee \zeta_1)_{LR} \oplus ([11x_2; 11x_2 - min\{7x_2 - 7l_2, 17x_2 - 17l_2\}, max\{17x_2 + 17r_2, 7x_2 + 7r_2\} - 11x_2; 11x_2 - min\{5x_2 - 5l'_2, 19x_2 - 19l'_2\}, max\{19x_2 + 19r'_2, 5x_2 + 5r'_2\} - 11x_2; 11x_2 - min \{2x_2 - 2l''_2, 21x_2 - 21l''_2\}, max\{21x_2 + 21r''_2, 2x_2 + 2r''_2\} - 11x_2]; 0.8 \wedge \chi_2, 0.3 \vee \eta_2, 0.4 \vee \zeta_2)_{LR} = ([115; 70, 198; 101, 284; 112, 377]; 0.7, 0.5, 0.4)_{LR}$$

where $([x_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)$ *and* $([x_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *are* LR*-type SNNs.* $\chi_1, \eta_1, \zeta_1, \chi_2, \eta_2, \zeta_2 \in [0, 1]$.

Step 3: *Using the arithmetic operations which are defined in Section (2.1), the FSNLP, problem obtained in step 2, can be rewritten as:*

$$\textit{Minimize}([10x_1 + 16x_2; 10x_1 - min\{7x_1 - 7l_1, 15x_1 - 15l_1\} + 16x_2 - min\{12x_2 - 12l_2, 22x_2 - 22l_2\}, max\{15x_1 + 15r_1, 7x_1 + 7r_1\} - 10x_1 + max\{22x_2 + 22r_2, 12x_2 + 12r_2\} - 16x_2; 10x_1 - min\{6x_1 - 6l'_1, 16x_1 - 16l'_1\} + 16x_2 - min\{8x_2 - 8l'_2, 26x_2 - 26l'_2\}, max\{16x_1 + 16r'_1, 6x_1 + 6r'_1\} - 10x_1 + max\{26x_2 + 26r'_2, 8x_2 + 8r'_2\} - 16x_2; 10x_1 - min\{3x_1 - 3l''_1, 18x_1 - 18l''_1\} + 16x_2 - min\{4x_2 - 4l''_2, 30x_2 - 30l''_2\}, max\{18x_1 + 18r''_1, 3x_1 + 3r''_1\} - 10x_1 + max\{30x_2 + 30r''_2, 4x_2 + 4r''_2\} - 16x_2]; \wedge[(0.8 \wedge \chi_1) \wedge (0.7 \wedge \chi_2)], \vee[(0.4 \vee \eta_1) \vee (0.3 \vee \eta_2)], \vee[(0.5 \vee \zeta_1) \vee (0.2 \vee \zeta_2]))_{LR}$$

subject to

$$\begin{aligned}
&7x_1+9x_2=87, 10x_1+11x_2=115,\\
&7x_1-min\{4x_1-4l_1, 11x_1-11l_1\}+9x_2-min\{5x_2-5l_2, 14x_2-14l_2\}=56,\\
&max\{11x_1+11r_1, 4x_1+4r_1\}-7x_1+max\{14x_2+14r_2, 5x_2+5r_2\}-9x_2=149,\\
&7x_1-min\{3x_1-3l_1', 13x_1-13l_1'\}+9x_2-min\{3x_2-3l_2', 16x_2-16l_2'\}=75\\
&max\{13x_1+13r_1', 3x_1+3r_1'\}-7x_1+max\{16x_2+16r_2', 3x_2+3r_2'\}-9x_2=216,\\
&7x_1-min\{2x_1-2l_1'', 14x_1-14l_1''\}+9x_2-min\{x_2-l_2'', 18x_2-18l_2''\}=84,\\
&max\{14x_1+14r_1'', 2x_1+2r_1''\}-7x_1)+max\{18x_2+18r_2'', x_2+r_2''\}-9x_2)=297,\\
&\wedge[((0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6, \vee[(0.5\vee\eta_1)\vee(0.1\vee\eta_2)]=0.5, \vee[(0.4\vee\zeta_1)\vee(0.3\vee\zeta_2)]=0.4,\\
&10x_1-min\{6x_1-6l_1, 16x_1-16l_1\}+11x_2-min\{7x_2-7l_2, 17x_2-17l_2\}=70,\\
&max\{16x_1+16r_1, 6x_1+6r_1\}-10x_1+max\{17x_2+17r_2, 7x_2+7r_2\}-11x_2=198,\\
&10x_1-min\{2x_1-2l_1', 19x_1-19l_1'\}+11x_2-min\{5x_2-5l_2', 19x_2-19l_2'\}=101,\\
&max\{19x_1+19r_1', 2x_1+2r_1'\}-10x_1+max\{19x_2+19r_2', 5x_2+5r_2'\}-11x_2=284,\\
&10x_1-min\{x_1-l_1'', 20x_1-20l_1''\}+11x_2-min\{2x_2-2l_2'', 21x_2-21l_2''\}=112,\\
&max\{20x_1+20r_1'', x_1+r_1''\}-10x_1+max\{21x_2+21r_2'', 2x_2+2r_2''\}-11x_2=377,\\
&\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7, \vee[(0.2\vee\eta_1)\vee(0.3\vee\eta_2)]=0.5, \vee[(0.1\vee\zeta_1)\vee(0.4\vee\zeta_2)]=0.4,\\
&l_1\geq 0, r_1\geq 0, l_1'-l_1\geq 0, r_1'-r_1\geq 0, l_1''-l_1'\geq 0, r_1''-r_1'\geq 0, l_2\geq 0, r_2\geq 0, l_2'-l_2\geq 0, r_2'-r_2\geq 0,\\
&l_2''-l_2'\geq 0, r_2''-r_2'\geq 0, \chi_1, \eta_1, \zeta_1, \chi_2, \eta_2, \zeta_2\in[0,1].
\end{aligned}$$

Step 4: *Using the ranking function which are defined in Section (3), the FSNLP, problem obtained in step 3, can be rewritten as:*

$$\begin{aligned}
&\textit{Minimize}\,\Re([10x_1+16x_2; 10x_1-min\{7x_1-7l_1, 15x_1-15l_1\}+16x_2-min\{12x_2-12l_2, 22x_2-22l_2\},\\
&max\{15x_1+15r_1, 7x_1+7r_1\}-10x_1+max\{22x_2+22r_2, 12x_2+12r_2\}-16x_2; 10x_1\\
&-min\{6x_1-6l_1', 16x_1-16l_1'\}+16x_2-min\{8x_2-8l_2', 26x_2-26l_2'\}, max\{16x_1+16r_1', 6x_1+6r_1'\}-10x_1\\
&+max\{26x_2+26r_2', 8x_2+8r_2'\}-16x_2; 10x_1-min\{3x_1-3l_1'', 18x_1-18l_1''\}+16x_2-min\{4x_2-4l_2'',\\
&30x_2-30l_2''\}, max\{18x_1+18r_1'', 3x_1+3r_1''\}-10x_1+max\{30x_2+30r_2'', 4x_2+4r_2''\}-16x_2]; \wedge[(0.8\wedge\chi_1)\\
&\wedge(0.7\wedge\chi_2)], \vee[(0.4\vee\eta_1)\vee(0.3\vee\eta_2)], \vee[(0.5\vee\zeta_1)\vee(0.2\vee\zeta_2]])_{LR}
\end{aligned}$$

subject to

$$
\begin{aligned}
&7x_1+9x_2=87, 10x_1+11x_2=115,\\
&7x_1-min\{4x_1-4l_1, 11x_1-11l_1\}+9x_2-min\{5x_2-5l_2, 14x_2-14l_2\}=56,\\
&max\{11x_1+11r_1, 4x_1+4r_1\}-7x_1+max\{14x_2+14r_2, 5x_2+5r_2\}-9x_2=149,\\
&7x_1-min\{3x_1-3l_1', 13x_1-13l_1'\}+9x_2-min\{3x_2-3l_2', 16x_2-16l_2'\}=75\\
&max\{13x_1+13r_1', 3x_1+3r_1'\}-7x_1+max\{16x_2+16r_2', 3x_2+3r_2'\}-9x_2=216,\\
&7x_1-min\{2x_1-2l_1'', 14x_1-14l_1''\}+9x_2-min\{x_2-l_2'', 18x_2-18l_2''\}=84,\\
&max\{14x_1+14r_1'', 2x_1+2r_1''\}-7x_1)+max\{18x_2+18r_2'', x_2+r_2''\}-9x_2)=297,\\
&\wedge[((0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6, \vee[(0.5\vee\eta_1)\vee(0.1\vee\eta_2)]=0.5, \vee[(0.4\vee\zeta_1)\vee(0.3\vee\zeta_2)]=0.4,\\
&10x_1-min\{6x_1-6l_1, 16x_1-16l_1\}+11x_2-min\{7x_2-7l_2, 17x_2-17l_2\}=70,\\
&max\{16x_1+16r_1, 6x_1+6r_1\}-10x_1+max\{17x_2+17r_2, 7x_2+7r_2\}-11x_2=198,\\
&10x_1-min\{2x_1-2l_1', 19x_1-19l_1'\}+11x_2-min\{5x_2-5l_2', 19x_2-19l_2'\}=101,\\
&max\{19x_1+19r_1', 2x_1+2r_1'\}-10x_1+max\{19x_2+19r_2', 5x_2+5r_2'\}-11x_2=284,\\
&10x_1-min\{x_1-l_1'', 20x_1-20l_1''\}+11x_2-min\{2x_2-2l_2'', 21x_2-21l_2''\}=112,\\
&max\{20x_1+20r_1'', x_1+r_1''\}-10x_1+max\{21x_2+21r_2'', 2x_2+2r_2''\}-11x_2=377,\\
&\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7, \vee[(0.2\vee\eta_1)\vee(0.3\vee\eta_2)]=0.5, \vee[(0.1\vee\zeta_1)\vee(0.4\vee\zeta_2)]=0.4,\\
&l_1\geq 0, r_1\geq 0, l_1'-l_1\geq 0, r_1'-r_1\geq 0, l_1''-l_1'\geq 0, r_1''-r_1'\geq 0, l_2\geq 0, r_2\geq 0, l_2'-l_2\geq 0, r_2'-r_2\geq 0,\\
&l_2''-l_2'\geq 0, r_2''-r_2'\geq 0, \chi_1, \eta_1, \zeta_1, \chi_2, \eta_2, \zeta_2\in[0,1].
\end{aligned}
$$

Step 5: *Using* $min\{a,b\}=\frac{a+b}{2}-|\frac{a-b}{2}|$, $max\{a,b\}=\frac{a+b}{2}+|\frac{a-b}{2}|$, *the FSNLP, problem obtained in step 4, can be rewritten as:*

$$
\begin{aligned}
&Minimize([\frac{(80+40\chi-40\eta-40\zeta+2\chi^2+2(\eta-1)^2+(\zeta-1)^2)}{12}x_1+\\
&\frac{(64+32\chi-32\eta-32\zeta+\chi^2+(\eta-1)^2+(\zeta-1)^2)}{6}x_2-\frac{11}{12}\chi^2 l_1-\frac{1}{3}\chi^2|x_1-l_1|-\frac{17}{12}\chi^2 l_2\\
&-\frac{5}{12}\chi^2|x_2-l_2|+\frac{11}{12}\chi^2 r_1+\frac{1}{3}\chi^2|x_1+r_1|+\frac{17}{12}\chi^2 r_2+\frac{5}{12}\chi^2|x_2+r_2|-\frac{11}{12}(\eta-1)^2 l_1'\\
&-\frac{5}{12}(\eta-1)^2|x_1-l_1'|-\frac{17}{12}(\eta-1)^2 l_2'-\frac{9}{12}(\eta-1)^2|x_2-l_2'|+\frac{11}{12}(\eta-1)^2 r_1'+\frac{5}{12}(\eta-1)^2|x_1+r_1'|\\
&+\frac{17}{12}(\eta-1)^2 r_2'+\frac{9}{12}(\eta-1)^2|x_2+r_2'|-\frac{21}{24}(\zeta-1)^2 l_1''-\frac{15}{24}(\zeta-1)^2|x_1-l_1''|-\frac{17}{12}(\zeta-1)^2 l_2''\\
&-\frac{13}{12}(\zeta-1)^2|x_2-l_2''|+\frac{21}{24}(\zeta-1)^2 r_1''+\frac{15}{24}(\zeta-1)^2|x_1+r_1''|+\frac{17}{12}(\zeta-1)^2 r_2''+\frac{13}{124}(\zeta-1)^2|x_2+r_2''|])
\end{aligned}
$$

subject to

$$7x_1+9x_2=87, 10x_1+11x_2=115,$$
$$-\frac{1}{2}x_1+\frac{15}{2}l_1+\frac{7}{2}|x_1-l_1|-\frac{x_2}{2}+\frac{19}{2}l_2+\frac{9}{2}|x_2-l_2|=56,$$
$$\frac{1}{2}x_1+\frac{15}{2}r_1+\frac{7}{2}|x_1+r_1|+\frac{x_2}{2}+\frac{19}{2}r_2+\frac{9}{2}|x_2+r_2|=149,$$
$$-x_1+8l'_1+5|x_1-l'_1|-\frac{x_2}{2}+\frac{19}{2}l'_2+\frac{13}{2}|x_2-l'_2|=75$$
$$x_1+8r'_1+5|x_1+r'_1|+\frac{x_2}{2}+\frac{19}{2}r'_2+\frac{13}{2}|x_2+r'_2|=216,$$
$$-x_1+8l''_1+6|x_1-l''_1|-\frac{x_2}{2}+\frac{19}{2}l''_2+\frac{17}{2}|x_2-l''_2|=84,$$
$$x_1+8r''_1+6|x_1+r''_1|+\frac{x_2}{2}+\frac{19}{2}r''_2+\frac{17}{2}|x_2+r''_2|=297,$$
$$\wedge[((0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6, \vee[(0.5\vee\eta_1)\vee(0.1\vee\eta_2)]=0.5, \vee[(0.4\vee\zeta_1)\vee(0.3\vee\zeta_2)]=0.4,$$
$$-x_1+11l_1+5|x_1-l_1|-x_2+12l_2+5|x_2-l_2|=70,$$
$$x_1+11r_1+5|x_1+r_1|+x_2+12r_2+5|x_2+r_2|=198,$$
$$-\frac{1}{2}x_1+\frac{21}{2}l'_1+\frac{17}{2}|x_1-l'_1|-x_2+12l'_2+7|x_2-l'_2|=101,$$
$$\frac{1}{2}x_1+\frac{21}{2}r'_1+\frac{17}{2}|x_1+r'_1|+x_2+12r'_2+7|x_2+r'_2|=284,$$
$$-\frac{1}{2}x_1+\frac{21}{2}l''_1+\frac{19}{2}|x_1-l''_1|-\frac{x_2}{2}+\frac{23}{2}l''_2+\frac{19}{2}|x_2-l''_2|=112,$$
$$\frac{1}{2}x_1+\frac{21}{2}r''_1+\frac{19}{2}|x_1+r''_1|+\frac{x_2}{2}+\frac{23}{2}r''_2+\frac{19}{2}|x_2+r''_2|=377,$$
$$\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7, \vee[(0.2\vee\eta_1)\vee(0.3\vee\eta_2)]=0.5, \vee[(0.1\vee\zeta_1)\vee(0.4\vee\zeta_2)]=0.4,$$
$$l_1\geq 0, r_1\geq 0, l'_1-l_1\geq 0, r'_1-r_1\geq 0, l''_1-l'_1\geq 0, r''_1-r'_1\geq 0, l_2\geq 0, r_2\geq 0, l'_2-l_2\geq 0, r'_2-r_2\geq 0,$$
$$l''_2-l'_2\geq 0, r''_2-r'_2\geq 0, \chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1].$$

Step 6:*By solving the crisp mathematical problem obtained in step 5, we get the optimal solution* $x_1=6, l_1=2, r_1=4, l'_1=4, r'_1=5, l''_1=5, r''_1=6, x_2=5, l_2=2, r_2=4, l'_2=3, r'_2=5, l''_2=4, r''_2=7, \chi_1=0.7, \eta_1=0.5, \zeta_1=0.3, \chi_2=0.9, \eta_2=0.3, \zeta_2=0.4$.
Step 7: *Substituting the values of* $x_1, l_1, r_1, l'_1, r'_1, l''_1, r''_1, x_2, l_2, r_2, l'_2, r'_2, l''_2, r''_2, \chi_1, \eta_1, \zeta_1, \chi_2, \eta_2$ *and* ζ_2 *in* $X_1=([x_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ *and* $X_2=([x_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *the exact* LR*-type single-valued neutrosophic optimal solution is* $X_1=([6; 2, 4; 4, 5; 5, 6]; 0.7, 0.5, 0.3)_{LR}, X_2=([5; 2, 4; 3, 5; 4, 7]; 0.9, 0.3, 0.4)_{LR}$.
Step 8: *By substituting the values of* X_1 *and* X_2*, obtained in Step 7, into the objective function, the* LR*-type single-valued neutrosophic optimal value is* $([140; 76, 208; 112, 296; 133, 436]; 0.7, 0.5, 0.5)_{LR}$.

5 Conclusion

In this paper, we have applied the concept of neutrosophic sets to the LPPs. We have defined unrestricted LR-type SNNs and their arithmetic operations. We have developed ranking function of the LR-type SNN. We have proposed a method to solve the FSNLP problems with equality constraints having unrestricted LR-type SNNs as right hand side, parameters and variables. We have solved numerical examples to explain it which satisfies the given constraints.

Conflicts of interest: The authors declare no conflicts of interest.

Acknowledgment: Author is thankful to Prof. Akram and Dr. Bashir for their valuable suggestions during the preparation of this paper

References

[1] Abdel-Basset, M., Gunasekaran, M., Mohamed, M., and Smarandache, F., *A novel method for solving the fully neutrosophic linear programming problems*, Neural Computing and Applications, Springer, **31**(6)(2018), 1595-1605.

[2] Abhishekh and Nishad, A. K., *A novel ranking approach to solving fully LR-intuitionistic fuzzy transportation problem*, New Mathematics and Natural Computation, 15(1)(2019), 95-112.

[3] Ahmad, J., Alharbi, M.G., Akram, M. and Bashir, S., *A new method to evaluate linear programming problem in bipolar single-valued neutrosophic environment*, Computer Modeling in Engineering and Sciences. 2021, DOI:10.32604/cmes.2021.017222.

[4] Akram, M., Ullah, I., Edalatpanah, S.A., and Allahviranloo, T., *Fully Pythagorean fuzzy linear programming problems with equality constraints*, Computational and Applied Mathematics, 2021, DOI: 10.1007/s40314-021- 01503-9.

[5] Akram, M., Ullah, I., Edalatpanah, S.A., and Allahviranloo, T., *LR-type Pythagorean fuzzy linear programming problems*, Journal of Intelligent and Fuzzy Systems, **2021**, DOI: 10.3233/JIFS-210655.

[6] Allahviranloo, T., Lotfi, F. H., Kiasary, M. K., Kiani, N. A., and Zadeh, L. A., *Solving fully fuzzy linear programming problem by the ranking function*, Applied Mathematical Sciences, **2**(1)(2008), 19-32.

[7] Angelov, P.P., *Optimization in an intuitionistic fuzzy environment*, Fuzzy Sets and Systems, **86**(3)(1997), 299-306.

[8] Atanassov, K.T., *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1986), 87-96.

[9] Behera, D., Peters, K., Edalatpanah, S. A., and Qiu, D. *New methods for solving imprecisely defined linear programming problem under trapezoidal fuzzy uncertainty*, Journal of Information and Optimization Sciences, (7)(2020), 603-629.

[10] Bellman, R. E. and Zadeh, L. A., *Decision making in a fuzzy environment*, Management Science, **17**(1970), 141-164.

[11] Bera, T., and Mahapatra, N., K., *Generalised Single Valued Neutrosophic Number and Its Application to Neutrosophic Linear Programming*, Neutrosophic Sets and Systems, **25**(1)(2019).

[12] Bera, T., and Mahapatra, N., K., *Neutrosophic linear programming problem and its application to real life*, Afrika Matematika, **31**(2020),709-726.

[13] Bharati, S. K., and Singh, S. R., *Solution of multiobjective linear programming problems in interval-valued intuitionistic fuzzy environment*, Soft Computing, **23**(1)(2019), 77-84.

[14] Chakraborty, A., Mondal, P., S., Alam, S., Dey, A., *Classifcation of trapezoidal bipolar neutrosophic number, de bipolarization technique and its execution in cloud service based MCGDM problem*, Complex and Intelligent Systems, **7**(8)(2020), 145-162.

[15] Das, S.K., Chakraborty, A., *A new approach to evaluate linear programming problem in pentagonal neutrosophic environment*, Complex Intell. Syst. **7**(2021), 101-110.

[16] Das, S. and Dash, J. K., *Modified solution for neutrosophic linear programming problems with mixed constraints*, International Journal of Research in Industrial Engineering, **9**(1)(2020), 13-24.

[17] Das, S., Mandal, T. and Edalatpanah, S. A., *A mathematical model for solving fully fuzzy linear programming problem with trapezoidal fuzzy numbers*, Applied Intelligence, **46**(2017), 509-519.

[18] Dubey, D., Mehra, A., *Linear programming with triangular intuitionistic fuzzy number*, Advances in Intelligent Systems Research, **1**(1)(2011), 563-569.

[19] Edalatpanah, S. A., *A data envelopment analysis model with triangular intuitionistic fuzzy numbers*, International Journal of Data Envelopment Analysis, **7**(4)(2019), 47-58.

[20] Edalatpanah, S. A., *A direct model for triangular neutrosophic linear programming*, International journal of neutrosophic science, **1**(1)(2020), 19-28.

[21] Garg, H., Rani, M., Sharma, S. P., Vishwakarma, Y., *Intuitionistic fuzzy optimization technique for solving multi-objective reliability optimization problems in interval environment*, Expert Systems with Applications, **41**(7)(2014), 3157-3167.

[22] Hussian, A.N., Mohamed, M., Baset, M.A., Smarandache, F., *Neutrosophic linear programming problem* Math. Sci. Lett. **6**(3),(2017) 319-324.

[23] Kabiraj, Nayak, A., P.K., and Raha, S., *Solving intuitionistic fuzzy linear programming problem*, International Journal of Intelligence Science, **9**(2019), 44-58.

[24] Kaur, J., and Kumar, A., *An introduction to fuzzy linear programming problems*, Springer Science and Business Media LLC, (2016).

[25] Kaur, J., and Kumar, A., *Exact fuzzy optimal solution of fully fuzzy linear programming problems with unrestricted fuzzy variables.*, Applied Intelligence, **37**(2012), 145-154.

[26] Kaur, J. and Kumar, A., *Mehar's method for solving fully fuzzy linear programming problems with LR fuzzy parameters*, Applied Mathematical Modelling, **37**(2013), 7142-7153.

[27] Khalifa, Abd El-Wahed., H., and Kumar, P., *Solving fully neutrosophic linear programming problem with application to stock portfolio selection*, Croatian Operational Research Review, 11(2020), 165-176.

[28] Kumar, A., Kaur, J., and Singh, P., *A new method for solving fully fuzzy linear programming problems*, Applied Mathematical Modelling, **35**(2011),817-823.

[29] Lotfi, F.H., Allahviranloo, T., Jondabeh, M.A. and Alizadeh, L., *Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution*, Applied Mathematical Modelling, **33**(7),(2009) 3151-3156.

[30] Malathi, C. and Umadevi, P., A new procedure for solving linear programming problems in an intuitionistic fuzzy environment, IOP Conf. Series: Journal of Physics: Conf. Series, 1139(2018)012079.

[31] Mahapatra, G.S. and Roy, T.K., *Reliability evaluation using triangular intuitionistic fuzzy numbers arithmetic operations*, World Acadmy of Science, Engineering and Technology, **50**(2009),574-581.

[32] Mahapatra, G. S. and Roy, T. K., *Intuitionistic fuzzy number and its arithmetic operation with application on system failure*, Journal of Uncertain Systems, **7**(2)(2013), 92-107.

[33] Mehmood, M. A., Akram, M., Alharbi, M. S., and Bashir, S., *Solution of fully bipolar fuzzy linear programming models*, Mathematical Problems in Engineering, **2021**(2021), 31 pages.

[34] Mehmood, M. A., Akram, M., Alharbi, M. S., and Bashir, S., *Optimization of LR-type fully bipolar fuzzy linear programming problems*, Mathematical Problems in Engineering, **2021**(2021), Article ID 1199336, 36 pages.

[35] Moloudzadeh, S., Allahviranloo, T., Darabi, P., *A new method for solving an arbitrary fully fuzzy linear system*, Soft Computing, **17**(9)(2013), 1725-1731.

[36] Nagoorgani, A., and Ponnalagu, K., *A new approach on solving intuitionistic fuzzy linear programming problem*, Applied Mathematical Sciences, **6** (70)(2012), 3467-3474.

[37] Nagoorgani, A., Kavikumar, J. and Ponnalagu, K., *The knowledge of expert opinion in intuitionistic fuzzy linear programming problem*, Mathematical Problems in Engineering, (2015), Article ID 875460.

[38] Najafi, H. S., and Edalatpanah, S. A., *A note on a new method for solving fully fuzzy linear programming problems*, Applied mathematical modelling, **37**(14-15)(2013), 7865-7867.

[39] Najafi, H. S., Edalatpanah, S. A., and Dutta, H., *A nonlinear model for fully fuzzy linear programming with fully unrestricted variables and parameters*, Alexandria engineering journal, **55**(3)(2016), 2589-2595.

[40] Parvathi, R., Malathi, C., Akram, M. and Atanassov, K.T., *Intuitionistic fuzzy linear regression analysis*, Fuzzy Optimization and Decision Making, **12**(2)(2013), 215-229.

[41] Parvathi, R. and Malathi, C., *Intuitionistic fuzzy linear programming problems*, World Applied Sciences Journal, **17**(12)(2012),1787-1791.

[42] Prez-Caedo, B. and Concepcin-Morales, E.R., *On LR-type fully intuitionistic fuzzy linear programming with inequality constraints*, Solutions with unique optimal values, Expert Systems with Applications, **128**(2019), 246-255.

[43] Prez-Caedo, B., Concepcin-Morales, E. R. and Edalatpanah, S. A., *A revised version of a lexicographical-based method for solving fully fuzzy linear programming problems with inequality constraints*, Fuzzy Information and Engineering, (2020), 1-20. https://doi.org/10.1080/16168658.2020.1761511

[44] Singh, V. and Yadav S. P., *Development and optimization of unrestricted LR-type intuitionistic fuzzy mathematical programming problems*, Expert Systems With Applications, **80**(2017), 147-161.

[45] Singh, S. K. and Yadav, S. P., *Intuitionistic fuzzy multi-objective linear programming problem with various membership functions, Annals of operations research*, **269**(1-2)(2018), 693-707.

[46] Smarandache, F., *Neutrosophy, neutrosophic probability, set and logic*, American Research Press, Rehoboth, USA, (1998).

[47] Smarandache, F., *A unifying field in logics., Neutrosophic probability, set and logic*, Rehoboth: American Research Press, (1999).

[48] Subas, Y., *Neutrosophic numbers and their application to multi-attribute decision making problems*, Masters Thesis, Kilis 7 Aralik University, Graduate School of Natural and Applied Science, (in Turkish), (2015).

[49] Suresh, M., Vengataasalam, S. and Arun, P.K., *Solving intuitionistic fuzzy linear programming problems by ranking function*, Journal of Intelligent and Fuzzy Systems, **27**(6)(2014), 3081-3087.

[50] Tanaka, H., Okudu, T. and Asai, K., *On fuzzy-mathematical programming, Journal of Cybernetics*, **3**(4)(1973), 37-46.

[51] Veeramani, P. and Ganesan, K., *Fuzzy linear programs with trapezoidal fuzzy numbers*, Annals of Operations Research, **143**(1),(2006) 305-315.

[52] Wang, H., Smarandache, F., Zhang, Y.Q. and Sunderraman, R., *Single-valued neutrosophic sets*, Multispace and Multistructure, **4**(2010), 410-413.

[53] Zadeh, L.A., *Fuzzy sets*, Information and Control, **8**(3)(1965), 338-353.

[54] Zimmerman, H. J., *Fuzzy programming and linear programming with several objective functions*, Fuzzy Sets and Systems, **1**(1)(1978), 45-55.

Received: May 20, 2021. Accepted: October 3, 2021

NSS

Neutrosophic Sets and Systems, Vol. 46, 2021

University of New Mexico

Introduction to NeutroNearrings

Vadiraja Bhatta G. R.[1], Manasa K. J. [2], Gautham Shenoy B. [3], Prasanna Poojary[4] *, Chaithra B. J.[5]

[1] Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India; vadiraja.bhatta@manipal.edu;

[1] Center for Cryptography, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India

[2] Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru, Karnataka, India; manasakj123@gmail.com

[3,4,5] Department of Mathematics, Poornaprajna College Udupi, Karnataka India; gautham.shenoy1996@gmail.com; poojaryprasanna34@gmail.com; chaithra.yelchithaya@gmail.com

[4] Department of Mathematics, Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education, Karnataka, India

* Correspondence: poojaryprasanna34@gmail.com

Abstract. Algebraic concepts and structures are enriched with the special types of operations and axioms known as NeutroOperations and NeutroAxioms. Various types of NeutroAlgebras are studied using several such defined concepts. The objective of this paper is to introduce the concept of NeutroNearrings. Several interesting results and examples of NeutroNearrings, NeutroSubRings, NeutroQuotientNearrings and NeutroNearringHomomorphisms are presented.

Keywords: Nearring; NeutroRings; NeutroNearring; Neutrosophy.

---

1. Introduction

The NeutroDefined and AntiDefined Laws, as well as the NeutroAxioms and AntiAxioms was first time introduced in 2019 by Smarandache [3, 5]. This concept has given birth to new fields of research called NeutroStructures and AntiStructures. For basic and recent results on Neutrosophy, NeutroAlgebraic structures and AntiAlgebraic structures we refer [4–8].

In [2],Agboola formally presented the notion of NeutroGroups. In this he showed that in general, Lagrange's theorem and first isomorphism theorem of the classical groups do not hold

in the NeutroGroups. Also in [1], Agboola studied NeutroRing, NeutroSubring, NeutroQuotientRings and he proved the 1st isomorphism theorem of the classical rings for this class of NeutroRing.
The present paper will be concerned with the introduction of NeutroNearrings.

2. Preliminaries

Definition 2.1. [7]

(i) A classical operation is an operation well defined for all the set's elements while a NeutroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set. An AntiOperation is an operation that is outer defined for all the set's elements.
(ii) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no AntiOperation or AntiAxiom. An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

Definition 2.2. A NeutroGroup is a nonempty set G with binary operation $*$ satisfying following conditions:

(i) The $*$ is **NeutroAssociative** if there exists atleast one triplet $(a, b, c) \in G$ such that

$$a * (b * c) = (a * b) * c$$

and there exists atleast one triplet $(x, y, z) \in G$ such that

$$x * (y * z) \neq (x * y) * z$$

(ii) There exists a **NeutroNeutral element** in G if at least one of the below statements occurs:
- There exists at least one element x that has no unit-element.
- There exists at least one element $b \in G$ that has at least two distinct unit-elements $e_1, e_2 \in G, e_1 \neq e_2$ such that:

$$b * e_1 = e_1 * b = b$$

$$b * e_2 = e_2 * b = b$$

- There exists at least two different elements $r, s \in G, r \neq s$, such that they have different unit elements $e_r, e_s \in G, e_r \neq e_s$, with $e_r * r = r * e_r = r$ and $e_s * s = s * e_s = s$

(iii) There exists a **NeutroInverse element** in G if there is an element $a \in G$ that has an inverse $b \in G$ with respect to a unit element $e \in G$ that is

$$b * a = a * b = e$$

or there exists atleast one element $b \in G$ that has two or more inverses $c, d \in G$ with respect to some unit element $u \in G$ that is

$$b * c = c * b = u$$

$$b * d = d * b = u$$

In addition, if $*$ is **NeutroCommutative** that is there exists atleast a duplet $(a, b) \in G$ such that

$$a * b = b * a$$

and there exists atleast a duplet $(c, d) \in G$ such that

$$c * d \neq d * c$$

then $(G, *)$ is called a NeutroCommutative group or NeutroAbelian group.
If condition (i) is satisfied, then $(G, *)$ is called a NeutroSemiGroup and if conditions (i) and (ii) are satisfied, then $(G, *)$ is called a NeutroMonoid [1].

Definition 2.3. Let R be a nonempty set and let $+, \cdot : R \times R \to R$ be binary operations of ordinary addition and multiplication on R. Then $\cdot$ is both left and right NeutroDistributive over $+$ that is there exists atleast a triplet $(a, b, c) \in R$ and atleast a triplet $(d, e, f) \in R$ such that

$$a.(b + c) = a.b + a.c$$

$$d.(e + f) \neq d.e + d.f$$

then . is **left NeutroDistributive** over + on a set$R$.
Suppose if there exists atleast a triplet $(p, q, r) \in R$ and atleast a triplet $(x, y, z) \in R$ such that

$$(p + q).r = p.r + q.r$$

$$(x + y).z \neq x.z + y.z$$

then the binary operation . is said to be **right NeutroDistributive** over + on a set R.

A **right Nearring** is a set N together with two binary operations $+$ and $\cdot$ such that:

(1) $(N, +)$ is a group (not necessarily abelian)
(2) $(N, \cdot)$ is a semigroup

(3) For all $n_1, n_2, n_3 \in N : (n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$ (right distribution law).

If $n_1 \cdot (n_2 + n_3) = n_1 \cdot n_2 + n_1 \cdot n_3$ instead of condition (3) then set N is a **left Nearring**.

NeutroNearring and their properties

A NeutroNearring is a Nearring that has either a Neutro-operation or a Neutro-axiom. In this paper we define NeutroNearing as below.

Definition 2.4. Let N be a nonempty set and let $+, \cdot : N \times N \to N$ be binary operations of ordinary addition and multiplication on N. The triple $(N, +, \cdot)$ is called a **left NeutroNearring** if the following conditions are satisfied:

(i) $(N, +)$ is a NeutroGroup (not necessarily abelian)

(ii) $(N, \cdot)$ is a NeutroSemiGroup

(iii) the left NeutroDistributive law holds in N: that is there exists atleast one triplet $(a, b, c) \in N$ and atleast one triplet $(d, e, f) \in N$ such that:

- $a \cdot (b + c) = a \cdot b + a \cdot c$
- $d \cdot (e + f) \neq d \cdot e + d \cdot f$

Remark 2.5. If right NeutroDistributive law holds in N: that is there exists atleast one triplet $(p, q, r) \in N$ and atleast one triplet $(s, t, u) \in N$ such that:

- $(p + q) \cdot r = p \cdot r + q \cdot r$
- $(s + t) \cdot u \neq s \cdot u + t \cdot u$

then N is called **right NeutroNearring**.

Example 2.6. Let $X = \mathbb{Z}_{12}$ and let $\oplus$ and $\odot$ be two binary operations on X defined by $x \oplus y = x + 2y$ and $x \odot y = x + 4y$ for all $x, y \in X$ where "+" is addition modulo 12. Then $(X, \oplus, \odot)$ is a NeutroNearring.

Example 2.7. Let $X = \{a, b, c\}$ with "+" and "·" be binary operations defined on X as shown in the Cayley tables below:

+	a	b	c
a	c	c	b
b	c	b	c
c	c	c	b

·	a	b	c
a	b	a	a
b	a	c	a
c	a	a	b

It is clear from the table that :

$$(a+b)+c = a+(b+c) = b,$$
$$(c+a)+b = c, \text{ but } c+(a+b) = b \neq c$$

This shows that $(X,+)$ is a NeutroSemiGroup.

Next, let N_x and I_x represent additive neutral and additive inverse element respectively with respect to any element $x \in X$. Then

$$N_b = b$$
$$I_b = b$$
$$N_a \text{ does not exist,}$$
$$I_b \text{ does not exist.}$$

Hence, $(X,+)$ is a NeutroGroup.

Next, consider

$$b(cb) = (bc)b$$
$$a(bc) = b \text{ but } (ab)c = a \neq b$$

This shows that $(X,\cdot)$ is NeutroAssociative.

Lastly, consider

$$b.(b+b) = b.b + b.b = c,$$
$$a.(b+c) = a, \text{but } a.b + a.c = c \neq a$$

This shows that "$\cdot$" is left distributive over "$+$". Hence, $(X,+,\cdot)$ is a left NeutroNearring.

Note 2.8. *Every NeutroRing is a NeutroNearring.*

Notation: Let N be a NeutroNearring $d \in N$ is called NeutroDistributive if there exist atleast two pairs (n_1,n_2) *and* $(m_1,m_2) \in N$ such that $(n_1+n_2)d = n_1d+n_2d$ *and* $(m_1+m_2)d \neq m_1d+m_2d$. *Let* $N_d = \{d \in N | \text{d is NeutroDistributive }\}$.

Remark 2.9. Let $(N,+,\cdot)$ be left NeutroNearring

(i) If $(N,+)$ is NeutroAbelian, then N is a NeutroAbelian NeutroNearring.
(ii) If $(N,.)$ is NeutroCommutative then N is a NeutroCommutative NeutroNearring.
(iii) If $N = N_d$ then N is said to be NeutroDistributive.
(iv) If $(N^*,\cdot)$ where $N^* = N \setminus \{0\}$ is a NeutroGroup then N is called NeutroNearfield.

Theorem 2.10. *Let* $(N_i,+,), i = 1,2,\ldots,n$ *be a family of NeutroNearrings. Then*

(1) $N = \cap_{i=1}^{n} N_i$ *is a NeutroNearring.*

(2) $N = \prod_{i=1}^{n} N_i$ *is a NeutroNearring.*

(1) *Proof.* Obvious □

(2) *Proof.* Proof is by induction on n.

For $n = 1$ result is trivial. Let $n = 2$. Consider $N = N_1 \times N_2$ then is closed with respect to coordinate wise addition and coordinate wise multiplication. Note that there exist $n_1 \in N_1$ such that $n_1 + e_1 = n_1$ and there exist $n_2 \in N_2$ such that $n_2 + e_2 = n_2$.
Also, there doesnot exist additive identity for $n_1^{'} \in N_1$ and $n_2^{'} \in N_2$.
But $(n_1, n_2) \in N$ such that $(n_1, n_2) + (e_1, e_2) = (n_1, n_2)$ and there doesnot exist additive identity for $(n_1^{'}, n_2^{'}) \in N$.
Similarly one can observe the existence of NeutroAdditive inverse in N.
Since $(N_1, +)$ and $(N_2, +)$ are NeutroAssociative, there exist $a_1, b_1, c_1, a_1^{'}, b_1^{'}, c_1^{'} \in N_1$ and $a_2, b_2, c_2, a_2^{'}, b_2^{'}, c_2^{'} \in N_2$ such that $a_1 + (b_1 + c_1) = (a_1 + b_1) + c_1$
$a_2 + (b_2 + c_2) = (a_2 + b_2) + c_2$
$a_1^{'} + (b_1^{'} + c_1^{'}) \neq (a_1^{'} + b_1^{'}) + c_1^{'}$
$a_2^{'} + (b_2^{'} + c_2^{'}) \neq (a_2^{'} + b_2^{'}) + c_2^{'}$
Now, $(a_1, a_2), (b_1, b_2), (c_1, c_2), (a_1^{'}, a_2^{'}), (b_1^{'}, b_2^{'}), (c_1^{'}, c_2^{'}) \in N$ such that
$(a_1, a_2) + [(b_1, b_2) + (c_1, c_2)] = (a_1, a_2) + [(b_1 + c_1, b_2 + c_2)] = (a_1 + (b_1 + c_1), a_2 + (b_2 + c_2))$
$= ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2) = (a_1 + b_1, a_2 + b_2) + (c_1, c_2)$
$= [(a_1, a_2) + (b_1, b_2)] + (c_1, c_2)$
and $(a_1^{'}, a_2^{'}) + [(b_1{'}, b_2^{'}) + (c_1^{'}, c_2^{'})] \neq [(a_1^{'}, a_2^{'}) + (b_1{'}, b_2^{'})] + (c_1^{'}, c_2^{'})$
Similarly we prove $(N, .)$ is NeutroAssociative.
Further there exist $x_1, y_1, z_1, x_1^{'}, y_1^{'}, z_1^{'} \in N_1$ and $x_2, y_2, z_2, x_2^{'}, y_2^{'}, z_2^{'} \in N_2$ such that
$x_1.(y_1 + z_1) = x_1.y_1 + x_1.z_1$
$x_2.(y_2 + z_2) = x_2.y_2 + x_2.z_2$
$x_1^{'}.(y_1^{'} + z_1^{'}) \neq x_1^{'}.y_1^{'} + x_1^{'}.z_1^{'}$
$x_2^{'}.(y_2^{'} + z_2^{'}) \neq x_2^{'}.y_2^{'} + x_2^{'}.z_2^{'}$
But then, $(x_1, x_2), (y_1, y_2), (z_1, z_2), (x_1^{'}, x_2^{'}), (y_1^{'}, y_2^{'}), (z_1^{'}, z_2^{'}) \in N$ such that
$(x_1, x_2).[(y_1, y_2) + (z_1, z_2)] = (x_1, x_2).(y_1 + z_1, y_2 + z_2)$
$= (x_1(y_1 + z_1), x_2(y_2 + z_2)) = (x_1.y_1 + x_1.z_1, x_2.y_2 + x_2.z_2)$
$= (x_1.y_1, x_2.y_2) + (x_1.z_1, x_2.z_2) = (x_1, x_2).(y_1, y_2) + (x_1, x_2).(z_1, z_2)$
and $(x_1^{'}, x_2^{'}).[(y_1^{'}, y_2^{'}) + (z_1^{'}, z_2^{'})] \neq (x_1^{'}, x_2^{'}).(y_1^{'}, y_2^{'}) + (x_1^{'}, x_2^{'}).(z_1^{'}, z_2^{'})$
$\therefore$ N is a NeutroNearring for $n = 2$.
Let $n > 2$. Assume the result for $n - 1$. Note that $M = \prod_{i=1}^{n-1} N_i$ forms a NeutroNearring with respect to coordinate wise addition and coordinate wise multiplication.
But then, $N = M \times N_n$ forms a NeutroNearring with respect to coordinate wise addition and coordinate wise multiplication. □

Definition 2.11. Let $(N, +, .)$ be a NeutroNearring. A nonempty subset S of N is called a **NeutroNearSubring** of N if $(S, +, .)$ is also a NeutroNearring. The only trivial NeutroNearSubring of N is N.

Theorem 2.12. *Let $(N, +, .)$ be a NeutroNearring and let $\{S_i\}, i = 1, 2, \ldots, n$ be a family of NeutroNearSubrings of N. Then*

(1) *$S = \cap_{i=1}^{n} S_i$ is a NeutroNearSubring of N.*
(2) *$S = \prod_{i=1}^{n} S_i$ is a NeutroNearSubring of N.*

Proof. Both result follows directly from Theorem 2.10. □

Definition 2.13. Let $(N, +, .)$ be a NeutroNearring. A nonempty subset I of R is called a **left NeutroNearIdeal** of N if the following conditions hold:

(1) I is a NeutroNearSubring of N.
(2) There exist $x \in I$ such that $xr \in I, \forall r \in N$.

Definition 2.14. Let $(N, +, .)$ be a NeutroNearring. A nonempty subset I of N is called a **right NeutroNearIdeal** of N if the following conditions hold:

(1) I is a NeutroNearSubring of N.
(2) There exist $x \in I$ such that $rx \in I, \forall r \in N$

Definition 2.15. Let $(N, +, .)$ be a NeutroNearring. A nonempty subset I of N is called a **NeutroNearIdeal** of N if the following condition hold:

(1) I is a NeutroNearSubring of N.
(2) There exist $x \in I$ such that $xr, rx \in I, \forall r \in N$

Theorem 2.16. *Let $(N, +, .)$ be a NeutroNearring and let $\{I_i\}, i = 1, 2, \ldots, n$ be a family of NeutroNearIdeals of N. Then*

(1) *$I = \cap_{i=1}^{n} I_i$ is a NeutroNearIdeal of N.*
(2) *$I = \sum_{i=1}^{n} I_i$ is NeutroNearIdeal of N.*

(1) *Proof.* Since each $I_i, 1 \leq i \leq n$ is a NeutroNearSubring of N, it follows from Theorem 1.8 that $I = \cap_{i=1}^{n} I_i$ is a NeutroNearSubring of N.

Note that there exist $x_i \in I_i$ such that $x_i r, r x_i \in I_i,\ \forall r \in N$ and $\forall i, 1 \leq i \leq n$.
Let $y = x_1^2 x_2^2 \cdots x_n^2$. Then $y \in I_i,\ \forall i, 1 \leq i \leq n$.
For any $r \in N$ $ry, yr \in I_i,\ \forall i, 1 \leq i \leq n$
$\therefore y \in I$ with $ry, yr \in I$. □

(2) Obivous.

Definition 2.17. Let $(N,+,.)$ be a NeutroNearring and let I be a NeutroNearIdeal of N. The set N/I is defined by

$$N/I = \{x + I : x \in N\}$$

.

for $x + I, y + I \in N/I$ with at least a pair $(x, y) \in N$, let $\oplus$ and $\odot$ be binary operations on N/I defined as follows:

$$(x + I) \oplus (y + I) = (x + y) + I$$

$$(x + I) \odot (y + I) = xy + I$$

Then it can be shown that the tripple $(N/I, \oplus, \odot)$ is a NeutroNearring which we call Neutro-QuotientNearring.

Theorem 2.18. *Let I be a NeutroNearIdeal of the NeutroNearring $(N,+,.)$. Suppose N is NeutroCommutative NeutroNearring with Neutro unity then so is N/I.*

Proof. There exist $a, b, c, d \in N$ such that $ab = ba$ and $cd \neq dc$.
But then $a + I, b + I, c + I, d + I \in N/I$ such that $(a + I)(b + I) = ab + I$
$= ba + I = (b + I)(a + I)$ and $(c + I)(d + I) = cd + I \neq dc + I = (d + I)(c + I)$
Let e_y be a Neutro unity of N. Then there exist $y \in N$ such that $ye_y = e_y y = y$
But then $y + I \in N/I$ such that $(y + I)(e_y + I) = ye_y + I$
$= y + I = (e_y + I)(y + I)$
$\therefore N/I$ is NeutroCommutative NeutroNearring with Neutro unity $e_y + I$. □

Definition 2.19. Let $(N,+,.)$ and $(S,\oplus,\odot)$ be any two NeutroNearrings. The mapping $\phi : N \to S$ is called a NeutroNearringHomomorphism if ϕ preserves the binary operations of N and S that is if for at least a pair $(x, y) \in N$, we have:

$$\phi(x + y) = \phi(x) \oplus \phi(y)$$

$$\phi(x.y) = \phi(x) \odot \phi(y)$$

The kernel of ϕ denoted by $\ker\phi$ is defined as $\ker\phi = \{x \in N : \phi(x) = e_N\}$
Where $e_N \in N$ is a neutral element for at least one $n \in N$. The image of ϕ denoted by $Im\phi$ is defined as

$$Im\phi = \{y \in S : y = \phi(x)\}$$

for at least one $x \in N$. If in addition ϕ is a NeutroBijection, then ϕ is called a NeutroNearringIsomorphism and we write $N \cong S$. NeutroNearringEpimorphism, NeutroNearringMonomorphism, NeutroNearringEndomorphism and NeutroNearringAutomorphism are defined similarly.

Theorem 2.20. *Let R and S be two NeutroNearrings. Let $N_x = e_R$ for at least one $x \in R$ and let $N_y = e_S$ for at least one $y \in S$. Suppose that $\phi : R \to S$ is a NeutroNearHomomorphism. Then:*

(1) *$\phi(e_R)$ is not necessarily equals e_S.*
(2) *Ker ϕ is a NeutroNearSubring of R.*
(3) *Im ϕ is not necessarily a Neutro Near Subring of S.*
(4) *ϕ is NeutroInjective if and only if Ker $\phi = \{e_R\}$ for at least one $e_R \in R$.*

Theorem 2.21. *Let I be a NeutroNearIdeal of a NeutroNearring $(N, +, .)$. Then the mapping $\phi : N \to N/I$ defined by $\phi(x) = x + I$ is a NeutroNearringIsomorphism with $Ker\phi = I$*

Proof. For atleast one pair x, y in N,
$\phi(x + y) = (x + y) + I = (x + I) + (y + I) = \phi(x) + \phi(y)$
and $\phi(xy) = xy + I = (x + I)(y + I) = \phi(x)\phi(y)$
$Ker\phi = \{x \in N|\phi(x) = e_{N/I}\}$, where $e_{N/I} \in N/I$ such that $N_r = e_{N/I}$ for atleast one $r \in N/I$
$. = \{x \in N|x + I = e_{N/I}\} = \{x \in N|x \in I\} = I$ □

Theorem 2.22. *Let $\phi : R \to S$ be a NeutroNearringHomomorphism and let $K = Ker\phi$. Then the mapping $\psi : R/K \to Im\phi$ defined by $\psi(x + K) = \phi(x)$ is a NeutroNearringIsomorphism.*

Proof. For atleast one pair $x, y \in R$
$\psi((x + K) + (y + K)) = \psi((x + y) + K) = \phi(x + y) = \phi(x) + \phi(y)$
$= \psi(x + K) + \psi(y + K)$ and $\psi((x + K)(y + K)) = \psi((xy) + K) = \phi(xy) = \phi(x)\phi(y)$
$= \psi(x + K)\psi(y + K)$
Also $Ker\psi = \{x + K \in R/K : \psi(x + K) = e_{Im\phi}\}$where $e_{Im\phi} \in Im\phi$ such that $N_r = e_{Im\phi}$ for atleast $r \in Im\phi$.
$= \{x + K \in R/K : \phi(x) = e_{Im\phi}\} = \{e_{R/K}\}$
Thus ψ is a NeutroBijectiveNearringHomomorphism. □

Note 2.23. *The above map ϕ is an epimorphism. So, we can treat ϕ as NeutroNearringEpimorphism.*

Theorem 2.24. *NeutroNearringIsomorphism of NeutroNearrings is an equivalence relation.*

Proof. Define a relation $\sim$ as follows:
For any two NeutroNearrings N and $N^{'}$, we say $N \sim N^{'}$ iff there exist a NeutroNearringIsomorphism between N and $N^{'}$. Clearly $\sim$ is reflexive.
Suppose NeutroNearrings $N, N^{'}$ are such that $N \sim N^{'}$, let $f : N \to N^{'}$ be NeutroNearringIsomorphism.
Then there exist $x, y \in N$ such that $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.
Now, $f(x), f(y) \in N^{'}$ and
$f^{-1}(f(x) + f(y)) = f^{-1}(f(x + y)) = x + y = f^{-1}(f(x)) + f^{-1}(f(y))$
$f^{-1}(f(x)f(y)) = f^{-1}(f(xy)) = xy = f^{-1}(f(x))f^{-1}(f(y))$
$\therefore N^{'} \sim N$ and f^{-1} is a NeutroNearringIsomorphism.
Let $f : N \to N^{'}$ and $g : N^{'} \to N^{''}$ be NeutroNearringIsomorphisms.
Then $g \circ f : N \to N^{''}$ is bijective and there exist $x^{'}, y^{'} \in N^{'}$ such that $g(x^{'} + y^{'}) = g(x^{'}) + g(y^{'})$ and $g(x^{'}y^{'}) = g(x^{'})g(y^{'})$
Now, $x^{'} = f(x), y^{'} = f(y)$ for some $x, y \in N$ with $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$
Consider,
$g \circ f(x + y) = g(f(x + y)) = g(f(x) + f(y)) = g(x^{'} + y^{'}) = g(x^{'}) + g(y^{'}) = g(f(x)) + g(f(y))$
$= g \circ f(x) + g \circ f(y)$
and $g \circ f(xy) = g(f(xy)) = g(f(x)f(y)) = g(x^{'}y^{'}) = g(x^{'})g(y^{'}) = g(f(x))g(f(y))$
$= g \circ f(x)g \circ f(y)$
$\therefore N \sim N^{''}$ □

3. Conclusion

We have introduced in this paper the concept of NeutroNearrings by considering three NeutroAxioms(NeutroGroup(additive)), NeutroSemigroup(multiplicative) and left and right NeutroDistributive laws(multiplication over addition). Several intresting results and examples on NeutroNearrings, NeutroSubrings, NeutroQuotientNearrings and NeutroNearringHomomorphisms are presented.

Funding: “Fourth author and fifth author received funding from VGST/KSTePS and DST, GoK, India”

Acknowledgments: Fourth author and fifth author thanks VGST/KSTePS and DST, GoK, India for sanctioning the project [VGST/GRD-897/2019-20/2020-21]. The authors are grateful to Prof. Dr. Florentin Smarandache from University of New Mexico for his valuable ideas and guidance.

Conflicts of Interest: “The authors declare no conflict of interest.”

References

1. Agboola, A. A. A. (2020). Introduction to NeutroRings. International Journal of Neutrosophic Science, 7(2), 62-73.
2. Agboola, A. A. A. (2020). Introduction to NeutroGroups. International Journal of Neutrosophic Science, 6(1), 41-47.
3. Smarandache, Florentin,(2019). Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures, in his book "Advances of Standard and Nonstandard Neutrosophic Theories", Pons Ed., Brussels, European Union.
4. Agboola, A. A. A., Davvaz, B. (2014). Introduction to Neutrosophic Nearrings, ViXra, 1-14.
5. Smarandache, F. (2020). Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited). Neutrosophic Sets and Systems, 31, 1-16.
6. Smarandache, F. (1998). Neutrosophy. / Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105.
7. Smarandache, F. (2020). NeutroAlgebra is a Generalization of Partial Algebra, International Journal of Neutrosophic Science, 2(1), 08-17.
8. Kandasamy V. W. B. (2002). Smarandache Near-Rings. Infinite Study.

Received: May 9, 2021. Accepted: October 5, 2021

NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries (revisited)

Florentin Smarandache
University of New Mexico
Mathematics, Physical and Natural Science Division
705 Gurley Ave., Gallup, NM 87301, USA
Email: smarand@unm.edu

Abstract

In this paper we extend the NeutroAlgebra & AntiAlgebra to the geometric spaces, by founding the NeutroGeometry & AntiGeometry.

While the Non-Euclidean Geometries resulted from the total negation of one specific axiom (Euclid's Fifth Postulate), the AntiGeometry results from the total negation of any axiom or even of more axioms from any geometric axiomatic system (Euclid's, Hilbert's, etc.) and from any type of geometry such as (Euclidean, Projective, Finite, Affine, Differential, Algebraic, Complex, Discrete, Computational, Molecular, Convex, etc.) Geometry, and the NeutroGeometry results from the partial negation of one or more axioms [and no total negation of no axiom] from any geometric axiomatic system and from any type of geometry. Generally, instead of a classical geometric Axiom, one may take any classical geometric Theorem from any axiomatic system and from any type of geometry, and transform it by NeutroSophication or AntiSophication into a NeutroTheorem or AntiTheorem respectively in order to construct a NeutroGeometry or AntiGeometry. Therefore, the NeutroGeometry and AntiGeometry are respectively alternatives and generalizations of the Non-Euclidean Geometries.

In the second part, we recall the evolution from Paradoxism to Neutrosophy, then to NeutroAlgebra & AntiAlgebra, afterwards to NeutroGeometry & AntiGeometry, and in general to NeutroStructure & AntiStructure that naturally arise in any field of knowledge. At the end, we present applications of many NeutroStructures in our real world.

Keywords: Non-Euclidean Geometries, Euclidean Geometry, Lobachevski-Bolyai-Gauss Geometry, Riemannian Geometry, NeutroManifold, AntiManifold, NeutroAlgebra, AntiAlgebra, NeutroGeometry, AntiGeometry, NeutroAxiom, AntiAxiom, NeutroTheorem, AntiTheorem, Partial Function, NeutroFunction, AntiFunction, NeutroOperation, AntiOperation, NeutroAttribute, AntiAttribute, NeutroRelation, AntiRelation, NeutroStructure, AntiStructure

1. Introduction

In our real world, the spaces are not homogeneous, but mixed, complex, even ambiguous. And the elements that populate them and the rules that act upon them are not perfect, uniform, or complete - but fragmentary and disparate, with unclear and conflicting information, and they do not apply in the same degree to each element. The perfect, idealistic ones exist just in the theoretical sciences. We live in a multi-space endowed with a multi-structure [35]. Neither the space's elements nor the regulations that govern them are egalitarian, all of them are characterized by degrees of diversity and variance. The indeterminate (vague, unclear, incomplete, unknown, contradictory etc.) data and procedures are surrounding us.

That's why, for example, the classical algebraic and geometric spaces and structures were extended to more realistic spaces and structures [1], called respectively NeutroAlgebra & AntiAlgebra [2019] and respectively NeutroGeometry & AntiGeometry [1969, 2021], whose elements do not necessarily behave the same, while the operations and rules onto these spaces may only be partially (not totally) true.

While the Non-Euclidean Geometries resulted from the total negation of only one specific axiom (Euclid's Fifth Postulate), the AntiGeometry results from the total negation of any axiom and even of more axioms from any geometric axiomatic system (Euclid's five postulates, Hilbert's 20 axioms, etc.), and the NeutroAxiom results from the partial negation of one or more axioms [and no total negation of no axiom] from any geometric axiomatic system.
Therefore, the NeutroGeometry and AntiGeometry are respectively alternatives and generalizations of the Non-Euclidean Geometries.
In the second part, we recall the evolution from Paradoxism to Neutrosophy, then to NeutroAlgebra & AntiAlgebra, afterwards to NeutroGeometry & AntiGeometry, and in general to NeutroStructure & AntiStructure that naturally arise in any field of knowledge. At the end, we present applications of many NeutroStructures in our real world.

On a given space, a classical Axiom is totally (100%) true. While a NeutroAxiom is partially true, partially indeterminate, and partially false. Also, an AntiAxiom is totally (100%) false.

A classical Geometry has only totally true Axioms. While a NeutroGeometry is a geometry that has at least one NeutroAxiom and no AntiAxiom. Also, an AntiGeometry is a geometry that has at least one AntiAxiom.

Below we introduce, in the first part of this article, the construction of NeutroGeometry & AntiGeometry, together with the Non-Euclidean geometries, while in the second part we recall the evolution from paradoxism to neutrosophy, and then to NeutroAlgebra & AntiAlgebra, culminating with the most general form of NeutroStructure & AntiStructure in any field of knowledge.

A classical (100%) true statement on a given classical structure, may or may not be 100% true on its corresponding NeutroStructure or AntiStructure, it depends on the neutrosophication or antisophication procedures [1 – 24].

Further on, the neutrosophic triplet (Algebra, NeutroAlgebra, AntiAlgebra) was restrained or extended to all fuzzy and fuzzy extension theories (FET) triplets of the form (Algebra, Neutro$_{FET}$Algebra, Anti$_{FET}$Algebra), where FET may be: Fuzzy, Intuitionistic Fuzzy, Inconsistent Intuitionistic Fuzzy (Picture Fuzzy, Ternary Fuzzy), Pythagorean Fuzzy (Atanassov's Intuitionistic Fuzzy of second type), q-Rung Orthopair Fuzzy, Spherical Fuzzy, n-HyperSpherical Fuzzy, Refined Neutrosophic, etc.

1.1. Concept, NeutroConcept, AntiConcept

Let us consider on a given geometric space a *classical geometric concept* (such as: axiom, postulate, operator, transformation, function, theorem, property, theory, etc.).

We form the following geometric neutrosophic triplet:

Concept(1, 0, 0), *NeutroConcept*(T, I, F), *AntiConcept* (0, 0, 1),

where (T, I, F) ∉ {(1, 0, 0), (0, 0, 1)}.

{ Of course, we consider only the neutrosophic triplets (Concept, NeutroConcept, AntiConcept) that make sense in our everyday life and in the real world. }

Concept(1, 0, 0) means that the degree of truth of the concept is T = 1, I = 0, F = 0, or the Concept is 100% true, 0% indeterminate, and 0% false in the given geometric space.

NeutroConcept (T, I, F) means that the concept is T% true, I% indeterminate, and 0% false in the given geometric space, with (T, I, F) ∈ [0, 1], and (T, I, F) ∉ {(1, 0, 0), (0, 0, 1)}.

AntiConcept (0, 0, 1) means that T = 0, I = 0, and F = 1, or the Concept is 0% true, 0% indeterminate, and 100% false in the given geometric space.

1.2. Geometry, NeutroGeometry, AntiGeometry

We go from the neutrosophic triplet (*Algebra, NeutroAlgebra, AntiAlgebra*) to a similar neutrosophic triplet (*Geometry, NeutroGeometry, AntiGeometry*), in the same way.

Correspondingly from the algebraic structuires, with respect to the geometries, one has:

In the *classical (Euclidean) Geometry*, on a given space, all classical geometric *Concepts* are 100% true (i.e. true for all elements of the space).

While in a *NeutroGeometry*, on a given space, there is at least one *NeutroConcept* (and no *AntiConcept*).

In the *AntiGeometry*, on a given space, there is at least one *AntiConcept*.

1.3. Geometric NeutroSophication and Geometric AntiSophication

Similarly, as to the algebraic structures, using the process of *NeutroSophication* of a classical geometric structure, a *NeutroGeometry* is produced; while through the process of *AntiSophication* of a classical geometric structure produces an *AntiGeometry*.

Let *S* be a classical *geometric space*, and <A> be a *geometric concept* (such as: postulate, axiom, theorem, property, function, transformation, operator, theory, etc.). The <antiA> is the opposite of <A>, while <neutA> (also called <neutroA>) is the neutral (or indeterminate) part between <A> and <antiA>.

The neutrosophication tri-sections *S* into three subspaces:

- the first subspace, denoted just by <A>, where the *geometric concept* is totally true [degree of truth $T = 1$]; we denote it by *Concept(1,0,0).*

- the second subspace, denoted by <neutA>, where the *geometric concept* is partially true [degree of truth *T*], partially indeterminate [degree of indeterminacy *I*], and partially false [degree of falsehood *F*], denoted as *NeutroConcept(T,I,F)*, where $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$;

- the third subspace, denoted by <antiA>, where the *geometric concept* is totally false [degree of falsehood $F = 1$], denoted by *AntiConcept(0,0,1).*

The three subspaces may or may not be disjoint, depending on the application, but they are exhaustive (their union equals the whole space *S*).

1.4. Non-Euclidean Geometries

1.4.1. The *Lobachevsky* (also known as *Lobachevsky-Bolyai-Gauss*) *Geometry*, and called *Hyperbolic Geometry*, is an AntiGeometry, because the Fifth Euclidean Postulate (in a plane, through a point outside a line, only one parallel can be drawn to that line) is 100% invalidated in the following AntiPostulate (first version) way: in a plane through a point outside of a line, there can be drawn infinitely many parallels to that line. Or (T, I, F) = (0, 0, 1).

1.4.2. The *Riemannian Geometry*, which is called *Elliptic Geometry*, is an AntiGeometry too, since the Fifth Euclidean Postulate is 100% invalidated in the following AntiPostulate (second version) way: in a place, through a point outside of a line, no parallel can be drawn to that line. Or (T, I, F) = (0, 0, 1).

1.4.3. The *Smarandache Geometries (SG)* are more complex [30 – 57]. Why this type of mixed non-Euclidean geometries, and sometimes partially Non-Euclidean and partially Euclidean? Because the real geometric spaces are not pure but hybrid, and the real rules do not uniformly apply to all space's elements, but they have degrees of diversity – applying to some geometrical concepts (point, line, plane, surface, etc.) in a smaller or bigger degree.

From Prof. Dr. Linfan Mao's arXiv.org paper *Pseudo-Manifold Geometries with Applications* [57], Cornell University, New York City, USA, 2006, https://arxiv.org/abs/math/0610307 :

> "A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom (1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways and a Smarandache n-manifold is a n-manifold that support a Smarandache geometry.
>
> Iseri provided a construction for Smarandache 2-manifolds by equilateral triangular disks on a plane and a more general way for Smarandache 2-manifolds on surfaces, called map geometries was presented by the author (…).

However, few observations for cases of n ≥ 3 are found on the journals. As a kind of Smarandache geometries, a general way for constructing dimensional n pseudo-manifolds are presented for any integer n ≥ 2 in this paper. Connection and principal fiber bundles are also defined on these manifolds. Following these constructions, nearly all existent geometries, such as those of Euclid geometry, Lobachevshy-Bolyai geometry, Riemann geometry, Weyl geometry, Kahler geometry and Finsler geometry, etc. are their sub-geometries."

Iseri ([34], [39 - 40]) has constructed some Smarandache Manifolds (S-manifolds) that topologically are piecewise linear, and whose geodesics have elliptic, Euclidean, and hyperbolic behavior. An SG geometry may exhibit one or more types of negative, zero, or positive curvatures into the same given space.

1.4.3.1) If at least one axiom is validated (partially true, T > 0) and invalidated (partially false, F > 0), and no other axiom is only invalidated (*AntiAxiom*), then this first class of SG geometry is a *NeutroGeometry*.

1.4.3.2) If at least one axiom is only invalidated (or F = 1), no matter if the other axioms are classical or *NeutroAxioms* or *AntiAxioms* too, then this second class of SG geometry is an *AntiGeometry*.

1.4.3.3) *The model of an SG geometry that is a NeutroGeometry:*

Bhattacharya [38] has constructed the following SG model:

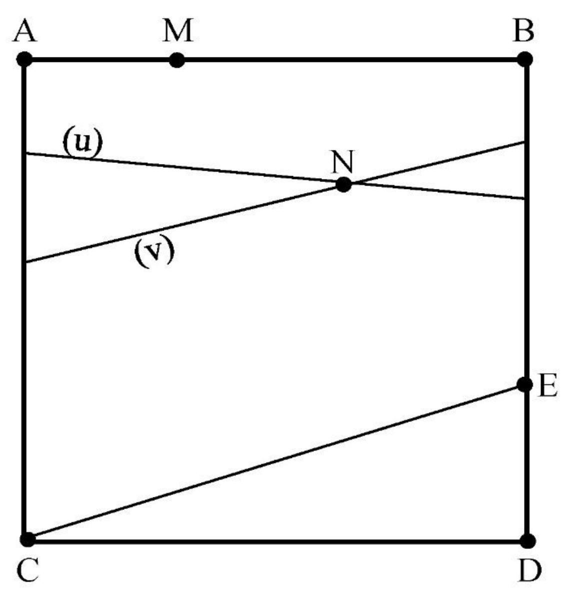


Fig. 1. Bhattacharya's Model for the SG geometry as a NeutroGeometry

The geometric space is a square ABCD, comprising all points inside and on its edges.

"Point" means the classical point, for example: A, B, C, D, E, N, and M.

"Line" means any segment of line connecting two points on the opposite square sides AC and BD, for example: AB, CD, CE, (u), and (v).

"Parallel lines" are lines that do not intersect.

Let us take a line CE and an exterior point N to it. We observe that there is an infinity of lines passing through N and parallel to CE [all lines passing through N and in between the lines (u) and (v) for example] – the *hyperbolic case*.

Also, taking another exterior point, D, there is no parallel line passing through D and parallel to CE because all lines passing through D intersects CE – the *elliptic case*.

Taking another exterior point M ∈ AB, then we only have one line AB parallel to CE, because only one line passes through the point M – the *Euclidean case*.

Consequently, the Fifth Euclidean Postulate is twice invalidated, but also once validated.

Being partially hyperbolic Non-Euclidean, partially elliptic Non-Euclidean, and partially Euclidean, therefore we have here a SG.

This is not a Non-Euclidean Geometry (since the Euclid's Fifth Postulate is not totally false, but only partially), but it is a NeutroGeometry.

Theorem 1.4.3.3.1

If a statement (proposition, theorem, lemma, property, algorithm, etc.) is (totally) true (degree of truth T = 1, degree of indeterminacy I = 0, and degree of falsehood F = 0) in the classical geometry, the statement may get any logical values (i.e. T, I, F may be any values in [0, 1]) in a NeutroGeometry or in an AntiGeometry

Proof.

The logical value the statement gets in a NeutroGeometry or in an AntiGeometry depends on what classical axioms the statement is based upon in the classical geometry, and how these axioms behave in the NeutroGeometry or AntiGeometry models.

Let's consider the below classical geometric proposition P(L1, L2, L3) that is 100% true:

> *In a 2D-Euclidean geometric space, if two lines L1 and L2 are parallel with the third line L3, then they are also parallel (i.e. L1 // L2).*

In Bhattacharya's Model of an SG geometry, this statement is partially true and partially false.

For example, in Fig. 1:

- degree of truth: the lines AB and (u) are parallel to the line CE, then AB is parallel to (u);
- degree of falsehood: the lines (u) and (v) are parallel to the line CE, but (u) and (v) are not parallel since they intersect in the point N.

1.4.3.4) The Model of a SG geometry that is an AntiGeometry

Let us consider the following rectangular piece of land PQRS,

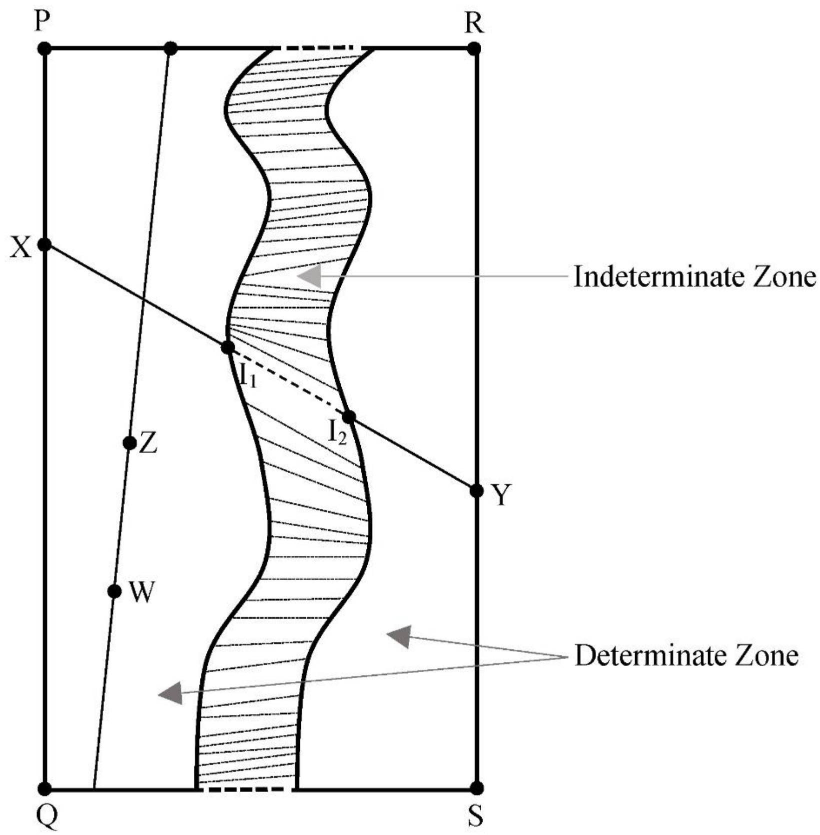


Fig. 2. Model for an SG geometry that is an AntiGeometry

whose middle (shaded) area is an indeterminate zone (a river, with swamp, canyons, and no bridge) that is impossible to cross over on the ground. Therefore, this piece of land is composed from a determinate zone and an indeterminate zone (as above).

"Point" means any classical (usual) point, for example: P, Q, R, S, X, Y, Z, and W that are determinate well-known (classical) points, and I_1, I_2 that are indeterminate (not well-known) points [in the indeterminate zone].

"Line" is any segment of line that connects a point on the side PQ with a point on the side RS. For example, PR, QS, XY. However, these lines have an indeterminate (not well known, not clear) part that is the indeterminate zone. On the other hand, ZW is not a line since it does not connect the sides PQ and RS.

The following geometric classical axiom: *through two distinct points there always passes one single line*, is totally (100%) denied in this model in the following two ways:

through any two distinct points, in this given model, either no line passes (see the case of ZW), or only one partially determinate line does (see the case of XY) - therefore no fully determinate line passes. Thus, this SG geometry is an AntiGeometry.

1.5. Manifold, NeutroManifold, AntiManifold

1.5.1. Manifold

The classical **Manifold** [29] is a topological space that, on the small scales, near each point, resembles the classical (Euclidean) Geometry Space [i.e. in this space there are only *classical Axioms (totally true)*]. Or each point has a neighborhood that is homeomorphic to an open unit ball of the Euclidean Space R^n (where R is the set of real numbers). Homeomorphism is a continuous and bijective function whose inverse is also continuous.

"In general, any object that is near 'flat' on the small scale is a manifold" [29].

1.5.2. NeutroManifold

The **NeutroManifold** is a topological space that, on the small scales, near each point, resembles the NeutroGeometry Space [i.e. in this space there is *at least a NeutroAxiom (partially true, partially indeterminate, and partially false) and no AntiAxiom*].

For example, Bhattacharya's Model for a SG geometry (Fig. 1) is a NeutroManifold, since the geometric space ABCD has a NeutroAxiom (i.e. the Fifth Euclidean Postulate, which is partially true and partially false), and no AntiAxiom.

1.5.3. AntiManifold

The Anti**Manifold** is a topological space that, on the small scales, near each point, resembles the AntiGeometry Space [i.e. in this space there is *at least one AntiAxiom (totally false)*].

For example, the Model for a SG geometry (Fig. 2) is an AntiManifold, since the geometric space PQRS has an AntiAxiom (i.e., through two distinct points there always passes a single line - which is totally false).

2. Evolution from Paradoxism to Neutrosophy then to NeutroAlgebra/AntiAlgebra and now to NeutroGeometry/AntiGeometry

Below we recall and revise the previous foundations and developments that culminated with the introduction of NeutroAlgebra & AntiAlgebra as new field of research, extended then to NeutroStructure & AntiStructure, and now particularized to NeutroGeometry & AntiGeometry that are extensions of the Non-Euclidean Geometries.

2.1. From Paradoxism to Neutrosophy

Paradoxism [58] is an international movement in science and culture, founded by Smarandache in 1980s, based on excessive use of antitheses, oxymoron, contradictions, and paradoxes. During three decades (1980-2020) hundreds of authors from tens of countries around the globe contributed papers to 15 international paradoxist anthologies.

In 1995, he extended the *paradoxism* (based on opposites) to a new branch of philosophy called *neutrosophy* (based on opposites and their neutral) [59], that gave birth to many scientific branches, such as: neutrosophic logic, neutrosophic set, neutrosophic probability, neutrosophic statistics, neutrosophic algebraic structures, and so on with multiple applications in engineering, computer science, administrative work, medical research, social sciences, etc.

Neutrosophy is an extension of Dialectics that have derived from the Yin-Yan Ancient Chinese Philosophy.

2.2. From Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures

In 2019 Smarandache [1] generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or NeutroAlgebras) {whose operations (or laws) and axioms (or theorems) are partially true, partially indeterminate, and partially false} as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebras) {whose operations (or laws) and axioms (or theorems) are totally false} and on 2020 he continued to develop them [2,3,4].

Generally, instead of a classical Axiom in a field of knowledge, one may take a classical Theorem in that field of knowledge, and transform it by NeutroSophication or AntiSophication into a NeutroTheorem or AntiTheorem in order to construct a NeutroStructure or AntiStructure in that field of knowledge.

The NeutroAlgebras & AntiAlgebras are a *new field of research*, which is inspired from our real world. As said ahead, we may also get a NeutroAlgebra & AntiAlgebra by transforming, instead of an Axiom, a classical algebraic Theorem into a NeutroTheorem or AntiTheorem; the process is called NeutroSophication or respectively AntiSophication.

In classical algebraic structures, all operations are 100% well-defined, and all axioms are 100% true, but in real life, in many cases these restrictions are too harsh, since in our world we have things that only partially verify some operations or some laws.

By substituting Concept with Operation, Axiom, Theorem, Relation, Attribute, Algebra, Structure etc. respectively, into the above (Concept, NeutroConcept, AntiConcept), we get the below neutrosophic triplets:

2.3. Operation, NeutroOperation, AntiOperation

When we define an operation on a given set, it does not automatically mean that the operation is well-defined. There are three possibilities:

1) The operation is well-defined (also called inner-defined) for all set's elements [degree of truth T = 1] (as in classical algebraic structures; this is a classical Operation). Neutrosophically we write: Operation(1,0,0).

2) The operation if well-defined for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and outer-defined for the other elements [degree of falsehood F], where (T,I,F) is different from (1,0,0) and from (0,0,1) (this is a NeutroOperation). Neutrosophically we write: NeutroOperation(T,I,F).

3) The operation is outer-defined for all set's elements [degree of falsehood F = 1] (this is an AntiOperation). Neutrosophically we write: AntiOperation(0,0,1).

An operation * on a given non-empty set *S* is actually a *n-ary* function, for integer $n \geq 1$, $f : S^n \to S$.

2.4. Function, NeutroFunction, AntiFunction

Let *U* be a universe of discourse, *A* and *B* be two non-empty sets included in *U*, and *f* be a function: $f : A \to B$

Again, we have three possibilities:

1) The function is well-defined (also called inner-defined) for all elements of its domain *A* [degree of truth T = 1] (this is a classical Function), i.e. $\forall x \in A, f(x) \in B$. Neutrosophically we write: Function(1,0,0).

2) The function if well-defined for some elements of its domain, i.e. $\exists x \in A, f(x) \in B$ [degree of truth T], indeterminate for other elements, i.e. $\exists x \in A, f(x) = indeterminate$ [degree of indeterminacy I], and outer-defined for the other elements, i.e. $\exists x \in A, f(x) \notin B$ [degree of falsehood F], where (T,I,F) is different from (1,0,0) and from (0,0,1). This is a NeutroFunction. Neutrosophically we write: NeutroFunction(T,I,F).

3) The function is outer-defined for all elements of its domain *A* [degree of falsehood F = 1] (this is an AntiFunction), i.e. $\forall x \in A, f(x) \notin B$ (all function's values are outside of its codomain *B*; they may be outside of the universe of discourse too). Neutrosophically we write: AntiFunction(0,0,1).

2.5. NeutroFunction & AntiFunction vs. Partial Function

We prove that the NeutroFunction & AntiFunction are extensions and alternatives of the Partial Function.

Definition of Partial Function [60]

A function f: A→B is sometimes called a *total function*, to signify that f(a) is defined for every a ∈ A. If C is any set such that C ⊇ A then f is also a *partial function* from C to B.

Clearly if f is a function from A to B then it is a partial function from A to B, but a partial function need not be defined for every element of its domain. The set of elements of A for which f is defined is sometimes called the *domain of definition*.

From other sites, the Partial Function means: for any a ∈ A one has: f(a) ∈ B or f(a) = undefined.

Comparison

i) "Partial" is mutually understood as there exist at least one element $a_1 \in A$ such that $f(a_1) \in B$, or the Partial Function is well-defined for at least one element (therefore T > 0).
The Partial Function does not allow the well-defined degree T = 0 (i.e. no element is well-defined), while the NeutroFunction and AntiFunction do**.**
Example 1.
Let's consider the set of positive integers *Z = {1, 2, 3, …}*, included into the universe of discourse *R*, which is the set of real numbers. Let's define the function
$f_1 : Z \to Z$, $f_1(x) = \frac{x}{0}$, for all $x \in Z$.
Clearly, the function f_1 is *100%* undefined, therefore the indeterminacy I = 1, while T = 0 and F = 0.
Hence f_1 is a NeutroFunction, but not a Partial Function.
Example 2.
Let's take the set of odd positive integers *D = {1, 3, 5, …}*, included in the universe of discourse *R*. Let's define the function $f_2 : D \to D$, $f_2(x) = \frac{x}{2}$, for all $x \in D$.
The function f_2 is *100%* outer-defined, since $\frac{x}{2} \notin D$ for all $x \in D$. Whence F = 1, T = 0, and I = 0. Hence this is an AntiFunction, but not a partial Function.

ii) ***The Partial Function does not catch all types of indeterminacies*** that are allowed in a NeutroFunction. Indeterminacies may occur with respect to: the function's domain, codomain, or relation that connects the elements in the domain with the elements in the codomain.
Example 3.
Let's consider the function *g: {1, 2, 3, …, 9, 10, 11}* → *{12, 13, …, 19}*, about whom we only have vague, unclear information as below:
g(1 or 2) = 12, i.e. we are not sure if g(1) = 12 or g(2) = 12;
g(3) = 18 or 19, i.e. we are not sure if g(3) = 18 or g(3) = 19;
g(4 or 5 or 6) = 13 or 17;
g(7) = unknown;
g(unknown) = 14.
All the above values represent the function's degree of indeterminacy (I > 0).

g(10) = 20 that does not belong to the codomain; (outer-defined, or degree of falsehood F > 0);
g(11) = 15 that belongs to the codomain; (inner-defined, or degree of truth, hence T > 0).
Function *g* is a NeutroFunction (with I > 0, T > 0, F > 0), but not a Partial Function since such types of indeterminacies are not characteristic to it.

iii) ***The Partial Fraction does not catch the outer-defined values***.

Example 4.

Let S = {0, 1, 2, 3} be a subset included in the set of rational numbers Q that serves as universe of discourse. The function h: S → S, $h(x) = \frac{2}{x}$ is a NeutroFunction, since h(0) = 2/0 = undefined, and h(3) = 2/3 $\notin S$ (outer-defined, 2/3 $\in Q - S$), but is not a Partial Function.

2.6. Axiom, NeutroAxiom, AntiAxiom

Similarly for an axiom, defined on a given set, endowed with some operation(s). When we define an axiom on a given set, it does not automatically mean that the axiom is true for all set's elements. We have three possibilities again:

1) The axiom is true for all set's elements (totally true) [degree of truth T = 1] (as in classical algebraic structures; this is a classical Axiom). Neutrosophically we write: Axiom(1,0,0).

2) The axiom if true for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and false for other elements [degree of falsehood F], where (T,I,F) is different from (1,0,0) and from (0,0,1) (this is NeutroAxiom). Neutrosophically we write NeutroAxiom(T,I,F).

3) The axiom is false for all set's elements [degree of falsehood F = 1](this is AntiAxiom). Neutrosophically we write AntiAxiom(0,0,1).

2.7. Theorem, NeutroTheorem, AntiTheorem

In any science, a classical Theorem, defined on a given space, is a statement that is 100% true (i.e. true for all elements of the space). To prove that a classical theorem is false, it is sufficient to get a single counter-example where the statement is false. Therefore, the classical sciences do not leave room for *partial truth* of a theorem (or a statement). But, in our world and in our everyday life, we have many more examples of statements that are only partially true, than statements that are totally true. The NeutroTheorem and AntiTheorem are generalizations and alternatives of the classical Theorem in any science.

Let's consider a theorem, stated on a given set, endowed with some operation(s). When we construct the theorem on a given set, it does not automatically mean that the theorem is true for all set's elements. We have three possibilities again:

1) The theorem is true for all set's elements [totally true] (as in classical algebraic structures; this is a classical Theorem). Neutrosophically we write: Theorem(1,0,0).

2) The theorem if true for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and false for the other elements [degree of falsehood F], where (T,I,F) is different from (1,0,0) and from (0,0,1) (this is a NeutroTheorem). Neutrosophically we write: NeutroTheorem(T,I,F).

3) The theorem is false for all set's elements (this is an AntiTheorem). Neutrosophically we write: AntiTheorem(0,0,1).

And similarly for (Lemma, NeutroLemma, AntiLemma), (Consequence, NeutroConsequence, AntiConsequence), (Algorithm, NeutroAlgorithm, AntiAlgorithm), (Property, NeutroProperty, AntiProperty), etc.

2.8. Relation, NeutroRelation, AntiRelation

1) A classical Relation is a relation that is true for all elements of the set (degree of truth T = 1). Neutrosophically we write Relation(1,0,0).

2) A NeutroRelation is a relation that is true for some of the elements (degree of truth T), indeterminate for other elements (degree of indeterminacy I), and false for the other elements (degree of falsehood F). Neutrosophically we write Relation(T,I,F), where (T,I,F) is different from (1,0,0) and (0,0,1).

3) An AntiRelation is a relation that is false for all elements (degree of falsehood F = 1). Neutrosophically we write Relation(0,0,1).

2.9. Attribute, NeutroAttribute, AntiAttribute

1) A classical Attribute is an attribute that is true for all elements of the set (degree of truth T = 1). Neutrosophically we write Attribute(1,0,0).

2) A NeutroAttribute is an attribute that is true for some of the elements (degree of truth T), indeterminate for other elements (degree of indeterminacy I), and false for the other elements (degree of falsehood F). Neutrosophically we write Attribute(T,I,F), where (T,I,F) is different from (1,0,0) and (0,0,1).

3) An AntiAttribute is an attribute that is false for all elements (degree of falsehood F = 1). Neutrosophically we write Attribute(0,0,1).

2.10. Algebra, NeutroAlgebra, AntiAlgebra

1) An algebraic structure who's all operations are well-defined and all axioms are totally true is called a classical Algebraic Structure (or Algebra).
2) An algebraic structure that has at least one NeutroOperation or one NeutroAxiom (and no AntiOperation and no AntiAxiom) is called a NeutroAlgebraic Structure (or NeutroAlgebra).
3) An algebraic structure that has at least one AntiOperation or one Anti Axiom is called an AntiAlgebraic Structure (or AntiAlgebra).
Therefore, a neutrosophic triplet is formed: <Algebra, NeutroAlgebra, AntiAlgebra>, where "Algebra" can be any classical algebraic structure, such as: a groupoid, semigroup, monoid, group, commutative group, ring, field, vector space, BCK-Algebra, BCI-Algebra, etc.

2.11. Algebra, Neutro$_{FET}$Algebra, Anti$_{FET}$Algebra

The neutrosophic triplet (Algebra, NeutroAlgebra, AntiAlgebra) was further on restrained or extended to all fuzzy and fuzzy extension theories (FET), making triplets of the form: (Algebra, Neutro$_{FET}$Algebra, Anti$_{FET}$Algebra), where FET may be: Fuzzy, Intuitionistic Fuzzy, Inconsistent Intuitionistic Fuzzy (Picture Fuzzy, Ternary Fuzzy), Pythagorean Fuzzy (Atanassov's Intuitionistic Fuzzy of second type), q-Rung Orthopair Fuzzy, Spherical Fuzzy, n-HyperSpherical Fuzzy, Refined Neutrosophic, etc. See several examples below.

2.11.1. The Intuitionistic Fuzzy Triplet (Algebra, Neutro$_{IF}$Algebra, Anti$_{IF}$Algebra)

Herein "IF" stands for intuitionistic fuzzy.

When Indeterminacy (I) is missing, only two components remain, T and F.

1) The Algebra is the same as in the neutrosophic environment, i.e. a classical Algebra where all operations are totally well-defined and all axioms are totally true (T = 1, F = 0).
2) The Neutro$_{IF}$Algebra means that at least one operation or one axiom is partially true (degree of truth T) and partially false (degree of partially falsehood F),
   with $T, F \in [0,1], 0 \le T + F \le 1,$ with $(T, F) \ne (1,0)$ that represents the classical Axiom, and $(T, F) \ne (0,1)$ that represents the Anti$_{IF}$Axiom,
   and no Anti$_{IF}$Operation (operation that is totally outer-defined) and no Anti$_{IF}$Axiom.
3) The Anti$_{IF}$Algebra means that at least one operation or one axiom is totally false (T = 0, F = 1), no matter how the other operations or axioms are.

Therefore, one similarly has the triplets: (Operation, Neutro$_{IF}$Operation, Anti$_{IF}$Operation) and (Axiom, Neutro$_{IF}$Axiom, Anti$_{IF}$Axiom).

2.11.2. The Fuzzy Triplet (Algebra, Neutro$_{Fuzzy}$Algebra, Anti$_{Fuzzy}$Algebra)

When the Indeterminacy (I) and the Falsehood (F) are missing, only one component remains, T.

1) The Algebra is the same as in the neutrosophic environment, i.e. a classical Algebra where all operations are totally well-defined and all axioms are totally true (T = 1).
2) The Neutro$_{Fuzzy}$Algebra means that at least one operation or one axiom is partially true (degree of truth T), with $T \in (0,1)$,
   and no Anti$_{Fuzzy}$Operation (operation that is totally outer-defined) and no Anti$_{Fuzzy}$Axiom.
3) The Anti$_{IF}$Algebra means that at least one operation or one axiom is totally false (F = 1), no matter how the other operations or axioms are.

Therefore, one similarly has the triplets: (Operation, Neutro$_{Fuzzy}$Operation, Anti$_{Fuzzy}$Operation) and (Axiom, Neutro$_{Fuzzy}$Axiom, Anti$_{Fuzzy}$Axiom).

2.12. Structure, NeutroStructure, AntiStructure in any field of knowledge

In general, by NeutroSophication, Smarandache extended any classical *Structure*, in no matter what field of knowledge, to a *NeutroStructur*e, and by AntiSophication to an *AntiStructure*.

i) A classical Structure, in any field of knowledge, is composed of: a non-empty space, populated by some elements, and both (the space and all elements) are characterized by some relations among themselves (such as: operations, laws, axioms, properties, functions, theorems, lemmas, consequences, algorithms, charts, hierarchies, equations, inequalities, etc.), and by their attributes (size, weight, color, shape, location, etc.).

Of course, when analysing a structure, it counts with respect to what relations and what attributes we do it.

ii) A NeutroStructure is a structure that has at least one NeutroRelation or one NeutroAttribute, and no AntiRelation and no AntiAttribute.

iii) An AntiStructure is a structure that has at least one AntiRelation or one AntiAttribute.

2.13. Almost all real Structures are NeutroStructures

The Classical Structures in science mostly exist in theoretical, abstract, perfect, homogeneous, idealistic spaces - because in our everyday life almost all structures are NeutroStructures, since they are neither perfect nor applying to the whole population, and not all elements of the space have the same relations and same attributes in the same degree (not all elements behave in the same way).

The indeterminacy and partiality, with respect to the space, to their elements, to their relations or to their attributes are not taken into consideration in the Classical Structures. But our Real World is full of structures with indeterminate (vague, unclear, conflicting, unknown, etc.) data and partialities.

There are exceptions to almost all laws, and the laws are perceived in different degrees by different people.

2.14. Applications of NeutroStructures in our Real World

(i) In the Christian society the marriage law is defined as the union between a male and a female (*degree of truth*).

But, in the last decades, this law has become less than 100% true, since persons of the same sex were allowed to marry as well (*degree of falsehood*).

On the other hand, there are transgender people (whose sex is indeterminate), and people who have changed the sex by surgical procedures, and these people (and their marriage) cannot be included in the first two categories (*degree of indeterminacy*).

Therefore, since we have a NeutroLaw (with respect to the Law of Marriage) we have a Christian NeutroStructure.

(ii) In India, the law of marriage is not the same for all citizen: Hindi religious men may marry only one wife, while the Muslims may marry up to four wives.

(iii) Not always the difference between good and bad may be clear, from a point of view a thing may be good, while from another point of view bad. There are things that are partially good, partially neutral, and partially bad.

(iv) The laws do not equally apply to all citizens, so they are NeutroLaws. Some laws apply to some degree to a category of citizens, and to a different degree to another category. As such, there is an American folkloric joke: All people are born equal, but some people are more equal than others!

- There are powerful people that are above the laws, and other people that benefit of immunity with respect to the laws.

- For example, in the court of law, privileged people benefit from better defense lawyers than the lower classes, so they may get a lighter sentence.

- Not all criminals go to jail, but only those caught and proven guilty in the court of law. Nor the criminals that for reason of insanity cannot stand trail and do not go to jail since they cannot make a difference between right and wrong.

- Unfortunately, even innocent people went and may go to jail because of sometimes jurisdiction mistakes...

- The Hypocrisy and Double Standard are widely spread: some regulation applies to some people, but not to others!

(v) Anti-Abortion Law does not apply to all pregnant women: the incest, rapes, and women whose life is threatened may get abortions.

(vi) Gun-Control Law does not apply to all citizen: the police, army, security, professional hunters are allowed to bear arms.

Etc.

Conclusion

In this paper we have extended the Non-Euclidean Geometries to AntiGeometry (a geometric space that has at least one AntiAxiom) and to NeutroGeometry (a geometric space that has at least one NeutroAxiom and no AntiAxiom) both in any axiomatic system and any type of geometry), similarly to the NeutroAlgebra and AntiAlgebra. Generally, instead of a geometric Axiom, one may take any classical geometric Theorem in any axiomatic system and in any type of geometry and transform it by NeutroSophication or AntiSophication into a NeutroTheorem or AntiTheorem in order to construct a NeutroGeometry or AntiGeometry respectively.

A NeutroAxiom is an axiom that is partially true, partially indeterminate, and partially false in the same space. While the AntiAxiom is an axiom that is totally false in the given space.

While the Non-Euclidean Geometries resulted from the total negation of one specific axiom (Euclid's Fifth Postulate), the AntiGeometry (1969) resulted from the total negation of any axiom and even of more axioms from any geometric axiomatic system (Euclid's, Hilbert's, etc.) and from any type of geometry such as (Euclidean, Projective, Finite, Affine, Differential, Algebraic, Complex, Discrete, Computational, Molecular, Convex, etc.) Geometry, and the NeutroGeometry resulted from the partial negation of one or more axioms [and no total negation of no axiom] from any geometric axiomatic system and from any type of geometry.

Therefore, the NeutroGeometry and AntiGeometry are respectively alternatives and generalizations of the Non-Euclidean Geometries.
In the second part, we recall the evolution from Paradoxism to Neutrosophy, then to NeutroAlgebra & AntiAlgebra, afterwards to NeutroGeometry & AntiGeometry, and in general to NeutroStructure & AntiStructure that naturally arise in any field of knowledge.
At the end, we present applications of many NeutroStructures in our real world.

Further on, we have recalled and reviewed the evolution from Paradoxism to Neutrosophy, and from the classical algebraic structures to NeutroAlgebra and AntiAlgebra structures, and in general to the NeutroStructure and AntiStructure in any field of knowledge. Then many applications of NeutroStructures from everyday life were presented.

References

1. F. Smarandache, Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures [http://fs.unm.edu/NA/NeutroAlgebraicStructures-chapter.pdf], in his book *Advances of Standard and Nonstandard Neutrosophic Theories*, Pons Publishing House Brussels, Belgium, *Chapter 6, pages 240-265*, 2019; http://fs.unm.edu/AdvancesOfStandardAndNonstandard.pdf

2. Florentin Smarandache: NeutroAlgebra is a Generalization of Partial Algebra. International Journal of Neutrosophic Science (IJNS), Volume 2, 2020, pp. 8-17. DOI: http://doi.org/10.5281/zenodo.3989285 http://fs.unm.edu/NeutroAlgebra.pdf

3. Florentin Smarandache: Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited). Neutrosophic Sets and Systems, vol. 31, pp. 1-16, 2020. DOI: 10.5281/zenodo.3638232 http://fs.unm.edu/NSS/NeutroAlgebraic-AntiAlgebraic-Structures.pdf

4. Florentin Smarandache, Generalizations and Alternatives of Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures, Journal of Fuzzy Extension and Applications (JFEA), J. Fuzzy. Ext. Appl. Vol. 1, No. 2 (2020) 85–87, DOI: 10.22105/jfea.2020.248816.1008 http://fs.unm.edu/NeutroAlgebra-general.pdf

5. A.A.A. Agboola, M.A. Ibrahim, E.O. Adeleke: Elementary Examination of NeutroAlgebras and AntiAlgebras viz-a-viz the Classical Number Systems. International Journal of Neutrosophic Science (IJNS), Volume 4, 2020, pp. 16-19. DOI: http://doi.org/10.5281/zenodo.3989530 http://fs.unm.edu/ElementaryExaminationOfNeutroAlgebra.pdf

6. A.A.A. Agboola: Introduction to NeutroGroups. International Journal of Neutrosophic Science (IJNS), Volume 6, 2020, pp. 41-47. DOI: http://doi.org/10.5281/zenodo.3989823
http://fs.unm.edu/IntroductionToNeutroGroups.pdf

7. A.A.A. Agboola: Introduction to NeutroRings. International Journal of Neutrosophic Science (IJNS), Volume 7, 2020, pp. 62-73. DOI: http://doi.org/10.5281/zenodo.3991389
http://fs.unm.edu/IntroductionToNeutroRings.pdf

8. Akbar Rezaei, Florentin Smarandache: On Neutro-BE-algebras and Anti-BE-algebras. International Journal of Neutrosophic Science (IJNS), Volume 4, 2020, pp. 8-15. DOI: http://doi.org/10.5281/zenodo.3989550
http://fs.unm.edu/OnNeutroBEalgebras.pdf

9. Mohammad Hamidi, Florentin Smarandache: Neutro-BCK-Algebra. International Journal of Neutrosophic Science (IJNS), Volume 8, 2020, pp. 110-117. DOI: http://doi.org/10.5281/zenodo.3991437
http://fs.unm.edu/Neutro-BCK-Algebra.pdf

10. Florentin Smarandache, Akbar Rezaei, Hee Sik Kim: A New Trend to Extensions of CI-algebras. International Journal of Neutrosophic Science (IJNS) Vol. 5, No. 1 , pp. 8-15, 2020; DOI: 10.5281/zenodo.3788124
http://fs.unm.edu/Neutro-CI-Algebras.pdf

11. Florentin Smarandache: Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra. Neutrosophic Sets and Systems, Vol. 33, pp. 290-296, 2020. DOI: 10.5281/zenodo.3783103
http://fs.unm.edu/NSS/n-SuperHyperGraph-n-HyperAlgebra.pdf

12. A.A.A. Agboola: On Finite NeutroGroups of Type-NG. International Journal of Neutrosophic Science (IJNS), Volume 10, Issue 2, 2020, pp. 84-95. DOI: 10.5281/zenodo.4277243,
http://fs.unm.edu/IJNS/OnFiniteNeutroGroupsOfType-NG.pdf

13. A.A.A. Agboola: On Finite and Infinite NeutroRings of Type-NR. International Journal of Neutrosophic Science (IJNS), Volume 11, Issue 2, 2020, pp. 87-99. DOI: 10.5281/zenodo.4276366,
http://fs.unm.edu/IJNS/OnFiniteAndInfiniteNeutroRings.pdf

14. A.A.A. Agboola, Introduction to AntiGroups, International Journal of Neutrosophic Science (IJNS), Vol. 12, No. 2, PP. 71-80, 2020, http://fs.unm.edu/IJNS/IntroductionAntiGroups.pdf

15. M.A. Ibrahim and A.A.A. Agboola, Introduction to NeutroHyperGroups, Neutrosophic Sets and Systems, vol. 38, 2020, pp. 15-32. DOI: 10.5281/zenodo.4300363, http://fs.unm.edu/NSS/IntroductionToNeutroHyperGroups2.pdf

16. Elahe Mohammadzadeh and Akbar Rezaei, On NeutroNilpotentGroups, Neutrosophic Sets and Systems, vol. 38, 2020, pp. 33-40. DOI: 10.5281/zenodo.4300370, http://fs.unm.edu/NSS/OnNeutroNilpotentGroups3.pdf

17. F. Smarandache, Structure, NeutroStructure, and AntiStructure in Science, International Journal of Neutrosophic Science (IJNS), Volume 13, Issue 1, PP: 28-33, 2020; http://fs.unm.edu/IJNS/NeutroStructure.pdf

18. Diego Silva Jiménez, Juan Alexis Valenzuela Mayorga, Mara Esther Roja Ubilla, and Noel Batista Hernández, NeutroAlgebra for the evaluation of barriers to migrants' access in Primary Health Care in Chile based on PROSPECTOR function, Neutrosophic Sets and Systems, vol. 39, 2021, pp. 1-9. DOI: 10.5281/zenodo.4444189;
http://fs.unm.edu/NSS/NeutroAlgebraForTheEvaluationOfBarriers1.pdf

19. Madeleine Al-Tahan, F. Smarandache, and Bijan Davvaz, NeutroOrderedAlgebra: Applications to Semigroups, Neutrosophic Sets and Systems, vol. 39, 2021, pp.133-147. DOI: 10.5281/zenodo.4444331, http://fs.unm.edu/NSS/NeutroOrderedAlgebra11.pdf

20. F. Smarandache, Universal NeutroAlgebra and Universal AntiAlgebra, Chapter 1, pp. 11-15, in the collective book NeutroAlgebra Theory, Vol. 1, edited by F. Smarandache, M. Sahin, D. Bakbak, V. Ulucay, A. Kargin, Educational Publ., Grandview Heights, OH, United States, 2021, http://fs.unm.edu/NA/UniversalNeutroAlgebra-AntiAlgebra.pdf

21. Madeleine Al-Tahan, NeutroOrderedAlgebra: Theory and Examples, 3rd International Workshop on Advanced Topics in Dynamical Systems, University of Kufa, Iraq, March 1st, 2021, http://fs.unm.edu/NA/NeutroOrderedAlgebra.pdf

22. F. Smarandache A. Rezaei A.A.A. Agboola Y.B. Jun R.A. Borzooei B. Davvaz A. Broumand Saeid M. Akram M. Hamidi S. Mirvakilii, On NeutroQuadrupleGroups, 51st Annual Mathematics Conference Kashan, February 16-19, 2021, http://fs.unm.edu/NA/OnNeutroQuadrupleGroups-slides.pdf

23. Madeleine Al-Tahan, Bijan Davvaz, Florentin Smarandache, and Osman Anis, On Some NeutroHyperstructures, Symmetry 2021, 13, 535, pp. 1-12, https://doi.org/10.3390/sym13040535; http://fs.unm.edu/NeutroHyperstructure.pdf

24. A. Rezaei, F. Smarandache, and S. Mirvakili, Applications of (Neutro/Anti)sophications to Semihypergroups, Journal of Mathematics, Hindawi, vol. 2021, Article ID 6649349, pp. 1-7, 2021; https://doi.org/10.1155/2021/6649349, http://fs.unm.edu/NA/Neutro-Anti-sophications.pdf

25. F. Smarandache, Neutrosophy. / Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998, http://fs.unm.edu/eBook-Neutrosophics6.pdf

26. Serkan Karatas and Cemil Kuru, Neutrosophic Topology, Neutrosophic Sets Syst, Vol. 13, 90-95, 2016, http://fs.unm.edu/NSS/NeutrosophicTopology.pdf

27. Florentin Smarandache, Neutrosophic Precalculus and Neutrosophic Calculus, EuropaNova, Brussels, Belgium, 154 p., 2015; https://arxiv.org/ftp/arxiv/papers/1509/1509.07723.pdf .

28. F. Smarandache, Indeterminacy in Neutrosophic Theories and their Applications, International Journal of Neutrosophic Science (IJNS), Vol. 15, No. 2, PP. 89-97, 2021, http://fs.unm.edu/Indeterminacy.pdf

29. Rowland, Todd. "Manifold." From *MathWorld*--A Wolfram Web Resource, created by Eric W. Weisstein. https://mathworld.wolfram.com/Manifold.html

30. L. Mao, Smarandache Geometries & Map Theories with Applications (I), Academy of Mathematics and Systems, Chinese Academy of Sciences, Beijing, P. R. China, 2006, http://fs.unm.edu/CombinatorialMaps.pdf

31. Linfan Mao, Automorphism Groups of Maps, Surfaces and Smarandache Geometries (first edition - postdoctoral report to Chinese Academy of Mathematics and System Science, Beijing, China; and second editions - graduate textbooks in mathematics), 2005 and 2011, http://fs.unm.edu/Linfan.pdf, http://fs.unm.edu/Linfan2.pdf

32. L. Mao, Combinatorial Geometry with Applications to Field Theory (second edition), graduate textbook in mathematics, Chinese Academy of Mathematics and System Science, Beijing, China, 2011, http://fs.unm.edu/CombinatorialGeometry2.pdf

33. Yuhua Fu, Linfan Mao, and Mihaly Bencze, Scientific Elements - Applications to Mathematics, Physics, and Other Sciences (international book series): Vol. 1, ProQuest Information & Learning, Ann Arbor, MI, USA, 2007, http://fs.unm.edu/SE1.pdf

34. Howard Iseri, Smarandache Manifolds, ProQuest Information & Learning, Ann Arbor, MI, USA, 2002, http://fs.unm.edu/Iseri-book.pdf

35. Linfan Mao, Smarandache Multi-Space Theory (partially post-doctoral research for the Chinese Academy of Sciences), Academy of Mathematics and Systems Chinese Academy of Sciences Beijing, P. R. China, 2006, http://fs.unm.edu/S-Multi-Space.pdf

36. Yanpei Liu, Introductory Map Theory, ProQuest Information & Learning, Michigan, USA, 2010, http://fs.unm.edu/MapTheory.pdf

37. L. Kuciuk & M. Antholy, An Introduction to the Smarandache Geometries, JP Journal of Geometry & Topology, 5(1), 77-81, 2005, http://fs.unm.edu/IntrodSmGeom.pdf

38. S. Bhattacharya, A Model to A Smarandache Geometry, http://fs.unm.edu/ModelToSmarandacheGeometry.pdf

39. Howard Iseri, A Classification of s-Lines in a Closed s-Manifold, http://fs.unm.edu/Closed-s-lines.pdf

40. Howard Iseri, Partially Paradoxist Smarandache Geometries, http://fs.unm.edu/Howard-Iseri-paper.pdf

41. Chimienti, Sandy P., Bencze, Mihaly, "Smarandache Paradoxist Geometry", Bulletin of Pure

and Applied Sciences, Delhi, India, Vol. 17E, No. 1, 123-1124, 1998.

42. David E. Zitarelli, Reviews, Historia Mathematica, PA, USA, Vol. 24, No. 1, p. 114,

#24.1.119, 1997.

43. Marian Popescu, "A Model for the Smarandache Paradoxist Geometry", Abstracts of Papers Presented to the American Mathematical Society Meetings, Vol. 17, No. 1, Issue 103, 1996, p. 265.

44. Popov, M. R., "The Smarandache Non-Geometry", Abstracts of Papers Presented to the American Mathematical Society Meetings, Vol. 17, No. 3, Issue 105, 1996, p. 595.

45. Brown, Jerry L., "The Smarandache Counter-Projective Geometry", Abstracts of Papers Presented to the American Mathematical Society Meetings, Vol. 17, No. 3, Issue 105, 595, 1996.

46. Florentin Smarandache, "Collected Papers" (Vol. II), University of Kishinev Press, Kishinev, pp. 5-28, 1997.

47. Florentin Smarandache, "Paradoxist Mathematics" (lecture), Bloomsburg University, Mathematics Department, PA, USA, November 1995.

48. F. Smarandache, Degree of Negation of Euclid's Fifth Postulate, http://fs.unm.edu/DegreeOfNegation.pdf

49. M. Antholy, An Introduction to the Smarandache Geometries, New Zealand Mathematics Colloquium, Palmerston North Campus, Massey University, 3-6 December 2001. http://fs.unm.edu/IntrodSmGeom.pdf

50. L. Mao, Let's Flying by Wing — Mathematical Combinatorics & Smarandache Multi-Spaces / 让我们插上翅膀飞翔 -- 数学组合与Smarandache重叠空间, English Chinese bilingual, Academy of Mathematics and Systems, Chinese Academy of Sciences, Beijing, P. R. China, http://fs.unm.edu/LetsFlyByWind-ed3.pdf

51. Ovidiu Sandru, Un model simplu de geometrie Smarandache construit exclusiv cu elemente de geometrie euclidiană, http://fs.unm.edu/OvidiuSandru-GeometrieSmarandache.pdf

52; L. Mao, A new view of combinatorial maps by Smarandache's notion, Cornell University, New York City, USA, 2005, https://arxiv.org/pdf/math/0506232

53. Linfan Mao, A generalization of Stokes theorem on combinatorial manifolds, Cornell University, New York City, USA, 2007, https://arxiv.org/abs/math/0703400

54. Linfan Mao, Combinatorial Speculations and the Combinatorial Conjecture for Mathematics, Cornell University, New York City, USA, 2006, https://arxiv.org/pdf/math/0606702

55. Linfan Mao, Geometrical Theory on Combinatorial Manifolds, Cornell University, New York City, USA, 2006, https://arxiv.org/abs/math/0612760

56. Linfan Mao, Parallel bundles in planar map geometries, Cornell University, New York City, USA, 2005, https://arxiv.org/pdf/math/0506386

57. Linfan Mao, Pseudo-Manifold Geometries with Applications, Cornell University, New York City, USA, 2006, Paper's abstract: https://arxiv.org/abs/math/0610307, Full paper: https://arxiv.org/pdf/math/0610307

58. Cornel Gingăraşu, Arta Paradoxistă, Sud-Est Forum, Magazine for photography, culture and visual arts, https://sud-est-forum.ro/wp/2020/06/paradoxismul-curent-cultural-romanesc-sau-cand-profu-de-mate-isi-gaseste-libertatea-in-poezie

59. F. Smarandache, Neutrosophy, A New Branch of Philosophy, Multiple Valued Logic / An International Journal, United States, Vol. 8, No. 3, pp. 297-384, 2002.

60. Editors, Partial Function, Planet Math, https://planetmath.org/partialfunction, February 8th, 2018.

Neutrosophic Sets and Systems (NSS) is an academic journal, published quarterly online and on paper, that has been created for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics etc. and their applications in any field.

All submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

ISSN (print): 2331-6055, ISSN (online): 2331-608X
Impact Factor: 1.739

NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, all NSS articles are indexed in Scopus.

NSS is also indexed by Google Scholar, Google Plus, Google Books, EBSCO, Cengage Thompson Gale (USA), Cengage Learning, ProQuest, Amazon Kindle, DOAJ (Sweden), University Grants Commission (UGC) - India, International Society for Research Activity (ISRA), Scientific Index Services (SIS), Academic Research Index (ResearchBib), Index Copernicus (European Union),CNKI (Tongfang Knowledge Network Technology Co., Beijing, China), etc.

Google Dictionary has translated the neologisms "neutrosophy" (1) and "neutrosophic" (2), coined in 1995 for the first time, into about 100 languages.

FOLDOC Dictionary of Computing (1, 2), Webster Dictionary (1, 2), Wordnik (1), Dictionary.com, The Free Dictionary (1),Wiktionary (2), YourDictionary (1, 2), OneLook Dictionary (1, 2), Dictionary / Thesaurus (1), Online Medical Dictionary (1, 2), and Encyclopedia (1, 2) have included these scientific neologisms.

DOI numbers are assigned to all published articles.

Registered by the Library of Congress, Washington DC, United States, https://lccn.loc.gov/2013203857.

Recently, NSS was also approved for Emerging Sources Citation Index (ESCI) available on the Web of Science platform, starting with Vol. 15, 2017.

Editors-in-Chief:

Prof. Dr. Florentin Smarandache, Postdoc, Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA, Email: smarand@unm.edu.
Dr. Mohamed Abdel-Basset, Faculty of Computers and Informatics, Zagazig University, Egypt, Email: mohamed.abdelbasset@fci.zu.edu.eg.
Dr. Said Broumi, Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco, Email: s.broumi@flbenmsik.ma.

23316055

$39.95