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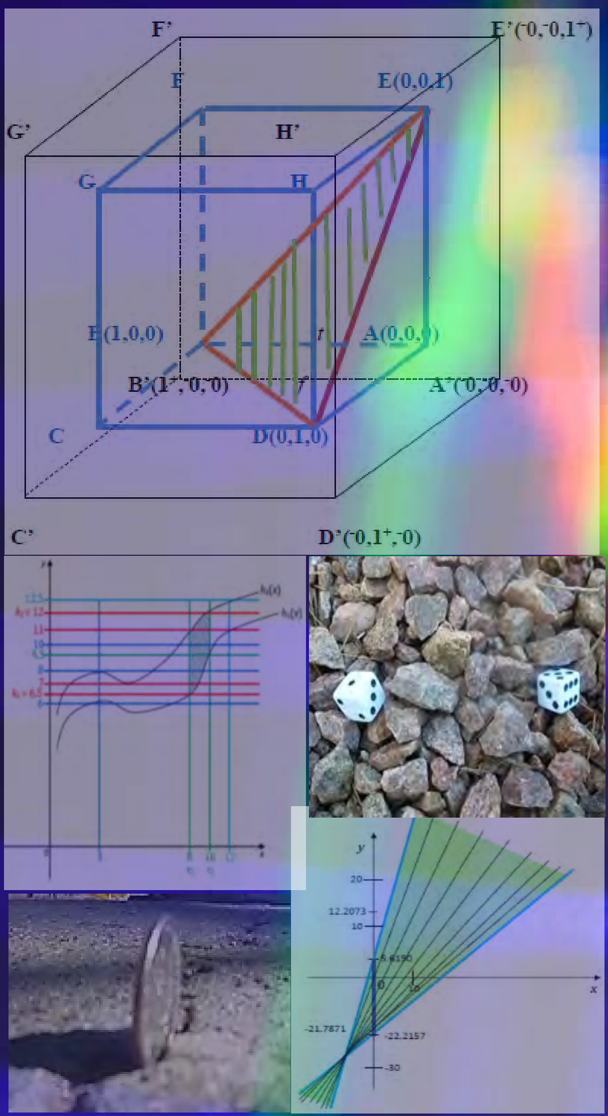

<A> <neutA> <antiA>

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Contents

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Medical Diagnosis Problems Based on Neutrosophic Sets and Their Hybrid Structures: A Survey

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Abstract

The investigation of a person's symptoms can be evaluated through medical diagnosis to diagnose the diseases. To medical clinicians, a large amount of data is available for diagnosis, which comprises uncertainty, inconsistency, and indeterminacy. The field of medicine is one of the best areas of application for neutrosophic set theory. The main intention of this article is to deal with some of the applications of neutrosophic sets and their hybrid structures to solve medical diagnosis problems.

Keywords: Neutrosophic sets; interval valued neutrosophic sets, simplified neutrosophic sets; medical diagnosis problem

I. Introduction

The concept of neutrosophic set theory was first developed by Smarandache [1]. Smarandache [1-2] developed the notions of neutrosophic set (NS) and neutrosophic logic as a generalization of fuzzy sets [3], intuitionistic fuzzy sets [4]. Certain kinds of uncertainty, such as incomplete, indeterminate, and inconsistent information seen in the real world and not handled by fuzzy sets, as well as intuitionistic fuzzy sets, can be easily handled by neutrosophic sets. Three independent membership degrees characterize the concept of a neutrosophic set: truth-membership degree (T), indeterminacy-membership degree (I), and falsity-membership degree (F).

Smarandache [5] developed the concept of a single-valued neutrosophic set (SVNS), which is a subclass of neutrosophic sets in which the values of the three membership functions T, I, and F are in the unit interval [0, 1]. Smarandache [6] extended the neutrosophic set to include neutrosophic precalculus, neutrosophic calculus, neutrosophic measure, neutrosophic probability (chance that an event occurs, indeterminate-chance of occurrence of the event, and chance that the event does not occur), and neutrosophic statistic (statistics that have indeterminacy) were carried out by Smarandache [6]. Many researchers have proposed extensions to the notion of neutrosophic sets since it was first introduced.interval-valued neutrosophic sets [7], simplified neutrosophic sets [8], trapezoidal neutrosophic sets [9], single-valued neutrosophic hesitant sets [10], neutrosophic overset, underset, and offset [11], bipolar neutrosophic sets [12], interval-valued bipolar neutrosophic sets [13], single-valued neutrosophic multi-sets [14], rough neutrosophic sets [15], bipolar neutrosophic refined sets [16] and refined neutrosophic sets [17]. All of the newly presented notions have been thoroughly investigated, and attempts to apply them to multiple-attribute decision-making issues and other disciplines have been explored.[18–30] contains a lot of study in this area. In clinical medicine, medical diagnosis is critical in determining diseases based on a set of symptoms.Many academics have undertaken studies linked to medical diagnosis difficulties in fuzzy and intuitionistic fuzzy settings, according to the literature review [31-47]. Later, so many of the fuzzy models based on soft sets were quickly investigated and applied to medical diagnosis issues [48-58].The purpose of all investigations is to establish an adequate medical diagnosis method for determining whether a patient has a specific disease. The medical diagnosis is determined in relation to a specific ailment under certain assumptions. Due to the presence of indeterminacy data, the approaches employed to solve the medical diagnosis problem in fuzzy environments and intuitionistic fuzzy environments are not suitable to neutrosophic related problems.As a result, a number of methods and algorithms for dealing with the medical diagnosis problem in a neutrosophic environment have been created.The purpose of this paper is to show how the neutrosophic set and its hybrid structures can be used to solve medical diagnosis issues.

The paper is organized as follows: Section 1 is introductory in nature. Section 2 deals with some preliminary definitions that are required in subsequent sections. Section 3 gives a literature survey of different neutrosophic models for solving a medical diagnosis problem, and Section 4 describes the conclusions.

II. Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic sets, refined neutrosophic sets, soft sets, bipolar neutrosophic refined sets and rough neutrosophic sets relevant to the present work. See especially [1, 2, 5, 6, 15, 17, 48] for further details and background

Definition 2.1 [1-2]. Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$ >, x $\in$ X}, where the functions T, I, F: X→]$^-$0,1$^+$[define respectively the truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element x $\in$ X to the set A with the condition:

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+. \qquad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of]$^-$0,1$^+$[.

Since it is difficult to apply NSs to practical problems, Smarandache [5] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [5]. Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x) >, x \in X\} \quad (2)$$

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 2.3 [6]. Let X be a space of points (objects) with generic elements in X denoted by x. An interval-valued neutrosophic set A (IVNS A) is characterized by an interval truth-membership function $T_A(x) = \left[T_A^L, T_A^U\right]$, an interval indeterminacy-membership function $I_A(x) = \left[I_A^L, I_A^U\right]$, and an interval falsity-membership function $F_A(x) = \left[F_A^L, F_A^U\right]$. For each point x in X $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$. An IVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x) >, x \in X\} \quad (3)$$

In some practical situations, there is the possibility of each element having different membership, indeterminacy and non-membership functions. For this purposeSmarandache [16] proposed the concept of:

Definition 2. 4 [17] (neutrosophic refined sets)

Let E be a universe, a neutrosophic refined set (NRS) A on E can be defined as follows

$$A = \left\{ \begin{matrix} < x, (T_A^1(x), T_A^2(x), ..., T_A^p(x)), (I_A^1(x), I_A^2(x), ..., I_A^p(x)), \\ (F_A^1(x), F_A^2(x), ..., F_A^p(x)) \end{matrix} \right\} \quad (4)$$

where $T_A^1(x), T_A^2(x), ..., T_A^p(x) : E \to [0, 1]$, $I_A^1(x), I_A^2(x), ..., I_A^p(x) : E \to [0, 1]$ and

$F_A^1(x), F_A^2(x), ..., F_A^p(x) : E \to [0, 1]$ such that

$0 \leq T_A^i(x) + I_A^i(x) + F_A^i(x) \leq 3$ (i=1, 2, 3,…, p)

Definition 2.5 [48] soft sets

Let U be an initial set and E be a set of parameters. Let P(U) denote the power set of U, and let A → E. A pair (F, A) is called a soft set over U, where F is a mapping given by F: A→ P(U).

Definition 2.6 [16] bipolar neutrosophic refined sets

Let E be a universe, A bipolar neutrosophic refined set (BNRS) A on E can be defined as follows

$$A=\left\{\begin{array}{l} <x,(T_A^{1+}(\mathrm{x}),T_A^{2+}(\mathrm{x}),...,T_A^{p+}(\mathrm{x}),T_A^{1-}(\mathrm{x}),T_A^{2-}(\mathrm{x}),...,T_A^{p-}(\mathrm{x})), \\ (\mathrm{I}_A^{1+}(\mathrm{x}),\mathrm{I}_A^{2+}(\mathrm{x}),...,\mathrm{I}_A^{p+}(\mathrm{x}),\mathrm{I}_A^{1-}(\mathrm{x}),\mathrm{I}_A^{2-}(\mathrm{x}),...,\mathrm{I}_A^{p-}(\mathrm{x})), \\ (\mathrm{F}_A^{1+}(\mathrm{x}),\mathrm{F}_A^{2+}(\mathrm{x}),...,\mathrm{F}_A^{p+}(\mathrm{x}),\mathrm{F}_A^{1-}(\mathrm{x}),\mathrm{F}_A^{2-}(\mathrm{x}),...,\mathrm{F}_A^{p-}(\mathrm{x}))>: x\in X \end{array}\right\} \quad (5)$$

Where

$(T_A^{1+}(\mathrm{x}),T_A^{2+}(\mathrm{x}),...,T_A^{p+}(\mathrm{x}),T_A^{1-}(\mathrm{x}),T_A^{2-}(\mathrm{x}),...,T_A^{p-}(\mathrm{x})): \mathrm{E} \rightarrow [0, 1]$,

$(\mathrm{I}_A^{1+}(\mathrm{x}),\mathrm{I}_A^{2+}(\mathrm{x}),...,\mathrm{I}_A^{p+}(\mathrm{x}),\mathrm{I}_A^{1-}(\mathrm{x}),\mathrm{I}_A^{2-}(\mathrm{x}),...,\mathrm{I}_A^{p-}(\mathrm{x})): \mathrm{E} \rightarrow [0, 1]$ and

$(\mathrm{F}_A^{1+}(\mathrm{x}),\mathrm{F}_A^{2+}(\mathrm{x}),...,\mathrm{F}_A^{p+}(\mathrm{x}),\mathrm{F}_A^{1-}(\mathrm{x}),\mathrm{F}_A^{2-}(\mathrm{x}),...,\mathrm{F}_A^{p-}(\mathrm{x})): \mathrm{E} \rightarrow [0, 1]$ such that $0 \le T_A^i(x) + I_A^i(x) + F_A^i(x) \le 3$

(i=1,2,3,…,p)

$(T_A^{1+}(\mathrm{x}),T_A^{2+}(\mathrm{x}),...,T_A^{p+}(\mathrm{x}),T_A^{1-}(\mathrm{x}),T_A^{2-}(\mathrm{x}),...,T_A^{p-}(\mathrm{x}))$ (6) $(\mathrm{I}_A^{1+}(\mathrm{x}),\mathrm{I}_A^{2+}(\mathrm{x}),...,\mathrm{I}_A^{p+}(\mathrm{x}),\mathrm{I}_A^{1-}(\mathrm{x}),\mathrm{I}_A^{2-}(\mathrm{x}),...,\mathrm{I}_A^{p-}(\mathrm{x}))$

(7)

$$(\mathrm{F}_A^{1+}(\mathrm{x}),\mathrm{F}_A^{2+}(\mathrm{x}),...,\mathrm{F}_A^{p+}(\mathrm{x}),\mathrm{F}_A^{1-}(\mathrm{x}),\mathrm{F}_A^{2-}(\mathrm{x}),...,\mathrm{F}_A^{p-}(\mathrm{x})) \quad (8)$$

is the truth membership sequence, indeterminacy membership sequence and falsity membership sequence of the element, x respectively. Also, P is called the dimension of BNR-set. The set of all bipolar neutrosophic refined sets on E is denoted by BNRS(E).

Definition 2.7 [15] rough neutrosophic sets.

Let *Z* be a non-null set and *R* be an equivalence relation on *Z*. Let P be a neutrosophic set in *Z* with the membership function T_P, indeterminacy function I_P and non-membership function F_P. The lower and the upper approximations of *P* in the approximation (*Z*, *R*) denoted by $\underline{N}(P)$ and $\overline{N}(P)$ are respectively defined as follows

$$\left\langle \left\langle x, \mathrm{T}_{\underline{N}(P)}(x), \mathrm{I}_{\underline{N}(P)}(x), F_{\underline{N}(P)}(x) \right\rangle / z \in [x]_R, x \in Z \right\rangle$$

$$\left\langle \left\langle x, T_{\overline{N}(P)}(x), \mathrm{I}_{\overline{N}(P)}(x), \mathrm{F}_{\overline{N}(P)}(x) \right\rangle / z \in [x]_R, x \in Z \right\rangle \quad (9)$$

Where $T_{\underline{N}(P)}(x) = \wedge_z T_P(z) \in [x]_R$, $I_{\underline{N}(P)}(x) = \wedge_z I_P(z) \in [x]_R$, $F_{\underline{N}(P)}(x) = \wedge_z F_P(z) \in [x]_R$,

$T_{\overline{N}(P)}(x) = \vee_z T_P(z) \in [x]_R$, $I_{\overline{N}(P)}(x) = \vee_z I_P(z) \in [x]_R$, $F_{\overline{N}(P)}(x) = \vee_z F_P(z) \in [x]_R$ So

$0 \le \sup T_{\underline{N}(P)}(x) + \sup I_{\underline{N}(P)}(x) + \sup \mathrm{F}_{\underline{N}(P)}(x) \le 3$ and $0 \le \sup T_{\overline{N}(P)}(x) + \sup \mathrm{I}_{\overline{N}(P)}(x) + \sup \mathrm{F}_{\overline{N}(P)}(x) \le 3$

And $\wedge$ and $\vee$ denote "min" and "max'' operators respectively, $T_P(z)$, $I_P(z)$ and $F_P(z)$ are the membership, indeterminacy and non-membership of Z with respect to P.

Thus NS mapping $\underline{N}$, $\overline{N}$: $N(Z) \rightarrow N(Z)$ are, respectively, referred to as the lower and upper rough neutrosophic approximation operators, and the pair ($\underline{N}(P)$, $\overline{N}(P)$) is called the rough neutrosophic set in Z.

III. REVIEW OF LITTERATURE

In this section, medical diagnosis under different neutrosophic hybrid environments is discussed since the medical field seems to be the most suitable for its applicability. Researchers concerned with neutrosophic sets have found that they needed to be developed for solving complex problems that occur most often in medical diagnosis. Some methods are as below:

3.1 Medical diagnosis using the single valued neutrosophic environment.

To achieve better results, Ansari et al. [59-60] introduced neutrosophic logic into the medical arena. Kharal [61] expanded Sanchez's method of medical diagnosis to neutrosophic sets. The proposed approach of diagnosis allows the decision maker to attribute ambiguous notions to degrees of satisfiability, non-satisfiability, and indeterminacy of symptoms. Shahzadi et al.[84] developed two algorithms for medical diagnosis based on distance and similarity measures in a neutrosophic environment, and discovered that the results achieved using the suggested technique are identical to those obtained using normalized Hamming and normalized Euclidean distance. Kharal [62] suggested a multi-criteria decision-making system based on further extensions of neutrosophic sets (MCDM).The mathematical aspects of the approach, as well as the vis neut-MCDM algorithm, are investigated. The algorithm of viz. neut-MCDM is provided, along with some noteworthy mathematical aspects of the method. The suggested method provides the MCDM community with the principles of neutrosophic set theory. With the use of the neutrosophic membership values of truth, indeterminacy, and falseness, De and Mishra [63] proposed a novel technique of decision making. The major goal was to come to a reasonable conclusion about the illness of a patient who was suffering from a condition utilizing neutrosophic notions. Sanchez's approach of medical diagnostics in the arena of fuzzy neutrosophic composition relations was examined by Jenny and Arockiarani [64]. The steps of proposed algorithm are as follows

Step 1: Determination of symptoms of the patients .i.e. the relation Q(R $\rightarrow$ S) between the patients and symptoms are noted.

Step 2: The medical knowledge relating the symptoms with the set of diseases under consideration are noted in table II i.e. the relation of symptoms and diseases R(S $\rightarrow$ D) are given.

Step 3: Compute the composition relation of patients and diseases T (P $\rightarrow$ D). Using the membership function given by

$$\mu_T(p_i,d) = \vee_{s\in S}\left[\mu_Q(p_i,s) \wedge \mu_R(s,d)\right], \tag{10}$$

the indeterminacy membership function given by $\nu_T(p_i,d) = \vee_{s\in S}\left[\nu_Q(p_i,s) \wedge \nu_R(s,d)\right]$

(11)

and non- membership function given by

$$\omega_T(p_i,d)=\underset{s\in S}{\wedge}\left[\omega_Q(p_i,s)\vee\omega_R(s,d)\right] \qquad (12)$$

and noted in Table III.

Step 4: Compute the value function using the

$$V(A)=\mu_A+(1-v_A)-\omega_A \qquad (13)$$

for Table III and is given in Table IV.

Step 5: Compute the score function for the table III using the

$$S_2=\mu_i-v_i\omega_i \qquad (14)$$

and it is given in Table V.

Step 6: The higher the score, higher is the possibility of the patient affected with the respective disease.

Ye [65] later produced the tangent function-based similarity measure for SVNSs and the weighted tangent similarity measure for SVNSs, which were introduced by first assessing the relevance of each element and then investigating their features.

The author developed a multi medical diagnosis technique based on the proposed similarity measure and weighted aggregation of multi-period data.

The diagnosis steps are given as follows:

Step1: Compute the similarity measure between a patients P_s and the considered Diseases $D_i\,(i=1,2,...,\mathrm{n})$ in each period $t_k\,(k=1,2,...,q)$ by the following formula:

$$T_{w_i}(P_S,\mathrm{t}_k)=$$
$$1-\sum_{j=1}^{m}\left\{w_j\tan\left[\frac{\pi}{12}\left(\left|T_j(w_i)-T_{ij}\right|+\left|I_j(w_i)-I_{ij}\right|+\left|F_j(w_i)-F_{ij}\right|\right)\right]\right\} \qquad (15)$$

Steps 2: Obtain the weighted aggregation values of $M_{T_i}(\mathrm{P}_S)=\sum_{k=1}^{q}T_{w_i}(\mathrm{P}_S,t_k)\,\omega(t_k)$

(16)

Steps 3: Obtain a proper diagnosis for the patient P_s according to the maximum weighted aggregation value.

Step 4: Last step.

According to [69], the multi-period medical diagnosis method is superior to the single-period medical diagnosis method because the latter can be difficult to give a proper diagnosis of a specific patient with a specific disease in some situations, whereas the former must examine the patient over multiple periods and take into account the weighted information aggregation of multiple periods in order to reach a proper conclusion for the patient.

The concept of fuzzy ontology was expanded to neutrosophic ontology by Bhutani and Aggarwal [66].On the appendicitis dataset, the authors used Fuzzy Ontology and Neutosophic Ontology.

Furthermore, the authors determined that categorization using neutrosophic ontology, as opposed to fuzzy ontology, produces more practical findings because it divides data into appendicitis, non-appendicitis, and uncertainty classes.

Prem Kumar Singh [67] has recently explored how the features of the three-way fuzzy idea lattice and neutrosophic graph presented by Broumi et al [28] can be used to analyze uncertainty and ambiguity in medical data sets. Using the vertices and edges of a neutrosophic graph, this study gave a precise description of medical diagnosis difficulties. Furthermore, using neutrosophic graphs and component-wise Godelresiduated lattice to enrich the knowledge, three-way fuzzy concept creation and hierarchical order visualization in the idea lattice are provided. The proposed method is also used to examine the multi-criteria decision-making process in one application.

3.2 Medical diagnosis under the interval neutrosophic environment.

The notion of interval neutrosophic linguistic numbers (INLNs) was developed by Ma et al. [68], and certain related properties were examined.The authors selected medical therapies based on interval neutrosophic linguistic information using interval neutrosophic linguistic prioritized harmonic.

In addition, the authors conclude that interval neutrosophic linguistic numbers can be utilized to analyze information more successfully than fuzzy sets during the medical treatment selection process.

3.3 Medical diagnosis under the simplified neutrosophic environment.

Ye [69] proposed an improved cosine similarity measure of simplified neutrosophic sets (SNSs) based on the cosine function, including single-valued neutrosophic cosine similarity measures and interval neutrosophic cosine similarity measures, to overcome some of the shortcomings of existing cosine similarity measures of SNSs.

The author then presented a medical diagnosis approach for solving medical diagnosis problems utilizing simplified neutrosophic information based on improved cosine similarity measurements.

To demonstrate the efficacy and rationale of the increased cosine similarity measures-based diagnosis technique, two medical diagnosis challenges were supplied.

3.4. Medical diagnosis under the neutrosophic refined environment.

Broumi and Smarandache [70] examined some of the basic properties of a new distance measure between neutrosophic refined sets based on the extended Hausdorff distance of a neutrosophic set.

A medical diagnosis problem is solved using the extended Hausdorff distance or similarity measurements.

Broumi and Smarandache [71] extended the enhanced cosine similarity measure of single-valued neutrosophic sets provided by Ye [21] to neutrosophic refined sets, and investigated some of their basic features.

Furthermore, using the formulas below, the concept of similarity is applied to medical diagnosis problems:

$$C_{NRS}(A,B)=\frac{1}{p}\sum_{j=1}^{p}\left\{\frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi\left(\left|T_A^j(x_i)-T_B^j(x_i)\right|+\left|I_A^j(x_i)-I_B^j(x_i)\right|+\left|F_A^j(x_i)-F_B^j(x_i)\right|\right)}{6}\right]\right\} \quad (17)$$

Mondal and Pramanik [73] suggested a tangent similarity measure for the neutrosophic refined set, and some of the features of tangent similarity measures were investigated. A tangent similarity measure of single-valued neutrosophic refined sets is a variant of the tangent similarity measure of single-valued neutrosophic sets. The proposed refined tangent similarity measure of single-valued neutrosophic sets is applied to solve a problem in medical diagnosis.

The notion of neutrosophic refined sets (NRS) has been used in medical diagnostics by Deli et al. [14]. The symptoms of each disease can be used to determine the distance and similarity of each patient to that disease.The suggested technique is unusual in that it takes into account multi-membership, indeterminacy, and non-membership.There may be some inaccuracies in diagnosis if you only do a one-time inspection.As a result, in the multi-time inspection procedure, obtaining samples from the same patient at different periods yields the most accurate diagnosis.

3.5 Medical diagnosis under the bipolar neutrosophic refined environment

Deli and ubaş [16] proposed the concept of a bipolar neutrosophic refined set, and further research was conducted into some of the basic properties of this bipolar neutrosophic refined set that generalize the fuzzy set, fuzzy multiset, bipolar fuzzy set, intuitionistic fuzzy multiset, and neutrosophic multisets. Two bipolar neutrosophic refined sets are compared using the score certainty and accuracy functions.Using bipolar neutrosophic refined sets, a new algorithm for solving a medical diagnosis problem was developed.

Ngan et al.[94] established a new distance measure based on the H-max distance measure of intuitionistic fuzzy sets and single valued neutrosophic sets, and then used the H-max distance measure of bipolar neutrosophic sets to introduce a technique of medical diagnosis.

3.6 Medical diagnosis under the single valued neutrosophicmultisets environment.

As a generalization of intuitionistic fuzzy multisets (IFM), Ye et al. [74] proposed a new theory of single-valued neutrosophic multisets (SVNMS), combining the concepts of single-valued neutrosophic sets with the theory of multisets. Then the dice similarity measure between SVNMs is discussed, and then the same measure is applied to medical diagnosis problems.

A generalized distance measure and similarity measures between single-valued neutrosophic multisets (SVNMs) were proposed by Ye et al. [75]. Then the similarity measures obtained in the process are applied to a medical diagnosis problem with incomplete, indeterminate, and inconsistent information. The diagnosis method deals with the diagnosis problem with indeterminate and inconsistent information, which cannot be handled by the diagnosis method based on intuitionistic fuzzy multisets (IFMs).

The notion of SVNMS is redefined by Chatterjee et al. [14] and several set theoretic and algebraic operations on SVNMS are also discussed. Distance and similarity measures between two single-valued neutrosophic multisets were introduced, and single-valued neutrosophic multisets were used to solve medical diagnosis problems.

Samuel et al. [85] used cosine logarithmic distance among single-valued neutrosophic sets to investigate relationships between sets of symptoms found in patients and sets of diseases affecting patients.

In another work, Samuel et al. [86] provided a new approach called the tangent inverse similarity measure by using single-valued neutrosophic sets and applied this newly introduced technique to diagnose which patient is suffering from which disease.

3.7 Medical diagnosis under the rough neutrosophic set environment.

Medical diagnosis necessitates a great deal of data from modern medical technologies, and this data is sometimes partial and inconclusive due to the complexities and ambiguity of disease symptoms.

A rough neutrosophic set has been shown to be effective in dealing with medical diagnosis, which often comprises of imperfect and partial information.

Pramanik and Mondal [76] defined a rough cosine similarity measure between two rough neutrosophic sets and investigated some of their basic features.

The following formula was used to apply these notions to a medical diagnosis problem:

$$C_{RNS}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2}\sqrt{(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2}} \quad (18)$$

Where $\delta T_A(x_i)=\left(\frac{\underline{T}_A(x_i)+\bar{T}_A(x_i)}{2}\right)$, $\delta I_A(x_i)=\left(\frac{\underline{I}_A(x_i)+\bar{I}_A(x_i)}{2}\right)$,

$\delta F_A(x_i)=\left(\frac{\underline{F}_A(x_i)+\bar{F}_A(x_i)}{2}\right)$

Pramanik and Mondal [77] established a rough cotangent similarity measure between two rough neutrosophic sets. In 3D-vector space, the concept of a rough neutrosophic set is used as a vector representation.The upper and lower approximation operators, as well as the pair of neutrosophic sets, are used to represent the rating of all elements in a rough neutrosophic set, which are characterized by truth-membership degree, indeterminacy-membership degree, and falsity-membership degree.

Cotangent similarity was used to solve a medical diagnosis challenge by the author. Pramanik and Mondal [78] introduced more rough dice and Jaccard similarity measures for rough neutrosophic sets, as well as some of their basic features.

The following notions were then applied to a medical diagnosis problem, and an algorithm was created to analyze the situation as follows:

Step1: Determination the relation between patients and symptoms

Step 2: Determination of the relation between Symptoms) and Diseases.

Step 3: Determination the relation between patients and Diseases

$$DIC_{RNS}(A,B)=$$
$$\frac{1}{n}\sum_{i=1}^{n}\frac{2\{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)\}}{\left\{\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\right\}} \quad (19)$$

$$JAC_{RNS}(A,B)=$$
$$\frac{1}{n}\sum_{i=1}^{n}\frac{\{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]+\\ \left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]-\\ \left[\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)\right]\end{array}\right\}} \quad (20)$$

Step 4: Ranking the alternative

The major feature of these proposed approaches is that they take a single time inspection to diagnose the truth, indeterminate, and false membership of each element between two approximations of neutrosophic sets.

The order function of rough neutrosophic sets is proposed in [87], and this method is then applied in the field of medical diagnosis to determine the sickness affecting the patient in question.

In [88] the authors proposes and discusses tangent logarithmic distance and cosecant similarity metrics between rough neutrosophic sets, as well as some of their features.

Following then, the use of this technology in medical diagnostics was discussed.Alias et al. [90] proposed a distance-based similarity measure for approximate neutrosophic sets as a means of medical diagnostics.

In [91], Olgun et al. presented 2-additive choquet similarity measures for multi-period medical diagnosis in single-valued neutrosophic set settings.

Ye et al. [92] introduced a generalized distance measure and similarity measures between single-valued neutrosophic multisets. This method of distance-based similarity measure of single-valued neutrosophic multisets is then applied in medical diagnosis to find which patient is suffering from which type of disease.

Habib et al. [93] presented a single-valued neutrosophic decision-making model for medical diagnosis.

3.8 Medical diagnosis problems under the neutrosophic soft sets

The concept of a neutrosophic soft matrix was introduced by Basu and Mondal [79]. (NS-Matrix).

Different forms of NS-Matrices were also discussed, as well as numerous operations. To handle neutrosophic soft set-based real-life group decision-making problems, a new methodology termed the NSM-Algorithm based on certain of these matrix operations was introduced. The NSM-Algorithm created can be used to solve problems with disease diagnosis based on a variety of symptoms.

Mukherjee and Sarkar [80] introduced a new approach for determining the degree of similarity and weighted similarity between two neutrosophic soft sets, as well as some features of the similarity measure.Similarity measures were used to construct further algorithms for pattern identification problems in neutrosophic soft sets.The proposed method can be used in a variety of situations, such as determining whether or not a sick person with obvious symptoms is suffering from cancer.
The following steps are required for the proposed algorithms.

Step1: Construction of NSS(s) $\hat{N}_i$ (i =1, 2, 3........, n) as ideal pattern(s).

Step2: Construction of NSS(s) $\hat{M}_j$ (j=1, 2, 3........, m) for sample pattern(s) which is/are to be recognized.

Step3: Compute the similarity measure between NSS(s) for ideal pattern(s) and sample pattern(s) using the following formulas:

$$Sim(N_1,N_2)= \frac{1}{3mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{pmatrix}3-\left|T_{N_1}(x_i)(e_j)-T_{N_2}(x_i)(e_j)\right|-\left|I_{N_1}(x_i)(e_j)-I_{N_2}(x_i)(e_j)\right|\\-\left|F_{N_1}(x_i)(e_j)-F_{N_2}(x_i)(e_j)\right|\end{pmatrix} \quad (21)$$

$$WSim(N_1,N_2)= \frac{1}{3m}\sum_{i=1}^{n}\sum_{j=1}^{m}w_i\begin{pmatrix}3-\left|T_{N_1}(x_i)(e_j)-T_{N_2}(x_i)(e_j)\right|-\left|I_{N_1}(x_i)(e_j)-I_{N_2}(x_i)(e_j)\right|\\-\left|F_{N_1}(x_i)(e_j)-F_{N_2}(x_i)(e_j)\right|\end{pmatrix} \quad (22)$$

Where $w_i \in [0,1]$.

Step 4: Consider sample pattern(s) under certain predefined conditions.
If the measure of similarities between the two NSSs considered is greater than or equal to 0.75 then the ill person is possibly suffering from the diseases.
For fuzzy neutrosophic soft sets, Sumathi and Arockiarani [81] developed various types of matrix operations. Furthermore, using fuzzy neutrosophic matrices, a composition approach for creating the decision matrix for medical diagnosis is described.
The proposed method is composed of the following steps:
Step1: Input the fuzzy neutrosophic sets (F, S) over P (the set of m patients) where F is a mapping F: S $\rightarrow FNS(P)$ gives a collection of an approximate description of patient symptoms and (G, D) over S (the set of n symptoms) where G is a mapping G:D $\rightarrow FNS(S)$ gives a collection of an approximate description of disease and their symptoms. In addition, find their corresponding fuzzy neutrosophic soft matrices A and B.

Step2: Compute max-min composition $A*B$ and max-min average composition $A\psi B$ of fuzzy neutrosophic soft matrices A and B.

Where $A*B$ =

$$\left\{\begin{array}{l}\max\left\{\min\limits_{j}\left[T_{ij}^{A},T_{jk}^{B}\right]\right\},\max\left\{\min\limits_{j}\left[I_{ij}^{A},I_{jk}^{B}\right]\right\},\\ \min\left\{\max\limits_{j}\left[F_{ij}^{A},\mathrm{F}_{jk}^{B}\right]\right\}\end{array}\right\} \quad (23)$$

$A\psi B$ =

$$\left\{\max\left\{\frac{T_{ij}^{A}\cdot T_{jk}^{B}}{2}\right\},\max\left\{\frac{I_{ij}^{A}\cdot I_{jk}^{B}}{2}\right\},\min\left\{\frac{F_{ij}^{A}\cdot F_{jk}^{B}}{2}\right\}\right\} \quad (24)$$

Step3: Compute the score matrix S for $A*B$ and $A\psi B$ using the following formulas:

(i) $S_1 = T_j - I_j \cdot F_j$ (ii) $S_2 = T_j + (1 - I_j) - F_j$ (25)

Step4: Identification of the maximum score S_{ij} for each patient P_i. Conclude that the patient P_i is suffering from disease D_j.

With the goal of developing an expert system for patient diagnosis, Arockiarani [82] presented the concept of fuzzy neutrosophic soft relations and the new score function. Some novel methodologies and measures, such as hamming distances and similarity measures, have been proposed, and their properties are now being investigated. A decision-making system based on similarity measures is developed. The author next proceeds through the concept of mappings on fuzzy neutrosophic soft sets and their characteristics.

Later, Celik [83] suggested a new method for medical diagnosis based on fuzzy neutrosophic soft sets and established a mechanism for determining which patient has which illness.

Jafar et al. [89] employed neutrosophic soft matrices and their complements to determine which patient was more likely to have which disease.

As an expansion of the neutrosophic soft matrix, Debnath [92] presented the notion of an interval neutrosophic soft matrix and studied various algebraic operations.In addition, utilizing an interval neutrosophic soft matrix, a new method to group decision-making problems has been proposed.

IV. Conclusions

A medical diagnosis is the process of identifying diseases based on a person's symptoms. To medical clinicians, a large amount of data is available for diagnosis, which comprises uncertainty, inconsistency, and indeterminacy. This paper emphasizes the use of neutrosophic sets and some of their hybrid structures for medical diagnosis problems, with the expectation that they will provide an effective method of diagnosing problem.

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Neutrosophic Fuzzy Pairwise Local Function and Its Application

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Abstract: In this paper we introduce the notion of neutrosophic fuzzy bitopological ideals. The concept of neutrosophic fuzzy pairwise local function is also introduced here by utilizing the neutrosophic quasi-coincident neighbourhood (i.e. $Nq-nbd$) structure in a neutrosophic fuzzy topological space. As well as, the concepts of neutrosophic fuzzy bitopologies and several relations between different neutrosophic fuzzy bitopological ideals have been explored.

Keywords: Neutrosophic Fuzzy Bitopological Space; Neutrosophic Fuzzy Ideals; Neutrosophic Fuzzy Pairwise Local Function.

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1. Introduction: The concept of neutrosophic fuzzy sets and neutrosophic fuzzy set operations was first introduced by Florentin [17]. Subsequently, Salama defined the notion of neutrosophic fuzzy topology [1]. Since then various aspects of bitopological spaces were investigated and carried out in neutrosophic fuzzy by several authors. The notions of neutrosophic fuzzy ideal and neutrosophic fuzzy local function were introduced and studied in [2-8]. Salama was the first researcher who initiated the study of neutrosophic fuzzy bitopological spaces where a neutrosophic fuzzy set equipped with two neutrosophic fuzzy topologies is called a neutrosophic fuzzy bitopological space. Concepts of the neutrosophic fuzzy ideals and the neutrosophic fuzzy local function were introduced and studied in [9-13]. The purpose of this paper is to suggest the neutrosophic fuzzy ideals in neutrosophic fuzzy bitopological spaces. The concept of neutrosophic fuzzy pairwise local function is also introduced here by utilizing the Nq-neighborhood structure [20], for more details of these concepts and other concepts, the readers can return to [14-19, 20,21].

2. Preliminaries

Throughout this paper, by (X, τ_1, τ_2) we mean a neutrosophic fuzzy bitopological space ($nfbts$ in short) in the sense of Salama [6]. A neutrosophic fuzzy point in X with support $x \in X$ and the value $\varepsilon_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>}$ $(0 < \varepsilon \leq 1)$ is denoted by $x\varepsilon =< \varepsilon_1, \varepsilon_2, \varepsilon_3 >$,[9]. A neutrosophic fuzzy point $x\varepsilon$ is said to be contained in a neutrosophic fuzzy set $\mu =< \mu_1, \mu_2, \mu_3 > \in I^X$ iff $\varepsilon \leq \mu$ and this will be denoted by $x\varepsilon\ in\ \mu$ [9] . For a neutrosophic fuzzy set μ in a $nfbts\ (X, \tau_1, \tau_2)$, $\tau_i - Ncl(\mu)$, $\tau_i - NInt(\mu)$, $i \in \{1,2\}$, and μ^c will respectively denote closure, interior and complement of μ The constant neutrosophic fuzzy sets that taking the values $0\ and\ 1$ on X are denoted by $0_N, 1_N$ respectively. A neutrosophic fuzzy set μ in $nfts$ is said to be neutrosophic quasi-coincident [9] with a neutrosophic fuzzy set $\eta =< \eta_1, \eta_2, \eta_3 >$, denoted by $\mu\ Nq\ \eta$, if there exists $x\ in\ X$ such that $\mu(x) + \eta(x) > 1$. A neutrosophic fuzzy set $v =< v_1, v_2, v_3 >$ in a $nfts\ (X, \tau)$ is called a $Nq - nbd$ [1,9] of a neutrosophic fuzzy point $x\varepsilon$ iff there exists a neutrosophic fuzzy open set μ such that $x\varepsilon\ Nq\ \mu \subseteq$ v we will denoted the set of all $Nq - nbd$ of $x\varepsilon$ in (X, τ) by $N\ (X, \tau)$. A nonempty collection of neutrosophic fuzzy sets L of a set X may be called neutrosophic fuzzy ideal [16,8,13] on X iff

(i) $\mu\ in\ L$ and $\eta \subseteq \mu \Rightarrow \eta\ in\ L$ (heredity),

(ii) $\mu\ in\ L\ and\ \eta\ in\ L \Rightarrow \mu \vee \eta\ in\ L$ (Finite additivity).

The neutrosophic fuzzy local function [8] $\mu^* \in (L, \tau)$ of a neutrosophic fuzzy set μ may be the union of all neutrosophic fuzzy points $x\varepsilon$ such that if $v\ in\ N\ (x\varepsilon)$ and $\rho =< \rho_1, \rho_2, \rho_3 >$ in L then there is at least one $r\ in\ X$ for which $v(r) + \mu(r) - 1 > \rho(r)$. For a $nfts\ (X, \tau)$ with neutrosophic fuzzy ideal $L\ ncl^*(\mu) = \mu \vee \mu^{*}$ [8,16] for any neutrosophic fuzzy set $\mu\ of\ X$ and $\tau^*(L)$ be the neutrosophic fuzzy topology generated by ncl^* [16].

3. Neutrosophic Fuzzy Pairwise Local Functions.

Definition 3.1. A neutrosophic fuzzy set $\mu =< \mu_1, \mu_2, \mu_3 >$ in a $nfbts\ (X, \tau_i), i\epsilon\{1,2\}$ is called neutrosophic Pairwise Quasi-coincident with a neutrosophic fuzzy set $\eta =< \eta_1, \eta_2, \eta_3 >$ and is denoted by $P(\mu\ Nq\ \eta)$, if there exists $x\epsilon X$ such that, either type 1 conditions satisfy, $\mu_1(x) + \eta_1(x) > 1$,$\mu_2(x) + \eta_2(x) > 1$,$\mu_3(x) + \eta_3(x) < 1$. Or type 2 conditions satisfied, $\mu_1(x) + \eta_1(x) > 1$,$\mu_2(x) + \eta_2(x) < 1$,$\mu_3(x) + \eta_3(x) < 1$.

It is obviously that for any two neutrosophic fuzzy sets μ and η, $NP(\mu\ Nq\ \eta)$ is identical to $NP(\eta\ Nq\ \mu)$.

Definition 3. 2. A neutrosophic fuzzy set $\mu =< \mu_1, \mu_2, \mu_3 >$ in a $nfbts\ (X, \tau_i), i\epsilon\{1,2\}$ is called neutrosophic pairwise quasi-neighborhood of the point $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ if and only if there exists a neutrosophic fuzzy τ_i-open, $i\epsilon\{1,2\}$ set $\rho =< \rho_1, \rho_2, \rho_3 >$ such that $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}\ Nq\ \rho \subseteq \mu$. We will denote the set of all pairwise $Nq - nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}\ in\ (X, \tau_i), i\epsilon\{1,2\}$ by $P(x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}, \tau_i), i\epsilon\{1,2\}$.

Definition 3.3. Let $(X, \tau_i), i\epsilon\{1,2\}$ be a $nfbts$ with neutrosophic fuzzy ideal L on X, and $\mu =< \mu_1, \mu_2, \mu_3 >$ in 1_N. Then the neutrosophic fuzzy pairwise local function $NP\mu^*(L, \tau_i), i\epsilon\{1,2\}$ of $\mu =< \mu_1, \mu_2, \mu_3 >$ is the union of all neutrosophic fuzzy points $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ such that for $\rho =< \rho_1, \rho_2, \rho_3 >$ in $NPN(x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}, \tau_i), i\epsilon\{1,2\}$ and λ in L then there is at least one r in X for which $\rho_1(r) + \mu_1(r) - 1 > \lambda(r)$, $\rho_2(r) + \mu_2(r) - 1 > \lambda(r)$, $\rho_3(r) + \mu_3(r) - 1 < \lambda(r)$ or $\rho_1(r) + \mu_1(r) - 1 > \lambda(r)$, $\rho_2(r) + \mu_2(r) - 1 < \lambda(r)$, $\rho_3(r) + \mu_3(r) - 1 < \lambda(r)$ where $NPN(x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}, \tau_i), i\epsilon\{1,2\}$ is the set of all $Nq - nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$. Therefore, any $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L, \tau_i), i\epsilon\{1, 2\}$(for any $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin \mu$ (any neutrosophic fuzzy set) implies hereafter, $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ maybe not contained in the neutrosophic fuzzy set μ, i.e. $x < \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 >> NP\mu^*(x)$, $\mu =< \mu_1, \mu_2, \mu_3 > (x)$ implies there is at least one ρ in $NPN(x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}, \tau_i)$ such that for every r in X, $\rho_1(r) + \mu_1(r) - 1 \leq \lambda(r)$, $\rho_2(r) + \mu_2(r) - 1 \leq \lambda(r)$, $\rho_3(r) + \mu_3(r) - 1 > \lambda(r)$, for some λ in L. We will occasionally write $NP\mu^*$ or $NP\mu^*(L)$ for $NP\mu^*(L, \tau_i)$.We define P^*-neutrosophic fuzzy closure operator, denoted by $Npcl^*$ for fuzzy bitopology $\tau^*{}_i(L)$finer than τ_i as follows: $Npcl^*(\mu) = \mu \vee NP\mu^*$ for every fuzzy set $\mu =< \mu_1, \mu_2, \mu_3 >$ on X. When there is no ambiguity, we will simply write the symbols $NP\mu^*$and $\tau^*{}_i$ for $NP\mu^*(L, \tau_i)$and $\tau^{i^*}(L)$, respectively.

Definition 3.4. Let $(X,\tau_i), i\epsilon\{1,2\}$ be a $nfbts$ with neutrosophic fuzzy ideal L on X, a neutrtosophic fuzzy pairwise local function $NP\mu^*(L, \tau_1 \vee \tau_2), i\epsilon\{1,2\}$ of $\mu =< \mu_1, \mu_2, \mu_3 >$ in 1_N is the union of all neutrosophic fuzzy points $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ such that for $\rho =< \rho_1 , \rho_2, \rho_3 >$ in $NPN(x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}, \tau_i)$ and λ in L. Then there is at least one $r\ in\ X$ may be for two types which:

type1, $\rho_1(r) + \mu_1(r) - 1 > \lambda(r),\ \rho_2(r) + \mu_2(r) - 1 > \lambda(r),\ \rho_3(r) + \mu_3(r) - 1 < \lambda(r),$

type 2, $\rho_1(r) + \mu_1(r) - 1 < \lambda(r),\ \rho_2(r) + \mu_2(r) - 1 < \lambda(r),\ \rho_3(r) + \mu_3(r) - 1 > \lambda(r),$

where $NPN(x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}, \tau_i)$ is the set of all $Nq - nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in $\tau_1 \vee \tau_2$(where $\tau_1 \vee \tau_2$ is the neutrosophic fuzzy topology generated by τ_1, τ_2 .

Example 3.1. One may easily noticed

i- Consider $L = \{0_N\}$, then $NP\mu^*(L, \tau_i) = \tau_i - Ncl\big(\mu =< \mu_1, \mu_2 , \mu_3 >\big)$, for any $\mu =< \mu_1, \mu_2 , \mu_3 > \in 1_N, i\{1,2\}$.

ii- Consider $L = \{1_N\}$, then $NP\mu^*(L, \tau_i) = 0_N$, for any $\mu =< \mu_1, \mu_2 , \mu_3 > \in 1_N, i\{1,2\}$.

Note 3.1. In a $nfbts$ $(X, \tau_i), i\epsilon\{1,2\}$ with neutrosophic fuzzy ideal L on X, we will denote by $\sigma - Ncl(\mu =< \mu_1, \mu_2 , \mu_3 >)$ for the neutrosophic closure, and $\sigma - Nint(\mu)$ for the neutrosophic interior of a neutrosophic fuzzy subset $\mu =< \mu_1, \mu_2 , \mu_3 >$ in 1_N with respect to the neutrosophic fuzzy topology $\sigma = \tau_1 \vee \tau_2$.

The following theorems give some general properties of neutrosophic fuzzy pairwise-local function.

Theorem 3.1. Let $(X,\tau_i), i\epsilon\{1,2\}$ be a $nfbts$ with neutrosophic fuzzy ideal L on $X, \mu =< \mu_1, \mu_2 , \mu_3 >$ $, \eta =< \eta_1, \eta_2, \eta_3 >$ in 1_N. Then we have:

i- $NP\mu^*(L, \sigma) \subseteq NP\mu^*(L, \tau_i); i\epsilon\{1,2\}$.

ii- If $\mu =< \mu_1, \mu_2 , \mu_3 > \subseteq \eta =< \eta_1, \eta_2, \eta_3 >$ then $NP\mu^*(L, \sigma) \subseteq NP\eta^*(L, \tau_i); i\epsilon\{1,2\}$.

iii- $NP\mu^*(L, \sigma) \subseteq \sigma - Ncl(\mu) \subseteq \tau_i - Ncl(\mu)$.

iv- $NP\mu^{**}(L, \sigma) \subseteq NP\mu^*(L, \tau_i); i\epsilon\{1,2\}$.

Proof

i- Let $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L, \tau_i)$ i.e.$\varepsilon =< \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 >> NP\mu^*(x)$so $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ is not contained in $NP\mu^*$, this implies there is at least one $\rho =< \rho_1 , \rho_2, \rho_3 > \in NPN(x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>})$ in τ_i such that for every r in X,

type 1, $\rho_1(r) + \mu_1(r) - 1 \leq \lambda(r)\ , \rho_2(r) + \mu_2(r) - 1 \leq \lambda(r)\ ,\ \rho_3(r) + \mu_3(r) - 1 > \lambda(r),$

type 2, $\rho_1(r) + \mu_1(r) - 1 \leq \lambda(r)\ , \rho_2(r) + \mu_2(r) - 1 > \lambda(r)\ ,\ \rho_3(r) + \mu_3(r) - 1 > \lambda(r),$

for some λ in L . Hence ρ in $NPN(x_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>},\sigma)$ and so $x_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>} \notin NP\mu^*(L,\sigma)$. Therefore $NP\mu^*(L,\sigma) \subseteq NP\mu^*(L,\tau_i); i\epsilon\{1,2\}$.

ii- Let $x_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>} \in NP\eta^*(L,\tau_i); i\epsilon\{1,2\}$, . This implies there is at least one $Nq - nbd$ $\rho = <\rho_1, \rho_2, \rho_3>$ in $NPN(x_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>},\tau_i)$ such that every $r\epsilon X$, $\rho_1(r) + \eta_1(r) - 1 > \lambda(r)$, $\rho_2(r) + \eta_2(r) - 1 > \lambda(r)$, $\rho_3(r) + \eta_3(r) - 1 < \lambda(r)$, or $\rho_1(r) + \eta_1(r) - 1 > \lambda(r)$, $\rho_2(r) + \eta_2(r) - 1 < \lambda(r)$, $\rho_3(r) + \eta_3(r) - 1 < \lambda(r)$, λ in L. Hence $\rho = <\rho_1, \rho_2, \rho_3>$ in $NPN(x_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>},\sigma)$. Since $\mu = <\mu_1, \mu_2, \mu_3> \subseteq \eta = <\eta_1, \eta_2, \eta_3>$, by the heredity property $\rho_1(r) + \mu_1(r) - 1 > \lambda(r)$, $\rho_2(r) + \mu_2(r) - 1 > \lambda(r)$, $\rho_3(r) + \mu_3(r) - 1 < \lambda(r)$ or $\rho_1(r) + \mu_1(r) - 1 > \lambda(r)$, $\rho_2(r) + \mu_2(r) - 1 < \lambda(r)$, $\rho_3(r) + \mu_3(r) - 1 < \lambda(r)$. Therefore $x_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>} \in NP\mu^*(L,\sigma)$.

iii- ,(iv)Obvious .

Theorem 3.2. Let $(X,\tau_i), i\epsilon\{1,2\}$ be a $nfbts$ with neutrosophic fuzzy ideal L on X, $\mu = <\mu_1, \mu_2, \mu_3>$, $\eta = <\eta_1, \eta_2, \eta_3>$ are two neutrosophic fuzzy sets, if $\tau_1 \subseteq \tau_2$, then

i- $NP\mu^*(L,\tau_2) \subseteq NP\mu^*(L,\tau_1)$, for every neutrosophic fuzzy set μ ,

ii- ${\tau_1}^* \subseteq {\tau_2}^*$.

Proof. i- Since every $Nq - nbd$ in τ_1 of any neutrosophic fuzzy point $x_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>}$ maybe also $Nq - nbd$ in τ_2. Therefore, $NP\mu^*(L,\tau_2) \subseteq NP\mu^*(L,\tau_1)$ as there may be other $Nq - nbd$ in τ_2 of $x_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>}$ where is the condition for $x_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>}$ to be in $NP\mu^*(L,\tau_2)$ may be not hold true, although $x_{<\varepsilon_1,\varepsilon_2,\varepsilon_3>}$ in $NP\mu^*(L,\tau_1)$.

ii- Clearly ,${\tau_1}^* \subseteq {\tau_2}^*$ as $NP\mu^*(L,\tau_2) \subseteq NP\mu^*(L,\tau_1)$.

Theorem 3.3. Let $(X,\tau_i), i\epsilon\{1,2\}$ be a nfbts and L, J be two neutrosophic fuzzy ideals with neutrosophic fuzzy ideal L on X. Then for any neutrosophic fuzzy sets $\mu = <\mu_1, \mu_2, \mu_3>$ and $\rho = <\rho_1, \rho_2, \rho_3>$. The following statements are satisfied:

i- $\mu = <\mu_1, \mu_2, \mu_3> \subseteq \rho = <\rho_1, \rho_2, \rho_3> \Longrightarrow NP\mu^*(L,\tau_i) \subseteq NP\rho^*(L,\tau_i), i\epsilon\{1,2\}$.

ii- $L \subseteq J \Longrightarrow NP\mu^*(L,\tau_i) \subseteq NP\mu^*(J,\tau_i), i\epsilon\{1,2\}$.

iii- $NP\mu^* = \tau_i - Ncl(NP\mu^*) \subseteq \tau_i - Ncl(\mu), i\epsilon\{1,2\}$.

iv- $NP\mu^{**}(L,\tau_i) \subseteq NP\mu^*(L,\tau_i), i\epsilon\{1,2\}$.

v- $NP(\mu \cup \rho)^*(L,\tau_i) = NP\mu^*(L,\tau_i) \cup \rho^*(L,\tau_i)$.

vi- $\rho =< \rho_1 , \rho_2, \rho_3 >$ in L $\Longrightarrow NP(\mu \cup \rho)^*(L, \tau_i) = NP\mu^*(L, \tau_i)$.

Proof.

i- Since $\mu \subseteq \rho$ implies $\mu \leq \rho$ for every x in X , therefore by Definition 3.1 $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in $NP\mu^*(L, \tau_i)$ implies $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in $NP\rho^*(L, \tau_i)$, which complete the proof of (i).

ii- Cleary, $L \subseteq J \Longrightarrow NP\mu^*(L, \tau_i) \subseteq NP\mu^*(J, \tau_i), i\epsilon\{1,2\}$ as there may be other neutrosophic fuzzy sets which belong toJso that for a neutrosophic fuzzy point $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in $NP\mu^*(J, \tau_i)$ but $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ may be not contained $NP\mu^*(L, \tau_i), i\epsilon\{1,2\}$.

iii- Since $\{0_N\} \subseteq L$ for any neutrosophic fuzzy ideal L on X, Therefore by (ii) and Example 3.1, $NP\mu^*(L, \tau_i) \subseteq NP\mu^*(\{0_N\}, \tau_i) = \tau_i - Ncl(\mu =< \mu_1, \mu_2 , \mu_3 >)$ for any neutrosophic fuzzy set $\mu =< \mu_1, \mu_2 , \mu_3 >$ of X. Suppose, $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in $\tau_i - Ncl(\mu =< \mu_1, \mu_2 , \mu_3 >^*)$, so there is at least one $r\epsilon X$ for which $NP\mu_1{}^* + \nu_1(r) - 1 > \lambda(r)$, $NP\mu_2{}^* + \nu_2(r) - 1 > \lambda(r)$, $NP\mu_3{}^* + \nu_3(r) - 1 < \lambda(r)$ or $NP\mu_1{}^* + \nu_1(r) - 1 > \lambda(r)$, $NP\mu_2{}^* + \nu_2(r) - 1 < \lambda(r)$, $NP\mu_3{}^* + \nu_3(r) - 1 < \lambda(r)$, for each $Nq - nbd\ \nu =< \nu_1, \nu_2, \nu_3 >$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$. Hence $NP\mu^* \neq \{0_N\}$. Let $S = NP\mu^*(r)$. Cleary $r_{t=<t_1,t_2,t_3>}$ in $NP\mu^*(L, \tau_i)$ and $t_1 + \nu_1(r) > 1,\ t_2 + \nu_2(r) > 1,\ t_3 + \nu_3(r) < 1$ or $t_1 + \nu_1(r) > 1,\ t_2 + \nu_2(r) < 1,\ t_3 + \nu_3(r) < 1$ so there is $\nu =< \nu_1, \nu_2, \nu_3 >$ is also $Nq - nbd$ of $r_{t=<t_1,t_2,t_3>}$ in τ_i. Now $r_{t=<t_1,t_2,t_3>}$ in $NP\mu^*(L, \tau_i)$, so there may be at least one $r^{/}$ in X for which $\eta_1(r^{/}) + \mu_1(r^{/}) - 1 > \lambda(r^{/})$, $\eta_2(r^{/}) + \mu_2(r^{/}) - 1 > \lambda(r^{/})$, $\eta_3(r^{/}) + \mu_3(r^{/}) - 1 < \lambda(r^{/})$ or $\mu_1(r^{/}) - 1 > \lambda(r^{/})$, $\eta_2(r^{/}) + \mu_2(r^{/}) - 1 < \lambda(r^{/})$, $\eta_3(r^{/}) + \mu_3(r^{/}) - 1 < \lambda(r^{/})$ for each $Nq - nbd\ \eta$ of $r_{t=<t_1,t_2,t_3>}$ and λ in L.This may be true for $\nu =< \nu_1, \nu_2, \nu_3 >$ so there is at least one $r^{//}$ in X such that $\nu_1(r^{//}) + \mu_1(r^{//}) - 1 > \lambda(r^{//})$, $\nu_2(r^{//}) + \mu_2(r^{//}) - 1 > \lambda(r^{//})$, $\nu_3(r^{//}) + \mu_3(r^{//}) - 1 < \lambda(r^{//})$ or $\nu_1(r^{//}) + \mu_1(r^{//}) - 1 > \lambda(r^{//})$, $\nu_2(r^{//}) + \mu_2(r^{//}) - 1 < \lambda(r^{//})$, $\nu_3(r^{//}) + \mu_3(r^{//}) - 1 < \lambda(r^{//})$ for each λ in L. Since $\nu =< \nu_1, \nu_2, \nu_3 >$ may be an arbitrary $Nq - nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i therefore $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in $NP\mu^*(L, \tau_i)$ hence $NP\mu^* = \tau_i - Ncl(NP\mu^*) \subseteq \tau_i - Ncl(\mu), i\epsilon\{1,2\}$,

iv- Clear

v- Suppose, $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L, \tau_i) \cup \rho^*(L, \tau_i)$ i.e. $\varepsilon = < \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 >, \varepsilon > (NP\mu^* \vee NP\rho^*)(x) = \max\{NP\mu^*(x), NP\rho^*\}$. So $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ is not contained in both $NP\mu^*$ and $NP\rho^*$. This implies that there is at least one $Nq - nbd\ \ \nu_1$ in τ_i, of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ such that for every r in X, $\nu_1(r) + \mu_1(r) - 1 \leq \lambda_1(r)$,

$\nu_1(r)+\mu_2(r)-1\le\lambda_1(r)$, $\nu_1(r)+\mu_3(r)-1>\lambda_1(r)$, for some λ_1 in L and similarly, there is at least one $Nq-nbd$ ν_2 of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i such that, for every r in X, $\nu_2(r)+\rho_1(r)-1\le\lambda_2(r)$, $\nu_2(r)+\rho_2(r)-1\le\lambda_2(r)$, $\nu_2(r)+\rho_3(r)-1>\lambda_2(r)$ for some λ_2 in L. Also, there is at least one $Nq-nbd$ ν_3 of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i such that, for every r in X, $\nu_3(r)+\eta_1(r)-1\le\lambda_3(r)$, $\nu_3(r)+\eta_2(r)-1\le\lambda_3(r)$, $\nu_3(r)+\eta_3(r)-1>\lambda_3(r)$ for some λ_3 in L . Let $\nu=\nu_1\wedge\nu_2\wedge\nu_3$, so ν is also $Nq-nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i and $\nu_1(r)+(\mu_1\vee\rho_1)(r)-1\le(\lambda_1\vee\lambda_2\vee\lambda_3)(r)$, $\nu_2(r)+(\mu_2\vee\rho_2)(r)-1\le(\lambda_1\vee\lambda_2\vee\lambda_3)(r)$, $\nu_3(r)+(\mu_3\vee\rho_3)(r)-1>(\lambda_1\vee\lambda_2\vee\lambda_3)(r)$, for every r in X. Therefore, by finite additively of neutrosophic fuzzy ideal as $\lambda_1\vee\lambda_2\vee\lambda_3$ in L, $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}\notin(\mu\vee\rho)^*$. Hence $P(\mu\cup\rho)^*(L,\tau_i)\subseteq P\mu^*(L,\tau_i)\cup\rho^*(L,\tau_i)$. Clearly, both μ and $\rho\subseteq\mu\cup\rho$ which implies $NP\mu^*(L,\tau_i)\cup\rho^*(L,\tau_i)\subseteq NP\big(\mu=<\mu_1,\mu_2,\mu_3>\cup\rho=<\rho_1,\rho_2,\rho_3>\big)^*(L,\tau_i)$ and this the proof .

vi- Clear.

4. Basic Structure of Generated Neutrosophic Fuzzy Bitopology.

Let (X,τ_i), $i\epsilon\{1,2\}$ be a $nfbts$ with neutrosophic fuzzy ideal L on X. Let us define $\tau_i-Npcl^*\big(\mu=<\mu_1,\mu_2,\mu_3>\big)=\mu=<\mu_1,\mu_2,\mu_3>\cup NP\mu^*(L,\tau_i)$, $i\epsilon\{1,2\}$ for any neutrosophic fuzzy set $\mu=<\mu_1,\mu_2,\mu_3>$ in 1_N . Clearly $\tau_i-Npcl^*(\mu=<\mu_1,\mu_2,\mu_3>)$ represent a neutrosophic fuzzy closure operator. Let $\tau^*{}_i(L)$ be the neutrosophic fuzzy bitopology generated by $\tau_i-Npcl^*\big(\mu=<\mu_1,\mu_2,\mu_3>\big)$, i.e. $\tau^*{}_i(L)=\{\mu=<\mu_1,\mu_2,\mu_3>:\tau_i-Npcl^*(\mu^c)=\mu^c\}$. Now, let $L=\{0_N\}\Rightarrow\tau_i-Ncl^*\big(\mu=<\mu_1,\mu_2,\mu_3>\big)=\mu\cup NP\mu^*(L,\tau_i)=\mu\cup\tau_i-Ncl(\mu)=\tau_i-Ncl(\mu)$ $i\epsilon\{1,2\}$, for every $\mu=<\mu_1,\mu_2,\mu_3>$ in 1_N, so $\tau^*{}_i(\{0_N\})=\tau_i$, $i\epsilon\{1,2\}$. Again let $L=\{1_N\}\Rightarrow\tau_i-Ncl^*\big(\mu=<\mu_1,\mu_2,\mu_3>\big)=\mu\cup P\mu^*(L,\tau_i)=\mu\cup\{0_N\}=\mu$, so $\tau^*{}_i(1_N)$, $i\epsilon\{1,2\}$ is neutrosophic fuzzy discrete bitopology on X. We can conclude by Theorem 3.1 (ii), $\tau^*{}_i(\{0_N\})\subseteq\tau^*{}_i(L)\subseteq\tau^*{}_i(1_N)$, i.e. $\tau_i\subseteq\tau^*{}_i$, $L\subseteq J\Rightarrow\tau^*{}_i(L)\subseteq\tau^*{}_i(J)$. Let $\mu=<\mu_1,\mu_2,\mu_3>$ be a $Nq-nbd$ of a neutrosophic fuzzy point $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in $\tau^*{}_i$ – neutrosophic fuzzy bitopology . Therefore , there exist $\rho=<\rho_1,\rho_2,\rho_3>$ in $\tau^*{}_i$, $i\epsilon\{1,2\}$ such that b, $\varepsilon_1+\rho_1(x)>1$, $\varepsilon_2+\rho_2(x)>1$, $\varepsilon_3+\rho_3(x)<1$ or $\varepsilon_1+\rho_1(x)>1$, $\varepsilon_2+\rho_2(x)<1$, $\varepsilon_3+\rho_3(x)<1$and $\rho=<\rho_1,\rho_2,\rho_3>\subseteq\mu=<\mu_1,\mu_2,\mu_3>$. Now , $\mu=<\mu_1,\mu_2,\mu_3>$ in $\tau^*{}_i\Leftrightarrow\mu^c$ is $\tau^*{}_i$ -closed $\Leftrightarrow\tau_i-Ncl^*(\mu)=\mu^c\Leftrightarrow NP(\mu^c)^*\subseteq\mu^c\Leftrightarrow\mu\subseteq(NP(\mu^c)^*)^c$. Therefore

$\varepsilon_1 + \mu_1(x) > 1 \Longrightarrow \varepsilon_1 + \{(\mu_1{}^c)^*\}(x) > 1 \Rightarrow \varepsilon_1 + 1 - NP(\mu_1{}^c)^*(x) > 1, \varepsilon_1 > (\mu_1{}^c)^*(x) \Rightarrow x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin \left(\mu = < \mu_1, \mu_2, \mu_3 >^c\right)^*$, $\varepsilon_2 + \mu_2(x) > 1 \Longrightarrow \varepsilon_2 + \{(\mu_2{}^c)^*\}(x) > 1 \Rightarrow \varepsilon_2 + 1 - NP(\mu_2{}^c)^*(x) > 1, \varepsilon_2 > (\mu_2{}^c)^*(x) \Rightarrow x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin \left(\mu =< \mu_1, \mu_2, \mu_3 >^c\right)^*$, $\varepsilon_3 + \mu_3(x) < 1 \Longrightarrow \varepsilon_3 + \{(\mu_3{}^c)^*\}(x) < 1 \Rightarrow \varepsilon_3 + 1 - NP(\mu_3{}^c)^*(x) < 1, \varepsilon_3 \leq (\mu_3{}^c)^*(x) \Rightarrow x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin \left(\mu =< \mu_1, \mu_2, \mu_3 >^c\right)^*$. This implies there exists at least one $Nq - nbd$ ν_1, of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \in \tau_i$ such that for every r in $X, \nu_1(r) + \mu_1{}^c(r) - 1 \leq \lambda_1(x)$, $\nu_1(r) + \mu_2{}^c(r) - 1 \leq \lambda_1(x)$, $\nu_1(r) + \mu_3{}^c(r) - 1 > \lambda_1(x)$ for some λ_1 in L. i.e. $\nu_1(r) - \lambda_1(r) \leq \lambda_1(x)$ for every r in X, there exists at least one $Nq - nbd$ ν_2, of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \in \tau_i$ such that for every r in $X, \nu_2(r) + \mu_1{}^c(r) - 1 \leq \lambda_1(x)$, $\nu_2(r) + \mu_2{}^c(r) - 1 \leq \lambda_1(x)$, $\nu_2(r) + \mu_3{}^c(r) - 1 > \lambda_1(x)$ for some λ_1 in L . i.e. $\nu_2(r) - \lambda_1(r) \leq \lambda_1(x)$ for every r in X, there exists at least one $Nq - nbd$ ν_3, of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ (in τ_i) such that for every r in $X, \nu_3(r) + \mu_1{}^c(r) - 1 \leq \lambda_1(x)$, $\nu_3(r) + \mu_2{}^c(r) - 1 \leq \lambda_1(x)$, $\nu_3(r) + \mu_3{}^c(r) - 1 > \lambda_1(x)$ for some λ_1 in L. i.e. $\nu_3(r) - \lambda_1(r) \leq \lambda_1(x)$ for every r in X, . Therefore, as ν_1 $Nq - nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \in \tau_i$, ν_2 $Nq - nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \in \tau_i$, ν_2 $Nq - nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \in \tau_i$, there is a $\nu =< \nu_1, \nu_2, \nu_3 >$ in τ_i such that $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} Nq\ \nu =< \nu_1, \nu_2, \nu_3 > \subseteq \nu_1$, $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} Nq\ \nu =< \nu_1, \nu_2, \nu_3 > \subseteq \nu_2$, $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} Nq\ \nu =< \nu_1, \nu_2, \nu_3 > \subseteq \nu_3$ and by heredity property of neutrosophic fuzzy ideal we have λ in L for which $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} Nq \left(\nu =< \nu_1, \nu_2, \nu_3 > -\lambda\right) \subseteq \mu$, where $\left(\nu =< \nu_1, \nu_2, \nu_3 > -\lambda\right)(r) = \max\{\nu(r) - \lambda(r), 0\}$ for every r in X. Hence , for $\mu =< \mu_1, \mu_2, \mu_3 >$ in $\tau^*{}_i$, we have a $\nu =< \nu_1, \nu_2, \nu_3 >$ τ_i and λ in L such that ,$(\nu =< \nu_1, \nu_2, \nu_3 > -\lambda) \subseteq \mu$. Let us denote $\beta(L, \tau_i) = \{\nu - \lambda: \nu \text{ in } \tau_i, \lambda \text{ in } L\}$. Then we have the following Theorem.

Theorem 4.1: $\beta(L, \tau_i)$ from a basis for the generated neutrosophic fuzzy bitopology $\tau^*{}_i(L)$ of the nfbts (X, τ_i), i∈{1,2} with neutrosophic fuzzy ideal L on X, the class $\beta(L, \tau_i) = \{\{\mu - \lambda\}: \mu \text{ in } \tau_i, \lambda \text{ in } L, i \in \{1,2\}\}$ may be the base for the neutrosophic fuzzy bitopology $\tau^*{}_i$.

Proof: Straightforward

Theorem 4.2. If $L_1$and L_2 are two neutrosophic fuzzy ideals on nfbts (X, τ_i), i∈{1,2}, μ in 1_N,then,

i- $NP\mu^*(L_1, \tau_i) \geq NP\mu^*(L_2, \tau_i)$ for every neutrosophic fuzzy set μ and $L_1 \leq L_2$.

ii- $\tau^*{}_i(L_1) \leq \tau^*{}_i(L_2)$ and $L_1 \leq L_2$.

iii- $NP\mu^*(L_1 \cap L_2, \tau_i) = NP\mu^*(L_1, \tau_i) \cup NP\mu^*(L_2, \tau_i)$.

iv- $NP\mu^*(L_1 \vee L_2, \tau_i) = NP\mu^*(L_1, \tau^*{}_i(L_2)) \cap NP\mu^*(L_2, \tau^*{}_i(L_1))$.

Proof. i and ii are clear.

iii- Let $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L_1, \tau_i) \cup x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L_2, \tau_i)$. So $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ is not contained in both $NP\mu^*(L_1,\tau_i)$ and $NP\mu^*(L_2,\tau_i)$. Now $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L_1,\tau_i)$ implies there is at least one $Nq-nbd$ ν_1 in τ_i, of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ such that for every r in X, $\nu_1(r)+\mu_1(r)-1 \le \lambda_1(r)$, $\nu_1(r)+\mu_2(r)-1 \le \lambda_1(r)$, $\nu_1(r)+\mu_3(r)-1 > \lambda_1(r)$ for some λ_1 in L. Again $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L_2,\tau_i)$ and similarly, there is at least one $Nq-nbd$ ν_2 of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i such that, for every r in X, $\nu_2(r)+\mu_1(r)-1 \le \lambda_2(x)$, $\nu_2(r)+\mu_2(r)-1 \le \lambda_2(x), \nu_2(r)+\mu_3(r)-1 > \lambda_2(x))$ for some λ_2 in L, similarly, there is at least one $Nq-nbd$ ν_3 of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i such that, for every r in X, $\nu_3(r)+\mu_1(r)-1 \le \lambda_3(x)$, $\nu_3(r)+\mu_2(r)-1 \le \lambda_3(x), \nu_3(r)+\mu_3(r)-1 > \lambda_3(x))$ for some λ_3 in L. Therefore, we have $\nu=\nu_1\cap\nu_2\cap\nu_3$, so $(\nu=<\nu_1,\nu_2,\nu_3>$ may be also $Nq-nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i and $\nu_1(r)+\mu_1(r)-1 \le \lambda_1\cap\lambda_2\cap\lambda_3(r)$, $\nu_2(r)+\mu_2(r)-1 \le \lambda_1\cap\lambda_2\cap\lambda_3(r)$, $\nu_3(r)+\mu_3(r)-1 > \lambda_1\cap\lambda_2\cap\lambda_3(r)$, for every r in X. Since $\nu=<\nu_1,\nu_2,\nu_3>$ may be also $Nq-nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i and $\lambda_1\cap\lambda_2\cap\lambda_3$ in ν, therefore $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L_1\cap L_2,\tau_i)$, so that $NP\mu^*(L_1\cap L_2,\tau_i) \subseteq NP\mu^*(L_1,\tau_i)\cup NP\mu^*(L_2,\tau_i)$. Also $(L_1\cap L_2)$ included in both L_1 and L_2, so by Theorem 3.1. (ii), reverse inclusion is obvious, which completes the proof of (iii).

iv) Let $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin, NP\mu^*(L_1\vee L_2,\tau_i)$ implies there is at least one $Nq-nbd$ ν_1 of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i such that for every r in X, $\nu_1(r)+\mu_1(r)-1 \le \lambda_1(r)$, $\nu_1(r)+\mu_2(r)-1 \le \lambda_1(r)$, $\nu_1(r)+\mu_3(r)-1 > \lambda_1(r)$ for some λ_1 in $L_1\vee L_2$, there is at least one $Nq-nbd$ ν_2 of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i such that for every r in X, $\nu_2(r)+\mu_1(r)-1 \le \lambda_2(r)$, $\nu_2(r)+\mu_2(r)-1 \le \lambda_2(r)$, $\nu_2(r)+\mu_3(r)-1 > \lambda_2(r)$ for some λ_2 in $L_1\vee L_2$, there is at least one $Nq-nbd$ ν_3 of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in τ_i such that for every r in X, $\nu_3(r)+\mu_1(r)-1 \le \lambda_3(r)$, $\nu_3(r)+\mu_2(r)-1 \le \lambda_3(r)$, $\nu_3(r)+\mu_3(r)-1 > \lambda_3(r)$ for some λ_3 in $L_1\vee L_2$. Therefore, by heredity of the neutrosophic fuzzy ideals and considering the structure of neutrosophic fuzzy τ_i-open sets generated neutrosophic fuzzy bitopology, we can find ν_1,ν_2,ν_3 the $Nq-nbd$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in $\tau^*{}_i(L_1)$ or $\tau^*{}_i(L_2)$ respectively, such that, for every r in X, $\nu_1(r)+\mu(r)-1 \le \lambda_1(r)$ or $\nu_2(r)+\mu(r)-1 \le \lambda_2(r)$ or $\nu_3(r)+\mu(r)-1 > \lambda_3(r)$ for some λ_2 in L_2 or λ_1 in L_1 or λ_2 in L_2 or λ_3 in L_1 for every r in X. This implies $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L_1,\tau^*{}_i(L_2))$ or $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L_2,\tau^*{}_i(L_1))$. Thus we have $NP\mu^*(L_1,\tau^*{}_i(L_2)) \cap NP\mu^*(L_2,\tau^*{}_i(L_1)) \subseteq NP\mu^*(L_1 \vee$

$L_2, \tau_i)$. Conversely, let $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin NP\mu^*(L_1, \tau^*{}_i(L_2))$. This implies there may be least one on $Nq-nbd\ \nu =< \nu_1, \nu_2, \nu_3 >$ of $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>}$ in $\tau^*{}_i$ such that for every r in X, $\nu(r) + \mu(r) - 1 \leq \lambda_1 \cup \lambda_2 \cup \lambda_2(r)$, for some λ_1 in L_1 and for some λ_2 in L_2, λ_3 in L_1.i.e., $x_{<\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3>} \notin, NP\mu^*(L_1 \vee L_2, \tau_i)$. Thus,

$NP\mu^*(L_1 \vee L_2, \tau_i) \subseteq NP\mu^*(L_1, \tau^*{}_i(L_2))$ and $NP\mu^*(L_2, \tau^*{}_i(L_1))$. Then

$NP\mu^*(L_1 \vee L_2, \tau_i) \subseteq NP\mu^*(L_1, \tau^*{}_i(L_2)) \cap NP\mu^*(L_2, \tau^*{}_i(L_1))$ and this completes the proof.

An important result follows from the above theorem that $\tau^*{}_i(L)$ and $\tau^{**}{}_i(L)$ are Equal for any neutrosophic fuzzy ideal on X.

Corollary 4.1: Let (X,τ_i), i϶{1,2} be a nfbts with neutrosophic fuzzy ideal L. Then $\tau^*{}_i(L) = \tau^{**}{}_i(L)$

Proof. By taking $L_1 = L_2 = L$ in the above Theorem, we have the required result .

Corollary 3.2: If L_1 and L_2 are two neutrosophic fuzzy ideals on nfbt (X, τ_i) then,

i- $\tau^*{}_i(L_1 \vee L_2, \tau_i) = [\tau^{**}{}_i(L_2, \tau_i)](L_1) = [\tau^{**}{}_i(L_1, \tau_i)](L_2)$,

ii- $\tau^*{}_i(L_1 \vee L_2, \tau_i) = [\tau^*{}_i(L_1, \tau_i)] \vee [\tau^*{}_i(L_2, \tau_i)]$,

iii-$\tau^*{}_i(L_1 \cap L_2, \tau_i) = [\tau^*{}_i(L_1, \tau_i)] \cap [\tau^*{}_i(L_2, \tau_i)]$.

5. Some Applications in Neutrosophic Fuzzy Ideal Function.

Application 5.1. In this example we illustrate the neutrosophic degrees, it produces three types of chips that are represented $X = \{ x_1 < 1,1,1 > \}$, it represents the total production of the plant, where $A = \{ x_1 < 0.6,0.3,0.4 > \}$ represents the neutrosophic component of the first type production, $B = \{ x_1 < 0.3,0.5,0.7 > \}$ represents the neutrosophic component of the second type production, $C = \{ x_1 < 0.1,0.7,0.9 > \}$ represents the neutrosophic component of the third type production. We defined the $N\tau_{T_1}$ is a neutrosophic bitopological space of the total production $N\tau_{T_i} = \{ O_N, X_N A, B , C \}, i \in \{1,2\}$, , NL is a neutrosophic ideal space of the total production $FNL = \{ O_N, A, B , C \}$, $A^* = B^* = C^* = \{ < 0,0,0 > \}$. Let $D = \{ x_1 < 0.6,0.1,0.9 > \} \notin N\tau_{T_i}, i \in \{1,2\}$, then $D^* = \{ < 0.6,0.3,0.9 > \}$, FNInt(D)=A, we, compute the complement of a neutrosophic bitopological space $co(N\tau_{T_i}) = \{ X_N, O_N, co(A), co(B), co(C) \}, i \in \{1,2\}$, $co(A) =< 0.4,0.7,0.6 >$, $co(B) =< 0.7,0.5,0.3 >$, $co(C) =< 0.4,0.7,0.6 >$, $co(D) =< 0.4,0.9,0.1 >$, NCL(D) = co(C). In the above Example, we conclude and add a new production with the new type D such that D^* as generalized of the production neutrosophic ideal subspace D, the following Table 5.1. represent the new Matrix for the type for projections.

N-type	NINT	*	NCL
A	$< 0.6,0.3,0.4 >$	$< 0,0,0 >$	$< 0.4,0.7,0.6 >$
B	$< 0.3,0.5,0.7 >$	$< 0,0,0 >$	$< 0.7,0.5,0.3 >$
C	$< 0.1,0.7,0.9 >$	$< 0,0,0 >$	$< 0.9,0.3,0.1 >$
Proposed D new type	$< 0.6,0.3,0.4 >$	$< 0.6,0.3,0.9 >$	$< 0.9,0.3,0.1 >$

Table 5.1. Neutrosophic Matrix for Projections.

Note That: $\mathrm{Nint}\ (D) \leq D \leq D^* \leq \mathrm{Ncl}(D)$

Application 5.2. The following example illustrates a construction of the neutrosophic topological space for an aircraft with two engines and we study the degrees of wear on the two engines by building a neutrosophic topological space to support and make the right decision, we defined universal set $X = \{x_1 < 1,1,1 >\}$, degrees of damage in the first engine $A = \{x_1 < 0.01,0.05,0.99 >\}$, degrees of damage in the second engine $B = \{x_1 < 0.1,0.7,0.9 >\}$, Degrees of damage in the two engines together $A \cap B = \{x_1 < 0.001,0.007,0.999 >\}$, degrees of damage in the first A or second B engine $A \cup B = \{x_1 < 0.1,0.7,0.9 >\}$, neutrosophic topological space to degrees damages $NT_{T_i} = \{0_N, X_N, A, B, A \cup B, A \cap B\}, i \in \{1,2\}$, we defined neutrosophic topological space to degrees the right competence $co(NT_{T_i}) = \{X_N, 0_N, co(A), co(B), co(A \cup B), co(A \cap B)\}, i \in \{1,2\}$, we introduce $co(A) = \{x_1 < 0.99,0.05,0.01 >\}$, $co(B) = \{x_1 < 0.9,0.3,0.1 >\}$, $co(A \cap B) = \{x_1 < 0.999,0.993,0.001 >\}$, $co(A \cup B) = \{x_1 < 0.9,0.3,0.1 >\}$, we defined neutrosophic bitopological ideal space to degrees the right competence $NL = \{0_N, co(A), co(B), , co(A \cap B)\}$, and

$(co(A))^* = \{x_1 < 0.99,0.993,0.01 >\}$, $(co(B))^* = \{x_1 < 0.9,0.993,0.1 >\}$, $(co(A \cap B))^* = \{x_1 < 0.99,0.993,0.01 >\}$.

From the above information, we found that the efficiency of the second engine type2 is correct and it is less than the certainty for the correct first engine type1. The degree of diffraction for the two motors is equal, the degree of uncertainty of the second plane's proper motion is greater than the degree of uncertainty of the first correct aircraft movement.

Also, we found that the degree of certainty of the efficiency of the two flying engines together is very high, we find that the degree of uncertainty of the efficiency of the two engines together is very small. From the above, we conclude that the degree of efficiency of the aircraft while operating the two engines together is of a high degree of efficiency.

6. Conclusion

There is no doubt that the neutrosophic fuzzy topology and bitopological spaces were unfathomable aspects, except the activity of some brilliant authors in publishing dozens of papers related to the structural of neutrosophic fuzzy bitopological spaces, neutrosophic fuzzy ideals, neutrosophic fuzzy local function, neutrosophic fuzzy pairwise local function. In this paper the authors suggested new theorems that give some general properties of the above mentioned concepts. Finally, some applied problems in neutrosophic fuzzy ideals function have been introduced.

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Pentapartitioned Neutrosophic Probability Distributions

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Abstract: In this manuscript, we introduce and study some pentapartitioned neutrosophic probability distributions. The study is done through the generalization of some classical probability distributions as Poisson distribution, Exponential distribution, Uniform distribution etc. This study opens the way for dealing with issues that follow the classical distributions and at the same time contains data not specified accurately.

Keywords: Neutrosophic Set; Probability Distributions; Pentapartitioned Neutrosophic Probability.

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1. Introduction: The term "Neutrosophy" was first proposed by Prof. Florentin Smarandache [5] in the year 1995. Neutrosophy is a new branch of philosophy, where one can study origin, nature and scope of neutralities. This theory considers every notion or idea <A> together with its opposite or negation <Anti-A>. The <neut-A> and <Anti-A> ideas together called as a <non-A>. Neutrosophic logic is a general framework for unification of many existing logics, fuzzy logic, intuitionistic logic, paraconsistent logic etc. The core objective of neutrosophic logic is to characterize each logical statement in a 3D-neutrosophic space, where each dimension of space represents respectively the truth(T), falsehood(F) and indeterminacies (I) of the statements under consideration, where T, I, F are standard or non-standard real subset of]-0,1+[without necessary connection between them. The classical distribution is extended neutrosophically. Which means that there is some indeterminacy related to the probabilistic experiment. Each experimental observation can result in an outcome of each trial labelled by failure (F) or some indeterminacies (I), in addition to some truthiness (T). Neutrosophic statistics is an extended form of classical statistics, dealing with values holding some vague, or indeterminacy, or incompleteness information. The fundamental concepts of neutrosophic set, introduced by Smarandache, et al [5-9] and Salama et al [10-14]. Recently, using neutrosophic theory, dozens of applications were re-analyzed and re-evaluated, including but not limited to the E-Learning that was raised due to quarantine situations of Coved-19 and its Omicron mutation, the integration system of renewable energy using various resources such as (Photovoltaic panels and Wind Turbines), and the neutrosophic treatment of the static model for inventory management with a safety reserve…etc. [15-27]. In this article, we will discuss a discrete random distribution such as Binomial distribution by approaching neutrosophically. Before shed the light on this context, we should familiar with the following notions: Neutrosophic statistical number 'N' has the form $N = a + I$; where the component a refers to the determinate part of N, while I refers to the indeterminate part of N. Recently, Mallick and Pramanik [2] introduced the concept of pentapartitioned neutrosophic set as an extension of neutrosophic set.

2. Some Relevant Definitions:

In this section, we recall some basic preliminaries and definitions which are relevant to the main results of this paper.

Definition 2.1. [1] Assume that 'w' be a continuous variable. A neutrosophic uniform distribution of w, is a classical uniform distribution, with imprecise distribution parameters c or d $(c < d)$.

Example 2.1. Assume that w be a variable represents a man waiting time lift (in minutes), lift arrival time is not specified, another man said:

1. the lift arrival time is either from now to 3 minutes [0,3] or will arrive after 13 to 17 minutes [13,17] , then $c = [0, 3]$, $d = [13, 17]$

Then, the probability density function:

$f_N(W) = \frac{1}{d-c} = \frac{1}{[13,17]-[0,3]} = \frac{1}{[10,17]}$ = [0.059, 0.1].

2. The lift arrives after seven minutes or will arrive after 13 to 17 minutes [13, 17]

Then, $c = 7, d = [13,17]$

Hence, the probability density function:

$f_N(w) = \frac{1}{d-c} = \frac{1}{[13,17]-7} = \frac{1}{[6,10]} = [0.1,0.167]$

Definition 2.2.[4] Assume that w be a random variable, which represents the number of success when events performs more than or equal to one times. Then, the corresponding probability distribution of w, is called a neutrosophic binomial distribution.

i. Neutrosophic Binomial Random Variable: The random variable 'w' represents the number of success more than or equal to one times.

ii. Neutrosophic Binomial Probability Distribution: The neutrosophic binomial probability distribution of 'w' is represented by n.b.p.d.

iii. Indeterminacy: It is not sure about the success or failure of an experiment output.

iv. Indeterminacy Threshold: Outcome of an event are indeterminate form**.** Where $th \in \{0,1,2 \dots m\}$, m is the sample size. Consider, $P(S)$ = The chance of a particular event outcome in the case of success. $P(F)$ = The chance of a particular event outcome in the case of failure, for both S and F different from indeterminacy. $P(I)$ = The chance of a particular event outcome in the case of an indeterminacy.

Let, $w \in \{0,1,2,\dots,m\}, NP = (T_w, I_w, F_w)$ with

T_w: Chances of 'w' success and the value of $(n-w)$ represents the number of failures and indeterminacy, such that the number of indeterminacy is less than or equal to the indeterminacy threshold. Where, n represents the population size.

F_w: Chances of 'v' success, with $v \neq w$ and the value of $(n-v)$ represents the number of failures and indeterminacy, and it is less than the indeterminacy threshold.

I_w: Chances of 'u' indeterminacy, where 'u' is strictly greater than the indeterminacy threshold.

$T_w + I_w + F_w = (P(S) + P(I) + P(F))^m$

For complete probability we have $P(S) + P(I) + P(F) = 1$; while if the probability was incomplete then, $0 \leq P(S) + P(I) + P(F) < 1$; however, for the paraconsistent probability we have $1 < P(S) + P(I) + P(F) \leq 3$;

$$T_w = \frac{m!}{w!\,(m-w)!}\, P(S)^w \sum_{r=0}^{th} \frac{r!}{(m-w)!\ (r-m+w)!}\, P(I)^r P(F)^{m-w-r}$$

$$= \frac{m!}{w!\,(m-x)!}\, P(S)^w \sum_{r=0}^{th} \frac{(m-w)!}{(m-w+r)!}\, P(I)^r P(F)^{m-w-r}$$

$$= \frac{m!}{w!}\, P(S)^w \sum_{r=0}^{th} \frac{\mathrm{P(I)}^r\ \mathrm{P(F)}^{\mathrm{m-w-r}}}{r!\ (m-w+r)!}$$

$$F_w = \sum_{v=0}^{m} T_v = \sum_{v=0, v\neq w}^{th} \frac{m!}{v!} \mathrm{P(S)}^v \sum_{r=0}^{th} \frac{\mathrm{P(S)}^r\ \mathrm{P(F)}^{\mathrm{m-v-r}}}{r!\ (m-v+r)!}$$

$$I_w = \sum_{u=th+1}^{m} \frac{m!}{u!\,(m-u)!} \mathrm{P(I)}^u \sum_{r=0}^{m-u} \frac{(m-u)!}{(m-u)!\ (m-u+r)!} \mathrm{P(S)}^r\ \mathrm{P(F)}^{\mathrm{m-u-r}}$$

$$= \sum_{u=th+1}^{m} \frac{m!}{u!}\, P(I)^u \sum_{r=0}^{m-u} \frac{\mathrm{P(S)}^r\ \mathrm{P(F)}^{\mathrm{m-u-r}}}{r!\ (m-u+r)!}$$

It is worthy to mention that $T_w, I_w, F_w, P(S), P(I), P(F)$ have their usual meaning.

Definition 2.3. [3] Neutrosophic Poisson distribution of a discrete variable 'w' is a classical Poisson distribution of w, but its parameter is imprecise. For example, λ can be a set contains two or more elements. The most common such distribution can be defined as follow:

$NP(w) = e^{-\lambda_N} \frac{(\lambda_N)^w}{w!}$; where λ_N is a set, and $w = 0,1, \ldots$

λ_N : Is a parameter of the distribution, also, λ_N represents the mean (the expectation) of the distribution , and at the same time it represents the variance value of the distribution. In symbols we can write, $NE(w) = NV(w) = \lambda_N$; where $N = d + I$; is a neutrosophic statistical number.

Definition 2.4. [1] Let 'w' be a continuous random variable is said to be neutrosophic exponential distribution, with parameter λ_N having some imprecise events which represent intervals, then the neutrosophic probability distribution function is given by

$W_N \sim \exp(\lambda_N) = \mathrm{f_N}(w) = \lambda_N . e^{-w.\lambda_N}$; $0 < w < \infty$,

$\exp(\lambda_N)$: Neutrosophic Exponential Distribution.

λ_N: Neutrosophic distribution parameter.

Definition 2.5. [7,8] Let the continuous random variable 'w' be a classical normal distribution, is said to be neutrosophic normal distribution with mean μ_N and variance σ_N, both contain intervals. Which probability distribution function is given by

$$W_N \sim N_N(\mu_N, \sigma_N) = \frac{1}{\sigma_N \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{w-\mu_N}{\sigma_N}\right)^2}$$

N_N: Neutrosophic Normal Distribution.

W_N: w Neutrosophic Random Variable.

3. A New Concepts:

In this section, we introduce new pentapartitioned neutrosophic probability distributions, which are first introduced and well defined supported by concrete examples.

Definition 3.1. Let w be a continuous random variable, and w is followed classical uniform distribution, with imprecise distribution parameters r and s ($r < s$), this kind of distribution is said to be a pentapartitioned neutrosophic uniform distribution, where pentapartitioned neutrosophic probability distribution function is given by

$f_{PN}(w) = \frac{1}{s-r}$, where $r \leq w \leq s$

Example 3.2.

Assume that w be a variable represents a person waiting time lift (in minutes/ seconds), where the lift arrival time is not specified, another person said:

1. The lift arrival time is either from now to 3 minutes $[0, 3]$, or will arrive after 13 to 17 minutes [13, 17], then: r= [0, 3], s = [13, 17]; Then, the probability density function:

   $$f_{PN}(w) = \frac{1}{s-r} = \frac{1}{[13,17]-[0,3]} = \frac{1}{[10,17]} = [0.059, 0.1000]$$

2. The lift arrives after seven minutes or will arrive after 13 to 17 minutes $[13, 17]$,

   then: $r = 7$, $s = [13,17]$

Then, the probability density function:

$f_{PN}(w) = \frac{1}{s-r} = \frac{1}{[13,17]-7} = \frac{1}{[6,10]} = [0.1, 0.167]$.

Now, solving this problem by using pentapartitioned neutrosophic uniform distribution,

1. The lift arrival time is either from now to 3 minutes $[0, 3]$ with contradiction $C \in [0, 0.2]$, ignorance $G \in [0, 0.04]$, unknown $U \in [0, 0.03]$, or will arrive after 13 to 17 minutes (i.e. [13,17]), then, $r = [0, 3] + \frac{[0,0.2]+[0,0.04]+[0,0.03]}{3} = [0, 3] + [0, 0.09] = [0, 3.09]$, $s = [13, 17]$ Then, the probability density function:

$f_{PN}(w) = \frac{1}{s-r} = \frac{1}{[13,17]-[0,3.09]} = \frac{1}{[9.91,17]} = [0.059, 0.1009]$

2. The lift arrives after seven minutes along with contradiction $C \in [0, 0.2]$, ignorance $G \in [0, 0.04]$, unknown $U \in [0, 0.03]$ or will arrive after 13 to 17 minutes [13, 17],

Then, $r = (7 + 0.09) = 7.09$, $s = [13, 17]$, and the probability density function:

$f_{PN}(w) = \frac{1}{s-r} = \frac{1}{[13,17]-7.09} = \frac{1}{[5.91,9.91]} = [0.1009, 0.1692]$.

Definition 3.3. A pentapartitioned neutrosophic random variable w is said to follow the pentapartition neutrosophic binomial distribution, if it is assuming non-negative variable and the number of success of an experiment is more than or equal to one time.

i. **Pentapartitioned Neutrosophic Binomial Random Variable:** is the random variable 'w' represents the number of success is more than or equal to one time.

ii. **Pentapartitioned Neutrosophic Binomial Probability Distribution:** The pentapartitioned neutrosophic probability distribution of w with pentapartitioned neutrosophic probability density function.

iii. **Contradiction:** it is a contradiction part of success and failure in which the event results cannot be confined.

iv. **Ignorance:** it is an ignorance part of success and failure in which the event results cannot be confined.

v. **Unknown:** it is an unknown part of success and failure in which the event results cannot be confined.

vi. **C.G.U. Threshold:** represents the number of events whose outcome is imprecise. In this study C. G. U. is $th \in \{0,1,2 \dots m\}$. In other words, C. G. U. is the number of events whose outcomes belong to contradiction, ignorance and unknown events.

Let P(S) = The scope of a particular event in which the output will be fully successful.

P(C) = The scope of a particular event in which the output will be a contradiction.

P(G) = The scope of a particular event in which the output will be ignored.

P(U) = The scope of a particular event in which the output will be unknown.

P(F) = The scope of a particular event in which the output will be failure, for both S and F, except the indeterminacy (I).

Assume that $w \in \{0,1,2,\dots,m\}$, where m represents sample size, $NP = (T_w, C_w, G_w, U_w, F_w)$ with

T_w: Chances of 'w' success, and $(n-w)$ is the number of failures, contradiction, ignorance, and unknown such that the events summation of contradiction, ignorance and unknown is less than or equal to C.G.U. Threshold. It is well known that n represents population size.

F_w: Chances of 'z' success, with $z \neq w$, and $(m-z)$ is the number of failures and contradiction, while the summation of ignorance and unknown events is less than the C.G.U. threshold.

C_w: Chances of 'u' contradiction, where 'u' is strictly greater than C.G.U. threshold.

G_w: Chances of 'v' ignorance, where 'v' is strictly greater than C.G.U. threshold.

U_w: Chances of 't' unknown, where 't' is strictly greater than C.G.U. threshold.

$T_w + F_w + C_w + G_w + U_w = (P(S) + P(C) + P(G) + P(U) + P(F))^m$.

For the complete probability, we have $P(S) + P(C) + P(G) + P(U) + P(F) = 1$;

for incomplete probability, $0 \leq P(S) + P(C) + P(G) + P(U) + P(F) < 1$;

for paraconsistent probability, $1 < P(S) + P(C) + P(G) + P(U) + P(F) \leq 5$;

$$T_w = \frac{m!}{w!\,(m-w)!}\, P(S)^w \sum_{r=0}^{th} \frac{r!}{(m-w)!\,(r-m+w)!}\, (P(C)+P(G)+P(U))^r\ \ \mathrm{P(F)}^{m-w-r}$$

$$= \frac{m!}{w!\,(m-w)!}\, P(S)^w \sum_{r=0}^{th} \frac{(m-w)!}{(m-w+r)!}\, (P(C)+P(G)+P(U))^r\ \ \mathrm{P(F)}^{m-w-r}$$

$$= \frac{m!}{w!}\, \mathrm{P(S)}^w \sum_{r=0}^{th} \frac{(P(C)+P(G)+P(U))^r\ \ \mathrm{P(F)}^{m-w-r}}{r!\ (m-w+r)!}$$

$$F_w = \sum_{z=0}^{m} \mathrm{T}_z = \sum_{z=0, z\neq w}^{th} \frac{m!}{z!}\mathrm{P(S)}^z \sum_{r=0}^{th} \frac{\mathrm{P(S)}^r\ \ \mathrm{P(F)}^{\mathrm{m-z-r}}}{r!\ (m-z+r)!}$$

$$C_w = \sum_{u=th+1}^{m} \frac{m!}{m!\,(m-u)!}\mathrm{P(C)}^u \sum_{r=0}^{n-u} \frac{(m-u)!}{(n-u)!\ (n-u+r)!}\mathrm{P(S)}^r\ \ \mathrm{P(F)}^{\mathrm{m-u-r}}$$

$$= \sum_{u=th+1}^{m} \frac{m!}{u!}\ P(C)^u \sum_{r=0}^{m-u} \frac{\mathrm{P(S)}^r\ \ \mathrm{P(F)}^{\mathrm{m-u-r}}}{r!\ (m-u+r)!}$$

$$G_w = \sum_{v=th+1}^{m} \frac{m!}{v!\,(m-v)!}\mathrm{P(G)}^v \sum_{r=0}^{m-v} \frac{(m-v)!}{(m-v)!\ (m-v+r)!}\mathrm{P(S)}^r\ \ \mathrm{P(F)}^{\mathrm{m-v-r}}$$

$$= \sum_{v=th+1}^{m} \frac{m!}{v!}\ P(G)^v \sum_{r=0}^{m-v} \frac{\mathrm{P(S)}^r\ \ \mathrm{P(F)}^{\mathrm{m-v-r}}}{r!\ (m-v+r)!}$$

$$U_w = \sum_{t=th+1}^{m} \frac{m!}{t!\,(m-t)!}\mathrm{P(U)}^t \sum_{r=0}^{m-t} \frac{(m-t)!}{(m-t)!\ (m-t+r)!}\mathrm{P(S)}^r\ \ \mathrm{P(F)}^{\mathrm{m-t-r}}$$

$$= \sum_{t=th+1}^{m} \frac{m!}{t!}\ P(U)^t \sum_{r=0}^{m-t} \frac{\mathrm{P(S)}^k\ \ \mathrm{P(F)}^{\mathrm{m-t-r}}}{r!\ (m-t+r)!}$$

Where, $T_w,\ I_w, F_w, P(S), P(C), P(G), P(U), P(F)$ have their usual meaning.

Example 3.4.

In a certain hospital, there are (6) patients suffering a particular disease, monitoring cases showed that 70% of patients are die, and 20% of patients recover, due to medicine, inexperienced doctors, the contradiction of availability of oxygen occurs 8% percentage, ignorance occurs 5% percentage, and unknown reasons occurs 2%. What is the probability that from three random selection patients, two will recover, with C.G.U. Threshold 3.

Solution:

$$T_w = \frac{6!}{3!(6-3)!}(0.7)^3 \sum_{r=0}^{3} \frac{r!}{(6-3)!(r-3)!} (0.08+0.05+0.02)^r (0.2)^{6-3-r} =$$

$$\frac{6!}{3!\,3!}(0.7)^3 \sum_{r=0}^{3} \frac{r!}{(6-3)!(r-3)!} (0.15)^r (0.2)^{3-r} = 0.023153$$

$$C_w = \sum_{u=4}^{6} \frac{6!}{4!\,2!}(0.08)^u \sum_{r=0}^{2} \frac{2!}{2!\ (2-r)!}(0.7)^r (0.2)^{2-\mathrm{r}} = 0.000327$$

$$G_w = \sum_{v=4}^{6} \frac{6!}{4!\,2!}(0.05)^v \sum_{r=0}^{2} \frac{2!}{2!\ (2-r)!}(0.7)^r (0.2)^{2-\mathrm{r}} = 0.0000000657$$

$$U_w = \sum_{t=4}^{6} \frac{6!}{4!\,2!}(0.02)^t \sum_{r=0}^{2} \frac{2!}{2!\ (2-r)!}(0.7)^r\ (0.2)^{2-\mathrm{r}} = 0.0000000199$$

$$F_w = \sum_{z=0}^{m} \mathrm{T}_z = \sum_{z=0, z\neq w}^{th} \frac{n!}{z!} \mathrm{P(S)}^z \sum_{r=0}^{th} \frac{\mathrm{P(S)}^r\ \mathrm{P(F)}^{\mathrm{m-z-r}}}{r!\,(m-z+r)!} = \big(P(S)+P(C)+P(G)+P(U)+P(F)\big)^n - T_w -$$

$$C_w - G_w - U_w = (0.7+0.08+0.05+0.02+0.2)^6 - 0.023153 - 0.000327 - 0.0000000657 -$$

$$0.0000000199 = 1.317269726$$

Definition 3.5. Consider a random variable $'w'$ follows Poisson distribution with imprecise parameter λ_{PN} represented by an interval is said to be pentapartitioned neutrosophic Poisson distribution , if the probability mass function is given by:

$$\mathrm{NP(w)} = e^{-\lambda_{PN}} . \frac{(\lambda_{PN})^w}{w!}\ ;\ \mathrm{w} = 0,1,\ldots$$

The mean and the variance of this distribution are:

$NE(w) = NV(w) = \lambda_{PN}$.

Example 3.6.

The rate numbers of cars crossing over the bridge are $\lambda_{PN} = [4, 6]$ cars per minute. We want to calculate the probability that only one car crosses through a particular minute.

Solution: Assume z be the number of cars passing within minutes.

$$\mathrm{NP(w=1)} = e^{-\lambda_{PN}} \frac{(\lambda_{PN})^1}{1!} = e^{-\lambda_{PN}} . \lambda_{PN} = \lambda_{PN} . e^{-[4,6]}$$

When $\lambda_{PN} = 4$;   NP(w = 1) $= e^{-\lambda_{PN}} \frac{(\lambda_{PN})^1}{1!} = \lambda_{PN}. e^{-4} = 0.0182\,.(4) = 0.0733$

When $\lambda_{PN} = 6$;   NP(w=1) = $e^{-\lambda_{PN}} \frac{(\lambda_{PN})^1}{1!}$ $= \lambda_{PN}. e^{-6} = 0.0025.(6) = 0.0148$

Therefore, the probability that only one car crossed in a minute be within ranges between $[0.0148, 0.0733]$.

Definition 3.7. Assume that 'w' be a continuous random variable , and it follows exponential distribution with imprecise distribution parameter Θ_{PN} represented by an interval is said to be pentapartitioned neutrosophic exponential distribution, if the probability density function is given by:

$W_N \sim \exp(\Theta_{PN}) = f_N(w) = \Theta_{PN}. e^{-w.\Theta_{PN}}; 0 < w < \infty,$

$exp(\Theta_{PN})$: pentapartitioned neutrosophic exponential distribution, Θ_{PN} : pentapartitioned neutrosophic distribution parameter.

3.8 Properties of Pentapartitioned Neutrosophic Exponential Distribution:

1. The values of the expectation and variance are: $E(w) = \frac{1}{\Theta_{PN}},\ Var(w) = \frac{1}{(\Theta_{PN})^2}$;
2. The distribution function: $NF(w) = NP(W \le w) = (1 - e^{-w.\Theta_{PN}})$

Example 3.9.

The time required to terminate a taxi service in a particular taxi online taxi booking app follows an exponential distribution, with an average of one minute, let us write a density function that represents the time required for terminating taxi service, and then calculate the probability of terminating taxi service in less than one minute.

Solution: Assume w the time required to terminating taxi service per minute:

The average $\frac{1}{\Theta} = 1 \Rightarrow \Theta = \mathbf{1}$

The probability density function: $f(w) = e^{-w}; 0 < w < \infty$

The possibility of taxi terminated in less than a minute:

$P(W \le 1) = (1 - e^{-w}) = (1 - e^{-(1)}) = 0.63$

Practically, the above one is a simple example, if we change it in the neutrosophic form then we get the following context:

The time required to terminate to taxi service follows an exponential distribution, with an average of $[0.69, 3]$ minutes. We know that when data are accurate then classical exponential distribution is performed. Remember the average here is an interval value, now, try to solve this problem by a neutrosophic exponential distribution with an average $[0.69,3]$ minutes, we get:

$\frac{1}{\Theta_N} = [0.69,3] \Rightarrow \Theta_N = \frac{1}{[0.69,\ 3]} = [0.33, 1.45]$

The probability density function become:

$f_N(\mathrm{w}) = \Theta_N.\ e^{-w.\Theta_N}\ ; 0 < \mathrm{w} < \infty,$

$f_N(\mathrm{w}) = [0.33, 1.45].e^{-[0.33,\ 1.45]w};\ 0 < \mathrm{w} < \infty,$

Now, the probability to terminate the taxi service in less than one minute:

$NF(w) = NP(W \leq w) = (1 - e^{-w.\Theta_N})$

$NP(W \leq 1) = \left(1 - e^{-[0.33,\ 1.45](1)}\right) = 1 - e^{-[0.33,\ 1.45]}$

Noticed that, for $\Theta = 0.33$,

$NP(W \leq 1) = 1 - e^{-0.33} = 0.281,$

For $\Theta = 1.45$,

$NP(W \leq 1) = 1 - e^{-1.45} = 0.765$

Therefore, the probability of terminating the taxi service in less than one minute is within the range $[0.281, 0.765]$.

Hence, the value of classical probability to terminate the taxi service in less than one minute is one of the domain values of neutrosophic probability.

$P(W \leq 1) = 0.63 \in [0.281, 0.765] = NP(\mathrm{W} \leq 1).$

Now, applying the pentapartitioned neutrosophic exponential distribution with contradiction $C \in [0, 0.1]$, ignorance $G \in [0, 0.06]$, unknown $U \in [0, 0.05]$, and $\frac{[0,0.1]+[0,0.06]+[0,0.05]}{3} = [0, 0.07]$; average become $[0.7, 3]$, so we get:

$\frac{1}{\Theta_{PN}} = [0.7, 3] \Rightarrow \Theta_{PN} = \frac{1}{[0.7,\ 3]} = [0.33, 1.43]$

The probability density function become:

$f_N(w) = \Theta_N.e^{-w.\Theta_N}; 0 < w < \infty,$

$f_N(w) = [0.33, 1.43].e^{-[0.33,\ 1.43]\ w}$; $0 < w < \infty$,

Now, the probability to terminate the taxi service in less than one minute is:

$NF(w) = NP(W \leq w) = (1 - e^{-w.\theta_N})$

$NP(W \leq 1) = (1 - e^{-[0.33,\ 1.43](1)}) = 1 - e^{-[0.33,\ 1.43]}$

Noticed that, for $\theta = 0.33$,

$NP(W \leq 1) = 1 - e^{-0.33} \cong 0.281$

For $\theta = 1.43$,

$NP(W \leq 1) = 1 - e^{-1.43} \cong 0.761$

Therefore, the probability of terminating the taxi service in less than one minute is within the range [0.281, 0.761].

The value of the classical probability to terminate the taxi service in less than one minute is one of the domain values of pentapartitioned neutrosophic probability, and it is quite closer to classical probability than neutrosophic probability.

Definition 3.10. Assume that w be a continuous random variable is follows classical normal distribution, with imprecise distribution parameters μ and σ , where they may contain some particular events such as contradiction, or ignorance, or unknown , all of these parameters represent intervals, this kind of distributions is said to be pentapartitioned neutrosophic normal distribution if the probability density function is given by:

$$W_{PN} \sim N_{PN}(\mu_{PN}, \sigma_{PN}) = \frac{1}{\sigma_{PN}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{w-\mu_{PN}}{\sigma_{PN}}\right)^2}$$

Where μ_{PN}, σ_{PN} both are set contain two or more elements.

N_{PN}: Pentapartitioned Neutrosophic Normal Distribution.

W_{PN}: Pentapartitioned Neutrosophic Continuous Random Variable.

Example 3.11.

1- In a shopping mall 55% shirt was not sell in Christmas, the average price of shirt is 55, and the standard deviation is 7 with contradiction $C \in [0,0.02]$, ignorance $G \in [0,0.03]$, unknown $U \in [0,0.05]$, the manager decide to give discount to show the owner that the percentage of sells will raise to 70% . Find the lowest discount amounts to show the percentage of sell is 70%. (Discount are normally distributed).

Solution: Given $\mu = 55, \sigma = 7$ with contradiction $C \in [0, 0.03]$, ignorance $G \in [0, 0.02]$, unknown $U \in [0, 0.04]$, so $\sigma = 7 + [[0, 0.02] + [0, 0.03] + [0, 0.05] = [7, 7.1]$. therefore, $\mu \pm \sigma = 55 \pm [7,7.1] = [55 - 7.1, 55 + 7] = [47.9, 62]$.

Now, $0.7 = P(W_{PN} \geq \alpha_{NP})$

$= 1 - \mathrm{P}(\mathrm{W_{PN}} \leq \alpha_{NP})$

$= 1\text{- P}\left(\frac{\mathrm{W_{NP}} - \mu_{\mathrm{NP}}}{\sigma_{NP}} \leq \frac{\alpha_{\mathrm{NP}} - \mu_{\mathrm{NP}}}{\sigma_{NP}} \right)$

$=1\text{- P}\left(\mathrm{Z_{NP}} \leq \frac{\alpha_{\mathrm{NP}} - 55}{[7,\ 7.1]} \right)$

Therefore, $\mathrm{P}\left(\mathrm{Z_{NP}} \leq \frac{\alpha_{\mathrm{NP}} - 55}{[7,\ 7.1]} \right) = 0.3$ clearly $\frac{\alpha_{\mathrm{NP}} - 55}{[7,\ 7.1]} < 0$; so, $\mathrm{P}(\mathrm{Z_{NP}} \leq \mathrm{Z}_{0.3}) = 0.7$

$\mathrm{Z}_{0.3} = \left(\frac{\alpha_{\mathrm{NP}} - 55}{[7,\ 7.1]} \right)$

2-The monthly electricity bill of a certain university is follows pentapartitioned neutrosophic normal distribution with mean 30,000 and standard deviation 5,000. find the following: $\mu \pm \sigma$, $\mu \pm 2\sigma$, where, $C \in [0, 0.08], G \in [0, 0.07], U \in [0, 0.05]$,

Solution: Given μ = 30,000, σ = 5,000 + [0, 0.08] + [0, 0.07]+ [0, 0.05] = [5000, 5000.2]

So, μ ± σ= 55± [7, 7.1] = [55-7.1, 55+7] = [47.9, 62].

μ ± σ= 30000± [5000, 5000.2] = [30000-5000.2, 30000+5000] = [24999.8, 35000].

μ ±2σ= 30000± 2[5000, 5000.2] = [30000-10000.4, 30000+10000] = [19999.6, 40000].

4. Conclusion:

In this article, we introduce different types of pentapartitioned neutrosophic probability distributions as an extension to the neutrosophic probability distributions. Three original events parameters are ignorance, unknown and contradiction, have been presented in this article, in addition to the fully successes and fully failures. Depending upon this new vision, we got new five probability density/ mass functions are T_w, F_w, C_w, G_w, U_w, which led to more accurate in analyzing practical problems that have been explained by well explained examples. We hope that, based on the notion of pentapartitioned neutrosophic probability distributions so many new investigations can be carried out in future.

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A multi-objective Shortage Follow Inventory (SFI) Model Involving Ramp-Type Demand, Time Varying Holding Cost and a Marketing Cost Under Neutrosophic Programming Approach

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Abstract: In the research paper we have discussed a multi-objective shortage follow deterministic inventory model where demand is ramp type and holding cost along with deterioration is time dependent. Nowadays, due to the online marketing facilities it comes to notice in different fields that various companies run a lucrative advertisement of their products through many online platforms like amazon, flipkart, snapdeal and so on. Even pre-booking goes on before the product is launched and a date for selling is fixed. Taking back-order first is highly appropriate for the seasonal items of newly launched devices like mobiles,cars, laptops, computers, automobiles etc. For the advertisement and booking the online platform a huge cost is spent by the companies. This additional charge is ultimately added to the total cost of a particular product with required proportion.In this paper we have taken all of the cost parameters as Intuitionistic triangular fuzzy numbers due to uncertainty. The minimization of total average cost is the main purpose of this model. To minimize this proposed model, we will use different methods like Fuzzy Non-Linear Programming Problem (FNLP), Fuzzy Additive Goal Programming Problem (FAGP), Intuitionistic Fuzzy Non-Linear Programming Technique (IFNLP) and Neutrosophic Non-Linear Programming Technique (NSNLP). To illustrate this proposed model the solution procedure and numerical examples have been given and sensitivity analysis for various parameters have been demonstrated lastly.

Keywords: Deterministic inventory model, multi-item, ramp-type demand, Time-varying holding cost, time-varying deterioration, neutrosophic triangular number, neutrosophic programming technique.

Introduction: Of late, the proper utilization of the inventories seeks great attention for the growth of business and every organization is trying this method to achieve their goal. Therefore, maintaining and controlling the inventories have also become a big challenge for them. They need an appropriate methodology for controlling the inventories to run their business successfully. That is why they must keep in mind the important factor of deterioration.The deterioration being a natural phenomena, its integral parts are change, damage, decay, spoilage etc. As the deterioration is most effective on food items, photographic films, pharmaceuticals, electronic components, drugs etc., we must count the loss caused by the deterioration in the processes of upgrading the model.

Nowadays many companies are found to run online advertisements for their products before they launch the product for sale. Pre-booking for a particular product is also invited through some on-line business platforms like amazon, snap-deal etc. This extra effect of cost must be included while developing the model. Pre-booking has great importance in understanding the demand of a particular product in the market. It has been a phenomenon in online marketing for recent years. So here we have considered all of these factors in this suggested model.

On considering the fluctuated economic circumstances, the basic assumptions of the Inventory Model (EOQ) should be upgraded at a regular interval.

Within 1957, many researchers discovered the impact of deterioration of many food items just after expiry. That is why they have taken the deterioration while developing their model. In 2007, Deng, P.S [16] developed inventory models for deterioration of items where demand is ramp-type. In 1999, Chang, H.J and Dye. C.Y [15] developed an inventory model with time varying demand and partial backlogging also included the deterioration.

It is practically impossible to take demand rate as a growing function depending on time for this in 2012, Skouri, K.; Konstantaras, I.; Manna, S.K.; Chaudhuri, K.S. [18] studied an inventory model with general ramp-type demand, the Weibull deterioration, and partial backlogging. P.S Deng in 2005 [17] considered an inventory model with ramp-type demand for items with Weibull deterioration. In 2006, S.K Manna and K.S Chaudhuri [20] described an EOQ model with time dependent deterioration rate, ramp-type demand rate, shortages and unit production cost. S. Jain and M. Kumar [5] in 2007 discovered an inventory Model with Weibull distribution deterioration, ramp-type demand, and starting with shortage. In 2010, Kun Shan Wu [2] developed an EOQ inventory model for items with ramp type demand rate, Weibull distribution deterioration and partial backlogging. In 2011, K.C Tripathy, U. Misra [3] described an EOQ model with ramp type demand and time dependent Weibull deterioration. In 2016, M. Valliathal, R. Uthayakumar [4] developed a desirable replenishment policy of an EOQ model with ramp-type demand under shortages along with non-instantaneous Weibull deteriorating items . In 2011, C.N Hung [19] developed an inventory model with deterioration, generalized type demand and back order rates.

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In this manuscript, we proposed an inventory deterministic model where demand acts as ramp-type, deterioration and holding cost both of them have been considered as time dependent. Due to uncertainty all the cost parameters have been taken as triangular fuzzy number and triangular intuitionistic fuzzy numbers.To minimize we solved the proposed model using Fuzzy Non-Linear Programming Problem (FNLP), Fuzzy Additive Goal Programming Problem (FAGP), Intuitionistic Fuzzy Non-Linear Programming Technique (IFNLP) and Neutrosophic Non-Linear Programming Technique (NSNLP). Finally, to illustrate this proposed model the solution procedure and numerical examples have been given.

2. Mathematical Preliminaries:

2.1 Fuzzy set:

Let X be a universe of discourse. A fuzzy set which is denoted by Ã∈X and is defined with the ordered pairs Ã= {(x,$T_{\tilde{A}}$(x)): x∈ X}.

Here $T_{\tilde{A}}$: X→[0,1] is a function known as truth membership function of the fuzzy set Ã.

2.2 Triangular Fuzzy Number (TFN) [28]:

Let F(R) be the set of all TFN in the set of real number R. A TFN $\tilde{A} \in F(R)$, where $\tilde{A} = (a, b, c)$ having the membership grade $\mu_{\tilde{A}}: R \to [0,1]$ is defined by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & For\ a \le x \le b \\ \dfrac{c-x}{c-b} & For\ b \le x \le c \\ 0 & otherwise \end{cases}$$

Where c and a are the upper and lower limit of sustain of the set $\tilde{A}$.

2.3 Intuitionistic fuzzy set [29]:

Let X be an fixed set.The intuitionistic fuzzy set $\tilde{A}^I \in X$ is given by

$$\tilde{A}^I = \{< x, T_A(x), F_A(x) > | x \in X\}$$

Where $T_A: X \to [0,1]$the truth membership is grade and $F_A: X \to [0,1]$ is the falsity membership grade. Here $0 \le T_A + F_A \le 1\ \forall\ x \in X$.

2.4 Triangular Intuitionistic Fuzzy Number (TIFN) [30]

Let a number $\tilde{A}^I = (a, b, c)(a', b, c')$ is an intuitionistic fuzzy number in R having the membership grade $\mu_{\tilde{A}^I}(x) \in [0,1]$and non-membership grade $\partial_{\tilde{A}^I}(x) \in [0,1]$ and it is defined as

$$\mu_{\tilde{A}^I}(x) = \begin{cases} 0 & for\ x < a \\ \frac{x-a}{b-a} & for\ a \le x \le b \\ 1 & for\ x = b \\ \frac{c-x}{c-b} & for\ b \le x \le c \\ 0 & x > c \end{cases}$$

$$\partial_{\tilde{A}^I}(x) = \begin{cases} 1 & for\ x < a' \\ \frac{b-x}{b-a'} & for\ a' \le x \le b \\ 0 & for\ x = b \\ \frac{x-b}{c'-b} & for\ b \le x \le c' \\ 1 & x > c' \end{cases}$$

Where $a' \le a \le b \le c \le c'\ \ and\ 0 \le \mu_{\tilde{A}^I}(x) + \partial_{\tilde{A}^I}(x) \le 1$.

2.5 Neutrosophic set:

Let X be the universe of discourse. The neutrosophic set $\tilde{A}^n \in X$ is given by

$\tilde{A}^n = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}$

Where $T_A(x)$ is the truth membership grade, $I_A(x)$ be the indeterminacy membership grade and $F_A(x)$ be falsity membership grade. These membership functions are defined by

$T_A(x)$: $X \rightarrow (0^-, 1^+)$

$I_A(x)$: $X \rightarrow (0^-, 1^+)$

$F_A(x)$: $X \rightarrow (0^-, 1^+)$

With $0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$

2.6 Single valued neutrosophic set: [31]

Let X be a collection of objects called the universe of discourse. The single valued neutrosophic set $\tilde{A}^n \in X$ is defined by $\tilde{A}^n = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}$

Here $T_A(x)$, $I_A(x)$, $F_A(x)$ are called truth, indeterminacy and falsity membership grade respectively. These membership functions are defined by

$T_A(x)$: $X \rightarrow [0, 1]$

$I_A(x)$: $X \rightarrow [0, 1]$

$F_A(x)$: $X \rightarrow [0,1]$

With $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$; $\forall\, x \in X$.

2.7 Triangular Neutrosophic Fuzzy Number (TNFN) [31].

Let $e_{\tilde{a}}, f_{\tilde{a}}, g_{\tilde{a}} \in [0,1]$ and $a_1, a_2, a_3 \in R$ such that $a_1 \le a_2 \le a_3$. then the single valued triangular neutrosophic number is given by $\tilde{a} =< (a_1, a_2, a_3); e_{\tilde{a}}, f_{\tilde{a}}, g_{\tilde{a}} >$, whose truth-membership, indeterminacy-membership and falsity-membership functions are given as follows:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} 0 & for\ x < a_1 \\ e_{\tilde{a}} \dfrac{x - a_1}{b_1 - a_1} & for\ a_1 \leq x \leq b_1 \\ e_{\tilde{a}} & for\ x = b_1 \\ e_{\tilde{a}} \dfrac{c_1 - x}{c_1 - b_1} & for\ b_1 \leq x \leq c_1 \\ 0 & for\ x > c_1 \end{cases}$$

$$\rho_{\tilde{A}^I}(x) = \begin{cases} 0 & for\ x < a_1 \\ \dfrac{(b_1 - x) + f_{\tilde{a}}(x - a_1)}{b_1 - a_1} & for\ a_1 \leq x \leq b_1 \\ f_{\tilde{a}} & for\ x = b_1 \\ \dfrac{(x - b_1) + f_{\tilde{a}}(c_1 - x)}{c_1 - b_1} & for\ b_1 \leq x \leq c_1 \\ 0 & for\ x > c_1 \end{cases}$$

$$\partial_{\tilde{A}^I}(x) = \begin{cases} 0 & for\ x < a_1 \\ \dfrac{(b_1 - x) + g_{\tilde{a}}(x - a_1)}{b_1 - a_1} & for\ a_1 \leq x \leq b_1 \\ g_{\tilde{a}} & for\ x = b_1 \\ \dfrac{(x - b_1) + g_{\tilde{a}}(c_1 - x)}{c_1 - b_1} & for\ b_1 \leq x \leq c_1 \\ 0 & for\ x > c_1 \end{cases}$$

Where $g_{\tilde{a}}, f_{\tilde{a}}\ and\ e_{\tilde{a}}$ denote the minimum falsity-membership degree, minimum indeterminacy-membership degree and maximum truth-membership degree respectively. The single valued triangular neutrosophic number $\tilde{a} =< (a_1, a_2, a_3); e_{\tilde{a}}, f_{\tilde{a}}, g_{\tilde{a}} >$ may express an ill-defined quantity about a_1, which is approximately equal to a_1.

2.8 Methods of defuzzification of TFN and TIFN.

2.8.1. Defuzzification of Triangular Fuzzy Number (TFN) [32]

If $\tilde{A} = (a_1, a_2, a_3)$ be a triangular fuzzy number then the total λ -integer value of $\tilde{A} = (a_1, a_2, a_3)$- is given by

$$I_\lambda(\tilde{A}) = \lambda \frac{(a_1+a_2)}{2} + (1 - \lambda) \frac{(a_2+a_3)}{2}. \tag{1}$$

Taking $\lambda = 0.5$ we have $I_{0.5}(\tilde{A}) = \frac{a_1+2a_2+a_3}{4}$, is the approximate value of TFN number $\tilde{A} = (a_1, a_2, a_3)$.

2.8.2. Defuzzification of Triangular Intuitionistic Fuzzy Number (TIFN) [32]

Let $\tilde{A}^I = (a_1, a_2, a_3)(a_1', a_2, a_3')$ be a TIFN and $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$.

For defuzzification we define a score membership grade of $\tilde{A}^I$ is given by

$$S_\mu(\tilde{A}^I) = \frac{a_1 + 2a_2 + a_3}{4}$$

The score non-membership function of $\tilde{A}^I$ is given by

$$S_\partial(\tilde{A}^I) = \frac{a_1' + 2a_2 + a_3'}{4}$$

Then the accuracy function of $\tilde{A}^I$ is represented as Acc ($\tilde{A}^I$) and is defined by

$$\mathrm{Acc}(\tilde{A}^I) = \frac{S_\mu(\tilde{A}^I) + S_\partial(\tilde{A}^I)}{2} = \frac{a_1 + 2a_2 + a_3 + a_1' + 2a_2 + a_3'}{8} \tag{2}$$

is the deffuzified value of the triangular intuitionistic fuzzy number $\tilde{A}^I = (a_1, a_2, a_3)(a_1', a_2, a_3')$.

3. Mathematical Formulation:

Some notations and assumptions are given below for the formulation of the model for i'th item (i=1, 2, 3,…., n):.

3.1 Notations

C_{Ai} : Ordering cost per unit of time.

C_{Hi}: Holding cost per unit of time.

θ_i : Deterioration rate is depending on time.

D_i : Ramp-type Demand.

C_{pi} : the purchase cost per unit of time.

C_{di} : Deterioration cost for each item per unit of time.

C_{si} : Back order cost(Shortage cost) per unit of time.

μ_i:Parameter for demand function (ramp-type) (break point).

C_{Mi}:The marketing cost depends on demand.

T_i : Length of cycle time , $T_i \geq 0$.

t_{1i}: Procurement time, $t_{1i} \geq 0$.

I_{1i} (t): The negative inventory level in the time $[0, \mu_i]$.

I_{2i} (t): The positive inventory level in the time $[\mu_i, t_{1i}]$

I_{3i} (t): The Inventory level with the time $[t_{1i}, T_i]$.

I_{max}: The level of maximum inventory per ordering cycle.

B_i:During the stock-out period the maximum backlogged quantity.

$I_{0i}(I_{max} + B_i)$: The order quantity for the duration of a cycle of length T_i for i'th item.

$TAC_i(t_{1i}, T_i)$: The total average cost for each of the items.

w_i: The space of storage per unit of time.

W_i: The total space of the area.

C_{Ai}^I: Intuitionistic cost for order per unit of time.

C_{pi}^I: Intuitionistic cost for purchase per unit of time.

C_{di}^I: Intuitionistic cost for deterioration per unit of time.

C_{si}^I: Intuitionistic cost for shortage per unit of time.

C_{Hi}^I: Intuitionisticcost for holding items per unit of time.

TAC_i^I: Total Intuitionistic average cost per unit of time.

3.2 Assumptions:

i. The proposed inventory model deals with multi-item. Each item is considered as an objective function.

ii. The occurrence of replenishment is instantaneously with an infinite rate.

iii. Here we neglect the lead time. That is we are neglecting the time interval between placing the order and receiving.

iv. The demand is deterministic and it is a ramp-type function as follows

D (t) $=d_i[t-(t-\mu_i)f(t-\mu_i)]$, $d_i>0$, $\mu_i > 0$ and Heaviside's function is given by

$f(t-\mu_i) = \begin{cases} 0, & t < \mu_i \\ 1, & t \geq \mu_i \end{cases}$ where d_i represents an initial demand and μ_i represents the fixed point with respect to time.

v. Deterioration rate is followed by $\theta_i(t) = \theta_i t. \quad 0 < \theta_i < 1.$

vi. Inventory model starts with shortages adequate to the procurement time.

vii. The units of deterioration are not repaired or replaced during the period.

viii. The inventory holding cost, $C_{Hi}=h_i.t$.

ix. The ordering cost is taken as constant.

x. The marketing cost **C**$_{Mi}$= $\alpha_i D_i^{\beta_i}, where\ \alpha_i > 0\ , \beta_i > 0$.

3.3 Mathematical Formulation

At time t=0 the system does not have any inventory at all. In the time interval t=0 to t=t_{1i} the back order (shortages) is permitted and at time t=t_{1i} the inventory is being replenished. And the quantity received, meets the back-order that has already been occurred up to the time t=t_{1i}. In the time t∈ $[t_{1i}, T_i]$ the inventory level is reduced for demand along with deterioration. Finally at time t=T_i inventory level falls down to zero. So, we can see that at time t=0 and t=T_i the reducing system does not have any inventory at all. The graph of the above-mentioned inventory system has been given in the figure-1 given below.

Figure-1

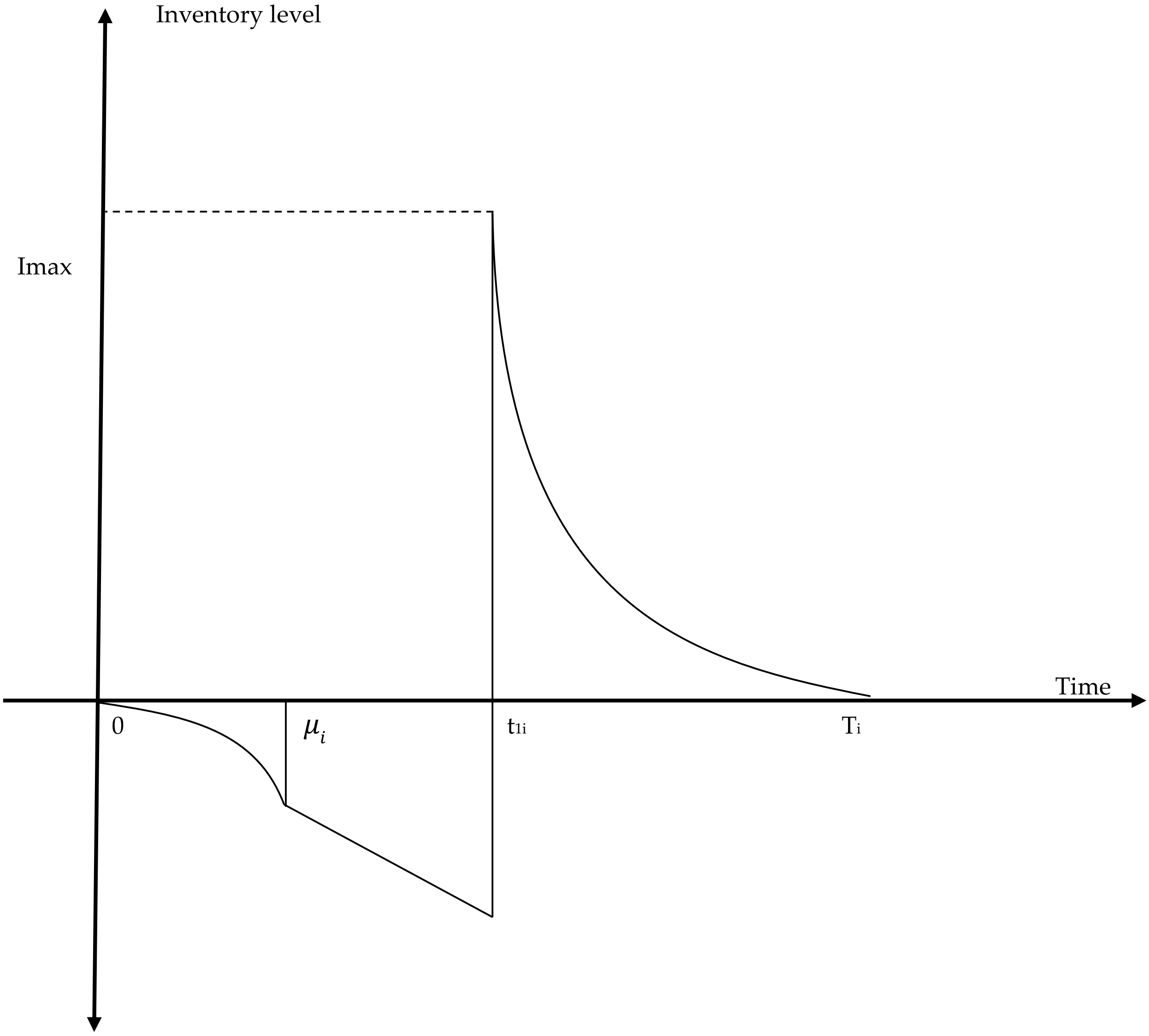


Figure-1: Representation of Inventory model

During the negative stock period $[0,\mu_i]$ and $[\mu_i, t_{1i}]$ and the positive stock period $[t_{1i},T_i]$, the rate of change of inventory are followed by the following differential equations.

$$\frac{dI_{1i}}{dt} = -d_i t \quad 0 \le t \le \mu_i; \qquad (3)$$

$$\frac{dI_{2i}}{dt} = -d_i\mu_i \quad \mu_i \le t \le t_{1i}; \qquad (4)$$

$$\frac{dI_{3i}}{dt} + \theta_i(t)I_{3i}(t) = -d_i\mu_i \quad t_{1i} \le t \le T_i; \qquad (5)$$

Here the boundary conditions are as follow

$$I_{1i}(0) = 0 \quad , I_{2i}(t_{1i}) = I_{max} \quad , I_{3i}(T_i) = 0 \;; \theta_i(t) = \theta_i t \quad 0 < \theta_i < 1$$

We have from (3)

$$\int_0^t dI_{1i} = -\int_0^t d_i t\, dt \quad where\ I_{1i}(0) = 0$$

$$I_{1i}(t) = -\frac{d_i t^2}{2}, \qquad 0 \le t \le \mu_i \qquad (6)$$

We have from (4)

$$\int_{\mu_i}^{t} dI_{1i} = -\int_{\mu_i}^{t} d_i\mu_i\, dt, \qquad where\ I_{1i}(\mu_i) = \frac{d_i{\mu_i}^2}{2}\ by\ (6)$$

$$I_{2i}(t) = d_i\mu_i\left(\frac{\mu_i}{2} - t\right), \qquad \mu_i \le t \le t_{1i} \qquad (7)$$

Now we integrate (5) w.r.t't' from the limit t=t to t=T and neglecting higher power of θ we get

$$I_{3i}(t) = d_i\mu_i\left[(T_i - t) + \frac{\theta_i}{6}(T_i^3 - t^3)\right]e^{-\frac{\theta_i t^2}{2}}, \qquad t_{1i} \le t \le T_i \qquad (8)$$

Now by putting $t=t_{1i}$ we get the maximum inventory level I_{max} as follows

$$I_{max} = d_i\mu_i\left[(T_i - t_{1i}) + \frac{\theta_i}{6}(T_i^3 - t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}} \qquad (9)$$

Total backlogged is given by

$$B=\int_0^{\mu_i} d_i t\, dt \quad + \int_{\mu_i}^{t_{1i}} d_i\, \mu_i dt$$

$$= \frac{d_i{\mu_i}^2}{2} + d_i\mu_i(t_{1i} - \mu_i)$$

$$= d_i\mu_i(\frac{\mu_i}{2} + t_{1i} - \mu_i)$$

$$= d_i\mu_i\left(t_{1i} - \frac{\mu_i}{2}\right) \qquad (10)$$

So, the initial ordering quantity is given by the following expression

$$I_0 = d_i\mu_i\left[\left[(T_i - t_{1i}) + \frac{\theta_i}{6}(T_i^3 - t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}} + \left(t_{1i} - \frac{\mu_i}{2}\right)\right] \qquad (11)$$

The types of cost are as follows

3.3.1. The ordering cost per cycle

$$O.C_i=C_{Ai} \qquad (12)$$

3.3.2. The inventory holding cost per cycle

$$IHC_i=\int_{t_{1i}}^{T} h_i.t_i.d_i.\mu_i\left[(T_i - t) + \frac{\theta_i}{6}(T_i^3 - t^3)\right]e^{-\frac{\theta_i t^2}{2}}dt$$

$d_i\mu_i h_i\left[\frac{T_i}{2}(T_i^2-t_{1i}^2)-\frac{1}{3}(T_i^3-t_{1i}^3)+\frac{\theta_i}{6}\left\{\frac{T_i^3}{2}(T_i^2-t_{1i}^2)-\frac{1}{5}(T_i^5-t_{1i}^5)\right\}-\frac{\theta_i}{2}\left\{\left\{\frac{T_i}{4}(T_i^4-t_{1i}^4)-\frac{1}{5}(T_i^5-t_{1i}^5)+\frac{\theta_i}{6}\left\{\frac{T_i^3}{4}(T_i^4-t_{1i}^4)-\frac{1}{7}(T_i^7-t_{1i}^7)\right\}\right\}\right\}\right]$ (Neglecting the higher power of t for the expression $e^{-\frac{\theta_i t^2}{2}}$ only)

(13)

3.3.3. Deterioration cost for the time interval $[t_{1i}, T_i]$ for i'th item per cycle

D.Ci == $C_{di}\left[d_i\mu_i\left[(T_i-t_{1i})+\frac{\theta_i}{6}(T_i^3-t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}}-\int_{t_{1i}}^{T} d_i\mu_i\,dt\right]$

$= C_{di}d_i\mu_i\left[\left[(T_i-t_{1i})+\frac{\theta_i}{6}(T_i^3-t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}}+(t_{1i}-T_i)\right]$ (14)

3.3.4. The Back-order cost (shortage cost) during the time period $[0, t_{1i}]$ for i'th item per cycle

S.Ci= $-C_{si}[\int_0^{\mu_i}\left(-\frac{d_i t^2}{2}\right)dt+\int_{\mu_i}^{t_{1i}}\left[d_i\mu_i\left(\frac{\mu_i}{2}-t\right)\right]dt$

$= \frac{d_i\mu_i C_{si}}{6}[\mu_i^2-3\mu_i(t_{1i}-\mu_i)+3(t_{1i}^2-\mu_i^2)]$ (15)

Where C_{si} is a shortage cost.

3.3.5. Total cost for purchasing an item per cycle that is max inventory as well as backlogged quantity.

P.Ci= $d_i\mu_i C_{pi}\left[\left[(T_i-t_{1i})+\frac{\theta_i}{6}(T_i^3-t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}}+\left(t_{1i}-\frac{\mu_i}{2}\right)\right]$ (From (9) and (10)) (16)

Where C_{pi} is the purchase cost.

3.3.6. The marketing cost which is basically depends on demand for the time interval $[0, T_i]$ for i'th item per cycle is as follows

CMi= $\int_0^{\mu_i}\alpha_i D_i^{\beta_i}\,dt \quad +\int_{\mu_i}^{T_i}\alpha_i D_i^{\beta_i}\,dt$

$= \alpha_i\int_0^{\mu_i}(d_i t)^{\beta_i}\,dt+\ \alpha_i\int_{\mu_i}^{T_i}(d_i\mu_i)^{\beta_i}\,dt$

$= \alpha_i d_i^{\beta_i}\left[\frac{\mu_i^{\beta_i+1}}{\beta_i+1}+\mu_i^{\beta_i}(T_i-\mu_i)\right]$ (17)

$where\ \alpha_i > 0\ , \beta_i > 0.$

Therefore, the total average cost per unit of time for each of the item per cycle is given below

TACi(t1i,Ti)= $\frac{1}{T_i}$[CAi+$d_i\mu_i h_i\left[\frac{T_i}{2}(T_i^2-t_{1i}^2)-\frac{1}{3}(T_i^3-t_{1i}^3)+\frac{\theta_i}{6}\left\{\frac{T_i^3}{2}(T_i^2-t_{1i}^2)-\frac{1}{5}(T_i^5-t_{1i}^5)\right\}-\frac{\theta_i}{2}\left\{\left\{\frac{T_i}{4}(T_i^4-t_{1i}^4)-\frac{1}{5}(T_i^5-t_{1i}^5)+\frac{\theta_i}{6}\left\{\frac{T_i^3}{4}(T_i^4-t_{1i}^4)-\frac{1}{7}(T_i^7-t_{1i}^7)\right\}\right\}\right\}\right]+C_{di}d_i\mu_i\left[\left[(T_i-t_{1i})+\frac{\theta_i}{6}(T_i^3-t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}}+(t_{1i}-T_i)\right]+\frac{d_i\mu_i C_{si}}{6}[\mu_i^2-3\mu_i(t_{1i}-\mu_i)+3(t_{1i}^2-\mu_i^2)]+d_i\mu_i C_{pi}\left[\left[(T_i-t_{1i})+\frac{\theta_i}{6}(T_i^3-t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}}+\left(t_{1i}-\frac{\mu_i}{2}\right)\right]+\alpha_i d_i^{\beta_i}\left[\frac{\mu_i^{\beta_i+1}}{\beta_i+1}+\mu_i^{\beta_i}(T_i-\mu_i)\right]$] (18)

We consider the average total cost of multi objective inventory model (MOIM) as follows

Minimize {TAC1(t11, T1), TAC2(t12, T2),…………………., TACn(t1n,Tn)} for i=1,2,3, 4,……,n

Subject to: $\sum_{i=i}^{n} \quad w_i Q_i \le W$

Where,

TAC$_i$(t_{1i}, T_i) represented by (18) and $Q_i = d_i\mu_i\left[\left[(T_i - t_{1i}) + \frac{\theta_i}{6}(T_i^3 - t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}} + \left(t_{1i} - \frac{\mu_i}{2}\right)\right]$ (19)

4. Fuzzy Model:

To deal with Uncertainties let us take all the cost parameters as triangular Intuitionistic fuzzy numbers.

Here we consider the cost parameters as TIFN

$\widetilde{C_{Ai}}^I = (C_{Ai1}^I, C_{Ai2}^I, C_{Ai3}^I)(C'^I_{Ai1}, C_{Ai2}^I, C'^I_{Ai3})$ Where $C'^I_{Ai1} \le C_{Ai1}^I \le C_{Ai2}^I \le C_{Ai3}^I \le C'^I_{Ai3}$.

$\widetilde{C_{Pi}}^I = (C_{pi1}^I, C_{pi2}^I, C_{pi3}^I)(C'^I_{pi1}, C_{pi2}^I, C'^I_{pi3})$ Where $C'^I_{pi1} \le C_{pi1}^I \le C_{pi2}^I \le C_{pi3}^I \le C'^I_{pi3}$.

$\widetilde{C_{di}}^I = (C_{di1}^I, C_{di2}^I, C_{di3}^I)(C'^I_{di1}, C_{di2}^I, C'^I_{di3})$ Where $C'^I_{di1} \le C_{di1}^I \le C_{di2}^I \le C_{di3}^I \le C'^I_{di3}$.

$\widetilde{C_{si}}^I = (C_{si1}^I, C_{si2}^I, C_{si3}^I)(C'^I_{si1}, C_{si2}^I, C'^I_{si3})$ Where $C'^I_{si1} \le C_{si1}^I \le C_{si2}^I \le C_{si3}^I \le C'^I_{si3}$.

$\widetilde{h_i}^I = (h_{i1}^I, h_{i2}^I, h_{i3}^I)(h'^I_{i1}, h_{i2}^I, h'^I_{i3})$ Where $h'^I_{i1} \le h_{i1}^I \le h_{i2}^I \le h_{i3}^I \le h'^I_{i3}$.

$\widetilde{d_i}^I = (d_{i1}^I, d_{i2}^I, d_{i3}^I)(d'^I_{i1}, d_{i2}^I, d'^I_{i3})$ Where $d'^I_{i1} \le d_{i1}^I \le d_{i2}^I \le d_{i3}^I \le d'^I_{i3}$

Our multi-objective inventory model (3.3.17) becomes Intuitionistic Fuzzy model as follows

Minimize { $(\widetilde{TAC_1^I}(t_{11}, T_1), \widetilde{TAC_2^I}(t_{12}, T_2), \ldots\ldots\ldots\ldots\ldots.., \widetilde{TAC_n^I}(t_{1n}, T_n)$}

Subject to:$\sum_{i=i}^{n} \quad w_i Q_i \le W$ for i=1, 2, 3, 4,………., n

Where,

$$\widetilde{TAC}_i(t_{1i}, T_i) = \frac{1}{T_i}[\widetilde{C_{Ai}}^I + \widetilde{d_i}^I\mu_i\widetilde{h_i}^I\left[\frac{T_i}{2}(T_i^2 - t_{1i}^2) - \frac{1}{3}(T_i^3 - t_{1i}^3) + \frac{\theta_i}{6}\left\{\frac{T_i^3}{2}(T_i^2 - t_{1i}^2) - \frac{1}{5}(T_i^5 - t_{1i}^5)\right\} - \frac{\theta_i}{2}\left\{\left\{\frac{T_i}{4}(T_i^4 - t_{1i}^4) - \frac{1}{5}(T_i^5 - t_{1i}^5) + \frac{\theta_i}{6}\left\{\frac{T_i^3}{4}(T_i^4 - t_{1i}^4) - \frac{1}{7}(T_i^7 - t_{1i}^7)\right\}\right\}\right\}\right] + \widetilde{C_{di}}^I\widetilde{d_i}^I\mu_i\left[\left[(T_i - t_{1i}) + \frac{\theta_i}{6}(T_i^3 - t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}} + (t_{1i} - T_i)\right] + \frac{\widetilde{d_i}^I\mu_i\widetilde{C_{si}}^I}{6}[\mu_i^2 - 3\mu_i(t_{1i} - \mu_i) + 3(t_{1i}^2 - \mu_i^2)] + \widetilde{d_i}^I\mu_i\widetilde{C_{Pi}}^I\left[\left[(T_i - t_{1i}) + \frac{\theta_i}{6}(T_i^3 - t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}} + \left(t_{1i} - \frac{\mu_i}{2}\right)\right] + \alpha_i(\widetilde{d_i}^I)^{\beta_i}\left[\frac{\mu_i^{\beta_i+1}}{\beta_i+1} + \mu_i^{\beta_i}(T_i - \mu_i)\right]]$$

And $Q_i = \widetilde{d_i}^I\mu_i\left[\left[(T_i - t_{1i}) + \frac{\theta_i}{6}(T_i^3 - t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}} + \left(t_{1i} - \frac{\mu_i}{2}\right)\right]$ (20)

Using the defuzzification technique (1) our Intuitionistic fuzzy parameters ($\widetilde{C_{Ai}}^I, \widetilde{C_{Pi}}^I, \widetilde{C_{di}}^I, \widetilde{C_{si}}^I, \widetilde{h_i}^I, \widetilde{d_i}^I$) transforming into crisp value ($\widehat{C_{Ai}^I}, \widehat{C_{pi}^I}, \widehat{C_{di}^I}, \widehat{C_{si}^I}, \widehat{h_i^I}, \widehat{d_i^I}$)

With these our Fuzzy Intuitionistic model transforming into crisp model as given as below

Minimize { $(\widehat{TAC_1^I}(t_{11}, T_1), \widehat{TAC_2^I}(t_{12}, T_2), \ldots\ldots\ldots\ldots\ldots.., \widehat{TAC_n^I}(t_{1n}, T_n)$}

Subject to:$\sum_{i=i}^{n} \quad w_i Q_i \le W$ for i=1,2,3,4,……….,n

Where,

$$\widehat{TAC_i^I}(t_{1i}, T_i) = \frac{1}{T_i}[\widehat{C_{Ai}^I} + \widehat{d_i^I}\mu_i\widehat{h_i^I}\left[\frac{T_i}{2}(T_i^2 - t_{1i}^2) - \frac{1}{3}(T_i^3 - t_{1i}^3) + \frac{\theta_i}{6}\left\{\frac{T_i^3}{2}(T_i^2 - t_{1i}^2) - \frac{1}{5}(T_i^5 - t_{1i}^5)\right\} - \frac{\theta_i}{2}\left\{\left\{\frac{T_i}{4}(T_i^4 - t_{1i}^4) - \frac{1}{5}(T_i^5 - t_{1i}^5) + \frac{\theta_i}{6}\left\{\frac{T_i^3}{4}(T_i^4 - t_{1i}^4) - \frac{1}{7}(T_i^7 - t_{1i}^7)\right\}\right\}\right\}\right] + \widehat{C_{di}^I}\widehat{d_i^I}\mu_i\left[\left[(T_i - t_{1i}) + \right.\right.$$

$$\frac{\theta_i}{6}\left(T_i^3 - t_{1i}^3\right)\Big] e^{-\frac{\theta_i t_{1i}^2}{2}} + \left(t_{1i} - T_i\right)\Bigg] + \frac{\widehat{d_i^I}\mu_i\widehat{C_{si}^I}}{6}\left[\mu_i^2 - 3\mu_i\left(t_{1i} - \mu_i\right) + 3\left(t_{1i}^2 - \mu_i^2\right)\right] + \widehat{d_i^I}\mu_i\widehat{C_{pi}^I}\Bigg[\Big[\left(T_i - t_{1i}\right) +$$

$$\frac{\theta_i}{6}\left(T_i^3 - t_{1i}^3\right)\Big] e^{-\frac{\theta_i t_{1i}^2}{2}} + \left(t_{1i} - \frac{\mu_i}{2}\right)\Bigg] + \alpha_i\left(\widehat{d_i^I}\right)^{\beta_i}\left[\frac{\mu_i^{\beta_i+1}}{\beta_i+1} + \mu_i^{\beta_i}\left(T_i - \mu_i\right)\right]]$$

And $Q_i = \widehat{d_i^I}\mu_i\left[\left[\left(T_i - t_{1i}\right) + \frac{\theta_i}{6}\left(T_i^3 - t_{1i}^3\right)\right] e^{-\frac{\theta_i t_{1i}^2}{2}} + \left(t_{1i} - \frac{\mu_i}{2}\right)\right]$ (21)

Here, w_i and W_i represent space per unit of time and the total area space for i'th item respectively for storing inventory.

5. New Techniques to Solve a Multi-Objective Inventory Model.

To solving the above multi objective inventory (21) problem we consider single objective at a time and the others objectives are ignored.

Applying this technique we find out the value of each objective function separately and by tracking this technique we will formulate the following pay-of-matrix.

$$\begin{array}{c} \\ (t_{11}^1, T_1^1) \\ (t_{12}^2, T_2^2) \\ \dots\dots\dots \\ (t_{1n}^n, T_n^n) \end{array} \begin{array}{cccc} TAC_1(t_{11}, T_1) & TAC_2(t_{12}, T_2) & \dots\dots\dots & TAC_n(t_{1n}, T_n) \\ \left[\begin{array}{c} TAC_1^*(t_{11}^1, T_1^1) \\ TAC_1(t_{12}^2, T_2^2) \\ \dots\dots\dots \\ TAC_1 t_{1n}^n, T_n^n) \end{array}\right. & \begin{array}{c} TAC_2(t_{11}^1, T_1^1) \\ TAC_2^*(t_{12}^2, T_2^2) \\ \dots\dots\dots \\ TAC_2(t_{1n}^n, T_n^n) \end{array} & \begin{array}{c} \dots\dots.. \\ \dots\dots\dots \\ \dots\dots\dots\dots.. \\ \dots\dots\dots\dots \end{array} & \left.\begin{array}{c} TAC_n(t_{11}^1, T_1^1) \\ TAC_n(t_{12}^2, T_2^2) \\ \dots\dots\dots\dots \\ TAC_n^*(t_{1n}^n, T_n^n) \end{array}\right] \end{array}$$

Now we set $U_r^T = \max\{TAC_r\left(t_{2i}^i, T_i^i\right), i = 1,2,3, \dots\dots, n\}$, for r= 1,2,3,…..,n

And $L_r^T = \{TAC_r^*(t_{1r}^r, T_r^r), r = 1,2,3, \dots\dots, n\}$

Where $L_r^T \le TAC_r\left(t_{2i}^i, T_i^i\right) \le U_r^T$; for $i = 1,2,3, \dots\dots, n$; and k= 1,2,3,…..,n; (22)

5.1. Fuzzy Non-Linear Programming Problems (FNLP) and Fuzzy Additive Goal Programming Problems (FAGP)

Now we take for simplicity a linear fuzzy membership function $\mu_{TAC_r}(TAC_r(t_{1r}, T_r))$ for the r'th objective function $TAC_r(t_{1r}, T_r)$ as follows.

$$\mu_{TAC_r}(TAC_r(t_{1r}, T_r)) = \begin{cases} 1 & for \quad TAC_r(t_{1r}, T_r) \le L_r^T \\ \frac{U_r^T - TAC_r(t_{1r}, T_r)}{U_r^T - L_r^T} & for \;\; L_r^T \le TAC_r(t_{1r}, T_r) \le U_r^T \\ 0 & for \quad TAC_r(t_{1r}, T_r) \ge U_r^T \end{cases} \qquad (23)$$

For r=1,2,3,…..,n;

Using (23) we established the fuzzy non-linear programming problems (FNLP).

Max= p

Subject to,

$p(U_r^T - L_r^T) + TAC_r(t_{1r}, T_r) \le U_r^T$ For r=1,2,3,……,n

$0 \le p \le 1, \;\; t_{1r} \ge 0, T_r \ge 0;$ (24)

And the same restriction and constraints as in the problem (21)

Now we formulated Fuzzy additive goal programming (FAGP) based on max-additive operator as given below:

$$\text{Max} \sum_{r=1}^{n} \frac{U_r^T - TAC_r(t_{1r},T_r)}{U_r^T - L_r^T}$$

$$\text{Subject to,} 0 \le \mu_{TAC_r}\big(TAC_r(t_{1r},T_r)\big) \le 1, \text{ for r=1,2,3,.......,n} \qquad (25)$$

And the same restriction and constraints as in the problem (21)

Now we are finding the optimal solution for the above reduced problem (24) and (25) with the help of above FNLP and FAGP method.

5.2. Weighted Fuzzy Non-Linear Programming technique and Weighted Fuzzy Goal Programming Technique (WFNLP AND WFAGP):

We are taking here a positive weight ω_r for every objective $\big(TAC_r(t_{1r},T_r)\big)$

(Where r=1,2,3,…..,n) and $\sum_{r=1}^{n} \omega_r = 1$.

Having these normalized weights and the membership function (23), the FNLP technique becomes

Max p

Subject to,

$$\omega_r.\mu_{TAC_r}\big(TAC_r(t_{1r},T_r)\big) \ge \text{p} \quad \text{For r=1, 2, 3,….., n}$$

$$0 \le p \le 1, \quad t_{1r} \ge 0, T_r \ge 0 \text{ and } \sum_{r=1}^{n} \omega_r = 1. \qquad (26)$$

And the same restriction and constraints as in the problem (21)

Having these normalized weights and the membership function (23), the FAGP technique becomes

$$\text{Max} \sum_{r=1}^{n} \omega_k.\mu_{TAC_r}\big(TAC_r(t_{1r},T_r)\big)$$

$$\text{Subject to,} 0 \le \mu_{TAC_r}\big(TAC_r(t_{1r},T_r)\big) \le 1, \text{ for r=1,2,3,.......,n and}$$

$$t_{1r} \ge 0, T_r \ge 0; \; \sum_{r=1}^{n} \omega_r = 1 \qquad (27)$$

And the same restriction and constraints as in the problem (21)

Now we are finding the optimal solution with the help of above WFNLP and WFAGP method.

5.3. Intuitionistic Fuzzy Non-Linear Programming (IFNLP) Method:

Using (5.1) here we have considered a linear membership grade (Truth membership) and a nonlinear membership grade (Falsity membership).

$$\mu_{TAC_r}(TAC_r(t_{1r},T_r)) = \begin{cases} 1 & for \quad TAC_r(t_{1r},T_r) \le L_r^T \\ \frac{U_r^T - TAC_r(t_{1r},T_r)}{U_r^T - L_r^T} & for \; L_r^T \le TAC_r(t_{1r},T_r) \le U_r^T \\ 0 & for \quad TAC_r(t_{1r},T_r) \ge U_r^T \end{cases}$$

$$\partial_{TAC_r}(TAC_r(t_{1r},T_r)) = \begin{cases} 1 & for \quad TAC_r(t_{1r},T_r) \ge U_r^T \\ \frac{TAC_r(t_{1r},T_r) - L_r^T}{U_r^T - L_r^T} & for \; L_r^T \le TAC_r(t_{1r},T_r) \le U_r^T \\ 0 & for \quad TAC_r(t_{1r},T_r) \le L_r^T \end{cases} \qquad (28)$$

For r=1,2,3,…..,n;

After getting the membership function (truth membership) and non-membership functions (falsity membership value) for every objective function and using (22), the original problem (21) can also be formulated as a crisp model as given below.

$$\text{Max } \alpha_1, \text{ Min} \beta_1$$

$$\text{Subject to} \quad \mu_{TAC_r}\big(TAC_r(t_{1r},T_r)\big) \ge \alpha_1$$

$$\partial_{TAC_r}\big(TAC_r(t_{1r},T_r)\big) \le \beta_1$$

$$\alpha_1+\beta_1 \le 1;\ \alpha_1 \ge \beta_1; \alpha_1,\beta_1 \ge 0;\ \ t_{1r} \ge 0, T_r \ge 0; \qquad (29)$$

And the same restriction and constraints as in the problem (21)

For r=1,2,3,…..,n

Where $\alpha_1$denotes the minimal accepting degree of the objectives and the constraints and $\beta_1$is the maximal rejection degree of the objectives and constraints. The above IFNLP model transforms into the following crisp (non-fuzzy) model

$$\text{Max } (\alpha_1-\beta_1)$$

$$\text{Subject to, } \mu_{TAC_r}\big(TAC_r(t_{1r},T_r)\big) \ge \alpha_1$$

$$\partial_{TAC_r}\big(TAC_r(t_{1r},T_r)\big) \le \beta_1$$

$$\alpha_1+\beta_1 \le 1;\ \alpha_1 \ge \beta_1\alpha_1,\beta_1 \in [0,1];\quad t_{1r} \ge 0, T_r \ge 0; \qquad (30)$$

And the same restriction and constraints as in the problem (21)

For r=1,2,3,…..,n

5.4. Neutrosophic Non-Linear Programming (NSNLP) technique.

By using (22), we define a linear type truth membership, indeterminacy membership, falsity membership functions as follows

$$\mu_{TAC_r}(TAC_r(t_{1r},T_r)) = \begin{cases} 1 & for \quad TAC_r(t_{1r},T_r) \le L_r^T \\ \frac{U_r^T - TAC_r(t_{1r},T_r)}{U_r^T - L_r^T} & for \ L_r^T \le TAC_r(t_{1r},T_r) \le U_r^T \\ 0 & for \quad TAC_r(t_{1r},T_r) \ge U_r^T \end{cases}$$

$$\rho_{TAC_r}(TAC_r(t_{1r},T_r)) = \begin{cases} 1 & for \quad TAC_r(t_{1r},T_r) \le L_r^I \\ \frac{U_r^I - TAC_r(t_{1r},T_r)}{U_r^I - L_r^I} & for \ L_r^I \le TAC_r(t_{1r},T_r) \le U_r^I \\ 0 & for \quad TAC_r(t_{1r},T_r) \ge U_r^I \end{cases}$$

$$\partial_{TAC_r}(TAC_r(t_{1r},T_r)) = \begin{cases} 1 & for \quad TAC_r(t_{1r},T_r) \ge U_r^F \\ \frac{TAC_r(t_{1r},T_r) - L_r^F}{U_r^F - L_r^F} & for \ L_r^F \le TAC_r(t_{1r},T_r) \le U_r^F \\ 0 & for \quad TAC_r(t_{1r},T_r) \le L_r^F \end{cases} \qquad (31)$$

For r=1,2, 3, ….. ,n

Where,

$$U_r^F = U_r^T \ \ and \ L_r^F = L_r^T + t(U_r^T - L_r^T)$$

$$L_r^I = L_r^T \ \ and \ U_r^I = L_r^T + s(U_r^T - L_r^T)\ ; s,t \ \epsilon \ [0,1]$$

After getting the membership function (truth membership), non-membership function (falsity membership value) and indeterminacy membership function for every objective function and using (22), our original problem (21) transform into a crisp model as given by

$$\text{Max } \alpha, \text{Min}\beta, \text{Max}\,\gamma$$

$$\text{Subject to } \ \mu_{TAC_r}\big(TAC_r(t_{1r},T_r)\big) \ge \alpha;$$

$$\partial_{TAC_r}\big(TAC_r(t_{1r},T_r)\big) \le \beta;$$

$$\rho_{TAC_r}\big(TAC_r(t_{1r},T_r)\big) \ge \gamma;$$

$$\alpha+\beta+\gamma \le 3;\ \alpha \ge \beta; \alpha \ge \gamma; \alpha,\beta,\gamma \in [0,1]; t_{1r} \ge 0, T_r \ge 0; \qquad (32)$$

And by taking into consideration the same restrictions and constraints as in the equation (21)

For r=1,2,3,…..,n

Where α denotes the minimal accepting degree for the objectives as well as for the constraints and β is stands for maximal rejection degree for the objectives as well as for the constraints and γ is the degree of indeterminacy. Above NSNLP model transforms into a crisp (non-fuzzy) model as follows:

Maximize $(\alpha - \beta + \gamma)$

Subject to $TAC_r(t_{1r}, T_r) + (U_r^T - L_r^T)\alpha \le U_r^T$;

$TAC_r(t_{1r}, T_r) + (U_r^I - L_r^I)\gamma \le U_r^I$;

$TAC_r(t_{1r}, T_r) - (U_r^F - L_r^F)\beta \le L_r^F$;

$\alpha + \beta + \gamma \le 3$; $\alpha \ge \beta$; $\alpha \ge \gamma$; $\alpha, \beta, \gamma \in [0,1]$; $t_{1r} \ge 0, T_r \ge 0$; For r=1,2,3,…..,n (33)

And the same restriction and constraints in the equation (21)

Where,

$$U_r^F = U_r^T \ \ and\ L_r^F = L_r^T + t(U_r^T - L_r^T)$$
$$L_r^I = L_r^T \ \ and\ U_r^I = L_r^T + s(U_r^T - L_r^T)$$

Where $s, t \in [0,1]$

6. Numerical Examples

To illustrate the multi objective inventory model where demand is ramp-type, deterioration and back-order are variable, we have considered the following example. Here we have taken the cost parameters as TIFN and some parameters are taken as crisp.

Let total area space W=5000 m²

Minimize { $(\widehat{TAC_1^I}(t_{11}, T_1), \widehat{TAC_2^I}(t_{12}, T_2))$}

Subject to:$\sum_{i=i}^{n} \quad w_i Q_i \le W$ where i=1,2

$$\widehat{TAC_i^I}(t_{1i}, T_i) = \frac{1}{T_i}[\widehat{C_{Ai}^I} + \widehat{d_i^I}\mu_i\widehat{h_i^I}\left[\frac{T_i}{2}(T_i^2 - t_{1i}^2) - \frac{1}{3}(T_i^3 - t_{1i}^3) + \frac{\theta_i}{6}\left\{\frac{T_i^3}{2}(T_i^2 - t_{1i}^2) - \frac{1}{5}(T_i^5 - t_{1i}^5)\right\} - \frac{\theta_i}{2}\left\{\left\{\frac{T_i}{4}(T_i^4 - t_{1i}^4) - \frac{1}{5}(T_i^5 - t_{1i}^5) + \frac{\theta_i}{6}\left\{\frac{T_i^3}{4}(T_i^4 - t_{1i}^4) - \frac{1}{7}(T_i^7 - t_{1i}^7)\right\}\right\}\right\}\right] + \widehat{C_{di}^I}\widehat{d_i^I}\mu_i\left[\left[(T_i - t_{1i}) + \frac{\theta_i}{6}(T_i^3 - t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}} + (t_{1i} - T_i)\right] + \frac{\widehat{d_i^I}\mu_i\widehat{C_{si}^I}}{6}\left[\mu_i^2 - 3\mu_i(t_{1i} - \mu_i) + 3(t_{1i}^2 - \mu_i^2)\right] + \widehat{d_i^I}\mu_i\widehat{C_{pi}^I}\left[\left[(T_i - t_{1i}) + \frac{\theta_i}{6}(T_i^3 - t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}} + \left(t_{1i} - \frac{\mu_i}{2}\right)\right] + \alpha_i(\widehat{d_i^I})^{\beta_i}\left[\frac{\mu_i^{\beta_i+1}}{\beta_i+1} + \mu_i^{\beta_i}(T_i - \mu_i)\right]]$$

And $Q_i = \widehat{d_i^I}\mu_i\left[\left[(T_i - t_{1i}) + \frac{\theta_i}{6}(T_i^3 - t_{1i}^3)\right]e^{-\frac{\theta_i t_{1i}^2}{2}} + \left(t_{1i} - \frac{\mu_i}{2}\right)\right]$ for i=1,2

Here we take the crisp values of cost parameters

For the first objective function (i.e., 1st item),$\alpha_1 = 0.55, \beta_1 = 0.45, \mu_1 = 0.26, \theta_2 = 0.08, w_1 = 4$

For the second objective function (i.e., 2nd item),$\alpha_2 = 0.55, \beta_2 = 0.45, \mu_2 = 0.26, \theta_2 = 0.08$, $w_2 = 4$, Q=5000

Here we take the cost parameters as triangular intuitionistic fuzzy numbers.

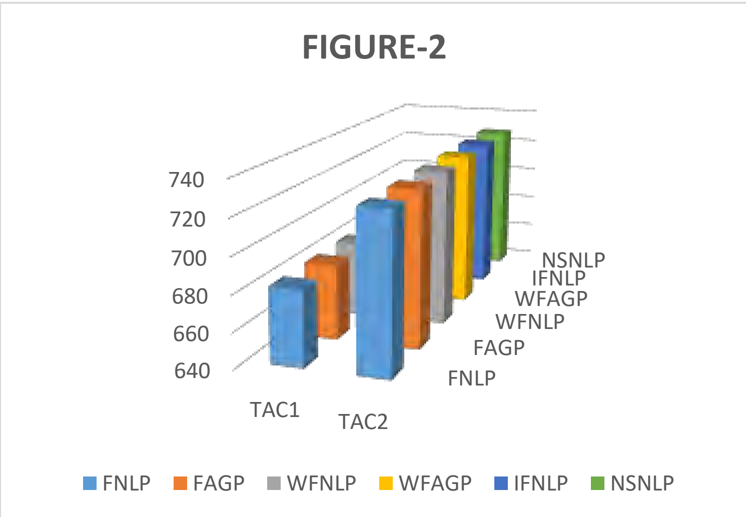
Table-1

Cost parameters	Items	
	1st item's cost (TIFN)	2nd item's cost (TIFN)
$\widetilde{C_{A\iota}}^I$	((125, 130, 135) (122,130,140))	((130, 135, 140) (126,135,142))
$\widetilde{C_{P\iota}}^I$	((17,19,23) (15,19,25))	((18,20,22) (16,20,24))
$\widetilde{C_{d\iota}}^I$	((12,15,18) (10,15,20))	((13,16,17) (11,16,22))
$\widetilde{C_{s\iota}}^I$	((5,7,10) (4,7,13))	((7,9,12) (6,9,14))
$\widetilde{h_\iota}^I$	((0.50,1,2) (0.25,1,3))	((1,1.5,2) (0.5,1.5,4))
$\widetilde{d_\iota}^I$	((115, 120, 125) (112,120,130))	((120, 125, 130) (118,125,135))

Optimum solution by different methods (FNLP, FAGP, WFNLP, WFAGP, IFNLP, NSNLP)

Table-2

METHODS	$TAC_1^*(t_{11}^*, T_1^*)$	t_{11}^*	T_1^*	$TAC_2^*(t_{12}^*, T_2^*)$	t_{12}^*	T_2^*
FNLP	682.3808	0.393911	1.085939	728.2010	0.215405	1.015091
FAGP	682.3808	0.393911	1.085939	728.2010	0.2149483	1.015632
WFNLP	682.3809	0.24049	1.086356	728.2010	0.2141508	1.015686
WFAGP	682.3809	0.24049	1.086356	728.2010	0.2141508	1.015686
IFNLP	682.3808	0.2393911	1.085939	728.2010	0.2143809	1.016013
NSNLP	682.3808	0.2393911	1.085939	728.2010	0.2153121	1.014994



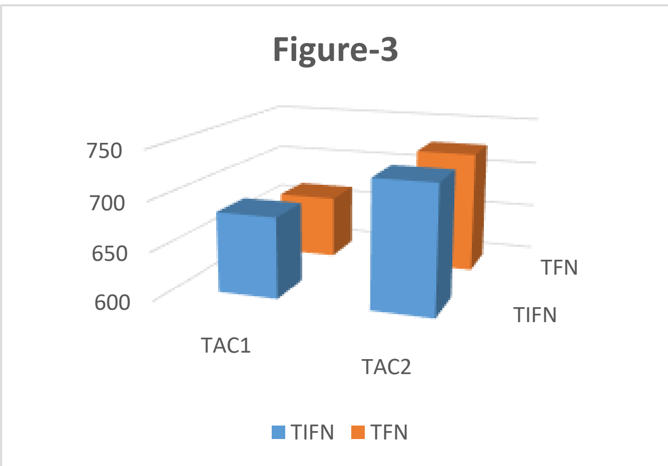
From the figure-2 it is easy to say the values of the average total cost of two-items are almost same. FNLP, FAGP, IFNLP and NSNLP gives the equal value of the total cost but WFNLP and WFAGP gives slide difference value of total cost.

Graph for the average total cost in new methods.

Now we are willing to find what different happen in total average cost if we take the cost parameters as triangular fuzzy numbers instead of the triangular intuitionistic numbers.

Table-3

Methods	Cost parameters	Triangular Intuitionistic fuzzy numbers (1st Item)	Triangular fuzzy numbers (1st Item)	Triangular Intuitionistic fuzzy numbers (2nd Item)	Triangular fuzzy numbers (2nd Item)	Total average cost (1st Item) (For TIFN)	Total average cost (2nd Item) (For TIFN)	Total average cost (1st Item) (For TFN)	Total average cost (2nd Item) (For TFN)
FNLP FAGP IFNLP NSNLP	$\widetilde{C_{A\iota}}^I$	(125,130, 135) (122,130,140)	(125,130,135)	(130, 135, 140) (126,135,142)	(130,135, 140)	682.3808	728.2010	665.6314	725.6821
	$\widetilde{C_{P\iota}}^I$	(17,19,23) (15,19,25)	(17,19,21)	(18,20,22) (16,20,24)	(18,20,22)				
	$\widetilde{C_{d\iota}}^I$	(12,15,18) (10,15,20)	(12,15,18)	(13,16,17) (11,16,22)	(14,16,18)				
	$\widetilde{C_{s\iota}}^I$	(5,7,10) (4,7,13)	(5,7,9)	(7,9,12) (6,9,14)	(7,9,11)				
	$\widetilde{h_\iota}^I$	(0.50,1,2) (0.25,1,3)	(0.7,1,1.3)	(1,1.5,2) (0.5,1.5,4)	(1,1.5,2)				
	$\widetilde{d_\iota}^I$	(115, 120, 125) (112,120,130)	(115,120,125)	(120, 125, 130) (118,125,135)	(120,125, 130)				



For FNLP, FAGP, IFNLP, NSNLP technique

In this section (**Figure-3)** we analyze the average of total minimum cost for triangular fuzzy number and Intuitionistic fuzzy number.

Here we observe that the total average cost of two item is bigger when we are taking triangular Intuitionistic fuzzy number instead of the triangular fuzzy number.

Figure-3 Graph for the total average costs of two items With TIFN and TFN.

8. Sensitivity Analysis

Now we will discuss how the total average cost changing on the basis of change of ordering cost, purchasing cost by FNLP, FAGP, WFNLP, WFAGP, IFNLP, NSFNLP technique.

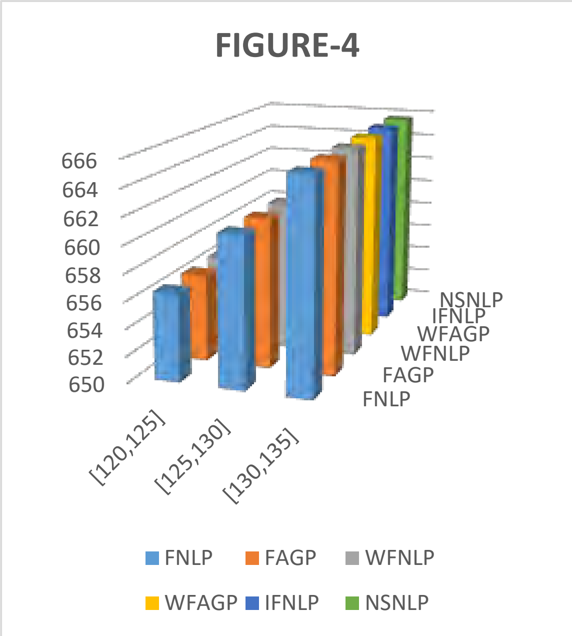
For first objective function (i.e., 1st item) we are taking, $\alpha_1 = 0.55, \beta_1 = 0.45, \mu_1 = 0.26, \theta_2 = 0.08, w_1 = 4, C_{P1} = 19, C_{d1} = 15, C_{s1} = 7, d_1 = 120, h_1 = 1$

For second objective function (i.e., 2nd item) we are taking, $\alpha_2 = 0.55, \beta_2 = 0.45, \mu_2 = 0.26, \theta_2 = 0.08, C_{P2} = 20, C_{d2} = 16, C_{s2} = 9, d_2 = 125, h_2 = 1.5, w_2 = 4, Q = 5000$

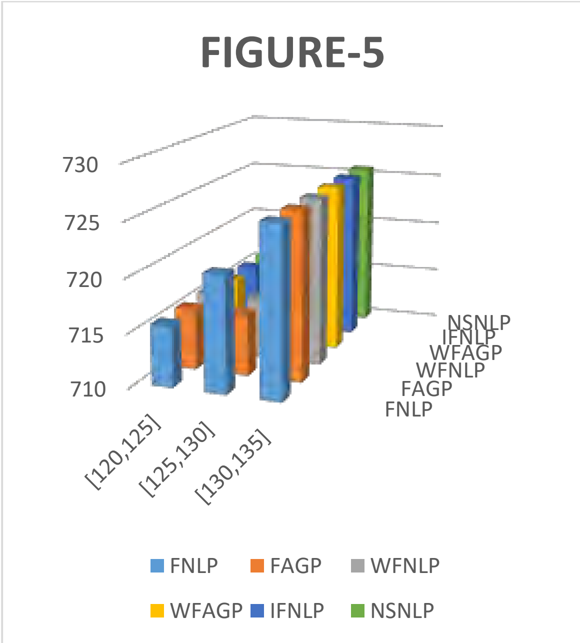
TABLE-4

Methods	ORDERING COST FOR 1st ITEM	ORDERING COST FOR 2nd ITEM	TAC1$(t_{01}^*, t_{11}^*, T_1^*)$	TAC2$(t_{02}^*, t_{12}^*, T_2^*)$
FNLP	120	125	656.5169	715.6611
	125	130	661.1160	720.7620
	130	135	665.6314	725.6820
FAGP	120	125	656.5169	715.6611
	125	130	661.1160	720.7620
	130	135	665.6314	725.6820
WFNLP	120	125	656.5245	715.7559
	125	130	661.1160	720.7620
	130	135	665.6314	725.6820
WFAGP	120	125	656.5245	715.7559
	125	130	661.1160	720.7620
	130	135	665.6314	725.6820
IFNLP	120	125	656.5245	715.7559
	125	130	661.1160	720.7620
	130	135	665.6314	725.6820
NSNLP	120	125	656.5169	715.6611
	125	130	661.1160	720.7620
	130	135	665.6314	725.6820

Here we have considered the weighs (0.7, 0.3) for both of the methods WFNLP, WFAGP.



Graph for the average of total cost of 1st item by different Technique with different ordering cost.



Graph for the average of total cost of 2nd item by different Technique with different ordering cost.

From the above figures (Figure-4, Figure-5), we can see that when ordering costs increase, the corresponding total average cost also increases.

For the first objective function (i.e., 1st item) we are taking, $\alpha_1 = 0.55, \beta_1 = 0.45, \mu_1 = 0.26, \theta_2 = 0.08, w_1 = 4, C_{A1} = 130, C_{d1} = 15, C_{s1} = 7, d_1 = 120, h_1 = 1$

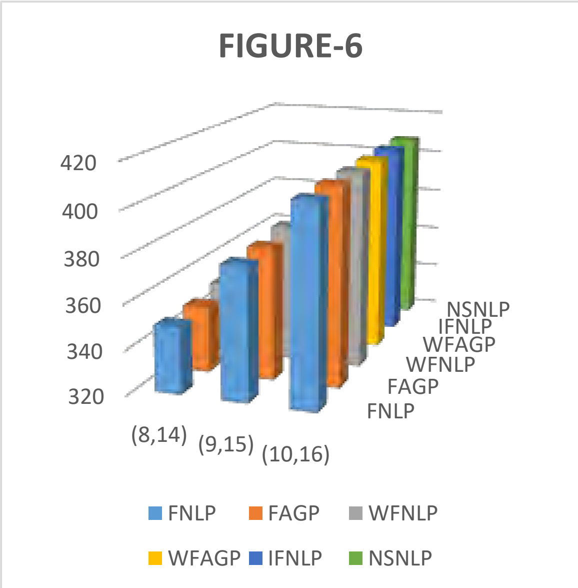
For the second objective function (i.e., 2nd item) we are taking, $\alpha_2 = 0.55, \beta_2 = 0.45, \mu_2 = 0.26, \theta_2 = 0.08, C_{A2} = 135, C_{d2} = 16, C_{s2} = 9, d_2 = 125, h_2 = 1.5$, $w_2 = 4, Q = 5000$

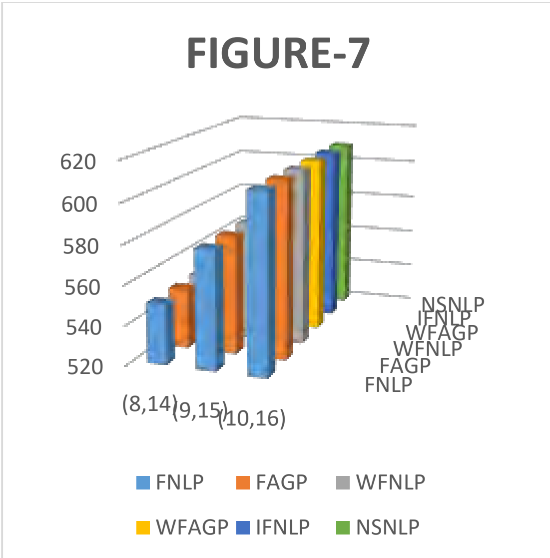
TABLE-5

Methods	PURCHASE COST FOR 1st ITEM	PURCHASE COST FOR 2nd ITEM	TAC1$(t_{01}^*, t_{11}^*, T_1^*)$	TAC2$(t_{02}^*, t_{12}^*, T_2^*)$
FNLP	8	14	349.5078	550.1597
	9	15	378.7996	579.7397
	10	16	407.9882	609.2553
FAGP	8	14	349.5078	550.1597
	9	15	378.7996	579.7397
	10	16	407.9882	609.2553
WFNLP	8	14	351.6440	550.1597
	9	15	380.9453	579.7397
	10	16	410.1524	609.2553

WFAGP	8	14	349.5078	550.1597
	9	15	378.7996	579.7397
	10	16	407.9882	609.2553
IFNLP	8	14	349.5078	550.1597
	9	15	378.7996	579.7397
	10	16	407.9882	609.2553
NSNLP	8	14	349.5078	550.1597
	9	15	378.7996	579.7397
	10	16	407.9882	609.2553

Here we have considered the weighs (0.7, 0.3) for both of the methods WFNLP, WFAGP.





Graph for the average of total cost of 1st item by different methods with different purchase costs. **Graph for the average of total cost of 2nd item by different methods with different purchase costs.**

From the figures (Figure-6, Figure-7), we can see that when purchase cost increases, the corresponding average of total cost also increases.

9. Conclusion

In the manuscript we have considered a multi-objective inventory-model under the restriction of the limited storage space. Here we have taken demand as ramp type, deceleration and holding cost as time dependent. In this shortage follow Inventory model we have taken an additional cost known as marketing cost which makes the proposed model more realistic. Shortage follow inventory model is very much attractive for any types of online companies where the companies take pre-booking for their product. It has been seen that if companies take their pre-booking in advance, then heir total average cost is much lower. The main purpose of the model is what would be the specific technique for minimizing the total inventory cost. The SFI model gives us more lesser cost in comparison to other traditional inventory model. In this paper we have taken all of the cost parameters as

Intuitionistic triangular fuzzy numbers due to uncertainty. The main objective of the model is to minimize the total average cost. Finally, this model is verified by different optimization techniques as FNLP, FAGP, WFNLP, WFAGP, IFNLP, and NSNLP.

For future research, the model proposed here can be execute by more practical presupposition like power-demand, probabilistic demand etc. The uncertainty can be controlled by taking triangular fuzzy number, trapezoidal fuzzy number, pentagonal fuzzy number, neutrosophic pentagonal number etc.

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Intuitionistic Plithogenic graph and it's $\left\{d_{(\alpha_1,\alpha_2)}, c_\beta\right\}$ -cut for knowledge processing tasks

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Abstract: Recently, properties of single-valued Plithogenic set is introduced for dealing with several opposite, non-opposite and neutral side of a multi-valued attribute. In this case, a problem arises due to conflict among the experts and their opinions. It is an indeed problem while dealing with single-valued Plithogenic membership. The reason is to deal with conflict or contradictions membership and non-membership values required. To deal with this problem intuitionistic Plithogenic context and its graphical structure visualization is introduced in this paper. In addition, $\left\{d_{(\alpha_1,\alpha_2)}, c_\beta\right\}$ -cut is introduced for dealing with intuitionistic degree of appurtenance and contradiction for multi-decision process.

Keywords: Knowledge representation; Neutrosophic set; Plithogenic set; Plithogenic graph; Intuitionistic fuzzy set; Multi-granulation

1. Introduction

Recently, properties of Plithogenic set are introduced for dealing with several opposite and non-opposite or indeterminant conditions [1]. It is considered one of the useful set to deal with dark data like doctor's prescription, sports analytics and other fields [2]. In this process, a problem arises while dealing with vague attributes. One of the suitable examples is a cricket match in which several times people intuition changes towards win, draw or loss of an India-Pakistan match [3]. It used to observe in a democratic country like India where people intuition changes several time towards or against the given leader [4]. Same time the prescription of one doctor differ from other doctors while disease and symptoms is also same [5]. It creates contradiction in human intuition while preference analysis for multi-decision process in case of bipolarity [6]. In this case, the first problem arises while representing these types of vague attributes as addressed recently [7-8]. Another problem arises while processing the contradiction among human intuition at given multi-granulation to take a conclusive decision [9]. To tackle this problem current paper focused on dealing with intuitionistic Plithogenic set based context and its zoom in and zoom out at user defined granules for the knowledge discovery tasks.

Recently, some of the authors paid attention towards data with intuitionistic Plithogenic attributes [10-11] and its extensive properties [12-13] for multi-decision process at different granulation [15]. The problem arises while visualization of intuitionistic Plithogenic attributes [16-17] as discussed recently [18-19]. Motivated from these studies current paper put forward effort for dealing data with intuitionistic Plithogenic set, its graphical visualization. In addition, another method is proposed to refine the intuitionistic Plithogenic context at defined Plithogenic granules $\left\{d_{(\alpha_1,\alpha_2)}, c_{\beta}\right\}$. The goal is to find some hidden pattern in data with intuitionistic Plithogenic set based on its defined degree of appurtenance and contradiction as shown in Figure 1.

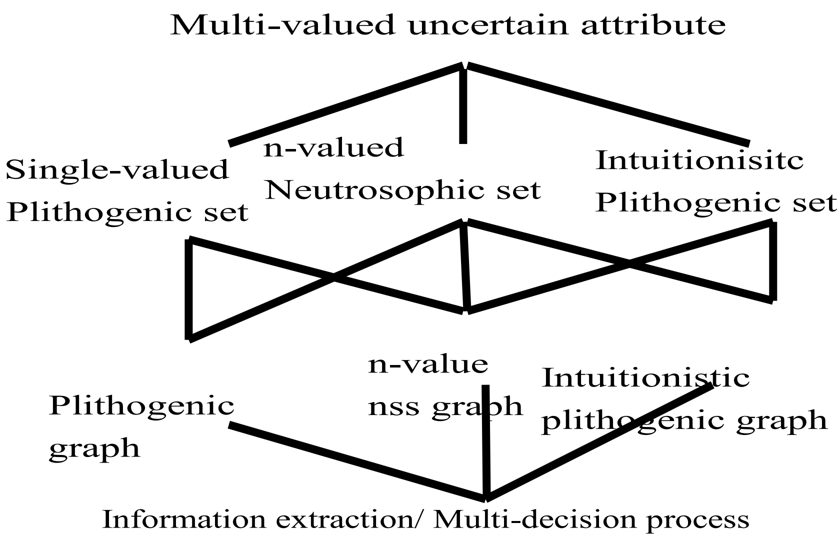

Figure 1: The graphical objective of this paper

The motivation of is to deal with opposite and non-opposite side of intuitionistic attributes for multi-decision process. The objective is to find some useful pattern in intuitionistic Plithogenic context for decision making process. One of the significant outcomes of the proposed method is that it provides a way to deal with contradiction degrees exists in intuitionistic Plithogenic set for conflict analysis.

Rest of the paper is constituted as follows: Section 2 provides basic background about Plithogenic set for data representation. Section 3 provides the proposed method for handling intuitionistic Plithogenic context for knowledge discovery and representation tasks with its illustration in Section 4. Section 5 contains conclusions followed by acknowledgements and references.

2. Data with Plithogenic Set

This section provides preliminaries about Plithogenic set and its examples for understanding of intuitionistic Plithogenic set:

Definition 1. Plithogenic Set [1-2]: This set contains five parts to represents the multi-valued attributes of the given data sets. Let us suppose, ξ be a universe of discourse, P be a subset of this universe of discourse, "a" a multi-valued attribute, V is the range of the multi-valued attribute, "d" be the known (fuzzy, intuitionistic fuzzy, or neutrosophic) degree of appurtenance with regard to some generic of element x's attribute value to the set P, and c is the (fuzzy, intuitionistic fuzzy, neutrosophic) degree of contradiction (dissimilarity) among the attribute values as (<*A*, Neutral *A*, Anti *A*>; <*B*, Neutral *B*, Anti *B*>; <*C*, Neutral *C*, Anti *C*>). It can be represented as a set (P, a, V, d, c) which named as a Plithogenic Set (**P**). The Plithogenic set is a set **P**(P, a, V, d, c) in which each element $x \in P$ is characterized by all attribute's (a) values in $V = \{v_1, v_2, \ldots, v_n\}$, for $n \geq 1$ for the degree of appurtenance (d). The contradiction degree function (c) distinct the Plithogenic set from all of the above set. It represents the between the attribute values in form of fuzzy t-norm and fuzzy t-conorm as:

(i) $c: V \times V \rightarrow [0, 1]$ represents the contradiction degree function among v_1 and v_2.

It used be noted as $c(v_1, v_2)$, and satisfies the following axioms:

(ii) $c(v_1, v_1) = 0$ *i.e.* the contradiction among v_1 and $v_2$is zero.

(iii) $c(v_1, v_2) = c(v_2, v_1)$,the contradiction among v_1 and v_2 or v_2 and v_1 used to be considered as per the commutative properties. In this paper author focuses on single-valued fuzzy membership to handle the Plithogenic set.

Example 1: Let us suppose, two experts or commentator (y_1) and (y_2) given an opinion towards the player (x_1). The expert (y_1)agreed that player (x_1) is 60 percent suitable TEST match whereas expert (y_2) agreed on 70 percent with zero contradiction. The expert (y_1)agreed that player (x_1) is 20 percent suitable for one day match whereas the expert (y_2) agreed on 40 percent which created $\frac{1}{3}$ contradiction. The expert (y_1) agreed that player (x_1) is 70 percent suitable for T20 match whereas the expert (y_2) agreed 60 percent which created $\frac{2}{3}$ contraction on this attribute. The reason given by expert (y_1) that player (x_1) is consistent at 80 percent matches whereas the expert (y_2) agreed on it 60 percent without any contradiction. Another reason given by expert (y_1) that player (x_1) is consistent due to 50 percent suitable health conditions whereas expert (y_2) agreed 40 percent on this attribute with $\frac{1}{2}$ contradiction. This type of complex or large information can be written using the properties of Plithogenic set as shown in Table 1 and Table 2 The Table 1represents the opinion of expert 1 towards the player (x_1) whereas Table 2 represents opinion of expert 2 towards player (x_1). The union and intersection among expert opinion can be computed as given below.

Table 1: The expert (y_1) opinion towards a player (x_1)

Contradiction degree	0	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{2}$
Multi-attributes	TEST Player	ODI Player	T20 Player	*Consistent*	*Health*
Fuzzy degree	0.6	0.2	0.7	0.8	0.5

Table 2: The expert (y_2) opinion towards a player (x_1)

Contradiction degree	0	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{2}$
Multi-attributes	TEST Player	ODI Player	T20 Player	*Consistent*	*Health*
Fuzzy degree	0.7	0.4	0.6	0.6	0.4

Definition 2. Intersection of PlithogenicSet [1]: Let us suppose two Plithogenic set (P_1, P_2) then the intersection can be computed as follows:

$$d_{p_1}(a_p,v_p)\wedge d_{p_2}(a_p,v_p)=(1-c_p)\times(d_{p_1}(a_p,v_p)\wedge_f d_{p_2}(a_p,v_p))+c_p(d_{p_1}(a_p,v_p)\vee_f d_{p_2}(a_p,v_p))$$

where *dp* represents degree of appurtenance, $c_p$represents contradiction degrees for the multi-valued attributes a_p. Others are fuzzy t-norms to define the intersection.

Example 2: Let us suppose, the example shown in Table 1 and 2 to find the intersection using above defined Plithogenic operator. Table 3 represents the intersection of expert opinion shown in Table 1 and 2 using the above operations. It shows the Plithogenic degree that on what level both the expert are maximal common point convinced each other on the given contraction.

Table 3: Intersection of Table 1 and 2 using Plithogenic operator

Contradiction degree	0	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{2}$
Multi-attributes	TEST Player	ODI Player	T20 Player	*Consistent*	*Health*
$y_1 \wedge_x y_1$	0.42	0.23	0.73	0.48	0.45

Definition 3. Union of Plithogenic Set [1]: Let us suppose two Plithogenic set (P_1, P_2) then the union can be computed as follows:

$$d_{p_1}(a_p,v_p) \vee d_{p_2}(a_p,v_p) = (1-c_p)\times(d_{p_1}(a_p,v_p) \vee_f d_{p_2}(a_p,v_p)) + c_p(d_{p_1}(a_p,v_p) \wedge_f d_{p_2}(a_p,v_p))$$

where *dp* represents degree of appurtenance, $c_p$represents contradiction degrees for the multi-valued attributes a_p. Others are fuzzy t-conorms to define the intersection.

Example 3: Let us suppose, the example shown in Table 1 and 2 to find the union using above defined Plithogenic operator. Table 4 represents the union of expert opinion shown in Table 1 and 2 using the above operations. It shows the Plithogenic degree that on what level both the expert convinced each other in the infimum way on the given contraction.

Table 4: Union of Table 1 and 2 using Plithogenic operator

Contradiction degree	0	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{2}$
Multi-attributes	TEST Player	ODI Player	T20 Player	*Consistent*	*Health*
$y_1 \vee_x y_1$	0.88	0.37	0.57	0.92	0.45

Definition 4. Complement of Plithogenic Set [1]: The complement can be computed as follows: $(d_p(a_p,v_p))' = (1-c_p)\times d_p(a_p,v_p)$ where *dp* represents degree of appurtenance, c_p represents contradiction degrees for the multi-valued attributes a_p.In case of conflict or quanta information of human cognition can be represented using intuitionistic fuzzy set.

Definition 5: Intuitionistic Fuzzy Set [13-14]: The intuitionistic fuzzy set is a generalization of fuzzy set. It represents the acceptation, rejection part of any attributes simultaneously. The intuitionistic fuzzy set *A* can be defined by $A = \{x, \mu_X(x), \nu_X(x) / x \in X\}$ where

$\mu_A(x): E \rightarrow [0,1], \nu_A(x): E \rightarrow [0,1]$ for each $x \in E$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. Here $\mu_A(x): E \rightarrow [0,1]$ denote degrees of membership and $\nu_A(x): E \rightarrow [0,1]$ denotes non-membership of x ∈ A, respectively.

Example 5: Let us suppose the above examples that an expert *(y_1)* gives opinion about a player *(x_1)* that the given player is 60 percent suitable for ODI whereas 30 percent not suitable based on his/her performance towards the given team. This type of data can be written using the Intuitionistic Plithogenic set as shown in Table 5.

Table 5: The expert (y_1) opinion towards a player (x_1) based on Intuitionistic set

Contradiction degree	0	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{2}$
Multi-attributes	ODI Player	TEST Player	T20 Player	*Consistent*	*Health*
Fuzzy degree	(0.6, 0.3)	(0.2, 0.6)	(0.7, 0.1)	(0.8, 0.1)	(0.5, 0.3)

It can be observed that, the Plithogenic set provides a chance to deal with multi-valued attributes and contradiction among expert opinion [15]. The problem arises when the expert agree or disagree for the same Plithogenic attribute in case of multi-decision process. It creates conflict among them. To deal with it based on membership and non-membership values the mathematics of intuitionistic fuzzy set is connected with Plithogenic set in this paper. Same time a new graph to visualize the data with intuitionistic Plithogenic context is introduced motivated from [4, 16]. Same time another method is introduced to zoom in and zoom out the intuitionistic Plithogenic context based on defined neutrosophic multi-granulation motivated from [9, 15-18]. In the next section one of the methods is proposed for intuitionistic Plithogenic graph and its processing to deal with conflict analysis arises due to contradiction.

3. Proposed method:

In this section, two methods are proposed the first one focused on graphical structure visualization of intuitionistic Plithogenic attributes and another one focused on decomposition of intuitionistic Plithogenic context. The computation time for the proposed method is also discussed.

3.1 A method for processing data with Intuitionistic Plithogenic Attribute

Let us suppose any data set having Intuitionistic Plithogenic attribute and need to process for multi-decision tasks. It can be done as follows:

Step 1. Let us consider, data with Intuitionistic Plithogenic attributes. Try to represent them in contextual format as shown in Table 6.

Table 6: Data with Intuitionistic Plithogenic attributes and its context representation

Contradiction degree	c_1	c_2	$\ldots$	c_k	c_{k+1}	$\ldots$	c_m
Attribute values	a_1	a_2	$\cdots$	a_k	a_{k+1}	$\cdots$	a_m
a_1	$d_{1,1}(\mu,\nu)$	$d_{1,2}(\mu,\nu)$	$\cdots$	$d_{1,k}(\mu,\nu)$	$d_{1,k+1}(\mu,\nu)$	$\cdots$	$d_{1,m}(\mu,\nu)$
a_2	$d_{2,1}(\mu,\nu)$	$d_{2,2}(\mu,\nu)$	$\cdots$	$d_{2,k}(\mu,\nu)$	$d_{2,k+1}(\mu,\nu)$	$\cdots$	$d_{2,m}(\mu,\nu)$
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
a_n	$d_{n,1}(\mu,\nu)$	$d_{n,2}(\mu,\nu)$	$\cdots$	$d_{n,k}(\mu,\nu)$	$d_{n,k+1}(\mu,\nu)$	$\cdots$	$d_{n,m}(\mu,\nu)$

Step 2. Write all the Plithogenic attributesas(*P*,*a*,*V*,*d*,*c*), where *P* is a set, *a*is the set of multi-valued attributes, *V is* the defined range of the attributes, *d* is the intuitionistic set based degree of appurtenance and *c* is the single-valued degree of contradiction. It means the intuitionistic degree of appurtenance and its contradiction value for the given attribute can be determined with respect to the dominant value of the attribute.

Step 3. Let us consider the,Plithogenic graph G= $\left\{V_p, E_p, a_p, (\mu_{d_p}, \nu_{d_p}), c_p\right\}$ can be called as intuitionistic Plithogenic graph where(*Vp*)represents Intuitionistic Plithogenic attributes as vertex, (*Ep*) represents the intuitionistic Plithogenic set based edges, (a_p)represents the multi-valued i.e. one or more attributes of distinct values. The intuitionistic degree of appurtenance (*dp*) says that at what level the given multi-valued attributes belongs to the set or does not belongs to the set. The (c_p) represents the contradiction degrees as single-valued fuzzy membership.

Step 4. Write each of the vertexes using the Intuitionistic Plithogenic set as: $\frac{\left\{a_p,(\mu_{d_p},\nu_{d_p}),c_p\right\}}{V_p}$

where (a_p)represents multi-valued attributes defines theIntuitionistic Plithogenic vertex(Vp). The degree of appurtenance (dp) represents the belongingness and non-belongingness of multi-valued attributes via intuitionistic Plithogenic set. The contradiction degree is represented using single-valued fuzzy membership as (c_p).

Step 5. Write edges for each of the Plithogenic vertexes as: $\frac{\left\{a_{pq},(\mu_{d_{pq}},\nu_{d_{pq}}),c_{pq}\right\}}{E_{pq}(V_pV_q)}$ where (a_{pq}) represents one or more attributes which defines the Intuitionistic Plithogenic edges (E_{pq}). The degree of appurtenance (d_{pq}) represents the belongingness and non-belongingness of multi-valued edges with its single-valued contradiction degrees (c_{pq}) for the given edge.

Step 6. The contradiction among v_1 and v_2 (or v_2 and v_1) satisfies commutative property as follows: $c(v_1, v_2) = c(v_2, v_1)$. It means the Intuitionistic Plithogenic set based edges (E_{pq}) and (E_{qp}) represents same edge.

Step 7. The contradiction degrees $c(v_1, v_1) = 0$ due to which the edges can be edges can be represented as $(E_{pq} \subseteq V_p \times V_q - V_p \times V_p - V_q \times V_q)$.

Step 8. The computation of relations for the Intuitionistic Plithogenic graph and its edges can be computed using extensive properties of union and intersection of single-valued Plithogenic set as follows:

(a) Intersection of single-valued Plithogenic set as

$$d_{p_1}(a_p,v_p)\wedge d_{p_2}(a_p,v_p)=(1-c_p)\times(d_{p_1}(a_p,v_p)\wedge_f d_{p_2}(a_p,v_p))+c_p(d_{p_1}(a_p,v_p)\vee_f d_{p_2}(a_p,v_p))$$

(b) Union of single-valued Plithogenic set as

$$d_{p_1}(a_p,v_p)\vee d_{p_2}(a_p,v_p)=(1-c_p)\times(d_{p_1}(a_p,v_p)\vee_f d_{p_2}(a_p,v_p))+c_p(d_{p_1}(a_p,v_p)\wedge_f d_{p_2}(a_p,v_p))$$

In case of Intuitionistic Plithogenic sets degree of appurtenance can be represented as:

$V_1 = \left\{ v_1, \mu_{v_1}(x), \nu_{v_1}(x) / x \in X \right\}$ and $V_2 = \left\{ v_2, \mu_{v_2}(x), \nu_{v_2}(x) / x \in X \right\}$ he union and intersection can be computed as follows:

$(A).\ V_1 \vee_p V_2 = \left(\mu_{v_1} \vee_p \mu_{v_2}, \nu_{v_1} \wedge_p \nu_{v2} \right)$

$(B).\ V_1 \wedge_p V_2 = \left(\mu_{v_1} \wedge_p \mu_{v_2}, \nu_{v_1} \vee_p \nu_{v2} \right)$

Otherwise the relation can be as follows:

$$d_{p_1}(a_p, v_p) \wedge d_{p_2}(a_p, v_p) \geq (1 - c_p) \times (d_{p_1}(a_p, v_p) \wedge_f d_{p_2}(a_p, v_p)) + c_p (d_{p_1}(a_p, v_p) \vee_f d_{p_2}(a_p, v_p))$$

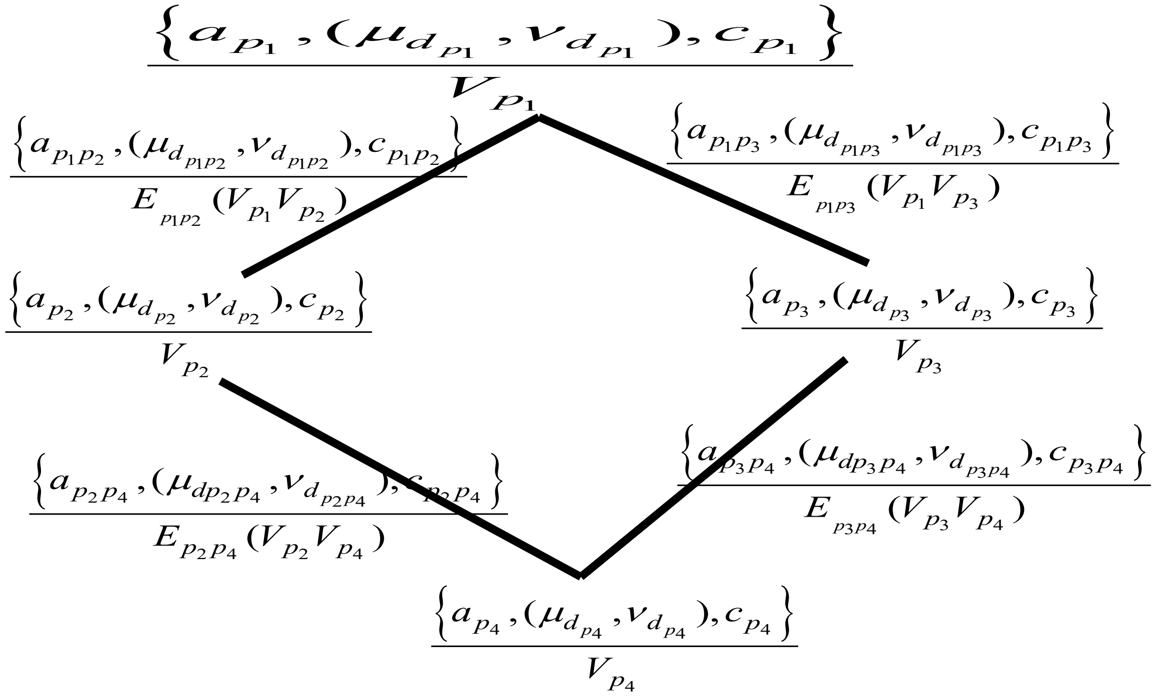

Figure 2. The graphical structure visualization of Plithogenic graph

Step 9. In this way, the data with Intuitionistic Plithogenic set can be analyzed. It can be visualized as Intuitionistic Plithogenic set of vertex and its edges as computed above.

Step 10. The Intuitionistic Plithogenic graph and its visualization is shown in Figure 2.

Step 11. In this way, the proposed method provides a visualization of data with Intuitionistic Plithogenic set which will help in adequate decision making process.

Time complexity: Let us suppose, there are *n*-number of Intuitionistic Plithogenic attribute in the given data set with *m*-number of multi-valued appurtenance degree of attributes. In this case, the time complexity taken in drawing the Intuitionistic Plithogenic graph can be takeO(*nm*). The intuitionistic degree of appurtenance can take maximum O($n.m^3$).

3.2 A method for $\left\{d_{(\alpha_1,\alpha_2)}, c_\beta\right\}$ -cut for Intuitionistic Plithogenic context:

In this section, a method is proposed to decompose the Plithogenic context for precise analysis of pattern based on user or expert requirements as shown in Table 6.

Step 1. Let us consider the Intuitionistic Plithogenic graph G= $\left\{V_p, E_p, a_p, (\mu_{d_p}, \nu_{d_p}), c_p\right\}$.

Step 2. The Intuitionistic Plithogenic context can be processed based on (α_1, α_2) -cut defined for the appurtenance degree (*dp*) as $d_{(\alpha_1,\alpha_2)}$ where $0 \leq \alpha_1 + \alpha_2 \leq 1$.

Step 3. The β -cut can be defined on contradiction degree (c_p) for measuring the conflict and its liabilities as c_β where $0 \leq \beta \leq 1$.

Step 4. Let us suppose, expert wants to analyze the Intuitionistic Plithogenic context based on defined $d_{(\alpha_1,\alpha_2)}$ -cut for degree of appurtenance and c_β for contradiction as $\left\{d_{(\alpha_1,\alpha_2)}, c_\beta\right\}$.

Step 5. In this case the expert wants that the given Intuitionistic Plithogenic context contain more degree of appurtenance from chosen $d_{(\alpha_1,\alpha_2)}$ -cut with less contradiction for the chosen c_β -cut.

$$P_{\{(\alpha_1,\alpha_2),\beta\}} = \left\{\left\{V_p, E_p, a_p, (\mu_{d_p}, \nu_{d_p}), c_p\right\} \mid (\mu_{d_p} \geq \alpha_1), (\nu_{d_p} \leq \alpha_2), c_p \leq \beta, \forall a_p \in P\right\}$$

where $0 \leq \alpha_1 + \alpha_2 \leq 1$ and $0 \leq \beta \leq 1$.

Step 6. In case the given Intuitionistic Plithogenic relation satisfies the –cut defined at step 5 then represent as 1 at particular entry of the attributes otherwise write as 0.

Step 7. In this way all the entries of given Intuitionistic context can be decomposed into 1 and 0 based on defined $\left\{d_{(\alpha_1,\alpha_2)}, c_\beta\right\}$ -cut.

Step 8. The $\left\{d_{(\alpha_1,\alpha_2)}, c_\beta\right\}$ -cut can be changed based on user or expert requirement to zoom in and zoom out the given Intuitionistic context for adequate information extraction.

Step 9. In this way, the proposed method provides intuitionistic level of granulation to deal with Intuitionistic Plithogenic context for knowledge processing tasks. In case the expert unable to draw its graph.

Time complexity: Let us suppose, the given Intuitionistic Plithogenic context contains n-number of attributes having m-number of multi-attributes. In this case, the $\left\{d_{(\alpha_1,\alpha_2)}, c_\beta\right\}$-cut may takeO($nm$) time for the membership and non-membership value traversal, independently. In this case it may cost maximum O(n^2m) and vice versa for decomposition of intuitionistic degree of appurtenance. The consideration of contradiction degree for decomposing m-number of multi-valued attributes brings the complexity as O($n^2.m^2$).

In the next section both of the method is illustrated for handling Intuitionistic Plithogenic context for multi-decision process. Same time the obtained results are compared for validation.

4. Illustration

The uncertainty and vagueness in Plithogenic attributes creates major issues with its representation and analysis [10-11]. The reason is Plithogenic set represents each multi-valued attributes as a generic element x characterized by one attribute only (appurtenance) [1-2]. In this case, intuitionistic fuzzy set can be helpful to represent the degree of appurtenance based on membership and non-membership. Recently, intuitionistic Plithogenic set is received attention of some of the researchers [10-12]. This paper focused on precise representation of data with intuitionistic Plithogenic attributes. In addition, zoom in and zoom out of Intuitionistic Plithogenic attributes for knowledge processing tasks. Same time the knowledge discovered from them is compared for validation of result. To achieve this goal, two methods are proposed in Section 3.1 and 3.2.

Section 4.1: The illustration of Intuitionistic Plithogenic context and its visualization

In this section, the proposed method shown in Section 3.1 using the cricket data set motivated from [3].

Example6: Let us consider the cricket data set[1]. An expert wants to give opinion on performance of Cheteshwar Pujara that he is good batsman for Test, ODI or T20 or selection in the team. The expert can give opinion based on his performance available on time as shown in Table 7. The expert (y_1) wants to give opinion that Pujara is 40 percent good player for test due to his 80 percent ball faced and 40 percent strike rate. Same time the expert wants to express that Pujara is 30 percent not good for some Test due to his 20 percent wrong played ball and 50 percent non-strike or slow rate with 50 percent contradiction. In similar way the expert (y_1) can give opinion about the Pujara based on his performance shown in Table 7 for ODI or T20. This type of data can be written precisely using the Intuitionistic Plithogenic set as shown in Table 8. In case, the selection committee unable to take decision based on expert (y_1) opinion. Then the committee can ask other experts having contradictory with expert (y_1) as shown in Table 9. The problem is to discover comprehensive decision for selecting the Pujara for Test, ODI or T20. This type of data can be solved using the proposed method shown in Section 3.1. The Table 10 represents intersection and union among the expert opinions about performance of Pujara.

Table 7. The Batting performance of Cheteshwar Pujara in various format

	Mat	Inns	NO	Runs	HS	Ave	BF	SR	100	50	4s	6s	Ct	St
Test	86	144	8	6267	206*	46.08	14038	44.64	18	29	740	14	57	0
ODI	5	5	0	51	27	10.19	130	39.23	0	0	4	0	0	0
FC	214	351	37	16311	352	51.94			50	65			139	0
List A	103	101	19	4445	158*	54.20			11	29			39	0
T20	64	56	10	1356	100*	29.47	1240	109.35	1	7	158	20	32	0

Table 8.An Expert(y_1) opinion about Pujara on various format

Contradiction degree	0	0.33	0.66	0.0	0.5
Attribute values	Test	One Day	T20	Ball faced	Strike rate
Puajara	(0.4, 0.5)	(0.1, 0.2)	(0.0, 0.3)	(0.8, 0.2)	(0.4, 0.5)

Table 9. An Expert (y_2) opinion about Pujara on various format

Contradiction degree	0	0.33	0.66	0.0	0.5
Attribute values	Test	One Day	T20	Ball faced	Strike rate
Puajara	(0.6, 0.3)	(0.4, 0.3)	(0.2, 0.5)	(0.6, 0.1)	(0.5, 0.3)

Table 10. The Intuitionistic Plithogenic context representation of Table 8 and 9

Contradiction degree	0	0.33	0.66	0.0	0.5
Attribute values	Test	One Day	T20	Ball faced	Strike rate
Expert $y_1$opinion about Pujara	(0.4, 0.5)	(0.1, 0.2)	(0.0, 0.3)	(0.8, 0.2)	(0.4, 0.5)
Expert $y_2$opinion about Pujara	(0.6, 0.3)	(0.4, 0.3)	(0.2, 0.5)	(0.6, 0.1)	(0.5, 0.3)
$y_1 \wedge_p y_2$ as per step 7 of Section 3.1	(0.24, 0.65)	(0.18, 0.31)	(0.13, 0.32)	(0.48, 28)	(0.45, 0.40)
$y_1 \vee_p y_2$ as per step 7 of Section 3.1	(0.76, 0.15)	(0.32, 0.19)	(0.07, 0.48)	(0.92, 0.02)	(0.45, 0.40)

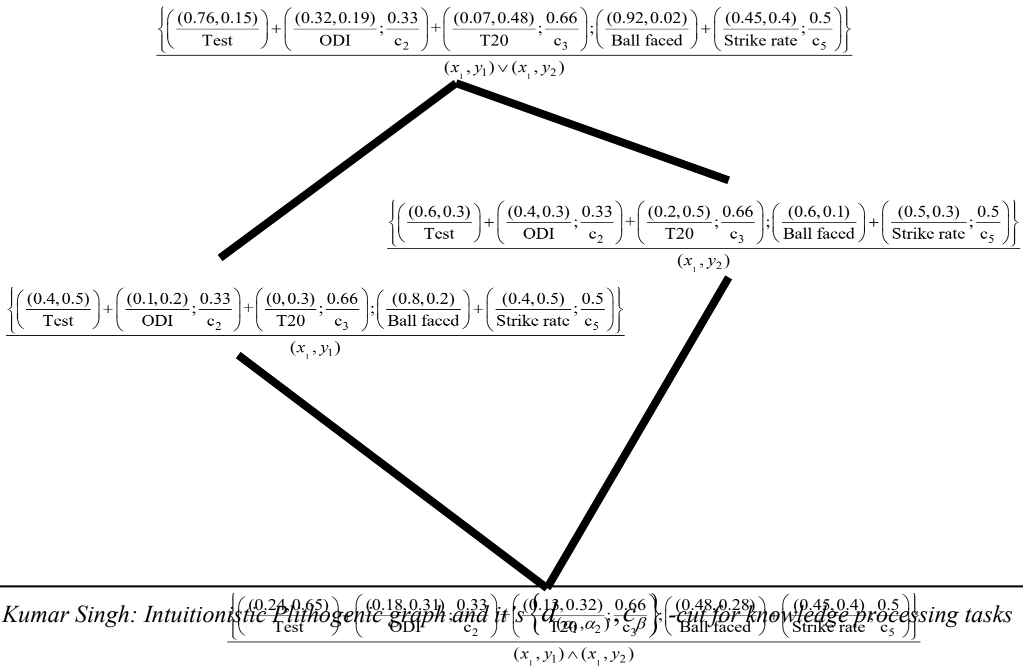

Figure 3. The Intuitionistic Plithogenic graph visualization of Table 10.

Figure 3 represents Intuitionistic Plithogenic graph for the context shown in Table 10 which reflect following information:

(i) The top node represents the infimum among expert's opinion (y_1) and (y_2). It represents that, the player (x_1) is 76 percent suitable for Test without any contradiction, 32 percent suitable for ODI with 30 percent contradiction, 7 percent for T20 with 66 percent contradiction due to his 92 percent ball faced and 45 percent strike rate with contradiction 0.5.

(ii) The last node represents supremum among the expert opinion. It represents that, the player (x_1) is 24 percent suitable for Test without any contradiction, 18 percent suitable for ODI with 30 percent contradiction, 13 percent for T20 with 66 percent contradiction due to his 48 percent ball faced and 45 percent strike rate with contradiction 0.5.

It can be observed that, both of the expert agreed about player (x_1) and its suitability maximally for the Test when compared to other parameters based on his performance available at Crickinfo[1]. The conflict among them is about his suitability for the ODI as 33 percent. To deal with it another method is proposed in Section 3.2 which is illustrated in the next section.

Section 4.1: The illustration of $\left\{d_{(\alpha_1,\alpha_2)}, c_\beta\right\}$ -cut for Intuitionistic Plithogenic context

The precise analysis of Intuitionistic Plithogenic context as per user requirement and its traversal is another concern. One of the reason is dealing the conflict among expert arises by contradiction degrees. To resolve this issue, current paper tries to introduce the properties of multi-granulation in this paper as shown in Section 3.2. The proposed method illustrated using the Intuitionistic

Plithogenic context shown in Table 10. Some potential level of $\left\{d_{(\alpha_1,\alpha_2)}, c_{\beta}\right\}$ -cut is shown in Table 11 to process the given Intuitionistic Plithogenic context.

Example 7: Let us supposethe Intuitionistic Plithogenic context shown in Table 10 and decomposition using the defined granulation as shown in Table 11. The selection committee required an average player with less than 33 percent contradiction. This is shows as Level 3 in the Table 11. The decomposed context at {(0.4, 0.3), 0.33}-cut is shown in Table 12 for knowledge processing tasks. The value 1 means satisfies the chosen information granules and o means does not satisfy the information granulation. In this way, the expert can select the player for which section his/her performance satisfies maximum rows as 1.

Table 11. Level of Intuitionistic Plithogenic granulation and its interpretation

Level	Degree of appurtenance $d_{(\alpha_1,\alpha_2)}$ where $0<\alpha_1+\alpha_2<1$	Contradiction degree c_{β} where $0<\beta<1$	Interpretation
1	(0.8, 0.1)	0.1	Top Player
2	(0.6, 0.3)	0.2	Good Player
3	(0.4, 0.3)	0.3	Average Player
4	(0.3, 0.2)	0.4	Player
5	(0.2, 0.1)	0.5	Last player/ Bowler

Table 12. Table 10 at {(0.4, 0.3), 0.33}-cutfor degree of appurtenance and contradiction

Contradiction degree	0	0.33	0.66	0.0	0.5
Attribute values	Test	One Day	T20	Ball faced	Strike rate
Expert $y_1$opinion about Pujara	1	0	0	1	1
Expert $y_2$opinion about Pujara	1	1	0	1	1

It can be observed that both reviewer agreed that Pujara is good player for the Test when compared to ODI and T20 as per given {(0.4, 0.3), 0.33}-cut shown in table 12 due to his ball faced and strike rate. The expert y_2 agreed that due to this reason Pujara can play some of the ODI match also. However, none of the expert agreed that Pujara is good player for T20. In this way, the selection committee can prefer Pujara for the Test as first preference which echo with results obtained from the Intuitionistic Plithogenic context graphshown in Figure 3 as per Section 4.1.

Table 13: The comparison of proposed method with recent approaches

	Plithogenic set [3]	Intuitionistic Plithogenic Set [18]	The proposed method
Plithogenic attributes	Yes	No	Yes
Vagueness measurement	No	Yes	Yes
Intersection and Union	Yes	Yes	Yes
Graph	No	No	Yes
Information granulation	No	No	Yes
Time complexity	Not given	Not given	O($n.m^3$) or O(n^2m^2)

Table 13 represents comparison of the proposed method with recently available methods on Intuitionistic Plithogenic set. It shows that, the proposed method distinct from each approach in various ways and provides an extensive version to deal with intuitionistic Plithogenic context.

In this way, the proposed method is helpful while dealing with data with Plithogenic attribute. The proposed method does not provide any clue about dealing with uncertainty [18] and its changes [19] arises due to conflict among experts. Hence the author will focus on tackling this problem in near future.

4. Conclusions

This paper focused on handling data with Intuitionistic Plithogenic data and its graphical visualization as shown in Figure 3. Same time the decomposition of Intuitionistic Plithogenic context based on user required intuition degree of appurtenance and contradiction as shown in Section 3.2. The knowledge discovered from both of the proposed methods is compared with each other and recently available methods as shown in Table 13. It is shown that, the proposed method is distinct from any of the available approaches in Intuitionistic Plithogenic context. In near future author will focus on dealing with uncertainty in Intuitionistic Plithogenic attributes and its precise measurement for knowledge processing tasks.

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Footnotes:

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Fundamentals of Neutrosophical Simulation for Generating Random Numbers Associated with Uniform Probability Distribution

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Abstract: The simulation process depends on generating a series of random numbers subject to the uniform probability distribution in the interval [0, 1]. The generation of these numbers is starting from the cumulative distribution function of the uniform distribution. Through previous studies in classical logic, we found any random number R_0, met with a cumulative distribution function value equal to R_0, but these specific numbers may not have sufficient accuracy, which leads to obtaining results that are not sufficiently accurate when doing the simulation. To bypass this case, in this paper, we present a study that enables us to generate as accurate as possible random numbers, using neutrosophic logic ' this Logic given by American mathematician Florentin Smarandache in 1995'. The first step in the study is, define the cumulative distribution function of the neutrosophic uniform distribution, depending on definition of the neutrosophic integral and definition of the neutrosophic uniform distribution. We used the new definition to generate random numbers subject to a neutrosophic uniform distribution on the interval [0, 1]. The result was that each random number R_0 corresponds to a interval of the distribution function related to R_0, So that it preserves enough precision for the random numbers, and thus we get a more accurate simulation of any system we want to simulate.

Keywords: Neutrosophic uniform distribution; Simulations; Cumulative distribution function of neutrosophic uniform distribution; neutrosophic random numbers.

1. Introduction

Because of the great difficulty that can face us when studying the work of any real system. As well as the high cost of studying. In addition, some systems we cannot be directly studied. Here comes the importance of the simulation process in all branches of science. The simulation depends on the application of the study on systems similar to the real systems, and then projecting these results if they are appropriate on the real system.

The simulation based on generating a series of random numbers that are subject to a uniform probability distribution in the interval [0, 1]. Then converting these numbers into random variables subject to the probability distribution in which the system to be simulated operates, based on the cumulative distribution function of the probability distribution [9].

In front of the great revolution brought about by neutrosophic logic in all fields of science, after the American philosopher and mathematician Florentin Smarandache laid its foundations in 1995 [2,4,5,6,8], who put it forward as a generalization of fuzzy logic, especially Intuitionistic fuzzy logic [3], and an extension of the theory of fuzzy sets presented by Lotfi Zadeh 1965 [1] . By extension to that, A.A. Salama presented the neutrosophic classical set theory as a generalization of the classical set theory and developed, introduced and formulated new concepts in the fields of mathematics, statistics, computer science and classical information systems using the neutrosophic [7,15,16]. The neutrosophic is the logic that studies the origin, nature and field of indeterminacy, taking into account every idea with its opposite and with the spectrum of indeterminacy.

This logic has developed in recent years, and most of the known concepts in classical logic have been reformulated according to neutrosophic logic [10,12,13,17,19,20,21,22,23,24]. Among these concepts is the study and formulation of most of the known probability distributions in classical logic [18]. In this paper, we present a study of the cumulative distribution function of the neutrosophic uniform distribution on the interval [a, b]. Depending on what researchers have found in the field of neutrosophic [11,14], such as the definition of the neutrosophic uniform distribution on the interval [a, b] and the definition of the neutrosophic integration [25]. Especially in the case where the indeterminacy is related to the upper and lower bounds of the interval [a, b]. We used the results as a basis for generating random numbers subject to a neutrosophic uniform distribution on the interval [0, 1].

The importance of this research stems from the importance of simulation in all fields of science, especially when we need accuracy in the results during the simulation process for any system. This accuracy not provided by classical logic. The limits of this study include all scientific fields that may contain indefinite cases and need simulation to represent them, and aim to reach the most accurate possible results

2. Experimental and Theoretical Part:

Based on the importance of simulation in all fields of science. Since the simulation depends primarily on generating a series of random numbers that are subject to a uniform distribution on the interval [0, 1]. It was necessary to keep pace with the neutrosophic revolution by presenting a study that generates neutrosophic random numbers that are subject to a neutrosophic uniform distribution on the interval [0, 1]. That is by finding mathematical relationships that describe the cumulative distribution function of the neutrosophic uniform distribution on the interval [a, b], especially when the indeterminacy relates to both ends of the interval. Where we studied the following three cases:

The first case: if the indeterminacy related to the lower limit of interval.

The second case: if the indeterminacy related to the upper limit of interval.

The third case: if the indeterminacy related to both the lower and upper limits of interval.

In addition, we arrived at mathematical formulas for the cumulative distribution function for each of the previous cases. We then used these formulas to generate random numbers that are subject to a neutrosophic uniform distribution. Which provides us with the most accurate results possible while performing the simulation process for any system.

2.1. In classical logic:

To generate the random numbers subject to a uniform distribution in the interval [0, 1] (according to the Monte–Carlo method), we assume R is the continuous random variable that is subject to the continuous uniform distribution defined on the interval [a, b], which is given by:

$$f(x) = \frac{1}{b-a} \quad ; \quad a \le x \le b$$

Then it would be:

$$f(R) = \frac{1}{1-0} = 1 \qquad 0 \le R \le 1$$

The distribution function is of the form:

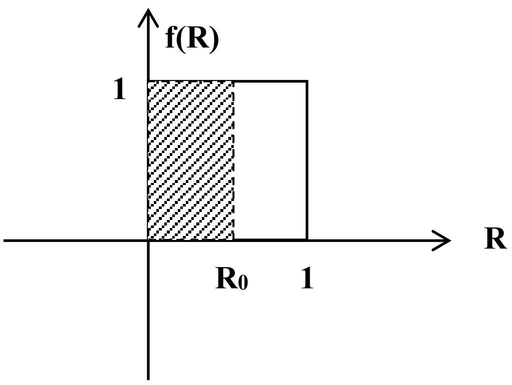

Figure (1)

We take the cumulative distribution function for this distribution. We will call it F(R), then:

$$P(R < R_0) = F(R_0) = \int_0^R f(R)dR = \int_0^{R_0} 1.dR = R_0$$

That is, every random number R_0 corresponds to a value of the distribution function equal to R_0.

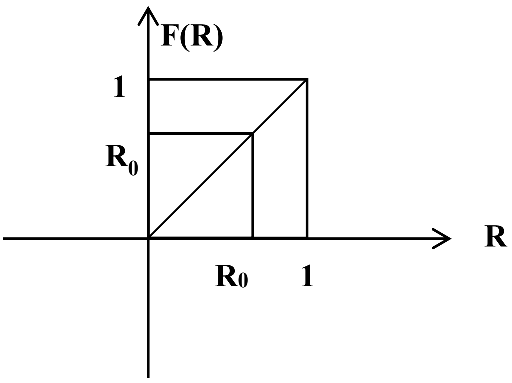

Figure (2)

2.2. In Neutrosophic logic:

The uniform distribution in neutrosophic logic takes the following form:

$$f_N(x) = \frac{1}{b-a}$$

Where b, a: one or both of them is not precisely defined, we find it in the form of set or interval, etc., we can consider all possible cases for b, a. While keeping the condition a < b.

Accordingly, we can write the interval [a, b] in one of the following forms **(*)**:

1. $[a_1 + \varepsilon , b]$; $a = a_N = a_1 + \varepsilon$

2. $[a , b_1 + \varepsilon]$; $b = b_N = b_1 + \varepsilon$

3. $[a_1 + \varepsilon , b_1 + \varepsilon]$; $a = a_N = a_1 + \varepsilon$ & $b = b_N = b_1 + \varepsilon$

Where $\varepsilon \in [0, n]$, with $a < n < b$.

In order to define the cumulative distribution function of the neutrosophic uniform distribution on the interval [a, b], it is necessary to define the neutrosophic integral.

We define the neutrosophic integral in the case of indeterminacy related to the lower limit of interval as follows:

Suppose we want to integrate the function **f: X → R** on the interval [a, b] of X , but we are not sure of the lower limit **a** , since **a** has a finite part (a_1) and an indefinite part (ε), i.e.:

$a = a_N = a_1 + \varepsilon$; $\varepsilon \in [0, n]$; $a < n < b$

Then we write the integral of f(x) on the interval [a, b], as follows:

$$\int_a^b f(x)dv = \int_a^b f(x)dx - i_1$$

Where: $i_1 \in \left[0, \int_{a_1}^{a_1+n} f(x)dx\right]$

In another way: $\int_a^b f(x)dv = \int_{a_1+n}^b f(x)dx + i_2$

Where: $i_2 \in \left[0, \int_{a_1}^{a_1+n} f(x)dx\right]$

2.3. Neutrosophical cumulative distribution function with states (*):

First case:

The indeterminate is in the lower limit of the interval $a = a_1 + \varepsilon$; $\varepsilon \in [0, n]$; $a < n < b$

The interval becomes $[a_1 + \varepsilon , b]$, and we have:

$$a = a_N = a_1 + \varepsilon = a_1 + [0, n] = [a_1, \quad a_1 + n]$$

The cumulative distribution function for a classical uniform distribution given as:

$$P(X < x_s) = F(x_s) = \int_a^{x_s} f(x)dx$$

The Cumulative distribution function for a neutrosophic uniform distribution:

$$NP(X < x_s) = NF(x_s) = \int_a^{x_s} f(x)dv = \int_{a_1}^{x_s} f(x)dx - i$$

Where: $i \in \left[0, \int_{a_1}^{a_1+n} f(x)dx\right]$

In the case of a neutrosophic uniform distribution, we get:

$$NF(x_s) = \int_{a_1}^{x_s} \frac{1}{b-a}dx - i = \frac{x_s - a_1}{b-a} - i$$

We define the interval for **i** , let's calculate the integral:

$$\int_{a_1}^{a_1+n} \frac{1}{b-a}dx = \frac{1}{b-a}[x]_{a_1}^{a_1+n} = \frac{a_1+n-a_1}{b-a} = \frac{n}{b-a}$$

This means that the indeterminate **i** belongs to the interval:

$$i \in \left[0, \frac{n}{b-a_N}\right]$$

Second case:

The indeterminate is in the upper limit of the interval $b = b_1 + \varepsilon$; $\varepsilon \in [0, n]$; $a < n < b$

The interval becomes **[a , b1 + ε]**, and we have:

$$b = b_N = b_1 + \varepsilon = b_1 + [0, n] = [b_1, \quad b_1 + n]$$

The cumulative distribution function for a classical uniform distribution given as:

$$P(X < x_s) = F(x_s) = \int_a^{x_s} f(x)dx$$

The Cumulative distribution function for a neutrosophic uniform distribution:

$$NP(X < x_s) = NF(x_s) = \int_a^{x_s} f(x)dv$$

$$NF(x_s) = \int_a^{x_s} \frac{1}{b-a}dx = \frac{x_s - a}{b_N - a}$$

Third case:

The Indeterminacy exists in the lower and upper limit of the interval

$=[a_N, b_N]$ $[a_1 + \varepsilon, \quad b_1 + \varepsilon]$; $\varepsilon \in [0, n]$; $a < n < b$. i.e.:

$$a_N = a_1 + \varepsilon = a_1 + [0, n] = [a_1, a_1 + n]$$

$$b_N = b_1 + \varepsilon = b_1 + [0, n] = [b_1, \ b_1 + n]$$

So we write:

$$NP(X < x_s) = NF(x_s) = \int_{a}^{x_s} f(x)dv = \int_{a_1}^{x_s} f(x)dx - i$$

Where: $i \in \left[0, \int_{a_1}^{a_1+n} f(x)dx\right]$

In addition, it will be:

$$NF(x_s) = \int_{a_1}^{x_s} \frac{1}{b-a}dx - i = \frac{x_s - a_1}{b-a} - i$$

Where: $i \in \left[0, \frac{n}{b_N - a_N}\right]$

3. Results and Discussion

Generation of neutrosophic random numbers that are subject to a uniform distribution in the interval [0, 1]:

Based on what we have put forward, we get the following forms of the interval [0,1] with cases of the Indeterminacy:

1. $[0 + \varepsilon \ , \ 1]$
2. $[0 \ , \ 1 + \varepsilon]$
3. $[0 + \varepsilon \ , \ 1 + \varepsilon]$

Where: $\varepsilon \in [0, n]$; $0 < n < 1$.

First case:

$$NP(R < R_0) = NF(R_0) = \frac{R_0 - \varepsilon}{1 - \varepsilon} \quad ; \varepsilon \in [0, n]$$

$$\Rightarrow NP(R < R_0) = NF(R_0) = \frac{R_0 - \varepsilon}{1 - \varepsilon} = \frac{[R_0, R_0 - n]}{[1, 1-n]} = \left[R_0, \frac{R_0 - n}{1-n}\right]$$

Second case:

$$NP(R < R_0) = NF(R_0) = \frac{R_0}{1 + \varepsilon} \quad ; \varepsilon \in [0, n]$$

$$\Rightarrow NP(R < R_0) = NF(R_0) = \frac{R_0}{1 + \varepsilon} = \frac{R_0}{[1, n+1]} = \left[R_0, \frac{R_0}{n+1}\right]$$

Third case:

$NP(R < R_0) = NF(R_0) = R_0 - \varepsilon \quad ; \ \varepsilon \in [0, n]$

$\Rightarrow NP(R < R_0) = NF(R_0) = R_0 - \varepsilon = [R_0 , R_0 - n]$

From the above we conclude that each random number R_0 corresponds to an interval of the cumulative distribution function related to R_0.

4. Application example:

Using "mean of the square" method (for von Neumann), we generate the random numbers R_0, R_1, R_2, R_3, R_4.

Method explanation: We choose a fractional random number R_0. Consisting of four places (called the seed) and does not contain zero. Then we square that number (R0). Choose the middle four digits of the fractional part then put a new fraction and consider it the random number R_1. And so on... until we get the required random numbers. i.e. we apply the rule:

$$R_{i+1} = \text{Mid}\left[R_i^2\right] \quad i = 0, 1, 2, \ldots$$

We denote by "Mid" the middle four ranks of R_i^2.

For example, we choose, $R_0 = 0.1276$ then:

$R_1 = \text{Mid}\left[R_0^2\right] = \text{Mid}\,[0.01628176] = 0.6281$

$R_2 = 0.4509, \ R_3 = 0.3310, \ R_4 = 0.0951$

We use these numbers to generate neutrosophic random numbers that follow a uniform distribution in the interval [0, 1], according to the three cases.

First case: In this case, each random number R_0 corresponds to $\dfrac{R_0 - \varepsilon}{1-\varepsilon}$

We found that:

$$R_0 \to \frac{R_0 - \varepsilon}{1-\varepsilon} \quad ; \ \varepsilon \in [0, n]$$

We take, for example: $[0, n] = [0, 0.03]$

$$R_0 \to \frac{0.1276-[0, 0.03]}{1-[0, 0.03]} = \frac{[0.1276, 0.1276-0.03]}{[0.97,1]} = \frac{[0.0976, 0.1276]}{[0.971]} = [0.1006, 0.1276]$$

$$R_1 \to \frac{R_1-\varepsilon}{1-\varepsilon} = \frac{0.6281-[0,0.03]}{[1,1\text{-}0.03]} = \frac{[0.5981,0.6281]}{[0.97,1]} = [0.6281, 0.6169]$$

$$R_2 \to \frac{R_2-\varepsilon}{1-\varepsilon} = \frac{0.4509-[0,0.03]}{[1,1\text{-}0.03]} = \frac{[0.4209,0.4509]}{[0.97,1]} = [0.4339, 0.4509]$$

$$R_3 \to \frac{R_3-\varepsilon}{1-\varepsilon} = \frac{0.3310-[0,0.03]}{[0.97,1]} = \frac{[0.301,0.3310]}{[0.97,1]} = [0.3103, 0.3310]$$

$$R_4 \to \frac{R_4-\varepsilon}{1-\varepsilon} = \frac{0.0956-[0,0.03]}{[0.97,1]} = \frac{[0.0656,0.956]}{[0.97,1]} = [0.0676, 0.956]$$

Second case: In this case, each random number R_0 corresponds to $\frac{R_0}{1+\varepsilon}$

Where $\varepsilon = [0, 0.03]$, $R_0 = 0.1276$

$$R_0 \to \frac{0.1276}{1+[0,0.03]} = \frac{0.1276}{[1,1.03]} = [0.1238, 0.1276]$$

$$R_1 \to \frac{0.6281}{[1,1.03]} = [0.6098, 0.6281]$$

$$R_2 \to \frac{0.4509}{[1,1.03]} = [0.4377, 0.4509]$$

$$R_3 \to \frac{0.0956}{[1,1.03]} = [0.0928, 0.0956]$$

$$R_4 \to \frac{0.0956}{[1,1.03]} = [0.0928, 0.0956]$$

Third case: In this case, each random number R_0 corresponds to $R_0 - \varepsilon$, where:

$\varepsilon \in [0, n]$; $0 < n < 1$

For $\varepsilon = [0, 0.03]$ and $R_0 = 0.1276$, we have:

$R_0 \rightarrow 0.1276 - [0, 0.03] = [0.0976, 0.1276]$

$R_1 \rightarrow 0.6281 - [0, 0.03] = [0.5981, 0.6281]$

$R_2 \rightarrow 0.4509 - [0, 0.03] = [0.4209, 0.4509]$

$R_3 \rightarrow 0.3310 - [0, 0.03] = [0.301, 0.3310]$

$R_4 \rightarrow 0.0956 - [0, 0.03] = [0.0656, 0.0956]$

5. **Conclusions**:

Through this paper, we found that when we use neutrosophic logic to generate random numbers, we get a series of numbers that are more accurate than the numbers we get when using classical logic. This is due to the margin of freedom offered by neutrosophic logic through the indeterminacy spectrum.

We are looking forward in the near future , to preparing a study in which we can generate the random numbers that are subject to non-uniform distributions, by converting the regular random numbers in the interval [0, 1] into random numbers that are subject to the appropriate probability distribution for the case under study.

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Neutrosophic Separation Axioms

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Abstract:

Neutrosophic set, developed by Smarandache, is characterized by a truth membership function, an indeterminacy function and a falsity membership function. Neutrosophic sets have been employed to model uncertainty in several areas of application such as decision making, pattern recognition, image segmentation, etc. Neutrosophic separation axioms are interesting concepts via neutrosophic topology. In this paper, we introduce the notion of neutrosophic T_i-spaces (i = 0, 1, 2, 3, 4) via neutrosophic topological spaces, and investigate their different properties. By defining neutrosophic T_i-spaces (i = 0, 1, 2, 3, 4), we prove some interesting results on neutrosophic separation axioms via neutrosophic topological spaces.

Keywords: Neutrosophic Set; Neutrosophic Topological Spaces; Neutrosophic Separation Axioms.

1. Introduction:

Based on neutrosophy [1], Neutrosophic Set (NS) was grounded by Smarandache [1], which is the generalization of Fuzzy Set (FS) [2] and intuitionistic FS [3]. Later on, Salama and Alblowi [4] presented the notion of Neutrosophic Topological Space (NTS). Arokiarani et al. [5] introduced the concept of Neutrosophic Point (NP) in NTSs. AL-Nafee et al. [6] studied some separation axioms on neutrosophic crisp topological spaces. Das and Pramanik [7] presented the generalized neutrosophic b-open sets in NTS. Das and Pramanik [8] developed the neutrosophic Φ-open sets and neutrosophic Φ-continuous functions in NTSs. Maji [9] grounded the idea of neutrosophic soft sets. Bera and Mahapatra [10] introduced the neutrosophic soft topological space. Das and Pramanik [11] presented the neutrosophic simply soft open set in Neutrosophic Soft Topological Space (NSTS). Gunnuz Aras et al. [12] presented the separation axioms on neutrosophic soft topological spaces. Mehmood et al. [13] worked on generalized neutrosophic separation axioms in NSTS in this article the worked on Neutrosophic soft p-separation structures are the most imperative and fascinating notions in neutrosophic soft topology. Acikgoz and Esenbel [14] studied on separation axioms in NTS by defining neutrosophic quasi-coincidence and neutrosophic Ri-spaces, i= 0, 1 and established some basic results. Khattak et al. [15] presented soft b-separation axioms in NSTS. Suresh and Palaniammal [16] worked on "NS(WG) separation axioms in NTS

Research Gap: No investigation on neutrosophic separation axioms [neutrosophic T_i-spaces, I = 0, 1, 2, 3, 4] on NTSs has been reported in the neutrosophic literature.

Motivation: To fill the research gap, we present the notion of neutrosophic separation axioms [neutrosophic T_i-spaces, I = 0, 1, 2, 3, 4] on NTSs.

The rest of the article has been split into following sections:

In section 2, we recall the basic definitions on NSs and NTSs. In section 3, we present the notion of neutrosophic separation axioms [neutrosophic T_i-spaces, i = 0, 1, 2, 3, 4] on NTSs, and examine several relationships between them. Section 4 presents the concluding remarks. In this section, we also state some future scope of research in this direction.

Throughout this article, we use the acronym for the clarity of the presentation (see Table 1).

Table 1. List of Short terms

String of words	acronym/ *abbreviation*
Neutrosophic Set	NS
Neutrosophic Topology	NT
Neutrosophic Topological Space	NTS
Neutrosophic Soft Topological Space	NSTS
Neutrosophic Open Set	NOS
Neutrosophic Closed Set	NCS
Neutrosophic Point	NP
Neutrosophic T_0-Space	N-T_0-S
Neutrosophic T_1-Space	N-T_1-S
Neutrosophic T_2-Space	N-T_2-S

2. Some Relevant Definitions:

In this section, we recall some basic definitions and results on NSs and NTSs.

Definition 2.1. An NS [1] R over a non-empty fixed set X is defined as follows:

$$R = \{(r, T_R(r), I_R(r), F_R(r)): r \in X\},$$

where T, I, F : $X \to]^-0, 1^+[$ are the truth, indeterminacy and false membership functions respectively.

Definition 2.2. The null NS (0_N) [1] and absolute NS (1_N) over X are defined as follows:

(i) $0_N = \{(r, 0, 1, 1) : r \in X\}$;

(ii) $1_N = \{(r, 1, 0, 0) : r \in X\}$.

Definition 2.3. Let $H = \{(r, T_H(r), I_H(r), F_H(r)) : r \in X\}$ and $K = \{(r, T_K(r), I_K(r), F_K(r)) : r \in X\}$ be two NSs over a fixed set X. Then, the following results [1] hold:

(i) $H^c = \{(r, 1-T_H(r), 1-I_H(r), 1-F_H(r)) : r \in X\}$;

(ii) $H \subseteq K$ if and only if $T_H(r) \leq T_K(r)$, $I_H(r) \geq I_K(r)$, $F_H(r) \geq F_K(r)$, for all $r \in X$.

(iii) $H \cup K = \{(r, T_H(r) \vee T_K(r), I_H(r) \wedge I_K(r), F_H(r) \wedge F_K(r)) : r \in X\}$;

(iv) $H \cap K = \{(r, T_H(r) \wedge T_K(r), I_H(r) \vee I_K(r), F_H(r) \vee F_K(r)) : r \in X\}$.

Definition 2.3. A non-empty collection τ of NSs over a fixed set X is called a neutrosophic topology (NT) [4] on X if the following three axioms hold:
(i) 0_N and 1_N are the members of τ;
(ii) $R_1, R_2 \in \tau \Rightarrow R_1 \cap R_2 \in \tau$;
(iii) $\cup \{R_i : i \in \Delta\} \in \tau$, for every $\{R_i : i \in \Delta\} \subseteq \tau$.

If τ is an NT on X, then the structure (X, τ) is called a neutrosophic topological space (NTS) [4]. Every member of τ is said to be a neutrosophic open set (NOS). If $R \in \tau$, then R^c is called a neutrosophic closed set (NCS).

Definition 2.4. Suppose that p, q, r be real standard and non-standard subsets of]-0, 1+[. An NS $z_{p.q.r}$ is called a Neutrosophic Point (NP) [5] over a fixed set X defined by

$$z_{p.q.r}(y) = \begin{cases} (p,q,r), & \text{if } z=y \\ (0,1,1), & \text{if } z \neq y) \end{cases}$$

where p, q, r ($\in]^-0, 1^+[$) are the truth, indeterminacy and falsity membership values of z. Undoubtedly, every NS is the union of its NPs.

Example 2.1. Suppose that $X = \{r_1, r_2\}$ be a fixed set. Clearly, ${r_1}_{0.2,0.3,0.7}$ and ${r_2}_{0.6,0.5,0.5}$ are two NPs over X. Then, the neutrosophic set R = {(r_1, 0.2, 0.3, 0.7), (r_2, 0.6, 0.5, 0.5)} is the union of neutrosophic points ${r_1}_{0.2,0.3,0.7}$ and ${r_2}_{0.6,0.5,0.5}$.

Definition 2.5. An NP $z_{p.q.r}$ is contained in a neutrosophic set R (i.e., $z_{p.q.r} \in R$) [5] if and only if $p \leq T_R(z)$, $q \geq I_R(z)$, $r \geq F_R(z)$.

Definition 2.6. A one to one and onto function $\xi : (X, \tau_1) \rightarrow (Y, \tau_2)$ is called a neutrosophic continuous mapping [5] if $\xi^{-1}(K)$ is an NOS in X, whenever K is an NOS in Y.

Definition 2.7. A function $\xi : (X, \tau_1) \rightarrow (Y, \tau_2)$ is called a neutrosophic open mapping [5] if ξ(K) is an NOS in Y, whenever K is an NOS in X.

3. Neutrosophic T_i–Spaces:

In this section, we present the notion of neutrosophic separation axioms via NTSs, and investigate different relationships among them.

Definition 3.1. An NTS (X, τ) is called a neutrosophic T_0-space (N-T_0-S) if for any pair of NPs $x_{\alpha,\beta,\gamma}$, $y_{\theta,\lambda,\mu}$ ($x \neq y$) in X, there exists an NOS R such that $x_{\alpha,\beta,\gamma} \in R$, $y_{\theta,\lambda,\mu} \notin R$ or $x_{\alpha,\beta,\gamma} \notin R$, $y_{\theta,\lambda,\mu} \in R$.

Example 3.1. Suppose that X = {x, y} & τ ={0_N, 1_N, {<x, 0.5, 0.4, 0.7>, <y, 0.3, 0.4, 0.3>}, { <x, 0.5, 0.4, 0.7>}}. Clearly, (X, τ) is a neutrosophic T_0-space.

Theorem 3.1. Suppose that $\xi : (X, \tau_1) \rightarrow (Y, \tau_2)$ is both one-one and neutrosophic continuous function from an NTS (X, τ_1) to another NTS (Y, τ_2). If (Y, τ_2) be an N-T_0-S, then (X, τ_1) is also an N-T_0-S.

Proof. Assume that (Y, τ_2) is an N-T_0-S. Also, let $x_{\alpha,\beta,\gamma}$, $y_{\theta,\lambda,\mu}$ ($x \neq y$) be any two NPs in (X, τ_1). Since, $\xi : (X, \tau_1) \rightarrow (Y, \tau_2)$ is a one-one function, so $\xi(x_{\alpha,\beta,\gamma})$, $\xi(y_{\theta,\lambda,\mu})$ are also distinct NPs in (Y, τ_2). Since, (Y, τ_2) is an N-T_0-S, so there exists an NOS R in Y such that $\xi(x_{\alpha,\beta,\gamma}) \in R$, $\xi(y_{\theta,\lambda,\mu}) \notin R$ or $\xi(x_{\alpha,\beta,\gamma}) \notin R$, $\xi(y_{\theta,\lambda,\mu}) \in R$. Therefore, $x_{\alpha,\beta,\gamma} \in \xi^{-1}(R)$, $y_{\theta,\lambda,\mu} \notin \xi^{-1}(R)$ or $x_{\alpha,\beta,\gamma} \notin \xi^{-1}(R)$, $y_{\theta,\lambda,\mu} \in \xi^{-1}(R)$. Since, ξ is a neutrosophic continuous function, so $\xi^{-1}(R)$ is an NOS in (X, τ_1). Therefore, for any pair of distinct NPs $x_{\alpha,\beta,\gamma}$, $y_{\theta,\lambda,\mu}$ in (X, τ_1), there exists an NOS $\xi^{-1}(R)$ such that $x_{\alpha,\beta,\gamma} \in \xi^{-1}(R)$, $y_{\theta,\lambda,\mu} \notin \xi^{-1}(R)$ or $x_{\alpha,\beta,\gamma} \notin \xi^{-1}(R)$, $y_{\theta,\lambda,\mu} \in \xi^{-1}(R)$. Hence, (X, τ_1) is an N-T_0-S.

Definition 3.2. An NTS (X, τ) is called a neutrosophic T_1-space (N-T_1-S) if for any pair of NPs $x_{\alpha,\beta,\gamma}$, $y_{\theta,\lambda,\mu}$ ($x \neq y$) in X, there exist two NOSs R and S such that $x_{\alpha,\beta,\gamma} \in R$, $x_{\alpha,\beta,\gamma} \notin S$ and $y_{\theta,\lambda,\mu} \notin R$, $y_{\theta,\lambda,\mu} \in S$.

Obviously, every neutrosophic T_1-space is also a neutrosophic T_0-space.

Example 3.2. Suppose that X = {x, y}. Let τ = {0_N, 1_N, {<x, 0.5, 0.5, 0.1>, <y, 0.7, 0.2, 0.3>}, { <x, 0.5, 0.5, 0.1>}{<y, 0.7, 0.2, 0.3>} be an NT on X. Clearly, (X, τ) is a neutrosophic T_1-space.

Theorem 3.2. Assume that $\xi : (X, \tau_1) \rightarrow (Y, \tau_2)$ be both one-one and neutrosophic continuous function from an NTS (X, τ_1) to another NTS (Y, τ_2). If (Y, τ_2) be an N-T_1-S, then (X, τ_1) is also an N-T_1-S.

Proof. Let (Y, τ_2) be an N-T_1-S. Also, let $x_{\alpha,\beta,\gamma}$, $y_{\theta,\lambda,\mu}$ ($x \neq y$) be any two NPs in X. Since, $\xi : (X, \tau_1) \rightarrow (Y, \tau_2)$ is a one-one function, so $\xi(x_{\alpha,\beta,\gamma})$, $\xi(y_{\theta,\lambda,\mu})$ are also distinct NPs in Y. Since, (Y, τ_2) is an N-T_1-S, so there exist two NOSs R, S in Y such that $\xi(x_{\alpha,\beta,\gamma}) \in R$, $\xi(x_{\alpha,\beta,\gamma}) \notin S$ or $\xi(y_{\theta,\lambda,\mu}) \notin R$, $\xi(y_{\theta,\lambda,\mu}) \in S$. Therefore, $x_{\alpha,\beta,\gamma} \in \xi^{-1}(R)$, $x_{\alpha,\beta,\gamma} \notin \xi^{-1}(S)$ or $y_{\theta,\lambda,\mu} \notin \xi^{-1}(R)$, $y_{\theta,\lambda,\mu} \in \xi^{-1}(S)$. Since, ξ is a neutrosophic continuous function, both $\xi^{-1}(R)$, $\xi^{-1}(R)$ are NOSs in X. Therefore, for any pair of distinct NPs $x_{\alpha,\beta,\gamma}$, $y_{\theta,\lambda,\mu}$ in X, there exist two NOSs $\xi^{-1}(R)$, $\xi^{-1}(S)$ such that $x_{\alpha,\beta,\gamma} \in \xi^{-1}(R)$, $x_{\alpha,\beta,\gamma} \notin \xi^{-1}(S)$ or $y_{\theta,\lambda,\mu} \notin \xi^{-1}(R)$, $y_{\theta,\lambda,\mu} \in \xi^{-1}(S)$. Hence, (X, τ_1) is an N-T_1-S.

Theorem 3.3. If an NTS (X, τ) is an N-T_1-S, then every NP in X is an NCS.

Proof. Suppose that (X, τ) is an N-T_1-S. Assume that $x_{\alpha,\beta,\gamma}$ is an arbitrary NP in X. Now, we can take an NP $y_{\theta,\lambda,\mu} \subseteq x^c_{\alpha,\beta,\gamma}$ ($y \neq x$) in X. Since, (X, τ) is an N-T_1-S, so there exist two NOSs R and S such that $x^c_{\alpha,\beta,\gamma} \in R$, $x^c_{\alpha,\beta,\gamma} \notin S$ and $y_{\theta,\lambda,\mu} \notin R$, $y_{\theta,\lambda,\mu} \in S$. Therefore, $x^c_{\alpha,\beta,\gamma} = \cup_{y_{\theta,\lambda,\mu} \subseteq x^c_{\alpha,\beta,\gamma}} \{R, S : x^c_{\alpha,\beta,\gamma} \in R, x^c_{\alpha,\beta,\gamma} \notin S$ and $y_{\theta,\lambda,\mu} \notin R, y_{\theta,\lambda,\mu} \in S\}$. Since, $\cup_{y_{\theta,\lambda,\mu} \subseteq x^c_{\alpha,\beta,\gamma}} \{R, S : x^c_{\alpha,\beta,\gamma} \in R, x^c_{\alpha,\beta,\gamma} \notin S$ and $y_{\theta,\lambda,\mu} \notin R, y_{\theta,\lambda,\mu} \in S\}$ is an NOS in X, so $x^c_{\alpha,\beta,\gamma}$ is an NOS in X. Hence, $x_{\alpha,\beta,\gamma}$ is an NCS in X.

Remark 3.1. Assume that (X, τ) is an NTS. Then, X is an N-T_1-S if and only if $x_{\alpha,\beta,\gamma} = \cap\{N_{cl}(R) : x_{\alpha,\beta,\gamma} \in N_{cl}(R)\}$.

Definition 3.3. An NTS (X, τ) is said to be a neutrosophic T_2-space (N-T_2-S) or neutrosophic Hausdorff space if for any pair of NPs $x_{\alpha,\beta,\gamma}$, $y_{\theta,\lambda,\mu}$ ($x \neq y$) in X, there exist two NOSs R and S such that $x_{\alpha,\beta,\gamma} \in R$, $x_{\alpha,\beta,\gamma} \notin S$ and $y_{\theta,\lambda,\mu} \notin R$, $y_{\theta,\lambda,\mu} \in S$ with $R \subseteq S^c$.

Obviously, every N-T_2-S is an N-T_1-S.

Example 3.3. Let X = {x, y, z} be a fixed set. Let τ = {0_N, 1_N, {<x, 0.5, 0.4, 0.7>, <y, 0.3, 0.4, 0.3>}, {<x, 0.5, 0.4, 0.7>, <z, 0.3, 0.5, 0.8>}, { <y, 0.3, 0.4, 0.3>, <z, 0.3, 0.5, 0.8>}, { <x, 0.5, 0.4, 0.7>} {<y, 0.3, 0.4, 0.3>}, { <z, 0.3,0.5,0.8>}} be an NT on X. Clearly, (X, τ) is an N-T_2-S.

Remark 3.2. In an N-T_2-S (X, τ), every NOS is an NCS.

Theorem 3.4. Assume that $\xi : (X, \tau_1) \rightarrow (Y, \tau_2)$ be both one-one and neutrosophic continuous function from an NTS (X, τ_1) to another NTS (Y, τ_2). If (Y, τ_2) is an N-T_2-S, then (X, τ_1) is also an N-T_2-S.

Proof. Since ξ is a neutrosophic continuous function, so inverse image of an NOS in (Y, τ_2) is also an NOS in (X, τ_1). Also, it is known that, the complement of NOS is NCS in an NTS. Here, since (Y, τ_2) is an N-T_2-S, so every NOS in (Y, τ_2) is also an NCS in (Y, τ_2).

Now, ξ is a neutrosophic continuous function

$\Rightarrow \xi(X) = Y$ is an NOS in τ_2.

$\Rightarrow \xi^{-1}(Y) = X$ is an NOS in τ_1.

Therefore, (Y, τ_2) is an N-T_2-S $\Rightarrow$ ($\xi^{-1}(Y)$, τ_1) is an N-T_2-S. Hence, (X, τ_1) is an N-T_2-S.

Theorem 3.5. Assume that (X, τ) is an N-T_1-S with the condition that complement of each NOS is also an NOS, then (X, τ) is an N-T_2-S.

Proof. Assume that (X, τ) is an N-T_1-S with the condition that complement of each NOS is also an NOS. That is for N, an NOS in (X, τ), $N^c = N$ (1)

Suppose that N is an NOS in (X, τ). Therefore, N^c is an NCS in (X, τ). Again, by equation (1), N^c is an NOS in (X, τ). So, $N^c = N$. Again $(N^c)^c = N^c = N$. Therefore, every NCS in (X, τ) is both NOS and NCS in (X, τ). Hence, by the Remark 3.2, (X, τ) is an N-T_2-S.

Definition 3.4. Assume that (X, τ) is an NTS. Then, X is called a neutrosophic regular-space if for any NP $x_{\alpha,\beta,\gamma}$ in X, and NCS Q with $x_{\alpha,\beta,\gamma} \in Q^c$, there exist two NOSs R and S such that $x_{\alpha,\beta,\gamma} \in R$, $Q \subseteq S$ and $R \subseteq S^c$.

Example 3.4. A zero-dimensional space (every finite open cover of the NT space has a refinement that is a finite open cover such that any NP point in the space is contained in exactly one NOS of this refinement.) with respect to the small inductive dimension has a base consisting of cl-open (NCS and NOS) sets. Every such space is neutrosophic regular-space.

Definition 3.5. An NTS (X, τ) is said to be a neutrosophic T_3-space (N-T_3-S) if it is an N-T_1-S and a neutrosophic regular space.

Obviously, every N-T_3-S is an N-T_2-S.

Example 3.5. The neutrosophic discrete topological space (X, τ) is a neutrosophic regular-space as well as N-T_1-S. Therefore, (X, τ) is an N-T_3-S.

Theorem 3.6. For any NTS (X, τ), the following results are equivalent:

(i) X is a neutrosophic regular-space.

(ii) For any NP $x_{\alpha,\beta,\gamma}$ and any NOS R containing $x_{\alpha,\beta,\gamma}$, there exists an NOS S such that $x_{\alpha,\beta,\gamma} \in S \subseteq N_{cl}(S) \subseteq R$.

Proof. (i)⇒(ii)

Suppose that (X, τ) iss a neutrosophic regular-space. So, for any NP $x_{\alpha,\beta,\gamma}$ in X, and an NCS Q with $x_{\alpha,\beta,\gamma} \in Q^c$, there exist two NOSs S and P such that $x_{\alpha,\beta,\gamma} \in S$, $Q \subseteq P$ and $S \subseteq P^c$.

Again, since R^c is an NCS so there exists an NCS H (say) such that $S \subseteq H$ and so $N_{cl}(S) \subseteq H$.

Again, for an NCS H there exists an NOS R such that $H \subseteq R$. Therefore, $x_{\alpha,\beta,\gamma} \in S \subseteq N_{cl}(S) \subseteq H \subseteq R$. This implies that, $x_{\alpha,\beta,\gamma} \in S \subseteq N_{cl}(S) \subseteq R$. Hence proved.

(ii)⇒(i)

The result is obvious for the neutrosophic regular space.

Definition 3.6. An NTS (X, τ) is said to be a neutrosophic normal-space if for any pair of NCSs G and H with $G \subseteq H^c$ in X, there exist two NOSs R and S in X such that $G \subseteq R$, $H \subseteq S$ and $R \subseteq S^c$.

Example 3.6. Let X = {x, y} be a fixed set. Let τ = {0_N, 1_N, {<x, 0.6, 0.4, 0.1>, <y, 0.7, 0.4, 0.3>}, {<x, 0.6, 0.4, 0.1>}, {<y, 0.7, 0.4, 0.3>}}. Consider the closed set = {0_N, 1_N, {<x, 0.4, 0.6, 0.9>, <y, 0.3, 0.6, 0.7>}, {<x, 0.4, 0.6, 0.9>}}. Clearly, (X, τ) is a neutrosophic normal-space.

Definition 3.7. An NTS (X, τ) is said to be a neutrosophic T_4-space (N-T_4-S) if it is both N-T_1-S and neutrosophic-normal-space.

Obviously, every N-T_4-S is also an N-T_1-S.

Example 3.7. To show the real-life example of separation axioms via NTS, we consider three department, namely, Mathematics = x, Physics = y, Chemistry = z of Tripura University. Based on different activities and departmental work, NAAC provides a degree of members as a neutrosophic set as given below:

{<x, 0.6, 0.3, 0.1>, <y, 0.7, 0.1, 0.3>, <z, 0.5, 0.3, 0.4>}. To analyze the comparison of different developmental work as well as future decision-making, we may consider different neutrosophic topological properties. Here, X = {x, y, z}, and consider the following NSs

A_1 = {<x, 0.6, 0.3, 0.1>}

A_2 = {<y, 0.7, 0.1, 0.3>}

A_3 = <z, 0.5, 0.3, 0.4>}.

A_4 = {<z, 0.5, 0.3, 0.4>, <y, 0.7, 0.1, 0.3>}

A_5= {<x, 0.6, 0.3, 0.1>, <z, 0.5, 0.3, 0.4>}

A_6= {<x, 0.6, 0.3, 0.1>, <y, 0.7, 0.1, 0.3>}

We consider, $\tau = \{0_N, 1_N, A_1, A_2, A_3, A_4, A_5, A_6\}$. Clearly, we can say that (X, τ) is a neutrosophic topological space, and it is an N-T_2-S.

Theorem 3.7. For any NTS (X, τ), the following results are equivalent:

(i) X is a neutrosophic normal-space.

(ii) For every NCS K and NOS U with $K \subseteq$ U, there exists an NOS V such that $K \subseteq V \subseteq N_{cl}(V) \subseteq U$.

Proof. (i)⇒(ii)

Assume that X is a normal-space neutrosophic. As a result, according to the concept of neutrosophic normal-space, there exist two NOSs U and V in X for any two NCSs K and H with K⊆ H^c in X, such that K⊆ U, H ⊆V, and U ⊆V^c.

For every NCS K and H with K⊆ H^c, we have K⊆ H^c ⇒K∩H=∅ (for any NCS K and H with K ⊆H^c).

Consider the NOS U that contains K and V that contains H, i.e., K⊆U and H⊆V. There are two NOS P and Q that are UP and VQ, where P and Q may or may not be disjoint NOS.

In terms of the first part, we have K⊆U and $\overline{U}$ ⊆P⇒ K⊆U ⊆ $\overline{U}$ ⊆P.

Assuming that U=V and P=U are both NOS, we obtain the following result.

(ii)⇒(i)

There exists an NOS V such that $K \subseteq V \subseteq N_{cl}(V) \subseteq U$ for any NCS K and NOS U with $K \subseteq$ U.

Now, it is necessary to demonstrate that X is a neutrosophic normal-space.

There are two NOSs U and V for any two NCSs K and H with K⊆H^c, such that K ⊆ U, H ⊆ V as K∩H=∅.

As a result, any two NCSs K and H with K⊆ H^c in X have two NOSs U and V, and we have K⊆ U, H ⊆V, and U ⊆V^c. As a result, X is a normal-space neutrosophic.

4. Conclusions:

In this study, we introduce the notion of neutrosophic T_i-spaces (i = 0, 1, 2, 3, 4) via NTS, and study their different properties. By defining neutrosophic T_i-spaces (i = 0, 1, 2, 3, 4), we prove some interesting results on neutrosophic separation axioms via NTSs. Further, we hope that, many new investigations can be done in the future based on the developed notions of neutrosophic separation axioms via NTSs. The notion of neutrosophic T_i-spaces (i = 0, 1, 2, 3, 4) can also be used for introducing the pairwise separation axioms under the neutrosophic bi-topological space. We further hope that the proposed theories can be explored in pentapartitioned neutrosophic set [17] environment.

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Neutrosophic Sociogram Approach to Neutrosophic Cognitive Maps in Swift Language

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Abstract: Neutrosophic Cognitive Maps (NCM) decision-making system encompasses sequential tasks of resolving the confrontations in finding the ideal solution. This paper introduces neutrosophic sociogram (NS) approach as an alternative to the conventional approach of NCM in finding the fixed point of the dynamical NCM system. A comparative analysis is made between two approaches and the efficiency of the proposed approach is validated with an application to find the interrelationship between the persuading eleven factors of Technopreneurship. It was observed that the results obtained using the proposed approach is more compatible, feasible and simple than the conventional approach in making decisions on the implications between the factors. The two approaches are programmed using SWIFT language to make the computations easier and the results are in consensus with the manual calculations.

Keywords: FCM; Neutrosophic sociogram; neutrosophic cognitive maps; Technopreneurship; SWIFT;

1. Introduction

Sociogram tools are significant in exploring the relationship between the members of the group. Jacob Levy Moreno made the first attempt in developing the sociogram techniques to investigate more on the interrelationship between the members and the factors persuading it. The extension of this classical sociogram to fuzzy sociogram was initiated to resolve the uncertainties prevailing in the social relationship. On profound study, it was found that the relationship involves indeterminacies inaddition to uncertainties and Gustavo et al [1] proposed the approach of the neutrosophic sociogram in group analysis. It was observed that neutrosophic approach has yielded more accurate results in enhancing the group dynamics. These sociogram approaches are predominantly applied to find the interrelationship between the members (m_i, i = 1 to n) of the group say G, where G = {m_1,m_2,..m_n}. The information on the interrelationship between the members is obtained using a questionnaire. In the neutrosophic sociogram (NS) approach, the most influential member is found and addressed as the leader of the group and the chances of strengthening the relationship between the members are also determined. The modalities of these approaches are in coherence with Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps. Kosko [2] introduced Fuzzy Cognitive maps (FCM) by incorporating the elements of fuzzy to the theory of Cognitive maps. FCM has extensive applications in various fields. To mention a few, Jason et al [3]developed FCM model to draw inferences on the impacts of learner's comprehension. Senniappan et al [4]proposed an FCM model to categorize concrete forms. Chrispen [5] applied FCM models to study the social aspects of livelihood. Abdollah [6] presented a review of different FCM models used in medicine.FCM decision-making models were constructed for making optimal decisions on agriculture by Makrinos et al [7] . Papageorgiou et al[8] developed a model for making decisions on environmental aspects and cotton yield. Song et al [9] Katarzyna et al [10] extended FCM models to make predictions, Antonie et al [11]made an attempt to extend FCM for further explorations. Papageorgiou et al [12] discussed the various methods and algorithms of FCM models. Felix et al [13] described the various software used for the computational purpose of FCM models.

FCM is a directed graph with nodes and arcs representing the decisive factors of the problems and their relationship respectively. The edge weights belong to {-1,0,1} states the influencing nature of the relationship between the factors. The value 1 denotes the positive kind of influence, -1 represents the negative kind of influence and the value 0 symbolizes null influence. The connection matrix of order n×n is derived from the FCM graphical representation of n factors. A general representation of FCM has nodes of the form $C_1,C_2...Cn$ and the instantaneous vector V of the form V= $(V_1,V_2,..V_n)$, where V_i takes the value 0 or 1 signifying ON or OFF position of the factors. For instance a decision-making problem on finding the factors causing gestational diabetics is considered with five factors say C1,C2,C3,C4,C5, here V = $(V_1\ V_2\ V_3\ V_4\ V_5)$ where vi takes the value 0 or 1 signifying the ON or OFF position of the factors if V= (1 0 0 0 0) then it signifies that the first factor is in ON position and other factors are in OFF position. FCM with a directed cycle formed by the edges has feedback and it becomes a dynamical system. The passing of this vector into the respective connection matrix M of order 5 × 5 results in another vector and the threshold function is used to update the values by assigning 1 to the values of the vector greater than 1, -1 to the values of the vector lesser than -1. The equilibrium state of the dynamical system is called the hidden pattern and it occurs when the vector with any of the factors kept in ON position is repeated after successive passing and threshold of the vectors. If the equilibrium state is a unique vector then it is called the fixed point and if it settles in the form of V1 →V2 →…V5 →V1 then it is termed as a limit cycle. Here Vi denotes the ON position of the i^{th} factor respectively and the OFF position of the remaining other factors. These are the underlying concepts of FCM involved in making optimal decisions.

Neutrosophic Cognitive maps (NCM) also involves the same aspects with the inclusion of indeterminacy. The edge weight set of NCM is {-1,0,*I*,1} and the elements of the instantaneous vector assume the values 0,1, *I*. NCM was first developed by Smarandache and Vasantha Kandasamy [14].NCM models are extended to combined overlap models and neutrosophic relational maps models for decision-making. Gaurav et al [15] used genetic algorithm in NCM models. NCM has extensive applications as FCM models in diagnosis [16], medicine[17], situational analysis[18,19], pandemic causative factors [20], decision-making [21-25],impact of imaginative play on children[26], religious impacts[27].Banerjee et al [28] compared FCM and NCM models and suggested NCM models be more compatible.NCM decision-making models are also developed to make optimal decisions on various dimensions of society, science and technology. On profound analysis, FCM and NCM model approaches appear to be similar to the approaches of Fuzzy and Neutrosophic Sociogram. FCM & NCM models determine the most influential factors and their interrelational impacts in which the latter considers indeterminacy, whereas Fuzzy and neutrosophic sociogram approaches determine the most influential person of the group and the extent of relational compatibility between the members of the group. The approach of Fuzzy sociogram was used in developing a new genre of FCM model by Jegan et al [29] to study the emotional intelligence of the students. FCM models with and without fuzzy sociogram approach were compared and the ranking of the factors in both the cases was the same and it was suggested to incorporate fuzzy sociogram approach in FCM model development. Based on this new sociogram approach in FCM models, this paper proposes the integration of the neutrosophic sociogram approach in the NCM model. In a NCM model, the indeterminacy between the factors is considered and the relational impacts between the factors are determined. The positive, negative or indeterminate influential status between the factors can be determined but there is no space to make a prediction on the extent of resolving indeterminate relational impacts between the factors. Also the indeterminacy between the factors on subjecting to computations gets retained as indeterminate values itself at certain instances. These are some of the constraints of decision making using NCM models. To overcome this shortcoming of NCM models, the neutrosophic sociogram approach shall be used as an alternative approach to the NCM model. The proposed approach is compared with the conventional NCM model. The efficiency of the two model approaches is tested by applying to the factors influencing Technopreneurship. SWIFT language is used to write coding for the two approaches to ease the computation and to draw the results instantly. The paper is structured as follows: Section 2 presents the origin and the development of the proposed NS integrated NCM approach; section 3 consists of the application of the proposed approach; section 4 discusses the results and the last section concludes the work.

2. Origin and Development of the Proposed Approach

This section presents the origin and the development of the proposedneutrosophic sociogram alternative approach to NCM.

A neutrosophic cognitive map is a neutrosophic directed graph (a directed graph with atleast one edge of indeterminate nature) with factors as nodes and their interrelationship as edges. The edge weight assume the values belonging to the set {-1,I,0,1}, where -1 indicated the negative interrelational impacts, I denotes the indeterminacy relation between the factors, 0 signifies void interrelational impacts and 1 represents the positive interrelational impacts.

Let us consider the factors contributing to environmental catastrophe, say E1,E2,E3,E4,E5. It is assumed that the following connection matrix (say M) is determined based on the questionnaire given to the experts.

M =

	E1	E2	E3	E4	E5
E1	0	1	0	I	0
E2	1	0	1	0	0
E3	I	1	0	1	0
E4	1	1	I	0	1
E5	0	0	1	1	0

Let us consider an instantaneous vector of the form V = (1 0 0 0 0), which states that the first factor E1 is in ON position and the other factors are in OFF position. The vector V is passed onto M and the resultant vector obtained is (0 1 0 I 0) and on updating the values the vector V, the new vector V1 = (1 1 0 I 0) is obtained.

$V \to V_1$, where $\to$ denotes the threshold of values (vi, i = 1,2..5) in a vector.

The threshold function T (x) is represented as

$$\begin{cases} 1 \; if \; v_i \geq 0 \\ 0 \; otherwise \end{cases}$$

On passing the vector V1 onto M again and repeating the steps as above the final vector obtained is (11111). The final vector thus obtained signifies the influence of the first factor over the other and it shows that the factor 1 is related to all other factors. The lastly obtained vector (11111) is called the fixed point or the limit cycle.

Suppose if the final vector obtained is of the form (1 1 0 I 0), it signifies that the first factor has positive influence over the second factor, null influences on third and fifth factors and indeterminate influence on the fourth factor. If such kinds of fixed points are obtained on keeping the factors in ON position, then the holistic decision on the influences between the factors cannot be made. The indeterminate influences remains as such and there is no scope for the possibilities of alleviating such indeterminacy.

Let us consider the same with two experts

Expert-I

	E1	E2	E3	E4	E5
E1	0	1	0	I	0
E2	1	0	1	0	0
E3	I	1	0	1	0
E4	1	1	I	0	1
E5	0	0	1	1	0

Expert II

	F1	F2	F3	F4	F5
F1	0	1	1	I	0
F2	1	0	I	0	0
F3	I	1	0	0	0
F4	I	1	I	0	1
F5	0	I	1	1	0

The combined connection matrix is

	F1	F2	F3	F4	F5
F1	0	1	0.5	2I	0
F2	1	0	0.5+I	0	I
F3	2I	1	0	0.5	0
F4	0.5+I	1	2I	0	1
F5	0	2I	1	1	0

On repeating the same NCM procedure to the above matrix with the initial step of keeping the first factor E1 in ON position we obtain the same fixed point (11111). This is one of the shortcomings of NCM . To handle the limitations of NCM, the neutrosophic sociogram approach shall be used as an alternative approach to the conventional method of NCM.

Neutrosophic sociogram approach aims in determining the social dynamics of the group. It considers the members of the group and the interest of the members in working or getting along with other members are determined based on questionnaire. Sometimes the responses are deterministic in nature and sometimes they are indeterminate. The neutrosophic sociogram approach enables to arrive at a conclusion of finding the possibilities of alleviating the indeterminacy along with the numerical range of extent. It also facilitates to find the leader of the group (the most preferred person). The neutrosophic sociogram as discussed by [] presents vividly the sequential steps and mathematical formulation of the decision-making model with a hypothetical example. On intense study NCM approach can be aligned in line with NS approach as both intends to find the associational impacts.

In NCM with NS approach, the factors of the group are considered as the members and based on the expert's opinion the deterministic and the indeterminate associational impact or influence between the factors is determined. The generic tabular representation is as follows.

	E1	**E2**	**…………….**	**Em**
Factors				
F1	$F^D_{11};F^N_{11}$	$F^D_{12};F^N_{12}$		$F^D_{1m};F^N_{1m}$
F2	$F^D_{21};F^N_{21}$	$F^D_{22};F^N_{22}$		$F^D_{2m};F^N_{2m}$
.	……….	………..	…………………..	………

.	……….	………..	…………………..	……….
Fn	F^D_{n1};F^N_{n1}	F^D_{n2};F^N_{n2}		F^D_{nm};F^N_{nm}

F^Dij⊂ F, where F is the set of factors (i = 1,2,..n), (j = 1,2,..m), the response represents the factors (Fi) of deterministic association in expert's (Ej) point of view where F^Nij represents the indeterminate associational response from the expert. Let us apply the proposed procedure to the above example of NCM with Factors Fi, i = 1,2,..5.

Let us consider the connection matrix given by two experts

Expert-I

	F1	F2	F3	F4	F5
F1	0	1	0	I	0
F2	1	0	1	0	I
F3	I	1	0	1	0
F4	1	1	I	0	1
F5	0	I	1	1	0

Expert II

	F1	F2	F3	F4	F5
F1	0	1	1	I	0
F2	1	0	I	0	0
F3	I	1	0	0	0
F4	I	1	I	0	1
F5	0	I	1	1	0

The fuzzy amicable degree f_{ij} and the neutrosophic amicable degree n_{ij} are determined by the below equations respectively together with the consideration of weight of the experts and the evaluation matrices of the experts.

$$\frac{2}{f_{ij}} = \frac{1}{d_{ij}} + \frac{1}{d_{ji}}$$

$$\frac{2}{n_{ij}} = \frac{1}{h_{ij}} + \frac{1}{h_{ji}}$$

here d_{ij} and h_{ij} denotes the deterministic and neutrosophic associations between the factors i and respectively.

The significance of the factor with fuzzy amicable degree shall be determined by using the following index

$$S\,(F_i) = \frac{\sum_j f_{ij}}{\sum_i \sum_j f_{ij}}$$

The competency of the factor together with neutrosophic amicable degree shall be determined by using the following index

$$C\,(F_i) = \frac{\sum_j n_{ij}}{\sum_i \sum_j n_{ij}}$$

The representation of the above two connection matrices in the newly proposed NS integrated NCM approach.

	Expert I	**Expert II**
F1	**F2;F4**	**F2,F3;F4**

F2	**F1,F3;F5**	**F1;F3**
F3	**F2,F4;F1**	**F2;F1**
F4	**F1,F2,F5;F3**	**F2,F5;F1,F3**
F5	**F3,F4;F2**	**F3,F4;F2**

Fuzzy Amicable degree Matrix

	F1	F2	F3	F4	F5
F1	0	1	0	0	0
F2	1	0	0.67	0	0
F3	0	0.67	0	0	0
F4	0	0	0	0	1
F5	0	0	0	1	0

Neutrosophic Amicable Degree Matrix

	F1	F2	F3	F4	F5
F1	0	1	0.67	1	0
F2	1	0	1	0	0.67
F3	0.5	1	0	0.67	0
F4	1	0	0.67	0	1
F5	0	0.67	0	1	0

F1-F2	[1,1]	F2-F3	[0.67,1]	F3-F4	[0,0.67]	F4-F5	[1,1]
F1-F3	[0,0.67]	F2-F4	[0,0]	F3-F5	[0,0]		
F1-F4	[0,1]	F2-F5	[0,0.67]				
F1-F5	[0,0]						

The values [1,1] and [0,0] indicates the strong influence and void influence between the factors and the other range of values signifies the possibilities of increasing the association between the factors. For instance the first factor has no influence on fifth factor, strong influence on second factor and the extent of influence over the factors third and fourth factor is determined by the given range of values. [Also F1-F2 is same as F2-F1]. This is a better result than the fixed point obtained from the conventional procedure of NCM.

3. Application of NS integrated NCM approach

This section presents the validation of the proposed approach to the decision-making on the factors influencing Technopreneurship. The decision-making environment is characterized by eleven factors say F1,F2,….F11.These factors are obtained from the experts in the field of Business administration and the respective stakeholders through a questionnaire. The initial input is presented in the table below

		Expert –I	Expert-II	Expert-III	Expert-IV
F1	Individual characteristic	F2,F3,F5,F6,F8,F10; F4	F2,F3,F4,F5,F8; F6,F10	F2,F3,F5,F6,F8,F10; F4,F7	F2,F3,F4,F5,F6,F7,F8,F9; F10

	factor				
F2	Motivation factor	F1,F6,F9,F10,F11; F3.	F1,F3,F8,F9,F10,F11; F5,F6	F1,F3,F6,F8,F10; F11	F3,F4,F5,F6,F8,F9; F7,F11
F3	Situational factors	F5,F6,F7,F9; F2,F4	F4,F6,F9,F10,F11; F7,F8	F10,F11; F1,F4	F1,F2,F5,F7,F8,F9,F10; F4
F4	Exogenous factors	F1; F3,F9	F1,F2,F3,F6,F9; F8,F5	F1,F5,F11;F3,F7	F1,F2,F3,F5,F8,F9,F10,F11; F6
F5	Social Factors	F9,F10,F11; F3,F6,F7	F7,F9; F10,F11	F10,F11; F1,F9	F1,F3,F4,F8,F9,F10,F11; F2,F6
F6	Financial factor	F10,F11; F7,F9	F2,F4,F10,F11; F3,F8	F3,F9,F10,F11; F1,F8	F1,F2,F7,F8,F10,F11; F3,F9
F7	Non-Financial Assistance factor	F2,F8,F9; F6,F10	F3,F6,F10,F11; F2,F9	F1,F2,F10; F6,F11	F2,F5,F8,F10; F1,F9
F8	Entrepreneurial and business skills factor	F1,F2,F3,F9; F4.	F1,F2,F3,F4,F5,F6, F8,F9,F10,F11; F7	F1,F2,F3,F5,F6,F7 F8,F9,F10,F11; F4	F1,F2,F3,F4,F6,F7,F9,F10,F11; F5
F9	Cultural factors	F1,F2,F4; F6,F8	F1,F2,F8; F4,F7	F1,F3,F5,F6; F4,F10	F1,F2,F3,F4,F5,F7,F8; F10,F11
F10	Socioeconomic conditions factor	F3,F6,F11; F1,F9	F1,F2,F3,F11; F5,F8	F1,F4,F5,F6,F11; F3.	F1,F2,F3,F4,F5,F7,F8,F11; F6
F11	Government policies and procedures factor	F2,F10; F1,F9	F2,F3,F7,F10; F6,F8	F1,F2,F4,F6,F10; F8,F9	F2,F3,F5,F6,F7,F8,F10; F1,F4

The evaluation matrix of each experts is as follows

Expert I	**F1**	**F2**	**F3**	**F4**	**F5**	**F6**	**F7**	**F8**	**F9**	**F10**	**F11**		**Expert II**	**F1**	**F2**	**F3**	**F4**	**F5**	**F6**	**F7**	**F8**	**F9**	**F10**	**F11**
F1	0	1	1	0	1	1	0	1	0	1	0		**F1**	0	1	1	1	1	0	0	1	0	0	0
F2	1	0	0	0	0	1	0	0	1	1	1		**F2**	1	0	1	0	0	0	0	1	1	1	1
F3	0	0	0	0	1	1	1	0	1	0	0		**F3**	0	0	0	1	0	1	0	0	1	1	1
F4	1	0	0	0	0	0	0	0	0	0	0		**F4**	1	1	1	0	0	1	0	0	1	0	0
F5	0	0	0	0	0	0	0	0	1	1	1		**F5**	0	0	0	0	0	0	1	0	1	0	0
F6	0	0	0	0	0	0	0	0	0	1	1		**F6**	0	1	0	1	0	0	0	0	0	1	1

F7	0	1	0	0	0	0	0	1	1	0	0		**F7**	0	0	1	0	0	1	0	0	0	1	1
F8	1	1	1	0	0	0	0	0	1	0	0		**F8**	1	1	1	1	1	1	0	0	1	1	1
F9	1	1	0	1	0	0	0	0	0	0	0		**F9**	1	1	0	0	0	0	0	1	0	0	0
F10	0	0	1	0	0	1	0	0	0	0	1		**F10**	1	1	1	0	0	0	0	0	0	0	1
F11	0	1	0	0	0	0	0	0	0	1	0		**F11**	0	1	1	0	0	0	1	0	0	1	0

Expert III	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11		Expert IV	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11
F1	0	1	1	0	1	1	0	1	0	1	0		F1	0	1	1	1	1	1	1	1	1	0	0
F2	1	0	1	0	0	1	0	1	0	1	0		F2	0	0	1	1	1	1	0	1	1	0	0
F3	0	0	0	0	0	0	0	0	0	1	1		F3	1	1	0	0	1	0	1	1	1	1	0
F4	1	0	0	0	1	0	0	0	0	0	1		F4	1	1	1	0	1	0	0	1	1	1	1
F5	0	0	0	0	0	0	0	0	0	1	1		F5	1	0	1	1	0	0	0	1	1	1	1
F6	0	0	1	0	0	0	0	0	1	1	1		F6	1	1	0	0	0	0	1	1	0	1	1
F7	1	1	0	0	0	0	0	0	0	1	0		F7	0	1	0	0	1	0	0	1	0	1	0
F8	1	1	1	0	1	1	1	0	1	1	1		F8	1	1	1	1	0	1	1	0	1	1	1
F9	1	0	1	0	1	1	0	0	0	0	0		F9	1	1	1	1	1	0	1	1	0	0	0
F10	1	0	0	1	1	1	0	0	0	0	1		F10	1	1	1	1	1	0	1	1	0	0	1
F11	1	1	0	1	0	1	0	0	0	1	0		F11	0	1	1	0	1	1	1	1	0	1	0

The final fuzzy amicable degrees representing the associational impacts between the factors

	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11
F1	0	.86	.4	.67	.4	.38	.25	1	.4	.6	0
F2	.86	0	.38	0.33	0	0.6	0	.86	.75	.5	.67
F3	.4	.38	0	0.33	.33	.33	.33	.4	0.6	.75	.5
F4	.67	0.33	.33	0	.33	.25	0	.33	.5	.33	.33
F5	.4	0	.33	.33	0	0	.25	.33	.6	.6	.38
F6	.38	0.6	.33	.25	0	0	.25	.38	.25	.67	.67
F7	.25	0	.33	0	.25	.25	0	.5	.25	.38	.33
F8	1	.86	.4	.33	.33	.38	.5	0	.67	0.38	.38
F9	.4	.75	.6	.5	.6	.25	.25	.67	0	0	0

F10	.6	.5	.75	.33	.6	.67	.38	.38	0	0	1
F11	0	.67	.5	.33	.38	.67	.33	.38	0	1	0

The final matrix representing the neutrosophic amicable degrees

	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11
F1	0	.86	.67	1	.67	.67	.5	1	.4	1	0
F2	.86	0	.67	.33	.33	.67	.4	.86	.75	.6	.86
F3	.67	.67	0	1	.5	.6	.38	.67	.6	.86	.5
F4	1	.33	1	0	.38	.25	0	.67	.86	.33	.5
F5	.67	.33	.5	.38	0	0	0.25	.38	.67	.86	.4
F6	.67	.67	.6	.25	0	0	.6	.75	.6	.86	.86
F7	.5	.4	.38	0	0.25	.6	0	.6	.6	.67	.6
F8	1	.86	.67	.67	.38	.75	.6	0	.86	.6	.6
F9	.4	.75	.6	.86	.67	.6	.6	.86	0	.5	.5
F10	1	.6	.86	.33	.86	.86	.67	.6	.5	0	1
F11	0	.86	.5	.5	.4	.86	.6	.6	.5	1	0

Associational Impact between the factors

F1-F2	[0.86,0.86]	F2-F3	[.38,.67]	F3-F4	[.33,1]	F4-F5	[.33,.38]	F5-F6	[0,0]
F1-F3	[.4,.67]	F2-F4	[.33,.33]	F3-F5	[.33,.5]	F4-F6	[.25,.25]	F5-F7	[0.25,.25]
F1-F4	[.67,1]	F2-F5	[0,.33]	F3-F6	[.33,.6]	F4-F7	[0,0]	F5-F8	[.33,.38]
F1-F5	[.4,.67]	F2-F6	[.6,.67]	F3-F7	[.33,.38]	F4-F8	[.33,.67]	F5-F9	[.6,.67]
F1-F6	[.38,.67]	F2-F7	[0,.4]	F3-F8	[.4,.67]	F4-F9	[.5,.6]	F5-F10	[.6,.86]
F1-F7	[.25,.5]	F2-F8	[.86,.86]	F3-F9	[.6,.6]	F4-F10	[.33,.33]	F5-F11	[.38,.4]
F1-F8	[1,1]	F2-F9	[.75,.75]	F3-F10	[.75,.86]	F4-F11	[.33,.5]		
F1-F9	[.4,.4]	F2-F10	[.5,.6]	F3-F11	[.5,.5]				
F1-F10	[.6,1]	F2-F11	[.67,.86]						

F1-F11	[0,0]

F6-F7	[.25,.6]	F7-F8	[.5,.6]	F8-F9	[.67,.86]	F9-F10	[0,.5]	F10-F11	[1,1]
F6-F8	[.38,.75]	F7-F9	[.25,.6]	F8-F10	[.38,.6]	F9-F11	[0,.5]		
F6-F9	[.25,.6]	F7-F10	[.38,.67]	F8-F11	[.38,.6]				
F6-F10	[.67,.86]	F7-F11	[.33,.6]						
F6-F11	[.67,.86]								

To ease the computation SWIFT language is used for programming the NS integrated NCM approach and the coding is presented in Appendix. The following figures represent the input data and output data.

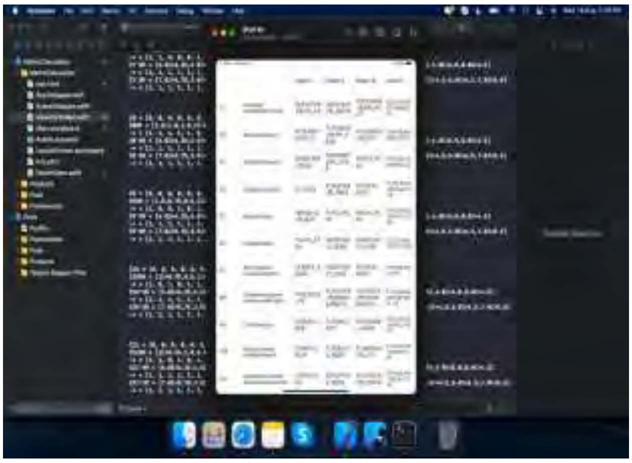
Fig 3a

Fig 3b

Fig 3c

Fig 3d

Fig 3a, 3b, 3c and 3d represents the input data in SWIFT language.



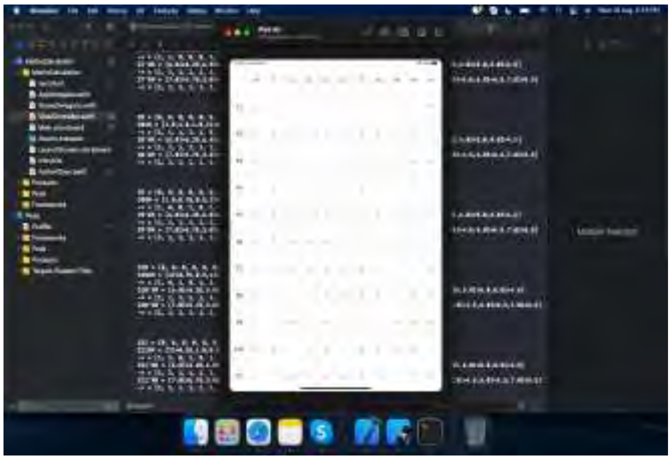
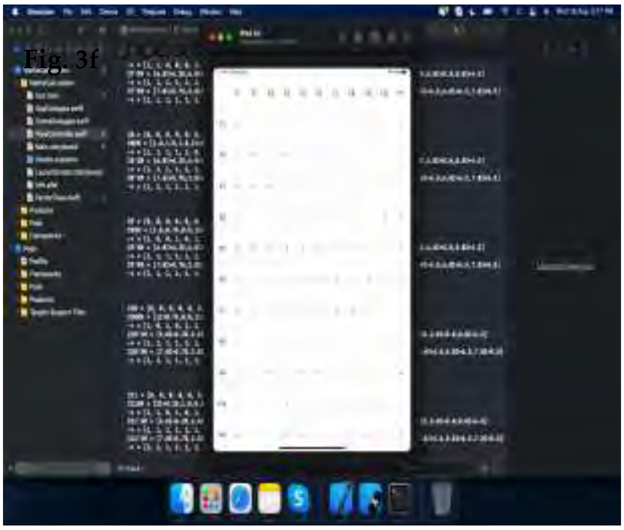


Fig. 3e **Fig. 3f**



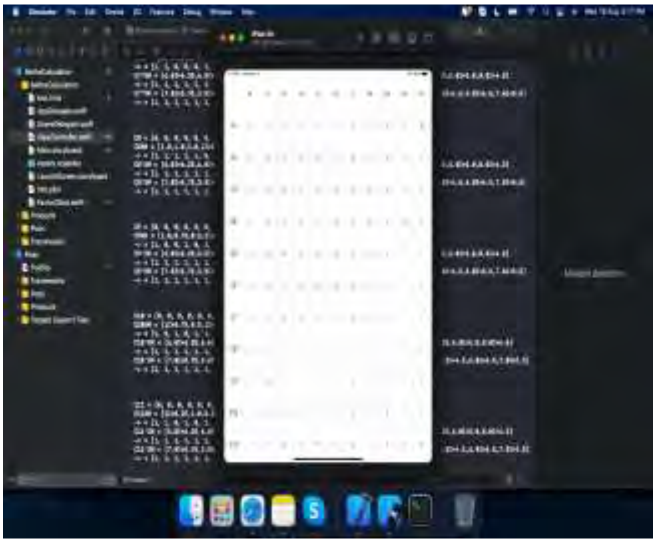


Fig. 3g **Fig. 3h**

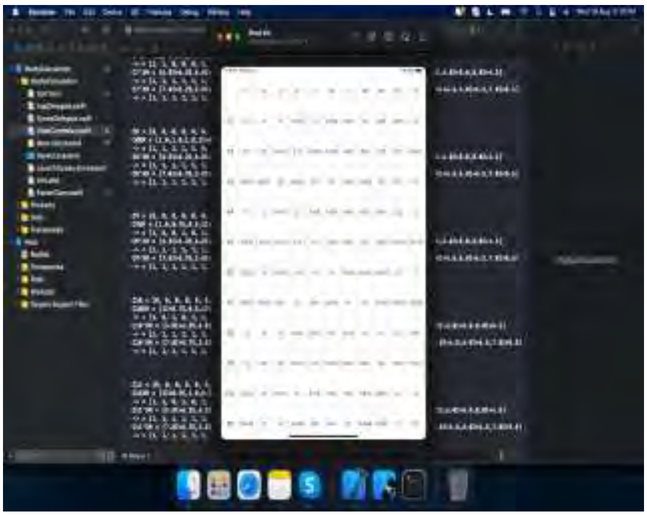


Fig 3i

Fig 3e, 3f, 3g, 3h and 3i represents the output of the associational impact values.

4. Discussion

The associational impact values between the factors clearly presents the extent of influence of one factor over the other , some of the values are deterministic in sense a real number that indicates the exact numerical value of influence and the range of values present the extent of influence between the factors and also it shows the existing possibilities of enhancing the influences between the factors for the holistic growth and development of the Technopreneurs and Technopreneurship. The results vividly shows that NS integrated NCM approach is highly accommodative in nature.

On applying the conventional NCM approach

Table 4.1. NCM Conventional approach results

On Position of the Factors	Fixed Point
(10000000000)	(11111111111)
(01000000000)	(11111111111)
(00100000000)	(11111111111)
(00010000000)	(11111111111)
(00001000000)	(11111111111)
(00000100000)	(11111111111)
(00000010000)	(11111111111)
(00000001000)	(11111111111)
(00000000100)	(11111111111)
(00000000010)	(11111111111)
(00000000001)	(11111111111)

The results obtained from conventional NCM approach (Table 4.1) has no provision for making specific analysis, the results shows that each factor influences others in a more general manner, but the actual extent of influence is not explored from the above obtained fixed points. For instance the fixed point obtained on keeping the first factor at ON position (10000000000) signifies that the first factor has impact on all the factors, but whereas the proposed approach gives the specific range of the associational impact between the factor F1 and all other factors say F2,F3,F4,F5,F6,F7,F8,F9,F10,F11. Thus the proposed approach is more efficient than the existing approach.

5. Conclusion

This paper has proposed Neutrosophic Sociogram integrated NMC approach as an alternative to the conventional NCM. The proposed approach facilitates to determine the specific associational impact between the factors rather in general manner as in the conventional method. The proposed decision-making alternative approach is highly compatible, flexible and simple in comparison with the existing conventional approach. This research work is an initiative to develop new approaches of finding the impact between the factors and this same approach shall be extended to Plithogenic Sociogram and Plithogenic cognitive maps (PCM). Just as FCM, NCM, PCM models shall be integrated with plithogenic sociogram approach.

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Appendix

NS INTEGRATED NCM CODING INPUT

```
//
//  ViewController.swift
//  MathsCalculation
//
//  Created by Hitasoft on 14/06/21.
//

import UIKit
import PDFGenerator
class ViewController: UIViewController { @IBOutlet weak var tableView: UITableView!
var headerArray = ["", "", "Expert –I", "Expert-II", "Expert-III", "Expert-IV"]
var factorsArray = ["Individual characteristic factor", "Motivation factor", "Situational factors", "Exogenous factors", "Social Factors", "Financial factor", "Non-Financial Assistance factor", "Entrepreneurial and business skills factor", "Cultural factors", "Socioeconomic conditions factor", "Government policies and procedures factor"]
// MARK: Expert I, Expert II, Expert III, Expert IV
var FFactors = [["F1", "F2", "F3", "F4", "F5", "F6", "F7", "F8", "F9", "F10", "F11"]]
var F1Factors = [["F2","F3","F5","F6","F8","F10", ";", "F4"],
["F2","F3","F4","F5","F8", ";", "F6", "F10"],
["F2","F3","F5","F6","F8","F10", ";", "F4","F7"],
["F2","F3","F4","F5","F6","F7","F8","F9",";","F10"]]
```

```
var F2Factors = [["F1","F6","F9","F10","F11",";","F3"],
["F1","F3","F8","F9","F10","F11",";","F5","F6"],
["F1","F3","F6","F8","F10",";","F11"],
["F3","F4","F5","F6","F8","F9",";","F7","F11"]]
var F3Factors = [["F5","F6","F7","F9",";","F2","F4"],
["F4","F6","F9","F10","F11",";","F7","F8"], ["F10","F11",";","F1","F4"],
["F1","F2","F5","F7","F8","F9","F10",";","F4"]]
var F4Factors = [["F1",";","F3","F9"], ["F1","F2","F3","F6","F9",";","F8","F5"],
["F1","F5","F11",";","F3","F7"], ["F1","F2","F3","F5","F8","F9","F10","F11",";", "F6"]]
var F5Factors = [["F9","F10","F11",";","F3","F6","F7"], ["F7","F9",";","F10","F11"],
["F10","F11",";","F1","F9"],
["F1","F3","F4","F8","F9","F10","F11",";" ,"F2","F6"]] var F6Factors = [["F10","F11",";","F7","F9"],
["F2","F4","F10","F11",";","F3","F8"],
["F3","F9","F10","F11",";","F1","F8"],
["F1","F2","F7","F8","F10","F11",";","F3","F9"]]
var F7Factors = [["F2","F8","F9",";","F6","F10"],
["F3","F6","F10","F11",";","F2","F9"], ["F1","F2","F10",";","F6","F11"], ["F2","F5","F8","F10",";","F1","F9"]]
var F8Factors = [["F1","F2","F3","F9",";","F4"],
["F1","F2","F3","F4","F5","F6", "F8","F9","F10","F11",";","F7"],["F1","F2","F3","F5","F6","F7",
"F8","F9","F10","F11",";","F4"], ["F1","F2","F3","F4","F6","F7","F9","F10","F11",";","F5"]]
var F9Factors = [["F1","F2","F4",";","F6","F8"], ["F1","F2","F8",";","F4","F7"],
["F1","F3","F5","F6",";","F4","F10"],
["F1","F2","F3","F4","F5","F7","F8",";", "F10","F11"]] var F10Factors = [["F3","F6","F11",";","F1","F9"],
["F1","F2","F3","F11",";","F5","F8"],
["F1","F4","F5","F6","F11",";","F3"],
["F1","F2","F3","F4","F5","F7","F8","F11",";","F6"]] var F11Factors = [["F2","F10",";","F1","F9"],
["F2","F3","F7","F10",";","F6","F8"],
["F1","F2","F4","F6","F10",";","F8","F9"],
["F2","F3","F5","F6","F7","F8","F10",";","F1","F4"]]

var factorArray = [[[String]]]() var finalArray = [String: String]() var tableTotalValue = Double(0)
override func viewDidLoad() { super.viewDidLoad() self.configUI()
// Do any additional setup after loading the view.
}
func configUI() {
```

```
factorArray.append(F1Factors) factorArray.append(F2Factors) factorArray.append(F3Factors)
factorArray.append(F4Factors) factorArray.append(F5Factors) factorArray.append(F6Factors)
factorArray.append(F7Factors) factorArray.append(F8Factors) factorArray.append(F9Factors)
factorArray.append(F10Factors) factorArray.append(F11Factors)
self.tableView.rowHeight = UITableView.automaticDimension
self.tableView.estimatedRowHeight = 45 self.tableView.sectionHeaderHeight =
UITableView.automaticDimension self.tableView.estimatedSectionHeaderHeight = 45
//    for row in 0..<11 {
//    for factor in 0..<factorArray.count {
//    // MARK: Calculate Expert I, Expert II, Expert III, Expert IV Value
//    finalArray["F\(row+1)\(factor+1)"] = (factorArray[row][0].contains("F\(factor+1)") ? 0.25 :
0)+(factorArray[row][1].contains("F\(factor+1)") ? 0.25 :
0)+(factorArray[row][2].contains("F\(factor+1)") ? 0.25 :
0)+(factorArray[row][3].contains("F\(factor+1)") ? 0.25 : 0)
//    }
//    }
for row in 0..<11 {
for factor in 0..<factorArray.count {
let factor1String = factorArray[row][0].split(separator: ";")
let factor2String = factorArray[row][1].split(separator: ";")
let factor3String = factorArray[row][2].split(separator: ";")
let factor4String = factorArray[row][3].split(separator: ";")
// MARK: Calculate Expert I, Expert II, Expert III, Expert IV Value
let FFirstVal = (((factor1String.first?.contains("F\(factor+1)") ?? false)
? 0.25 : 0) +
((factor2String.first?.contains("F\(factor+1)") ?? false)
? 0.25 : 0) +
((factor3String.first?.contains("F\(factor+1)") ?? false)
? 0.25 : 0) +
((factor4String.first?.contains("F\(factor+1)") ?? false)
? 0.25 : 0))
let FLastVal = ((factor1String.last?.contains("F\(factor+1)") ?? false) ?
1 : 0) + ((factor2String.last?.contains("F\(factor+1)") ?? false) ? 1 : 0) +
((factor3String.last?.contains("F\(factor+1)") ?? false) ?
1 : 0) +    ((factor4String.last?.contains("F\(factor+1)")
?? false) ? 1 : 0)
```

```
finalArray["F\(row+1)\(factor+1)"] = FLastVal == 0 ? "\(FFirstVal)" : "\(FLastVal)I\(FFirstVal == 0 ?
"" : "+\(FFirstVal)")"
//    print(finalArray["F\(row+1)\(factor+1)"] ?? "")
}
//    print("\n")
}
var totalRowArray = [String]() var totalColumnArray = [String]() for row in 0..<11 {
var c = [0,0,0,0,0,0,0,0,0,0,0]
c[row] = 1

var RowITotal = Double(0) var RowTotal = Double(0)
var ColumnITotal = Double(0) var ColumnTotal = Double(0)
for factor in 0..<factorArray.count {
// MARK: The fuzzy comatibility degree calculation
var rowTot = Double(0) var columnTot = Double(0)
if finalArray["F\(row+1)\(factor+1)"]?.contains("I") ?? false {
let component = finalArray
["F\(row+1)\(factor+1)"]?.components(separatedBy: "+") RowITotal = RowITotal+1
//            print("Component = \(component)") if (component?.count ?? 0) > 1 {
rowTot = (Double(component?.last ?? "0") ?? 0)
}
}
else {
//    print("Component =
\(Double(finalArray["F\(row+1)\(factor+1)"] ?? "0") ?? 0)")
rowTot = (Double(finalArray["F\(row+1)\(factor+1)"] ?? "0") ?? 0)
}

if finalArray["F\(factor+1)\(row+1)"]?.contains("I") ?? false {
let component = finalArray
["F\(factor+1)\(row+1)"]?.components(separatedBy: "+")
ColumnITotal = ColumnITotal+1 if (component?.count ?? 0) > 1 {
columnTot = (Double(component?.last ?? "0") ?? 0)
}
}
else {
```

```
columnTot = (Double(finalArray["F\(factor+1)\(row+1)"]
?? "0") ?? 0)
}
if !rowTot.isNaN {
RowTotal = RowTotal+rowTot
}
if !columnTot.isNaN {
ColumnTotal = ColumnTotal+columnTot
}
}

let totalRow = "\(RowITotal > Double(0) ? "\(RowITotal)I" : "")\(RowTotal > Double(0) ?
"+\(RowTotal)" : "")"
totalRowArray.append(totalRow)

let totalColumn = "\(ColumnITotal > Double(0) ? "\(ColumnITotal)I" : "")\(ColumnTotal >
Double(0) ? "+\(ColumnTotal)" : "")"
totalColumnArray.append(totalColumn)
}

for row in 0..<11 {
var totalRow = [Int]() var totalColumn = [Int]() var fArray = [String]() var totArray = [Int]()
print("\n\n")

var c = [0,0,0,0,0,0,0,0,0,0,0]
c[row] = 1 print("\tC\(row+1) = \(c)")
for factor in 0..<factorArray.count { if row == 0 && factor == 10 {
fArray.append("0")
totArray.append(0)
}
else {
fArray.append(finalArray["F\(row+1)\(factor+1)"] ?? "") let text =
((finalArray["F\(row+1)\(factor+1)"] ??
"").contains("I")) ? 1 : ((Double(finalArray["F\(row+1)\(factor+1)"] ?? "") ?? 0)>=1 ? 1 : 0)
let finalText = c[factor]+text totArray.append(finalText)
```

```
}
}
print(("\tC\(row+1)XM = [\(fArray.joined(separator: ","))]")) print(("\t-> = \(totArray)"))
for factor in 0..<totalRowArray.count {
let text = (totalRowArray[factor].contains("I")) ? 1 : ((Double(totalRowArray[factor]) ?? 0)>0 ? 1 : 0)
let finalText = c[factor]+text>=1 ? 1 : 0
totalRow.append(finalText)
}
print(("\tC\(row+1)'XM = [\(totalRowArray.joined(separator: ","))]"))
print(("\t-> = \(totalRow)"))
for factor in 0..<totalColumnArray.count {
let text = (totalColumnArray[factor].contains("I")) ? 1 : ((Double(totalColumnArray[factor]) ?? 0)>0 ?
1 : 0)
let finalText = (c[factor]+text)>=1 ? 1 : 0
totalColumn.append(finalText)
}
print(("\tC\(row+1)'XM = [\(totalColumnArray.joined(separator: ","))]"))
print(("\t-> = \(totalColumn)"))


}
}
func generatePDF() {
guard let v1 = self.tableView else { return }
//          v1.contentSize = CGSize(width: 100.0, height: 200.0) let dst = URL(fileURLWithPath:
NSTemporaryDirectory().appending("FinalFactorCalculation.pdf"))
// writes to Disk directly. do {
try PDFGenerator.generate(v1, to: dst)
} catch (let error) { print(error)
}
}
}
extension ViewController: UITableViewDelegate, UITableViewDataSource { func tableView(_
tableView: UITableView, numberOfRowsInSection section:
Int) -> Int {
return 12
}
```

```
func tableView(_ tableView: UITableView, heightForRowAt indexPath: IndexPath) -> CGFloat {
return self.view.frame.height/12
}
func numberOfSections(in tableView: UITableView) -> Int { return 8
}
func tableView(_ tableView: UITableView, heightForHeaderInSection section: Int) -> CGFloat {
return 45
}
func tableView(_ tableView: UITableView, titleForHeaderInSection section: Int) -> String? {
if section == 0 {
return ""
}
else if section == 1 { return "Expert - I"
}
else if section == 2 { return "Expert - II"
}
else if section == 3 { return "Expert - III"
}
else if section == 4 { return "Expert - IV"
}
else if section == 5 { return "Table 5"
}
else if section == 6 { return ""
}
return "Final Scores"
}
func tableView(_ tableView: UITableView, cellForRowAt indexPath: IndexPath) ->
UITableViewCell {
if indexPath.section == 0 {
let cell = tableView.dequeueReusableCell(withIdentifier: "FactorTableViewCell") as!
FactorTableViewCell
if indexPath.row == 0 { cell.expert1Label.text = "Expert –I" cell.expert2Label.text = "Expert-II"
cell.expert3Label.text = "Expert-III" cell.Expert4Label.text = "Expert-IV"
}
else if indexPath.row == 1 {
cell.titleLabel.text = "F1"
```

```
cell.factorLabel.text = "Individual characteristic factor" cell.expert1Label.text =
F1Factors[0].joined(separator: ",") cell.expert2Label.text = F1Factors[1].joined(separator: ",")
cell.expert3Label.text = F1Factors[2].joined(separator: ",") cell.Expert4Label.text =
F1Factors[3].joined(separator: ",")
}
else if indexPath.row == 2 { cell.titleLabel.text = "F2" cell.factorLabel.text = "Motivation factor"
cell.expert1Label.text = F2Factors[0].joined(separator: ",") cell.expert2Label.text =
F2Factors[1].joined(separator: ",") cell.expert3Label.text = F2Factors[2].joined(separator: ",")
cell.Expert4Label.text = F2Factors[3].joined(separator: ",")


}
else if indexPath.row == 3 { cell.titleLabel.text = "F3" cell.factorLabel.text = "Situational factors"
cell.expert1Label.text = F3Factors[0].joined(separator: ",") cell.expert2Label.text =
F3Factors[1].joined(separator: ",") cell.expert3Label.text = F3Factors[2].joined(separator: ",")
cell.Expert4Label.text = F3Factors[3].joined(separator: ",")


}
else if indexPath.row == 4 { cell.titleLabel.text = "F4" cell.factorLabel.text = "Exogenous factors"
cell.expert1Label.text = F4Factors[0].joined(separator: ",") cell.expert2Label.text =
F4Factors[1].joined(separator: ",") cell.expert3Label.text = F4Factors[2].joined(separator: ",")
cell.Expert4Label.text = F4Factors[3].joined(separator: ",")


}
else if indexPath.row == 5 { cell.titleLabel.text = "F5" cell.factorLabel.text = "Social Factors"
cell.expert1Label.text = F5Factors[0].joined(separator: ",") cell.expert2Label.text =
F5Factors[1].joined(separator: ",") cell.expert3Label.text = F5Factors[2].joined(separator: ",")
cell.Expert4Label.text = F5Factors[3].joined(separator: ",")


}
else if indexPath.row == 6 { cell.titleLabel.text = "F6" cell.factorLabel.text = "Financial factor"
cell.expert1Label.text = F6Factors[0].joined(separator: ",") cell.expert2Label.text =
F6Factors[1].joined(separator: ",") cell.expert3Label.text = F6Factors[2].joined(separator: ",")
cell.Expert4Label.text = F6Factors[3].joined(separator: ",")


}
else if indexPath.row == 7 { cell.titleLabel.text = "F7"
cell.factorLabel.text = "Non-Financial Assistance factor" cell.expert1Label.text =
F7Factors[0].joined(separator: ",") cell.expert2Label.text = F7Factors[1].joined(separator: ",")
cell.expert3Label.text = F7Factors[2].joined(separator: ",") cell.Expert4Label.text =
```

```
F7Factors[3].joined(separator: ",")
}
else if indexPath.row == 8 { cell.titleLabel.text = "F8"
cell.factorLabel.text = "Entrepreneurial and business skills factor"
cell.expert1Label.text = F8Factors[0].joined(separator: ",") cell.expert2Label.text =
F8Factors[1].joined(separator: ",") cell.expert3Label.text = F8Factors[2].joined(separator: ",")
cell.Expert4Label.text = F8Factors[3].joined(separator: ",")


}
else if indexPath.row == 9 { cell.titleLabel.text = "F9" cell.factorLabel.text = "Cultural factors"
cell.expert1Label.text = F9Factors[0].joined(separator: ",") cell.expert2Label.text =
F9Factors[1].joined(separator: ",") cell.expert3Label.text = F9Factors[2].joined(separator: ",")
cell.Expert4Label.text = F9Factors[3].joined(separator: ",")


}
else if indexPath.row == 10 { cell.titleLabel.text = "F10"
cell.factorLabel.text = "Socioeconomic conditions factor" cell.expert1Label.text =
F10Factors[0].joined(separator:
",")
cell.expert2Label.text = F10Factors[1].joined(separator: ",")
cell.expert3Label.text = F10Factors[2].joined(separator: ",")
cell.Expert4Label.text = F10Factors[3].joined(separator: ",")


}
else if indexPath.row == 11 { cell.titleLabel.text = "F11"
cell.factorLabel.text = "Government policies and procedures factor"
cell.expert1Label.text = F11Factors[0].joined(separator: ",")
cell.expert2Label.text = F11Factors[1].joined(separator: ",")
cell.expert3Label.text = F11Factors[2].joined(separator: ",")
cell.Expert4Label.text = F11Factors[3].joined(separator:
",")
}
return cell
}
else {
let cell = tableView.dequeueReusableCell(withIdentifier: "Factor1TableViewCell") as!
Factor1TableViewCell
```

```
//          cell.finalArray = self.finalArray cell.totalValue = self.tableTotalValue
if indexPath.section == 1 || indexPath.section == 2 || indexPath.section == 3 || indexPath.section ==
4 || indexPath.section == 5 || indexPath.section == 6 || indexPath.section == 7 {
if indexPath.row == 0 {
cell.loadData(indexPath.section, row: indexPath.row, factor: [[String]]())
}
else {
cell.loadData(indexPath.section, row: indexPath.row, factor: factorArray[indexPath.row-1])

}
}
//
return cell
}

}
}

// Display Value
class Factor1TableViewCell: UITableViewCell {

@IBOutlet weak var fLabel: UILabel! @IBOutlet weak var f1Label: UILabel! @IBOutlet weak var
f2Label: UILabel! @IBOutlet weak var f3Label: UILabel! @IBOutlet weak var f4Label: UILabel!
@IBOutlet weak var f5Label: UILabel! @IBOutlet weak var f6Label: UILabel! @IBOutlet weak var
f7Label: UILabel! @IBOutlet weak var f8Label: UILabel! @IBOutlet weak var f9Label: UILabel!
@IBOutlet weak var f10Label: UILabel! @IBOutlet weak var f11Label: UILabel! var finalArray =
[String: Double]() var totalValue = Double(0)
override func awakeFromNib() { super.awakeFromNib()
// Initialization code
}

func loadData(_ section: Int, row: Int, factor: [[String]]) { self.f2Label.isHidden = false
self.f3Label.isHidden = false self.f4Label.isHidden = false self.f5Label.isHidden = false
self.f6Label.isHidden = false self.f7Label.isHidden = false
self.f8Label.isHidden = false self.f9Label.isHidden = false self.f10Label.isHidden = false
self.f11Label.isHidden = false fLabel.text = ""
f1Label.text = "" f2Label.text = "" f3Label.text = "" f4Label.text = "" f5Label.text = "" f6Label.text = ""
```

```
f7Label.text = "" f8Label.text = "" f9Label.text = "" f10Label.text = "" f11Label.text = "" if row == 0 {
if section == 7 { fLabel.text = "Factors" f1Label.text = "Score"
self.f2Label.isHidden = true self.f3Label.isHidden = true self.f4Label.isHidden = true
self.f5Label.isHidden = true self.f6Label.isHidden = true self.f7Label.isHidden = true
self.f8Label.isHidden = true self.f9Label.isHidden = true self.f10Label.isHidden = true
self.f11Label.isHidden = true
}
else {
fLabel.text = "" f1Label.text = "F1" f2Label.text = "F2" f3Label.text = "F3" f4Label.text = "F4"
f5Label.text = "F5" f6Label.text = "F6" f7Label.text = "F7" f8Label.text = "F8" f9Label.text = "F9"
f10Label.text = "F10" f11Label.text = "F11"
}
}
else {
fLabel.text = "F\(row)" f1Label.text = "0"
f2Label.text = "0"
f3Label.text = "0"
f4Label.text = "0"
f5Label.text = "0"
f6Label.text = "0"
f7Label.text = "0"
f8Label.text = "0"
f9Label.text = "0"
f10Label.text = "0"
f11Label.text = "0"
f1Label.font = UIFont.systemFont(ofSize: 15) f2Label.font = UIFont.systemFont(ofSize: 15)
f3Label.font = UIFont.systemFont(ofSize: 15) f4Label.font = UIFont.systemFont(ofSize: 15)
f5Label.font = UIFont.systemFont(ofSize: 15) f6Label.font = UIFont.systemFont(ofSize: 15)
f7Label.font = UIFont.systemFont(ofSize: 15) f8Label.font = UIFont.systemFont(ofSize: 15)
f9Label.font = UIFont.systemFont(ofSize: 15) f10Label.font = UIFont.systemFont(ofSize: 15)
f11Label.font = UIFont.systemFont(ofSize: 15)
if section == 1 || section == 2 || section == 3 || section == 4{ if factor.count > (section-1) {
let factorString = factor[section-1].split(separator: ";")
if    factor[section-1].contains("F1") {


f1Label.text = "I"
f1Label.font = UIFont.boldSystemFont(ofSize:
```

```
15)

}    else {

}
if   factor[section-1].contains("F2") {

f2Label.text = "I"
     f2Label.font = UIFont.boldSystemFont(ofSize:   15)

}    else {

}
if   factor[section-1].contains("F3") {

f3Label.text = "I"
     f3Label.font = UIFont.boldSystemFont(ofSize:   15)

}    else {

}
if   factor[section-1].contains("F4") {

f4Label.text = "I"
     f4Label.font = UIFont.boldSystemFont(ofSize:   15)
     else {

}

if factorString.last?.contains("F1") ?? false {
```

```
}
f1Label.text = "1"

if factorString.last?.contains("F2") ?? false {

}
f2Label.text = "1"

if factorString.last?.contains("F3") ?? false {

}
f3Label.text = "1"

if factorString.last?.contains("F4") ?? false {

}
f4Label.text = "1"
}
if factor[section-1].contains("F5") {
if factorString.last?.contains("F5") ?? false { f5Label.text = "I"
```

```
f5Label.font = UIFont.boldSystemFont(ofSize: 15)
}
else {
f5Label.text = "1"
}
}
if factor[section-1].contains("F6") {
if factorString.last?.contains("F6") ?? false { f6Label.text = "I"
f6Label.font = UIFont.boldSystemFont(ofSize: 15)
}
else {
f6Label.text = "1"
}
}
if factor[section-1].contains("F7") {
if factorString.last?.contains("F7") ?? false { f7Label.text = "I"
f7Label.font = UIFont.boldSystemFont(ofSize: 15)
}
else {
f7Label.text = "1"
}
}
if factor[section-1].contains("F8") {
if factorString.last?.contains("F8") ?? false { f8Label.text = "I"
f8Label.font = UIFont.boldSystemFont(ofSize: 15)
}
else {
f8Label.text = "1"
}
}
if factor[section-1].contains("F9") {
if factorString.last?.contains("F9") ?? false { f9Label.text = "I"
f9Label.font = UIFont.boldSystemFont(ofSize: 15)
}
else {
```

```
f9Label.text = "1"
}
}
if factor[section-1].contains("F10") {
if factorString.last?.contains("F10") ?? false { f10Label.text = "I"
f10Label.font = UIFont.boldSystemFont(ofSize:
15)
}
else {
f10Label.text = "1"
}
}
if factor[section-1].contains("F11") {
if factorString.last?.contains("F11") ?? false { f11Label.text = "I"
f11Label.font = UIFont.boldSystemFont(ofSize:
15)
}
else {
f11Label.text = "1"
}
}
}
}
else if section == 5 || section == 6 || section == 7 { let factor1String = factor[0].split(separator: ";") let
factor2String = factor[1].split(separator: ";") let factor3String = factor[2].split(separator: ";") let
factor4String = factor[3].split(separator: ";")
let F1FirstVal = "\(((factor1String.first?.contains("F1")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F1") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F1") ?? false) ?
0.25 : 0) +       ((factor4String.first?.contains("F1") ?? false) ? 0.25 : 0))"
let F1LastVal = ((factor1String.last?.contains("F1") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F1") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F1") ?? false) ? 1 :
0) +((factor4String.last?.contains("F1") ?? false) ? 1 : 0)
self.f1Label.text = F1LastVal == 0 ? F1FirstVal : "\(F1LastVal)I+\(F1FirstVal)"
```

```
let F2FirstVal = "\(((factor1String.first?.contains("F2")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F2") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F2") ?? false) ?
0.25 : 0) +      ((factor4String.first?.contains("F2") ?? false) ? 0.25 : 0))"
let F2LastVal = ((factor1String.last?.contains("F2") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F2") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F2") ?? false) ? 1 :
0) +((factor4String.last?.contains("F2") ?? false) ? 1 : 0)
self.f2Label.text = F2LastVal == 0 ? F2FirstVal : "\(F2LastVal)I+\(F2FirstVal)"
let F3FirstVal = "\(((factor1String.first?.contains("F3")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F3") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F3") ?? false) ?
0.25 : 0) +      ((factor4String.first?.contains("F3") ?? false) ? 0.25 : 0))"
let F3LastVal = ((factor1String.last?.contains("F3") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F3") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F3") ?? false) ? 1 :
0) +((factor4String.last?.contains("F3") ?? false) ? 1 : 0)
self.f3Label.text = F3LastVal == 0 ? F3FirstVal : "\(F3LastVal)I+\(F3FirstVal)"
let F4FirstVal = "\(((factor1String.first?.contains("F4")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F4") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F4") ?? false) ?
0.25 : 0) +      ((factor4String.first?.contains("F4") ?? false) ? 0.25 : 0))"
let F4LastVal = ((factor1String.last?.contains("F4") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F4") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F4") ?? false) ? 1 :
0) +((factor4String.last?.contains("F4") ?? false) ? 1 : 0)
self.f4Label.text = F4LastVal == 0 ? F4FirstVal : "\(F4LastVal)I+\(F4FirstVal)"

let F5FirstVal = "\(((factor1String.first?.contains("F5")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F5") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F5") ?? false) ?
0.25 : 0) +      ((factor4String.first?.contains("F5") ?? false) ? 0.25 : 0))"
let F5LastVal = ((factor1String.last?.contains("F5") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F5") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F5") ?? false) ? 1 :
0) +((factor4String.last?.contains("F5") ?? false) ? 1 : 0)
self.f5Label.text = F5LastVal == 0 ? F5FirstVal : "\(F5LastVal)I+\(F5FirstVal)"
```

```
let F6FirstVal = "\(((factor1String.first?.contains("F6")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F6") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F6") ?? false) ?
0.25 : 0) +     ((factor4String.first?.contains("F6") ?? false) ? 0.25 : 0))"
let F6LastVal = ((factor1String.last?.contains("F6") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F6") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F6") ?? false) ? 1 :
0) +((factor4String.last?.contains("F6") ?? false) ? 1 : 0)
self.f6Label.text = F6LastVal == 0 ? F6FirstVal : "\(F6LastVal)I+\(F6FirstVal)"
let F7FirstVal = "\(((factor1String.first?.contains("F7")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F7") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F7") ?? false) ?
0.25 : 0) +     ((factor4String.first?.contains("F7") ?? false) ? 0.25 : 0))"
let F7LastVal = ((factor1String.last?.contains("F7") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F7") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F7") ?? false) ? 1 :
0) +((factor4String.last?.contains("F7") ?? false) ? 1 : 0)
self.f7Label.text = F7LastVal == 0 ? F7FirstVal : "\(F7LastVal)I+\(F7FirstVal)"
let F8FirstVal = "\(((factor1String.first?.contains("F8")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F8") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F8") ?? false) ?
0.25 : 0) +     ((factor4String.first?.contains("F8") ?? false) ? 0.25 : 0))"
let F8LastVal = ((factor1String.last?.contains("F8") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F8") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F8") ?? false) ? 1 :
0) +((factor4String.last?.contains("F8") ?? false) ? 1 : 0)
self.f8Label.text = F8LastVal == 0 ? F8FirstVal : "\(F8LastVal)I+\(F8FirstVal)"
let F9FirstVal = "\(((factor1String.first?.contains("F9")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F9") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F9") ?? false) ?
0.25 : 0) +     ((factor4String.first?.contains("F9") ?? false) ? 0.25 : 0))"
let F9LastVal = ((factor1String.last?.contains("F9") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F9") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F9") ?? false) ? 1 :
0) +((factor4String.last?.contains("F9") ?? false) ? 1 : 0)
self.f9Label.text = F9LastVal == 0 ? F9FirstVal : "\(F9LastVal)I+\(F9FirstVal)"
```

```
let F10FirstVal = "\(((factor1String.first?.contains("F10")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F10") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F10") ?? false) ?
0.25 : 0) +     ((factor4String.first?.contains("F10") ?? false) ? 0.25 : 0))"
let F10LastVal = ((factor1String.last?.contains("F10") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F10") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F10") ?? false) ? 1
: 0) +     ((factor4String.last?.contains("F10")
?? false) ? 1 : 0)
self.f10Label.text = F10LastVal == 0 ? F10FirstVal : "\(F10LastVal)I+\(F10FirstVal)"
let F11FirstVal = "\(((factor1String.first?.contains("F11")
?? false) ? 0.25 : 0) +
((factor2String.first?.contains("F11") ?? false) ? 0.25 :
0) + ((factor3String.first?.contains("F11") ?? false) ?
0.25 : 0) +     ((factor4String.first?.contains("F11") ?? false) ? 0.25 : 0))"
let F11LastVal = ((factor1String.last?.contains("F11") ?? false) ? 1 : 0) +
((factor2String.last?.contains("F11") ?? false) ? 1 : 0) + ((factor3String.last?.contains("F11") ?? false) ? 1
: 0) +     ((factor4String.last?.contains("F11")
?? false) ? 1 : 0)
self.f11Label.text = F11LastVal == 0 ? F11FirstVal : "\(F11LastVal)I+\(F11FirstVal)"
}


}
}


override func setSelected(_ selected: Bool, animated: Bool) { super.setSelected(selected, animated:
animated)
// Configure the view for the selected state
}


}
class FactorTableViewCell: UITableViewCell {


@IBOutlet weak var Expert4Label: UILabel! @IBOutlet weak var expert3Label: UILabel! @IBOutlet
weak var expert2Label: UILabel! @IBOutlet weak var expert1Label: UILabel! @IBOutlet weak var
factorLabel: UILabel! @IBOutlet weak var titleLabel: UILabel!
override func awakeFromNib() { super.awakeFromNib()
```

```
// Initialization code
}

override func setSelected(_ selected: Bool, animated: Bool) { super.setSelected(selected, animated: animated)
// Configure the view for the selected state
}

}
```

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Single Valued Bipolar Pentapartitioned Neutrosophic Set and Its Application in MADM Strategy

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Abstract

The main objective of this paper is to introduce the notion of single-valued bipolar pentapartitioned neutrosophic set (SVBPNS). We also present some supporting examples and prove some basic properties of SVBPNS. We define score function and accuracy function of SVBPNS, and establish their basic properties. We define the single-valued bipolar pentapartitioned neutrosophic arithmetic mean (SVBPNAM) operator and the single-valued bipolar pentapartitioned neutrosophic geometric mean (SVBPNGM) operator and prove their basic properties. We develop two Multi-Attribute Decision Making (MADM) strategies namely SVBPNS-MADM Strategy based on SVBPNAM operator and SVBPNS-MADM strategy based on SVBPNGM operator under SVBPNS environment. Finally, we present a real world numerical example to illustrate the developed strategies.

Keywords: Single-Valued Pentapartitioned Neutrosophic Set; SVBPNS; MADM-Strategy.

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1. Introduction

Smarandache [1] defined the Neutrosophic Set (NS) to deal with uncertainty, indeterminacy and inconsistency involved in this real world of mathematical objects. NS is the generalization of Fuzzy Set (FS) [2] and intuitionistic fuzzy set (IFS) [3] by incorporating degrees of indeterminacy and rejection (falsity or non-membership) as independent components. In 2010, Wang et al. [4] defined Single Valued NS (SVNS). The SVNSs, its variants and extensions have been utilized in many areas such as air surveillance [5], conflict resolution [6], decision making [7-12] fault diagnosis [13], image segmentation [14], and so on. Details applications and theoretical developments of NSs are depicted in the studies [15-20].

Deli et al. [21] introduced the Single Valued Bipolar NS (SVBNS). Later on, so many researchers applied the notion of SVBNS in the model formation for Multi Attribute Decision making (MADM) [22-26] problems. In 2020, Mallick and Pramanik [27] grounded the notion of Pentapartitioned Neutrosophic Set (PNS) in which five independent components were introduced. In 2021, Das et al. [28] established an MADM strategy using tangent similarity measure under single valued PNS Environment. Recently, Das et al. [29] proposed an MADM strategy based on Grey Relational Analysis (GRA) under the single valued PNS Environment.

Research gap: No report of the investigation dealing with the combination of bipolar neutrosophic set and PNS has been appeared in the literature.

Motivation of the study: The research gap motives us to investigate the possible combination of bipolar neutrosophic set and PNS.

In this study, we introduce the Single-Valued Bipolar Pentapartitioned Neutrosophic Set (SVBPNS) by combing SVBNS and PNS. Then, we establish some basic properties of SVBPNS. Also, few illustrative examples on the SVBPNS are provided. Further, we propose some aggregation operators and prove their basic properties. Also, we develop two new MADM strategies under the SVBPNS environment.

The organization of the remaining part of this article is described as follows:
Section 2 presents some relevant results on PNS. Section 3 devotes to introduce the SVBPNS. In Section 4, we introduce two aggregation operators, namely, single-valued bipolar pentapartitioned neutrosophic arithmetic mean operator and single-valued bipolar pentapartitioned neutrosophic geometric mean operator under the SVBPNS environment. In Section 5, we procure the notion of score function and accuracy function under SVBPNS Environment. In Section 6, we develop an MADM strategy using the single-valued bipolar pentapartitioned neutrosophic arithmetic mean operator under SVBPNS environment. Further, in Section 7, we establish an MADM strategy using the single-valued bipolar pentapartitioned neutrosophic geometric mean operator under SVBPNS environment. In Section 8, we validated the proposed MADM strategies by providing a real world numerical example, and also comparing both the MADM strategies. Finally, in Section 9, we conclude the paper by stating future scope research in newly defined set environment.

2. Some Preliminary Results

We recall some basic definitions on NS, Bipolar NS, and PNS, which are relevant to the main results of this paper.

Definition 2.1.[1]. An NS V over a fixed set ψ is defined as follows:

$$V=\{(\mu,T_V(\mu),I_V(\mu),F_V(\mu)):\mu\in\psi\},$$

where T, I, F : $\psi \rightarrow]^-0, 1^+[$ are the truth, indeterminacy and falsity membership functions respectively and

.

Example 2.1. Suppose that $\psi = \{x, y\}$ be a fixed set. Then, U={(x,0.2,0.8,0.8), (y,0.3,0.2,0.4)} is an NS over ψ.

Definition 2.2.[21]. A BNS U over a non-empty set ψ is defined as follows:

$$U=\{(\mu,\ T_U^+(\mu),\ I_U^+(\mu),\ F_U^+(\mu),\ T_U^-(\mu),\ I_U^-(\mu),\ F_U^-(\mu)):\mu\in\psi\},$$

where $T_U^+(\mu)$, $I_U^+(\mu)$, $F_U^+(\mu) \in [0, 1]$, and $T_U^-(\mu)$, $I_U^-(\mu)$, $F_U^-(\mu) \in [-1, 0]$.

Here, $T_U^+(\mu)$, $I_U^+(\mu)$, and $F_U^+(\mu)$ denote the positive degree of truth-membership, indeterminacy-membership, falsity-membership respectively for $\mu\in\psi$ corresponding to the BNS U and $T_U^-(\mu)$, $I_U^-(\mu)$,and $F_U^-(\mu)$ denote the negative degree of truth-membership, indeterminacy-membership, falsity-membership respectively of $u\in\psi$ corresponding to the BNS U.

Example 2.2. Suppose that $\psi=\{x, y\}$ be a fixed set. Then, U={(x,0.1,0.6,0.8,-0.3,-0.4,-0.7), (y,0.3,0.4,0.6,-0.5,-0.4,-0.5)} is a bipolar neutrosophic set over ψ.

Definition 2.3.[21]. Assume that $U=\{(\mu,\ T_U^+(\mu),\ I_U^+(\mu),\ F_U^+(\mu),\ T_U^-(\mu),\ I_U^-(\mu),\ F_U^-(\mu)):\mu\in\psi\}$ be a BNS. Then, for each $\mu\in\psi$, $[T_U^+(\mu),\ I_U^+(\mu),\ F_U^+(\mu),\ T_U^-(\mu),\ I_U^-(\mu),\ F_U^-(\mu)]$ is called a Single Valued Bipolar Neutrosophic Number (SVBNN).

Definition 2.4.[27]. Assume that ψ be a fixed set. A PNS Z over ψ is defined by:

$$Z = \{(\mu,\ T_Z(\mu),\ C_Z(\mu),\ G_Z(\mu),\ U_Z(\mu),\ F_Z(\mu)):\mu\in\psi\},$$

where $T_Z(\mu)$, $C_Z(\mu)$, $G_Z(\mu)$, $U_Z(\mu)$, and $F_Z(\mu) \in [0, 1]$ are the truth, contradiction, ignorance, unknown and falsity membership values for each $\mu\in\psi$. So,

$$0 \le T_Z(\mu)+C_Z(\mu)+G_Z(\mu)+U_Z(\mu)+F_Z(\mu) \le 5.$$

Definition 2.5.[27]. Suppose that $M = \{(\mu,\ T_M(\mu),\ C_M(\mu),\ G_M(\mu),\ U_M(\mu),\ F_M(\mu)):\mu\in\psi\}$ and $N = \{(\mu,\ T_N(\mu),\ C_N(\mu),\ G_N(\mu),\ U_N(\mu),\ F_N(\mu)):\mu\in\psi\}$ be any two PNSs over ψ. Then, $M \subseteq N \Leftrightarrow T_M(\mu)\le T_N(\mu)$, $C_M(\mu)\le C_N(\mu)$, $G_M(\mu)\ge G_N(\mu)$, $U_M(\mu)\ge U_N(\mu)$, $F_M(\mu)\ge F_N(\mu)$, for all $\mu\in\psi$.

Definition 2.6.[27]. The null PNS (0_{PN}) and the absolute PNS (1_{PN}) over ψ are defined as follows:

(i) $0_{PN}=\{(\mu, 0, 0, 1, 1, 1): \mu\in\psi\}$;

(ii) $1_{PN}=\{(\mu, 1, 1, 0, 0, 0): \mu\in\psi\}$;

It is clearly seen that, $0_{PN} \subseteq X \subseteq 1_{PN}$, where X is a PNS over ψ.

Example 2.3. Consider a PNS X={(n,0.3,0.4,0.5,0.7,0.3), (m,0.3,0.6,0.4,0.8,0.4)} and Y={(n,0.4,0.7,0.1,0.5, 0.2), (m,0.8,0.9,0.2,0.1,0.2)} over $\psi=\{n, m\}$. Then, $X\subseteq Y$.

Definition 2.7.[27]. Suppose that $M = \{(\mu,\ T_M(\mu),\ C_M(\mu),\ G_M(\mu),\ U_M(\mu),\ F_M(\mu)): \mu\in\psi\}$ and $N = \{(\mu,\ T_N(\mu),\ C_N(\mu),\ G_N(\mu),\ U_N(\mu),\ F_N(\mu)): \mu\in\psi\}$ be any two PNSs over ψ. Then, their intersection $X\cap Y = \{(\mu, \min\{T_M(\mu), T_N(\mu)\}, \min\{C_M(\mu), C_N(\mu)\}, \max\{G_M(\mu), G_N(\mu)\}, \max\{U_M(\mu), U_N(\mu)\}, \max\{F_M(\mu), F_N(\mu)\}): \mu\in\psi\}$.

Example 2.4. Consider two PNSs X= {(n,0.4,0.3,0.7,0.4,0.9), (m,0.5,0.6,0.3,0.8,0.4)} and Y={(n,0.6,0.2,0.8, 0.7,0.8), (m,0.5,0.8,0.7,0.4,0.8)} over $\psi = \{n, m\}$. Then, their intersection is:

$X \cap Y$ = {(n,0.4,0.2,0.8,0.7,0.9), (m,0.5,0.6,0.7,0.8,0.8)}.

Definition 2.8. [27]. Assume that M = {(μ, T_M(μ), C_M(μ), G_M(μ), U_M(μ), F_M(μ)):μ∈ψ} and N = {(μ, T_N(μ), C_N(μ), G_N(μ), U_N(μ), F_N(μ)):μ∈ψ} be two PNSs over ψ. Then, the union of X and Y is defined by:

$X \cup Y$ = {(μ, max {T_M(μ), T_N(μ)}, max {C_M(μ), C_N(μ)}, min {G_M(μ), G_N(μ)}, min {U_M(μ), U_N(μ)}, min {F_M(μ), F_N(μ)}):μ∈ψ}.

Example 2.5. Consider two PNSs X={(n,0.4,0.5,0.6,0.8,0.9), (m,0.8,0.5,0.9,1.0,0.5)} and Y={(n,0.6,0.7,0.0, 0.5,0.3), (m,1.0,0.9,0.4,0.0,0.1)} over ψ={n, m}. Then, their union is:

$X \cup Y$ = {(n,0.6,0.7,0.0,0.5,0.3), (m,1.0,0.9,0.4,0.0,0.1)}.

Definition 2.9.[27]. Suppose that M = {(μ, T_M(μ), C_M(μ), G_M(μ), U_M(μ), F_M(μ)): μ∈ψ} a PNS over ψ. Then, the complement of M is defined by:

M^c = {(μ, F_M(μ), U_M(μ), 1-G_M(μ), C_M(μ), T_M(μ)): μ∈ψ}.

Example 2.6. Suppose that M={(n,0.5,0.7,0.9,0.7,0.9), (m,0.8,0.1,0.5,0.7,0.0)} be an PNS over a fixed set ψ={n, m}. Then, M^c={(n,0.9,0.7,0.1,0.7,0.5), (m,0.0,0.7,0.5,0.1,0.8)}.

Definition 2.10. Suppose that u_1, u_2,…, u_n be n real numbers. Then, the arithmetic mean (AM) of u_1, u_2,…, u_n is defined by AM (u_1, u_2,…, u_n) = $\frac{1}{n}\sum_{i=1}^{n} u_i$.

Definition 2.11. Suppose that u_1, u_2,…, u_n be n real numbers. Then, the geometric mean (GM) of u_1, u_2,…, u_n is defined by GM (u_1, u_2,…, u_n) = $\left(\prod_{i=1}^{n} u_i\right)^{\frac{1}{n}}$.

3. Single-Valued Bipolar Pentapartitioned Neutrosophic Set

In this section, we procure the notion of SVBPNS. Also, we investigate some different properties of these kind of sets. Also, few illustrative examples are given.

Definition 3.1. A single-valued bipolar pentapartitioned neutrosophic set N over a non-empty set ψ is defined as:

N={(μ, T_N^-(μ), C_N^-(μ), G_N^-(μ), U_N^-(μ), F_N^-(μ), T_N^+(μ), C_N^+(μ), G_N^+(μ), U_N^+(μ), F_N^+(μ)):μ∈ψ}, where T_N^- (μ), C_N^-(μ), G_N^- (μ), U_N^- (μ), F_N^-(μ) ∈ [-1,0] and T_N^+ (μ), C_N^+(μ), G_N^+ (μ), U_N^+ (μ), F_N^+(μ) ∈ [0,1].

The negative membership degrees T_N^-(μ), C_N^-(μ), G_N^-(μ), U_N^-(μ), *and* F_N^-(μ) indicate the degree of truth-membership, contradiction-membership, ignorance-membership, unknown-membership, falsity-membership respectively for μ∈ψ corresponding to an SVBPNS N. Again, the positive membership degrees, T_N^+ (μ), C_N^+ (μ), G_N^+ (μ), U_N^+ (μ), and F_N^+ (μ) indicate the degree of truth-membership, contradiction-membership, ignorance-membership, unknown-membership, falsity-membership respectively for n∈ψ corresponding to an SVBPNS N.

Example 3.1. Let ψ={n, m} be a fixed set. Then, U={(n,-0.2,-0.4,-0.3,-0.4,-0.7,0.1,0.6,0.8,0.4,0.1), (y,-0.5, -0.4,-0.5,-0.3,-0.2,0.5,0.1,0.3,0.4,0.6)} is an SVBPNS over ψ.

Definition 3.2. Let N={(μ, $T_N^-(\mu)$,$C_N^-(\mu)$,$G_N^-(\mu)$,$U_N^-(\mu)$,$F_N^-(\mu)$,$T_N^+(\mu)$,$C_N^+(\mu)$,$G_N^+(\mu)$,$U_N^+(\mu)$,$F_N^+(\mu)$):μ∈ψ} be an SVBPNS. Then, [$T_N^-(\mu)$, $C_N^-(\mu)$, $G_N^-(\mu)$, $U_N^-(\mu)$, $F_N^-(\mu)$, $T_N^+(\mu)$, $C_N^+(\mu)$, $G_N^+(\mu)$, $U_N^+(\mu)$, $F_N^+(\mu)$] is called a single-valued bipolar pentapartitioned neutrosophic number (SVBPNN), for each μ∈ψ.

Definition 3.3. Suppose that A = {(μ, $T_A^-(\mu)$, $C_A^-(\mu)$, $G_A^-(\mu)$, $U_A^-(\mu)$, $F_A^-(\mu)$, $T_A^+(\mu)$, $C_A^+(\mu)$, $G_A^+(\mu)$, $U_A^+(\mu)$, $F_A^+(\mu)$): μ∈ψ} and B = {(μ, $T_B^-(\mu)$, $C_B^-(\mu)$, $G_B^-(\mu)$, $U_B^-(\mu)$, $F_B^-(\mu)$, $T_B^+(\mu)$, $C_B^+(\mu)$, $G_B^+(\mu)$, $U_B^+(\mu)$, $F_B^+(\mu)$): μ∈ψ} be any two SVBPNSs over ψ. Then, $A \subseteq B$ if and only if $T_A^-(\mu) \leq T_B^-(\mu)$, $C_A^-(\mu) \geq C_B^-(\mu)$, $G_A^-(\mu) \geq G_B^-(\mu)$, $U_A^-(\mu) \geq U_B^-(\mu)$, $F_A^-(\mu) \geq F_B^-(\mu)$, $T_A^+(\mu) \leq T_B^+(\mu)$, $C_A^+(\mu) \geq C_B^+(\mu)$, $G_A^+(\mu) \geq G_B^+(\mu)$, $U_A^+(\mu) \geq U_B^+(\mu)$, $F_A^+(\mu) \geq F_B^+(\mu)$, for all μ∈ψ.

Example 3.2. Consider two SVBPNSs X={(x,-0.2,-0.5,-0.3,-0.4,-0.3,0.3,0.4,0.5,0.7,0.3), (y,-0.3,-0.5,-0.4, -0.2,-0.4,0.3,0.6,0.4,0.8,0.4)} and Y={(x,-0.2,-0.6,-0.7,-0.5,-0.5,0.4,0.3,0.1,0.5,0.2), (y,-0.2,-0.6,-0.6,-0.3,-0.5, 0.8,0.5,0.2,0.1,0.2)} over ψ = {x, y}. Then, $X \subseteq Y$.

Definition 3.4. Suppose that A = {(μ, $T_A^-(\mu)$, $C_A^-(\mu)$, $G_A^-(\mu)$, $U_A^-(\mu)$, $F_A^-(\mu)$, $T_A^+(\mu)$, $C_A^+(\mu)$, $G_A^+(\mu)$, $U_A^+(\mu)$, $F_A^+(\mu)$): μ∈ψ} and B = {(μ, $T_B^-(\mu)$, $C_B^-(\mu)$, $G_B^-(\mu)$, $U_B^-(\mu)$, $F_B^-(\mu)$, $T_B^+(\mu)$, $C_B^+(\mu)$, $G_B^+(\mu)$, $U_B^+(\mu)$, $F_B^+(\mu)$):μ∈ψ} are any two SVBPNSs over ψ. Then, the intersection of X and Y is defined by:

$X \cap Y$ = {(μ, min {$T_A^-(\mu)$, $T_B^-(\mu)$}, max {$C_A^-(\mu)$, $C_B^-(\mu)$}, max {$G_A^-(\mu)$, $G_B^-(\mu)$}, max {$U_A^-(\mu)$, $U_B^-(\mu)$}, max {$F_A^-(\mu)$, $F_B^-(\mu)$}, min {$T_A^+(\mu)$, $T_B^+(\mu)$}, max {$C_A^+(\mu)$, $C_B^+(\mu)$}, max {$G_A^+(\mu)$, $G_B^+(\mu)$}, max {$U_A^+(\mu)$, $U_B^+(\mu)$}, max {$F_A^+(\mu)$, $F_B^+(\mu)$}) : μ∈ψ}.

Example 3.3. Suppose that X and Y are two SVBPNSs over ψ= {x, y} such that X = {(x,-0.3,-0.7,-0.5, -0.1,-0.5,0.5,0.7,0.2,0.4,0.2), (y,-0.5,-0.1,-0.5,-0.3,-0.4,0.4,0.7,0.5,0.7,0.3)} and Y = {(x,-0.1,-0.7,-0.5,-0.4, -0.3,0.2,0.5,0.3,0.5,0.4), (y,-0.4,-0.5,-0.5,-0.2,-0.3,0.4,0.5,0.3,0.4,0.3)}. Then, their intersection is $X \cap Y$ = {(x,-0.3,-0.7,-0.5,-0.1,-0.3,0.2,0.7,0.3,0.5,0.4), (y,-0.5,-0.1,-0.5,-0.2,-0.3, 0.4,0.7,0.5,0.7,0.3)}.

Definition 3.5. Suppose that A = {(μ, $T_A^-(\mu)$, $C_A^-(\mu)$, $G_A^-(\mu)$, $U_A^-(\mu)$, $F_A^-(\mu)$, $T_A^+(\mu)$, $C_A^+(\mu)$, $G_A^+(\mu)$, $U_A^+(\mu)$, $F_A^+(\mu)$):μ∈ψ }and B = {(μ, $T_B^-(\mu)$, $C_B^-(\mu)$, $G_B^-(\mu)$, $U_B^-(\mu)$, $F_B^-(\mu)$, $T_B^+(\mu)$, $C_B^+(\mu)$, $G_B^+(\mu)$, $U_B^+(\mu)$, $F_B^+(\mu)$): μ∈ψ } are any two SVBPNSs over ψ. Then, the union of X and Y is defined by:

$X \cup Y$ = {(μ, max {$T_A^-(\mu)$, $T_B^-(\mu)$}, min {$C_A^-(\mu)$, $C_B^-(\mu)$}, min {$G_A^-(\mu)$, $G_B^-(\mu)$}, min {$U_A^-(\mu)$, $U_B^-(\mu)$}, min {$F_A^-(\mu)$, $F_B^-(\mu)$}, max {$T_A^+(\mu)$, $T_B^+(\mu)$}, min {$C_A^+(\mu)$, $C_B^+(\mu)$}, min {$G_A^+(\mu)$, $G_B^+(\mu)$}, min {$U_A^+(\mu)$, $U_B^+(\mu)$}, min {$F_A^+(\mu)$, $F_B^+(\mu)$}) : μ∈ψ}.

Example 3.4. Suppose that X and Y be two SVBPNSs over ψ = {x, y} such that X = {(x,-0.4,-0.7,-0.5,-0.6,-0.7,0.5,0.7,0.5,0.2,0.3), (y,-0.1,-0.3,-0.7,-0.7,-0.4,0.4,0.7,0.8,0.6,0.4)} and Y = {(x, -0.2, -0.3,-0.4,-0.7,-0.6,0.3,0.8,0.5,0.4,0.7), (y,-0.7,-0.1,-0.4,-0.7,-0.6,0.7,0.8,0.6,0.7,0.9)}. Then, their union is $X \cup Y$ = {(x,-0.2,-0.7,-0.5,-0.7,-0.7,0.5,0.7,0.5,0.2,0.3), (y,-0.1,-0.3,-0.7,-0.7,-0.6,0.7,0.7,0.6,0.6,0.4).

Definition 3.6. Let A = {(μ,$T_A^-(\mu)$,$C_A^-(\mu)$,$G_A^-(\mu)$,$U_A^-(\mu)$,$F_A^-(\mu)$,$T_A^+(\mu)$,$C_A^+(\mu)$,$G_A^+(\mu)$,$U_A^+(\mu)$,$F_A^+(\mu)$) : μ∈ψ} be an SVBPNSs over ψ. Then, the complement of A is defined as follows:

A^c= {(μ,-1-$T_A^-(\mu)$,-1-$C_A^-(\mu)$,-1-$G_A^-(\mu)$,-1-$U_A^-(\mu)$,-1-$F_A^-(\mu)$,1-$T_A^+(\mu)$,1-$C_A^+(\mu)$,1-$G_A^+(\mu)$,1-$U_A^+(\mu)$,1-$F_A^+(\mu)$):μ∈ψ}.

Example 3.5. Suppose that A = {(x,-0.4,-0.7,-0.5,-0.6,-0.7,0.5,0.7,0.5,0.2,0.3), (y,-0.1,-0.3,-0.7,-0.7,-0.4,0.4, 0.7,0.8,0.6,0.4)} be an SVBPNS over ψ = {x, y}. Then, the complement of A is A^c ={(x,-0.6,-0.3,-0.5,-0.4, -0.3,0.5,0.3,0.5,0.8,0.7), (y,-0.9,-0.7,-0.3,-0.3,-0.6,0.6,0.3,0.2,0.4,0.6)}.

Definition 3.7. The null SVBPNS (0_{BPN}) and the absolute SVBPNS (1_{BPN}) over ψ are defined as follows:

(i) 0_{BPN} = {(μ, -1, 0, 0, 0, 0, 0, 1, 1, 1, 1) : μ∈ψ};

(ii) 1_{BPN} = {(μ, 0, -1, -1, -1, -1, 1, 0, 0, 0, 0) : μ∈ψ};

It is clearly seen that,

(i) $0_{BPN} \subseteq X \subseteq 1_{BPN}$, where X is an SVBPNS over ψ;

(ii) $0_{\mathbf{BPN}}^{c}$ = 1_{BPN} & $1_{\mathbf{BPN}}^{c}$ = 0_{BPN};

(iii) $0_{BPN} \cup 1_{BPN} = 1_{BPN}$;

(iv) $0_{BPN} \cap 1_{BPN} = 0_{BPN}$.

Definition 3.8. Suppose that μ=[T_ψ^-(μ),C_ψ^-(μ),G_ψ^-(μ),U_ψ^-(μ),F_ψ^-(μ),T_ψ^+(μ),C_ψ^+(μ),G_ψ^+(μ),U_ψ^+(μ),F_ψ^+(μ)] and v=[$T_\psi^-(v)$,$C_\psi^-(v)$,$G_\psi^-(v)$,$U_\psi^-(v)$,$F_\psi^-(v)$,$T_\psi^+(v)$,$C_\psi^+(v)$,$G_\psi^+(v)$,$U_\psi^+(v)$,$F_\psi^+(v)$] be two SVBPNNs. Then,

(i) k.μ=[-(-T_ψ^-(μ))^k, -(-C_ψ^-(μ))^k, -(-G_ψ^-(μ))^k, -(-U_ψ^-(μ))^k, -(1-(1-(-F_ψ^-(μ)))^k), 1-(1-T_ψ^+(μ))^k, (C_ψ^+(μ))^k, (G_ψ^+(μ))^k, (U_ψ^+(μ))^k, (F_ψ^+(μ))^k], where k >0.

(ii) μ^k=[-(1-(1-(-T_ψ^-(μ)))^k), -(-C_ψ^-(μ))^k, -(-G_ψ^-(μ))^k, -(-U_ψ^-(μ))^k, -(-F_ψ^-(μ))^k, (T_ψ^+(μ))^k, 1-(1-C_ψ^+(μ))^k, 1-(1-G_ψ^+(μ))^k, 1-(1-U_ψ^+(μ))^k, 1-(1-F_ψ^+(μ))^k], where k >0.

(iii) μ+η=[- T_ψ^- (μ). T_ψ^- (η), -(- C_ψ^- (μ)- C_ψ^- (η)- C_ψ^- (μ) . C_ψ^- (η)), -(- G_ψ^- (μ)- G_ψ^- (η)- G_ψ^- (μ) . G_ψ^- (η)), -(-U_ψ^-(μ)-U_ψ^-(η)- U_ψ^-(μ). U_ψ^-(η)), -(-F_ψ^-(μ)-F_ψ^-(η)- F_ψ^-(μ). F_ψ^-(η)), T_ψ^+(μ)+ T_ψ^+(η)-T_ψ^+(μ).T_ψ^+(η), C_ψ^+(μ). C_ψ^+(η), G_ψ^+(μ). G_ψ^+(η), U_ψ^+(μ). U_ψ^+(η), F_ψ^+(μ).F_ψ^+(η)];

(iv) μ.η=[-(-T_ψ^-(μ)-T_ψ^-(η)-T_ψ^-(μ).T_ψ^-(η)), -C_ψ^-(μ). C_ψ^-(η), -G_ψ^-(μ). G_ψ^-(η), -U_ψ^-(μ). U_ψ^-(η), -F_ψ^-(μ). F_ψ^-(η), T_ψ^+(μ). T_ψ^+(η), C_ψ^+(μ)+ C_ψ^+(η)- C_ψ^+(μ). C_ψ^+(η), G_ψ^+(μ)+ G_ψ^+(η)- G_ψ^+(μ). G_ψ^+(η), U_ψ^+(μ)+ U_ψ^+(η)- U_ψ^+(μ). U_ψ^+(η), F_ψ^+(μ)+F_ψ^+(η)-F_ψ^+(μ).F_ψ^+(η)].

4. Single-Valued Bipolar Pentapartitioned Neutrosophic Aggregation Operators

Definition 4.1. Assume that u_i=[$T_\psi^-(u_i)$,$C_\psi^-(u_i)$,$G_\psi^-(u_i)$,$U_\psi^-(u_i)$,$F_\psi^-(u_i)$,$T_\psi^+(u_i)$,$C_\psi^+(u_i)$,$G_\psi^+(u_i)$,$U_\psi^+(u_i)$,$F_\psi^+(u_i)$], i=1, 2, 3,…, n, be a collection of SVBPNNs over ψ. Then, the single-valued bipolar pentapartitioned neutrosophic arithmetic mean (SVBPNAM) operator is defined as follows:

$$\text{SVBPNAM}(u_1, u_2,\ldots, u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i \quad (1)$$

Theorem 4.1. Assume that u_i=[$T_\psi^-(u_i)$,$C_\psi^-(u_i)$,$G_\psi^-(u_i)$,$U_\psi^-(u_i)$,$F_\psi^-(u_i)$,$T_\psi^+(u_i)$,$C_\psi^+(u_i)$,$G_\psi^+(u_i)$,$U_\psi^+(u_i)$,$F_\psi^+(u_i)$], i=1, 2, 3,…, n, be a collection of SVBPNNs over ψ. Then, the aggregated value SVBPNAM (u_1, u_2,…, u_n) is also an SVBPNN.

Proof. Assume that u_i=[$T_\psi^-(u_i)$,$C_\psi^-(u_i)$,$G_\psi^-(u_i)$,$U_\psi^-(u_i)$,$F_\psi^-(u_i)$,$T_\psi^+(u_i)$,$C_\psi^+(u_i)$,$G_\psi^+(u_i)$,$U_\psi^+(u_i)$,$F_\psi^+(u_i)$], i=1, 2, 3,…, n, be a finite collection of SVBPNNs over ψ. Therefore, u_1 is an SVBPNN.

Now,

$\sum_{i=1}^{2} u_i$ = (u_1 + u_2)

=[- T_ψ^- (u_1). T_ψ^- (u_2), -(- C_ψ^- (u_1)- C_ψ^- (u_2)- C_ψ^- (u_1) . C_ψ^- (u_2)), -(- G_ψ^- (u_1)- G_ψ^- (u_2)- G_ψ^- (u_1) . G_ψ^- (u_2)), -(- U_ψ^- (u_1)- U_ψ^- (u_2)- U_ψ^- (u_1) . U_ψ^- (u_2)), -(- F_ψ^- (u_1)- F_ψ^- (u_2)- F_ψ^- (u_1) . F_ψ^- (u_2)), T_ψ^+ (u_1)+ T_ψ^+ (u_2)- T_ψ^+ (u_1). T_ψ^+ (u_2), C_ψ^+(u_1). C_ψ^+(u_2), G_ψ^+(u_1). G_ψ^+(u_2), U_ψ^+(u_1). U_ψ^+(u_2), F_ψ^+(u_1).F_ψ^+(u_2)]

= [T_ψ^-(u_1, u_2), C_ψ^-(u_1, u_2), G_ψ^-(u_1, u_2), U_ψ^-(u_1, u_2), F_ψ^-(u_1, u_2), T_ψ^+(u_1, u_2), C_ψ^+(u_1, u_2), G_ψ^+(u_1, u_2), U_ψ^+(u_1, u_2), F_ψ^+(u_1, u_2)] (say), which is an SVBPNN.

Assume that, $\sum_{i=1}^{n} u_i$ is an SVBPNN over ψ for n = m, i.e. $\sum_{i=1}^{m} u_i$= [T_ψ^-(u_1, u_2,…,u_m), C_ψ^-(u_1, u_2,…,u_m), G_ψ^-(u_1, u_2,…,u_m), U_ψ^-(u_1, u_2,…,u_m), F_ψ^-(u_1, u_2,…,u_m), T_ψ^+(u_1, u_2,…,u_m), C_ψ^+(u_1, u_2,…,u_m), G_ψ^+(u_1, u_2,…,u_m), U_ψ^+(u_1, u_2,…,u_m), F_ψ^+(u_1, u_2,…,u_m)] is an SVBPNN.

Now,

$\sum_{i=1}^{m+1} u_i$

= $\sum_{i=1}^{m} u_i$ + u_{m+1}

= [T_ψ^-(u_1, u_2,…,u_m), C_ψ^-(u_1, u_2,…,u_m), G_ψ^-(u_1, u_2,…,u_m), U_ψ^-(u_1, u_2,…,u_m), F_ψ^-(u_1, u_2,…,u_m), T_ψ^+(u_1, u_2,…,u_m), C_ψ^+(u_1, u_2,…,u_m), G_ψ^+(u_1, u_2,…,u_m), U_ψ^+(u_1, u_2,…,u_m), F_ψ^+(u_1, u_2,…,u_m)]

+ [T_ψ^-(u_{m+1}),C_ψ^-(u_{m+1}),G_ψ^-(u_{m+1}),U_ψ^-(u_{m+1}),F_ψ^-(u_{m+1}),T_ψ^+(u_{m+1}),C_ψ^+(u_{m+1}),G_ψ^+(u_{m+1}),U_ψ^+(u_{m+1}),F_ψ^+(u_{m+1})].

=[- T_ψ^- (u_1, u_2,…,u_m). T_ψ^- (u_{m+1}), -(- C_ψ^- (u_1, u_2,…,u_m)- C_ψ^- (u_{m+1})- C_ψ^- (u_1, u_2,…,u_m) . C_ψ^- (u_{m+1})), -(- G_ψ^- (u_1, u_2,…,u_m)-G_ψ^-(u_{m+1})- G_ψ^-(u_1, u_2,…, u_m). G_ψ^-(u_{m+1})), -(-U_ψ^-(u_1, u_2,…,u_m)-U_ψ^-(u_{m+1})- U_ψ^-(u_1, u_2,…,u_m). U_ψ^-(u_{m+1})), -(- F_ψ^- (u_1, u_2,…,u_m)- F_ψ^- (u_{m+1})- F_ψ^- (u_1, u_2,…,u_m) . F_ψ^- (u_{m+1})), T_ψ^+ (u_1, u_2,…,u_m)+ T_ψ^+ (u_{m+1})- T_ψ^+ (u_1, u_2,…,u_m). T_ψ^+ (u_{m+1}), C_ψ^+ (u_1, u_2,…,u_m). C_ψ^+ (u_{m+1}), G_ψ^+ (u_1, u_2,…,u_m). G_ψ^+ (u_{m+1}), U_ψ^+ (u_1, u_2,…,u_m). U_ψ^+ (u_{m+1}), F_ψ^+(u_1, u_2,…,u_m).F_ψ^+ (u_{m+1})]

=[T_ψ^-(u_1, u_2,…, u_{m+1}), C_ψ^-(u_1, u_2,…, u_{m+1}), G_ψ^-(u_1, u_2,…, u_{m+1}), U_ψ^-(u_1, u_2,…, u_{m+1}), F_ψ^-(u_1, u_2,…, u_{m+1}), T_ψ^+(u_1, u_2,…, u_{m+1}), C_ψ^+(u_1, u_2,…, u_{m+1}), G_ψ^+(u_1, u_2,…, u_{m+1}), U_ψ^+(u_1, u_2,…, u_{m+1}), F_ψ^+(u_1, u_2,…, u_{m+1})] (say), which is an SVBPNN.

Therefore, $\sum_{i=1}^{m+1} u_i$ is an SVBPNN. This implies, $\sum_{i=1}^{n} u_i$ is an SVBPNN for n= m+1.

Hence, $\sum_{i=1}^{n} u_i$ is an SVBPNN for n=1 and 2. Again, $\sum_{i=1}^{n} u_i$ is an SVBPNN for n=m+1, whenever it is an SVBPNN for n=m. Therefore, by the principle of mathematical induction, we can say that $\sum_{i=1}^{n} u_i$ is an SVBPNN for each n. Now, from Definition 3.8. we can say that $\frac{1}{n}\sum_{i=1}^{n} u_i$ is an SVBPNN. Hence,

SVBPNAM (u_1,u_2,…,u_n) = $\frac{1}{n}\sum_{i=1}^{n} u_i$ is an SVBPNN.

Example 4.1. Assume that u=(-0.3,-0.5,-0.3,-0.2,-0.5,0.5,0.3,0.6,0.5,0.2) and v=(-0.8,-0.5,-0.5,-0.3, -0.7,0.3,0.6,0.2,0.5,0.4) be two SVBPNNs. Then, SVBPNAM(u, v) = 0.5 (u+v) = 0.5 (-0.24, -0.75,-0.65, -0.44,-0.85,0.65,0.18,0.12,0.25,0.08) = (-0.49,-0.87,-0.81,-0.66,-0.61,0.41,0.42,0.35,0.5,0.28). It is also an SVBPNN.

Definition 4.2. Assume that u_i=[$T_\psi^-(u_i)$,$C_\psi^-(u_i)$,$G_\psi^-(u_i)$,$U_\psi^-(u_i)$,$F_\psi^-(u_i)$,$T_\psi^+(u_i)$,$C_\psi^+(u_i)$,$G_\psi^+(u_i)$,$U_\psi^+(u_i)$,$F_\psi^+(u_i)$], i=1, 2, 3,…, n, be the family of SVBPNNs over ψ. Then, the Single-Valued Bipolar Pentapartitioned Neutrosophic Geometric Mean (SVBPNGM) operator is defined as follows:

SVBPNGM (u_1, u_2,…, u_n) = $\left(\prod_{i=1}^{n} u_i\right)^{\frac{1}{n}}$ (2)

Theorem 4.2. Assume that u_i=[$T_\psi^-(u_i)$,$C_\psi^-(u_i)$,$G_\psi^-(u_i)$,$U_\psi^-(u_i)$,$F_\psi^-(u_i)$,$T_\psi^+(u_i)$,$C_\psi^+(u_i)$,$G_\psi^+(u_i)$,$U_\psi^+(u_i)$,$F_\psi^+(u_i)$], i=1, 2, 3,…, n, be a family of SVBPNNs over ψ. Then the aggregated value SVBPNGM (u_1, u_2,…, u_n) is also an SVBPNN.

Proof. Assume that u_i=[$T_\psi^-(u_i)$,$C_\psi^-(u_i)$,$G_\psi^-(u_i)$,$U_\psi^-(u_i)$,$F_\psi^-(u_i)$,$T_\psi^+(u_i)$,$C_\psi^+(u_i)$,$G_\psi^+(u_i)$,$U_\psi^+(u_i)$,$F_\psi^+(u_i)$], i=1, 2, 3,…, n, be a finite collection SVBPNNs over ψ. Therefore, u_1 is an SVBPNN.

Now, $\prod_{i=1}^{2} u_i$ = $u_1 \,.\, u_2$ = [-(- $T_\psi^-(u_1)$- $T_\psi^-(u_2)$- $T_\psi^-(u_1)$. $T_\psi^-(u_2)$), - $C_\psi^-(u_1)$. $C_\psi^-(u_2)$, - $G_\psi^-(u_1)$. $G_\psi^-(u_2)$, - $U_\psi^-(u_1)$. $U_\psi^-(u_2)$, - $F_\psi^-(u_1)$. $F_\psi^-(u_2)$, $T_\psi^+(u_1)$. $T_\psi^+(u_2)$, $C_\psi^+(u_1)$+ $C_\psi^+(u_2)$- $C_\psi^+(u_1)$. $C_\psi^+(u_2)$, $G_\psi^+(u_1)$+$G_\psi^+(u_2)$-$G_\psi^+(u_1)$.$G_\psi^+(u_2)$, $U_\psi^+(u_1)$+$U_\psi^+(u_2)$-$U_\psi^+(u_1)$.$U_\psi^+(u_2)$, $F_\psi^+(u_1)$+$F_\psi^+(u_2)$-$F_\psi^+(u_1)$.$F_\psi^+(u_2)$]

= [$T_\psi^-(u_1, u_2)$, $C_\psi^-(u_1, u_2)$, $G_\psi^-(u_1, u_2)$, $U_\psi^-(u_1, u_2)$, $F_\psi^-(u_1, u_2)$, $T_\psi^+(u_1, u_2)$, $C_\psi^+(u_1, u_2)$, $G_\psi^+(u_1, u_2)$, $U_\psi^+(u_1, u_2)$, $F_\psi^+(u_1, u_2)$] (say), which is an SVBPNN.

Suppose that, $\prod_{i=1}^{n} u_i$is an SVBPNN over ψ for n = m, i.e. $\prod_{i=1}^{m} u_i$ = [$T_\psi^-(u_1, u_2,…,u_m)$, $C_\psi^-(u_1, u_2,…,u_m)$, $G_\psi^-(u_1, u_2,…,u_m)$, $U_\psi^-(u_1, u_2,…,u_m)$, $F_\psi^-(u_1, u_2,…,u_m)$, $T_\psi^+(u_1, u_2,…,u_m)$, $C_\psi^+(u_1, u_2,…,u_m)$, $G_\psi^+(u_1, u_2,…,u_m)$, $U_\psi^+(u_1, u_2,…,u_m)$, $F_\psi^+(u_1, u_2,…,u_m)$] is an SVBPNN.

Now,

$\prod_{i=1}^{m+1} u_i$

= u_{m+1} . $\prod_{i=1}^{m} u_i$

= [$T_\psi^-(u_{m+1})$,$C_\psi^-(u_{m+1})$,$G_\psi^-(u_{m+1})$,$U_\psi^-(u_{m+1})$,$F_\psi^-(u_{m+1})$,$T_\psi^+(u_{m+1})$,$C_\psi^+(u_{m+1})$,$G_\psi^+(u_{m+1})$,$U_\psi^+(u_{m+1})$,$F_\psi^+(u_{m+1})$]. [$T_\psi^-(u_1, u_2,…,u_m)$, $C_\psi^-(u_1, u_2,…,u_m)$, $G_\psi^-(u_1, u_2,…,u_m)$, $U_\psi^-(u_1, u_2,…,u_m)$, $F_\psi^-(u_1, u_2,…,u_m)$, $T_\psi^+(u_1, u_2,…,u_m)$, $C_\psi^+(u_1, u_2,…,u_m)$, $G_\psi^+(u_1, u_2,…,u_m)$, $U_\psi^+(u_1, u_2,…,u_m)$, $F_\psi^+(u_1, u_2,…,u_m)$]

=[-(-$T_\psi^-(u_{m+1})$-$T_\psi^-(u_1, u_2,…,u_m)$-$T_\psi^-(u_{m+1})$.$T_\psi^-(u_1, u_2,…,u_m)$), -$C_\psi^-(u_{m+1})$. $C_\psi^-(u_1, u_2,…,u_m)$, -$G_\psi^-(u_{m+1})$. $G_\psi^-(u_1, u_2,…,u_m)$, -$U_\psi^-(u_{m+1})$. $U_\psi^-(u_1, u_2,…,u_m)$, -$F_\psi^-(u_{m+1})$. $F_\psi^-(u_1, u_2,…,u_m)$, $T_\psi^+(u_{m+1})$.$T_\psi^+(u_1, u_2,…,u_m)$, $C_\psi^+(u_{m+1})$+$C_\psi^+(u_1, u_2,…,u_m)$-$C_\psi^+(u_{m+1})$.$C_\psi^+(u_1, u_2,…,u_m)$, $G_\psi^+(u_{m+1})$+$G_\psi^+(u_1, u_2,…,u_m)$-$G_\psi^+(u_{m+1})$.$G_\psi^+(u_1, u_2,…,u_m)$, $U_\psi^+(u_{m+1})$+$U_\psi^+(u_1, u_2,…,u_m)$-$U_\psi^+(u_{m+1})$.$U_\psi^+(u_1, u_2,…,u_m)$, $F_\psi^+(u_{m+1})$+$F_\psi^+(u_1, u_2,…,u_m)$-$F_\psi^+(u_{m+1})$.$F_\psi^+(u_1, u_2,…,u_m)$]

=[$T_\psi^-(u_1, u_2,…, u_{m+1})$, $C_\psi^-(u_1, u_2,…, u_{m+1})$, $G_\psi^-(u_1, u_2,…, u_{m+1})$, $U_\psi^-(u_1, u_2,…, u_{m+1})$, $F_\psi^-(u_1, u_2,…, u_{m+1})$, $T_\psi^+(u_1, u_2,…, u_{m+1})$, $C_\psi^+(u_1, u_2,…, u_{m+1})$, $G_\psi^+(u_1, u_2,…, u_{m+1})$, $U_\psi^+(u_1, u_2,…, u_{m+1})$, $F_\psi^+(u_1, u_2,…, u_{m+1})$] (say), which is an SVBPNN.

Therefore, $\prod_{i=1}^{m+1} u_i$ is an SVBPNN. This implies, $\prod_{i=1}^{n} u_i$ is an SVBPNN for n=m+1.

Hence, $\prod_{i=1}^{n} u_i$ is an SVBPNN for n=1 and 2. Again, $\prod_{i=1}^{n} u_i$ is an SVBPNN for n=m+1, whenever it is an SVBPNN for n=m. Therefore, by the principle of mathematical induction, we can say that $\prod_{i=1}^{n} u_i$

is an SVBPNN for each n. Now, from Definition 3.8. we can say that $(\prod_{i=1}^{n} u_i)^{\frac{1}{n}}$ is an SVBPNN.

Hence, SVBPNGM ($u_1,u_2,\ldots,u_n$) = $(\prod_{i=1}^{n} u_i)^{\frac{1}{n}}$ is an SVBPNN.

Example 4.2. Let u=(-0.3,-0.5,-0.3,-0.2,-0.5,0.5,0.3,0.6,0.5,0.2), v=(-0.8,-0.5,-0.5,-0.3,-0.7,0.3,0.6,0.2,0.5, 0.4) be two SVBPNNs as shown in Example 4.1. Then, SVBPNGM (u, v) = $(u+v)^{0.5}$ = (-0.86,-0.25,-0.15, -0.06,-0.35,0.15,0.72,0.68,0.75,0.52)$^{0.5}$ = (-0.63,-0.5,-0.39,-0.24,-0.59,0.39,0.47,0.43,0.5,0.31). It is also an SVBPNN.

5. Score & Accuracy Functions under the SVBPNS Environment

Definition 5.1. Suppose that μ = $[T_\psi^-(\mu),C_\psi^-(\mu),G_\psi^-(\mu),U_\psi^-(\mu),F_\psi^-(\mu),T_\psi^+(\mu),C_\psi^+(\mu),G_\psi^+(\mu),U_\psi^+(\mu),F_\psi^+(\mu)]$ be an SVBPNN over ψ. Then, the score function and accuracy function are defined by:

$$S_f(\mu) = \frac{[\,1+T_\psi^-(\mu)-C_\psi^-(\mu)-G_\psi^-(\mu)-U_\psi^-(\mu)-F_\psi^-(\mu)+T_\psi^+(\mu)+1-C_\psi^+(\mu)+1-G_\psi^+(\mu)+1-U_\psi^+(\mu)+1-F_\psi^+(\mu)]}{10} \tag{3}$$

$$A_f(\mu) = \frac{[T_\psi^-(\mu)-C_\psi^-(\mu)-F_\psi^-(\mu)+T_\psi^+(\mu)-C_\psi^+(\mu)-F_\psi^+(\mu)]}{3} \tag{4}$$

Example 5.1. Suppose that μ=(-0.3,-0.5,-0.3,-0.2,-0.5,0.5,0.3,0.6,0.5,0.2) be an SVBPNN as defined in Example 4.1. Then, $S_f(\mu)$=0.51 and $A_f(\mu)$=0.233.

Definition 5.2. Suppose that μ=$[T_\psi^-(\mu),C_\psi^-(\mu),G_\psi^-(\mu),U_\psi^-(\mu),F_\psi^-(\mu),T_\psi^+(\mu),C_\psi^+(\mu),G_\psi^+(\mu),U_\psi^+(\mu),F_\psi^+(\mu)]$ and v=$[T_\psi^-(v),C_\psi^-(v),G_\psi^-(v),U_\psi^-(v),F_\psi^-(v),T_\psi^+(v),C_\psi^+(v),G_\psi^+(v),U_\psi^+(v),F_\psi^+(v)]$ be any two SVBPNNs over ψ. Then,

(i) $S_f(\mu) > S_f(\eta) \Rightarrow \mu > \eta$;

(ii) $S_f(\mu) = S_f(\eta)$, $A_f(\mu) > A_f(\eta) \Rightarrow \mu > \eta$;

(iii) $S_f(\mu) = S_f(\eta)$, $A_f(\mu) = A_f(\eta)$, $T_\psi^+(\mu) > T_\psi^+(\eta)$, $T_\psi^-(\mu) < T_\psi^-(\eta) \Rightarrow \mu > \eta$.

Theorem 5.1. The score function and accuracy function of an SVBPNN are bounded.

Proof. Suppose that η=$[T_\psi^-(\eta),C_\psi^-(\eta),G_\psi^-(\eta),U_\psi^-(\eta),F_\psi^-(\eta),T_\psi^+(\eta),C_\psi^+(\eta),G_\psi^+(\eta),U_\psi^+(\eta),F_\psi^+(\eta)]$ be an SVBPNN.

Therefore, $-1\le T_\psi^-(\eta)\le 0$, $-1\le C_\psi^-(\eta)\le 0$, $-1\le G_\psi^-(\eta)\le 0$, $-1\le U_\psi^-(\eta)\le 0$, $-1\le F_\psi^-(\eta)\le 0$, $0\le T_\psi^+(\eta)\le 1$, $0\le C_\psi^+(\eta)\le 1$, $0\le G_\psi^+(\eta)\le 1$, $0\le U_\psi^+(\eta)\le 1$, $0\le F_\psi^+(\eta)\le 1$.

This implies, $0 \le 1+T_\psi^-(\eta)+T_\psi^+(\eta) \le 2$, $0 \le -C_\psi^-(\eta)+1-C_\psi^+(\eta) \le 2$, $0 \le -G_\psi^-(\eta)+1-G_\psi^+(\eta) \le 2$, $0 \le -U_\psi^-(\eta)+1-U_\psi^+(\eta) \le 2$, $0 \le -F_\psi^-(\eta)+1-F_\psi^+(\eta) \le 2$.

Therefore,

$0 \le 1+T_\psi^-(\eta)+T_\psi^+(\eta)-C_\psi^-(\eta)+1-C_\psi^+(\eta)-G_\psi^-(\eta)+1-G_\psi^+(\eta)-U_\psi^-(\eta)+1-U_\psi^+(\eta)-F_\psi^-(\eta)+1-F_\psi^+(\eta) \le 10$

$\Rightarrow 0 \le 1+T_\psi^-(\eta)-C_\psi^-(\eta)-G_\psi^-(\eta)-U_\psi^-(\eta)-F_\psi^-(\eta)+T_\psi^+(\eta)+1-C_\psi^+(\eta)+1-G_\psi^+(\eta)+1-U_\psi^+(\eta)+1-F_\psi^+(\eta) \le 10$

$$\Rightarrow 0 \le \frac{[\,1+T_\psi^-(\eta)-C_\psi^-(\eta)-G_\psi^-(\eta)-U_\psi^-(\eta)-F_\psi^-(\eta)+T_\psi^+(\eta)+1-C_\psi^+(\eta)+1-G_\psi^+(\eta)+1-U_\psi^+(\eta)+1-F_\psi^+(\eta)]}{10} \le 1$$

$\Rightarrow 0 \le S_f(u) \le 1$.

Hence, the score function is bounded.

Again, -1 ≤ $T_{\psi}^{-}(\eta)+T_{\psi}^{+}(\eta)$ ≤ 1, -1 ≤ $-C_{\psi}^{-}(\eta)-C_{\psi}^{+}(\eta)$ ≤ 1, -1 ≤ $-F_{\psi}^{-}(\eta)-F_{\psi}^{+}(\eta)$ ≤ 1

This implies,

$-3 \le T_{\psi}^{-}(\eta)+T_{\psi}^{+}(\eta)-C_{\psi}^{-}(\eta)-C_{\psi}^{+}(\eta)-F_{\psi}^{-}(\eta)-F_{\psi}^{+}(\eta) \le 3$

$\Rightarrow -1 \le \frac{T_{\psi}^{-}(\eta)-C_{\psi}^{-}(\eta)-F_{\psi}^{-}(\eta)+T_{\psi}^{+}(\eta)-C_{\psi}^{+}(\eta)-F_{\psi}^{+}(\eta)}{3} \le 1$

$\Rightarrow -1 \le A_f(\eta) \le 1.$

Hence, the accuracy function is bounded.

Theorem 5.2. The score function and accuracy function of an SVBPNN are monotonic increasing.

Proof. Suppose that $\mu=[T_{\psi}^{-}(\mu), C_{\psi}^{-}(\mu), G_{\psi}^{-}(\mu), U_{\psi}^{-}(\mu), F_{\psi}^{-}(\mu), T_{\psi}^{+}(\mu), C_{\psi}^{+}(\mu), G_{\psi}^{+}(\mu), U_{\psi}^{+}(\mu), F_{\psi}^{+}(\mu)]$ and $\eta=[T_{\psi}^{-}(\eta), C_{\psi}^{-}(\eta), G_{\psi}^{-}(\eta), U_{\psi}^{-}(\eta), F_{\psi}^{-}(\eta), T_{\psi}^{+}(\eta), C_{\psi}^{+}(\eta), G_{\psi}^{+}(\eta), U_{\psi}^{+}(\eta), F_{\psi}^{+}(\eta)]$ be two SVBPNNs over ψ such that $\mu \subseteq \eta$.

Therefore, $T_{\psi}^{-}(\mu) \le T_{\psi}^{-}(\eta)$, $C_{\psi}^{-}(\mu) \ge C_{\psi}^{-}(\eta)$, $G_{\psi}^{-}(\mu) \ge G_{\psi}^{-}(\eta)$, $U_{\psi}^{-}(\mu) \ge U_{\psi}^{-}(\eta)$, $F_{\psi}^{-}(\mu) \ge F_{\psi}^{-}(\eta)$, $T_{\psi}^{+}(\mu) \le T_{\psi}^{+}(\eta)$, $C_{\psi}^{+}(\mu) \ge C_{\psi}^{+}(\eta)$, $G_{\psi}^{+}(\mu) \ge G_{\psi}^{+}(\eta)$, $U_{\psi}^{+}(\mu) \ge U_{\psi}^{+}(\eta)$, $F_{\psi}^{+}(\mu) \ge F_{\psi}^{+}(\eta)$.

It is known that,

$$S_f(\mu) = \frac{[1+T_{\psi}^{-}(\mu)-C_{\psi}^{-}(\mu)-G_{\psi}^{-}(\mu)-U_{\psi}^{-}(\mu)-F_{\psi}^{-}(\mu)+T_{\psi}^{+}(\mu)+1-C_{\psi}^{+}(\mu)+1-G_{\psi}^{+}(\mu)+1-U_{\psi}^{+}(\mu)+1-F_{\psi}^{+}(\mu)]}{10};$$

$$S_f(\eta) = \frac{[1+T_{\psi}^{-}(\eta)-C_{\psi}^{-}(\eta)-G_{\psi}^{-}(\eta)-U_{\psi}^{-}(\eta)-F_{\psi}^{-}(\eta)+T_{\psi}^{+}(\eta)+1-C_{\psi}^{+}(\eta)+1-G_{\psi}^{+}(\eta)+1-U_{\psi}^{+}(\eta)+1-F_{\psi}^{+}(\eta)]}{10};$$

$$A_f(\mu) = \frac{[T_{\psi}^{-}(\mu)-C_{\psi}^{-}(\mu)-F_{\psi}^{-}(\mu)+T_{\psi}^{+}(\mu)-C_{\psi}^{+}(\mu)-F_{\psi}^{+}(\mu)]}{3};$$

$$A_f(\eta) = \frac{[T_{\psi}^{-}(\eta)-C_{\psi}^{-}(\eta)-F_{\psi}^{-}(\eta)+T_{\psi}^{+}(\eta)-C_{\psi}^{+}(\eta)-F_{\psi}^{+}(\eta)]}{3};$$

Now,

$S_f(\eta) - S_f(\mu)$

$$= \frac{[1+T_{\psi}^{-}(\eta)-C_{\psi}^{-}(\eta)-G_{\psi}^{-}(\eta)-U_{\psi}^{-}(\eta)-F_{\psi}^{-}(\eta)+T_{\psi}^{+}(\eta) +1-C_{\psi}^{+}(\eta)+1-G_{\psi}^{+}(\eta)+1-U_{\psi}^{+}(\eta)+1-F_{\psi}^{+}(\eta)]}{10} - \frac{[1+T_{\psi}^{-}(\mu)-C_{\psi}^{-}(\mu)-G_{\psi}^{-}(\mu)-U_{\psi}^{-}(\mu)-F_{\psi}^{-}(\mu)+T_{\psi}^{+}(\mu) +1-C_{\psi}^{+}(\mu)+1-G_{\psi}^{+}(\mu)+1-U_{\psi}^{+}(\mu)+1-F_{\psi}^{+}(\mu)]}{10}$$

≥ 0 [since $\mu \subseteq \eta$]

This implies, $S_f(\eta) \ge S_f(\mu)$, i.e. the score function is monotonic increasing.

Now,

$A_f(\eta) - A_f(\mu)$

$$= \frac{[T_{\psi}^{-}(\eta)-C_{\psi}^{-}(\eta)-F_{\psi}^{-}(\eta)+T_{\psi}^{+}(\eta)-C_{\psi}^{+}(\eta)-F_{\psi}^{+}(\eta)]}{3} - \frac{[T_{\psi}^{-}(\mu)-C_{\psi}^{-}(\mu)-F_{\psi}^{-}(\mu)+T_{\psi}^{+}(\mu)-C_{\psi}^{+}(\mu)-F_{\psi}^{+}(\mu)]}{3}$$

≥ 0 [since $\mu \subseteq \eta$]

This implies, $A_f(\eta) \ge A_f(\mu)$, i.e., the accuracy function is monotonic increasing.

Hence, the score and accuracy functions are monotonic increasing functions.

6. SVBPNS-MADM Strategy Based on SVBPNAM Operator

Suppose that $A = \{A_1, A_2, ..., A_n\}$ be a fixed set of alternatives, and $P = \{P_1, P_2, ..., P_m\}$ be a family of attributes. The decision maker involves in the decision making provides his/her evaluation information of each alternative Q_i (i = 1, 2,…, n) over the attribute P_j (j = 1, 2,..., m) in terms of SVBPNNs. The whole evaluation information of all alternatives can be expressed by a decision matrix.

The proposed SVBPNS-MADM strategy (see Figure 1)is described using the following steps:

Step-1: Construct the decision matrix using SVBPNSs.

The whole evaluation information of each alternative A_i (i = 1, 2,..., n) based on the attributes P_j (j = 1, 2,..., m) is expressed in terms of SVBPNS $E_{A_i} = \{(P_j, T^-_{ij}(A_i, P_j), C^-_{ij}(A_i, P_j), G^-_{ij}(A_i, P_j), U^-_{ij}(A_i, P_j), F^-_{ij}(A_i, P_j), T^+_{ij}(A_i, P_j), C^+_{ij}(A_i, P_j), G^+_{ij}(A_i, P_j), U^+_{ij}(A_i, P_j), F^+_{ij}(A_i, P_j)): P_j \in P\}$, where $(T^-_{ij}(A_i, P_j), C^-_{ij}(A_i, P_j), G^-_{ij}(A_i, P_j), U^-_{ij}(A_i, P_j), F^-_{ij}(A_i, P_j), T^+_{ij}(A_i, P_j), C^+_{ij}(A_i, P_j), G^+_{ij}(A_i, P_j), U^+_{ij}(A_i, P_j), F^+_{ij}(A_i, P_j))$ denote the evaluation information of A_i (i = 1, 2,..., n) based on P_j (j = 1, 2,..., m).

Then the Decision Matrix (DM[$A|P$]) can be expressed as:

DM[$A|P$] =

	P_1	P_2	…	…	P_m
A_1	$[T^-_{11}(A_1, P_1), C^-_{11}(A_1, P_1),$ $G^-_{11}(A_1, P_1), U^-_{11}(A_1, P_1),$ $F^-_{11}(A_1, P_1), T^+_{11}(A_1, P_1),$ $C^+_{11}(A_1, P_1), G^+_{11}(A_1, P_1),$ $U^+_{11}(A_1, P_1), F^+_{11}(A_1, P_1)]$	$[T^-_{12}(A_1, P_2), C^-_{12}(A_1, P_2),$ $G^-_{12}(A_1, P_2), U^-_{12}(A_1, P_2),$ $F^-_{12}(A_1, P_2), T^+_{12}(A_1, P_2),$ $C^+_{12}(A_1, P_2), G^+_{12}(A_1, P_2),$ $U^+_{12}(A_1, P_2), F^+_{12}(A_1, P_2)]$	…	…	$[T^-_{1m}(A_1, P_m), C^-_{1m}(A_1, P_m),$ $G^-_{1m}(A_1, P_m), U^-_{1m}(A_1, P_m),$ $F^-_{1m}(A_1, P_m), T^+_{1m}(A_1, P_m),$ $C^+_{1m}(A_1, P_m), G^+_{1m}(A_1, P_m),$ $U^+_{1m}(A_1, P_m), F^+_{1m}(A_1, P_m)]$
A_2	$[T^-_{21}(A_2, P_1), C^-_{21}(A_2, P_1),$ $G^-_{21}(A_2, P_1), U^-_{21}(A_2, P_1),$ $F^-_{21}(A_2, P_1), T^+_{21}(A_2, P_1),$ $C^+_{21}(A_2, P_1), G^+_{21}(A_2, P_1),$ $U^+_{21}(A_2, P_1), F^+_{21}(A_2, P_1)]$	$[T^-_{22}(A_2, P_2), C^-_{22}(A_2, P_2),$ $G^-_{22}(A_2, P_2), U^-_{22}(A_2, P_2),$ $F^-_{22}(A_2, P_2), T^+_{22}(A_2, P_2),$ $C^+_{22}(A_2, P_2), G^+_{22}(A_2, P_2),$ $U^+_{22}(A_2, P_2), F^+_{22}(A_2, P_2)]$	…	…	$[T^-_{2m}(A_2, P_m), C^-_{2m}(A_2, P_m),$ $G^-_{2m}(A_2, P_m), U^-_{2m}(A_2, P_m),$ $F^-_{2m}(A_2, P_m), T^+_{2m}(A_2, P_m),$ $C^+_{2m}(A_2, P_m), G^+_{2m}(A_2, P_m),$ $U^+_{2m}(A_2, P_m), F^+_{2m}(A_2, P_m)]$
…	…	…	…	…	…
A_n	$[T^-_{n1}(A_n, P_1), C^-_{n1}(A_n, P_1),$ $G^-_{n1}(A_n, P_1), U^-_{n1}(A_n, P_1),$ $F^-_{n1}(A_n, P_1), T^+_{n1}(A_n, P_1),$ $C^+_{n1}(A_n, P_1), G^+_{n1}(A_n, P_1),$ $U^+_{n1}(A_n, P_1), F^+_{n1}(A_n, P_1)]$	$[T^-_{n2}(A_n, P_2), C^-_{n2}(A_n, P_2),$ $G^-_{n2}(A_n, P_2), U^-_{n2}(A_n, P_2),$ $F^-_{n2}(A_n, P_2), T^+_{n2}(A_n, P_2),$ $C^+_{n2}(A_n, P_2), G^+_{n2}(A_n, P_2),$ $U^+_{n2}(A_n, P_2), F^+_{n2}(A_n, P_2)]$	… … … …	… … … …	$[T^-_{nm}(A_n, P_m), C^-_{nm}(A_n, P_m),$ $G^-_{nm}(A_n, P_m), U^-_{nm}(A_n, P_m),$ $F^-_{nm}(A_n, P_m), T^+_{nm}(A_n, P_m),$ $C^+_{nm}(A_n, P_m), G^+_{nm}(A_n, P_m),$ $U^+_{nm}(A_n, P_m), F^+_{nm}(A_n, P_m)]$

Step-2: In this step, the decision maker determines the aggregation values (A_i | P_1, P_2,…, P_m) = SVBPNAM (P_1, P_2,…, P_m) of all the attributes for each alternative by using the eq. (1). After the determination of aggregation values SVBPNAM (P_1, P_2,…, P_m), the decision maker makes an aggregate decision matrix aggregate-D_M.

Step-3: In this step, the decision maker determines the score and accuracy values of each alternative by using the eqs. (3) and (4).

Step-4: In this step, the decision maker ranks the alternatives by using Definition 5.1. and Definition 5.2.

Step-5: End.

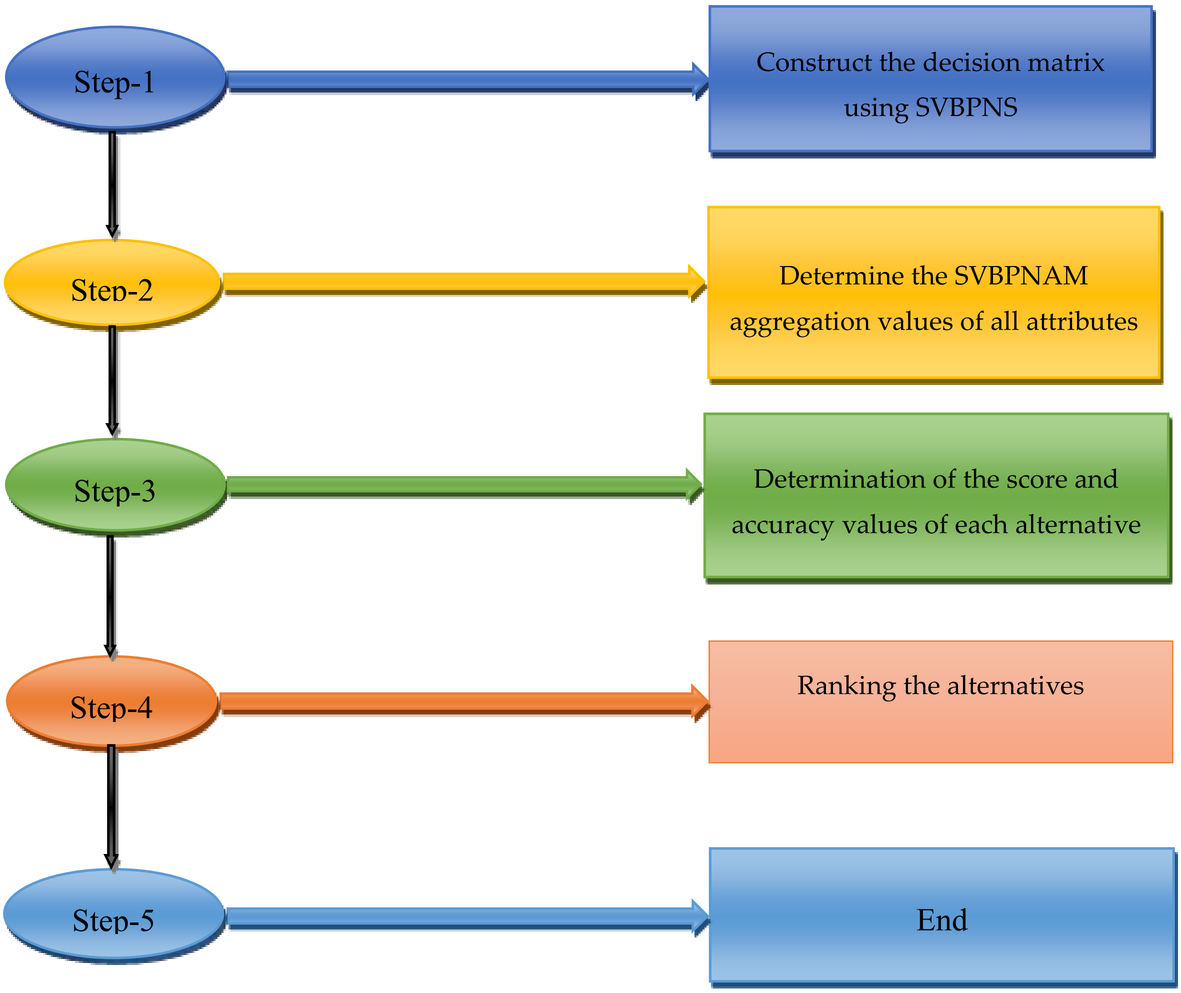

Figure 1: Flow chart of the SVBPNS-MADM Strategy based on SVBPNAM operator

7. SVBPNS-MADM Strategy Based on SVBPNGM Operator

Consider the same MADM problem which is considered in section 6. Then the proposed SVBPNS-MADM strategy (see Figure 2) can be described by the following steps:

Step-1: Construct the decision matrix using SVBPNSs.

It is similar to the step-1 of the section 6.

Step-2: In this step, the decision makers determine the aggregation values (A_i | $P_1, P_2,\ldots, P_m$) = SVBPNGM ($P_1, P_2,\ldots, P_m$) of all the attributes for each alternative by using the eq. (1). After the determination of aggregation values SVBPNGM ($P_1, P_2,\ldots, P_m$), the decision maker makes an aggregate decision matrix aggregate-D_M.

Step-3: In this step, the decision maker determines the score and accuracy values of each alternative by using the eqs. (3) and (4).

Step-4: In this step, the decision maker ranks the alternatives by using Definition 5.1. and Definition 5.2.

Step-5: End.

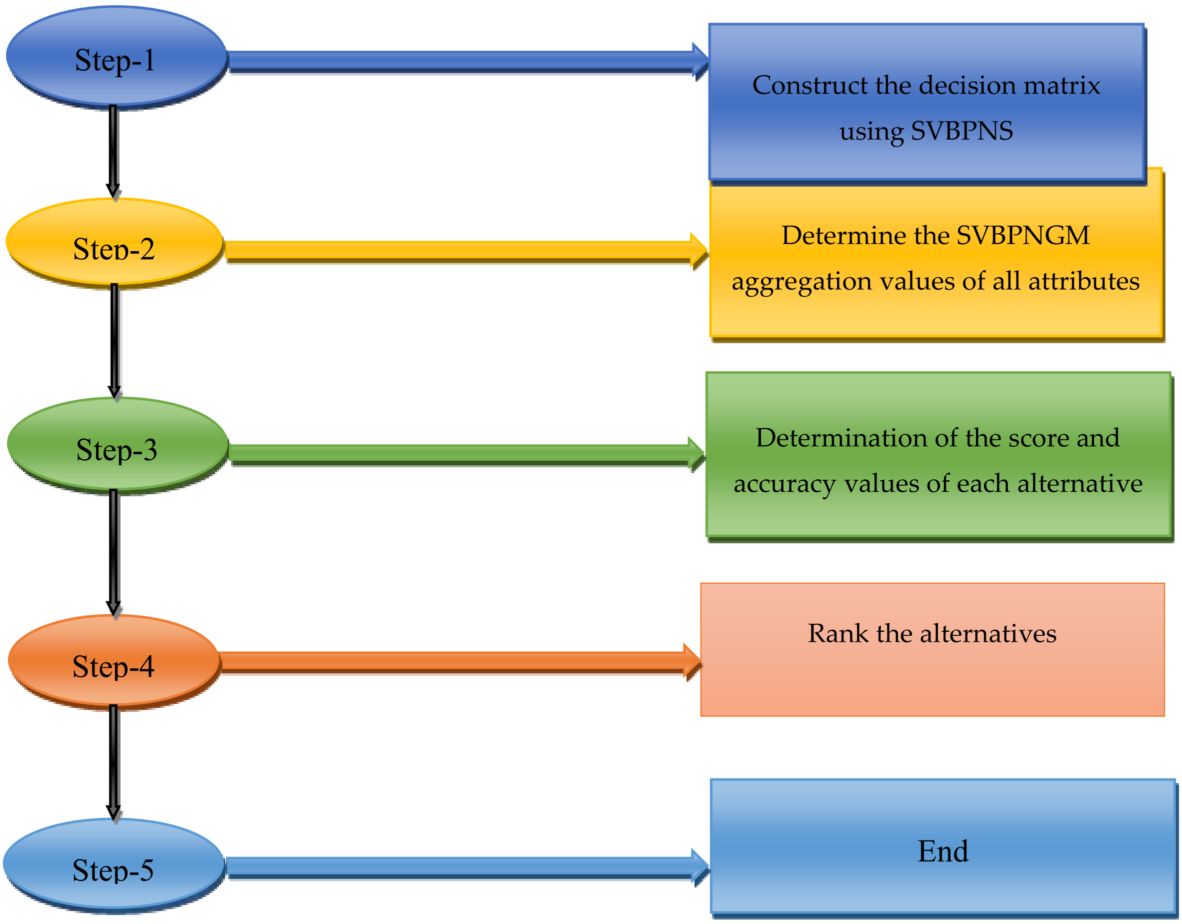

Figure 2: Flow chart of the SVBPNS-MADM Strategy based on SVBPNGM operator

8. Validation of the Proposed SVBPNS-MADM Strategies:

In this section, we present a realistic example of "University Selection for Admission into Various Degree Course" to validate the proposed SVBPNS-MADM strategies based on both SVBPNAM operator and SVBPNGM operator.

8.1. Example: "University Selection for Admission into Various Degree Course".

The selection of university for getting admission for higher education by the students who just have passed the higher secondary or college from any stream can be considered as an MADM problem. To select the best university for higher education, the students must need to select some attributes based on which they select the best university. After the initial screening, the decision maker (student) chooses three alternatives (Universities) for further screening. Suppose the alternatives (Universities) are A_1, A_2, A_3. After the consultation with experts the decision makers (students) can choose three major attributes namely

P_1 **(Faculty):-** In an educational institution, faculty has the most important role for the system as well as students. The number of faculty members and the quality of the faculty members that is the profile of faculty is too important. Only faculty can help and find the creative students for the success of the social. A good quality Teacher encourages students to come to class from time to time with work interest.

P_2 **(NAAC-Grade):-** In India, UGC gives different grades based on their different performance. Higher learning institutes in India are graded for each key aspect/ parameter under different categories such as 'A', 'B', 'C', and 'D'. The NAAC grade indicates the overall performance of an institution such as very good, good, satisfactory, and unsatisfactory.

P_3 **(Government University / Private University):**- Most of the time a central University certificate has more value than a state university. It's generally seen that the government universities charge a lower tuition fee than private universities. There are also more opportunities for a fee reduction in government universities with scholarships and/or quota-based benefits (SC/ST/OBC/EWS, etc.). So there are many issues on this regard that is why we are taking a criterion on this objective.

P_4 **(Infrastructure):** A high-grade university infrastructures [30] must have a dynamic facility. The infrastructure criteria for being a world-class university are:

(1) Physical infrastructure,

(2) Digital infrastructure,

(3) Innovative academic & training Infrastructure for confidence building,

(4) Intellectual property infrastructure,

(5) Emotional infrastructure, and

(6) Network infrastructure,

Based on the rating of the alternatives in terms of SVBPNNs, the decision matrix D_M (see Table-1) is constructed as follows:

Table-1:

D_M	P_1	P_2	P_3	P_4
A_1	(-0.3,-0.5,-0.4,-0.6,-0.3, 0.3,0.6,0.5,0.4,0.2)	(-0.7,-0.2,-0.6,-0.5-0.6, 0.4,0.5,0.5,0.3,0.7)	(-0.4,-0.6,-0.3,-0.5,-0.5, 0.7,0.8,0.5,0.6,0.5)	(-0.1,-0.2,-0.8,-0.1,-0.8, 0.9,0.2,0.8,0.4,0.1)
A_2	(-0.3,-0.7,-0.5,-0.5,-0.3, 0.3,0.5,0.4,0.3,0.2)	(-0.6,-0.6,-0.5,-0.4,-0.5, 0.5,0.4,0.5,0.6,0.5)	(-0.5,-0.5,-0.6,-0.4,-0.2, 0.8,0.6,0.4,0.2,0.6)	(-0.2,-0.2,-0.5,-0.8,-0.9, 1.0,0.7,0.5,0.4,0.4)
A_3	(-0.5,-0.5,-0.7,-0.5,-0.8, 0.6,0.3,0.4,0.6,0.8)	(-0.5,-0.4,-0.7,-0.5,-0.4, 0.8,0.5,0.6,0.5,0.4)	(-0.5,-0.5,-0.2,-0.4,-0.8, 1.0,0.6,0.4,0.2,0.5)	(-0.1,-0.5,-0.4,-0.1,-0.7, 1.0,0.7,0.4,0.3,0.3)

In Table 2, we calculate the aggregation values (A_i | P_1, P_2, P_3) of all attributes for each alternative A_i, by using the SVBPNAM operator.

Table-2: Aggregate-D_M

	(A_i | P_1, P_2, P_3)
A_1	(-0.30274,-0.96634,-0.99149,-0.9767,-0.59094,0.664963,0.468069,0.562341,0.411953,0.289251)
A_2	(-0.36628,-0.98778,-0.98726,-0.99088,-0.59094,1.00000,0.538356,0.447214,0.34641,0.393598)
A_3	(-0.33437,-0.9807,-0.98902,-0.96439,-0.7087,1.00000,0.500997,0.442673,0.366284,0.468069)

By using eq (2), we get $S_f(A_1)$= 0.7156079; $S_f(A_2)$= 0.7465002; $S_f(A_3)$= 0.7530417.

Therefore, $S_f(A_1) < S_f(A_2) < S_f(A_3)$.

The ranking order is obtained as: $A_1 < A_2 < A_3$.

Hence, A_3 is the best university for getting admission among the set of alternatives (universities).

In table 3, we calculate the aggregation values (A_i | P_1, P_2, P_3) of all attributes for each alternative A_i, by using the SVBPNGM operator.

Table-3: Aggregate-D_M

	(A_i | P_1, P_2, P_3)
A_1	(-0.4197,-0.33098,-0.4899,-0.34996,-0.518,0.524361,0.577051,0.602365,0.436537,0.426734)
A_2	(-0.4215,-0.4527,-0.52332,-0.50297,-0.40536,0.588566,0.564412,0.452277,0.39452,0.443368)
A_3	(-0.42085,-0.47287,-0.44496,-0.31623,-0.65063,0.832358,0.547298,0.457839,0.421498,0.547298)

By using eq. (2), we get $S_f(A_1)$= 0.4750814; $S_f(A_2)$= 0.5196839; $S_f(A_3)$= 0.5322265.

Therefore, $S_f(A_1) < S_f(A_2) < S_f(A_3)$.

The ranking order is obtained as: $A_1 < A_2 < A_3$.

Hence, A_3 is the best university for getting admission.

Table 4: Ranking order of alternatives

Strategies	Ranking order	Best alternative
SVBPNS-MADM strategy based on BPNAM operator.	$A_1 < A_2 < A_3$	A_3
SVBPNS-MADM strategy based on BPNAM operator.	$A_1 < A_2 < A_3$	A_3

Both the SVBPNS-MADM strategies offer the same ranking order of the alternatives (See table 4) and A_3 is the best university for getting admission.

9. Conclusions

In this paper, we introduce the notion of SVBPNS, and prove its basic properties and operations. We define the score and accuracy functions of SVBPNNs, and prove their basic properties. Besides, we define two aggregation operators namely, single-valued bipolar pentapartitioned neutrosophic arithmetic mean operator and the single-valued bipolar pentapartitioned neutrosophic geometric mean operator, and prove their basic properties. Based on these two operators, we develop two new MADM strategies and present a numerical example in SVBPNS environment to show the applicability of SVBPNS in MADM. The developed strategies can be further used for the other MADM problems [31-34], medical diagnosis [35-36] , risk analysis [37], and so on.

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Ensemble Classifiers for Acute Leukemia Classification Using Microarray Gene Expression Data under uncertainty

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Abstract: One of the most prevalent cancers in children and adults, acute leukemia has the potential to lead to death if left untreated. Within a few weeks after diagnosis, childhood ALL has spread throughout the body, posing a serious health risk to the patient. Evaluation of acute leukemia contains uncertainty and incomplete information. Due to the subjective nature of the expectations, this rating procedure incorporates ambiguity and inaccuracy. To illustrate the ambiguity of our subjective judgments, we can use the triplet T, F, and I, truth, falsity, and indeterminacy (I). Therefore, a Single-Valued Neutrosophic Sets (SVNSs) approach based on AHP, TOPSIS, and VIKOR is designed and implemented in this article. Neutrosophic AHP is used to determine the weighting of criteria in this methodology. A neutrosophic TOPSIS and VIKOR model are used to rank alternatives. There is further validation and verification of the proposed methodology in the application. To demonstrate the adaptability of the offered decisions under various circumstances, sensitivity assessments and comparative analyses were carried out.

Keywords: AHP; TOPSIS; VIKOR; Acute Leukemia; Neutrosophic; MCDM

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1. Introduction and Background

There are a wide variety of blood-related diseases known as acute leukemia, which are defined by aberrant growth of blast cells in bone marrow, which results in the replacement of healthy cells and a decrease in the 3 hematopoietic types in peripheral blood.

Approximately 300,000 people are expected to die from them in 2018, making them the 11th and 10th greatest common causes of cancer in the world, respectively. There are 3.7 new cases of acute myeloid leukemia per 100,000 residents in Europe each year, with only 19 percent of those patients surviving for five years[1]. A precise and appropriate diagnosis is essential to successful disease control. In the bone marrow, immature lymphocytes cause acute lymphoblastic leukemia (ALL), also known as acute lymphocytic leukemia [2], [3]. Upon entering the bloodstream, leukemic cells move rapidly to several organs and tissues, including the spleen, liver, lymph nodes, brain, and the neurological system. The bone marrow and blood are primarily affected by ALL, which is a disease of the immune system [4], [5]. It is also known as acute pediatric leukemia because it is the most prevalent kind of leukemia in children since chronic and myeloid leukemias are rare in children.

For an accurate diagnosis of acute leukemia, the World Health Organization (WHO) recommends combining morphology with additional tests like immunophenotype, cytogenetics, and molecular biology[6]. As a result, finding blasts in the blood is still the first step in their diagnosis. It is true that smear review takes a long time, requires well-trained staff, and is subject to base on inter variability, which is especially important when dealing with the blast. Indeed, leukemia types have small interclass morphological variations, which results in low specificity scores during routine screening[7]. There's little doubt that clinical pathologists have difficulty distinguishing between different types of blasts and the subjective nature of their morphological identification. Leukemia lineage identification is critical since the prognosis and acute treatment effects are heavily dependent on this differentiation. Although automated blood cell image analyzers tend to underestimate the amount of blast cells, this complex topic hasn't been addressed in the literature[8], [9].

Medical diagnosis has been refrained by statistical approaches, pattern recognition, artificial intelligence, and neural networks[10]–[14]. To make medical diagnoses easier, another tool called a "MCDA" was developed. The MCDA approach uses the preference relational system proposed by Roy in 1996 [15] and Vincke in 1992 [16] to compare the individuals to be categorized and prototypes (prototypes are the reference points of classes).

It is so possible to use both qualitative and quantitative criteria in the MCDA approach. Additionally, it aids in overcoming some of the challenges associated with expressing data in several units.

In addition, several researchers provide certain improvements and enhancements to improve acute leukemia classification performance by better representing and reflecting acute leukemia data. Reviewing the above expansions reveals that the various types of uncertainty in the data set are primarily to blame for these additional versions. Different forms of fuzzy set extensions are used to address the uncertainty in the data set since it may contain vagueness, imprecision, indeterminacy, and hesitant information. There is no middle ground in classical set theory, optimization, and Boolean logic. An element can either belong to a set or not, and a statement can only be true or false[17]. The problem is that in the real world, hardly anything is accurate and it's all a relative term that cannot be characterized by classical reasoning. This type of ambiguity was addressed by Zadeh's fuzzy sets theory [18]. Since its inception in 1965, it has been reimagined in some ways. By introducing type-2 fuzzy sets, the mathematical procedures of Zadeh were able to better depict their imprecision [19]. A concept called "intuitional fuzzy sets" (also known as "membership degrees") was first developed by Atanassov in 1986 [20].

Afterward, Smarandache presents neutrosophic sets that have three distinct subsets to reflect different sorts of uncertainty [21]. Each element in the cosmos has a degree of truthiness, indeterminacy, and falsehood between 0 and 1, and these degrees are independent subsets of the neutrosophic sets [21]. To discriminate between degrees of belonging and non-belongingness and to depict absoluteness from relativeness, indeterminacy functions are used in neutrosophic sets. Neutrosophic sets use this notation to deal with the system's uncertainty and lessen the indecision caused by conflicting data. The neutrosophic sets have the most essential benefit over other fuzzy extensions in this regard. Three functions of neutrosophic sets give a domain area that can be used to undertake mathematical operations with varying degrees of uncertainty.

Analytic Hierarchy Process (AHP), developed by Saaty [22], is a well-known technique for solving complicated problems by breaking them down into subproblems and then combining the solutions of these subproblems. It is critical to ensure that the judgments are consistent in this procedure, which uses pairwise comparisons of experts. According to the literature [23]–[32], AHP is frequently utilized as a standard procedure.

When faced with uncertainty and incomplete information, one popular decision-making technique is the TOPSIS approach, which allows for a wide range of alternatives and criteria to be considered in the decision-making process [33]. Consequently, TOPSIS is an excellent method for determining the predicted usefulness of a scenario that is ambiguous, lacking information, or vague. Using the TOPSIS technique, it is possible to identify a short distance from the ideal solution and a long distance from the negative-ideal solution, but these distances are not reflected in their proportionate significance.

Serafim Opricovic (1998) first developed the VIKOR technique, which was first applied in 2004 by Opricovic and Tzeng to solve multicriteria decision-making problems. To begin, there is a compromise solution, which is closer to an ideal answer than any other option available.

Many studies employ the SVNS technique. Distance measurement for SVNSs was first proposed by ahin and Küçük [34] using the neutrosophic subset idea. Several steps in the analysis of Ye [35] were shown to be unrealistic by Peng et al. [36]. Making decisions using machine learning methods has recently become popular [37]. There is also a growing usage of deep learning and other types of learning-based methodologies in the field of decision-making [38] in engineering research [39]–[44]. Machine learning, on the other hand, has its drawbacks, such as the fact that it requires a distinct training phase each time and is only applicable to the data it is trained on. The current scoring function and distance measure utilized in many research with SVNSs yielded erroneous results, according to an analysis of the literature. As a result of this research, we have devised a new score function and a new distance measure.

The remainder of this paper is structured as follows: Section 2 outlines the method that will be taken and lays the groundwork for it. Applicability is shown in Section 3, which includes problem definitions, computations, and results. Section 4 provides comparative assessments and section 5 provides sensitivity analysis. Section 6 concludes with some final thoughts and ideas for future research.

2. Methodology

By Saaty, the AHP approach was invented, which allows for comparisons between two variables. This study proposes a single-valued neutrosophic (SVN) AHP approach.

Step 1: Build the comparison matrix between criteria as:

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1b} \\ \vdots & \ddots & \vdots \\ X_{a1} & \cdots & X_{ab} \end{pmatrix} \quad (1)$$

Where $a = 1,2,3 \ldots, e\ (alternatives), b = 1,2,3 \ldots, f\,(criteria)$

Step 2: Compute the score function as:

$$S(X) = \frac{2+a-b-c}{3} \quad (2)$$

Which, a, b and c present truth, indeterminacy, and falsity values.

Step 3: Normalize the comparison matrix as:

$$N(A) = \frac{A_a}{\sum_{a=1}^{e} A_a} \quad (3)$$

where, A_a value in comparison matrix and $\sum_{a=1}^{e} A_a$ sum all values in each column.

Step 4: Compute the weights of criteria by taking the average row.

Step 5: Check the consistency ratio (CR)

Apply the Steps of the TOPSIS method

The steps of the TOPSIS approach are as follows:

A. A decision matrix should be built.
B. Make the decision-making matrix uniform.
C. Make a decision matrix that is normalised and weighted.
D. Decide on the ideal remedies for the positive and negative scenarios you're dealing with.

E. The ideal solutions, both positive and negative, are at a certain distance from each alternative.

F. Distance measurements can be used to calculate the relative closeness coefficients.

G. Rank alternatives

Step 6: Build the decision matrix between criteria and alternatives as Eq. (1), then convert the neutrosophic values to one value by Eq. (2).

Step 7: Normalize the decision matrix as:

$$Nor_{ef} = \frac{X_{ef}}{\sqrt{\sum_{f=1}^{b} X_{ef}^2}} \quad (4)$$

Step 8: Compute the weighted normalized decision matrix as:

$$WN_{ef} = Nor_{ef} * W_f \quad (5)$$

where W_f the weights of the criteria

Step 9: Compute the positive and cost ideal solution as:

$$PI_e^+ = \max_e WN_{ef} \text{ for positive criteria} \quad (6)$$

$$PI_e^- = \min_e WN_{ef} \text{ for cost criteria} \quad (7)$$

$$PI_e^- = \min_e WN_{ef} \text{ for positive criteria} \quad (8)$$

$$PI_e^- = \max_e WN_{ef} \text{ for cost criteria} \quad (9)$$

Step 10: Compute the distance of each alternative from the positive and cost criteria as:

$$DI_f^+ = \sqrt{\sum_e^b (WN_{ef} - PI_e^+)^2} \quad (10)$$

$$DI_f^- = \sqrt{\sum_e^b (WN_{ef} - PI_e^-)^2} \quad (11)$$

Step 11: Compute the closeness coefficient as:

$$CC_f = \frac{DI_f^-}{DI_f^- + DI_f^+} \quad (12)$$

Step 12: Rank alternatives according to descending values of CC_f

Apply the steps of the VIKOR method

Step 13: Compute the positive and cost ideal solution as:

$$CI_e^+ = \max_e X_{ef} \text{ for positive criteria} \quad (13)$$

$CI_e^- = \min_e X_{ef}$ for cost criteria (14)

$CI_e^- = \min_e X_{ef}$ for positive criteria (15)

$CI_e^- = \max_e X_{ef}$ for cost criteria (16)

Step 14: Compute the value of C_e and D_e as:

$$C_e = \sum_{f=1}^{b} W_f * \frac{CI_e^+ - X_{ef}}{CI_e^+ - CI_e^-} \quad (17)$$

$$D_e = \max_f W_f * \frac{CI_e^+ - X_{ef}}{CI_e^+ - CI_e^-} \quad (18)$$

Step 15: Compute the value of G_e as:

$$G_e = i * \left(\frac{C_e - \min_e C_e}{\max_e C_e - \min_e C_e}\right) + (1 - i) * \left(\frac{D_e - \min_e D_e}{\max_e D_e - \min_e D_e}\right) \quad (19)$$

Where the value of i is in the range 0 to 1. It refers to the utility degree. We use the i =0.5.

Step 16: Rank alternatives according to ascending value of G_e.

3. Application

To validate the steps of the methodology, we apply them with the application. We collected the criteria and alternatives from previous studies as in Fig 1. The decision-makers are selected according to their experts in this field to evaluate the criteria and alternatives by using the single-valued neutrosophic numbers (SVNNs) as [45]. Then we convert the SVNNs into one value by applying Eq. (2) score function. This matrix is called a comparison matrix where data between criteria. Then compute the normalized comparison matrix in Table 1. Then compute the weights of criteria where $w_1 = 0.138656, w_2 = 0.163386, w_3 = 0.168678, w_4 = 0.227536, w_5 = 0.301744$. According to [46] the opinions of experts are consistent. Then go-ahead to apply the steps of the TOPSIS method.

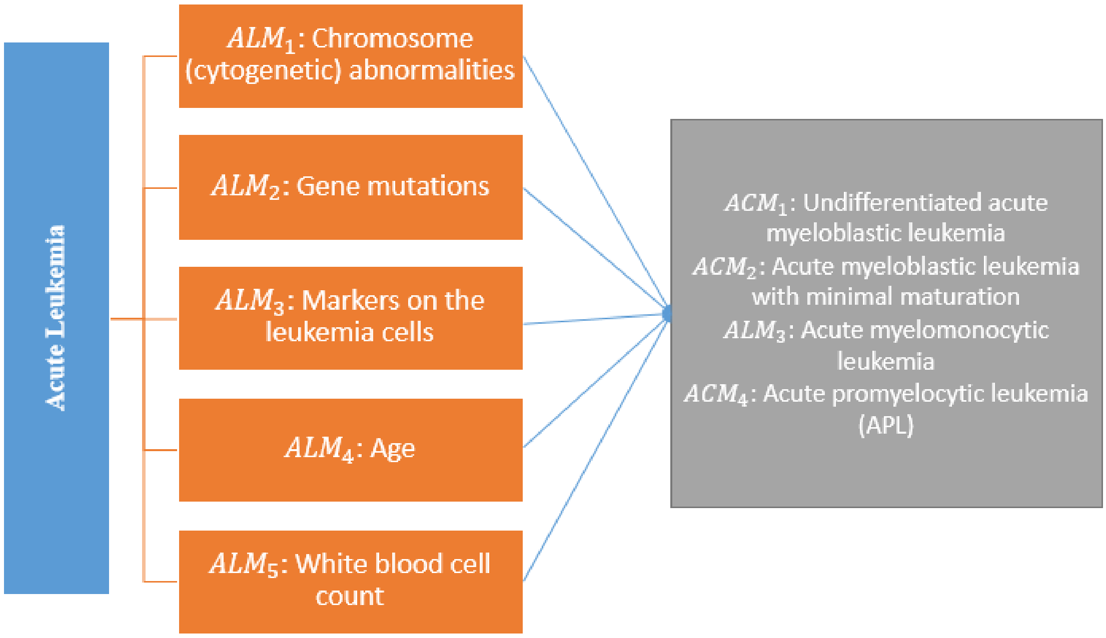

Fig 1. The criteria and alternatives in this study.

Table 1. Normalized comparison matrix by the AHP method.

Criteria	ALM_1	ALM_2	ALM_3	ALM_4	ALM_5
ALM_1	0.076252	0.128138	0.113528	0.242515	0.132848
ALM_2	0.186664	0.07842	0.048313	0.22015	0.283385
ALM_3	0.16945	0.409503	0.063071	0.103204	0.098162
ALM_4	0.16945	0.19197	0.329354	0.134731	0.312175
ALM_5	0.398184	0.19197	0.445734	0.299401	0.17343

Let experts build the decision matrix between criteria and alternatives. Then normalize the decision matrix as Eq. (4) in Table 2. Then compute the weighted normalized decision matrix as Eq. (5). All criteria are positive so, we apply Eqs. (6 and 8) to obtain a positive and cost ideal solution. Then compute the distance of each alternative from the positive and cost ideal solution as Eq. (10) in Table 3. Then compute the closeness coefficient as Eq. (12). Finally rank alternatives according to the biggest value of the closeness coefficient as $ACM_3 \succ ACM_1 \succ ACM_4 \succ ACM_2$. Fig 2. Show the rank of alternatives by the TOPSIS method.

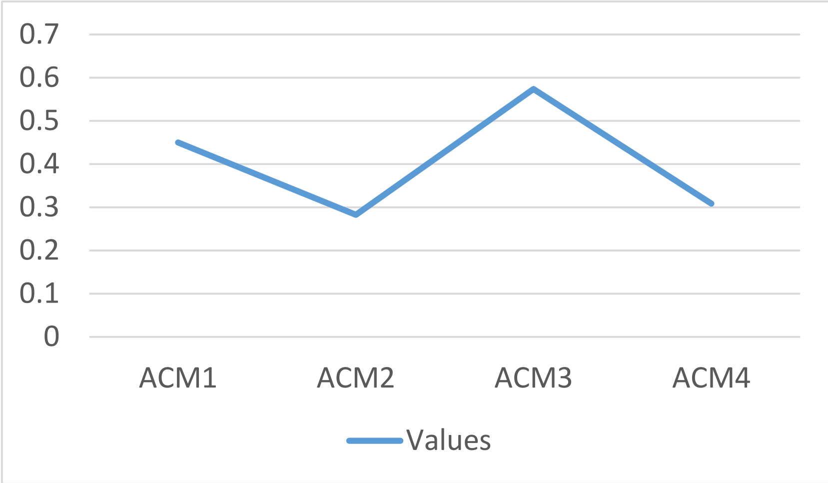

Fig 2. The rank of alternatives by the TOPSIS method.

Table 2. Normalized decision matrix by the TOPSIS method.

Criteria/alternatives	ALM_1	ALM_2	ALM_3	ALM_4	ALM_5
ACM_1	0.310344828	0.166521739	0.268576544	0.461775269	0.160249151
ACM_2	0.097586207	0.355217391	0.243807819	0.145202668	0.216874292
ACM_3	0.310344828	0.123043478	0.243807819	0.196511031	0.462627407
ACM_4	0.281724138	0.355217391	0.243807819	0.196511031	0.160249151

Table 3. Distance of each alternative from positive and cost criteria.

Criteria/alternatives	ALM_1	ALM_2	ALM_3	ALM_4	ALM_5
ACM_1	0	0.000950507	0	0	0.008324884
ACM_2	0.000870268	0	1.74551E-05	0.005188551	0.005498891
ACM_3	0	0.001438991	1.74551E-05	0.003642981	0
ACM_4	1.57485E-05	0	1.74551E-05	0.003642981	0.008324884

By using the decision matrix from the TOPSIS method, the VIKOR method used the Eqs. (13 and 15) to compute the positive and cost ideal solution in Table 4. Eqs. (17 and 18) are used to compute the values of C_e, D_e. Then Eq. (19) is used to compute the values of G_e. Then rank alternatives according to ascending values of G_e. $ACM_3 \succ ACM_1 \succ ACM_2 \succ ACM_4$. Fig 3 shows the rank of alternatives by the VIKOR method.

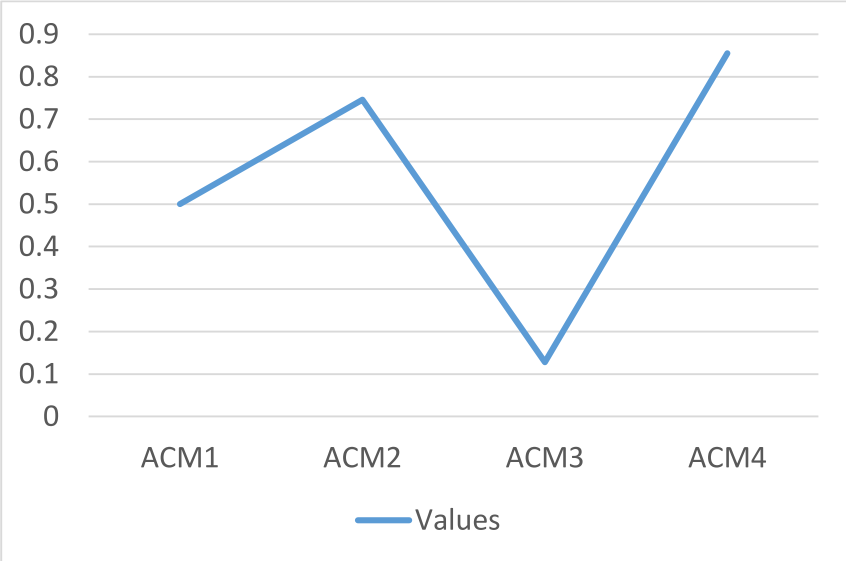

Fig 3. Rank of alternatives by the VIKOR method.

Table 4. The positive and cost ideal solution by the VIKOR method.

Criteria/alternatives	ALM_1	ALM_2	ALM_3	ALM_4	ALM_5
ACM_1	0	0.13279	0	0	0.301744
ACM_2	0.138656	0	0.168678	0.227536	0.245238
ACM_3	0	0.163386	0.168678	0.190658	0
ACM_4	0.018652	0	0.168678	0.190658	0.301744

4. Comparative analysis

In this section, we compare our methods (SVNNs TOPSIS and VIKOR) with Bipolar Neutrosophic Numbers (BNNs VIKOR and TOPSIS) [47] to show the validity of our proposed model. We used the same weights. Fig 4. Show the rank of alternatives under four methods. Table 5 shows the best and worst alternatives. All four methods show the ACM_3 is the best alternative. The SVNNs TOPSIS show ACM_2 is the worst alternative and other three methods show ACM_4 is the worst alternative. Table 5 show the correlation between four methods. The correlation between the four methods is strong.

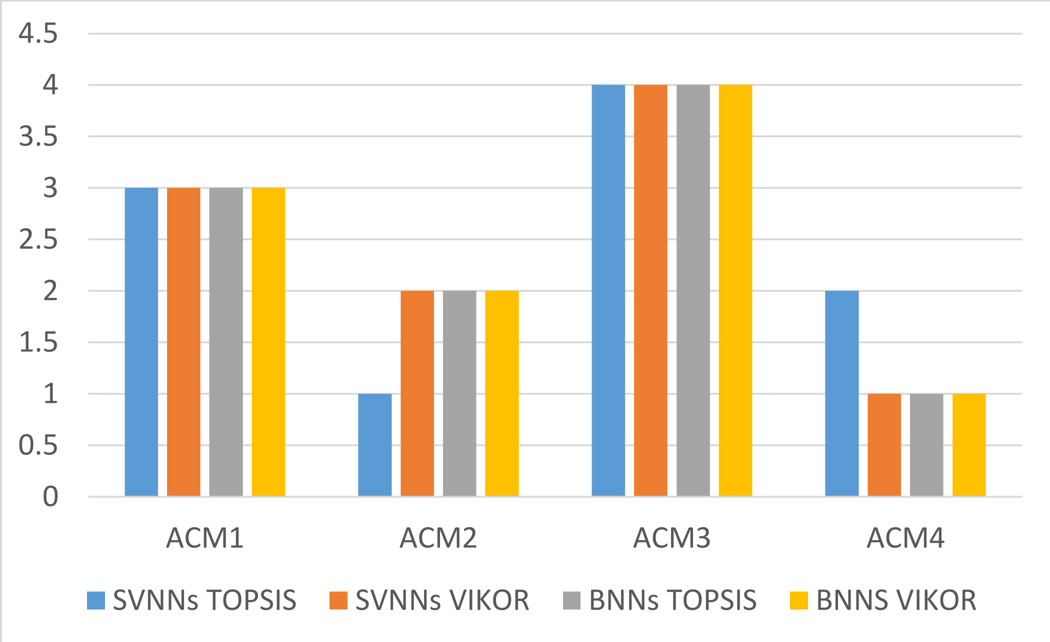

Fig 4. The rank of alternative under comparative analysis.

Table 5. The rank of alternatives by the four methods.

Methods	*Rank*
SVNNs TOPSIS	$ACM_3 \succ ACM_1 \succ ACM_4 \succ ACM_2$
SVNNs VIKOR	$ACM_3 \succ ACM_1 \succ ACM_2 \succ ACM_4$
BNNs TOPSIS	$ACM_3 \succ ACM_1 \succ ACM_2 \succ ACM_4$
BNNS VIKOR	$ACM_3 \succ ACM_1 \succ ACM_2 \succ ACM_4$

Table 6. Pearson correlation between methods.

Methods	Correlation
SVNNs TOPSIS and SVNNs VIKOR	0.8
SVNNs TOPSIS and BNNs TOPSIS	0.8
SVNNs TOPSIS and BNNs VIKOR	0.8
SVNNs VIKOR and BNNs TOPSIS	1
SVNNs VIKOR and BNNs VIKOR	1

5. Sensitivity Analysis

In this section, we change the weights of criteria and then compute the rank of alternatives. Table 7 shows the five cases in changing weights of criteria. In Fig 5. The weights of criteria under five cases. In each case we put the weight by 0.5 and the rest of 0.5 is disrupted to all other criteria. For example, in case 1, the first criteria is 0.5 and the other criteria have 0.125 weights. Fig 6. Show the rank of alternatives under five cases by the TOPSIS method. Fig 7. Show the rank of alternatives under five cases by the VIKOR method. Table 8 and Table 9. Show the rank of alternatives by the TOPSIS and VIKOR methods under five cases. In case 1, we put the first criteria with 0.5 weight and the other four criteria have 0.125. In case 2, the second criteria have 0.5 and the other four criteria have 0.125. In case 3, the third criteria have 0.5 and the other four criteria have 0.125. In case 4, the fourth criteria has 0.5 and the other four criteria have 0.125. In case 5, the fifth criteria have 0.5 and the other four criteria have 0.125.

Table 7. The five cases change the weights of the criteria.

Criteria	Case 1	Case 2	Case 3	Case 4	Case 5
ALM_1	0.5	0.125	0.125	0.125	0.125
ALM_2	0.125	0.5	0.125	0.215	0.125
ALM_3	0.125	0.125	0.5	0.125	0.125
ALM_4	0.125	0.125	0.125	0.5	0.125
ALM_5	0.125	0.125	0.125	0.125	0.5

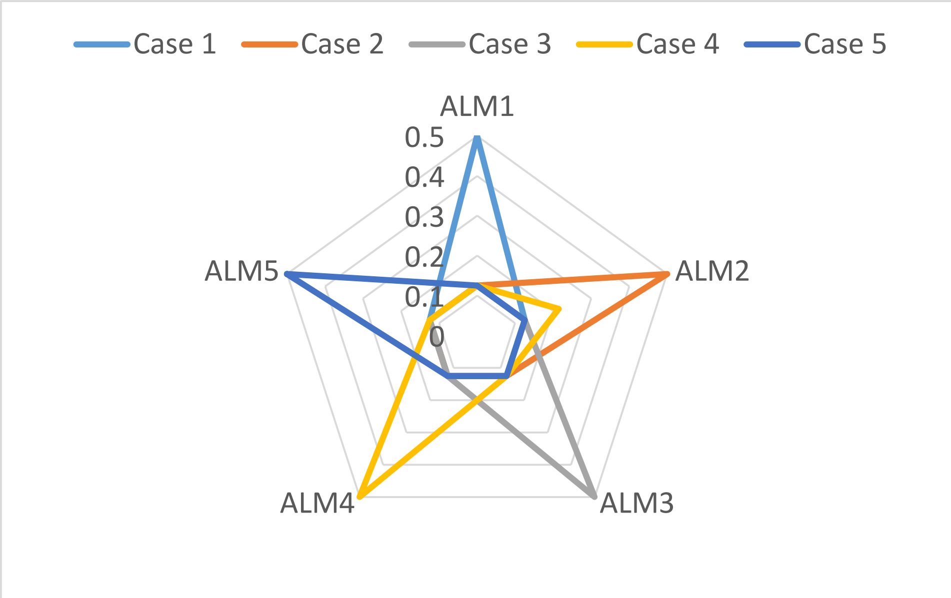

Fig 5. The weights of criteria under five cases.

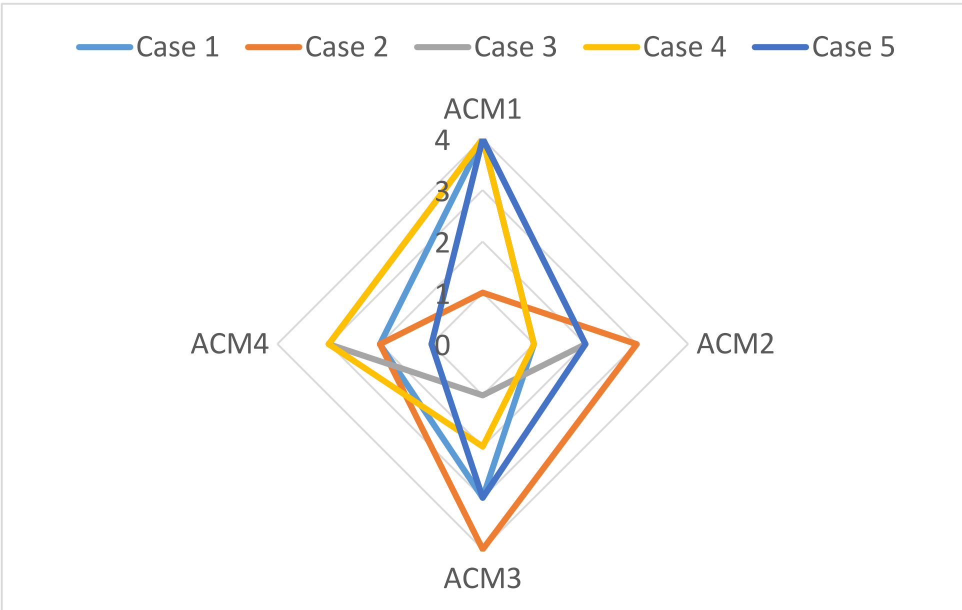

Fig 6. The rank of the TOPSIS method under five cases.

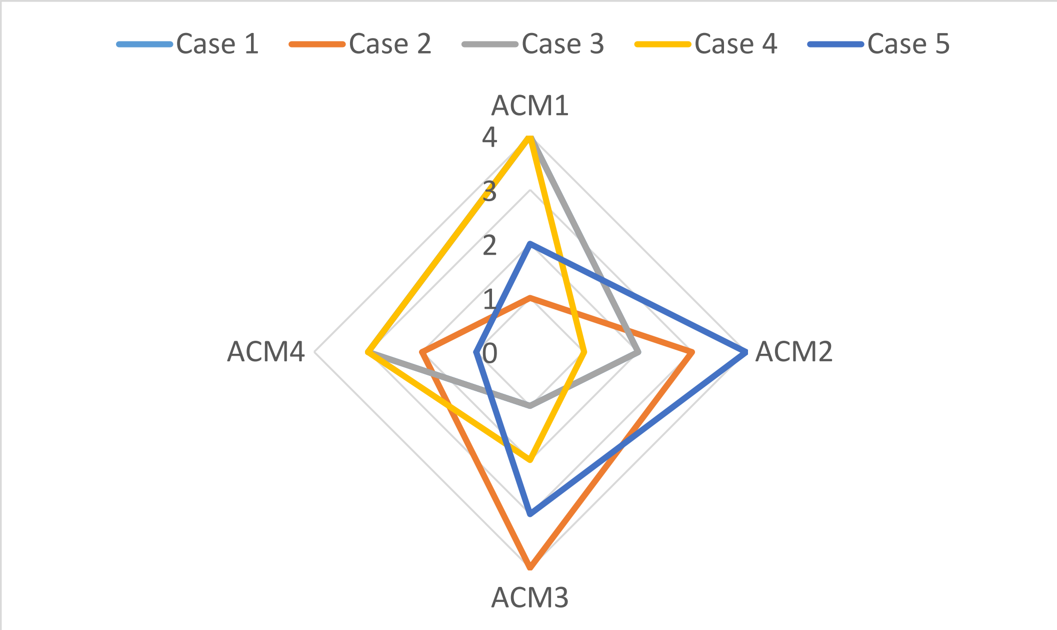

Fig 7. The rank of the VIKOR method is under five cases.

Table 8. The rank of alternatives by the TOPSIS method under five cases.

Cases	Rank by the TOPSIS method
Case 1	$ACM_1 \succ ACM_3 \succ ACM_4 \succ ACM_2$
Case 2	$ACM_4 \succ ACM_2 \succ ACM_1 \succ ACM_3$
Case 3	$ACM_1 \succ ACM_3 \succ ACM_4 \succ ACM_2$
Case 4	$ACM_1 \succ ACM_4 \succ ACM_3 \succ ACM_2$
Case 5	$ACM_1 \succ ACM_3 \succ ACM_2 \succ ACM_4$

Table 9. The rank of alternatives by the VIKOR method under five cases.

Cases	Rank by the TOPSIS method
Case 1	$ACM_1 \succ ACM_3 \succ ACM_4 \succ ACM_2$
Case 2	$ACM_4 \succ ACM_2 \succ ACM_1 \succ ACM_3$
Case 3	$ACM_1 \succ ACM_3 \succ ACM_4 \succ ACM_2$
Case 4	$ACM_1 \succ ACM_4 \succ ACM_3 \succ ACM_2$
Case 5	$ACM_3 \succ ACM_1 \succ ACM_2 \succ ACM_4$

6. Conclusions

According to the results of this study, a new method to prioritize acute leukemia based on their weight is proposed. Neutrosophic AHP and neutrosophic TOPSIS and VIKOR are used in the suggested approach to rank alternatives in acute leukemia. Because the relationships between acute leukemia data are likewise represented using neutrosophic numbers, the integration of all these components is also carried out using neutrosophic operations.

This methodology can be employed in future research on any other MCDM problems. Furthermore, additional forms of fuzzy sets, such as intuitionistic, hesitant, and Pythagorean fuzzy sets, which reflect uncertainty in different ways, can be added to this strategy. The use of many decision-making methodologies, such as multi-criteria decision-making, can also be used for acute leukemia.

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Neutrosophic Multi Fuzzy Ideals of γ Near Ring

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Abstract: The theory of neutrosophic multi fuzzy ideals of γ near ring is dispensed in this work and various algebraic properties such as intersection, union of neutrosophic multi fuzzy ideals of γ near ring are examined.

Keywords: Neutrosophic fuzzy set, γ near ring, Neutrosophic multi fuzzy set, neutrosophic multi fuzzy ideal of γ near ring.

1. Introduction

In 1965, Zadeh[25] proposed the notion of fuzzy set. Later A. Rosenfeld[16] developed fuzzy groups. The numerous authors like Bh. Satyanarayana[3,4,5] proposed the concept of fuzzy γ near ring. The authors S. Ragamai, Y. Bhargavi, T. Eswarlal[19] developed theory of fuzzy and L fuzzy ideals of γ near ring. Later the properties of γ near ring in multi fuzzy sets were extended by K. Hemabala and Srinivasa kumar[13]. Florentin Smarandache[7,8] established as a new field of philosophy which is a neutrosophic theory,in 1995. The main base of neutrosophic logic is neutrosophy that includes indeterminacy. It is an augmentation of fuzzy set and intuitionstic fuzzy set. In neutrosophic logic each proposition is estimated by three components T,I,F. The neutrosophic set theory have seen great triumph in several fields such as image processing ,medical diagnosis, robotic, decision making problem and so on. I. Arockiarani[3] extended the theory of neutrosophic fuzzy set. A. Solairaju and S. Thiruveni[2] verified the algebraic properties of fuzzy neutrosophic set in near rings. In fuzzy neutrosophic set, the three components T,I,F can take single values between 0 and 1. There is some ambiquity irrespective of the distance to the element is. The neutrosophic fuzzy set theory on its own is not sufficient to study real world problems. F. Smarandache[9] developed notion of neutrosophic multi sets, an extension of neutrosophic set, in 2016. Authors like Vakkas Ulucay and Memet sahin[23] verified the concepts of neutrosophic multi fuzzy set in groups and verified the group properties. We carry the neutrosophic multi fuzzy notion in γ near ring and hence some properties of algebra are verified.

2. Preliminaries:

Basic definitions of fuzzy set, multi fuzzy set, neutrosophic set and neutrosophic multi set, γ near ring are presenting in this section. Fuzzy set can take a single value between [0,1].

2.1 Definition:

Let $\mathcal{H}$ be a non empty set and $\mathcal{T}$ be a fuzzy set over $\mathcal{H}$ is defined by[25]

$\mathcal{T}=\{\mathcal{T}(x)/\ x \in \mathcal{H}\}$where $\mathcal{T}: \mathcal{H} \to [0,1]$.

2.2 Definition:

Let $\mathcal{H}$ be a non empty set and $\mathcal{S}$ be a multi fuzzy set over $\mathcal{H}$ is defined as[20,21]

$\mathcal{S}$={<$\mathfrak{x}$,$\mathcal{S}_1(\mathfrak{x})$,$\mathcal{S}_2(\mathfrak{x})$,$\mathcal{S}_3(\mathfrak{x})$,…$\mathcal{S}_s(\mathfrak{x})$>:$\mathfrak{x} \in \mathcal{H}$} where $\mathcal{S}_m:\mathcal{H} \to [0,1]$ for all m$\in$ {1,2,…s} and $\mathfrak{x} \in \mathcal{H}$

2.3 Definition:

Let $\mathcal{H}$ be a non empty set then neutrosophic fuzzy set $\mathcal{J}$[7] in $\mathcal{H}$ is defined as

$\mathcal{J}$={$\mathfrak{x}$,$t_{\mathcal{J}}(\mathfrak{x})$,$i_{\mathcal{J}}(\mathfrak{x})$,$f_{\mathcal{J}}(\mathfrak{x})$>:$\mathfrak{x} \in \mathcal{H}$ and $t_{\mathcal{J}}(\mathfrak{x})$,$i_{\mathcal{J}}(\mathfrak{x})$,$f_{\mathcal{J}}(\mathfrak{x}) \in [0,1]$}

Where $t_{\mathcal{J}}(\mathfrak{x})$is the truth membership function, $i_{\mathcal{J}}(\mathfrak{x})$ is the indeterminancy membership function and $f_{\mathcal{J}}(\mathfrak{x})$ falsity membership function and $0 \le t_{\mathcal{J}}(\mathfrak{x})+i_{\mathcal{J}}(\mathfrak{x})+f_{\mathcal{J}}(\mathfrak{x}) \le 1$.

2.4 Definition:

Let $\mathcal{H}$ be a non empty set. A neutrosophic multi fuzzy set $\mathcal{L}$ on $\mathcal{H}$ can be defined as follows

$\mathcal{L}$={< $\mathfrak{x}$,($t_{\mathcal{L}}^1(\mathfrak{x}),t_{\mathcal{L}}^2(\mathfrak{x}),..t_{\mathcal{L}}^s(\mathfrak{x})$),($i_{\mathcal{L}}^1(\mathfrak{x}),i_{\mathcal{L}}^2(\mathfrak{x}),\dots\dots,i_{\mathcal{L}}^s(\mathfrak{x})$),($f_{\mathcal{L}}^1(\mathfrak{x}),f_{\mathcal{L}}^2(\mathfrak{x}),\dots..f_{\mathcal{L}}^s(\mathfrak{x})$)>:$\mathfrak{x} \in \mathcal{H}$}

Where $t_{\mathcal{L}}^1(\mathfrak{x}),t_{\mathcal{L}}^2(\mathfrak{x}),..t_{\mathcal{L}}^s(\mathfrak{x}):\mathcal{H} \to [0,1]$

$i_{\mathcal{L}}^1(\mathfrak{x}),i_{\mathcal{L}}^2(\mathfrak{x}),\dots\dots,i_{\mathcal{L}}^s(\mathfrak{x}):\mathcal{H} \to [0,1]$

$f_{\mathcal{L}}^1(\mathfrak{x}),f_{\mathcal{L}}^2(\mathfrak{x}),\dots..f_{\mathcal{L}}^s(\mathfrak{x}):\mathcal{H} \to [0,1]$

$0 \le \sup t_{\mathcal{L}}^m(\mathfrak{x})+\sup i_{\mathcal{L}}^m(\mathfrak{x})+\sup f_{\mathcal{L}}^m(\mathfrak{x}) \le 1$ for m=1 to s

($t_{\mathcal{L}}^1(\mathfrak{x}),t_{\mathcal{L}}^2(\mathfrak{x}),..t_{\mathcal{L}}^s(\mathfrak{x})$) , ($i_{\mathcal{L}}^1(\mathfrak{x}),i_{\mathcal{L}}^2(\mathfrak{x}),\dots\dots,i_{\mathcal{L}}^s(\mathfrak{x})$) , ($f_{\mathcal{L}}^1(\mathfrak{x}),f_{\mathcal{L}}^2(\mathfrak{x}),\dots..f_{\mathcal{L}}^s(\mathfrak{x})$ are the sequences of truth membership values, indeterminacy membership values and falsity membership values. In addition s is called the dimension of neutrosophic multi fuzzy set $\mathcal{L}$ denoted by d($\mathcal{L}$). The sequence of truth membership values are arranged in decreasing order, but the corresponding indeterminacy membership and falsity membership values may not be in any order.

2.5 Definition:

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy sets where$\mathcal{L}$={($t_{\mathcal{L}}^1(\mathfrak{x}),t_{\mathcal{L}}^2(\mathfrak{x}),..t_{\mathcal{L}}^s(\mathfrak{x})$), ($i_{\mathcal{L}}^1(\mathfrak{x}),i_{\mathcal{L}}^2(\mathfrak{x}),\dots\dots,i_{\mathcal{L}}^s(\mathfrak{x})$), ($f_{\mathcal{L}}^1(\mathfrak{x}),f_{\mathcal{L}}^2(\mathfrak{x}),\dots..f_{\mathcal{L}}^s(\mathfrak{x})$)} and $\mathcal{R}$={($t_{\mathcal{R}}^1(\mathfrak{x}),t_{\mathcal{R}}^2(\mathfrak{x}),..t_{\mathcal{R}}^s(\mathfrak{x})$), ($i_{\mathcal{R}}^1(\mathfrak{x}),i_{\mathcal{R}}^2(\mathfrak{x}),\dots\dots,i_{\mathcal{R}}^s(\mathfrak{x})$), (($f_{\mathcal{R}}^1(\mathfrak{x}),f_{\mathcal{R}}^2(\mathfrak{x}),\dots..f_{\mathcal{R}}^s(\mathfrak{x})$)} then we have the following relations and operations

1. $\mathcal{L} \subseteq \mathcal{R}$ iff $t_{\mathcal{L}}^m(\mathfrak{x}) \le t_{\mathcal{R}}^m(\mathfrak{x})$, $i_{\mathcal{L}}^m(\mathfrak{x}) \ge i_{\mathcal{R}}^m(\mathfrak{x})$, $f_{\mathcal{L}}^m(\mathfrak{x}) \ge f_{\mathcal{R}}^m(\mathfrak{x})$,$\mathfrak{x} \in \mathcal{H}$ and m= 1 to s.
2. $\mathcal{L} = \mathcal{R}$ iff $t_{\mathcal{L}}^m(\mathfrak{x}) = t_{\mathcal{R}}^m(\mathfrak{x})$, $i_{\mathcal{L}}^m(\mathfrak{x}) = i_{\mathcal{R}}^m(\mathfrak{x})$, $f_{\mathcal{L}}^m(\mathfrak{x}) = f_{\mathcal{R}}^m(\mathfrak{x})$, $x \in \mathcal{H}$ and m=1 to s.
3. $\mathcal{L} \cup \mathcal{R}$ = {$\mathfrak{x}$, max ($t_{\mathcal{L}}^m(\mathfrak{x}),t_{\mathcal{R}}^m(\mathfrak{x})$), min($i_{\mathcal{L}}^m(\mathfrak{x}),i_{\mathcal{R}}^m(\mathfrak{x})$), min($f_{\mathcal{L}}^m(\mathfrak{x}),f_{\mathcal{R}}^m(\mathfrak{x})$)}, $\mathfrak{x} \in \mathcal{H}$ and

   m= 1 to s
4. $\mathcal{L} \cap \mathcal{R}$= {$\mathfrak{x}$, min($t_{\mathcal{L}}^m(\mathfrak{x}) \le t_{\mathcal{R}}^m(\mathfrak{x})$),max ($i_{\mathcal{L}}^m(\mathfrak{x}),i_{\mathcal{R}}^m(\mathfrak{x})$), max ($f_{\mathcal{L}}^m(\mathfrak{x}),f_{\mathcal{R}}^m(\mathfrak{x})$)}, $\mathfrak{x} \in \mathcal{H}$ and

   m= 1 to s

2.6 Definition:

A non empty set $\mathcal{H}$ with the binary operations '+'(addition) and '.'(multiplication) is called a near ring[3] if the following conditions hold:

1. $(\mathcal{H}, +)$ is a group
2. $(\mathcal{H}, .)$ is a semigroup
3. $(e_1 + e_2).e_3 = e_1.e_3 + e_2.e_3$ for all $e_1, e_2, e_3 \in \mathcal{H}$

To be precise, it is called right near ring .Since it satisfies the right distributive law. But the word near ring is intended to mean right near ring. We use gh instead of $g.h$

A γ near ring is a triple (ξ,+, γ) where

1. (ξ,+) is a group
2. γ is a non empty set of binary operations on ξ such that $\tau \in \gamma$, (ξ,+,τ) is a near ring.
3. $e_1\tau(e_2\sigma e_3) = (e_1\tau e_2)\sigma e_3$ for all $e_1, e_2, e_3 \in \xi$ and $\tau, \sigma \in \gamma$.

4. Neutrosophic multi fuzzy set of γ near ring

In this section, we introduce the definition of neutrosophic multi fuzzy sets of γ near ring. We proved that union of two neutrosophic multi fuzzy ideals $\mathcal{L}$ and $\mathcal{R}$ is neutrosophic multi fuzzy ideal whenever $\mathcal{L} \subseteq \mathcal{R}$. We also prove that the intersection of two neutrosophic multi fuzzy ideals $\mathcal{L}$ and $\mathcal{R}$ is also a neutrosophic multi fuzzy ideal.

3.1 Definition:

A neutrosophic multi fuzzy set $\mathcal{L}$= {$(t_{\mathcal{L}}^1(x), t_{\mathcal{L}}^2(x), .. t_{\mathcal{L}}^s(x))$,$(i_{\mathcal{L}}^1(x), i_{\mathcal{L}}^2(x), \dots\dots, i_{\mathcal{L}}^s(x))$, $(f_{\mathcal{L}}^1(x), f_{\mathcal{L}}^2(x), \dots.. f_{\mathcal{L}}^s(x))$} in a γ near ring ξ is called neutrosophic multi fuzzy sub γ near ring of ξ if

i) $t_{\mathcal{L}}^m(x - y) \geq \min(t_{\mathcal{L}}^m(x) , t_{\mathcal{L}}^m(y))$,

$i_{\mathcal{L}}^m(x - y) \leq \max(i_{\mathcal{L}}^m(x) , i_{\mathcal{L}}^m(y))$,

$f_{\mathcal{L}}^m(x - y) \leq \max(f_{\mathcal{L}}^m(x), f_{\mathcal{L}}^m(y))$, m= 1 to s.

ii) $t_{\mathcal{L}}^m(x\tau y) \geq \min(t_{\mathcal{L}}^m(x)$,

$t_{\mathcal{L}}^m(y))$,$i_{\mathcal{L}}^m(x\tau y) \leq \max(i_{\mathcal{L}}^m(x) , i_{\mathcal{L}}^m(y))$,

$f_{\mathcal{L}}^m(x\tau y) \leq \max(f_{\mathcal{L}}^m(x), f_{\mathcal{L}}^m(y))$, m= 1 to s.

3.2 **Definition:**

Let ξ be a γ near ring. A neutrosophic multi fuzzy set $\mathcal{L}$in a γ near ring ξ is called neutrosophic multi fuzzy ideal left(resp. right) of ξ if for all $\mathfrak{x}, \mathfrak{z}, \theta_1, \theta_2 \in \xi, \tau \in \gamma$

i) $t_{\mathcal{L}}^{m}(\mathfrak{x} - \mathfrak{z}) \geq \min(t_{\mathcal{L}}^{m}(\mathfrak{x}), t_{\mathcal{L}}^{m}(\mathfrak{z}))$, m= 1 to s

$i_{\mathcal{L}}^{m}(\mathfrak{x} - \mathfrak{z}) \leq \max(i_{\mathcal{L}}^{m}(\mathfrak{x}), i_{\mathcal{L}}^{m}(\mathfrak{z}))$, m= 1 to s

$f_{\mathcal{L}}^{m}(\mathfrak{x} - \mathfrak{z}) \leq \max(f_{\mathcal{L}}^{m}(\mathfrak{x}), f_{\mathcal{L}}^{m}(\mathfrak{z}))$, m= 1 to s.

ii) $t_{\mathcal{L}}^{m}(\mathfrak{z} + \mathfrak{x} - \mathfrak{z}) \geq t_{\mathcal{L}}^{m}(\mathfrak{x})$, m= 1 to s

$i_{\mathcal{L}}^{m}(\mathfrak{z} + \mathfrak{x} - \mathfrak{z}) \leq i_{\mathcal{L}}^{m}(\mathfrak{x})$, m= 1 to s

$f_{\mathcal{L}}^{m}(\mathfrak{z} + \mathfrak{x} - \mathfrak{z}) \leq f_{\mathcal{L}}^{m}(\mathfrak{x})$, m= 1 to s

iii) $t_{\mathcal{L}}^{m}(\theta_1\tau(\mathfrak{x} + \theta_2) - \theta_1\tau\theta_2) \geq t_{\mathcal{L}}^{m}(\mathfrak{x})$, m= 1 to s

$i_{\mathcal{L}}^{m}(\theta_1\tau(\mathfrak{x} + \theta_2) - \theta_1\tau\theta_2) \leq i_{\mathcal{L}}^{m}(\mathfrak{x})$, m= 1 to s

$f_{\mathcal{L}}^{m}(\theta_1\tau(\mathfrak{x} + \theta_2) - \theta_1\tau\theta_2) \leq f_{\mathcal{L}}^{m}(\mathfrak{x})$, m= 1 to s

[resp. right

$t_{\mathcal{L}}^{m}(\mathfrak{x}\tau\theta_1) \geq t_{\mathcal{L}}^{m}(\mathfrak{x})$, m= 1 to s

$i_{\mathcal{L}}^{m}(\mathfrak{x}\tau\theta_1) \leq i_{\mathcal{L}}^{m}(\mathfrak{x})$, m= 1 to s

$f_{\mathcal{L}}^{m}(\mathfrak{x}\tau\theta_1) \leq f_{\mathcal{L}}^{m}(\mathfrak{x})$, = 1 to s]

$\mathcal{L}$ is called a neutrosophic multi fuzzy ideal of ξ if $\mathcal{L}$ both left and right neutrosophic multi fuzzy ideal of ξ.

Example:

Let ξ be the set of the 2x2 matrices over the set of integers and $I_{2x2} \in \gamma$,Then ξ is a γ near ring, Define a neutrosophic multi fuzzy subset $\mathcal{L}$ of ξ as follows

$$\xi(x) = \begin{cases} \{(1,1,1),(0,0,0),(0,0,0)\} & \text{if } x \in \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} \\ \{(0.6,0.7,0.7),(0.1,0.2,0.1),(0.3,0.1,0.2)\} & \text{otherwise} \end{cases}$$

Then clearly $\mathcal{L}$ is a neutrosophic multi fuzzy ideal of γ near ring ξ.

3.1 **Theorem:**

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy left ideal of ξ. If $\mathcal{L} \subset \mathcal{R}$ then $\mathcal{L} \cup \mathcal{R}$ is a neutrosophic multi fuzzy left ideal of ξ.

Proof:

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy left ideal of ξ.

Let $\mathfrak{x}, \mathfrak{z}, \theta_1, \theta_2 \in \xi$, $\tau \in \gamma$

i) $t^m_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}-\mathfrak{z}) = \max\{t^m_{\mathcal{L}}(\mathfrak{x}-\mathfrak{z}), t^m_{\mathcal{R}}(\mathfrak{x}-\mathfrak{z})\}$

$\geq \max\{\{\min(t^m_{\mathcal{L}}(\mathfrak{x}), t^m_{\mathcal{L}}(\mathfrak{z})), \min(t^m_{\mathcal{R}}(\mathfrak{x}), t^m_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\geq \min\{\{\max(t^m_{\mathcal{L}}(\mathfrak{x}), t^m_{\mathcal{L}}(\mathfrak{z})), \max(t^m_{\mathcal{R}}(\mathfrak{x}), t^m_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\geq \min\{\{\max(t^m_{\mathcal{L}}(\mathfrak{x}), t^m_{\mathcal{R}}(\mathfrak{x})), \max(t^m_{\mathcal{L}}(\mathfrak{z}), t^m_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\geq \min(t^m_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}), t^m_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}))$

$i^m_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}-\mathfrak{z}) = \min\{i^m_{\mathcal{L}}(\mathfrak{x}-\mathfrak{z}), i^m_{\mathcal{R}}(\mathfrak{x}-\mathfrak{z})\}$

$\leq \min\{\{\max(i^m_{\mathcal{L}}(\mathfrak{x}), i^m_{\mathcal{L}}(\mathfrak{z})), \max(i^m_{\mathcal{R}}(\mathfrak{x}), i^m_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\leq \max\{\{\min(i^m_{\mathcal{L}}(\mathfrak{x}), i^m_{\mathcal{R}}(\mathfrak{x})), \min(i^m_{\mathcal{L}}(\mathfrak{z}), i^m_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\leq \max(i^m_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}), i^m_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}))$

$f^m_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}-\mathfrak{z}) = \min\{f^m_{\mathcal{L}}(\mathfrak{x}-\mathfrak{z}), f^m_{\mathcal{R}}(\mathfrak{x}-\mathfrak{z})\}$

$\leq \min\{\{\max(f^m_{\mathcal{L}}(\mathfrak{x}), f^m_{\mathcal{L}}(\mathfrak{z})), \max(f^m_{\mathcal{R}}(\mathfrak{x}), f^m_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\leq \max\{\{\min(f^m_{\mathcal{L}}(\mathfrak{x}), f^m_{\mathcal{L}}(\mathfrak{z})), \min(f^m_{\mathcal{R}}(\mathfrak{x}), f^m_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\leq \max\{\{\min(f^m_{\mathcal{L}}(\mathfrak{x}), f^m_{\mathcal{R}}(\mathfrak{x})), \min(f^m_{\mathcal{L}}(\mathfrak{z}), f^m_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\leq \max(f^m_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}), f^m_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}))$

ii) $t^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z}) = \max\{t^{m}_{\mathcal{L}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\ ,\ t^{m}_{\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\}$

$\geq \max\{t^{m}_{\mathcal{L}}(\mathfrak{x}), t^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\geq t^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$

$i^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z}) = \min\{i^{m}_{\mathcal{L}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\ ,\ i^{m}_{\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\}$

$\leq \min\{i^{m}_{\mathcal{L}}(\mathfrak{x}), i^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\leq i^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$

$f^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z}) = \min\{f^{m}_{\mathcal{L}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\ ,\ f^{m}_{\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\}$

$\leq \min\{f^{m}_{\mathcal{L}}(\mathfrak{x}), f^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\leq f^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$

iii) $t^{m}_{\mathcal{L}\cup\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)$

$= \max\{t^{m}_{\mathcal{L}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big), t^{m}_{\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)\}$

$\geq \max\{t^{m}_{\mathcal{L}}(\mathfrak{x}), t^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\geq t^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$

$i^{m}_{\mathcal{L}\cup\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)$

$= \min\{i^{m}_{\mathcal{L}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)\ ,\ i^{m}_{\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)\}$

$\leq \min\{i^{m}_{\mathcal{L}}(\mathfrak{x}), i^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\leq i^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$

$f^{m}_{\mathcal{L}\cup\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)$

$= \min\{f^{m}_{\mathcal{L}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)\ ,\ f^{m}_{\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)\}$

$\leq \min\{f^{m}_{\mathcal{L}}(\mathfrak{x}), f^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\leq f^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$

$\therefore\ \mathcal{L}\cup\mathcal{R}$ is a neutrosophic multi fuzzy left ideal of ξ.

3.2 **Theorem:**

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy right ideal of ξ. If $\mathcal{L}\subset\mathcal{R}$ then $\mathcal{L}\cup\mathcal{R}$ is a neutrosophic multi fuzzy right ideal of ξ.

Proof:

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy right ideal of ξ.

Let $\mathfrak{x}, \mathfrak{z}, \theta_1, \theta_2 \in \xi$, $\tau \in \gamma$

i) $t^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}-\mathfrak{z}) = \max\{t^{m}_{\mathcal{L}}(\mathfrak{x}-\mathfrak{z}),\ t^{m}_{\mathcal{R}}(\mathfrak{x}-\mathfrak{z})\}$

$\geq \max\{\{\min(t^{m}_{\mathcal{L}}(\mathfrak{x}), t^{m}_{\mathcal{L}}(\mathfrak{z})), \min(t^{m}_{\mathcal{R}}(\mathfrak{x}), t^{m}_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\geq \min\{\{\max(t^{m}_{\mathcal{L}}(\mathfrak{x}), t^{m}_{\mathcal{L}}(\mathfrak{z})), \max(t^{m}_{\mathcal{R}}(\mathfrak{x}), t^{m}_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\geq \min\{\{\max(t^{m}_{\mathcal{L}}(\mathfrak{x}), t^{m}_{\mathcal{R}}(\mathfrak{x})), \max(t^{m}_{\mathcal{L}}(\mathfrak{z}), t^{m}_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\geq \min(t^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}),\ t^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}))$

$i^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}-\mathfrak{z}) = \min\{i^{m}_{\mathcal{L}}(\mathfrak{x}-\mathfrak{z}),\ i^{m}_{\mathcal{R}}(\mathfrak{x}-\mathfrak{z})\}$

$\leq \min\{\{\max(i^{m}_{\mathcal{L}}(\mathfrak{x}), i^{m}_{\mathcal{L}}(\mathfrak{z})), \max(i^{m}_{\mathcal{R}}(\mathfrak{x}), i^{m}_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\leq \max\{\{\min(i^{m}_{\mathcal{L}}(\mathfrak{x}), i^{m}_{\mathcal{R}}(\mathfrak{x})), \min(i^{m}_{\mathcal{L}}(\mathfrak{z}), i^{m}_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\leq \max(i^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}),\ i^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}))$

$f^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}-\mathfrak{z}) = \min\{f^{m}_{\mathcal{L}}(\mathfrak{x}-\mathfrak{z}),\ f^{m}_{\mathcal{R}}(\mathfrak{x}-\mathfrak{z})\}$

$\leq \min\{\{\max(f^{m}_{\mathcal{L}}(\mathfrak{x}), f^{m}_{\mathcal{L}}(\mathfrak{z})), \max(f^{m}_{\mathcal{R}}(\mathfrak{x}), f^{m}_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\leq \max\{\{\min(f^{m}_{\mathcal{L}}(\mathfrak{x}), f^{m}_{\mathcal{L}}(\mathfrak{z})), \min(f^{m}_{\mathcal{R}}(\mathfrak{x}), f^{m}_{\mathcal{R}}(\mathfrak{z}))\}\}$

$\leq \max\{\{\min(f^{m}_{\mathcal{L}}(\mathfrak{x}), f^{m}_{\mathcal{R}}(\mathfrak{x})), \min(f^{m}_{\mathcal{L}}(\mathfrak{z}), f^{m}_{\mathcal{R}}(\mathfrak{z}))\}\}$

$$\leq \max(f^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}), f^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}))$$

ii) $t^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z}) = \max\{t^{m}_{\mathcal{L}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z}), t^{m}_{\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\}$

$$\geq \max\{t^{m}_{\mathcal{L}}(\mathfrak{x}), t^{m}_{\mathcal{R}}(\mathfrak{x})\}$$

$$\geq t^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$$

$$i^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z}) = \min\{i^{m}_{\mathcal{L}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z}), i^{m}_{\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\}$$

$$\leq \min\{i^{m}_{\mathcal{L}}(\mathfrak{x}), i^{m}_{\mathcal{R}}(\mathfrak{x})\}$$

$$\leq i^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$$

$$f^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z}) = \min\{f^{m}_{\mathcal{L}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z}), f^{m}_{\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\}$$

$$\leq \min\{f^{m}_{\mathcal{L}}(\mathfrak{x}), f^{m}_{\mathcal{R}}(\mathfrak{x})\}$$

$$\leq f^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$$

iii) $t^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}\,\tau\theta_1) = \max\{t^{m}_{\mathcal{L}}(\mathfrak{x}\,\tau\theta_1), t^{m}_{\mathcal{R}}(\mathfrak{x}\,\tau\theta_1)\}$

$$\geq \max\{t^{m}_{\mathcal{L}}(\mathfrak{x}), t^{m}_{\mathcal{R}}(\mathfrak{x})\}$$

$$\geq t^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$$

$$i^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}\,\tau\theta_1) = \min\{i^{m}_{\mathcal{L}}(\mathfrak{x}\,\tau\theta_1), i^{m}_{\mathcal{R}}(\mathfrak{x}\,\tau\theta_1)\}$$

$$\leq \min\{i^{m}_{\mathcal{L}}(\mathfrak{x}), i^{m}_{\mathcal{R}}(\mathfrak{x})\}$$

$$\leq i^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$$

$$f^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x}\,\tau\theta_1) = \min\{f^{m}_{\mathcal{L}}(\mathfrak{x}\,\tau\theta_1), f^{m}_{\mathcal{R}}(\mathfrak{x}\,\tau\theta_1)\}$$

$$\leq \min\{f^{m}_{\mathcal{L}}(\mathfrak{x}), f^{m}_{\mathcal{R}}(\mathfrak{x})\}$$

$$\leq f^{m}_{\mathcal{L}\cup\mathcal{R}}(\mathfrak{x})$$

$\therefore$ $\mathcal{L} \cup \mathcal{R}$ is a neutrosophic multi fuzzy right ideal of ξ.

3.3 **Theorem:**

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy ideal of ξ. If $\mathcal{L} \subset \mathcal{R}$ then $\mathcal{L} \cup \mathcal{R}$ is a neutrosophic multi fuzzy ideal of ξ.

Proof: It is clear.

3.4 **Theorem:**

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy left ideal of ξ then $\mathcal{L} \cap \mathcal{R}$ is a neutrosophic multi fuzzy left ideal of ξ.

Proof:

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy left ideal of ξ.

Let $\mathfrak{x}, \mathfrak{z}, \theta_1, \theta_2 \in \xi, \tau \in \gamma$

i) $t^m_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{x} - \mathfrak{z})$ = min$\{t^m_{\mathcal{L}}(\mathfrak{x} - \mathfrak{z})\ ,\ t^m_{\mathcal{R}}(\mathfrak{x} - \mathfrak{z})\}$

$\geq$ min{{min($t^m_{\mathcal{L}}(\mathfrak{x}), t^m_{\mathcal{L}}(\mathfrak{z})$) ,min($t^m_{\mathcal{R}}(\mathfrak{x}), t^m_{\mathcal{R}}(\mathfrak{z})$)}}

$\geq$ min{{ min($t^m_{\mathcal{L}}(\mathfrak{x}), t^m_{\mathcal{R}}(\mathfrak{x})$) , min ($t^m_{\mathcal{L}}(\mathfrak{z}), t^m_{\mathcal{R}}(\mathfrak{z})$)}}

$\geq$min($t^m_{\mathcal{L}\cap\mathcal{R}}((\mathfrak{x}))\ ,\ t^m_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{z})$)

$i^m_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{x} - \mathfrak{z})$= max $\{i^m_{\mathcal{L}}(\mathfrak{x} - \mathfrak{z})\ ,\ i^m_{\mathcal{R}}(\mathfrak{x} - \mathfrak{z})\}$

$\leq$ max {{ max ($i^m_{\mathcal{L}}(\mathfrak{x}), i^m_{\mathcal{L}}(\mathfrak{z})$) , max ($i^m_{\mathcal{R}}(\mathfrak{x}), i^m_{\mathcal{R}}(\mathfrak{z})$)}}

$\leq$ max ($i^m_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{x})\ ,\ i^m_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{z})$)

$f^m_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{x} - \mathfrak{z})$ = max $\{f^m_{\mathcal{L}}(\mathfrak{x} - \mathfrak{z})\ ,\ f^m_{\mathcal{R}}(\mathfrak{x} - \mathfrak{z})\}$

$\leq$ max {{ ma x ($f^m_{\mathcal{L}}(\mathfrak{x}), f^m_{\mathcal{L}}(\mathfrak{z})$) , max ($f^m_{\mathcal{R}}(\mathfrak{x}), f^m_{\mathcal{R}}(\mathfrak{z})$)}}

$\leq$ max {{ max ($f^m_{\mathcal{L}}(\mathfrak{x}), f^m_{\mathcal{R}}(\mathfrak{x})$) , max ($f^m_{\mathcal{L}}(\mathfrak{z}), f^m_{\mathcal{R}}(\mathfrak{z})$)}}

$\leq$ max ($f^m_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{x})\ ,\ f^m_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{z})$)

ii) $\mathfrak{t}^{m}_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})$ = min$\{\mathfrak{t}^{m}_{\mathcal{L}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})$, $\mathfrak{t}^{m}_{\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\}$

$\geq$ min$\{\mathfrak{t}^{m}_{\mathcal{L}}(\mathfrak{x}),\mathfrak{t}^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\geq \mathfrak{t}^{m}_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{x})$

$\mathfrak{i}^{m}_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})$ = max $\{\mathfrak{i}^{m}_{\mathcal{L}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})$, $\mathfrak{i}^{m}_{\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\}$

$\leq$ max $\{\mathfrak{i}^{m}_{\mathcal{L}}(\mathfrak{x}),\mathfrak{i}^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\leq \mathfrak{i}^{m}_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{x})$

$\mathfrak{f}^{m}_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})$ = max $\{\mathfrak{f}^{m}_{\mathcal{L}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})$, $\mathfrak{f}^{m}_{\mathcal{R}}(\mathfrak{z}+\mathfrak{x}-\mathfrak{z})\}$

$\leq$ max $\{\mathfrak{f}^{m}_{\mathcal{L}}(\mathfrak{x}),\mathfrak{f}^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\leq \mathfrak{f}^{m}_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{x})$

iii) $\mathfrak{t}^{m}_{\mathcal{L}\cap\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\,\theta_2\big)$

= min$\{\mathfrak{t}^{m}_{\mathcal{L}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau,\theta_2\big)$, $\mathfrak{t}^{m}_{\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\,\theta_2\big)\}$

$\geq$ min$\{\mathfrak{t}^{m}_{\mathcal{L}}(\mathfrak{x}),\mathfrak{t}^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\geq \mathfrak{t}^{m}_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{x})$

$\mathfrak{i}^{m}_{\mathcal{L}\cap\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\,\theta_2\big)$

= max $\{\mathfrak{i}^{m}_{\mathcal{L}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)$, $\mathfrak{i}^{m}_{\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)\}$

$\leq$ max $\{\mathfrak{i}^{m}_{\mathcal{L}}(\mathfrak{x}),\mathfrak{i}^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\leq \mathfrak{i}^{m}_{\mathcal{L}\cap\mathcal{R}}(\mathfrak{x})$

$\mathfrak{f}^{m}_{\mathcal{L}\cap\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)$

= max $\{\mathfrak{f}^{m}_{\mathcal{L}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)$, $\mathfrak{f}^{m}_{\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\,\theta_2\big)\}$

$\leq$ max $\{\mathfrak{f}^{m}_{\mathcal{L}}(\mathfrak{x}),\mathfrak{f}^{m}_{\mathcal{R}}(\mathfrak{x})\}$

$\leq f^{m}_{\mathcal{L}\cap\mathcal{R}}(x)$

$\therefore$ $\mathcal{L}\cap\mathcal{R}$ is a neutrosophic multi fuzzy left ideal of ξ.

4.5 Theorem:

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy right ideal of $\mathcal{E}$then $\mathcal{L}\cap\mathcal{R}$is a neutrosophic multi fuzzy right ideal of ξ.

Proof:

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy right ideal of ξ.

Let $x, y, \theta_1, \theta_2 \in \xi$, $\tau \in \gamma$

i) $t^{m}_{\mathcal{L}\cap\mathcal{R}}(x - y)$ = min{$t^{m}_{\mathcal{L}}(x - y)$, $t^{m}_{\mathcal{R}}(x - y)$}

$\geq$ min{{min($t^{m}_{\mathcal{L}}(x)$,$t^{m}_{\mathcal{L}}(y)$) , min($t^{m}_{\mathcal{R}}(x)$,$t^{m}_{\mathcal{R}}(y)$)}}

$\geq$ min{{ min($t^{m}_{\mathcal{L}}(x)$,$t^{m}_{\mathcal{R}}(x)$) , min ($t^{m}_{\mathcal{L}}(y)$,$t^{m}_{\mathcal{R}}(y)$)}}

$\geq$ min($t^{m}_{\mathcal{L}\cap\mathcal{R}}((x))$, $t^{m}_{\mathcal{L}\cap\mathcal{R}}(y)$)

$i^{m}_{\mathcal{L}\cap\mathcal{R}}(x - y)$ = max {$i^{m}_{\mathcal{L}}(x - y)$, $i^{m}_{\mathcal{R}}(x - y)$}

$\leq$ max {{ max ($i^{m}_{\mathcal{L}}(x)$,$i^{m}_{\mathcal{L}}(y)$) , max ($i^{m}_{\mathcal{R}}(x)$,$i^{m}_{\mathcal{R}}(y)$)}}

$\leq$ max {{ max ($i^{m}_{\mathcal{L}}(x)$,$i^{m}_{\mathcal{R}}(x)$) , max ($i^{m}_{\mathcal{L}}(y)$,$i^{m}_{\mathcal{R}}(y)$)}}

$\leq$ max ($i^{m}_{\mathcal{L}\cap\mathcal{R}}(x)$, $i^{m}_{\mathcal{L}\cap\mathcal{R}}(y)$)

$f^{m}_{\mathcal{L}\cap\mathcal{R}}(x - y)$ = max {$f^{m}_{\mathcal{L}}(x - y)$, $f^{m}_{\mathcal{R}}(x - y)$}

$\leq$ max {{ ma x ($f^{m}_{\mathcal{L}}(x)$,$f^{m}_{\mathcal{L}}(y)$) , max ($f^{m}_{\mathcal{R}}(x)$,$f^{m}_{\mathcal{R}}(y)$)}}

$\leq$max {{ max ($f^{m}_{\mathcal{L}}(x)$,$f^{m}_{\mathcal{L}}(y)$) , max ($f^{m}_{\mathcal{R}}(x)$,$f^{m}_{\mathcal{R}}(y)$)}}

$\leq$max {{ max ($f^{m}_{\mathcal{L}}(x)$,$f^{m}_{\mathcal{R}}(x)$) , max ($f^{m}_{\mathcal{L}}(y)$,$f^{m}_{\mathcal{R}}(y)$)}}

$\leq$max ($f^{m}_{\mathcal{L}\cap\mathcal{R}}(x)$, $f^{m}_{\mathcal{L}\cap\mathcal{R}}(y)$)

ii) $t^{m}_{\mathcal{L}\cap\mathcal{R}}(y + x - y)$ = min{$t^{m}_{\mathcal{L}}(y + x - y)$, $t^{m}_{\mathcal{R}}(y + x - y)$}

$$\geq \min\{t_{\mathcal{L}}^{m}(x), t_{\mathcal{R}}^{m}(x)\}$$

$$\geq t_{\mathcal{L}\cap\mathcal{R}}^{m}(x)$$

$$i_{\mathcal{L}\cap\mathcal{R}}^{m}(z + x - z) = \max\{i_{\mathcal{L}}^{m}(z + x - z)\ ,\ i_{\mathcal{R}}^{m}(z + x - z)\}$$

$$\leq \max\{i_{\mathcal{L}}^{m}(x), i_{\mathcal{R}}^{m}(x)\}$$

$$\leq i_{\mathcal{L}\cap\mathcal{R}}^{m}(x)$$

$$f_{\mathcal{L}\cap\mathcal{R}}^{m}(z + x - z) = \max\{f_{\mathcal{L}}^{m}(z + x - z)\ ,\ f_{\mathcal{R}}^{m}(z + x - z)\}$$

$$\leq \max\{f_{\mathcal{L}}^{m}(x), f_{\mathcal{R}}^{m}(x)\}$$

$$\leq f_{\mathcal{L}\cap\mathcal{R}}^{m}(x)$$

iii) $t_{\mathcal{L}\cap\mathcal{R}}^{m}(x\ \tau\theta_1) = \min\{t_{\mathcal{L}}^{m}(x\ \tau\theta_1)\ ,\ t_{\mathcal{R}}^{m}(x\ \tau\theta_1)\}$

$$\geq \min\{t_{\mathcal{L}}^{m}(x), t_{\mathcal{R}}^{m}(x)\}$$

$$\geq t_{\mathcal{L}\cap\mathcal{R}}^{m}(x)$$

$$i_{\mathcal{L}\cap\mathcal{R}}^{m}(x\ \tau\theta_1) = \max\{i_{\mathcal{L}}^{m}(x\ \tau\theta_1)\ ,\ i_{\mathcal{R}}^{m}(x\ \tau\theta_1)\}$$

$$\leq \max\{i_{\mathcal{L}}^{m}(x), i_{\mathcal{R}}^{m}(x)\}$$

$$\leq i_{\mathcal{L}\cap\mathcal{R}}^{m}(x)$$

$$f_{\mathcal{L}\cap\mathcal{R}}^{m}(x\ \tau\theta_1) = \max\{f_{\mathcal{L}}^{m}(x\ \tau\theta_1)\ ,\ f_{\mathcal{R}}^{m}(x\ \tau\theta_1)\}$$

$$\leq \max\{f_{\mathcal{L}}^{m}(x), f_{\mathcal{R}}^{m}(x)\}$$

$$\leq f_{\mathcal{L}\cap\mathcal{R}}^{m}(x)$$

$\therefore$ $\mathcal{L}\cap\mathcal{R}$ is a neutrosophic multi fuzzy right ideal of ξ.

4.6 Theorem:

Let $\mathcal{L}$ and $\mathcal{R}$neutrosophic multi fuzzy ideal of ξthen $\mathcal{L}\cap\mathcal{R}$is a neutrosophic multi fuzzy ideal of ξ.

Proof: It is clear.

5. Conclusion:

To conclude, the notion of neutrosophic multi fuzzy gamma near-ring, neutrosophic multi fuzzy ideals of gamma near-rings have been discussed. The proof for the theorem that states Union and Intersection of two neutrosophic multi fuzzy ideals of gamma near-ring is also a Neutrosophic multi fuzzy ideal of gamma near-ring has been provided.

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On Neutrosophic Multiplication Module

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Abstract: In this article, we investigate some new results of Neutrosophic multiplication module E (shortly Ne(E)). We additionally introduce some light points about some concepts in which have relationship with Neutrosophic multiplication module. We prove that if E is Neutrosophic Artinian multiplication module and Neutrosophic Jacobson radical of E is a Neutrosophic small submodule of Ne(E), then Ne(E) is a Neutrosophic cyclic module. Finally, we show that if E is a Neutrosophic divisible module over Neutrosophic integral domain, then E is a Neutrosophic multiplication module if and only if E is a Neutrosophic cyclic module.

Keywords: Cyclic module; multiplication module; neutrosophic sets; neutrosophic submodule; neutrosophic multiplication module.

1. Introduction

Multiplication module is one of the important concepts in module theory. Several researchers have studied this module in an abstract way, but in this paper, we will present an indeterminacy to study some properties of this module. In 1999, the neutrosophy introduced by Smarandache [1] as a generalization of intuitionistic fuzzy set. Accordingly, he introduced the concept of neutrosophic logic and neutrosophic set where all notion in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic especially to intuitionistic fuzzy logic. In fact, neutrosophic set is the generalization of fuzzy set [2], classical set [3], intuitionistic fuzzy set [4], while neutrosophic group and neutrosophic ring are the generalizations of fuzzy group and ring classical group. In the same way, by the generalization of the classical module, we get the neutrosophic multiplication module. By using the idea of neutrosophic theory, several researchers have studied neutrosophic algebraic structures by inserting an indeterminate element in the algebraic structure. Modules are so much important in algebraic structures as they are in almost all algebraic structures theory [5, 6]. Modules are thought as old algebra due to its rich structure compared to other notions. A few researchers [7- 10] have studied certain type of modules with favourable results. Hence, we will use neutrosophic groups [11] to study the neutrosophic notions formation. In this paper, we will introduce a new hyper algebraic concept that is neutrosophic multiplication module.

The paper is organized as follows. After the literature review in section 1, the preliminaries are reviewed in section 2. The neutrosophic multiplication module is introduced in section 3 along with several relevant section 3, and conclusion in section 4.

2. Preliminaries

In this section, we recall some definitions to be used in this paper.

Definitional 2.1 [12] Suppose that T is a commutative ring with unity. We say that E is a *T-module* if:

$T \times E \to E\ (r, v) \to rv\ such\ that\ E$ is a commutative group with T and satisfies the following.

1. $(rv_1)v = r(v_1 v)$
2. $(r_1 + r_2)v = r_1 v + r_2 v$
3. $r(v_1 + v_2) = rv_1 + rv_2$
4. $1.v = v = v.1$

Definition 2.2 [12] A subset E_1 is called the submodule of E $(E_1 \leq E)$ if closed with (+) and scalar multiplication, that is

(*) $a + b \in E_1,\ \forall a, b \in E_1$

(*) $r\ a \in E_1,\ \forall r \in T,\ a \in E_1$

Definition 2.3 [13] Let U be a universal set. The *neutrosophic U*, in short *Ne (U)* is defined as

$$H=\{(\xi,\ t_H(\xi),\ i_H(\xi),\ f_H(\xi) : \xi \in U\} \ni t_H,\ i_H,\ f_H : U \to [0, 1].$$

Remark 2.4 t_H denotes the percentage of truth, i_H denotes the percentage of indeterminacy and f_H denotes the percentage of falsity.

Remark 2.5 V^U denotes the set of all neutrosophic subsets of U.

Definition 2.6 [13] Let U be an initial universe and if we take Ne(H_1) and Ne(H_2) be two neutrosophic subsets of U. Then Ne(H_1)⊆Ne(H_2) $H_1 \subseteq H_2$ if and only if

$$t_{Ne}(H_1) \leq t_{Ne}(H_2),\ i_{Ne}(H_1) \leq i_{Ne}(H_2),\ f_{Ne}(H_1) \geq f_{Ne}(H_2).$$

Definition 2.7 [13] Let $(T, +, .)$ be a ring and let *Ne(T)* be a neutrosophic set by T and I. So $Ne(T) = \{T(I), +, .\}$ is a neutrosophic ring.

i.e. the set $< T \cup I > = \{t_1 + t_2 I : t_1, t_2 \in T\}$ is a neutrosophic ring generated by T and I with operation of T such that I represented the percentage of determinacy.

Definition 2.8 [13] If we have $N_1e(T)$ as a neutrosophic ring and if we take $N_2e(T)$ as a subset of $N_1e(T)$, we define $N_1e(T)$ as a neutrosophic subring precisely when

1)$N_2e(T)\neq\varphi$

2)$N_2e(T)$ itself is a neutrosophic ring.

3) $N_2e(T)$ must has a proper subset which is a ring.

We know that if *Ne(T)* is a neutrosophic ring and such that *J* is an ideal of *T*. Hence *Ne(J)* is called the neutrosophic ideal of neutrosophic ring *T* if :

$$j_1 - j_2 \in Ne(J) \ni j_1 \in Ne(j) \text{ and } j_2 \in Ne(J).$$

$$rj\,, jr \in Ne\,(J) \ni r \in Ne(T) \text{ and } j \in Ne(J).$$

Definition 2.9 [14]. Let (*E*,+,.) be a module over the ring *T*. Then (*E*(*I*),+,.) is called a weak neutrosophic module over the ring *T*, and it is called a strong neutrosophic module if it is a module over the neutrosophic ring *T*(*I*).

Definition 2.10 [15]. Let P={($t_P(\eta)$, $i_P(\eta)$, $f_P(\eta)$): $\eta \in$ R} be an Ne(R). Then P is called a neutrosophic ideal of R if it satisfies the following conditions $\forall\ \eta,\ \theta \in$ R *Ne*(*E*) be a neutrosophic of module over *Ne*(*T*). Then any neutrosophic subset *Ne(K)* of *Ne(E)* is called neutrosophic submodule if:

(1) $t_P(\eta-\theta)\geq t_P(\eta)\wedge t_P(\theta)$

(2) $i_P(\eta-\theta)\geq i_P(\eta)\wedge i_P(\theta)$

(3) $f_P(\eta-\theta)\leq f_P(\eta)\vee f_P(\theta)$

(4) $t_P(\eta\theta)\geq t_P(\eta)\vee t_P(\theta)$

(5) $i_P(\eta\theta)\geq i_P(\eta)\vee i_P(\theta)$

(6) $f_P(\eta\theta)\leq f_P(\eta)\wedge f_P(\theta)$

Note that any neutrosophic set *Ne(K)* in *E* is called a neutrosophic submodule if

$K(0) = U : t_k(0) = 1, i_k(0) = 1 \text{ and } f_k(0) = 0.$

$K(a + b) \geq k(a) \wedge k(b)\ a, b \in E :$

$t_k(a + b) \geq t_k(a) \wedge t_k(b), i_k(a + b) \geq i_k(b) \text{ and } f_k(a + b) \leq f_k(a) \vee f(_k(b).$

$k(ra) \geq k(a)\,, a \in E, r \in T :$

$t_k(ra) \geq t_k(a), i_k(ra) \geq i_k(a)$ and $f(ra) \leq f(a)$.

Remark 2.11 More details on neutrosophic module and neutrosophic submodule are discussed by Ameri [16].

3. Neutrosophic Multiplication Module

In this section, we define the concept of a neutrosophic multiplication module over a neutrosophic ring. We investigate and obtain some results on the relationship between neutrosophic multiplication module and other concepts.

Definition 3.1 Let *E* be a neutrosophic *T-module*. Then *E* is called the neutrosophic multiplication module in case for every *Ne(K)* of *Ne(E)*, $\exists$ $Ne(J)$ an neutrosophic ideal of *Ne(T)* such that

$$Ne(K) = Ne(J)\, Ne(E).$$

Here, we consider neutrosophic multiplication module *E* over neutrosophic invariant rings *Ne(T)*.

Definition 3.2 A ring *T* is called the neutrosophic invariants ring if every right (left) neutrosophic ideal is a neutrosophic ideal *Ne(J)*.

Theorem 3.3 Let *E* be a neutrosophic multiplication module *Ne(E)* over neutrosophic ring *T*. If *K* is a neutrosophic submodule of *Ne(E)* such that

$$Ne(K) \cap Ne(E) Ne(J) = Ne(K) Ne(J)$$

and *Ne(J)* is a neutrosophic ideal of *Ne(E)*, then *Ne(K)* is a neutrosophic multiplication module.

Proof:

Let $Ne(H) \leq Ne(K)$. Since Ne(E) is a neutrosophic multiplications module, there exists a *Ne(J)* of *Ne(T)* $\ni$ *Ne(H) = Ne(E) Ne(J)*.

We have *Ne(K)* $\cap$ *Ne(E) Ne (J) = Ne(K) Ne(J)*.

Then

$$Ne(H) = Ne(E)Ne(J) \subseteq Ne((K) \cap Ne(E)Ne(J)$$

$$= Ne(K)Ne(J) \subseteq Ne(E)Ne(J)$$

$$= Ne(H)$$

Thus

$$Ne(H) = Ne(K)Ne(J)$$

Hence K is a neutrosophic multiplication module.

Definition 3.4 A T-module E is called neutrosophic cyclic module if $Nec(E) = Ne(E)x\ (I) \ni x(I)$ is a neutrosophic $(x(I) = y + ZI)$.

Theorem 3.5. Let E be a neutrosophic multiplication module over neutrosophic ring T and let J be a neutrosophic maximal ideal of T. Then $\frac{Ne(E)}{Ne(E)Ne(J)}$ is a neutrosophic cyclic module with at most two neutrosophic submodules and *Ne(E)=Ne(E)Ne(J)* or *Ne(E)Ne(T)* is a neutrosophic maximal submodule of *Ne(E)*.

Proof:

We know that $\frac{Ne(E)}{Ne(E)Ne(J)}$ is a neutrosophic multiplication module over simple neutrosophic ring of $\frac{T}{J}\left(Ne\left(\frac{T}{J}\right)\right)$. If $\frac{Ne(E)}{Ne(E)Ne(I)} = 0$, then the $\frac{Ne(E)}{Ne(E)Ne(I)}$ is a cyclic with only one neutrosophic submodule. If $\frac{Ne(E)}{Ne(E)Ne(I)} \neq 0$ then *Ne(E)Ne(J)* is a neutrosophic maximal submodule of neutrosophic module E (*Ne(E)*). Note that $\frac{Ne(E)}{Ne(E)Ne(J)}$ is a neutrosophic cyclic module having only two neutrosophic submodules.

Theorem 3.6 Let T be a neutrosophic ring with commutative neutrosophic multiplication ideals, *Ne(E)* be a neutrosophic multiplication T-module and J be a neutrosophic maximal ideal of *Ne(T)*. If J does not contain neutrosophic annihilator of any neutrosophic cyclic submodule of *Ne(E)*, then *Ne(K) = Ne(K) Ne(J)* for every neutrosophic cyclic submodule of *Ne(E).*

Proof:

Suppose that K be a neutrosophic cyclic submodule of neutrosophic module E, i.e. $Ne(K) \leq Ne(E)$. We have $Ne(r)(Ne(K)) \not\subset J$ where J is a neutrosophic maximal ideal, and $T = J + Ne(r)(Ne\ (K))$.

Thus

$$\begin{aligned} Ne(K) &= Ne(K)\ Ne(T) \\ &= Ne(K)\ (Ne(J) + Ne(r)(Ne(K)) \\ &= Ne(K)\ Ne(J) + Ne(K)Ne(r)(Ne(K)) \\ &= Ne(K)\ (Ne(J). \end{aligned}$$

Corollary 3.7 For a neutrosophic module *E* over a neutrosophic ring, if for every neutrosophic submodule *K* of a neutrosophic module *E* ($Ne(K) \leq Ne(E)$), there exists a set $\{k_i\};\ i \in I$ of neutrosophic ideals of *T* such that $Ne(K) = \sum_{i\in I} Ne(K_i)$ and $Ne(K_c) = Ne(E)Ne(J);\ i \in I,$ then E is a neutrosophic multiplication module.

Proof:

Suppose that *Ne(K)* is a submodule of *Ne(E).* There exists *Ne*$\{k_i\}$ and *Ne*(J_i) of

$$Ne(T) \ni Ne(Ki) = \sum Ne(k_i)\ and\ K_i = Ne(E)Ne(J) \forall i \in I.$$

Let $Ne(J) = \sum Ne(J_i)$.

Hence

$$Ne(K) = \sum Ne(K_i) = \sum Ne(E)Ne(J_i) = Ne(t)(\sum Ne(J_i))$$

$$= Ne(E)Ne(J).$$

Thus *E* is a neutrosophic multiplication module.

Recall that a module *E* is called neutrosophic artinian module if *E* satisfy neutrosophic descending chain condition. *E* is neutrosophic divisible module if *Ne(r)Ne(E) = Ne(E)* for every $0 \neq$ r $\in$ *Ne(T).*

Theorem 3.8 Let *E* be n neutrosophic artinan multiplication module. Then if *Ne(J(E))* is a small neutrosophic submodule of *Ne(E)*, then *Ne(E)* is a neutrosophic cyclic module.

Proof:

Since $\left(\frac{Ne(E)}{Ne(J(E))}\right)$ is a neutrosophic cyclic module over neutrosophic submodule *K* of neutrosophic module *E* (*Ne(K)* $\leq$ *Ne(E)*) $\ni$ *Ne(E) = Ne(K) + Ne(J(E)),* so *Ne(J(E))* is a small neutrosophic of *Ne(E)* (*Ne(J(E))* << *Ne(E)*). Hence *Ne(E) = Ne(K*). Then *E* is a neutrosophic cyclic module**.**

Corollary 3.9 For a neutrosophic artinian multiplication module *E*, if *Ne(E)* is a neutrosophic finitely generated module, then *Ne(E)* is a neutrosophic cyclic module.

Proof:

Suppose that E is a neutrosophic finitely generated module. Then *Ne(J(E))* is a neutrosophic small submodule of *Ne(E)* (*Ne(J(E))* ≤ *Ne(E)*). Thus from Theorem 3.8, *Ne(E)* is a neutrosophic cyclic module.

Note that a module E is called neutrosophic semi-prime submodule if for each *Ne(r)* ∈ *Ne(T)*, *Ne(x)* ∈ *Ne(E)*, *Ne(s)* ∈ *Ne(Z⁺)* with *Ne* (r^k) *Ne (x)* ∈ *K* implies that *Ne (r) Ne (x)* ∈ *Ne(K).*

Proposition 3.10 Let E be a neutrosophic multiplication module. Then K is a neutrosophic semi-prime submodule of E if and only if *Ne(r) Ne(K)* = *Ne(K)*.

Proof:

⇒

We know that *Ne(k)* ⊆ *Ne(r) (Ne(K))*, where K is a neutrosophic submodule of E. Suppose that K is a neutrosophic semi-prime submodule of E and let *Ne(a)* ∈ *N(r) (Ne(K))*. Thus, for some $k \in Z^+$; $(Ne(a))^k \subseteq Ne(K)$. Now for some *Ne(a)* ∈ *Ne(K)* and *Ne(K)* being a neutrosophic semi-prime submodule, we then obtain *Ne(K)* = *Ne(r) (Ne(K)).*

⇐

Suppose that *Ne(r)* / *Ne(K)* = *Ne(k)* and let $(Ne(a))^n \subseteq Ne(K); n \in Z^+$. Therefore some *Ne(a)* ∈ *Ne(K)*. Thus, we get *Ne(K)* to be a neutrosophic semi-prime submodule of Ne(E).

Corollary 3.11 Let E be a neutrosophic divisible module over neutrosophic integral domain. Then E is a neutrosophic multiplication module if and only if E is a neutrosophic cyclic module.

Proof:

⇒

It is clear that every neutrosophic cyclic module is neutrosophic multiplication module.

⇐

Assume that E is a neutrosophic multiplication module. Let $0 \neq K$ be a neutrosophic submodule of E. So there exists a neutrosophic *Ne(J)* such that

$$Ne(K) = Ne(J)Ne(E) = Ne(E)$$

Definition 3.12 Let U be an initial universe. If *Ne*(H_1) and *Ne*(H_2) are two neutrosophic subsets of U, then $Ne(s) = Ne(H_1) \cap Ne(H_2)$ is also neutrosophic defined as follows.

$t_{Ne(s)}(K) = min(t_{Ne(H_1)}(K), t_{Ne(H_2)}(K))$

$I_{Ne(s)}(K) = min(I_{Ne(H_1)}(K), I_{Ne(H_2)}(K))$

$f_{Ne(s)}(K) = min(f_{Ne(H_1)}(K), f_{Ne(H_2)}(K))$

$\forall k \in U, t(K)_{Ne(H_1)}, I(K)_{Ne(H_1)}, f(K)_{Ne(H_1)} \in [0,1]$,

$\forall k \in U, t(K)_{Ne(H_2)}, I(K)_{Ne(H_2)}, f(K)_{Ne(H_2)} \in [0,1][1,0]$.

Theorem 3.13 Let E be a neutrosophic multiplication T-module and let K be a neutrosophic prime submodule of E. If $\mathrm{K}_1, \mathrm{K}_2, \ldots, \mathrm{K}_\mathrm{n}$ are neutrosophic submodules of E, then the following are equivalent.

(1) *Ne*(K_j) ⊆ *Ne*(K), $1 \leq j \leq K$ $Ne(K_j) \subseteq Ne(K)$.

(2) Neutrosophic of the intersect of $K_j \subseteq Ne(K)$.

(3) $Ne(\pi_{i=1}^{n}(K_r) \subseteq Ne(K)$.

Proof:

(1) ⟹(2): Obvious.

(2) ⟹(3): We know that from (2), $Ne(\pi_{i=1}^{n}(K_r) \subseteq Ne(\cap(K_i)) \subseteq Ne(k)$

(3) ⟹(1): For some J_i , $1 \leq i \leq n$ such that I_j an ideal of T, we have $Ne(k_i) = Ne(J_i)Ne(E)$. Hence $Ne(k_1,k_2,\ldots.k_n) = Ne(J_1J_2\ldots\ldots J_n\ Ne\ (E) \subseteq Ne(k)$. Then $Ne\ (J_1J_2\ldots..J_n) \subseteq Ne\ (k_iE)$. But $Ne\ (k_iE)$ is a prime neutrosophic ideal of T, i.e. $Ne(P.I) \subseteq Ne\ (k_iE)$ for some $1 \leq i \leq n$. Thus $Ne(k_i) = Ne(J_i)Ne(E) \subseteq Ne(k)$ for some i , $1 \leq i \leq n$.

Definition 3.14. Let E be a neutrosophic multiplication T-module. A non-empty neutrosophic subset S^* of *Ne*(E) is called neutrosophic multiplicatively closed, *Ne*(*MC*).

Theorem 3.15. Suppose that E is a neutrosophic T- module. Then the following are equivalent.

(1) K is a proper neutrosophic prime submodule of *Ne*(E).

(2) $Ne(\frac{E}{k})$ is a *Ne*(*M.C*).

Proof:

Suppose that condition (1) is true. Let m_1, $m_2 \in Ne(\frac{E}{k})$. From condition (1), we have k is a neutrosophic prime submodule and m_1, $m_2 \not\subset Ne(k)$. So m_1, $m_2 \cap Ne(\frac{E}{k}) \neq \emptyset$.

Now suppose that condition (2) is true. Let m_1, $m_2 \notin Ne(k)$. Hence $m_1, m_2 \in Ne(\frac{E}{k})$. But $Ne(\frac{E}{k})$ is a $Ne(M.C)$, $m_1, m_2 \cap Ne(\frac{E}{k}) \neq \emptyset$. Thus $m_1, m_2 \not\subset Ne(k)$ (see[10]).

4. Conclusion

Neutrosophic module is one of many important concepts in module theory. In this paper we have defined neutrosophic multiplication *T*-module as an algebraic structure. Some basic properties have been introduced. It has been shown that if a neutrosophic Artinian multiplication module is a neutrosophic cyclic, then it is a neutrosophic finitely generated. The main result is if E is a neutrosophic divisible module over neutrosophic integral domain, then E is a neutrosophic multiplication module if and only if E is a neutrosophic cyclic module. Our future research is to further develop more types of neutrosophic multiplication modules, such as those on Q-fuzzy [17-20], Q-neutrosophic [21-28], soft intuitionistic [29], multiparameterized soft set [30], vague soft set [31-32], neutrosophic bipolar [33], neutrosophic cubic [34] and to be used in neurogenetic algorithms [35], numerical analysis for root convergence [36-41] interval complex neutrosophic [42,43] and some algebraic structures [44-46].

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Neutrosophic Entropy Based Fluoride Contamination Indices for Community Health Risk Assessment from Groundwater of Kangra County, North India

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Abstract: The underlying study intends to evaluate community health risk assessment from fluoride contamination of groundwater samples employing the proposed entropy variants of single valued neutrosophic sets. The symmetric fuzzy cross entropy numbers, which can represent the macroscopic view of fluoride contamination more effectively, are constructed in this study and then deployed to rank the seasonal parameters (pre-monsoon, rainy season and pre-monsoon) responsible for fluoride contamination in the study area. To quantify the non-linear relationship between seasonal parameters and sampling spots, the proposed neutrosophic entropy variants are fascinated for assigning weights to the monitored concentration reading of each seasonal parameter with respect to various sampling spots. Thereafter, these weights are coupled with the quality rating scale of each parameter, intended to establish new fuzzy and single valued neutrosophic entropy weighted fluoride contamination indices (FEFCI & NEFCI) respectively. The maximum (or minimum) FEFCI or NEFCI score at a particular sampling spot is designated to the "most (or least) contaminated" sampling spot accordingly.The underlying fluoride contaminated sampling spot identification methodology is efficacious for providing a better insight in assessing the community health risk from ground water of the study area.

Keywords: Fuzzy Entropy, Neutrosophic Entropy, Cross Entropy, Fluoride, Ground Water, Community health.

1. Introduction

Fluoride contamination in groundwater affects public health and its excess is responsible for the spread of incurable but preventable disease called as fluorosis. Many health problems are associated with drinking of water contaminated with elevated level of fluoride ($2mg$ /day) and are responsible for various common diseases such as arthritis, brittle bones, Alzheimer, skeletal malformation etc. Excess concentration of fluoride in drinking water leads to low calcium, high alkalinity, fluoride poisoning and thus affects the individual.

The recommended concentration of fluoride **[9]** in drinking water quality $1.5\ mg/l$. However, optimum concentration of fluoride varies between 0.5 - $1.0\ mg/l$ **[10]** according to climatic conditions. Public across world dependent mostly on groundwater resources have been encountering issues with increased concentration of fluoride. Fluorosis has mostly affected India and China, the two most populated countries of the world. In Pakistan, fluoride analysis on 29 main cities **[12]** showed 34% of the cities with elevated fluoride levels having mean value greater than $1.5\ mg/l$. In this study, Lahore, Quetta and Tehsil Mailsi were found with highest fluoride level values of 23.60, 24.48, 5.5 mg/l respectively. Fluoride epidemic has been reported in upwards of 19 Indian states and union regions. India is among the 23 countries in the world where fluoride sullied ground water is making medical issues. Recent studies from state Andhra Pradesh (India) have shown that fluoride level ranges from 0.4-5.8 mg/l with a mean value of 1.98 mg/l and villagers have been suffering severely from Fluorosis **[3]**. The province of Art Report of UNICEF affirms the fluoride issues in 177 locales of 20 states in India **[1]**. Fluoride content in local water well-springs of Dungapur area of Rajasthan was examined by Choubisa **[2]** and they revealed the fluoride content in open wells up to 10 mg/l. Fluoride focus in ground water of Prakasham area (Andhra Pradesh) in India was observed by **[4]** and they found the convergence of fluoride in surface and ground water tests shifted between 0.5 mg/l to 9.0 mg/l. A detailed instance of fluoride was in the Tekelangjun region, Karbi Anglong area, where fluoride fixations, in May 2019, were ranged between 5-23 mg/l. The profoundly fluoride influenced zones of Assam viz. Kamrup, Nagaon were investigated. Fluoride focuses in these zones were accounted for to be substantially higher than the BIS reasonable cutoff points of 1.5 mg/l. Extreme sullying of fluoride in groundwater of Karbi Anglong and Nagaon locale of Assam and its appearance has been accounted for fluorosis **[5-7]**.

Recently, Adimalla et al. **[19]** constructed entropy water quality index (EWQI) and assessed the overall quality of ground water for irrigation and domestic purposes. Singh et al. **[20]** deployed Shannon's information entropy for constructing entropy weighted heavy metal contamination index (EHCI) and performed spatial assessment of water quality in some tributaries of Brahmaputra River. Unfortunately, the additive and probabilistic Shannon's entropy is facing a major drawback as it is based on the fancy presumption $0 \times log 0 = 0$ and hence indicates major conflicts in water treatment strategies. Under such problematic situation, Zadeh's **[17]** fuzzy set theory can handle the complexity of contamination level from macroscopic point of view. Dubois and Prade **[16]** developed the first non-additive and non-probabilistic entropy measure for elaborating some measurements of membership functions, intended to develop uncertainty modeling. In the existing literature, many equivalents of fuzzy sets are available and can be deployed to tackle fluoride contamination issues for quality evaluation. Subsequently, Smarandache **[18]** neutrosophic set theory can represent the macroscopic state of fluoride contamination of ground water in a broader way. A neutrosophic set contains more quantified information than any fuzzy set and can be characterized by the forms of truism membership, indeterminacy membership and fallacy membership functions respectively. To the our best knowledge, no neutrosophic entropy measure, till so far, has been developed and deployed for quantifying the non-linear relationship of fluoride contamination between seasonal parameters and sampling spots. Subsequently, an effort has been accomplished in this pathway by constructing symmetric fuzzy cross entropy numbers (SFCNs) followed by fuzzy entropy and single valued neutrosophic entropy weighted fluoride contamination indices (FEFCI and NEFCI) consecutively. A schematic flow chart of the underlying methodology is depicted in **Fig 1** and the rest of the proposed research work is organized as follows.

Section 2 provides the details of study area, materials and the procedure employed in collecting ground water samples under investigation. **Section 3** discusses in brief the basic terminology of Information theory, required for understanding the underlying study. **Section 4** deals with the establishment of a novel hyperbolic fuzzy entropy measure followed by symmetric fuzzy cross entropy (FCE) as well as single valued neutrosophic entropy measures. The efficaciousness of the proposed symmetric fuzzy cross entropy numbers (SFCNs) is validated in **Section 5** by classifying each seasonal parameter responsible for fluoride contamination in ground water samples. **Section 6** introduces a novel HFE and HNE based methodology for constructing fuzzy entropy and single valued neutrosophic entropy weighted fluoride contamination indices (FEFCI and NEFCI). **Section 7** provides the applicability and effectiveness of the underlying methodology by reckoning the most contaminated sampling spot along with community health risk assessment related to ground water quality whereas **Section 8** finally summarizes the concrete conclusions of this study.

2. Materials and Procedure

2.1 Study Area The area under investigation lies between 78.91 and 79.13 N longitude and 18.00 and Kangra-the most populated district of Himachal Pradesh, India, with a population of 15,07,223, (2011 Census)- is located on the southern ridge of the Himalaya between 31°2 to 32°5 N and 75° to 77°45 E. This district is surrounded by mountain altitude of the Shivaliks, Dhauladhar and the Himalayas from north-west to south-east. The district has a geographical area of 5,739 km. The altitude varies from 500 meters above the average sea level to 5000 meters. Due to its ideal location, Kangra is renowned for tourism activities and therefore the district's economy is centered mainly on tourism apart from agriculture and industrial resources. While Kangra is gifted with ample freshwater resources such as River Beas, Dal and Kareri Lakes, Pong reservoir etc.; along with numerous ground water sources such as dug wells, hand pumps, tube wells and springs. Due to environmental degradation, population overgrowth, pollution, tourism and various developmental activities affect overall water quality (MWR, 2016). CGWB (2018) surveyed annual fluctuation in water level of GWMS during different monitoring periods were analyzed. The climate of the district Kangra varies from sub-tropical to sub-humid. Winter varies from December to February and summer extends from March to June while July to September are rainy months. The average rainfall in the district during 2005 was 1765.1 mm. Snow fall is received in the higher reaches of Dhauladhar mountain ranges. Average minimum and maximum temperature ranges from 3°C and 45°C. In this study, the details of the sampling spots along with sampling codes are mentioned below:

S_1 = Shahpur, S_2 = Samlana Jawali, S_3 = Indora, S_4 = Mata Rani Chowk, Haripur, S_5 = Sukka Talab Chowk, Haripur, S_6 = Garli, S_7 = Sapadi, S_8 = Jawalaji, S_9 = Dehra, S_{10} = Nagrota Bagwan, S_{11} = Dharamsala, S_{12} = Bod, S_{13} = Thural, S_{14} = Baijnath, S_{15} = Chougan, Bir, S_{16} = Palampur and S_{17} = Main Bazar Kangra.

2.2 Spectrophotometric Method A compound of a metal such as aluminum, iron, thorium, zirconium, lanthanum or cerium reacts with an indicator dye to form a complex of low dissociation constant. This complex reacts with fluoride to give a new complex. Due to the change in the structure of the complex, the absorption spectrum also shifts relative to the spectrum for the fluoride-free reagent solutions. This change can be detected by using a spectrophotometer. One of the important dyes used is trisodium 2-(parasulfophenylazo-1), 8- dihydroxy-3, 6- naphthalene disulfonate, commonly known as SPADNS. The dye reacts with metal ions to give a colored complex. In the SPADNS method, zirconium reacts with SPADNS to form a red coloured complex.

Fluoride bleaches the red color of the complex and hence the change in absorbance can be measured using a spectrophotometer.

2.3 Procedure Preparation of the reagent: 958 mg of SPADNS was dissolved in distilled water and diluted to 500 ml. 133 mg of zirconyl chloride octahydrate ($ZrCl_2.8H_2O$) was dissolved in 25 ml distilled water and 350 ml of conc. HCl was added and diluted to 500 ml with distilled water. SPADNS solution and Zirconyl acid solutions were mixed in equal volume.

To prepare the calibration curve, 0.221 g of anhydrous sodium fluoride was dissolved in water and diluted up to one liter and further diluted to get standard solution having 10 mg per liter of fluoride.1, 2, 3, 4, 5 and 6 ml of this solution was pipetted out into 50 ml standard flasks. 10 ml of Zirconyl-SPADNS reagent and one drop of $NaAsO_2$ were added to each of the solutions and was diluted up to the mark and mixed well.

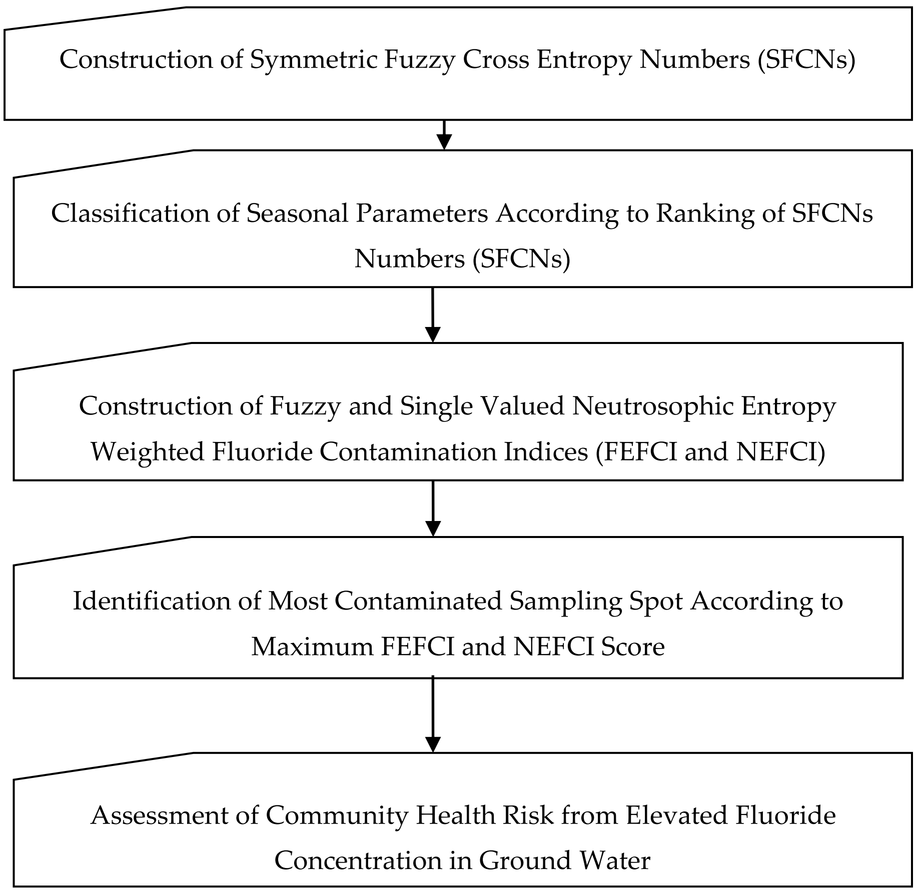


Fig.1 Schematic Flow Chart For Identifying the Most Contaminated Sampling Spot Responsible for Fluoride Contamination in Ground Water

The absorbance of the solutions was measured at 570 nm against a reagent blank and a calibration plot was constructed by plotting absorbance against concentrations using colorimeter. Suitable aliquot of water sample was taken and repeated the step. The concentration of F^-/l was calculated after using the calibration curve.

3. Preliminaries: -

Def.3.1 Fuzzy Entropy Measure [16] Let $W(U)$ represents the collection of all fuzzy sets in a space of discourse U generated by generic elements $(x_1, x_2, x_3, .., x_n)$. Let

$R_1=\left(\prec x_i,\mu_R\left(x_i\right)\succ\forall x_i\in U\right)(i=1,2,...n)$ be any fuzzy set in U which is quantified by its truth membership functions $\mu_R\left(x_i\right):U\rightarrow[0,1]$ and satisfy $0\le\mu_R\left(x_i\right)\le 1\forall i$ .Then a function $T\left(R_1\right):W(U)\rightarrow R^+$ (the set of non-negative real numbers) is called as fuzzy entropy measure if $(i)\mathrm{T}\left(R_1\right)\ge 0\forall R_1\in W(U)$ with equality if $\mu_R\left(x_i\right)=0$ or 1. $(ii)\mathrm{T}\left(R_1\right)$ is a concave function with respect to each $\mu_R\left(x_i\right)$ and $(iii)\mathrm{T}\left(R_1^c\right)=\mathrm{T}\left(R_1\right)\forall R_1\in W(U)$ where R_1^c denotes the complement of the fuzzy set R_1 and is defined as $R_1^c=\left(\prec x_i,1-\mu_R\left(x_i\right)\succ\forall x_i\in U\right)$.

Def.3.2 Symmetric Cross Entropy Measure [16] Let $R_1=\left(\prec x_i,\mu_{R_1}\left(x_i\right)\succ\forall x_i\in U\right)$ and $R_2=\left(\prec x_i,\mu_{R_2}\left(x_i\right)\succ\forall x_i\in U\right)$ represent two fuzzy sets in U, quantified by their truth membership functions $\mu_{R_1}\left(x_i\right),\mu_{R_2}\left(x_i\right):U\rightarrow[0,1]$ and satisfy $0\le\mu_{R_1}\left(x_i\right),\mu_{R_2}\left(x_i\right)\le 1$. Then a function $F\left(R_1,R_2\right):W(U)\times W(U)\rightarrow R^+$ is called as symmetric fuzzy cross entropy (FCE) or discrimination information measure between two FSs R_1 and R_2 if $(i)F\left(R_1,R_2\right)\ge 0$ with equality if $R_1=R_2$ $(ii)F\left(R_1,R_2\right)=F\left(R_2,R_1\right)$. In other words, $F\left(R_1,R_2\right)$ is symmetric in nature $(iii)F\left(R_1^c,R_2^c\right)=F\left(R_1,R_2\right)\forall R_1,R_2\in W(U)$ which means $F\left(R_1,R_2\right)$ remains unchanged on interchanging $\mu_{R_1}\left(x_i\right),\mu_{R_2}\left(x_i\right)$ with their counter parts $1-\mu_{R_1}\left(x_i\right),1-\mu_{R_2}\left(x_i\right)$.

Def.3.3 Single Valued Neutrosophic Set [18] A single valued neutrosophic set S_1 in $U=\left(x_1,x_2,x_3,..x_n\right)$ is an entity of the form $S_1=\left(\prec x_i,\mu_{S_1}\left(x_i\right),i_{S_1}\left(x_i\right),f_{S_1}\left(x_i\right)\succ\forall x_i\in U\right)$ where each $\mu_{S_1}\left(x_i\right),i_{S_1}\left(x_i\right),f_{S_1}\left(x_i\right):U\rightarrow[0,1]$ satisfy $0\le\mu_{S_1}\left(x_i\right)+i_{S_1}\left(x_i\right)+f_{S_1}\left(x_i\right)\le 3$ and are characterized by (i) truth membership function $\mu_{S_1}\left(x_i\right)$ (ii) indeterminacy function $i_{S_1}\left(x_i\right)$ and (iii) falsity membership function $f_{S_1}\left(x_i\right)$ respectively where each $x_i\in U$ is associated to a unique real number in the closed interval [0,1].

Def.3.4 Single Valued Neutrosophic Entropy Measure [18] Let $T(U)$ represents the collection of all single valued neutrosophic sets (SVNSs) in U. Then a function $T\left(S_1\right):S(U)\rightarrow R^+$ is called as single valued neutrosophic entropy measure if

$(i)T\left(S_1\right)\ge 0\forall S_1\in T(U)$ $(ii)T\left(S_1\right)=0$ whenever either $\mu_{S_1}\left(x_i\right)=1,i_{S_1}\left(x_i\right)=0,f_{S_1}\left(x_i\right)=0$ or $\mu_{S_1}\left(x_i\right)=0,i_{S_1}\left(x_i\right)=0,f_{S_1}\left(x_i\right)=1$. $(iii)T\left(S_1^c\right)=T\left(S_1\right)$ where S_1^c denotes the complement of S_1 and is defined as $S_1^c=\left(\prec x_i,f_{S_1}\left(x_i\right),1-i_{S_1}\left(x_i\right),\mu_{S_1}\left(x_i\right)\succ\forall x_i\in U\right)$ and (iv) $T\left(S_1\right)$ exhibits the concavity property with respect to each $\mu_{S_1}\left(x_i\right),i_{S_1}\left(x_i\right),f_{S_1}\left(x_i\right)$. Also, $T\left(S_1\right)$ admits its maximum value, which arises when $\mu_{S_1}\left(x_i\right)=i_{S_1}\left(x_i\right)=f_{S_1}\left(x_i\right)=\frac{1}{2}$ and the maximum value is an increasing function of *n.*

4. Establishment of Single Valued Neutrosophic Entropy Measure

Our endeavor will be to develop a novel fuzzy entropy measure followed by symmetric fuzzy cross entropy measure hinged on two fuzzy sets. The aftermaths of which will be a backbone for the construction of proclaimed symmetric fuzzy cross entropy numbers (SFCNs), required for classifying the seasonal parameters responsible for fluoride contamination in ground water.

4.1 A Novel Hyperbolic Fuzzy Entropy Measure

We shall propose a novel hyperbolic fuzzy entropy (HFE) measure (**Theorem 4.1)**, the aftermaths of which will be a backbone for the proposed symmetric fuzzy cross entropy measure (**Theorem 4.2)**.

Theorem.4.1 Let $R_1 = \left(\prec x_i, \mu_{R_1}(x_i) \succ \forall x_i \in U\right)$ be any fuzzy set in U. Then

$$F(R_1) = -\sum_{i=1}^{n} \left[\tanh\left(\frac{1+\sqrt{\mu_{R_1}^2(x_i) + \left(1-\mu_{R_1}(x_i)\right)^2}}{2+\sqrt{\mu_{R_1}(x_i)} + \sqrt{1-\mu_{R_1}(x_i)}} \right) - \tanh\left(\frac{2}{3}\right) \right] \quad \ldots (1)$$

represents an authentic hyperbolic fuzzy entropy measure with minimum value zero and maximum value as $\left(\tanh\frac{2}{3} - \tanh\frac{1}{2}\right)n$. Here, the generic element 'x_i' denotes the 'i^{th}' macroscopic level of fluoride contamination and $F(R_1)$ represents the fuzziness of fluoride contamination indicated by the fuzzy set R_1.

Proof (i) In view of **Def. 3.1,** $F(R_1) \geq 0$ since $0 \leq \mu_{R_1}(x_i) \leq 1 (i = 1, 2, ..., n)$. Also, $F(R_1)$ vanishes whenever $\mu_{R_1}(x_i) = 0 \text{ or } 1$.

(ii) $F(R_1)$ remains unchanged after replacing $\mu_R(x_i)$ with $1-\mu_R(x_i)$.

(iii) **Concavity:** The fact that hyperbolic fuzzy entropy $F(R_1)$ exhibits its concavity property with respect to each $\mu_R(x_i)$, can be seen from its 3-D rotational plot displayed in **Fig. 2.** Next, the positive term finite series **(1)** converges absolutely which motivates $F(R_1)$ to possess first order partial differentiation with respect to each $\mu_R(x_i)$ Set $T_0 = \sqrt{\mu_{R_1}^2(x_i) + \left(1-\mu_{R_1}(x_i)\right)^2}$, $T_1 = \sqrt{\mu_{R_1}(x_i)}$, and $T_2 = \sqrt{\left(1-\mu_{R_1}(x_i)\right)}$. Due to concavity, $F(R_1)$ affirms its maximum value which arises only when $\frac{\partial F(R_1)}{\partial \mu_{R_1}(x_i)} = 0$ which implies

$$Sech^2\left(\frac{1+T_0}{2+T_1+T_2}\right) \times \left(\frac{2\left(1-2T_1^2\right)}{2T_0\left(2+T_1+T_2\right)^2} + \frac{\left(T_1-T_2\right)\left(1+T_0\right)}{2T_1T_2\left(2+T_1+T_2\right)^2} \right) = 0 \quad \ldots (2)$$

The resulting expression **(2)** yields $\mu_{R_1}(x_i) = \frac{1}{2}$ and hence (1) returns

$$\text{Max.}F(R_1) = F(R_1)\Big|_{\mu_A(x_i)=\frac{1}{2}} = \left(\tanh\frac{2}{3} - \tanh\frac{1}{2}\right)n. \quad \ldots (3)$$

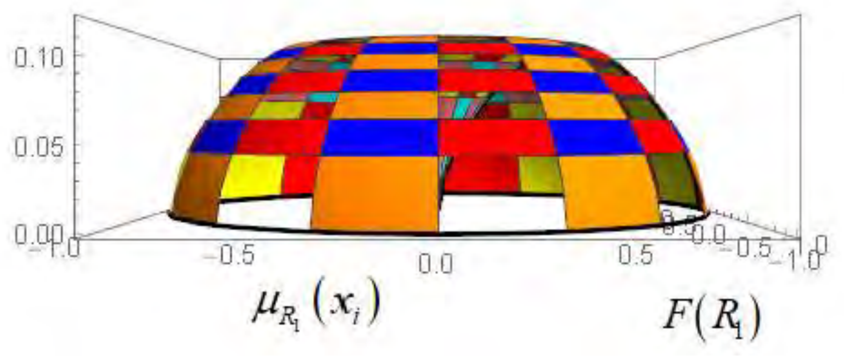


Fig.2 Concavity of $F(R_1)$ with respect to $\mu_{R_1}(x_i)$

Theorem.4.2 Let $R_1 = (\prec x_i, \mu_{R_1}(x_i) \succ \forall x_i \in U)$ and $R_2 = (\prec x_i, \mu_{R_2}(x_i) \succ \forall x_i \in U)$ be any two fuzzy sets in U. Then $F^{\mu}(R_1, R_2)$ is an authentic symmetric hyperbolic fuzzy cross entropy measure **(Def. 3.2)** hinged on two fuzzy sets R_1 and R_2 where

$$F^{\mu}(R_1,R_2)=\sum_{i=1}^{n}\left[\begin{array}{l}-6Tanh\,\frac{1}{2}+\left(2+\mu_{R_1}(x_i)+\mu_{R_2}(x_i)\right)Tanh\left(\frac{1+\sqrt{\mu_{R_1}^2(x_i)+\mu_{R_2}^2(x_i)}}{2+\left(\sqrt{\mu_{R_1}(x_i)}+\sqrt{\mu_{R_2}(x_i)}\right)\left(\sqrt{\mu_{R_1}(x_i)+\mu_{R_2}(x_i)}\right)}\right)\\ +\left(4-\mu_{R_1}(x_i)-\mu_{R_2}(x_i)\right)Tanh\left(\frac{1+\sqrt{\left(1-\mu_{R_1}(x_i)\right)^2+\left(1-\mu_{R_2}(x_i)\right)^2}}{2+\left(\sqrt{1-\mu_{R_1}(x_i)}+\sqrt{1-\mu_{R_2}(x_i)}\right)\left(\sqrt{2-\mu_{R_1}(x_i)-\mu_{R_2}(x_i)}\right)}\right)\end{array}\right] \quad \ldots (4)$$

Here $F^{\mu}(R_1, R_2)$ indicates the mathematical value of true membership degree of symmetric discrimination of the fuzzy set R_1 against R_2

Proof. The conditions (ii) and (iii) of **Def. 3.2** are obvious. We shall, equally well, establish the following **Lemma 4.1,** intended to establish that non-negativity of symmetric fuzzy cross entropy measure $F^{\mu}(R_1, R_2)$

Lemma 4.1 If $P=\sqrt{\frac{\mu_{R_1}^2(x_i)+\mu_{R_2}^2(x_i)}{2}}$, $N=\left(\frac{\sqrt{\mu_{R_1}(x_i)}+\sqrt{\mu_{R_2}(x_i)}}{2}\right)\left(\sqrt{\frac{\mu_{R_1}(x_i)+\mu_{R_2}(x_i)}{2}}\right)$.

Then, there exists the inequality $P\left(\mu_{R_1}(x_i),\mu_{R_2}(x_i)\right)\geq N\left(\mu_{R_1}(x_i),\mu_{R_2}(x_i)\right)$ with equality if $\mu_{R_1}(x_i)=\mu_{R_2}(x_i)\forall\mu_{R_1}(x_i),\mu_{R_2}(x_i)\in[0,1]$.

Proof. The undergoing inequality can be made true if

$$P^2 \geq N^2 \Rightarrow \frac{\mu_{R_1}^2(x_i)+\mu_{R_2}^2(x_i)}{2} \geq \left(\frac{\sqrt{\mu_{R_1}(x_i)}+\sqrt{\mu_{R_2}(x_i)}}{2}\right)^2 \left(\frac{\mu_{R_1}(x_i)+\mu_{R_2}(x_i)}{2}\right) \text{ or if}$$

$$3\mu_{R_1}^2(x_i)+3\mu_{R_2}^2(x_i)-2\mu_{R_1}(x_i)\mu_{R_2}(x_i) \geq 2\sqrt{\mu_{R_1}(x_i)}\sqrt{\mu_{R_1}(x_i)}\left(\mu_{R_1}(x_i)+\mu_{R_2}(x_i)\right) \text{ or if}$$

$$\left(3\mu_{R_1}^2(x_i)+3\mu_{R_2}^2(x_i)-2\mu_{R_1}(x_i)\mu_{R_2}(x_i)\right)^2 \geq 4\mu_{R_1}(x_i)+\mu_{R_2}(x_i)\left(\mu_{R_1}(x_i)+\mu_{R_2}(x_i)\right)^2 \text{ or if}$$

$9\mu_{R_1}^4(x_i)+9\mu_{R_2}^4(x_i)+14\mu_{R_1}^2(x_i)\mu_{R_2}^2(x_i) \geq 16\mu_{R_1}(x_i)\mu_{R_2}(x_i)\left(\mu_{R_1}^2(x_i)+\mu_{R_2}^2(x_i)\right)$ which is obviously true for each $\mu_{R_1}(x_i),\mu_{R_2}(x_i) \in [0,1]$.

Thus, in view of **Lemma 3.1,** the oncoming inequality can be re-scheduled as

$$P\left(\mu_{R_1}(x_i),\mu_{R_2}(x_i)\right)+1 \geq N\left(\mu_{R_1}(x_i),\mu_{R_2}(x_i)\right)+1$$

$$\Rightarrow \sqrt{\frac{\mu_{R_1}^2(x_i)+\mu_{R_2}^2(x_i)}{2}}+1 \geq \left(\frac{\sqrt{\mu_{R_1}(x_i)}+\sqrt{\mu_{R_1}(x_i)}}{2}\right)\left(\sqrt{\frac{\mu_{R_1}(x_i)+\mu_{R_2}(x_i)}{2}}\right)+1$$

$$\Rightarrow \frac{1+\sqrt{\mu_{R_1}^2(x_i)+\mu_{R_2}^2(x_i)}}{2+\left(\sqrt{\mu_{R_1}(x_i)}+\sqrt{\mu_{R_2}(x_i)}\right)\left(\sqrt{\mu_{R_1}(x_i)+\mu_{R_2}(x_i)}\right)} \geq \frac{1}{2} \qquad \ldots (5)$$

Since, the hyperbolic functions over $[0,1]$ are monotonic in nature, the foregoing inequality **(5)** can be re-designed as

$$\left(2+\mu_{R_1}(x_i)+\mu_{R_2}(x_i)\right)\tanh\left(\frac{1+\sqrt{\mu_{R_1}^2(x_i)+\mu_{R_2}^2(x_i)}}{2+\left(\sqrt{\mu_{R_1}(x_i)}+\sqrt{\mu_{R_2}(x_i)}\right)\left(\sqrt{\mu_{R_1}(x_i)+\mu_{R_2}(x_i)}\right)}\right) \geq \left(2+\mu_{R_1}(x_i)+\mu_{R_2}(x_i)\right)\tanh\left(\frac{1}{2}\right) \qquad \ldots (6)$$

After replacement of $\mu_{R_1}(x_i),\mu_{R_2}(x_i)$ with their counter parts $1-\mu_{R_1}(x_i),1-\mu_{R_2}(x_i)$ into **(6)** yields

$$\left(4-\mu_{R_1}(x_i)-\mu_{R_2}(x_i)\right)\tanh\left(\frac{1+\sqrt{\left(1-\mu_{R_1}(x_i)\right)^2+\left(1-\mu_{R_2}(x_i)\right)^2}}{2+\left(\sqrt{1-\mu_{R_1}(x_i)}+\sqrt{1-\mu_{R_2}(x_i)}\right)\left(\sqrt{2-\mu_{R_1}(x_i)-\mu_{R_2}(x_i)}\right)}\right) \geq \left(4-\mu_{R_1}(x_i)-\mu_{R_2}(x_i)\right)\tanh\left(\frac{1}{2}\right) \qquad \ldots (7)$$

with equality if $\mu_{R_1}(x_i)=\mu_{R_2}(x_i)\,\forall i=1,2,\ldots,n.$

Simply adding the resulting inequalities **(6)** and **(7)** and summing over $i=1$ to $i=n$ yields $F^{\mu}(R_1,R_2)\,\forall \mu_{R_1}(x_i),\mu_{R_2}(x_i) \in [0,1]$ with equality if $\mu_{R_1}(x_i)=\mu_{R_2}(x_i)\,\forall i=1,2,\ldots,n.$

We next divert our attention to discuss the situation under which the proposed symmetric fuzzy cross entropy $F^{\mu}(R_1,R_2)$ admists its maximum and minimum values as follows.

Theorem 4.3 Let $R_1 = \left(\prec x_i, \mu_{R_1}(x_i) \succ \forall x_i \in U \right)$ and $R_2 = \left(\prec x_i, \mu_{R_2}(x_i) \succ \forall x_i \in U \right)$ be two fuzzy sets with same cardinality as of U. Then there exists the inequality: $0 \le F^{\mu}(R_1, R_2) \le 6\left(\tanh\frac{2}{3} - \tanh\frac{1}{2} \right)n,$ where n is the cardinality of U.

Proof. Replacement of R_2 with R_1^c into the resulting equality **(4)** yields

$$F^{\mu}(R_1, R_1^c) = \sum_{i=1}^{n} \left[-6\tanh\frac{1}{2} + 6\tanh\left(\frac{1 + \sqrt{\mu_{R_1}^2(x_i) + \left(1 - \mu_{R_2}(x_i)\right)^2}}{2 + \left(\sqrt{\mu_{R_1}(x_i)} + \sqrt{1 - \mu_{R_2}(x_i)} \right)} \right) \right]$$

$$= \sum_{i=1}^{n} \left[6\tanh\frac{2}{3} - 6\tanh\frac{1}{2} - 6\left(\tanh\frac{2}{3} - \tanh\left(\frac{1 + \sqrt{\mu_{R_1}^2(x_i) + \left(1 - \mu_{R_2}(x_i)\right)^2}}{2 + \left(\sqrt{\mu_{R_1}(x_i)} + \sqrt{1 - \mu_{R_2}(x_i)} \right)} \right) \right) \right]$$

$$= 6\text{Max.}F(R_1) - 6F(R_1) \qquad \dots (8)$$

Since $F(R_1) \ge 0 \forall \mu_{R_1}(x_i)$, the oncoming equality **(8)** yields

$$F(R_1) = \text{Max.}F(R_1) - \frac{1}{6}F^{\mu}(R_1, R_1^c) \ge 0 \Rightarrow 0 \le F^{\mu}(R_1, R_1^c) \le 6\left(\tanh\frac{2}{3} - \tanh\frac{1}{2} \right)n \qquad \dots (9)$$

Discussion. The undergoing inequality **(9)** clarifies the finiteness of $F^{\mu}(R_1, R_1^c)$ whenever n is a fixed natural number. On the same pattern, the users can establish that, $F^{\mu}(R_1, R_1^c)$ will also be finite and has the range value $0 \le F^{\mu}(R_1, R_2) \le 6\left(\tanh\frac{2}{3} - \tanh\frac{1}{2} \right)n.$ In view of **Theorem 4.2**, we have $\text{Max.}F^{\mu}(R_1, R_2) = 6\left(\tanh\frac{2}{3} - \tanh\frac{1}{2} \right)n$ for a fixed n and this value completely depends upon the cardinality of U. Also, the three-dimensional plot depicted in **Fig 3(a, b)** exhibits that $F^{\mu}(R_1, R_2)$ admits its minimum value zero. Furthermore, $F^{\mu}(R_1, R_2)$ increases with the increase in $|R_1 - R_2|$, attains it maximum value at the points $(1,0)$ & $(0,1)$ and minimum value zero whenever $R_1 = R_2$.

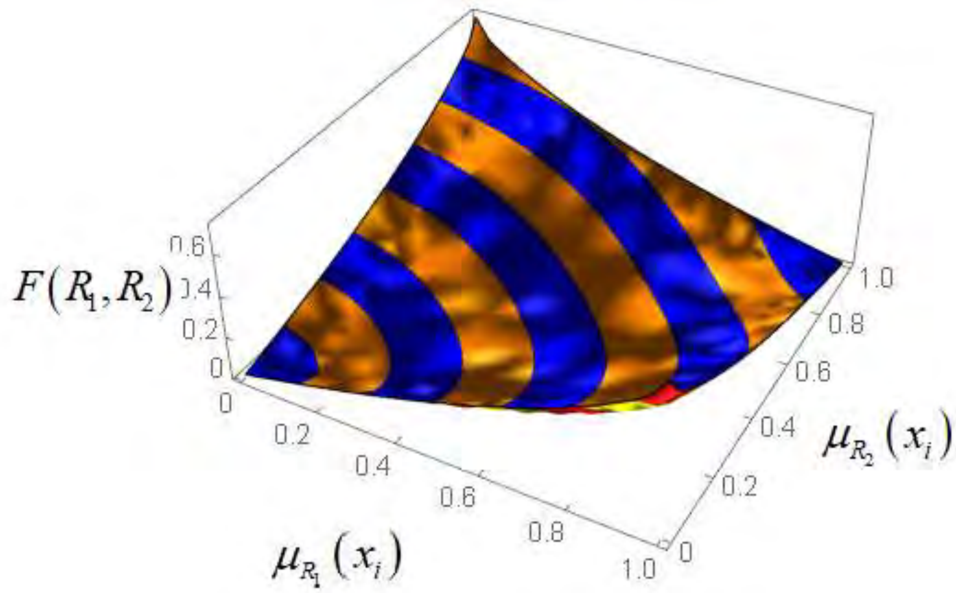
We next switch to establish the proclaimed single valued neutrosophic entropy measure hinged on two single valued neutrosophic sets, the aftermaths of which will be utilized to understand the macroscopic state of fluoride contamination in ground water.

4.2 A Novel Hyperbolic Single Valued Neutrosophic Entropy Measure

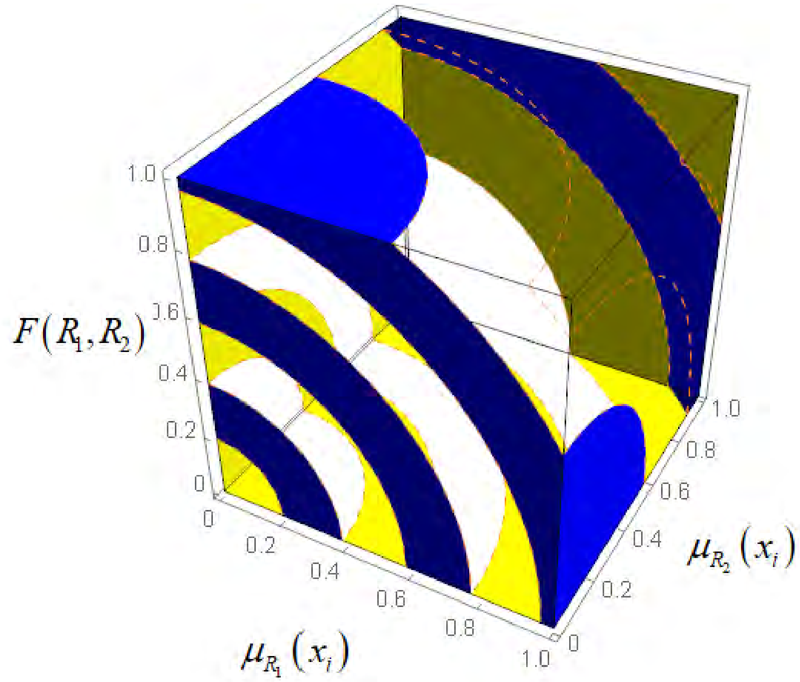
To meet the desired goal, we shall first extend the resulting symmetric hyperbolic fuzzy cross entropy measure **(Theorem 4.2)** hinged on two fuzzy sets to this measure hinged on two single valued neutrosophic sets as follows.

Def. 4.1 Let $S_1 = \left(\prec x_i, \mu_{S_1}(x_i), i_{S_1}(x_i), f_{S_1}(x_i) \succ \forall x_i \in U \right); S_2 = \left(\prec x_i, \mu_{S_2}(x_i), i_{S_2}(x_i), f_{S_2}(x_i) \succ \forall x_i \in U \right)$ be any two single valued neutrosophic sets **(Def. 3.3).** In view of **Theorem 4.2,** the mathematical value of true membership degree of symmetric discrimination of S_1 against S_2 is given as

$$F^{\mu}(S_1, S_2) = \sum_{i=1}^{n} \left[\begin{array}{l} -6\tanh\frac{1}{2} + \left(2 + \mu_{S_1}(x_i) + \mu_{S_2}(x_i)\right)\tanh\left(\frac{1+\sqrt{\mu_{S_1}^2(x_i)+\mu_{S_2}^2(x_i)}}{2+\left(\sqrt{\mu_{S_1}(x_i)}+\sqrt{\mu_{S_2}(x_i)}\right)\left(\sqrt{\mu_{S_1}(x_i)+\mu_{S_2}(x_i)}\right)}\right) \\ +\left(4 - \mu_{S_1}(x_i) - \mu_{S_2}(x_i)\right)\tanh\left(\frac{1+\sqrt{\left(1-\mu_{S_1}(x_i)\right)^2+\left(1-\mu_{S_2}(x_i)\right)^2}}{2+\left(\sqrt{1-\mu_{S_1}(x_i)}+\sqrt{1-\mu_{S_2}(x_i)}\right)\left(\sqrt{2-\mu_{S_1}(x_i)-\mu_{S_2}(x_i)}\right)}\right) \end{array} \right] \quad \ldots (10)$$



(a)



(b)

Fig. 3 Maximum and Minimum Value of $F^{\mu}(R_1,R_2)$

Similarly, the mathematical values of indeterminacy and falsity membership degrees of symmetric discrimination of S_1 against S_2 are given as

$$F^{i}(S_1,S_2)=\sum_{i=1}^{n}\left[\begin{array}{l}-6\tanh\frac{1}{2}+\left(2+i_{S_1}(x_i)+i_{S_2}(x_i)\right)\tanh\left(\frac{1+\sqrt{i_{S_1}^2(x_i)+i_{S_2}^2(x_i)}}{2+\left(\sqrt{i_{S_1}(x_i)}+\sqrt{i_{S_2}(x_i)}\right)\left(\sqrt{i_{S_1}(x_i)+i_{S_2}(x_i)}\right)}\right)\\+\left(4-i_{S_1}(x_i)-i_{S_2}(x_i)\right)\tanh\left(\frac{1+\sqrt{\left(1-i_{S_1}(x_i)\right)^2+\left(1-i_{S_2}(x_i)\right)^2}}{2+\left(\sqrt{1-i_{S_1}(x_i)}+\sqrt{1-i_{S_2}(x_i)}\right)\left(\sqrt{2-i_{S_1}(x_i)-i_{S_2}(x_i)}\right)}\right)\end{array}\right]$$

… (11)

$$F^{f}(S_1,S_2)=\sum_{i=1}^{n}\left[\begin{array}{l}-6\tanh\frac{1}{2}+\left(2+f_{S_1}(x_i)+f_{S_2}(x_i)\right)\tanh\left(\frac{1+\sqrt{f_{S_1}^2(x_i)+f_{S_2}^2(x_i)}}{2+\left(\sqrt{f_{S_1}(x_i)}+\sqrt{f_{S_2}(x_i)}\right)\left(\sqrt{f_{S_1}(x_i)+f_{S_2}(x_i)}\right)}\right)\\+\left(4-f_{S_1}(x_i)-f_{S_2}(x_i)\right)\tanh\left(\frac{1+\sqrt{\left(1-f_{S_1}(x_i)\right)^2+\left(1-f_{S_2}(x_i)\right)^2}}{2+\left(\sqrt{1-f_{S_1}(x_i)}+\sqrt{1-f_{S_2}(x_i)}\right)\left(\sqrt{2-f_{S_1}(x_i)-f_{S_2}(x_i)}\right)}\right)\end{array}\right]$$

… (12)

Hence, the proclaimed single valued neutrosophic cross entropy measure hinged on two SVNSs S_1 and S_2 can be easily established by adding the resulting expressions **(10), (11)** and **(12).** Thus,

$$T(S_1,S_2)=F^{\mu}(S_1,S_2)+F^{i}(S_1,S_2)+F^{f}(S_1,S_2) \qquad \ldots (13)$$

Here, $T(S_1,S_2)$ represents the true, indeterminacy and falsity membership degrees indicated by the symmetric discrimination of SVNS S_1 against S_2

Theorem.4.4 Let $S_1=\left(\prec x_i,\mu_{S_1}(x_i),i_{S_1}(x_i),f_{S_1}(x_i)\succ\right)$ and $S_2=\left(\prec x_i,\mu_{S_2}(x_i),i_{S_2}(x_i),f_{S_2}(x_i)\succ\right)$ be any two single valued neutrosophic sets, with same cardinality as of U. Then there exists the inequality $0\le T(S_1,S_2)\le 18\left(\tanh\frac{2}{3}-\tanh\frac{1}{2}\right)n.$

Proof. Replacement of S_2 with its counterpart S_1^c into the expression **(13)** yields

$$T(S_1,S_1^c)=\sum_{i=1}^{n}\left[\begin{array}{l}18\tanh\frac{2}{3}-18\tanh\frac{1}{2}\\ -6\left(\begin{array}{l}3\tanh\frac{2}{3}-\left(\frac{2+\mu_{S_1}(x_i)+f_{S_1}(x_i)}{3}\right)\tanh\left(\frac{1+\sqrt{\mu_{S_1}^2(x_i)+f_{S_1}^2(x_i)}}{2+\left(\sqrt{\mu_{S_1}(x_i)}+\sqrt{f_{S_1}(x_i)}\right)\left(\sqrt{\mu_{S_1}(x_i)+f_{S_1}(x_i)}\right)}\right)\\ -\left(\frac{4-\mu_{S_1}(x_i)-f_{S_1}(x_i)}{3}\right)\tanh\left(\frac{1+\sqrt{\left(1-\mu_{S_1}(x_i)\right)^2+\left(1-f_{S_1}(x_i)\right)^2}}{2+\left(\sqrt{1-\mu_{S_1}(x_i)}+\sqrt{1-f_{S_1}(x_i)}\right)\left(\sqrt{2-\mu_{S_1}(x_i)-f_{S_1}(x_i)}\right)}\right)\\ -\tanh\left(\frac{1+\sqrt{i_{S_1}^2(x_i)+\left(1-i_{S_1}(x_i)\right)^2}}{2+\sqrt{i_{S_1}(x_i)}+\sqrt{1-i_{S_1}(x_i)}}\right)\end{array}\right)\end{array}\right]$$

$$=6\text{Max.T}(S_1)-6T(S_1); \quad \text{where} \qquad \ldots (14)$$

$$T(S_1)=\sum_{i=1}^{n}\left[\begin{array}{l}3\tanh\frac{2}{3}-\left(\frac{2+\mu_{S_1}(x_i)+f_{S_1}(x_i)}{3}\right)\tanh\left(\frac{1+\sqrt{\mu_{S_1}^2(x_i)+f_{S_1}^2(x_i)}}{2+\left(\sqrt{\mu_{S_1}(x_i)}+\sqrt{f_{S_1}(x_i)}\right)\left(\sqrt{\mu_{S_1}(x_i)+f_{S_1}(x_i)}\right)}\right)\\ -\left(\frac{4-\mu_{S_1}(x_i)-f_{S_1}(x_i)}{3}\right)\tanh\left(\frac{1+\sqrt{\left(1-\mu_{S_1}(x_i)\right)^2+\left(1-f_{S_1}(x_i)\right)^2}}{2+\left(\sqrt{1-\mu_{S_1}(x_i)}+\sqrt{1-f_{S_1}(x_i)}\right)\left(\sqrt{2-\mu_{S_1}(x_i)-f_{S_1}(x_i)}\right)}\right)\\ -\tanh\left(\frac{1+\sqrt{i_{S_1}^2(x_i)+\left(1-i_{S_1}(x_i)\right)^2}}{2+\sqrt{i_{S_1}(x_i)}+\sqrt{1-i_{S_1}(x_i)}}\right)\end{array}\right] \qquad \ldots (15)$$

The mathematical expression **(15)** is the desired hyperbolic single valued neutrosophic entropy measure since it meets all the essential conditions laid down in **Def. 3.4.** With the aid of non-negativity of $S(R_1)$, the equality **(14)** can be re-scheduled as

$$T(S_1)=\text{Max.T}(S_1)-\frac{1}{6}T(S_1,S_1^c)\geq 0\Rightarrow 0\leq T(S_1,S_1^c)\leq 18\left(\tanh\frac{2}{3}-\tanh\frac{1}{2}\right)n \quad \ldots (16)$$

The resulting inequality equality **(14)** clarifies that $T(S_1,S_1^c)$ is a finite quantity for a fixed $n\in N$. Following the similar pattern, the users can easily establish that $0\leq T(S_1,S_2)\leq 18\left(Tanh\frac{2}{3}-Tanh\frac{1}{2}\right)n$ where $n\in N$ is the cardinality of S_1. Thus,

$\text{Max}.T(S_1,S_2)=18\left(Tanh\frac{2}{3}-Tanh\frac{1}{2}\right)n$, $\text{Min}.T(S_1,S_2)=0$ The fact that $T(S_1)$ affirms its minimum value zero can also been experiences from its three-dimensional contour plot shown in **Fig. 4.**

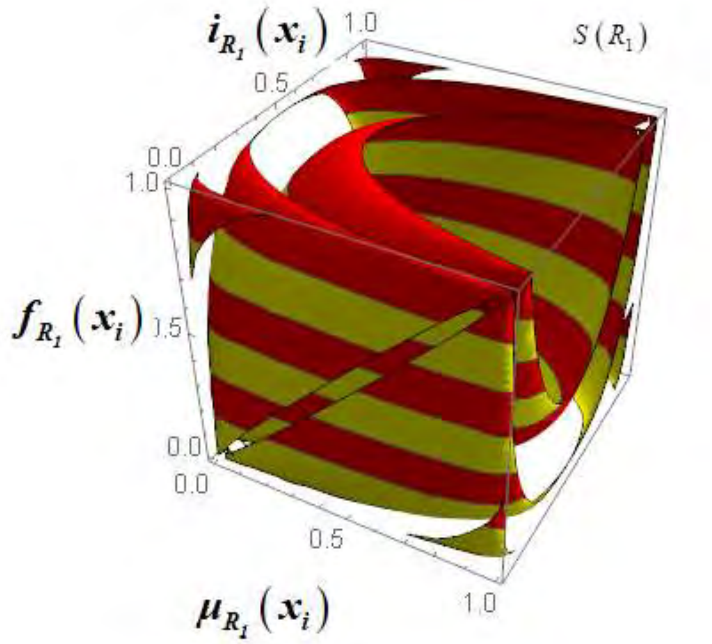


Fig. 4 Three- Dimensional Contour Plot Exhibiting the Minimum Value of $T(S_1)$

To evaluate the impact of elevated levels of fluoride concentration, we shall first customize or rank seasonal parameters employing the proposed possibility fuzzy cross entropy degree measure as follows.

5. Ranking of Seasonal Parameters

To reckon the quality of river water for drinking or irrigation purposes, it is mandatory to represent fluoride concentration of seasonal parameters by the set $P=(P_1,P_2,P_3,...,P_n)$. A symmetric fuzzy cross entropy number(SFCN), denoted b f_{rs}, is an object of the from $f_{rs}=\prec F(P_r,P_1),F(P_r,P_2),....,F(P_r,P_s)\succ;1\leq r,s\leq n$ under the assumption $F(P_r,P_s)=0\Leftrightarrow r=s.$

where each pair $F(P_r,P_s)$ indicates the mathematical value of true membership degree of symmetric discrimination of the seasonal parameter P_r against P_s and can be evaluated employing **(4).** Let f_{rs} and f_{ts} be any two symmetric fuzzy cross entropy numbers (SFCNs). Then the inclusion-comparison fuzziness of two SFCNs $f_{rs} \geq f_{ts}$ for $r = 1,2,...,n$ and fixed s, is denoted by $\eta(f_{rs} \geq f_{ts})$ and is known as possibility fuzzy cross entropy degree measure. Let the matrix representation of $\eta(f_{rs} \geq f_{ts})$ is denoted by $N = (\eta_{rt})_{n\times n}$ where $\eta_{rt} = \eta(f_{rs} \geq f_{ts})$ and

$$N = \begin{pmatrix} \eta_{11} & \eta_{12} \cdots & \eta_{1n} \\ \eta_{21} & \eta_{22} \cdots & \eta_{2n} \\ \vdots & \ddots & \vdots \\ \eta_{m1} & \eta_{n2} \cdots & \eta_{nn} \end{pmatrix} \quad \ldots (17)$$

Then N is called as possibility fuzzy cross entropy degree measure matrix. The optimal fuzzy cross entropy membership degree, denoted by s_k, is defined

$$s_k = \frac{1}{n(n+1)}\left(\sum_{t=1}^{n} \eta_{kt} + \frac{n}{2} - 1\right); k \in N \quad \ldots (18)$$

The ranking of each seasonal parameter $P_k\ (k = 1,2,..,n)$ is obtained according to the corresponding decreasing ordered value of s_k. For convenience, the symmetric fuzzy cross entropy numbers f_{rs} for $r = 1,2,3$ and $s = 3$ are given as

$$f_{13} = \prec F(R_1,R_1), F(R_1,R_2), F(R_1,R_3) \succ \quad \ldots (19)$$

$$f_{23} = \prec F(R_2,R_1), F(R_2,R_2), F(R_2,R_3) \succ \quad \ldots (20)$$

$$f_{31} = \prec F(R_3,R_1), F(R_3,R_2), F(R_3,R_3) \succ \quad \ldots (21)$$

The corresponding possibility fuzzy cross entropy degree measures are proposed as

$$\eta(f_{13} \geq f_{23}) = \text{Min}\left(\text{Max}\left(\frac{F(R_1,R_2)+F(R_2,R_2)}{1+F(R_1,R_1)-2F(R_2,R_1)-F(R_2,R_3)}, 0\right), 1\right) \quad \ldots (22)$$

$$\eta(f_{13} \geq f_{33}) = \text{Min}\left(\text{Max}\left(\frac{F(R_1,R_2)+F(R_3,R_2)}{1+F(R_1,R_1)-2F(R_3,R_1)-F(R_3,R_3)}, 0\right), 1\right) \quad \ldots (23)$$

$$\eta(f_{23} \geq f_{33}) = \text{Min}\left(\text{Max}\left(\frac{F(R_2,R_2)+F(R_3,R_2)}{1+F(R_2,R_1)-2F(R_3,R_1)-F(R_3,R_3)}, 0\right), 1\right) \quad \ldots (24)$$

After collecting ground water samples during sampling year 2014-15 and 2015-16, we have done a lot of data comparison and experimental investigations to extract the lower and upper bounds from monitored fluoride concentration reading of each $P_K\ (K = 1,2,3)$ where P_1 = Pre-Monsoon, P_2 = Rainy Season and P_3 = Post-Monsoon respectively. Suppose $\mu_{P_K}(x)$ denotes the lower bound of K^{th} seasonal parameter, then the set $P = (P_1, P_2, P_3)$ can be constructed for both the sampling years under study and the results are displayed in **Table.1.**

Table 1 Possibility Fuzzy Cross Entropy Degree Measure Values (2014-15, 2015-16)

	2015-16				2014-15			
Parameter	Lower Bound	FCE Measure Values P_1	P_2	P_3	Lower Bound	FCE Measure Values P_1	P_2	P_3
P_1	0.0282	0.0000	0.0112	0.0187	0.0301	0.0000	0.0119	0.0125
P_2	0.0000	0.0112	0.0000	0.0472	0.0000	0.0119	0.0000	0.0395
P_3	0.1162	0.0187	0.0472	0.0000	0.0978	0.0125	0.0395	0.0000

For the sampling year **2015-16,** the various symmetric fuzzy cross entropy numbers

$$f_{13} = \prec 0.0000, 0.0112, 0.0187 \succ, f_{23} = \prec 0.0112, 0.0000, 0.0472 \succ, f_{33} = \prec 0.0187, 0.0472, 0.0000 \succ \quad (25)$$

can be evaluated employing equations **(19-21)** and the results are shown in the first row of **Table.1.** Next, the various possibility fuzzy cross entropy degree measures can be computed employing **(22-24)** as follows.

$$\eta_{11} = 0, \eta_{12} = \eta\left(f_{13} \geq f_{23}\right) = \text{Min}\left(\text{Max}\left(\frac{0.0112+0}{1+0-2\times0.0112-0.0472}, 0\right), 1\right)$$

$$= \text{Min}\left(\text{Max}\left(\frac{0.0112}{0.9304}, 0\right), 1\right) = \text{Min}\left(0.0120, 1\right) = 0.0120$$

$$\eta_{13} = 0.0606, \eta_{21} = 0.0113, \eta_{22} = 0, 0.0000,$$

$$\eta_{23} = \eta\left(f_{23} \geq f_{33}\right) = \text{Min}\left(\text{Max}\left(\frac{0+0.0472}{1+0.0112-2\times0.0187-0}, 0\right), 1\right)$$

$$= \text{Min}\left(\text{Max}\left(\frac{0.0472}{0.9738}, 0\right), 1\right) = \text{Min}\left(0.0485, 1\right) = 0.0485.$$

$$\eta_{31} = \eta\left(f_{33} \geq f_{13}\right) = \text{Min}\left(\text{Max}\left(\frac{0.0472+0.0112}{1+0.0187-2\times0.0000-0.0187}, 0\right), 1\right)$$

$$= \text{Min}\left(\text{Max}\left(0.0584, 0\right), 1\right) = \text{Min}\left(0.0584, 1\right) = 0.0584.$$

$$\eta_{32} = 0.0497, \eta_{33} = 0.0000.$$

Hence, the required possibility fuzzy cross entropy measure degree matrix in this case is given as

$$N = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} 0.0000 & 0.0120 & 0.0606 \\ 0.0113 & 0.0000 & 0.0485 \\ 0.0584 & 0.0497 & 0.0000 \end{pmatrix} \quad \ldots (26)$$

For the sampling year **2014-15,** the various symmetric fuzzy cross entropy numbers

$$f_{13} = \prec 0.0000, 0.0119, 0.0125 \succ, f_{23} = \prec 0.0119, 0.0000, 0.0395 \succ, f_{33} = \prec 0.0125, 0.0395, 0.0000 \succ \quad (27)$$

can also be evaluated employing **(19-21)** and the results are shown in the first row of **Table.1.** The corresponding possibility fuzzy cross entropy measure degree matrix, say M, is given as

$$M = \begin{pmatrix} 0.0000 & 0.0127 & 0.0527 \\ 0.0119 & 0.0000 & 0.0400 \\ 0.0514 & 0.0416 & 0.0000 \end{pmatrix} \quad \ldots (28)$$

For the sampling year **2015-16, t**he optimal fuzzy cross entropy membership degrees $s_k\,(k=1,2,3)$ for $n=3$ can be computed employing **(18)** and the results are as under.

$s_1 = \frac{1}{12}\left(\sum_{t=1}^{3}\eta_{kt} + \frac{n}{2} - 1\right) = \frac{1}{12}(0+0.0120+0.0606+0.5) = 0.0477, s_2 = 0.0466, s_3 = 0.0507.$ For the sampling year **2014-15, t**he corresponding values of $s_k\,(k=1,2,3)$ for $n=3$ are $s_1 = 0.0471, s_2 = 0.0460, s_3 = 0.0494.$

Since the ranking order of $s_k\,(k=1,2,3)$ for both the sampling years 2014-15 and 2015-16 is $s_3 \succ s_1 \succ s_2$, therefore, the classification of seasonal parameters should be $P_3 \succ P_1 \succ P_2$.

Results and Discussions.

Based upon experimental investigations, it has been found that during 2015-16, fluoride concentration of groundwater samples varied from 0.065 to 0.91 mg/l during pre-monsoon season. Fluoride concentration varied from 0.025 to 0.42 mg/l (lowest)during rainy season whereas during post-monsoon season it varied from 0.19 to 1.42 mg/l(highest). During 2014-15, fluoride concentration varied from 0.06 to 0.85 mg/l during pre-monsoon season. In rainy season, fluoride ranged from 0.02 to 0.36 mg/l (lowest)whereas during post-monsoon season it varied from 0.15 to 1.33 mg/l(highest). The classification of seasonal parameters $P_3 \succ P_1 \succ P_2$ also exhibit that the fluoride concentration was highest in post monsoon season, owing to the highest fuzzy cross entropy membership degree (0.0507,0.0494).

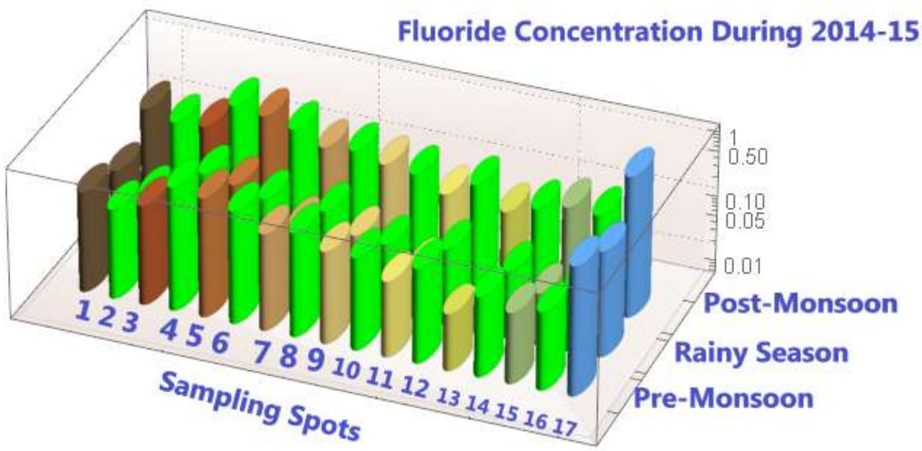


Fig. 5 Seasonal Variations in Fluoride Concentration of Groundwater (2014-15)

5.1 Experimental Assessment of Fluoride Concentration (2014-15)

The results depicted in **Fig.5** dictate that in during **2014-15**, fluoride concentration varied from 0.06 to 0.85 mg/l during pre-monsoon season. In rainy season, it ranged from 0.02 to 0.36 mg/l

whereas during post-monsoon season it varied from 0.15 to 1.33 mg/l It was found to be highest at sampling spot $S_{17}(0.85\,mg/l)$ followed by $S_4(0.70\ mg/l)$, $S_5(0.60\ mg/l)$, $S_6(0.47\ mg/l)$ and lowest concentration was observed at $S_{13}(0.06\ mg/l)$. During rainy season, fluoride concentration was found to be highest at sampling spot $S_{17}(0.36\,mg/l)$ followed by $S_5(0.25\,mg/l)$ and lowest concentration was observed at $S_{13}(0.02\ mg/l)$. During post-monsoon season, fluoride has shown highest concentration at $S_{17}(1.33\ mg/l)$ followed by $S_4(0.94\ mg/l)$ & $S_4(0.80\ mg/l)$. Likewise during pre-monsoon & rainy season, fluoride has shown lowest concentration at $S_{13}(0.15\ mg/l)$ during post-monsoon also (**Fig.5)**.

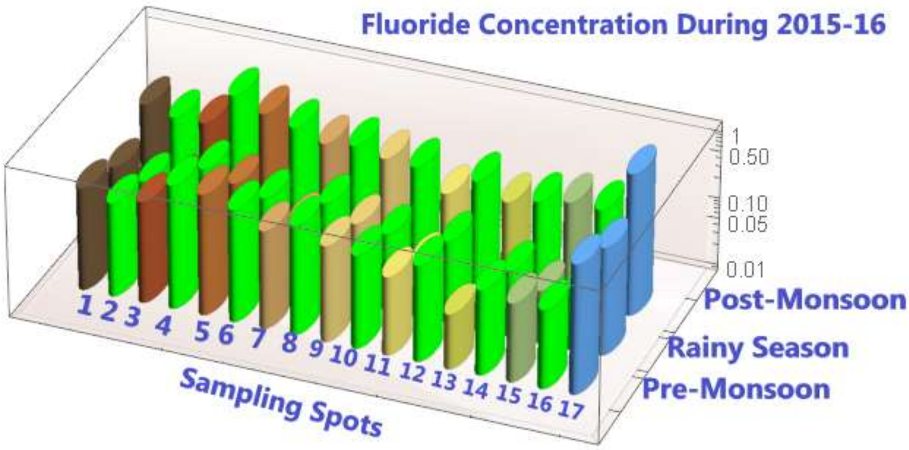


Fig. 6 Seasonal Variations in Fluoride Concentration of Groundwater (2015-16)

5.2 Experimental Assessment of Fluoride Concentration (2015-16)

The results depicted in **Fig.6** indicate that during **2015-16**, fluoride concentration of groundwater samples varied from 0.065 to 0.91 mg/l during pre-monsoon season. Fluoride concentration varied from 0.025 to 0.42 mg/l during rainy season whereas during post-monsoon season it varied from 0.19 to 1.42 mg/l. It was observed highest at sampling spot $S_{17}(0.91\ mg/l)$ followed by $S_4(0.75\ mg/l)$ & $S_5(0.68\,mg/l)$ and lowest concentration was observed at $S_{13}(0.065\,mg/l)$. During rainy season, fluoride concentration was found to be highest at sampling spot $S_{17}(0.42\ mg/l)$ followed by $S_4(0.36\ mg/l)$ & $S_5(0.27\ mg/l)$ and lowest was observed at $S_{13}(0.025\ mg/l)$. During post-monsoon, fluoride concentration has shown highest concentration at $S_{17}(1.42\ mg/l)$

followed by $S_4(1.30\ mg/l)$, $S_5(0.85\ mg/l)$, $S_6(0.65\ mg/l)$ & lowest fluoride concentration was observed at $S_{13}(0.19\ mg/l)$ **(Fig.6)**. Furthermore, if fluoride concentration is below 0.6 mg/l drinking water should be rejected. Maximum limit of fluoride is extended up to 1.5 mg/l. During these investigations, fluoride concentration was found to be highest during post-monsoon season followed by pre-monsoon and rainy season. The most contaminated sampling spot was identified as S_{17} and least contaminated site was discovered as S_{13}. Most of the sampling spots have shown fluoride concentration below 0.6 mg/l in **2014-15.** Fluoride concentration increased during **2015-16** at sampling spots S_4, S_5, S_6, S_{11} and S_{17} but was found to be with in permissible limits.

7. Methodology for the Identification of "most" contaminated sampling spot

We next switch to construct the proclaimed fuzzy entropy and single valued neutrosophic entropy weighted fluoride contamination indices (FEFCI and NEFCI), intended to identify the most contaminated sampling spot responsible for fluoride contamination in ground water samples as follows.

Step: -1 Collection of Ground Water Samples

Present investigations were carried out in District Kangra, Himachal Pradesh. The reason for this area selection was because of its position in relation to groundwater morphometric. In this study, seventeen sampling spots of groundwater were sampled in pre-monsoon, rainy and post-monsoon season in the specified area for two sampling years **2014-15** and **2015-16.**

Step: -2 Normalization of Monitored Fluoride Concentration Reading

Suppose the number of seasonal parameters (seasons) to be studied is denoted by "n". Let the number of sampling spots under study is denoted by "m". Let l_{ji} denotes the monitored fluoride concentration reading of j^{th} season at i^{th} sampling spot. The normalization of concentration reading is essential for the purpose of reducing the errors created by various factors. If p_{ji} denotes the normalization construction function for j^{th} season at i^{th} sampling spot, then

$$p_{ji} = \frac{l_{ji} - \text{Min}.l_{ji}}{\text{Max}.l_{ji} - \text{Min}.l_{ji}};\ j = 1,2,...,n, i = 1,2,...,m. \quad \dots (29)$$

Step:- 3 Determination of Fuzzy Entropy Weights

Deluca and Termini **[16]** suggested the following first non-additive and non-probabilistic equivalent associate of Shannon's entropy.

$$H(R_1) = -\frac{1}{\log m}\sum_{j=1}^{n}\left[\mu_{R_1}(x_j)\log\mu_{R_1}(x_j) + \left(1-\mu_{R_1}(x_j)\right)\log\left(1-\mu_{R_1}(x_j)\right)\right] \quad \dots (30)$$

where $R_1 = \left(< x_j, \mu_{R_1}(x_j) >| x_j \in U\right)$ is a fuzzy set (**Def. 3.1)**

Let T_{ji} denotes the amount of fuzziness based on the true membership concentration of j^{th} seasonal parameter at i^{th} sampling spot. Then,

$$T_{ji} = \frac{p_{ji}}{\sum_{j=1}^{n} p_{ji}} \quad \dots (31)$$

(a) The fuzzy entropy weights $w_{ji}^{(0)}$ of j^{th} seasonal parameter at i^{th} sampling spot employing Deluca and Termini **(30)** can be evaluated as follows; Let "m" be the number of sampling spots, then

$$w_{ji}^{(0)} = \frac{1 - E_{ji}^{(0)}}{\sum_{j=1}^{n} E_{ji}^{(0)}}, \text{ where} \quad \dots (32)$$

$$E_{ji}^{(0)} = -\frac{1}{\log m} \sum_{j=1}^{n} \left[T_{ji} \log T_{ji} + \left(1 - T_{ji}\right) \log\left(1 - T_{ji}\right) \right] \quad \dots (33)$$

However, the fuzzy entropy measure **(30)** is facing a major drawback as it is based on the fancy presumption $0 \times log 0 = 0$ and hence indicates major conflicts in water treatment strategies. To overcome these barricades and problematic situations, the proposed hyperbolic fuzzy and single valued neutrosophic entropy measures (HFE and HNE) can play a crucial role for handling the complexity of contamination level in a macroscopic point of view.

(b) The fuzzy entropy weights $w_{ji}^{(1)}$ of j^{th} seasonal parameter at i^{th} sampling spot employing the proposed hyperbolic fuzzy entropy measure (**1)** can be evaluated as follows: Let "m" be the number of sampling spots, then

$$w_{ji}^{(1)} = \frac{1 - E_{ji}^{(1)}}{\sum_{j=1}^{n} E_{ji}^{(1)}}, \text{ where } E_{ji}^{(1)} = -\tanh\left(m^{-1}\right) \sum_{j=1}^{n} \left[\tanh\left(\frac{1 + \sqrt{T_{ji}^2 + \left(1 - T_{ji}\right)^2}}{2 + \sqrt{T_{ji}} + \sqrt{1 - T_{ji}}} \right) - \tanh\left(\frac{2}{3}\right) \right] \quad \dots (34)$$

(c) The fuzzy entropy weights $w_{ji}^{(2)}$ of j^{th} seasonal parameter at i^{th} sampling spot employing the proposed single valued neutrosophic entropy measure **(15)** can be evaluated as follows: Let $F_{ji} = 1 - T_{ji}$ and $I_{ji} = 1 - T_{ji} - F_{ji}$ denote the amount of fuzziness based on the indeterminacy and falsity membership concentration of j^{th} seasonal parameter at i^{th} sampling. Here, the values of I_{ji} are restricted to $0 \cdot 001$ if it is less than or equal to zero. Then,

$$w_{ji}^{(2)} = \frac{1 - E_{ji}^{(2)}}{\sum_{j=1}^{n} E_{ji}^{(2)}} \quad \dots (35)$$

$$E_{ji}^{(2)} = \tanh\left(m^{-1}\right)\sum_{j=1}^{n}\left[\begin{array}{l} 3\tanh\frac{2}{3} - \tanh\left(\frac{1+\sqrt{I_{ji}^2+\left(1-I_{ji}\right)^2}}{2+\sqrt{I_{ji}}+\sqrt{1-I_{ji}}}\right) - \left(\frac{2+T_{ji}+F_{ji}}{3}\right)\tanh\left(\frac{1+\sqrt{T_{ji}^2+F_{ji}^2}}{2+\left(\sqrt{T_{ji}}+\sqrt{F_{ji}}\right)\left(\sqrt{T_{ji}+F_{ji}}\right)}\right) \\ -\left(\frac{4-T_{ji}-F_{ji}}{3}\right)\tanh\left(\frac{1+\sqrt{\left(1-T_{ji}\right)^2+\left(1-F_{ji}\right)^2}}{2+\left(\sqrt{1-T_{ji}}+\sqrt{1-F_{ji}}\right)\left(\sqrt{2-T_{ji}-F_{ji}}\right)}\right) \end{array}\right] \quad \ldots (36)$$

Step: -4 Quality Rating Scales of Seasonal Parameters

To describe the quality of ground water parameters, eminent researchers have been employing two types of quality rating scales-absolute and relative. Since absolute quality rating does not depend upon water quality standards, therefore, relative quality rating approach has been empowered in this study. Let $Q_{ji} =$ Relative Quality Scale, $S_{ji} =$ Maximum permissible fluoride concentration limit and $l_{ji} =$ Monitored fluoride concentration reading, of j^{th} seasonal parameter at i^{th} sampling spot consecutively. Then

$$Q_{ji} = \left(\frac{l_{ji}}{S_{ji}}\times 100\right)\left(\frac{7-S_{pH}}{7-l_{pH}}\right); j = 1,2,...,n, i = 1,2..,m. \quad \ldots (37)$$

where $(i) S_{ji} = 1.5\left(mg/L\right)$ is the maximum permissible limit of fluoride concentration (WHO Standards) of j^{th} seasonal parameter at i^{th} sampling spot. (ii) S_{pH} is the permissible limit of pH (varies from 6.5 to 8.5) values and is defined as $S_{pH} = \begin{cases} 6.5, \text{if } l_{pH} < 7 \\ 8.5, \text{if } l_{pH} > 7 \end{cases}$

(iii) l_{pH} is the pH value in ground water samples (Table)

Step: -5 Construction of f FEFCI and NEFCI

The existing Deluca and Termini fuzzy entropy **(33)** and the proposed hyperbolic fuzzy entropy and single valued neutrosophic entropy weighted fluoride contamination indices (DEFCI, FEFCI and NEFCI) can be computed as follows:

$$\text{DEFCI at } i^{th} \text{ Sampling Spot} = \sum_{j=1}^{n} w_{ji}^{(0)} Q_{ji} \quad \ldots (38)$$

$$\text{FEFCI at } i^{th} \text{ Sampling Spot} = \sum_{j=1}^{n} w_{ji}^{(1)} Q_{ji} \quad \ldots (39)$$

$$\text{NEFCI at } i^{th} \text{ Sampling Spot} = \sum_{j=1}^{n} w_{ji}^{(2)} Q_{ji} \quad \ldots (40)$$

Step: -6 Identifying the Most Contaminated Sampling Spot

The maximum (or minimun) DEFCI, FEFCI or NEFCI scores among various sampling spot is designated to the "most (or least) contaminated" sampling spot.

7. Application of HFE and HNE Based Method

To predict the contamination impact of each sampling spot, the DEFCI, FEFCI and NEFCI score at various sampling spots $S_1, S_2, ..., S_{17}$ can be evaluated employing as follows.

7.1 Identification of Most Contaminated Sampling Spot Based on DEFCI

Based upon Deluca and Termini entropy **(30)**, the existing fuzzy entropy weighted fluoride contamination index (DEFCI) scores at 17 sampling spots can be calculated employing the proposed methodology explained in **Section. 6.** The steps involved in the calculation of DEFCI scores at various sampling spots during **2014-15** and **2015-16** are depicted in **Table 2(a, b).** The monitored fluoride concentration readings of each seasonal parameter are expressed in terms of mg/l. The number of seasonal parameters (seasons) in this study, is three $(n=3)$ and the number of sampling spots is seventeen $(m=17)$ .The normalization construction function $p_{ji}\left(j=1,2,3; i=1,2,..,17\right)$ of all the three seasonal parameters at 17 sampling spots is calculated employing **(29).**

Observations The tabulated values of **Table 2(a, b)** as well as trend of DEFCI score **(Fig. 7) i**ndicate that during **2014-15**, the sampling spot S_{17} was found to be most contaminated owing to its maximum DEFCI score $\left(882\right)$ whereas the least contaminated sampling spot was observed as $S_{16}\left(54\right)$. During **2015-16**, the sampling spot S_{17} was again found to be most contaminated owing to its maximum DEFCI score $\left(1244\right)$ and $S_{16}\left(73\right)$ was the least contaminated **(Fig 8)**.

7.2 Identification of Most Contaminated Sampling Spot Based on FEFCI

The proposed fuzzy entropy weighted fluoride contamination index (FEFCI) scores at 17 sampling spots can be calculated employing the proposed methodology explained in **Section. 6.** The steps involved in the calculation of FEFCI scores at various sampling spots during **2014-15** and **2015-16** are depicted in **Table 3(a, b).**

Observations The resulting values of **Table 3(a, b)** and trend of FEFCI score **(Fig 7) i**ndicate that during **2014-15**, the sampling spot S_{17} was found to be most contaminated owing to its maximum

Table 2: Calculation of DEFCI Score Employing Deluca and Termini Entropy [ ] (C.F.=Construction Function, FVs=Fuzzy Values, EVS=Entropy Values, Aws=Assigned Weights, RSIs=Relative Sub-Indices)

Seasons		C.F.	FVs	EVs	AWs	RSIs	DEFCI Score	C.F.	FVs	EVs	AWs	RSIs	DEFCI Score
		p_{ji}	T_{ji}	$E_{ji}^{(0)}$	$w_{ji}^{(0)}$	Q_{ji}		p_{ji}	T_{ji}	$E_{ji}^{(0)}$	$w_{ji}^{(0)}$	Q_{ji}	
		2014-2015						2015-2016					
Pre-M	S_1	0.21	0.33	0.22	1.22	76.92	299	0.23	0.33	0.22	1.22	145.8	522
RS		0.11	0.18	0.17	1.31	60.71		0.12	0.18	0.17	1.31	133.3	
Post-M		0.31	0.49	0.24	1.19	104.88		0.33	0.49	0.24	1.19	142.8	
Pre-M	S_2	0.14	0.32	0.22	1.30	34.48	147	0.17	0.32	0.22	1.24	57.78	230
RS		0.05	0.13	0.13	1.45	25.71		0.09	0.17	0.16	1.33	51.72	

Post-M		0.23	0.55	0.24	1.27	50.77		0.26	0.50	0.24	1.20	75.00	
Pre-M	S_3	0.20	0.40	0.24	1.25	31.52	102	0.22	0.38	0.23	1.20	47.22	147
RS		0.06	0.12	0.13	1.43	19.61		0.10	0.17	0.16	1.31	37.78	
Post-M		0.25	0.49	0.24	1.24	27.56		0.26	0.45	0.24	1.18	34.78	
Pre-M	S_4	0.51	0.36	0.23	1.23	85.37	318	0.51	0.31	0.22	1.29	104.1	490
RS		0.21	0.15	0.15	1.36	66.67		0.24	0.14	0.15	1.41	109.0	
Post-M		0.69	0.49	0.24	1.21	101.08		0.90	0.55	0.24	1.25	162.5	
Pre-M	S_5	0.44	0.36	0.23	1.24	82.19	302	0.46	0.38	0.23	1.23	103.0	363
RS		0.17	0.14	0.15	1.37	49.02		0.17	0.14	0.14	1.37	60.00	
Post-M		0.59	0.49	0.24	1.21	109.59		0.58	0.48	0.24	1.21	126.8	
Pre-M	S_6	0.34	0.37	0.23	1.24	47.96	172	0.33	0.37	0.23	1.24	61.73	209
RS		0.13	0.14	0.14	1.38	33.93		0.12	0.14	0.14	1.39	42.55	
Post-M		0.45	0.49	0.24	1.22	53.91		0.44	0.49	0.24	1.22	60.75	
Pre-M	S_7	0.18	0.35	0.23	1.32	29.89	104	0.19	0.35	0.23	1.31	39.47	138
RS		0.05	0.10	0.12	1.51	12.33		0.06	0.11	0.12	1.49	16.42	
Post-M		0.29	0.55	0.24	1.29	36.04		0.31	0.55	0.24	1.28	48.42	
Pre-M	S_8	0.24	0.35	0.23	1.20	73.91	264	0.30	0.36	0.23	1.18	132.3	418
RS		0.13	0.19	0.17	1.29	86.36		0.17	0.20	0.18	1.26	136.8	
Post-M		0.32	0.46	0.24	1.18	54.32		0.36	0.43	0.24	1.17	76.81	
Pre-M	S_9	0.15	0.33	0.22	1.28	26.83	91	0.18	0.32	0.22	1.24	35.90	117
RS		0.06	0.13	0.14	1.42	13.51		0.10	0.17	0.16	1.33	21.62	
Post-M		0.24	0.53	0.24	1.25	29.82		0.28	0.50	0.24	1.20	36.84	
Pre-M	S_{10}	0.15	0.36	0.23	1.26	20.95	148	0.16	0.32	0.22	1.25	19.38	180
RS		0.05	0.13	0.13	1.42	6.12		0.08	0.16	0.16	1.35	9.52	
Post-M		0.21	0.51	0.24	1.24	90.91		0.26	0.52	0.24	1.21	118.1	
Pre-M	S_{11}	0.08	0.34	0.23	1.32	17.91	116	0.09	0.38	0.23	1.26	22.39	128
RS		0.02	0.10	0.12	1.50	20.00		0.03	0.12	0.13	1.43	26.00	
Post-M		0.12	0.55	0.24	1.29	48.65		0.12	0.50	0.24	1.24	51.35	
Pre-M	S_{12}	0.17	0.35	0.23	1.22	16.00	56	0.18	0.34	0.23	1.20	62.22	171
RS		0.08	0.16	0.16	1.34	8.00		0.10	0.19	0.17	1.28	48.57	
Post-M		0.23	0.48	0.24	1.20	21.33		0.24	0.46	0.24	1.17	29.84	
Pre-M	S_{13}	0.03	0.24	0.19	2.10	10.07	462	0.03	0.17	0.16	2.61	46.43	341
RS		0.00	0.00	0.00*	2.60	4.63		0.00	0.00	0.00*	3.11	2.53	
Post-M		0.10	0.76	0.19	2.10	204.69		0.14	0.83	0.16	2.61	81.48	
Pre-M	S_{14}	0.09	0.32	0.22	1.24	24.14	75	0.12	0.34	0.23	1.26	41.30	95
RS		0.05	0.18	0.17	1.32	7.44		0.05	0.14	0.14	1.39	7.36	
Post-M		0.14	0.50	0.24	1.20	29.17		0.18	0.52	0.24	1.23	26.67	
Pre-M	S_{15}	0.06	0.22	0.18	1.64	25.64	75	0.09	0.26	0.20	1.53	44.12	165
RS		0.02	0.08	0.10	1.81	6.67		0.02	0.07	0.09	1.74	5.26	

Post-M		0.20	0.70	0.21	1.58	13.15		0.22	0.66	0.23	1.48	59.65	
Pre-M	S_{16}	0.08	0.27	0.21	1.36	9.15	54	0.09	0.26	0.20	1.43	11.03	72
RS		0.04	0.14	0.14	1.47	9.15		0.04	0.11	0.12	1.57	11.11	
Post-M		0.18	0.59	0.24	1.30	21.49		0.21	0.63	0.23	1.37	28.95	
Pre-M	S_{17}	0.62	0.33	0.22	1.27	146.55	882	0.62	0.33	0.22	1.26	178.4	1244
RS		0.26	0.14	0.14	1.41	150.00		0.28	0.15	0.15	1.38	233.3	
Post-M		0.98	0.53	0.24	1.24	391.18		0.98	0.52	0.24	1.23	568.0	
		***At S_{13}, the entropy value of Rainy Season is based on the assumption:** $0\times\log 0=0$.											

FEFCI score (38904) whereas the least contaminated sampling spot was observed as $S_{16}(2356)$. During **2015-16**, the sampling spot S_{17} was again found to be most contaminated owing to its maximum FEFCI score (57943) and $S_{16}(3165)$ was the least contaminated **(Fig 8).**

7.3 Identification of Most Contaminated Sampling Spot Based on NEFCI

The steps involved in the computation of single valued neutrosophic entropy weighted fluoride contamination index (NEFCI) scores at 17 sampling spots during **2014-15** and **2015-16** are depicted in **Table 4(a, b).**

Observations The tabulated values exhibited by **Table 4(a, b)** and trend of NEFCI score **(Fig 7) i**ndicate that during **2014-15**, the sampling spot S_{17} was found to be most contaminated owing to its maximum NEFCI score (81596) whereas the least contaminated sampling spot was observed as $S_{16}(4773)$.

During **2015-16**, the sampling spot S_{17} was again found to be most contaminated owing to its maximum NEFCI score (115995) and $S_{16}(6193)$ was the least contaminated **(Fig 8).**

Discussions. The accumulated trend of DEFCI, FEFCI and NEFCI scores at 17 sampling spot has finally put us in a culminative situation to wind-up the conclusion that the quality of ground water was " impeccable" and " favourable".

Table 3: Calculation of FEFCI Score Employing Proposed Fuzzy Entropy Measure (C.F.=Construction Function, FVs=Fuzzy Values, EVS=Entropy Values, Aws=Assigned Weights, RSIs=Relative Sub-Indices)

Seasons		**C.F.**	**FVs**	**EVs**	**AWs**	**RSIs**	**FEFCI Score**	**C.F.**	**FVs**	**EVs**	**AWs**	**RSIs**	**FEFCI Score**
		p_{ji}	T_{ji}	$E_{ji}^{(1)}$	$w_{ji}^{(1)}$	Q_{ji}		p_{ji}	T_{ji}	$E_{ji}^{(1)}$	$w_{ji}^{(1)}$	Q_{ji}	
		2014-2015						**2015-2016**					
Pre-M	S_1	0.21	0.33	0.0064	54.51	76.92	13222	0.23	0.33	0.0064	54.47	145.83	22997
RS		0.11	0.18	0.0047	54.61	60.71		0.12	0.18	0.0047	54.57	133.33	
Post-M		0.31	0.49	0.0071	54.48	104.88		0.33	0.49	0.0071	54.44	142.86	

Pre-M	S_2	0.14	0.32	0.0063	57.76	34.48	6412	0.17	0.32	0.0064	55.00	57.78	10150
RS		0.05	0.13	0.0038	57.91	25.71		0.09	0.17	0.0046	55.10	51.72	
Post-M		0.23	0.55	0.0070	57.73	50.77		0.26	0.50	0.0071	54.96	75.00	
Pre-M	S_3	0.20	0.40	0.0068	56.33	31.52	4435	0.22	0.38	0.0067	54.00	47.22	6473
RS		0.06	0.12	0.0037	56.51	19.61		0.10	0.17	0.0046	54.12	37.78	
Post-M		0.25	0.49	0.0071	56.31	27.56		0.26	0.45	0.0070	53.99	34.78	
Pre-M	S_4	0.51	0.36	0.0066	55.28	85.37	13997	0.51	0.31	0.0062	56.97	104.17	21414
RS		0.21	0.15	0.0042	55.41	66.67		0.24	0.14	0.0042	57.09	109.09	
Post-M		0.69	0.49	0.0071	55.25	101.08		0.90	0.55	0.0070	56.93	162.50	
Pre-M	S_5	0.44	0.36	0.0067	55.43	82.19	13351	0.46	0.38	0.0067	55.31	103.03	16040
RS		0.17	0.14	0.0042	55.57	49.02		0.17	0.14	0.0041	55.45	60.00	
Post-M		0.59	0.49	0.0071	55.40	109.59		0.58	0.48	0.0071	55.29	126.87	
Pre-M	S_6	0.34	0.37	0.0067	55.62	47.96	7557	0.33	0.37	0.0067	55.66	61.73	9191
RS		0.13	0.14	0.0041	55.77	33.93		0.12	0.14	0.0041	55.81	42.55	
Post-M		0.45	0.49	0.0071	55.60	53.91		0.44	0.49	0.0071	55.64	60.75	
Pre-M	S_7	0.18	0.35	0.0065	58.52	29.89	4580	0.19	0.35	0.0065	58.18	39.47	6070
RS		0.05	0.10	0.0034	58.71	12.33		0.06	0.11	0.0035	58.36	16.42	
Post-M		0.29	0.55	0.0070	58.49	36.04		0.31	0.55	0.0070	58.15	48.42	
Pre-M	S_8	0.24	0.35	0.0066	53.85	73.91	11563	0.30	0.36	0.0067	53.20	132.35	18417
RS		0.13	0.19	0.0048	53.94	86.36		0.17	0.20	0.0050	53.28	136.84	
Post-M		0.32	0.46	0.0071	53.82	54.32		0.36	0.43	0.0070	53.18	76.81	
Pre-M	S_9	0.15	0.33	0.0064	56.83	26.83	3988	0.18	0.32	0.0064	55.01	35.90	5191
RS		0.06	0.13	0.0040	56.97	13.51		0.10	0.17	0.0046	55.10	21.62	
Post-M		0.24	0.53	0.0071	56.79	29.82		0.28	0.50	0.0071	54.97	36.84	
Pre-M	S_{10}	0.15	0.36	0.0066	56.38	20.95	6651	0.16	0.32	0.0063	55.59	19.38	8173
RS		0.05	0.13	0.0039	56.54	6.12		0.08	0.16	0.0045	55.70	9.52	
Post-M		0.21	0.51	0.0071	56.36	90.91		0.26	0.52	0.0071	55.55	118.18	
Pre-M	S_{11}	0.08	0.34	0.0065	58.47	17.91	5063	0.09	0.38	0.0067	56.42	22.39	5631
RS		0.02	0.10	0.0034	58.65	20.00		0.03	0.12	0.0038	56.59	26.00	
Post-M		0.12	0.55	0.0070	58.44	48.65		0.12	0.50	0.0071	56.40	51.35	
Pre-M	S_{12}	0.17	0.35	0.0066	54.83	16.00	2486	0.18	0.34	0.0065	53.72	62.22	7558
RS		0.08	0.16	0.0044	54.95	8.00		0.10	0.19	0.0049	53.80	48.57	
Post-M		0.23	0.48	0.0071	54.80	21.33		0.24	0.46	0.0071	53.69	29.84	
Pre-M	S_{13}	0.03	0.24	0.0055	91.03	10.07	19973	0.03	0.17	0.0046	108.66	46.43	14174
RS		0.00	0.00	0.0000	91.53	4.63		0.00	0.00	0.0000	109.16	2.53	
Post-M		0.10	0.76	0.0055	91.03	204.69		0.14	0.83	0.0046	108.66	81.48	
Pre-M	S_{14}	0.09	0.32	0.0063	54.91	24.14	3335	0.12	0.34	0.0065	56.17	41.30	4232
RS		0.05	0.18	0.0047	55.00	7.44		0.05	0.14	0.0041	56.30	7.36	
Post-M		0.14	0.50	0.0071	54.87	29.17		0.18	0.52	0.0071	56.14	26.67	
Pre-M	S_{15}	0.06	0.22	0.0052	69.36	25.64		0.09	0.26	0.0058	65.95	44.12	

RS		0.02	0.08	0.0030	69.52	6.67	3153	0.02	0.07	0.0028	66.15	5.26	7189
Post-M		0.20	0.70	0.0061	69.30	13.15		0.22	0.66	0.0065	65.91	59.65	
Pre-M		0.08	0.27	0.0059	59.22	9.15		0.09	0.26	0.0057	61.95	11.03	
RS	S_{16}	0.04	0.14	0.0040	59.33	9.15	2356	0.04	0.11	0.0036	62.08	11.11	3165
Post-M		0.18	0.59	0.0069	59.16	21.49		0.21	0.63	0.0067	61.89	28.95	
Pre-M		0.62	0.33	0.0064	56.56	146.55		0.62	0.33	0.0064	56.07	178.43	
RS	S_{17}	0.26	0.14	0.0041	56.70	150.00	38904	0.28	0.15	0.0042	56.19	233.33	54943
Post-M		0.98	0.53	0.0071	56.52	391.18		0.98	0.52	0.0071	56.03	568.00	

Table 4: Calculation of NEFCI Score (C.F.=Construction Function, FVs=Fuzzy Values, EVS=Entropy Values, Aws=Assigned Weights, RSIs=Relative Sub-Indices)

Seasons		C.F.	FVs	EVs	AWs	RSIs	NEFCI Score	C.F.	FVs	EVs	AWs	RSIs	NEFCI Score
		p_{ji}	T_{ji}	$E_{ji}^{(2)}$	$w_{ji}^{(2)}$	Q_{ji}		p_{ji}	T_{ji}	$E_{ji}^{(2)}$	$w_{ji}^{(2)}$	Q_{ji}	
		2014-2015						2015-2016					
Pre-M		0.21	0.33	0.0029	117.5	76.92		0.23	0.33	0.0029	117.54	145.83	
RS	S_1	0.11	0.18	0.0027	117.5	60.71	28429	0.12	0.18	0.0027	117.56	133.33	49606
Post-M		0.31	0.49	0.0029	117.5	104.88		0.33	0.49	0.0029	117.53	142.86	
Pre-M		0.14	0.32	0.0029	117.8	34.48		0.17	0.32	0.0029	117.82	57.78	
RS	S_2	0.05	0.13	0.0026	117.8	25.71	13233	0.09	0.17	0.0027	117.85	51.72	21739
Post-M		0.23	0.55	0.0029	117.8	50.77		0.26	0.50	0.0029	117.81	75.00	
Pre-M		0.20	0.40	0.0029	117.2	31.52		0.22	0.38	0.0029	117.28	47.22	
RS	S_3	0.06	0.12	0.0025	117.3	19.61	9328	0.10	0.17	0.0027	117.31	37.78	14050
Post-M		0.25	0.49	0.0029	117.2	27.56		0.26	0.45	0.0029	117.28	34.78	
Pre-M		0.51	0.36	0.0029	118.8	85.37		0.51	0.31	0.0028	118.85	104.17	
RS	S_4	0.21	0.15	0.0026	118.8	66.67	28697	0.24	0.14	0.0026	118.88	109.09	44661
Post-M		0.69	0.49	0.0029	118.8	101.08		0.90	0.55	0.0029	118.84	162.50	
Pre-M		0.44	0.36	0.0029	117.9	82.19		0.46	0.38	0.0029	117.99	103.03	
RS	S_5	0.17	0.14	0.0026	118.0	49.02	28510	0.17	0.14	0.0026	118.03	60.00	34207
Post-M		0.59	0.49	0.0029	117.9	109.59		0.58	0.48	0.0029	117.99	126.87	
Pre-M		0.34	0.37	0.0029	118.1	47.96		0.33	0.37	0.0029	118.18	61.73	
RS	S_6	0.13	0.14	0.0026	118.2	33.93	16047	0.12	0.14	0.0026	118.22	42.55	19504
Post-M		0.45	0.49	0.0029	118.1	53.91		0.44	0.49	0.0029	118.17	60.75	
Pre-M		0.18	0.35	0.0029	119.4	29.89		0.19	0.35	0.0029	119.46	39.47	
RS	S_7	0.05	0.10	0.0025	119.5	12.33	9361	0.06	0.11	0.0025	119.50	16.42	12462
Post-M		0.29	0.55	0.0029	119.4	36.04		0.31	0.55	0.0029	119.45	48.42	
Pre-M		0.24	0.35	0.0029	116.8	73.91		0.30	0.36	0.0029	116.83	132.35	
RS	S_8	0.13	0.19	0.0027	116.8	86.36	25152	0.17	0.20	0.0027	116.85	136.84	40427
Post-M		0.32	0.46	0.0029	116.8	54.32		0.36	0.43	0.0029	116.83	76.81	
Pre-M		0.15	0.33	0.0029	117.8	26.83		0.18	0.32	0.0029	117.82	35.90	
RS	S_9	0.06	0.13	0.0026	117.8	13.51	8335	0.10	0.17	0.0027	117.85	21.62	11118
Post-M				0.0029	117.8	29.82		0.28	0.50	0.0029	117.81	36.84	

		0.24	0.53										
Pre-M	S_{10}	0.15	0.36	0.0029	118.1	20.95	13987	0.16	0.32	0.0029	118.14	19.38	17375
RS		0.05	0.13	0.0026	118.1	6.12		0.08	0.16	0.0026	118.16	9.52	
Post-M		0.21	0.51	0.0029	118.1	90.91		0.26	0.52	0.0029	118.13	118.18	
Pre-M	S_{11}	0.08	0.34	0.0029	118.5	17.91	10353	0.09	0.38	0.0029	118.58	22.39	11827
RS		0.02	0.10	0.0025	118.6	20.00		0.03	0.12	0.0026	118.62	26.00	
Post-M		0.12	0.55	0.0029	118.5	48.65		0.12	0.50	0.0029	118.57	51.35	
Pre-M	S_{12}	0.17	0.35	0.0029	117.1	16.00	5337	0.18	0.34	0.0029	117.12	62.22	16472
RS		0.08	0.16	0.0026	117.1	8.00		0.10	0.19	0.0027	117.14	48.57	
Post-M		0.23	0.48	0.0029	117.1	21.33		0.24	0.46	0.0029	117.12	29.84	
Pre-M	S_{13}	0.03	0.24	0.0028	134.4	10.07	29861	0.03	0.17	0.0027	134.47	46.43	17540
RS		0.00	0.00	0.0021	134.5	4.63		0.00	0.00	0.0021	134.55	2.53	
Post-M		0.10	0.76	0.0028	134.4	204.69		0.14	0.83	0.0027	134.47	81.48	
Pre-M	S_{14}	0.09	0.32	0.0029	118.4	24.14	7154	0.12	0.34	0.0029	118.44	41.30	8923
RS		0.05	0.18	0.0027	118.4	7.44		0.05	0.14	0.0026	118.47	7.36	
Post-M		0.14	0.50	0.0029	118.4	29.17		0.18	0.52	0.0029	118.43	26.67	
Pre-M	S_{15}	0.06	0.22	0.0027	122.9	25.64	5648	0.09	0.26	0.0028	122.94	44.12	13403
RS		0.02	0.08	0.0025	122.9	6.67		0.02	0.07	0.0024	122.98	5.26	
Post-M		0.20	0.70	0.0028	122.9	13.15		0.22	0.66	0.0029	122.93	59.65	
Pre-M	S_{16}	0.08	0.27	0.0028	121.2	9.15	4773	0.09	0.26	0.0028	121.22	11.03	6193
RS		0.04	0.14	0.0026	121.2	9.15		0.04	0.11	0.0025	121.25	11.11	
Post-M		0.18	0.59	0.0029	121.2	21.49		0.21	0.63	0.0029	121.21	28.95	
Pre-M	S_{17}	0.62	0.33	0.0029	118.3	146.55	81596	0.62	0.33	0.0029	118.39	178.43	115995
RS		0.26	0.14	0.0026	118.4	150.00		0.28	0.15	0.0026	118.42	233.33	
Post-M		0.98	0.53	0.0029	118.3	391.18		0.98	0.52	0.0029	118.38	568.00	

A careful analysis of tabulated values of **Table.2 (a, b)** reveals that, while calculating DEFCI score, the values $E_{13}^{(0)}$ at sampling spots S_{13} is based on the fancy assumption $0 \times \log 0 = 0$ which creates uncertainty in the quantification of information contained in fluoride concentration of ground water samples. However, the identification of most and least contaminated sampling spots based on Deluca and Termini **[16]** and proposed fuzzy and single valued neutrosophic entropy measures identical. This justifies the feasibility and compatibility of the proposed methodology of identifying the most and least contaminated sampling spots.

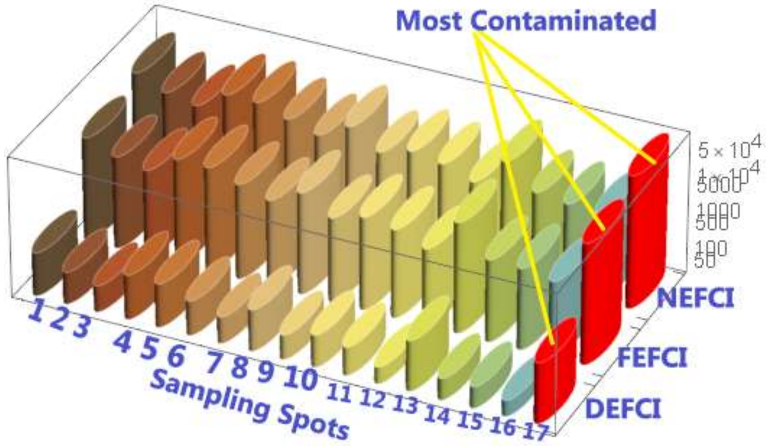


Fig.7 Identification of Most Contaminated Sampling Spot Based on DEFCI, FEFCI and NEFCI Scores at Seventeen Sampling Spots (2014-15)

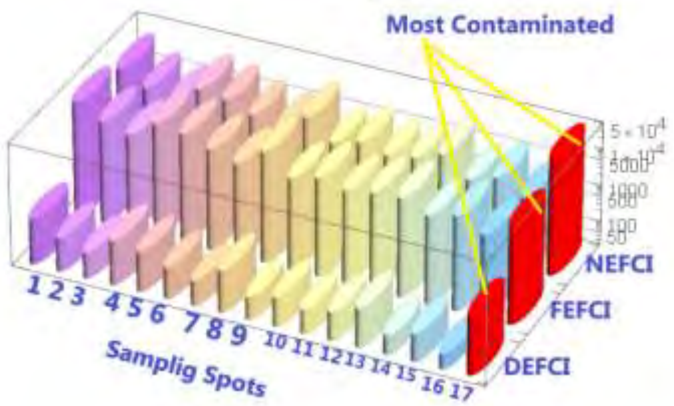


Fig.8 Identification of Most Contaminated Sampling Spot Based on DEFCI, FEFCI and NEFCI Scores at Seventeen Sampling Spots (2015-16)

7.4 Impact of elevated Fluoride concentration on Community Health

According to **[15]** and **[13]**, drinking water containing high concentrations of fluoride is one of the main sources of fluorosis. As per American Dental Association (ADA), fluoride in water is beneficial to people as it protects against cavities and reduces tooth decay by 20-40%. On contrary, just like any other substance we are exposed to in our everyday lives, fluoride carries toxic effects in certain quantities. Acute toxicity can occur after ingesting one or more doses of fluoride over a short time period which then leads to poisoning. The stomach is the first organ that is affected. First signs and symptoms are nausea, abdominal pain, bloody vomiting and diarrhea. Based on extensive studies, probable toxic dose (PTD) was defined at 5 mg/kg of body mass. The PTD is the minimal dose that could trigger serious and life-threatening signs and symptoms and requires immediate treatment and hospitalization **[11]**.

To evaluate the impact of elevated levels of Fluoride on public health, a survey was conducted in the selected areas and interaction with the public was done. To verify the facts, local Hospitals/Clinics and public health department were visited and authorities were consulted to understand the nature of health problem people have been suffering. During these investigations, it was found out that residents, who have been using unfiltered/untreated groundwater for drinking, have been suffering from dental Fluorosis or skeletal fluorosis, which mostly damage their bones & joints. Many residents were observed with white streaks or specks in their teeth enamel. In skeletal fluorosis, bones become hardened and less elastic that increases the risk of fracture. Residents were found to be complaining about pain in bones and joints. Though this data could not be considered as a base for medical investigations yet it could be measured as a connecting link between fluorosis and drinking water with higher fluoride concentration. Similar kind of studies were conducted on Factors influencing the relationship between fluoride in drinking water and dental fluorosis and results of the systematic review have shown that dental fluorosis affects individuals of all ages, with the highest prevalence below 11, while the impact of other factors (gender, environmental conditions, diet and dental caries) was inconclusive. Meta-regression analysis, based on information collected through systematic review, indicates that both fluoride in drinking water and temperature influence dental fluorosis significantly and that these studies might be affected by publication bias. Findings show that fluoride negatively affects people's health in less developed countries **[14]**.

Besides, fluoride acts as neurotoxin that could carry adverse impact on human development. As per, International Association of Oral medicine and Toxicology (IAOMT), excessive use of added fluoride may create skin problems, arteriosclerosis, arterial calcification, high blood pressure, myocardial damage and some reproductive issues such as lower fertility and early puberty in girls.

CONCLUSION

It has been concluded that in Kangra district fluoride concentration in groundwater has been increased since 2014 to 2016 and higher concentration has been observed during post-monsoon season consecutively for both years. Although elevated levels of fluoride in drinking water have shown adverse impact on people residing in this region; however no consistent pattern has been observed during these studies for these health problems. Many other factors like nutrition can play a significant role in weakening health condition also. By considering elevated levels of fluoride in drinking water & health related issues, it is advisable for the public to treat the water before drinking to avoid any health complications. State pollution control board should intervene in this matter and to make sure that guidelines laid down by pollution board has been followed up regularly by the industries before disposing off any wastewater in to any adjacent water body or open field.

In 2014-15, fluoride concentration varied from 0.06 to 0.85 mg/l during pre-monsoon; varied from 0.02 to 0.36 mg/l during rainy season; varied from 0.15 to 1.33 mg/l during post-monsoon. In 2015-16, fluoride concentration of groundwater samples varied from 0.065 to 0.91 mg/l during pre-monsoon; varied from 0.025 to 0.42 mg/l during rainy season; varied from 0.19 to 1.42 mg/l during post-monsoon. Most of the sampling spots during all seasons have shown marginal value of fluoride. Elevated levels of fluoride in groundwater for prolonged time cause many negative impacts on public health such as fluorosis, discoloration, osteoporosis, cardiovascular disorders and skeletal deformities. These studies have shown that local residents have been suffering from these kinds of health issues due to elevated fluoride level in groundwater and advised to use proper water purification techniques to avoid any health complications.

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Annexure.1Table 1 pH value of Seasonal Parameters Collected from Various Sampling Spots

Sampling	**Pre-Monsoon**		**Rainy Season**		**Post-Monsoon**	
Spots	*2014-2015*	2015-2016	*2014-2015*	2015-2016	*2014-2015*	2015-2016
S_1	7.39	7.24	7.28	7.15	7.41	7.35
S_2	7.58	7.45	7.35	7.29	7.65	7.52
S_3	7.92	7.72	7.51	7.45	8.27	8.15
S_4	7.82	7.72	7.45	7.33	7.93	7.8
S_5	7.73	7.66	7.51	7.45	7.73	7.67
S_6	7.98	7.81	7.56	7.47	8.15	8.07
S_7	7.87	7.76	7.73	7.67	8.11	7.95
S_8	7.46	7.34	7.22	7.19	7.81	7.69
S_9	7.82	7.78	7.74	7.74	8.14	8.14
S_{10}	6.65	6.57	6.51	6.51	6.89	6.89
S_{11}	7.67	7.67	7.25	7.25	7.37	7.37
S_{12}	7.58	7.45	7.45	7.35	8.31	8.24
S_{13}	7.24	7.14	6.73	6.67	7.02	6.91
S_{14}	7.58	7.46	6.61	6.57	6.76	6.65
S_{15}	6.87	7.34	6.75	6.62	6.29	6.81
S_{16}	8.42	8.36	7.82	7.72	8.21	8.14
S_{17}	7.58	7.51	7.24	7.18	7.34	7.25

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Reliability Measures in Neutrosophic Soft Graphs

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Abstract: This paper introduces the concept of reliable nodes in neutrosophic soft graphs by evaluating path-based parameters. A reliable node is defined as one which is least susceptible to changes that are quantized by the indeterminacy and falsity values in a neutrosophic tuple. A new path measure called farness and three novel reliability measures which make use of the same are presented. Farness is defined in terms of a novel score function. The first, proximity reliability of a node, computes the farness of a node to its neighbours. The second, intermediate reliability of a node, computes the fraction of paths of minimal farness that pass through it. The third, crisis reliability of a node, is a hybrid of the two previously defined. It considers the farness of a node to its neighbours taking into account the farness of the neighbours to other nodes in the graph.
Keywords: Neutrosophic soft graphs, Strong arcs, Score functions, Proximity reliability, Intermediate reliability, Crisis reliability.

1. Introduction

Graphs have been used to model and solve real-world problems in social and information systems [1]. In the field of computer science, graphs are used to represent networks of communication, data organization, computational devices, and the flow of computation to name a few. Graphs have also been used extensively to model scenarios for path-based applications. For example, shortest path problems used in route planning model the real world as a graph with the nodes representing destinations and the edges representing connections between destinations through some mode of transport. Some prevalent algorithms which make use of the graph model are Dijkstra's shortest path algorithm and the Floyd Warshall algorithm [2].

An edge connecting two nodes in a classical graph is binary in nature: It either exists or it doesn't. Therefore, stochastic optimization problems cannot be modelled using classical graphs. An extended version of the classical set is the fuzzy set where the elements have a value ranging from 0 to 1 indicating the degree of membership. Zadeh [3] introduced the degree of membership/truth (T) in 1965 and defined the fuzzy set. The concept of fuzziness in graph theory was described by Kaufmann [4] using the fuzzy relation. Rosenfeld [5] introduced some concepts such as bridges, cycles, paths, trees, the connectedness of fuzzy graphs and described some of the properties of the fuzzy graph. Samanta and Pal [6] and Rashmanlou and Pal [7] presented the concept of irregular and regular fuzzy graphs. Intuitionistic fuzzy sets (IFS) consider not only the membership grade (degree) but also independent membership grade and non-membership grade for any entity. The only requirement is that the sum of non-membership and membership degree values be no greater than one. The idea of the intuitionistic fuzzy set (IFS) as a modified version of the classical fuzzy set was introduced by Atanassov [8–10]. The idea of the IFS relation and intuitionistic fuzzy graphs (IFG) was presented by Shannon and Atanassov [11]. In real-world problems, uncertainties due to inconsistent and indeterminate information about a problem cannot be represented properly by the fuzzy graph or IFG. To overcome this situation, a new concept was introduced which is called the neutrosophic sets. Smarandache [12] introduced the degree of indeterminacy/neutrality (I) as an independent component in 1995 and defined the neutrosophic set on three components (T, I, F) = (Truth, Indeterminacy, Falsity). Neutrosophic soft graphs [13] based on the soft set theory [14] is a parameterized family of neutrosophic graphs. The class of all neutrosophic soft graphs is denoted by NS(G*).

The concept of centrality measures in graphs has been given a lot of attention as well. Nodal centrality measures are used to quantify the influence of a node with respect to other nodes within the network. Some of the more well-known centrality measures include the degree centrality [15] eigenvector centrality [16], closeness centrality [17], and betweenness centrality [18]. The utilization of centrality measures on networks to identify influential nodes can lead to a more comprehensive understanding of the dynamics and behaviour of real-world systems. Past applications of the four well-known centralities, along with various generalizations of the measures, on real-world networks include the Internet, transportation systems and social systems. One of the most recent works in this area is that of Heatmap centrality [19]. The heatmap centrality compares the distance of a node with the average sum of the distance of its adjacent nodes in order to identify influential nodes within the network. The readers can use the ideas in [24-25] to add more reliability measures. The readers can use the applications in [26-33] to extend the ideas presented.

The motivation behind this paper was to develop path-based measures for neutrosophic soft graphs to identify important nodes. The new measures developed would be a natural extension of centrality measures to the neutrosophic domain. A new path measure for neutrosophic soft graphs is presented in this work. The concept of reliability, an extension of centrality to neutrosophic soft graphs, is defined. Three reliability measures based on the newly introduced path measure are also elucidated.

The rest of this paper is as follows. Section 2 lists the preliminaries required for the study of reliability measures in neutrosophic soft graphs. Section 3 introduces the concept of reliability and proximity, intermediate and crisis reliabilities with examples. Section 4 illustrates a real-world application of the reliability measures. Section 5 concludes the paper.

2. Preliminaries

2.1. Definition [20]

A neutrosophic graph is defined as a pair $G = (V, E)$ where:

1. V = $\{v1, v2, .., vn\}$ such that $T = V \rightarrow [0,1],\ I = V \rightarrow [0,1]\ and\ F = V \rightarrow [0,1]$ denotes the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and
2. $0 \leq TA(x) + IA(x) + FA(x) \leq 3$

2.2 Definition [21]

Let U be an initial universe set and E a set of parameters or attributes with respect to U. Let P(U) denote the power set of U and A $\subseteq$ E. A pair (F, A) is called a soft set over U, where F is a mapping given by F: A $\rightarrow$P(U). In other words, a soft set (F, A) over U is a parameterized family of subsets of

U. For e∈A, F(e) may be considered as the set of e-elements or e-approximate elements of the soft sets (F, A).

2.3 Definition [22]

Let U be an initial universe and P be the set of all parameters. ϱ(U) denotes the set of all neutrosophic sets of U. Let A be a subset of P. A pair (J, A) is called a neutrosophic soft set over U. Let ϱ(V) denotes the set of all neutrosophic sets of V and ϱ(E) denotes the set of all neutrosophic sets of E.

2.4 Definition [23]

A neutrosophic soft graph G = (G∗, J, K, A) is an ordered four tuple, if it satisfies the following conditions:

(i) G∗ = (V, E) is a neutrosophic graph

(ii) A is a non-empty set of parameters

(iii) (J, A) is a neutrosophic soft set over V

(iv) (K, A) is a neutrosophic soft set over E,

(v) (J(e), K(e)) is a neutrosophic graph of G∗, then

$$T_{K(e)}(xy) \le \{T_{J(e)}(x) \wedge T_{J(e)}(y)\},$$

$$I_{K(e)}(xy) \le \{I_{J(e)}(x) \wedge I_{J(e)}(y)\},$$

$$F_{K(e)}(xy) \le \{F_{J(e)}(x) \vee F_{J(e)}(y)\}, such\ that$$

$$0 \le T_{K(e)}(xy) + I_{K(e)}(xy) + F_{K(e)}(xy) \le 3\ for\ all\ e \in A\ and\ x, y \in V.$$

2.5 Definition [23]

Consider a neutrosophic graph G. Let (u, v) be any arc in G. An arc (u, v) is said to be strong arc, if $T_{K(e)}(u,v) \ge T^{\infty}_{K(e)}(u,v)$ and $I_{K(e)}(u,v) \ge I^{\infty}_{K(e)}(u,v)$and $F_{K(e)}(u,v) \ge F^{\infty}_{K(e)}(u,v)$.

2.6 Definition [23]

Consider a neutrosophic graph G. Let vi , vj be any two vertices in G and if they are connected means of a path then the strength of that path is defined as $(min_{i,j}T_{K(e)}(vi,vj), min_{i,j}I_{K(e)}(vi,vj), max_{i,j}F_{K(e)}(vi,vj))$ where $min_{i,j}T_{K(e)}(vi,vj)$ is the$T_{K(e)}$- strength of weakest arc and $min_{i,j}I_{K(e)}(vi,vj)$is the $I_{K(e)}$- strength of weakest arc and is the $max_{i,j}F_{K(e)}(vi,vj)$ is the $F_{K(e)}$- strength of strong arc.

3. Reliability Measures

3.1 *Definition (Strong-arced graph)*

Consider a Neutrosophic graph $G*$. The underlying strong-arced graph G' of $G*$ is defined as the spanning subgraph of $G*$ with only strong arcs as edges. A strong-arc graph needn't be connected even if $G*$ is connected.

3.2 *Definition (Score function of the strength of a path)*

Consider a neutrosophic soft graph H(e) corresponding to a parameter e. Consider a path in the graph from u to v. Let the strength of the path be represented as a tuple (T, I, F). Then, the score of the strength of the path is given by the function:

$$Str(u,v) = (1 + \mathrm{t} + \mathrm{i} - \mathrm{f})/2$$

3.3 Definition (farness)

Consider a neutrosophic graph G. The farness of a node v in G is defined as

$$\sum_{u!=v} Str(u,v)$$

3.4 Definition (Reliability)

Reliability can be considered as an extension of centrality to neutrosophic graphs. A reliable node is one that is least susceptible to changes that are quantized by the indeterminacy and falsity values in a neutrosophic tuple. Reliability talks about the robustness of the system, and its configuration to avert failure. It gives the designer of a system scope to focus on unreliable nodes. In the context of real-world applications, it is the node that remains intact/functional to a large extent.

3.5 Definition (Proximity Reliability)

Consider a strong-arced graph G`(e) of a neutrosophic soft graph H(e) corresponding to a parameter e. The proximity reliability for node v in G'(e) denoted as Pr(v), is defined as the reciprocal of farness, where farness is defined as the sum of the strength of the minimised strong arcs between node v_i and all other nodes in the network [5]. Generally, the proximity reliability is a measure of how fast data spreads from the node v_i, by taking into consideration that a node is close to all nodes in the network and not just to its neighbours. In other words, proximity reliability denotes the connectivity of the network.

$$\Pr(v) = \frac{1}{\sum_{u!=v} Str(u,v)}$$

Algorithm

1. Begin
2. For all edges in the graph do:
    a. Check if the edge is a strong arc
    b. If true, add the edge to the list of strong arcs
3. For all the strong arcs in the graph do:
    a. Obtain all the paths from one vertex of the edge to the other
    b. Obtain the aggregate tuple of (T, I, F) values using definition 2.6
    c. Assess the strength of the path by applying the formula Str(.) to these aggregate tuples
    d. Retain the maximum strength reliable paths
4. For each vertex in the strong-arced graph calculate the following:
    i. $\Pr(v)$
    ii. Number of unreachable vertices
5. Sort the resultant tuples $t = (\Pr(v), number\ of\ unreachable\ vertices)$ according to min(t[1]) and min(t[0]) (for tuples with the same t[1] values).
6. End.

3.6 Examples for Proximity Reliability

Consider two parameters describing the universe U: e1, e2.

Consider the graph H(e1) shown in Figure 1.

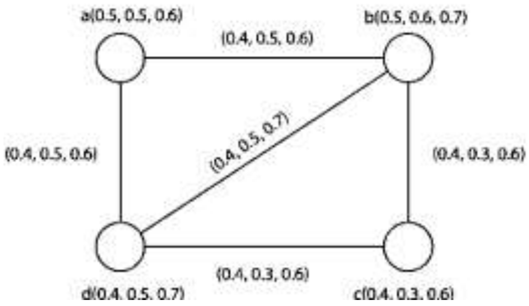

Figure 1

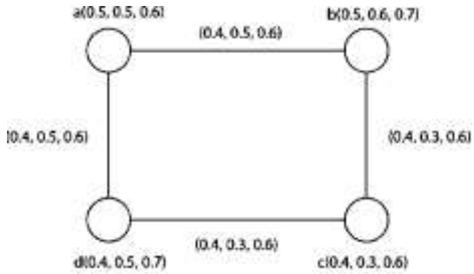

Figure 2

1. **Obtain underlying strong-arced graph.**

AB: (0.4,0.5,0.6) A-D-C-B: (0.4,0.3,0.6) A-D-B: (0.4,0.5,0.7) By property, AB is a strong arc.	AD: (0.4,0.5,0.6) A-B-C-D: (0.4,0.3,0.6) A-B-D: (0.4,0.5,0.7) By property, AD is a strong arc.	BC: (0.4,0.3,0.6) B-A-D-C: (0.4,0.3,0.6) B-D-C: (0.4,0.3,0.7) By property, BC is a strong arc.	BD: (0.4,0.5,0.7) B-A-D: (0.4,0.5,0.6) B-C-D: (0.4,0.3,0.6) By property, BD is not a strong arc.	CD: (0.4,0.3,0.6) C-B-A-D: (0.4,0.3,0.6) C-B-D: (0.4,0.3,0.7) By property, CD is a strong arc.

Table 1

2. **Obtain all the shortest paths.**

Reliable path from A to B: Paths: A-B and A-D-B-C (0.4,0.5,0.6) and (0.4,0.3,0.6) Applying the formula: (1+T+I-F)/2 we have, (1+0.4+0.5-0.6)/2 = 0.65 (1+0.4+0.3-0.6)/2 = 0.55 Reliable path from A to B: A-B	Reliable path from A to C: Paths: A-B-C and A-D-C (0.4,0.3,0.6) and (0.4,0.3,0.6) Applying the formula: (1+T+I-F)/2 we have, (1+0.4+0.3-0.6)/2 = 0.55 (1+0.4+0.3-0.6)/2 = 0.55 Reliable path from A to C: A-B-C and A-D-C	Reliable path from A to D: Paths: A-D and A-B-C-D (0.4,0.5,0.6) and (0.4,0.3,0.6) Applying the formula: (1+T+I-F)/2 we have, (1+0.4+0.5-0.6)/2 = 0.65 (1+0.4+0.3-0.6)/2 = 0.55 Reliable path from A to D: A-D
Reliable path from B to C: Paths: B-C and B-A-D-C (0.4,0.3,0.6) and (0.4,0.3,0.6) Applying the formula: (1+T+I-F)/2 we have, (1+0.4+0.3-0.6)/2 = 0.55 (1+0.4+0.3-0.6)/2 = 0.55 Reliable path from B to C: B-C and B-A-D-C	Reliable path from B to D: Paths: B-A-D and B-C-D (0.4,0.5,0.6) and (0.4,0.3,0.6) Applying the formula: (1+T+I-F)/2 we have, (1+0.4+0.5-0.6)/2 = 0.65 (1+0.4+0.3-0.6)/2 = 0.55 Reliable path from B to D: B-A-D	Reliable path from C to D: Paths: C-D and C-B-A-D (0.4,0.3,0.6) and (0.4,0.3,0.6) Applying the formula: (1+T+I-F)/2 we have, (1+0.4+0.3-0.6)/2 = 0.55 (1+0.4+0.3-0.6)/2 = 0.55 Reliable path from C to D: C-D and C-B-A-D

Table 2

3. Compute the proximity reliability.

VERTEX	Pr (VERTEX)	Number of unreachable nodes
A	$Pr(A) = \frac{1}{(0.55+0.65+0.65)} = 0.540$	0
B	$Pr(B) = \frac{1}{(0.55+0.65+0.65)} = 0.540$	0
C	$Pr(C) = \frac{1}{(0.55+0.55+0.55)} = 0.606$	0
D	$Pr(D) = \frac{1}{(0.55+0.65+0.65)} = 0.540$	0

Table 3

Hence in H(e1) C is reliable as per the proximity reliability criterion.

Now consider H(e2),

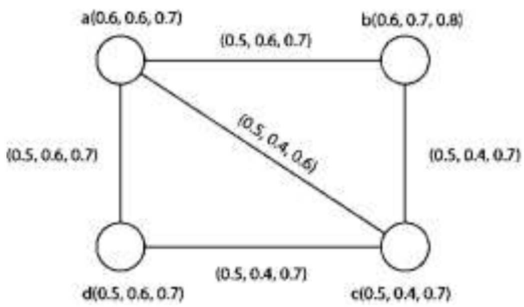

Figure 3

1. **Obtain underlying strong-arced graph.** It is the same as figure 3.
2. **Obtain all Reliable paths.**

Reliable path from A to B: A-B

Reliable path from A to C: A-B-C and A-D-C

Reliable path from A to D: A-D

Reliable path from B to C: B-A-C, B-A-D-C and B-C

Reliable path from B to D: B-A-D

Reliable path from C to D: C-D, C-B-A-D and C-A-D

3. Compute the proximity reliability.

VERTEX	Pr (VERTEX)	Number of unreachable nodes
A	$Pr(A) = \frac{1}{(0.7+0.6+0.7)} = 0.50$	0
B	$Pr(B) = \frac{1}{(0.7+0.6+0.7)} = 0.50$	0
C	$Pr(C) = \frac{1}{(0.6+0.6+0.6)} = 0.555$	0
D	$Pr(D) = \frac{1}{(0.7+0.65+0.7)} = 0.4878$	0

Table 4

Hence in H(e2) D is reliable as per the proximity reliability criterion.

3.7 Definition (Intermediate Reliability)

Consider a strong-arced graph G\`(e) of a neutrosophic soft graph H(e) corresponding to a parameter e. The intermediate reliability of a vertex v in G\`(e) is defined as the fraction of all the Reliable paths between any two vertices that passes through v. Mathematically it is defined as follows:

$$Int(v) = \frac{Number\ of\ shortest\ paths\ passing\ through\ v}{Total\ number\ of\ shortest\ paths\ between\ any\ two\ vertices}$$

Inferences:

1. The higher the value of $Int(v)$, the more reliable is the node.
2. $0 \leq Int(v) \leq 1$

Algorithm:

1. Begin
2. For all edges in the graph do:
   a. Check if the edge is a strong arc
   b. If true, add the edge to the list of strong arcs
3. For all the strong arcs in the graph do:
   a. Obtain all the paths from one vertex of the edge to the other
   b. Obtain the aggregate tuple of (T, I, F) values using definition 2.6
   c. Assess the strength of the path by applying the formula Str(.) to these aggregate tuples
   d. Retain the maximum strength reliable paths
4. For each vertex in the strong-arced graph calculate Int(v).
5. End.

Note: Intermediate reliability doesn't maintain a count of the number of unreachable nodes. The reason is that Intermediate reliability is a relative measure, it is the fraction of Reliable paths passing through a particular vertex *relative* to the number of paths present.

3.8 Definition **Sufficient criteria for G\` = G***

Consider a neutrosophic graph G* based on parameter e. The underlying strong-arced graph G\` is the same as G* if the following criteria are met:

1. G\` has the same number of nodes as G*.
2. G\` has the same number of edges as G*.
3. Every edge in G* connecting nodes u and v is constructed as:

   (Min(T(u), T(v)), Min(I(u), I(v)), Max(F(u), F(v))**)**

Proof: The proof here is direct as every edge constructed using condition 3 will be a strong arc. Every path in consideration will reduce to **(**Min(T(u), T(v)), Min(I(u), I(v)), Max(F(u), F(v))**)** of all the edges along the path, which is the basis for constructing an edge in the first place.

3.9 Examples for Intermediate Reliability:

Consider the same universe as described in Example 3.6.

Compute intermediate reliability for H(e1).

Total number of Reliable paths: 9

$Int(A) = 3/9$, $Int(B) = 2/9$, $Int(C) = 0/9$, $Int(D) = 2/9$

Hence in H(e1), A is the intermediate reliable node.

Compute intermediate reliability for H(e2).

Total number of Reliable paths: 11

$Int(A) = 5/11$, $Int(B) = 2/11$, $Int(C) = 0/11$, $Int(D) = 2/11$

Hence in H(e2), A is the intermediate reliable node.

3.10 Definition (Crisis Reliability)

Consider a strong-arced graph $G'(e)$ of a neutrosophic soft graph $H(e)$ corresponding to a parameter e. The crisis reliability of a vertex v in $G'(e)$ is defined as the difference between the farness of v and the average farness of the neighbors of v. When the strong-arced graph contains unreachable vertices resulting in un-connectedness, then, we take the maximum number of unreachable neighbors when computing crisis reliability. The most reliable node is the one with minimum farness and the minimum number of unreachable neighbors.

Algorithm:

Crisis Reliability:

1. Begin
2. For each parameter produced in the neutrosophic soft graph
    a. For all edges in the graph
        i. Find all paths between the vertices of the selected edge and reduce the path to a neutrosophic tuple
        ii. If the current edge has larger or equal T, I value and smaller of equal F value than all the known path reduces, then the current edge is strong
    b. For each vertex in the graph
        i. For every other vertex
            1. find all the paths
            2. compute the path cost for each path
            3. obtain the max of all computed path costs to be the cost of reaching that vertex
            4. note the number of unreachable neighbors
        ii. Farness = Sum all the path costs for reaching every other vertex
    c. Reliability score
        i. For each vertex
            1. compute the neighbor farness by computing the average of farness of neighbors
            2. Score is computed as the difference between the farness of the current vertex and the neighbor farness
            3. note the maximum number of unreachable neighbors

3. For each vertex in the graph, find the minimum reliability score and the maximum number of unreachable neighbors as an aggregate across all the parameters produced neutrosophic graph generated and tabulate the result.
4. Sort the tabulated result by the number of disconnected vertices. Within the same number of disconnected nodes, sort by lower heatmap value.
5. Vertex at the top of the tabulated result is the most reliable during a crisis.
6. End

3.11 Examples for Crisis Reliability

Consider two parameters describing the universe U: e1, e2.

Consider the graph H(e1):

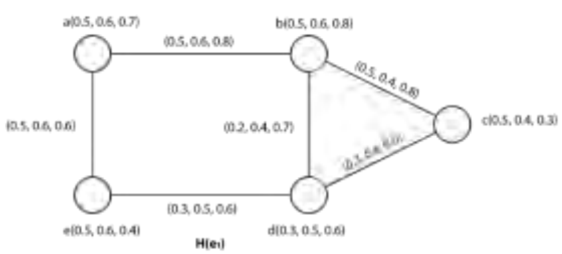

Figure 4

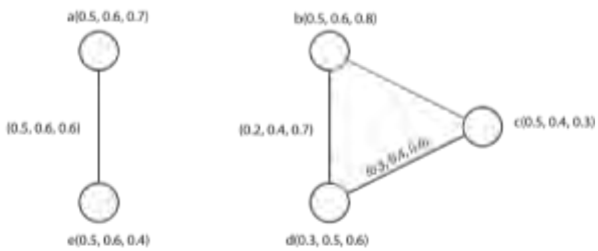

Figure 5

1. Find strong arcs.

Test if ab(0.5, 0.6, 0.8) is strong a-e-d-b: (0.2, 0.4,0 .7) a-e-d-c-b: (0.3, 0.4, 0.8) ab is not a strong arc.	Test if bc(0.5, 0.4, 0.8) is strong b-d-c: (0.2, 0.4, 0.7) b-a-e-d-c: (0.3, 0.4, 0.8) bc is not a strong arc.	Test if cd(0.3, 0.4, 0.6) is strong c-b-d: (0.2, 0.4, 0.8) c-b-a-e-d: (0.3, 0.4, 0.8) cd is a strong arc.
Test if de(0.3, 0.5, 0.6) is strong d-b-a-e: (0.2, 0.4, 0.8) d-c-b-a-e: (0.3, 0.4, 0.8) de is a strong arc.	Test if ea(0.5, 0.6, 0.6) is strong e-d-b-a: (0.2, 0.4, 0.8) e-d-c-b-a: (0.3, 0.4, 0.8) ea is a strong arc.	Test if bd(0.2, 0.4, 0.7) is strong b-a-e-d: (0.3, 0.5, 0.8) b-c-d: (0.3, 0.4, 0.8) bd is not a strong arc.

Table 5

The underlying strong-arced graph is shown in Figure 5.

2. Obtain farness measures of each vertex.

Vertex a: To b: Unreachable To c: Unreachable To d: Unreachable To e: a-e (0.5, 0.6, 0.6) Farness = (1+0.5+0.6-0.6)/2 = 0.75 Farness(a) = 0.75 Unreachable neighbors(a) = 3	Vertex b: To a: Unreachable To c: b-d-c (0.2, 0.4, 0.7) Farness = (1+0.2+0.4-0.7)/2 = 0.45 To d: b-d (0.2, 0.4, 0.7) Farness = (1+0.2+0.4-0.7)/2 = 0.45 To e: Unreachable Farness(b) = 0.9 Unreachable neighbors(b) = 2	Vertex c: To a: Unreachable To b: c-d-b (0.2, 0.4, 0.7) Farness = (1+0.2+0.4-0.7)/2 = 0.45 To d: c-d (0.3, 0.4, 0.6) Farness = (1+0.3+0.4-0.6)/2 = 0.55 To e: Unreachable Farness(c) = 1.0 Unreachable neighbors(c) = 2
Vertex d: To a: Unreachable	Vertex e: To a: e-a (0.5, 0.6, 0.6) Farness = (1+0.5+0.6-0.6)/2 = 0.75	

To b: d-b (0.2, 0.4, 0.7) Farness = (1+0.2+0.4-0.7)/2 = 0.45 To c: d-c (0.3, 0.4, 0.6) Farness = (1+0.2+0.4-0.7)/2 = 0.55 To e: Unreachable Farness(d) = 1.0 Unreachable neighbors(d) = 2	To b: Unreachable To c: Unreachable To d: Unreachable Farness(e) = 0.75 Unreachable neighbors(e) = 3	

Table 6

3. Compute crisis reliability.

Farness(a) = 0.75 Unreachable neighbors(a) = 3 Neighbor Farness(a) = Farness(e) = 0.75 Unreachable neighbors from neighbors(a) = Unreachable neighbors(e) = 3 **Aggregate unreachable neighbors = max (3, 3) = 3** **Score = Farness(a) – Neighbor Farness(a) = 0.75 – 0.75 = 0**	Farness(b) = 0.9 Unreachable neighbors(b) = 2 Neighbor Farness(b) = Farness(d) = 1.0 Unreachable neighbors from neighbors(b) = Unreachable neighbors(d) = 2 **Aggregate unreachable neighbors = max (2, 2) = 2** **Score = Farness(b) – Neighbor Farness(b) = 0.9 – 1.0 = -0.1**	Farness(c) = 1.0 Unreachable neighbors(c) = 2 Neighbor Farness(c) = Farness(d) = 1.0 Unreachable neighbors from neighbors(c) = Unreachable neighbors(d) = 2 **Aggregate unreachable neighbors = max (2, 2) = 2** **Score = Farness(c) – Neighbor Farness(c) = 1.0 – 1.0 = 0**
Farness(d) = 1 Unreachable neighbors(d) = 2 Neighbor Farness(d) = Mean (Farness(b), Farness(c)) = (0.9+1.0)/2 = 0.95 Unreachable neighbors from neighbors(d) = max (Unreachable neighbors(b), Unreachable neighbors(c)) = max(2, 2) = 2 **Aggregate unreachable neighbors = max (2, 2) = 2** **Score = Farness(d) – Neighbor Farness(d) = 1-0.95 = 0.05**	Farness(e) = 0.75 Unreachable neighbors(e) = 3 Neighbor Farness(e) = Farness(a) = 0.75 Unreachable neighbors from neighbors(e) = 3 **Aggregate unreachable neighbors = max (3, 3) = 3** **Score = Farness(e) – Neighbor Farness(e) = 0.75 – 0.75 = 0**	

Table 7

Now consider the graph H(e2):

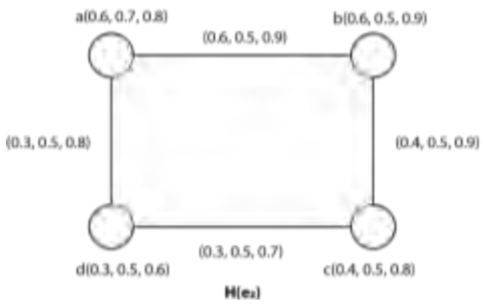

Figure 6

1. **Find strong arcs.** The strong-arc graph is the same as Figure 6.

2. Obtain farness measures of each vertex.

Vertex a: To b: a-b = **0.6**, To c: a-b-c = **0.5** a-d-c = **0.5**, To d: a-d = **0.5** Farness(a) = 0.6 + 0.5 + 0.5 = 1.6; Unreachable neighbors(a) = 0	Vertex b: To a: b-a = **0.6,** To c: b-c = **0.5**, To d: b-c-d = **0.45** b-a-d = **0.45** Farness(b) = 0.6 + 0.5 + 0.45 = 1.55; Unreachable neighbors(b) = 0	Vertex c: To c: c-b-a = **0.5** c-d-a = **0.5**, To b: c-b = **0.5**, To d: c-d = **0.55** Farness(c) = 0.5 + 0.5 + 0.55 = 1.55; Unreachable neighbors(c) = 0	Vertex d: To a: d-a = **0.5**, To b: d-c-b = **0.45,** d-a-b = **0.45**, To c: d-c = **0.55** Farness(d) = 0.5 + 0.45 + 0.55 = 1.5; Unreachable neighbors(d) = 0

Table 8

3. Compute crisis reliability.

	H(e_1)		H(e_2)		Aggregate	
	Score	Unreachable	Score	Unreachable	Score	Unreachable
A	0	3	0.075	0	0	3
B	-0.1	2	-0.025	0	-0.1	2
C	0	2	0.025	0	0	2
D	0.05	2	-0.075	0	-0.075	2
E	0	3	-	-	0	3

Table 9

4. Applications

We consider a neutrosophic set of five countries: Germany, China, USA, Brazil and Mexico. Suppose we want to travel between these countries through an airline journey. The airline companies aim to facilitate their passengers with high quality of services.

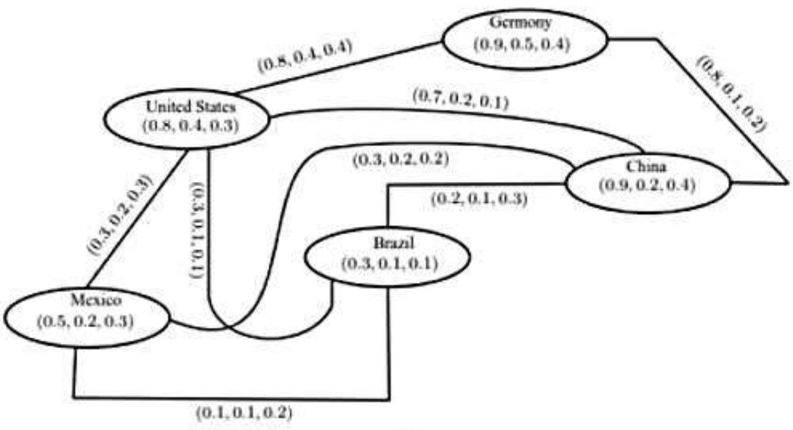

Figure 7

The reliability measures presented in this paper can be applied to the above graph to obtain the central nodes:

Proximity Reliability:

On applying proximity reliability, we obtain the following statistic:

$Pr(United\ States) =$ 0.328, Number of unreachable nodes: 0

$Pr(Mexico) =$ 0.4, Number of unreachable nodes: 0

$Pr(Germany) =$ 0.322, Number of unreachable nodes: 0

$Pr(Brazil) =$ 0.408, Number of unreachable nodes: 0

$Pr(China) =$ 0.328, Number of unreachable nodes: 0

Conclusion: 'China' is the node that is the most reliably connected to all other nodes, and is connected with every other node as well. It can thus be used as a fail-safe airport in case an airplane needs to make an emergency landing.

Intermediate Reliability:

On applying intermediate reliability, we obtain the following statistic:

Total number of Reliable paths: 10

$Int(United\ States) = 3/10$, $Int(Mexico) = 0/10$, $Int(Germany) = 0/10$, $Int(Brazil) = 0/10$

$Int(China) = 5/10$.

Conclusion: The node 'China' lies on 50% (half) of the reliable paths between the nodes. It can thus be used as a connecting terminal for long-distance flights.

Crisis Reliability:

United States: (0.175,0), Mexico: (-0.45,0), Germany: (-0.1,0), Brazil (-0.4,0), United States: (0.2,0).

Conclusion: The node 'Mexico' can reach other destinations more reliably than other nodes. Airplane companies can therefore make a strategic decision to dock planes in Mexico or to start journeys from Mexico for flights that go to multiple destinations.

5. Conclusions

In this paper, the concept of strong-arced graphs and reliability measures were introduced. Three pertinent reliability measures, namely, proximity, intermediate and crisis reliability were discussed. The first, proximity reliability of a node, computes the farness of a node to its neighbors. The second, intermediate reliability of a node, computes the fraction of paths of minimal farness that pass through it. The third, crisis reliability of a node, is a hybrid of the two previously defined. It considers the farness of a node to its neighbors taking into account the farness of the neighbors to other nodes in the graph. These reliability measures were applied to a real-world airplane application to determine the important nodes.

A summary of the new notations presented in this paper is shown below:

Notation	Description
$Str(u,v)$	Then, the score of the strength of the path from node u to node v.
$\Pr(v)$	Proximity reliability of node v.
$Int(v)$	Intermediate reliability of node v.
$Cr(v)$	Crisis reliability of node v.

6. Future Scope

All three measures make use of the same score function. A score function tailored to each measure can be developed in the future. The algorithms used in this paper find all possible paths between pairs of nodes and then only eliminate the paths which are not required. There is great scope for improvement in this area. One could try to develop a heuristics-based algorithm to improve the efficiency of finding reliable nodes and paths by eliminating certain paths. A few fields that can benefit from this research work are: supply chain management, logistics, network management and warfare planning.

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Algebraic Properties of Finite Neutrosophic Fields

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Abstract: We explore a finite Neutrosophic field $\boldsymbol{F_p(I)}$ and its Neutrosophic multiplicative group $\boldsymbol{F_p(I)^\times}$ in this study. We first show $|\boldsymbol{F_p(I)^\times}| = (\boldsymbol{p-1})^2$ and then its algebraic properties are studied. The Neutrosophic Fermat's and Little Fermat's theorems over $\boldsymbol{F_p(I)^\times}$ are then proved. Finally, this paper investigates some applications of Neutrosophic Fermat's theorem over $\boldsymbol{F_p(I)^\times}$ with various illustrations.

Keywords: Neutrosophic Field; Neutrosophic Group; Neutrosophic Fermat's Theorem, Neutrosophic Little Fermat's Theorem.

1. Introduction

In the algebraic sense, finite field theory deals with the algebraic concepts and related systems with the properties of different sets of complete residue system $Z_n = \{0, 1, 2, \ldots\ldots, n-1\}$ of integers modulo n. In this paper, we consider some particularly important sets of numbers $Z_p = \{0, 1, 2, \ldots\ldots, p-1\}$ under addition and multiplication modulo a prime p. The theory of these numbers is concerned, at least in its elementary aspects, with properties of the scalars and more particularly with the numbers in Z_p and their related concepts. We shall make no attempt to construct the set of numbers axiomatically, assuming instead that they are already well-known and that any reader of this paper is familiar with many elementary concepts and results about finite fields. Among these some are defined and stated to refresh in algebraic terminology. We can generally define a field F as an abelian group under addition together with multiplicative operation such that the structure $(F-\{0\},\cdot)$ is also an abelian group satisfies the distributive axioms: $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$. Now we shift our attention to the finite field F_p, we are considering in this paper [1]. For a prime p, we represent the number of elements in the field F_p is p. Also, in any finite field of order p, we have $ap = 0$ for every a in F_p. This means that the characteristic of F_p is p. Further, all fields of order p are isomorphic, that is, there is a unique field up to isomorphism of order p.

Classic algebra, control systems, neural networks, decision and estimation issues have all been reformed and adapted to adhere to Neutrosophic logic and systems in recent years [2-7]. In 1980, Smarandache developed his Neutrosophic sets philosophical theory to address many forms of uncertainties in a variety of real-world challenges, and it has since been successfully implemented in a variety of study domains. However, Neutrosophic theory is an extension theory of Fuzzy logic theory in which indeterminacy $\boldsymbol{I}$ is included with $\boldsymbol{I}$ follows some algebraic properties, namely $\boldsymbol{I}+$

$\boldsymbol{I = 2I}$, $\boldsymbol{I^2 = I}$, $\boldsymbol{I - I = 0}$, $\boldsymbol{0I = 0}$, $\boldsymbol{1I = I}$ but $\boldsymbol{I^{-1}}$ does not exist. The concept of Neutrosophic field structure was introduced by F. Smarandache and W.B. Vasantha Kandasamy in 2006.

Now we give a brief introduction to Neutrosophic field structures. For any classical field, there exists a Neutrosophic field $F(I)$. The structure $F(I) = (F(I), +, \cdot)$ is called a Neutrosophic field under Neutrosophic operations $+$ and $\cdot$, which are defined as

$(a + bI) + (c + dI) = (a + c) + (b + d)I$ and

$(a + bI)(c + dI) = ac + (ad + bc + bd)I$

for every Neutrosophic elements $a + bI$ and $c + dI$ in $F(I)$. Note that, $F(I)$ is generated by F and I, and it is represented by $F(I) = \langle F \cup I \rangle = F + FI$. If F is a finite field then $F(I)$ is also finite. Otherwise, $F(I)$ is an infinite Neutrosophic field. For example, $Q(I)$, $R(I)$ and $C(I)$ for all infinite fields but $F_p(I)$ is a finite field, where F_p is isomorphic to Z_p. However, for further details about Neutrosophic field, the reader should see [8-13].

This manuscript makes three contributions. To begin, we propose using finite fields to investigate the algebraic features of the corresponding Neutrosophic field through several cases. Second, we characterise in detail the Neutrosophic Fermat's and Little Fermat's Theorems over finite Neutrosophic fields. Additionally, we present certain necessary and sufficient conditions for the Neutrosophic elements' abilities in the Neutrosophic field $F_p(I)$. Finally, and most importantly, to illustrate three alternative implementations of the Neutrosophic Fermat's Theorem. Additionally, we developed a table comparing classical and neutrosophic fields.

2. Properties of Finite Neutrosophic Fields

Most of the researchers in abstract algebra show how to represent a finite field F_p over its prime characteristic p by clearly representing its additive structure as an abelian group, or a quotient ring of polynomials over F_p. In this section, we represent a Neutrosophic set representation of finite Neutrosophic field that naturally and simply displays both the Neutrosophic additive and multiplicative structures of the finite Neutrosophic field $F_p(I)$ over the classical field F_p under its prime characteristic p.

Let p be a prime number. Then we specify the finite field of order p^2, $F_p(I)$ also denoted by $F_p + F_p(I)$ as follows:

$F_p(I) = \{a + bI : a, b \epsilon F_p, I^2 = I\}$

where the Neutrosophic operations $(a + bI) + (c + dI)$ and $(a + bI)(c + dI)$ are both performed modulo p, this means that $(a + bI) + (c + dI)$ is the remainder of the division $\frac{(a+bI)+(c+dI)}{p}$ and similarly for $(a + bI) + (c + dI)$ remainder of $\frac{(a+bI)(c+dI)}{p}$.

Generally, the following result is well-known with respect to the classical field F_p.

Theorem 2.1[15]: For every element u in $F_p^{\times}$ there exists v in $F_p^{\times}$ such that $uv \equiv 1 \ (mod\ p)$.

This result is very useful for studying every result in F_p. But, this result is not true in the Neutrosophic field $F_p(I)$, that is, $(a + bI)(c + dI) \not\equiv 1 \ (mod\ p)$ for some elements $(a + bI)$ and $(c + dI)$ in $F_p(I)$, since $I(1 + (p - 1)I) \equiv 0 \ (mod\ p)$. A fairly natural question presents itself. Is it possible to enumerate the number of multiplicative inverse elements in the Neutrosophic field $F_p(I)$? The answer is yes and it is contained in the following Theorems. ■

Theorem 2.2: Let $u + vI$ be an element in $F_p(I)^{\times}$ then there exists its multiplicative inverse $(u + vI)^{-1}$ in $F_p(I)^{\times}$ such that $(u + vI)^{-1} = u^{p-2} - vu^{p-2}(u + v)^{p-2}I$.

Proof. Suppose $u + vI$ be an element in $F_p(I)^{\times}$. Then $u, u + v \epsilon F_p$. If possible assume that $u \neq 0$ and $u + v \neq 0$, then there exists u^{-1} and $(u + v)^{-1}$ in $F_p^{\times}$ such that $u^{-1} = u^{p-2}$ and $(u + v)^{-1} = (u + v)^{p-2}$. Because $(u + vI)(u^{p-2} - vu^{p-2}(u + v)^{p-2}I) = 1$, so the definition of multiplication inverse elements yields the inverse $(u+vI)^{-1}$ of $(u+vI)$ exists in $F_p(I)^{\times}$ such that $(u+vI)^{-1} = u^{p-2} - vu^{p-2}(u+v)^{p-2}I$. ■

Example 2.3: In $F_5(I)^{\times}$, $(4+5I)^{-1} = 4^{5-2} - 2(4^{5-2})(6^{5-2})I = 4-3I = 4+2I$.

Here is another basic fact extracted from [14] regarding mutual additive inverse elements. Consider an ordered pair (u,v) in$F_p{}^{\times}$. An ordered pair (u,v) in $F_p{}^{\times}$ is called mutual additive pair if $u+v=0$ in $F_p{}^{\times}$. The set of all mutual additive pairs in $F_p{}^{\times}$ is denoted by $M(F_p{}^{\times})$, particularly, $M(F_p{}^{\times}) = \{(u+vI): u+v=0\}$.
Note that $|F_p(I)^*| = p-1$, where $F_p(I)^* = F_p(I) - \{0\}$. If $u+v=0$ inF_p, then $uv \not\equiv 1 \ (mod\ p)$. Let us see how all this works in a specific instance.

Example 2.4: The following table exhibits the cardinality of the set $F_p(I)^{\times}$ for $p = 2,3,5$.

Prime	2	3	5
$F_p(I)^{\times}$	1	4	16

Here, we observe that the cardinality of $F_5(I)^{\times}$ is 16, whereas the cardinality of $F_2(I)^{\times}$ and $F_3(I)^{\times}$ are 1 and 3 respectively. It is easy to verify that $F_2(I)^{\times} = \{1\}$, $F_3(I)^{\times} = \{1,2,1+I,2+2I\}$, $F_5(I)^{\times} = \{1,2,3,4,1+I,1+2I,1+3I,2+I,2+2I,2+4I,3+I,3+3I,3+4I,4+2I,4+3I,4+4I\}$

One consequence of what has just been proved is that, in those cases in which a multiplicative inverse exists in $F_p(I)^{\times}$, we can now state exactly how many there are.

Theorem 2.5: If $u+iv$ is a multiplicative inverse in $F_p(I)$ has exactly $(p-1)^2$ of them. Particularly,$|F_p(I)^{\times}| = (p-1)^2$.

Proof. Because $\boldsymbol{p}$ is a prime, surely the Neutrosophic field $\boldsymbol{F_p(I)}$ is a disjoint union of the sets $\{\boldsymbol{0}\}, \boldsymbol{F^*I}, \boldsymbol{M(F_p(I)^*)}$ and$\boldsymbol{F_p(I)^{\times}}$, that is, $\boldsymbol{F_p(I) = \{0\} \cup F_p^*I \cup M(F_p(I)^*) \cup F_p(I)^{\times}}$, where $\boldsymbol{F_p^* = F - \{0\}}$ and $\boldsymbol{F_p^*I = \{uI: u\epsilon F_p^*\}}$. Raise both sides of this relation to the cardinality and expand to obtain the relation

$$\boldsymbol{F_p(I) = |\{0\}| + |F_p^*I| + |M(F_p(I)^*)| + |F_p(I)^{\times}|}$$
$$\boldsymbol{\Rightarrow p^2 = 1 + (p-1) + (p-1) + |F_p(I)^{\times}|}$$
$$\boldsymbol{\Rightarrow |F_p(I)^{\times}| = p^2 - 1 - (p-1) - (p-1)}$$
$$\boldsymbol{\Rightarrow |F_p(I)^{\times}| = (p-1)^2}.$$

For an illustration of these ideas, let us demonstrate the cardinality of $F_3(I)^{\times}$. Using the Neutrosophic elements in $F_3(I)$, we observe that
$F_3(I) = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$,
$F_3^*I = \{I,2I\}$, and $M(F_3(I)^*) = \{u+vI: u+v=0\}$
$= \{1+2I, 2+I\}$.

Therefore,

$$|F_3(I)| = |\{0\}| + |F_3^*I| + |M(F_3(I)^*)| + |F_3(I)^{\times}|$$
$$\Rightarrow 3^2 = 1 + (3-1) + (3-1) + |F_3(I)^{\times}|$$
$$\Rightarrow |F_3(I)^{\times}| = 3^2 - (3-1) - (3-1) = (3-1)^2 = 4,$$

which are listed below

$$F_3(I)^{\times} = \{1,2,1+I,2+2I\}.$$

In view of classical algebraic sense, well-known that $a^{\varphi(n)} \equiv 1 \ (mod\ n)$ whenever $(a,n) = 1$, where $\varphi(n)$ is the Euler totient function of n. This supports the following definition in the classical field F_p.

Definition 2.6: Let $u \in F_p$, then there exists a least positive integer k such that $O(u) = k$ with respect to multiplication defined over F_p if and only if $u^k \equiv 1 \ (mod\ n)$.
For instance, $2^3 \equiv 1 \ (mod\ 7)$ in the field F_7, so that the integer 2 has order 3 modulo 7. According to this classical field systems, we know that every non-zero element in F_p has unique order with respect to multiplication. However, it is not true in the Neutrosophic sense. Now let us see how all this works in the following specific instances.

Example 2.7: The following table exhibits the order of the non-zero elements in the Neutrosophic field
$F_3(I) = \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$
under Neutrosophic multiplication modulo 3.

Element in $F_3(I)$	1	2	I	$2I$	$1 + I$	$1 + 2I$	2+I	2+2I
Order	1	2	d.e	d.e	2	d.e	d.e	2

where "d.e" represents does not exist.

Particularly, the following table illustrates the orders of each element in $F_3(I)^\times$ exits.

Element in $F_3(I)^\times$	1	2	$1 + I$	$2 + 2I$
Order	1	2	2	2

Theorem 2.9: Let $u, v \in F_p$. Then $u + vI$ has a multiplicative inverse in $F_p(I)$ if and only if $u \neq 0$ and $u + v \neq 0$ in F_p.

Proof. We denote multiplicative identity in F_p by 1. Consider a nonzero pair of elements u, v in F_p and write it in the form $(u + vI)$ in $F_p(I)$. Then

$(u + vI)$ has a multiplicative inverse $\Leftrightarrow$ $(u + vI)(x + yI) = 1$ has a solution in $F_p(I)$

$\Leftrightarrow$ $ux \equiv 1 \ (mod\ p)$ has a solution in Z and
$vx + (u + v)y \equiv 0 \ (mod\ p)$ has a solution in Z.

$\Leftrightarrow$ $u \neq 0$ and $u + v \neq 0$ in F_p. ■

Let us now employ the unique technique of this section to enumerate the number of elements in $F_p(I)^\times$ of the form $(u + vI)^2 = 1$. To start, we know that there is only one element 1 in $F_2(I)^\times$ with $1^2 = 1$. Now, our enumeration starts from $p > 2$, which explore the following theorem.

Theorem 2.10: If $p > 2$ is a prime number, then the congruence$(u + vI)^2 - 1 \equiv 0 \ (modulo\ p)$ has exactly 4 solutions in $F_p(I)^\times$.

Proof. Because p is an odd prime, it follows that $F_p(I)^\times$ contains at least one element of order 2. Suppose that $u + vI$ is an element in $F_p(I)^\times$ of order 2, then the Neutrosophic multiplication inverse of $u + vI$ is itself $u + vI$ in $F_p(I)^\times$. Therefore,

$(u + vI)^2 = 1 \ \Leftrightarrow \ u^2 + v^2 I + 2uvI = 1 + 0I$

$\Leftrightarrow u^2 = 1, \ v^2 + 2uv = 0$

$\Leftrightarrow u^2 = 1, v^2 = 4$, since $v^2 \neq 0$

$\Leftrightarrow u = 1, p - 1, v = 2, p - 2$ in F_p.

So, there exists six $\left[\binom{4}{2} = 6\right]$Neutrosophic elements, namely
$1 + 0I, (p - 1) + 0I, 1 + 2I, 1 + (p - 2)I, (p - 1) + 2I$

and $(p-1)+(p-2)I$ in $F_p(I)^{\times}$.
Out of these six elements, four elements $1, p-1, 1+(p-2)I$ and $(p-1)+2I$ satisfies the Neutrosophic equation $(u+vI)^2=1$ in $F_p(I)^{\times}$, because $(1+2I)^2 \neq 1$ and $\left((p-1)+(p-2)I\right)^2 \neq 1$is true in $F_p(I)^{\times}$. ■
As an immediate consequence of Theorem [2.10], we deduce the following corollary.
Corollary 2.11: The set $\mathcal{I}_p(I)=\{u+vI\epsilon F_p(I)^{\times}: (u+vI)^2=1\}$ is a Neutrosophic subgroup of the Neutrosophic group $F_p(I)^{\times}$.

Proof. It is clear from the well-known result:
$(u+vI)^2=1,\ (u'+v'I)^2=1$ implies that $[(u+vI)(u'+v')]^2=1$ in $F_p(I)^{\times}$. ■

Remark 2.12: (1) $\mathcal{I}_p(I)=F_p(I)^{\times} \Leftrightarrow p=3$.
(2) $|\mathcal{I}_p(I)| \leq |F_p(I)^{\times}|$ for every $p \geq 3$.
Let us see what happens if $0(u+vI)=2$ is evaluated for each $u+vI$ in $F_p(I)^{\times}$ of $p \geq 3$ and the required results are added. In the case $p=3$, the answer is easy; here

$$0(u+vI)=2 \Leftrightarrow u+vI\epsilon F_p(I)^{\times}-\{1\}.$$

Suppose that $p>3$, then the non-empty subset

$$H_p(I)=\{u+vI\epsilon F_p(I)^{\times}:\ 0(u+vI)=2\}$$

exists in $F_p(I)^{\times}$ but it is not a Neutrosophic subgroup of $F_p(I)^{\times}$ because $1 \notin H_p(I)$ (since $o(1)=1 \neq 2$).

3. Neutrosophic Fermat's and Little Fermat's Theorems

The above information of the Neutrosophic field $F_p(I)$ seems the opportune moment to mention the Fermat's and Little Fermat's Theorems gave an essentially valid proof of Neutrosophic filed Theory. First of all, we state classical Fermat's and Little Fermat's Theorems in the classical field F_p as follows.

Theorem3.1 [15]: (Fermat's Theorem)
For every u in F_p, we have $u^{p-1} \equiv 1\ (mod\ p)$.

Theorem3.2 [15]: (Fermat's Little Theorem)
For every u in F_p, we have $u^p \equiv u\ (mod\ p)$.

Classical Fermat's theorem contains many applications and it plays a central role in much of what is done in many applied and engineering sciences. However, now we introduce Neutrosophic Fermat's theorem over the Neutrosophic field $F_p(I)$.
We now proceed to state and prove Neutrosophic Fermat's Theorem in $F_p(I)$.

Theorem 3.3: (Neutrosophic Fermat's Theorem for $\boldsymbol{F_p(I)}$)
Let p be a prime and let $u+vI \in F_p(I)$. Then

$$(u+vI)^{p-1} \equiv 1\ (modulo\ p).\ ■$$

Before we proceed to the proof of this theorem, we observe that the congruence

$$(u+vI)^{p-1} \equiv 1\ (modulo\ p)$$

fails to hold for some choice of $u+vI$ in $F_p(I)$. As an illustration of this approach, let us look $p=3$. The determination is kept under control by selecting a suitable Neutrosophic element for $u+vI$, say, $u+vI=1+2I$. Because $(1+2I)^{p-1}$ maybe written as, $(1+2I)^{3-1}=(1+2I)^2=1+4I+4I=1+2I\ (mod\ 3)$, but $(1+2I) \not\equiv 1\ (mod3$. Combining these congruences, we finally obtain

$(1+2I)^{3-1} \not\equiv 1\ (mod\ 3)$. So, Theorem **[3.3]** is not true in $F_p(I)$. However, the upshot of all this is the following Theorem.

Theorem3.4: (Neutrosophic Fermat's Theorem for $F_p(I)^{\times}$)
Let $p > 2$ be a prime. For every Neutrosophic element $u+vI$ in $F_p(I)^{\times}$ such that
$(u+vI)^{p-1} \equiv 1\ (mod\ p)$.

Proof. Let $u+vI$ in $F_p(I)^{\times}$. Then we begin by assuming the first $(p-1)$ multiples of $u+vI$, that is, $u+vI, 2(u+vI), 3(u+vI), \ldots\ldots, (p-1)(u+vI)$. None of these Neutrosophic elements in $F_p(I)^{\times}$ is congruent modulo p to any other element in $F_p(I)^{\times}$. To see this, we consider $r(u+vI) \equiv s(u+vI)\ (mod\ p)$ for some r and s such that $1 \le r < s \le p-1$. Since $u+vI \in F_p(I)^{\times}$, there exists a multiplicative inverse of $u+vI$ in $F_p(I)^{\times}$, so $u+vI$ could be cancelled in $r(u+vI) \equiv s(u+vI)\ (mod\ p)$ to give $r \equiv s\ (mod\ p)$, which is not true because $1 \le r < s \le p-1$. Therefore, the set $u+vI, 2(u+vI), 3(u+vI), \ldots\ldots, (p-1)(u+vI)$ of Neutrosophic elements in $F_p(I)^{\times}$ must be congruent modulo p under the following bijection:

$$r \mapsto (u+vI)r$$

for every r in $\{0,1,2,3,\ldots\ldots,p-1\}$. Now multiply all these elements together, we obtain that

$$(u+vI)2(u+vI)3(u+vI)\cdots(p-1)(u+vI) \equiv 1\ 2\ 3 \cdots (p-1)\ (mod\ p)$$
$$\Longrightarrow (u+vI)^{p-1}(p-1)! \equiv (p-1)!\ (mod\ p)$$
$$\Longrightarrow (u+vI)^{p-1} \equiv 1\ (mod\ p), \text{ since } gcd(p,(p-1)!) = 1. \ \blacksquare$$

An application of Neutrosophic Fermat's Theorem leads to the congruences $(1+I)^2 \equiv 1\ (mod\ 3), (1+I)^6 \equiv 1\ (mod\ 7), (1+I)^{10} \equiv 1\ (mod\ 11)$ and, in turn, to solve the following example.

Example 3.5: In the Neutrosophic multiplicative group $F_{101}(I)^{\times}$, we have
$(1+I)^{100} \equiv 1\ (mod\ 101)$.
Solution. It is easy to see that
$(1+I)^2 \equiv (1+3I)\ (mod\ 101), (1+I)^{10} \equiv (1+13I)\ (mod\ 101)$.
However, we conclude that,

$$(1+I)^{100} = [(1+I)^{10}]^{10} = (1+13I)^{10}$$
$$\equiv (1+83I)(1+94I)\ (mod\ 101$$
$$\equiv 1\ (mod\ 101).$$

Now, starts the greatest advances in this direction were made by this manuscript called Neutrosophic Fermat's Little Theorem. We state this more precisely in the following theorem.

Theorem 3.6: (Neutrosophic Little Fermat's Theorem)
Let p be a prime. Then for every $u+vI$ in the Neutrosophic field $F_p(I)$,

$$(u+vI)^p \equiv (u+vI)\ (mod\ p).$$

Proof In light of the Binomial theorem, $(u+vI)^p = \binom{p}{0}u^p(vI)^0 + \binom{p}{1}u^{p-1}(vI)^1 + \cdots + \binom{p}{2}u^{p-2}(vI)^2 + \cdots + \binom{p}{p-1}u^{p-(p-1)}(vI)^{p-1} + \binom{p}{p}u^0(vI)^p$.
Because $u+vI \epsilon F_p(I)$, we have $u, v, I \epsilon F_p$. So, by the classical Fermat's Little Theorem [3.2],

$$u^p \equiv u\ (mod\ p),\ u^p \equiv u\ (mod\ p) \text{ and } I^p \equiv I\ (mod\ p).$$

Since $p|\binom{p}{1}$, $p|\binom{p}{2}$,...... $p|\binom{p}{p-1}$. In this sequence, we can obtain easily as
$(u+vI)^p \equiv (u+vI)\ (mod\ p)$. ■

At this stage, when $p = 2$, $2(u + vI) \equiv 0(mod\ 2)$ for any $u + vI \in F_2(I)$, so $u + vI = -(u + vI)$ for any $u + vI \in F_2(I)$. Therefore, we also have

$$(u - vI)^2 \equiv u^2 - v^2 I \equiv (u^2 + v^2 I)(mod\ 2).$$

Corollary 3.7:

Let p be an odd prime. Then for every $u + vI$ in the Neutrosophic field $F_p(I)$,

$$(u - vI)^p \equiv (u - vI)\ (mod\ p).$$

Proof. By Theorem [3.6], we have

$$(u - vI)^p = (u + (-vI))^p = u^p + (-vI)^p$$
$$= u^p + (-1)^p (v)^p (I)^p$$
$$= u^p + (-1)^p v^p I.$$

When $p > 2$, p is odd, we have $(-1)^p \equiv -1(mod\ p)$, and $I^p \equiv I(mod\ p)$. Hence

$$(u - vI)^p \equiv (u - vI)\ (mod\ p). \blacksquare$$

4. Applications of Neutrosophic Fermat's Theorem

Already, it is well known that the Quadratic congruence $(u + vI)^2 - 1 \equiv 0\ (mod\ p)$ has exactly four solutions whenever p is an odd prime. From this result, we can pass simply to the following application of Neutrosophic Fermat's theorem.

Theorem 4.1: Let $p > 3$ be an odd prime and let $u + vI \in F_p(I)^\times$. If $4|(p-1)$ and $4d|(p-1)^2$ then the congruence $(u + vI)^{4d} - 1 \equiv 0\ (mod\ p)$ has exactly $4d$ solutions.

Proof. Since $|F_p(I)^\times| = (p-1)^2$. Suppose $u + vI$ be any element in $F_p(I)^\times$. But by hypothesis, $4d|(p-1)^2$, so we have $(p-1)^2 = 4dq$ for some positive integer q. Then the expression $(u + vI)^{(p-1)^2} - 1 = (u + vI)^{4dq} - 1$

$$= ((u + vI)^{4d})^q - 1^q$$
$$= ((u + vI)^{4d} - 1) f(u + vI),$$

where

$$f(u + vI) = (u + vI)^{4d(q-1)} + (u + vI)^{4d(q-2)} + \cdots + (u + vI)^{4d} + 1$$

is a polynomial of degree

$$4d(q-1) = 4dq - 4d = (p-1)^2 - 4d.$$

We know that any solution $u + vI \equiv (a + bI)(mod\ p)$ of the congruence $(u + vI)^{(p-1)^2} - 1 \equiv 0(mod\ p)$ that is not a solution of $f(u + vI) \equiv 0(mod\ p)$ must satisfy the congruence$(u + vI)^{4d} - 1 \equiv 0\ (mod\ p)$.

For the element $a + bI$ in $F_p(I)^\times$, we have

$$0 \equiv (a + bI)^{(p-1)^2} - 1 = ((a + bI)^{4d} - 1) f(a + bI)(mod\ p)$$

with the condition $p \nmid f(a + bI)$, which implies that $p|((a + bI)^{4d} - 1)$. It follows that the required congruence $(u + vI)^{4d} - 1 \equiv 0\ (mod\ p)$ must have

$$(p-1)^2 - ((p-1)^2 - 4d) = 4d \text{ solutions. } \blacksquare$$

Example 4.2: For an illustration of these facts, let us solve the congruence

$$(u + vI)^4 - 1 \equiv 0\ (mod\ 5).$$

A table of powers of Neutrosophic elements in $F_5(I)^\times$ can be constructed once a modulo 5 is fixed. Using this modulo 5, we simply calculate the powers of elements in $F_5(I)^\times$ as follows.

$$1^4 \equiv 1\ (mod\ 5),\ 2^4 \equiv 1\ (mod\ 5),\ 3^4 \equiv 1\ (mod\ 5), 4^4 \equiv 1\ (mod\ 5),$$
$$(1 + I)^4 \equiv 1\ (mod\ 5),\ (1 + 2I)^4 \equiv 1\ (mod\ 5),\ (1 + 3I)^4 \equiv 1\ (mod\ 5),$$
$$(2 + I)^4 \equiv 1\ (mod\ 5),\ (2 + 2I)^4 \equiv 1\ (mod\ 5),\ (2 + 4I)^4 \equiv 1\ (mod\ 5),$$
$$(3 + I)^4 \equiv 1\ (mod\ 5),\ (3 + 3I)^4 \equiv 1\ (mod\ 5),\ (3 + 4I)^4 \equiv 1\ (mod\ 5),$$
$$(4 + 2I)^4 \equiv 1\ (mod\ 5),\ (4 + 3I)^4 \equiv 1\ (mod\ 5),\ (4 + 4I)^4 \equiv 1\ (mod\ 5).$$

Consulting the above list of powers of 4 in each element of $F_5(I)^\times$, we obtain that the original congruence $(u + vI)^4 - 1 \equiv 0\ (mod\ 5)$ possesses the $4d = 4 \cdot 4 = 16$ solutions, namely

$u + vI \equiv 1,2,3,4, 1 + I, 1 + 2I, \ldots,$ and $4 + 4I \ (mod\ 5)$.

Remark 4.3: The congruence $(u + vI)^4 - 1 \equiv 0\ (mod\ 11)$ is not solvable in $F_{11}(I)^\times$, because $4 \nmid (11 - 1)$.

We would like to close this paper with another application of Neutrosophic Fermat's theorem to the study of quadratic congruence $(u + vI)^2 \equiv 0\ (mod\ p)$.

Theorem 4.4: Let $u + vI \in F_p(I)^\times$ and let $p > 3$ be a prime. If the quadratic congruence $(u + vI)^2 + 1 \equiv 0\ (mod\ p)$ has a solution, the prime $p \equiv 1\ (mod\ 4)$.

Proof: Suppose $a + bI \in F_p(I)^\times$ be any solution of $(u + vI)^2 + 1 \equiv 0\ (mod\ p)$. Then

$$(a + bI)^2 \equiv -1\ (mod\ p).$$

By the Neutrosophic Fermat's theorem [15],

$$1 \equiv (a + bI)^{p-1} \equiv [(a + bI)^2]^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}}\ (mod\ p).$$

If possible assume that $p = 4q + 3$ for some q, then

$$(-1)^{\frac{p-1}{2}} = (-1)^{2q+1} = -1, \text{ hence } 1 \equiv -1\ (mod\ p).$$

This implies that $p|2$, which is not true because p is an odd prime. Consequently, our assumption that $p = 4q + 3$ is not true, and hence p must be of the form $4q + 1$. ■

The converse of the preceding theorem may not be true. That is if $p = 4q + 1$, then $(u + vI)^2 + 1 \equiv 0\ (mod\ p)$ is not solvable in $F_p(I)^\times$. For instance, $p = 5$, the congruence $(u + vI)^2 + 1 \equiv 0\ (mod\ 5)$ is not solvable in $F_5(I)^\times$.

Example 4.5: Consider the case $p = 13$, which is a prime of form $4q + 1$. It is easy to see that $(3 + 4I)^2 + 1 \equiv 0\ (mod\ 13)$. Thus the congruence $(u + vI)^2 + 1 \equiv 0\ (mod\ 13)$ is solvable in $F_{13}(I)^\times$.

Finally, the difference table for F_p and $F_p(I)$ is displayed below:

Classical Field F_p	Neutrosophic Field $F_p(I)$
1. $\lvert F_p \rvert = p$.	1. $\lvert \boldsymbol{F_p(I)} \rvert = \boldsymbol{p^2}$.
2. $\lvert F_p^\times \rvert = p - 1$.	2. $\lvert \boldsymbol{F_p(I)^\times} \rvert == \boldsymbol{(p-1)^2}$.
3. For each u in F_p^*, there exists v in F_p^* such that $uv \equiv 1\ (mod\ p)$.	3. For some $\boldsymbol{a + bI}$ and $\boldsymbol{c + dI}$ in$\boldsymbol{F_p(I)^*}$, we have $\boldsymbol{(a + bI)(c + dI) \not\equiv 1\ (mod\ p)}$.
4. $F_p^\times$ is a cyclic group.	4. $\boldsymbol{F_p^\times}$ is not a cyclic group.
5. The product of all elements in F_p^* is non-zero.	5. The product of all elements in $\boldsymbol{F_p(I)^*}$ is zero.
6. The congruence $x^2 + 1 \equiv 0\ (mod\ p)$ has a solution $\Leftrightarrow$ $p \equiv 1\ (mod\ 4)$.	6. If $\boldsymbol{(u + vI)^2 + 1 \equiv 0\ (mod\ p)}$ has a solution in $\boldsymbol{F_p(I)^\times}$ then $\boldsymbol{p \equiv 1\ (mod\ 4)}$. But converse need not be true.

5. Conclusions

In this manuscript, we turn to close to another milestone of the development of Fermat's theorem under the Neutrosophic sense. In this regard, we constructed a table to differentiate the field F_p and Neutrosophic field $F_p(I)$. Also, we have given necessary and sufficient conditions for solving Neutrosophic quadratic congruences like

$$(u + vI)^2 + 1 \equiv 0\ (mod\ p),$$

$$(u + vI)^2 - 1 \equiv 0\ (mod\ p) \text{ and } (u + vI)^{4d} - 1 \equiv\ (mod\ p)$$

with various illustrations in the Neutrosophic field $F_p(I)$.

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A new Cramèr–von Mises Goodness-of-fit test under Uncertainty

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Abstract: The Cramer-von Mises test is commonly used to determine how well-observed sample data fits a given model. The existing Cramer-von Mises test under traditional statistics is commonly used when sample data in reliability work are resolute and precise. In this paper, we introduced a Neutrosophic Cramer-von Mises (NCVM) test under neutrosophic statistics. The necessary measures and procedures are presented to perform the test. For the application purpose, we consider the real-life data sets of failure time batteries and ball bearings. It is inferred that the NCVM test is more instructive than the classical CVM test under indeterminacy.

Keywords: Cramer-von Mises; Neutrosophic Weibull; Neutrosophic Rayleigh; Goodness of fit

1. Introduction

The statistical techniques have been utilized in every practical field for modeling data sets, prediction, and forecasting purposes. The application of these modeling statistical techniques/tests is made under specific suppositions, and infringement of these assumptions could prompt deluding interpretation and dependable outcomes [1, 2]. One of the fundamental presumptions that various statistical techniques are associated with the distribution of observed data follows a specified distribution. Typically, it is expected that the obtained information follows the normal distribution. In some viable situations, data sets don't need to be normally distributed. Therefore, researchers planned a few tests to valuation some hypotheses about the distribution of the information being scrutinized. Various tests, for the most part, known as "goodness-of-fit" are employed to evaluate whether an example of observations can be considered as a sample from a given distribution. The frequently utilized goodness-of-fit tests are; Kolmogorov–Smirnov [3, 4], Anderson–Darling [5, 6], Pearson's chi-square [7], Cramèr–von Mises [8, 9], Shapiro–Wilk [10], Jarque–Bera [11, 12], D'Agostino–Pearson [13] and Lilliefors [14].

The Cramèr–von Mises (CVM) test is a criterion utilized for the evaluation of the goodness of fit. The CVM test is the generalization of the Anderson-Darling test. The CVM test is the assessment of the minimum distance between hypothetical and sample probability distribution. Stephens [16] utilized the CVM goodness-of-fit test based on the experimental distribution function considering normal and exponential distributions. It was found that the CVM test appears more powerful test than chi-square. Al-zahrani [17] introduced the CVM goodness of fit test for Topp-Leone distribution.

The classical Cramèr–von Mises test can't be applied when the sample observations are neutrosophic numbers. So the principle motivation behind this study is to present another Cramèr–von Mises goodness-of-fit test within the sight of indeterminacy. We will introduce the technique to fit the neutrosophic Weibull and Rayleigh distributions on the lifetime of batteries and ball-bearings data sets.

2. Preliminaries

Suppose that $X_N = X_L + X_U I_N; I_N \in [I_L, I_U]$ denotes the neutrosophic number (NN) that follows the neutrosophic Weibull distribution with neutrosophic shape parameter $\beta_N = \beta_L + \beta_U I_N; I_N \in [I_L, I_U]$ and the neutrosophic scale parameter $\alpha_N = \alpha_L + \alpha_U I_N; I_N \in [I_L, I_U]$. Here X_L is the determinate part and $X_U I_N$ is the indeterminate part with an indeterminacy constant $I_N \in [I_L, I_U]$. Note that neutrosophic Weibull random variable X_N reduces to classical Weibull distribution when $I_N = 0$.

Neutrosophic statistics is the augmentation of classical statistics. This field acquires significance because of dealing with the data sets of values more specifically an interval, for more detail reader can consult the following references [18-20]. For the presentation of the neutrosophic environment, normally a subsequent "N" is utilized such as X_N.

3. Neutrosophic Weibull distribution

The neutrosophic Weibull (NW) distribution was introduced by [21]. The cumulative distribution function of NW distribution is

$$F(X_N) = 1 - \exp\left\{-\left(\frac{X_N}{\alpha_N}\right)^{\beta_N}\right\}, \quad X_N > 0. \tag{1}$$

where $\alpha_N \in [\alpha_L, \alpha_U], \beta_N \in [\beta_L, \beta_U]$

4. Neutrosophic Rayleigh distribution

The Neutrosophic Rayleigh (NR) distribution was introduced by [22]. The cumulative distribution function of NR distribution is

$$F(X_N) = 1 - \exp\left\{-\frac{1}{2}\left(\frac{X_N}{\alpha_N}\right)^{2}\right\}, \quad X_N > 0. \tag{2}$$

where $\alpha_N \in [\alpha_L, \alpha_U]$

5. Neutrosophic Cramèr-von-Mises

The CVM test is a non-parametric test of the hypothesis. It is utilized to test whether an example comes from a particular distribution when the observed data set is precise or determined. When the data set is imprecise then the exiting CVM test cannot be used to test the goodness of fit due to indeterminacy in the data. We modify the classical CVM test and proposed the neutrosophic Cramer-von-Mises (NCVM) test for the data having neutrosophic numbers. The proposed test will bring about terms of indeterminacy interval which will be more successful when compared to the classical CVM test. The assumption for the NCVM test are

- The data consists of imprecise observations.
- The observations in the interval are mutually independent.

"Suppose $X_{1N}, X_{2N}, X_{3N}, \ldots, X_{nN}$ is a neutrosophic random sample from a neutrosophic population having a neutrosophic cumulative distribution function, say $F(X_N)$". Then the NCVM is given by

$$CVM_N = \frac{1}{12n_N} + \sum_{i=1}^{n_N}\left\{\frac{2i-1}{2n_N} - \left[1 - \exp\left\{-M_N(i)\right\}\right]\right\}^2; \; CVM_N \in [CVM_L, CVM_U] \tag{3}$$

where $n_N \in [n_L, n_U]$ are the neutrosophic random samples and $M_N(i) = \left(\frac{X_N(i)}{\alpha_N}\right)^{\beta_N}$.

6. Applications of Neutrosophic Cramèr-von-Mises Test

This section examines the use of the newly introduced test. For the application purposed we consider two real-life data sets.

6.1. Application on data Set I (lifetime in 100 h of 23 batteries

The first data set is regarding the lifetime of batteries also utilized by [21]. The lifetime in 100 h of 23 batteries is given in Table 1.

Table 1. The lifetime of batteries

Sr. No	X_N	Sr. No	X_N	Sr. No	X_N	Sr. No	X_N
1	[2.9,3.99]	7	[12.65,17.4]	13	[17.4,23.93]	19	[26.07,35.84]
2	[5.24,7.2]	8	[13.24,18.21]	14	[17.8,24.48]	20	[30.29,41.65]
3	[6.56,9.02]	9	[13.67,18.79]	15	[19.01,26.14]	21	[43.97,60.46]
4	[7.14,9.82]	10	[13.88,19.09]	16	[19.34,26.59]	22	[48.09,66.13]
5	[11.6,15.96]	11	[15.64,21.51]	17	[23.13,31.81]	23	[73.48,98.04]
6	[12.14,16.69]	12	[17.05,23.45]	18	[23.34,32.09]		

The mechanical investigators are intrigued to test either the given informational collection follows Weibull distribution or not. It is not difficult to take note that the data observations are given in indeterminacy intervals instead of the specific observation. So the classical CVM test is not appropriate. Therefore, we will utilize the option NCVM test proposed in section 5 is used for these neutrosophic numbers.

Assume that we need to test the following hypothesis:

H_0=The sample observation follows to neutrosophic Weibull distribution.

H_1=The distribution of sample observation is not neutrosophic Weibull distribution.

The numerical computations are listed in Table 2. The parameters of Neutrosophic Weibull distribution are estimated using the maximum likelihood estimation method. The estimated values are $\hat{\alpha}_N \in [22.936, 31.427]$ and $\hat{\beta}_N \in [1.465, 1.481]$. The test statistic values of the proposed CVM_N test for the considered lifetime of batteries data are shown as

$$CVM_N = \frac{1}{12[n_L, n_U]} + \sum_{i=1}^{n_N} \left\{ \frac{2i-1}{2[n_L, n_U]} - \left[1 - \exp\{-M_N(i)\}\right] \right\}^2$$

$$CVM_N = \frac{1}{12[23, 23]} + \sum_{i=1}^{n_N} \left\{ \frac{2i-1}{2[23, 23]} - \left[1 - \exp\{-M_N(i)\}\right] \right\}^2$$

$CVM_N \in [0.1112, 0.3617]$

Table 2. The necessary calculation of the NCVM test for the first data

ith	X_N	$M_N(i)$	$F_N(X_N)$	ith-term
1	[2.9,3.99]	[0.0484, 0.0772]	[0.0472, 0.0743]	[0.0006, 0.0028]
2	[5.24,7.2]	[0.1150, 0.1832]	[0.1087, 0.1674]	[0.0019, 0.0104]
3	[6.56,9.02]	[0.1599, 0.2549]	[0.1477, 0.2250]	[0.0015, 0.0135]
4	[7.14,9.82]	[0.1810, 0.2887]	[0.1656, 0.2507]	[0.0002, 0.0097]
5	[11.6,15.96]	[0.3684, 0.5879]	[0.3082, 0.4445]	[0.0127, 0.0619]
6	[12.14,16.69]	[0.3938, 0.6277]	[0.3255, 0.4662]	[0.0075, 0.0516]
7	[12.65,17.4]	[0.4183, 0.6672]	[0.3418, 0.4869]	[0.0035, 0.0417]
8	[13.24,18.21]	[0.4472, 0.7132]	[0.3606, 0.5099]	[0.0012, 0.0338]
9	[13.67,18.79]	[0.4686, 0.7467]	[0.3741, 0.5261]	[0.0000, 0.0245]
10	[13.88,19.09]	[0.4792, 0.7643]	[0.3807, 0.5343]	[0.0010, 0.0147]
11	[15.64,21.51]	[0.5708, 0.9103]	[0.4349, 0.5976]	[0.0005, 0.0199]
12	[17.05,23.45]	[0.6477, 1.0330]	[0.4767, 0.6441]	[0.0005, 0.0208]
13	[17.4,23.93]	[0.6672, 1.0641]	[0.4869, 0.6550]	[0.0032, 0.0124]
14	[17.8,24.48]	[0.6898, 1.1001]	[0.4983, 0.6672]	[0.0079, 0.0064]
15	[19.01,26.14]	[0.7596, 1.2111]	[0.5321, 0.7021]	[0.0097, 0.0051]
16	[19.34,26.59]	[0.7790, 1.2418]	[0.5411, 0.7111]	[0.0176, 0.0014]
17	[23.13,31.81]	[1.0124, 1.6146]	[0.6367, 0.8010]	[0.0065, 0.0070]
18	[23.34,32.09]	[1.0259, 1.6354]	[0.6415, 0.8051]	[0.0142, 0.0020]
19	[26.07,35.84]	[1.2063, 1.9228]	[0.7007, 0.8538]	[0.0107, 0.0024]
20	[30.29,41.65]	[1.5028, 2.3961]	[0.7775, 0.9089]	[0.0049, 0.0037]
21	[43.97,60.46]	[2.5941, 4.1359]	[0.9253, 0.9840]	[0.0012, 0.0086]
22	[48.09,66.13]	[2.9577, 4.7161]	[0.9481, 0.9911]	[0.0002, 0.0032]
23	[73.48,98.04]	[5.5034, 8.3957]	[0.9959, 0.9998]	[0.0003, 0.0005]

6.2. Application on data Set I (lifetime of ball-bearings)

The second data set is about the service life of ball-bearing data [23]. The second data set is listed in below Table 3.

Table 3. The failure life of 21 ball bearings

Sr. No	X_N	Sr. No	X_N	Sr. No	X_N	Sr. No	X_N
1	[0.70, 0.81]	7	[0.85, 1.03]	13	[0.23, 0.74]	19	[0.34, 1.11]
2	[0.63, 0.81]	8	[0.67, 0.73]	14	[0.76, 0.95]	20	[0.07, 1.17]
3	[0.35, 0.41]	9	[0.96, 1.04]	15	[0.80, 0.86]	21	[0.41, 0.44]
4	[0.70, 0.72]	10	[1.07, 1.26]	16	[1.06, 1.21]		
5	[1.12, 1.43]	11	[0.95, 1.35]	17	[0.60, 0.70]		
6	[0.47, 1.39]	12	[0.82, 1.02]	18	[0.85, 1.01]		

For the second application, we utilized the ball-bearing failure time data test either it follows Neutrosophic Raleigh or not.

Assume that we need to test the following hypothesis:

H_0 =The sample observation follows to neutrosophic Raleigh distribution.
H_1 =The distribution of sample observation is not neutrosophic Raleigh distribution.

The maximum likelihood estimates for NR distribution are $\hat{\alpha}_N \in [0.52447, 0.70812]$. The numerical computation of NCVM is presented in Table 4. The test statistic values of the proposed CVM_N test for the considered lifetime of batteries data are shown as

$$CVM_N = \frac{1}{12[21,21]} + \sum_{i=1}^{n_N} \left\{ \frac{2i-1}{2[21,21]} - \left[1 - \exp\left\{-M_N(i)\right\}\right] \right\}^2$$

$CVM_N \in [0.1564, 0.3550]$

Table 4. The necessary calculation of the NCVM test for second data

ith	X_N	$M_N(i)$	$F_N(X_N)$	ith-term
1	[0.70, 0.81]	[0.0178, 0.3352]	[0.0089, 0.1543]	[0.0002, 0.0170]
2	[0.63, 0.81]	[0.1923, 03861]	[0.0917, 0.1756]	[0.0004, 0.0108]
3	[0.35, 0.41]	[0.4203, 0.9772]	[0.1895, 0.3865]	[0.0050, 0.0715]
4	[0.70, 0.72]	[0.4453, 1.0338]	[0.1996, 0.4036]	[0.0011, 0.0562]
5	[1.12, 1.43]	[0.6111, 1.0628]	[0.2633, 0.4122]	[0.0024, 0.0392]
6	[0.47, 1.39]	[0.8031, 1.0921]	[0.3307, 0.4208]	[0.0047, 0.0252]
7	[0.85, 1.03]	[1.3088, 1.3084]	[0.4802, 0.4802]	[0.0291, 0.0291]
8	[0.67, 0.73]	[1.4429, 1.3084]	[0.5140, 0.4802]	[0.0246, 0.0151]
9	[0.96, 1.04]	[1.6320, 1.4750]	[0.5578, 0.5217]	[0.0234, 0.0137]
10	[1.07, 1.26]	[1.7814, 1.7998]	[0.5896, 0.5934]	[0.0188, 0.0199]
11	[0.95, 1.35]	[1.7814, 2.0344]	[0.5896, 0.6384]	[0.0080, 0.0192]
12	[0.82, 1.02]	[2.0998, 2.0748]	[0.6500, 0.6456]	[0.0105, 0.0096]
13	[0.23, 0.74]	[2.3267, 2.1157]	[0.6876, 0.6528]	[0.0085, 0.0033]
14	[0.76, 0.95]	[2.4445, 2.1570]	[0.7054, 0.6599]	[0.0039, 0.0003]
15	[0.80, 0.86]	[2.6266, 2.4572]	[0.7311, 0.7073]	[0.0016, 0.0003]
16	[1.06, 1.21]	[2.6266, 2.7300]	[0.7311, 0.7446]	[0.0000, 0.0000]
17	[0.60, 0.70]	[3.2810, 2.9198]	[0.8061, 0.7677]	[0.0004, 0.0003]
18	[0.85, 1.01]	[3.3504, 3.1661]	[0.8127, 0.7947]	[0.0004, 0.0015]
19	[0.34, 1.11]	[4.0848, 3.6346]	[0.8703, 0.8375]	[0.0001, 0.0019]
20	[0.07, 1.17]	[4.1622, 3.8531]	[0.8752, 0.8544]	[0.0028, 0.0055]
21	[0.41, 0.44]	[4.5603, 4.0781]	[0.8977, 0.8698]	[0.0062, 0.0113]

7. Discussion and Conclusion

In this section, we will compare the efficiency of the proposed NCVM test under the neutrosophic environment with the existing classical CVM test. The proposed test is more efficient when dealing with data having imprecise observation or indeterminacy as the proposed method provides results in the form of indeterminacy. For comparison purposes, we use the same data set for classical CVM. Note that the data given in Tables 1 and 3 have a determinate part as well as the indeterminate part. The determinate part will be used for the existing CVM test and the same data set is used for the NCVM test. The critical value at 1% and 5% are $CVM_{1\%,23} = 0.267$ and $CVM_{5\%,23} = 0.187$, respectively. From Table 2 it is unmistakably that the proposed test gives the results in the form of indeterminacy interval rather than determinate part only. Utilizing Equation (3) the value of statistic as indeterminacy interval can be written as $0.1112 + 0.3617I; I_N \in [0, 0.6926]$. Note that the proposed test gives a decent portion of indeterminacy. At a 1% level of significance, the probability of accepting the true null hypothesis is 0.99, the probability of rejecting the true null hypothesis is 0.01 and the probability of indeterminacy is 0.69. For instance, $CVM = 0.3617$ is the value of classical CVM and $CVM_U = 0.3617$ gives the indeterminate part under uncertainty. By contrasting with crucial values, we can see that the determinant component of the information follows the Weibull distribution, but the uncertain part does not. Similarly, for the failure of ball bearings data the value of statistics as indeterminacy interval can be written as $0.1564 + 0.3550I; I_N \in [0, 0.5994]$. By contrasting with critical values, we note that the determinant part follows the Rayleigh distribution, yet the uncertain part of the information doesn't follow the Rayleigh distribution.

It is concluded that the proposed NCVM test under neutrosophic statistics provides information about the measure of indeterminacy, but the classical CVM test does not. Furthermore, the existing test delivers accurate statistics values, which are not necessary for uncertainty. As a result, under neutrosophic statistics, the proposed NCVM goodness-of-fit test is particularly efficacious when used under uncertainty.

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Neutrosophic Kumaraswamy Distribution with Engineering Application

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Abstract: In this study, Neutrosophic Kumaraswamy (NKw) distribution was proposed to analyze bounded data sets under an indeterminacy environment. Mathematical properties of NKw distribution are derived including, moments, mean, variance, Shannon entropy, reliability measures. We also present the graphical representation of density curves, cumulative distribution function, and hazard rate function. The parameters of NKw distribution are estimated using the maximum likelihood technique. We perform a simulation study to see the performance of maximum likelihood estimates. Eventually, the proposed model is applied to real data set. It has been concluded that NKw distribution provides better results than Neutrosophic beta distribution.

Keywords: Neutrosophic; indeterminacy; Kumaraswamy distribution; MLE; Simulation

1. Introduction

In 1995, Neutrosophic statistics was originally introduced by [1]. It is a new branch of philosophy, presented as a generalization for fuzzy logic and as a generalization for intuitionistic fuzzy logic. Neutrosophic statistics can be applied in an uncertain environment. Neutrosophic statistics acquire significance because of their ability to manage sets of values more explicitly an interval. To be more exact, when the values or parameters have disarray attached with them, then that particular value or parameter is replaced with a set of values [2, 3]. Further, [4-9] presented some more interesting fundamental concepts of the neutrosophic set.

Nowadays, authors contributed to the field of neutrosophic statistics methodologically as well as applied it in various fields. Alhabib and Salama [10] introduced time-series theory under indeterminacy. Aslam [11—13] extend the neutrosophic statistics in the field of total quality control. He proposed control charts under an indeterminacy environment. He presented several neutrosophic sampling plans.

The classical probability distributions applicable when the sample is selected from the population having uncertain observations. So there is an essential need to introduced probability models under an indeterminacy environment. Several authors introduced neutrosophic probability distributions, for example, Neutrosophic Weibull by [14], Neutrosophic Uniform, Neutrosophic exponential, and Neutrosophic Poisson [15], Normal distribution and binomial distribution by [16], Neutrosophic Raleigh distribution by [17] and Neutrosophic Beta distribution by [18].

1.1. Neutrosophic Approach

Neutrosophic statistics is the extended form of classical statistics. In classical statistics, we are dealing with specific values or crisp values but in neutrosophic statistics, the sample observations are taken from a population having uncertainty in observations. In the field of neutrosophic statistics, the data information might be vague, imprecise, ambiguous, uncertain, incomplete, even unknown. The shape of the neutrosophic number has a standard form in terms of the extension of the classical statistics and is shown below

$$X_N = E + i$$

where E is the exact or determined part of data information and i is the uncertain, inexact, or indeterminacy part of data. To differentiate the neutrosophic random variable the subscript N is used such as XN.

1.2. Kumaraswamy distribution

The Kumaraswamy (Kw) distribution is one of the most important and flexible distribution to analyze unit interval (0, 1) data sets. The Kw distribution was originally introduced by Kumaraswamy in 1980 [19]. The Kw distribution contained two positive shape parameters. The Kw distribution is applicable in the fields of reliability analysis, atmosphere temperatures, scores acquired in the test, hydrological, and economic data, etc. Jones [20] derived mathematical properties of Kumaraswamy distribution.

2. Neutrosophic Kumaraswamy distribution

A neutrosophic Kumaraswamy distribution (NKw) of a continuous variable X is a classical Kumaraswamy distribution of x, but such parameters are imprecise. The probability density function (pdf) of NKw distribution is

$$f_N(X) = \alpha_N \beta_N X^{\alpha_N - 1}(1 - X^{\alpha_N})^{\beta_N - 1}, \;\; X \in (0,1) \tag{1}$$

where α_N and β_N are the shape parameters. Figure 1 shows the pdf plots for various values of parameters.

The NKw distribution is flexible due to its variable shapes of the density function. The PDF curves showing exponentially decreasing behavior and start from the infinite point for $\alpha_N < 1$. For $\alpha_N = 1$, its behavior is exponentially decreasing but starts from a specific point on the y-axis. For $\alpha_N > 1$, the density curves showing unimodal behavior.

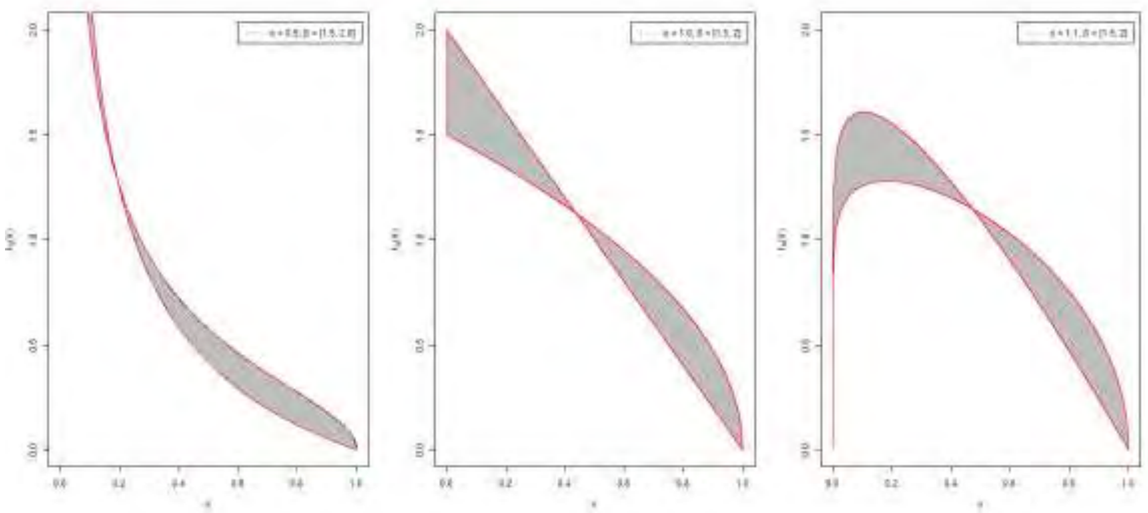

Figure 1(a). Density function plots of NKw distribution

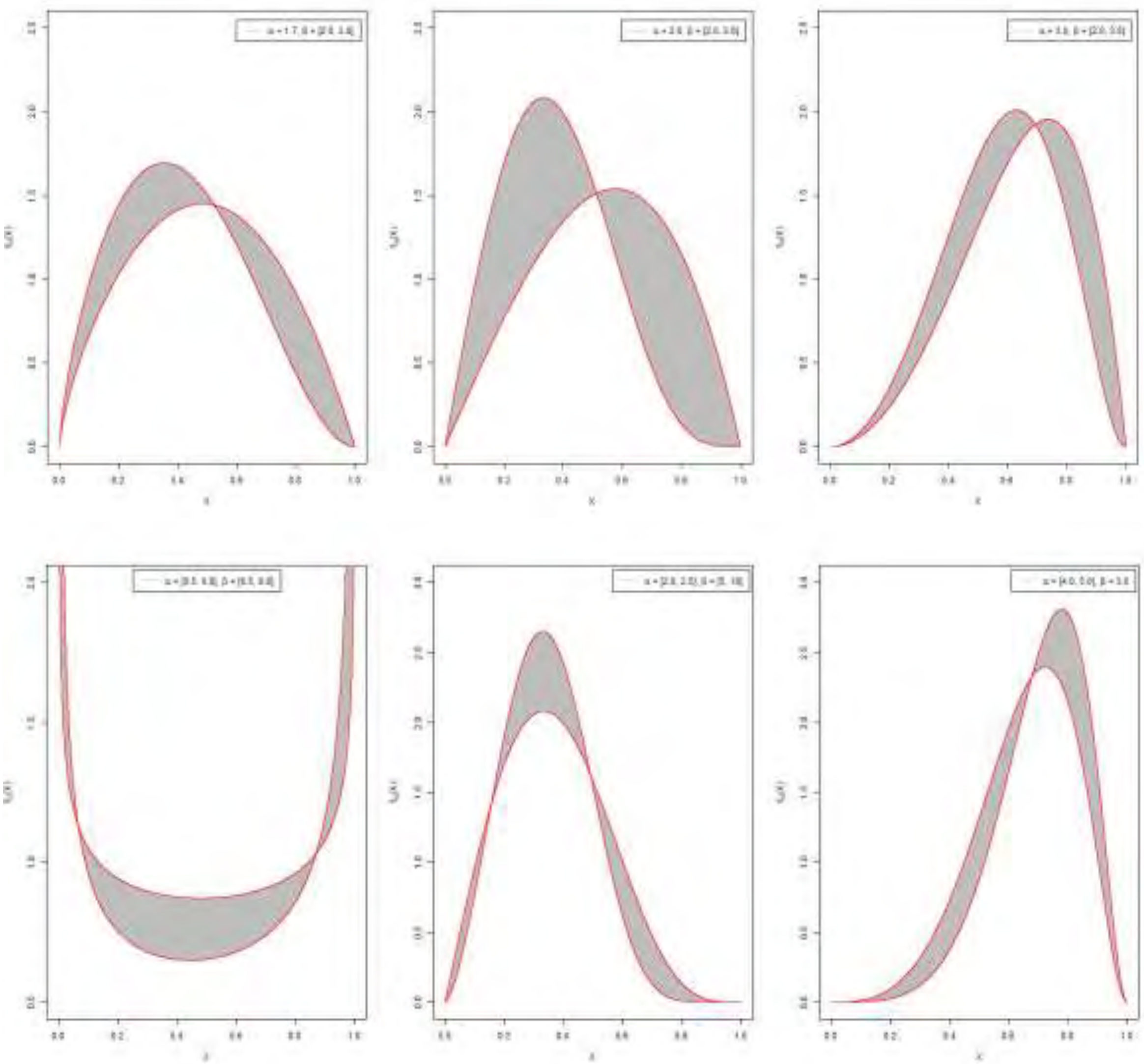

Figure 1(b). Density function plots of NKw distribution

The cumulative distribution function (CDF) of NKw distribution is

$$F_N(X) = 1 - (1 - X^{\alpha_N})^{\beta_N} \quad (2)$$

We plot CDF curves for some selected values of parameters, see Figure 2.

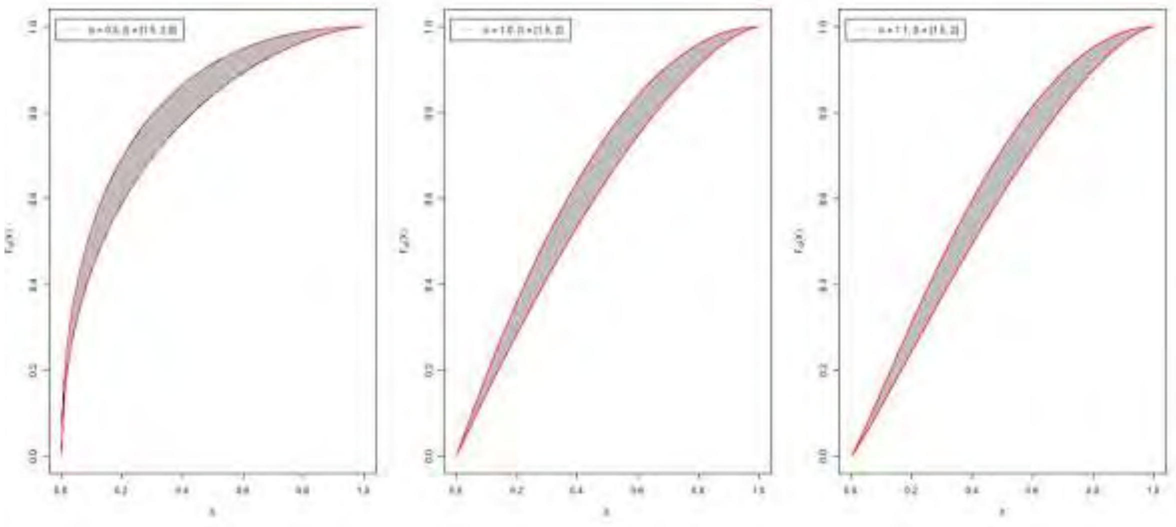

Figure 2. The CDF curves of NKw distribution

The survival function and hazard function of NKw distribution expressed as

$$S_N(X) = (1 - X^{\alpha_N})^{\beta_N} \tag{3}$$

and

$$F_N(X) = \frac{\alpha_N \beta_N X^{\alpha_N - 1}}{(1 - X^{\alpha_N})} \tag{4}$$

The HRF curves of the NKw distribution are presented in Figure 3.

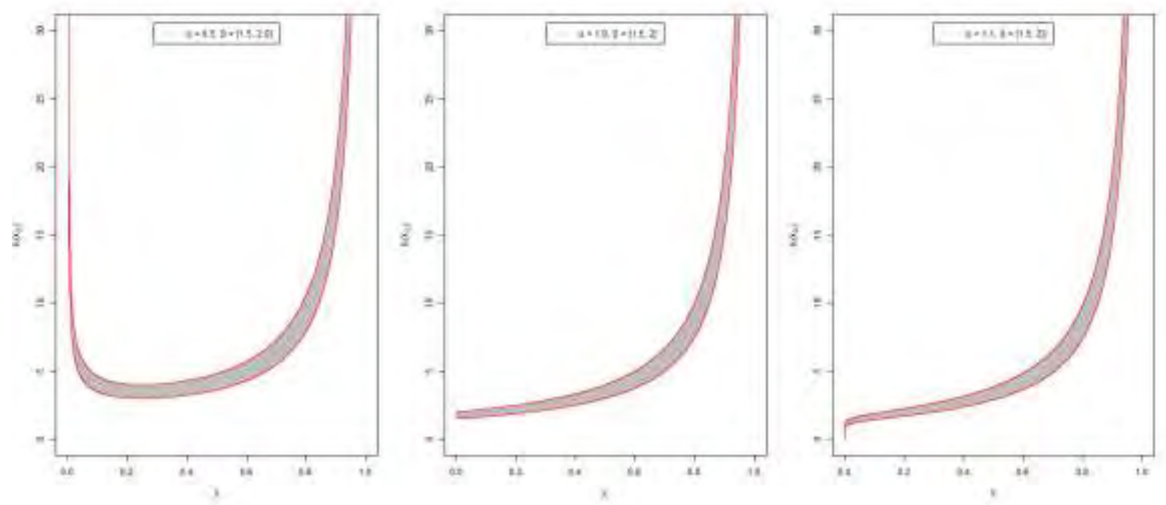

Figure 3. HRF curves for NKw distribution

From Figure 3, it is interesting to note the NKw distribution has variable shapes. The failure rate of NKw distribution is a bathtub and increasing behavior, which is very important to analyze data sets in various fields.

3. Mathematical Properties

In this section, we discussed some mathematical properties of the NKw distribution.

3.1. The rth moments:

$$E_N(X^r) = \beta_N \mathrm{B}\left[1 + \frac{r}{\alpha_N}, \beta_N\right]$$

3.2. The mean and variance:

$$E_N(X) = \beta_N \mathrm{B}\left[1 + \frac{1}{\alpha_N}, \beta_N\right]$$

and

$$Var_N(X) = \beta_N \mathrm{B}\left[1 + \frac{2}{\alpha_N}, \beta_N\right] - \left\{\beta_N \mathrm{B}\left[1 + \frac{1}{\alpha_N}, \beta_N\right]\right\}^2$$

3.3. Median:

$$Q_{0.25} = \left(1 - \frac{1}{\sqrt[\beta_N]{2}}\right)^{\frac{1}{\alpha_N}}$$

3.4. Shannon entropy of NKw distribution is

$$H_N(X) = \int_0^1 f_N(X) \log\big(f_N(X)\big)\, dx$$

$$H_N(X) = \int_0^1 \left\{\alpha_N \beta_N X^{\alpha_N - 1}(1 - X^{\alpha_N})^{\beta_N - 1}\right\} \log\left\{\alpha_N \beta_N X^{\alpha_N - 1}(1 - X^{\alpha_N})^{\beta_N - 1}\right\} dx$$

$$H_N(X) = \left(1 - \frac{1}{\alpha_N}\right) + \left(1 + \frac{1}{\beta_N}\right) H_{\beta_N} - \log(\alpha_N \beta_N)$$

where H_i is the harmonic number.

4. Maximum Likelihood Estimation

The model parameters are estimated using the famous maximum likelihood approach. Let $X_{N1}, X_{N2}, \dots, X_{Nn}$ be a random sample of NKw distribution. The log-likelihood function can be written as

$$l(\alpha_N, \beta_N) = n_N \log(\alpha_N \beta_N) + (\alpha_N - 1)\sum_{i=1}^{n_N} \log(X_{Ni}) + (\beta_N - 1)\sum_{i=1}^{n_N} \log\left(1 - X_{Ni}^{\alpha_N}\right) \quad (5)$$

The MLEs, $\hat{\alpha}_N \in [\hat{\alpha}_L, \hat{\alpha}_U]$ and $\hat{\beta}_N \in \left[\hat{\beta}_L, \beta_U\right]$, can be obtained by maximizing the above log-likelihood function equation.

$$\frac{\partial l(\alpha_N, \beta_N)}{\partial l(\alpha_N)} = \frac{n_N}{\alpha_N} + \sum_{i=1}^{n_N} \log(X_{Ni}) - (\beta_N - 1)\sum_{i=1}^{n_N} \frac{X_{Ni}^{\alpha_N} \log(X_{Ni})}{\left(1 - X_{Ni}^{\alpha_N}\right)} \quad (6)$$

$$\frac{\partial l(\alpha_N,\beta_N)}{\partial l(\beta_N)}=\frac{n_N}{\beta_N}+\sum_{i=1}^{n_N}\log\left(1-X_{Ni}^{\alpha_N}\right) \tag{7}$$

The maximum likelihood estimates can be obtained using the above equations.

5. Simulation Study

In this section, we carry out a simulation study to check the behavior of proposed estimators for NKw distribution. We generate 10,000 samples of sizes, $n = 30, 50, 100, 200,$ and 250 from NKw distribution with different combinations of parameters. The sample we generated from a random number generator. The average bias and Mean Square Error (MSEs) are used to check the properties of the best estimator. The results of the simulation study are listed in Tables 1-2.

Table 1. Parameter Estimates for $\alpha_N = 0.5,\ \beta_N = [1.5,2.0]$.

n	AEs		Avg. Biases		MSEs	
	$\hat{\alpha}_N$	$\hat{\beta}_N$	$\hat{\alpha}_N$	$\hat{\beta}_N$	$\hat{\alpha}_N$	$\hat{\beta}_N$
30	0.5334	[1.6469, 2.2859]	0.0334	[0.1469, 0.2859]	0.0141	[0.2436, 0.6027]
50	0.5180	[1.6045, 2.1481]	0.0180	[0.1045, 0.1481]	0.0071	[0.1293, 0.2584]
100	0.5102	[1.5492, 2.0636]	0.0102	[0.0492, 0.0636]	0.0036	[0.0524, 0.1129]
200	0.5059	[1.5173, 2.0368]	0.0059	[0.0173, 0.0368]	0.0017	[0.0236, 0.0483]
250	0.5033	[1.5256, 2.0378]	0.0033	[0.0256, 0.0378]	0.0013	[0.0214, 0.0403]

Table 2. Parameter Estimates for$\alpha_N = [1.5,2.0],\ \beta_N = 0.5$

n	AEs		Avg. Biases		MSEs	
	$\hat{\alpha}_N$	$\hat{\beta}_N$	$\hat{\alpha}_N$	$\hat{\beta}_N$	$\hat{\alpha}_N$	$\hat{\beta}_N$
30	[1.6603, 2.2442]	0.5346	[0.1603, 0.2442]	0.0346	[0.2798, 0.5358]	0.0171
50	[1.6107, 2.1833]	0.5249	[0.1107, 0.1833]	0.0249	[0.1624, 0.3348]	0.0091
100	[1.5449, 2.0835]	0.5091	[0.0449, 0.0835]	0.0091	[0.0647, 0.1206]	0.0039
200	[1.5314, 2.0356]	0.5062	[0.0314, 0.0356]	0.0062	[0.0332, 0.0550]	0.0019
250	[1.5209, 2.0233]	0.5058	[0.0209, 0.0233]	0.0058	[0.0251, 0.0400]	0.0015

From the above tables, it is seen that the ML estimators are consistent. The average bias and MSE decrease with an increase in sample size.

6. Application

In this section, a data set is analyzed to demonstrate the applicability and flexibility of the newly neutrosophic probability distribution over well-known existing probability distribution. The considered data set is about the ball-bearing data and from [21]. The selection of the best fit model shall be considered using the following model selection standards, log-likelihood value (Log-Lik.), and Akaike Information Criteria (AIC), and Kolmogorov Smirnov test. The maximum values of log-Likelihood and minimum values of AIC and KS statistic indicate that the model provides the best fit. The ML estimates along with goodness of fit measures are presented in Table 3.

Table 3. MLEs and model adequacy measures for ball bearing data.

Model		Estimates	LogLik.	AIC	KS
NKw	$\hat{\alpha}$	[2.0758, 2.2871]	[43.828, 41.623]	[-83.656, -79.246]	[0.600, 0.270]
	$\hat{\beta}$	[207.91, 164.94]			
NBD	$\hat{\alpha}$	[3.5523, 11.570]	[42.533, 39.262]	[-81.066, -74.525]	[0.510, 0.192]
	$\hat{\beta}$	[48.377, 103.14]			

From the above Table 3, it is tracked down that the new proposed neutrosophic distribution gives more efficient results than the neutrosophic beta distribution.

7. Conclusion

In this work, a new generalization of classical Kumaraswamy distribution is proposed for interval form of data sets. The proposed distribution is known as Neutrosophic Kumaraswamy distribution. Some mathematical properties of the NKw distribution are derived. The parameters are estimated using the maximum likelihood method. In the end, a real data set have been utilized to demonstrate the usefulness of the proposed distribution over Neutrosophic beta distribution. Numerical findings show that the NKw distribution provides better results than the Neutrosophic beta distribution.

Our future research will use the neutrosophic probability distributions NP of an event E defined as follows: NP(E)=(chance that the event E occurs (T), indeterminate-chance that the event E occurs (I), chance that the event E does not occur (F))

where T,I,F in [0, 1] and 0 <= T+I+F <= 3.

Therefore, we'll need to graph three curves for each neutrosophic probability distribution.

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The definite neutrosophic integrals and its applications

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Abstract: the purpose of this article is to study the definite neutrosophic integrals, where the neutrosophic integrals are defined, in addition, set of theories and properties related to them were discussed, also, applications of the definite neutrosophic integrals were introduced, such as area of neutrosophic curves, length of neutrosophic curve and volumes of neutrosophic revolution. Where detailed examples were given to clarify each case.

Keywords: definite neutrosophic integrals; area of neutrosophic curves; length of neutrosophic volumes of neutrosophic revolution.

1. Introduction

As an alternative to the existing logics, Smarandache proposed the Neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3-13]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist, he defined the standard form of neutrosophic complex number, and found root index $n \geq 2$ of a neutrosophic real and complex number [2-4], studying the concept of the Neutrosophic probability [3-5], the Neutrosophic statistics [4][6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1-8]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [9]. Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side represented with triangular neutrosophic numbers [10]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [11-12]. Y.Alhasan studied the concepts of neutrosophic complex numbers, the general exponential form of a neutrosophic complex, the neutrosophic integrals and integration methods, and the neutrosophic integrals by parts [7-14-18-20]. On the other hand, M.Abdel-Basset presented study in the science of neutrosophic about an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number [15].

Also, neutrosophic sets played an important role in applied science such as health care, industry, and optimization [16-17].. Smarandache, F, and Khalid, H are studied the neutrosophic precalculus and neutrosophic calculus (second enlarged edition)[19].

Integration is important in human life, and one of its most important applications is the calculation of area, size and arc length. In our reality we find things that cannot be precisely defined, and that contain an indeterminacy part.

Paper consists of 5 sections. In 1th section, provides an introduction, in which neutrosophic science review has given. In 2th section, some definitions and theories of the neutrosophic integrals and are discussed. The 3th section frames the definite neutrosophic integrals, in which set of theories and properties related to them were discussed. In 4th section, applications of the definite neutrosophic integrals were introduced. In 5th section, a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic integration by substitution method [18]

Definition2.1.1

$Let\ \ f: D_f \subseteq R \rightarrow R_f \cup \{I\}$, to evaluate $\int f(x)dx$

Put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

By substitution, we get:

$$\int f(x)dx = \int f\big(g(u)\big)\acute{g}(u)du$$

then we can directly integral it.

Theorme2.1.1:

If $\int f(x,I)dx = \varphi(x,I)\ \ then,$

$$\int f\big((a+bI)x+c+dI)\big)\,dx = \left(\frac{1}{a} - \frac{b}{a(a+b)}I\right)\varphi\big((a+bI)x+c+dI)\big) + C$$

where C is an indeterminate real constant, $a \neq 0\,, a \neq -b\ and\ b, c, d$ are real numbers, while I = indeterminacy.

Theorme2.1.2:

Let $f: D_f \subseteq R \rightarrow R_f \cup \{I\}$ then:

$$\int \frac{\acute{f}(x,I)}{f(x,I)}dx = ln|f(x,I)| + C$$

where C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while I = indeterminacy).

Theorme2.1.3:

Let $f: D_f \subseteq R \rightarrow R_f \cup \{I\}$, then:

$$\int \frac{\acute{f}(x,I)}{\sqrt{f(x,I)}}dx = 2\sqrt{f(x,I)} + C$$

where C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while I = indeterminacy).

Theorme2.1.4:

$f: D_f \subseteq R \rightarrow R_f \cup \{I\}$, then:

$$\int [f(x,I)]^n \acute{f}(x)\, dx = \frac{[f(x,I)]^{n+1}}{n+1} + C$$

Where n is any rational number. C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while I = indeterminacy).

2.2. Integrating products of neutrosophic trigonometric function [18]

I. $\int \sin^m(a+bI)x \cos^n(a+bI)x \ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following two cases:

- Case $n\ is$ odd:
  - Split of $\cos(a+bI)x$
  - Apply $\cos^2(a+bI)x = 1 - \sin^2(a+bI)x$
  - We substitution $u = \sin(a+bI)x$
- Case $m\ is$ odd:
  - Split of $\sin(a+bI)x$
  - Apply $\sin^2(a+bI)x = 1 - \cos^2(a+bI)x$
  - We substitution $u = \cos(a+bI)x$

II. $\int \tan^m(a+bI)x \sec^n(a+bI)x \ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following cases:

- Case $n\ is$ even:
  - Split of $\sec^2(a+bI)x$
  - Apply $\sec^2(a+bI)x = 1 + \tan^2(a+bI)x$
  - We substitution $u = \tan(a+bI)x$
- Case $m\ is$ odd:
  - Split of $\sec(a+bI)x \tan(a+bI)x$
  - Apply $\tan^2(a+bI)x = \sec^2(a+bI)x - 1$
  - We substitution $u = ses(a+bI)x$
- Case m even and n odd:
  - Apply $\tan^2(a+bI)x = \sec^2(a+bI)x - 1$
  - We substitution $u = \sec(a+bI)x$ or $u = \tan(a+bI)x$, depending on the case.

III. $\int \cot^m(a+bI)x \csc^n(a+bI)x \ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following cases:

- Case $n\ is$ even:
  - Split of $\csc^2(a+bI)x$
  - Apply $\csc^2(a+bI)x = 1 + \cot^2(a+bI)x$
  - We substitution $u = \cot(a+bI)x$
- Case $m\ is$ odd:
  - Split of $\csc(a+bI)x \cot(a+bI)x$
  - Apply $\cot^2(a+bI)x = \csc^2(a+bI)x - 1$
  - We substitution $u = \csc(a+bI)x$

- Case m even and n odd:
  - Apply $\cot^2(a+bI)x = \csc^2(a+bI)x - 1$
  - We substitution $u = \csc(a+bI)x$ or $u = \cot(a+bI)x$, depending on the case.

2.3. Neutrosophic trigonometric identities [18]

1) $sin(a+bI)x\cos(c+dI)\,x = \frac{1}{2}\,[\sin(a+bI+c+dI)x + \sin(a+bI-c-dI)x]$

2) $cos(a+bI)x\sin(c+dI)\,x = \frac{1}{2}\,[\sin(a+bI+c+dI)x - \sin(a+bI-c-dI)\,x]$

3) $cos(a+bI)x\cos(c+dI)\,x = \frac{1}{2}\,[\cos(a+bI+c+dI)x + \cos(a+bI-c-dI)x]$

4) $sin(a+bI)x\sin(c+dI)\,x = \frac{-1}{2}\,[\cos(a+bI+c+dI)x - \cos(a+bI-c-dI)x]$

Where $a \neq c\ (not\ zero)\ and\ b,d$ are real numbers, while I = indeterminacy.

3. The definite neutrosophic integrals

We will choose $I \in]0,1[$, because the undefined(indeterminacy) part in the case of the drawing is usually located in]0,1[. Look at pp.20-22 [19]

Theorem 3.1 (Fundamental theorem of neutrosophic integral calculus)
Let be $f(x,I)$ a continuous function defined in the closed interval $[a+a_0I, b+b_0I]$, and let $F(x,I)$ be the anti-derivative of $f(x,I)$, that is $\int f(x,I)dx = F(x,I)$. Then:

$$\int_{a+a_0I}^{b+b_0I} f(x,I)dx = F(\,b+b_0I) - F(a+a_0I)$$

Where a, a_0, b, b_0 are real number, I represent indeterminacy and $I \in]0,1[$.

Example3.1:

1) $$\int_{1+2I}^{3-5I} (2x+7I)dx = [x^2+7Ix]_{1+2I}^{3-5I}$$

$$= [(3-5I)^2 + 7I(3-5I)] - [(1+2I)^2 + 7I(1+2I)] = 8 - 38I$$

2) $$\int_{0}^{\pi+3I} cos(x-3I)dx = [sin(x-3I)]_0^{\pi+3I}$$

$$= [sin(\pi+3I-3I)] - [sin(-3I)] = sin(3I)$$

3) $$\int_{2I}^{3+I} 2x(x^2+5I)^2dx = \left[\frac{(x^2+5I)^3}{3}\right]_{2I}^{3+I}$$

$$= \left[\frac{((3+I)^2+5I)^3}{3}\right] - \left[\frac{81I}{3}\right] = \frac{81+279I}{3} = 27+93I$$

4) $\int_{4}^{9+7I} \frac{1}{2\sqrt{x}} dx = \left[\sqrt{x}\right]_{4}^{9+7I}$

$$= \left[\sqrt{9+7I}\right] - [2] \qquad (*)$$

Let's find $\sqrt{9+7I}$

$$\sqrt{9+7I} = \alpha + \beta I$$
$$9+7I = \alpha^2 + 2\alpha\beta I + \beta^2 I$$

$$9+7I = \alpha^2 + (2\alpha\beta + \beta^2)I$$

then:

$$\begin{cases} \alpha^2 = 9 \\ 2\alpha\beta + \beta^2 = 7 \end{cases}$$

$$\begin{cases} \alpha = \pm 3 \\ \beta^2 + 2\alpha\beta - 7 = 0 \end{cases}$$

Find β:

- When $\alpha = 3 \quad \Rightarrow \beta^2 + 6\beta - 7 = 0$

$$(\beta+7)(\beta-1) = 0 \quad \Rightarrow \beta = -7, \beta = 1$$

$$(3,-7), (3,1)$$

- When $\alpha = -3 \quad \Rightarrow \beta^2 - 6\beta - 7 = 0$

$$(\beta-7)(\beta+1) = 0 \quad \Rightarrow \beta = 7, \beta = -1$$

$$(-3,7), (-3,-1)$$

$$(\alpha,\beta) = (3,-7), (3,1), (-3,7), (-3,-1)$$

$$\sqrt{9+7I} = 3-7I \; or \; 3+I \; or -3+7I \; or -3-I$$

By substitution in (*), we get the following cases:

$$\int_{4-3I}^{9+7I} \frac{1}{2\sqrt{x}} dx = \left[\sqrt{x}\right]_{4-3I}^{9+7I}$$

$$= \left[\sqrt{9+7I}\right] - [2] = 3-7I-2 = 1-7I$$

$$or \; = \left[\sqrt{9+7I}\right] - \left[\sqrt{4-3I}\right] = 3+I-2 = 1+I$$

$$or \; = \left[\sqrt{9+7I}\right] - \left[\sqrt{4-3I}\right] = -3+7I-2 = -5+7I$$

$$or \; = \left[\sqrt{9+7I}\right] - \left[\sqrt{4-3I}\right] = -3-I-2 = -5-I$$

Theorem 3.2 (The mean- value theorem of neutrosophic integral calculus_ part I)

We say that $f(x,I)$ has an anti- derivative on an interval, if $f(x,I)$ is continuous on that interval, then. In specific, if $a+a_0I$ is any point in the interval, then the function $f(x,I)$ defined by:

$$1)\ \frac{d}{dx}\left[\int_{a+a_0I}^{x} f(t,I)dt\right] = f(x,I)$$

$$2)\ \frac{d}{dx}\left[\int_{x}^{a+a_0I} f(t,I)dt\right] = -f(x,I)$$

Example3.2:

$$1)\ \frac{d}{dx}\left[\int_{3I}^{x} (t^2+5I)dt\right] = x^2+5I$$

$$2)\ \frac{d}{dx}\left[\int_{\pi+\frac{\pi}{2}I}^{x} \frac{\sin(t+3I)}{t}dt\right] = \frac{\sin(x+3I)}{x}$$

$$3)\ \frac{d}{dx}\left[\int_{x}^{5-3I} (2It^2+4It)dt\right] = 2Ix^2+4Ix$$

Remarks 3.1:

$$\mathbf{1)}\ \frac{d}{dx}\left[\int_{a+a_0I}^{g(x,I)} f(t,I)dt\right] = f\big(g(x,I)\big)\acute{g}(x,I)$$

Proof:

$$\frac{d}{dx}\left[\int_{a+a_0I}^{g(x,I)} f(t,I)dt\right] = \frac{d}{dx}\big[F\big(g(x,I)\big)\big]$$

$$= \acute{F}\big(g(x,I)\big)\acute{g}(x,I)$$

$$= f\big(g(x,I)\big)\acute{g}(x,I)$$

$$\mathbf{2)}\ \frac{d}{dx}\left[\int_{g(x,I)}^{a+a_0I} f(t,I)dt\right] = -f\big(g(x,I)\big)\acute{g}(x,I)$$

Proof:

$$\frac{d}{dx}\left[\int_{g(x,I)}^{a+a_0I} f(t,I)dt\right] = \frac{d}{dx}\left[-F\big(g(x,I)\big)\right]$$

$$= -\acute{F}\big(g(x,I)\big)\acute{g}(x,I)$$

$$= -f\big(g(x,I)\big)\acute{g}(x,I)$$

3) $\frac{d}{dx}\left[\int_{g_1(x,I)}^{g_2(x,I)} f(t,I)dt\right] = f(g_2(x,I))\,\acute{g}_2(x,I) - f(g_1(x,I))\,\acute{g}_1(x,I)$

Proof:

$$\frac{d}{dx}\left[\int_{g_1(x,I)}^{g_2(x,I)} f(t,I)dt\right] = \frac{d}{dx}\left[\int_{g_1(x,I)}^{0+0I} f(t,I)dt + \int_{0+0I}^{g_2(x,I)} f(t,I)dt\right]$$

$$= f(g_2(x,I))\,\acute{g}_2(x,I) - f(g_1(x,I))\,\acute{g}_1(x,I)$$

Example3.3:

1) $\frac{d}{dx}\left[\int_{1+I}^{sin(x+2I)} (3I + t^2)dt\right] = (3I + sin^2(x+2I))cos(x+2I)$

2) $\frac{d}{dx}\left[\int_{4+2I}^{\sqrt{3x+7I}} (t - 2I)dt\right] = (\sqrt{3x+7I} - 2I)\frac{3}{2\sqrt{3x+7I}}$

3) $\frac{d}{dx}\left[\int_{tan(2x+4I)}^{7-6I} \frac{t^2}{1+t^2}dt\right] = -\frac{tan^2(2x+4I)}{1+tan^2(2x+4I)}tan^2(2x+4I) = -tan^2(2x+4I)$

4) $\frac{d}{dx}\left[\int_{3x+I}^{x^2+2I} \frac{4-5I}{t+2I}dt\right] = \frac{4-5I}{x^2+2I+2I}(2x) - \frac{4-5I}{3x+I+2I}(3)$

$$= \frac{(8-10I)x}{x^2+4I} - \frac{12-15I}{3x+3I}$$

Theorem 3.3 (The mean- value theorem of neutrosophic integral calculus_ part II)

If $f(x,I)$ is continuous on a closed interval $[a + a_0I, b + b_0I]$, then there is at least one point $x^* = x_0 + x_1I$ in $[a + a_0I, b + b_0I]$ such that:

$$\int_{a+a_0I}^{b+b_0I} f(x,I)dx = f(x^*,I)\big(b + b_0I - (a + a_0I)\big)$$

Where x_0, x_1 are real numbers, I represent indeterminacy and $I \in]0,1[$

Example3.4:

Findx^* that satisfy The Mean-Value Theorem of Integral Calculus for $f(x,I) = 2x + 3I$ on the interval $[1 + 2I, 3 + 4I]$.

Solution:

$$\int_{a+a_0I}^{b+b_0I} f(x,I)dx = f(x^*,I)\big(b + b_0I - (a + a_0I)\big)$$

$$\int_{1+2I}^{3+4I} (2x + 3I)dx = f(x^*,I)(2 + 2I)$$

$$[x^2 + 3Ix]_{1+2I}^{3+4I} = (2x^* + 3I)(2 + 2I)$$

$$8 + 44I = (2x^* + 3I)(2 + 2I)$$

$$2x^* + 3I = \frac{8 + 44I}{2 + 2I}$$

$$2x^* + 3I = \frac{4 + 22I}{1 + I}$$

$$2x^* + 3I = 4 + 9I$$

$$2x^* = 4 + 6I$$

$$x^* = 2 + 3I \quad \in [1 + 2I, 3 + 4I]$$

Example3.4:

Find x^* that satisfy The Mean-Value Theorem of Integral Calculus for $f(x,I) = \sqrt{x}$ on the interval $[0 + 0I, 3 + 2I]$.

Solution:

$$\int_{a+a_0I}^{b+b_0I} f(x,I)dx = f(x^*,I)\big(b + b_0I - (a + a_0I)\big)$$

$$\int_{0}^{3+2I} \sqrt{x}\,dx = (3 + 2I)\sqrt{x^*}$$

$$\left[\frac{2}{3}x\sqrt{x}\right]_0^{3+2I} = (3+2I)\sqrt{x^*}$$

$$\frac{2}{3}(3+2I)\sqrt{3+2I} = (3+2I)\sqrt{x^*}$$

$$\sqrt{x^*} = \frac{2\sqrt{3+2I}}{3}$$

By squared, we get:

$$x^* = \frac{4(3+2I)}{9} = \frac{12+8I}{9}$$

$$x^* = \frac{4}{3} + \frac{8}{9}I \quad \in [0+0I, 3+2I]$$

If we take several values of I in the $]0,1[$, we find:

$\boldsymbol{I}$	$[\mathbf{0}+\mathbf{0}\boldsymbol{I}, \mathbf{3}+\mathbf{2}\boldsymbol{I}]$	$\boldsymbol{x}^*$	$\boldsymbol{x}^* \in [\mathbf{0}+\mathbf{0}\boldsymbol{I}, \mathbf{3}+\mathbf{2}\boldsymbol{I}]$
0.1	$[0, 3.2]$	1.43	Satisfied
0.3	$[0, 3.6]$	1.61	Satisfied
0.5	$[0, 4]$	1.78	Satisfied

3.1 Properties of definite neutrosophic integrals.

Let $f: D_f \subseteq R \to R_f \cup \{I\}$, and $g: D_g \subseteq R \to R_f \cup \{I\}$ then:

1) $$\int_{a+a_0I}^{b+b_0I} f(x,I)dx = \int_{a+a_0I}^{b+b_0I} f(t,I)dt$$

2) $$\int_{a+a_0I}^{b+b_0I} f(x,I)dx = \int_{a+a_0I}^{c+c_0I} f(x,I)dx + \int_{c+c_0I}^{b+b_0I} f(x,I)dx \quad ; \quad a+a_0I \le c+c_0I \le b+b_0I$$

3) $$\int_{a+a_0I}^{a+a_0I} f(x,I)dx = 0$$

4) $$\int_{a+a_0I}^{b+b_0I} f(x,I)dx = -\int_{b+b_0I}^{a+a_0I} f(x,I)dx$$

5) $$\int_{a+a_0I}^{b+b_0I} (c+c_0I)f(x,I)dx = (c+c_0I)\int_{a+a_0I}^{b+b_0I} f(x,I)dx$$

6) $$\int_{a+a_0I}^{b+b_0I} [f(x,I) \pm g(x.I)]dx = \int_{a+a_0I}^{b+b_0I} f(x,I)dx \pm \int_{a+a_0I}^{b+b_0I} g(x,I)dx$$

7) $$\int_{-(a+a_0I)}^{a+a_0I} f(x,I)dx = \begin{cases} 2\int_{0}^{a+a_0I} f(x,I)dx & ; if\ f(x,I)\ is\ even\ function \\ 0 & ; if\ f(x,I)\ is\ odd\ function \end{cases}$$

Example3.1.1:

1) $$\int_{5+6I}^{5+6I} (x^4 + 2Ix - 4I)dx = 0$$

2) $$\int_{5+6I}^{9+2I} 5I \sin^5 x\, cos^4 x\, dx$$

Let $f(x,I) = 5I \sin^5 x\, cos^4 x$, then:

$f(-x,I) = 5I \sin^5(-x) cos^4(-x)$

$$= 5I(\sin(-\mathrm{x}))^5(\cos(-x))^4 = -5I \sin^5 x\, cos^4 x$$

$$= -f(x,I)$$

Thus $f(x,I)$ is an odd function and so by property 7, we get:

$$\int_{5+6I}^{9+2I} 5I \sin^5 x\, cos^4 x\, dx = 0$$

4. Applications of the definite neutrosophic integrals

4.1 The area under neutrosophic curves

Theorem 4.1.1

Let $f(x,I)$ be a continuous function defined in the interval $[a + a_0I, b + b_0I]$. Then the area of the region below the neutrosophic curve of $f(x,I)$, above the $x-\ axis$, between $x = a + a_0I\ and$ $x = b + b_0I$ $(b > a)$, is given by formula:

$$A = \int_{a+a_0I}^{b+b_0I} |f(x,I)|\, dx$$

Where a, a_0, b, b_0 are real numbers, I represent indeterminacy and $I \in]0,1[$

Theorem 4.1.2

Let $f(y,I)$ be a continuous function defined in the interval $[c+c_0I, d+d_0I]$. Then the area of the region below the neutrosophic curve of $f(y,I)$, above the $y-$ $axis$, between $y = c+c_0I$ and $y = d+d_0I$ $(d > c)$, is given by formula:

$$A = \int_{c+c_0I}^{d+d_0I} |f(y,I)|\,dy$$

Where c, c_0, d, d_0 are real numbers, I represent indeterminacy and $I \in]0,1[$

Example4.1.1:

Find the area of the region bounded by the line $f(x,I) = x+4-3I$, the $x-axis$ and the lines $x = 2+3I$ and $x = 4+I$.

Solution:

$$A = \int_{a+a_0I}^{b+b_0I} |f(x,I)|\,dx = \int_{2+3I}^{4+I} |x+4-3I|\,dx$$

$$x+4-3I > 0 \ \ on\ [2+3I, 4+I] \ \text{ for } I \in]0,1[$$

$$\Longrightarrow \qquad A = \int_{2+3I}^{4+I} (x+4-3I)dx = \left[\frac{x^2}{2} + (4-3I)x\right]_{2+3I}^{4+I} = 8-4I$$

Clearly that: $8-4I > 0$ for $I \in]0,1[$

4.2 Area between two neutrosophic curves

Theorem 4.2.1 (Area between two neutrosophic curves (attributed to $x-axis$))

The area A of the region bounded by the curves $f(x,I), g(x,I)$, and the lines $x = a+a_0I$ and $x = b+b_0I$ $(b > a)$, where f and g are continuous and $f(x,I) \geq g(x,I)$ for all x in $[a+a_0I, b+b_0I]$, is given by formula:

$$A = \int_{a+a_0I}^{b+b_0I} [f(x,I) - g(x.I)]dx$$

Where a, a_0, b, b_0 are real numbers, I represent indeterminacy and $I \in]0,1[$

Theorem 4.2.2 (Area between two neutrosophic curves (attributed to $y-axis$))

The area A of the region bounded by the curves $f(y,I), g(y,I)$, and the lines $y = c+c_0I$ and $y = d+d_0I$ $(d > c)$, where f and g are continuous and $f(y,I) \geq g(y,I)$ for all x in $[c+c_0I,\ d+d_0I]$, is given by formula:

$$A = \int_{c+c_0I}^{d+d_0I} [f(y,I) - g(y.I)]dy$$

Where c, c_0, d, d_0 are real numbers, I represent indeterminacy and $I \in]0,1[$

Example4.2.1:

Evaluate the area of the region bounded by $y = e^{x+7I}$, $y = x - 3I$, and the lines $x = 0$, $x = 1 + I$

Solution:

$y = e^{x+7I} > y = x - 3I$ on $[0, 1 + I]$ for $I \in]0,1[$, then:

$$A = \int_{0}^{1+I} [e^{x+7I} - x + 3I]dx = \left[e^{x+7I} - \frac{x^2}{2} + 3Ix\right]_0^{1+I}$$

$$= e^{1+8I} - \frac{1}{2} + \frac{13I}{2} - e^{7I}$$

Clearly that $e^{1+8I} - \frac{1}{2} + \frac{13I}{2} - e^{7I} > 0$ for $I \in]0,1[$

Example4.2.1:

Evaluate the area of the region bounded by $x = y^2$, $x = (-2 - I)x + 2I$, and the lines
N $x = -2 + I$, $x = -2I$

Solution:

$x = y^2 \geq x = (-2 - I)x + 2I$ on $[-2 + I, -2I]$ for $I \in]0,1[$, then:

$$A = \int_{-2+\mathrm{I}}^{-2I} [y^2 + (2 + I)x - 2I]dy = \left[\frac{y^3}{3} + \frac{(2 + I)}{2}y^2 - 2Iy\right]_{-2+I}^{-2I}$$

$$= \left[\frac{(-2I)^3}{3} + \frac{(2 + I)}{2}(-2I)^2 - 2I(-2I)\right] - \left[\frac{(-2 + I)^3}{3} + \frac{(2 + I)}{2}(-2 + I)^2 - 2I(-2 + I)\right]$$

$$= \frac{-4}{3} + \frac{68}{15}I$$

Clearly that: $\frac{-4}{3} + \frac{68}{15}I > 0$ for $I \in]0,1[$

4.3 Length of neutrosophic curve

Definition 4.3.1

If $y = f(x, I)$ is a smooth curve on the interval $[a + a_0I, b + b_0I]$, then the arc length L of this curve over $[a + a_0I, b + b_0I]$ is defined as:

$$L = \int_{a+a_0I}^{\mathrm{b}+b_0I} \sqrt{1 + [\acute{f}(x, I)]^2}\ dx$$

Where a, a_0, b, b_0 are real numbers, I represent indeterminacy and $I \in]0,1[$

Definition 4.3.2

If $x = g(y, I)$ is a smooth curve on the interval $[c + c_0 I,\ d + d_0 I]$, then the arc length L of this curve over $[c + c_0 I,\ d + d_0 I]$is defined as:

$$L = \int_{c+c_0 I}^{d+d_0 I} \sqrt{1 + [\acute{g}(y, I)]^2}\, dy$$

Example4.3.1:

Find the arc length of the curve of $y = f(x, I) = ln(secx)$ on the interval $\left[0,\ \frac{\pi}{4} + 3I\right]$.

Solution:

$f(x, I) = ln(sec(x - 3I)) \implies \acute{f}(x, I) = tan(x - 3I)$

$$L = \int_{a+a_0 I}^{b+b_0 I} \sqrt{1 + \left[\acute{f}(x, I)\right]^2}\ dx$$

$$L = \int_{0}^{\frac{\pi}{4}+3I} \sqrt{1 + tan^2(x - 3I)}\ dx$$

$$= \int_{0}^{\frac{\pi}{4}+3I} \sqrt{sec^2(x - 3I)}\ dx$$

$$= \int_{0}^{\frac{\pi}{4}+3I} \sec(x - 3I)\ dx = [ln|sec(x - 3I) + \tan(x - 3I)|]_0^{\frac{\pi}{4}+3I}$$

$$= \left[ln\left|sec\left(\frac{\pi}{4}\right) + \tan(\frac{\pi}{4})\right|\right] - [ln|sec(-3I) + \tan(-3I)|]$$

$$= ln\left|\sqrt{2} + 1\right| - [ln|sec(-3I) + \tan(-3I)|]$$

$$= ln\left|\sqrt{2} + 1\right| - ln|sec(3I) + \tan(3I)|$$

Clearly that: $ln\left|\sqrt{2} + 1\right| - ln|sec(3I) + \tan(3I)| > 0$ for $I \in]0,1[$

4.4 Volumes of neutrosophic revolution

Definition 4.4.1

Suppose that $f(x, I) \geq 0$ and it is continuous on the interval $[a + a_0 I, b + b_0 I]$, the volume of the resulting solid of revolution the region under the curve $y = f(x, I)$ for the interval $[a + a_0 I, b + b_0 I]$ about the $x - axis$ is given by:

$$V = \int_{a+a_0 I}^{b+b_0 I} \pi[f(x, I)]^2\, dx$$

Where a, a_0, b, b_0 are real numbers, I represent indeterminacy and $I \in]0,1[$

Definition 4.4.2

Suppose that $g(y, I) \geq 0$ and it is continuous on the interval $[c + c_0 I,\ d + d_0 I]$, the volume of the resulting solid of revolution the region under the curve $x = g(y, I)$ for the interval $[c + c_0 I,\ d + d_0 I]$ about the $y-axis$ is given by:

$$V = \int_{c+c_0 I}^{d+d_0 I} \pi[g(y, I)]^2\, dy$$

Example4.4.1:

Find the volume of the solid resulting from rotating the region bounded by the curves $y = f(x, I) = \sqrt{x + 2 + 3I}$ from $x = 0\ to\ x = 4 + 5I$ about the $x - axis$.

Solution:

$$V = \int_{a+a_0 I}^{\mathrm{b}+b_0 I} \pi[f(x, I)]^2\, dx = \int_{0}^{4+5I} \pi\left[\sqrt{x + 2 + 3I}\,\right]^2 dx$$

$$= \int_{0}^{4+5I} \pi[x + 2 + 3I\,]\, dx = \pi\left[\frac{x^2}{2} + (2 + 3I)x\right]_0^{4+5I}$$

$$= \pi\left[\frac{(4 + 5I)^2}{2} + (2 + 3I)(4 + 5I)\ \right] - [0\,]$$

$$= \left(16 + \frac{139}{2} I\right)\pi$$

Example4.4.2:

Find the volume of the solid resulting from rotating the region bounded by the curves $x = g(y, I) = \sqrt{4 + 6I - y}$ from $y = 1 + \mathrm{I}\ to\ y = 4 + 4I$ about the $y - axis$.

Solution:

$$V = \int_{c+c_0 I}^{d+d_0 I} \pi[g(y, I)]^2\, dy = \int_{1+I}^{4+4I} \pi\left[\sqrt{4 + 6I - y}\right]^2 dy$$

$$= \int_{1+I}^{4+4I} \pi[4 + 6I - y]dy = \pi\left[(4 + 6I)y - \frac{y^2}{2}\right]_{1+I}^{4+4I}$$

$$= \pi\left[(4 + 6I)(4 + 4I) - \frac{(4 + 4I)^2}{2}\right] - [0\,]$$

$$= \left(\frac{9}{2} + \frac{51}{2} I\right)\pi$$

Definition 4.4.3

Suppose that $f(x,I), g(x,I)$ are continuous and non-negative on the interval $[a + a_0 I, b + b_0 I]$, and $f(x,I) \geq g(x,I)$ for all x in the interval $[a + a_0 I, b + b_0 I]$, the volume of the resulting solid of revolution the region bounded between tow the curves $f(x,I), g(x,I)$ for the interval $[a + a_0 I, b + b_0 I]$ about the $x - axis$ is given by:

$$V = \int_{a+a_0 I}^{b+b_0 I} \pi([f(x,I)]^2 - [g(x,I)]^2)\, dx$$

Definition 4.4.4

Suppose that $w(y,I), v(y,I)$ are continuous and non-negative on the interval $[c + c_0 I,\ d + d_0 I]$, and $w(y,I) \geq v(y,I)$ for all x in the interval $[c + c_0 I,\ d + d_0 I]$, the volume of the resulting solid of revolution the region bounded between tow the curves $w(y,I), v(y,I)$ for the interval $[c + c_0 I,\ d + d_0 I]$ about the $y - axis$ is given by:

$$V = \int_{c+c_0 I}^{d+d_0 I} \pi([w(y,I)]^2 - [v(y,I)]^2)\, dy$$

Example4.4.3:

Find the volume of the solid resulting from rotating the region bounded between tow the curves $f(x,I) = x^2 + 3I$ *and* $g(x,I) = 3I + x$ from $x = 1 + I$ *to* $x = 4 + 2I$ about the $x - axis$.

Solution:

$f(x,I) = x^2 + 3I > g(x,I) = 3I + x$ *on* $[1 + I, 4 + 2I]$ for $I \in]0,1[$, then:

$$V = \int_{a+a_0 I}^{b+b_0 I} \pi([f(x,I)]^2 - [g(x,I)]^2)\, dx$$

$$= \int_{1+I}^{4+2I} \pi([x^2 + 3I]^2 - [3I + x]^2)\, dx$$

$$= \int_{1+I}^{4+2I} \pi([x^4 + 6Ix^2 + 9I] - [9I + 6Ix + x^2])\, dx$$

$$= \int_{1+I}^{4+2I} \pi(x^4 + (6I - 1)x^2 - 6Ix)\, dx$$

$$= \left[\pi\left(\frac{x^5}{5} + (6I-1)\frac{x^3}{3} - 3Ix^2\right)\right]_{1+I}^{4+2I}$$

$$= \pi\left[\left(\frac{(4+2I)^5}{5} + (6I-1)\frac{(4+2I)^3}{3} - 3I(4+2I)^2\right) - \left(\frac{(1+I)^5}{5} + (6I-1)\frac{(1+I)^3}{3} - 3I(1+I)^2\right)\right]$$

$$= \pi\left[\left(\frac{2752}{15} + \frac{3424}{3}I\right) - \left(\frac{-2}{15} - \frac{32}{15}I\right)\right]$$

$$= \pi\left(\frac{2754}{15} + \frac{3456}{3}I\right)$$

5. Conclusions

This paper is an extension of the papers I presented in the field of neutrosophic integrals. Integrals are important in our life, as they facilitate many mathematical operations in our reality, and this is what led us to study the definite neutrosophic integrals, and its applications, the most important of which are area of neutrosophic curves, length of neutrosophic curve and volumes of neutrosophic revolution. In addition, this paper is considered important in continuing the study of neutrosophic integrals.

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Symmetric Neutrosophic Cross Entropy Based Fault Recognition of Turbine

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Abstract: This study introduces a novel fault recognition methodology for turbine faults through symmetric trigonometric fuzzy and neutrosophic cross entropy measures (FCEM and NCEM) consequently. After knowing the nethermost (lowest) and uppermost (highest) energy bounds of each real fault conditions, the energy interval ranges are constructed and then transformed into the form of single valued neutrosophic (SN) sets. Thereafter, the proposed symmetric trigonometric cross -entropy measures are deployed to recognize faults of turbine. The nethermost FCEM and NCEM values between familiar and unfamiliar fault conditions indicates that the unfamiliar fault condition is closer to the familiar one. The applicability of the proposed methodology is validated by taking into consideration the example of fault diagnosis of turbine. The repercussions of this study yield that the proposed symmetric trigonometric FCEM and NCEM cannot only recognize optimal fault, they can also provide meaningful and remarkable fault information. A comparison of the underlying FCEM and NCEM (based on SN sets) with the enduring cosine measures (based on vague sets) conclude that the latter sets may hide some fruitful fault information., when experimented under sensitive and intuitive criteria and thus resulting an incomplete fault evaluation criterion.

Keywords: Fuzzy Sets, Neutrosophic Sets, Cross Entropy, Fault Diagnosis, Turbine.

1. Introduction

A Turbine generator is an important mechanical device and is being widely used for converting heat energy of steam into electrical energy in thermal plants. It is a natural process for a huge steam turbine generator to have vibrations produced by many factors such as misalignment, heating of rotor element and lubricant oil etc. When a fault occurs, it not only damages the generator set but also disturbs the continuous and safe operation of internal machinery. Careful analysis of vibration signals of the generator set can reveal some useful evaluation information, which in turn, can avoid catastrophic mechanical disorder as well as huge economic losses. It is, therefore, necessary to reckon and fix the actual cause of the fault as early as possible. Over the past few years, researchers have developed some fault recognition methods, that works on cross entropy measures, for quantifying the non-linear relationship between unfamiliar and familiar turbine fault conditions. Recently, Ren et al. **[1]** extracted the fuzzy entropy of a series of mode components for observing the complexity of working condition and thereafter improved the fault diagnosis accuracy of wind turbine. Lilian and Ye **[2,3]** modified the vague sets of enduring similarity measure and observed the non-linear and complex relationship between vibration signals and various fault conditions. Recently, Lilian Shi **[4]** constructed simplified neutrosophic sets by exercising the enduring Karl Pearson's coefficient of correlation and combined it with wavelet packet transforms for reckoning faults of rolling bearing. Tian et al. **[5]** established a systematic and comprehensive approach based on permutation entropy for automatic testimony of bearing defects under and time varying conditions. Recently, Martinez et al. **[6]** utilized Shannon's Information entropy for quantifying and extracting the fault information available in the vibration signals of broken bars in induction motors. Under multi fault severities and time-varied complexities, Fu et al. **[7]** combined approximate entropy and wavelet packet transforms for decomposing deterministic and stochastic power signals. Zhao et al. **[8]** extended the existing wavelet entropy to instantaneous wavelet singular entropy for extracting the sensor fault characteristics of a gas turbine. Zhao et al. **[8]** deployed multiscale fuzzy distribution entropy for understanding the nonlinear and non-consistent fault characteristics signals. Zhang et al. **[9]** understood the irregularity and complexity of vibration signals by extending Shannon's entropy to wavelet entropy and concluded that whenever wavelet entropy increases, the tightness conditions of bolted joints diverge to looseness. Leite et al. **[10]** deployed Shannon's entropy and Jenson-Renyi's directed divergence (JRDD) for constructing discrete probability mass function of a known time waveform and utilized it for identifying faults of rolling bearing elements. Many times, the approaches based on variants of Shannon's probabilistic entropy and JRDD have been found inefficacious in providing semantic output due to the difficulty in transforming fault characteristics of cumbersome signals. Hence, the above-mentioned fault diagnosis techniques may not be capable for extracting remarkable and accurate fault information from faults conditions of turbine. This reinforces the exigency for an effective fault diagnosis procedure which can make precise and fruitful analysis for a fault that occurs in turbine generator set because the same symptom of a fault may have variety of fault causations. A single valued neutrosophic (SN) set **[11]** is mainly portrayed by truth, indeterminacy and falsity membership functions and inherits its indeterminacy into the form of truth and falsity values. Kumar et al. **[12,13]** effectively identified bearing faults by decomposing vibrational signals into eight different frequency modes under neutrosophic environment. However, the enduring research on neutrosophic sets and systems have mainly dealt with its theoretical or asymmetrical aspects and ignores those engineering problems which may exhibit symmetrical phenomenon or return inconsequential results under neutrosophic treatments. Neutrosophic cross entropy approach has been found significantly efficacious in tackling complex engineering problems under multi-faults severities. Till so far, no symmetric neutrosophic cross entropy measure has been developed and utilized for improving fault identification accuracy of turbine. Subsequently, an effort is accomplished in this direction which can overcome the above-mentioned shortcomings and effectively diagnose the faults of a huge steam turbine generator set. Moreover, the underlying symmetric trigonometric cross entropy measure of neutrosophic sets provides meaningful fault information whereas the enduring similarity measure may hide some fruitful fault information and thus resulting an ambiguous phenomenon. In addition,

Section 2 deals with pre-requisites of neutrosophic entropy measure, needed for the successive growth of the proposed research. **Section 3** is devoted to establish a novel symmetric trigonometric FCEM whereas **Section 4** expands the outcomes of **Section 3** to another novel symmetric cross entropy measure, hinged on two single valued neutrosophic sets. **Section 5** inaugurates the proposed neutrosophic cross entropy-based fault recognition methodology, the applicability and remarkability of which are exemplified in **Section 6.** Finally, **Section 6** contributes the concrete conclusions extracted from this study.

2. Preliminaries: -

This section deals with the introduction of some familiar apprehensions as follows:

Def. 2.1 SN Entropy Measure [11-13] A SN set, in any universal set X with its generic elements $x_1, x_2, ..., x_n$, is an entity of the form: $A = \left(< x_i, \mu_A(x_i), i_A(x_i), f_A(x_i) > | x_i \in X\right)$ where each $\mu_A(x_i): X \to [0,1], i_A(x_i): X \to [0,1], f_A(x_i): X \to [0,1]$ satisfy $0^- \le \mu_A(x_i) + i_A(x_i) + f_A(x_i) \le 3^+$. Suppose $T(X)$ represents the collection of all SN sets in X , Then $T_N(A): T(X) \to R$ is called as SN entropy measure if

$(i) T_N(A) \geq 0 \forall 0 \leq \mu_A(x_i), i_A(x_i), f_A(x_i) \in [0,1]$ with equality if either $\mu_A(x_i)=1, i_A(x_i)=0,$ $f_A(x_i)=0$ or $\mu_A(x_i)=0, i_A(x_i)=0, f_A(x_i)=1$. $(ii) T_N(A^c)=T_N(A)$. If A^c denotes the complement of A, then $A^c = (< x_i, f_A(x_i), 1-i_A(x_i), \mu_A(x_i) > | x_i \in X)$.

(iii) $T_N(A)$ possesses concavity property for each $\mu_A(x_i), i_A(x_i), f_A(x_i)$.

(iv) $T_N(A)$ admits its maximum value which arises when $\mu_A(x_i)=i_A(x_i)=f_A(x_i)=\frac{1}{2}$.

3. A Novel Symmetric Trigonometric FCEM (Fuzzy Cross Entropy Measure)

We first establish the following **Theorem 3.1,** the out coming of which will be a backbone for the proposed symmetric trigonometric fuzzy cross entropy measure, hinged on two fuzzy sets (**Theorem 3.2).**

Theorem.3.1 Set $T_0=\sqrt{\mu_A(x_i)}, T_1=\sqrt{1-\mu_A(x_i)}, T_2=\sqrt{\mu_A(x_i)(1-\mu_A(x_i))}$. Let $A \subseteq X$ be any fuzzy set **[14]**.Then

$$T_{FS}(A)=\sum_{i=1}^{n}\left[\tan\left(\frac{3\sqrt{2}}{3\sqrt{2}+2}\right)-\tan\left(\frac{3\sqrt{2}}{3\sqrt{2}+2(T_0+T_1)-\sqrt{2}T_2}\right)\right] \quad \text{... (1)}$$

represents a valid measure of fuzzy entropy with $Max.T_{FS}(A)=\left(\tan\frac{3\sqrt{2}}{3\sqrt{2}+2}-\tan\frac{2}{3}\right)n$ and minimum value as zero.

Proof (i) The expressions denoted by T_0, T_1, T_2 are non0negative because $\mu_A(x_i) \in [0,1]$. This justifies that $T_F(A) \geq 0 \ \forall \mu_A(x_i) \in [0,1]$ with equality $T_0=0, T_1=1, T_2=1$ or $T_0=1, T_1=0, T_2=0$. In other words, $T_F(A)$ vanishes whenever $\mu_A(x_i)=0 \text{ or } 1$.

(ii) If we replace $\mu_A(x_i)$ with its counterpart $1-\mu_A(x_i)$, then T_0 changes to T_1, $T_1 \to T_0, T_2 \to T_2$, which means $T_F(A^c)=T_F(A)$.

(iii) **Concavity:** To establish the concavity of $T_{FS}(A)$, differentiating **(1)** partially with respect to $\mu_A(x_i)$ to get

$$\frac{\partial T_{FS}(A)}{\partial \mu_A(x_i)}=\frac{3\sqrt{2}\left(\frac{T_1^2-T_0^2}{T_0T_1(T_0+T_1)}-\frac{1-2T_0^2}{\sqrt{2}T_2}\right)\sec^2\frac{3\sqrt{2}}{3\sqrt{2}+2(T_0+T_1)-\sqrt{2}T_2}}{\left(3\sqrt{2}+2(T_0+T_1)-\sqrt{2}T_2\right)^2} \quad \ldots (2)$$

It is informative to point out that $T_1^2-T_0^2=1-2\mu_A(x_i)=1-2T_0^2$. With this information in hand, the above equality simplifies to

$$\frac{\partial T_{FS}(A)}{\partial \mu_A(x_i)}=\frac{3\sqrt{2}(1-2\mu_A(x_i))\left(\frac{1}{T_0T_1(T_0+T_1)}-\frac{2}{\sqrt{2}T_2}\right)\sec^2\frac{3\sqrt{2}}{3\sqrt{2}+2(T_0+T_1)-\sqrt{2}T_2}}{\left(3\sqrt{2}+2(T_0+T_1)-\sqrt{2}T_2\right)^2} \quad \ldots (3)$$

Again, partial differentiation of **(2)** with respect to $\mu_A(x_i)$ yields

$$\frac{\partial^2 T_{FS}(A)}{\partial \mu_A^2(x_i)}=-\frac{9\left(\begin{array}{c}\frac{\left(\sqrt{2}-2T_1+2T_1T_0^2-2T_0^3\right)\left(3\sqrt{2}+2T_1+2T_0-\sqrt{2}T_2\right)^2}{2\sqrt{2}T_1\left(T_0^2-1\right)T_0^3}\\+\left(\frac{1}{T_0}-\frac{1}{T_1}-\frac{1-2T_0^2}{\sqrt{2}T_2}\right)^2\left(\begin{array}{c}2\sqrt{2}\left(3\sqrt{2}+2(T_0+T_1)-\sqrt{2}T_2\right)\\+12\tan\frac{3\sqrt{2}}{3\sqrt{2}+2(T_0+T_1)-\sqrt{2}T_2}\end{array}\right)\end{array}\right)\sec^2\frac{3\sqrt{2}}{3\sqrt{2}+2(T_0+T_1)-\sqrt{2}T_2}}{\left(3\sqrt{2}+2(T_0+T_1)-\sqrt{2}T_2\right)^2}\le 0$$

for each $\mu_A(x_i)\in[0,1]$, This establishes that $T_{FS}(A)$ exhibits the concavity property with respect to $\mu_A(x_i)$. This motivates $T_F(A)$ to admit its maximum value which can occur if $\frac{\partial T_{FS}(A)}{\partial \mu_A(x_i)}=0$ and hence **(1)** yields $\mu_A(x_i)=\frac{1}{2}$. Thus,

$$\text{Max.}T_{FS}(A)=T_{FS}(A)\big|_{\mu_A(x_i)=\frac{1}{2}.}=\left(\tan\frac{3\sqrt{2}}{3\sqrt{2}+2}-\tan\frac{2}{3}\right)n. \quad \ldots (4)$$

Also, the graphical representation of $T_{FS}(A)$ as shown in **Fig 2** justifies that it admits its minimum value as zero.

Theorem.3.2 Set $E_0=\sqrt{\mu_B(x_i)}, E_1=\sqrt{1-\mu_B(x_i)}$. Let A and B belongs to $T(X)$ X, then $T_{FS}^{\mu}(A,B)$ is a correct symmetric trigonometric FCEM (fuzzy cross entropy measure **[15-16]**) given as

$$T_{FS}^{\mu}(A,B)$$
$$=\sum_{i=1}^{n}\left[\begin{array}{l}-6\tan\left(\frac{2}{3}\right)\\+\left(2+T_0^2+E_0^2\right)\tan\left(\frac{2+T_0^2+E_0^2}{3+2\left(T_0+E_0\right)\sqrt{\frac{T_0^2+E_0^2}{2}}-T_0E_0}\right)+\left(4-T_0^2-E_0^2\right)\tan\left(\frac{4-T_0^2-E_0^2}{3+2\left(T_1+E_1\right)\sqrt{\frac{2-T_0^2-E_0^2}{2}}-T_1E_1}\right)\end{array}\right]$$

… (5)

Here, $T_{FS}^{\mu}(A,B)$ represents the subjective value of symmetric discrimination of A against B.

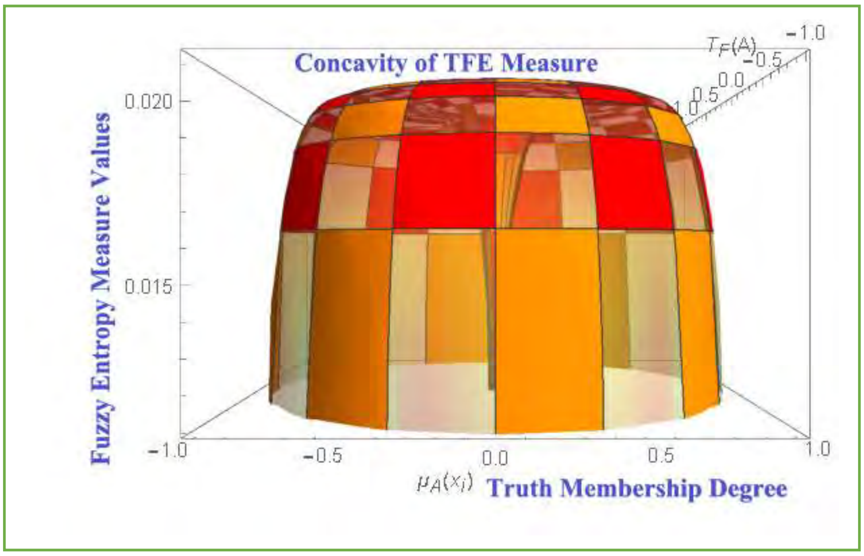


Fig. 2 A Revolution Plot for the Concavity Property and Minimum value of $T_{FS}(A)$

Proof. (i) Since $T_{FS}^{\mu}(A,B)$ does not change after the replacement of $\mu_A(x_i)$ with $\mu_B(x_i)$, this validates the symmetric nature of $T_{FS}^{\mu}(A,B)$.

(ii) Since $T_{FS}^{\mu}(A,B)$ remains unchanged after the replacement of A,B with A^c,B^c, this suggests that $T_{FS}^{\mu}(A^c,B^c)=T_{FS}^{\mu}(A,B)$. The fact, that $T_{FS}^{\mu}(A,B)$ is non-negative, can be established if we first inculcate the following **Lemma 3.1.**

Lemma 3.1 Define $N=\left(\frac{T_0+E_0}{2}\right)\left(\sqrt{\frac{T_0^2+E_0^2}{2}}\right), A=\frac{T_0^2+E_0^2}{2}, G=T_0E_0$. There exists the inequality: $4N\le 3A+G$ with equality if $T_0^2=\mu_A(x_i)=E_0^2=\mu_B(x_i)$. … (6)

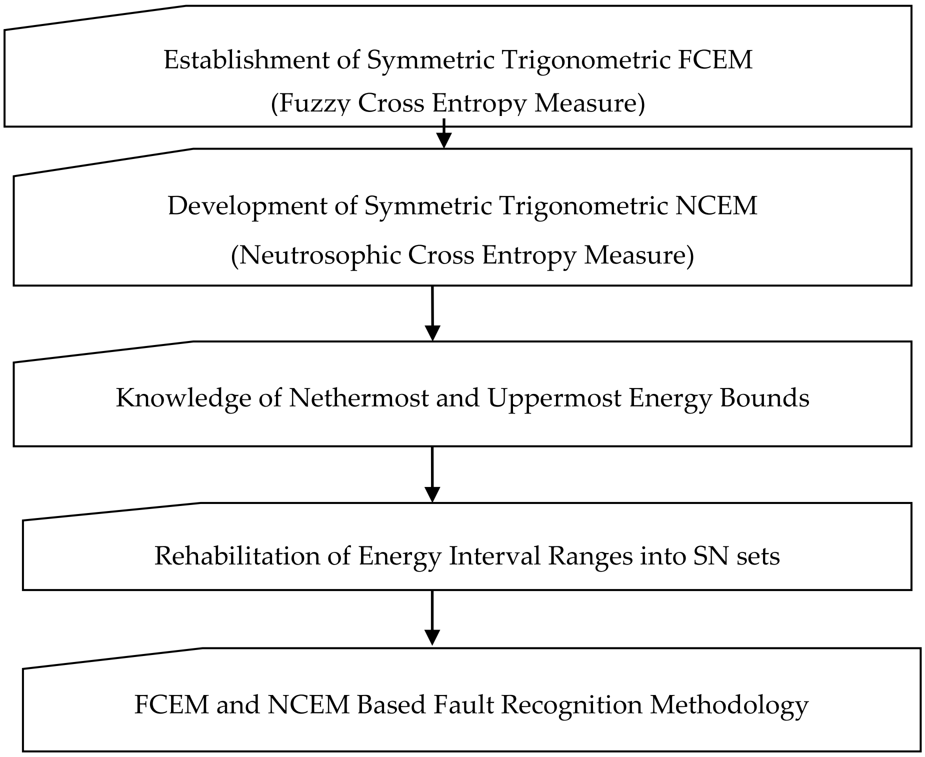


Fig. 1 A Step-wise Flow Chart of FCEM and NCEM Based urbine Fault Recognition Methodology

Proof. In our notations, we have

$$N = \left(\frac{T_0 + E_0}{2}\right)\left(\sqrt{\frac{T_0^2 + E_0^2}{2}}\right), A = \frac{T_0^2 + E_0^2}{2}, G = T_0 E_0.$$

The undergoing inequality **(6)** could be true if $2\left(T_0 + E_0\right)\left(\sqrt{\frac{T_0^2 + E_0^2}{2}}\right) \le \frac{3}{2}\left(T_0^2 + E_0^2\right) + T_0 E_0$

$$\Leftrightarrow 8\left(T_0^2 + E_0^2 + 2T_0E_0\right)\left(T_0^2 + E_0^2\right) \le 9\left(T_0^2 + E_0^2\right)^2 + 4T_0^2E_0^2 + 12\left(T_0^2 + E_0^2\right)T_0E_0$$

$$\Leftrightarrow 8\left(T_0^2 + E_0^2\right)^2 + 16T_0E_0\left(T_0^2 + E_0^2\right) \le 9\left(T_0^2 + E_0^2\right)^2 + 4T_0^2E_0^2 + 12\left(T_0^2 + E_0^2\right)T_0E_0$$

$$\Leftrightarrow \left(T_0^2 + E_0^2\right)^2 - 4\left(T_0^2 + E_0^2\right)T_0E_0 + 4T_0^2E_0^2 \ge 0$$

$\Leftrightarrow \left(T_0^2 + E_0^2 - 2T_0E_0\right)^2 \ge 0 \Leftrightarrow \left(T_0 - E_0\right)^4 \ge 0$ which is obviously true.

Thus, in view of the resulting **Lemma 3.1,** the inequality **(6)** can be rescheduled as

$$\frac{4N - G}{3} \le A \Rightarrow \frac{4N - G}{3} + 1 \le A + 1 \quad \Rightarrow \frac{2\left(T_0 + E_0\right)\left(\sqrt{\frac{T_0^2 + E_0^2}{2}}\right) - T_0E_0}{3} + 1 \le \frac{T_0^2 + E_0^2}{2} + 1$$

$$\Rightarrow \frac{2+T_0^2+E_0^2}{3+2\left(T_0+E_0\right)\sqrt{\frac{T_0^2+E_0^2}{2}}-T_0E_0} \geq \frac{2}{3} \quad ..(7)$$

Since tangent function exhibits the monotonicity property over $[0,1]$, the resulting inequality **(7)** can be rescheduled as

$$\left(2+T_0^2+E_0^2\right)\tan\frac{2+T_0^2+E_0^2}{3+2\left(T_0+E_0\right)\sqrt{\frac{T_0^2+E_0^2}{2}}-T_0E_0} \geq \left(2+T_0^2+E_0^2\right)\tan\frac{2}{3} \quad \ldots(8)$$

With the replacement of $\mu_A(x_i)$ with $1-\mu_A(x_i)$ and of $\mu_B(x_i)$ with $1-\mu_B(x_i)$ into **(8),** we observe that

T_0^2 changes to $T_1^2=1-T_0^2;\ E_0^2 \to E_1^2=1-E_0^2;\ T_0+E_0 \to T_1+E_1;\ \ T_0E_0 \to T_1E_1.$ Thus, **(8)** yields

$$\left(4-T_0^2-E_0^2\right)\tan\frac{4-T_0^2-E_0^2}{3+2\left(T_1+E_1\right)\sqrt{\frac{2-T_0^2-E_0^2}{2}}-T_1E_1} \geq \left(4-T_0^2-E_0^2\right)\tan\frac{2}{3} \quad \ldots(9)$$

We can simply add the resulting inequalities **(8, 9)** and then take the summation over $i=1$ to n to yield $T_{FS}^{\mu}(A,B)\geq 0$ as desired. Moreover, when $\mu_A(x_i)=\mu_B(x_i)$, then $T_0=E_0, T_1=E_1, 1-T_0^2=T_1^2, \sqrt{1-T_0^2}=T_1.$ Also,

$$T_{FS}^{\mu}(A,A)=\sum_{i=1}^{n}\left[-6\tan\left(\frac{2}{3}\right)+\left(2+2T_0^2\right)\tan\left(\frac{2+2T_0^2}{3+3T_0^2}\right)+\left(4-2T_0^2\right)\tan\left(\frac{4-2T_0^2}{6-3T_0^2}\right)\right]=0 \quad \ldots(10)$$

The equality (10) justifies that $T_{FS}^{\mu}(A,B)=0$ whenever $\mu_A(x_i)=\mu_B(x_i)$ as desired.

After the establishment of proposed fuzzy cross entropy measure $T_{FS}^{\mu}(A,B)$, the next Theorem 3.3 argues the urgent situation under which it will admits its extreme values.

Theorem 3.3 If $n\in N$ is the cardinality of X, then

$$0\leq T_{FS}^{\mu}(\mathrm{A,B})\leq 6\left(\tan\frac{3\sqrt{2}}{3\sqrt{2}+2}-\tan\frac{2}{3}\right)n. \quad \ldots(11)$$

Proof. If we replace the fuzzy set B with A^c, we observe that E_0^2 changes to $1-T_0^2;\ E_1\to T_0, E_0\to T_1, T_0T_1\to T_2.$

Thus, after the replacement of $\mu_B(x_i)$ with $1-\mu_A(x_i)$, the undergoing equality measure **(5)** yields

$$T_{FS}^{\mu}(A,A^c)=\sum_{i=1}^{n}\left|6\tan\frac{3\sqrt{2}}{3\sqrt{2}+2}-6\tan\left(\frac{2}{3}\right)-6\left(\tan\left(\frac{3\sqrt{2}}{3\sqrt{2}+2}\right)-\tan\left(\frac{3\sqrt{2}}{3\sqrt{2}+2\left(T_0+T_1\right)-\sqrt{2}T_2}\right)\right)\right|$$

$= 6\text{Max.T}_F\left(A\right) - 6\text{T}_F\left(A\right)$ … (12)

Because T_F(A) is non-negative (**Theorem 3.1),** this motivates **(12)** to yield

$$T_{\text{F}}(\text{A})=\text{Max.T}_F\left(A\right) - \frac{1}{6}T_{FS}^{\mu}(\text{A,A}^c) \geq 0 \Rightarrow 0 \leq T_{FS}^{\mu}(\text{A,A}^c) \leq 6\left(\tan\frac{3\sqrt{2}}{3\sqrt{2}+2} - \tan\frac{2}{3}\right)n \quad \dots (13)$$

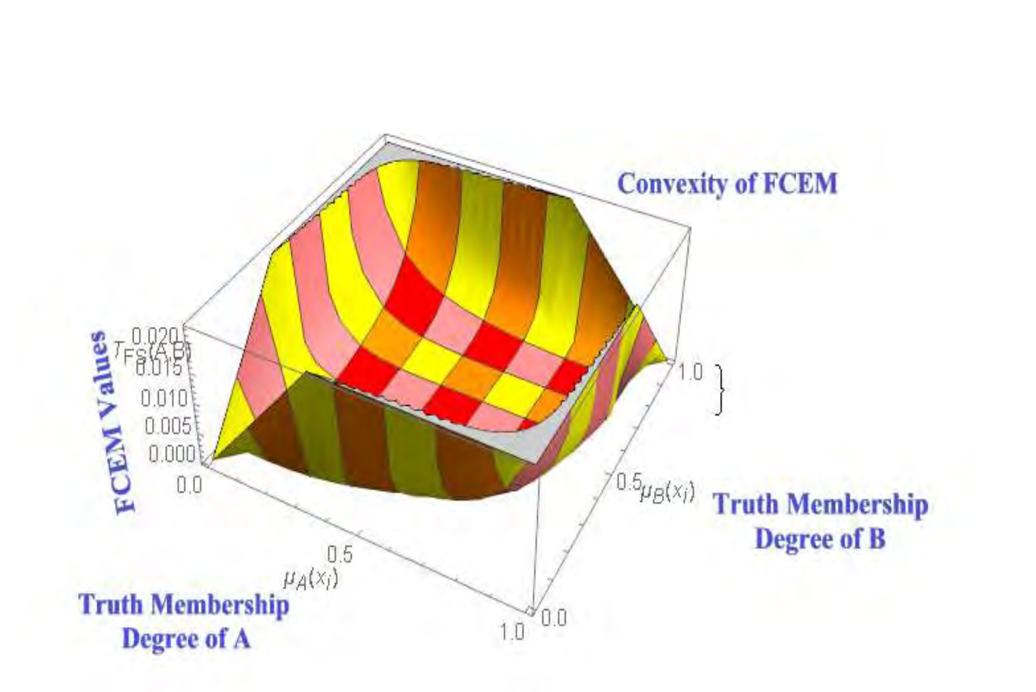
With the establishment of resulting inequality **(13),** it is informative to know that $T_{FS}\left(A, A^c\right)$ is finite for a fixed n. This justifies the finiteness of our proposed symmetric trigonometric FCEM (fuzzy cross entropy measure) which ranges as $0 \leq T_{FS}^{\mu}(\text{A,B}) \leq 6\left(\tan\frac{3\sqrt{2}}{3\sqrt{2}+2} - \tan\frac{2}{3}\right)n.$ Thus, $\text{Max}T_{FS}^{\mu}\left(A, B\right) = \left(6\left(\tan\frac{3\sqrt{2}}{3\sqrt{2}+2} - \tan\frac{2}{3}\right)n\right)$ which clarifies that this maximum value does not depend upon its truth membership degree, but completely depends upon the cardinality of X. Also, the surface plot of $T_{FS}^{\mu}(A,B)$, represented by **Fig 3(a, b),** justifies the fact that this measure, because of its convexity, admits its $\text{Min.}T_{FS}^{\mu}\left(A, B\right) = 0$ Also, it is evident that $T_{FS}^{\mu}(A,B)$ gets increased as soon as $\left|A - B\right|$ increases, attains $\text{Max.}T_{FS}^{\mu}\left(A, B\right) = 6\left(\tan\frac{3\sqrt{2}}{3\sqrt{2}+2} - \tan\frac{2}{3}\right)n.$

The establishment of FCEM $T_{FS}^{\mu}(A,B)$, resulted from **Theorem 3.2** in the overhead discussion, will lead to develop the proposed NCEM (represented by $T_{\text{NC}}\left(\text{A,B}\right)$), the repercussions of which will be utilized to meet our goal of recognizing fault conditions of turbine.

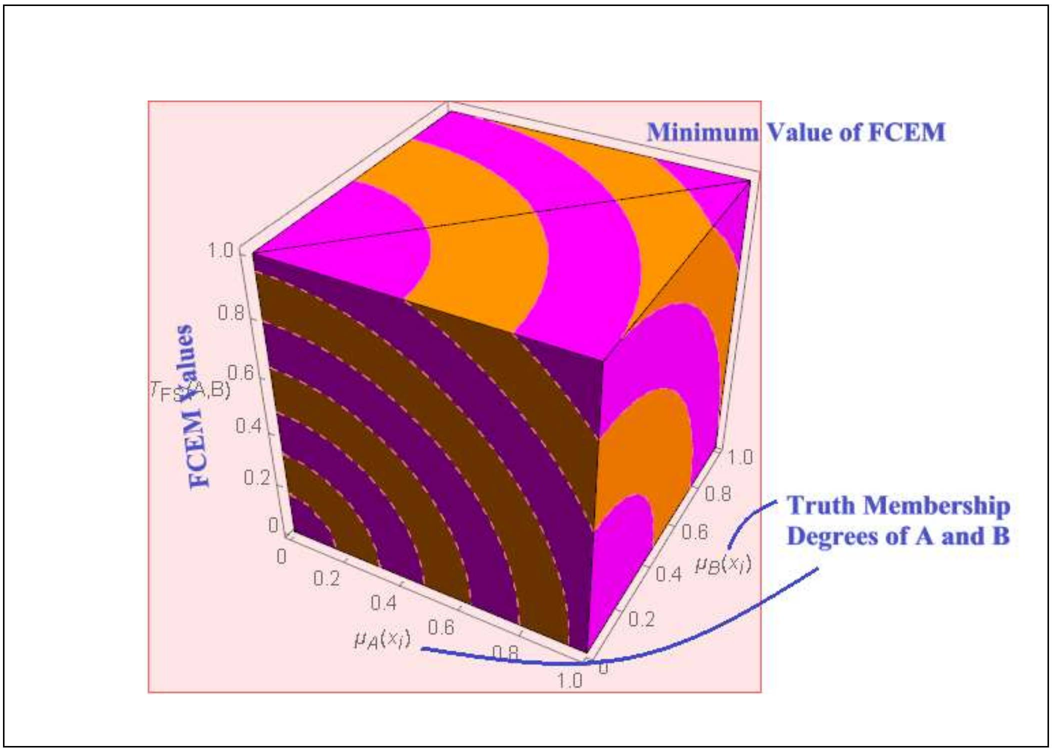
4. A Novel Symmetric Trigonometric NCEM (Neutrosophic Cross Entropy Measure)

In the resulting **Theorem 3.2,** the entity $T_{FS}^{\mu}(A,B)$ represents the amount of fuzziness which arises due to true membership degree for symmetric discrimination of fuzzy set A against B. Similarly, if we set $I_0 = \sqrt{i_A\left(x_i\right)}, I_1 = \sqrt{1 - i_A\left(x_i\right)}, J_0 = \sqrt{i_B\left(x_i\right)}, J_1 = \sqrt{1 - i_B\left(x_i\right)},$ then the amount of fuzziness which arises due to indeterminancy membership degree of A against B is established as

$$T_{FS}^{i}(A,B) = \sum_{i=1}^{n}\left[\begin{array}{l} -6\tan\left(\frac{2}{3}\right) \\ +\left(2+I_0^2+J_0^2\right)\tan\left(\frac{2+I_0^2+J_0^2}{3+2\left(I_0+J_0\right)\sqrt{\frac{I_0^2+J_0^2}{2}}-I_0J_0}\right)+\left(4-I_0^2-J_0^2\right)\tan\left(\frac{4-I_0^2-J_0^2}{3+2\left(I_1+J_1\right)\sqrt{\frac{2-I_0^2-J_0^2}{2}}-I_1J_1}\right) \end{array}\right] \quad \ldots (14)$$



(a)



(b)

Fig 3 (a) Convexity and (b) Minimum Value of the proposed of FCEM $T_{FS}^{\mu}(A,B)$

Again, we set $F_0=\sqrt{f_A(x_i)}, F_1=\sqrt{1-f_A(x_i)}, H_0=\sqrt{f_B(x_i)}, H_1=\sqrt{1-f_B(x_i)}$, then the amount of fuzziness which arises due to falsity membership degree for symmetric discrimination of A against B can also be established as

$$T_{FS}^{f}(A,B) = \sum_{i=1}^{n}\left[\begin{array}{l}-6\tan\left(\frac{2}{3}\right)\\ +\left(2+F_0^2+H_0^2\right)\tan\left(\frac{2+F_0^2+H_0^2}{3+2\left(F_0+H_0\right)\sqrt{\frac{F_0^2+H_0^2}{2}}-F_0H_0}\right)+\left(4-F_0^2-H_0^2\right)\tan\left(\frac{4-F_0^2-H_0^2}{3+2\left(F_1+H_1\right)\sqrt{\frac{2-F_0^2-H_0^2}{2}}-F_1H_1}\right)\end{array}\right] \quad \text{... (15)}$$

Def.4.1 Let A and B are any two SN sets in $X=(x_1,x_2,...,x_n)$. The desired NCEM (symmetric trigonometric neutrosophic cross entropy measure of SN sets can be constructed by simply adding **(12), (14)** and **(15)** as below.

$$T_{NC}(A,B)=T_{FS}^{\mu}(A,B)+T_{FS}^{i}(A,B)+T_{FS}^{f}(A,B) \quad \text{... (16)}$$

Following same procedure as deployed in **Theorem 3.3,** readers can easily establish that if A and B are any SN sets with same cardinality n, then there exists the inequality:

$$0 \le T_{\mathrm{NC}}(\mathrm{A,B}) \le 18\left(\tan\frac{3\sqrt{2}}{3\sqrt{2}+2} - \tan\frac{2}{3}\right)n.$$

We shall now authenticate the applicability of our newly discovered NCEM $T_{\mathrm{NC}}(\mathrm{A,B})$ by recognizing the optimal fault condition of some huge steam turbine generator as follows.

6. FCEM and NCEM Based Fault Recognition Methodology

To achieve the desired goal, we shall, equally well, establish a neutrosophic cross entropy-based methodology which has the necessary capability of identifying various fault conditions of some turbine. A schematic flow chart explaining our fault recognition methodology has been provided in **Fig. 1** and discussed as below.

Step:-1 Construction of Energy Interval Ranges

The applicability of the underlying methodology is exemplified by taking into consideration the illustration **[12].** Suppose the ten familiar fault conditions experienced by some huge steam turbine generator set is represented by $B_K = (B_1, B_2, B_3, ...B_{10})$ where the fault condition "Unbalance" is abbreviated as B_1. Similarly, the conditions $B_2, B_3, ..., B_{10}$ have been provided in **[2,3].** Also, the nine frequency intervals $C_1 = [0.01, 0.3f], C_2 = [0.4, 0.49f], ..., C_9 = \text{higher frequency} > 5f$, of frequency spectrum , resulted from vibration signals of turbine, are available in **[2,3].**

Let $\mu_{B_K}(x_i)$ (lower bound) and $U_{B_K}(x_i)$ (upper bound) represent the amount of fuzziness resulted from the truth membership degree of K^{th} fault condition at i^{th} range of frequency spectrum. Then

$$B_K = \left(< x_1, \left[\mu_{B_K}(x_1), U_{B_K}(x_1)\right] >, < x_2, \left[\mu_{B_K}(x_2), U_{B_K}(x_2)\right] >, ... < x_9, \left[\mu_{B_K}(x_9), U_{B_K}(x_9)\right] >\right); K = 1,2,...,10.$$

… (17)

Generally, the acquitted vibration data may be non-commensurate and conflicting, it becomes essential for us to transform the energy interval ranges **(17)** into the form of SN sets. This conversion, however may be problematic, but can be done as follows.

Step:-2 Transformation of Interval Ranges (energy) by the Form of Neutrosophic Sets

The amount of fuzziness based on falsity membership degree of K^{th} familiar fault condition at i^{th} range of frequency spectrum is denoted by $f_{B_K}(x_i)$ where $f_{B_K}(x_i) = 1 - U_{B_K}(x_i)$. Similarly, the amount of fuzziness based on indeterminacy membership degree of K^{th} familiar fault condition at i^{th} range of frequency spectrum is denoted by $i_{B_K}(x_i)$ where $i_{B_K}(x_i)=1-f_{B_K}(x_i)-U_{B_K}(x_i)$. We have restricted the value of $i_{B_K}(x_i)$ to $0\cdot 001$ in case it if returns any other value less than or equal to $0\cdot 001$ .Then, the interval ranges (energy), represented by **(17),** for each B_K can be transformed into the forms of SN sets is described below.

$$B_K=\begin{pmatrix}\left(<x_1,\left[\mu_{B_K}(x_1),i_{B_K}(x_1),f_{B_K}(x_1)\right]>\right),\left(<x_2,\left[\mu_{B_K}(x_2),i_{B_K}(x_2),f_{B_K}(x_2)\right]>\right),\\ \left(<x_3,\left[\mu_{B_K}(x_3),i_{B_K}(x_3),f_{B_K}(x_3)\right]>\right),...\left(<x_9,\left[\mu_{B_K}(x_9),i_{B_K}(x_9),f_{B_K}(x_9)\right]>\right)\end{pmatrix};K=1,2,...,10 \quad \ldots (18)$$

Also, the unfamiliar fault conditions, represented by F_{T_j}, can also be transformed into the forms of SN sets as below:

$$F_{T_J}=\begin{pmatrix}\left(<x_1,\left[\mu_{F_{T_J}}(x_1),i_{F_{T_J}}(x_1),f_{F_{T_J}}(x_1)\right]>\right),\left(<x_2,\left[\mu_{F_{T_J}}(x_2),i_{F_{T_J}}(x_2),f_{F_{T_J}}(x_2)\right]>\right),\\ \left(<x_3,\left[\mu_{F_{T_J}}(x_3),i_{F_{T_J}}(x_3),f_{F_{T_J}}(x_3)\right]>\right),...,\left(<x_9,\left[\mu_{F_{T_J}}(x_9),i_{F_{T_J}}(x_9),f_{F_{T_J}}(x_9)\right]>\right)\end{pmatrix} \quad \ldots (19)$$

Step: -3 Computation of FCEM and NCEM Values between familiar and unfamiliar fault conditions

The cross-entropy values $T_{\text{NC}}(B_K,F_{T_J}),T^{\mu}_{FS}(B_K,F_{T_J})$ between each B_K and F_{T_J} can be evaluated as follows. Replacement of introduced notations $T_0,F_0,H_0...$, etc., with their original values and then taking $i=1,2,...,9$ into **(5,16)** yields

$$T^{\mu}_{FS}(B_K,F_{T_J})=\sum_{i=1}^{9}\left[\begin{array}{l}-6\tan\frac{2}{3}+\left(2+\mu_{B_K}(x_i)+\mu_{F_{T_J}}(x_i)\right)\tan\left(\frac{2+\mu_{B_K}(x_i)+\mu_{F_{T_J}}(x_i)}{3+2\left(\sqrt{\mu_{B_K}(x_i)}+\sqrt{\mu_{F_{T_J}}(x_i)}\right)\left(\frac{\sqrt{\mu_{B_K}(x_i)+\mu_{F_{T_J}}(x_i)}}{2}\right)-\sqrt{\mu_{B_K}(x_i)\mu_{F_{T_J}}(x_i)}}\right)\\ -\left(4-\mu_{B_K}(x_i)-\mu_{F_{T_J}}(x_i)\right)\tan\left(\frac{4-\mu_{B_K}(x_i)-\mu_{F_{T_J}}(x_i)}{3+2\left(\sqrt{1-\mu_{B_K}(x_i)}+\sqrt{1-\mu_{F_{T_J}}(x_i)}\right)\left(\frac{\sqrt{2-\mu_{B_K}(x_i)-\mu_{F_{T_J}}(x_i)}}{2}\right)-\sqrt{\left(1-\mu_{B_K}(x_i)\right)\left(1-\mu_{F_{T_J}}(x_i)\right)}}\right)\end{array}\right]$$

... (20)

$$T_{NC}\left(B_K,F_{T_J}\right)$$

$$=\sum_{i=1}^{9}\left[\begin{array}{l}-6\tan\frac{2}{3}+\left(2+\mu_{B_K}(x_i)+\mu_{F_{T_J}}(x_i)\right)\tan\left(\frac{2+\mu_{B_K}(x_i)+\mu_{F_{T_J}}(x_i)}{3+2\left(\sqrt{\mu_{B_K}(x_i)}+\sqrt{\mu_{F_{T_J}}(x_i)}\right)\left(\frac{\sqrt{\mu_{B_K}(x_i)+\mu_{F_{T_J}}(x_i)}}{2}\right)-\sqrt{\mu_{B_K}(x_i)\mu_{F_{T_J}}(x_i)}}\right)\\ -\left(4-\mu_{B_K}(x_i)-\mu_{F_{T_J}}(x_i)\right)\tan\left(\frac{4-\mu_{B_K}(x_i)-\mu_{F_{T_J}}(x_i)}{3+2\left(\sqrt{1-\mu_{B_K}(x_i)}+\sqrt{1-\mu_{F_{T_J}}(x_i)}\right)\left(\frac{\sqrt{2-\mu_{B_K}(x_i)-\mu_{F_{T_J}}(x_i)}}{2}\right)-\sqrt{\left(1-\mu_{B_K}(x_i)\right)\left(1-\mu_{F_{T_J}}(x_i)\right)}}\right)\end{array}\right]$$

$$+\sum_{i=1}^{9}\left[\begin{array}{l}-6\tan\frac{2}{3}+\left(2+i_{B_K}(x_i)+i_{F_{T_J}}(x_i)\right)\tan\left(\frac{2+i_{B_K}(x_i)+i_{F_{T_J}}(x_i)}{3+2\left(\sqrt{i_{B_K}(x_i)}+\sqrt{i_{F_{T_J}}(x_i)}\right)\left(\frac{\sqrt{i_{B_K}(x_i)+i_{F_{T_J}}(x_i)}}{2}\right)-\sqrt{i_{B_K}(x_i)i_{F_{T_J}}(x_i)}}\right)\\ -\left(4-i_{B_K}(x_i)-i_{F_{T_J}}(x_i)\right)\tan\left(\frac{4-i_{B_K}(x_i)-i_{F_{T_J}}(x_i)}{3+2\left(\sqrt{1-i_{B_K}(x_i)}+\sqrt{1-i_{F_{T_J}}(x_i)}\right)\left(\frac{\sqrt{2-i_{B_K}(x_i)-i_{F_{T_J}}(x_i)}}{2}\right)-\sqrt{\left(1-i_{B_K}(x_i)\right)\left(1-i_{F_{T_J}}(x_i)\right)}}\right)\end{array}\right]$$

$$+\sum_{i=1}^{9}\left[\begin{array}{l}-6\tan\frac{2}{3}+\left(2+f_{B_K}(x_i)+f_{F_{T_J}}(x_i)\right)\tan\left(\frac{2+f_{B_K}(x_i)+f_{F_{T_J}}(x_i)}{3+2\left(\sqrt{f_{B_K}(x_i)}+\sqrt{f_{F_{T_J}}(x_i)}\right)\left(\frac{\sqrt{f_{B_K}(x_i)+f_{F_{T_J}}(x_i)}}{2}\right)-\sqrt{f_{B_K}(x_i)f_{F_{T_J}}(x_i)}}\right)\\ -\left(4-f_{B_K}(x_i)-f_{F_{T_J}}(x_i)\right)\tan\left(\frac{4-f_{B_K}(x_i)-f_{F_{T_J}}(x_i)}{3+2\left(\sqrt{1-f_{B_K}(x_i)}+\sqrt{1-f_{F_{T_J}}(x_i)}\right)\left(\frac{\sqrt{2-f_{B_K}(x_i)-f_{F_{T_J}}(x_i)}}{2}\right)-\sqrt{\left(1-f_{B_K}(x_i)\right)\left(1-f_{F_{T_J}}(x_i)\right)}}\right)\end{array}\right]$$

… (21)

Step: -4 Identification of Turbine Faults

The Smallest value of $T_{NC}(B_K,F_{T_J});T_{FS}^{\mu}(B_K,F_{T_J})$ indicate that the familiar fault condition B_K is closer to the unfamiliar fault condition F_{T_J}. In other words, a typical selection of turbine fault will be designated as optimal fault type selection owing to the smallest NCEM $T_{NC}(B_K,F_{T_J})$ or FCEM $T_{FS}^{\mu}(B_K,F_{T_J})$ value.

7. APPLICATION TO FAULT DIAGNOSIS OF TURBINE

In order to validate the applicability of FCEM and NCEM based fault recognition methodology, the energy interval ranges for each familiar fault condition at various ranges of frequency spectrum is provided in **Table 1.**

Table 1. The Nethermost and Uppermost Energy Bounds of Each B_K at Nine Ranges of frequency Spectrum

B_K	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
B_1	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.85,1.00]	[0.04,0.06]	[0.04,0.07]	[0.00,0.00]	[0.00,0.00]
B_2	[0.00,0.00]	[0.03,0.31]	[0.90,0.12]	[0.55,0.70]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.08,0.13]
B_3	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.30,0.58]	[0.40,0.62]	[0.08,0.13]	[0.00,0.00]	[0.00,0.00]
B_4	[0.09,0.11]	[0.78,0.82]	[0.00,0.00]	[0.08,0.11]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,1.00]
B_5	[0.09,0.12]	[0.09,0.11]	[0.08,0.12]	[0.09,0.12]	[0.18,0.21]	[0.08,0.13]	[0.08,0.13]	[0.08,0.22]	[0.08,0.12]
B_6	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.18,0.22]	[0.12,0.17]	[0.37,0.45]	[0.00,0.00]	[0.22,0.28]
B_7	[0.00,0.00]	[0.00,0.00]	[0.08,0.12]	[0.86,0.93]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]
B_8	[0.00,0.00]	[0.27,0.32	[0.08,0.12]	[0.54,0.62]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]
B_9	[0.85,0.93]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.08,0.12]	[0.00,0.00]
B_{10}	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.77,0.83]	[0.19,0.30]	[0.00,0.00]	[0.00,0.00]

Table 2. Transforming B_K into the forms of Single valued neutrosophic (SN) Sets

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.85,0.15,0.00]	[0.04,0.02,0.94]	[0.04,0.03,0.93]	[0.00,0.01,1.00]	[0.00,0.01,1.00]
[0.00,0.01,1.00]	[0.03,0.01,0.69]	[0.90,0.03,0.88]	[0.55,0.15,0.30]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.08,0.05,0.87]
[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.30,0.28,0.42]	[0.40,0.22,0.38]	[0.08,0.05,0.87]	[0.00,0.01,1.00]	[0.00,0.01,1.00]
[0.09,0.02,0.89]	[0.78,0.04,0.18]	[0.00,0.01,1.00]	[0.08,0.03,0.89]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]
[0.09,0.03,0.88]	[0.09,0.02,0.89]	[0.08,0.04,0.88]	[0.09,0.03,0.88]	[0.18,0.03,0.79]	[0.08,0.05,0.87]	[0.08,0.05,0.87]	[0.08,0.04,0.88]	[0.08,0.04,0.88]
[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.18,0.04,0.78]	[0.12,0.05,0.83]	[0.37,0.08,0.55]	[0.00,0.01,1.00]	[0.22,0.06,0.72]
[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.08,0.04,0.88]	[0.86,0.07,0.07]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]
[0.00,0.01,1.00]	[0.27,0.05,0.68]	[0.08,0.04,0.88]	[0.54,0.08,0.38]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]
[0.85,0.08,0.07]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.08,0.04,0.88]	[0.00,0.01,1.00]
[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.00,0.01,1.00]	[0.77,0.06,0.17]	[0.19,0.04,0.77]	[0.00,0.01,1.00]	[0.00,0.01,1.00]

Step:-2 The nethermost (lowest) and uppermost (highest) energy bounds of each real fault conditions (B_K) have been extracted and thereafter rehabilitated into the forms of SN sets as shown in **Table 2.** The fault testing samples $F_{T_J}\ (J=1,2)$ in this study can also be transformed into the forms of SN sets as follows.

$$F_{T_1}=\left\langle\begin{matrix}[0.000,0.010,1.000],[0.000,0.010,1.000],[0.100,0.010,0.900],[0.000,0.010,1.000],\\ [0.000,0.010,1.000],[0.000,0.010,1.000],[0.000,0.010,1.000],[0.000,0.010,1.000],\\ [0.000,0.010,1.000]\end{matrix}\right\rangle \quad \ldots (22)$$

$$F_{T_2}=\left\langle\begin{matrix}[0.390,0.010,0.610],[0.070,0.010,0.930],[0.000,0.010,1.000],[0.060,0.010,0.940],\\ [0.000,0.010,1.000],[0.130,0.010,0.870],[0.000,0.010,1.000],[0.000,0.010,1.000],\\ [0.350,0.010,0.650]\end{matrix}\right\rangle \quad \ldots (23)$$

Step:- 3 The FCEM $T_{FS}^{\mu}(B_K,F_{T_J})$ and NCEM $T_{NC}(B_K,F_{T_J})$ values between each B_K (provided in **Table 2**) and F_{T_J} (represented by **(22,23)**) can be computed employing the resulting equations **(20,21)**. The fault diagnosis order obtained through the proposed FCEM and NCEM as well as by the existing cosine similarity measure **[3]** is represented in **Table 3.**

Diagnosis Result 1. The fuzzy as well as neutrosophic cross entropy values between each familiar fault condition B_K and the first testing sample F_{T_1}, as can be seen from **Table 3,** are

$$T_{FS}^{\mu}(B_1,F_{T_1})=0.1382, T_{FS}^{\mu}(B_2,F_{T_1})=0.0082, T_{FS}^{\mu}(B_3,F_{T_1})=0.1222; T_{FS}^{\mu}(B_4,F_{T_1})=0.0830, T_{FS}^{\mu}(B_5,F_{T_1})=0.0595,$$

$$T_{FS}^{\mu}(B_6,F_{T_1})=0.1289, T_{FS}^{\mu}(B_7,F_{T_1})=0.0000, T_{FS}^{\mu}(B_8,F_{T_1})=0.0184, T_{FS}^{\mu}(B_9,F_{T_1})=0.1382, T_{FS}^{\mu}(B_{10},F_{T_1})=0.1368.$$

$$T_{NC}(B_1,F_{T_1})=0.3491, T_{NC}(B_2,F_{T_1})=0.0385, T_{NC}(B_3,F_{T_1})=0.2913; T_{NC}(B_4,F_{T_1})=0.1676, T_{NC}(B_5,F_{T_1})=0.1327,$$

$$T_{NC}(B_6,F_{T_1})=0.2737, T_{NC}(B_7,F_{T_1})=0.0006, T_{NC}(B_8,F_{T_1})=0.0401, T_{NC}(B_9,F_{T_1})=0.2931, T_{NC}(B_{10},F_{T_1})=0.2824.$$

In view of Minimum Argument Principle, the minimum symmetric trigonometric FCEM and NCEM values are 0.0000 and 0.0006 respectively. Clearly, these values confirm that vibration fault in turbine occurs due to the defect in anti-thrust bearing (B_7), which is an optimal turbine fault selection, as it can also be experienced from **Fig. 4(a).** The next smallest FCEM and NCEM values are 0.0082,0.0184 and 0.0385,0.0401 respectively which correspond to the fault types B_2 and B_8. This indicates that there is a high possibility of pneumatic force couple and surge faults in the generator. The fault type B_5 (radial impact friction of rotor) has low possibility owing to the next smaller

FCEM and NCEM values (0.0595,0.1327). Similarly, the fault types B_4, B_6, B_{10}, B_3, B_9 and B_1 have very low possibility owing to their smaller FCEM and NCEM entropy values.

Table 3. Fault Recognition of Turbine employing (a) FCEM (b) NCEM and (b) Existing Cosine Similarity Measure **[3]**

Description	Measure Values	Recognized Fault Condition	Actual Fault Condition
FCEM	**FCEM Values**		
$T_{FS}^{\mu}\left(B_K, F_{T_1}\right)$	0.1382 0.0082 0.1222 0.0830 0.0595 0.1289 **0.0000** 0.0184 0.13821 0.1368	Antithrust Bearing	Antithrust Bearing
$T_{FS}^{\mu}\left(B_K, F_{T_2}\right)$	0.1170 0.0445 0.0787 0.0424 **0.0282** 0.0670 0.0818 0.0651 0.0448 0.0720	Radial Impact Friction	Radial Impact Friction
NCEM	**NCEM Values**		
$T_{NC}\left(B_K, F_{T_1}\right)$	0.3491 0.0385 0.2913 0.1676 0.1327 0.2737 **0.0006** 0.0401 0.2931 0.2824	Antithrust Bearing	Antithrust Bearing
$T_{NC}\left(B_K, F_{T_2}\right)$	0.3053 0.0901 0.1916 0.0867 **0.0584** 0.1428 0.1679 0.1284 0.0967 0.1493	Radial Impact Friction	Radial Impact Friction
Cosine	**Cosine Similarity Measure Value []**		
$C_{VS}\left(B_K, F_{T_1}\right)$	0.7891 0.9799 0.8282 0.8236 0.9057 0.8714 **0.9995** 0.9774 0.7974 0.8099	Antithrust Bearing	Antithrust Bearing
$C_{VS}\left(B_K, F_{T_2}\right)$	0.8563 0.9128 0.9066 0.8953 **0.9738** 0.9567 0.8720 0.9201 0.9403 0.8968	Radial Impact Friction	Radial Impact Friction

Thus, the optimal fault recognition order is

$B_7 \succ B_2 \succ B_8 \succ B_5 \succ B_4 \succ B_3 \succ B_6 \succ B_{10} \succ B_1 \succ B_9$ (Obtained from FCEM)

$B_7 \succ B_2 \succ B_8 \succ B_5 \succ B_4 \succ B_6 \succ B_{10} \succ B_3 \succ B_9 \succ B_1$ (Obtained from NCEM)

Diagnosis Result 2. The FCEM and NCEM values between second real testing sample F_{T_2} and B_K are

$$T_{FS}^{\mu}\left(B_1,F_{T_2}\right)=0.1170, T_{FS}^{\mu}\left(B_2,F_{T_2}\right)=0.0445, T_{FS}^{\mu}\left(B_3,F_{T_2}\right)=0.0787, T_{FS}^{\mu}\left(B_4,F_{T_2}\right)=0.0424, T_{FS}^{\mu}\left(B_5,F_{T_2}\right)=0.0282,$$

$$T_{FS}^{\mu}\left(B_6,F_{T_2}\right)=0.0670, T_{FS}^{\mu}\left(B_7,F_{T_2}\right)=0.0818, T_{FS}^{\mu}\left(B_8,F_{T_2}\right)=0.0651, T_{FS}^{\mu}\left(B_9,F_{T_2}\right)=0.0448, T_{FS}^{\mu}\left(B_{10},F_{T_2}\right)=0.0720.$$

$$T_{NC}\left(B_1,F_{T_2}\right)=0.3053, T_{NC}\left(B_2,F_{T_2}\right)=0.0901, T_{NC}\left(B_3,F_{T_2}\right)=0.1916, T_{NC}\left(B_4,F_{T_2}\right)=0.0867, T_{NC}\left(B_5,F_{T_2}\right)=0.0584,$$

$$T_{NC}\left(B_6,F_{T_2}\right)=0.1428, T_{NC}\left(B_7,F_{T_2}\right)=0.1679, T_{NC}\left(B_8,F_{T_2}\right)=0.1284, T_{NC}\left(B_9,F_{T_2}\right)=0.0967, T_{NC}\left(B_{10},F_{T_2}\right)=0.1493.$$

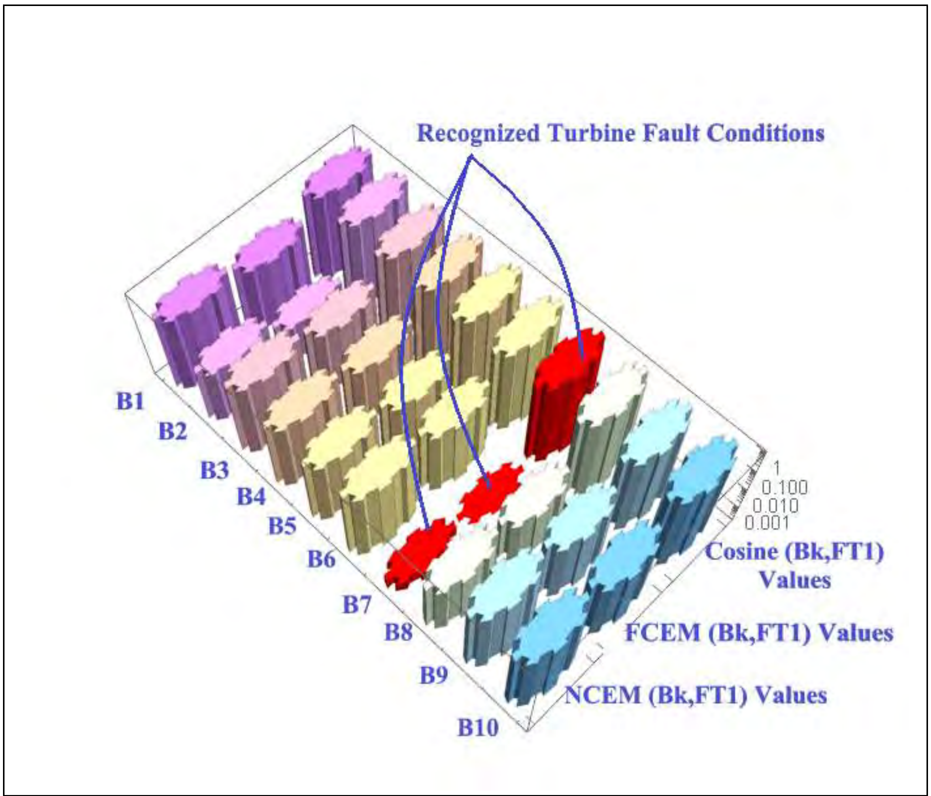


Fig.4(a) Recognized Optimal Fault Condition Employing Proposed Fuzzy, Neutrosophic Cross Entropy and Existing Cosine Similarity Measures **[3]**

In this case, the minimum symmetric trigonometric FCEM and NCEM values are 0.0282 and 0.0584 respectively. Clearly, these values confirm that vibration fault in turbine occurs due to the defect in radial impact friction of the rotor $\left(B_5\right)$, which is an optimal turbine fault selection, as it can also be experienced from **Fig. 4(b).** The next smallest FCEM and NCEM values are 0.0424,0.0445 and 0.0584,0.0901 respectively which correspond to the fault types B_4 and B_2. This indicates that there is a high possibility of pneumatic force couple and oil membrane oscillation. The fault type B_9 (looseness of bearing block) has low possibility owing to its smaller FCEM and NCEM values (0.0448,0.0967). Similarly, the fault types B_8, B_6, B_{10}, B_7, B_3 and B_1 have very low possibility owing to their smaller cross entropy values. Thus, the optimal fault recognition order is $B_5 \succ B_4 \succ B_2 \succ B_9 \succ B_8 \succ B_6 \succ B_{10} \succ B_7 \succ B_3 \succ B_1$.

Validity Test: In order to perform the validity of NCEM under validity criteria **[11]**, we inter-change the degree of true and falsity membership of non-optimal $\left(B_9\right)$ alternative and worse $\left(B_1\right)$ alternatives. The new symmetric trigonometric FCEM and NCEM values can be recalculated employing (21) and are given below.

$$T_{NC}\left(B_1,F_{T_1}\right)=1.4895, T_{NC}\left(B_2,F_{T_1}\right)=0.0385, T_{NC}\left(B_3,F_{T_1}\right)=0.2913; T_{NC}\left(B_4,F_{T_1}\right)=0.1676, T_{NC}\left(B_5,F_{T_1}\right)=0.1327,$$

$$T_{NC}\left(B_6,F_{T_1}\right)=0.2737, T_{NC}\left(B_7,F_{T_1}\right)=0.0006, T_{NC}\left(B_8,F_{T_1}\right)=0.0401, T_{NC}\left(B_9,F_{T_1}\right)=1.5678, T_{NC}\left(B_{10},F_{T_1}\right)=0.2824.$$

$T_{NC}\left(B_1,F_{T_2}\right)=1.1789, T_{NC}\left(B_2,F_{T_2}\right)=0.0976, T_{NC}\left(B_3,F_{T_2}\right)=1.1882, T_{NC}\left(B_4,F_{T_2}\right)=0.0867, T_{NC}\left(B_5,F_{T_2}\right)=0.0660,$

$T_{NC}\left(B_6,F_{T_2}\right)=0.1428, T_{NC}\left(B_7,F_{T_2}\right)=0.1754, T_{NC}\left(B_8,F_{T_2}\right)=0.1360, T_{NC}\left(B_9,F_{T_2}\right)=0.0967, T_{NC}\left(B_{10},F_{T_2}\right)=0.1493.$

The results clearly indicate that the optimal fault selection does not change whenever we interchange the non-optimal and worse alternatives. This justifies that our proposed NCEM is capable of holding the best fault selection whenever worse and non-optimal are interchanged. However, the existing measures **[2,3]** are insufficient for holding the best fault selection. This indicates some ambiguity in the enduring fault recognition methods

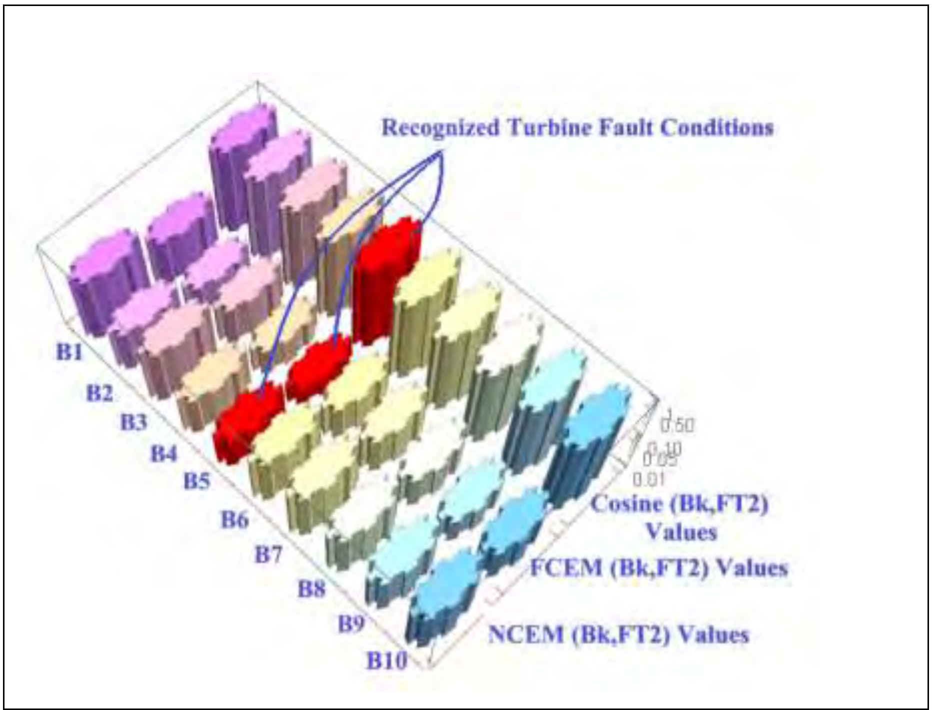


Fig.4(b) Recognized Optimal Fault Condition Employing Proposed Fuzzy, Neutrosophic Cross Entropy and Existing Cosine Similarity Measures

Table 4. Fault Recognition of Turbine employing (a) FCEM (b) NCEM and (b) Existing Cosine Similarity Measure **[3]** Under Sensitive Analysis

Description	Measure Values					Optimal, Worse Alternatives Under Sensitive Analysis	
NCEM	**NCEM Values**					**Before**	**After**
$T_{NC}\left(B_K,F_{T_1}\right)$	0.3497	0.0391	0.2920	0.1682	0.1284	B_9, B_1	B_3, B_1
	0.2743	**0.0012**	0.0407	0.2887	0.2831		
$T_{NC}\left(B_K,F_{T_2}\right)$	0.3059	0.0907	0.1922	0.0873	**0.0541**	B_3, B_1	B_3, B_1
	0.1434	0.1685	0.1290	0.0924	0.1499		

Sensitive Analysis In order to demonstrate the effectiveness of NCEM under sensitive analysis**[11]** ,we slightly change the value $\left(< x_8, \left[0.00, 0.01, 1.00\right] >\right)$of F_{T_1} to $\left(< x_8, \left[0.010, 0.010, 1.000\right] >\right)$.

Next, we again compute $T_{NC}\left(B_K, F_{T_J}\right)\left(J = 1, 2\right)$ employing **(21)** and represent the results in ranking order of all ten knowledge of system faults is provided in **Table 4**. A The comparison of the results depicted in **Table 3** and **Table 4** indicate that the optimal and worse alternatives remain unchanged whenever there is a small change in the SN set $\left(< x_8, \left[0.00, 0.01, 1.00\right] >\right)$. This clarifies that our symmetric trigonometric NCEM is an insensitive measure when subjected to a little change in the evaluation values. However, the enduring measures **[2,3]** have been found sensitive under this experiment.

Intuitive Analysis For the performance of FCEM, NCEM and existing measures **[12]** under intuitive analysis, we have assumed two fuzzy sets $\left(F_1, F_2\right)$ and SN sets $\left(T_1, T_2\right)$ as depicted in **Table**. In this experiment, we have fixed the value of F_2 as $\left[1.000\right]$, T_2 as $\left[1.000, 0.010, 0.000\right]$ meanwhile, the value of F_1, T_1 are increased gradually as presented in Table. The FCEM and NCEM values along with existing measure values **[]** are calculated using **(20,21)** and the results are presented **Table 5.** The tabulated results reveal that $T_{FS}^{\mu}\left(F_1, F_2\right); T_{NC}\left(T_1, T_2\right)$ values decrease whenever there is a slight increase in the values of F_1, T_1. However, a constant or undefined trend was experienced while repeating this phenomenon with the enduring measures **[2,3]**. This justifies that fault information conveyed by proposed cross entropy measures are feasible and meaningful. Moreover, this also justifies the superiority and remarkability of proposed methodology over the enduring methods **[2,3],** under intuitive analysis.

Table 5. Intuitive analysis of (a) FCEM (b) NCEM (c) Existing Measures **[2,3]**

Gp.	**Fuzzy Set**		**SN Set**		**FCEM**	**NCEM**	**Cosine[3]**	**Measure [3]**
No.	F_1	F_2	T_1	T_2	**Values**	**Values**	**Values**	**Values**
1	0.000	1.000	[0.000,0.010,0.000]	[1.000,0.010,0.000]	0.1272	0.1272	0.0009	#NUM!
2	0.100	1.000	[0.100,0.010,0.000]	[1.000,0.010,0.000]	0.0672	0.0672	0.0905	1.1371
3	0.200	1.000	[0.200,0.010,0.000]	[1.000,0.010,0.000]	0.0544	0.0544	0.0908	1.1262
4	0.300	1.000	[0.300,0.010,0.000]	[1.000,0.010,0.000]	0.0457	0.0457	0.0909	1.1163

5	0.400	1.000	[0.400,0.010,0.000]	[1.000,0.010,0.000]	0.0384	0.0384	0.0909	1.1071
6	0.500	1.000	[0.500,0.010,0.000]	[1.000,0.010,0.000]	0.0316	0.0316	0.0909	1.0984
7	0.600	1.000	[0.600,0.010,0.000]	[1.000,0.010,0.000]	0.0251	0.0251	0.0909	1.0901
8	0.700	1.000	[0.700,0.010,0.000]	[1.000,0.010,0.000]	0.0188	0.0188	0.0909	1.0822
9	0.800	1.000	[0.800,0.010,0.000]	[1.000,0.010,0.000]	0.0124	0.0124	0.0909	1.0745
10	0.900	1.000	[0.900,0.010,0.000]	[1.000,0.010,0.000]	0.0062	0.0062	0.0909	1.0671
11	1.000	1.000	[1.000,0.010,0.000]	[1.000,0.010,0.000]	0.0000	0.0000	0.0909	#NUM!

Conclusion

This study has propounded the establishment of novel symmetric trigonometric fuzzy as well as single valued neutrosophic cross entropy measures (FCEM and NCEM). To overcome the shortcomings faced by non-fuzzy and asymmetrical cross entropy measures and to obtain meaningful fault information, the proposed symmetric FCEM and NCEM has the necessary capability for recognizing the optimal fault conditions such as antithrust bearing and radial impact of friction of rotor, of a huge steam turbine generator. The proposed variants of neutrosophic cross entropy measures are compatible for further mathematical treatments under sensitive and intuitive analysis because of their symmetric quintessence whereas the enduring measures exhibit inconsequential results indicating ambiguity in the evaluation information of fault features

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A Note on μ_N P Spaces

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Abstract: In this article we introduce a new concept called μ_N D Baire spaces and μ_N P spaces, their properties were contemplated.

Keywords: μ_N D Baire space, D/μ_N space, μ_N P space, $\mu_N F_\sigma$ set, $\mu_N G_\delta$ set.

1.Introduction

The concept fuzziness had a great impact in all branches of mathematics which was put forth by Zadeh [16]. Later on the idea of fuzziness and topological spaces were put together by C.L.Chang[3] and laid a foundation to the theory of fuzzy topological spaces. By focussing the membership and non-membership of the elements, K.T.Attanasov[1] made out intuitionistic fuzzy sets and he extended his research towards and gave out a generalization to intuitionistic L-fuzzy sets with his friend Stoeva. F.Smarandache[6,7,8] put his thoughts towards the degree of indeterminacy and bring forth the neutrosophic sets. Subsequently, the neutrosophic topological spaces with the help of neutrosophic sets were found out by A.A.Salama and S.A.Alblowi[11,12,13]. By making all the works together as inspiration, we[9] made Generalized topological spaces via neutrosophic sets and named it as μ_N topological space (μ_NTS). The μ_N nowhere dense sets in μ_NTS were put forth by us [10]. Here by making use of the concepts of μ_N nowhere dense sets, in this paper we introduce a new concept called μ_N D Baire Spaces and μ_N P Spaces, their properties were contemplated.

2.Necessities

Definition 2.1[13] Let X be a non-empty fixed set. A neutrosophic set [NS for short] A is an object having the form$A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function, the degree of indeterminacy and the degree of non-membership function respectively of each element $x \in X$ to the set A.

Remark 2.4.[13] Every intuitionistic fuzzy set A is a non empty set in X is obviously on neutrosophic sets having the form $A = \{\langle \mu_A(x), 1 - \mu_A(x) + \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$. Since our main purpose is to construct the tools for developing neutrosophic set and neutrosophic topology, we must introduce the neutrosophic sets 0_N and 1_N in X as follows:

0_N may be defined as follows

$0_N = \{\langle x, 0,1,1\rangle : x\epsilon X\}$

1_N may be defined as follows

$1_N = \{\langle x, 1,0,0\rangle : x\epsilon X\}$

Definition 2.5.[13] Let $A = \{\langle \mu_A, \sigma_A, \gamma_A\rangle\}$ be a neutrosophic set on X, then the complement of the set A [C(A) for short] may be defined and denoted by C(A) or $\bar{A}$

$C(A) = \{\langle x, \gamma_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle : x \in X\}$

Definition 2.6.[13] Let X be a non-empty set and the neutrosophic sets A and B are in the form of $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle : x \in X\}$.
$A \subseteq B$ may be defined as:
$(A \subseteq B) \Leftrightarrow \mu_A(x) \leq \mu_B(x), \sigma_A(x) \geq \sigma_B(x), \gamma_A(x) \geq \gamma_{\boldsymbol{B}}(x)\ \forall x \in X$
Proposition 2.7. [13] For any neutrosophic set A, the following conditions holds:
$0_N \subseteq$ A, $A \subseteq 1_N$.
Definition 2.8. [13] Let X be a non empty set and $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$
$B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle : x \in X\}$ are neutrosophic sets. Then $A \cap B$ may be defined as:
$A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle$
$A \cup B$ may be defined as:
$A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle$
Definition 2.9[12]. A μ_N topology on a non - empty set X is a family of neutrosophic subsets in X $satisfying\ the\ following\ axioms$:
$(\mu_{N_1}) 0_N \in \mu_N$
$(\mu_{N_2})\ G_1 \cup G_2 \in \mu_N$ for any $G_1, G_2 \in \mu_N$.
Throughout this paper, the pair of (X, μ_N) is known as μ_N topological space (μ_N TS)
Remark 2.10.[12] The elements of μ_N are μ_N open sets and their complement of μ_N open sets are called μ_N closed sets.
Definition 2.11.[12] The $\mu_N -$ Closure of A is the intersection of all μ_N closed sets containing A.
Definition 2.12.[12] The $\mu_N -$ Interior of A is the union of all μ_N open sets contained in A.
Definition 2.13.[11]. A neutrosophic set A in neutrosophic topological space is called neutrosophic dense if there exists no neutrosophic closed sets B in (X, T) such that $A \subset B \subset 1_N$.
Definition 2.14.[10]. The neutrosophic topological spaces is said to be μ_N Baire space if $N\ Int(\bigcup_{i=1}^{\infty} G_i) = 0_N$ where $G_i's$ are neutrosophic nowhere dense set in (X, T).
Theorem 2.15.[10]: Let (X, μ_N) be a μ_NTS. Then the following are equivalent.

(i) (X, μ_N) is μ_N Baire space.
(ii) $\mu_N\ Int(A) = 0_N$, for all μ_N first category set in (X, μ_N).
(iii) $\mu_N\ Cl(A) = 1_N$, μ_N Residual set in (X, μ_N).

3. μ_N *D* Baire spaces

Proposition 3.1: If $\wp$ is a μ_N first category set in a μ_NTS (X, μ_N) such that $\mu_N\ Int(\mu_N Cl\ \wp) = 0_N$,then (X, μ_N) is a μ_N Baire space.

Proof: Let $\wp$ be a μ_N first category set in μ_NTS (X, μ_N) that implies $\wp = \bigvee_{i=1}^{\infty} \wp_i$ where $\wp_i's$ are μ_N nowhere dense sets in (X, μ_N). We know that $\mu_N Int(\mu_N Cl\ \wp) = 0_N$. Also, $\mu_N Int\ \wp \subseteq \mu_N Int\ (\mu_N\ Cl\ \wp)$ that entails us $\mu_N Int(\wp) = 0_N \Rightarrow \mu_N Int(\bigcup_{i=1}^{\infty} \wp_i) = 0_N$, $\wp_i's$ are μ_N nowhere dense sets in $(X, \mu_N) \Rightarrow (X, \mu_N)$ is a μ_N Baire space.

Proposition 3.2: If a μ_N first category set η in a μ_N Baire space (X, μ_N) is a μ_N closed set, then $\mu_N Int(\mu_N Cl\ \eta) = 0_N$ in (X, μ_N).

Proof: Let η be a μ_N first category set in $\mu_N TS$. Owing to the fact that (X, μ_N) is a μ_N Baire space, we have that for every μ_N first category set η in (X, μ_N), $\mu_N Int(\eta) = 0_N$. Now, η is μ_N closed in (X, μ_N) that implies us that $\mu_N\ Cl(\eta) = \eta$. Now we have that $\mu_N Int(\mu_N Cl\ \eta) = \mu_N Int \eta = 0_N \Rightarrow \mu_N Int(\mu_N Cl\ \eta) = 0_N$.

Definition 3.3: A $\mu_N TS$ is called μ_N D Baire space if every μ_N first category set in (X, μ_N) is a μ_N nowhere dense set in (X, μ_N).

Example 3.4: Let $X = \{a, b\}$ and $0_N = \{\langle 0,1,1\rangle\langle 0,1,1\rangle\}, A = \{\langle 0.6,0.4,0.8\rangle\langle 0.8,0.6,0.9\rangle\}, B = \{\langle 0.6,0.3,0.8\rangle\langle 0.9,0.2,0.7\rangle\}$, $C = \{\langle 0.5,0.4,0.9\rangle\langle 0.7,0.8,0.9\rangle\}$, $D = \{\langle 0.4,0.6,0.9\rangle\langle 0.6,0.8,0.9\rangle\}, E = \{\langle 0.3,0.7,0.9\rangle\langle 0.5,0.9,0.9\rangle\}$, $1_N = \{\langle 1,0,0\rangle\langle 1,0,0\rangle\}$. We define a $\mu_N\, TS$ by $\{0_N, A, B, C, D\}$. The μ_N closed sets are $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}, 1_N\}$. Here, 0_N and $\bar{B} = \{\langle 0.8,0.7,0.6\rangle\langle 0.7,0.8,0.9\rangle\}$ are μ_N first category sets and $0_N, E, \bar{B}$ are μ_N nowhere dense. Hence, every μ_N first category is μ_N nowhere dense. Thus, (X, μ_N) is μ_N D Baire space.

Proposition 3.5: If (X, μ_N) is μ_N D Baire Space, then (X, μ_N) is a μ_N Baire space.

Proof: Let ζ be a μ_N first category set in μ_N D Baire space (X, μ_N). Then $\zeta = \bigcup_{i=1}^{\infty} \zeta_i$ where $\zeta_i's$ are μ_N nowhere dense sets and ζ is a μ_N nowhere dense set in (X, μ_N). Thereupon, we obtain that $\mu_N\, Int\,(\ \mu_N Cl\, \zeta) = 0_N$. Hence, $\mu_N\, Int\, \zeta \subseteq\ \mu_N\, Int\,(\ \mu_N Cl\, \zeta) \Rightarrow\ \mu_N\, Int\, \zeta = 0_N$ which entails us that $\mu_N\, Int(\bigcup_{i=1}^{\infty} \zeta_i) = 0_N$ where $\zeta_i's$ are μ_N nowhere dense sets in (X, μ_N). Hence (X, μ_N) is a μ_N Baire space.

Remark 3.6: Converse of the above proposition need not be true. Every μ_N Baire space need not be μ_N D Baire space.This can be explained in the following example.

Example 3.7: Let $X = \{a\}, 0_N = \{\langle 0,1,1\rangle\}, A = \{\langle 0.3,0.3,0.5\rangle\}, B = \{\langle 0.1,0.2,0.3\rangle\}, C = \{\langle 0.3,0.2,0.3\rangle\}, D = \{\langle 0.3,0.6,0.2\rangle\}, E = \{\langle 0.3,0.8,0.5\rangle\}, 1_N = \{\langle 1,0,0\rangle\}$ and we define a μ_NTS as $\{0_N, A, B, C\}$.Here (X, μ_N) is a μ_N Baire space. The μ_N first category sets are 0_N and $\bar{E}$ and the μ_N nowhere dense sets are $0_N, E, \bar{A}, \bar{B}, \bar{C}, \bar{D}$ .Here $\bar{E}$ is μ_N first category set but not μ_N Nowhere dense set. Hence (X, μ_N) is not a μ_N D Baire space.

Proposition 3.8: If δ is an arbitrary μ_N first category set in μ_N Baire space and $\delta\ is\ \mu_N\ Closed$ then (X, μ_N) is μ_N D Baire space.

Proof: Let $\delta = \bigcup_{i=1}^{\infty} \delta_i$, where $\delta_i's$ are μ_N nowhere dense sets. From this we obtain that $\mu_N\, Int(\bigcup_{i=1}^{\infty}\ \delta_i) = 0_N$ because of the fact that (X, μ_N) is μ_N Baire Space. Also we have that $\mu_N Cl(\delta) = \delta$ from this we get that $\mu_N\, Int\big(\mu_N Cl(\delta)\big) = \mu_N\, Int\delta \Rightarrow \mu_N\, Int\big(\mu_N Cl(\delta)\big) = 0_N$. Since δ is μ_N first category set in (X, μ_N). Thus (X, μ_N) is μ_N D Baire space.

Definition 3.9: A neutrosophic set in a μ_NTS (X, μ_N) is called $\mu_N F_\sigma$ set in (X, μ_N) if $\theta = \bigcup_{i=1}^{\infty} \theta_i$, where $\overline{\theta_i} \epsilon\ \mu_N$.

Definition 3.10: A neutrosophic set in a μ_NTS (X, μ_N) is called $\mu_N G_\delta$ set in (X, μ_N) if $\theta = \bigcap_{i=1}^{\infty} \theta_i$, where $\theta_i \epsilon\ \mu_N$.

Proposition 3.11: If α is μ_N dense and $\mu_N G_\delta$ set in (X, μ_N) in a $\mu_N TS$ then $\overline{\alpha}$ is a μ_N first category set in (X, μ_N).

Proof: Since α is $\mu_N G_\delta$ set in $(X, \mu_N), \alpha = \bigcap_{i=1}^{\infty} \alpha_i$ where $\alpha_i\ \epsilon\ \mu_N$ and also α is μ_N dense so we get $\mu_N Cl(\alpha) = 1_N$. Thereupon we get $\mu_N Cl(\bigcap_{i=1}^{\infty} \alpha_i) = 1_N$. But we know that $\mu_N Cl(\bigcap_{i=1}^{\infty} \alpha_i) \subseteq \bigcap_{i=1}^{\infty}\big(\mu_N Cl(\alpha_i)\big)$. Hence, we retrieve that $1_N \subseteq \bigcap_{i=1}^{\infty}\big(\mu_N Cl(\alpha_i)\big) \Rightarrow \bigcap_{i=1}^{\infty}\big(\mu_N Cl(\alpha_i)\big) = 1_N$. Thus we have that for each $\alpha_i\ \epsilon\ \mu_N$, $\mu_N Cl(\alpha_i) = 1_N$. Now, $\mu_N Cl(\mu_N\ Int\ \alpha_i\ ) = 1_N \Rightarrow \overline{\mu_N Cl(\mu_N\ Int\ \alpha_i\ )} = 0_N \Rightarrow \mu_N\ Int\big(\mu_N Cl(\overline{\alpha_i})\big) = 0_N$ which entails us that $\overline{\alpha_i}$ is μ_N nowhere dense sets in (X, μ_N) that implies us $\overline{\alpha} = \bigcup_{i=1}^{\infty}(\overline{\alpha_i})$ is μ_N nowhere dense sets in $(X, \mu_N) \Rightarrow \overline{\alpha}$ is μ_N first category set in (X, μ_N).

Proposition 3.12: If β is μ_N dense and $\mu_N G_\delta$ set in (X, μ_N) in a $\mu_N TS$ then β is a μ_N residual set in (X, μ_N).

Proof: Owing to the fact that β is μ_N dense and $\mu_N G_\delta$ set in (X, μ_N) by using Proposition 3.11 we obtain that $\bar{\beta}$ is μ_N first category set in (X, μ_N). From this we conclude that β is a μ_N residual set in (X, μ_N).

Proposition 3.13: If ν is both μ_N dense and $\mu_N G_\delta$ set, then $\mu_N Int\overline{\nu_i} = 0_N$ where $\nu_i's$ are μ_N nowhere dense sets such that $\overline{\nu} = \bigcup_{i=1}^{\infty}(\overline{\nu_i})$.

Proof: Let ν be a μ_N dense and $\mu_N G_\delta$ set in (X,μ_N). Then by Proposition 3.11, $\overline{\nu}$ is μ_N first category set in (X,μ_N) and $\overline{\nu} = \cup_{i=1}^{\infty}(\overline{\nu_\iota})$ where $\overline{\nu_\iota}$'s are μ_N nowhere dense sets in (X,μ_N). But $\mu_N Int(\overline{\nu}) = \overline{\mu_N Cl\, \nu} = \overline{1_N} = 0_N$. Then $\mu_N Int(\cup_{i=1}^{\infty}(\overline{\nu_\iota})) = \mu_N Int\overline{\nu_\iota} = 0_N \Rightarrow \mu_N Int\overline{\nu_\iota} = 0_N$,where $\overline{\nu_\iota}$'s are μ_N nowhere dense sets in (X,μ_N).

Proposition 3.14: If ξ is μ_N first category set in (X,μ_N) then there is a $\mu_N F_\sigma$ set ℘ in (X,μ_N) such that ξ⊆ ℘.

Proof: Let ξ be a μ_N first category set in (X,μ_N) then ξ= $\cup_{i=1}^{\infty} \xi_i$, where ξ_i's are μ_N nowhere dense sets in (X,μ_N). Now $\overline{\mu_N Cl(\xi)}$ is μ_N open in (X,μ_N) thereupon $\cap_{i=1}^{\infty}\left(\overline{\mu_N Cl(\xi_\iota)}\right)$ is $\mu_N G_\delta$ set in (X,μ_N). Let κ= $\cap_{i=1}^{\infty}\left(\overline{\mu_N Cl(\xi_\iota)}\right)$. On considering κ= $\cap_{i=1}^{\infty}\left(\overline{\mu_N Cl(\xi_\iota)}\right) = \overline{\cup_{\iota=1}^{\infty} \mu_N Cl(\xi_\iota)} \subseteq \overline{\cup_{\iota=1}^{\infty} \xi_\iota} = \bar{\xi} \Rightarrow \kappa \subseteq \bar{\xi} \Rightarrow \xi \subseteq \bar{\kappa}$. Let ℘= $\bar{\kappa}$. Since κ is $\mu_N G_\delta$ set in (X,μ_N) and ℘ is $\mu_N F_\sigma$ set in (X,μ_N). Thus, we obtain "If ξ is μ_N first category set in (X,μ_N) then there is a $\mu_N F_\sigma$ set ℘ in (X,μ_N) such that ξ⊆ ℘".

Remark 3.15: If $\mu_N\, Int\,(\wp) = 0_N$ in Proposition 3.14 then (X,μ_N) is μ_N Baire space. For $\mu_N Int\xi \subseteq \mu_N\, Int(\wp) = 0_N \Rightarrow \mu_N Int\xi = 0_N \Rightarrow (X,\mu_N)$ is μ_N Baire space.

Proposition 3.16: If $\mu_N Cl(\mu_N\, Int\, \gamma) = 1_N$ for every μ_N *dense* and $\mu_N G_\delta$ set in (X,μ_N) then (X,μ_N) is μ_N D Baire space.

Proof: let γ be a μ_N *dense* and $\mu_N G_\delta$ set in (X,μ_N). Then by Proposition 3.11 we obtain that $\bar{\gamma}$ is a μ_N first category set in (X,μ_N). Now, $\mu_N Cl(\mu_N\, Int\, \gamma) = 1_N \Rightarrow \overline{\mu_N Cl(\mu_N\, Int\, \gamma)} = 0_N \Rightarrow \mu_N\, Int(\mu_N Cl\, \bar{\gamma}) = 0_N$. For the μ_N first category set $\bar{\gamma}$ in (X,μ_N) we have that $\mu_N\, Int(\mu_N Cl\, \bar{\gamma}) = 0_N$. Thus, (X,μ_N) is μ_N D Baire space.

Proposition 3.17: If a $\mu_N\, TS$ (X,μ_N) has a μ_N *dense and* $\mu_N G_\delta$ set in (X,μ_N) then (X,μ_N) is not a μ_N D Baire space.

Proof: Let γ be a μ_N *dense* and $\mu_N G_\delta$ set in (X,μ_N). Then by Proposition 3.11 we obtain that $\bar{\gamma}$ is a μ_N first category set in (X,μ_N). Now, $\overline{\mu_N Cl(\mu_N\, Int\, \gamma)} \supseteq \overline{\mu_N Cl\, \gamma} \supseteq \overline{1_N} = 0_N$. Hence, $\mu_N Int(\mu_N Cl\bar{\gamma}) \supseteq 0_N \neq 0_N$, for the μ_N first category set $\bar{\gamma}$ in (X,μ_N). Clearly we get that (X,μ_N) is not a μ_N D Baire space.

Proposition 3.18: If $\mu_N Cl(\mu_N\, Int\, \vartheta) = 1_N$,for every μ_N residual set ϑ in a $\mu_N TS$ (X,μ_N) then (X,μ_N) is a μ_N D Baire space.

Proof: Let ϑ ba a μ_N residual set in a $\mu_N TS$ (X,μ_N). Thereupon $\bar{\vartheta}$ is a μ_N first category set in (X,μ_N). Now, $\mu_N Cl(\mu_N\, Int\, \vartheta) = 1_N \Rightarrow \overline{\mu_N Cl(\mu_N\, Int\, \vartheta)} = 0_N \Rightarrow$ $\mu_N\, Int\, (\mu_N Cl\, \overline{\vartheta}) = 0_N$ that entails us that for a μ_N first category set $\overline{\vartheta}$ in(X,μ_N), $\mu_N\, Int\, (\mu_N Cl\, \overline{\vartheta}) = 0_N$ which leads us into that (X,μ_N) is a μ_N D Baire space.

Proposition 3.19: If a μ_NTS (X,μ_N) is a μ_N D Baire space then there is no non void μ_N dense set is a μ_N first category set in (X,μ_N).

Proof: Suppose that ω is a non-void μ_N first category set and μ_N dense set in (X,μ_N). Owing to the fact that (X,μ_N) is a μ_N D Baire space and ω is a μ_N first category set in (X,μ_N). From this we get that $\mu_N\, Int\, (\mu_N Cl\, \omega) = 0_N$. By our assumption we get $\mu_N Cl\, \omega = 1_N$ that implies us $\mu_N\, Int\, (\mu_N Cl\, \omega) = \mu_N\, Int\, (1_N) \neq 1_N$ which is a contradiction to (X,μ_N) is a μ_N D Baire space. Hence, we must have $\mu_N Cl\, \omega \neq 1_N$. Thereupon we get no non zero μ_N dense set is a μ_N first category set in μ_N D Baire space.

4. μ_N *P Spaces*

Definition 4.1: A μ_NTS is called D/μ_N space if for all non-empty neutrosophic set η in (X,μ_N), $\mu_N Cl\, \eta = 1_N$.

Definition 4.2: A μ_NTS is called μ_N P space if the countable intersection of μ_N open sets is μ_N open. That is every non zero $\mu_N G_\delta$ set in (X, μ_N) is a μ_N open set in (X, μ_N).

Example 4.3: Let $X = \{a\}$. We define neutrosophic sets as $A_1 = \{\langle 0.1.0.4,0.6\rangle\}, A_2 = \{\langle 0.2,0.3,0.5\rangle\}$ and we define a μ_NTS as $\{0_N, A_1, A_2\}$. Here the countable intersection of μ_N open sets are μ_N open. Hence, (X, μ_N) is a μ_N P space.

Example 4.4: Let $X = \{a\}$. We define a μ_NTS as $\{0_N, \omega_1, \omega_2, \omega_3\}$ where $\omega_1 = \{\langle 0.3,0.3,0.5\rangle\}, \omega_2 = \{\langle 0.1,0.2,0.3\rangle\}, \omega_3 = \{\langle 0.3,0.2,0.3\rangle\}, \omega_4 = \{\langle 0.3,0.6,0.2\rangle\}, \omega_5 = \{\langle 0.3,0.8,0.5\rangle\}$. Here the countable intersection of μ_N open set is not μ_N open set in (X, μ_N).

Proposition 4.5: If $\wp$ is a non-zero $\mu_N F_\sigma$ set in a μ_N P space (X, μ_N), then $\wp$ is a μ_N closed set in (X, μ_N).

Proof: Since $\wp$ is a non-zero $\mu_N F_\sigma$ set in (X, μ_N), $\wp = \bigcup_{i=1}^{\infty} \wp_i$ where the neutrosophic sets $\wp_i's$ are μ_N closed in (X, μ_N). Then $\overline{\wp} = \overline{\bigcup_{\iota=1}^{\infty} \wp_\iota} = \bigcap_{i=1}^{\infty} \overline{\wp_\iota}$. Now, $\wp_i's$ are μ_N closed in (X, μ_N) that entails $\overline{\wp} = \bigcap_{i=1}^{\infty} \overline{\wp_\iota}$ where $\overline{\wp_\iota} \epsilon \mu_N$.Thereupon $\overline{\wp}$ is a $\mu_N G_\delta$ set in (X, μ_N).Since (X, μ_N) is a μ_N P space, $\overline{\wp}$ is μ_N open. Therefore, $\wp$ is μ_N closed set in (X, μ_N).

Proposition 4.6: If the μ_N TS (X, μ_N) is a μ_N P space and if $\wp$ is a μ_N first category set in (X, μ_N) then $\wp$ is not a μ_N dense.

Proof: Let us assume that the contrary statement. Suppose that $\wp$ is a μ_N first category set in (X, μ_N) such that $\mu_N Cl(\wp) = 1_N$ where $\wp = \bigcup_{i=1}^{\infty} \wp_i$ and $\wp_i's$ are μ_N nowhere dense sets in (X, μ_N). Now, $\overline{\mu_N Cl(\wp_\iota)}$ is μ_N open in (X, μ_N). Let $\xi = \bigcap_{i=1}^{\infty} \overline{\mu_N Cl(\wp_\iota)}$. Thereupon ξ is a non-zero $\mu_N G_\delta$ set in (X, μ_N). Now we have $\bigcap_{i=1}^{\infty} \overline{\mu_N Cl(\wp_\iota)} = \overline{\bigcup_{\iota=1}^{\infty} \mu_N Cl(\wp_\iota)} \subseteq \overline{\bigcup_{\iota=1}^{\infty} \wp_\iota} = \overline{\wp}$. Thus we obtain that $\xi \subseteq \overline{\wp}$. From this we obtain that $\mu_N Int\,(\xi) \subseteq \mu_N Int\,(\overline{\wp}) = \overline{\mu_N Cl\,\wp} = \overline{1_N} = 0_N$. Since (X, μ_N) is a μ_N P spaces, $\mu_N Int\,(\xi) = \xi$ that yields us that $\xi = 0_N$ which is a strict opposite statement to a non- zero $\mu_N G_\delta$ set in μ_N P space (X, μ_N) that implies us that $\mu_N Cl(\wp) \neq 1_N$. Thereupon we conclude that $\wp$ is not a μ_N dense.

Proposition 4.7: If λ is a μ_N first category set in μ_N P space such that $\sigma \subseteq \bar{\lambda}$ where σ is a non-zero μ_N dense and $\mu_N G_\delta$ set in (X, μ_N) then λ is a μ_N nowhere dense set in (X, μ_N).

Proof: Let λ be a μ_N first category set in (X, μ_N). Then $\lambda = \bigcup_{i=1}^{\infty} \lambda_i$ where λ_i's are μ_N nowhere dense set in (X, μ_N). Now, $\overline{\mu_N Cl(\lambda_\iota)}$ is μ_N open in (X, μ_N). Let $\sigma = \bigcap_{i=1}^{\infty} \overline{\mu_N Cl(\lambda_\iota)} = \overline{\bigcup_{\iota=1}^{\infty} \mu_N Cl(\lambda_\iota)} \subseteq \overline{\bigcup_{\iota=1}^{\infty} \lambda_\iota} = \bar{\lambda}$. Hence, we get that $\sigma \subseteq \bar{\lambda}$. From this we get that $\lambda \subseteq \overline{\sigma}$. Now $\mu_N Int(\mu_N Cl\,\lambda) \subseteq \mu_N Int(\mu_N Cl\,\overline{\sigma})$ which implies us that $\mu_N Int(\mu_N Cl\,\lambda) \subseteq \overline{\mu_N Cl(\mu_N Int\,\sigma)}$. Now owing to the fact that (X, μ_N) is a μ_N P space, the $\mu_N G_\delta$ set σ is $\mu_N$open in (X, μ_N) and μ_N $Int(\sigma) = \sigma$. Therefore we get that $\mu_N Int(\mu_N Cl\,\lambda) \subseteq \overline{\mu_N Cl(\mu_N Int\,\sigma)} = \overline{\mu_N Cl\,\sigma} = \overline{1_N} = 0_N$. Thereupon $\mu_N Int(\mu_N Cl\,\lambda) = 0_N$. Hence λ is μ_N nowhere dense set in (X, μ_N).

Proposition 4.8: If λ is a μ_N first category set in μ_N P space such that $\sigma \subseteq \bar{\lambda}$ where σ is a non-zero μ_N dense and $\mu_N G_\delta$ set in (X, μ_N) then (X, μ_N) is μ_N Baire space.

Proof: Let λ be a μ_N first category set in (X, μ_N). As in the above Proposition 4.7 we have $\mu_N Int(\mu_N Cl\,\lambda) = 0_N$. Thereupon $\mu_N Int\,\lambda \subseteq \mu_N Int(\mu_N Cl\,\lambda)$ which entails us that $\mu_N Int\,\lambda = 0_N$. Thus, we obtain that $\mu_N Int\,\lambda = 0_N$ for every μ_N first category set in (X, μ_N). Hence (X, μ_N) is μ_N Baire space.

Proposition 4.9: If the μ_NTS (X, μ_N) is a μ_N P space and λ is a non-zero μ_N dense and μ_N first category set in (X, μ_N) then there is no non-zero $\mu_N G_\delta$ set in (X, μ_N) such that $\sigma \subseteq \bar{\lambda}$.

Proof: Let λ be a non-zero μ_N first category set in (X, μ_N). Suppose there exists a $\mu_N G_\delta$ set σ in (X, μ_N) such that $\sigma \subseteq \bar{\lambda}$. Thereupon we get $\mu_N Int\, \sigma \subseteq \mu_N Int \bar{\lambda}$ that implies us that $\mu_N Int\, \sigma \subseteq \overline{\mu_N Cl\, \lambda} = 0_N$ *because* λ is μ_N dense. Now we have $\mu_N Int\, \sigma = 0_N$. Since (X, μ_N) is a μ_N P space, $\mu_N Int\, \sigma = \sigma$ and so we obtain $\sigma = 0_N$. Hence we conclude that if λ is μ_N dense and μ_N first category set in (X, μ_N) then there is no non-zero $\mu_N G_\delta$ set in (X, μ_N) such that $\sigma \subseteq \bar{\lambda}$.

Proposition 4.10: If η is a non-empty μ_N residual set in μ_N P space (X, μ_N) then $\mu_N Int\, \eta \neq 0_N$.

Proof: Let η be a non-empty μ_N residual set in μ_N P space (X, μ_N) then $\bar{\eta}$ is a μ_N first category set in (X, μ_N) and hence by proposition 4.6 we obtain that $\bar{\eta}$ is not a μ_N dense set in (X, μ_N). From this we obtain that $\mu_N Cl(\bar{\eta}) \neq 1_N$ which entails us $\overline{\mu_N Int \eta} \neq 1_N \Rightarrow \mu_N Int \eta \neq 0_N$.

Proposition 4.11: If **κ** is a μ_N dense and $\mu_N G_\delta$ set in a μ_N P space (X, μ_N) then $\mu_N Int\, \kappa \neq 0_N$.

Proof: Let κ be a μ_N dense and $\mu_N G_\delta$ set in a μ_N P space (X, μ_N) then by using proposition 3.11 $\bar{\kappa}$ is a μ_N first category set in (X, μ_N). Since (X, μ_N) is μ_N P space by proposition 4.6 $\bar{\kappa}$ is not a μ_N dense set in (X, μ_N) and so $\mu_N Cl(\bar{\kappa}) \neq 1_N \Rightarrow$ that $\overline{\mu_N Int \kappa} \neq 1_N \Rightarrow \mu_N Int \kappa \neq 0_N$.

5. μ_N P space & μ_N Submaximal Space

Proposition 5.1: If each non-zero μ_N G_δ set is a μ_N dense set in a μ_N submaximal space (X, μ_N) then (X, μ_N) is a μ_N P space.

Proof: Let λ be a μ_N G_δ set in a μ_N submaximal space (X, μ_N) then by hypothesis λ is a μ_N dense set in (X, μ_N). Since (X, μ_N) is a μ_N submaximal space, the μ_N dense set λ in (X, μ_N) is μ_N open in (X, μ_N). That is every μ_N G_δ set in (X, μ_N) is μ_N open in (X, μ_N). Thus (X, μ_N) is a μ_N P space.

Proposition 5.2: If μ_N $Int(\lambda) = 0_N$, where λ is a μ_N F_σ set in a μ_N submaximal space (X, μ_N) then (X, μ_N) is a μ_N P space.

Proof: Let λ be a μ_N G_δ set in a μ_N submaximal space (X, μ_N). Then $\bar{\lambda}$ is a μ_N F_σ set in (X, μ_N). By hypothesis μ_N $Int(\bar{\lambda}) = 0_N$, for the μ_N F_σ set $\bar{\lambda}$ in (X, μ_N) which entails us that $\mu_N Cl(\lambda) = 1_N$. Then λ is a μ_N dense set in (X, μ_N). Since (X, μ_N) is a μ_N submaximal space, the μ_N dense set λ in (X, μ_N) is μ_N open in (X, μ_N). Henceforth every μ_N G_δ set in (X, μ_N) is μ_N open in (X, μ_N). Thus we conclude that (X, μ_N) is a μ_N P space.

Proposition 5.3: If each μ_N F_σ set except 1_N is a μ_N nowhere dense set in a μ_N submaximal space (X, μ_N) then (X, μ_N) is a μ_N P space.

Proof: Let λ be a μ_N F_σ set in a μ_N submaximal space (X, μ_N) such that $\mu_N Int(\mu_N Cl\, \lambda) = 0_N$. Then $\mu_N Int(\lambda) \subseteq \mu_N Int(\mu_N Cl\, \lambda) \Rightarrow \mu_N Int(\lambda) = 0_N$. Now, $\mu_N Int(\lambda) = 0_N$ for the μ_N F_σ set λ in μ_N submaximal space (X, μ_N),then by proposition 5.2 (X, μ_N) is a μ_N P space.

Proposition 5.4: If $\mu_N Cl(\mu_N Int\, \lambda) = 1_N$ for each non-empty μ_N G_δ set in a μ_N submaximal space (X, μ_N) then (X, μ_N) is a μ_N P space.

Proof: Let λ be a μ_N F_σ set in a μ_N submaximal space (X, μ_N), then $\bar{\lambda}$ is a μ_N G_δ set in a μ_N submaximal space (X, μ_N). By the given condition $\mu_N Cl\big(\mu_N Int\, \bar{\lambda}\big) = 1_N \Rightarrow \overline{\mu_N Cl\big(\mu_N Int\, \bar{\lambda}\big)} = 0_N$ and hence we retrieve that λ

is a μ_N nowhere dense set in (X, μ_N). Thus the μ_N F_σ set λ is a μ_N nowhere dense set in a μ_N submaximal space (X, μ_N). Hence by proposition 5.3 we derive that (X, μ_N) is a μ_N P space.

Proposition 5.5: If λ is a μ_N residual set in a μ_N submaximal space (X, μ_N) then λ is a μ_N G_δ set in (X, μ_N).

Proof: Let λ be a μ_N residual set in a μ_N submaximal space (X, μ_N) then $\bar{\lambda}$ is a μ_N first category set in (X, μ_N) and so $\bar{\lambda} = \bigcup_{i=1}^{\infty} \lambda_i$,where $\lambda_i's$ are μ_N nowhere dense set in (X, μ_N). By owing to the fact that $\lambda_i's$ are μ_N nowhere dense set in (X, μ_N), $\mu_N Int(\mu_N Cl\, \lambda_i) = 0_N$.Then $\mu_N Int(\lambda_i) \subseteq \mu_N Int(\mu_N Cl\ \lambda_i) \Rightarrow \mu_N Int(\lambda_i) = 0_N \Rightarrow \overline{\mu_N Int(\lambda_i)} = 1_N \Rightarrow \mu_N Cl\, \overline{\lambda_i} = 1_N \Rightarrow \overline{\lambda_i}'s$ are μ_N dense set in (X, μ_N). Since, (X, μ_N) is μ_N submaximal space, $\overline{\lambda_i}'s$ are μ_N open in (X, μ_N) that entails us that $\lambda_i's$ are μ_N closed in (X, μ_N). Hence $\bar{\lambda} = \bigcup_{i=1}^{\infty} \lambda_i$,where $\lambda_i's$ are μ_N closed in (X, μ_N). Thus we retrieve that $\bar{\lambda}$ is μ_N F_σ set in (X, μ_N). Thus, λ is a μ_N G_δ set in (X, μ_N).

Proposition 5.6: If λ is μ_N nowhere dense set in a μ_N submaximal space (X, μ_N) then λ is μ_N closed set in (X, μ_N).

Proof: Let λ be a μ_N nowhere dense set in (X, μ_N) where (X, μ_N) is μ_N submaximal space. Thereupon we obtain that $\mu_N Int(\mu_N Cl(\lambda)) = 0_N$ and $\mu_N Int(\lambda) \subseteq \mu_N Int(\mu_N Cl(\lambda))$ which implies us that $\mu_N Int(\lambda) = 0_N$. Hence $\overline{\mu_N Int(\lambda)} = 1_N$ that yields us $\mu_N Cl(\bar{\lambda}) = 1_N \Rightarrow \bar{\lambda}$ is μ_N dense in (X, μ_N). Since (X, μ_N) is μ_N submaximal space, $\bar{\lambda}$ is μ_N open in (X, μ_N) that yields us λ is μ_N closed set in (X, μ_N).

Proposition 5.7: If a μ_N TS (X, μ_N) is μ_N submaximal space and also μ_N Baire space then (X, μ_N) is μ_N D Baire space.

Proof: Let (X, μ_N) be a μ_N submaximal space and μ_N Baire space. Let λ be the μ_N first category set in (X, μ_N). Since (X, μ_N) is a μ_N Baire space, $\mu_N Int(\lambda) = 0_N$. Thereupon $\overline{\mu_N Int(\lambda)} = 1_N$ that yields us $\mu_N Cl(\bar{\lambda}) = 1_N \Rightarrow \bar{\lambda}$ is μ_N dense in (X, μ_N). Since (X, μ_N) is μ_N submaximal space, $\bar{\lambda}$ is μ_N open in (X, μ_N) that yields us λ is μ_N closed set in (X, μ_N). Now $\mu_N Int(\mu_N Cl(\lambda)) = \mu_N Int(\lambda) = 0_N$. Then λ is μ_N nowhere dense set in (X, μ_N). Hence each μ_N first category set in (X, μ_N) is μ_N nowhere dense set in (X, μ_N). Therefore (X, μ_N) is μ_N D Baire space.

Conclusion: In this article we have listed many new aspects of μ_N topological space with respect to μ_N D-Baire space and μ_N space. In future μ_N filter, μ_N-ultrafilter can be implemented and further the applications of μ_N topological space can be found out.

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Attribute based Double Bounded Rough Neutrosophic Sets in Facial Expression Detection

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Abstract: In this paper, a hybrid intelligent structure called "Double Bounded Rough Neutrosophic Sets" is defined, which is a combination of Neutrosophic sets theory and Rough sets theory. Further, the Attribute based Double Bounded Rough Neutrosophic Sets was implemented using this hybrid intelligent structure for Facial Expression Detection on real time data. Facial expression detection is becoming increasingly important to understand one's emotion automatically and efficiently and is rich in applications. This paper implements some of these applications of facial expression such as: differentiating between Genuine and Fake smiles, prediction of Depression, determining the Degree of Closeness to a particular Attribute/Expression and detection of fake expression during an examination. With the onset of COVID – 19 pandemic, majority of people are choosing to wear masks. A suitable method to detect Facial Expression with and without mask is also implemented. Double Bounded Rough Neutrosophic Sets proposed in this paper is found to yield better results as compared to that of individual structures (Neutrosophic sets theory or Rough sets theory)

Keywords: Double Bounded Rough Neutrosophic Sets, Facial Expression Detection, Facial key points, Neutrosophic sets, Fuzzy set, Rough Set

1. Introduction

Non-verbal communication constitutes a key part of understanding one's emotion, thought process and mentality. Facial expressions, body language and movements/gestures primarily make up non-verbal communication. Hence, biometrics like facial recognition are essential for conversational user experience. Facial recognition is being employed as a standard safety feature in various applications. With latest developments, it is getting increasingly efficient to detect emotions and sentiment through the facial expression of a person. These expressions can further be used to differentiate between different emotions, such as sad, angry, happy, etc.

Counselling systems, lie detection, etc are some among the wide array of applications that automatic facial expression detection has. Facial expressions form a critical aspect of how we communicate, interact and develop impressions of people who we observe and are influenced by. Behavioural scientists like Darwin in 1872 [1,2,3] and Suwa *et al* in 1978 presented an early attempt to automatically analyse facial expressions by tracking the motion of 20 identified spots on an image sequence.

Following this, computer systems were developed which helped us understand and use this natural form of human communication. Research carried out by psychologists [4] indicates that only 7% of the actual information is transmitted orally, and 38% by auxiliary language, such as the rhythm and speed of speech, tone, etc. 55% of information is transmitted by the expression of face. Thus, most of the valuable information can be obtained by facial expression recognition and it provides the best way to judge a person's mental state.

Having said this, there have been numerous methodologies to determine facial expressions. Some of these methodologies involve Neutrosophic sets theory and Rough sets theory which have been implemented in "Facial Expression Recognition Based on Rough Set Theory and SVM" [5], "Face Recognition with Triangular Fuzzy Set-Based Local Cross Patterns in Wavelet Domain" [6], "Facial Expression Recognition based on Fuzzy Networks" [7], etc. These methods are certainly emerging as powerful tools for managing uncertainty, indeterminate, incomplete and imprecise information. This paper mainly focuses on a hybrid intelligent structure called "Rough Neutrosophic Sets" and also introduces "Double Bounded Rough Neutrosophic Sets" which are used for facial expression recognition. The significance of introducing these hybrid set structures is that the computational techniques based on any one of these individual structures will not always yield the best results, but a fusion of two or more of these often provide better results.

2. Materials and Methods

In this section, we give the definitions that are required to study the forth coming sections.

The source code for detection of Facial Expression is publicly available at:

https://github.com/Nethraasivakumar/Facial-Expression-Detection-Using-Double-Bounded-Rough-Neutrosophic-Sets-

https://github.com/poojasrini/Facial-Expression-Detection-using-Double-Bounded-Rough-Neutrosophic-Sets

2.1 Preliminaries:

Definition 2.1.1: Fuzzy set [8]

Fuzzy sets can be considered as an extension and gross oversimplification of classical sets. If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, \mu_a(x)) | x \in X\}$$

μ_a is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in A that maps X to the membership space M (when M contains only the two points 0 and 1, A is nonfuzzy and $\mu_a(x)$ is identical to the characteristic function of a nonfuzzy set). The range of the membership function is a subset of the non-negative real numbers whose supremum is finite. Elements with a zero degree of membership are normally not listed.

Definition 2.1.2: Rough set [9]

Let $I = (U, A)$ be an information system, where U is a non-empty set of finite objects, called the universe and A is a non-empty finite set of fuzzy attributes defined by $\mu_a: U \to [0,1]$, $a \in A$, is a fuzzy set. Formally for any set $P \subseteq A$, there is an associated equivalence relation called Indiscernibility relation defined as follows:

$$IND(P) = \{(x, y) \in U^2 \mid \forall\, a \in P, \mu_a(x) = \mu_a(y)\}$$

The partition induced by $IND(P)$ consists of equivalence classes defined by:

$$[x]_p = \{y \in U \mid (x, y) \in IND(P)\}$$

For any $X \subseteq U$, define the lower approximation space $p_-(X)$ such that

$$p_-(X) = \{x \in U \mid [x]_P \subseteq X\}$$

Also, define the upper approximation space $p^-(X)$ such that

$$p^-(X) = \{x \in U \mid [x]_p \cap X \neq \emptyset\}$$

A rough set corresponding to X, where X is an arbitrary subset of U in the approximation space P, we mean the ordered pair $\{p_-(x), p^-(X)\}$ and it is denoted by $RS(X)$.

Definition 2.1.3: Neutrosophic set [10]

Neutrosophic sets are described by three functions: a membership function, indeterminacy function and a non-membership function that are independently related. The Rough Neutrosophic Set takes the form:

$$N = \{(x, \alpha N\,(x), \beta N\,(x), \gamma N\,(x)) \mid x \in X\}$$

which is characterized by a truth-membership function αN, an indeterminacy-membership function βN and falsity-membership function γN where the functions $\alpha N : X \to]0-, 1+[$, $\beta N : X \to]0-, 1+[$ and $\gamma N : X \to]0-, 1+[$ are real standard or non-standard subsets of $]0-, 1+[$. There is no restriction on the sum of $\alpha N\,(x), \beta N\,(x)$ and $\gamma N\,(x)$, therefore $0- \leq \alpha N\,(x) + \beta N\,(x) + \gamma N\,(x) \leq 3+$.

2.2 Attribute based Double Bounded Rough Neutrosophic Sets

In this section, we define Double Bounded Rough Neutrosophic Sets and some operations on these sets.

Let $I = (U, A)$ be an information system where U is a non-empty finite set of objects and A is a finite set of attributes possessed by the objects in view.

Let $F: A \to \rho(U)$ be a mapping such that for each $a \in A$, $F(a) \subseteq U$, containing those elements of U possessing the attribute a, we assume that $U\, F(a) = U, a \in A$.

Also let $N: U \to \rho(U)$ is a mapping that associates each $x \in U$ to a subset $N(x)$ consisting of the neighbours of x.

Note that the functions F and N are defined according to the systems under consideration and also using the expert knowledge. The function N can also be defined using the relation that prevails among the elements of U.

Now $I = (U, A, F, N)$ is called as a covering based N-information system. Throughout this section we consider this covering based N-information system.

Definition 2.2.1:

Let $I = (U, A, F, N)$ be a covering based N-information system. For any subset X of U define $N(X) = U\, N(x)\,, x \in X$

Definition 2.2.2:

Let $I = (U, A, F, N)$ be a covering based N-information system. For any subset X of U define:

$$DR_{-}(a{\sim}X) = N(F(a) \cap N(x))\,,$$

$$^{-}DR(a{\sim}X) = N(X) \cup (N(F(a)\,) \cap N(X)) \text{ and}$$

$$DR^{-}(a{\sim}X) = N(F(a)) \cup (N\big(F(a)\big) \cap N(X))$$

$DR_{-}(a{\sim}X)$ is called as the lower approximation of X with respect to the attribute a;

$^{-}DR(a{\sim}X)$ is called the left upper approximation of X with respect to the attribute a;

$DR^{-}(a{\sim}X)$ is called the right upper approximation of X with respect to the attribute a;

Definition 2.2.3:

For any subset $X(U)$ define $DRS(a{\sim}X) = (DR_{-}(a{\sim}X),\ ^{-}DR(a{\sim}X), DR^{-}(a{\sim}X))$ is called as the Double Bounded Rough Set of X with respect to the attribute a.

This rough set gives the definite, possible and unascertainable elements of X possessing the attribute a. Note that for each $a \in A, DRS(a{\sim}X)$ can be attained. This method of defining the Attribute based Double Bounded Rough Set will play a significant role in analysing the elements of X with respect to A.

Also, by evaluating the attribute based DBRS for various subsets of U with respect to a single attribute $a \in A$, the significance of $a \in A$ on the subsets can be easily compared.

This DBRS is called as the Attribute based Double Bounded Rough Set of X. Further if there is a set of parameters P defining the attributes and let for each

$p \in P, \mu_p : U \to [0,1]$ be a fuzzy set describing the degree of existence of the parameters on the elements of U . Then a Neutrosophic set can be defined for each $DRS(a{\sim}X)$ as follows,

Let,

$DR = \{DRS(a{\sim}X) | X \subseteq U, a \in A\}$

$DR_{-} = \{DR_{-}(a{\sim}X)\ | X \subseteq U, a \in A\}$

$^{-}DR = \{\ ^{-}DR(a{\sim}X)\ | X \subseteq U, a \in A\}$

$DR^{-} = \{DR^{-}(a{\sim}X)\ | X \subseteq U, a \in A\}$

Definition 2.2.4:

Define a fuzzy set $\mu_{-}: DR_{-} \to [0,1]$ as follows,

$$\mu_{\_}(DR_{\_}(a{\sim}X)) = max\{min(\mu_p(x)\} \ , x \in DR_{\_}(a{\sim}X)$$

similarly, $^{-}\mu$: $^{-}DR \rightarrow [0,1]$ by

$$^{-}\mu(^{-}DR(a{\sim}X)) = max\{min(\mu_p(x)\} \ , x \in \ ^{-}DR(a{\sim}X) \ and$$

$\mu^{-} : DR^{-} \rightarrow [0,1]$ by

$$\mu^{-} \ (DR^{-}(a{\sim}X)) = max\{min(\mu_p(x)\} \ , x \in DR^{-}(a{\sim}X)$$

Hence fuzzy set,

$\bar{\mu}$: DR → [0,1] X [0,1] X [0,1] defined by

$$\bar{\mu} \ (DRS(a{\sim}X)) = \ (\mu_{\_}(DR_{\_}(a{\sim}X)) \ , \ ^{-}\mu(^{-}DR(a{\sim}X)) \ , \mu^{-} \ (DR^{-}(a{\sim}X)))$$

constitutes a Neutrosophic fuzzy set on the set of all Attribute based Double Bounded N- rough sets.

Definition 2.2.5:

From the Neutrosophic fuzzy set, it is possible to predict the facial expression of the object/image.

The attribute value can be calculated using the following expression:

Let: $\mu_{\_}(DR_{\_}(a{\sim}X))$ be denoted by T_a

$^{-}\mu(^{-}DR(a{\sim}X))$ be denoted by I_a

$\mu^{-} \ (DR^{-}(a{\sim}X))$ be denoted by F_a

General Formula to calculate Attribute "a" Value: [12]

$$V(A) = 2\left(\max\left(\left(\frac{T_A + I_A}{2}\right), \left(\frac{1 + I_A - F_A}{2}\right)\right) - \min\left(\left(\frac{T_A + I_A}{2}\right), \left(\frac{1 + I_A - F_A}{2}\right)\right)\right)$$

Example 2.1:

Let $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = \{a_1, a_2, a_3\}$

$F: A \rightarrow P(U)$ is defined by $F(a_1) = \{x_1, x_3\}, F(a_2) = \{x_2, x_4\}, F(a_3) = \{x_5\}$

$Let\ P = \{P1, P2\}$.The fuzzy set μ_{p_1} and μ_{p_2} are tabulated below

$\mu_p \backslash U$	x_1	x_2	x_3	x_4	x_5
μ_{p_1}	0.1	0.3	0.2	0.4	0.7
μ_{p_2}	0.4	0.3	0.8	0.6	0.9

Table 1: Fuzzy set values for objects x_1, x_2, x_3, x_4, x_5

$N(x_1) = \{x_1, x_2, x_3\}$

Lower approximation:

- $DR_-(a_1 \sim x_1) = F(a_1) \cap N(x_1) = \{x_1, x_3\} \cap \{x_1, x_2, x_3\} = \{x_1, x_3\}$

  $\mu_-(DR_-(a_1 \sim x_1)) = max\{min\{\mu_{p_1}(x_1), \mu_{p_1}(x_3)\}, min\{\mu_{p_2}(x_1), \mu_{p_2}(x_3)\}\}$

  $= max\{min\{0.1,0.2\}, min\{0.4,0.8\}\} = max\{0.1,0.4\} = 0.4$

- $DR_-(a_2 \sim x_1) = F(a_2) \cap N(x_1) = \{x_2\}$

  $\mu_-(DR_-(a_2 \sim x_1)) = max\{min\{\mu_{p_1}(x_2)\}, min\{\mu_{p_2}(x_2)\}\} = 0.3$

- $DR_-(a_1 \sim x_1) = F(a_3) \cap N(x_1) = \{\ \}$

  $\mu\_(DR\_(a_3 \sim x_1)) = max\{min\{\ \}, min\{\ \}\} = 0$

Left upper approximation:

- ${}^-DR(a_1 \sim x_1) = N(x_1) \cup (F(a_1) \cap N(x_1))$
  $= \{x_1, x_2, x_3\} \cup (\{x_1, x_3\} \cap \{x_1, x_2, x_3\})$
  $= \{x_1, x_2, x_3\} \cup \{x_1, x_3\} = \{xx_1, x_2, x_3\}$

  ${}^-\mu({}^-DR(a_1 \sim x_1)) = max\{min\{\mu_{p_1}(x_1), \mu_{p_1}(x_2), \mu_{p_1}(x_3)\}, min\{\mu_{p_2}(x_1), \mu_{p_2}(x_2), \mu_{p_2}(x_3)\}\}$

  $= max\{min\{0.1,0.3,0.2\}, min\{0.4,0.3,0.8\}\} = max\{0.1,0.3\} = 0.3$

- ${}^-DR(a_2 \sim x_1) = N(x_1) \cup ((F(a_2)) \cap N(x_1)) = \{x_1, x_2, x_3\}$

  ${}^-\mu({}^-DR(a_2 \sim x_1)) = max\{min\{\mu_{p_1}(x_1), \mu_{p_1}(x_2), \mu_{p_1}(x_3)\}, min\{\mu_{p_2}(x_1), \mu_{p_2}(x_2), \mu_{p_2}(x_3)\}\}$

  $= 0.3$

- ${}^-DR(a_3 \sim x_1) = N(x_1) \cup ((F(a_3)) \cap N(x_1)) = \{x_1, x_2, x_3\}$

  ${}^-\mu\left({}^-DR(a_3 \sim x_1)\right) = max\{min\{\mu_{p_1}(x_1), \mu_{p_1}(x_2), \mu_{p_1}(x_3)\}, min\{\mu_{p_2}(x_1), \mu_{p_2}(x_2), \mu_{p_2}(x_3)\}\}$

  $= 0.3$

Right upper approximation:

- $DR^-(a_1 \sim x_1) = (F(a_1)) \cup ((F(a_1)) \cap N(x_1))$
  $= \{x_1, x_3\} \cup (\{x_1, x_3\} \cap \{x_1, x_2, x_3\})$
  $= \{x_1, x_3\} \cup \{x_1, x_3\} = \{x_1, x_3\}$

  $\mu^-(DR^-(a_1 \sim x_1)) = max\{min\{\mu_{p_1}(x_1), \mu_{p_1}(x_3)\}, min\{\mu_{p_2}(x_1), \mu_{p_2}(x_3)\}\}$

$= max\{min\{0.1,0.2\}, min\{0.4,0.8\}\} = max\{0.1,0.4\} = 0.4$

- $DR^{-}(a_2 \sim x_1) = (F(a_2)) \cup ((F(a_2)) \cap N(x_1)) = \{x_2, x_4\}$
  $\mu^{-}(DR^{-}(a_2 \sim x_1)) = max\{min\{\mu_{p_1}(x_2), \mu_{p_1}(x_4)\}, min\{\mu_{p_2}(x_2), \mu_{p_2}(x_4)\}\} = 0.3$

- $DR^{-}(a_3 \sim x_1) = (F(a_3)) \cup ((F(a_3)) \cap N(x_1)) = \{x_5\}$

  $\mu^{-}(DR^{-}(a_3 \sim x_1)) = max\{min\{\mu_{p_1}(x_5)\}, min\{\mu_{p_2}(x_5)\}\}$

  $= 0.7$

Result:

Table 2: Attributes versus Double Bounded Rough Neutrosophic Sets

a\approximation	$\mu_{-}(DR_{-}(a \sim x))$	$^{-}\mu(^{-}DR(a \sim x))$	$\mu^{-}(DR^{-}(a \sim x))$
a_1	0.4	0.3	0.4
a_2	0.3	0.3	0.3
a_3	0	0.3	0.7

2.3 Implementing Attribute based Double Bounded Rough Neutrosophic Sets to Detect Facial Expressions

The concepts of Double Bounded Rough Neutrosophic Sets were implemented in the decision-making process of detecting facial expressions of humans on real time data.

Objective: To determine the facial expression of a person by classifying into 4 expressions: Sad, Angry, Happy and Surprised.

Data: A is a finite set of attributes possessed by the objects in view. The image of the person's face constitutes an object. Any object possesses one of the four attributes present in A: Sad, Angry, Happy and Surprised.

$$A = \{S, A, H, SU\}$$

Where: S represents Sad

A represents Angry

H represents Happy

SU represents Surprised

U is a non-empty finite set of objects/images. In this illustration, we have taken 200 objects as Universal set, U. The n[th] object is denoted x_n.

The images and the respective parameter values were obtained from the Kaggle Dataset provided by Dr Yoshua Bengio of the University of Montreal. [11]

$$U = \{x_1, x_2, x_3, \dots x_{200}\}$$

The real time data constituted the position of 15 feature points located at pivotal parts of the face/object. Each of these 15 feature points were divided into their respective x and y coordinates, hence resulting in a set of 30 parameters. These 30 parameters were represented by P.

Where:

$$P = \{ P_1, P_2, P_3, \dots P_{30} \}$$

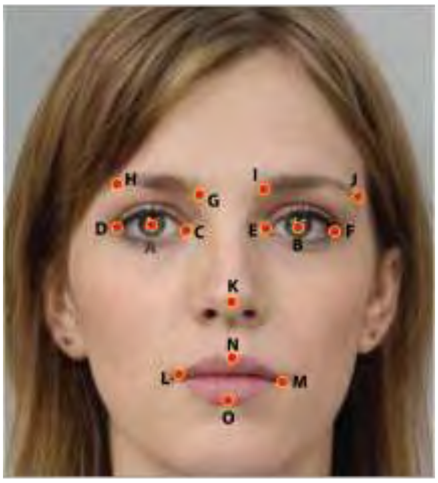

Figure 1: Location of facial feature points on the face

The 15 Facial Feature Points are:

$A\ (P_1\ , P_2\ )$
$B\ (P_3\ , P_4\ )$
$C\ (P_5\ , P_6\ )$
$D\ (P_7\ , P_8\ )$
$E\ (P_9\ , P_{10}\ )$
$F\ (P_{11}\ , P_{12}\ )$
$G\ (P_{13}\ , P_{14}\ )$
$H\ (P_{15}\ , P_{16}\ )$
$I\ (P_{17}\ , P_{18}\ )$
$J\ (P_{19}\ , P_{20}\ )$
$K\ (P_{21}\ , P_{22}\ )$
$L\ (P_{23}\ , P_{24}\ )$
$M\ (P_{25}\ , P_{26}\ )$
$N\ (P_{27}\ , P_{28}\ )$
$O\ (P_{29}\ , P_{30}\ )$

Each of the 200 objects consists of these 30 parameters which are used to define their attribute. The tabulated form of the objects and their respective parameter values are given below. The values of the 30 attributes lie between [0,1].

Table 3: The parameter values for x_1 and x_2

Parameter	Name	$\mu_{P_i}(x_1)$	$\mu_{P_i}(x_2)$
P_1	left_eye_center_x	0.6701	0.6680
P_2	left_eye_center_y	0.3643	0.3572
P_3	right_eye_center_x	0.3120	0.3081
P_4	right_eye_center_y	0.3484	0.3452
P_5	left_eye_inner_corner_x	0.6131	0.6021
P_6	left_eye_inner_corner_y	0.3674	0.3662
P_7	left_eye_outer_corner_x	0.7367	0.7190
P_8	left_eye_outer_corner_y	0.3769	0.3572
P_9	right_eye_inner_corner_x	0.3754	0.3621
P_{10}	right_eye_inner_corner_y	0.3579	0.3512
P_{11}	right_eye_outer_corner_x	0.2549	0.2511
P_{12}	right_eye_outer_corner_y	0.3453	0.3452
P_{13}	left_eyebrow_inner_end_x	0.5624	0.6111
P_{14}	left_eyebrow_inner_end_y	0.2945	0.2822
P_{15}	left_eyebrow_outer_end_x	0.8191	0.7940
P_{16}	left_eyebrow_outer_end_y	0.3167	0.3032
P_{17}	right_eyebrow_inner_end_x	0.4451	0.4191

P_{18}	right_eyebrow_inner_end_y	0.2724	0.2822
P_{19}	right_eyebrow_outer_end_x	0.1757	0.2031
P_{20}	right_eyebrow_outer_end_y	0.2819	0.2882
P_{21}	nose_tip_x	0.5021	0.4881
P_{22}	nose_tip_y	0.5798	0.5521
P_{23}	mouth_left_corner_x	0.5877	0.5811
P_{24}	mouth_left_corner_y	0.7953	0.7351
P_{25}	mouth_right_corner_x	0.3659	0.3531
P_{26}	mouth_right_corner_y	0.7922	0.7321
P_{27}	mouth_center_top_lip_x	0.4863	0.4701
P_{28}	mouth_center_top_lip_y	0.7319	0.6781
P_{29}	mouth_center_bottom_lip_x	0.4736	0.4731
P_{30}	mouth_center_bottom_lip_y	0.8904	0.8131

The parameter values for x_1 and x_2 are given above. All the values lie in [0,1].

Let $F: A \rightarrow \rho(U)$ be a mapping such that for each $a \in A$, $F(a) \subseteq U$. $F(a)$ constitutes those images which possess attribute 'a' such that $a \in A$. Therefore, the 200 images in U are categorised into the 4 attributes present in A. The four attributes are S (Sad), A(Angry), H (Happy) and SU (Surprised).

Table 4: $F(a)$ versus the attribute a

$\boldsymbol{a}$	$\boldsymbol{F(a)}$
$\boldsymbol{S}$	$\{x_{19}, x_{29}, x_{30}, x_{36}, x_{38}, x_{39}, x_{42}, x_{47}, x_{51}, x_{59}, x_{65}, x_{79}, x_{87}$, $x_{91}, x_{97}, x_{107}, x_{117}, x_{126}, x_{127}, x_{129}, x_{145}, x_{146}, x_{147}, x_{150}, x_{156}, x_{158}$, $x_{163}, x_{167}, x_{168}, x_{170}, x_{171}, x_{173}, x_{177}, x_{181}, x_{182}, x_{183}, x_{184}, x_{185}, x_{188}, x_{189}, x_{190}$, $x_{191}, x_{192}, x_{194}, x_{195}, x_{196}, x_{197}, x_{198}, x_{199}, x_{200}\}$
$\boldsymbol{A}$	$\{x_{24}, x_{28}, x_{31}, x_{33}, x_{34}, x_{35}, x_{40}, x_{44}, x_{50}, x_{52}, x_{54}, x_{56}, x_{58}, x_{70}, x_{80}, x_{83}, x_{92}, x_{93}, x_{94}$, $x_{95}, x_{96}, x_{99}, x_{100}, x_{105}, x_{108}, x_{111}, x_{112}, x_{114}, x_{115}, x_{120}, x_{121}, x_{128}, x_{132}, x_{133}, x_{134}$, $x_{135}, x_{136}, x_{137}, x_{138}, x_{139}, x_{142}, x_{143}, x_{152}, x_{153}, x_{160}, x_{174}, x_{180}, x_{186}, x_{187}, x_{193}\}$
$\boldsymbol{H}$	$\{x_3, x_4, x_5, x_9, x_{25}, x_{32}, x_{37}, x_{41}, x_{46}, x_{53}, x_{55}, x_{67}, x_{72}, x_{78}, x_{84}, x_{85}, x_{86}, x_{103}$, $x_{104}, x_{109}, x_{110}, x_{113}, x_{116}, x_{122}, x_{123}, x_{124}, x_{125}, x_{130}, x_{131}, x_{140}, x_{141}, x_{144}, x_{148}, x_{149}$, $x_{151}, x_{154}, x_{155}, x_{157}, x_{159}, x_{161}, x_{162}, x_{164}, x_{165}, x_{166}, x_{169}, x_{172}, x_{175}, x_{176}, x_{178}, x_{179}\}$
$\boldsymbol{SU}$	$\{x_1, x_2, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}$, $x_{20}, x_{21}, x_{22}, x_{23}, x_{26}, x_{27}, x_{43}, x_{45}, x_{48}$, $x_{49}, x_{57}, x_{60}, x_{61}, x_{62}, x_{63}, x_{64}, x_{66}, x_{68}, x_{69}, x_{71}, x_{73}, x_{74}, x_{75}$, $x_{76}, x_{77}, x_{81}, x_{82}, x_{88}, x_{89}, x_{90}, x_{98}, x_{101}, x_{102}, x_{106}, x_{118}, x_{119}\}$

For an image x_n from the Universal Set, the neighbours of x_n are denoted by $N(x_n)$.

$N(x_n)$ is a subset of the Universal Set and consists of images from the Universal set which lie in the neighbourhood of the given image x_n.

In order to compute $N(x_n)$, following steps were implemented.

Algorithm:

For each image $x_m \in$ U,

And for each parameter P_i $(i = 1,2,3 \ldots,30)$,

1. Let ‘q’ be the absolute difference between the value of P_i for image x_n , and the mean value of P_i , where x_n is the given image under consideration.
2. Let ‘r’ be the absolute difference between the value of P_i for image x_m , and the mean value of P_i.
3. The absolute difference of ‘q’ and ‘r’ is computed and is denoted by ‘s’.
4. The value of ‘s’ is compared with the threshold value for the parameter P_i .

The image x_m is said to fall in the neighbourhood of image x_n if at least 25 out of the 30 values of ‘s’ fall within the threshold.

Threshold and number of parameters are subject to the system under study.

In this manner, the neighbourhood of a given image x_n is computed by carrying out the above steps for each image in the Universal Set. Hence, $N(x_n)$ is determined and is a subset of the Universal Set.

In the following table, we give examples for calculating the neighbourhood set.

Table 5: Neighbourhood sets $N(x_n)$ versus the object x_n

x_n	$N(x_n)$
x_1	$\{x_1, x_{36}, x_{41}, x_{62}, x_{66}, x_{83}, x_{178}, x_{189}\}$
x_2	$\{x_2, x_6, x_{16}, x_{111}\}$
x_3	$\{x_3, x_{12}, x_{24}, x_{59}, x_{80}, x_{111}, x_{122}\}$
x_4	$\{x_4, x_7, x_{13}, x_{15}, x_{17}, x_{20}, x_{21}, x_{22}, x_{25}, x_{27}, x_{36}, x_{38}, x_{49}, x_{50}, x_{71}, x_{76}, x_{81}, x_{88}, x_{89}, x_{90}, x_{93}, x_{94}, x_{106}, x_{108}, x_{109}, x_{114}, x_{118}, x_{124}, x_{138}, x_{147}, x_{152}, x_{153}, x_{163}, x_{168}, x_{171}, x_{178}, x_{186}, x_{189}\}$

Now $I = (U, A, F, N)$ is a covering based N-information system.

When $X = \{x_3\}$,

Figure 2: Image/object x_3

$$N(X) = \{x_3, x_{12}, x_{24}, x_{59}, x_{80}, x_{111}, x_{122}\}$$

Following table shows the Double Bounded Rough Sets with respect to X for each attribute.

Table 6: Double Bounded Rough Sets with respect to X versus attribute a

	$DR\_(a\sim X)$	$^{-}DR(a\sim X)$	$DR^{-}(a\sim X)$
S	$\{x_3, x_6, x_{12}, x_{14}, x_{27}, x_{36}, x_{59}, x_{87}, x_{91}, x_{109}, x_{117}, x_{122}, x_{127}, x_{142}, x_{163}, x_{180}, x_{188}, x_{189}\}$	$\{x_3, x_6, x_{12}, x_{14}, x_{24}, x_{27}, x_{36}, x_{59}, x_{80}, x_{87}, x_{91}, x_{109}, x_{111}, x_{117}, x_{122}, x_{127}, x_{142}, x_{163}, x_{180}, x_{188}, x_{189}\}$	$\{x_1, x_3, x_4, x_5, x_6, x_7, x_{12}, x_{13}, x_{14}, x_{15} \dots, x_{191}, x_{192}, x_{193}, x_{194}, x_{195}, x_{196}, x_{197}, x_{198}, x_{199}, x_{200}\}$
A	$\{x_2, x_3, x_6, x_8, x_{12}, x_{16}, x_{24}, x_{30}, x_{35}, x_{46}, x_{52}, x_{58}, x_{80}, x_{99}, x_{111}, x_{116}, x_{128}, x_{130}, x_{131}, x_{142}\}$	$\{x_2, x_3, x_6, x_8, x_{12}, x_{16}, x_{24}, x_{30}, x_{35}, x_{46}, x_{52}, x_{58}, x_{59}, x_{80}, x_{99}, x_{111}, x_{116}, x_{122}, x_{128}, x_{130}, x_{131}, x_{142}\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}, \dots, x_{189}, x_{190}, x_{191}, x_{192}, x_{193}, x_{194}, x_{195}, x_{196}, x_{198}, x_{199}\}$
H	$\{x_3, x_6, x_{12}, x_{24}, x_{36}, x_{44}, x_{59}, x_{70}, x_{80}, x_{82}, x_{101}, x_{111}, x_{118}, x_{122}, x_{127}, x_{130}, x_{138}, x_{158}\}$	$\{x_3, x_6, x_{12}, x_{24}, x_{36}, x_{44}, x_{59}, x_{70}, x_{80}, x_{82}, x_{101}, x_{111}, x_{118}, x_{122}, x_{127}, x_{130}, x_{138}, x_{158}\}$	$\{x_1, x_3, x_4, x_5, x_6, x_7, x_9, x_{11}, x_{12}, x_{13} \dots, x_{186}, x_{187}, x_{189}, x_{190}, x_{192}, x_{193}, x_{194}, x_{195}, x_{198}, x_{199}\}$
SU	$\{x_3, x_6, x_{12}, x_{13}, x_{17}, x_{20}, x_{21}, x_{22}, x_{24}, x_{27}, x_{36}, x_{59}, x_{67}, x_{69}, x_{70}, x_{77}, x_{107}, x_{116}, x_{118}, x_{122}, x_{137}, x_{161}, x_{186}, x_{192}, x_{197}\}$	$\{x_3, x_6, x_{12}, x_{13}, x_{17}, x_{20}, x_{21}, x_{22}, x_{24}, x_{27}, x_{36}, x_{59}, x_{67}, x_{69}, x_{70}, x_{77}, x_{80}, x_{107}, x_{111}, x_{116}, x_{118}, x_{122}, x_{137}, x_{161}, x_{186}, x_{192}, x_{197}\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \dots, x_{190}, x_{191}, x_{192}, x_{193}, x_{194}, x_{195}, x_{196}, x_{197}, x_{198}, x_{199}\}$

From the Double Bounded Rough Sets, the elements of Neutrosophic set are obtained as follows:

Table 7: Neutrosophic sets versus attribute a

	μ_($DR(a\sim X)$)	$^{-}$μ($^{-}DR(a\sim X)$)	μ^{-} (DR^{-} $(a\sim X)$)
S	0.7816	0.7786	0.7597
A	0.7627	0.7627	0.7597
H	0.7786	0.7786	0.7676
SU	0.786	0.7786	0.7597

The Neutrsophic fuzzy set on the set of all attribute for the image x_3 is given by:

μ($DRS(S\sim X)$) = { 0.7816, 0.7786, 0.7597}
μ($DRS(A\sim X)$) = { 0.7627, 0.7627, 0.7597}
μ($DRS(H\sim X)$) = { 0.7786, 0.7786, 0.7676}
μ($DRS(SU\sim X)$) = { 0.7860, 0.7786, 0.7597}

From this Neutrosophic fuzzy set, it is possible to predict the facial expression of the object/image.

The attribute value can be calculated using the following expressions:

$$V(S) = 2\left(\max\left(\left(\frac{T_S + I_S}{2}\right),\left(\frac{1 + I_S - F_S}{2}\right)\right) - \min\left(\left(\frac{T_S + I_S}{2}\right),\left(\frac{1 + I_S - F_S}{2}\right)\right)\right)$$

$$V(A) = 2\left(\max\left(\left(\frac{T_A + I_A}{2}\right),\left(\frac{1 + I_A - F_A}{2}\right)\right) - \min\left(\left(\frac{T_A + I_A}{2}\right),\left(\frac{1 + I_A - F_A}{2}\right)\right)\right)$$

$$V(H) = 2\left(\max\left(\left(\frac{T_H + I_H}{2}\right),\left(\frac{1 + I_H - F_H}{2}\right)\right) - \min\left(\left(\frac{T_H + I_H}{2}\right),\left(\frac{1 + I_H - F_H}{2}\right)\right)\right)$$

$$V(SU) = 2\left(\max\left(\left(\frac{T_{SU} + I_{SU}}{2}\right),\left(\frac{1 + I_{SU} - F_{SU}}{2}\right)\right) - \min\left(\left(\frac{T_{SU} + I_{SU}}{2}\right),\left(\frac{1 + I_{SU} - F_{SU}}{2}\right)\right)\right)$$

Substituting Values from the fuzzy Neutrosophic Set, the following are obtained:

$V(S)$ = 0.5413

$V(A)$ = 0.5224

$V(H)$ = 0.5462

$V(SU)$ = 0.5457

The attribute having the highest value is most likely to be the attribute possessed by the image.

Conclusion**:** The Person is Happy.

3. Results

Implication of Attribute Based Double Bounded Rough Neutrosophic Sets to Detect Facial Expressions:

3.1 By implementing Attribute based Double Bounded Rough Neutrosophic Sets, it is possible to detect the expression of a person with real time data.

$\boldsymbol{V(S) = 0.5466}$
$\boldsymbol{V(A) = 0.2403}$
$\boldsymbol{V(H) = 0.2324}$
$\boldsymbol{V(SU) = 0.5343}$
Person is Sad.

$\boldsymbol{V(S) = 0.5272}$
$\boldsymbol{V(A) = 0.5852}$
$\boldsymbol{V(H) = 0.5366}$
$\boldsymbol{V(SU) = 0.5577}$
Person is Angry.

$\boldsymbol{V(S) = 0.5272}$
$\boldsymbol{V(A) = 0.5303}$
$\boldsymbol{V(H) = 0.5422}$
$\boldsymbol{V(SU) = 0.5272}$
Person is Happy**.**

$V(S) = 0.2403$
$V(A) = 0.2403$
$V(H) = 0.2324$
$V(SU) = 0.7164$
Person is Surprised.

Figure 3: Values of attributes and predicted facial expression for each image

3.2 Clinicians realize that making an accurate diagnosis relies on the provision of reliable information by patients and their family members and that timely, astute, and compassionate care depends on effective bidirectional communications (between the patient and the physician) [13]. Unfortunately, both patients and physicians are often challenged by complicated communications; each group withholds, distorts, obfuscates, fabricates, or lies about information that is crucial to the doctor-patient relationship and to effective treatment. Such untruths and manipulation of information can damage relationships and compromise clinical care.

Facial cues lead to detection of lies and hence can be incorporated in order to detect any sort of miscommunication by the patient.

It is possible to differentiate between genuine smiles and fake smiles using our proposed method. This is often not obvious when seen with naked eye. The advantage of this is that we get a deeper and more realistic insight about the patient's emotion. Below are two images of a patient taken at different instances. A lower (indeterminacy + non-membership) value indicates a realistic smile. As the (indeterminacy + non- membership) value increases, the smile becomes fake.

$V(S) = 0.2403$
$V(A) = 0.2403$
$V(H) = 0.5422$
$V(SU) = 0.2403$
The patient is smiling.

$D(S) = T_S + I_S = 1.5571$
$D(A) = T_A + I_A = 1.5571$
$D(H) = T_H + I_H = 1.5422$
$D(SU) = T_{SU} + I_{SU} = 1.5571$

$D(H)$ is NOT the Highest.
GENUINE SMILE!

$V(S) = 0.5272$
$V(A) = 0.5343$
$V(H) = 0.5352$
$V(SU) = 0.5343$
The patient is smiling.

$D(S) = T_S + I_S = 1.5272$
$D(A) = T_A + I_A = 1.5343$
$D(H) = T_H + I_H = 1.5352$
$D(SU) = T_{SU} + I_{SU} = 1.5343$

$D(H)$ is Highest.
FAKE SMILE!

Figure 4: Illustration showing the distinction between detection of genuine and fake smile

3.3 Sadness is most often the primary emotion that gets transformed into anger. As a result of suppressing their full expression, the energy "becomes" anger. Sadness turns into anger when we realize all our sadness won't resolve the problem. The combination of sadness and anger generally indicates depression. This kind of emotion can be detected when the $V(S) = V(A)$.

$V(S) = 0.6141$
$V(A) = 0.6141$
$V(H) = 0.2324$
$V(SU) = 0.2403$

Person is Sad and Angry
(Possibly Depressed)

Figure 5 : Detection based on combination of expressions

3.4 While detecting facial expressions, it is very important to know how closely the person's expression resembles the detected expression, i.e., the surety/precision of the output.
Using Double Bounded Neutrosophic Sets, we can predict how closely an image resembles any expression. This degree of closeness is denoted by Q(a).

$V(S) = 0.2403$
$V(A) = 0.5659$
$V(H) = 0.2324$
$V(SU) = 0.2403$
The Preson is Angry.

$Q(S) = 25.03$ %
$Q(A) = 58.94$ %
$Q(H) = 24.21$ %
$Q(SU) = 25.03$ %

The degree of closeness to Anger is 58.94 %

Figure 6: Calculation of degree of closeness to the detected attribute

3.5 With the onset of the corona virus pandemic, most people are choosing to wear masks on a regular basis. Thus, many of the feature points on the face will be hidden, which makes it difficult to detect the person's actual expression. However, by using Attribute based Double Bounded Rough Neutrosophic Sets the person's true expression can be detected just by using the feature points in and around the eyes. The image below shows that the prediction of the person's actual expression is possible with and without the mask.




$V(S) = 0.5652$	$V(S) = 0.5376$	$V(S) = 0.2403$	$V(S) = 0.2403$
$V(A) = 0.5413$	$V(A) = 0.2403$	$V(A) = 0.6452$	$V(A) = 0.5193$
$V(H) = 0.2324$	$V(H) = 0.2403$	$V(H) = 0.2324$	$V(H) = 0.2403$
$V(SU) = 0.2403$	$V(SU) = 0.2403$	$V(SU) = 0.5681$	$V(SU) = 0.2403$
Person is Sad	Person is Sad	Person is Angry	Person is Angry

$V(S) = 0.5272$	$V(S) = 0.2403$	$V(S) = 0.2403$	$V(S) = 0.2403$
$V(A) = 0.5303$	$V(A) = 0.2403$	$V(A) = 0.2403$	$V(A) = 0.2403$
$V(H) = 0.5422$	$V(H) = 0.5413$	$V(H) = 0.2324$	$V(H) = 0.2403$
$V(SU) = 0.5272$	$V(SU) = 0.2403$	$V(SU) = 0.5457$	$V(SU) = 0.5275$
Person is Happy	Person is Happy	Person is Surprised	Person is Surprised

Figure 7: Detection of facial expression with and without mask

3.6 All over the world various educational institutes are now slowly moving towards conducting exams online, even competitive exams like GRE, GMAT and English language tests like TOEFL. As more and more exams are conducted online, students tend to involve themselves in various malpractices. Proctors find it difficult to assess each and every student's movement and expression because some might be faking it. But, using this Attribute based Double Bounded Rough Neutrosophic Sets, it becomes easy for the invigilators to detect if the student is actually faking an expression or not. Thus, it ensures that the students don't cheat and helps the universities in getting quality results.

$$\mu(DRS(S\sim X)) = \{0.7869, 0.7869, 0.7597\}$$
$$\mu(DRS(A\sim X)) = \{0.8256, 0.8256, 0.7597\}$$
$$\mu(DRS(H\sim X)) = \{0.7911, 0.7911, 0.7676\}$$
$$\mu(DRS(SU\sim X)) = \{0.8109, 0.8109, 0.7597\}$$

$$V(S) = 0.5465$$
$$V(A) = 0.5852$$
$$V(H) = 0.5587$$
$$V(SU) = 0.5705$$

From above values, we can say that person is Angry.

$$D(S) = T_S + I_S = 1.5465$$
$$D(A) = T_A + I_A = 1.5852$$
$$D(H) = T_H + I_H = 1.5587$$
$$D(SU) = T_{SU} + I_{SU} = 1.5706$$

However, Sum of Falsity and Non-Membership Value is maximum for Anger. Hence, it can be concluded that person is faking the expression.

Figure 8: Cheating detection

4. Applications

Human beings have continually been seeking personal possessions (like nourishment, garments, vehicles, houses, fundamental information and data), ever since the birth of first mankind. It is turning out to be progressively significant that such important resources be preserved and protected by methods for security control. The types of technologies used in the access control systems are countless, throughout history. Traditional methodologies include security guard checks, elementary keypads, locks, passwords and entry codes. However, organisations now seek more progressed technologies with greater security and suitability. They seek an economical way for property protection, particularly in today's multifaceted society.

Fingerprint recognition, iris recognition, voice recognition, and facial recognition systems are some of the popular biometric systems in use today. These systems are being used in various organizations like banks, airports, social services offices, blood banks and other highly sensitive organizations. Biometrics play a very crucial role in today's society as they offer the most accurate authentication solution, and hence as a result of fast increasing technology, facial expression recognition becomes very important. The expressions that we emote are signals that carry high biological value. The key job that these facial articulations perform is that they transmit flags about the expresser's feeling, aims and conditions which are effective in social connection. It has always been a topic of discussion that the evolution of facial expression signalling systems have assisted adaptation. Hence the creditable transmission and decoding of such signals by human operators are of much significance.

Nonverbal communication cues such as facial expressions and other gestures play an important role in interpersonal relations. These cues assist speech by helping the listener to interpret the intended meaning of spoken words. Data from the images or any other visual feed are used in a variety of fields especially for Human Computer Interaction like computer vision, biometric security, social interaction, emotional and social intelligence.

5. Conclusions

A hybrid intelligent structure called "Double Bounded Rough Neutrosophic Sets" was defined. The Attribute based Double Bounded Rough Neutrosophic Sets was implemented for Facial Expression Detection and the following implications were discussed:

1. Detecting the facial expression of a person using real time data
2. Differentiating between Genuine and Fake smiles
3. Predicting if person might be Depressed
4. Determining the Degree of Closeness to a particular Attribute/Expression
5. With the onset of the corona virus pandemic, most people are choosing to wear masks on a regular basis. By using Attribute based Double Bounded Rough Neutrosophic Sets the person's true expression can be detected just by using the feature points in and around the eyes.
6. To check if a person is faking an expression or trying to cheat during an examination.

The results from our work helped us to understand the importance of Attribute based Double Bounded Rough Neutrosophic Sets and we were able to apply it for Facial Expression Detection and its various implications. The future work in this direction is to explore various other applications of double bounded rough Neutrosophic sets and detection of facial expressions using various other concepts.

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Conflicts of Interest

The Authors declare no conflict of interest.

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Entropy and Correlation Coefficients of Neutrosophic and Interval-Valued Neutrosophic Hypersoft Set with application of Multi-Attributive Problems

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Abstract: In computational intelligence, machine learning, image processing, neural networks, medical diagnostics, and decision analysis, the ideas of correlation coefficients and entropy have practical applications. By applying hypersoft set (HSS) in neutrosophic environment provides a good model for describing and addressing uncertainties. In statistics, the correlation coefficient between two variables is crucial. Furthermore, the accuracy of the correlation assessment is dependent on data from the discourse set. The main focus of this study, is to develop entropy for (NHSS) and generalized correlation coefficient interval-valued neutrosophic hypersoft-set (IVNHSS). We proposed some theorems on entropy along with algorithms based on correlation and weighted correlation coefficients in the context of NHSs and IVNHSS. The validity and superiority are presented along with application and also comparison is made with existing approaches.

Keywords: Entropy, Correlation Coefficients, Fuzziness, Soft Set, Hypersoft Set, Neutrosophic Set, Neutrosophic hypersoft set, Interval-valued neutrosophic hypersoft set, MCDM.

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1. Introduction

The joint connection of two variables may be used to analyses the interdependence of two variables using correlation analysis, which is important in statistics and engineering. Despite the fact that probabilistic approaches have been used to a variety of actual engineering issues, probabilistic solutions still face considerable challenges. The probability of a procedure, for example, is determined by based on the enormous amount of random data obtained However, because huge complex systems contain numerous fuzzy uncertainties, obtaining exact probability events is challenging. As a result, outcomes based on probability theory may not always give relevant information for specialists due to a lack of quantitative data. Furthermore, in real applications, there is sometimes insufficient data to make a decision. Experts do not always have access to results based on probability theory due to the aforementioned limitations. As a result, probabilistic approaches are frequently insufficient to resolve data with inherent uncertainties. Many scholars throughout the world have presented and suggested various ways for resolving situations involving ambiguity. To begin, Zadeh created the notion of a fuzzy set (FS) [1], which he used to handle problems involving uncertainty and ambiguity. It is clear that in some circumstances, FS is unable to resolve the matter. To deal with these problems, Turksen [2] created the concept of interval-valued fuzzy sets (IVFS). In some circumstances, membership as a non-member value must be carefully considered in the right representation of objects that cannot be handled by FS or IVFS under unknown conditions. Atanasov suggested the notion of intuitionistic fuzzy sets (IFSs) to resolve these challenges [3]. Atanassov's theory only deals with inadequate data owing to membership and non-membership values; nevertheless, IFS is unable to cope with incompatible and imprecise data. Soft sets were introduced by Molodtsov [4] as a broad mathematical tool for dealing with uncertain, ambiguous, and indeterminate substances (SS). Maji and colleagues [5] SS's work was expanded, and several enterprises with properties were established. They employ SS theory to make judgments in [6]. Ali and others [7] tweaked the SS Maji technique and created some additional operations utilizing its features. They proved De Morgan's rules [8] in the SS environment by utilizing different operators. Cagman and Enginoglu [9] introduced and studied the notion of soft matrices with operations, as well as their attributes. They also devised a decision-making strategy for dealing with unclear circumstances. They adjusted Molodtsov's SS's suggested operation in [10]. By merging FS and SS, Maji et al. [11] established the notion of fuzzy soft set (FSS). They also suggested the Intuitionistic Fuzzy Soft Set (IFSS) [12], which includes fundamental operations and properties. The idea of IFS was developed by Atanassov and Gargov [13], who introduced a new notion called Interval Valued Intuitionistic Fuzzy Set (IVIFS). For illness diagnosis, Jafar et al. [14] used intuitionistic fuzzy soft matrices. Yang et al. [15] presented the idea of interval-valued fuzzy soft sets with operations (IVFSS) and demonstrated several key findings by merging IVFS and SS, as well as applying the established notions to decision-making. By expanding IVIFS, Jiang et al. [16] developed the notion of interval-valued intuitionistic fuzzy soft sets (IVIFSS). They also offered IVIFSS's need and possible operations, as well as their features. Jafar et.al [17-19] suggested a new technique using neutrosophic soft sets and used it in agriculture sciences, applied sanchez approach for medical diagnosis and proposed an algorithm for neutrsphic soft matrices. Ma and Rani [20] built an algorithm based on IVIFSS and utilized it to make decisions. The aggregation operations for bipolar neutrosophic soft sets were developed by Jafar et.al [21]. Naveed et al [22] developed similarity measures of cosine, tangent and cotangent functions in neutrosophic soft sets environments. Maji [23] proposed a neutrosophic soft set (NSS) with all of the required operations and attributes. Karaaslan [24] proposed the potential NSS, which provided the prospect of a neutrosophic soft decision-making approach to tackle situations with uncertainty based on And-product. Broumi [25] created a generic NSS with certain operations and characteristics and utilized it to make decisions. Deli and Subas [26] introduced the notion of cut sets of SVNNs to handle MCDM issues with single-valued Neutrosophic numbers (SVNNs). The term CC of SVNSs [27] was coined based on the IFS correlation, Simplified NSs were introduced along with various operational rules and aggregation operators including weighted arithmetic and weighted geometric average operators. On the basis of proposed aggregation operators, they developed an MCDM technique. A fuzzy logic controller using neutrsphic soft sets presented by jafar et.al. [28]. Hung and Wu [30] introduced the centroid approach for calculating the CC of IFSs and applied it to IVIFS. The correlation and CC of IVIFS were presented by Bustince and Burillo [31], who also established the decomposition theorems on the correlation of IVIFS. The CCs for IFSs and IVIFSs were also created by Hong [32] and Mitchell [33]. Garg and Arora created the TOPSIS methodology using derived correlation metrics and brought them to the IFSS [34]. With these properties, Huang and Guo [35] enhanced the CC on IFS, as well as establishing the IVIFS coefficient. Singh et al. [36] constructed a one- and two-parameter generalization of CC on IFS and used it to multi-attribute group decision-making situations. Naveed et.al [37] devised a decision-making technique for handling multi-criteria decision-making issues by proposing IVFSS. Experts have been known to evaluate the sub-traits of certain attributes when making decisions. In such cases, none of the aforementioned theories can offer experts with knowledge regarding sub-qualities of the specified attributes. Smarandache [38] expanded the notion of soft sets to hypersoft sets (HSS) by substituting the single-parameter function F with a multi-parameter function based on the Cartesian product of n distinct qualities. The well-established HSS is more adaptable than soft sets and better suited to decision-making situations. Crisp HSS, fuzzy HSS, intuitionistic fuzzy HSS, NHSS, and Plithogenic HSS are some of the other HSS extensions he discussed. Today, the HSS theory and its extensions are quickly progressing, and many academics have produced many operators and characteristics based on the HSS theory and its extensions [39-42]. Abdel-Basset et al. [43] employed Plithogenic set theory to cope with uncertainty and analyses the manufacturing industry's financial performance. To attain this purpose, they employed the VIKOR and TOPSIS techniques to calculate the weight of the financial ratio, followed by the AHP approach. Abdel-Basset et al. [44] proposed a successful combination of Plithogenic aggregate operations and quality feature selection. This combination has the benefit of increasing accuracy, which summarizes the decision-makers. Jafar et al. [45] intuitionistic fuzzy hypersoft matrices and proposed an algorithm for solving MADM problems. To overcome the MADM problem, they also devised a decision-making technique based on created TOPSIS. The type 2 neutrosophic numbers were proposed by Basset et al. [46], along with several operational rules. They also created aggregation operators for type 2 neutrosophic numbers and a decision-making methodology to tackle the MADM issue based on the created operators. Basset et al. [47] developed the AHP and VIKOR techniques for calculating neutrosophic numbers and used them to pick suppliers. Basset et al. [48] proposed a robust ranking methodology for managing green supply chains in a neutrosophic setting. Basset et al. [49] developed a neutrosophic multi-criteria decision-making methodology to assist patients and physicians in determining if a patient has heart failure.

The NHSS in Smarandache is incapable of resolving these issues. The object of any sub-truthiness, attribute's indeterminacy, and falsity is supplied in interval form. We know that values change in general; for example, when medical specialists provide a report for a patient, we can see that the HP level of blood ranges between 0 and 17.5; these values are not handled by NHSS. The concept of HSS was extended by Saqlain et. al. [50] he proposed the concept of NHSS with aggregate operators with application to MCDM problems. Then after this concept of NHSS was extended to single and multi-valued neutrosophic hypersoft set with similarity measures and distances [51]. The concept of Interval-valued neutrosophic hypersoft set, m-polar and m-polar neutrosophic hypersoft set was proposed by [52]. The MCDM techniques are also proposed to deal with many daily life issues based on hypersoft set environment theory [53-60]. Jafar et al [61] proposed Trigonometric Similarity measures in NHSs and applied it in Renewable energy source selection. Jafar et.al [62] proposed distance and similarity measures using Max-Min operators and applied it solid waste management system. Many other MCDM techniques used in computer applications by Muslim et al [63] implemented TWOFISH algorithm for data security using activex encryption. Prasetiyo et al [64-65] evaluated about the credit card detection using SMOTE oversampling technique.

To handle the above-discussed environment we need to develop IVNHSS based correlation coefficients and Entropy. The developed IVNHSS Correlations deals with uncertain problems comparative to fuzzy and intuitionistic hypersoft set studies.

The paper is organized as follows: In Section 2, we review some basic definitions used in the following sequels, such as SS, NSS, NHSS, and IVNHSS, etc. In Section 3, Entropy for IVNHSS is proposed along with an algorithm to solve decision-making problem. In Section 4, established the notions of generalized CC and WCC under IVNHSS and discussed their desirable properties with algorithm to solve MCDM. Result Discussion and Comparison are added in section 5. Finally, the current research is concluded with future directions.

2. Preliminaries

In this chapter, some important definitions are listed which will be helpful to understand the thesis and the calculations made.

Definition 2.1: Soft Set [4]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathsf{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathsf{A})$ is called a **soft set** over $\mathcal{U}$ and its mapping is given as;

$$\mathcal{F}: \mathsf{A} \rightarrow \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, \mathsf{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \text{ } if \text{ } e \notin \mathsf{A}\}$$

Definition 2.2 Hypersoft Set [38]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}, (n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of multi-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}})$ is said to be hypersoft set over $\mathcal{U}$ and its mapping is defined as;

$$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \rightarrow \mathcal{P}(\mathcal{U}).$$

It is also defined as

$$(\mathcal{F}, \ddot{\mathsf{A}}) = \{\breve{a}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}): \breve{a} \in \ddot{\mathsf{A}}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) \in \mathcal{P}(\mathcal{U})\}$$

Definition 2.3: Neutrosophic Soft Set [23]

Let ξ be the universal set and € be the set of attributes with respect to ξ. Let P(ξ) be the set of Neutrosophic values of ξ and Ą ⊆ € . A pair (F, Ą) is called a Neutrosophic soft set over ξ and its mapping is given as

$$₣: Ą \rightarrow P(\xi)$$

Definition 2.4: Neutrosophic Hypersoft Set (NHSS) [38]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}, (n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $NS^{\mathcal{U}}$ be a collection of all neutrosophic subsets over $\mathcal{U}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}})$ is said to be Neutrosophic Hypersoft Set over $\mathcal{U}$ and its mapping is defined as

$$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \rightarrow NS^{\mathcal{U}}.$$

It is also defined as;

$(\mathcal{F}, \ddot{\mathsf{A}}) = \{(\breve{a}, \mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a})): \breve{a} \in \ddot{\mathsf{A}}, \mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a}) \in NS^{\mathcal{U}}\}$, where $\mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a}) = \{\langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta), \gamma_{\mathcal{F}(\breve{a})}(\delta) \rangle: \delta \in \mathcal{U}\}$, where $\sigma_{\mathcal{F}(\breve{a})}(\delta)$, $\tau_{\mathcal{F}(\breve{a})}(\delta)$, and $\gamma_{\mathcal{F}(\breve{a})}(\delta)$ represent the truth, indeterminacy, and falsity grades of the attributes such as $\sigma_{\mathcal{F}(\breve{a})}(\delta)$, $\tau_{\mathcal{F}(\breve{a})}(\delta)$, $\gamma_{\mathcal{F}(\breve{a})}(\delta) \in [0, 1]$, and

$0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta) + \tau_{\mathcal{F}(\breve{a})}(\delta) + \gamma_{\mathcal{F}(\breve{a})}(\delta) \leq 3$.

Definition 2.7: Interval-valued Neutrosophic Hypersoft Number (IVNHSN) [42]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}, (n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $IVNS^{\mathcal{U}}$ be a collection of all interval-valued neutrosophic subsets over $\mathcal{U}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}})$ is said to be IVNHSS over $\mathcal{U}$ and its mapping is defined as,

$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \rightarrow IVNS^{\mathcal{U}}$.

It is also defined as;

$(\mathcal{F},\ddot{A}) = \{(\breve{a}_k, \mathcal{F}_{\ddot{A}}(\breve{a}_k)): \breve{a}_k \in \ddot{A},\ \mathcal{F}_{\ddot{A}}(\breve{a}_k) \in NS^{\mathcal{U}}\}$,

where $\mathcal{F}_{\ddot{A}}(\breve{a}) = \{\langle\delta, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta), \tau_{\mathcal{F}(\breve{a}_k)}(\delta), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta)\rangle: \delta \in \mathcal{U}\}$, where $\sigma_{\mathcal{F}(\breve{a}_k)}(\delta)$, $\tau_{\mathcal{F}(\breve{a}_k)}(\delta)$, and $\gamma_{\mathcal{F}(\breve{a}_k)}(\delta)$ represent the interval truth, indeterminacy, and falsity grades of the attributes such as

$\sigma_{\mathcal{F}(\breve{a}_k)}(\delta) = [\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)]$, $\tau_{\mathcal{F}(\breve{a}_k)}(\delta) = [\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)]$,

$\gamma_{\mathcal{F}(\breve{a}_k)}(\delta) = [\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)]$, where $\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)$, $\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)$, $\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)$ $\subseteq [0,1]$, and $0 \le \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta) \le 3$.

Simply an interval-valued neutrosophic hypersoft number (IVNHSN) can be expressed as

$\mathcal{F} = \{[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)], [\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)], [\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)]\}$,

$$\text{Where } 0 \le \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta) \le 3.$$

3. Entropy of NHSS and IVNHSS

In this section, we propose the entropy of neutrosophic hypersoft set (NHSS) and interval-valued neutrosophic hypersoft set (IVNHSS).

- Entropy for NHSS and theorems.
- Entropy for IVNHSS and theorems.

In decision making measure of fuzziness is an important factor. The measurement of fuzziness in neutrosophic environment plays a vital role, since neutrosophic numbers and its decision-making approaches are used in many daily life issues like HR personnel selection, equipment selection, shortest path selection, engineering and medical etc. The validity and superiority can be measure by considering the value of fuzziness, when this value of fuzziness is less, then it can be considered as the best modelling and more accurate.

Definition 3.1: Entropy for NHSS

Let $\mathcal{H}$ *and* $\mathbb{E}$ be defined as;

$\mathcal{H} : \mathcal{H}^1 \times \mathcal{H}^2 \times \mathcal{H}^3 \times \ldots \times \mathcal{H}^n \longrightarrow P(U)$ be neutrosophic hypersoft set, $\mathbb{E} : \mathcal{H} \longrightarrow [0,1]$ such that $\omega \in \mathcal{H}$ and $\omega := ([p: p \in [0,1]], [q: q \in [0,1]], [r: r \in [0,1]])$. Then $\mathbb{E}(\omega)$ is said to be an entropy of neutrosophic hypersoft set if, $\mathbb{E}$ satisfies the following axioms.

(1) $\mathbb{E}(\omega) = 0 \Leftrightarrow (p = q = r = 0)$

(2) $\mathbb{E}(\omega) = 3 \Leftrightarrow (p = q = r = 1)$

(3) $\mathbb{E}(\omega) = \mathbb{E}(\omega^c) \Leftrightarrow (p = q = r = 0.5)$

(4) Let $\omega, \mu \in \mathcal{H}$ then $\mathbb{E}(\omega) \le \mathbb{E}(\mu)$ if $\omega \le_{\mathcal{H}} \mu$.

Where $\mathbb{E}(\omega)$ is defined as;

$$\mathbb{E}(\omega) = \begin{cases} 3 - (P^c + q^c + r^c) & when \quad p, q, r \in [0,1] \\ 0 & \quad otherwise \end{cases} \tag{1}$$

Theorem 3.2 $\mathbb{E}(\omega)$ introduced as (1) is entropy for ω.

Proof: It is easy to see that,

(1) $\mathbb{E}(\omega) = 0 \Leftrightarrow 3 - (P^c + q^c + r^c)$

$$= 3 - (0^c + 0^c + 0^c)$$

$$= 3 - (1 + 1 + 1) = 3 - 3 = 0$$

(2) $\mathbb{E}(\omega) = 3 \Leftrightarrow 3 - (P^c + q^c + r^c)$

$$= 3 - (1^c + 1^c + 1^c)$$

$$= 3 - (0 + 0 + 0) = 3 - 0 = 3$$

(3) $\mathbb{E}(\omega) = \mathbb{E}(\omega^c) \Leftrightarrow 3 - (P^c + q^c + r^c)\ clearly\ satisfied.$

(4) Let $\omega, \mu \in \mathcal{H}$ and $\omega^c = 1 - \omega$ also $\mu^c = 1 - \mu$. If $\omega \leq_{\mathcal{H}} \mu$ when $\omega \leq_{\mathcal{H}} \omega^c$ or $\omega \leq_{\mathcal{H}} \mu$ when $\mu^c \leq_{\mathcal{H}} \mu$ then $\mathbb{E}(\omega) \leq \mathbb{E}(\mu)$. ■

Definition 3.3: Entropy for IVNHSS

Let $\mathcal{H}$ and $\mathbb{E}$ be defined as;

$\mathcal{H} : \mathcal{H}^1 \times \mathcal{H}^2 \times \mathcal{H}^3 \times \ldots \times \mathcal{H}^n \longrightarrow P(U)$ be interval-valued neutrosophic hypersoft set, $\mathbb{E} : \mathcal{H} \longrightarrow [0,1]$ such that $\omega \in \mathcal{H}$ and $\omega := ([p^l, p^u] \in [0,1], [q^l, q^u] \in [0,1], [r^l, r^u] \in [0,1])$. Then $\mathbb{E}(\omega)$ is said to be an entropy of neutrosophic hypersoft set if, $\mathbb{E}$ satisfies the following axioms.

(5) $\mathbb{E}(\omega) = 0 \Leftrightarrow [p^l, p^u] = [0, 0]$ or $[1, 1]$ and $[r^l, r^u] = [0, 0]$ or $[1, 1]$

(6) $\mathbb{E}(\omega) = 1 \Leftrightarrow [p^l, p^u] = [q^l, q^u] = [r^l, r^u] = [0.5, 0.5]$

(7) $\mathbb{E}(\omega) = \mathbb{E}(\omega^c)$

(8) Let $\omega, \mu \in \mathcal{H}$ then $\mathbb{E}(\omega) \leq \mathbb{E}(\mu)$ if $\omega \leq_{\mathcal{H}} \mu$.

Where $\mathbb{E}(\omega)$ is defined as;

$$\mathbb{E}(\omega) = \begin{cases} 1 - \frac{|q^l + q^u - 1|}{2}, [p^l, p^u] = [r^l, r^u] = [0.5, 0.5] \\ \frac{1}{2} - \frac{1}{2}\{\max\{|p^l - r^l|, |p^u - r^u|\}\} \qquad otherwise \end{cases} \tag{2}$$

Theorem 3.4

$\mathbb{E}(\omega)$ Introduced as (2) is entropy for ω.

Proof: It is easy to see that,

(5) $\mathbb{E}(\omega) = 0 \Leftrightarrow \frac{1}{2} - \frac{1}{2}\{\max\{|p^l - r^l|, |p^u - r^u|\}\}$

$\Leftrightarrow [p^l, p^u] = [0, 0]$ or $[1, 1]$, $[r^l, r^u] = [0, 0]$ or $[1, 1]$.

(6) $\mathbb{E}(\omega) = 1 \Leftrightarrow [p^l, p^u] = [q^l, q^u] = [r^l, r^u] = [0.5, 0.5]$

(7) $\mathbb{E}(\omega) = \mathbb{E}(\omega^c)\ clearly\ satisfied.$

(8) Let $\omega := ([p^l, p^u] \in [0,1], [q^l, q^u] \in [0,1], [r^l, r^u] \in [0,1]), \mu := ([p^{2l}, p^{2u}] \in [0,1], [q^{2l}, q^{2u}] \in [0,1], [r^{2l}, r^{2u}] \in [0,1]) \in \mathcal{H}$

then $\mu^c = ([r^{2l}, r^{2u}], [1 - r^{2u}, 1 - r^{2l}], [p^{2l}, p^{2u}])$

If $\omega \leq_{\mathcal{H}} \mu$ when $\mu \leq_{\mathcal{H}} \mu^c$ or $\omega \leq_{\mathcal{H}} \mu$ when $\mu^c \leq_{\mathcal{H}} \mu$ then $\mathbb{E}(\omega) \leq \mathbb{E}(\mu)$.

■

4. Generalized Correlation Coefficients of IVNHSS

In this section we propose the generalized correlation coefficients of IVNHSS.

4.1: Calculations

Assume that there are two interval valued neutrosophic hypersoft Set A and B in the universe of discourse U = {u[1], u[2], u[3]… u[n]}

$$A = \sum_{1}^{n} \langle \begin{Bmatrix} \{A_{1^a}[\inf T_A(x_i), \sup T_A(x_i)], [\inf I_A(x_i), \sup I_A(x_i)], [\inf F_A(x_i), \sup F_A(x_i)]\} \\ \{A_{2^b}[\inf T_A(x_i), \sup T_A(x_i)], [\inf I_A(x_i), \sup I_A(x_i)], [\inf F_A(x_i), \sup F_A(x_i)]\} \\ \vdots \\ \{A_{n^z}[\inf T_A(x_i), \sup T_A(x_i)], [\inf I_A(x_i), \sup I_A(x_i)], [\inf F_A(x_i), \sup F_A(x_i)]\} \end{Bmatrix} \rangle$$

$$B = \sum_{1}^{n} \langle \begin{Bmatrix} \{B_{1^a}[\inf T_B(x_i), \sup T_B(x_i)], [\inf I_B(x_i), \sup I_B(x_i)], [\inf F_B(x_i), \sup F_B(x_i)]\} \\ \{B_{2^b}[\inf T_B(x_i), \sup T_B(x_i)], [\inf I_B(x_i), \sup I_B(x_i)], [\inf F_B(x_i), \sup F_B(x_i)]\} \\ \vdots \\ \{B_{n^z}[\inf T_B(x_i), \sup T_B(x_i)], [\inf I_B(x_i), \sup I_B(x_i)], [\inf F_B(x_i), \sup F_B(x_i)]\} \end{Bmatrix} \rangle$$

Where

E = set of attributes

$A \subseteq E$ and $A_{1^a}, A_{2^b}, A_{3^c}, \dots, A_{n^z}$ are bifurcated attributes of A

B⊆ E and $B_{1^a}, B_{2^b}, B_{3^c}, \dots, B_{n^z}$ are bifurcated attributes of B.

Correlation of IVNHSS ($\boldsymbol{C_{IVNHSS}}$)

$$\boldsymbol{C_{IVNHSS}} = \sum_{1}^{n} \begin{Bmatrix} \begin{bmatrix} \{(A_{1^a} \inf T_A(x_i) . B_{1^a} \inf T_B(x_i) + A_{1^a} \sup T_A(x_i) . B_{1^a} \sup T_B(x_i))\} \\ \{(A_{2^a} \inf T_A(x_i) . B_{2^a} \inf T_B(x_i) + A_{2^a} \sup T_A(x_i) . B_{2^a} \sup T_B(x_i))\} \\ \vdots \\ \{(A_{n^z} \inf T_A(x_i) . B_{n^z} \inf T_B(x_i) + A_{n^z} \sup T_A(x_i) . B_{n^z} \sup T_B(x_i))\} \end{bmatrix} \\ + \\ \begin{bmatrix} \{(A_{1^a} \inf I_A(x_i) . B_{1^a} \inf I_B(x_i) + A_{1^a} \sup I_A(x_i) . B_{1^a} \sup I_B(x_i))\} \\ \{(A_{2^a} \inf I_A(x_i) . B_{2^a} \inf I_B(x_i) + A_{2^a} \sup I_A(x_i) . B_{2^a} \sup I_B(x_i))\} \\ \vdots \\ \{(A_{n^z} \inf I_A(x_i) . B_{n^z} \inf I_B(x_i) + A_{n^z} \sup I_A(x_i) . B_{n^z} \sup I_B(x_i))\} \end{bmatrix} \\ + \\ \begin{bmatrix} \{(A_{1^a} \inf F_A(x_i) . B_{1^a} \inf F_B(x_i) + A_{1^a} \sup F_A(x_i) . B_{1^a} \sup F_B(x_i))\} \\ \{(A_{2^a} \inf F_A(x_i) . B_{2^a} \inf F_B(x_i) + A_{2^a} \sup F_A(x_i) . B_{2^a} \sup F_B(x_i))\} \\ \vdots \\ \{(A_{n^z} \inf F_A(x_i) . B_{n^z} \inf F_B(x_i) + A_{n^z} \sup F_A(x_i) . B_{n^z} \sup F_B(x_i))\} \end{bmatrix} \end{Bmatrix}$$

E (A) → Informational energy of A

$$E(A) = \sum_{1}^{n} \begin{Bmatrix} \left[\left(A_{1^a}T_{AL}^2(x_i) + A_{1^a}T_{AU}^2(x_i)\right) + \left(A_{2^a}T_{AL}^2(x_i) + A_{2^a}T_{AU}^2(x_i)\right) + \cdots \left(A_{n^z}T_{AL}^2(x_i) + A_{n^z}T_{AU}^2(x_i)\right)\right] \\ + \\ \left[\left(A_{1^a}I_{AL}^2(x_i) + A_{1^a}I_{AU}^2(x_i)\right) + \left(A_{2^a}I_{AL}^2(x_i) + A_{2^a}I_{AU}^2(x_i)\right) + \dots \left(A_{n^z}I_{AL}^2(x_i) + A_{n^z}I_{AU}^2(x_i)\right)\right] \\ + \\ \left[\left(A_{1^a}F_{AL}^2(x_i) + A_{1^a}F_{AU}^2(x_i)\right) + \left(A_{2^a}F_{AL}^2(x_i) + A_{2^a}F_{AU}^2(x_i)\right) + \cdots \left(A_{n^z}F_{AL}^2(x_i) + A_{n^z}F_{AU}^2(x_i)\right)\right] \end{Bmatrix}$$

E (B) → Informational energy of B

$$E(B) = \sum_{1}^{n} \begin{Bmatrix} \left[\left(B_{1^a}T_{AL}^2(x_i) + B_{1^a}T_{AU}^2(x_i)\right) + \left(B_{2^a}T_{AL}^2(x_i) + B_{2^a}T_{AU}^2(x_i)\right) + \cdots \left(B_{n^z}T_{AL}^2(x_i) + B_{n^z}T_{AU}^2(x_i)\right)\right] \\ + \\ \left[\left(B_{1^a}I_{AL}^2(x_i) + B_{1^a}I_{AU}^2(x_i)\right) + \left(B_{2^a}I_{AL}^2(x_i) + B_{2^a}I_{AU}^2(x_i)\right) + \dots \left(B_{n^z}I_{AL}^2(x_i) + B_{n^z}I_{AU}^2(x_i)\right)\right] \\ + \\ \left[\left(B_{1^a}F_{AL}^2(x_i) + B_{1^a}F_{AU}^2(x_i)\right) + \left(B_{2^a}F_{AL}^2(x_i) + B_{2^a}F_{AU}^2(x_i)\right) + \cdots \left(B_{n^z}F_{AL}^2(x_i) + B_{n^z}F_{AU}^2(x_i)\right)\right] \end{Bmatrix}$$

Where

T_{AL} = infimum (lower bound) of truthness value of A

I_{AL} = infimum (lower bound) of indeterminacy value of A

I_{AU} = supremum (upper bound) of indeterminacy value of A

F_{AL}= infimum (lower bound) of Falsity value of A

F_{AU} = supremum (upper bound) of Falsity value of B

And

T_{BL} = infimum (lower bound) of truthiness value of B

T_{BU} = supremum (upper bound) of truthiness value of B

I_{BL}= infimum (lower bound) of indeterminacy value of B

I_{BU} = supremum (upper bound) of indeterminacy value of B

F_{BL}= infimum (lower bound) of Falsity value of B

U = supremum (upper bound) of Falsity value of B

Correlation coefficient of IVNHSS

Let A and B be the two IVNHSS then the correlation coefficient of A and B is denoted by $\mathcal{R}$(A, B) and defined as

$$R(A,B) = \frac{CIVNHSS}{(E(A))^{\frac{1}{2}}.(E(B))^{\frac{1}{2}}} \in [0, 1^{+} [$$

$\mathcal{R}$(A, B) Satisfies the following properties

(1) $0 \leq \mathcal{R}(A, B) \leq 1$

(2) $\mathcal{R}$(A, B) = $\mathcal{R}$(B, A)

(3) $\mathcal{R}$(A, B) = 1 if A = B

Also the value of T, I, F should be independent of each other, i.e

$$0 \leq \sup T_A(x_i) + \sup I_A(x_i) + \sup F_A(x_i) \leq 3$$

Information about IVNHSS (A)	Information about IVNHSS (B)
$\inf T_A(x_i) \leq \sup T_A(x_i)$	$\inf T_B(x_i) \leq \sup T_B(x_i)$
$\inf I_A(x_i) \leq \sup I_A(x_i)$	$\inf I_B(x_i) \leq \sup I_B(x_i)$
$\inf F_A(x_i) \leq \sup F_A(x_i)$	$\inf F_B(x_i) \leq \sup F_B(x_i)$
$\inf T_A(x_i), \inf I_A(x_i), \inf F_A(x_i) \in [0,1]$	$\inf T_B(x_i), \inf I_B(x_i), \inf F_B(x_i) \in [0,1]$
$\sup T_A(x_i), \sup I_A(x_i), \sup F_A(x_i) \in [0,1]$	$\sup T_B(x_i), \sup I_B(x_i), \sup F_B(x_i) \in [0,1]$

4.2: Case Study

To discuss the

- Validity
- Applicability

of the proposed algorithm, best school selection is considered as a MCDM problem.

Numerical Example:

Let U be the set of different schools nominated for best school given as U = {s^1, s^2, s^3, s^4, s^5} and consider the set of attributes as E = {Teaching standard, organization, ongoing evaluation, Goals}, consider the subset of attributive set A ⊆ E which is $A = \{A_1, A_2, A_3, A_4\}$ where

$$A = \begin{cases} A_1 = \text{teaching standard} \\ A_2 = \text{organization} \\ A_3 = \text{ongoing evaluation} \\ A_4 = \text{goals} \end{cases}$$

These attributes are further bifurcated as

$$A=\begin{cases}A_1^a = A_1 = \text{teaching standard} = \langle \text{High, mediocre, low}\rangle \\ A_1^b = A_2 = \text{organization} = \langle \text{good, average, poor}\rangle \\ A_1^c = A_3 = \text{ongoing evaluation} = \langle \text{yes, no}\rangle \\ A_1^d = A_4 = \text{goals} = \langle \text{effective, committed, up to date}\rangle\end{cases}$$

For discussion we suppose a **SIVNHSS** F (high, average, yes, effective) $= \{ s^1, s^5\}$ then

$$A=\sum_1^n\left\langle\begin{matrix}\left\{\begin{matrix}s^1\ \{\textbf{high}\ ([0.6, 0.8], [0.2, 0.3], [0.1, 0.2])\} + \{\textbf{average}([0.7, 0.8], [0.2, 0.4], [0.1, 0.3])\} \\ +\{\textbf{yes}([0.4, 0.6], [0.3, 0.5], [0.1, 0.3])\} + \{(\textbf{effective}[0.5, 0.7], [0.2, 0.4], [0.2, 0.3])\}\end{matrix}\right\} \\ \left\{\begin{matrix}s^5\ \{\textbf{high}\ ([0.6, 0.8], [0.4, 0.5], [0.2, 0.4])\} + \{\textbf{average}([0.8, 0.9], [0.4, 0.6], [0.2, 0.3])\} \\ +\{\textbf{yes}([0.5, 0.7], [0.1, 0.3], [0.1, 0.4])\} + \{(\textbf{effective}[0.6, 0.8], [0.2, 0.5], [0.2, 0.4])\}\end{matrix}\right\}\end{matrix}\right\rangle$$

Let B ⊆ E, B = {B_1, B_2, B_3, B_4} and

$$B=\begin{cases}B_1 = \text{teaching standard} \\ B_2 = \text{organization} \\ B_3 = \text{ongoing evaluation} \\ B_4 = \text{goals}\end{cases}$$

Further bifurcated attributes of B are

$$A=\begin{cases}B_1^a = A_1 = \text{teaching standard} = \langle \text{High, mediocre, low}\rangle \\ B_1^b = A_2 = \text{organization} = \langle \text{good, average, poor}\rangle \\ B_1^c = A_3 = \text{ongoing evaluation} = \langle \text{yes, no}\rangle \\ B_1^d = A_4 = \text{goals} = \langle \text{effective, committed, up to date}\rangle\end{cases}$$

For discussion we suppose a **SIVNHSS** F (high, good, yes, up − to date) $= \{ s^2, s^3\}$ then

$$B=\sum_1^n\left\langle\begin{matrix}\left\{\begin{matrix}s^2\ \{\textbf{high}\ ([0.7, 0.9], [0.2, 0.4], [0.1, 0.3])\} + \{\textbf{good}([0.6, 0.8], [0.4, 0.5], [0.2, 0.3])\} \\ +\{\textbf{yes}([0.7, 0.9], [0.4, 0.5], [0.2, 0.3])\} + \{(\textbf{up}-\textbf{to date}[0.8, 0.9], [0.5, 0.6], [0.3, 0.4])\}\end{matrix}\right\} \\ \left\{\begin{matrix}s^3\ \{\textbf{high}\ ([0.6, 0.8], [0.2, 0.4], [0.1, 0.3])\} + \{\textbf{good}([0.7, 0.9], [0.3, 0.5], [0.1, 0.3])\} \\ +\{\textbf{yes}([0.8, 0.9], [0.1, 0.3], [0.2, 0.4])\} + \{(\textbf{up}-\textbf{to date}[0.6, 0.8], [0.3, 0.4], [0.3, 0.5])\}\end{matrix}\right\}\end{matrix}\right\rangle$$

C_{VNHSS}

$$=\sum_1^n\left\{\left[\begin{matrix}((0.6)(0.7) + (0.8)(0.9)\} + \{(0.7)(0.6) + (0.8)(0.8)\} + \{(0.4)(0.7) + (0.6)(0.9)\}) \\ +\{(0.5)(0.8) + ((0.7)(0.9))\}] + [\{(0.2)(0.2)) + (0.3)(0.4))\} + \{((0.2)(0.4)) + ((0.4)(0.5)\} \\ +\{(0.3)(0.4)) + (0.5)(0.6))\} + \{(0.2)(0.5) + (0.4)(0.6)\}] + [\{(0.1)(0.1) + (0.2)(0.3)\} \\ +\{(0.1)(0.2) + (0.3)(0.3)\} + \{(0.1)(0.4) + (0.3)(0.5)\} + \{(0.2)(0.3) + (0.3)(0.4)\}] \\ +\{[\{(0.6)(0.6) + (0.8)(0.8)\} + \{(0.8)(0.7) + (0.9)(0.9)\} + \{(0.5)(0.8) + (0.7)(0.9)\} + \\ +\{(0.6)(0.6) + (0.8)(0.8)\}] + [\{(0.4)(0.2) + (0.5)(0.4)\} + \{(0.4)(0.3) + (0.6)(0.5)\} \\ +\{(0.1)(0.1) + (0.3)(0.3)\} + \{(0.2)(0.3) + (0.5)(0.4)\}] + [\{(0.2)(0.1) + (0.4)(0.3)\} \\ +\{(0.2)(0.1) + (0.3)(0.3)\} + \{(0.1)(0.2) + (0.4)(0.4)\} + \{(0.2)(0.3) + (0.4)(0.5)\}\end{matrix}\right]\right\}$$

C_{VNHSS}

$$=\sum_1^n\left\{\left[\begin{matrix}\{0.42 + 0.72\} + \{0.42 + 0.64\} + \{0.28 + 0.54\} + \{0.4 + 0.63\}] + [\{0.04 + 0.12\} \\ +\{0.08 + 0.2\} + \{0.12 + 0.3\} + \{0.7 + 0.24\}] + [\{0.01 + 0.06\} + \{0.02 + 0.09\} \\ +\{0.04 + 0.15\} + \{0.06 + 0.12\}]\} + \{[\{0.36 + 0.64\} + \{0.56 + 0.8\} + \{0.4 + 0.63\} \\ +\{0.36 + 0.64\}] + [\{0.08 + 0.2\} + \{0.12 + 0.3\} + \{0.01 + 0.09\} + \{0.06 + 0.2\}] \\ +[\{0.02 + 0.12\} + \{0.02 + 0.09\} + \{0.02 + 0.16\} + \{0.06 + 0.2\}\end{matrix}\right]\right\}$$

$$C_{VNHSS}=\sum_1^n\left\{\left[\begin{matrix}[\{1.14 + 1.06 + 0.82 + 0.252\} + \{0.16 + 0.28 + 0.42 + 1.36\} \\ +\{0.07 + 0.11 + 0.19 + 0.18\}] + [\{1 + 1.36 + 0.252 + 1\} \\ +\{0.28 + 0.42 + 0.1 + 0.26\} + \{0.14 + 0.11 + 0.18 + 0.26\}\end{matrix}\right]\right\}$$

$$C_{VNHSS} = \sum_{1}^{n}\{[3.72 + 2.22 + 0.55] + [3.162 + 1.06 + 0.69]\} = 11.404$$

E(A) = {{[(0.6)² + (0.8)² + (0.7)² + (0.8)² + (0.4)² + (0.6)² + (0.5)2+(0.7)²] + [(0.2)² + (0.3)² + (0.2)² + (0.4)² + (0.3)² + (0.5)² + (0.2)² + (0.4)²] + [(0.1)² + (0.2)² + (0.1)² + (0.3)² + (0.1)² + (0.3)² + (0.2)² + (0.3)²]} + {[(0.6)²+ (0.8)² + (0.8)² + (0.9)² + (0.5)² + (0.7)² + (0.6)² + (0.8)²] + [(0.4)² + (0.5)² + (0.4)² + (0.6)² + (0.1)² + (0.3)² + (0.2)² + (0.5)²] + [(0.2)² + (0.4)² + (0.2)² + (0.3)² + (0.1)² + (0.4)² + (0.2)² + (0.4)²Type equation here.]}

$$\textbf{E (A)} = \sum_{1}^{n}\{\{ 3.64 + 0.87 + 0.38\} + \{4.19 + 1.32 + 0.7\}\}$$

$$\textbf{E (A)} = \sum_{1}^{n}\{\{ 4.89 + 6.21\}$$

$$\textbf{E (A)} = 11.1$$

E(B) = $\sum_{1}^{n}$ {{[(0.7)² + (0.9)² + (0.6)² + (0.8)² + (0.7)² + (0.9)² + (0.8)2+(0.9)²] + [(0.2)² + (0.4)² + (0.4)² + (0.5)² + (0.4)² + (0.6)² + (0.5)² + (0.6)²] + [(0.1)² + (0.3)² + (0.2)² + (0.3)² + (0.4)² + (0.5)² + (0.3)² + (0.4)²]} + {[(0.6)²+ (0.8)² + (0.7)² + (0.9)² + (0.8)² + (0.9)² + (0.6)² + (0.8)²] + [(0.2)² + (0.4)² + (0.3)² + (0.5)² + (0.1)² + (0.3)² + (0.3)² + (0.4)²] + [(0.1)² + (0.3)² + (0.1)² + (0.3)² + (0.2)² + (0.4)² + (0.3)² + (0.5²]}

$$\textbf{E (B)} = \sum_{1}^{n}\{\{5.05 + 1.74 + 0.89\} + \{4.75 + 0.89 + 0.74\}\}$$

$$\textbf{E (B)} = \sum_{1}^{n}\{7.68 + 6.38\}$$

$$\textbf{E (B)} = 14.06$$

$$\mathcal{R}(A, B) = \frac{CIVNHSS}{(E(A))^{\frac{1}{2}}.(E(B))^{\frac{1}{2}}} \in [0, 1^{+}[$$

$$\mathcal{R}(A, B) = \frac{11.404}{(11.1)^{\frac{1}{2}}.(14.06)^{\frac{1}{2}}}$$

$$\mathcal{R}(A, B) = 0.912 \in [0, 1^{+}[$$

It shows that the IVNHSS A and B have a good positive relation.

4. Result Discussion

In this section, we discuss the results obtained by using the proposed algorithms. We have proposed generalized-CC for interval-valued neutrosophic hypersoft set, in which we merged two existing theories i.e. interval-valued neutrosophic set theory (IVNSS) and hypersoft set theory (HSS). As we know interval-valued neutrosophic set theories are more accurate, superior and valid. Whereas, the hypersoft set structure is valid in the environment where attributes are further divided into n-terms. Thus, by merging these theories our new decision-making and optimization environment IVNHSS becomes more efficient and faster. Hence, the correlation coefficients-CC's and weighted correlation coefficients-WCC's are used to design an algorithm which can be utilized to solve decision-making problems which have more than one attribute and are further-bifurcated in interval-valued neutrosophic hypersoft environment.

Comparison of Results

It can be concluded from the current investigation, as well as the comparison analysis in Table 1 below, that the results obtained by the suggested technique overlap with those obtained by other approaches. The fundamental benefit of the suggested method in relation to accessible decision-making strategies, however, is that it includes more information. The information about the thing can be evaluated more properly and objectively among them. In the DM process, it's also a great tool for resolving erroneous and imprecise data. In addition, the created method's calculating methodology differs from existing methodologies. As a result, the motivation for the score value corresponding to each parameter will have no effect on other values, resulting in predictable information loss.

As a result, it's a good technique for combining erroneous and ambiguous data in the DM process. As a result, our proposed methodologies are effective, adaptable, and simple.

	Set	**Truthness**	**Indeterminacy**	**Falsity**	**Parameterization**	**Attributes**	**Sub-attributes**
Zadeh [1]	FS	✓	×	×	×	✓	×
Atanassov [2]	IFS	✓	×	✓	×	✓	×
Maji [21]	FSS	✓	×	×	✓	✓	×
Maji [22]	IFSS	✓	×	✓	✓	✓	×
Proposed	IVNHSS	✓	✓	✓	✓	✓	✓

Table: 1 The Comparison of proposed techniques with existing-ones

Based on the findings, it is reasonable to conclude that the suggested technique provides greater stability and usability for decision-makers in the DM procedure.

5. Conclusions

In decision making measure of fuzziness (entropy) is an important factor. The measurement of fuzziness in neutrosophic environment plays a vital role, since neutrosophic numbers and its decision-making approaches are used in many daily life issues like HR personnel selection, equipment selection, shortest path selection, engineering and medical etc. The validity and superiority can be measure by considering the value of fuzziness, when this value of fuzziness is less, then it can be considered as the best modelling and more accurate. Under the IVNHSS context, we introduced entropy, and generalized correlation coefficients. Based on the established correlation coefficient, a decision-making strategy has been constructed. Finally, a numerical example of best school selection is solved. In future, this concept of entropy and correlation coefficients can be extended to m-polar NHSS.

Conflicts of Interest

The authors declare no conflict of interest.

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The neutrosophic differentials calculus

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Abstract: the purpose of this article is to study the neutrosophic differentials and rules of the neutrosophic derivative, Where the neutrosophic differentiable is defined, and properties of neutrosophic differentiation are introduced, where we discussed how to find the derivatives of addition, subtraction, multiplication, and division of two neutrosophic functions. Also, derivative of composite neutrosophic functions is studied by method of chain rule, in addition to studying derivatives of inverse neutrosophic trigonometric functions, differentiation of implicit neutrosophic functions, logarithmic neutrosophic differentiation, higher order neutrosophic derivatives, and differentiation of parametric neutrosophic functions. Where detailed examples were given to clarify each case.

Keywords: the neutrosophic differentials; neutrosophic functions; indeterminacy; derivative neutrosophic functions.

1. Introduction

As an alternative to the existing logics, Smarandache proposed the Neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3-13]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist, he defined the standard form of neutrosophic complex number, and found root index $n \geq 2$ of a neutrosophic real and complex number [2-4], studying the concept of the Neutrosophic probability [3-5], the Neutrosophic statistics [4][6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1-8]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [9]. Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side represented with triangular neutrosophic numbers [10]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [11-12]. Y.Alhasan studied the concepts of neutrosophic complex numbers and the general exponential form of a neutrosophic complex [7-

14]. On the other hand, M.Abdel-Basset presented study in the science of neutrosophic about an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number [15]. Also the neutrosophic integrals are introduced by Y.Alhasan [16-17].

Paper consists of 5 sections. In 1th section, provides an introduction, in which neutrosophic science review has given. In 2th section, some definitions and examples of neutrosophic real number neutrosophic. The 3th section frames the neutrosophic differentiable and rules of the neutrosophic derivative, properties of neutrosophic differentiation are introduced, where we discussed how to find the derivatives of addition, subtraction, multiplication, and division of two neutrosophic functions, derivative of composite neutrosophic functions is studied by method of chain rule. The 4th section introduces the derivatives of inverse neutrosophic trigonometric functions, differentiation of implicit neutrosophic functions, logarithmic neutrosophic differentiation, higher order neutrosophic derivatives, and differentiation of parametric neutrosophic functions. In 5th section, a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic Real Number [4]

Suppose that w is a neutrosophic number, then it takes the following standard form: $w = a + bI$ where a , b are real coefficients, and I represent indeterminacy, such $0.\mathrm{I} = 0$ and $\mathrm{I}^n = \mathrm{I}$, for all positive integers n.

2.2. Division of neutrosophic real numbers [4]

Suppose that w_1, w_2 are two neutrosophic numbers, where

$$w_1 = a_1 + b_1 I \ , \qquad w_2 = a_2 + b_2 I$$

To find $(a_1 + b_1 I) \div (a_2 + b_2 I)$, we can write:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} \equiv x + yI$$

where x and y are real unknowns.

$$a_1 + b_1 I \equiv (a_2 + b_2 I)(x + yI)$$

$$a_1 + b_1 I \equiv a_2 x + (b_2 x + a_2 y + b_2 y) I$$

by identifying the coefficients, we get

$$a_1 = a_2 x$$

$$b_1 = b_2 x + (a_2 + b_{2)} y$$

We obtain unique one solution only, provided that:

$$\begin{vmatrix} a_2 & 0 \\ b_2 & a_2 + b_2 \end{vmatrix} \neq 0 \quad \Rightarrow \quad a_2(a_2 + b_2) \neq 0$$

Hence: $a_2 \neq 0 \ \ and \ a_2 \neq -b_2$ are the conditions for the division of two neutrosophic real numbers to exist.

Then:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} . I$$

3. The neutrosophic differentials

Definition3.1
Let $f: D_f \subseteq R \to R_f \cup \{I\}$, if:

$$\lim_{h+h_0I\to 0+0I} \frac{f(x+h+h_0I)-f(x,I)}{h+h_0I}$$

exist, then we say that the function $f(x,I)$ is differentiable with respect to x and it is given by the formula:

$$\acute{f}(x,I) = \lim_{h+h_0I\to 0+0I} \frac{f(x+h+h_0I)-f(x,I)}{h+h_0I}$$

Where $h+h_0I$ is amount of indetermined small change in x, and h, h_0 are real numbers, while $I =$ indeterminacy.

Note:

1) The tangent slop to $f(x,I)$ at $x_0 = a+bI$ is $m_I = \acute{f}(a+bI)$.
2) The equation of the tangent to $f(x,I)$ at $x_0 = a+bI$ is:

$$y - f(a+bI) = \acute{f}(a+bI)(x-a-bI)$$

where a, b are real numbers, while $I =$ indeterminacy.

Example3.1
Differentiate $f(x,I) = Ix^2$ with respect to x using definition, and find an equation of the tangent line to the curve at $x_0 = 3+3I$.

Solution:

$$\acute{f}(x,I) = \lim_{h+h_0I\to 0+0I} \frac{f(x+h+h_0I)-f(x,I)}{h+h_0I}$$

$$\acute{f}(x,I) = \lim_{h+h_0I\to 0+0I} \frac{I(x+h+h_0I)^2 - Ix^2}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} \frac{I(x^2+2(h+h_0I)x+(h+h_0I)^2) - Ix^2}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} \frac{Ix^2+2(h+h_0I)xI+(h+h_0I)^2I - Ix^2}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} \frac{2(h+h_0I)xI+(h+h_0I)^2I}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} \frac{(h+h_0I)[2xI+(h+h_0I)I]}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} [2xI+(h+h_0I)I]$$

$$\Longrightarrow \qquad \acute{f}(x,I) = 2xI$$

Finding the tangent equation:

$$m_I = \acute{f}(3+3I) = 2I(3+3I) = 12I$$

$$f(3+3I) = I(3+3I)^2 = 24I$$

Then:

$$y - f(a+bI) = \acute{f}(a+bI)(x-a-bI)$$

$$y - 24I = 12I\,(x-3-3I)$$

$$y = 12I\,x - 72I + 24I$$

$$y = 12I\,x - 48I$$

Example3.2

Differentiate $f(x,I) = sin(5x+3I)$ with respect to x using definition.

Solution:

$$\acute{f}(x,I) = \lim_{h+h_0I \to 0+0I} \frac{f(x+h+h_0I) - f(x,I)}{h+h_0I}$$

$$\acute{f}(x,I) = \lim_{h+h_0I \to 0+0I} \frac{sin(5(x+h+h_0I)+3I) - sin(5x+3I)}{h+h_0I}$$
$$= \lim_{h+h_0I \to 0+0I} \frac{sin\big(5x+3I+5(h+h_0I)\big) - sin(5x+3I)}{h+h_0I}$$

$$= \lim_{h+h_0I \to 0+0I} \frac{cos\left(5x+3I+\frac{5}{2}(h+h_0I)\right) sin(\frac{5}{2}(h+h_0I))}{\frac{h+h_0I}{2}}$$

$$= \lim_{h+h_0I \to 0+0I} cos\left(5x+3I+\frac{5}{2}(h+h_0I)\right) \lim_{h+h_0I \to 0+0I} \frac{sin(\frac{5}{2}(h+h_0I))}{\frac{h+h_0I}{2}}$$

$$= \lim_{h+h_0I \to 0+0I} cos\left(5x+3I+\frac{5}{2}(h+h_0I)\right) \lim_{h+h_0I \to 0+0I} \frac{5\,sin(\frac{5}{2}(h+h_0I))}{\frac{5(h+h_0I)}{2}}$$

$$\Rightarrow \qquad \acute{f}(x,I) = 5\,cos(5x+3I)$$

Example3.3

Differentiate $f(x,I) = \sqrt{3Ix+4I}$ with respect to x using definition.

Solution:

$$\acute{f}(x,I) = \lim_{h+h_0I \to 0+0I} \frac{f(x+h+h_0I) - f(x,I)}{h+h_0I}$$

$$\acute{f}(x,I) = \lim_{h+h_0I \to 0+0I} \frac{\sqrt{3I(x+h+h_0I)+4I} - \sqrt{3Ix+4I}}{h+h_0I}$$

$$= \lim_{h+h_0I \to 0+0I} \frac{\sqrt{3I(x+h+h_0I)+4I} - \sqrt{3Ix+4I}}{h+h_0I} \frac{\sqrt{3I(x+h+h_0I)+4I} + \sqrt{3Ix+4I}}{\sqrt{3I(x+h+h_0I)+4I} + \sqrt{3Ix+4I}}$$

$$= \lim_{h+h_0I \to 0+0I} \frac{3I(x+h+h_0I)+4I-3Ix-4I}{(h+h_0I)\left(\sqrt{3I(x+h+h_0I)+4I} + \sqrt{3Ix+4I}\right)}$$

$$= \lim_{h+h_0I \to 0+0I} \frac{3Ix+3I(h+h_0I)+3Ix}{(h+h_0I)\left(\sqrt{3I(x+h+h_0I)+4I} + \sqrt{3Ix+4I}\right)}$$

$$= \lim_{h+h_0I \to 0+0I} \frac{3I(h+h_0I)}{(h+h_0I)\left(\sqrt{3I(x+h+h_0I)+4I} + \sqrt{3Ix+4I}\right)}$$

$$= \lim_{h+h_0I \to 0+0I} \frac{3I}{\left(\sqrt{3I(x+h+h_0I)+4I} + \sqrt{3Ix+4I}\right)}$$

$$\Rightarrow \qquad \acute{f}(x,I) = \frac{3I}{2\sqrt{3Ix+4I}}$$

3.1 The rules of the neutrosophic derivative

We can prove each of the following, using the Definition3.1:

1) $\frac{d}{dx}(c+dI) = 0+0I$; where c, d are real numbers, while $I =$ indeterminacy.

2) $\frac{d}{dx}[(a+bI)x+c+dI] = a+bI$; where c, d are real numbers, while $I =$ indeterminacy.

3) $\frac{d}{dx}[(a+bI)x^n] = n(a+bI)x^{n-1}; n\ is$ real number.

4) $\frac{d}{dx}\left[e^{(a+bI)x+c+dI}\right] = (a+bI)e^{(a+bI)x+c+dI}$

5) $\frac{d}{dx}(c+dI)^x = (c+dI)^x \ln(c+dI)$

Where $c > 0\,, d > 0\ and\ I \geq 0\ \ or\ c > 0\,, d < 0\ and\ I \leq 0$

6) $\frac{d}{dx}\left[log_{a+bI}\, x\right] = \frac{1}{x\ln(a+bI)}$

Where $a > 0\,, b > 0\ and\ I \geq 0\ \ or\ a > 0\,, b < 0\ and\ I \leq 0$

7) $\frac{d}{dx}[\ln((a+bI)x+c+dI)] = \frac{a+bI}{(a+bI)x+c+dI}$

8) $\frac{d}{dx}\left[\sqrt{(a+bI)x+c+dI}\right]=\frac{a+bI}{2\sqrt{(a+bI)x+c+dI}}$

9) $\frac{d}{dx}[sin((a+bI)x+c+dI)]=(a+bI)cos((a+bI)x+c+dI)$

10) $\frac{d}{dx}[cos((a+bI)x+c+dI)]=-(a+bI)sin((a+bI)x+c+dI)$

11) $\frac{d}{dx}[tan((a+bI)x+c+dI)]=(a+bI)sec^2((a+bI)x+c+dI)$

12) $\frac{d}{dx}[cot((a+bI)x+c+dI)]=-(a+bI)csc^2((a+bI)x+c+dI)$

13) $\frac{d}{dx}[sec((a+bI)x+c+dI)]=(a+bI)sec((a+bI)x+c+dI)tan((a+bI)x+c+dI)$

14) $\frac{d}{dx}[csc((a+bI)x+c+dI)]=-(a+bI)csc((a+bI)x+c+dI)cot((a+bI)x+c+dI)$

Proof (3):

$$\frac{d}{dx}[(a+bI)x^n]=\lim_{h+h_0I\to 0+0I}\frac{f(x+h+h_0I)-f(x,I)}{h+h_0I}$$

$$=\lim_{h+h_0I\to 0+0I}\frac{(x+h+h_0I)^n-(a+bI)x^n}{h+h_0I}$$

$$=\lim_{h+h_0I\to 0+0I}\frac{\left[(a+bI)x^n+n(a+bI)x^{n-1}(h+h_0I)+\frac{n(n-1)}{2!}(a+bI)x^{n-2}(h+h_0I)^2+\cdots+n(a+bI)x(h+h_0I)^{n-1}+(h+h_0I)^n\right]-(a+bI)x^n}{h+h_0I}$$

$$=\lim_{h+h_0I\to 0+0I}\left[\frac{n(a+bI)x^{n-1}(h+h_0I)+\frac{n(n-1)}{2!}(a+bI)x^{n-2}(h+h_0I)^2+\cdots+n(a+bI)x(h+h_0I)^{n-1}+(h+h_0I)^n}{h+h_0I}\right]$$

$$=\lim_{h+h_0I\to 0+0I}\left[n(a+bI)x^{n-1}+\frac{n(n-1)}{2!}(a+bI)x^{n-2}(h+h_0I)+\cdots+n(a+bI)x(h+h_0I)^{n-2}+(h+h_0I)^{n-1}\right]$$

$$=n(a+bI)x^{n-1}+0+\cdots\cdots+0+0$$

$$=n(a+bI)x^{n-1}$$

Example3.1.1

1) $\frac{d}{dx}(5-6I)=0+0I=0$

2) $\frac{d}{dx}[(4+2I)x-8I]=4+2I$

3) $\frac{d}{dx}[(7+3I)x^4]=(28+12I)x^3$

4) $\frac{d}{dx}\left[e^{(3+I)x+5I}\right]=(3+I)e^{(3+I)x+5I}$

5) $\frac{d}{dx}(5+2I)^x=(5+2I)^x\, ln(5+2I)\;;\; case\ I\geq 0$

6) $\frac{d}{dx}(3-I)^x=(3-I)^x\, ln(3-I)\;; case\ I\leq 0$

7) $\frac{d}{dx}\left[ln\big((3+2I)x+6+7I\big)\right]=\frac{3+2I}{(3+2I)x+6+7I}$

8) $\frac{d}{dx}\left[\sqrt{(5+4I)x+9+I}\right]=\frac{5+4I}{2\sqrt{(5+4I)x+9+I}}$

9) $\frac{d}{dx}\left[sin\big((6-2I)x+9I\big)\right]=(6-2I)cos\big((6-2I)x+9I\big)$

10) $\frac{d}{dx}\left[cos\big((3-3I)x+2-I\big)\right]=(-3+3I)sin\big((3-3I)x+2-I\big)$

11) $\frac{d}{dx}[tan((8+9I)x+6I)]=(8+9I)sec^2((8+9I)x+6I)$

12) $\frac{d}{dx}\left[csc\big((3-4I)x+6+I\big)\right]=(-3+4I)csc\big((3-4I)x+6+I\big)cot\big((3-4I)x+6+I\big)$

13) $\frac{d}{dx}\left[log_{3+5I}\, x\right]=\frac{1}{x\ln(3+5I)}\;;\; case\ I\ \geq\ 0$

3.2 Properties of neutrosophic differentiation:

3.2.1 Derivative of sum or difference of neutrosophic functions.

Suppose that $f(x,I)$ and $g(x,I)$ are any two differentiable neutrosophic functions, then:

$$\frac{d}{dx}[f(x,I)\ \pm g(x,I)]=\frac{d}{dx}[f(x,I)\,]\pm\frac{d}{dx}[g(x,I)]$$

Proof:

$\frac{d}{dx}[f(x,I)+g(x,I)]=$

$$=\lim_{h+h_0I\to 0+0I}\frac{f(x+h+h_0I)\pm g(x+h+h_0I)-[f(x,I)+g(x,I)]}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} \frac{[f(x+h+h_0I)-f(x,I)] \pm [g(x+h+h_0I)-g(x,I)]}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} \left[\frac{[f(x+h+h_0I)-f(x,I)]}{h+h_0I} \pm \frac{[g(x+h+h_0I)-g(x,I)]}{h+h_0I}\right]$$

$$= \lim_{h+h_0I\to 0+0I} \frac{[f(x+h+h_0I)-f(x,I)]}{h+h_0I} \pm \lim_{h+h_0I\to 0+0I} \frac{[g(x+h+h_0I)-g(x,I)]}{h+h_0I}$$

$$= \frac{d}{dx} f(x,I) \pm \frac{d}{dx} g(x,I)$$

Example3.2.1

1) $\frac{d}{dx}[3Ix^3 + tan((8+9I)x)] = 9\mathrm{I}x^2 + (8+9I)sec^2((8+9I)x)$

2) $\frac{d}{dx}\left[8Ix + ln\big((3+2I)x\big)\right] = 8I + \frac{3+2I}{(3+2I)x}$

3.2.2 Derivative of product of a neutrosophic constant & neutrosophic function

$$\frac{d}{dx}[(c+dI)f(x,I)] = (c+dI)\frac{d}{dx}\,[f(x,I)]$$

where c, d are real numbers, while $I =$ indeterminacy.

Proof:

$$\frac{d}{dx}[(c+dI)f(x,I)] = \lim_{h+h_0I\to 0+0I} \frac{(c+dI)f(x+h+h_0I)-(c+dI)f(x,I)}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} (c+dI)\left[\frac{f(x+h+h_0I)-f(x,I)}{h+h_0I}\right]$$

$$= (c+dI)\lim_{h+h_0I\to 0+0I} \left[\frac{f(x+h+h_0I)-f(x,I)}{h+h_0I}\right]$$

$$= (c+dI)\frac{d}{dx}\,[f(x,I)]$$

3.2.3 Derivative of product of two neutrosophic functions

$$\frac{d}{dx}[f(x,I).\,g(x,I)] = f(x,I)\frac{d}{dx}[g(x,I)] + g(x,I)\frac{d}{dx}[f(x,I)]$$

Proof:

$$\frac{d}{dx}[f(x,I).\,g(x,I)] =$$

$$= \lim_{h+h_0I\to 0+0I} \frac{f(x+h+h_0I).\,g(x+h+h_0I) - f(x,I).\,g(x,I)}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} \frac{f(x+h+h_0I).\,g(x+h+h_0I) - f(x+h+h_0I)g(x,I) + f(x+h+h_0I)g(x,I) - f(x,I).\,g(x,I)}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} \left[f(x+h+h_0I)\frac{g(x+h+h_0I)-g(x,I)}{h+h_0I} + g(x,I)\frac{f(x+h+h_0I)-f(x,I)}{h+h_0I}\right]$$

$$= \lim_{h+h_0I\to 0+0I} f(x+h+h_0I) \lim_{h+h_0I\to 0+0I} \frac{g(x+h+h_0I)-g(x,I)}{h+h_0I} + \lim_{h+h_0I\to 0+0I} g(x,I) \lim_{h+h_0I\to 0+0I} \frac{f(x+h+h_0I)-f(x,I)}{h+h_0I}$$

$$= f(x,I)\frac{d}{dx}[g(x,I)] + g(x,I)\frac{d}{dx}[f(x,I)]$$

Example3.2.2

1) $\frac{d}{dx}[-7Ix^2 sin((8+9I)x)] = -14x.sin\big((8+9I)x\big) - 119Icos\big((8+9I)x\big)$

2) $\frac{d}{dx}\left[2Ix\sqrt{(5+4I)x+9+I}\right] = 2I\sqrt{(5+4I)x+9+I} + \frac{9Ix}{\sqrt{(5+4I)x+9+I}}$

3.2.3 Derivative of quotient of two neutrosophic functions

$$\frac{d}{dx}\left[\frac{f(x,I)}{g(x,I)}\right] = \frac{f(x,I)\frac{d}{dx}[g(x,I)] - g(x,I)\frac{d}{dx}[f(x,I)]}{(g(x,I))^2}$$

Proof:

$$\frac{d}{dx}\left[\frac{f(x,I)}{g(x,I)}\right] = \lim_{h+h_0I\to 0+0I} \frac{\frac{f(x+h+h_0I)}{g(x+h+h_0I)} - \frac{f(x,I)}{g(x,I)}}{h+h_0I}$$

$$= \lim_{h+h_0I\to 0+0I} \frac{f(x+h+h_0I).g(x,I) - f(x,I).g(x,I) - f(x,I).g(x+h+h_0I) + f(x,I).g(x,I)}{(h+h_0I)g(x,I).g(x+h+h_0I)}$$

$$= \lim_{h+h_0I\to 0+0I} \left[\frac{g(x,I)\frac{f(x+h+h_0I)-f(x,I)}{h+h_0I} - f(x,I)\frac{g(x+h+h_0I)-g(x,I)}{h+h_0I}}{g(x,I).g(x+h+h_0I)}\right]$$

$$= \frac{\lim_{h+h_0I\to 0+0I}\left[g(x,I)\frac{f(x+h+h_0I)-f(x,I)}{h+h_0I}\right] - \lim_{h+h_0I\to 0+0I}\left[f(x,I)\frac{g(x+h+h_0I)-g(x,I)}{h+h_0I}\right]}{\lim_{h+h_0I\to 0+0I}[g(x,I).g(x+h+h_0I)]}$$

$$= \frac{\lim_{h+h_0I\to 0+0I} g(x,I) \lim_{h+h_0I\to 0+0I}\frac{f(x+h+h_0I)-f(x,I)}{h+h_0I} - \lim_{h+h_0I\to 0+0I} f(x,I) \lim_{h+h_0I\to 0+0I}\frac{g(x+h+h_0I)-g(x,I)}{h+h_0I}}{\lim_{h+h_0I\to 0+0I} g(x,I).\lim_{h+h_0I\to 0+0I} g(x+h+h_0I)}$$

$$= \frac{d}{dx}\left[\frac{f(x,I)}{g(x,I)}\right] = \frac{f(x,I)\frac{d}{dx}[g(x,I)] - g(x,I)\frac{d}{dx}[f(x,I)]}{g(x,I).g(x,I)}$$

$$= \frac{d}{dx}\left[\frac{f(x,I)}{g(x,I)}\right] = \frac{f(x,I)\frac{d}{dx}[g(x,I)] - g(x,I)\frac{d}{dx}[f(x,I)]}{(g(x,I))^2}$$

Example3.2.3

1) $\frac{d}{dx}\left[\frac{e^{(3+I)x+5I}}{(3+4I)x}\right]=\frac{(3+I)(3+4I)xe^{(3+I)x+5I}-(3+4I)e^{(3+I)x+5I}}{(3+4I)^2x^2}$

$$=\frac{(3+I)xe^{(3+I)x+5I}-e^{(3+I)x+5I}}{(3+4I)x^2}$$

$$=(\frac{1}{3}-\frac{4}{21}I)\left[\frac{(3+I)xe^{(3+I)x+5I}-e^{(3+I)x+5I}}{x^2}\right]$$

2) $\frac{d}{dx}\left[\frac{5I}{(1+I)x}\right]=\frac{-5I}{(1+I)x^2}=\left(-5-\frac{5}{2}I\right)\frac{1}{x^2}$

3.3 Derivative of composite neutrosophic functions

Chain Rule:
if $y=f(u,I)$ $and u=g(x,I)$, then:

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx}=\acute{f}(u,I).\acute{g}(x,I)$$

Remarks:

1) $\frac{d}{dx}[f(g(x,I))]=f'(g(x,I)).g'(x,I)$
2) $\frac{d}{dx}\left[f\left(g(h(x,I))\right)\right]=f'\left(g(h(x,I))\right).g'(h(x,I)).h'(x,I)$
3) $\frac{d}{dx}[f(x,I)]^n=n[f(x,I)]^{n-1}.[f'(x,I)]\,;n\in R-\{0,1\}$

Example3.3.1

1) $\frac{d}{dx}\left((2+I)x^2+2Ix-5+6I\right)^7=7\left((2+I)x^2+2Ix-5+6I\right)^6\left((4+2I)x+2I\right)$

2) $\frac{d}{dx}sin^5\left((3+4I)x+5I\right)=4(3+4I)sin^4\left((3+4I)x+5I\right)\left(cos\left((3+4I)x+5I\right)\right)$

$$=(12+16I)sin^4\left((3+4I)x+5I\right)\left(cos\left((3+4I)x+5I\right)\right)$$

3) $\frac{d}{dx}\left[\sqrt{tan\left((6+4I)x-2+7I\right)}\right]=\frac{(6+4I)sec^2\left((6+4I)x-2+7I\right)}{2\sqrt{tan\left((6+4I)x-2+7I\right)}}$

$$=\frac{(3+2I)sec^2\left((6+4I)x-2+7I\right)}{\sqrt{tan\left((6+4I)x-2+7I\right)}}$$

Example3.3.2

Find $\frac{dy}{dx}$ of each the following:

a) $y = f(t,I) = 3It^2 + 5 - 6I$, $t = g(x,I) = sin\big((6+4I)x - 2 + 7I\big) + tan\big((6+4I)x - 2 + 7I\big)$

Solution:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \acute{f}(t,I).\acute{g}(x,I)$$

$$= 6It \,. \Big((6+4I)cos\big((6+4I)x - 2 + 7I\big) + (6+4I)sec^2\big((6+4I)x - 2 + 7I\big)\Big)$$

$$= 6I(6+4I)t \,. \Big(cos\big((6+4I)x - 2 + 7I\big) + sec^2\big((6+4I)x - 2 + 7I\big)\Big)$$

$$= 60Isin\big((6+4I)x - 2 + 7I\big) + tan\big((6+4I)x - 2 + 7I\big) \,. \Big(cos\big((6+4I)x - 2 + 7I\big) + sec^2\big((6+4I)x - 2 + 7I\big)\Big)$$

b) $y = f(t,I) = (t + 1 - 4I)^2$, $t = g(x,I) = \sqrt{(4-2I)x + 5I}$

Solution:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \acute{f}(t,I).\acute{g}(x,I)$$

$$= 2(t + 1 - 4I)\frac{4-2I}{2\sqrt{(4-2I)x + 5I}}$$

$$= \frac{(4-2I)(t + 1 - 4I)}{\sqrt{(4-2I)x + 5I}}$$

$$= \frac{(4-2I)\left(\sqrt{(4-2I)x + 5I} + 1 - 4I\right)}{\sqrt{(4-2I)x + 5I}}$$

$$= \frac{(4-2I)\sqrt{(4-2I)x + 5I} + (4-2I)(1-4I)}{\sqrt{(4-2I)x + 5I}}$$

$$= (4-2I) + \frac{4-10I}{\sqrt{(4-2I)x + 5I}}$$

4. Derivatives of inverse neutrosophic trigonometric functions

9) $$\frac{d}{dx}[sin^{-1}((a+bI)x + c + dI)] = \frac{a+bI}{\sqrt{1 - \big((a+bI)x + c + dI\big)^2}}$$

10) $$\frac{d}{dx}[cos^{-1}((a+bI)x + c + dI)] = -\frac{a+bI}{\sqrt{1 - \big((a+bI)x + c + dI\big)^2}}$$

11) $\frac{d}{dx}[tan^{-1}((a+bI)x+c+dI)] = \frac{a+bI}{1+\left((a+bI)x+c+dI\right)^2}$

12) $\frac{d}{dx}[cot^{-1}((a+bI)x+c+dI)] = -\frac{a+bI}{1+\left((a+bI)x+c+dI\right)^2}$

13) $\frac{d}{dx}[sec^{-1}((a+bI)x+c+dI)] = \frac{a+bI}{|(a+bI)x+c+dI|\sqrt{\left((a+bI)x+c+dI\right)^2-1}}$

14) $\frac{d}{dx}[csc^{-1}((a+bI)x+c+dI)] = -\frac{a+bI}{|(a+bI)x+c+dI|\sqrt{\left((a+bI)x+c+dI\right)^2-1}}$

Proof (1):

$$y = sin^{-1}\left((a+bI)x+c+dI\right)$$

$$sin\, y = (a+bI)x+c+dI$$

$$\Rightarrow cos\, y\frac{dy}{dx} = a+bI$$

$$\frac{dy}{dx} = \frac{a+bI}{cos\, y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a+bI}{\sqrt{1-sin^2 y}} = \frac{a+bI}{\sqrt{1-\left((a+bI)x+c+dI\right)^2}}$$

Note:
In the same way we can prove the rest of the rules.

Example4.1

1) $\frac{d}{dx}tan^{-1}\left((5+6I)x+4-7I\right) = \frac{5+6I}{1+\left((5+6I)x+4-7I\right)^2}$

2) $\frac{d}{dx}\left[(4-2I)x^2+sec^{-1}\left((1+4I)x-3I\right)\right] = (8-4I)x+\frac{1+4I}{|(1+4I)x-3I|\sqrt{\left((1+4I)x-3I\right)^2-1}}$

3) $\frac{d}{dx}[(-4Ix+6-I)cos^{-1}(9Ix+5-3I)] = -4Icos^{-1}(9Ix+5-3I)-\frac{9I(-4Ix+6-I)}{\sqrt{1-(9Ix+5-3I)^2}}$

$$= -4Icos^{-1}(9Ix+5-3I)+\frac{36Ix-45I}{\sqrt{1-(9Ix+5-3I)^2}}$$

4.1 Differentiation of implicit neutrosophic functions

$y = f(x,I)$ can be directly expressed as a function of (x,I), such functions are known as explicit neutrosophic functions. But the relations of the form $f(x,y,I) = 0+0I,\ y$ is not directly expressed as a function of (x, I) and also it is not easily solvable for y, In such case functions are known as implicit neutrosophic functions.

To find differentiation of implicit neutrosophic functions, we follow the following steps:

- Differentiate two sides of the given equation with respect to (x, I).
- We isolate d on one side and the other terms on the other side to get the following equation:

$$\varphi(x, y, I)\frac{dy}{dx} = \omega(x, y, I)$$

Hence:

$$\frac{dy}{dx} = \frac{\omega(x, y, I)}{\varphi(x, y, I)}$$

Example4.1.1

If $3Ixy^3 - (2 + 5I)x^2y = 7Ix + 3 + 8I$, find $\frac{dy}{dx}$.

Solution:

$$\frac{d}{dx}(3Ixy^3 - (2 + 5I)x^2y) = \frac{d}{dx}(7Ix + 3 + 8I)$$

$$9Ixy^2\frac{dy}{dx} - (4 + 10I)xy - (2 + 5I)x^2\frac{dy}{dx} = 7I$$

$$(9Ixy^2 - (2 + 5I)x^2)\frac{dy}{dx} = (4 + 10I)xy - 3Iy^3 + 7I$$

$$\frac{dy}{dx} = \frac{(4 + 10I)xy - 3Iy^3 + 7I}{9Ixy^2 - (2 + 5I)x^2}$$

Example4.1.2

If $(3 + 5I)xy - siny = (5 + I)y - 1 + 2I$, find $\frac{dy}{dx}$.

Solution:

$$\frac{d}{dx}\big((3 + 5I)xy - siny\big) = \frac{d}{dx}\big((5 + I)y - 1 + 2I\big)$$

$$(3 + 5I)y + (3 + 5I)x\frac{dy}{dx} - cosy\,\frac{dy}{dx} = (5 + I)\frac{dy}{dx}$$

$$(3 + 5I)x\frac{dy}{dx} - cosy\,\frac{dy}{dx} - (5 + I)\frac{dy}{dx} = -(3 + 5I)y$$

$$\big((3 + 5I)x - cosy - (5 + I)\big)\frac{dy}{dx} = -(3 + 5I)y$$

$$\frac{dy}{dx} = \frac{-(3 + 5I)y}{(3 + 5I)x - cosy - (5 + I)}$$

4.2 Logarithmic neutrosophic differentiation

We use the logarithmic neutrosophic differentiation for differentiating neutrosophic functions of the form $y = f(x, I)^{g(x,I)}$ and for neutrosophic function which contains product and quotient of two or more neutrosophic functions.

We will discuss the steps for solving the first case, and in the same way the second case is done.

Solution steps:

- Take logarithmic of the two sides

$$lny = lnf(x,I)^{g(x,I)}$$

$$lny = g(x,I).lnf(x,I)$$

- Now differentiate of the two sides

$$\frac{1}{y}\frac{dy}{dx} = g(x,I)\frac{d}{dx}lnf(x,I) + lnf(x,I)\frac{d}{dx}g(x,I)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = g(x,I)\frac{1}{f(x,I)}\acute{f}(x,I) + lnf(x,I)\acute{g}(x,I)$$

$$\Rightarrow \frac{dy}{dx} = y\left(g(x,I)\frac{1}{f(x,I)}\acute{f}(x,I) + lnf(x)\acute{g}(x,I)\right)$$

$$\frac{dy}{dx} = f(x,I)^{g(x,I)}\left(g(x,I)\frac{1}{f(x,I)}\acute{f}(x,I) + lnf(x)\acute{g}(x,I)\right)$$

Example4.2.1

If $y = (ln(2+3I)x + 9I)^{\sqrt{(4-2I)x}}$, find $\frac{dy}{dx}$.

Solution:

$$lny = ln(ln(2+3I)x + 9I)^{\sqrt{(4-2I)x}}$$

$$lny = \sqrt{(4-2I)x}\, ln(ln(2+3I)x + 9I)$$

$$\frac{d}{dx}lny = \frac{d}{dx}\left(\sqrt{(4-2I)x}\, ln(ln(2+3I)x + 9I)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4-2I}{2\sqrt{(4-2I)x}}\, ln(\ln(2+3I)x + 9I) + \frac{\frac{2+3I}{ln(2+3I)x}}{(ln(2+3I)x + 9I)}\sqrt{(4-2I)x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2-I}{\sqrt{(4-2I)x}}\, ln(\ln(2+3I)x + 9I) + \frac{2+3I}{ln(2+3I)x\,(ln(2+3I)x + 9I)}\sqrt{(4-2I)x}$$

$$\frac{dy}{dx} = y\left(\frac{2-I}{\sqrt{(4-2I)x}}\, ln(\ln(2+3I)x + 9I) + \frac{2+3I}{ln(2+3I)x\,(ln(2+3I)x + 9I)}\sqrt{(4-2I)x}\right)$$

$$\frac{dy}{dx} = (ln(2+3I)x + 9I)^{\sqrt{(4-2I)x}}\left(\frac{2-I}{\sqrt{(4-2I)x}}\, ln(\ln(2+3I)x + 9I) + \frac{2+3I}{ln(2+3I)x\,(ln(2+3I)x + 9I)}\sqrt{(4-2I)x}\right)$$

Example4.2.2

If $y=\frac{(3x^2-1+6I)^4 tan^{-1}(5Ix+2+9I)}{(3-5I)x-4I}$, find $\frac{dy}{dx}$.

Solution:

$$lny = ln\left(\frac{(3x^2-1+6I)^4 tan^{-1}(5Ix+2+9I)}{(3-5I)x-4I}\right)$$

$$lny = ln\big((3x^2-1+6I)^4 tan^{-1}(5Ix+2+9I)\big) - ln\big((3-5I)x-4I\big)$$

$$lny = 4ln(3x^2-1+6I) + ln\big(tan^{-1}(5Ix+2+9I)\big) - ln\big((3-5I)x-4I\big)$$

$$\frac{d}{dx}lny = \frac{d}{dx}\Big(4ln(3x^2-1+6I) + ln\big(tan^{-1}(5Ix+2+9I)\big) - ln\big((3-5I)x-4I\big)\Big)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{12}{3x^2-1+6I} + \frac{\frac{5I}{1+(5Ix+2+9I)^2}}{tan^{-1}(5Ix+2+9I)} - \frac{3-5I}{(3-5I)x-4I}$$

$$\frac{dy}{dx} = y\left[\frac{12}{3x^2-1+6I} + \frac{5I}{(1+(5Ix+2+9I)^2)tan^{-1}(5Ix+2+9I)} - \frac{3-5I}{(3-5I)x-4I}\right]$$

$$\frac{dy}{dx} = \frac{(3x^2-1+6I)^4 tan^{-1}(5Ix+2+9I)}{(3-5I)x-4I}\left[\frac{12}{3x^2-1+6I} + \frac{5I}{(1+(5Ix+2+9I)^2)tan^{-1}(5Ix+2+9I)} - \frac{3-5I}{(3-5I)x-4I}\right]$$

4.3 Higher order neutrosophic derivatives

Let $f: D_f \subseteq R \to R_f \cup \{I\}$, then $\frac{dy}{dx} = \acute{f}(x,I)$ is also a neutrosophic function, which can be again differentiated with respect to x. The derivative of $\frac{dy}{dx}$ is denoted by $\frac{d^2y}{dx^2}$ and is called (second order derivative) of the neutrosophic function $y = f(x,I)$, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = f''(x,I)$ is called (third order derivative).

Similarly, fourth, fifth and so on. In general the n^{th} derivative of neutrosophic function $y = f(x,I)$ is denoted by $\frac{d^ny}{dx^n} = f^{(n)}(x,I)$.

Example4.3.1

Find the second derivative of $f(x,I) = cos\big((4-5I)x-7I\big)$

Solution:

$$\acute{f}(x,I) = -(4-5I)sin\big((4+5I)x-7I\big)$$

$$f''(x,I) = -(4-5I)^2 cos\big((4+5I)x-7I\big)$$

$$f''(x,I) = (-16+15I)cos\big((4+5I)x-7I\big)$$

Example4.3.2

Let $f(x,I) = 6Ix^3 - (2-I)x^2 + 5Ix + 2 - 7I$, find $f''(1-3I)$.

Solution:

$$\acute{f}(x,I) = 12Ix^2 - (4-2I)x + 5I$$

$$f''(x,I) = 24Ix - 4 + 2I$$

$$\Rightarrow f''(1-3I) = 24I(1-3I) - 4 + 2I$$

$$= 24I - 72I - 4 + 2I = -4 - 46I$$

4.4 Differentiation of parametric neutrosophic functions

Definition4.4.1

Let $y = \phi(x,I)$ neutrosophic function, it can be represented by means of some parametric equations such as $y = f(t,I)$ *and* $x = g(t,I)$, where t is some parameter. We call $y = \phi(x,I)$ a parametric neutrosophic functions.

Then:

$$\acute{\phi}(x,I) = \frac{\acute{f}(x,I)}{\acute{g}(x,I)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example4.4.1

Let $x = (1-3I)t^2$ *and* $y = (2-2I)t$, find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2-2I}{2(1-3I)t} = \frac{1+I}{(1-3I)t}$$

$$= (1-2I)\frac{1}{t}$$

Example4.4.2

Let $x = (4+I)\,(\theta + sin(2\theta + 4 - 6I))$ *and* $y = (2-2I)(1 - cos(2\theta + 4 - 6I))$, find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{(2-2I)\,sin(2\theta + 4 - 6I)}{(4+I)\,(1 + cos(2\theta + 4 - 6I))} = \frac{2-2I}{4+I}\,\frac{sin(2\theta + 4 - 6I)}{1 + cos(2\theta + 4 - 6I)}$$

$$= \left(\frac{1}{2} - \frac{1}{2}I\right)\frac{2sin(\theta + 2 - 3I)\,cos(\theta + 2 - 3I)}{2\,cos^2(\theta + 2 - 3I)}$$

$$= \left(\frac{1}{2} - \frac{1}{2}I\right) \frac{sin(\theta + 2 - 3I)}{cos(\theta + 2 - 3I)}$$

$$= \left(\frac{1}{2} - \frac{1}{2}I\right) tan(\theta + 2 - 3I)$$

Example4.4.2

Differentiate $e^{(3+I)x+5I}$ with respect to $\sqrt{(5 + 4I)x + 9 + I}$.

Solution:

Let $u = e^{(3+I)x+5I}$ and $v = \sqrt{(5 + 4I)x + 9 + I}$

Then, the required derivative is:

$$\frac{du}{dv} = \frac{du/_{dx}}{dv/_{dx}}$$

$$= \frac{(3 + I)e^{(3+I)x+5I}}{\frac{5 + 4I}{2\sqrt{(5 + 4I)x + 9 + I}}}$$

$$= \frac{6 + 2I}{5 + 4I} \sqrt{(5 + 4I)x + 9 + I}\, e^{(3+I)x+5I}$$

$$= \left(\frac{6}{5} - \frac{14}{45}I\right) \sqrt{(5 + 4I)x + 9 + I}\, e^{(3+I)x+5I}$$

5. Conclusions

The derivatives are important in our lives, such as calculating the function of velocity, displacement and acceleration as a function of time for rectilinear motion and others, and calculating any rate of change of any variable in relation to another variable or variables such as the rate of fuel consumption or the rate of decreasing or increasing any variable by changing any other. This led us to study the neutrosophic differentials for neutrosophic functions from that contain indeterminacy. Where the neutrosophic differentiable is defined, and properties of neutrosophic differentiation are introduced. In addition to studying derivative of composite neutrosophic functions, derivatives of inverse neutrosophic trigonometric functions, differentiation of implicit neutrosophic functions, logarithmic neutrosophic differentiation, higher order neutrosophic derivatives, and differentiation of parametric neutrosophic functions. The importance of this paper lies in the field of the neutrosophic integrals.

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Neutrosophic αGS Closed Sets in Neutrosophic Topological Spaces

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Abstract: The notion of Neutrosophic sets naturally plays a significant role in the study of Neutrosophic topology which was introduced by A.A. Salama. Chang also studied fuzzy continuity which was proved to be of fundamental importance in the realm of Neutrosophic topology. Since then various notions in classical topology have been extended to Neutrosophic topological spaces. Aim of this paper is to initiate and examine about new type of Neutrosophic closed set called Neutrosophic α-GS closed sets and Neutrosophic α-GS open sets. Further some of their properties are discussed.

Keywords: Neutrosophic, Topological, Closed Sets, $\boldsymbol{\alpha}$GS

1. Introduction

In 1980s the international movement called paradoxism based on contradictions in science and literature, was founded by Smarandache[18,19], who then extended it to neutrosophy, based on contradictions and their neutrals.Smarandache's[18,19] Neutrosophic sets have the components T, I, F which symbolize the membership, indeterminacy and non-membership values in that order. A.A. Salama [32] introduced Neutrosophic topological spaces by using Smarandache's Neutrosophic sets. Each year different kinds of Neutrosophic closed sets have been introduced by researchers.The concept of Neutrosophic semiopen sets and Neutrosophic semiclosed sets were first introduced in Neutrosophic topological space by P. Ishwarya [20] in 2016 and also studied the concept of Neutrosophic semi interior and closure properties in Neutrosophic topological spaces. V.VenkateswaraRao & Y.SrinivasaRao [35] extended the concepts of Neutrosophic preopen sets and Neutrosophic pre closed sets in Neutrosophic setting in 2017. I. Arokiarani [7] et al., introduced Neutrosophic α closed sets in Neutrosophic topological spaces.
R. Dhavaseelan, and S.Jafari (2018)[17] introduced and studied the concept of Generalized Neutrosophic closed sets. V.K.Shanthi and S.Chandrasekar[34] et al (2018) introduced and established Neutrosophic Generalized semi closed sets. Another important Neutrosophic closed sets Neutrosophic $\boldsymbol{\alpha}$-generalized closed sets initiated by R. Dhavaseelan[17] et al., Aim of this paper is , We introduce the concepts of Neutrosophic $\boldsymbol{\alpha}$-generalized semi-closed sets and Neutrosophic α -generalized semi-open sets. we get results Every Neutrosophic closed set, Neutrosophic α-closed sets, Neutrosophic regular closed sets are Neutrosophic $\boldsymbol{\alpha}$-generalized

semi-closed sets. Also, every Neutrosophic α-generalized semi-closed sets is Neutrosophic α-generalized closed sets, Neutrosophic generalized α- closed sets, and Neutrosophic generalized semi-closed sets. Neutrosophic α-generalized semi-closed sets independent with Neutrosophic pre-closed sets, Neutrosophic b closed sets, Neutrosophic semi pre-closed sets and Neutrosophic generalized closed sets. We obtain their properties and relationship between other Neutrosophic closed sets.. Also, we discussed their properties and relationships.

2. Preliminaries

Definition 1.1 [18,19] Let N^X be a non-empty fixed set. A Neutrosophic set V_1^* in N^X is a object having the form$V_1^* = \{\langle x, \mu_{V_1^*}(x), \sigma_{V_1^*}(x), \nu_{V_1^*}(x)\rangle | x \in N^x\}$ where the function $\mu_{V_1^*}(x): N^X \to [0,1]$ degree of membership (namely $\mu_{V_1^*}(x)$), $\sigma_{V_1^*}(x)$ denotes the indeterminancy and the function $\nu_{V_1^*}(x): N^X \to [0,1]$ denotes the degree of non-membership (namely $\nu_{V_1^*}(x)$) of each element $x \in N^X$ to the set V_1^* respectively.

Definition 1.2 [18,19]. Let V_1^* and V_2^* be NSs of the form $V_1^* = \{\langle x, \mu_{V_1^*}(x), \sigma_{V_1^*}(x), \nu_{V_1^*}(x)\rangle | x \in N^x\}$ and $V_2^* = \{\langle x, \mu_{V_2^*}(x), \sigma_{V_2^*}(x), \nu_{V_2^*}(x)\rangle | x \in N^x\}$. Then

1. $V_1^* \subseteq V_2^*$ iff $\mu_{V_1^*}(x) \le \mu_{V_2^*}(x)$, $\sigma_{V_1^*}(x) \le \sigma_{V_2^*}(x)$ and $\nu_{V_1^*}(x) \ge \nu_{V_2^*}(x)$ for all $x \in N^x$
2. $V_1^* = V_2^*$ iff $V_1^* \subseteq V_2^*$ and $V_2^* \subseteq V_1^*$
3. $V_1^{*c} = \{\langle x, \nu_{V_1^*}(x), 1 - \sigma_{V_1^*}(x), \mu_{V_1^*}(x)\rangle | x \in N^x\}$
4. $V_1^* \cap V_2^*$=$\{\langle x, \mu_{V_1^*}(x) \wedge \mu_{V_2^*}(x), \sigma_{V_1^*}(x) \wedge \sigma_{V_2^*}(x), \nu_{V_1^*}(x) \vee \nu_{V_2^*}(x)\rangle | x \in N^x\}$
5. $V_1^* \cup V_2^*$=$\{\langle x, \mu_{V_1^*}(x) \vee \mu_{V_2^*}(x), \sigma_{V_1^*}(x) \vee \sigma_{V_2^*}(x), \nu_{V_1^*}(x) \wedge \nu_{V_2^*}(x)\rangle | x \in N^x\}$

Definition 1.3 [32]. A Neutrosophic topology (NT in short) on N^x is a family $N^\tau$of NS in N^x satisfying the following axioms.

1. $0_N, 1_N \in N^\tau$
2. $J_1 \cap J_2 \in N^\tau$ for any $J_1, J_2 \in N^\tau$
3. $\cup J_i \in N^\tau$for any family $\{J_i | i \in j\} \subseteq N^\tau$

In this case, the pair (N^X, N^τ) is called a Neutrosophic topological space (NTS in short) and any NS in N^τ is known as an Neutrosophic open set (NOS) in N^X. The complement V_1^{*c} of a NOS V_1^* in a NTS (N^X, N^τ) is called a Neutrosphic closed set (NCS) in N^X.

Definition 1.4 [32]. For any NSs V_1^* and V_2^* in (N^X, N^τ), we have

1. $N^{int}(V_1^*) \subseteq V_1^*$
2. $V_1^* \subseteq N^{cl}(V_1^*)$
3. $V_1^* \subseteq V_2^* \Longrightarrow N^{int}(V_1^*) \subseteq N^{int}(V_2^*)$ and $N^{cl}(V_1^*) \subseteq N^{cl}(V_2^*)$
4. $N^{int}(N^{int}(V_1^*)) = N^{int}(V_1^*)$
5. $N^{cl}(N^{cl}(V_1^*)) = N^{cl}(V_1^*)$
6. $N^{cl}(V_1^* \cup V_2^*) = N^{cl}(V_1^*) \cup N^{cl}(V_2^*)$
7. $N^{int}(V_1^* \cap V_2^*) = N^{int}(V_1^*) \cap N^{int}(V_2^*)$

Proposition 1.5 [32]. For any NS V_1^* in (N^X, N^τ), we have

1. $N^{int}(0_N) = 0_N$ and $N^{cl}(0_N) = 0_N$

2. $N^{int}(1_N) = 1_N$ and $N^{cl}(1_N) = 1_N$
3. $(N^{int}(V_1^*))^c = N^{cl}(V_1^{*c})$
4. $(N^{cl}(V_1^*))^c = N^{int}(V_1^{*c})$

Definition 1.6. A NS $V_1^* = \langle x, \mu_{V_1^*}, \sigma_{V_1^*}, \nu_{V_1^*} \rangle$ in a NTS (N^X, N^τ) is called as
1. Neutrosophic regular closed set [7] (N(R)CS in short) if $V_1^* = N^{cl}(N^{int}(V_1^*))$
2. Neutrosophic α-closed set [7] (N(α)CS in short) if $N^{cl}(N^{int}(N^{cl}(V_1^*))) \subseteq V_1^*$
3. Neutrosophic semi closed set [20] (N(S)CS in short) if $N^{int}(N^{cl}(V_1^*)) \subseteq V_1^*$
4. Neutrosophic pre-closed set [35] (N(P)CS in short) if $N^{cl}(N^{int}(V_1^*)) \subseteq V_1^*$
5. Neutrosophic b-closed set [23] (N(b)CS in short) if $N^{cl}(N^{int}(V_1^*)) \cap N^{int}(N^{cl}(V_1^*)) \subseteq V_1^*$

Definition 1.7. A NS V_1^* of a NTS (N^X, N^τ) is a
1. Neutrosophic semi preopen set [17] (N(SP)OS) if there exists a N(P)OS V_2^* such that $V_1^* \subseteq (V_1^*) \subseteq N^{cl}(V_2^*)V_1^*$
2. Neutrosophic semi pre closed set (N(SP)CS) if there exists a N(P)CS V_2^* such that $N^{int}(V_2^*) \subseteq V_1^* \subseteq V_2^*$

Definition 1.8. Let V_1^* be a NS in (N^X, N^τ), then Neutrosophic semi interior of V_1^* ($N^{Sint}(V_1^*)$ in short) and Neutrosophic semi closure of V_1^* ($N^{Scl}(V_1^*)$ in short) are defined as
1. $N^{Sint}(V_1^*) = \cup \{H | H \text{ is a } N(S)OS \text{ in } N^X \text{ and } H \subseteq V_1^*\}$
2. $N^{Scl}(V_1^*) = \cap \{G | G \text{ is a } N(S)CS \text{ in } N^X \text{ and } V_1^* \subseteq G\}$

Definition 1.9. Let V_1^* be a NS in (N^X, N^τ), then Neutrosophic semi pre interior of V_1^* ($N^{SPint}(V_1^*)$ in short) and Neutrosophic semi preclosure of V_1^* ($N^{SPcl}(V_1^*)$ in short) are defined as
1. $N^{SPint}(V_1^*) = \cup \{E | E \text{ is a } N(S)POS \text{ in } N^X \text{ and } E \subseteq V_1^*\}$
2. $N^{SPcl}(V_1^*) = \cap \{K | K \text{ is a } N(S)PCS \text{ in } N^X \text{ and } V_1^* \subseteq K\}$

Definition 1.10. Let V_1^* be an NS of a NTS (N^X, N^τ). Then
1. $N^{\alpha cl}(V_1^*) = \cap \{I | I \text{ is a } N(\alpha)CS \text{ in } N^X \text{ and } V_1^* \subseteq I\}$
2. $N^{\alpha int}(V_1^*) = \cup \{I | I \text{ is a } N(\alpha)OS \text{ in } N^X \text{ and } I \subseteq V_1^*\}$

Definition 1.11. A NS V_1^* of a NTS (N^X, N^τ) is a
1. Neutrosophic generalized closed set [15] (N(G)CS in short) if $N^{cl}(V_1^*) \subseteq \Psi$ whenever $V_1^* \subseteq \Psi$ and Ψ is a NOS in N^X.
2. Neutrosophic generalized semi closed set [34](N(GS)CS in short) if $N^{Scl}(V_1^*) \subseteq \Psi$ whenever $V_1^* \subseteq \Psi$ and Ψ is a NOS in N^X.
3. Neutrosophic alpha generalized closed set [21](N(α)GCS in short) if $N^{\alpha cl}(V_1^*) \subseteq \Psi$ whenever $V_1^* \subseteq \Psi$ and Ψ is a NOS in N^X.
4. Neutrosophic generalized alpha closed set [16](NGαCS in short) if $N^{\alpha cl}(V_1^*) \subseteq \Psi$ whenever $V_1^* \subseteq \Psi$ and Ψ is a NαOS in N^X.

The complement of the above mentioned Neutrosophic closed sets are called their relevant open sets.

Remark 1.12. Let V_1^* be a NS in (N^X, N^τ). Then

1. $N^{S-cl}(V_1^*) = V_1^* \cap N^{int}(N^{cl}(V_1^*))$
2. $N^{S-int}(V_1^*) = V_1^* \cup N^{cl}(N^{int}(V_1^*))$

If V_1^* is a NS of N^X then $N^{Scl}(V_1^{*c}) = (N^{Scl}(V_1^*))^c$

Definition 1.13. Let V_1^* be a NS in (N^X, N^τ). Then

1. $N^{\alpha cl}(V_1^*) = V_1^* \cup N^{cl}\left(N^{int}(N^{cl}(V_1^*))\right)$

2. $N^{\alpha int}(V_1^*) = V_1^* \cap N^{int}\left(N^{cl}(N^{int}(V_1^*))\right)$

2. Neutrosophic α Generalized Semi-Closed Sets

Definition 2.1. A NS V_1^* in (N^X, N^τ) is said to be a Neutrosophic α generalized semi-closed set (N(αGS)CS in short) if $N^{\alpha cl}(V_1^*) \subseteq \Psi$ whenever $V_1^* \subseteq \Psi$ and Ψ is a N(S)OS in (N^X, N^τ).

Example 2.2. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ be a NT on N^X, where $J_1^* = \langle x, \left(\frac{6}{10}, \frac{1}{2}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right)\rangle$. Let us consider the NS $V_1^* = \langle x, \left(\frac{1}{10}, \frac{1}{2}, \frac{9}{10}\right), \left(\frac{1}{5}, \frac{1}{2}, \frac{4}{5}\right)\rangle$.

Since $N^{\alpha cl}(V_1^*) = V_1^*$, V_1^* is N(αGS)CS in (N^X, N^τ).

Theorem 2.3 Every NCS in (N^X, N^τ) is a N(αGS)CS.

Proof: Assume that V_1^* is a NCS in (N^X, N^τ). Let us consider a NS $V_1^* \subseteq \Psi$ and Ψ is a N(S)OS in N^X. Since $N^{\alpha cl}(V_1^*) \subseteq N^{cl}(V_1^*)$ and V_1^* is a NCS in N^X, $N^{\alpha cl}(V_1^*) \subseteq N^{cl}(V_1^*) = V_1^* \subseteq \Psi$ and Ψ is N(S)OS. That is $N^{\alpha cl}(V_1^*) \subseteq \Psi$. Therefore V_1^* is N(αGS)CS in N^X.

Example 2.4. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ be a NT on N^X. Here $J_1^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{5}, \frac{1}{2}, \frac{7}{10}\right)\rangle$.

Then the NS $V_1^* = \langle x, \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right), \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{5}\right)\rangle$ is N(αGS)CS but not NCS. Since $N^{\alpha cl}(V_1^*) = 1_N$ and possible $\Psi = 1_N$.

Theorem 2.5 Every NαCS in (N^X, N^τ) is a N(αGS)CS in (N^X, N^τ).

Proof: Let V_1^* be a NαCS in N^X. Let us consider a NS $V_1^* \subseteq \Psi$ and Ψ be a N(S)OS in (N^X, N^τ). Since V_1^* is a NαCS,$N^{\alpha cl}(V_1^*) = V_1^*$. Hence $N^{\alpha cl}(V_1^*) \subseteq \Psi$ whenever $V_1^* \subseteq \Psi$ and Ψ is N(S)OS. Therefore V_1^* is a N(αGS)CS in N^X.

Example 2.6. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ be a NT on N^X. Here $J_1^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\rangle$.

Consider NS $V_1^* = \langle x, \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{5}\right), \left(\frac{3}{5}, \frac{1}{2}, \frac{2}{5}\right)\rangle$ is N(αGS)CS but not NαCS since $N^{cl}\left(N^{int}(N^{cl}(V_1^*))\right) = 1_N \nsubseteq V_1^*$.

Theorem 2.7 Every N(R)CS in (N^X, N^τ) is a N(αGS)CS in (N^X, N^τ).

Proof: Let V_1^* be a N(R)CS in (N^X, N^τ). Since every N(R)CS is a NCS ,V_1^* is a NCS in N^X. By Theorem 2.3, V_1^* is a N(αGS)CS in N^X.

Example 2.8. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ be a NT on N^X.

Here $J_1^* = \langle x, \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{5}\right), \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right)\rangle$. Consider a NS $V_1^* = \langle x, \left(0, \frac{1}{2}, \frac{9}{10}\right), \left(\frac{1}{5}, \frac{1}{2}, \frac{4}{5}\right)\rangle$ which is a N(αGS)CS but not N(R)CS in N^X as $N^{cl}(N^{int}(V_1^*)) = 0_N \neq V_1^*$.

Remark 2.9. A N(G) closedness is independent of a N(αGS) closedness.

Example 2.10. Let $N^X = \{v_1, v_2\}$.Let $N^\tau = \{0_N, J_1^*, 1_N\}$ be a NT on N^X.

Here $J_1^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{5}, \frac{1}{2}, \frac{4}{5}\right)\rangle$ is a N(αGS)CS but not NGCS in N^X as $N^{cl}(V_1^*) \nsubseteq G$ eventhough $V_1^* \subseteq G$ and G is a GSOS in N^X.

Example 2.11. Let $N^X = \{v_1, v_2\}$.Let $N^\tau = \{0_N, J_1^*, 1_N\}$ be a NT on N^X.

Here $J_1^* = \langle x, \left(\frac{3}{5}, \frac{1}{2}, \frac{2}{5}\right), \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{5}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right), \left(\frac{9}{10}, \frac{1}{2}, \frac{1}{10}\right)\rangle$ is a NGCS but not N(αGS)CS since $N^{cl}(V_1^*) = 1_N \nsubseteq V_2^* = \langle x, \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{5}\right), \left(\frac{9}{10}, \frac{1}{2}, \frac{1}{10}\right)\rangle$ whenever $V_1^* \subseteq V_2^*$ and V_2^* is a N(S)OS in N^X.

Theorem 2.12. Every N(αGS)CS in (N^X, N^τ) is a NGSCS in (N^X, N^τ).

Proof: Assume that V_1^* is a N(αGS)CS in (N^X, N^τ). Let a NS $V_1^* \subseteq \Psi$ and Ψ be a NOS in N^X. By hypothesis $N^{cl}(V_1^*) \subseteq \Psi$, that is $V_1^* \cup N^{cl}\left(N^{int}(N^{cl}(V_1^*))\right) \subseteq \Psi$. This implies $V_1^* \cup N^{int}(N^{cl}(V_1^*)) \subseteq \Psi$. But $N^{scl}(V_1^*) = V_1^* \cup N^{int}(N^{cl}(V_1^*))$. Therefore $N^{scl}(V_1^*) = V_1^* \cup N^{int}(N^{cl}(V_1^*)) \subseteq \Psi$ whenever $V_1^* \subseteq \Psi$ and Ψ is NOS. Hence V_1^* is NGSCS.

Example 2.13. Let $N^X = \{v_1, v_2\}$.Let $N^\tau = \{0_N, J_1^*, 1_N\}$ be a NT on N^X.Here $J_1^* = \langle x, \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right), \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{10}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{5}\right), \left(\frac{4}{5}, \frac{1}{2}, 0\right)\rangle$ is a NGCS but not N(αGS)CS as $N^{\alpha cl}(V_1^*) = 1_N \nsubseteq V_2^* = \langle x, \left(\frac{9}{10}, \frac{1}{2}, \frac{1}{10}\right), \left(\frac{9}{10}, \frac{1}{2}, 0\right)\rangle$ whenever $V_1^* \subseteq V_2^*$ and V_2^* is a N(S)OS in N^X.

Remark 2.14. A NP closedness is independent of NαGS closedness.

Example 2.15. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ be a NT on N^X. Here $J_1^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{1}{10}, \frac{1}{2}, \frac{4}{5}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{1}{5}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{1}{10}, \frac{1}{2}, \frac{4}{5}\right)\rangle$ is a N(P)CS but not N(αGS)CS. Since $N^{\alpha cl}(V_1^*) \nsubseteq G$ even though $V_1^* \subseteq G$ and G is N(S)OS.

Example 2.16. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, J_2^*, 1_N\}$ is NT on N^X . Here $J_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right)\rangle$ and $J_2^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{3}{5}, \frac{1}{2}, \frac{2}{5}\right)\rangle$ is a N(αGS)CS. Since $N^{cl}(N^{int}(V_1^*)) \subseteq V_1^*$,$V_1^*$ is not a N(P)CS.

Remark 2.17. N(SP) closedness is independent of a NαGS closedness.

Example 2.18. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X. Here $J_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right)\rangle$.

Then the NS $V_1^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{10}, \frac{1}{2}, \frac{3}{5}\right)\rangle$ is a NSPCS but not N(αGS)CS. Since $N^{\alpha cl}(V_1^*) \nsubseteq V_2^*$, $V_2^* = \langle x, \left(\frac{1}{2}, \frac{1}{2}, \frac{2}{5}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right)\rangle$ where $V_1^* \nsubseteq V_2^*$ and V_2^* is N(S)OS.

Example 2.19. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, J_2^*, 1_N\}$ is NT on N^X . Here $J_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right)\rangle$ and $J_2^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{3}{5}, \frac{1}{2}, \frac{2}{5}\right)\rangle$ is a N(αGS)CS but not NSPCS as $N^{int}\left(N^{cl}(N^{int}(V_1^*))\right) \nsubseteq V_1^*$.

Diagram- I

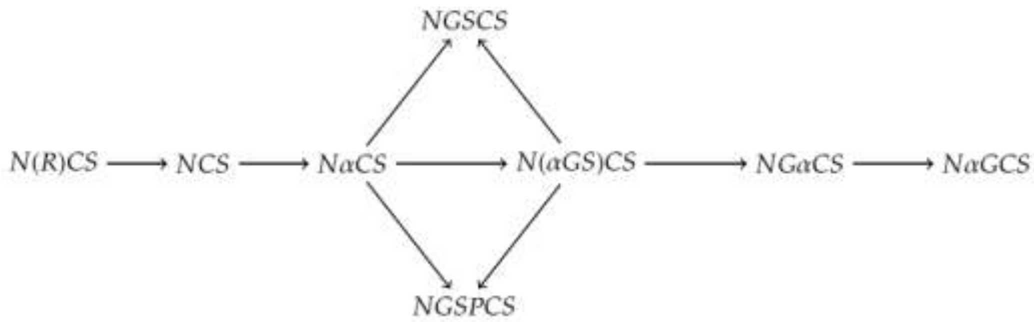
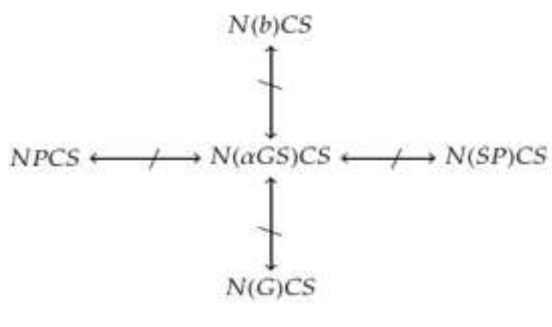


Diagram-II



Remark 2.20. *Nb* closedness is independent of NαGS closedness.

Example 2.21. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X,Here $J_1^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{4}{5}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right)\rangle$.

Then the NS $V_1^* = \langle x, \left(\frac{3}{5}, \frac{1}{2}, \frac{2}{5}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\rangle$ is NbCS but not N(α GS)CS. Since $N^{\alpha cl}(V_1^*) \nsubseteq V_2^* =$

$\langle x, \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right), \left(\frac{3}{5}, \frac{1}{2}, \frac{2}{5}\right)\rangle$, where $V_1^* \nsubseteq V_2^*$and V_2^* is N(S)OS in N^X.

Example 2.22. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, J_2^*, 1_N\}$ is NT on N^X ,Here $J_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right)\rangle$ and $J_2^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{3}{5}, \frac{1}{2}, \frac{2}{5}\right)\rangle$ is a N(αGS)CS but not NbCS as $N^{int}\left(N^{cl}(V_1^*)\right) \cap N^{cl}(N^{int}(V_1^*)) \nsubseteq V_1^*$.

Theorem 2.23. Every N(αGS)CS in (N^X, N^τ) is a NαGCS in (N^X, N^τ).

Proof: Assume that V_1^* is a N(αGS)CS in (N^X, N^τ). Let us consider a NS $V_1^* \subseteq \Psi$ and Ψ is NOS in (N^X, N^τ). By hypothesis, $N^{\alpha cl}(V_1^*) \nsubseteq \Psi$ whenever, $V_1^* \subseteq \Psi$ and Ψ is N(S)OS. This implies $N^{\alpha cl}(V_1^*) \nsubseteq \Psi$ whenever $V_1^* \subseteq \Psi$ and Ψ is NOS. Therefore V_1^* is a N(αG)CS in (N^X, N^τ).

Example 2.24. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X ,Here $J_1^* = \langle x, \left(\frac{1}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{3}{10}, \frac{1}{2}, \frac{3}{5}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right)\rangle$ is NαGCS but not N(αGS)CS. Since

$N^{\alpha cl}(V_1^*) \nsubseteq V_2^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{2}{5}\right)\rangle$, eventhough $V_1^* \nsubseteq V_2^*$and V_2^* is N(S)OS .

Theorem 2.25. Every N(αGS)CS in (N^X, N^τ) is a NGαCS in (N^X, N^τ).

Proof: Assume that V_1^* is a N(αGS)CS in (N^X, N^τ). Let $V_1^* \subseteq \Psi$ and Ψ is NαOS in N^X. By hypothesis, $N^{\alpha cl}(V_1^*) \subseteq \Psi$ whenever, $V_1^* \subseteq \Psi$ and Ψ is N(S)OS. This implies $N^{\alpha cl}(V_1^*) \subseteq \Psi$ whenever $V_1^* \subseteq \Psi$ and Ψ is NαOS. Therefore V_1^* is a NGαCS in (N^X, N^τ).

Example 2.26. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X.

Here $J_1^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{5}, \frac{1}{2}, \frac{4}{5}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{5}, \frac{1}{2}, \frac{2}{5}\right)\rangle$ is NGαCS but not

N(αGS)CS. Since $N^{\alpha cl}(V_1^*) \nsubseteq V_2^* = \langle x, \left(\frac{11}{20}, \frac{1}{2}, \frac{9}{20}\right), \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right)\rangle$, eventhough $V_1^* \nsubseteq V_2^*$and V_2^* is N(S)OS .

Remark 2.27. The intersection of any two N(αGS)CS is not a N(αGS)CS in general as seen from the following example.

Example 2.28. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X.

Here $J_1^* = \langle x, \left(\frac{1}{5}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{3}{10}, \frac{1}{2}, \frac{3}{5}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{1}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{5}\right)\rangle$, $V_2^* =$

$\langle x, \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{10}\right), \left(\frac{1}{5}, \frac{1}{2}, \frac{7}{10}\right)\rangle$ are N(αGS)CS . Now $V_1^* \cap V_2^* = \langle x, \left(\frac{1}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{1}{5}, \frac{1}{2}, \frac{7}{10}\right)\rangle$. Since $N^{\alpha cl}(V_1^* \cap V_2^*) \nsubseteq G$, eventhough $V_1^* \subseteq G$ and G is N(S)OS in N^X, $V_1^* \cap V_2^*$ is not a N(αGS)CS in N^X.

Theorem 2.29. Every (N^X, N^τ) is a NTS. Then for every $V_1^* \in$ N(αGS)C(N^X) and for every $V_2^* \in$ NS(N^X),$V_1^* \subseteq V_2^* \subseteq N^{\alpha cl}(V_1^*)$ implies $V_2^* \in$ N(αGS)C(N^X).

Proof: Let $V_2^* \subseteq \Psi$ and Ψ is N(S)OS in N^X. Since $V_1^* \subseteq V_2^*$, $V_1^* \subseteq \Psi$ and V_1^* is a N(α GS)CS, $N^{\alpha cl}(V_1^*) \subseteq \Psi$. By hypothesis, $V_2^* \subseteq N^{\alpha cl}(V_1^*)$, $N^{\alpha cl}(V_2^*) \subseteq N^{\alpha cl}(V_1^*) \subseteq \Psi$. Therefore $N^{\alpha cl}(V_2^*) \subseteq \Psi$. Hence V_2^* is N(αGS)CS of N^X.

Theorem 2.30. If V_1^* **is** both N(S)OS and N(αGS)CS in (N^X, N^τ), then V_1^* is a N(α)CS in N^X.

Proof: Let V_1^* is N(S)OS in N^X. Since $V_1^* \subseteq V_1^*$, by hypothesis $N^{\alpha cl}(V_1^*) \subseteq V_1^*$. But $V_1^* \subseteq N^{\alpha cl}(V_1^*)$. Therefore $N^{\alpha cl}(V_1^*) = V_1^*$. Hence V_1^* is a NαCS in N^X.

Theorem 2.31. The union of two N(αGS)CS is a N(αGS)CS in (N^X, N^τ), if they are NCS in (N^X, N^τ).

Proof: Assume that V_1^* and V_2^* are N(αGS)CS in (N^X, N^τ). Since V_1^* and V_2^* are NCS in N^X, $N^{cl}(V_1^*) = V_1^*$ and $N^{cl}(V_2^*) = V_2^*$. Let $V_1^* \cup V_2^* \subseteq \Psi$ and Ψ is N(S)OS in N^X. Then $N^{cl}\left(N^{int}(N^{cl}(V_1^* \cup V_2^*))\right) = N^{cl}\left(N^{int}(V_1^* \cup V_2^*)\right) \subseteq N^{cl}(V_1^* \cup V_2^*) = V_1^* \cup V_2^* \subseteq \Psi$, i.e., $N^{\alpha cl}(V_1^* \cup V_2^*) \subseteq \Psi$. Therefore $V_1^* \cup V_2^*$ is N(αGS)CS.

Theorem 2.32. Let (N^X, N^τ) is NTS and V_1^* is NS in N^X. Then V_1^* is a N(αGS)CS if and only if $V_1^* \bar{q}$F implies $N^{\alpha cl}(V_1^*)\overline{q}$F for every N(S)CS of N^X.

Proof: Necessary Part: Let F_1^* is N(S)CS in N^X and let $V_1^* \bar{q} F_1^*$. Then $V_1^* \subseteq F_1^{*c}$, Here F_1^{*c} is a N(S)OS in N^X. Therefore by hypothesis, $N^{\alpha cl}(V_1^*) \subseteq F_1^{*c}$. Hence $N^{\alpha cl}(V_1^*)\overline{q} F_1^*$.

Sufficient Part: Let F_1^* is N(S)CS in N^X and let V_1^* is NS in N^X. By hypothesis, $V_1^* \bar{q}$F implies $N^{\alpha cl}(V_1^*)\overline{q} F_1^*$. Then $N^{\alpha cl}(V_1^*) \subseteq F_1^{*c}$ whenever $V_1^* \subseteq F_1^{*c}$ and F_1^{*c} is a N(S)OS in N^X. Hence V_1^* is a N(αGS)CS in N^X.

3.Neutrosophic α Generalized Semi-Open Sets

In this section we introduce Neutrosophic α Generalized Semi-Open Sets and study some of its properties.

Definition 3.1. A NS V_1^* is said to be Neutrosophic α generalized semi-open set (NαGSOS in short) in (N^X, N^τ), if the complement V_1^{*c} is a N(αGS)CS in N^X. The family of all N(αGS)OS of a NTS (N^X, N^τ) is denoted by NαGSO(N^X).

Theorem 3.2 For any NTS (N^X, N^τ), every NOS is a N(αGS)OS.

Proof: Let V_1^* is NOS in N^X. Then V_1^{*c} is a NCS in N^X, By Theorem 2.3, V_1^{*c} is a N(αGS)CS in N^X. Hence V_1^* is a N(αGS)OS in N^X.

Example 3.3. Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X. Here $J_1^* = \langle x, \left(\frac{1}{5}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{1}{10}, \frac{1}{2}, \frac{4}{5}\right) \rangle$.

Then the NS $V_1^* = \langle x, \left(\frac{1}{10}, \frac{1}{2}, \frac{4}{5}\right), \left(0, \frac{1}{2}, \frac{9}{10}\right) \rangle$. Sinc e V_1^{*c} is a N(αGS)CS, V_1^* is a

N(αGS)OS, but not NOS.

Theorem 3.4 For any NTS (N^X, N^τ),every N(α)OS is a N(αGS)OS.
Proof: Let V_1^* is N(α)OS in N^X. Then V_1^{*c} is a N(α)CS in N^X, By Theorem 2.5, V_1^{*c} is a N(αGS)CS in N^X.

Example 3.5 Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X,Here $J_1^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \rangle$.

Then the NS $V_1^* = \langle x, \left(\frac{1}{5}, \frac{1}{2}, \frac{4}{5}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right) \rangle$ is a N(αGS)OS in N^X , V_1^* is not a NαOS in N^X.

Theorem 3.6 For any NTS (N^X, N^τ),every N(R)OS is a N(αGS)OS.
Proof: Let V_1^* is N(R)OS in N^X. Then V_1^{*c} is a N(R)CS in N^X, By Theorem 2.7, V_1^{*c} is a N(αGS)CS in N^X.

Example 3.7 Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X.

Here $J_1^* = \langle x, \left(\frac{3}{5}, \frac{1}{2}, \frac{3}{10}\right), \left(\frac{7}{10}, \frac{1}{2}, \frac{1}{10}\right) \rangle$. Then the NS $V_1^* = \langle x, \left(\frac{7}{10}, \frac{1}{2}, \frac{1}{5}\right), \left(\frac{4}{5}, \frac{1}{2}, 0\right) \rangle$ is a N(αGS)OS in N^X ,

V_1^* is not a N(R)OS in N^X.

Remark 3.8. N(αGS)OS and N(G)OS are independent in general.

Example 3.9 Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X.

Here $J_1^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right) \rangle$. Then the NS $V_1^* = \langle x, \left(\frac{3}{5}, \frac{1}{2}, \frac{3}{10}\right), \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{5}\right) \rangle$ is a N(αGS)OS in N^X , V_1^*

is not a NGOS in N^X.

Example 3.10 Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X. Here $J_1^* = \langle x, \left(\frac{3}{5}, \frac{1}{2}, \frac{2}{5}\right), \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{5}\right) \rangle$.

Then the NS $V_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{1}{10}, \frac{1}{2}, \frac{9}{10}\right) \rangle$ is a NGOS in N^X , but V_1^* is not a N(αGS)OS in N^X.

Theorem 3.11 Every N(αGS)OS in (N^X, N^τ) is a N(αGS)OS in (N^X, N^τ).
Proof: Let V_1^* is N(αGS)OS in N^X. Then V_1^{*c} is a N(αGS)CS in N^X, By Theorem 2.12, V_1^{*c} is a NGSCS in N^X.

Example 3.12 Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X.

Here $J_1^* = \langle x, \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right), \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{10}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{1}{5}, \frac{1}{2}, \frac{4}{5}\right), \left(0, \frac{1}{2}, \frac{4}{5}\right)\rangle$ is a NGSOS in N^X , but V_1^* is not a N(αGS)OS in N^X.

Remark 3.13. N(SP)OS is independent of N(αGS)OS.

Example 3.14 Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X.

Here $J_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{1}{2}, \frac{1}{2}, \frac{2}{5}\right), \left(\frac{3}{5}, \frac{1}{2}, \frac{3}{10}\right)\rangle$ is a N(SP)OS in N^X, but V_1^* is not a N(αGS)OS in N^X.

Example 3.15 Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X.

Here $J_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right)\rangle$ and $J_2^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\rangle$. Then the NS $V_1^* =$

$\langle x, \left(\frac{7}{10}, \frac{1}{2}, \frac{3}{10}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right)\rangle$ is a N(αGS)OS in N^X but V_1^* is not a N(SP)OS in N^X.

Theorem 3.16 Every N(αGS)OS in (N^X, N^τ) is a NαGOS in (N^X, N^τ).

Proof: Let V_1^* is N(αGS)OS in N^X. Then V_1^{*c} is a N(αGS)CS in N^X, By Theorem 2.23, V_1^{*c} is a N(αG)CS in N^X. Hence V_1^* is a NαGOS in N^X.

Example 3.17 Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X.

Here $J_1^* = \langle x, \left(\frac{1}{10}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{3}{10}, \frac{1}{2}, \frac{3}{5}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{3}{5}, \frac{1}{2}, \frac{3}{10}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{2}{5}\right)\rangle$ is NαGOS in N^X, but not N(αGS)OS in N^X.

Theorem 3.18 Every N(αGS)OS in (N^X, N^τ) is a NGαOS in (N^X, N^τ).

Proof: Let V_1^* is N(αGS)OS in N^X. Then V_1^{*c} is a N(αGS)CS in N^X, By Theorem 2.25, V_1^{*c} is a NGαCS in N^X. Hence V_1^* is a NGαOS in N^X.

Example 3.19 Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X.

Here $J_1^* = \langle x, \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{5}, \frac{1}{2}, \frac{4}{5}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}\right)\rangle$ is NGαOS in N^X, but not N(αGS)OS in N^X.

Theorem 3.2 Let (N^X, N^τ) is NTS. If V_1^* is a NS of N^X followed by consequences are equal:

1. $V_1^* \in \mathrm{N\alpha GSO}(N^X)$

2. $V \subseteq N^{int}\left(N^{cl}(N^{int}(V_1^*))\right)$ whenever $V \subseteq V_1^*$ and V is a N(S)CS in N^X

3. There exists NOS $G_1 \subseteq V \subseteq N^{int}(N^{cl}(G))$ where $G = N^{int}(V_1^*); V \subseteq V_1^*$ and V is a N(S)CS in N^X

Proof: (1)⟹ (2) Let $V_1^* \in \mathrm{N(\alpha GS)O}(N^X)$. Then V_1^{*c} is a N(αGS)CS in N^X, Therefore $N^{\alpha cl}(V_1^{*c}) \subseteq \Psi$, whenever $V_1^{*c} \subseteq \Psi$ and Ψ is a N(S)OS in N^X. i.e.,$N^{cl}\left(N^{int}(N^{cl}(V_1^{*c}))\right) \subseteq \Psi$. Taking complement on both sides, we get $\left(N^{cl}\left(N^{int}(N^{cl}(V_1^{*c}))\right)\right)^c = \left(N^{int}\left(N^{int}(N^{cl}(V_1^{*c}))\right)\right)^c = N^{int}N^{int}(N^{cl}\left(N^{cl}(V_1^{*c})\right)^c =$ $N^{int}(N^{cl}(N^{int}(V_1^{*c})^c) = N^{int}\left(N^{cl}(N^{int}(V_1^{*}))\right) \supseteq \Psi^c$. This implies $\Psi^c \subseteq N^{int}\left(N^{cl}(N^{int}(V_1^{*}))\right)$ whenever $\Psi^c \subseteq V_1^*$ and Ψ^c is a N(S)CS in N^X. Replace Ψ^c by V, $V \subseteq N^{int}\left(N^{cl}(N^{int}(V_1^{*}))\right)$ whenever $V \subseteq V_1^*$ and V is a N(S)CS in N^X.

(2) ⟹ (3) Let $V \subseteq N^{int}\left(N^{cl}(N^{int}(V_1^{*}))\right)$ whenever $V \subseteq V_1^*$ and V is a N(S)CS in N^X. Hence $N^{int}(V) \subseteq V \subseteq N^{int}\left(N^{cl}(N^{int}(V_1^{*}))\right)$. Then there exists NOS J_1^* in N^X such that $G_1 \subseteq V \subseteq N^{int}(N^{cl}(G))$ where $G = N^{int}(V_1^*)$ and $J_1^* = N^{int}(V)$.

(3) ⟹ (1) Suppose that there exists NOS J_1^* such that $J_1^* \subseteq V \subseteq N^{int}(N^{cl}(G))$ where $G = N^{int}(V_1^*)$; $V \subseteq V_1^*$ and V is a N(S)CS in N^X. It is clear that $(N^{int}(N^{cl}(G)))^c \subseteq V^c$. That is $\left(N^{int}\left(N^{cl}(N^{int}(V_1^{*}))\right)\right)^c \subseteq V^c$. This implies $N^{cl}\left(N^{cl}(N^{int}(V_1^{*}))\right)^c \subseteq V^c$. Therefore $N^{cl}\left(N^{int}(N^{int}(V_1^{*c}))\right) \subseteq V^c, V_1^{*c} \subseteq V^c$ and V^c is N(S)OS in N^X. Hence $\alpha N^{cl}(V_1^{*c}) \subseteq V^c$.i.e, V_1^{*c} is a N(αGS)CS in N^X. This implies $V_1^* \in$ NαGSO(N^X).

Theorem 3.21 Let (N^X, N^τ) is NTS. Then for every $V_1^* \in \mathrm{N\alpha GSO}(N^X)$ and for every $V_1^* \in NS(N^X), N^{\alpha int}(V_1^*) \subseteq V_2^* \subseteq V_1^*$ implies $V_2^* \in \mathrm{N\alpha GSO}(N^X)$.

Proof: By hypothesis, $N^{\alpha int}(V_1^*) \subseteq V_2^* \subseteq V_1^*$. Taking complement on both sides, we get $V_1^{*c} \subseteq V_2^{*c} \subseteq (N^{\alpha int}(V_1^*))^c$. Let $V_2^{*c} \subseteq \Psi$ and Ψ is N(S)OS in N^X. Since $V_1^{*c} \subseteq V_2^{*c}.V_1^{*c} \subseteq \Psi$. Since V_1^{*c} is a $\mathrm{N(\alpha GS)CS}, N^{\alpha int}(V_1^{*c}) \subseteq \Psi$. Also $V_2^{*c} \subseteq (N^{\alpha int}(V_1^*))^c = N^{\alpha cl}(V_1^{*c})$. Therefore $N^{\alpha cl}(V_2^{*c}) \subseteq N^{\alpha cl}(V_1^{*c}) \subseteq \Psi$. Hence V_2^{*c} is a $\mathrm{N(\alpha GS)CS}$ in N^X. This implies V_2^* is a $\mathrm{N(\alpha GS)OS}$ in N^X.i.e.,$V_2^* \in \mathrm{N\alpha GSO}(N^X)$.

Remark 3.22. The union of any two N(αGS)OS in (N^X, N^τ) is not a N(αGS)OS in (N^X, N^τ).

Example 3.23 Let $N^X = \{v_1, v_2\}$. Let $N^\tau = \{0_N, J_1^*, 1_N\}$ is NT on N^X.

Here $J_1^* = \langle x, \left(\frac{3}{10}, \frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{5}, \frac{1}{2}, \frac{7}{10}\right)\rangle$. Then the NS $V_1^* = \langle x, \left(\frac{3}{5}, \frac{1}{2}, \frac{1}{5}\right), \left(\frac{1}{10}, \frac{1}{2}, \frac{4}{5}\right)\rangle$ and $V_2^* = \langle x, \left(\frac{1}{5}, \frac{1}{2}, \frac{7}{10}\right), \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{10}\right)\rangle$ are N(αGS)OS in N^X, but $V_1^* \cup V_2^* = \langle x, \left(\frac{3}{5}, \frac{1}{2}, \frac{1}{5}\right), \left(\frac{4}{5}, \frac{1}{2}, \frac{1}{10}\right)\rangle$ is not an N(αGS)OS in N^X.

Theorem 3.24 A NS V_1^* of a NTS (N^X, N^τ) is a N(αGS)OS if and only if $F \subseteq N^{\alpha int}(V_1^*)$ whenever $F \subseteq V_1^*$ and F is a N(S)CS in N^X.

Proof: Necessary Part: Suppose V_1^* is a N(αGS)OS in N^X. Let F is N(S)CS in N^X and $F \subseteq V_1^*$. Then F^c is a N(S)OS in N^X such that $V_1^{*c} \subseteq F^c$. Since V_1^{*c} is a N(αGS)CS, we have $N^{\alpha\mathrm{cl}}(V_1^{*c}) \subseteq F^c$. Hence $\left(N^{\alpha\mathrm{int}}(V_1^*)\right)^c \subseteq F^c$. Therefore $F \subseteq N^{\alpha int}(V_1^*)$.

Sufficient Part: Let V_1^* is NS in N^X and let $F \subseteq N^{\alpha\mathrm{int}}(V_1^*)$ whenever F is a N(S)CS in N^X and $F \subseteq V_1^*$. Then $V_1^{*c} \subseteq F^c$ and F^c is a N(S)OS. By hypothesis , $\left(N^{\alpha\mathrm{int}}(V_1^*)\right)^c \subseteq F^c$,which implies $N^{\alpha\mathrm{cl}}(V_1^{*c}) \subseteq F^c$. Therefore V_1^{*c} is a N(αGS)CS in N^X. Hence V_1^* is a N(αGS)OS in N^X.

4. Conclusion

In this paper, Neutrosophic αGS closed sets and Neutrosophic αGS open sets are introduced and discussed some of its basic properties and their relationships with existing Neutrosophic closed and open sets. In future, this set can be extended with various results and their applications. this is a very initial work it can be applicable in Neutrosophic supra topological spaces, Neutrosophic crisp topological spaces and Neutrosophic n -topological spaces

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Pentapartitioned Neutrosophic Pythagorean Strongly Irresolvable Spaces

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Abstract: The aim of this paper is to develop many characterizations of Pentapartitioned Neutrosophic Pythagorean (PNP) strongly irresolvable spaces and its properties is also studied. Several characterizations of Pentapartitioned Neutrosophic Pythagorean strongly irresolvable spaces are investigated in this study. Also examined are the conditions under which Pentapartitioned Neutrosophic Pythagorean strongly irresolvable spaces become Pentapartitioned Neutrosophic Pythagorean first category spaces, Pentapartitioned Neutrosophic Pythagorean Baire spaces, and Pentapartitioned Baire spaces.

Keywords: Pentapartitioned neutrosophic pythagorean set, pentapartitioned neutrosophic pythagorean resolvable space, pentapartitioned neutrosophic pythagorean irresolvable spaces, pentapartitioned neutrosophic pythagorean strongly irresolvable spaces.

1. Introduction

Zadeh [17] proposed the fuzzy set concept in 1965 as a new technique to modelling uncertainties. Researches revealed the value of the fuzzy concept and have effectively used it to all fields of mathematics. Topology provides the most natural framework for fuzzy set theories to flourish in mathematics. Chang [3] first suggested the method of fuzzy topological space in 1968. Chang's paper established the stage for the tremendous growth of several fuzzy topological concepts that followed. Several mathematicians have continued to integrate all of the key notions of general topology to fuzzy circumstances, resulting in the development of a current theory of fuzzy topology. Today, fuzzy topology has been firmly established as one of the basic disciplines of fuzzy mathematics. Atanassov and plenty of researchers [1] worked on intuitionistic fuzzy sets within the literature. Florentin Smarandache [14] introduced the idea of Neutrosophic set in 1995 that provides the information of neutral thought by introducing the new issue referred to as uncertainty within the set. thus neutrosophic set was framed and it includes the parts of truth membership function(T), indeterminacy membership function(I), and falsity membership function(F) severally. Neutrosophic sets deals with non normal interval of]−0 1+[. Pentapartitioned neutrosophic set and its properties were introduced by Rama Malik and Surpati Pramanik [13]. In this case, indeterminacy is divided into three components: contradiction, ignorance, and an unknown membership function. The concept of Pentapartitioned neutrosophic pythagorean sets was initiated by R. Radha and A. Stanis Arul Mary[7]. The concept of intuitionistic fuzzy almost resolvable spaces and irresolvable spaces was introduced by Sharmila s [15].R. Radha and A.Stanis Arul Mary introduced Pentapartitioned neutrosophic pythagorean resolvable and irresolvable spaces. Also we have studied the concept of Pentapartitioned neutrosophic pythagorean almost resolvable and irresolvable spaces. Now we extend the concepts to pentapartitioned neutrosophic pythagorean strongly irresolvable spaces and studied relations with other Pentapartitioned neutrosophic pythagorean baire spaces, first category set, second category set and hyper connected spaces.

2. Preliminaries

2.1 Definition [14]

Let X be a universe. A Neutrosophic set A on X can be defined as follows:

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}$$

Where $T_A, I_A, F_A: U \to [0,1]$ *and* $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$

Here, $T_A(x)$ is the degree of membership, $I_A(x)$ is the degree of inderminancy and $F_A(x)$ is the degree of non-membership.

2.2 Definition [7]

Let X be a universe. A Pentapartitioned neutrosophic pythagorean [PNP] set A with T, F, C and U as dependent neutrosophic components and I as independent component for A on X is an object of the form

$$A = \{< x, T_A, C_A, I_A, U_A, F_A >: x \in X\}$$

Where $T_A + F_A \leq 1, C_A + U_A \leq 1$ *and*

$$(T_A)^2 + (C_A)^2 + (I_A)^2 + (U_A)^2 + (F_A)^2 \leq 3$$

Here, $T_A(x)$ is the truth membership, $C_A(x)$ is contradiction membership, $U_A(x)$ is ignorance membership,

$F_A(x)$ is the false membership and $I_A(x)$ is an unknown membership.

2.3 Definition [13]

Let P be a non-empty set. A Pentapartitioned neutrosophic set A over P characterizes each element p in P a truth -membership function T_A , a contradiction membership function C_A, an ignorance membership function G_A, unknown membership function U_A and a false membership function F_A , such that for each p in P

$$T_A + C_A + G_A + U_A + F_A \leq 5$$

2.4 Definition [7]

The complement of a pentapartitioned neutrosophic pythagorean set A on R Denoted by A^C or A^* and is defined as

$A^C = \{< x, F_A(x), U_A(x), 1 - G_A(x), C_A(x), T_A(x) > : x \in X\}$

2.5 Definition [7]

Let A = $< x, T_A(x), C_A(x), G_A(x), U_A(x), F_A(x) >$ and B = $< x, T_B(x), C_B(x), G_B(x), U_B(x), F_B(x) >$ are pentapartitioned neutrosophic pythagorean sets. Then

A $\cup$ B = $< x, max(T_A(x), T_B(x)), max(C_A(x), C_B(x)), \min(G_A(x), G_B(x)), \min(U_A(x), U_B(x)),$
$\min(F_A(x), F_B(x)), >$

A $\cap$ B = $< x, min(T_A(x), T_B(x)), min(C_A(x), C_B(x)), \max(G_A(x), G_B(x))$

$$, max(U_A(x), U_B(x)), max(F_A(x), F_{B}(x)) >$$

2.6 Definition [7]

A PNP topology on a nonempty set R is a family of a PNP sets in R satisfying the following axioms

1) $0,1 \in \tau$
2) $R_1 \cap R_2 \in \tau$ for any $R_1, R_2 \in \tau$
3) $\cup R_i \in \tau$ for any $R_i : i \in I \subseteq \tau$

The complement R* of PNP open set (PNPOS, in short) in PNP topological space [PNPTS] (R,$\boldsymbol{\tau}$), is called a PNP closed set [PNPCS].

2.7 Definition [7]

Let (R,$\boldsymbol{\tau}$) be a PNPTS and L be a PNPTS in R. Then the PNP interior and PNP Closure of R denoted by

Cl(L) = $\cap\{\boldsymbol{K}$: K is a PNPCS in R and L⊆ K}.

Int(L) = ⋃{G: G is a PNPOS in R and G⊆ L}.

2.8 Definition [11]

Let (R,$\boldsymbol{\tau}$) be a PNPTS and K be a PNP set in (R,$\boldsymbol{\tau}$). Then the PNP closure operator satisfy the following properties.

1-PNPCl(K) = PNPInt(1-K)

1-PNPInt(K) = PNPCl(1-K)

2.9 Definition [11]

A PNP A in PNPTS (R,$\boldsymbol{\tau}$) is called PNP dense if there exists no PNPCS L in (R,$\boldsymbol{\tau}$) such that K ⊆ **L** ⊆ **1.** That is PNPCl(K) = 1.

2.10 Definition [11]

A PNP A in PNPTS (R,$\boldsymbol{\tau}$) is called PNP nowhere dense if there exists no nonzero PNPOS L in (R,$\boldsymbol{\tau}$) such that **L** ⊆ PNPCl(K). That is PNPInt (PNPCl(K)) = 0.

2.11 Definition [11]

A PNPTS (R,$\boldsymbol{\tau}$) is called PNP resolvable if there exists a PNP dense set K in (R,$\boldsymbol{\tau}$) such that PNPCl (1-K) = 1. Otherwise (R,$\boldsymbol{\tau}$) is called PNP irresolvable.

2.12 Definition [11]

A PNPTS (R,$\boldsymbol{\tau}$) is called PNP submaximal if PNPCl(K) = 1 for any non-zero PNP set K in (R,$\boldsymbol{\tau}$).

2.13 Definition [11]

A PNPTS (R,$\boldsymbol{\tau}$) is called a PNP open hereditarily resolvable if PNPInt (PNPCl(K)) $\neq \mathbf{0}$ for any PNP set K in (R,$\boldsymbol{\tau}$).

2.14 Definition [11]

APNPTS (R,$\boldsymbol{\tau}$) is called PNP first category if $\bigcup_{i=1}^{\infty} \boldsymbol{K_i}$, where $\mathbf{K_i's}$ are PNP nowhere dense sets in (R,$\boldsymbol{\tau}$). A PNPTS which is not first category is said to be PNP second category.

2.15 Definition [11]

A PNPTS (R,$\boldsymbol{\tau}$) is called a PNP baire space if PNPInt $(\bigcup_{i=1}^{\infty} \mathbf{K_i}) = \mathbf{0}$, where $\boldsymbol{K_i's}$ are PNP nowhere dense sets in (R,$\boldsymbol{\tau}$).

2.16 Definition [12]
A PNP K in a PNPTS (R,τ) is called PNPR_1 if K = $\bigcap_{1=1}^{\infty} K_i$ where each $K_i \in \tau$.

2.17 Definition [12]
A PNP K in a PNPTS (R,τ) is called PNPR_2 if K = $\bigcup_{i=1}^{\infty} K_i$ where each $K_i \in \tau$.

2.18 Definition [12]
A PNPTS (R,τ) is called a PNP hyper- connected space if every PNP open set is PNP dense in (R,τ). That is PNPCl $(K_i) = 1$ for all $K_i \in \tau$.

2.19 Definition [12]
A PNPTS (R,τ) is called Pentapartitioned Neutrosophic Pythagorean nodec space, if every non-zero PNP nowhere dense set in (R,τ) is PNP closed.

3. **Pentapartitioned Neutrosophic Pythagorean (PNP) Strongly Irresolvable Spaces**

3.1 Definition
A Pentapartitioned Neutrosophic Pythagorean topological space PNPTS (R, τ) is called a Pentapartitioned Neutrosophic Pythagorean strongly irresolvable space if PNPCl (K) = 1 for any non-zero Pentapartitioned neutrosophic pythagorean set K in (R,τ) implies that PNPCl (PNPInt(K)) = 1 .

3.2 Example
Let R = {p}. Let A and B be the PNP sets defined on R as follows.
A = {p,0.4,0.3,0.3,0.5,0.4}
B = {p,0.5,0.6,0.5,0.2,0.3}
Then, clearly τ = {0, A, 1} is a PNP topology on R.
Then, PNPCl(A) = 1 and PNPCl (PNPInt(A) = 1,
PNPCl(B) = 1 and PNPCl (PNPInt (B) = 1.
Hence (R,τ) is a PNP strongly irresolvable space.

3.3 Theorem
If (R,τ) is a PNP strongly irresolvable space and if PNPInt(K) = 0 for any non-zero PNP set K in (R,τ) , then PNPInt (PNPCl(K)) = 0.
Proof
Let K be a non-zero PNP set in (R,τ) such that PNPInt(K) = 0. Then 1 – PNPInt(K) = 1 which implies that PNPCl(1-K) = 1. Since (R,τ) is PNP strongly irresolvable space, we have PNPCl (PNPInt(1-k)) = 1 which implies that 1 – PNPInt (PNPCl(K)) = 1. Therefore PNPInt (PNPCl(K)) = 0.

3.4 Theorem

If (R,τ) is a PNP strongly irresolvable space and if PNPInt(PNPCl(K)) $\neq 0$ for any non-zero PNP set K in (R,τ) then PNPInt (K) $\neq 0$.

Proof

Let K be a non-zero PNP set in (R,τ) such that PNPInt(PNPCl(K)) $\neq 0$. We claim that PNPInt(K) $\neq$ 0. Supp ose that PNPInt(K) $= 0$. Then 1 – PNPInt(K) $= 1$. Now PNPCl$(1 - \text{K}) = 1$. Since (R, τ) is a PNP strongly irresolvable space, we have PNPCl(PNPInt(1-K)) = 1. Hence 1 – PNPInt(PNPCl(K)) = 1 implies that PNPInt(PNPCl(K)) = 0, which is a contradiction. Hence we must have PNPInt(K) $\neq$ 0.

3.5 Theorem

If (R,τ) is a PNP strongly irresolvable space, then (R,τ) is a PNP irresolvable space.

Proof

Let K be a non-zero PNP set in (R,τ) such that PNPCl(K) = 1. We claim that PNPInt(K) $\neq$ 0. Suppose that PNPInt(K) = 0, then 1 – PNPInt(K) = 1, which implies that PNPCl(1-K) = 1. Then PNPInt(PNPCl(1-K)) = PNPInt(1) = 1. This implies that 1 – PNPCl(PNPInt(K)) = 1. Then we have PNPCl(PNPInt(K)) = 0 which is a contradiction to (R,τ) is a PNP strongly irresolvable spaces. Hence our assumption PNPInt(K) = 0 is wrong,. Hence we must have PNPInt(K) $\neq$ 0 for all PNP dense sets K in (R,τ). Therefore (R,τ) must be a PNP irresolvable space.

3.6 Theorem

If (R,τ) is a PNP strongly irresolvable space , then PNPInt(K_1) $\subseteq 1 -$ PNPInt(K_2) for any two dense sets K_1, K_2 in (R,τ).

Proof

Let K_1 and K_2 be any two non-zero PNP dense sets in (R,τ). Then PNPCl(K_1) = 1 and PNPCl(K_2) = 1 which implies that PNPInt(PNPCl(K_1) $\neq$ 0 and PNPInt(PNPCl(K_2) $\neq$ 0. Since is a PNP strongly irresolvable space, by theorem 3.4, we have PNPInt(K_1) $\neq$ 0 and PNPInt(K_2) $\neq$ 0. By theorem 3.5, (R,τ) is a PNP irresolvable space, But (R,τ) is PNP irresolvable if has a pair of dense sets, K_1 & K_2 $\exists\, K_1 \subseteq K_{2,}$ Now PNPInt(K_1) $\subseteq K_1 \subseteq 1 - K_2 \subseteq 1 -$ PNPInt(K_2). Therefore we have PNPInt(K_1) $\subseteq$ $K_1 \subseteq 1 -$PNPInt(K_2) for any two PNP dense sets K_1, K_2 in (R,τ),

3.7 Theorem

If a PNPTS (R,τ) is a PNP submaximal space, then (R,τ) is a PNP strongly irresolvable space.

Proof

Let (R,τ) be a PNP submaximal space and K be a PNP dense set in (R,τ). Since K is a PNP dense set in (R,τ), PNPCl(K) = 1, which implies PNPInt(1-K) = 1-1 = 0.Therefore PNPCl(PNPInt(1-K)) = 0. That is 1 – PNPCl(PNPInt(K)) = 1, which implies 1 – PNPInt(PNPCl(1- K)) = 1. Hence PNPCl(PNPInt(K)) = 1. Therefore (R,τ) is a strongly irresolvable space

3.8 Theorem

If K is a PNP nowhere dense set in a PNP topological space (R,τ), then (1 – K) is a PNP dense set in (R,τ).

Proof

Let K be a PNP nowhere dense set in (R,τ). Then we have PNPInt(PNPCl(K)) = 0. Now 1 – PNPInt(PNPCl(K)) = 1- 0 = 1.Then PNPCl(1 – PNPCl(K)) = 1, which implies that PNPCl(1-PNPInt(1-K)) = 1.But PNPCl(1 – PNPInt(1- K)) $\subseteq$ PNPCl(PNPCl(1-K)). Hence 1 $\subseteq$ PNPCl(PNPCl(1-K)). Therefore PNPCl(1-PNPInt(1-K)) = 1. Also 1 – PNPInt(PNPCl(K)) = 1-0=1. Then we have PNPCl(1-PNPCl(K)) = 1, which implies that PNPCl(PNPInt(1-K)) = 1. But PNPCl(PNPInt(1-K)) $\subseteq$ PNPCl(PNPCl(1-K)). Hence $1 \subseteq$ PNPCl(PNPCl(1-K)). That is PNPCl(PNPCl(1-K)) = 1. Therefore 1- K is a PNP dense set in (R,τ).

3.9 Theorem

If a PNPTS (R,τ) is a PNP submaximal space, then (R,τ) is a PNP nodec space.

Proof

Let (R,τ) be a PNP submaximal space and K be a PNP nowhere dense set in (R,τ). Then by theorem 3.8, 1-K is a PNP dense set in (R,τ). Since (R,τ) is a PNP submaximal space, 1-K is a PNP open set in (R,τ). This implies that K is a PNP closed set in (R,τ). Hence each PNP nowhere dense set is a PNP closed set in (R,τ) and therefore (R,τ) is a PNP nodec space.

3.10 Theorem

If (R,τ) is a PNP strongly irresolvable then (R,τ) is a PNP Baire space if and only if PNPCl($\bigcap_{i=1}^{\infty} K_i$) = 1.

Proof

Let (R,τ) be a PNP strongly irresolvable space. Suppose that $K_i's$ are PNP dense set in (R,τ), then PNPCl(PNPInt(K_i)) = 1. Now 1 – PNPCl(PNPInt(K_i)) = 1-1 = 0. Then we have PNPInt(PNPCl(1-K_i)) = 0. Hence (1-K_i)'s are PNP nowhere dense sets in (R,τ). Now PNPCl($\bigcap_{i=1}^{\infty} K_i$) = 1 implies that 1 – PNPCl($\bigcap_{i=1}^{\infty} K_i$) = 0 and hence PNPInt(1 – ($\bigcap_{i=1}^{\infty} K_i$)) = 0 and hence PNPInt($\bigcup_{i=1}^{\infty}$(1-$K_i$)) = 0 ,where (1-$K_i$)'s are PNP nowhere dense sets in (R,τ) and therefore (R,τ) is a PNP baire space.

Conversely, Let K_i's be PNP nowhere dense sets in a PNP strongly irresolvable space and PNP baire space (R,τ). Since (R,τ) is a PNP baire space, PNPInt($\bigcup_{i=1}^{\infty} K_i$) = 0. Then 1 – PNPInt($\bigcup_{i=1}^{\infty} K_i$) = 1. This implies that

$$\text{PNPCl}\left(\bigcap_{i=1}^{\infty}(1-K_i)\right) = 1 \qquad (1)$$

Since K_i's be PNP nowhere dense sets in a PNP strongly irresolvable space then by theorem3.8, (1-$K_i)'s$ are PNP dense sets in . Let B_i = 1-K_i. Then from(1), PNPCl($\bigcap_{i=1}^{\infty} K_i$) = 1, where $K_i's$ are PNP nowhere dense sets in (R,τ).

3.11 Theorem

If (R,τ) is a PNP strongly irresolvable and K = $\bigcap_{i=1}^{\infty} K_i$ be a PNP dense set in (R,τ). Then 1 - K is a PNP first category set in (R,τ),

Proof

Let K = be a PNP dense set in (R,τ). Then PNPCl($\bigcap_{i=1}^{\infty} K_i$) = 1. But PNPCl($\bigcap_{i=1}^{\infty} K_i$) $\subseteq$ $\bigcap_{i=1}^{\infty} P$ NPCl(K_i). Thus 1 $\subseteq$ PNPCl($\bigcap_{i=1}^{\infty} K_i$) $\subseteq$ $\bigcap_{i=1}^{\infty} P$ NPCl(K_i) . Then $\bigcap_{i=1}^{\infty}(PNPCl(K_i)$= 1. This implies that PNPCl(K_i) = 1. Thus K_i's are PNP dense set in (R,τ). Since (R,τ) is PNP strongly irresolvable, by theorem 3.8,(1- K_i)'s are PNP dense sets in (R,τ). Therefore , we have 1 – K = $\bigcup_{i=1}^{\infty}$(1-K_i), where (1-K_i)'s are PNP nowhere dense sets. Hence 1 – K is a PNP first category set in (R,τ).

3.12 Theorem

If is a PNP strongly irresolvable space and K = $\bigcap_{i=1}^{\infty} K_i$ be a PNP dense set in (R,τ). Then K is a PNP residual set in (R,τ).

Proof

Let K = $\bigcap_{i=1}^{\infty} K_i$ be a PNP dense set in (R,τ). Since (R,τ) is a PNP strongly irresolvable space, by theorem 3.11, 1 – K is a PNP first category set in (R,τ). Therefore K is a PNP residual set in (R,τ).

3.13 Theorem

Let (R,τ) be a PNP strongly irresolvable space. If K is a PNP dense set in (R,τ), then 1 – K is a PNP nowhere dense set.

Proof

Let K be a PNP dense set in (R,τ). Since (R,τ) is a PNP strongly irresolvable space, PNPCl(PNPInt(K)) = 1. This implies that 1 – PNPCl(PNPInt(K)) = 0. Therefore PNPInt(PNPCl(1-K)) = 0 and hence 1 – K is a PNP nowhere dense set in (R,τ).

3.14 Theorem

If (R,τ) is a PNP strongly irresolvable and PNP nodec space, then (R,τ) is a PNP submaximal space,

Proof

Let (R,τ) be a PNP strongly and PNP nodec space. Let K be a PNP dense set in (R,τ). Since (R,τ) is a PNP strongly irresolvable, by theorem 3.13, 1-K is a PNP nowhere dense set in (R,τ). Since (R,τ) is a PNP nodec space, 1-K is a PNP closed set in (R,τ). Then K is a PNP open set in (R,τ). Hence every PNP dense set is PNP open set in (R,τ). Therefore (R,τ) is a PNP submaximal space.

3.15 Theorem

If (R,τ) is a PNP strongly irresolvable and PNP second category space, then $\bigcap_{i=1}^{\infty} K_i \neq 0$ where $K_i's$ are PNP dense sets in (R,τ).

Proof

Let (R,τ) be a PNP second category space. Let us assume that $\bigcap_{i=1}^{\infty} K_i \neq 0$. Since K_i's are PNP dense sets in (R,τ), by theorem 3.12, $(1-K_i)'s$ are PNP nowhere dense sets in (R,τ). Now $1 - \bigcap_{i=1}^{\infty} K_i$ = 1, implies that $\bigcup_{i=1}^{\infty}(1-K_i)$ and (1 - K_i)'s are PNP nowhere dense sets in (R,τ).Hence (R,τ) is a PNP first category space, which is a contradiction. Therefore $\bigcap_{i=1}^{\infty} K_i \neq 0$, where $K_i's$ are PNP dense sets in (R,τ).

3.16 Theorem

If (R,τ) is a PNP submaximal space and K is a PNP first category set, then 1 – K is a PNPR_2 set in (R,τ).

Proof

Let K be a PNP first category set in (R,τ). Then K = $\bigcup_{i=1}^{\infty} K_i$, where K_i's are PNP nowhere dense sets in (R,τ). Therefore, by theorem 3.8, $(1- K_i)$'s are PNP dense sets in (R,τ). Since (R,τ) is a PNP submaximal space, $(1- K_i)$'s are open set in (R,τ). Also 1-K = 1-($\bigcup_{i=1}^{\infty} K_i$) = $\bigcap_{i=1}^{\infty}(1-K_i)$, where (1-$K_i$)'s are PNP open sets in (R,τ). Therefore 1-K is a PNPR_2set in (R,τ).

3.17 Theorem

If (R,τ) is a PNP submaximal space, then every PNP first category set is a PNPR_1 set in (R,τ).

Proof

Let K be a PNP first category set in (R,τ). Since (R,τ) is a PNP submaximal space, by theorem 3.16, 1-K is a PNPR_2 set in (R,τ) and hence K is a PNPR_1 set in (R,τ).

3.18 Theorem

If (R,τ) is a PNP submaximal space, then every PNP residual set is a PNPR_1 set in (R,τ).

Proof

Let K be a PNP residual set in (R,τ). Then 1-K is a PNP first category set in **3.17 Theorem**

If (R,τ) is a PNP submaximal space, then every PNP first category set is a PNPR_1 set in (R,τ).

Proof

Let K be a PNP first category serin (R,τ). Since (R,τ) is a PNP submaximal space, by theorem 3.16, 1-K is a PNPR_2 set in (R,τ) and hence K is a PNPR_1 set in (R,τ). Since (R,τ) is a PNP submaximal space, by theorem 3.17, 1-K is a PNPR_1 set in (R,τ) and hence K is a PNPR_2 set in (R,τ).

5. Conclusion

In this paper, it is established that in PNP strongly irresolvable spaces, the condition under which PNP topological spaces become PNP strongly irresolvable spaces is obtained by means of the PNP denseness of PNP open sets. It is proved that PNP first category sets are PNP closed sets in a PNP Baire, PNP nodec and PNP strongly irresolvable spaces. It is established that PNP resolvable and PNP *ir*resolvable spaces are not PNP strongly irresolvable spaces. The conditions under which PNP strongly irresolvable spaces become PNP Baire spaces are also obtained. In future study, we cans study about filters and ultra filters in PNP irresolvable space.

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Hypersoft Topological Spaces

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Abstract. Smarandache [48] introduced the concept of hypersoft set which is a generalization of soft set. This notion is more adaptable than soft set and more suited to challenges involving decision-making. Consequently, topology defined on the collection of hypersoft sets will be in great importance. In this paper, we introduce hypersoft topological spaces which are defined over an initial universe with a fixed set of parameters. The notions of hypersoft open sets, hypersoft closed sets, hypersoft neighborhood, hypersoft limit point, and hypersoft subspace are introduced and their basic properties are investigated. Finally, we introduce the concepts of hypersoft closure, hypersoft interior, hypersoft exterior, and hypersoft boundary and the relationship between them are discussed.

Keywords: hypersoft sets; hypersoft topology; hypersoft open sets; hypersoft closed sets; hypersoft neighborhood; hypersoft limit point; hypersoft closure; hypersoft interior; hypersoft exterior; hypersoft boundary.

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1. **Introduction**

In 1999, Molodtsov [30] developed the concept of a soft set to handle difficult problems in economics, engineering, and the environment, where no mathematical methods could effectively deal with the many types of uncertainty. Maji et al. in [25] developed various operators for soft set theory and conducted a more detailed theoretical analysis of soft set theory. Various operations analogous to union, intersection, complement, difference etc. in set theory have been discussed in the context of soft sets (see [5, 6, 10, 46]).

It is known that Topology is a branch of mathematics that has numerous applications in the physical and computer sciences. Topology is the study of qualitative properties of particular objects, known as topological spaces, that are invariant under specific transformations, known as continuous mappings. Open sets are commonly used to describe these characteristics. By replacing open sets with more general ones, the concept of topological space is frequently generalized. A classic example of this form of generalization is fuzzy topology, proposed by Chang [12] and later improved by Lowen [24]. Topological structures on soft sets, in a similar manner, are more generalized methods that can be used to measure the similarities and differences between the objects in a universe which are soft sets.

There are two versions of soft topology defined on soft sets, one by Shabir [47] and other by Čagman et al. [11]. The main difference between these approaches is that the first investigates a subcollection of all soft sets in an initial universe with a fixed set of parameters, whereas the second considers a subcollection of all soft subsets of a specific soft set in a universe. Based on these two definitions on soft topology, some concepts such as soft interior, soft closure, soft continuity, soft separation axioms etc. were introduced and studied by many authors (see for example [2–4, 7–9, 13–23, 26–29, 33–37, 40, 42–45, 49–54]).

In 2018, Smarandache [48] expanded the notion of a soft set to a hypersoft set by substituting the function with a multi-argument function described in the cartesian product with a different set of parameters. This concept is more adaptable than the soft set and more useful when it comes to making decisions. Recently, Musa and Asaad ([31, 32]) introduced a new idea of hypersoft sets called bipolar hypersoft sets and they investigated some of their bipolar hypersoft topological structures. Researchers have been drawn to hypersoft set structure because it is better suited to decision-making difficulties than soft set structure. Despite the fact that it is a new concept, numerous studies have been conducted, and the field of study continues to grow [1, 38, 39, 41].

Our paper is organized as follows: Section 2 contains some basic definitions related to hypersoft set which are required in our work. In section 3, we introduce hypersoft topological spaces which are defined over an initial universe with a fixed set of parameters and investigate the concepts of hypersoft neighborhood and hypersoft limit points. In section 4, the notions of hypersoft closure, hypersoft interior, hypersoft exterior, and hypersoft boundary are introduced associated with some of their properties. Furthermore, the relationships between all of the preceding concepts are studied, as well as several illustrated examples. The conclusion, on the other hand, is included in Section 5.

2. Hypersoft Sets

Here we recall some basic terminologies and results regarding hypersoft sets. For more details, the reader could refer to [39, 41].

Throughout the paper, let $\mathcal{U}$ be an initial universe, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$, and $E_1, E_2, ..., E_n$ the pairwise of disjoint sets of parameters. Let $A_i, B_i \subseteq E_i$ for $i = 1, 2, ..., n$.

Definition 2.1. [48] A pair $(\mathbb{F}, A_1 \times A_2 \times ... \times A_n)$ is called a hypersoft set over $\mathcal{U}$, where $\mathbb{F}$ is a mapping given by $\mathbb{F} : A_1 \times A_2 \times ... \times A_n \to \mathcal{P}(\mathcal{U})$.

Simply, we write the symbol $\mathcal{E}$ for $E_1 \times E_2 \times ... \times E_n$, and for the subsets of $\mathcal{E}:$ the symbols $\mathcal{A}$ for $A_1 \times A_2 \times ... \times A_n$, and $\mathcal{B}$ for $B_1 \times B_2 \times ... \times B_n$. Clearly, each element in $\mathcal{A}, \mathcal{B}$ and $\mathcal{E}$ is an n-tuple element.

We can represent a hypersoft set $(\mathbb{F}, \mathcal{A})$ as an ordered pair,

$$(\mathbb{F}, \mathcal{A}) = \{(\alpha, \mathbb{F}(\alpha)) : \alpha \in \mathcal{A}\}.$$

Definition 2.2. [39] For two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$, we say that $(\mathbb{F}, \mathcal{A})$ is a hypersoft subset of $(\mathbb{G}, \mathcal{B})$ if

(1) $\mathcal{A} \subseteq \mathcal{B}$, and

(2) $\mathbb{F}(\alpha) \subseteq \mathbb{G}(\alpha)$ for all $\alpha \in \mathcal{A}$.

We write $(\mathbb{F}, \mathcal{A}) \tilde{\subseteq} (\mathbb{G}, \mathcal{B})$.

$(\mathbb{F}, \mathcal{A})$ is said to be a hypersoft superset of $(\mathbb{G}, \mathcal{B})$, if $(\mathbb{G}, \mathcal{B})$ is a hypersoft subset of $(\mathbb{F}, \mathcal{A})$. We denote it by $(\mathbb{F}, \mathcal{A}) \tilde{\supseteq} (\mathbb{G}, \mathcal{B})$.

Definition 2.3. [39] Two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$ are said to be hypersoft equal if $(\mathbb{F}, \mathcal{A})$ is a hypersoft subset of $(\mathbb{G}, \mathcal{B})$ and $(\mathbb{G}, \mathcal{B})$ is a hypersoft subset of $(\mathbb{F}, \mathcal{A})$.

Definition 2.4. [39] Let $\mathcal{A} = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a set of parameters. The NOT set of $\mathcal{A}$ denoted by $\neg\mathcal{A}$ is defined by $\neg\mathcal{A} = \{\neg\alpha_1, \neg\alpha_2, ..., \neg\alpha_n\}$ where $\neg\alpha_i =$ not α_i for $i = 1, 2, ..., n$.

Proposition 2.5. *[31] For any subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{E}$.*

(1) $\neg(\neg\mathcal{A}) = \mathcal{A}$.

(2) $\neg(\mathcal{A} \cup \mathcal{B}) = \neg\mathcal{A} \cup \neg\mathcal{B}$.

(3) $\neg(\mathcal{A} \cap \mathcal{B}) = \neg\mathcal{A} \cap \neg\mathcal{B}$.

Definition 2.6. [39] The complement of a hypersoft set $(\mathbb{F}, \mathcal{A})$ is denoted by $(\mathbb{F}, \mathcal{A})^c$ and is defined by $(\mathbb{F}, \mathcal{A})^c = (\mathbb{F}^c, \mathcal{A})$ where $\mathbb{F}^c : \mathcal{A} \to \mathcal{P}(\mathcal{U})$ is a mapping given by $\mathbb{F}^c(\alpha) = \mathcal{U} \setminus \mathbb{F}(\alpha)$ for all $\alpha \in \mathcal{A}$.

Definition 2.7. [41] A hypersoft set $(\mathbb{F}, \mathcal{A})$ over $\mathcal{U}$ is said to be a relative null hypersoft set, denoted by $(\Phi, \mathcal{A})$, if for all $\alpha \in \mathcal{A}$, $\mathbb{F}(\alpha) = \phi$.

Definition 2.8. [41] A hypersoft set $(\mathbb{F}, \mathcal{A})$ over $\mathcal{U}$ is said to be a relative whole hypersoft set, denoted by $(\Psi, \mathcal{A})$, if for all $\alpha \in \mathcal{A}$, $\mathbb{F}(\alpha) = \mathcal{U}$.

Definition 2.9. [41] Difference of two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$, is a hypersoft set $(\mathbb{H}, C)$, where $C = \mathcal{A} \cap \mathcal{B}$ and for all $\alpha \in C$, $\mathbb{H}(\alpha) = \mathbb{F}(\alpha) \setminus \mathbb{G}(\alpha)$. We write $(\mathbb{F}, \mathcal{A}) \setminus (\mathbb{G}, \mathcal{B}) = (\mathbb{H}, C)$.

Definition 2.10. [41] Union of two hypersoft sets $(\mathbb{F},\mathcal{A})$ and $(\mathbb{G},\mathcal{B})$ over a common universe $\mathcal{U}$, is a hypersoft set $(\mathbb{H},C)$, where $C=\mathcal{A}\cap\mathcal{B}$ and for all $\alpha\in C$, $\mathbb{H}(\alpha)=\mathbb{F}(\alpha)\cup\mathbb{G}(\alpha)$. We write $(\mathbb{F},\mathcal{A})\,\widetilde{\sqcup}\,(\mathbb{G},\mathcal{B})=(\mathbb{H},C)$.

Definition 2.11. [39] Intersection of two hypersoft sets $(\mathbb{F},\mathcal{A})$ and $(\mathbb{G},\mathcal{B})$ over a common universe $\mathcal{U}$, is a hypersoft set $(\mathbb{H},C)$, where $C=\mathcal{A}\cap\mathcal{B}$ and for all $\alpha\in C$, $\mathbb{H}(\alpha)=\mathbb{F}(\alpha)\cap\mathbb{G}(\alpha)$. We write $(\mathbb{F},\mathcal{A})\,\widetilde{\sqcap}\,(\mathbb{G},\mathcal{B})=(\mathbb{H},C)$.

3. **Hypersoft Topological Spaces**

Let $\mathcal{U}$ be an initial universe set and $\mathcal{E}$ be the non-empty set of parameters.

Definition 3.1. Let $(\mathbb{F},\mathcal{E})$ be a hypersoft set over $\mathcal{U}$ and $u\in\mathcal{U}$. Then $u\in(\mathbb{F},\mathcal{E})$ if $u\in\mathbb{F}(\alpha)$ for all $\alpha\in\mathcal{E}$. Note that for any $u\in\mathcal{U}$, $u\notin(\mathbb{F},\mathcal{E})$, if $u\notin\mathbb{F}(\alpha)$ for some $\alpha\in\mathcal{E}$.

Definition 3.2. Let $\mathcal{Y}$ be a non-empty subset of $\mathcal{U}$. Then $(\Upsilon,\mathcal{E})$ denotes the hypersoft set over $\mathcal{U}$ defined by $\Upsilon(\alpha)=\mathcal{Y}$ for all $\alpha\in\mathcal{E}$.

Definition 3.3. Let $(\mathbb{F},\mathcal{E})$ be a hypersoft set over $\mathcal{U}$ and $\mathcal{Y}$ be a non-empty subset of $\mathcal{U}$. Then the sub hypersoft set of $(\mathbb{F},\mathcal{E})$ over $\mathcal{Y}$ denoted by $(\mathbb{F}_{\mathcal{Y}},\mathcal{E})$, is defined as $\mathbb{F}_{\mathcal{Y}}(\alpha)=\mathcal{Y}\cap\mathbb{F}(\alpha)$ for each $\alpha\in\mathcal{E}$.

In other words $(\mathbb{F}_{\mathcal{Y}},\mathcal{E})=(\Upsilon,\mathcal{E})\,\widetilde{\sqcap}\,(\mathbb{F},\mathcal{E})$.

Definition 3.4. Let $\mathcal{T}_{\mathcal{H}}$ be the collection of hypersoft sets over $\mathcal{U}$, then $\mathcal{T}_{\mathcal{H}}$ is said to be a hypersoft topology on $\mathcal{U}$ if

(1) $(\Phi,\mathcal{E})$, $(\Psi,\mathcal{E})$ belong to $\mathcal{T}_{\mathcal{H}}$,
(2) the intersection of any two hypersoft sets in $\mathcal{T}_{\mathcal{H}}$ belongs to $\mathcal{T}_{\mathcal{H}}$,
(3) the union of any number of hypersoft sets in $\mathcal{T}_{\mathcal{H}}$ belongs to $\mathcal{T}_{\mathcal{H}}$.

Then $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ is called a hypersoft topological space over $\mathcal{U}$.

Definition 3.5. Let $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ be a hypersoft space over $\mathcal{U}$, then the members of $\mathcal{T}_{\mathcal{H}}$ are said to be hypersoft open sets in $\mathcal{U}$.

Example 3.6. Let $\mathcal{U}=\{h_1,h_2\}$, $E_1=\{e_1,e_2\}$, $E_2=\{e_3\}$, and $E_3=\{e_4\}$. Let $\mathcal{T}_{\mathcal{H}}=\{(\Phi,\mathcal{E}),(\Psi,\mathcal{E}),(\mathbb{F}_1,\mathcal{E}),(\mathbb{F}_2,\mathcal{E}),(\mathbb{F}_3,\mathcal{E})\}$ where $(\mathbb{F}_1,\mathcal{E})$, $(\mathbb{F}_2,\mathcal{E})$, and $(\mathbb{F}_3,\mathcal{E})$ are hypersoft sets over $\mathcal{U}$, defined as follows

$$(\mathbb{F}_1,\mathcal{E})=\{((e_1,e_3,e_4),\{h_1\}),((e_2,e_3,e_4),\{h_2\})\}.$$
$$(\mathbb{F}_2,\mathcal{E})=\{((e_1,e_3,e_4),\{h_1\}),((e_2,e_3,e_4),\mathcal{U}\}.$$
$$(\mathbb{F}_3,\mathcal{E})=\{((e_1,e_3,e_4),\mathcal{U}),((e_2,e_3,e_4),\{h_2\})\}.$$

Then the collection $\mathcal{T}_{\mathcal{H}}$ forms a hypersoft topology on $\mathcal{U}$.

Definition 3.7. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. A hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is said to be a hypersoft closed set in $\mathcal{U}$, if its complement $(\mathbb{F}, \mathcal{E})^c$ belongs to $\mathcal{T}_{\mathcal{H}}$.

Proposition 3.8. *Let* $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ *be a hypersoft space over* $\mathcal{U}$*. Then*

(1) $(\Phi, \mathcal{E})$, $(\Psi, \mathcal{E})$ *are hypersoft closed set over* $\mathcal{U}$*,*

(2) the union of any two hypersoft closed sets is a hypersoft closed set over $\mathcal{U}$*,*

(3) the intersection of any number of hypersoft closed sets is a hypersoft closed set over $\mathcal{U}$*.*

Proof. Follows from the definition of hypersoft topological spaces and De Morgan's laws.

Definition 3.9. Let $\mathcal{U}$ be an initial universe, $\mathcal{E}$ be the set of parameters, and $\mathcal{T}_{\mathcal{H}} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E})\}$. Then $\mathcal{T}_{\mathcal{H}}$ is called the hypersoft indiscrete topology on $\mathcal{U}$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is said to be a hypersoft indiscrete space over $\mathcal{U}$.

Definition 3.10. Let $\mathcal{U}$ be an initial universe, $\mathcal{E}$ be the set of parameters, and $\mathcal{T}_{\mathcal{H}}$ be the collection of all hypersoft sets which can be defined over $\mathcal{U}$. Then $\mathcal{T}_{\mathcal{H}}$is called the hypersoft discrete topology on $\mathcal{U}$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is said to be a hypersoft discrete space over $\mathcal{U}$.

Definition 3.11. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ be two hypersoft topological spaces over $\mathcal{U}$. If $\mathcal{T}_{\mathcal{H}_1} \tilde{\sqsubseteq} \mathcal{T}_{\mathcal{H}_2}$, then $\mathcal{T}_{\mathcal{H}_2}$ is said to be finer than $\mathcal{T}_{\mathcal{H}_1}$. If $\mathcal{T}_{\mathcal{H}_1} \tilde{\sqsubseteq} \mathcal{T}_{\mathcal{H}_2}$ or $\mathcal{T}_{\mathcal{H}_2} \tilde{\sqsubseteq} \mathcal{T}_{\mathcal{H}_1}$, then $\mathcal{T}_{\mathcal{H}_1}$ and $\mathcal{T}_{\mathcal{H}_2}$ are said to be comparable hypersoft topologies over $\mathcal{U}$.

Proposition 3.12. *Let* $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ *and* $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ *be two hypersoft topological spaces on* $\mathcal{U}$*, then* $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ *is a hypersoft topological space over* $\mathcal{U}$*.*

Proof.

i. $(\Phi, \mathcal{E})$, $(\Psi, \mathcal{E})$ belong to $\mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}$.

ii. Let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}$. Then $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_1}$ and $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_2}$. Since $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_1}$ and $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_2}$, so $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}$.

iii. Let $\{(\mathbb{F}_i, \mathcal{E}) \mid i \in I)\}$ be a family of hypersoft sets in $\mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}$. Then $(\mathbb{F}_i, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_1}$ and $(\mathbb{F}_i, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_2}$, for all $i \in I$, so $\tilde{\sqcup}_{i \in I} (\mathbb{F}_i, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_1}$ and $\tilde{\sqcup}_{i \in I} (\mathbb{F}_i, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_2}$. Therefore, $\tilde{\sqcup}_{i \in I}(\mathbb{F}_i, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}$.

Thus $\mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}$ defines a hypersoft topology on $\mathcal{U}$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is a hypersoft topological space over $\mathcal{U}$.

Remark 3.13. The union of two hypersoft topologies on $\mathcal{U}$ may not be a hypersoft topology on $\mathcal{U}$.

Example 3.14. Let $\mathcal{U} = \{h_1, h_2, h_3, h_4\}$, $E_1 = \{e_1, e_2\}$, $E_2 = \{e_3\}$, and $E_3 = \{e_4\}$. Let $\mathcal{T}_{\mathcal{H}_1} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}_1, \mathcal{E}), (\mathbb{F}_2, \mathcal{E}), (\mathbb{F}_3, \mathcal{E})\}$ and $\mathcal{T}_{\mathcal{H}_2} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{G}_1, \mathcal{E}), (\mathbb{G}_2, \mathcal{E}),$

$(\mathbb{G}_3, \mathcal{E})\}$ be two hypersoft topologies defined on $\mathcal{U}$ where $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$, $(\mathbb{F}_3, \mathcal{E})$, $(\mathbb{G}_1, \mathcal{E})$, $(\mathbb{G}_2, \mathcal{E})$, and $(\mathbb{G}_3, \mathcal{E})$ are hypersoft sets over $\mathcal{U}$, defined as follows

$(\mathbb{F}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_3, h_4\}), ((e_2, e_3, e_4), \{h_2, h_3\})\}$.
$(\mathbb{F}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_1, h_2, h_3\}), ((e_2, e_3, e_4), \{h_1, h_4\})\}$.
$(\mathbb{F}_3, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_3\}), ((e_2, e_3, e_4), \phi)\}$.

and

$(\mathbb{G}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_3, h_4\}), ((e_2, e_3, e_4), \{h_1, h_3, h_4\})\}$.
$(\mathbb{G}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_1, h_2\}), ((e_2, e_3, e_4), \{h_2, h_4\})\}$.
$(\mathbb{G}_3, \mathcal{E}) = \{((e_1, e_3, e_4), \phi), ((e_2, e_3, e_4), \{h_4\}\}$.

Then $\mathcal{T}_{\mathcal{H}_1} \widetilde{\sqcup} \mathcal{T}_{\mathcal{H}_2} = \{(\Phi, \mathcal{E})$, $(\Psi, \mathcal{E})$, $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$, $(\mathbb{F}_3, \mathcal{E})$,$(\mathbb{G}_1, \mathcal{E})$, $(\mathbb{G}_2, \mathcal{E})$, $(\mathbb{G}_3, \mathcal{E})\}$.

If we take

$(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{G}_1, \mathcal{E}) = (\mathbb{H}, \mathcal{E})$,

then

$(\mathbb{H}, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_3, h_4\}), ((e_2, e_3, e_4), \mathcal{U})\}$,

but $(\mathbb{H}, \mathcal{E}) \widetilde{\notin} \mathcal{T}_{\mathcal{H}_1} \widetilde{\sqcup} \mathcal{T}_{\mathcal{H}_2}$. Hence, $\mathcal{T}_{\mathcal{H}_1} \widetilde{\sqcup} \mathcal{T}_{\mathcal{H}_2}$ is not a hypersoft topology on $\mathcal{U}$.

Definition 3.15. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$ and $u \in \mathcal{U}$. Then $(\mathbb{F}, \mathcal{E})$ is said to be a hypersoft neighborhood of u if there exists a hypersoft open set $(\mathbb{G}, \mathcal{E})$ such that $u \in (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})$.

Proposition 3.16. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, then*

(1) If $(\mathbb{F}, \mathcal{E})$ is a hypersoft neighborhood of $u \in \mathcal{U}$, then $u \in (\mathbb{F}, \mathcal{E})$.
(2) Each $u \in \mathcal{U}$ has a hypersoft neighborhood.
(3) If $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})$ are hypersoft neighborhoods of some $u \in \mathcal{U}$, then $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{G}, \mathcal{E})$ is also a hypersoft neighborhood of u.
(4) If $(\mathbb{F}, \mathcal{E})$ is a hypersoft neighborhood of $u \in \mathcal{U}$ and $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{G}, \mathcal{E})$, then $(\mathbb{G}, \mathcal{E})$ is also a hypersoft neighborhood of $u \in \mathcal{U}$.

Proof.

(1) Follows from Definition 3.15.
(2) For any $u \in \mathcal{U}$, $u \in (\Psi, \mathcal{E})$ and since $(\Psi, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}$, so $u \in (\Psi, \mathcal{E}) \widetilde{\sqsubseteq} (\Psi, \mathcal{E})$. Thus $(\Psi, \mathcal{E})$ is a hypersoft neighborhood of u.
(3) Let $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})$ be the hypersoft neighborhoods of $u \in \mathcal{U}$, then there exist $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}$ such that $u \in (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})$ and $u \in (\mathbb{F}_2, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{G}, \mathcal{E})$. Now $u \in (\mathbb{F}_1, \mathcal{E})$ and $u \in (\mathbb{F}_2, \mathcal{E})$ implies that $u \in (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})$ and $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}$. So we have $u \in (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{G}, \mathcal{E})$. Thus, $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{G}, \mathcal{E})$ is a hypersoft neighborhood of u.

(4) Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft neighborhood of $u \in \mathcal{U}$ and $(\mathbb{F}, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{G}, \mathcal{E})$. By definition, there exists a hypersoft open set $(\mathbb{F}_1, \mathcal{E})$ such that $u \in (\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{G}, \mathcal{E})$. Thus, $u \in (\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{G}, \mathcal{E})$. Hence, $(\mathbb{G}, \mathcal{E})$ is a hypersoft neighborhood of u.

Proposition 3.17. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. For any hypersoft open set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$, $(\mathbb{F}, \mathcal{E})$ is a hypersoft neighborhood of each point of $\cap_{\alpha \in \mathcal{E}} \mathbb{F}(\alpha)$, that is, of each of its points.*

Proof. Let $(\mathbb{F}, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}}$. For any $u \in \cap_{\alpha \in \mathcal{E}} \mathbb{F}(\alpha)$, we have $u \in \mathbb{F}(\alpha)$ for each $\alpha \in \mathcal{E}$. Thus $u \in (\mathbb{F}, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})$ and so $(\mathbb{F}, \mathcal{E})$ is a hypersoft neighborhood of u.

Remark 3.18. The following example shows that the converse of Proposition 3.17 is not true in general.

Example 3.19. Consider $\mathcal{T}_{\mathcal{H}_1}$ given in Example 3.14 and let $(\mathbb{F}, \mathcal{E})$ be any hypersoft set defined as follows:

$(\mathbb{F}, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_1, h_3, h_4\}), ((e_2, e_3, e_4), \{h_2, h_3\})\}$.

Then $(\mathbb{F}, \mathcal{E})$ is a hypersoft neighborhood of each point of $\cap_{\alpha \in \mathcal{E}} \mathbb{F}(\alpha)$, that is, of each of its points, but it is not a hypersoft open set.

Definition 3.20. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. A point $u \in \mathcal{U}$ is called a hypersoft limit point of $(\mathbb{F}, \mathcal{E})$ if $(\mathbb{F}, \mathcal{E}) \tilde{\sqcap} ((\mathbb{G}, \mathcal{E}) \setminus \{u\})$ $\neq (\Phi, \mathcal{E})$ for every hypersoft open set $(\mathbb{G}, \mathcal{E})$ containing u. The set of all hypersoft limit points of $(\mathbb{F}, \mathcal{E})$ is denoted by $(\mathbb{F}, \mathcal{E})^d$.

Proposition 3.21. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$ *implies* $(\mathbb{F}_1, \mathcal{E})^d \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^d$.
(2) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^d \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^d \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^d$.
(3) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^d = (\mathbb{F}_1, \mathcal{E})^d \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^d$.

Proof.

(1) Let $u \in (\mathbb{F}_1, \mathcal{E})^d$ so that u is a hypersoft limit point of $(\mathbb{F}_1, \mathcal{E})$. Then, by definition $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} ((\mathbb{G}, \mathcal{E}) \setminus \{u\}) \neq (\Phi, \mathcal{E})$ for every hypersoft open set $(\mathbb{G}, \mathcal{E})$ containing u. But $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$, it follows that $(\mathbb{F}_2, \mathcal{E}) \tilde{\sqcap} ((\mathbb{G}, \mathcal{E}) \setminus \{u\}) \neq (\Phi, \mathcal{E})$. Thus, $u \in (\mathbb{F}_2, \mathcal{E})^d$. Therefore, $(\mathbb{F}_1, \mathcal{E})^d \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^d$.
(2) Since $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$. It follows from (1) that, $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^d \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^d$ and $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^d \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^d$. Hence, $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^d \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^d \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^d$.

(3) Since $(\mathbb{F}_1,\mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1,\mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E})$ and $(\mathbb{F}_2,\mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1,\mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E})$. By (1) we have $(\mathbb{F}_1,\mathcal{E})^d \tilde{\sqsubseteq} ((\mathbb{F}_1,\mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d$ and $(\mathbb{F}_2,\mathcal{E})^d \tilde{\sqsubseteq} ((\mathbb{F}_1,\mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d$. So, $(\mathbb{F}_1,\mathcal{E})^d \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E})^d \tilde{\sqsubseteq} ((\mathbb{F}_1,\mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d$. Now, let $u \in ((\mathbb{F}_1,\mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d$. Then, $((\mathbb{F}_1,\mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E})) \tilde{\sqcap} ((\mathbb{G},\mathcal{E}) \setminus \{u\}) \neq (\Phi,\mathcal{E})$ for every hypersoft open set $(\mathbb{G},\mathcal{E})$ containing u. Therefore, $(\mathbb{F}_1,\mathcal{E}) \tilde{\sqcap} ((\mathbb{G},\mathcal{E}) \setminus \{u\}) \neq (\Phi,\mathcal{E})$ or $(\mathbb{F}_2,\mathcal{E}) \tilde{\sqcap} ((\mathbb{G},\mathcal{E}) \setminus \{u\}) \neq (\Phi,\mathcal{E})$. Thus, $u \in (\mathbb{F}_1,\mathcal{E})^d$ or $u \in (\mathbb{F}_2,\mathcal{E})^d$ and then $u \in (\mathbb{F}_1,\mathcal{E})^d \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E})^d$. Therefore, $((\mathbb{F}_1,\mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d \tilde{\sqsubseteq} (\mathbb{F}_1,\mathcal{E})^d \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E})^d$. Now, we have $((\mathbb{F}_1,\mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d = (\mathbb{F}_1,\mathcal{E})^d \tilde{\sqcup} (\mathbb{F}_2,\mathcal{E})^d$.

Remark 3.22. The following example shows that the equality in Proposition 3.21 (2) does not hold in general.

Example 3.23. Let us consider the hypersoft topological space $(\mathcal{U},\mathcal{T}_{\mathcal{H}_1},\mathcal{E})$ in Example 3.14 and let $(\mathbb{F},\mathcal{E})$ and $(\mathbb{G},\mathcal{E})$ are hypersoft sets defined as follows:

$(\mathbb{F},\mathcal{E}) = \{((e_1,e_3,e_4),\phi),((e_2,e_3,e_4),\{h_4\})\}$.
$(\mathbb{G},\mathcal{E}) = \{((e_1,e_3,e_4),\{h_2\}),((e_2,e_3,e_4),\{h_3\}$.

Then $(\mathbb{F},\mathcal{E})^d \tilde{\sqcap} (\mathbb{G},\mathcal{E})^d = \{h_1\}$. But, $(\mathbb{F},\mathcal{E}) \tilde{\sqcap} (\mathbb{G},\mathcal{E}) = (\Phi,\mathcal{E})$ and $((\mathbb{F},\mathcal{E}) \tilde{\sqcap} (\mathbb{G},\mathcal{E}))^d = (\Phi,\mathcal{E})^d = \phi$. Hence, $((\mathbb{F}_1,\mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2,\mathcal{E}))^d \neq (\mathbb{F}_1,\mathcal{E})^d \tilde{\sqcap} (\mathbb{F}_2,\mathcal{E})^d$.

Definition 3.24. Let $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and $\mathcal{Y}$ be a non-empty subset of $\mathcal{U}$. Then

$$\mathcal{T}_{\mathcal{H}_{\mathcal{Y}}} = \{(\mathbb{F}_{\mathcal{Y}},\mathcal{E}) \mid (\mathbb{F},\mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}}\}$$

is said to be the relative hypersoft topology on $\mathcal{Y}$ and $(\mathcal{Y},\mathcal{T}_{\mathcal{H}_{\mathcal{Y}}},\mathcal{E})$ is called a hypersoft subspace of $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$

It is easy to verify that $T_{\mathcal{H}_{\mathcal{Y}}}$ is a hypersoft topology on $\mathcal{Y}$.

Example 3.25. Any hypersoft subspace of a hypersoft indiscrete topological space is a hypersoft indiscrete topological space.

Example 3.26. Any hypersoft subspace of a hypersoft discrete topological space is a hypersoft discrete topological space.

Proposition 3.27. *Let $(\mathcal{Y},\mathcal{T}_{\mathcal{H}_{\mathcal{Y}}},\mathcal{E})$ be a hypersoft subspace of a hypersoft topological space $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ and $(\mathbb{F}_{\mathcal{Y}},\mathcal{E})$ be a hypersoft open set in $\mathcal{Y}$. If $(\Upsilon,\mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}}$ then $(\mathbb{F}_{\mathcal{Y}},\mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}}$.*

Proof. Let $(\mathbb{F}_{\mathcal{Y}},\mathcal{E})$ be a hypersoft open set in $\mathcal{Y}$, then there exists a hypersoft open set $(\mathbb{F},\mathcal{E})$ in $\mathcal{U}$ such that $(\mathbb{F}_{\mathcal{Y}},\mathcal{E}) = (\Upsilon,\mathcal{E}) \tilde{\sqcap} (\mathbb{F},\mathcal{E})$. Now, if $(\Upsilon,\mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}}$ then $(\Upsilon,\mathcal{E}) \tilde{\sqcap} (\mathbb{F},\mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}}$. Hence, $(\mathbb{F}_{\mathcal{Y}},\mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}}$.

Proposition 3.28. *Let $(\mathcal{Y}, \mathcal{T}_{\mathcal{H}_\mathcal{Y}}, \mathcal{E})$ and $(\mathcal{Z}, \mathcal{T}_{\mathcal{H}_\mathcal{Z}}, \mathcal{E})$ be two hypersoft subspaces of $(\mathcal{U}, \mathcal{T}_\mathcal{H}, \mathcal{E})$ and let $\mathcal{Y} \tilde{\sqsubseteq} \mathcal{Z}$. Then $(\mathcal{Y}, \mathcal{T}_{\mathcal{H}_\mathcal{Y}}, \mathcal{E})$ is a hypersoft subspace of $(\mathcal{Z}, \mathcal{T}_{\mathcal{H}_\mathcal{Z}}, \mathcal{E})$.*

Proof. Let $(\mathbb{F}_\mathcal{Y}, \mathcal{E})$ be a hypersoft open set in $\mathcal{Y}$, then there exists a hypersoft open set $(\mathbb{F}, \mathcal{E})$ in $\mathcal{U}$ such that $(\mathbb{F}_\mathcal{Y}, \mathcal{E}) = (\Upsilon, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}, \mathcal{E})$, or equivalently, for each $\alpha \in \mathcal{E}$, $\mathbb{F}_\mathcal{Y}(\alpha) = \mathcal{Y} \cap \mathbb{F}(\alpha)$. Since $\mathcal{Y} \tilde{\sqsubseteq} \mathcal{Z}$ then $\mathcal{Y} = \mathcal{Y} \tilde{\sqcap} \mathcal{Z}$. Now, $\mathbb{F}_\mathcal{Y}(\alpha) = \mathcal{Y} \cap \mathbb{F}(\alpha) = (\mathcal{Y} \tilde{\sqcap} \mathcal{Z})$ $\cap$ $\mathbb{F}(\alpha) = \mathcal{Y} \tilde{\sqcap} (\mathcal{Z} \cap \mathbb{F}(\alpha)) = \mathcal{Y} \cap \mathbb{F}_\mathcal{Z}(\alpha)$ so we have $\mathbb{F}_\mathcal{Y}(\alpha) = \mathcal{Y} \cap \mathbb{F}_\mathcal{Z}(\alpha)$, or equivalently, $(\mathbb{F}_\mathcal{Y}, \mathcal{E}) = (\Upsilon, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_\mathcal{Z}, \mathcal{E})$ where $(\mathbb{F}_\mathcal{Z}, \mathcal{E})$ is a hypersoft open set in $\mathcal{Z}$. Hence, $(\mathcal{Y}, \mathcal{T}_{\mathcal{H}_\mathcal{Y}}, \mathcal{E})$ is a hypersoft subspace of $(\mathcal{Z}, \mathcal{T}_{\mathcal{H}_\mathcal{Z}}, \mathcal{E})$.

4. Hypersoft Closure, Hypersoft Interior, Hypersoft Exterior, and Hypersoft Boundary

Definition 4.1. Let $(\mathcal{U}, \mathcal{T}_\mathcal{H}, \mathcal{E})$ be a hypersoft space and $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. The intersection of all hypersoft closed supersets of $(\mathbb{F}, \mathcal{E})$ is called the hypersoft closure of $(\mathbb{F}, \mathcal{E})$ and is denoted by $\overline{(\mathbb{F}, \mathcal{E})}$.

In other words, $\overline{(\mathbb{F}, \mathcal{E})} = \tilde{\sqcap} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E})^c \tilde{\in} \mathcal{T}_\mathcal{H}, (\mathbb{G}, \mathcal{E}) \tilde{\sqsupseteq} (\mathbb{F}, \mathcal{E})\}$.

Proposition 4.2. *Let $(\mathcal{U}, \mathcal{T}_\mathcal{H}, \mathcal{E})$ be a hypersoft space and $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then*

(1) $\overline{(\mathbb{F}, \mathcal{E})}$ is the smallest hypersoft closed set containing $(\mathbb{F}, \mathcal{E})$.
(2) $(\mathbb{F}, \mathcal{E})$ is a hypersoft closed set if and only if $(\mathbb{F}, \mathcal{E}) = \overline{(\mathbb{F}, \mathcal{E})}$.

Proof.

(1) Follows from Definition 4.1.
(2) Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft closed set. So, $(\mathbb{F}, \mathcal{E})$ itself is the smallest hypersoft closed set over $\mathcal{U}$ containing $(\mathbb{F}, \mathcal{E})$ and hence $(\mathbb{F}, \mathcal{E}) = \overline{(\mathbb{F}, \mathcal{E})}$. Conversely, suppose that $(\mathbb{F}, \mathcal{E}) = \overline{(\mathbb{F}, \mathcal{E})}$. By (1.) $\overline{(\mathbb{F}, \mathcal{E})}$ is a hypersoft closed, so $(\mathbb{F}, \mathcal{E})$ is also a hypersoft closed set over $\mathcal{U}$

Proposition 4.3. *Let $(\mathcal{U}, \mathcal{T}_\mathcal{H}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $\overline{(\Phi, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\Psi, \mathcal{E})} = (\Psi, \mathcal{E})$.
(2) $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})}$.
(3) $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$ implies $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$.
(4) $\overline{((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))} = \overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$.
(5) $\overline{((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))} \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})}$.
(6) $\overline{\overline{(\mathbb{F}_1, \mathcal{E})}} = \overline{(\mathbb{F}_1, \mathcal{E})}$.

Proof.

(1) Since $(\Phi, \mathcal{E})$ and $(\Psi, \mathcal{E})$ are hypersoft closed sets, then by Proposition 4.2 (2), we have $\overline{(\Phi, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\Psi, \mathcal{E})} = (\Psi, \mathcal{E})$.

(2) By Proposition 4.2 (1), $\overline{(\mathbb{F}_1, \mathcal{E})}$ is the smallest hypersoft closed set containing $(\mathbb{F}_1, \mathcal{E})$ and so $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})}$.

(3) By (2.), $(\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$. Since $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$, we have $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$. But $\overline{(\mathbb{F}_2, \mathcal{E})}$ is a hypersoft closed set. Thus, $\overline{(\mathbb{F}_2, \mathcal{E})}$ is a hypersoft closed set containing $(\mathbb{F}_1, \mathcal{E})$. Since $\overline{(\mathbb{F}_1, \mathcal{E})}$ is the smallest hypersoft closed set over $\mathcal{U}$ containing $(\mathbb{F}_1, \mathcal{E})$, so we have $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$.

(4) Since $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$. By (3.), we have $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqsubseteq} \overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))}$ and $\overline{(\mathbb{F}_2, \mathcal{E})} \tilde{\sqsubseteq} \overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))}$. Hence, $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})} \tilde{\sqsubseteq} \overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))}$. Now, since $\overline{(\mathbb{F}_1, \mathcal{E})}$ and $\overline{(\mathbb{F}_2, \mathcal{E})}$ are hypersoft closed sets, $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$ is also hypersoft closed. Also, $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})}$ and $(\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$ implies that $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$. Thus, $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$ is a hypersoft closed containing $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$. Since $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))}$ is the smallest hypersoft closed set containing $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$, we have $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))} \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$. Hence, $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))} = \overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$.

(5) Since $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$, then $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcap}(\mathbb{F}_2, \mathcal{E}))} \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})}$ and $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcap}(\mathbb{F}_2, \mathcal{E}))} \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$. Therefore, $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcap}(\mathbb{F}_2, \mathcal{E}))} \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})}$.

(6) Since $\overline{(\mathbb{F}_1, \mathcal{E})}$ is a hypersoft closed set, therefore by Proposition 4.2 (2), we have $\overline{\overline{(\mathbb{F}_1, \mathcal{E})}} = \overline{(\mathbb{F}_1, \mathcal{E})}$.

Remark 4.4. The following example shows that the equality does not hold in Proposition 4.3 (5).

Example 4.5. Let us consider the hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ in Example 3.14 and let $(\mathbb{F}, \mathcal{E})$, $(\mathbb{G}, \mathcal{E})$ in Example 3.23. Then

$\overline{(\mathbb{F}, \mathcal{E})} = (\mathbb{F}_1, \mathcal{E})^c$ and $\overline{(\mathbb{G}, \mathcal{E})} = (\mathbb{F}_3, \mathcal{E})^c$ and $\overline{(\mathbb{F}, \mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{G}, \mathcal{E})} = (\mathbb{F}_1, \mathcal{E})^c$. Now, $(\mathbb{F}, \mathcal{E}) \tilde{\sqcap} (\mathbb{G}, \mathcal{E}) = (\Phi, \mathcal{E})$ and $\overline{((\mathbb{F}, \mathcal{E})\tilde{\sqcap}(\mathbb{G}, \mathcal{E}))} = \overline{(\Phi, \mathcal{E})} = (\Phi, \mathcal{E})$. Hence, $\overline{((\mathbb{F}, \mathcal{E})\tilde{\sqcap}(\mathbb{G}, \mathcal{E}))} \neq \overline{(\mathbb{F}, \mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{G}, \mathcal{E})}$.

Definition 4.6. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$ and $u \in \mathcal{U}$. Then u is said to be a hypersoft interior point of $(\mathbb{F}, \mathcal{E})$ if there exists a hypersoft open set $(\mathbb{G}, \mathcal{E})$ such that $u \in (\mathbb{G}, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})$.

Definition 4.7. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. Then hypersoft interior of hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is denoted by $(\mathbb{F}, \mathcal{E})^o$ and is defined as the union of all hypersoft open set contained in $(\mathbb{F}, \mathcal{E})$.

In other words, $(\mathbb{F}, \mathcal{E})^o = \widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})\}$.

Proposition 4.8. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then*

(1) $(\mathbb{F}, \mathcal{E})^o$ is the largest hypersoft open set contained in $(\mathbb{F}, \mathcal{E})$.

(2) $(\mathbb{F}, \mathcal{E})$ is a hypersoft open set if and only if $(\mathbb{F}, \mathcal{E}) = (\mathbb{F}, \mathcal{E})^o$.

Proof.

(1) Follows from Definition 4.7.

(2) Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft open set. Then $(\mathbb{F}, \mathcal{E})$ is surely identical with the largest hypersoft open subset of $(\mathbb{F}, \mathcal{E})$. But by (1.), $(\mathbb{F}, \mathcal{E})^o$ is the largest hypersoft open subset of $(\mathbb{F}, \mathcal{E})$. Hence, $(\mathbb{F}, \mathcal{E}) = (\mathbb{F}, \mathcal{E})^o$. Conversely, let $(\mathbb{F}, \mathcal{E}) = (\mathbb{F}, \mathcal{E})^o$. By (1.), $(\mathbb{F}, \mathcal{E})^o$ is a hypersoft open set and therefore $(\mathbb{F}, \mathcal{E})$ is also hypersoft open set.

Proposition 4.9. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $(\Phi, \mathcal{E})^o = (\Phi, \mathcal{E})$ and $(\Psi, \mathcal{E})^o = (\Psi, \mathcal{E})$.

(2) $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$.

(3) $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$ implies $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^o$.

(4) $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^o = ((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o$.

(5) $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^o \widetilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^o$.

(6) $((\mathbb{F}_1, \mathcal{E})^o)^o = (\mathbb{F}_1, \mathcal{E})^o$.

Proof.

(1) Since $(\Phi, \mathcal{E})$ and $(\Psi, \mathcal{E})$ are hypersoft open sets, then by Proposition 4.8 (2), we have $(\Phi, \mathcal{E})^o = (\Phi, \mathcal{E})$ and $(\Psi, \mathcal{E})^o = (\Psi, \mathcal{E})$.

(2) Let $u \in (\mathbb{F}_1, \mathcal{E})^o$ then u is a hypersoft interior point of $(\mathbb{F}_1, \mathcal{E})$. This implies that $(\mathbb{F}_1, \mathcal{E})$ is a hypersoft neighborhood of u. Then $u \in (\mathbb{F}_1, \mathcal{E})$. Hence, $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$.

(3) Let $u \in (\mathbb{F}_1, \mathcal{E})^o$. Then u is a hypersoft interior point of $(\mathbb{F}_1, \mathcal{E})$ and so $(\mathbb{F}_1, \mathcal{E})$ is a hypersoft neighborhood of u. Since $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ is also is a hypersoft neighborhood of u. This implies that $u \in (\mathbb{F}_2, \mathcal{E})^o$. Thus, $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^o$.

(4) Since $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$, we have by (3.), $((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^o$ and $((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^o$. This implies that $((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^o$. Again let $u \in (\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^o$. Then $u \in (\mathbb{F}_1, \mathcal{E})^o$ and $u \in (\mathbb{F}_2, \mathcal{E})^o$. Hence u is a hypersoft interior point of each of the

hypersoft sets $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$. It follows that $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are hypersoft neighborhoods of u so that their intersection $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})$ is also is a hypersoft neighborhood of u. Hence, $u \in ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o$. Thus, $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o$. Therefore, $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^o = ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o$.

(5) By (3.), $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$ implies $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^o$ and $(\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$ implies $(\mathbb{F}_2, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^o$. Hence, $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^o$.

(6) By Proposition 4.8 (1), $(\mathbb{F}_1, \mathcal{E})^o$ is the hypersoft open set. Hence by (2.) of the same proposition $((\mathbb{F}_1, \mathcal{E})^o)^o = (\mathbb{F}_1, \mathcal{E})^o$.

Remark 4.10. The following example shows that the equality does not hold in Proposition 4.9 (5).

Example 4.11. Let us consider the hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ in Example 3.14 and let $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})$ are hypersoft sets defined as follows:

$$(\mathbb{F}, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_1, h_3, h_4\}), ((e_2, e_3, e_4), \{h_2, h_3\})\}.$$

$$(\mathbb{G}, \mathcal{E}) = \{((e_1, e_3, e_4), \mathcal{U}), ((e_1, e_3, e_4), \{h_1, h_4\})\}.$$

Then $(\mathbb{F}, \mathcal{E})^o = (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})^o = (\mathbb{F}_2, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E})^o \tilde{\sqcup} (\mathbb{G}, \mathcal{E})^o = (\mathbb{F}_3, \mathcal{E})$. Now, $(\mathbb{F}, \mathcal{E}) \tilde{\sqcup} (\mathbb{G}, \mathcal{E}) = (\Psi, \mathcal{E})$ and $((\mathbb{F}, \mathcal{E}) \tilde{\sqcup} (\mathbb{G}, \mathcal{E}))^o = (\Psi, \mathcal{E})^o = (\Psi, \mathcal{E})$. Hence, $((\mathbb{F}, \mathcal{E}) \tilde{\sqcup} (\mathbb{G}, \mathcal{E}))^o \neq (\mathbb{F}, \mathcal{E})^o \tilde{\sqcup} (\mathbb{G}, \mathcal{E})^o$.

Proposition 4.12. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then $(\mathbb{F}, \mathcal{E})^o \tilde{\sqsubseteq} (\mathbb{F}, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}, \mathcal{E})}$.*

Proof. Follows from Proposition 4.3 (2) and Proposition 4.9 (2).

Proposition 4.13. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $(\overline{(\mathbb{F}_1, \mathcal{E})})^c = ((\mathbb{F}_1, \mathcal{E})^c)^o$.
(2) $((\mathbb{F}_1, \mathcal{E})^o)^c = \overline{((\mathbb{F}_1, \mathcal{E})^c)}$.
(3) $\overline{(\mathbb{F}_1, \mathcal{E})} = (((\mathbb{F}_1, \mathcal{E})^c)^o)^c$.
(4) $(\mathbb{F}_1, \mathcal{E})^o = (\overline{((\mathbb{F}_1, \mathcal{E})^c)})^c$.
(5) $((\mathbb{F}_1, \mathcal{E}) \setminus (\mathbb{F}_2, \mathcal{E}))^o \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^o \setminus (\mathbb{F}_2, \mathcal{E})^o$.

Proof. From the definitions of hypersoft closure and hypersoft interior, we have

(1) $\overline{(\mathbb{F}_1, \mathcal{E})} = \widetilde{\sqcap} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E})^c \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsupseteq} (\mathbb{F}_1, \mathcal{E})\}$.
$(\overline{(\mathbb{F}_1, \mathcal{E})})^c = [\widetilde{\sqcap} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E})^c \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsupseteq} (\mathbb{F}_1, \mathcal{E})\}]^c$.
$(\overline{(\mathbb{F}_1, \mathcal{E})})^c = \widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E})^c \mid (\mathbb{G}, \mathcal{E})^c \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E})^c \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c\} = ((\mathbb{F}_1, \mathcal{E})^c)^o$.
(2) $(\mathbb{F}_1, \mathcal{E})^o = \widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})\}$.
$((\mathbb{F}_1, \mathcal{E})^o)^c = [\widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})\}]^c$.
$((\mathbb{F}_1, \mathcal{E})^o)^c = \widetilde{\sqcap} \{(\mathbb{G}, \mathcal{E})^c \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{F}_1, \mathcal{E})^c \widetilde{\sqsubseteq} (\mathbb{G}, \mathcal{E})^c\} = \overline{((\mathbb{F}_1, \mathcal{E})^c)}$.
(3) Obtained from (1.) by taking the hypersoft complement.
(4) Obtained from (2.) by taking the hypersoft complement.
(5) $((\mathbb{F}_1, \mathcal{E}) \setminus (\mathbb{F}_2, \mathcal{E}))^o = ((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^c)^o = (\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcap} ((\mathbb{F}_2, \mathcal{E})^c)^o = (\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcap}$ $(\overline{(\mathbb{F}_2, \mathcal{E})})^c = (\mathbb{F}_1, \mathcal{E})^o \setminus \overline{(\mathbb{F}_2, \mathcal{E})} \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^o \setminus (\mathbb{F}_2, \mathcal{E})^o$.

Definition 4.14. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. A point $u \in \mathcal{U}$ is said to be a hypersoft exterior point of $(\mathbb{F}, \mathcal{E})$ if and only if it is a hypersoft interior point of $(\mathbb{F}, \mathcal{E})^c$, that is, if and only if there exists a hypersoft open set $(\mathbb{G}, \mathcal{E})$ such that $u \in (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})^c$. The set of all hypersoft exterior points of $(\mathbb{F}, \mathcal{E})$ is called the hypersoft exterior of $(\mathbb{F}, \mathcal{E})$ and is denoted by $(\mathbb{F}, \mathcal{E})^e$.

Thus $(\mathbb{F}, \mathcal{E})^e = ((\mathbb{F}, \mathcal{E})^c)^o$. It follows that $((\mathbb{F}, \mathcal{E})^c)^e = (((\mathbb{F}, \mathcal{E})^c)^c)^o = (\mathbb{F}, \mathcal{E})^o$.

We also have $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}, \mathcal{E})^e = (\Phi, \mathcal{E})$, that is, no point of $(\mathbb{F}, \mathcal{E})$ can be a hypersoft exterior point of $(\mathbb{F}, \mathcal{E})$.

Example 4.15. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be the same as in Example 3.6. Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set defined as follows:

$$(\mathbb{F}, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_2\}), ((e_2, e_3, e_4), \{h_1\})\}.$$

Then $(\mathbb{F}, \mathcal{E})^e = \{((e_1, e_3, e_4), \{h_1\}), ((e_2, e_3, e_4), \{h_2\})\}$.

Remark 4.16. Since $(\mathbb{F}, \mathcal{E})^e$ is the hypersoft interior of $(\mathbb{F}, \mathcal{E})^c$, it follows that $(\mathbb{F}, \mathcal{E})^e$ is the hypersoft open and is the largest hypersoft open set contained in $(\mathbb{F}, \mathcal{E})^c$.

Proposition 4.17. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then*

$$(\mathbb{F}, \mathcal{E})^e = \widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})^c\}.$$

Proof. From the definitions of hypersoft interior and hypersoft exterior, we have
$((\mathbb{F}, \mathcal{E})^c)^o = \widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})^c\} = (\mathbb{F}, \mathcal{E})^e$.

Proposition 4.18. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $(\Psi, \mathcal{E})^e = (\Phi, \mathcal{E})$ *and* $(\Phi, \mathcal{E})^e = (\Psi, \mathcal{E})$.
(2) $(\mathbb{F}_1, \mathcal{E})^e \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$.
(3) $(\mathbb{F}_1, \mathcal{E})^e = (((\mathbb{F}_1, \mathcal{E})^e)^c)^e$.
(4) $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$ *implies* $(\mathbb{F}_2, \mathcal{E})^e \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^e$.
(5) $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E})^e)^e$.
(6) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^e = (\mathbb{F}_1, \mathcal{E})^e \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^e$.
(7) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^e \tilde{\sqsupseteq} (\mathbb{F}_1, \mathcal{E})^e \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^e$.

Proof.

(1) $(\Psi, \mathcal{E})^e = ((\Psi, \mathcal{E})^c)^o = (\Phi, \mathcal{E})^o = (\Phi, \mathcal{E})$.
$(\Phi, \mathcal{E})^e = ((\Phi, \mathcal{E})^c)^o = (\Psi, \mathcal{E})^o = (\Psi, \mathcal{E})$.

(2) By definition, $(\mathbb{F}_1, \mathcal{E})^e = ((\mathbb{F}_1, \mathcal{E})^c)^o$ and by Proposition 4.9 (2), we have $((\mathbb{F}_1, \mathcal{E})^c)^o \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$. Hence, $(\mathbb{F}_1, \mathcal{E})^e \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$.

(3) $(((\mathbb{F}_1, \mathcal{E})^e)^c)^e = ((((\mathbb{F}_1, \mathcal{E})^c)^o)^c)^e = (((((\mathbb{F}_1, \mathcal{E})^c)^o)^c)^c)^o = (((\mathbb{F}_1, \mathcal{E})^c)^o)^o = ((\mathbb{F}_1, \mathcal{E})^c)^o = (\mathbb{F}_1, \mathcal{E})^e$.

(4) $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$ then $(\mathbb{F}_2, \mathcal{E})^c \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$. Implies that $((\mathbb{F}_2, \mathcal{E})^c)^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E})^c)^o$. So, $(\mathbb{F}_2, \mathcal{E})^e \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^e$.

(5) By (2.), we have $(\mathbb{F}_1, \mathcal{E})^e \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$. Then (4.) gives $((\mathbb{F}_1, \mathcal{E})^c)^e \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E})^e)^e$. But $(\mathbb{F}_1, \mathcal{E})^o = ((\mathbb{F}_1, \mathcal{E})^c)^e$. Hence $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E})^e)^e$.

(6) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^e = (((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^c)^o = ((\mathbb{F}_1, \mathcal{E})^c \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^c)^o = ((\mathbb{F}_1, \mathcal{E})^c)^o \tilde{\sqcap} ((\mathbb{F}_2, \mathcal{E})^c)^o = (\mathbb{F}_1, \mathcal{E})^e \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^e$.

(7) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^e = (((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^c)^o = ((\mathbb{F}_1, \mathcal{E})^c \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^c)^o \tilde{\sqsupseteq} ((\mathbb{F}_1, \mathcal{E})^c)^o \tilde{\sqcup} ((\mathbb{F}_2, \mathcal{E})^c)^o = (\mathbb{F}_1, \mathcal{E})^e \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^e$.

Definition 4.19. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, then hypersoft boundary of hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is denoted by $(\mathbb{F}, \mathcal{E})^{\mathit{b}}$ and is defined as $(\mathbb{F}, \mathcal{E})^{\mathit{b}} = \overline{(\mathbb{F}, \mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{F}, \mathcal{E})^c}$.

Example 4.20. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E})$ be the same as in Example 4.15, then $(\mathbb{F}, \mathcal{E})^{\mathit{b}} = (\mathbb{F}, \mathcal{E})$.

Remark 4.21. From Definition 4.19 it follows that the hypersoft sets $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E})^c$ have the same hypersoft boundary.

Proposition 4.22. *Let* $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ *be a hypersoft space and let* $(\mathbb{F}, \mathcal{E})$ *be a hypersoft set over* $\mathcal{U}$*. Then*

(1) $(\mathbb{F}, \mathcal{E})^{\mathit{b}} \tilde{\sqsubseteq} \overline{(\mathbb{F}, \mathcal{E})}$.
(2) $(\mathbb{F}, \mathcal{E})^{\mathit{b}} = \overline{(\mathbb{F}, \mathcal{E})} \setminus (\mathbb{F}, \mathcal{E})^o$.
(3) $((\mathbb{F}, \mathcal{E})^{\mathit{b}})^c = (\mathbb{F}, \mathcal{E})^o \tilde{\sqcup} (\mathbb{F}, \mathcal{E})^e$.

(4) $(\mathbb{F},\mathcal{E})^{o} = (\mathbb{F},\mathcal{E}) \setminus (\mathbb{F},\mathcal{E})^{b}$.

(5) $((\mathbb{F},\mathcal{E})^{o})^{b} \widetilde{\sqsubseteq} (\mathbb{F},\mathcal{E})^{b}$.

(6) $(\overline{(\mathbb{F},\mathcal{E})})^{b} \widetilde{\sqsubseteq} (\mathbb{F},\mathcal{E})^{b}$.

Proof.

(1) By definition, $(\mathbb{F},\mathcal{E})^{b} = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{((\mathbb{F},\mathcal{E})^{c})}$. Hence, $(\mathbb{F},\mathcal{E})^{b} \widetilde{\sqsubseteq} \overline{(\mathbb{F},\mathcal{E})}$.

(2) $(\mathbb{F},\mathcal{E})^{b} = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{((\mathbb{F},\mathcal{E})^{c})} = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{o})^{c} = \overline{(\mathbb{F},\mathcal{E})} \setminus (\mathbb{F},\mathcal{E})^{o}$.

(3) $((\mathbb{F},\mathcal{E})^{b})^{c} = [\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{((\mathbb{F},\mathcal{E})^{c})}]^{c} = (\overline{(\mathbb{F},\mathcal{E})})^{c} \,\widetilde{\sqcup}\, (\overline{(\mathbb{F},\mathcal{E})^{c}})^{c} = ((\mathbb{F},\mathcal{E})^{c})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{o} = (\mathbb{F},\mathcal{E})^{e} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{o}$.

(4) $(\mathbb{F},\mathcal{E}) \setminus (\mathbb{F},\mathcal{E})^{b} = (\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{b})^{c} = (\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{e}) = ((\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{o}) \,\widetilde{\sqcup}\, ((\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{e}) = (\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\Phi,\mathcal{E}) = (\mathbb{F},\mathcal{E})^{o}$.

(5) $((\mathbb{F},\mathcal{E})^{o})^{b} = \overline{(\mathbb{F},\mathcal{E})^{o}} \,\widetilde{\sqcap}\, \overline{((\mathbb{F},\mathcal{E})^{o})^{c}} = \overline{(\mathbb{F},\mathcal{E})^{o}} \,\widetilde{\sqcap}\, \overline{\overline{(\mathbb{F},\mathcal{E})^{c}}} \widetilde{\sqsubseteq} \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\mathbb{F},\mathcal{E})^{b}$.

(6) $(\overline{(\mathbb{F},\mathcal{E})})^{b} = \overline{\overline{(\mathbb{F},\mathcal{E})}} \,\widetilde{\sqcap}\, \overline{(\overline{(\mathbb{F},\mathcal{E})})^{c}} \widetilde{\sqsubseteq} \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\mathbb{F},\mathcal{E})^{b}$.

Proposition 4.23. *Let $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1,\mathcal{E})$, $(\mathbb{F}_2,\mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $((\mathbb{F}_1,\mathcal{E}) \,\widetilde{\sqcup}\, (\mathbb{F}_2,\mathcal{E}))^{b} \widetilde{\sqsubseteq} (\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcup}\, (\mathbb{F}_2,\mathcal{E})^{b}$.

(2) $((\mathbb{F}_1,\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F}_2,\mathcal{E}))^{b} \widetilde{\sqsubseteq} (\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcup}\, (\mathbb{F}_2,\mathcal{E})^{b}$.

Proof.

(1) $((\mathbb{F}_1,\mathcal{E})\widetilde{\sqcup}(\mathbb{F}_2,\mathcal{E}))^{b} = [\overline{(\mathbb{F}_1,\mathcal{E})\widetilde{\sqcup}(\mathbb{F}_2,\mathcal{E})}]\widetilde{\sqcap}[\overline{((\mathbb{F}_1,\mathcal{E})\widetilde{\sqcup}(\mathbb{F}_2,\mathcal{E}))^{c}}] = [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcup}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcap}\, [\overline{(\mathbb{F}_1,\mathcal{E})^{c}\widetilde{\sqcap}(\mathbb{F}_2,\mathcal{E})^{c}}] \widetilde{\sqsubseteq} [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcup}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcap}\, [\overline{(\mathbb{F}_1,\mathcal{E})^{c}} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}] = [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, (\overline{(\mathbb{F}_1,\mathcal{E})^{c}} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}})] \,\widetilde{\sqcup}\, [\overline{(\mathbb{F}_2,\mathcal{E})} \,\widetilde{\sqcap}\, (\overline{(\mathbb{F}_1,\mathcal{E})^{c}} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}})] = [(\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_1,\mathcal{E})^{c}}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}] \,\widetilde{\sqcup}\, [(\overline{(\mathbb{F}_2,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_1,\mathcal{E})^{c}}] \widetilde{\sqsubseteq} [(\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}] \,\widetilde{\sqcup}\, [(\mathbb{F}_2,\mathcal{E})^{b} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_1,\mathcal{E})^{c}}] \widetilde{\sqsubseteq} (\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcup}\, (\mathbb{F}_2,\mathcal{E})^{b}$.

(2) $((\mathbb{F}_1,\mathcal{E})\widetilde{\sqcap}(\mathbb{F}_2,\mathcal{E}))^{b} = [\overline{(\mathbb{F}_1,\mathcal{E})\widetilde{\sqcap}(\mathbb{F}_2,\mathcal{E})}]\widetilde{\sqcap}[\overline{((\mathbb{F}_1,\mathcal{E})\widetilde{\sqcap}(\mathbb{F}_2,\mathcal{E}))^{c}}] \widetilde{\sqsubseteq} [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcap}\, [\overline{(\mathbb{F}_1,\mathcal{E})^{c}\widetilde{\sqcup}(\mathbb{F}_2,\mathcal{E})^{c}}] = [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcap}\, [\overline{(\mathbb{F}_1,\mathcal{E})^{c}} \,\widetilde{\sqcup}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}] = [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_1,\mathcal{E})^{c}}] \,\widetilde{\sqcup}\, [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}] = [(\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_1,\mathcal{E})^{c}}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcup}\, [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, (\overline{(\mathbb{F}_2,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}})] = [(\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcup}\, [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, (\mathbb{F}_2,\mathcal{E})^{b}] \widetilde{\sqsubseteq} (\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcup}\, (\mathbb{F}_2,\mathcal{E})^{b}$.

Proposition 4.24. *Let $(\mathbb{F},\mathcal{E})$ be a hypersoft set of hypersoft space over $\mathcal{U}$. Then the following hold.*

(1) $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{b} = \overline{(\mathbb{F},\mathcal{E})}$.

(2) $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{e} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{b} = (\Psi,\mathcal{E})$.

Proof.

(1) $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{b} = (\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, [\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}}] = [(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, \overline{(\mathbb{F},\mathcal{E})}] \,\widetilde{\sqcap}\, [(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, \overline{(\mathbb{F},\mathcal{E})^{c}}] = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, [(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, ((\mathbb{F},\mathcal{E})^{o})^{c}] = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, (\Psi,\mathcal{E}) = \overline{(\mathbb{F},\mathcal{E})}$.

(2) By Proposition 4.22 (3), $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{e} = ((\mathbb{F},\mathcal{E})^{b})^{c}$, then $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{e} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{b} = ((\mathbb{F},\mathcal{E})^{b})^{c} \,\widetilde{\sqcup}\, ((\mathbb{F},\mathcal{E})^{b}) = (\Psi,\mathcal{E})$.

Proposition 4.25. *Let $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ be a hypersoft space and let $(\mathbb{F},\mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then*

(1) $(\mathbb{F},\mathcal{E})$ is a hypersoft open set if and only if $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$.

(2) $(\mathbb{F},\mathcal{E})$ is a hypersoft closed set if and only if $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})$.

Proof.

(1) Let $(\mathbb{F},\mathcal{E})$ be a hypersoft open set. By Proposition 4.22 (3), $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqsubseteq}\, ((\mathbb{F},\mathcal{E})^{b})^{c}$. But $(\mathbb{F},\mathcal{E})^{o} = (\mathbb{F},\mathcal{E})$ since $(\mathbb{F},\mathcal{E})$ is a hypersoft open set. Hence, $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqsubseteq}\, ((\mathbb{F},\mathcal{E})^{b})^{c}$. This implies that $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$.
Conversely, let $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$. Then $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, [\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}}] = (\Phi,\mathcal{E})$ or $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\Phi,\mathcal{E})$ or $\overline{(\mathbb{F},\mathcal{E})^{c}} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})^{c}$, which implies $(\mathbb{F},\mathcal{E})^{c}$ is a hypersoft closed set and hence $(\mathbb{F},\mathcal{E})$ is a hypersoft open set.

(2) Let $(\mathbb{F},\mathcal{E})$ be a hypersoft closed set. By Proposition 4.22 (1), $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqsubseteq}\, \overline{(\mathbb{F},\mathcal{E})}$. Since $(\mathbb{F},\mathcal{E})$ is a hypersoft closed set, then $\overline{(\mathbb{F},\mathcal{E})} = (\mathbb{F},\mathcal{E})$. This implies that $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})$. Conversely, let $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})$. Then $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{c} = (\Phi,\mathcal{E})$. Since $(\mathbb{F},\mathcal{E})^{b} = ((\mathbb{F},\mathcal{E})^{b})^{c}$, then we have $((\mathbb{F},\mathcal{E})^{b})^{c} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{c} = (\Phi,\mathcal{E})$. By (1), $(\mathbb{F},\mathcal{E})^{c}$ is a hypersoft open set and hence $(\mathbb{F},\mathcal{E})$ is a hypersoft closed set.

Proposition 4.26. *let $(\mathbb{F},\mathcal{E})$ be a hypersoft set of a hypersoft space over $\mathcal{U}$. Then $(\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$ if and only if $(\mathbb{F},\mathcal{E})$ is a hypersoft open set and a hypersoft closed set.*

Proof. Suppose that $(\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$ then $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\Phi,\mathcal{E})$ implies $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqsubseteq}\, (\overline{((\mathbb{F},\mathcal{E})^{c})})^{c} = (\mathbb{F},\mathcal{E})^{o}$. Since $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})$ then $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})$ and hence $\overline{(\mathbb{F},\mathcal{E})} = (\mathbb{F},\mathcal{E})$. This implies that $(\mathbb{F},\mathcal{E})$ is a hypersoft closed set. Again, $(\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$ then $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\Phi,\mathcal{E})$ or $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{o})^{c} = (\Phi,\mathcal{E})$ then $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})^{o}$. This implies that $(\mathbb{F},\mathcal{E})^{o} = (\mathbb{F},\mathcal{E})$. Hence $(\mathbb{F},\mathcal{E})$ is a hypersoft open set.
Conversely, suppose that $(\mathbb{F},\mathcal{E})$ is a hypersoft open set and a hypersoft closed set. Then $(\mathbb{F},\mathcal{E})^{b} = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{o})^{c} = (\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{c} = (\Phi,\mathbb{G},\mathcal{E})$.

Proposition 4.27. *Let $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ be a hypersoft space and let $(\mathbb{F},\mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then*

(1) $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$.

(2) $(\mathbb{F},\mathcal{E})^{e} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$.

Proof.

(1) $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, [\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}}] = [(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})}] \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{o})^{c} = (\Phi,\mathcal{E})$.

(2) $(\mathbb{F},\mathcal{E})^{e} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = ((\mathbb{F},\mathcal{E})^{c})^{o} \,\widetilde{\sqcap}\, [\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}}] = (\overline{(\mathbb{F},\mathcal{E})})^{c} \,\widetilde{\sqcap}\, [\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}}]$
$= [(\overline{(\mathbb{F},\mathcal{E})})^{c} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})}] \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}}] = (\Phi,\mathcal{E}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\Phi,\mathcal{E})$.

5. Conclusions

In this paper, we have introduced the concept of hypersoft topological spaces which are defined over an initial universe with a fixed set of parameters. Some concepts such as hypersoft closure, hypersoft interior, hypersoft boundary, etc. which are based on our definition were introduced and studied and some relationships between them were discussed. For future trends, we can define the most important fundamental topological properties such as connectedness and compactness. Also, we can define hypersoft separation axioms by using ordinary point as well as hypersoft point.

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Products of Interval Neutrosophic Automata

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Abstract. In this paper, we introduced direct product, restricted direct product of interval neutrosophic automata and prove that direct, restricted direct product of cyclic and retrievable of interval neutrosophic automata are cyclic and retrievable interval neutrosophic automata.

Keywords: Cyclic, Retrievability, Direct product.

1. Introduction

Neutrosophic set is a part of neutrosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set is a powerful general formal framework that has been recently proposed. The theory of neutrosophy and neutrosophic set was introduced by Florentin Smarandache in 1999 [18]. The neutrosophic set is the generalization of classical sets, fuzzy set [22] and so on. The concept of fuzzy set and intuitionstic fuzzy set unsuccessful when the relation is indeterminate. Neutrosophic sets are powerful logics designed to facilitate understanding of indeterminate and inconsistent information. A neutrosophic set consider truth-membership, in-determinacy-membership and falsity-membership which are completely independent. A neutrosophic set N is classified by a Truth membership T_N, Indeterminacy membership I_N, and Falsity membership function F_N, where $T_N, I_N,$ andF_N are real standard and non-standard subsets of $]0^-, 1^+[$.

Wang $etal.$ [20] introduced the notion of interval-valued neutrosophic sets. The interval neutrosophic set are characterized by an interval memebership degree, interval indeterminacy degree, and interval nonmembership degree.

Neutrosophic sets and methods have recently gained popularity in a variety of domains and it has lot of applications. For example, on similarity and entropy in neutrosophic sets were discussed in [16]. Subsequently, on entropy and similarity measure of interval valued neutrosophic sets was discussed in [1]. Multi-criteria decision-making method based on a cross-entropy with interval neutrosophic sets were discussed in [19]. An interval neutrosophic linguistic multi-criteria group decision-making method and its application in selecting medical treatment options were discussed in [13].

The concept of single valued and interval valued neutrosophic set applied in automata theory. It was introduced by Tahir Mahmood et. al in [14, 15]. Consequently, J. Kavikumar et.al were introduced neutrosophic general finite automata and composite neutrosophic finite automata [11, 12]. Later, the concept interval valued neutrosophic automata applied in retrievability, subsystem, strong subsystem and characterizations of submachines were discussed in [4–7].

Products is important concept in automata theory since it produce a new automata with the existing automata by taking products. The Cartesian composition of automata was discussed by W. Dorfler in 1977 [3]. Cartesian product of fuzzy automata was discussed by D. S. Malik et.al [17]. Later number of authors have worked in these lines. Generalized products of directable fuzzy automata were discussed in [8]. Generalized products of Δ-synchronized fuzzy automata were discussed in [9]. Cartesian products of interval neutrosophic automata were discussed in [10].

In this paper, we introduce direct and restricted direct prodcut of interval neutrosophic automata and prove that direct and restricted direct product of cyclic, retirevable of interval neutrosophic automata are cyclic, retirevable interval neutrosophic automata.

2. Preliminaries

Definition 2.1. [18] Let U be the universe of discourse. A neutrosophic set (NS) N in U is defined by a truth membership T_N, indeterminacy membership I_N and a falsity membership F_N, where T_N, I_N, and F_N are real standard or non-standard subsets of $]0^-, 1^+[$. That is
$N = \{\langle x, (T_N(x), I_N(x), F_N(x))\rangle, x \in U,\ T_N, I_N, F_N \in\ ]0^-, 1^+[\ \}$ and
$0^- \leq sup\ T_N(x) + sup\ I_N(x) + sup\ F_N(x) \leq 3^+$. We use the interval $[0, 1]$ instead of $]0^-, 1^+[$.

Definition 2.2. [20] Interval neutrosophic set (INS for short) is of the form $N = \{\langle \alpha_N(x), \beta_N(x), \gamma_N(x)\rangle\, |x \in U\}$
$= \{\langle x, [\inf\ \alpha_N(x), \sup\ \alpha_N(x)], [\inf\ \beta_N(x), \sup\ \beta_N(x)], [\inf\ \gamma_N(x), \sup \gamma_N(x)]\rangle\}$,
$x \in U$, $\alpha_N(x),\ \beta_N(x),\ \gamma_N(x) \subseteq [0, 1]$ and
$0 \leq \sup\ \alpha_N(x) +\ \sup\ \beta_N(x) +\ \sup\ \gamma_N(x) \leq\ 3.$

Definition 2.3. [20] An INS N is empty if $\inf\, \alpha_N(x) = \sup\, \alpha_N(x) = 0,\ \inf\, \beta_N(x) = \sup\, \beta_N(x) = 1,\ \inf\, \gamma_N(x) = \sup\, \gamma_N(x) = 1$ forall $x \in U$.

Definition 2.4. [14] Interval neutrosophic automaton $M = (Q,\ \Sigma,\ N)$ $(INAforshort)$, where Q and Σ are non-empty finite sets called the set of states and input symbols respectively, and $N = \{\langle \alpha_N(x), \beta_N(x), \gamma_N(x)\rangle\}$ is an INS in $Q \times \Sigma \times Q$.

The set of all words of finite length of Σ is denoted by Σ^*. The empty word is denoted by ϵ, and the length of each $x \in \Sigma^*$ is denoted by $|x|$.

Definition 2.5. [14] Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton and extended interval neutrosophic set is defined as $N^* = \{\langle \alpha_{N^*}(x), \beta_{N^*}(x), \gamma_{N^*}(x)\rangle\}$ in $Q \times \Sigma^* \times Q$ by

$$\alpha_{N^*}(q_i,\ \epsilon,\ q_j) = \begin{cases} [1,1] & \text{if } q_i = q_j \\ [0,0] & \text{if } q_i \neq q_j \end{cases}$$

$$\beta_{N^*}(q_i,\ \epsilon,\ q_j) = \begin{cases} [0,0] & \text{if } q_i = q_j \\ [1,1] & \text{if } q_i \neq q_j \end{cases}$$

$$\gamma_{N^*}(q_i,\ \epsilon,\ q_j) = \begin{cases} [0,0] & \text{if } q_i = q_j \\ [1,1] & \text{if } q_i \neq q_j \end{cases}$$

$\alpha_{N^*}(q_i, w, q_j) = \alpha_{N^*}(q_i, xy, q_j) = \vee_{q_r \in Q}[\alpha_{N^*}(q_i, x, q_r) \wedge \alpha_{N^*}(q_r, y, q_j)]$,
$\beta_{N^*}(q_i, w, q_j) = \beta_{N^*}(q_i, xy, q_j) = \wedge_{q_r \in Q}[\beta_{N^*}(q_i, x, q_r) \vee \beta_{N^*}(q_r, y, q_j)]$,
$\gamma_{N^*}(q_i, w, q_j) = \gamma_{N^*}(q_i, xy, q_j) = \wedge_{q_r \in Q}[\gamma_{N^*}(q_i, x, q_r) \vee \gamma_{N^*}(q_r, y, q_j)], \forall q_i, q_j \in Q$,
$w = xy, x \in \Sigma^*$ and $y \in \Sigma$.

3. Products of Interval Neutrosophic Automata

Definition 3.1. Let $M_i = (Q_i, \Sigma_i, N_i), i = 1, 2$ be interval neutrosophic automata. Let $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma_1 \times \Sigma_2, N_1 \times N_2)$, where
$(\alpha_{N_1} \times \alpha_{N_2})((q_i, q_j), (a, b), (q_k, q_l)) = \alpha_{N_1}(q_i, a, q_k) \wedge \alpha_{N_2}(q_j, b, q_l)$
$(\beta_{N_1} \times \beta_{N_2})((q_i, q_j), (a, b), (q_k, q_l)) = \beta_{N_1}(q_i, a, q_k) \vee \beta_{N_2}(q_j, b, q_l)$
$(\gamma_{N_1} \times \gamma_{N_2})((q_i, q_j), (a, b), (q_k, q_l)) = \gamma_{N_1}(q_i, a, q_k) \vee \gamma_{N_2}(q_j, b, q_l)$.
$\forall (q_i, q_j), (q_k, q_l) \in Q_1 \times Q_2, (a, b) \in \Sigma_1 \times \Sigma_2$. Then $M_1 \times M_2$ is called direct product of interval neutrosophic automata.

Definition 3.2. Let $M_i = (Q_i, \Sigma, N_i), i = 1, 2$ be interval neutrosophic automata. Let $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma, N_1 \times N_2)$, where
$(\alpha_{N_1} \times \alpha_{N_2})((q_i, q_j), a, (q_k, q_l)) = \alpha_{N_1}(q_i, a, q_k) \wedge \alpha_{N_2}(q_j, a, q_l)$
$(\beta_{N_1} \times \beta_{N_2})((q_i, q_j), a, (q_k, q_l)) = \beta_{N_1}(q_i, a, q_k) \vee \beta_{N_2}(q_j, a, q_l)$
$(\gamma_{N_1} \times \gamma_{N_2})((q_i, q_j), a, (q_k, q_l)) = \gamma_{N_1}(q_i, a, q_k) \vee \gamma_{N_2}(q_j, a, q_l)$.
$\forall (q_i, q_j), (q_k, q_l) \in Q_1 \times Q_2, a, \in \Sigma$. Then $M_1 \times M_2$ is called restricted direct product of interval neutrosophic automata.

Definition 3.3. Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton. M is said to be cyclic if $\exists\ q_i \in Q$ such that $Q = S(q_i)$.

Definition 3.4. Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton. M is said to be connected if $\forall\ q_j, q_i$ and $\exists\ a \in \Sigma$ such that either
$\alpha_N(q_i, a, q_j) > [0,0],\ \beta_N(q_i, a, q_j) < [1,1],\ \gamma_N(q_i, a, q_j) < [1,1]$ or
$\alpha_N(q_j, a, q_i) > [0,0],\ \beta_N(q_j, a, q_i) < [1,1],\ \gamma_N(q_j, a, q_i) < [1,1]$.

Definition 3.5. Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton. M is said to be strongly connected if for every $q_i,\ q_j \in Q$, there exists $u \in \Sigma^*$ such that $\alpha^*_N(q_i,\ u,\ q_j) > [0,0]$, $\beta^*_N(q_i,\ u,\ q_j) < [1,1], \gamma^*_N(q_i,\ u,\ q_j) < [1,1]$. M is strongly connected if it has no proper subautomaton.

4. **Properties of Products of Interval Neutrosophic Automata**

Theorem 4.1. Let $M_i = (Q_i, \Sigma_i, N_i), i = 1, 2$ be interval neutrosophic automata. Let $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma_1 \times \Sigma_2, N_1 \times N_2)$ be the full direct product of M_1 and M_2. Then $\forall x_1 \in \Sigma_1^*, x_2 \in \Sigma_2^*, x_1, x_2 \neq \epsilon$
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (x_1, x_2)(q_k, q_l)) = \alpha_{N_1{}^*}(q_i, x_1, q_k) \wedge \alpha_{N_2{}^*}(q_j, x_2, q_l)$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (x_1, x_2)(q_k, q_l)) = \beta_{N_1{}^*}(q_i, x_1, q_k) \vee \beta_{N_2{}^*}(q_j, x_2, q_l)$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (x_1, x_2)(q_k, q_l)) = \gamma_{N_1{}^*}(q_i, x_1, q_k) \vee \gamma_{N_2{}^*}(q_j, x_2, q_l)$
$\forall (q_i, q_j), (q_k, q_l) \in Q_1 \times Q_2$.

Proof. Let $x_1 \in \Sigma_1^*,\ x_2 \in \Sigma_2^*, x_1, x_2 \neq \epsilon$. Let $|x_1| = |x_2| = m$. The result is trivial if $m = 1$. Suppose the result is true $\forall u_1 \in \Sigma_1^*,\ u_2 \in \Sigma_2^*, |u_1| = |u_2| = m - 1, m > 1$. Let $x_1 = a_1u_1,\ x_2 = a_2u_2$ where $a_1 \in \Sigma_1,\ a_2 \in \Sigma_2$ and $u_1 \in \Sigma_1^*, u_2 \in \Sigma^*$. Now,
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (a_1u_1, a_2u_2)(q_k, q_l)) =$
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), a_1u_1, (q_k, q_l)) \wedge (\alpha_1 \times \alpha_2)^*((q_i, q_j), a_2u_2, (q_k, q_l))$
$= \{\vee_{(q_r,q_s)\in Q_1\times Q_2}(\alpha_{N_1} \times \alpha_{N_2})((q_i, q_j), a_1, (q_r, q_s)) \wedge (\alpha_{N_1} \times \alpha_{N_2})^*((q_r, q_s), u_1, (q_k, q_l))\} \wedge$
$\{\vee_{(q_u,q_v)\in Q_1\times Q_2}\{(\alpha_{N_1} \times \alpha_{N_2})((q_i, q_j), a_2, (q_u, q_v)) \wedge (\alpha_{N_1} \times \alpha_{N_2})^*((q_u, q_v), u_2, (q_k, q_l))\}$
$= \{\vee_{q_r\in Q_1}\{\alpha_{N_1}(q_i, a_1, q_r) \wedge \alpha^*_{N_1}(q_r, u_1, q_k)\}\} \wedge \{\vee_{q_u\in Q_2}\{\alpha_{N_2}(q_j, a_2, q_u) \wedge \alpha^*_{N_2}(q_u, u_2, q_l)\}\}$
$= \{\alpha^*_{N_1}(q_i, a_1u_1, q_k \wedge \alpha^*_{N_2}(q_j, a_2u_2, q_l)\}$
$= \{\alpha^*_{N_1}(q_i, x_1, q_k \wedge \alpha^*_{N_2}(q_j, x_2, q_l)\}$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (a_1u_1, a_2u_2)(q_k, q_l)) =$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), a_1u_1, (q_k, q_l)) \vee (\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), a_2u_2, (q_k, q_l))$
$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}(\beta_{N_1} \times \beta_{N_2})((q_i, q_j), a_1, (q_r, q_s)) \vee (\beta_{N_1} \times \beta_{N_2})^*((q_r, q_s), u_1, (q_k, q_l))\} \vee$
$\{\wedge_{(q_u,q_v)\in Q_1\times Q_2}\{(\beta_{N_1} \times \beta_{N_2})((q_i, q_j), a_2, (q_u, q_v)) \vee (\beta_{N_1} \times \beta_{N_2})^*((q_u, q_v), u_2, (q_k, q_l))\}$
$= \{\wedge_{q_r\in Q_1}\{\beta_{N_1}(q_i, a_1, q_r) \vee \beta^*_{N_1}(q_r, u_1, q_k)\}\} \vee \{\wedge_{q_u\in Q_2}\{\beta_{N_2}(q_j, a_2, q_u) \vee \beta^*_{N_2}(q_u, u_2, q_l)\}\}$
$= \{\beta^*_{N_1}(q_i, a_1u_1, q_k \vee \beta^*_{N_2}(q_j, a_2u_2, q_l)\}$

$= \{\beta^*_{N_1}(q_i, x_1, q_k \vee \beta^*_{N_2}(q_j, x_2, q_l)\}$

$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (a_1u_1, a_2u_2)(q_k, q_l)) =$

$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), a_1u_1, (q_k, q_l)) \vee (\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), a_2u_2, (q_k, q_l))$

$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}(\gamma_{N_1} \times \gamma_{N_2})((q_i, q_j), a_1, (q_r, q_s)) \vee (\gamma_{N_1} \times \gamma_{N_2})^*((q_r, q_s), u_1, (q_k, q_l))\} \vee$

$\{\wedge_{(q_u,q_v)\in Q_1\times Q_2}\{(\gamma_{N_1} \times \gamma_{N_2})((q_i, q_j), a_2, (q_u, q_v)) \vee (\gamma_{N_1} \times \gamma_{N_2})^*((q_u, q_v), u_2, (q_k, q_l))\}$

$= \{\wedge_{q_r\in Q_1}\{\gamma_{N_1}(q_i, a_1, q_r) \vee \gamma^*_{N_1}(q_r, u_1, q_k)\}\} \vee \{\wedge_{q_u\in Q_2}\{\gamma_{N_2}(q_j, a_2, q_u) \vee \gamma^*_{N_2}(q_u, u_2, q_l)\}\}$

$= \{\gamma^*_{N_1}(q_i, a_1u_1, q_k \vee \gamma^*_{N_2}(q_j, a_2u_2, q_l)\}$

$= \{\gamma^*_{N_1}(q_i, x_1, q_k \vee \gamma^*_{N_2}(q_j, x_2, q_l)\}$

Theorem 4.2. Let $M_i = (Q_i, \Sigma, N_i), i = 1, 2$ be interval neutrosophic automata. Let $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma, N_1 \times N_2)$ be the restricted direct product of M_1 and M_2. Then $\forall x \in \Sigma^*$

$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), x(q_k, q_l)) = \alpha_{N_1{}^*}(q_i, x, q_k) \wedge \alpha_{N_2{}^*}(q_j, x, q_l)$

$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), x(q_k, q_l)) = \beta_{N_1{}^*}(q_i, x, q_k) \vee \beta_{N_2{}^*}(q_j, x, q_l)$

$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), x(q_k, q_l)) = \gamma_{N_1{}^*}(q_i, x, q_k) \vee \gamma_{N_2{}^*}(q_j, x, q_l)$

$\forall (q_i, q_j), (q_k, q_l) \in Q_1 \times Q_2$.

Proof. We prove the result by induction on $|x| = n$. If $n = 1$ then the result is obivious. Suppose the result is true for all $x \in \Sigma^*$. Let $x = au$, where $a \in \Sigma, u \in \Sigma^*, |u| = m - 1, m > 1$. Then

$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = (\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), au, (q_k, q_l))$

$= \{\vee_{(q_r,q_s)\in Q_1\times Q_2}\{(\alpha_{N_1} \times \alpha_{N_2})((q_i, q_j), a, (q_k, q_l))\} \wedge \{(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), u, (q_k, q_l))\}\}$

$= \{\vee_{(q_r,q_s)\in Q_1\times Q_2}\{\alpha_{N_1}(q_i, a, q_r) \wedge \alpha_{N_2}(q_j, a, q_s) \wedge \alpha^*_{N_1}(q_r, u, q_k) \wedge \alpha^*_{N_2}(q_s, u, q_l)\}\}$

$= \alpha^*_{N_1}(q_i, au, q_k) \wedge \alpha^*_{N_2}(q_j, au, q_l)$

$= \alpha^*_{N_1}(q_i, x, q_k) \wedge \alpha^*_{N_2}(q_j, x, q_l)$

$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = (\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), au, (q_k, q_l))$

$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}\{(\beta_{N_1} \times \beta_{N_2})((q_i, q_j), a, (q_k, q_l))\} \vee \{(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), u, (q_k, q_l))\}\}$

$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}\{\beta_{N_1}(q_i, a, q_r) \vee \beta_{N_2}(q_j, a, q_s) \vee \beta^*_{N_1}(q_r, u, q_k) \vee \beta^*_{N_2}(q_s, u, q_l)\}\}$

$= \beta^*_{N_1}(q_i, au, q_k) \vee \beta^*_{N_2}(q_j, au, q_l)$

$= \beta^*_{N_1}(q_i, x, q_k) \vee \beta^*_{N_2}(q_j, x, q_l)$

$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = (\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), au, (q_k, q_l))$

$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}\{(\gamma_{N_1} \times \gamma_{N_2})((q_i, q_j), a, (q_k, q_l))\} \vee \{(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), u, (q_k, q_l))\}\}$

$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}\{\gamma_{N_1}(q_i, a, q_r) \vee \gamma_{N_2}(q_j, a, q_s) \vee \gamma^*_{N_1}(q_r, u, q_k) \vee \gamma^*_{N_2}(q_s, u, q_l)\}\}$

$= \gamma^*_{N_1}(q_i, au, q_k) \vee \gamma^*_{N_2}(q_j, au, q_l)$

$= \gamma^*_{N_1}(q_i, x, q_k) \vee \gamma^*_{N_2}(q_j, x, q_l)$

Theorem 4.3. Let $M_i = (Q_i, \Sigma_i, N_i), i = 1, 2$ be interval neutrosophic automata. Then full direct product of $M_1 \times M_2$ is cyclic if and only if M_1 and M_2 are cyclic.

Proof. Let $\times$ be full direct product.Suppose M_1 and M_2 are cyclic, say $Q_1 = S(q_i)$ and $Q_2 = S(p_j)$ for some $q_i \in Q_1$, $p_j \in Q_2$. Let $(q_k, p_l) \in Q_1 \times Q_2$. Then $\exists x \in \Sigma_1^*$ and such

that $\alpha^*_{N_1}(q_i, x, q_k) > [0,0], \beta^*_{N_1}(q_i, x, q_k) < [1,1], \gamma^*_{N_1}(q_i, x, q_k) < [1,1]$ and
$\alpha^*_{N_2}(q_j, y, q_l) > [0,0], \beta^*_{N_2}(q_j, y, q_l) < [1,1], \gamma^*_{N_2}(q_j, y, q_l) < [1,1]$, Thus
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \alpha^*_{N_1}(q_i, x, q_k) \wedge \alpha^*_{N_2}(q_j, y, q_l) > [0,0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \beta^*_{N_1}(q_i, x, q_k) \vee \beta^*_{N_2}(q_j, y, q_l) < [1,1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \gamma^*_{N_1}(q_i, x, q_k) \vee \gamma^*_{N_2}(q_j, y, q_l) < [1,1]$.
Hence $(q_k, p_l) \in S((q_i, p_j))$. Thus $Q_1 \times Q_2 = S((q_i, p_j))$. Hence $M_1 \times M_2$ is cyclic. Conversely, suppose $M_1 \times M_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_i, p_j))$ for some $(q_i, p_j) \in Q_1 \times Q_2$. Let $q_k \in Q_1$ and $q_l \in Q_2$.
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \alpha^*_{N_1}(q_i, x, q_k) \wedge \alpha^*_{N_2}(q_j, y, q_l) > [0,0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \beta^*_{N_1}(q_i, x, q_k) \vee \beta^*_{N_2}(q_j, y, q_l) < [1,1]$ and
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \beta^*_{N_1}(q_i, x, q_k) \vee \beta^*_{N_2}(q_j, y, q_l) < [1,1]$.

Theorem 4.4. Let $M_i = (Q_i, \Sigma, N_i), i = 1, 2$ be interval neutrosophic automata. If restricted direct product of interval neutrosophic automata $M_1 \times M_2$ is cyclic, then M_1 and M_2 are cyclic.

Proof. Let $\times$ be restricted direct product. Suppose $M_1 \times M_2$ are cyclic. $Q_1 \times Q_2 = S((q_i, q_j))$ for some $q_i, q_j \in Q_1 \times Q_2$. Let $q_k \in Q_1, q_l \in Q_2$. Then $(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = \alpha^*_{N_1}(q_i, x, q_k) \wedge \alpha_{N_2})^*(q_j, x, q_l) > [0,0]$ $(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = \beta^*_{N_1}(q_i, x, q_k) \vee \beta_{N_2})^*(q_j, x, q_l) < [1,1]$ $(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = \gamma^*_{N_1}(q_i, x, q_k) \vee \gamma_{N_2})^*(q_j, x, q_l) < [1,1]$. Thus $q_k \in S(q_i)$ and $q_l \in S(q_j)$. Therefore $Q_1 = S(q_i)$ for some $q_i \in Q_1$ and $Q_2 = S(q_j)$. Hence M_1 and M_2 are cyclic.

Theorem 4.5. Let $M_i = (Q_i, \Sigma_i, N_i), i = 1, 2$ be interval neutrosophic automata. Then the full direct product of interval neutrosophic automata $M_1 \times M_2$ is retrievable if and only if M_1 and M_2 are interval neutrosophic retrievable automata.

Proof. Let $\times$ be full direct product. Suppose that M_1 and M_2 are interval neutrosophic retrievable.
Let $(q_i, q_j), (t_k, s_l) \in Q_1 \times Q_2$ and $(x, y) \in (\Sigma_1 \times \Sigma_2)^*$ be such that
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (x, y), (t_k, s_l)) = \alpha^*_{N_1}(q_i, x, t_k) \wedge \alpha^*_{N_2}(q_j, y, s_l) > [0,0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (x, y), (t_k, s_l)) = \beta^*_{N_1}(q_i, x, t_k) \vee \beta^*_{N_2}(q_j, y, s_l) < [1,1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (x, y), (t_k, s_l)) = \gamma^*_{N_1}(q_i, x, t_k) \vee \gamma^*_{N_2}(q_j, y, s_l) < [1,1]$
Since M_1 and M_2 are interval neutrosophic retrievable $\exists u_1 \in \Sigma_1^*, u_2 \in \Sigma_2^*$ such that
$\alpha^*_{N_1}(q_k, u_1, q_i) > [0,0], \beta^*_{N_1}(q_k, u_1, q_i) < [1,1], \gamma^*_{N_1}(q_k, u_1, q_i) < [1,1]$
$\alpha^*_{N_2}(q_l, u_2, q_j) > [0,0], \beta^*_{N_2}(q_l, u_2, q_l) < [1,1], \gamma^*_{N_2}(q_l, u_2, q_j) < [1,1]$.
$\alpha^*_{N_1}(q_k, u_1, q_i) \wedge \alpha^*_{N_2}(q_l, u_2, q_j)$ $(\alpha_{N_1} \times \alpha_{N_2})^*((q_k, q_l), (u_1, u_2), (q_i, q_j)) > [0,0]$ $\beta^*_{N_1}(q_k, u_1, q_i) \vee \beta^*_{N_2}(q_l, u_2, q_j)$ $(\beta_{N_1} \times \beta_{N_2})^*((q_k, q_l), (u_1, u_2), (q_i, q_j)) < [1,1]$ $\gamma^*_{N_1}(q_k, u_1, q_i) \vee \gamma^*_{N_2}(q_l, u_2, q_j)$ $(\gamma_{N_1} \times \gamma_{N_2})^*((q_k, q_l), (u_1, u_2), (q_i, q_j)) < [1,1]$. Thus, $M_1 \times M_2$ are interval neutrosophic retrievable.
Conversely, suppose $M_1 \times M_2$ are interval neutrosophic retrievable. Let $(q_i, q_j) \in Q_1 \times Q_2$ and

$(x,y) \in (\Sigma_1 \times \Sigma_2)^*, \exists (q_k, q_l) \in Q_1 \times Q_2$ such that
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i,q_j),(x,y),(q_k,q_l)) > [0,0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i,q_j),(x,y),(q_k,q_l)) < [1,1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i,q_j),(x,y),(q_k,q_l)) < [1,1]$.
Then $\exists (u_1,u_2) \in (\Sigma_1 \times \Sigma_2)^*$ such that $(\alpha_{N_1} \times \alpha_{N_2})^*((q_k,q_l),(u_1,u_2),(q_k,q_l)) > [0,0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_k,q_l),(u_1,u_2),(q_k,q_l)) < [1,1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_k,q_l),(u_1,u_2),(q_k,q_l)) < [1,1]$
$(\alpha^*_{N_1}(q_k,u_1,q_i) \wedge (\alpha^*_{N_2}(q_l,u_2,q_j) > [0,0]$
$(\beta^*_{N_1}(q_k,u_1,q_i) \vee (\beta^*_{N_2}(q_l,u_2,q_j) < [1,1]$
$(\gamma^*_{N_1}(q_k,u_1,q_i) \vee (\gamma^*_{N_2}(q_l,u_2,q_j) < [1,1]$.
Hence, M_1 and M_2 are interval neutrosophic retrievable.

Theorem 4.6. Let $M_i = (Q_i, \Sigma_i, N_i), i = 1,2$ be interval neutrosophic automata. Then restricted direct product of interval neutrosophic automata $M_1 \times M_2$ is interval neutrosophic retrievable then M_1 and M_2 are interval neutrosophic retrievable.

Proof. Let $\times$ be interval neutrosophic resticted direct product. suppose $M_1 \times M_2$ is interval neutrosophic retrievable. Let $(q_i,q_j),(q_k,q_l) \in Q_1 \times Q_2, x \in \Sigma$ such that
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i,q_j),x,(q_k,q_l)) > [0,0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i,q_j),x,(q_k,q_l)) < [1,1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i,q_j),x,(q_k,q_l)) < [1,1]$ Then $\exists u \in \Sigma^*$ such that
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_k,q_l),u,(q_i,q_j)) = \alpha^*_{N_1}(q_k,u,q_i) \wedge \alpha^*_{N_2}(q_l,u,q_j) > [0,0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_k,q_l),u,(q_i,q_j)) = \beta^*_{N_1}(q_k,u,q_i) \vee \beta^*_{N_2}(q_l,u,q_j) < [1,1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_k,q_l),u,(q_i,q_j)) = \gamma^*_{N_1}(q_k,u,q_i) \vee \gamma^*_{N_2}(q_l,u,q_j) < [1,1]$.
Hence M_1 and M_2 are interval neutrosophic retrievable.

5. **Conclusions**

In this paper, we introduced direct product, restricted direct product of interval neutrosophic automata and prove that direct, restricted direct product of cyclic and retrievable of interval neutrosophic automata are cyclic and retrievable interval neutrosophic automata.

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I-Valued Neutrosophic AHP:An Application To Assess Airline Service Quality After Covid-19 Pandemy

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Abstract. This study proposes the I-valued Neutrosophic AHP technique to evaluate airline service quality by determining importance priorities for passengers and generating recommendations to managers to allocate the most appropriate resource for increasing service quality and customer satisfaction. We also provide a list of what airline managers need to improve in resource allocation to increase service quality by taking customer satisfaction into account. This technique can be adapted for any industry where service quality depends on multiple attributes.

Keywords: Interval valued neutrosophic AHP; Euclidean tangent combine similarity; airline ; Covid 19 pandemy; hygiene, pandemy.

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1. **Introduction**

More than 209 countries that are desperate in the face of COVID-19, which first appeared in December 2019, have been struggling against the pandemic by focusing on the social distance and hygiene rules proposed by the World Health Organization (WHO), Boopathi et al. ([4]). Especially, after infected airline passengers spread Covid 19 to different regions and turning it into a pandemic, the vast majority of countries have taken measures in order to restrict international human mobility, such as closing their borders to international traffic, imposing visa restrictions or putting in quarantines to their citizens coming from abroad Chung, ([11]); Liu et al. ([25]). The number of passengers who prefer air transportation today has decreased by 80Air Transport Association (2020). The Airlines sector has become one of the service sectors which suffered the most damage from Covid-19 with their parked airplanes. However, the sectors such as trade, business and tourism are dependent upon airline transport since the basis of air travel consists of the concept of traveling quickly and safely to long distances. The new priorities and concerns that the Covid-19 pandemic induced will unquestionably cause to remarkable changes on the criteria of airline service quality Cao et al., ([6]). In the concerning literature, the only study exercised in point of this subject is the study that measures the preventive expectations of the passengers with regard to the airline service quality by using IPA after the influenza pandemic of novel swine Chou and Lu, ([10]). However, the studies emphasized the importance of the measures to be taken in airplanes and during flight, while it has been determined that their passengers pay attention to pandemic and they do not want to catch this pandemic Chou, ([9]; Khan et al., ([22]).

In some MCDM problems, situations where the degree of membership values are not a real numerical value but instead they are an interval. The service quality is also a combination of various properties in this sense, it contains many indefinite properties and it is difficult to measure with classical MCDM techniques. Fuzzy Sets (FS) and IVFSs have been used in various studies in order to eliminate this deficiency. However, the relationships are defined by membership degrees in fuzzy sets and they dont contain non-membership degree values (they dont contain non-membership degree values). Intuitionistic fuzzy sets are sets whose elements have degrees of membership and non-membership. (Intuitionistic fuzzy sets are sets whose elements have degrees of membership and non-membership). The totals of membership and non-membership values are 1 or less than 1 in this place. In fact, the components regarding the assesstments of the answerers are independent from each other, especially in the studies of service quality, where there is no complete information and mixed, ambiguous, close expectations and perceptions are observed.

Neutrosophic sets developed by Smarandache ([32]) approached these deficiencies of intuitionistic fuzzy sets (IFS's) with regard to uncertainty, impreciseness, inconsistency and vagueness from a different perspective. Its degree of indeterminacy/neutrality component was included in fuzzy sets and defined as three components.. Thus, it gives a more capable description of the indeterminacy parameter membership functions. In addition, since component membership degree and non-membership degree are not interconnect, the necessity of the total of membership function elements to be equal to for a certain event 1 is eliminated. There are just a few neutrosophic AHP papers containing applications. Namely, Radwan et al. ([31]) worked on a hybrid neutrosophic AHP approach in learning management systems and Abdel-Basset et al. ([1]). introduced the integration of AHP into Delphi under neutrosophic framework. The study by Bolturk et al. ([4]) used the interval valued neutrosophic AHP with the deneutrosophication method together with the cosine similarity measure.

On the other hand, most of the researchers concern in comparing two or more MCDM methodologies and analyzing the advantages or drawbacks of each method in the literature. However, what needs to be examined is the validity, reliability and / or consistency of the result that is analyzed and decided. The sensitivity analyzes do not reflect the reality or sensitivity results with different analyzes give different results most of the time. This study is almost the first work to determine the service quality attributes that the passengers will place emphasis on after pandemic. The method of the research gives better results than traditional approaches. Moreover, analyzes with regard to the validity, reliability and / or consistency of the results are applied for the first time in the concening literature. This study has a distinctive added value for all these reasons.

2. Service quality in the airline transportation industry in the light of pandemic issues

The issues such as fast diagnosing of sick passengers, implementation of isolation and quarantine applications in time constitute another dimension of airline service quality since passengers can spread contagious diseases to more distant regions by connection flights between countries on air travel (Nikolaou and Dimitriou, [29]). These factors of service quality, on one hand, relate to reducing the risk of pandemic to passengers, on the other hand, they associate with the crew, airline company managers, and also other passengers having responsible attitude pertinent to the pandemic (Baker et al., [?]).

Studies with regard to the airline sector concerning the pandemic are very restricted. These studies are with regard to the spread of the pandemic (Gold et al., [18]; Nikolaou and Dimitriou, [29]; Hsu and Shih, [21]; Tuncer and Le, [33]) and the economic impact of the pandemic on the sector (Chung, [12]). It has been observed that there is a single study where the priorities of service quality regarding the pandemic were assessed from a passenger perspective (Chou and Lu, [11]). While this study measures the perception of passengers in airlines against influenza (H1N1) preventive measures, (1) the cleaning of cabin and disinfection services, (2) the personal protective requirements and (3) influenza preventive equipment are determined as service quality dimensions (Chou and Lu, [11]). The directors of the sector sharing their views relating to the Covid 19 pandemic emphasize the need for social distance and hygiene measures in order to eliminate the concerns of the passengers and to make the air travel attractive.

Different airlines and airport operators try to respond to the concerns of the passengers regarding these issues by taking a variety of measures with COVID-19 pandemic. For example, the technological devices such as thermo-imaging cameras and temperature measurement equipment, filter and duct cleaning works in ventilation ducts and new hand sanitizer stations are being used in Istanbul airport (Istanbul Airport, 2020). EVA Air asks that the crew on airplanes wear sanitary masks, that the passengers collect their cafeteria trays by wearing gloves and it distributes brochures providing information on how to prevent the spread of the virus. China Airlines ensures that all passengers wear a face mask at check-in and during flight, they are checked for temperature before boarding, table cloth, Menu card / wine list are not presented and the cabins are sterilized after each flight. Tigerair Taiwan replaces the head restraints every time after journey and also canceled the on-board duty free service by removing the magazines in the airplane (Tigerair Taiwan, 2020).

3. **Interval Valued Neutrosophic Sets Logic and arithmetic**

In 1998, Smarandache ([32]) introduced a more generalized tool to handle uncertainty, imprecise, incomplete and inconsistent information, called as Neutrosophic logic and sets. Neutrosophic set is a generalization of fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. It is a logic, in which each proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F). Also an element x in a Neutrosophic set (NS) X has a truth membership, an indeterminacy membership and a falsity membership, which are independent and which lies between $[0, 1]$, and sum of them is less than or equal to 3.

Definition 3.1. [34]. Consider a given set $X = \{x_1, x_2, \ldots, x_n\}$. **The Interval valued neutrosophic set** S on X is defined as follows:

$$S = \{< x_i, [T_S^L(x_i), T_S^U(x_i)], [I_S^L(x_i), I_S^U(x_i)], [F_S^L(x_i), F_S^U(x_i)] >: x_i \in X\}$$

satisfying the condition

$$0 \leq \sup T_S^U(x_i) + \sup I_S^U(x_i) + \sup F_S^U(x_i) \leq 3$$

where $[T_S^L(x_i), T_S^U(x_i)]$ represent the truth-membership function and similarly $[I_S^L(x_i), I_S^U(x_i)]$ and $[F_S^L(x_i), F_S^U(x_i)]$ are the indeterminacy-membership function and the falsity - membership function respectively.

Definition 3.2. [25]. For any two interval valued neutrosophic sets

$$N_1 = \{< x_i, [T_{N_1}^L(x_i), T_{N_1}^U(x_i)], [I_{N_1}^L(x_i), I_{N_1}^U(x_i)], [F_{N_1}^L(x_i), F_{N_1}^U(x_i)] >: x_i \in X\}$$

and

$$N_2 = \{< x_i, [T_{N_2}^L(x_i), T_{N_2}^U(x_i)], [I_{N_2}^L(x_i), I_{N_2}^U(x_i)], [F_{N_2}^L(x_i), F_{N_2}^U(x_i)] >: x_i \in X\}$$

$$\begin{aligned} N_1 \subseteq N_2 \quad \iff \; & T_{N_1}^L(x_i) \leq T_{N_2}^L(x_i), T_{N_1}^U(x_i) \leq T_{N_2}^U(x_i), I_{N_1}^L(x_i) \geq I_{N_2}^L(x_i), \\ & I_{N_1}^U(x_i) \geq I_{N_2}^U(x_i),\; F_{N_1}^L(x_i) \geq F_{N_2}^L(x_i), F_{N_1}^U(x_i) \geq F_{N_2}^U(x_i) \end{aligned} \tag{1}$$

Note that if $T_S^L(x_i) = T_S^U(x_i)$; $I_S^L(x_i) = I_S^U(x_i)$ and $F_S^L(x_i) = F_S^U(x_i)$ then IVNS S is reduced to the single valued neutrosophic set S.

The purpose of deneutrosophication is to convert a neutrosophic number obtained by any Neutrosophic multi criteria decision method to a single real number, crisp number, which can be easily used for comparison. Similar to defuzzification (Klir and Yuan, [24]), there are several deneutrosophication methods according to different applications (Bolturk and Kahraman, [4]) There is no conclusion as to which of the different deneutrosophication methods in the literature is more appropriate to use. This research propose the following deneutrosophication process for IVN numbers as a modified version of the deneutrosophication definition given in the PhD Thesis of H. Wang ([34]).

Definition 3.3. (The deneutrosophication) For any interval valued neutrosophic number

$$A = \{< [T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U] >\}$$

the deneutrosophication of A is defined by the formula

$$D(A) = c_1.AvT_A + c_2.(1 - AvF_A) + \frac{c_3}{2}.AvI_A + c_4.(1 - \frac{1}{2}.AvI_A) \quad (2)$$

where $0 \leq c_i, i = 1, 2, 3, 4$ and $\sum_i^4 c_i = 1, c_3 \neq c_4$ with

$$AvT_A = \frac{T_A^L + T_A^U}{2}, \; AvI_A = \frac{I_A^L + I_A^U}{2}, \; AvF_A = \frac{F_A^L + F_A^U}{2}$$

Here the AvT_A term gives the direct information about the truth degree. For this reason it is used directly in the formula. On the other hand the AvF_A and AvI_A terms gives the indirect information about the truth-degree, so
$(1 - AvF_A)$ and $(1 - AvI_A)$ are used in the formula. The formula has to consider two potential truth values implicitly represented by I_A with different weights c_3 and c_4. In general, $c_1 > c_2 > c_3, c_4$; where c_3 and c_4 should be decided according to the available information in the numerical examples.

4. I-Valued Neutrosophic AHP Method

The I- Valued Neutrosophic with AHP and sensitivity analysis consists of 10 basic steps.

Step 1. Determine the I- valued neutrosophic scale (see Table 1).

Step 2. Write the alternatives; pre flight, in flight, post flight and criteria (social distance, (SC) ; C2: hygiene (H) and C3: corona pandemy awereness and concern (CAC))

Step 3. Collect evaluations by using a questionnaire form including pairwise comparisons of criteria and alternatives. Write it using the following proposed I- valued neutrosophic evaluation scale given in Table 1.

Table 1. Linguistic terms and neutrosophicated importance weights.

Linguistic Term	Neutrosophic Sets
Equal importance	¡[0.50,0.50], [0.50,0.50], [0,50,0.50]¿
Weakly more importance	¡[0.50,0.60], [0.35,0.45], [0.40,0.50]¿
Moderate importance	¡[0.55,0.65], [0.30,0.40], [0.35,0.45]¿
Moderately more importance	¡[0.60,0.70], [0.25,0.35], [0.30,0.40]¿
Strong importance	¡[0.65,0.75], [0.20,0.30], [0.25,0.35]¿
Strongly more importance	¡[0.70,0.80], [0.15,0.25], [0.20,0.30]¿
Very strong importance	¡[0.75,0.85], [0.10,0.20], [0.15,0.25]¿
Very strongly more importance	¡[0.80,0.90], [0.05,0.10], [0.10,0.20]¿
Extreme importance	¡[0.90,0.95], [0.00,0.05], [0.05,0.15]¿
Extremely high importance	¡[0.95,1.00], [0.00, 0.00],[0.00,0.10]¿
Absolutely more importance	¡[1.00,1.00], [0.00,0.00], [0.00,0.00]¿

Step 4. Compute the sum of the columns of the pairwise comparison matrix and obtain the normalized values.

$$\dddot{A}_{ij} =< [\sum_{k=1}^{n} T_{kj}^{L}, T_{kj}^{U}], [\sum_{k=1}^{n} I_{kj}^{L}, I_{kj}^{U}], [\sum_{k=1}^{n} F_{kj}^{L}, F_{kj}^{U}] > \quad (3)$$

Where $j = 1, 2, 3, \ldots, n$.

Step 5. Evaluate the average of elements in the rows and obtain $A\dddot{W}_{ij}$.

Step 6. Evaluate the corresponding crisp numbers for each of the neutrosophic values by applying the deneutrosophication process. (Eq.3.2)

Step 7. According to the AHP calculations, write comparison matrices for alternatives for the criteria.

Step 8. Obtain neutrosophic weight vectors for alternatives by repeating the previous steps according to the criteria. Repeat the same process for obtaining the priority weights of the alternatives.

Step 9. Apply the deneutrosophication formula in Eq. (3.2) in order to obtain the crisp weights of alternatives.

Step 10. For testing the order of the alternatives that is given in step 9, evaluate the Euclidean - Tangent combine similarity measures for different values of lambda (namely $\lambda = 0.2, \lambda = 0.3$ and $\lambda = 0.6$) of alternatives and the ideal solution.

5. Empirical study using I- Valued Neutrosophic with AHP in airline transportation industry

Following the 10 steps given in the section above, the mentioned technique is given step by step.

In the first step of the research, the I-valued scale was determined using Table 1.
In the second step, a literature study was conducted first and the factors of airline service quality that may be associated with the pandemic were researched in order to determine the necessary alternatives and criteria. Afterwards, these identified factors of service quality were brought up for discussion by conducting a focus group study. The passengers who prefer the airline at least once a year before the pandemic broke out are included in the focus group study on a voluntary basis. Half of the passengers participating in the study were selected from among those who usually choose the airplane travel for leisure trip and the remaining half for business trip. The purpose of the focus group was explained to the participants before the study began. The study was maintained until the group participants reached a consensus with regard to attributes in the accompaniment of a specialist in focus group and also a moderator working in service marketing issues. 10 people, 6 of them are women and 4 of them are men, were included in the Focus group. Focus group participants are between 35 and 60 years old.

The attributes identified by focus group were subjected to a classification assessing by four experts and afterwards, each class of attributes was named as a dimension. Two of the experts are designated as airline directors. One of the directors were elected as woman and one as a man with ages in the interval from 40 to 60. Each manager have over 17 years of experience in this industry of the related areas. The other two people are public health professionals who work on the pandemic. The moderator is the moderator managing the passenger focus group. The dimensions of service quality that meet the most important concerns and needs of the pandemic for passengers were stated as social distance (SC), hygiene (H) and corona pandemy awereness and concern (CAC) as a result of this study. The experts have pointed out that the attributes of service quality arising from the concerns and needs of passengers regarding the pandemic may change for the pre-flight phases consisting of activities such as check-in, passenger boarding, lounge facilities, direction; in flight phases with factors such as veseating, lavatories, catering, entertainment, the cabin crews attitudes and post flight phases that are basically shaped by the passenger movement in due course of the these assessments. The selection of alternatives was completed and alternatives were determined as pre flight, in flight and post flight based upon these assesstments.

In the third stage of the research, The survey was constructed by the researchers. The whole airline service process was visualised as follows: pre-flight, in flight, and post-flight phases. The participants were asked to assess the significance of social distance, hygiene and corona pandemy awereness and concern attributes in accordance with three pre-flight, in-flight and post-flight alternatives. The sample of current passengers consisted of 402 passengers, taking into account a 95% confidence level and a 5% error margin (DeVaus, [14]). Of the passengers, 48% were aged between 21 and 30, 49% were male, and 53% had an undergraduate degree. IVN scale used in this part of the study is shown in Table 1.

Table 2. Pairwise comparison matrix for criteria

	SC	H	CAC
SC	¡[0.5,0.5], [0.5,0.5],[0.5,0.5]¿	¡[0.2,0.3],[0.15,0.25],[0.7,0.8]¿	¡[0.0, 0.1],[0.1,0.0],[0.95,1.0]¿
H	¡[0.7,0.8],[0.15,0.25],[0.2,0.3]¿	¡[0.5,0.5],[0.5,0.5],[0.5,0.5]¿	¡[0.3,0.4], [0.25,0.35],[0.6,0.7]¿
CAC	¡[0.95,1.0],[0.1,0.0],[0.0,0.1]¿	¡[0.6,0.7],[0.25,0.35],[0.3,0.4]¿	¡[0.5 , 0.5],[0.5,0.5],[0.5,0.5]¿

In the fourth step, the sum of the columns of the pairwise comparison matrix is computed by Eq.3.3 and the normalized values are obtained (Table 3-Table 4).

Table 3. The column sums of the pairwise comparison matrix

	SC						H						CAC					
	TI	Tu	II	Iu	FI	Fu	TI	Tu	II	Iu	FI	Fu	Tl	Tu	Il	Iu	Fl	Fu
SC	0,50	0,50	0,50	0,50	0,50	0,50	0,20	0,30	0,15	0,25	0,70	0.80	0,00	0,10	0,10	0,00	0.95	1.00
H	0,70	0,80	0,15	0,25	0,20	0,30	0,5	0,5	0,5	0,5	0,5	0.5	0,30	0.40	0,25	0.35	0.60	0.70
CAC	0,95	1.00	0,1	0,00	0,00	0,10	0,6	0.7	0.25	0.35	0.3	0.4	0.5	0,5	0,5	0,5	0.5	0.5
SUM	2.15	2.3	0.75	0.75	0.7	0.9	1.3	1.5	0,9	1.1	1.5	1.7	0.80	1.00	0,85	0,85	2.05	2.20

Table 4. The normalized values of the pairwise comparison matrix

	TI	Tu	II	Iu	FI	Fu	TI	Tu	II	Uu	FI	Fu	TI	Tu	II	Uu	FI	Fu
SC	0,217	0,217	0,667	0,667	0,556	0,556	0,133	0,2	0,136	0,227	0,412	0,471	0	0,1	0,118	0	0,432	0,455
H	0,304	0,348	0,2	0,333	0,222	0,333	0,333	0,333	0,455	0,455	0,294	0,294	0,3	0,4	0,294	0,412	0,273	0,318
CAC	0,413	0,435	0,133	0	0	0,111	0,4	0,467	0,227	0,318	0,176	0,235	0,5	0,5	0,588	0,588	0,227	0,227

In the fifth step, the weights in the rows are evaluated by using Eq.3 (Table 5). In the

Table 5. The weights of criteria

	Tl	Tu	Il	Iu	Fl	Fu
SC	0,117	0,172	0,307	0,298	0,466	0.494
H	0,313	0,360	0,316	0,400	0,263	0.315
CAC	0,438	0,467	0,316	0,302	0,135	0.191

sixth step, the corresponding crisp numbers for each of the neutrosophic values are evaluate by applying the deneutrosophication process given in Eq.3.2 (Table 6). It was found that the most important dimension for the passengers among three identified criteria was the social

Table 6. Criteria deneutrosophication and ranking

Criteria	Results
SC	0,329
H	0,466
CAC	0,548

distance dimension, the second significant criterion was hygiene, and the third was the corona pandemy awereness and concern criterion in this step.

In the seventh step, according to the AHP calculations,comparison matrices for alternatives according to the criteria are writen in Table 7. Afterwards, its pairwise comparison matrix was formed on the basis of the Interval-valued neutrosophic evaluation scale of alternatives depending upon the criteria in line with AHP calculations, and presented in Table 7.

Table 7. Pairwise comparison matrices for alternatives with respect to the criteria.

	SC						H						CAC					
	TI	Tu	II	Iu	FI	Fu	TI	Tu	II	Iu	FI	Fu	Tl	Tu	Il	Iu	Fl	Fu
Pre Flight	0,357	0,313	0,455	0,385	0,357	0,313	0,235	0,286	0,20	0,263	0,359	0.372	0,314	0,333	0,316	0,348	0,333	0,36
In Flight	0,25	0,281	0,273	0,308	0,393	0,406	0,588	0,476	0,667	0,526	0,256	0.233	0,40	0.41	0,158	0.217	0.19	0.24
Post Flight	0,393	0,406	0,273	0,308	0,25	0,281	0,176	0.238	0.133	0.211	0.385	0.395	0.286	0,256	0,526	0,435	0.476	0.4

In the Eight step, the previous items according to criterions are repeated and calculated weights vectors. Then the same items are repeated (Table 8). Table 8 gives the weights of Alternatives.

Table 8. The Weights of Alternatives

	Tl	Tu	Il	Iu	Fl	Fu
Pre Flight	0,302	0,311	0,323	0,332	0,350	0,348
In Flight	0,413	0,389	0,366	0,350	0,280	0,293
Post Flight	0,285	0,300	0,311	0,318	0,370	0,359

At this step, the formula in Eq. (1) is applied for obtaining the crisp weights as in Table 9. According to this order, in flight, pre flight and post flight were listed (Table 9).

Table 9. Alternative deneutrosophication and ranking

	Results
Pre Flight	0,434
In Flight	0,492
Post Flight	0,423

At the last step, for testing the order of the alternatives that is given in Table 9, the Euclidean-Tangent combine similarity measures for different values of lambda (namely $\lambda = 0.2, \lambda = 0.3$ and $\lambda = 0.6$) of alternatives and the ideal solution are evaluated (Table 10, Table 11 and Table 12). Similarity tests were performed and compared with an ideal solution in order to test the alternative ranking shown in Table 9. These results are presented in Table 10, Table 11 and Table 12. The reason for presenting as three separate tables is that the Lambda values in the similarity test are shown to be different, although there is no change in the ranking.

TABLE 10. Euclidean - Tangent Combine Similarity ($\lambda = 0,2$)

Similarity between	SC	H	CAC	Overall
Pre Flight & In Flight	0,948	0,868	0,935	0,917
In Flight & Post Flight	0,945	0,825	0,890	0,887
Pre Flight & Post Flight	0,948	0,974	0,944	0,955
Pre Flight & IDEAL	0,752	0,755	0,760	0,756
In Flight & IDEAL	0,745	0,762	0,813	0,773
Post Flight & IDEAL	0,793	0,744	0,715	0,751

TABLE 11. Euclidean - Tangent Combine Similarity ($\lambda = 0,3$)

Similarity between	SC	H	CAC	Overall
Pre Flight & In Flight	0,947	0,869	0,935	0,917
In Flight & Post Flight	0,945	0,823	0,890	0,886
Pre Flight & Post Flight	0,947	0,974	0,945	0,955
Pre Flight & IDEAL	0,752	0,755	0,760	0,756
In Flight & IDEAL	0,745	0,761	0,813	0,773
Post Flight & IDEAL	0,793	0,744	0,714	0,751

TABLE 12. Euclidean - Tangent Combine Similarity ($\lambda = 0,6$)

Similarity between	SC	H	SAC	Overall
Pre Flight & In Flight	0,935	0,871	0,934	0,913
In Flight & Post Flight	0,945	0,821	0,892	0,886
Pre Flight & Post Flight	0,935	0,974	0,947	0,952
Pre Flight & IDEAL	0,740	0,756	0,761	0,752
In Flight & IDEAL	0,746	0,760	0,814	0,773
Post Flight & IDEAL	0,794	0,745	0,713	0,751

6. Discussion and Conclusion

Two of the three criteria specified as the criteria of service quality with regard to the established pandemic appeared as the two most highlighted measures concerning the pandemic rather than the sectoral basis in the study. The third criterion can be evaluated in a more sectoral basis. The criteria of service quality identified in consequence of the study correspond to the study results of Chou and Lu ([11]). Hygiene and social distance are determined as the elements on which the passengers focus most in different studies in a similar way. Similarly, tangible cues referred to as "airline tangibles" by Ekiz et al. ([15]) emphasizes similar necessities.

Farooq ([16]) and Gudmundsson ([19]) wasrecognize with quality of interior and exterior equipments, catering service, comfortable seats and cleanliness (Ali et al., [2]). Seat space and Legroom and seat comfort attributes have been introduced in the studies of Chen and Chang ([9]), Nejati et al. ([28]), Chang and Yeh ([8]), Liou and Tzeng ([26]), Gupta ([20]) have been previously emphasized among airline service quality issues in relation to social distance. The emphasis on hygiene and cleanliness has been demonstrated in several studies except from pandemic context. (Chen and Chang, [9]; Chang and Yeh, [8]; Liou and Tzeng, [26]; Gilbert and Wong, [17]; Nejati et al., [28]; Chen and Chang, [9]; Jiang and Zhang, [22]; Gupta, [20]). In particular, it is often mentioned as cleanliness of seats. Baker et al. ([4]) and Wu et al. ([36]) also drew attention to the corona pandemy awereness and concerns in their studies. Babbar and Koufteros [3] attracted attention to the significance of level of concern in the context of service of airline quality. The ranking of significance of these criteria as in-flight, pre-flight and post-flight (De Neufville, [13]; Camilleri, [6]) does not match up with the results of the study of Namukasa [38]. It has been found out that the service quality in the pre - flight, in - flight and post - flight phases has equally importance. This non-overlapping situation is considered to arise from the concentrate on the three elements in the context of the pandemic.

Intense requests for cancelation and various government measures owing to COVID-19 pandemic, which the aviation industry has never seen before, have forced many airline companies to call billions of dollars in emergency assistance whilst others have directed their crew to take voluntary leave in order not to dismiss their crew. It is considered that the recovery of the sector is possible only with medium and long term planning, by constantly following the effects of the pandemic and by fulfilling all the undertakings of governments and financial institutions. Above all, the passengers must be persuaded to use the airline again. The service quality is regarded as a competitive marketing strategy, particularly in the airline industry as Andotra et al. [3] stated. It has significance that the airline companies spend less time, effort and charges on the relatively less important elements, by focusing on service quality elements which are most important to their customers . The presence of a substantive relationship between service quality and customer satisfaction is frequently emphasized in the related literature (Ali et al., [2]). In this sense, the results of this study consist of several significant assessments for the airline companies in order to regain and convince their passengers. The ranking of alternatives in all of the similarity tests conducted in order to test the reliability of Interval Neutrosophic AHP at the same time is completely compatible with the ranking made with Interval Neutrosophic AHP (Table 10-11-12). The post Flight status is always closer to the ideal when the similarity test is analyzed on the basis of criteria, in terms of Social distance criterion. The general ranking is not disrupted in the ranking of the Hygiene and Corona Pandemic Awareness and Concern Criteria.

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The neutrosophic integrals by partial fraction

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Abstract: The purpose of this article is to study the neutrosophic integrals by partial fraction, where the neutrosophic fraction function is defined, in addition, four cases of the neutrosophic proper rational function were discussed, also, integral of the neutrosophic improper rational functions were introduced.Where detailed examples were given to clarify each case.

Keywords: neutrosophic partial fraction; neutrosophic proper rational function; neutrosophic integrals; neutrosophic improper rational functions.

1. Introduction

As an alternative to the existing logics, Smarandache proposed the Neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3-13]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist, he defined the standard form of neutrosophic complex number, and found root index $n \geq 2$ of a neutrosophic real and complex number [2-4], studying the concept of the Neutrosophic probability [3-5], the Neutrosophic statistics [4][6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1-8]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [9]. Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side represented with triangular neutrosophic numbers [10]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [11-12]. Y.Alhasan studied the concepts of neutrosophic complex numbers, the general exponential form of a neutrosophic complex, the neutrosophic integrals and integration methods, and the neutrosophic integrals by parts [7-14-21-22]. On the other hand, M.Abdel-Basset presented study in the science of neutrosophic about an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number [15].

Also, neutrosophic sets played an important role in applied science such as health care, industry, and optimization [16-17-18-19].

Integration is important in human life, and one of its most important applications is the calculation of area, size and arc length. In our reality we find things that cannot be precisely defined, and that contain an indeterminacy part.

Paper consists of 4 sections. In 1th section, provides an introduction, in which neutrosophic science review has given. In 2th section, some definitions and theories of The neutrosophic integrals and are discussed. The 3th section frames neutrosophic integrals by partial fraction, in which four cases of the neutrosophic proper rational function were discussed, also, integral of the neutrosophic improper rational functions were introduced. In 4th section, a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic integration by substitution method [24]

Definition2.1.1

$Let\ \ f: D_f \subseteq R \to R_f \cup \{I\}$, to evaluate $\int f(x)dx$

Put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

By substitution, we get:

$$\int f(x)dx = \int f(u)\acute{g}(u)du$$

then we can directly integral it.

Theorme2.1.1:

If $\int f(x, I)dx = \varphi(x, I)\ \ then,$

$$\int f\big((a + bI)x + c + dI\big)\, dx = \left(\frac{1}{a} - \frac{b}{a(a + b)} I\right) \varphi\big((a + bI)x + c + dI\big) + C$$

where C is an indeterminate real constant, $a \neq 0\,, a \neq -b\ and\ b, c, d$ are real numbers, while I = indeterminacy.

Theorme2.1.2:

Let $f: D_f \subseteq R \to R_f \cup \{I\}$ then:

$$\int \frac{\acute{f}(x, I)}{f(x, I)} dx = ln|f(x, I)| + C$$

where C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while I = indeterminacy).

Theorme2.1.3:

Let $f: D_f \subseteq R \to R_f \cup \{I\}$, then:

$$\int \frac{\acute{f}(x, I)}{\sqrt{f(x, I)}} dx = 2\sqrt{f(x, I)} + C$$

where C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while I = indeterminacy).

Theorme2.1.4:

$f: D_f \subseteq R \to R_f \cup \{I\}$, then:

$$\int [f(x,I)]^n \acute{f}(x)\, dx = \frac{[f(x,I)]^{n+1}}{n+1} + C$$

Where n is any rational number. C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while I = indeterminacy).

2.2. Integrating products of neutrosophic trigonometric function [24]

I. $\int \sin^m(a+bI)x \cos^n(a+bI)x \ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following two cases:

- Case $n\ is$ odd:
  - Split of $\cos(a+bI)x$
  - Apply $\cos^2(a+bI)x = 1 - \sin^2(a+bI)x$
  - We substitution $u = \sin(a+bI)x$
- Case $m\ is$ odd:
  - Split of $\sin(a+bI)x$
  - Apply $\sin^2(a+bI)x = 1 - \cos^2(a+bI)x$
  - We substitution $u = \cos(a+bI)x$

II. $\int \tan^m(a+bI)x \sec^n(a+bI)x \ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following cases:

- Case $n\ is$ even:
  - Split of $\sec^2(a+bI)x$
  - Apply $\sec^2(a+bI)x = 1 + \tan^2(a+bI)x$
  - We substitution $u = \tan(a+bI)x$
- Case $m\ is$ odd:
  - Split of $\sec(a+bI)x \tan(a+bI)x$
  - Apply $\tan^2(a+bI)x = \sec^2(a+bI)x - 1$
  - We substitution $u = \sec(a+bI)x$
- Case m even and n odd:
  - Apply $\tan^2(a+bI)x = \sec^2(a+bI)x - 1$
  - We substitution $u = \sec(a+bI)x$ or $u = \tan(a+bI)x$, depending on the case.

III. $\int \cot^m(a+bI)x \csc^n(a+bI)x \ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following cases:

- Case $n\ is$ even:
  - Split of $\csc^2(a+bI)x$
  - Apply $\csc^2(a+bI)x = 1 + \cot^2(a+bI)x$
  - We substitution $u = \cot(a+bI)x$
- Case $m\ is$ odd:
  - Split of $\csc(a+bI)x \cot(a+bI)x$
  - Apply $\cot^2(a+bI)x = \csc^2(a+bI)x - 1$
  - We substitution $u = \csc(a+bI)x$

- Case m even and n odd:
  - Apply $\cot^2(a+bI)x = \csc^2(a+bI)x - 1$
  - We substitution $u = \csc(a+bI)x$ or $u = \cot(a+bI)x$, depending on the case.

2.3. Neutrosophic trigonometric identities [24]

1) $sin(a+bI)x\cos(c+dI)\,x = \frac{1}{2}\,[\sin(a+bI+c+dI)x + \sin(a+bI-c-dI)x]$

2) $cos(a+bI)x\sin(c+dI)\,x = \frac{1}{2}\,[\sin(a+bI+c+dI)x - \sin(a+bI-c-dI)x]$

3) $cos(a+bI)x\cos(c+dI)\,x = \frac{1}{2}\,[\cos(a+bI+c+dI)x + \cos(a+bI-c-dI)\,x]$

4) $sin(a+bI)x\sin(c+dI)\,x = \frac{-1}{2}\,[\cos(a+bI+c+dI)x - \cos(a+bI-c-dI)x]$

Where $a \neq c$ $(not\ zero)$ $and\ b, d$ are real numbers, while I = indeterminacy.

3. The neutrosophic integrals by partial fraction

Definition3.1
A polynomial whose coefficients (at least one of them containing I) are neutrosophic numbers is called neutrosophic real polynomials, and take the form:

$$P(x,I) = (a_0+b_0I) + (a_1+b_1I)x + (a_2+b_2I)x^2 + \cdots + (a_n+b_nI)x^n$$

Where $a_0, b_0, a_1, b_1, a_2, b_2 \ldots a_n, b_n$ are real number, I represent indeterminacy and n.is positive integer.

Definition3.2
Neutrosophic fraction function is a function which can be written in the form of:

$$f(x,I) = \frac{P(x,I)}{Q(x,I)}$$

Where $P(x,I)$, $Q(x,I)$ are neutrosophic real polynomials and $Q(x,I) \neq 0$, the numerator or denominator, at least, can be a neutrosophic real polynomials.

Example3.1:

1) $f(x,I) = \dfrac{(3+7I)x^3 + 4Ix - 2}{2Ix + 8 - 5I}$

2) $f(x,I) = \dfrac{(7+2I)x}{(3+7I)x^2 + 4Ix - 2}$

3) $f(x,I) = \dfrac{1}{(3+7I)x^2 + 4Ix - 2}$

Remark3.1

- If degree of $P(x,I)$ is less than degree of $Q(x,I)$, then: $f(x,I) = \frac{P(x,I)}{Q(x,I)}$ is an neutrosophic proper rational function.
- If degree of $P(x,I)$ is greater than degree of $Q(x,I)$, then: $f(x,I) = \frac{P(x,I)}{Q(x,I)}$ is an neutrosophic improper rational function.

3.1 Integral of the neutrosophic proper rational functions

3.1.2 There are four cases of the neutrosophic proper rational function

- state1: When the denominator can be expressed as the product of non-repeated linear factors.

Let $Q(x,I)=\left((a_1+b_1I)x+c_1+d_1I\right)\left((a_2+b_2I)x+c_2+d_2I\right)\dots\left((a_n+b_nI)x+c_n+d_nI\right)$, then we can write:

$$\frac{P(x,I)}{Q(x,I)}=\frac{A_1}{(a_1+b_1I)x+c_1+d_1I}+\frac{A_2}{(a_2+b_2I)x+c_2+d_2I}+\dots+\frac{A_n}{(a_n+b_nI)x+c_n+d_nI}$$

Where $A_1,A_2,\dots,A_n$ are constants whose values are to be determined.

- State2: When the denominator can be expressed as the product of repeated linear factors.

Let $Q(x,I)=\left((a+bI)x+c+dI\right)\left((a+bI)x+c+dI\right)\dots\left((a+bI)x+c+dI\right)$
$=\left((a+bI)x+c+dI\right)^n$, then we can write:

$$\frac{P(x,I)}{Q(x,I)}=\frac{A_1}{(a+bI)x+c+dI}+\frac{A_2}{\left((a+bI)x+c+dI\right)^2}+\dots+\frac{A_n}{\left((a+bI)x+c+dI\right)^n}$$

Where $A_1,A_2,\dots,A_n$ are constants whose values are to be determined.

- State3: When the denominator can be expressed as the product of repeated and non-repeated linear factors.

Let $Q(x,I)=\left((a_1+b_1I)x+c_1+d_1I\right)\left((a_2+b_2I)x+c_2+d_2I\right)\dots\left((a_n+b_nI)x+c_n+d_nI\right)\left((a+bI)x+c+dI\right)^m$, then we can write:

$$\frac{P(x,I)}{Q(x,I)}=\frac{A_1}{(a_1+b_1I)x+c_1+d_1I}+\frac{A_2}{(a_2+b_2I)x+c_2+d_2I}+\dots+\frac{A_n}{(a_n+b_nI)x+c_n+d_nI}+\frac{B_1}{(a+bI)x+c+dI}+\frac{B_2}{\left((a+bI)x+c+dI\right)^2}+\dots+\frac{B_m}{\left((a+bI)x+c+dI\right)^m}$$

Where $A_1,A_2,\dots,A_n,\ B_1,B_2,\dots,B_m$ are constants whose values are to be determined.

- State4: When the denominator can be expressed as the product of non-repeated quadratic factors which cannot be further factorized to linear factors.

Let $Q(x,I)=\left((a_1+b_1I)x^2+(c_1+d_1I)x+e_1+k_1I\right)\left((a_2+b_2I)x^2+(c_2+d_2I)x+e_2+k_2I\right)\dots\left((a_n+b_nI)x^2+(c_n+d_nI)x+e_n+k_nI\right)$, then we can write:

$$\frac{P(x,I)}{Q(x,I)}=\frac{A_1x+B_1}{(a_1+b_1I)x^2+(c_1+d_1I)x+e_1+k_1I}+\frac{A_2x+B_2}{(a_2+b_2I)x^2+(c_2+d_2I)x+e_2+k_2I}+\dots+\frac{A_nx+B_n}{(a_n+b_nI)x^2+(c_n+d_nI)x+e_n+k_nI}$$

Where $A_1,A_2,\dots,A_n,\ B_1,B_2,\dots,B_n$ are constants whose values are to be determined.

3.1.2 Algorithm for finding Integral of the neutrosophic proper rational functions

To evaluate $\int \frac{P(x,I)}{Q(x,I)}$ we follow the following steps:

1) Reformulate the form of the function in one of the previous four cases according to the form of the denominator as sum of neutrosophic partial fractions.
2) Integrate both sides.

Example3.1.1:

Evaluate:

$$\int \frac{dx}{(x+2-I)(x+4+2I)}$$

Solution:

$$\frac{1}{(x+2-I)(x+4+2I)} = \frac{A}{x+2-I} + \frac{B}{x+4+2I} \quad (*)$$

To find value of A, We multiply both sides $(*)$ by $(x+2-I)$:

$$\frac{1}{x+4+2I} = A + \frac{(x+2-I)B}{x+4+2I} \quad (1)$$

by substitution $x = -2+I$ in (1), we get:

$$A = \frac{1}{-2+I+4+2I} = \frac{1}{2+3I}$$

To find value of B, We multiply both sides $(*)$ by $(x+4+2I)$:

$$\frac{1}{x+2-I} = \frac{(x+4+2I)A}{x+2-I} + B \quad (2)$$

by substitution $x = -4-2I$ in (2), we get:

$$B = \frac{1}{-4-2I+2-I} = \frac{1}{-2-3I}$$

by substitution in (*) we get:

$$\frac{1}{(x+2-I)(x+4+2I)} = \frac{\frac{1}{2+3I}}{x+2-I} + \frac{\frac{-1}{2+3I}}{x+4+2I}$$

$$\Rightarrow \int \frac{dx}{(x+2-I)(x+4+2I)} dx = \int \left(\frac{\frac{1}{2+3I}}{x+2-I} + \frac{\frac{-1}{2+3I}}{x+4+2I} \right) dx$$

$$= \frac{1}{2+3I} ln|x+2-I| - \frac{1}{2+3I} ln|x+4+2I|$$

$$= \frac{1}{2+3I} (ln|x+2-I| - ln|x+4+2I|)$$

$$= \frac{1}{2+3I} ln\left|\frac{x+2-I}{x+4+2I}\right|$$

$$= \left(\frac{1}{2} - \frac{3}{10} I\right) ln\left|\frac{x+2-I}{x+4+2I}\right| + C$$

Example3.1.2:

Evaluate:

$$\int \frac{5+2Idx}{((2+I)x+2-I)((-1+2I)x+4+2I)}$$

Solution:

$$\frac{5+2I}{((2+I)x+2-I)((-1+2I)x+4+2I)} = \frac{A}{(2+I)x+2-I} + \frac{B}{(-1+2I)x+4+2I} \quad (*)$$

To find value of A, We multiply both sides $(*)$ by $((2+I)x+2-I)$:

$$\frac{5+2I}{(-1+2I)x+4+2I} = A + \frac{((2+I)x+2-I)B}{(-1+2I)x+4+2I} \quad (1)$$

by substitution $x = \frac{-2+I}{2+I} = -1 + \frac{2}{3} I$ in (1), we get:

$$A = \frac{5+2I}{(-1+2I)(-1+\frac{2}{3}I)+4+2I} = \frac{5+2I}{5+\frac{2}{3}I} = 1 + \frac{4}{17} I$$

To find value of B, We multiply both sides $(*)$ by $((-1+2I)x+4+2I)$:

$$\frac{5+2I}{(2+I)x+2-I} = \frac{((-1+2I)x+4+2I)A}{(2+I)x+2-I} + B \quad (2)$$

by substitution $x = \frac{-4-2I}{-1+2I} = 4 - 10I$ in (2), we get:

$$B = \frac{5+2I}{(2+I)(4-10I)+2-I} = \frac{5+2I}{10-25I} = \frac{1}{2} - \frac{29}{30} I$$

by substitution in (*) we get:

$$\frac{5+2I}{((2+I)x+2-I)((-1+2I)x+4+2I)} = \frac{1+\frac{4}{17}I}{(2+I)x+2-I} + \frac{\frac{1}{2}-\frac{29}{30}I}{(-1+2I)x+4+2I}$$

$$\Rightarrow \quad \int \frac{5+2Idx}{((2+I)x+2-I)((-1+2I)x+4+2I)} = \int \left(\frac{1+\frac{4}{17}I}{(2+I)x+2-I} + \frac{\frac{1}{2}-\frac{29}{30}I}{(-1+2I)x+4+2I}\right) dx$$

$$= \frac{1+\frac{4}{17}I}{2+I} ln|(2+I)x+2-I| + \frac{\frac{1}{2}-\frac{29}{30}I}{-1+2I} ln|(-1+2I)x+4+2I|$$

$$= \left(\frac{1}{2} - \frac{3}{2} I\right) ln|(2+I)x+2-I| + \left(-\frac{1}{2} + \frac{1}{30} I\right) ln|(-1+2I)x+4+2I|$$

Example3.1.3:

Evaluate:

$$\int \frac{3x-5+4I}{(x+1+I)^2(x+3-2I)}dx$$

Solution:

$$\frac{3x-5+4I}{(x+1+I)^2(x+3-2I)} = \frac{A}{x+1+I} + \frac{B}{(x+1+I)^2} + \frac{D}{x+3-2I} \quad (*)$$

To find value of C, We multiply both sides $(*)$ by $(x+3-2I)$:

$$\frac{3x-5+4I}{(x+1+I)^2} = \frac{(x+3-2I)A}{x+1+I} + \frac{(x+3-2I)B}{(x+1+I)^2} + D \quad (1)$$

by substitution $x = -3+2I$ in (1), we get:

$$D = \frac{-9+6I-5+4I}{(3+2I+1+I)^2} = \frac{-14+10I}{(-2+3I)^2}$$

$$= \frac{-14+10I}{4-12I+9I} \Rightarrow \quad D = \frac{-14+10I}{4-3I}$$

To find value of B, We multiply both sides $(*)$ by $(x+1+I)^2$:

$$\frac{3x-5+4I}{x+3-2I} = \frac{(x+1+I)^2 A}{x+1+I} + B + \frac{(x+1+I)^2 C}{x+3-2I} \quad (2)$$

by substitution $x = -1-I$ in (2), we get:

$$B = \frac{-3-3I-5+4I}{-1-I+3-2I} = \frac{-8+I}{2-3I}$$

To find value of A, we substitute value of B, D and any value of x so that it does not nullify the denominator in (*), let it be $x = 0$, we get:

$$\frac{-5+4I}{(1+I)^2(3-2I)} = \frac{A}{1+I} + \frac{\frac{-8+I}{2-3I}}{(1+I)^2} + \frac{\frac{-14+10I}{4-3I}}{3-2I}$$

$$\frac{-5+4I}{3+I} = \frac{A}{1+I} + \frac{-8+I}{2-6I} + \frac{14+10I}{12-11I}$$

$$\frac{A}{1+I} = \frac{7}{2} - \frac{3}{2}I$$

$$A = (1+I)\left(\frac{7}{2} - \frac{3}{2}I\right) = \frac{7}{2} - \frac{1}{2}I$$

by substitution in (*) we get:

$$\frac{3x-5+4I}{(x+1+I)^2(x+3-2I)} = \frac{\frac{7}{2}-\frac{1}{2}I}{x+1+I} + \frac{\frac{-8+I}{2-3I}}{(x+1+I)^2} + \frac{\frac{-14+10I}{4-3I}}{x+3-2I}$$

$$\Rightarrow \int \frac{3x-5+4I}{(x+1+I)^2(x+3-2I)}dx = \int \left(\frac{\frac{7}{2}-\frac{1}{2}I}{x+1+I} + \frac{\frac{-8+I}{2-3I}}{(x+1+I)^2} + \frac{\frac{-14+10I}{4-3I}}{x+3-2I} \right) dx$$

$$= \left(\frac{7}{2} - \frac{1}{2}I\right) ln|x + 1 + I| + \frac{-8 + I}{2 - 3I} . \frac{-1}{x + 1 + I} + \frac{-14 + 10I}{4 - 3I} ln|x + 3 - 2I|$$

$$= \left(\frac{7}{2} - \frac{1}{2}I\right) ln|x + 1 + I| + (4 - 11I). \frac{-1}{x + 1 + I} - \left(\frac{7}{2} - \frac{1}{2}I\right) ln|x + 3 - 2I|$$

$$= \left(\frac{7}{2} - \frac{1}{2}I\right) ln\left|\frac{x + 1 + I}{x + 3 - 2I}\right| + (4 - 11I). \frac{-1}{x + 1 + I} + C$$

Example3.1.4:

Evaluate:

$$\int \frac{5I}{x^2 - 4 + 3I} dx$$

Solution:
to find the denominator factors

$$x^2 - 4 + 3I = x^2 - \left(\sqrt{4 - 3I}\right)^2$$

Let's find $\sqrt{4 - 3I}$

$$\sqrt{4 - 3I} = \alpha + \beta I$$
$$4 - 3I = \alpha^2 + 2\alpha\beta I + \beta^2 I$$

$$4 - 3I = \alpha^2 + (2\alpha\beta + \beta^2)I$$

then:

$$\begin{cases} \alpha^2 = 4 \\ 2\alpha\beta + \beta^2 = -3 \end{cases}$$

$$\begin{cases} \alpha = \pm 2 \\ \alpha^2 + 2\alpha\beta + 3 = 0 \end{cases}$$

Find β:

- When $\alpha = 2 \quad \Rightarrow \beta^2 + 4\beta + 3 = 0$

$$(\beta + 3)(\beta + 1) = 0 \quad \Rightarrow \beta = -3 , \beta = -1$$

$$(2, -3), (2, -1)$$

- When $\alpha = -2 \quad \Rightarrow \beta^2 - 4\beta + 3 = 0$

$$(\beta - 3)(\beta - 1) = 0 \quad \Rightarrow \beta = 3 , \beta = 1$$

$$(-2,3), (-2,1)$$

Thus, the denominator factors can be written in two cases:
Case1:

$$x^2 - 4 + 3I = (x - 2 + 3I)(x + 2 - 3I)$$

$$\frac{5I}{x^2 - 4 + 3I} = \frac{5I}{(x - 2 + 3I)(x + 2 - 3I)}$$

$$\frac{5I}{(x - 2 + 3I)(x + 2 - 3I)} = \frac{A}{x - 2 + 3I} + \frac{B}{x + 2 - 3I} \quad (*)$$

To find value of A, We multiply both sides $(*)$ by $(x - 2 + 3I)$:

$$\frac{5I}{x+2-3I}=A+\frac{(x-2+3I)B}{x+2-3I} \qquad (1)$$

by substitution $x=2-3I$ in (1), we get:

$$A=\frac{5I}{2-3I+2-3I}=\frac{I5}{4-6I}$$

To find value of A, We multiply both sides $(*)$ by $(x+2-3I)$:

$$\frac{5I}{x-2+3I}=\frac{(x+2-3I)A}{x-2+3I}+B \qquad (2)$$

by substitution $x=-2+3I$ in (2), we get:

$$B=\frac{5I}{-2+3I-2+3I}=\frac{I5}{-4+6I}=\frac{-I5}{4-6I}$$

by substitution in (*) we get:

$$\frac{5I}{(x-2+3I)(x+2-3I)}=\frac{\frac{I5}{4-6I}}{x-2+3I}+\frac{\frac{-I5}{4-6I}}{x+2-3I}$$

$$\Rightarrow\int\frac{5I}{(x-2+3I)(x+2-3I)}dx=\int\left(\frac{\frac{I5}{4-6I}}{x-2+3I}+\frac{\frac{-I5}{4-6I}}{x+2-3I}\right)dx$$

$$=\frac{I5}{4-6I}ln|x-2+3I|-\frac{I5}{4-6I}ln|x+2-3I|$$

$$=\frac{I5}{4-6I}(ln|x-2+3I|-ln|x+2-3I|)$$

$$=\frac{I5}{4-6I}ln\left|\frac{x-2+3I}{x+2-3I}\right|$$

$$=\left(-\frac{5}{2}I\right)ln\left|\frac{x-2+3I}{x+2-3I}\right|+C$$

Case2:

$$x^2-4+3I=(x-2+I)(x+2-I)$$

$$\frac{5I}{x^2-4+3I}=\frac{5I}{(x-2+I)(x+2-I)}$$

$$\frac{5I}{(x-2+I)(x+2-I)}=\frac{A}{x-2+I}+\frac{B}{x+2-I} \qquad (*)'$$

To find value of A, We multiply both sides $(*)$ by $(x-2+I)$:

$$\frac{5I}{x+2-I} = A + \frac{(x-2+I)B}{x+2-I} \quad (3)$$

by substitution $x = 2 - I$ in (3), we get:

$$A = \frac{5I}{2-I+2-I} = \frac{I5}{4-2I}$$

To find value of A, We multiply both sides $(*)$ by $(x+2-I)$:

$$\frac{5I}{x-2+I} = \frac{(x+2-I)A}{x-2+I} + B \quad (4)$$

by substitution $x = -2 + I$ in (4), we get:

$$B = \frac{5I}{-2+I-2+I} = \frac{I5}{-4+2I} = \frac{-I5}{4-2I}$$

by substitution in $(*)'$, we get:

$$\frac{5I}{(x-2+I)(x+2-I)} = \frac{\frac{I5}{4-2I}}{x-2+I} + \frac{\frac{-I5}{4-2I}}{x+2-I}$$

$$\Rightarrow \int \frac{5I}{(x-2+I)(x+2-I)}dx = \int \left(\frac{\frac{I5}{4-2I}}{x-2+I} + \frac{\frac{-I5}{4-2I}}{x+2-I} \right) dx$$

$$= \frac{I5}{4-2I} ln|x-2+I| - \frac{I5}{4-2I} ln|x+2-I|$$

$$= \frac{I5}{4-2I}(ln|x-2+I| - ln|x+2-I|)$$

$$= \frac{I5}{4-2I} ln\left|\frac{x-2+I}{x+2-I}\right|$$

$$= \left(\frac{5}{2}I\right) ln\left|\frac{x-2+I}{x+2-I}\right| + C$$

Hence:

$$\int \frac{5I}{x^2-4+3I}dx = \begin{cases} \left(\left(\frac{1}{4} - \frac{5}{2}I\right) ln\left|\frac{x-2+3I}{x+2-3I}\right| + C \right. \\ \left(\frac{5}{4} + \frac{5}{2}I\right) ln\left|\frac{x-2+I}{x+2-I}\right| + C \end{cases}$$

Example3.1.5:

Evaluate:

$$\int \frac{1+2I}{x^2+(4-I)x+2I}dx$$

Solution:

to find the denominator factors, we write it as an equation:

$$x^2 + (4 - I)x + 2I = 0$$
$$\Delta = (4 - I)^2 - 8I = 16 - 15I$$

$$x = \frac{-(4 - I) \pm \sqrt{16 - 15I}}{2} \qquad (**)$$

$$\sqrt{16 - 15I} = \alpha + \beta I$$

$$16 - 15I = \alpha^2 + 2\alpha\beta I + \beta^2 I$$

$$16 - 15I = \alpha^2 + (2\alpha\beta + \beta^2)I$$

then:

$$\begin{cases} \alpha^2 = 16 \\ 2\alpha\beta + \beta^2 = -15 \end{cases}$$

$$\begin{cases} \alpha = \pm 4 \\ \beta^2 + 2\alpha\beta + 15 = 0 \end{cases}$$

Find β:

- When $\alpha = 4 \quad \Rightarrow \beta^2 + 8\beta + 15 = 0$

$$(\beta + 3)(\beta + 5) \quad \Rightarrow \beta = -3\,, \beta = -5$$

$$(4, -3), (4, -5)$$

- When $\alpha = -4 \quad \Rightarrow \beta^2 - 8\beta + 15 = 0$

$$(\beta - 3)(\beta - 5) \quad \Rightarrow \beta = 3\,, \beta = 5$$

$$(-4,3), (-4,5)$$

Then:

$$(\alpha, \beta) = (4, -3), (4, -5), (-4,3), (-4,5)$$

$$\sqrt{16 - 15I} = 4 - 3I \; or \; 4 - 5I \; or \; -4 + 3I \; or \; -4 + 5I$$

We can note $4 - 3I$ and $-4 + 3I$ give the same values for x. Similarly, $4 - 5I$ and $-4 + 5I$.

So, we can now to find x in (**):

$$\begin{cases} x_1 = \dfrac{-(4 - I) + 4 - 3I}{2} = -I \\ x_2 = \dfrac{-(4 - I) - 4 + 3I}{2} = -4 + 2I \end{cases}$$

$$\begin{cases} x_3 = \dfrac{-(4 - I) + 4 - 5I}{2} = -2I \\ x_4 = \dfrac{-(4 - I) - 4 + 5I}{2} = -4 + 3I \end{cases}$$

Thus, the denominator factors can be written in two cases:

Case1:

$$x^2 + (4 - I)x + 2I = (x + I)(x + 4 - 2I)$$

$$\frac{1 + 2I}{x^2 + (4 - I)x + 2I} = \frac{1 + 2I}{(x + I)(x + 4 - 2I)}$$

$$\frac{1+2I}{(x+I)(x+4-2I)}=\frac{A}{x+I}+\frac{B}{x+4-2I} \quad (*)$$

To find value of A, We multiply both sides $(*)$ by $(x+I)$:

$$\frac{1+2I}{x+4-2I}=A+\frac{(x+I)B}{x+4-2I} \quad (1)$$

by substitution $x=-I$ in (1), we get:

$$A=\frac{1+2I}{-I+4-2I}=\frac{1+2I}{4-3I}$$

To find value of A, We multiply both sides $(*)$ by $(x+4-2I)$:

$$\frac{1+2I}{x+I}=\frac{(x+4-2I)A}{x+I}+B \quad (2)$$

by substitution $x=-4+2I$ in (2), we get:

$$B=\frac{1+2I}{-4+2I+I}=\frac{1+2I}{-4+3I}$$

by substitution in (*) we get:

$$\frac{1+2I}{(x+I)(x+4-2I)}=\frac{\frac{1+2I}{4-3I}}{x+I}+\frac{\frac{1+2I}{-4+3I}}{x+4-2I}$$

$$\Rightarrow \int \frac{1+2I}{(x+I)(x+4-2I)}dx=\int\left(\frac{\frac{1+2I}{4-3I}}{x+I}+\frac{\frac{1+2I}{-4+3I}}{x+4-2I}\right)dx$$

$$=\frac{1+2I}{4-3I}ln|x+I|-\frac{1+2I}{4-3I}ln|x+4-2I|$$

$$=\frac{1+2I}{4-3I}(ln|x+I|-ln|x+4-2I|)$$

$$=\frac{1+2I}{4-3I}ln\left|\frac{x+I}{x+4-2I}\right|$$

$$=\left(\frac{1}{4}-\frac{11}{4}I\right)ln\left|\frac{x+I}{x+4-2I}\right|+C$$

Case2:

$$x^2+(4-I)x+2I=(x+2I)(x+4-3I)$$

$$\frac{1+2I}{x^2+(4-I)x+2I}=\frac{1+2I}{(x+2I)(x+4-3I)}$$

$$\frac{1+2I}{(x+2I)(x+4-3I)}=\frac{A}{x+2I}+\frac{B}{x+4-3I} \quad (*)$$

To find value of A, We multiply both sides $(*)$ by $(x+2I)$:

$$\frac{1+2I}{x+4-3I}=A+\frac{(x+2I)B}{x+4-3I} \quad (1)$$

by substitution $x=-2I$ in (1), we get:

$$A=\frac{1+2I}{-2I+4-3I}=\frac{1+2I}{4-5I}$$

To find value of A, We multiply both sides $(*)$ by $(x+4-3I)$:

$$\frac{1+2I}{x+2I}=\frac{(x+4-3I)A}{x+2I}+B \quad (2)$$

by substitution $x=-4+3I$ in (2), we get:

$$B=\frac{1+2I}{-4+3I+2I}=\frac{1+2I}{-4+5I}$$

by substitution in (*) we get:

$$\frac{1+2I}{(x+2I)(x+4-3I)}=\frac{\frac{1+2I}{4-3I}}{x+2I}+\frac{\frac{1+2I}{-4+3I}}{x+4-3I}$$

$$\Rightarrow \int \frac{1+2I}{(x+2I)(x+4-3I)}dx=\int\left(\frac{\frac{1+2I}{4-5I}}{x+2I}+\frac{\frac{1+2I}{-4+5I}}{x+4-3I}\right)dx$$

$$=\frac{1+2I}{4-5I}ln|x+2I|-\frac{1+2I}{4-5I}ln|x+4-3I|$$

$$=\frac{1+2I}{4-5I}(ln|x+2I|-ln|x+4-3I|)$$

$$=\frac{1+2I}{4-5I}ln\left|\frac{x+2I}{x+4-3I}\right|$$

$$=\left(\frac{1}{4}-\frac{13}{4}I\right)ln\left|\frac{x+2I}{x+4-3I}\right|+C$$

Hence:

$$\int \frac{1+2I}{x^2+(4-I)x+2I}dx=\begin{cases}\left(\frac{1}{4}-\frac{11}{4}I\right)ln\left|\frac{x+I}{x+4-2I}\right|+C\\ \left(\frac{1}{4}-\frac{13}{4}I\right)ln\left|\frac{x+2I}{x+4-3I}\right|+C\end{cases}$$

Example3.1.6:

Evaluate:

$$\int \frac{3I}{(x-2+3I)(x^2+1+I)}dx$$

Solution:

We note that (x^2+1+I) cannot be analyzing, because:

$$x^2+1+I = x^2 - \left(\sqrt{-1-I}\right)^2$$

Let's find $\sqrt{-1-I}$

$$\sqrt{-1-I} = \alpha + \beta I$$

$$-1-I = \alpha^2 + 2\alpha\beta I + \beta^2 I$$

$$-1-I = \alpha^2 + (2\alpha\beta + \beta^2)I$$

then:

$$\alpha^2 = -1 \text{ (impossible in real number)}$$

So:

$$\frac{3I}{(x-2+3I)(x^2+1+I)} = \frac{A}{x-2+3I} + \frac{Bx+D}{x^2+1+I} \quad (*)$$

To find value of A, We multiply both sides $(*)$ by $(x-2+3I)$:

$$\frac{5I}{x-2+3I} = A + \frac{(x-2+3I)(Bx+C)}{x-2+3I} \qquad (1)$$

by substitution $x = 2-3I$ in (1), we get:

$$A = \frac{5I}{2-3I+2-3I} = \frac{5I}{4-6I}$$

To find value of B, We multiply both sides $(*)$ by x:

$$\frac{3Ix}{(x-2+3I)(x^2+1+I)} = \frac{Ax}{x-2+3I} + \frac{Bx^2+D}{x^2+1+I} \qquad (2)$$

By take limit both sides in (2), when $x \longrightarrow \infty$, we get:

$$0 = A + B \implies B = -A = \frac{-5I}{4-6I}$$

To find value of D, we substitute value of A, B and let be $x = 0$, in (*), we get:

$$\frac{3I}{(0-2+3I)(0^2+1+I)} = \frac{\frac{5I}{4-6I}}{0-2+3I} + \frac{\frac{-5I}{4-6I}(0)+D}{0^2+1+I} \implies D = 3I$$

$$\frac{3I}{-2+4I} = \frac{5I}{-4+6I} + \frac{D}{1+I}$$

$$\frac{D}{1+I} = \frac{-7}{8} + 4I \implies D = \frac{-7}{8} + \frac{57}{8}I$$

by substitution in (*) we get:

$$\frac{3I}{(x-2+3I)(x^2+1+I)}=\frac{\frac{5I}{4-6I}}{x-2+3I}+\frac{\frac{-5I}{4-6I}x+\frac{-7}{8}+\frac{57}{8}I}{x^2+1+I}$$

$$\Rightarrow\int\frac{3I}{(x-2+3I)(x^2+1+I)}dx=\int\left(\frac{\frac{5I}{4-6I}}{x-2+3I}+\frac{\frac{-5I}{4-6I}x+\frac{-7}{8}+\frac{57}{8}I}{x^2+1+I}\right)dx$$

$$=\int\frac{\frac{5I}{4-6I}}{x-2+3I}dx+\int\frac{\frac{-5I}{4-6I}x}{x^2+1+I}dx+\int\frac{\frac{-7}{8}+\frac{57}{8}I}{x^2+1+I}dx$$

$$=\frac{I5}{4-6I}ln|x-2+3I|-\frac{I5}{8-12I}ln|x^2+1+I|+\int\frac{\frac{-7}{8}+\frac{57}{8}I}{x^2+1+I}dx\quad(*)'$$

Let's now find:

$$\int\frac{\frac{-7}{8}+\frac{57}{8}I}{x^2+1+I}dx$$

$$x^2+9+7I=x^2+\left(\sqrt{1+I}\right)^2$$

Let's find $\sqrt{1+I}$

$$\sqrt{1+I}=\alpha+\beta I$$
$$1+I=\alpha^2+2\alpha\beta I+\beta^2 I$$

$$1+I=\alpha^2+(2\alpha\beta+\beta^2)I$$

then:

$$\begin{cases}\alpha^2=1\\2\alpha\beta+\beta^2=1\end{cases}$$

$$\begin{cases}\alpha=\pm1\\\beta^2+2\alpha\beta-1=0\end{cases}$$

Find β:

- When $\alpha=1\quad\Rightarrow\beta^2+2\beta-1=0$

$$\Rightarrow\ \beta=-1+\sqrt{2},\beta=-1-\sqrt{2}$$

$$\left(1,-1+\sqrt{2}\right),\left(1,-1-\sqrt{2}\right)$$

- When $\alpha=-1\quad\Rightarrow\beta^2-2\beta-1=0$

$$\Rightarrow\ \beta=1+\sqrt{2},\beta=1-\sqrt{2}$$

$$\left(1,1+\sqrt{2}\right),\left(1,1-\sqrt{2}\right)$$

$$(\alpha,\beta)=\left(1,-1+\sqrt{2}\right),\left(1,-1-\sqrt{2}\right),\left(1,1+\sqrt{2}\right),\left(1,1-\sqrt{2}\right)$$

$$\sqrt{1+I}=1+\left(-1+\sqrt{2}\right)I\ or\ 1+\left(-1-\sqrt{2}\right)I\ or-1+\left(1+\sqrt{2}\right)I\ or\ 1+(1-\sqrt{2})I$$

Thus, the denominator factors can be written in two cases:
Case1:

$$x^2+9+7I=x^2+\left(1+\left(-1+\sqrt{2}\right)I\right)^2$$

$$\int \frac{4+I}{x^2+9+7I}dx = \int \frac{4+I}{x^2+\left(1+\left(-1+\sqrt{2}\right)I\right)^2}dx$$

$$= \left(\frac{4+I}{1+\left(-1+\sqrt{2}\right)I}\right)tan^{-1}\left(\frac{x}{1+\left(-1+\sqrt{2}\right)I}\right)+C$$

$$= \left(4+\left(\frac{5-4\sqrt{2}}{\sqrt{2}}\right)I\right)tan^{-1}\left(\left(1+\left(\frac{-1+\sqrt{2}}{\sqrt{2}}\right)I\right)x\right)+C$$

$$= \left(4-\left(\frac{5\sqrt{2}}{2}-4\right)I\right)tan^{-1}\left(\left(1+\left(\frac{-\sqrt{2}}{2}+1\right)I\right)x\right)+C$$

Case2:

$$x^2-4+3I = x^2+\left(1+\left(-1-\sqrt{2}\right)I\right)^2$$

$$\int \frac{4+I}{x^2+9+7I}dx = \int \frac{4+I}{x^2+\left(1+\left(-1-\sqrt{2}\right)I\right)^2}dx$$

$$= \left(\frac{4+I}{1+\left(-1-\sqrt{2}\right)I}\right)tan^{-1}\left(\frac{x}{1+\left(-1-\sqrt{2}\right)I}\right)+C$$

$$= \left(4-\left(\frac{5+4\sqrt{2}}{\sqrt{2}}\right)I\right)tan^{-1}\left(\left(1+\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)I\right)x\right)+C$$

$$= \left(4-\left(\frac{5\sqrt{2}}{2}+4\right)I\right)tan^{-1}\left(\left(1+\left(\frac{\sqrt{2}}{2}+1\right)I\right)x\right)+C$$

Hence:

$$\int \frac{4+I}{x^2+9+7I}dx = \begin{cases} \left(4-\left(\frac{5\sqrt{2}}{2}-4\right)I\right)tan^{-1}\left(\left(1+\left(\frac{-\sqrt{2}}{2}+1\right)I\right)x\right)+C \\ \left(4-\left(\frac{5\sqrt{2}}{2}+4\right)I\right)tan^{-1}\left(\left(1+\left(\frac{\sqrt{2}}{2}+1\right)I\right)x\right)+C \end{cases}$$

by substitution in $(*)'$, we get:

$$\int \frac{4+I}{x^2+9+7I}dx =$$

$$\begin{cases} \frac{I5}{4-6I}ln|x-2+3I| - \frac{I5}{8-12I}ln|x^2+1+I| + \left(4-\left(\frac{5\sqrt{2}}{2}-4\right)I\right)tan^{-1}\left(\left(1+\left(\frac{-\sqrt{2}}{2}+1\right)I\right)x\right)+C \\ \frac{I5}{4-6I}ln|x-2+3I| - \frac{I5}{8-12I}ln|x^2+1+I| + \left(4-\left(\frac{5\sqrt{2}}{2}+4\right)I\right)tan^{-1}\left(\left(1+\left(\frac{\sqrt{2}}{2}+1\right)I\right)x\right)+C \end{cases}$$

Result3.1:

When decomposing the neutrosophic function into factors, it can give us more than one analysis, and thus we get more than one result in the case the integral of the neutrosophic rational functions, as example3.1.4 and example3.1.5

3.2 Integral of the neutrosophic improper rational functions

If the degree of the numerator is greater than the degree of the denominator, then we use long division method or using synthetic division method to facilitate the integration process.

Example3.2.1:

Evaluate:

$$\int \frac{x^3+(3+2I)x^2+(-5+I)x+8-4I}{x-1-2I}dx$$

Solution:

By using synthetic division method, we get:

$1+2I$	1	$3+2I$	$-5+I$	$8-4I$
		$1+2I$	$4+20I$	$-1+84I$
	1	$4+4I$	$-1+21I$	$-1+80I$

Then:

$$\frac{x^3+(3+2I)x^2+(-5+I)x+8-4I}{x-1-2I}=x^2+(4+4I)x+(-1+21I)+\frac{-1+80I}{x-1-2I}$$

$$\Rightarrow \int \frac{x^3+(3+2I)x^2+(-5+I)x+8-4I}{x-1-2I}dx=\int \left(x^2+(4+4I)x-1+21I+\frac{-1+80I}{x-1-2I}\right)dx$$

$$=\frac{x^3}{3}+(2+2I)x^2+(-1+21I)x+(-1+80I)ln|x-1-2I|+C$$

Example3.2.2:

Evaluate:

$$\int \frac{(1+I)x^2+(2-3I)x+4-5I}{x-2-7I}dx$$

Solution:

By using synthetic division method, we get:

$2+7I$	$1+I$	$2-3I$	$4-5I$
		$2+11I$	$8-44I$
	$1+I$	$4-8I$	$12-49I$

Then:

$$\frac{(1+I)x^2+(2-3I)x+4-5I}{x-2-7I}=(1+I)x+(4-8I)+\frac{12-49I}{x-2-7I}$$

$$\Rightarrow \int \frac{(1+I)x^2+(2-3I)x+4-5I}{x-2-7I}dx=\int \left((1+I)x+(4-8I)+\frac{12-49I}{x-2-7I}\right)dx$$

$$=\left(\frac{1}{2}+\frac{1}{2}I\right)x^2+(4-8I)x+(12-49I)ln|x-2-7I|+C$$

4. Conclusions

This paper is an extension of the papers I presented in the field of neutrosophic integrals. Integrals are important in our life, as they facilitate many mathematical operations in our reality, and this is what led us to study the neutrosophic integrals by partial fraction, and I concluded that more than one result can be obtained in the case of integration of the neutrosophic fraction function. In addition, this paper is considered an introduction to the applications in neutrosophic integrals.

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Tensor Product of Neutrosophic submodules of an R-module

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Abstract. In this paper, we develop a framework for tensor product with imprecise and indeterminate bounds as a neutrosophic submodule of an R-modules. The fundamental goal of this study is to extend the conventional tensor product in contemporary algebra to the most generalized domain of neutrosophic set algebraic structures. We discuss the construction of tensor product in neutrosophic submodules as a quotient space in this study and derives the universal uniqueness property of tensor product in neutrosophic domain .

Keywords: Neutrosophic set; Neutrosophic homomorphism; Direct producr;Cartesian product of neutrosophic set;Neutrosophic R-bi additive ; Neutrosophic tensor product

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1. Introduction

Algebra is a vital branch of Mathematics. The gist of Algebra lies in construction of fundamental mathematical structures and identification of relations between mathematical ideas. The group representation theory proposed by Frobenius [16] has a major role in advanced algebra and emphazises on the detailed study of symmetries in nature. The study of modules and group representation are inter-related and has several applications in different branches of physics and chemistry. Several researchers have looked into the algebraic structure underlying uncertainty in pure mathematics. When the novel set theories evolved as a result of the works of Zadeh [35], the concept of modules also underwent the subsequent tranformation. The major inventions in this direction include the research contributions of Negoita and Ralescue [21] and Mashinchi and Zahedi [17, 18] towards fuzzy modules. In 1986, Atanassov [2]came up with the intuitionistic version of fuzzy sets. In 2011, P. Isaac, P.P.John [14] characterised an intuitionistic fuzzy submodule

The notion of neutrosophy originally appeared in philosophy [26], and then as a mathematical tool. In 1995, Smarandache [19, 23] invented the neutrosophic set with the main goal of translating set theory into the the real world by bridging the gap with theoretical certainities and practical uncertainities. The intervention of neutrosophic sets to algebraic structures by Smarandache [34] paved the way to many innovative concepts in algebraic research field. Kandasamy and Smarandache [33] designed the fundamental algebraic neutrosophic systems. Smarandache and Ali [29] proposed the neutrosophic triplet group, a new algebraic structure that leads to evolutionary changes in the neutrosophic research domain. Gulistan and Naeem [11, 12] introduced the concept of neutrosophic triplet semi hypergroups and complex fuzzy hyper ideals in non-associative hyperrings. The works of Vidan Cetkin [5, 6] resulted in the creation of neutrosophic subgroups and modules. The idea of fuzzy G-modules and innovative group represenations was proposed by Sherry [9]. The further study in this area was carried out by Ursala by infusing the concepts of rough and fuzzy sets with module theory [15, 24]. The notion of neutrosophic submodules of an R-module and the additional characteristics in methodology were introduced by Binu [3, 4]

In equivalences of module categories, homomorphism functions and tensor product play a major role, and it is the more flexible generalisation of free module. The study of bilinear operations and the extension of scalars is made possible by the tensor product of modules. The universal multiplication of modules is achieved by tensor algebra of modules. In a neutrosophic context, the principal application of the tensor product is the management of large amounts of ambiguous data and the reduction of object dimensions in mathematical modelling. The tensor product in neutrosophic modules can be used to set up computational uncertainty quantification in probabilistic and deterministic models, allowing for easier interpretation of latent information and extraction of more components of information. The concept of tensor product in neutrosophic submodules of an R-modules generalises the concept of tensor product in modules and gives the multilinear operation more strength. We design the tensor product in neutrosophic submodules and characterised its features in this paper. The objects or entities used in this study for the algebraic creation of tensor products are neutrosophic submodules, which allow us deal with ambiguous data in imprecise bounds.

The following is the organization of this work, with the first section serving as an introductory concept. The Section 2 of this research article deals with the pre-requisite definitions and results. The concept of tensor product between neutrosophic submodules and the characteristic properties are presented in Section 3. Section 4 provides an overview of the further research work in this particular area of neutrosophic set theory.

2. Preliminaries

In this section, we examine some of the preliminary definitions and outcomes that are necessary for a thorough understanding of the subsequent sections.

Definition 2.1. [1] Let R be a commutative ring with unity. A *module* M over R is an 'Abelian' group with a law of composition written '+' and a scalar multiplication $R \times M \to M$, written $(r, x) \rightsquigarrow rx$, that satisfy these axioms

(1) $1x = x$
(2) $(rs)x = r(sx)$
(3) $(r + s)x = rx + sx$
(4) $r(x + y) = rx + ry \ \ \forall\ r, s \in R$ and $x, y \in M$.

Definition 2.2. [8, 10] Let R be a ring and M be an R-module. Let N be a submodule of M. The (additive, abelian) quotient group M/N can be made into an R-module by defining an action of elements of R by

$$r(x + N) = (rx) + N, \forall\ r \in R,\ x + N \in M/N$$

Remark 2.1. $[m]$ represents the coset $m + N, \forall\ m \in M$.

Definition 2.3. [8] A homomorphism $\Upsilon : M \to N$ of R-modules is a map compatible with the laws of composition

(1) $\Upsilon(x + y) = \Upsilon(x) + \Upsilon(y)$
(2) $\Upsilon(rx) = r\Upsilon(x)\ \forall\ x, y \in M,\ r \in R$.

Remark 2.2. $Hom_R(M, N)$ represent the set of all R-module homomorphisms of M into N.

Definition 2.4. [8] Let $M_1, M_2, ..., M_n$ be R-modules. Then the direct product is a collection of n-tuples, denoted and defined as $M_1 \times M_2 \times ... \times M_n = (m_1, m_2, ..., m_n),\ m_i \in M_i, 1 \leq i \leq n$ with addition and action of R defined component wise is again an R-module.

Definition 2.5. [22] Let M be an R-module and $S \subseteq M$ the set of finite formal linear combinations $L(S)$ of elements of S is a submodule of M. A typical element of $L(S)$ is $r_1m_1 + r_2m_2 + ... + r_nm_n,\ r_i \in R, m_i \in S,\ \forall\ i = 1, 2, ..., n$.

Remark 2.3. If $S \subseteq M$ and $L(S)$ is the set of all finite linear combination of elements of S, then $L(S)$ is the smallest submodule that contains S.

Definition 2.6. [13] Let M, N *and* P be an R-modules. A map $\varphi : M \times N \to P$ is said to be $R - bilinear$ if $\forall\ m_1, m_2 \in M,\ n_1, n_2 \in N$ and $r \in R$, the following conditions hold

(1) $\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)\ \forall\ m \in M$
(2) $\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)\ \forall\ n \in N$
(3) $\varphi(rm, n) = \varphi(m, nr) = r\varphi(m, n)\ \forall\ m \in M, n \in N, r \in R$.

Definition 2.7. [1, 13] The tensor product of R-modules M and N can be denoted and defined as $M \otimes N = (M \times N)/L(S)$ where S is the set of all formal sums of the following type

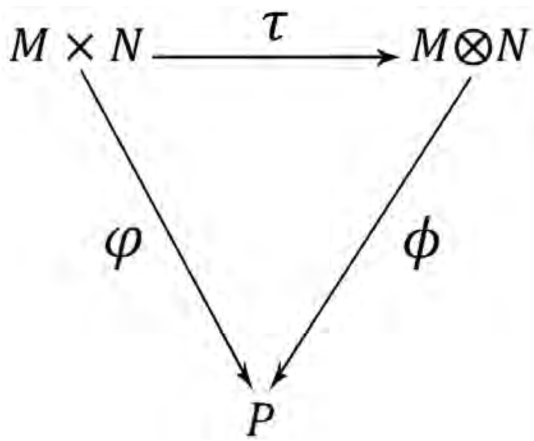

FIGURE 1. Tensor product

(1) $(rm, n) - r(m, n)$
(2) $(m, rn) - r(m, n)$
(3) $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$
(4) $(m, n_1 + n_2) - (m, n_1) - (m, n_2), \forall\ m, m_1, m_2 \in M; n, n_1, n_2 \in N; r \in R.$

Remark 2.4. (1) Being the quotient of R-module by a submodule, the tensor product $M \otimes N$ is an another R-module.
(2) $\exists$ a map $\tau : M \times N \to M \otimes N$ such that $\tau(m, n) = (m, n) + L(S)$, $\forall\ m \in M, n \in N$ and denote $\tau(m, n)$ by $m \otimes n$.

Definition 2.8. [1, 22] The tensor product $M \otimes N$ of R-modules M and N satisfies the following properties

(1) $(rm) \otimes n = r(m \otimes n)$
(2) $m \otimes (rn) = r(m \otimes n)$
(3) $(m_1 + m_2) \otimes n = (m_1 \otimes n) + (m_2 \otimes n)$
(4) $m \otimes (n_1 + n_2) = (m \otimes n_1) + (m \otimes n_2)$

where $\forall\ m, m_1, m_2 \in M,\ n, n_1, n_2 \in N,\ r \in R$

Definition 2.9. [8,22] Let M *and* N be two R-modules. A tensor product of M *and* N over R is an R-module $M \otimes N$ which is equipped with an $R-$ bilinear map

$$\tau : M \times N \to M \otimes N$$

such that for each R-module P and each R-bilinear map $\varphi : M \times N \to P$, there is a unique homomorphism $\phi : M \otimes N \to P,$ *that is* $\varphi = \phi \circ \tau$ (Refer Fig. 1)

Proposition 2.1. [8] "Let M *and* N be two R-modules. Tensor products of M *and* N over R are unique upto isomorphism .

Definition 2.10. [27, 31] A neutrosophic set P of the universal set X is defined as $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in X\}$ where $t_P, i_P, f_P : X \to (^-0, 1^+)$. The three components t_P, i_P *and* f_P represent membership value (Percentage of truth), indeterminacy (Percentage of indeterminacy) and non membership value (Percentage of falsity) respectively. These components are functions of non standard unit interval $(^-0, 1^+)$ [25].

Remark 2.5. [3, 28, 32]

(1) If $t_P, i_P, f_P : X \to [0, 1]$, then P is known as single valued neutrosophic set(SVNS).
(2) The algebraic structure R-module with SVNS as the underlying set is discussed in this work. SVNS will be referred to as a neutrosophic set for the sake of convenience.
(3) U^X denotes the set of all neutrosophic subset of X or neutrosophic power set of X.

Definition 2.11. [3, 20, 30] Let $P, Q \in U^X$. Then P is contained in Q, denoted as $P \subseteq Q$ if and only if $P(\eta) \leqslant Q(\eta)\ \forall \eta \in X$, this means that $t_P(\eta) \le t_Q(\eta), i_P(\eta) \le i_Q(\eta), f_P(\eta) \ge f_Q(\eta),\ \forall\ \eta \in X$.

Definition 2.12. [7] "Let M be an R module. Let $P \in U^M$ where U^Mdenotes the neutrosophic power set of R-module M. Then a neutrosophic subset $P = \{x, t_P(x), i_P(x),$ $f_P(x) : x \in M\}$ in M is called neutrosophic submodule of M if it satisfies the following;

(1) $t_P(0) = 1, i_P(0) = 1, f_P(0) = 0$
(2) $t_P(x + y) \ge t_P(x) \wedge t_P(y)$
$i_P(x + y) \ge i_P(x) \wedge i_P(y)$
$f_P(x + y) \le f_P(x) \vee f_P(y),\ \forall x, y \in M$
(3) $t_P(rx) \ge t_P(x),\ i_P(rx) \ge i_P(x),\ f_P(rx) \le f_P(x), \forall\ x \in M, \forall\ r \in R$

Remark 2.6. The set of all neutrosophic submodules of R-module M represented by $U(M)$.

Definition 2.13. [3] A homomorphism Υ of M into N is called a weak neutrosophic homomorphism of P onto Q if $\Upsilon(P) \subseteq Q$. If Υ is a **weak neutrosophic homomorphism** of P onto Q, then P is weakly homomorphic to Q and we write $P \sim Q$. A homomorphism Υ of M into N is called a **neutrosophic homomorphism** of P onto Q if $\Upsilon(P) = Q$ and we represent it as $P \approx Q$.

Definition 2.14. [3] If $P = \{m, t_P(m), i_P(m), f_P(m) : m \in M\} \in U(M)$ and N be a submodule of M, then define ω, a neutrosophic set in M/N as follows.

$$\omega = \{[m], t_\omega([m]), i_\omega([m]), f_\omega([m]) : m \in M\}$$

where

$$t_\omega([m]) = \vee\{t_P(u) : u \in [m]\}$$
$$i_\omega([m]) = \vee\{i_P(u) : u \in [m]\}$$

$$f_\omega([m]) = \wedge\{f_P(u) : u \in [m]\}$$

Then $\omega \in U(M/N)$.

3. **Tensor Products in Neutrosophic submodules**

The tensor product between R-modules M and N is a more general than the vector space tensor product. The construction of tensor products give a most characteristic strategy for joining two modules. This section describes the construction and properties of neutrosophic tensor products.

Definition 3.1. Let M *and* N be two R-modules. If $A \in U(M)$ *and* $B \in U(N)$, then the cartesian product $A \times B$ of A *and* B is a neutrosophic set of $M \times N$ drfined as $(A \times B)(x, y) = \{(x, y), t_{A\times B}(x, y), i_{A\times B}(x, y), f_{A\times B}(x, y) : (x, y) \in M \times N\}$ where

$$t_{A\times B}(x, y) = t_A(x) \wedge t_B(y)$$
$$i_{A\times B}(x, y) = i_A(x) \wedge i_B(y)$$
$$f_{A\times B}(x, y) = f_A(x) \vee f_B(y).$$

Proposition 3.1. If $A \in U(M)$ *and* $B \in U(N)$, then $A \times B \in U(M \times N)$.

Proof. We prove that $A \times B$ satisfies the following conditions :

(1) $t_{A\times B}(0, 0) = 1,\ i_{A\times B}(0, 0) = 1\ \& f_{A\times B}(0, 0) = 0$

(2) $t_{A\times B}((x_1, y_1) + (x_2, y_2)) \geq t_{A\times B}(x_1, y_1) \wedge t_{A\times B}(x_2, y_2)$
$i_{A\times B}((x_1, y_1) + (x_2, y_2)) \geq i_{A\times B}(x_1, y_1) \wedge i_{A\times B}(x_2, y_2)$
$f_{A\times B}((x_1, y_1) + (x_2, y_2)) \leq f_{A\times B}(x_1, y_1) \vee f_{A\times B}(x_2, y_2)\ \forall\ (x_1, y_1), (x_2, y_2) \in M \times N$

(3) $t_{A\times B}(r(x, y)) \geq t_{A\times B}(x, y)$
$i_{A\times B}(r(x, y)) \geq i_{A\times B}(x, y)$
$f_{A\times B}(r(x, y)) \leq f_{A\times B}(x, y)\ \forall\ (x, y) \in M \times N, r \in R$

1. From the definition 3.1,

$$t_{A\times B}(0, 0) = t_A(0) \wedge t_B(0) = 1$$
$$i_{A\times B}(0, 0) = i_A(0) \wedge i_B(0) = 1$$
$$f_{A\times B}(0, 0) = f_A(0) \vee f_B(0) = 0$$

2. Now $\forall\ (x_1, y_1), (x_2, y_2) \in M \times N$

$$\begin{aligned} t_{A\times B}((x_1, y_1) + (x_2, y_2)) &= t_{A\times B}(x_1 + x_2, y_1 + y_2) \\ &= t_A(x_1 + x_2) \wedge t_B(y_1 + y_2) \\ &\geq (t_A(x_1) \wedge t_A(x_2)) \wedge (t_B(y_1) \wedge t_B(y_2)) \\ &= (t_A(x_1) \wedge t_B(y_1)) \wedge (t_A(x_2) \wedge t_B(y_2)) \\ &= t_{A\times B}(x_1, y_1) \wedge t_{A\times B}(x_2, y_2) \end{aligned}$$

Similarly prove that $i_{A\times B}((x_1,y_1)+(x_2,y_2)) \geq i_{A\times B}(x_1,y_1)\wedge i_{A\times B}(x_2,y_2)$ and $f_{A\times B}((x_1,y_1)+(x_2,y_2)) \leq f_{A\times B}(x_1,y_1)\vee f_{A\times B}(x_2,y_2)$.

3. Consider $\forall\ (x,y)\in M\times N, r\in R$

$$\begin{aligned} t_{A\times B}(r(x,y)) &= t_{A\times B}(rx,ry) \\ &= t_A(rx)\wedge t_B(ry) \\ &\geq t_A(x)\wedge t_B(y) \\ &= t_{A\times B}(x,y) \end{aligned}$$

Similarly prove that $i_{A\times B}(r(x,y)) \geq i_{A\times B}(x,y)$ *and* $f_{A\times B}(r(x,y)) \leq f_{A\times B}(x,y)$. Hence $A\times B\in U(M\times N)$. □

Definition 3.2. Let $A\in U(M), B\in U(N)$ *and* $C\in U(P)$ where M,N *and* P are R modules. A map $\varphi : M\times N\rightarrow P$ is called neutrosophic R bi additive if the following conditions are hold $\forall\ (m,n)\in M\times N, m\in M, n\in N$

(1) The map $\varphi : M\times N\rightarrow P$ is R bi additive
(2) $t_C(\varphi((m,n)))\geq t_{A\times B}((m,n))$
(3) $i_C(\varphi((m,n)))\geq i_{A\times B}((m,n))$
(4) $f_C(\varphi((m,n)))\leq f_{A\times B}((m,n))$

Definition 3.3. Let $A\in U(L(M\times N)) = U(Y)$ and $Y(S)$ be a submodule of $L(M\times N) = Y$. Then the neutrosophic tensor product of R-modules M and N, $(M\otimes N)$, is a neutrosophic set Q of $Y/Y(S)$ defined as follows

$$Q((m,n)+Y(S)) = \{(m,n)+Y(S), t_Q((m,n)+Y(S), i_Q((m,n)+Y(S), f_Q((m,n)+Y(S)\}$$

$\forall\ (m,n)\in M\times N, m\in M, n\in N$ where

$$t_Q((m,n)+Y(S)) = \vee\{t_A((m,n)+y(S)) : y(S)\in Y(S)\}$$
$$i_Q((m,n)+Y(S)) = \vee\{i_A((m,n)+y(S)) : y(S)\in Y(S)\}$$
$$f_Q((m,n)+Y(S)) = \wedge\{f_A((m,n)+y(S)) : y(S)\in Y(S)\}$$

Remark: The coset $(m,n)+Y(S)$ is represented by $[(m,n)]$

Theorem 3.1. Let Q be the neutrosophic tensor product of $M\otimes N$, then $Q\in U(Y/Y(S))$.

Proof. We have

$Q([(m,n)]) = \{[(m,n)], t_Q([(m,n)]), i_Q([(m,n)]), f_Q([(m,n)]) : (m,n)\in M\otimes N, m\in M, n\in N\}$

and $A\in U(L(M\times N)) = U(Y)$ and $Y(S)$ be a submodule of $L(M\times N) = Y$. Also

$$t_Q([(m,n)]) = \vee\{t_A(x,y) : (x,y)\in[m,n]\}$$

$$i_Q([(m,n)]) = \vee\{i_A(x,y) : (x,y) \in [m,n]\}$$

$$f_Q([(m,n)]) = \wedge\{f_A(x,y) : (x,y) \in [m,n]\}$$

We have $t_Q([0,0]) = \vee\{t_A(x,y) : (x,y) \in [0,0]\} = t_A(0) = 1$, similarly $i_Q([0]) = 1$ and $f_Q([0,0]) = \wedge\{f_A(x,y) : (x,y) \in [0,0]\} = f_A(0) = 0$

Now for $(m_1,n_1),(m_2,n_2) \in M \otimes N$

$$\begin{aligned}
t_Q([(m_1,n_1)] + [(m_2,n_2)]) &= \vee\{t_A(x,y) : (x,y) \in [(m_1,n_1)] + [(m_2,n_2)]\} \\
&= \vee\{t_A((x_1,y_1) + (x_2,y_2)) : (x_1,y_1) + (x_2,y_2) \in [(m_1,n_1)] + [(m_2,n_2)]\} \\
&\geq \vee\{t_A((x_1,y_1) + (x_2,y_2)) : (x_1,y_1) \in [(m_1,n_1)], (x_2,y_2) \in [(m_2,n_2)]\} \\
&\geq \vee\{t_A(x_1,y_1) \wedge t_A(x_2,y_2) : (x_1,y_1) \in [(m_1,n_1)], (x_2,y_2) \in [m_2,n_2]\} \\
&= (\vee\{t_A((x_1,y_1)) : (x_1,y_1) \in [m_1,n_1]\}) \wedge \\
&\quad (\vee\{t_A((x_2,y_2)) : (x_2,y_2) \in [m_2,n_2]\}) \\
&= t_Q([m_1,n_1]) + t_Q([(m_1,n_2)])
\end{aligned}$$

Similarly we can prove that

$$i_Q([m_1,n_1] + [m_2,n_2]) \geq i_Q([(m_1,n_1)]) \wedge i_Q([(m_2,n_2)])$$

and

$$f_Q([(m_1,n_1)] + [(m_2,n_2)]) \leq f_Q([(m_1,n_1)]) \vee f_Q([(m_2,n_2)])$$

Now for any $r \in R, (m,n) \in M \otimes N$,

$$\begin{aligned}
t_Q(r[(m,n)]) &= t_Q([r(m,n)]) \\
&= \vee\{t_A(x,y) : (x,y) \in [r(m,n)]\} \\
&= \vee\{t_A(r(x,y) + y(s) : y(s) \in Y(S)\} \\
&\geq \vee\{t_A(r(x,y) + ry_1(S)) : y_1(S) \in Y(S)\} \\
&= \vee\{t_A(r((x,y) + y_1(S))) : y_1(S) \in Y(S)\} \\
&\geq \vee\{t_A((x,y) + y_1(S)) : y_1(S) \in Y(S)\} \\
&= \vee\{t_A(x_1,y_1) : (x_1,y_1) \in [m,n]\} \\
&= t_Q([m,n])
\end{aligned}$$

Similarly we can prove that

$$i_Q(r[(m,n)]) \geq i_Q([(m,n)]) \ and \ f_Q(r[(m,n)]) \geq f_Q([(m,n)])$$

Thus $Q \in U(Y/Y(S))$. □

Definition 3.4. A pair $(M \otimes N, \tau)$ or a map $\tau : M \times N \to M \otimes N$ is said to be the tensor product of A and B where $A \in U(M)$ *and* $B \in U(N)$ if for every neutosophic R bi additive map $\varphi : M \times N \to P$ of $A \times B$ to C, there is unique neutrosophic homomorphism $\phi : M \otimes N \to P$ of $A \otimes B$ onto C such that $\phi \circ \tau = \varphi$ where

$$t_C(\phi(m \otimes n)) \geq t_{A \times B}(m, n)$$

$$i_C(\phi(m \otimes n)) \geq i_{A \times B}(m, n)$$

$$f_C(\phi(m \otimes n)) \leq f_{A \times B}(m, n)$$

Theorem 3.2. The tensor product of two neutrosophic R modules exists and it is unique up to isomorphism.

Proof. Let $A \in U(M),\ B \in U(N)$ and $\tau : M \times N \to M \otimes N$ be the tensor product of R-modules M *and* N.Then by the definition of 3.4, for every neutosophic R bi additive map $\varphi : M \times N \to P$ where P be an R module, there is unique neutrosophic homomorphism $\phi : M \otimes N \to P$ such that $\phi \circ \tau = \varphi$.
Now define a map $A \otimes B : M \otimes N \to [0, 1]$ by putting

$$A \otimes B(m \otimes n) = \{(m \otimes n), t_{A \otimes B}(m \otimes n), i_{A \otimes B}(m \otimes n), f_{A \otimes B}(m \otimes n)\}$$

where

$$t_{A \otimes B}(m \otimes n) = \bigvee\{t_{A \times B}(m', n') : (m' \otimes n') = (m \otimes n)\}$$

$$i_{A \otimes B}(m \otimes n) = \bigvee\{i_{A \times B}(m', n') : (m' \otimes n') = (m \otimes n)\}$$

$$f_{A \otimes B}(m \otimes n) = \bigwedge\{f_{A \times B}(m', n') : (m' \otimes n') = (m \otimes n)\}$$

Let $C \in U(P)$, then to prove that, $\forall\ m \otimes n \in M \otimes N$

$$t_C(\phi(m \otimes n)) \geq t_{A \otimes B}(m \otimes n)$$

$$i_C(\phi(m \otimes n)) \geq i_{A \otimes B}(m \otimes n)$$

$$f_C(\phi(m \otimes n)) \leq f_{A \otimes B}(m \otimes n)$$

Suppose $(m' \otimes n') = (m \otimes n) \in M \otimes N$

$$\begin{aligned} t_C(\phi(m' \otimes n')) &\geq \bigvee\{t_C(\phi \circ \tau)(m', n')\} \\ &= \bigvee\{t_{A \times B}(m, , n')\} \\ &= t_{A \otimes B}(m \otimes n) \end{aligned}$$

□

Definition 3.5. Let A be left neutrosophic R-module of left R-module M and let B be right neutrosophic R-module of right R-module N. Let C be a neutrosophic abelian group and $\widetilde{g} : A \times B \to C$ be neutosophic biadditive. A pair $(C, \widetilde{g})$ is called a tensor product of A *and* B if for every fuzzy biadditive $F : A \times B \to H$ where H is a neutrosophic abelian group , there is a unique neutrosophic map $\theta \in Hom(C, H)$ such that $\theta \circ \widetilde{g} = F$.

Theorem 3.3. Let A be left neutrosophic R-module of left R-module M and let B be right neutrosophic R-module of right R-module N. The tensor product of the two neutrosophic R-modules A and B exist and it is unique up to isomorphism.

Proof. Let $\varphi : M \times N \to M \otimes N$ be the tensor products of R-modules A and B. Then we can define the maps $t, i, f : M \otimes N \to [0, 1]$ such that $\forall\ i$

$$t_{A\otimes B}(\sum(a_i \otimes b_i)) = \bigvee\{t_{A\times B}(\sum(a_i', b_i')) : \sum(a_i' \otimes b_i') = \sum(a_i \otimes b_i)\}$$
$$i_{A\otimes B}(\sum(a_i \otimes b_i)) = \bigvee\{i_{A\times B}(\sum(a_i', b_i')) : \sum(a_i' \otimes b_i') = \sum(a_i \otimes b_i)\}$$
$$f_{A\otimes B}(\sum(a_i \otimes b_i)) = \bigwedge\{f_{A\times B}(\sum(a_i', b_i')) : \sum(a_i' \otimes b_i') = \sum(a_i \otimes b_i)\}$$

Then $\varphi : A \times B \to A \otimes B$ is neutrosophic biadditive and $\widetilde{H}$ be a neutrosophic abelian group and $\psi : A \times B \to \widetilde{H}$ be a neutrosophic biadditive. Thn by definition of tensor product, $\exists$ a unique homomorphism $\theta : M \otimes N \to H$such that $\theta \circ \psi = \varphi$.

Now we have to show that $\sum(a_i \otimes b_i) \in A \otimes B$ and

$$t_H(\theta(\sum(a_i \otimes b_i))) \geq t_{A\otimes B}(\sum(a_i \otimes b_i))$$
$$i_H(\theta(\sum(a_i \otimes b_i))) \geq i_{A\otimes B}(\sum(a_i \otimes b_i))$$
$$f_H(\theta(\sum(a_i \otimes b_i))) \leq f_{A\otimes B}(\sum(a_i \otimes b_i))$$

Suppose $\sum(a_i' \otimes b_i') = \sum(a_i \otimes b_i) \in M \otimes N$

$$\begin{aligned}
t_H(\theta\sum(a_i' \otimes b_i') &= t_H(\sum(\theta(a_i' \otimes b_i'))) \\
&\geq \bigwedge\{t_H\theta(a_i' \otimes b_i')\} \\
&= \bigwedge\{t_H\theta\varphi(a_i', b_i')\} \\
&= \bigwedge\{t_H\psi(a_i', b_i')\} \\
&\geq \bigwedge\{t_{A\times B}(a_i', b_i')\} \\
&= t_{A\times B}(a_i', b_i') \\
&= t_{A\otimes B}(\sum(a_i' \otimes b_i'))
\end{aligned}$$

Similarly ,

$$i_H(\theta(\sum(a_i \otimes b_i))) \geq i_{A\otimes B}(\sum(a_i \otimes b_i))$$

$$f_H(\theta(\sum(a_i \otimes b_i))) \leq f_{A\otimes B}(\sum(a_i \otimes b_i))$$

This concludes that $\sum(a_i \otimes b_i) \in A \otimes B$ and $A \otimes B$ is a tensor product of A and B. Also it is obvious that tensor product is unique up to isomorphism. □

Remark 3.1 : Let A and B be two neutrosophic right and left R module, then $0_R \otimes A \cong A$ and $B \otimes 0_R \cong B$

4. Conclusion

The concept of tensor product is great significance in classical algebra, geometry and analysis. In the emerging algebraic research domain, the amalgamation of tensor product in a neutrosophic submodule context leads to the design of the most flexible version of algebraic product. In this research a neutrosophic quotient submodule of an R-module is constructed as a tensor product in neutrosophic submodules of an R-modules. It will definitely lead to the development of new theoretical and practical techniques for problem solving in the fields of classical and quantum mechanics, image processing and neural networks, artificial intelligence and machine learning. The concept of tensor product in neutrosophic submodules is an imminent tool for the vague multi dimensional real world big data processing and analysis. In our future research, we propose to extend the concept of exact sequences to use tensor factorization in the neutrosophic domain, as well as the associative property of the relationship between neutrosophic injective and projective modules.The above mentioned study leads to the homological properties of neutrosophic submodule tensor products and neutrosophic module category theory.

Conflicts of Interest: Declare conflicts of interest or state "The authors declare no conflict of interest."

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Bipolar neutrosophic soft generalized pre-closed sets and pre-open sets in topological space

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Abstract. Neutrosophy is one of the widely used tool to deal with uncertainty. In recent years, neutrosophic sets were applied in the field of topology. There are numerous types of neutrosophic topological spaces based on different kinds of neutrosophic sets are proposed by research community. Bipolar neutrosophic set and their topological spaces were proposed and analyzed by many researchers. In this paper, the generalization of bipolar neutrosophic soft set and their classifications are proposed. A bipolar neutrosophic soft generalized pre-closed sets and bipolar neutrosophic soft generalized pre-open sets are proposed along with their properties. Then, bipolar neutrosophic soft topology concepts are generalized to the proposed sets. Also, the relation between proposed sets and various conventional sets are discussed through theorems with examples.

Keywords: Bipolar neutrosophic soft set; BNGS-topology; BNGPCS; BNGPOS; Pre-closed set; Pre-open set.

1. Introduction

Most of the real life problems has some uncertain information which makes difficult to retrieve the solution. In earlier days, researcher did not take the uncertainty into account while solving problems. But, those information makes significant difference in the final decision. Zadeh [1] were introduced fuzzy theory in 1968. Fuzzy theory were very useful to deal with uncertainty in real life problems. Since the introduction of fuzzy theory, many researchers were proposed different types of fuzzy concepts by extending and modifying the original fuzzy theory and applied to science and engineering problems. But the main drawback of fuzzy theory is, its uncertainty is dependent on the certainty of the problem. In many situations, uncertainty information may be independent. Many years later, Florentin Smarandache [2, 3] introduced the novel concept neutrosophy in 1998. Neutrosophy has three independent components namely, truth membership, indeterminacy and false membership each has the value in the interval $]^{-}0, 1^{+}[$. Neutrosophic sets were derived from neutrosophy and which is powerful than fuzzy sets. Molodtsov [5] introduced soft set theory in 1999 which is also deal with uncertainty in a parametric wise. In 2013, Pabitra Kumar Maji [14, 15] proposed neutrosophic soft set which is the combination of both neutrosophic set and soft set. Neutrosophic soft sets were widely used in decision making problems by many researchers. Irfan Deli et al. [4] proposed bipolar neutrosophic sets and decision making technique in 2015. Mumtaz ali et al. [6] proposed bipolar neutrosophic soft sets and decision making method in 2017. After that, many different approaches on bipolar neutrosophic soft sets were proposed by several authors [7, 8, 16].

Neutrosophic sets were applied in almost all mathematics fields such as neutrosophic graph, neutrosophic statistics, neutrosophic algebra and so on. Neutrosophic sets were widely used many topology concepts; in particular, general topology. In 2012, A.A.Salama et al. [19] developed a new topological space namely, neutrosophic topology based on neutrosophic sets. Then, most of the general topology concepts were combined with neutrosophic sets and some new topologies were proposed [9–11, 16, 18]. In 1970's Norman Levine [12, 13] was defined generalized closed sets and many set theory concepts. In 1995, J.Dontchev [11] proposed generalized semi pre-open sets in topology. In 2018, Taha Yasin [20] have proposed some properties on bipolar soft topological space with appropriate examples; later, in 2020 [21], he introduced bipolar soft points which is very useful to investigate continuity, openness and closeness of topology mappings. In 2021, Simsekler Dizman and Taha Yasin [22] proposed a novel concept fuzzy bipolar soft topological spaces which is the extension of bipolar soft topology to fuzzy sets.

In this paper, the generalized set concept is applied to the bipolar neutrosophic soft set. As we discussed, the fusion of soft set and bipolar neutrosophic set gives bipolar neutrosophic soft set. In a similar manner, we take a fusion of generalized pre-sets (both open and closed) with bipolar neutrosophic soft set and defined new classes namely, bipolar neutrosophic soft generalized pre-cloesd sets and bipolar neutrosophic soft generalized pre-open set. Further, we investigate the reltion between former sets with the proposed sets.

This paper is organized as follows: section 1, gives introduction and previous works on the related topics. Section 2 consists required preliminary definitions. Section 3 deals with the notions on bipolar neutrosophic soft set topological space and their important results and properties. Section 4 and 5 deals with the proposed set, bipolar neutrosophic soft generalized pre-closed sets and related theorems and the following section consists, bipolar neutrosophic soft generalized pre-open sets and related theorems.

2. Preliminaries

Definition 2.1. [2,3] For a universal set X and every $x \in X$, the components $\mathcal{T}(x)$, $\mathcal{I}(x)$ and $\mathcal{F}(x)$ represents truth, indeterminate and false degrees of x. Then the Neutrosophic set (NS) over X be defined as follows.
$N = \{\mathcal{T}(x),\ \mathcal{I}(x),\ \mathcal{F}(x)\ :\ x \in X\}$
Here, $\mathcal{T}(x),\ \mathcal{I}(x),\ \mathcal{F}(x)$ ranges in the non-standard interval $]^{-}0, 1^{+}[$ and their sum $^{-}0 \leq T + I + F \leq 3^{+}$. Further, single valued neutrosophic set is defined by replacing the interval $]^{-}0, 1^{+}[$ with $[0, 1]$ in the definition of neutrosophic set.

Definition 2.2. [5] A soft set is a function which maps a parameter set to the power set of X. It is denoted by (f, E) and is defined by
$f : E \to P(x)$
Each member of X is parameterized with the parameter set E by the function f.

Definition 2.3. [4] For the universe set X and positive member values $T^{+},\ I^{+},\ F^{+}\ :\ E \to [0, 1]$, negative member values $T^{-},\ I^{-},\ F^{-}\ :\ E \to\ [-1, 0]$, A bipolar neutrosophic set (BNS) is defined by

$$B = \Big\{\langle x,\ \mathcal{T}^{+}(x),\ \mathcal{I}^{+}(x),\ \mathcal{F}^{+}(x),\ \mathcal{T}^{-}(x),\ \mathcal{I}^{-}(x),\ \mathcal{F}^{-}(x)\ \rangle :\ x \in\ X\Big\}$$

Definition 2.4. [6,7] A bipolar neutrosophic soft set (BNSS) is the fusion of soft set and bipolar neutrosophic set and is defined as follows.

$$BNS = (f_A, E) = \{\langle e, f_A(x)\rangle : e \in A \subset E, f_A(x) \in BNS(X)\}$$

Here $f_A(x) = \Big\{\Big\langle x, \mathcal{T}^{+}_{f_{A(e)}}(x), \mathcal{I}^{+}_{f_{A(e)}}(x), \mathcal{F}^{+}_{f_{A(e)}}(x), \mathcal{T}^{-}_{f_{A(e)}}(x), \mathcal{I}^{-}_{f_{A(e)}}(x), \mathcal{F}^{-}_{f_{A(e)}}(x)\Big\rangle : x \in X\Big\}$.

Definition 2.5. [6,7,16] Let B be a $BNSS$. Then the complement of B is defined as

$$B^{c} = \{\Big\langle e, \mathcal{F}^{+}_{f}(e), 1 - \mathcal{I}^{+}_{f}(e), \mathcal{T}^{+}_{f}(e), \mathcal{F}^{-}_{f}(e), -1 - \mathcal{I}^{-}_{f}(e), \mathcal{T}^{-}_{f}(e)\Big\rangle\}.$$

Definition 2.6. [6,7,16] Let $\phi_{\mathbb{B}}$ be a null $BNSS$ and is defined as

$$\phi_{\mathbb{B}} = \Big\{\ \langle e_i, \{x_i, 0, 1, 1, 0, -1, -1\}\rangle : x \in X, e \in E\Big\}$$

Definition 2.7. [6,7,16] Let $1_{\mathbb{B}}$ be a complete $BNSS$ and is defined as

$$1_{\mathbb{B}} = \Big\{\ \langle e_i, \{x_i, 1, 0, 0, -1, 0, 0\}\rangle : x \in X, e \in E\Big\}$$

Definition 2.8. [6,7,16] Let B_1 and B_2 be two $BNSS$s. Then their union $B_1 \cup B_2$ is defined as

$$B_1 \cup B_2 = \left\{ \left\langle e, \cup_i f^{(i)}(e) \right\rangle \right\}.$$

Here,

$$\bigcup_i f^{(i)}(e) = \Big\{ \langle x, max\left[\mathcal{T}^+_{f^i}(e)(x)\right], min\left[\mathcal{I}^+_{f^i}(e)(x)\right], min\left[\mathcal{F}^+_{f^i}(e)(x)\right], min\left[\mathcal{T}^-_{f^i}(e)(x)\right], max\left[\mathcal{I}^-_{f^i}(e)(x)\right], max\left[\mathcal{F}^-_{f^i}(e)(x)\right] \rangle \Big\}$$

Definition 2.9. [6,7,16] Let B_1 and B_2 be two $BNSS$s. Then their intersection $B_1 \cap B_2$ is defined as

$$B_1 \cap B_2 = \left\{ \left\langle e, \cap_i f^{(i)}(e) \right\rangle \right\}.$$

Here,

$$\bigcap_i f^{(i)}(e) = \Big\{ \langle x, min\left[\mathcal{T}^+_{f^i}(e)(x)\right], max\left[\mathcal{I}^+_{f^i}(e)(x)\right], max\left[\mathcal{F}^+_{f^i}(e)(x)\right], max\left[\mathcal{T}^-_{f^i}(e)(x)\right], min\left[\mathcal{I}^-_{f^i}(e)(x)\right], min\left[\mathcal{F}^-_{f^i}(e)(x)\right] \rangle \Big\}$$

Definition 2.10. [6,7,16] Let B_1 and B_2 be two $BNSS$s. Then B_1 is called subset of B_2 (i.e. $B_1 \subseteq B_2$) only if the following condition hold.
For every $x \in X$ and $e \in E$,

$$\left[\mathcal{T}^+_{B_1}(x) \leq \mathcal{T}^+_{B_2}(x)\right], \left[\mathcal{I}^+_{B_1}(x) \geq \mathcal{I}^+_{B_2}(x)\right], \left[\mathcal{F}^+_{B_1}(x) \geq \mathcal{F}^+_{B_2}(x)\right]$$
$$\left[\mathcal{T}^-_{B_1}(x) \geq \mathcal{T}^-_{B_2}(x)\right], \left[\mathcal{I}^-_{B_1}(x) \leq \mathcal{I}^-_{B_2}(x)\right], \left[\mathcal{F}^-_{B_1}(x) \leq \mathcal{F}^-_{B_2}(x)\right]$$

Definition 2.11. [13] Let (X, τ) be a topological space. For any subset $Y \in X$,

i). $cl(Y) = Y$, then Y is closed set
ii). $int(cl(Y)) \subseteq Y$, then Y is semi closed set (SCS)
iii). $cl(int(Y)) \subseteq Y$, then Y is pre-closed set (PCS)
iv). $int(cl(int(Y))) \subseteq Y$, then Y is semi pre-closed set (SPCS)
v). $cl(int(cl(Y))) \subseteq Y$, then Y is α-closed set (α-CS)
vi). $Y = cl(int(Y))$, then Y is regular closed set (RCS)

Definition 2.12. [12] Let (X, τ) be a topological space. For any subset $Y \in X$ and $Y \subseteq U$ and U is open in X,

i). $cl(Y) \subseteq U$, then Y is generalized closed set (g-closed).
ii). $scl(Y) \subseteq U$, then Y is generalized semi closed set (gs-closed).

iii). $pcl(Y) \subseteq U$, then Y is generalized pre-closed set (gp-closed).

iv). $spcl(Y) \subseteq U$, then Y is generalized semi pre-closed set (gsp-closed).

v). $\alpha cl(Y) \subseteq U$, then Y is α- generalized closed set (αg-closed).

Definition 2.13. [16] A bipolar neutrosophic soft topology (BNST) on X is a collection τ of bipolar neutrosophic soft sets (BNSS) in X satisfying the following conditions:

1). $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}}$

2). $\bigcup_{i\in n} \mathbb{B}_i \in \tau_{\mathbb{B}}$ for each $\mathbb{B}_i \in \tau_{\mathbb{B}}$

3). $\mathbb{B}_i \cap \mathbb{B}_j \in \tau_{\mathbb{B}}$ for any $\mathbb{B}_i, \mathbb{B}_j \in \tau_{\mathbb{B}}$

The pair $(X, \tau_{\mathbb{B}})$ is called $\mathbb{BNSS}$-topological space. The members of $\tau_{\mathbb{B}}$ are called bipolar neutrosophic soft open sets (BNOS) and their complements are called bipolar neutrosophic soft closed sets (BNCS).

The collection of all subsets of X [P(x)] along with null set and complete set, i.e. $\tau = \{\phi_{\mathbb{B}}, 1_{\mathbb{B}}, P(X)\}$ is called discrete topology on X. The collection of X and null set, i.e. $\tau = \{\phi_{\mathbb{B}}, X\}$ is called indiscrete topology.

Example 2.14. Let $X = x_1, x_2$ be set of alternatives and $E = e_1, e_2, e_3$ be a parameter set. Now let us define a topology on (X, E) as follows.

$$\tau_{\mathcal{B}} = \left\{\phi_{\mathbb{B}}, 1_{\mathbb{B}}, \mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4\right\}$$

Here $\phi_{\mathbb{B}}, 1_{\mathbb{B}}$ are null and complete $\mathbb{BNSS}$ respectively. Also,

$$\mathbb{B}_1 = \left\{ \begin{array}{c} \left\langle e_1, \{\langle x_1,\ 1,0,1,-1,0,0\rangle, \langle x_2,\ 0.5,0.2,0.4,-0.5,-0.4,-0.3\rangle\} \right\rangle, \\ \left\langle e_2, \{\langle x_1,\ 0.4,0.6,0.3,-0.4,-0.7,-0.2\rangle, \langle x_2,\ 0.7,0.2,0.1,-0.3,-0.5,-0.7\rangle\} \right\rangle, \\ \left\langle e_3, \{\langle x_1,\ 0.5,0.3,0.7,-0.2,-0.4,-0.8\rangle, \langle x_2,\ 0.4,0.3,0.5,-0.1,-0.4,-0.6\rangle\} \right\rangle \end{array} \right\}$$

$$\mathbb{B}_2 = \left\{ \begin{array}{c} \left\langle e_1, \{\langle x_1,\ 0.3,0.1,0.7,-0.5,-0.6,-0.3\rangle, \langle x_2,\ 0,1,1,-0.7,0,-1\rangle\} \right\rangle, \\ \left\langle e_2, \{\langle x_1,\ 0.2,0.5,0.7,-1,0,-0.2\rangle, \langle x_2,\ 0.9,0.1,0.3,-0.1,-0.6,-0.3\rangle\} \right\rangle, \\ \left\langle e_3, \{\langle x_1,\ 0.3,0.5,0.3,-0.2,0,-0.4\rangle, \langle x_2,\ 0.7,0.4,0.1,-0.3,-0.5,-0.1\rangle\} \right\rangle \end{array} \right\}$$

$$\mathbb{B}_3 = \left\{ \begin{array}{c} \left\langle e_1, \{\langle x_1,\ 1,0,0.7,-1,0,0\rangle, \langle x_2,\ 0.5,0.2,0.4,-0.7,0,-0.3\rangle\} \right\rangle, \\ \left\langle e_2, \{\langle x_1,\ 0.4,0.5,0.3,-1,0,-0.2\rangle, \langle x_2,\ 0.9,0.1,0.1,-0.3,-0.5,-0.3\rangle\} \right\rangle, \\ \left\langle e_3, \{\langle x_1,\ 0.5,0.3,0.3,-0.2,0,-0.4\rangle, \langle x_2,\ 0.7,0.3,0.1,-0.3,-0.4,-0.1\rangle\} \right\rangle \end{array} \right\}$$

$$\mathbb{B}_4 = \left\{ \begin{array}{c} \left\langle e_1, \{\langle x_1,\ 0.3, 0.1, 1, -0.5, -0.6, -0.3\rangle, \langle x_2,\ 0, 1, 1, -0.5, -0.4, -1\rangle\} \right\rangle, \\ \left\langle e_2, \{\langle x_1,\ 0.2, 0.6, 0.7, -0.4, -0.7, -0.2\rangle, \langle x_2,\ 0.7, 0.2, 0.3, -0.1, -0.6, -0.7\rangle\} \right\rangle, \\ \left\langle e_3, \{\langle x_1,\ 0.3, 0.5, 0.7, -0.2, -0.4, -0.8\rangle, \langle x_2,\ 0.4, 0.4, 0.5, -0.1, -0.5, -0.6\rangle\} \right\rangle \end{array} \right\}$$

The $\tau_{\mathbb{B}}$ satisfies all three conditions of topology. So $\tau_{\mathbb{B}}$ is a $\mathbb{BNSS}$-topology.

3. Notions of Bipolar neutrosophic soft topological spaces

Taha Yasin et al. [16] proposed bipolar neutrosophic soft topological space in 2019. Here, we defined some notions and properties of the bipolar neutrosophic soft topological spaces. However, we redefined some of the existing results in order to make suitable for the bipolar neutrosophic soft set which was defined by Arulpandy et al. [7] in 2019. Since the proposed bipolar neutrosophic soft set by Arulpandy et al. [7] is modified version of Mumtaz Ali's [6], there should be some changes in the corresponding topological spaces are also needed.

Definition 3.1. Let $(X, \tau_{\mathbb{B}})$ be a $BNST$ and $B = \left\{ \langle e, f(x)\rangle : e \in E, f(x) \in BNS(X) \right\}$ be $BNSS$ in X. Then the bipolar neutrosophic soft interior and bipolar neutrosophic soft closure are defined by

$BNint(B) = \bigcup \left\{ U\ :\ U\ is\ a\ BNOS\ in\ U \subseteq B \right\}$

$BNcl(B) = \bigcap \left\{ V\ :\ V\ is\ a\ BNCS\ in\ V \subseteq B \right\}$

Note 3.2. Let B be BNS of a $BNTS(X, \tau)$. Then

1. $BN\alpha cl(B) = B \cup BNcl(BNint(BNcl(B)))$
2. $BN\alpha int(B) = B \cap BNint(BNcl(BNint(B)))$

Remark 3.3. Following relations hold for any BNS set $B \in (X, \tau)$.

1. $BNcl(B^c) = (BNint(B))^c$ and $BNint(B^c) = (BNcl(B))^c$.
2. $BNcl(B)$ is a $BNCS$ and $BNint(B)$ is a $BNOS$ in X.
3. B is $BNCS$ in X if and only if $BNcl(B) = B$.
4. B is $BNOS$ in X if and only if $BNint(B) = B$.

Proposition 3.4. *Let (X, τ) be a $BNSTS$ and A, B be $BNSSs$ in X. Then the following relations hold.*

i). $BNint(A) \subseteq A$; $\quad A \subseteq BNcl(A)$

ii). $A \subseteq B \Rightarrow BNint(A) \subseteq BNint(B)$; $\quad A \subseteq B \Rightarrow BNcl(A) \subseteq BNcl(B)$

iii). $BNint(BNint(A)) = BNint(A)$; $\quad BNcl(BNcl(A)) = BNcl(A)$

iv). $BNint(A \cap B) = BNint(A) \cap BNint(B)$; $\quad BNcl(A \cup B) = BNcl(A) \cup BNcl(B)$

v). $BNint(1_{BN}) = 1_{BN}$; $\quad BNcl(0_{BN}) = 0_{BN}$

Definition 3.5. A $BNSS$set B in $BNSTS(X,\tau)$ is said to be

1). Bipolar neutrosophic soft semi closed set (BNSCS) if $BNint(BNcl(B)) \subseteq B$,

2). Bipolar neutrosophic soft semi open set (BNSOS) if $B \subseteq BNcl(BNint(B))$,

3). Bipolar neutrosophic soft pre-closed set (BNPCS) if $BNcl(BNint(B)) \subseteq B$,

4). Bipolar neutrosophic soft pre-open set (BNPOS) if $B \subseteq BNint(BNcl(B))$,

5). Bipolar neutrosophic soft α-closed set (BNαCS) if $BNcl(BNint(BNcl(B))) \subseteq B$,

6). Bipolar neutrosophic soft α-open set (BNαOS) if $B \subseteq BNint(BNcl(BNint(B)))$,

7). Bipolar neutrosophic soft semi pre-closed set (BNSPCS) if $BNint(BNcl(BNint(B))) \subseteq B$,

8). Bipolar neutrosophic soft semi pre-open set (BNSPOS) if $B \subseteq BNcl(BNint(BNcl(B)))$,

9). Bipolar neutrosophic soft regular open set (BNROS) if $B = BNint(BNcl(B))$,

10). Bipolar neutrosophic soft regular closed set (BNRCS) if $B = BNcl(BNint(B))$.

Definition 3.6. Let B be a $BNSS$ in $BNSTS(X,\tau)$. Then

1). Bipolar neutrosophic soft semi interior of B (BNsint(B)) is
$BNsint(B) = \cup\{U \mid U\ is\ a\ BNSOS\ in\ X\ and\ U \subseteq B\}$

2). Bipolar neutrosophic soft semi closure of B (BNscl(B)) is
$BNscl(B) = \cap\{V \mid V\ is\ a\ BNSCS\ in\ X\ and\ B \subseteq V\}$

3). Bipolar neutrosophic soft alpha interior of B (BNαint(B)) is
$BN\alpha int(B) = \cup\{U \mid U\ is\ a\ BN\alpha OS\ in\ X\ and\ U \subseteq B\}$

4). Bipolar neutrosophic soft alpha closure of B (BNαcl(B)) is
$BN\alpha cl(B) = \cap\{V \mid V\ is\ a\ BN\alpha CS\ in\ X\ and\ B \subseteq V\}$

5). Bipolar neutrosophic soft semi pre-interior of B (BNspint(B)) is
$BNspint(B) = \cup\{U \mid U\ is\ a\ BNSPOS\ in\ X\ and\ U \subseteq B\}$

6). Bipolar neutrosophic soft semi pre-closure of B (BNspcl(B)) is
$BNspcl(B) = \cap\{V \mid V\ is\ a\ BNSPCS\ in\ X\ and\ B \subseteq V\}$

7). Bipolar neutrosophic soft pre-interior of$B$ (BNpint(B)) is
$BNpint(B) = \cup\{U \mid U\ is\ a\ BNPOS\ in\ X\ and\ U \subseteq B\}$

8). Bipolar neutrosophic soft pre-closure of B (BNpcl(B)) is
$BNspcl(B) = \cap\{V \mid V\ is\ a\ BNPCS\ in\ X\ and\ B \subseteq V\}$

Remark 3.7. For a $BNSS$ B in (X,τ),

1. $BNscl(B) = B \cup BNint(BNcl(B))$
2. $BNsint(B) = B \cap BNcl(BNint(B))$
3. $BN\alpha cl(B) = B \cup BNcl(BNint(BNcl(B)))$
4. $BN\alpha int(B) = B \cap BNint(BNcl(BNint(B)))$
5. $BNpcl(B) = B \cup BNcl(BNint(B))$
6. $BNpint(B) = B \cap BNint(BNcl(B))$

Definition 3.8. A $BNSS$set B in $BNSTS(X, \tau)$ is said to be

1). Bipolar neutrosophic soft generalized closed set (BNGCS) if $BNcl(B) \subseteq U$ whenever $B \subseteq U$ and U is $BNOS$ in X.

2). Bipolar neutrosophic soft generalized semi closed set (BNGSCS) if $BNscl(B) \subseteq U$ whenever $B \subseteq U$ and U is $BNOS$ in X.

3). Bipolar neutrosophic soft α generalized closed set (BNαGCS) if $BN\alpha cl(B) \subseteq U$ whenever $B \subseteq U$ and U is $BNOS$ in X.

4. Bipolar neutrosophic soft generalized pre-closed sets

In this section, a new class of sets namely, bipolar neutrosophic soft generalized pre-closed sets are proposed. Also, we have investigated some properties of the proposed set with appropriate examples.

Definition 4.1. A $BNSS$set B is said to be bipolar neutrosophic soft generalized pre-closed set (BNGPCS) in (X, τ) if $BNpcl(B) \subseteq U$ whenever $B \subseteq U$ and U is $BNOS$ in X. The collection of all $BNGPCS$s of a $BNSTS$ (X, τ) is denoted by $BNGPC(X)$.

Example 4.2. Consider the BNS-topology (X, τ) in Example 2.14.
Let

$$B = \left\{ \begin{array}{c} \left\langle e_1, \{\langle x_1,\ 0.3, 0.1, 1, -0.5, -0.6, -0.3\rangle, \langle x_2,\ 0, 1, 1, -0.5, -0.4, -1\rangle\} \right\rangle, \\ \left\langle e_2, \{\langle x_1,\ 0.2, 0.6, 0.7, -0.4, -0.7, -0.2\rangle, \langle x_2,\ 0.7, 0.2, 0.3, -0.1, -0.6, -0.7\rangle\} \right\rangle, \\ \left\langle e_3, \{\langle x_1,\ 0.3, 0.5, 0.7, -0.2, -0.4, -0.8\rangle, \langle x_2,\ 0.4, 0.4, 0.5, -0.1, -0.5, -0.6\rangle\} \right\rangle \end{array} \right\}$$

Here, $BNint(B) = \phi_{\mathbb{B}}$ and $BNcl(BNint(B)) = B \subseteq \mathbb{B}_2$ whereas $\mathbb{B}_2$ is a $BNOS$ in (X, τ). Hence B is a $BNGPCS$ in X.

Theorem 4.3. *Every $BNCS$ is $BNGCS$ but converse not true.*

Proof. Let B be $BNCS$ in X. Suppose U in $BNOS$ in X, such that $B \subseteq U$. Then $BNcl(B) = B \subseteq U$. Hence B is $BNGCS$. Conversely, let B be a $BNGCS$; so $B \in U$ and U is some open set such that $cl(B) \subseteq U$. From this, $cl(B)$ only closed and B is not necessarily closed. Hence, B may or may not be $BNCS$. □

Example 4.4. Consider the BNS-topology in Example 2.14. Let

$$B = \left\{ \begin{array}{c} \left\langle e_1, \{\langle x_1,\ 0.2, 0.3, 0.8, 0.2, 0.7, 0.8\rangle, \langle x_2,\ 0, 1, 1, 0.5, 0.4, 1\rangle\} \right\rangle, \\ \left\langle e_2, \{\langle x_1,\ 0.1, 0.6, 0.8, 0.3, 0.5, 0.6\rangle, \langle x_2,\ 0.3, 0.4, 0.6, 0, 0.7, 0.5\rangle\} \right\rangle, \\ \left\langle e_3, \{\langle x_1,\ 0.1, 0.6, 0.5, 0.1, 0.4, 0.5\rangle, \langle x_2,\ 0.5, 0.5, 0.4, 0.2, 0.7, 0.4\rangle\} \right\rangle \end{array} \right\}$$

Then $BNcl(B) \neq B$. So B is not a $BNCS$.

Theorem 4.5. *Every $BNCS$ is $BNGPCS$ but converse not true.*

Proof. Let B be $BNCS$ in X and let $B \subseteq U$ and U be $BNOS$ in X. Since $BNpcl(B) \subseteq BNcl(B)$ and A is $BNCS$ in X, $BNpcl(B) \subseteq BNcl(B) = B \subseteq U$. So B is $BNGPCS$ in X. Conversely, if B is a $BNGPCS$, then $BNpcl(B) \subseteq U$. This means, only $BNpcl(B)$ is $BNCS$ and not necessarily B. Hence proved. □

Example 4.6. We proved earlier that every $BNCS$ is not necessarily be a $BNGCS$. By definition, every $BNGPCS$ must be a $BNGCS$ first. This implies that, every $BNCS$ not necessarily be a $BNGPCS$.

Theorem 4.7. *Every $BNGCS$ is $BNGPCS$ but converse not true.*

Proof. By definition of $BNGCS$, for some $BNOS$ U, $cl(B) \subseteq U$. Since B is closed by default, $cl(int(B)) = cl(B)$. So $cl(in(B)) \subseteq U$. Hence B is $BNGPCS$. Conversely, let B be $BNGPCS$ in X. Then, B is not necessarily closed. So B may or may not be $BNGCS$. □

Example 4.8. Consider the topology in Example 2.14 and $BNGCS$ in Example 4.4. Since B is closed set by default, $BNint(B) \neq B$ in most of the cases (equal in some cases). So, B is not $BNGCS$.

Theorem 4.9. *Every $BN\alpha CS$ is $BNGPCS$ but converse not true.*

Proof. Let B be a $BN\alpha CS$ in X and let $B \subseteq U$ and U be $BNOS$ in X. Since $B \subseteq BNcl(B)$, $BNcl(BNint(B)) \subseteq BNcl(BNint(BNcl(B))) \subseteq B$. Hence $BNpcl(B) \subseteq B \subseteq U$. So B is $BNGPCS$ in X. By converse, let B be $BNGPCS$ in X. By default, $BN\alpha CS$ is a subset of $BNpcs$. So it is obvious that every $BNGPCS$ is not necessarily be a $BN\alpha CS$. □

Example 4.10. Since every PCS is not necessarily be a $\alpha - CS$. By definition, every PCS must be a $GPCS$ first. From this, every, $GPCS$ not necessarily be a $\alpha - CS$. So that every $BNGPCS$ not necessarily a $BN\alpha CS$.

Theorem 4.11. *Every $BNPCS$ is $BNGPCS$ but converse not true.*

Proof. Let B be $BNPCS$ in X and let $B \subseteq U$ for some $BNOS$ U in X. By definition of $BNPCS$, $BNcl(BNint(B)) \subseteq B$. This gives, $BNpcl(B) = B \cup BNcl(BNint(B)) \subseteq B$. Hence $BNpcl(B) \subseteq U$. So B is $BNGPCS$ in X. □

Example 4.12. Since every $BNGPCS$ not necessarily be closed. But every $BNPCS$ is closed. So that, every $BNGPCS$ not necessarily be a $BNPCS$.

Theorem 4.13. *Every $BN\alpha GCS$ is $BNGPCS$ but converse not true.*

Proof. Let B be $BN\alpha GCS$ in X and let $B \subseteq U$ for some $BNOS$ U in (X, τ). From Note3.2, $B \cup BNcl(BNint(BNcl(A))) \subseteq U$. So $BNcl(BNint(BNcl(B))) \subseteq U$ and $BNcl(BNint(B)) \subseteq U$. Thus $BNpcl(B) = B \cup BNcl(BNint(A)) \subseteq U$. Hence B is $BNGPCS$ in X. □

Example 4.14. Since every $BNGPCS$ not necessarily a $BN\alpha CS$ and every $BN\alpha GCS$ not necessarily be a $BN\alpha CS$, so that every $BNGPCS$ not necessarily be $BN\alpha GCS$.

Theorem 4.15. *Every $BNGPCS$ is $BNSPCS$ but converse not true.*

Proof. Let B be $BNGPCS$ in X, then $BNpcl(B) \subseteq U$ when $B \subseteq U$ for some $BNOS$ U in X. By definition, $BNcl(BNint(B)) \subseteq B$. Therefore $BNint(BNcl(BNint(B))) \subseteq BNint(B) \subseteq B$. So $BNint(BNcl(BNint(B))) \subseteq B$. Hence B is $BNSPCS$ in X. □

Example 4.16. Since every $BNSPCS$ not necessarily a $BNPCS$ and every $BNPCS$ must be a $BNGPCS$, so that every $BNSPCS$ not necessarily be $BNGPCS$.

5. Bipolar neutrosophic soft generalized pre-open sets

In this section, bipolar neutrosophic soft generalized pre-open sets as the complement of bipolar neutrosophic soft generalized pre-closed sets are proposed. Also, we have investigated some properties of the proposed set with appropriate examples.

Definition 5.1. A $BNSS$ set B is said to bipolar neutrosophic soft generalized pre-open set (BNGPOS) in (X, τ) if the complement B^c is $BNGPCS$ in X. The collection of all $BNGPOS$s of $BNST(X, \tau)$ is denoted by $BNGPO(X)$.

Example 5.2. Consider the BNS-topology (X, τ) in Example 2.14. Let

$$B = \left\{ \begin{array}{c} \left\langle e_1, \{\langle x_1,\ 1, 0, 0.7, -1, 0, 0\rangle, \langle x_2,\ 0.5, 0.2, 0.4, -0.7, 0, -0.3\rangle\} \right\rangle, \\ \left\langle e_2, \{\langle x_1,\ 0.4, 0.5, 0.3, -1, 0, -0.2\rangle, \langle x_2,\ 0.9, 0.1, 0.1, -0.3, -0.5, -0.3\rangle\} \right\rangle, \\ \left\langle e_3, \{\langle x_1,\ 0.5, 0.3, 0.3, -0.2, 0, -0.4\rangle, \langle x_2,\ 0.7, 0.3, 0.1, -0.3, -0.4, -0.1\rangle\} \right\rangle \end{array} \right\}$$

Here, $BNcl(B) = 1_{\mathbb{B}}$ and $BNcl(BNint(B)) = 1_{\mathbb{B}} \supseteq B$ whereas $1_{\mathbb{B}}$ is a $BNOS$ in (X, τ). Hence, by definition, B is a $BNGPOS$ in X.

Theorem 5.3. *Let (X, τ) be a $BNSTS$. Then the following relations are hold.*

1). Every $BNOS$ is $BNGPOS$ but converse not true.

2). Every $BNROS$ is $BNGPOS$ but converse not true.

3). Every $BN\alpha OS$ is $BNGPOS$ but converse not true.

4). Every $BNPOS$ is $BNGPOS$ but not converse not true.

Theorem 5.4. *Let (X, τ) be a $BNSTS$. If $B \in BNGPO(X)$, then $V \subseteq BNint(BNcl(B))$ whenever $V \subseteq B$ and V is $BNCS$ in X.*

Proof. Let $B \in BNGPO(X)$. Then B^c be a $BNGPCS$ in X. So $BNpcl(B^c) \subseteq U$ whenever $B^c \subseteq U$ and U is $NVOS$ in X. Therefore $BNcl(BNint(B^c)) \subseteq U$. This implies that $U^c \subseteq BNint(BNcl(B))$ whenever $U^c \subseteq B$ and U^c is $BNCS$ in X. Substituting U^c by V, we get $V \subseteq BNint(BNcl(B))$ whenever $V \subseteq B$ and V is $BNCS$ in X. □

Theorem 5.5. *Let (X, τ) be $BNSTS$. Then for every $B \in BNGPO(X)$ and for every $N \in BNS(X)$, $BNpint(B) \subseteq N \subseteq B$ implies $N \in BNGPO(X)$.*

Proof. By hypothesis $B^c \subseteq N^c \subseteq (BNpint(B))^c$. Let $N^c \subseteq U$ and U be $BNOS$. Since $B^c \subseteq N^c \subseteq, B^c \subseteq U$. But B^c is $BNGPCS$, $BNpcl(B^c) \subseteq U$. Also $B^c \subseteq (BNpint(B))^c = BNpcl(B^c)$. Therefore $BNpcl(N^c) \subseteq BNpcl(B^c) \subseteq U$. Hence N^c is $BNGPCS$ which implies B is $BNGPOS$ in X. □

Theorem 5.6. *A BNS B of $BNSTS(X, \tau)$ is $BNGPOS$ if and only if $V \subseteq BNpint(B)$ whenever V is $BNCS$ and $V \subseteq B$.*

Proof. Suppose B is $BNGPOS$ in X. Let V be $BNCS$ and $V \subseteq B$. Then V^c is $BNOS$ in X such that $B^c \subseteq V^c$. Since B^c is $BNGPCS$, we have $BNpcl(B^c) \subseteq V^c$. Hence $(BNpint(B))^c \subseteq V^c$. Therefore $V \subseteq BNpint(B)$.

On the other hand, let B be BNS in X and let $V \subseteq BNpint(B)$ whenever V is $BNCS$ and

$V \subseteq B$. Then $B^c \subseteq N^c$ and N^c is $BNOS$. By hypothesis, $(BNpint(B))^c \subseteq N^c$ which implies $BNpcl(B^c) \subseteq N^c$. Therefore B^c is $BNGPCS$ of X. Hence B is $BNGPOS$ in X. □

Theorem 5.7. *A BNS B of a BNSTS(X, τ) is BNGPOS if and only if $V \subseteq BNint(BNcl(B))$ whenever V is BNCS and $V \subseteq B$.*

Proof. Suppose B is $BNGPOS$ in X. Let V be $NVCS$ and $V \subseteq B$. Then V^c is $BNOS$ in X such that $B^c \subseteq V^c$. Since B^c is $BNGPCS$, we have $BNpcl(B^c) \subseteq V^c$. Therefore $BNcl(BNint(B^c)) \subseteq V^c$. Hence $(BNint(BNcl(B)))^c \subseteq V^c$. This implies $V \subseteq BNint(BNcl(B))$.

On the other hand, let B be BNS of X and let $V \subseteq BNint(BNcl(B))$ whenever V is $BNCS$ and $V \subseteq B$. Then $B^c \subseteq V^c$ and V^c is $BNOS$. By hypothesis, $(BNint(BNcl(B)))^c \subseteq V^c$. Hence $BNcl(BNint(B^c)) \subseteq V^c$, which implies $BNpcl(B^c) \subseteq V^c$. Hence B is $BNGPOS$ of X. □

Theorem 5.8. *For any BNS B, B is BNOS and BNGPCS in X if and only if B is BNROS in X.*

Proof. Let B be $BNOS$ and $BNGPCS$ in X. Then $BNpcl(B) \subseteq B$. This implies $BNcl(BNint(B)) \subseteq B$. Since B is $BNOS$, it is $BNPOS$. Hence $B \subseteq BNint(BNcl(B))$. Therefore $B = BNint(BNcl(B))$. Hence B is $BNROS$ in X.

On the other hand, let B be $BNROS$ in X. So $B = BNint(BNcl(B))$. Let $B \subseteq U$ and U is $BNOS$ in X. This implies that $BNpcl(B) \subseteq B$. Hence B is $BNGPCS$ in X. □

Remark 5.9. There are few limitations of the proposed works. The proposed bipolar neutrosophic soft generalized pre-closed sets and pre-open sets are purely based on point set topology (i.e. general topology). So it is quite difficult to apply in real world problems unlike neutrosophic sets. On the other hand, along with neutrosophic topology, we can explore many applied mathematics problems such as decision making technique, image processing, data analytics and so on. Also, the soft sets are parametrized sets in nature. So obviously, the proposed topology and proposed sets are based on parameters. There are few drawbacks when applying soft sets in real world problems such as choosing correct number of parameters and choosing only the essential parameters. It will create an impact in final results. To overcome this, the user can decide the number of parameters and choice of parameters depends on the problem's nature.

6. Conclusion

Bipolar neutrosophic soft set is the base for many topological spaces. In topology, the topological structures such as closedness and openness are the important concepts. It helps to determine the continuity of a mapping between to topologies. Many researchers have proposed various types of closed and open sets for a specific topological space. In this paper, we introduced new family of sets namely, bipolar neutrosophic soft generalized pre-closed sets and bipolar neutrosophic soft generalized pre-open sets for the bipolar neutrosophic soft topological space. Further, some important relations between proposed sets and many other type of sets have been discussed through theorems. Development of bipolar neutrosophic soft generalized pre-sets is thought to contribute to the development of bipolar neutrosophic soft continuity in the topology as well as algebra, geometry and analysis of other sub-branches of mathematics. We expect that the proposed sets will serve contributions to some future works about bipolar neutrosophic soft topology. Our future work will consist applications of the proposed sets and topology in decision making problems. There are numerous neutrosophy based decision making algorithms available. In future, we will explore decision making scenarios and try to define novel algorithms by applying proposed concepts. Also, image processing is one of the field which uses neutrosophic logic. We will try to develop image processing algorithms based on proposed neutrosophic topology such as image denoising, segmentation, edge detection and so on.

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A Maple Code to Perform Operations on Single Valued Neutrosophic Matrices

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Abstract: In this paper, we present a maple programming code that helps useres and scientific researchers to input single valued neutrosophic matrices, checks whether inputted matrix is single valued neutrosohic matrix, finds the complement of a single valued neutrosophic matrix, calculates score matrix, accuracy matrix and certainty matrix, finds the union and intersection of two single valued neutrosophic matrices, finds addition and product of two single valued neutrosophic matrices, also finds transpose of single valued neutrosophic matrix. This code is very important and useful in decision making problems that depend on single valued neutrosophic data.

Keywords: Maple language; neutrosophic set; operations of matrices; single valued neutrosophic sets

1. Introduction

The idea of fuzzy set was introduced by Zadeh where every element has a degree of membership[1]. In [2], as a generalization of fuzzy set, Atanassov introduced intuitionistic fuzzy set with two degrees for each element namely degree of membership nad degree of non-memebrship. Henceforth, Smarandache introduced neutrosophic set which is based on three independent degrees namely truth membership, indeterminate membership and falsity membership [4]. Single valued and interval valued neutrosophic sets and numbers have many applications in many branches of science including pure mathematics, linear algebra, statistics, probability, operations research, etc., as they are fruitfully address uncertainities as a single numer and interval numbers in the unit interval [0,1] as well [9, 12-13]. Neutrosopihic matrices, a development of neutrosophic theory, are used to deal with uncertainties and have beautiful operations which are very useful in decision making [16]. Many researchers have been presented packages and programming codes to deal with single valued neutrosophic numbers. Various engineering and scientific problems can be solved by linear methods, ut non-trivial examples of these problems may require large amounts of memory to represent and even large amounts of computing time to solve. Memory demands of large arrays can be reduced by partitioning those arrays into smaller sections of processing and loading a few of those sections from disk into virtual memory as they are only needed. Using this way, the computation can run smaller and faster in a time-saved environment. Maple contributes, a good prototyping environment for addressing this problem [3]. Resultant matrices can be obtained by using Macaulay2 and Maple [5]. The Maple package called conley has been introduced in [6], to compute connection and C-connection matrices. Some of the definite integrals involving Residue theory has been evaluated using Maple code in [7]. In [8], special types of Maple codes namely Tan method maple code, Tanh method maple code, Sech method maple code, Cot method maple code and Coth method maple code have been introduced. In [10], Maple code of the cubic algorithm has been proposed for obtaining optimized result of multiobjective decision making problem with box constaints. Practical explanation of SCAToolbox is given in [11]. Minimum arc length of an intuitionistic fuzzy hyperpath is determined using Maple in [14]. In [15], some of the new operations on single-valued neutrosophic matrices have been proposed and applied in a decision making problem. A new Python toolbox for single valued neutrosophic matrices has been proposed in [16]. Orthogonal basis for a set of vectors or a matrix using Householder transformations has been constructed where only rational computtaions required with rational output using Maple in [17]. A Maple package for the symbolic computation of Drazin matrices with multivariate transcendental functions has been introduced in [18]. The approximate value of two Taylor series for the real or complex valued functions of a single variable has been obtained using Maple in [19] where the Maple implementation was stable and effective in evaluating blends using linear-cost Horner form. Maple DEtools have been introduced in [20]. A variety of approaches to study formal multivariate power series and univariate polynomials over such series was provided as a multivariate power series. Its implementation based on idle evaluation techniques and takes advantage of Maple aspect for object oriented programming [21]. The determinant and adjoint of neutrosophic matrix have been determined in [22]. Most of the jobs and proofs of Euclidean geometry can easily be carried out without sine and cosine functions and without introducing differential calculus as well. Using Maple, this concept has been accomplished in [23]. Complex neutrosophic soft matrices were introduced and some of the basic operations namely, complement, union and intersection on these matrices have been presented. Also, a novel algorithm has been developed using complex neutrosophic soft matrices and applied in signal processing [24]. Representation of neutrosophic matrices defined over a neutrosophic field using neutrosophic linear transformation between neutrosophic vector spaces and it was concluded that, every neutrosophic matrix can be represented uniquely by a neutrosophic linear transformation [25]. Neutrosophic matrices are widely used to handle with especially computer science problems in which the inputs are neutrosophic numbers. This kind of matrices and its properties have been proposed in [26]. A Maple package has been introduced for performing the operations on single-valued trapezoidal neutrosophic numbers using (α,β,γ)-cuts [27].The interrelation between the motion parameters and the configuration elements has been investigated by performing 6-degree-of-freedom simulations of the Autorotative flight of Maple seeds [28]. Selected tools offered by Maple and used support contributed by Maplesoft.Inc for professional and modern implementation in the field of scientific computation, modeling and visualizations in economics is mapped in [29]. In this paper, we presented a maple code that deals with single valued neutrosophic matrices which have many applications in various fields of science specially decision making. This code allows users and researchers to do many operations on single valued neutrosophic matrices like addition, product, union, intersection, transpose, etc. The rest of the paper is organized as follows. In section 2, background of single valued neutrosoohic sets and its operations have been presented for better understanding of the present work. In section 3, single valued matrix operatons have been computed using Maple programming. In section 4, conclusion of the present work is given with future directon.

2. Background and Single Valued Neutrosophic Sets

In this section, we will discuss some definitions regarding neutrosophic sets, single valued neutrosophic sets, the set-theoretic operators on single valued neutrosophic set, which will be used in the rest of the paper. However, for details on the single valued neutrosophic sets, one can see(Smarandache,1998,Wang et al, 2014**,** Zhang et al, 2014).

2.1. Definition [4]:Suppose ξ be an universal set. The neutrosophic set A on the universal set ξ categorized in to three membership functions called the true T_A(x), indeterminate I_A(x) and false F_A(x) contained in real standard or non-standard subset of]-0, 1+[respectively.

$$-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3+ \quad (1)$$

2.2. Definition [12]:suppose ξ be a space of points (objects)with a generic element in ξ denoted by x. A single valued neutrosophic set (SVNS) A in ξ is characterized by truth-membership function T_A, indeterminacy-membership functionI_A, andfalsity-membershipfunctionF_A. Foreachpoint x ∈ ξ, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0,1]$.

$$A_{\text{SVNS}} = \{\langle T_A(x), I_A(x), F_A(x)\rangle : x \in \xi\}$$

$$\text{with} \quad 0 \leq T_A(x)+I_A(x)+F_A(x) \leq 3 \quad (2)$$

2.3. Definition [12] : Suppose two interval valued neutrosophic sets

$$A_{\text{SVNS}} = \{\langle T_A(x), I_A(x), F_A(x)\rangle : x \in \xi\}$$

and

$$B_{\text{SVNS}} = \{\langle T_B(x), I_B(x), F_B(x)\rangle : x \in \xi\}$$

the set-theoretic operators on the interval neutrosophic set are defined as follow.

1. An single valued neutrosophic set A is contained in another single valued neutrosophic set B, $A_{SVNS} \subseteq B_{SVNS}$, if and only if

$T_A(x) \leq T_B(x)$,

$I_A(x) \geq I_B(x)$,

$F_A(x) \geq F_B(x)$, for all $x \in \xi$.

2. Two singlevalued neutrosophic sets A and B are equal, written as $A_{SVNS} = B_{SVNS}$, if and only if A $\subseteq$ B and B $\subseteq$ A, i.e.

$T_A(x) = T_B(x)$,

$I_A(x) = I_B(x)$,

$F_A(x) = F_B(x)$,

for all $x \in \xi$.

3. A single neutrosophic set A is empty if and only if

$T_A(x) = 0$,$I_A(x) = 1$ and$F_A(x) = 0$,for all $x \in \xi$.

The complement of a single neutrosophic set A is denoted by A^c and is defined by

$A_{SVNS}{}^c = \{x, [F_A(x)], [1 - I_A(x),], [T_A(x),] : x \in X\}$,

for all x in ξ.

4. The intersection of two singlevalued neutrosophic sets A and B is a singlevalued neutrosophic set$A \cap B$defined as follow

$$A_{SVNS} \cap B_{SVNS} = \{\langle x, [T_A(x) \wedge T_B(x)], [I_A(x) \vee I_B(x)], [F_A(x) \vee F_B(x)]\rangle : x \in \xi\}, \text{for all } x \text{ in } \xi. \quad (3)$$

5. The union of two single valued neutrosophic sets A and B is a single valued neutrosophic set $A_{IVNS} \cup B_{SVNS}$defined as follow:

$$A_{SVNS} \cup B_{SVNS} = \left\{ \begin{matrix} x, [T_A^L(x) \vee T_B^L(x), T_A^U(x) \vee T_B^U(x)], & \\ \langle \quad [I_A^L(x) \wedge I_B^L(x), I_A^U(x) \wedge I_B^U(x)], & \rangle : x \in \xi \\ [F_A^L(x) \wedge F_B^L(x), F_A^U(x) \wedge F_B^U(x)] & \end{matrix} \right\}, \text{for all } x \text{ in } \xi. \quad (4)$$

6. The difference of two single valued neutrosophic sets A and B is single valued neutrosophic set $A_{SVNS} \ominus B_{SVNS}$defined as follow: $A \ominus B = <[T_{A\ominus B}, T^U_{A\ominus B}], [I_{A\ominus B}, I^U_{A\ominus B}], [F_{A\ominus B}, F^U_{A\ominus B}]$ (5)

where

$T_{A\ominus B}$=min$(T_A(x), F_B(x))$,

$I_{\mathrm{A}\ominus_2\mathrm{B}}$= $\max(I_A(x), 1 - I_B(x))$,
$F_{\mathrm{A}\ominus_2\mathrm{B}}$=$\max(F_A(x), T_B(x))$,
[15] introduced a new difference operation for the single valued neutrosophic sets as follow:
$\mathrm{A}\ominus_2\mathrm{B}$=<$T_{\mathrm{A}\ominus_2\mathrm{B}}, I_{\mathrm{A}\ominus_2\mathrm{B}}, F_{\mathrm{A}\ominus_2\mathrm{B}}$]> (6)

where
$T_{\mathrm{A}\ominus_2\mathrm{B}}$=$T_A(x) - F_B(x)$,
$I_{\mathrm{A}\ominus_2\mathrm{B}}$= $\max(I_A(x), I_B(x))$,
$F_{\mathrm{A}\ominus_2\mathrm{B}}$=$F_A(x) - T_B(x)$,

for all x in ξ.

7. The scalar multiplication of single valued neutrosophic set A is $A_{\mathrm{SVNS}}.a$, defined as follow

$A_{\mathrm{SVNS}}.a = \{\langle x, \min(T_A^L(x).a, 1), \min(I_A^L(x).a, 1), \min(F_A^L(x).a, 1)\rangle : x \in \xi\}$for all $x \in \xi$, a ∈ R+.

8. The scalar division of single neutrosophic set A is A_{SVNS}/adefined as follow

$$A_{\mathrm{IVNS}}/a == \{\langle x, \min(T_A^L(x)/ a, 1), \min(I_A^L(x)/ a, 1), \min(F_A^L(x)/a, 1)\rangle : x \in \xi\}$$

for all $x \in \xi$, $a \in R^+$

the convenient method for comparing single valued neutrosophic and interval valued neutrosophic numbers can be done by using score function.

2.4 Definition [22]: Suppose A be an interval neutrosophic numberA_{IVNN}, the score function is defined as follow :

$$\tilde{S}_{IVNN}(x) = \frac{T_A^L(x)+T_A^U(x)+4-I_A^L(x)-I_A^U(x)-F_A^L(x)-F_A^U(x)}{6} \quad (7)$$

$$\tilde{S}_{SVNN}(x) = \frac{2+T_A(x)-I_A(x)-F_A(x)}{3}$$

$$\tilde{A}_{IVNN}(x) = \frac{T_A^L(x)+T_A^U(x)-F_A^L(x)-F_A^U(x)}{2}$$

$$\tilde{A}_{SVNN}(x) = T_A(x) - F_A(x)$$

$$\tilde{C}_{IVNN}(x) = \frac{T_A^L(x)+T_A^U(x)}{2}$$

$$\tilde{C}_{SVNN}(x) = T_A(x)$$

2.5 Definition [12]:A single valued valued neutrosophic matrix(SVNM) of order m× n is defined as

A_{SVNM}=$[< a_{ij}, a_{ij_T}, a_{ij_I}, a_{ij_F} >]_{\text{m}\times \text{n}}$ where

a_{ij_T} is the membership value of element a_{ij} in A.

a_{ij_I} is the indeterminate-membership value of element a_{ij} in A.

$a^{L}_{ij_F}$ is the non- membership value of element a_{ij} in A.

For simplicity, we write A as

A_{SVNM}= $[< a_{ij_T}, a_{ij_I}, a_{ij_F} >]_{\text{m}\times \text{n}}$ (8)

3. Computing the Single Valued Neutrosophic Matrix Operations using Maple Language

In this section, the Maple program is developed for inputting the single valued neutrosophic matrices as follows:

3.1. Inputting SVNM to Maple

Here, for inputting SVNM to Maple, simply call the function SVNMInput(m,n) where m, n are numbers of rows and columns respectively and the code is described as follows:

```
interface(warnlevel=0):with(Maplets[Elements]):with(Maplets):
SVNMInput:=proc(m::integer,n::integer)
local mat:=Matrix(m,n);
for i from 1 to m by 1 do
for j from 1 to n by 1 do
truth:=Maplet(InputDialog['x'](cat("Enter truth of element
",i,",",j),'onapprove'=Shutdown(['x']),'oncancel'=Shutdown()));
truth:=parse(op(Display(truth)));
indeterminacy:=Maplet(InputDialog['x'](cat("Enter indeterminacy of element
",i,",",j),'onapprove'=Shutdown(['x']),'oncancel'=Shutdown()));
indeterminacy:=parse(op(Display(indeterminacy)));
falsity:=Maplet(InputDialog['x'](cat("Enter falsity of element
",i,",",j),'onapprove'=Shutdown(['x']),'oncancel'=Shutdown()));
falsity:=parse(op(Display(falsity)));
mat(i,j):=convert([truth,indeterminacy,falsity],string);
end do;
end do;
mat;
```

```
end proc:
```

3.1.1. Checking the matrix is SVNM or not

To generate the Maple program for deciding if a given matrix (say mat) is single valued neutrosophic matrix or not, simply call the function **SVNMChecking (mat)** is defined as follow:

```
SVNMChecking:=proc(mat)
IsMembership:=proc(num)
if num<0 or num>1 then return false else return true end if;
end proc:
m,n:=LinearAlgebra[Dimension](mat);
result:=true;
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat(i,j));
truth:=x[1];
indeterminacy:=x[2];
falsity:=x[3];
result:= IsMembership(x[1]) and IsMembership(x[2]) and IsMembership(x[3]);
if not result then break; end if;
end do;
if not result then break; end if;
end do;
if result then cat("your matrix is a single valued neutrosophic matrix") else cat("your matrix is not a
single valued neutrosophic matrix") end if;
end proc:
```

Example 1. In this example we evaluate the checking the matrix is SVNM or not of the single valued neutrosophic matrix E of order 4X4:

E=

$$\begin{pmatrix} <.5,.7,.2> & <.4,.4,.5\ > & <.7,.7,.5> & <.1,.5,.7> \\ <.9,.7,.5\ > & <.7,.6,.8> & <.9,.4,.6> & <.5,.2,.7> \\ <.9,.4,.2,> & <.2,.2,.2> & <.9,.5,.5> & <.7,.5,.3> \\ <.9,.7,.2,> & <.3,.5,.2> & <.5,.4,.5> & <.2,.4,.8> \end{pmatrix}$$

The single valued neutrosophic matrix E can be inputted in Maple code like this:

E:=SVNMInput(4,4);

Then an input box dialogue is going to appear and lead you how to input elements.

The result of checking the matrix is SVNM or not E can be obtained by the call of the command SVNMChecking (E);

And the result will be:

"your matrix is a single valued neutrosophic matrix"

3.2. Determining complement of single valued neutrosophic matrix

For a given SVNM A= $[< T_{ij}, I_{ij}, F_{ij} >]_{\mathrm{m}\times\mathrm{n}}$, the complement of A is defined as follow:

$$A^c = [< \{1\} - T_{ij}, \{1\} - I_{ij}, \{-1\} - F_{ij}]_{\mathrm{m}\times\mathrm{n}} \quad (9)$$

$$A^c = [< F_{ij}, \{1\} - I_{ij}, T_{ij},]_{\mathrm{m}\times\mathrm{n}} \quad (10)$$

To generate the Maple program for finding complement of single valued neutrosophic matrix, simple call of the function**SVNMCompelementOf1 (mat)** is defined as follow:

The function SVNMCompelementOf1 (mat) the below returns the complement matrix of a given single valued neutrosophic matrix mat for (9).

```
SVNMCompelementOf1:=proc(mat::Matrix)

temp:=LinearAlgebra[Copy](mat);

m,n:=LinearAlgebra[Dimension](temp);

for i from 1 to m by 1 do

for j from 1 to n by 1 do

x:=parse(temp(i,j));

truth:=1-x[1];

indeterminacy:=1-x[2];
```

```
falsity:=1-x[3];
temp(i,j):=convert([truth,indeterminacy,falsity],string);
end do;
end do;
temp;
end proc:
```

Example 2. Evaluate the complement of matrix E in example 1.

So, the complement of single valued neutrosophic matrix E is portrayed as follow:

$$E^c = \begin{pmatrix} <.5,.3,.8> & <.6,.6,.5> & <.3,.3,.5> & <.9,.5,.3> \\ <.1,.3,.5> & <.3,.4,.2> & <.1,.6,.4> & <.5,.8,.3> \\ <.1,.6,.8> & <.8,.8,.8> & <.1,.5,.5> & <.3,.5,.7> \\ <.1,.3,.8,> & <.7,.5,.8> & <.5,.6,.5> & <.8,.6,.2> \end{pmatrix}$$

The result of the complement of single valued neutrosophic matrix E can be obtained by the call of the command SVNMCompelementOf1(E);

SVNMCompelementOf1(E);

$$\begin{bmatrix} \text{"[.5, .3, .8]"} & \text{"[.6, .6, .5]"} & \text{"[.3, .3, .5]"} & \text{"[.9, .5, .3]"} \\ \text{"[.1, .3, .5]"} & \text{"[.3, .4, .2]"} & \text{"[.1, .6, .4]"} & \text{"[.5, .8, .3]"} \\ \text{"[.1, .6, .8]"} & \text{"[.8, .8, .8]"} & \text{"[.1, .5, .5]"} & \text{"[.3, .5, .7]"} \\ \text{"[.1, .3, .8]"} & \text{"[.7, .5, .8]"} & \text{"[.5, .6, .5]"} & \text{"[.8, .6, .2]"} \end{bmatrix}$$

The function SVNMCompelementOf2(A) the below returns the complement matrix of a given single valued neutrosophic matrix A for (10).

```
SVNMCompelementOf2:=proc(mat::Matrix)
temp:=LinearAlgebra[Copy](mat);
m,n:=LinearAlgebra[Dimension](temp);
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(temp(i,j));
truth:=x[3];
indeterminacy:=1-x[2];
```

```
falsity:=x[1];
temp(i,j):=convert([truth,indeterminacy,falsity],string);
end do;
end do;
temp;
end proc:
```

The single valued neutrosophic matrix A is a simple example, one can create his/her SVNM and try it into the function **SVNMCompelementOf1** ( ); or **SVNMCompelementOf2** ( );

3.3. Determining the score, accuracy and certainty matrices of single valued neutrosophic matrix

To generate the Maple program for obtaining the score matrix, accuracy of single valued neutrosophic matrix, simple call of the functions**ScoreMatrix( )**, **AccuracyMatrix ( )** and **CertaintyMatrix ( )** are defined as follow:

```
ScoreMatrix:=proc(mat::Matrix)
m,n:=LinearAlgebra[Dimension](mat);
scoreMat:=Matrix(m,n);
fori from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat(i,j));
score:=(2+x[1]-x[2]-x[3])/3;
scoreMat(i,j):=score;
end do;
end do;
scoreMat;
endproc:
AccuracyMatrix:=proc(mat::Matrix)
m,n:=LinearAlgebra[Dimension](mat);
aMat:=Matrix(m,n);
fori from 1 to m by 1 do
for j from 1 to n by 1 do
```

```
x:=parse(mat(i,j));
a:=x[1]-x[3];
aMat(i,j):=a;
end do;
end do;
aMat;
endproc:
CertaintyMatrix:=proc(mat::Matrix)
m,n:=LinearAlgebra[Dimension](mat);
cMat:=Matrix(m,n);
fori from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat(i,j));
c:=x[1];
cMat(i,j):=c;
end do;
end do;
cMat;
endproc:
```

3.4. Computing union of two single valued neutrosophic matrices

The union of two single valued neutrosophic matrices A and Bis defined as follow:

$$A \cup B = C = \left[< c_{ij_T}, c_{ij_I}, c_{ij_F} >\right]_{m\times n} \quad (11)$$

where

$c_{ij_T} = a_{ij_T} \vee b_{ij_T}$,

$c_{ij_I} = a_{ij_I} \wedge b_{ij_I}$,

$c_{ij_F} = a_{ij_F} \wedge b_{ij_F}$

The union of two single valued neutrosophic matrices can be determined using the Maple program with simple call of the following function**Union(A, B)** is described as follows:

```
Union:=proc(mat1::Matrix,mat2::Matrix)

m1,n1:=LinearAlgebra[Dimension](mat1);

m2,n2:=LinearAlgebra[Dimension](mat2);

if (n1=n2) and (m1=m2) then

m:=m1;n:=n1;

unionMat:=Matrix(m,n);

for i from 1 to m by 1 do

for j from 1 to n by 1 do

x:=parse(mat1(i,j));

y:=parse(mat2(i,j));

truth:=max(x[1],y[1]);

indeterminacy:=min(x[2],y[2]);

falsity:=min(x[3],y[3]);

unionMat(i,j):=convert([truth,indeterminacy,falsity],string);

end do;

end do;

unionMat;

else

print("dimension of given matrices must be equal!");

end if;

end proc:
```

Example 3. Here, union of two single valued neutrosophic matrices E and F of order 4X4 has been obtained:

E=

$$\begin{pmatrix} <.5,.7,.2> & <.4,.4,.5\ > & <.7,.7,.5> & <.1,.5,.7> \\ <.9,.7,.5\ > & <.7,.6,.8> & <.9,.4,.6> & <.5,.2,.7> \\ <.9,.4,.2,> & <.2,.2,.2> & <.9,.5,.5> & <.7,.5,.3> \\ <.9,.7,.2,> & <.3,.5,.2> & <.5,.4,.5> & <.2,.4,.8> \end{pmatrix}$$

The single valued neutrosophic matrix E can be inputted in Maple code like this:

E:=SVNMInput(4,4);

F=

$$\begin{pmatrix} <.3,.4,.3> & <.1,.2,.7> & <.3,.2,.6> & <.2,.1,.3> \\ <.2,.2,.7> & <.3,.5,.6> & <.6,.5,.4,> & <.3,.4,.4> \\ <.5,.3,.1> & <.5,.4,.3> & <.5,.8,.6> & <.4,.6,.5> \\ <.6,.1,.7> & <.4,.6,.4> & <.4,.9,.3> & <.4,.5,.4> \end{pmatrix}$$

The single valued neutrosophic matrix F can be inputted in Maple code like this:

F:=SVNMInput(4,4);

So, the union matrix of two single valued neutrosophic matricesis portrayed as follow

$$E_{SVNM} \cup F_{SVNM} = \begin{pmatrix} <.5,.4,.2> & <.4,.2,.5> & <.7,.2,.5> & <.2,.1,.3> \\ <.9,.2,.5> & <.7,.5,.6> & <.9,.4,.4> & <.5,.2,.4> \\ <.9,.3,.1> & <.5,.2,.2> & <.9,.5,.5> & <.7,.5,.3> \\ <.9,.1,.2,> & <.4,.5,.2> & <.5,.4,.3> & <.4,.4,.4> \end{pmatrix}$$

The result of union matrix of two single valued neutrosophic matrices E and F can be obtained by the call of the command Union (E, F):

Union(E, F);

$$\begin{bmatrix} "[.5, .4, .2]" & "[.4, .2, .5]" & "[.7, .2, .5]" & "[.2, .1, .3]" \\ "[.9, .2, .5]" & "[.7, .5, .6]" & "[.9, .4, .4]" & "[.5, .2, .4]" \\ "[.9, .3, .1]" & "[.5, .2, .2]" & "[.9, .5, .5]" & "[.7, .5, .3]" \\ "[.9, .1, .2]" & "[.4, .5, .2]" & "[.5, .4, .3]" & "[.4, .4, .4]" \end{bmatrix}$$

3.5. Computing intersection of two single valued neutrosophic matrices

The union of two single valued neutrosophic matrices A and B is defined as follow:

$A \cap B = D = [< d_{ij_T}, d_{ij_I}, d_{ij_F} >]_{m\times n}$ (12)

where

$d_{ij_T} = a_{ij_T} \wedge b_{ij_T}$,

$d_{ij_I} = a_{ij_I} \vee b_{ij_I}$,

$d_{ij_F} = a_{ij_F} \vee b_{ij_F}$,

To develop the Maple program to find the intersection of two single valued neutrosophic matrices, simple call of the functionIntersection(,) is defined in the following manner.

```
Intersection:=proc(mat1::Matrix,mat2::Matrix)
m1,n1:=LinearAlgebra[Dimension](mat1);
```

```
m2,n2:=LinearAlgebra[Dimension](mat2);
if (n1=n2) and (m1=m2) then
m:=m1;n:=n1;
intersectMat:=Matrix(m,n);
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat1(i,j));
y:=parse(mat2(i,j));
truth:=min(x[1],y[1]);
indeterminacy:=max(x[2],y[2]);
falsity:=max(x[3],y[3]);
intersectMat(i,j):=convert([truth,indeterminacy,falsity],string);
end do;
end do;
intersectMat;
else
print("dimension of given matrices must be equal!");
end if;
end proc:
```

Example 4. Here, the intersection of two single valued neutrosophic matrices E and F of order 4X4 which are presented in example 3 is

So, the intersection matrix of two single valued neutrosophic matrices is portrayed as follow

$$E_{SVNM} \cap F_{SVNM} = \begin{pmatrix} <.3,.7,.3> & <.1,.4,.7> & <.3,.7,.6> & <.1,.5,.7> \\ <.2,.7,.7> & <.3,.6,.8> & <.6,.5,.6> & <.3,.4,.7> \\ <.5,.4,.2> & <.2,.4,.3> & <.5,.8,.6> & <.4,.6,.5> \\ <.6,.7,.7> & <.3,.6,.4> & <.4,.9,.5> & <.2,.5,.8> \end{pmatrix}$$

The result of intersection matrix of two single valued neutrosophic matrices E and F can be obtained by the call of the command Intersection (E, F):

Intersection (E, F)

$$\begin{bmatrix} "[.3, .7, .3]" & "[.1, .4, .7]" & "[.3, .7, .6]" & "[.1, .5, .7]" \\ "[.2, .7, .7]" & "[.3, .6, .8]" & "[.6, .5, .6]" & "[.3, .4, .7]" \\ "[.5, .4, .2]" & "[.2, .4, .3]" & "[.5, .8, .6]" & "[.4, .6, .5]" \\ "[.6, .7, .7]" & "[.3, .6, .4]" & "[.4, .9, .5]" & "[.2, .5, .8]" \end{bmatrix}$$

3.6. Computing addition operation of two single valued neutrosophic matrices.

The addition of two single valued neutrosophic matrices A and B is defined as follow:

$$A \oplus B = S = \left[< s_{ij_T}, s_{ij_I}, s_{ij_F} >\right]_{m\times n} \quad (13)$$

where

$s_{ij_T} = a_{ij_T} + b_{ij_T} - a_{ij_T}.b_{ij_T},$

$s_{ij_I} = a_{ij_I}.b_{ij_I},$

$s_{ij_F} = a_{ij_F}.b_{ij_F},$

To generate the Maple program for obtaining the addition of two single valued neutrosophic matrices, simple call of the function **Addition (A, B)**is defined as follow:

```
Addition:=proc(mat1::Matrix,mat2::Matrix)
m1,n1:=LinearAlgebra[Dimension](mat1);
m2,n2:=LinearAlgebra[Dimension](mat2);
if (n1=n2) and (m1=m2) then
m:=m1;n:=n1;
addMat:=Matrix(m,n);
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat1(i,j));
y:=parse(mat2(i,j));
truth:=x[1]+y[1]-x[1]*y[1];
indeterminacy:=x[2]*y[2];
falsity:=x[3]*y[3];
addMat(i,j):=convert([truth,indeterminacy,falsity],string);
end do;
end do;
```

```
addMat;
else
print("dimension of given matrices must be equal!");
end if;
end proc:
```

Example 5. In this example we evaluate the addition of the two single valued neutrosophic matrices E and F of order 4X4 presented in example 3:

So, the addition matrix of two single valued neutrosophic matricesis portrayed as follow

$$C_{SVNM} \oplus D_{SVNM} = \begin{pmatrix} <.65,.28,.06 > & <.46,.08,.35> & <.79,.14,.30> & <.28,.05,.21> \\ <.92,.14,.35> & <.79,.30,.48> & <.96,.20,.24> & <.65,.08,.28> \\ <.65,.12,.02> & <.60,.08,.06> & <.95,.40,.30> & <.82,.30,.15> \\ <.96,.07,.14> & <.58,.30,.08> & <.70,.36,.15> & <.52,.20,.3> \end{pmatrix}$$

The result of addition matrix of two single valued neutrosophic matrices E and F can be obtained by the call of the command addition (E, F):

```
Addition(E,F);
```

```
[ "[.65, .28, .6e-1]"   "[.46, .8e-1, .35]"    "[.79, .14, .30]"  "[.28, .5e-1, .21]" ]
[ "[.92, .14, .35]"     "[.79, .30, .48]"      "[.96, .20, .24]"  "[.65, .8e-1, .28]" ]
[ "[.95, .12, .2e-1]"   "[.60, .8e-1, .6e-1]"  "[.95, .40, .30]"  "[.82, .30, .15]"   ]
[ "[.96, .7e-1, .14]"   "[.58, .30, .8e-1]"    "[.70, .36, .15]"  "[.52, .20, .32]"   ]
```

3.7. Computing product of two single valued neutrosophic matrices

The product of two single valued neutrosophic matrices A and B is defined as follow:

$$A \odot B = R = [< r_{ij_T}, r_{ij_I}, r_{ij_F} >]_{m\times n} \tag{14}$$

where

$r_{ij_T} = a_{ij_T} . b_{ij_T},$

$r_{ij_I} = a_{ij_I} + b_{ij_I} - a_{ij_I} . b_{ij_I},$

$r_{ij_F} = a_{ij_F} + b_{ij_F} - a_{ij_F} . b_{ij_F},$

To generate the Maple program for finding the product operation of two single valued neutrosophic matrices, simple call of the function **Product (A, B)** is defined as follow:

```
Prod:=proc(mat1::Matrix,mat2::Matrix)
```

```
m1,n1:=LinearAlgebra[Dimension](mat1);
m2,n2:=LinearAlgebra[Dimension](mat2);
if (n1=n2) and (m1=m2) then
m:=m1;n:=n1;
prodMat:=Matrix(m,n);
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat1(i,j));
y:=parse(mat2(i,j));
truth:=x[1]*y[1];
indeterminacy:=x[2]+y[2]-x[2]*y[2];
falsity:=x[3]+y[3]-x[3]*y[3];
prodMat(i,j):=convert([truth,indeterminacy,falsity],string);
end do;
end do;
prodMat;
else
print("dimension of given matrices must be equal!");
end if;
end proc:
```

Example 6. In this example we evaluate the product of the two single valued neutrosophic matrices E and F of order 4X4 presented in example 3:

So, theproduct matrix of two single valued neutrosophic matricesis portrayed as follow

$$E_{SVNM} \odot F_{SVNM} = \begin{pmatrix} <.15,.82,.44> & <.04,.52,.85> & <.21,.76,.80> & <.02,.55,.79> \\ <.18,.76,.85> & <.21,.80,.92> & <.54,.70,.76> & <.15,.52,.82> \\ <.45,.58,.28> & <.10,.52,.44> & <.45,.90,.80> & <.28,.80,.65> \\ <.54,.73,.76> & <.12,.80,.52> & <.20,.94,.65> & <.08,.70,.88> \end{pmatrix}$$

The result of product matrix of two single valued neutrosophic matrices E and F can be obtained by the call of the command Product (E, F):

Product(E, F);

Product=

$$\begin{bmatrix} "[.15, .82, .44]" & "[.4e-1, .52, .85]" & "[.21, .76, .80]" & "[.2e-1, .55, .79]" \\ "[.18, .76, .85]" & "[.21, .80, .92]" & "[.54, .70, .76]" & "[.15, .52, .82]" \\ "[.45, .58, .28]" & "[.10, .52, .44]" & "[.45, .90, .80]" & "[.28, .80, .65]" \\ "[.54, .73, .76]" & "[.12, .80, .52]" & "[.20, .94, .65]" & "[.8e-1, .70, .88]" \end{bmatrix}$$

3.8. Computing transpose of single valued neutrosophic matrix

To generate the Maple program for finding the transpose of single valued neutrosophic matrix, simple call of the function **Transpose(A)** is defined as follow:

```
Transpose:=proc(mat::Matrix)

m,n:=LinearAlgebra[Dimension](mat);

temp:=Matrix(n,m);

for i from 1 to n by 1 do

for j from 1 to m by 1 do

temp(i,j):=mat(j,i);

end do;

end do;

temp;

end proc:
```

Example 7. In this example we evaluate the transpose of the single valued neutrosophic matrix E of order 4X4:

C=

$$\begin{pmatrix} < .5,.7,.2 > & < .4,.4,.5 > & < .7,.7,.5 > & < .1,.5,.7 > \\ < .9,.7,.5 > & < .7,.6,.8 > & < .9,.4,.6 > & < .5,.2,.7 > \\ < .9,.4,.2 > & < .2,.2,.2 > & < .9,.5,.5 > & < .7,.5,.3 > \\ < .9,.7,.2 > & < .3,.5,.2 > & < .5,.4,.5 > & < .2,.4,.8 > \end{pmatrix}$$

So, the transpose matrix of single valued neutrosophic matricesis portrayed as follow

$$C^T=\begin{pmatrix} <.5,.7,.2> & <.9,.7,.5> & <.9,.4,.2> & <.9,.7,.2> \\ <.4,.4,.5> & <.7,.6,.8> & <.2,.2,.2> & <.3,.5,.2> \\ <.7,.7,.5> & <.9,.4,.6> & <.9,.5,.5> & <.5,.4,.5> \\ <.1,.5,.7> & <.5,.2,.7> & <.7,.5,.3> & <.2,.4,.8> \end{pmatrix}$$

3.9. Computing determinant of single valued neutrosophic matrices

To generate the Maple program for finding the determinant of a single valued neutrosophic matrix, simply call this code, then call**det()**procedure:

```
AND:=proc(m1,m2)
n1:=parse(m1);
n2:=parse(m2);
if numelems(n1)<>3 then return convert(n2,string)
elifnumelems(n2)<>3   then return convert(n1,string)
else
t1:=n1[1];
i1:=n1[2];
f1:=n1[3];
t2:=n2[1];
i2:=n2[2];
f2:=n2[3];
t:=min(t1,t2);
i:=min(i1,i2);
f:=max(f1,f2);
return convert([t,i,f],string);
end if;
end proc:
OR:=proc(m1,m2)
n1:=parse(m1);
n2:=parse(m2);
```

```
if numelems(n1)<>3 then return convert(n2,string)
elifnumelems(n2)<>3   then return convert(n1,string)
else
t1:=n1[1];
i1:=n1[2];
f1:=n1[3];
t2:=n2[1];
i2:=n2[2];
f2:=n2[3];
t:=max(t1,t2);
i:=max(i1,i2);
f:=min(f1,f2);
return convert([t,i,f],string);
end if;
end proc:
subMat:=proc(mat,temp,p::integer ,q::integer ,n::integer)
    i:=1;j:=1;
    for row from 1 to n do
        for col from 1 to n do
            if row <> p and col <> q then
                temp(i,j) := mat(row,col);
                j:=j+1;
                if j = n then
                    j := 1;
                    i:=i+1;
                end if;
```

```
            end if;
        end do;
    end do;
end proc:
myDet:=proc(mat, n::integer)
    determinant:="[0]";
    if n = 1 then
        return mat(1,1);
    end if;
    if n = 2 then
        return OR(AND(mat(1,1) , mat(2,2)),AND(mat(1,2) , mat(2,1)));
    end if;
    for   i from 1 to n do
subMat(mat, temp, 1, i, n);
        determinant := OR(determinant,AND(mat(1,i),myDet(temp, n - 1)));
    end do;
    return determinant;
end proc:
det:=proc(mat::Matrix)
n:=LinearAlgebra[RowDimension](mat):
return myDet(mat,n);
end proc:
```

Example 8. In this example we evaluate the determinant of a single valued neutrosophicmatrixF of order 4X4by the call of the command det (F):

det (F)

"[.3, .4, .3]"

4. Conclusions

This paper proposed some new Maple programs for set-theoretic operations on single valued matrices. The package provides some programs such as complement, transpose, scalar multiplication of matrix, scalar division of matrix, computing the union, intersection addition, product, and difference and division operations for the single valued neutrosophic matrices. In future work, the interval valued neutrosophic matrices can be studied using Maple language.

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Single Valued Neutrosophic General Machine

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Abstract. In this paper, first of all, considering the notions of single-valued neutrosophic and general fuzzy automata we present the concept of single-valued neutrosophic general machine, to simplicity, SVNGM. Also, for a given SVNGM $\mathcal{M}$, we give the concept of single-valued neutrosophic sub-general machine (SVNSGM) of $\mathcal{M}$. Moreover, we show that if there exists a strong homomorphism between two SVNGM, then there is a connection between the SVNSGM of them. Further, we give the notion of single-valued neutrosophic strong sub-general machine (SVNSSGM). In addition, we show that for a given SVNGM $\mathcal{M}$ if $\mathcal{M}'$ is a SVNSSGM of $\mathcal{M}$, then $\mathcal{M}'$ is a SVNSGM of $\mathcal{M}$, but the converse does not hold.

Keywords: Neutrosophic set; Automata; Intuitionistic set; Submachine; General fuzzy automata

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1. Introduction

The idea of 'fuzzy' and a number of other notions in mathematics and other fields were fuzzified by Zadeh [18] in 1965. The concept of fuzzy automaton suggested by Wee [17] and Santos [10]. Doostfatemeh and Kremer [3] introduced the concept of general fuzzy automata.

An intuitionistic fuzzy set may be considered an alternative approach when the available information is not sufficient to define the vagueness of the conventional fuzzy set. In fuzzy sets the degree of acceptance is taken into account solely however intuitionistic fuzzy set is characterized by a membership function and a non-membership function, the only need is that the sum of both values is less and equal to one. Intuitionistic fuzzy set will solely contend with incomplete information but not the indeterminate information and inconsistent information that commonly exists within the certainty system.

In intuitionistic fuzzy sets, indeterminacy is its hesitation part by default. Neutrosophy is one of the helpful tools for managing uncertainty in concrete problems. Neutrosophy is a branch of philosophy that was introduced by Florentin Smarandache [4–6]. Afterwards, to generalise, Shamsizadeh and Zahedi introduced and studied the concept of intuitionistic general fuzzy automata [12]. For further details see recent literature such as [1,2,7,11,13,14].

Neutrosophy deals with the origin, nature and scope of neutralities, as well as their interactions with various ideational spectra. Neutrosophy is the foundation of neutrosophic sets (derivative of neutrosophy). Wang et al. [16] introduced single valued neutrosophic sets that is a neutrosophic set defined in the range [0, 1]. Wang et.al: [15] presented the concepts of interval-valued neutrosophic sets. Tahir Mahmood presented to the idea of interval neutrosophic finite state machine [9]. In 2019 [8] the idea of neutrosophic general fuzzy automata was presented by Kavikumar. The basic advantage of incorporating neutrosophic sets into general fuzzy automata is the ability to bring indeterminacy membership and nonmembership in every transition and active states that help us to overcome the uncertain situation at the time of predicting the next active state.

The present paper is organized as follows: Section 2 encompasses preliminary information pertaining to the content of the paper. In Section 3 by considering the notions of single-valued neutrosophic and general fuzzy automata we focus on the study of the concepts of single-valued neutrosophic general machine (SVNGM). Also, for a given SVNGM $\mathcal{M}$, we confer the concept of single-valued neutrosophic sub-general machine (SVNSGM) of $\mathcal{M}$. Moreover, we show that if there exists a strong homomorphism between two SVNGM, then there is a connection between SVNSGM of them. Section 4 is towards the study the notion of single-valued neutrosophic strong sub-general machine (SVNSSGM). Also, we show that for a given SVNGM $\mathcal{M}$ if $\mathcal{M}'$ is a SVNSSGM of $\mathcal{M}$, then $\mathcal{M}'$ is a SVNSGM of $\mathcal{M}$, but the converse is not true.

2. **Preliminaries**

In this section, some concepts and definitions related to single-valued neutrosophy and automata are introduced.

Definition 2.1. [3] A general fuzzy automaton (GFA) is considered as:

$$\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2),$$

where (i) Q is a finite set of states, (ii) Σ is a finite set of input symbols, (iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$, (iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$, (v) $\omega : Q \to Z$ is the output function, (vi) $\tilde{\tilde{\delta}} : (Q \times [0,1]) \times \Sigma \times Q \to [0,1]$ is the augmented transition function. (vii) Function $F_1 : [0,1] \times [0,1] \to [0,1]$ is called membership assignment function. Function $F_1(\mu, \delta)$, as is seen, is motivated by two parameters μ and δ, wherever μ is that the membership value of a predecessor and δ is that the value of a transition.

With this definition, the process that happens upon the transition from state q_i to q_j an input a_k is characterized by:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

It denote that membership value (mv) of the state q_j at time $t+1$ is calculated by function F_1 utilizing both the membership value of q_i at time t and the value of the transition.
(viii) $F_2 : [0,1]^* \to [0,1]$, is called multi-membership resolution function. The multi-membership resolution function determines the multi-membership active states and assigns them a unique membership value.

Definition 2.2. Let Σ be a space of points, with a generic element in Σ denoted by x. A neutrosophic set A in Σ is characterised by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is $T_A : \Sigma \to]0^-, 1^+[, I_A : \Sigma \to]0^-, 1^+[, F_A : \Sigma \to]0^-, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2.3. Single-valued neutrosophic set is the immediate results of neutrosophic set if it is defined over standard unit interval $[0,1]$ instead of the non-standard unit interval $]0^-, 1^+[$. A single-valued neutrosophic subset (SVNS) A of Q is defined by $SVNS(A) = \{(x, T_A(x), I_A(x), F_A(x)) \big| x \in \Sigma\}$, where $T_A(x), I_A(x), F_A(x) : \Sigma \to [0,1]$ such that $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$.

3. Single-valued neutrosophic general machine

Definition 3.1. A single-valued neutrosophy general machine (SVNGM) $\mathcal{M}$ is a six-tuple machine denoted by $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$, where

1. Q is a finite set of states,
2. Σ is a finite set of input symbols,
3. $\tilde{R} \subseteq \tilde{P}(Q)$ is the set of single-valued neutrosophic initial states,
4. $\tilde{\delta} : (Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma \times Q \to [0,1] \times [0,1] \times [0,1]$ is the single-valued neutrosophic augmented transition function,
5. $E_1 = (E_1^T, E_1^I, E_1^F)$, where $E_1^T : [0,1] \times [0,1] \to [0,1]$ is a t-norm and it is called the truth-membership assignment function. $E_1^T(T, T_\delta)$ is motivated by two parameters T and T_δ, where T is the truth-membership value of a predessor and T_δ is the truth-membership value of the transition. Also, $E_1^I : [0,1] \times [0,1] \to [0,1]$ is a t-norm and it is called the indeterminacy-membership function. $E_1^I(I, I_\delta)$ is motivated by two parameters I and I_δ, where I is the indeterminacy-membership value of a predessor and I_δ is the indeterminacy-membership value of the transition. Moreover, $E_1^F : [0,1] \times$

$[0,1] \to [0,1]$ is a t-conorm and it is called the falsity-membership function. $E_1^F(F, F_\delta)$ is motivated by two parameters F and F_δ, where F is the falsity-membership value of a predessor and F_δ is the falsity-membership value of the transition.
In this definition, the process that takes place upon the transition from the state q_i to q_j on an input a_k is represented by:

$$T^{t+1}(q_j) = \tilde{\delta}_1((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = E_1^T(T^t(q_i), \delta_1(q_i, a_k, q_j)),$$
$$I^{t+1}(q_j) = \tilde{\delta}_2((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = E_1^I(I^t(q_i), \delta_2(q_i, a_k, q_j)),$$
$$F^{t+1}(q_j) = \tilde{\delta}_3((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = E_1^F(T^t(q_i), \delta_3(q_i, a_k, q_j)),$$

where

$$\tilde{\delta}((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = (\tilde{\delta}_1((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j), \tilde{\delta}_2((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j), \tilde{\delta}_3((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j)),$$

and

$$\delta(q_i, a_k, q_j) = (\delta_1(q_i, a_k, q_j), \delta_2(q_i, a_k, q_j), \delta_3(q_i, a_k, q_j)).$$

6. $E_2 = (E_2^T, E_2^I, E_2^F)$, where $E_2^T : [0,1] \times [0,1] \to [0,1]$ is a T-conorm and it is called multi-truth-membership function, $E_2^I : [0,1] \times [0,1] \to [0,1]$ is a T-conorm and it is called multi-indetermincy-membership function, $E_2^F : [0,1] \times [0,1] \to [0,1]$ is a T-norm and it is called multi-falsity-membership function.

Example 3.2. Let the SVNGM $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ such that $Q = \{q_0, q_1, q_2\}, \Sigma = \{a\}$, $\tilde{R} = \{(q_0, 0.4, 0.7, 0.3)\}$ and δ is defined as follows:

$$\delta(q_0, a, q_0) = (0.6, 0.7, 1), \qquad \delta(q_0, a, q_1) = (0.7, 0.5, 0.5),$$
$$\delta(q_0, a, q_2) = (0.9, 0.7, 0.4), \qquad \delta(q_1, a, q_1) = (0.4, 0.5, 0.2),$$
$$\delta(q_1, a, q_2) = (0.3, 0.7, 0.6), \qquad \delta(q_2, a, q_0) = (0.7, 0.9, 0.6),$$
$$\delta(q_2, a, q_1) = (0.7, 1, 1), \qquad \delta(q_2, a, q_2) = (0.6, 0.9, 0.5).$$

Now, we can consider E_1 as follows:
1. $E_1^T = T \wedge T_\delta,\ E_1^I = I \wedge I_\delta,\ E_1^F = F \vee F_\delta$,

$$T^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^T(T^t(q_i), \delta_1(q_i, a_k, q_m)),$$
$$I^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^I(I^t(q_i), \delta_2(q_i, a_k, q_m)),$$
$$F^{t+1}(q_m) = \bigwedge_{i=1}^{n} E_3^F(F^t(q_i), \delta_3(q_i, a_k, q_m)),$$

2. $E_1^T = T.T_\delta,\ E_1^I = I.I_\delta,\ E_1^F = F + F_\delta - F.F_\delta,$

$$T^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^T(T^t(q_i), \delta_1(q_i, a_k, q_m)),$$
$$I^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^I(I^t(q_i), \delta_2(q_i, a_k, q_m)),$$
$$F^{t+1}(q_m) = \bigwedge_{i=1}^{n} E_3^F(F^t(q_i), \delta_3(q_i, a_k, q_m)),$$

3. $E_1^T = T \wedge T_\delta,\ E_1^I = I \wedge I_\delta,\ E_1^F = F \vee F_\delta,$

$$T^{t+1}(q_m) = T_p(T_p(T^t(q_i), \delta_1(q_i, a_k, q_m))),$$
$$I^{t+1}(q_m) = T_p(T_p(I^t(q_i), \delta_2(q_i, a_k, q_m)),$$
$$F^{t+1}(q_m)) = S_p(S_p E_3^F(F^t(q_i), \delta_3(q_i, a_k, q_m))),$$

where T_p is the product t-norm and S_p is the product t-conorm.
If we choose the case 1, then we have

$$T^{t_1}(q_0) = E_1^T(T^{t_0}(q_0), \delta_1(q_0, a, q_0)) = 0.4 \wedge 0.6 = 0.4,$$
$$I^{t_1}(q_0) = E_1^I(I^{t_0}(q_0), \delta_2(q_0, a, q_0)) = 0.7 \wedge 0.7 = 0.7,$$
$$F^{t_1}(q_0) = E_1^F(F^{t_0}(q_0), \delta_3(q_0, a, q_0)) = 0.3 \vee 1 = 1,$$
$$T^{t_1}(q_1) = E_1^T(T^{t_0}(q_0), \delta_1(q_0, a, q_1)) = 0.4 \wedge 0.7 = 0.4,$$
$$I^{t_1}(q_1) = E_1^I(I^{t_0}(q_0), \delta_2(q_0, a, q_1)) = 0.7 \wedge 0.5 = 0.5,$$
$$F^{t_1}(q_1) = E_1^F(F^{t_0}(q_0), \delta_3(q_0, a, q_1)) = 0.3 \vee 0.5 = 0.5,$$
$$T^{t_1}(q_2) = E_1^T(T^{t_0}(q_0), \delta_1(q_0, a, q_2)) = 0.4 \wedge 0.9 = 0.4,$$
$$I^{t_1}(q_2) = E_1^I(I^{t_0}(q_0), \delta_2(q_0, a, q_2)) = 0.7 \wedge 0.7 = 0.7,$$
$$F^{t_1}(q_2) = E_1^F(F^{t_0}(q_0), \delta_3(q_0, a, q_2)) = 0.3 \vee 0.4 = 0.4,$$

$$T^{t_2}(q_0) = E_1^T(T^{t_1}(q_0), \delta_1(q_0, a, q_0)) \vee E_1^T(T^{t_1}(q_2), \delta_1(q_2, a, q_0)) = (0.4 \wedge 0.6) \vee (0.4 \wedge 0.7) = 0.4,$$
$$I^{t_2}(q_0) = E_1^I(I^{t_1}(q_0), \delta_2(q_0, a, q_0)) \vee E_1^I(I^{t_1}(q_2), \delta_2(q_2, a, q_0)) = (0.7 \wedge 0.7) \vee (0.7 \wedge 0.9) = 0.7,$$
$$F^{t_2}(q_0) = E_1^F(F^{t_1}(q_0), \delta_3(q_0, a, q_0)) \wedge E_1^F(F^{t_1}(q_2), \delta_3(q_2, a, q_0)) = (1 \vee 1) \wedge (0.4 \vee 0.6) = 0.6,$$
$$T^{t_2}(q_1) = E_1^T(T^{t_1}(q_0), \delta_1(q_0, a, q_1)) \vee E_1^T(T^{t_1}(q_1), \delta_1(q_1, a, q_1)) \vee E_1^T(T^{t_1}(q_2), \delta_1(q_2, a, q_1))$$
$$= (0.4 \wedge 0.7) \vee (0.4 \wedge 0.4) \vee (0.4 \wedge 0.7) = 0.4,$$
$$I^{t_2}(q_1) = E_1^I(I^{t_1}(q_0), \delta_2(q_0, a, q_1)) \vee E_1^I(I^{t_1}(q_1), \delta_2(q_1, a, q_1)) \vee E_1^I(I^{t_1}(q_2), \delta_2(q_2, a, q_1))$$
$$= (0.7 \wedge 0.5) \vee (0.5 \wedge 0.5) \vee (0.7 \wedge 1) = 0.7,$$
$$F^{t_2}(q_1) = E_1^F(F^{t_1}(q_0), \delta_3(q_0, a, q_1)) \wedge E_1^F(F^{t_1}(q_1), \delta_3(q_1, a, q_1)) \wedge E_1^F(F^{t_1}(q_2), \delta_3(q_2, a, q_1))$$
$$= (1 \vee 0.5) \wedge (0.5 \vee 0.2) \wedge (0.4 \vee 1) = 0.5,$$
$$T^{t_2}(q_2) = E_1^T(T^{t_1}(q_0), \delta_1(q_0, a, q_2)) \vee E_1^T(T^{t_1}(q_1), \delta_1(q_1, a, q_2)) \vee E_1^T(T^{t_1}(q_2), \delta_1(q_2, a, q_2))$$
$$= (0.4 \wedge 0.9) \vee (0.4 \wedge 0.3) \vee (0.4 \wedge 0.6) = 0.4,$$
$$I^{t_2}(q_2) = E_1^I(I^{t_1}(q_0), \delta_2(q_0, a, q_2)) \vee E_1^I(I^{t_1}(q_1), \delta_2(q_1, a, q_2)) \vee E_1^I(I^{t_1}(q_2), \delta_2(q_2, a, q_2))$$
$$= (0.7 \wedge 0.7) \vee (0.5 \wedge 0.7) \vee (0.7 \wedge 0.9) = 0.7,$$
$$F^{t_2}(q_2) = E_1^F(F^{t_1}(q_0), \delta_3(q_0, a, q_2)) \wedge E_1^F(F^{t_1}(q_1), \delta_3(q_1, a, q_2)) \wedge E_1^F(F^{t_1}(q_2), \delta_3(q_2, a, q_2))$$
$$= (1 \vee 0.4) \wedge (0.5 \vee 0.6) \wedge (0.4 \vee 0.5) = 0.5.$$

Clearly, we can see that there are three simultaneous transition to the action states q_0, q_1 and q_2 at time t_2.

Definition 3.3. Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM. We define max-min SVNGM $\mathcal{M} = (Q, \Sigma, \tilde{\delta}^*, \tilde{R}, E_1, E_2)$ such that $\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to [0,1] \times [0,1] \times [0,1]$, where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), ...\}$ and for every $i \geq 0$,

$$\tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, p) = \begin{cases} 1 & \text{if p=q} \\ 0 & \text{otherwise} \end{cases}, \quad (1)$$

$$\tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, p) = \begin{cases} 1 & \text{if p=q} \\ 0 & \text{otherwise} \end{cases}, \quad (2)$$

$$\tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, p) = \begin{cases} 0 & \text{if p=q} \\ 1 & \text{otherwise} \end{cases}, \quad (3)$$

and for every $i \geq 1, \tilde{\delta}^*((q, T^t(q), I^t(q), F^t(q)), a, p) = \tilde{\delta}((q, T^t(q), I^t(q), F^t(q)), a, p)$ and recursively,

$$\tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), a_1 a_2 ... a_n, p) = \vee\{\tilde{\delta}_1((q, T^t(q), I^t(q), F^t(q)), a_1, p_1) \wedge ...$$
$$\wedge\, \tilde{\delta}_1((p_{n-1}, T^t(p_{n-1}), I^t(p_{n-1}), F^t(p_{n-1})), a_n, p) \big| p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), ..., p_{n-1} \in Q_{act}(t_{n-1})\},$$

$$\tilde{\delta}_2^*((q,T^t(q),I^t(q),F^t(q)),a_1a_2...a_n,p)=\vee\{\tilde{\delta}_2((q,T^t(q),I^t(q),F^t(q)),a_1,p_1)\wedge ... \\ \wedge\,\tilde{\delta}_2((p_{n-1},T^t(p_{n-1}),I^t(p_{n-1}),F^t(p_{n-1})),a_n,p)\big|p_1\in Q_{act}(t_1),p_2\in Q_{act}(t_2),...,p_{n-1}\in Q_{act}(t_{n-1})\},$$

$$\tilde{\delta}_3^*((q,T^t(q),I^t(q),F^t(q)),a_1a_2...a_n,p)=\wedge\{\tilde{\delta}_3((q,T^t(q),I^t(q),F^t(q)),a_1,p_1)\vee ... \\ \vee\,\tilde{\delta}_3((p_{n-1},T^t(p_{n-1}),I^t(p_{n-1}),F^t(p_{n-1})),a_n,p)\big|p_1\in Q_{act}(t_1),p_2\in Q_{act}(t_2),...,p_{n-1}\in Q_{act}(t_{n-1})\},$$

in which $a_i \in \Sigma$, for all $1 \leq i \leq n$ and assuming that the entered input at time t_i is a_i, for $1 \leq i \leq n-1$.

In the rest of paper, instead of max-min SVNGM we say that SVNGM.

Definition 3.4. Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM. Let N be a single-valued neutrosophic subset of Q. Then $\mathcal{M}' = (Q, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is called a single-valued neutrosophic subgeneral machine (SVNSGM) of $\mathcal{M}$ if

$$T_N(q) \geq T_N(p) \wedge \tilde{\delta}_1((p,T^t(p),I^t(p),F^t(p)),a,q),$$
$$I_N(q) \geq I_N(p) \wedge \tilde{\delta}_2((p,T^t(p),I^t(p),F^t(p)),a,q),$$
$$F_N(q) \leq F_N(p) \vee \tilde{\delta}_3((p,T^t(p),I^t(p),F^t(p)),a,q),$$

for every $p, q \in Q$ and $a \in \Sigma$.

Theorem 3.5. *Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM and N be a SVNS of Q. Then $\mathcal{M}' = (Q, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a SVNSGM of $\mathcal{M}$ if and only if*

$$T_N(q) \geq T_N(p) \wedge \tilde{\delta}_1^*((p,T^t(p),I^t(p),F^t(p)),x,q),$$
$$I_N(q) \geq I_N(p) \wedge \tilde{\delta}_2^*((p,T^t(p),I^t(p),F^t(p)),x,q),$$
$$F_N(q) \leq F_N(p) \vee \tilde{\delta}_3^*((p,T^t(p),I^t(p),F^t(p)),x,q),$$

for every $p, q \in Q$ and $x \in \Sigma^*$.

Proof. Let $\mathcal{M}'$ be a SVNSGM of $\mathcal{M}$. We prove the claim by induction on $|x| = n$. If $n = 0$, then $x = \Lambda$. Let $p = q$. Then

$$T_N(p) \wedge \tilde{\delta}_1^*((p,T^t(p),I^t(p),F^t(p)),\Lambda,q) \leq T_N(p) = T_N(q),$$
$$I_N(p) \wedge \tilde{\delta}_2^*((p,T^t(p),I^t(p),F^t(p)),\Lambda,q) \leq I_N(p) = I_N(q),$$
$$F_N(p) \vee \tilde{\delta}_3^*((p,T^t(p),I^t(p),F^t(p)),\Lambda,q) \geq F_N(p) = F_N(q),$$

for every $p, q \in Q$ and $x \in \Sigma^*$. Now, let $p \neq q$. Then

$$T_N(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), \Lambda, q) = 0 \leq T_N(q),$$
$$I_N(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), \Lambda, q) = 0 \leq I_N(q),$$
$$F_N(p) \vee \tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), \Lambda, q) = 1 \geq F_N(q).$$

So, the claim is true for $n = 0$. Now, let the claim holds for every $y \in \Sigma^*$ such that $|y| = n-1, n \geq 1$. Let $x = ya$, $|y| = n-1, y \in \Sigma^*$ and $a \in \Sigma$. Then we have

$$\begin{aligned}
T_N(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), x, q) &= T_N(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), ya, q) \\
&= T_N(p) \wedge \big(\vee \{\tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), y, r) \\
&\wedge \tilde{\delta}_1^*((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) \big| r \in Q\}\big) \\
&= \vee\{T_N(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), y, r) \wedge \\
&\tilde{\delta}_1^*((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) \big| r \in Q\} \\
&\leq \vee\{T_N(r) \wedge \tilde{\delta}_1^*((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) \big| r \in Q\} \\
&\leq T_N(q),
\end{aligned}$$

$$\begin{aligned}
I_N(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), x, q) &= I_N(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), ya, q) \\
&= I_N(p) \wedge \big(\vee \{\tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), y, r) \\
&\wedge \tilde{\delta}_2^*((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) \big| r \in Q\}\big) \\
&= \vee\{I_N(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), y, r) \\
&\wedge \tilde{\delta}_2^*((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) \big| r \in Q\} \\
&\leq \vee\{I_N(r) \wedge \tilde{\delta}_2^*((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) \big| r \in Q\} \\
&\leq I_N(q),
\end{aligned}$$

$$\begin{aligned}
F_N(p) \vee \tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), x, q) &= F_N(p) \vee \tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), ya, q) \\
&= F_N(p) \vee \big(\wedge \{\tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), y, r) \\
&\vee \tilde{\delta}_3^*((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) \big| r \in Q\}\big) \\
&= \wedge\{F_N(p) \vee \tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), y, r) \\
&\vee \tilde{\delta}_3^*((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) \big| r \in Q\} \\
&\geq \wedge\{F_N(r) \vee \tilde{\delta}_1^*((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) \big| r \in Q\} \\
&\geq F_N(q).
\end{aligned}$$

Hence, the claim holds. □

Example 3.6. Let $Q = \{p_1, p_2\}, \Sigma = \{a, b\}$, $\tilde{R} = \{(p_1, 0.7, 0.8, 0.4), (p_2, 0.7, 0.8, 0.4)\}$, $\delta(q, a, q') = (0.5, 0.6, 0.7)$, for every $q, q' \in Q$. Let $N = \{(p_1, 0.6, 0.8, 0.6), (p_2, 0.5, 0.6, 0.7)\}$. Then

$$\tilde{\delta}_1^*((q_1, T^{t_0}(q_1), I^{t_0}(q_1), F^{t_0}(q_1)), a, q_2) = 0.7 \wedge 0.5 = 0.5$$
$$\tilde{\delta}_2^*((q_1, T^{t_0}(q_1), I^{t_0}(q_1), F^{t_0}(q_1)), a, q_2) = 0.8 \wedge 0.6 = 0.6$$
$$\tilde{\delta}_3^*((q_1, T^{t_0}(q_1), I^{t_0}(q_1), F^{t_0}(q_1)), a, q_2) = 0.4 \vee 0.7 = 0.4,$$

so $\tilde{\delta}((q_1, T^{t_0}(q_1), I^{t_0}(q_1), F^{t_0}(q_1)), a, q_2) = (0.5, 0.6, 0.7)$, for every $q_1, q_2 \in Q$. Then

$$T_N(p_1) \wedge \tilde{\delta}_1^*((p_1, T^{t_0}(p_1), I^{t_0}(p_1), F^{t_0}(p_1)), a, p_2) = 0.5 = T_N(p_2)$$
$$I_N(p_1) \wedge \tilde{\delta}_2^*((p_1, T^{t_0}(p_1), I^{t_0}(p_1), F^{t_0}(p_1)), a, p_2) = 0.6 = I_N(p_2)$$
$$F_N(p_1) \vee \tilde{\delta}_3^*((p_1, T^{t_0}(p_1), I^{t_0}(p_1), F^{t_0}(p_1)), a, p_2) = 0.7 = F_N(p_2),$$

and

$$T_N(p_2) \wedge \tilde{\delta}_1^*((p_2, T^{t_0}(p_2), I^{t_0}(p_2), F^{t_0}(p_2)), a, p_1) = 0.5 \leq T_N(p_1)$$
$$I_N(p_2) \wedge \tilde{\delta}_2^*((p_2, T^{t_0}(p_2), I^{t_0}(p_2), F^{t_0}(p_2)), a, p_1) = 0.6 \leq I_N(p_1)$$
$$F_N(p_2) \vee \tilde{\delta}_3^*((p_2, T^{t_0}(p_2), I^{t_0}(p_2), F^{t_0}(p_2)), a, p_1) = 0.7 \geq F_N(p_1).$$

Therefore, $\mathcal{M}' = (Q, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a SVNSGM of $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$.

Theorem 3.7. *Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM and $\mathcal{M}_1 = (Q, N_1, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ and $\mathcal{M} = (Q, N_2, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be two SVNSGM of $\mathcal{M}$. Then the following hold:*

(1) *$\mathcal{M}_1 \cup \mathcal{M}_2 = (Q, N_1 \cup N_2, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a SVNSGM of $\mathcal{M}$, where*

$$N_1 \cup N_2 = (x, T_{N_1}(x) \vee T_{N_2}(x), I_{N_1}(x) \vee I_{N_2}(x), F_{N_1}(x) \wedge F_{N_2}(x)),$$

(2) *$\mathcal{M}_1 \cap \mathcal{M}_2 = (Q, N_1 \cap N_2, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a SVNSGM of $\mathcal{M}$, where*

$$N_1 \cap N_2 = (x, T_{N_1}(x) \wedge T_{N_2}(x), I_{N_1}(x) \wedge I_{N_2}(x), F_{N_1}(x) \vee F_{N_2}(x)).$$

Proof. The proofs 1 and 2 are clear. □

Definition 3.8. Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM and $\mathcal{M}' = (Q, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNSGM of $\mathcal{M}$. Define the SVNS N_x of Q, for every $x \in \Sigma^*$, as follows: $N_x(q) =$

$(q, T_{N_x}(q), I_{N_x}(q), F_{N_x}(q))$, where

$$T_{N_x}(q) = \bigvee_{p\in Q} T_N(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), a, q),$$
$$I_{N_x}(q) = \bigvee_{p\in Q} I_N(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), a, q)$$
$$F_{N_x}(q) = \bigwedge_{p\in Q} F_N(p) \vee \tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), a, q),$$

for every $q \in Q$.

Theorem 3.9. *Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM. Then for every SVNS N of Q and for every $x, y \in \Sigma^*$, we have $(N_x)_y = N_{xy}$.*

Proof. Let N be a SVNS of Q and $x, y \in \Sigma^*$. We prove the claim by induction on $|y| = n$. Let $n = 0$. Then $y = \Lambda$. Let $q \in Q$. Then

$$\begin{aligned}
T_{(N_x)_\Lambda}(q) &= \bigvee_{p\in Q} T_{N_x}(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), \Lambda, q)\\
&= T_{N_x}(q) \wedge \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, q) = T_{N_x}(q),\\
I_{(N_x)_\Lambda}(q) &= \bigvee_{p\in Q} I_{N_x}(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), \Lambda, q)\\
&= I_{N_x}(q) \wedge \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, q) = I_{N_x}(q),\\
F_{(N_x)_\Lambda}(q) &= \bigwedge_{p\in Q} F_{N_x}(p) \vee \tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), \Lambda, q)\\
&= F_{N_x}(q) \vee \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, q) = F_{N_x}(q).
\end{aligned}$$

Therefore, $(N_x)_\Lambda = N_{x\Lambda}$. Now, let the claim holds for every SVNS N and for every $y \in \Sigma^*$ such that $|y| = n-1, n \geq 1$. Let $y = wa$, where $w \in \Sigma^*, |w| = n-1$ and $a \in \Sigma$. Then

$$\begin{aligned}
T_{N_{xy}}(q) &= T_{N_{x(wa)}}(q) \\
&= T_{N_{(xw)a}}(q) \\
&= \bigvee_{p\in Q} T_{N_{xw}}(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), a, q) \\
&= \bigvee_{p\in Q} \bigvee_{r\in Q} T_{N_x}(r) \wedge \tilde{\delta}_1^*((r, T^{t-n+1}(r), I^{t-n+1}(r), F^{t-n+1}(r)), w, p) \\
&\wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), a, q) \\
&= \bigvee_{r\in Q} T_{N_x}(r) \wedge \bigvee_{p\in Q} \big(\tilde{\delta}_1^*((r, T^{t-n+1}(r), I^{t-n+1}(r), F^{t-n+1}(r)), w, p) \\
&\wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), a, q)\big) \\
&= \bigvee_{r\in Q} T_{N_x}(r) \wedge \tilde{\delta}_1^*((r, T^{t-n+1}(r), I^{t-n+1}(r), F^{t-n+1}(r)), wa, r) \\
&= T_{N_{x(wa)}}(q),
\end{aligned}$$

and

$$\begin{aligned}
I_{N_{xy}}(q) &= I_{N_{x(wa)}}(q) \\
&= I_{N_{(xw)a}}(q) \\
&= \bigvee_{p\in Q} I_{N_{xw}}(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), a, q) \\
&= \bigvee_{p\in Q} \bigvee_{r\in Q} I_{N_x}(r) \wedge \tilde{\delta}_2^*((r, T^{t-n+1}(r), I^{t-n+1}(r), F^{t-n+1}(r)), w, p) \\
&\wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), a, q) \\
&= \bigvee_{r\in Q} I_{N_x}(r) \wedge \bigvee_{p\in Q} \big(\tilde{\delta}_2^*((r, T^{t-n+1}(r), I^{t-n+1}(r), F^{t-n+1}(r)), w, p) \\
&\wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), a, q)\big) \\
&= \bigvee_{r\in Q} I_{N_x}(r) \wedge \tilde{\delta}_2^*((r, T^{t-n+1}(r), I^{t-n+1}(r), F^{t-n+1}(r)), wa, r) \\
&= I_{N_{x(wa)}}(q),
\end{aligned}$$

similarly, $F_{N_{xy}}(q) = I_{(N_x)_y}(q)$, for every $q \in Q$. Hence, the claim holds. □

Definition 3.10. Let A and B be two SVNS of Q. Then $A \subseteq B$ if and only if for every $q \in Q$, $T_A(q) \leq T_B(q), I_A(q) \leq I_B(q)$ and $F_A(q) \geq F_B(q)$.

Theorem 3.11. *Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM and $\mathcal{M}' = (Q, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNSGM of $\mathcal{M}$. Then $\mathcal{M}'$ is a SVNSGM of $\mathcal{M}$ if and only if $N_x \subseteq N$, for every $x \in \Sigma^*$.*

Proof. Let $\mathcal{M}'$ be a SVNSGM of $\mathcal{M}$. For every $x \in \Sigma^*$ and $q \in Q$, we have

$$T_{N_x}(q) = \bigvee_{p\in Q} T_N(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), x, q) \leq T_N(q),$$
$$I_{N_x}(q) = \bigvee_{p\in Q} I_N(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), x, q) \leq I_N(q),$$

and

$$F_{N_x}(q) = \bigwedge_{p\in Q} F_N(p) \vee \tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), x, q) \geq F_N(q).$$

So, $N_x \subseteq N$. Now, let $N_x \subseteq N$, for every $x \in \Sigma^*$. Then for every $x \in \Sigma^*$ and $q \in Q$, we have

$$\begin{aligned} T_N(q) \geq T_{N_x}(q) &= \bigvee_{p\in Q} T_N(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), x, q) \\ &\geq T_N(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), x, q), \\ I_N(q) \geq I_{N_x}(q) &= \bigvee_{p\in Q} I_N(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), x, q) \\ &\geq I_N(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), x, q), \end{aligned}$$

and

$$\begin{aligned} F_N(q) \leq F_{N_x}(q) &= \bigwedge_{p\in Q} F_N(p) \vee \tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), x, q) \\ &\leq F_N(p) \vee \tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), x, q), \end{aligned}$$

for every $p \in Q$. Hence, $\mathcal{M}' = (Q, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a SVNSGM of $\mathcal{M}$. □

Definition 3.12. Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM, $t \in (0, 1]$ and $q \in Q$. Define the SVNS $q_t\Sigma$ by $q_t\Sigma = (p, T_{q_t\Sigma}(p), I_{q_t\Sigma}(p), F_{q_t\Sigma}(p))$, where

$$T_{q_t\Sigma}(p) = \bigvee_{a\in\Sigma} t \wedge \tilde{\delta}_1((q, T^t(q), I^t(q), F^t(q)), a, p),$$
$$I_{q_t\Sigma}(p) = \bigvee_{a\in\Sigma} t \wedge \tilde{\delta}_2((q, T^t(q), I^t(q), F^t(q)), a, p),$$
$$F_{q_t\Sigma}(p) = \bigwedge_{a\in\Sigma} t \vee \tilde{\delta}_3((q, T^t(q), I^t(q), F^t(q)), a, p),$$

for every $p \in Q$.

Definition 3.13. Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM and $t \in (0, 1]$. For every $q \in Q$ define the SVNS $q_t\Sigma^*$ by $q_t\Sigma^* = (p, T_{q_t\Sigma^*}(p), I_{q_t\Sigma^*}(p), F_{q_t\Sigma^*}(p))$, where

$$T_{q_t\Sigma^*}(p) = \bigvee_{y\in\Sigma^*} t \wedge \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), y, p),$$
$$I_{q_t\Sigma^*}(p) = \bigvee_{y\in\Sigma^*} t \wedge \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), y, p),$$
$$F_{q_t\Sigma^*}(p) = \bigwedge_{y\in\Sigma^*} t \vee \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), y, p),$$

for every $p \in Q$.

Theorem 3.14. *Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM and $t \in (0, 1]$. Then $\mathcal{M} = (Q, q_t\Sigma^*, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a SVNSGM of $\mathcal{M}$.*

Proof. Let $p' \in Q$ and $x \in \Sigma^*$. First we show that $(q_t\Sigma^*)_x \subseteq q_t\Sigma^*$.

$$\begin{aligned}
T_{(q_t\Sigma^*)_x}(p') &= \bigvee_{p\in Q} T_{q_t\Sigma^*}(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), x, p') \\
&= \bigvee_{p\in Q} \Big(\bigvee_{y\in\Sigma^*} t \wedge \tilde{\delta}_1^*((q, T^{t-|y|}(q), I^{t-|y|}(q), F^{t-|y|}(q)), y, p) \\
&\wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), x, p')\Big) \\
&= \bigvee_{y\in\Sigma^*} t \wedge \tilde{\delta}_1^*((q, T^{t-|y|}(q), I^{t-|y|}(q), F^{t-|y|}(q)), yx, p') \\
&\leq \bigvee_{w\in\Sigma^*} t \wedge \tilde{\delta}_1^*((q, T^{t-|y|}(q), I^{t-|y|}(q), F^{t-|y|}(q)), w, p') \\
&= T_{q_t\Sigma^*}(p'),
\end{aligned}$$

$$\begin{aligned}
I_{(q_t\Sigma^*)_x}(p') &= \bigvee_{p\in Q} I_{q_t\Sigma^*}(p) \wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), x, p') \\
&= \bigvee_{p\in Q} \Big(\bigvee_{y\in\Sigma^*} t \wedge \tilde{\delta}_2^*((q, T^{t-|y|}(q), I^{t-|y|}(q), F^{t-|y|}(q)), y, p) \\
&\wedge \tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), x, p')\Big) \\
&= \bigvee_{y\in\Sigma^*} t \wedge \tilde{\delta}_2^*((q, T^{t-|y|}(q), I^{t-|y|}(q), F^{t-|y|}(q)), yx, p') \\
&\leq \bigvee_{w\in\Sigma^*} t \wedge \tilde{\delta}_2^*((q, T^{t-|y|}(q), I^{t-|y|}(q), F^{t-|y|}(q)), w, p') \\
&= I_{q_t\Sigma^*}(p'),
\end{aligned}$$

$$\begin{aligned} F_{(q_t\Sigma^*)_x}(p') &= \bigwedge_{p\in Q} F_{q_t\Sigma^*}(p) \vee \tilde{\delta}_3^*((p,T^t(p),I^t(p),F^t(p)),x,p') \\ &= \bigwedge_{p\in Q} \big(\bigwedge_{y\in\Sigma^*} t \vee \tilde{\delta}_3^*((q,T^{t-|y|}(q),I^{t-|y|}(q),F^{t-|y|}(q)),y,p) \\ &\vee \tilde{\delta}_3^*((p,T^t(p),I^t(p),F^t(p)),x,p')\big) \\ &= \bigwedge_{y\in\Sigma^*} t \vee \tilde{\delta}_3^*((q,T^{t-|y|}(q),I^{t-|y|}(q),F^{t-|y|}(q)),yx,p') \\ &\geq \bigwedge_{w\in\Sigma^*} t \vee \tilde{\delta}_3^*((q,T^{t-|y|}(q),I^{t-|y|}(q),F^{t-|y|}(q)),w,p') \\ &= F_{q_t\Sigma^*}(p'), \end{aligned}$$

Therefore, $(q_t\Sigma^*)_x \subseteq q_t\Sigma^*$. Then by Theorem 3.11, $\mathcal{M}'$ is a SVNSGM of $\mathcal{M}$. □

Theorem 3.15. *Let $\mathcal{M} = (Q,\Sigma,\tilde{\delta},\tilde{R},E_1,E_2)$ be a SVNGM and N ba a SVNS of Q. Then the following assertions are equivalent:*

(1) $\mathcal{M}' = (Q,N,\Sigma,\tilde{\delta},\tilde{R},E_1,E_2)$ *is a SVNSGM of* $\mathcal{M}$,

(2) $q_t\Sigma^* \subseteq N$, *for every* $q_t \subseteq N$, *where* $q \in Q$ *and* $t \in (0,1]$,

(3) $q_t\Sigma \subseteq N$, *for every* $q_t \subseteq N$, *where* $q \in Q$ *and* $t \in (0,1]$,

Proof. $1 \to 2$. Let $q_t \subseteq N$, where $q \in Q$ and $t \in (0,1]$. Let $p \in Q$ and $y \in \Sigma^*$. Then

$$\begin{aligned} \tilde{\delta}_1^*((q,T^t(q),I^t(q),F^t(q)),y,p) \wedge t &= \tilde{\delta}_1^*((q,T^t(q),I^t(q),F^t(q)),y,p) \wedge T_{(q_t)\Lambda}(q) \\ &\leq \tilde{\delta}_1^*((q,T^t(q),I^t(q),F^t(q)),y,p) \wedge T_N(q) \\ &\leq T_N(p), \end{aligned}$$

$$\begin{aligned} \tilde{\delta}_2^*((q,T^t(q),I^t(q),F^t(q)),y,p) \wedge t &= \tilde{\delta}_2^*((q,T^t(q),I^t(q),F^t(q)),y,p) \wedge I_{(q_t)\Lambda}(q) \\ &\leq \tilde{\delta}_2^*((q,T^t(q),I^t(q),F^t(q)),y,p) \wedge I_N(q) \\ &\leq I_N(p), \end{aligned}$$

$$\begin{aligned} \tilde{\delta}_3^*((q,T^t(q),I^t(q),F^t(q)),y,p) \vee t &= \tilde{\delta}_3^*((q,T^t(q),I^t(q),F^t(q)),y,p) \vee F_{(q_t)\Lambda}(q) \\ &\geq \tilde{\delta}_3^*((q,T^t(q),I^t(q),F^t(q)),y,p) \vee F_N(q) \\ &\geq F_N(p). \end{aligned}$$

Therefore, $q_t\Sigma^* \subseteq N$.

$2 \to 3$. It is clear.

$3 \to 1$. Let $p,q \in Q$ and $a \in \Sigma$. If $T_N(q) = 0$ or $\tilde{\delta}_1^*((q,T^t(q),I^t(q),F^t(q)),a,p) = 0$, then $T_N(p) \geq T_N(q) \wedge \tilde{\delta}_1^*((q,T^t(q),I^t(q),F^t(q)),a,p)$. Also, if $I_N(q) = 0$ or $\tilde{\delta}_2^*((q,T^t(q),I^t(q),F^t(q)),a,p) = 0$, then $I_N(p) \geq I_N(q) \wedge \tilde{\delta}_2^*((q,T^t(q),I^t(q),F^t(q)),a,p)$ and if $F_N(q) = 1$ or $\tilde{\delta}_3^*((q,T^t(q),I^t(q),F^t(q)),a,p) = 1$, then $F_N(p) \leq F_N(q) \vee$

$\tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), a, p)$. Now, let $T_N(q) = t$ and $\tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), a, p) \neq 0$. Then

$$\begin{aligned} T_N(p) \geq T_{q_t\Sigma}(p) &= \bigvee_{y\in\Sigma} t \wedge \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), y, p) \\ &\geq t \wedge \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), y, p) \\ &= T_N(q) \wedge \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), y, p), \end{aligned}$$

also, let $I_N(q) = t$ and $\tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), a, p) \neq 0$. Then

$$\begin{aligned} I_N(p) \geq I_{q_t\Sigma}(p) &= \bigvee_{y\in\Sigma} t \wedge \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), y, p) \\ &\geq t \wedge \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), y, p) \\ &= I_N(q) \wedge \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), y, p), \end{aligned}$$

and if $F_N(q) = t \neq 1$ and $\tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), a, p) \neq 1$, then

$$\begin{aligned} F_N(p) \leq F_{q_t\Sigma}(p) &= \bigwedge_{y\in\Sigma} t \vee \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), y, p) \\ &\leq t \vee \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), y, p) \\ &= F_N(q) \vee \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), y, p). \end{aligned}$$

Hence, $\mathcal{M}'$ is a SVNSGM of $\mathcal{M}$. □

Definition 3.16. Let $\mathcal{M}_1 = (Q_1, \Sigma_1, \tilde{\delta}, \tilde{R}, E_1, E_2)$ and $\mathcal{M}_2 = (Q_2, \Sigma_2, \tilde{\delta}', \tilde{R}', E_1, E_2)$ be two SVNGM. A pair (f, g) of mappings $f : Q_1 \to Q_2$ and $g : \Sigma_1 \to \Sigma_2$ is called a homomorphism, written $(f, g) : \mathcal{M}_1 \to \mathcal{M}_2$ if

(1) $\tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), a, p) \leq \tilde{\delta}_1'^*((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(p))$,
(2) $\tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), a, p) \leq \tilde{\delta}_2'^*((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(p))$,
(3) $\tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), a, p) \geq \tilde{\delta}_3'^*((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(p))$,
(4) for every $(q, T(q), I(q), F(q)) \in \tilde{R}$, we have $T(f(q)) \geq T(q), I(f(q)) \geq I(q)$ and $F(f(q)) \leq F(q)$,

for every $p, q \in Q_1$ and $x \in \Sigma_1$. The pair (f, g) is called strong homomorphism if

(1)

$$\begin{aligned} \tilde{\delta}_1'^*((f(q), &T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(p)) \\ &= \bigvee \{\tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), a, r) \big| r \in Q_1, f(r) = f(p)\}, \end{aligned}$$

(2)

$$\tilde{\delta}_2'^*((f(q),T^t(f(q)),I^t(f(q)),F^t(f(q))),g(a),f(p))$$
$$= \bigvee\{\tilde{\delta}_2^*((q,T^t(q),I^t(q),F^t(q)),a,r)\big|r \in Q_1, f(r) = f(p)\},$$

(3)

$$\tilde{\delta}_3'^*((f(q),T^t(f(q)),I^t(f(q)),F^t(f(q))),g(a),f(p))$$
$$= \bigwedge\{\tilde{\delta}_3^*((q,T^t(q),I^t(q),F^t(q)),a,r)\big|r \in Q_1, f(r) = f(p)\},$$

(4) for every $(q,T(q),I(q),F(q)) \in \tilde{R}$, we have $(f(q),T(f(q)),I(f(q)),F(f(q))) \in \tilde{R}'$,

where $p,q \in Q_1$ and $x \in \Sigma_1$.

Definition 3.17. Let $\mathcal{M}_1 = (Q_1,\Sigma_1,\tilde{\delta},\tilde{R},E_1,E_2)$ and $\mathcal{M}_2 = (Q_2,\Sigma_2,\tilde{\delta}',\tilde{R}',E_1,E_2)$ be two SVNGM. Let $(f,g) : \mathcal{M}_1 \to \mathcal{M}_2$ be a homomorphism and N be a SVNS of Q_1. Define the SVNS $f(N)$ of Q_2 as follows:

$$T_{f(N)}(q') = \begin{cases} \bigvee\{T_N(q)\big|q \in Q_1, f(q) = q'\} & if\ f^{-1}(q') \neq \emptyset \\ 0 & f^{-1}(q') = \emptyset \end{cases},$$

$$I_{f(N)}(q') = \begin{cases} \bigvee\{I_N(q)\big|q \in Q_1, f(q) = q'\} & if\ f^{-1}(q') \neq \emptyset \\ 0 & f^{-1}(q') = \emptyset \end{cases},$$

$$F_{f(N)}(q') = \begin{cases} \bigwedge\{F_N(q)\big|q \in Q_1, f(q) = q'\} & if\ f^{-1}(q') \neq \emptyset \\ 0 & f^{-1}(q') = \emptyset \end{cases},$$

for every $q' \in Q_2$.

Theorem 3.18. *Let $\mathcal{M}_1 = (Q_1,\Sigma,\tilde{\delta},\tilde{R},E_1,E_2)$ and $\mathcal{M}_2 = (Q_2,\Sigma,\tilde{\delta}',\tilde{R}',E_1,E_2)$ be two SVNGM and $f : \mathcal{M}_1 \to \mathcal{M}_2$ be an onto strong homomorphism. If $\mathcal{M}_1' = (Q_1,N,\Sigma,\tilde{\delta},\tilde{R},E_1,E_2)$ is a SVNSGM of $\mathcal{M}_1$, then $\mathcal{M}_2' = (Q_2,f(N),\Sigma,\tilde{\delta}',\tilde{R}',E_1,E_2)$ is a SVNSGM of $\mathcal{M}_2$.*

Proof. Let $p, q \in Q_1, p', q' \in Q_2, a \in X$ and $f(p) = p'$ and $f(q) = q'$. Then

$$\begin{aligned}
&T_{f(N)}(p') \wedge \tilde{\delta}_1'^*((p', T^t(p'), I^t(p'), F^t(p')), a, q') \\
&\quad = (\bigvee\{T_N(p)|p \in Q_1, f(p) = p'\}) \wedge \tilde{\delta}_1'^*((p', T^t(p'), I^t(p'), F^t(p')), a, q') \\
&\quad = \bigvee\{T_N(p) \wedge \tilde{\delta}_1'^*((p', T^t(p'), I^t(p'), F^t(p')), a, q')|p \in Q_1, f(p) = p'\} \\
&\quad = \bigvee\{T_N(p) \wedge \tilde{\delta}_1'^*((f(p), T^t(f(p)), I^t(f(p)), F^t(f(p))), a, f(q))|p \in Q_1, f(p) = p'\} \\
&\quad = \bigvee\{T_N(p) \wedge \bigvee\{\tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), a, r)|p, r \in Q_1, f(p) = p', f(r) = f(q)\}\} \\
&\quad = \bigvee\{\bigvee\{T_N(p) \wedge \tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), a, r)|r \in Q_1, f(r) = f(q)\}|p \in Q_1, f(p) = p'\} \\
&\quad \leq \bigvee\{\bigvee\{T_N(r)|r \in Q_1, f(r) = f(q)\}|p \in Q_1, f(p) = p'\} \\
&\quad = \bigvee\{T_{f(N)}(q')|p \in Q_1, f(p) = p'\} \\
&\quad = T_{f(N)}(q'),
\end{aligned}$$

also, similarly

$$I_{f(N)}(p') \wedge \tilde{\delta}_2'^*((p', T^t(p'), I^t(p'), F^t(p')), a, q') \leq I_{f(N)}(q'),$$

and

$$\begin{aligned}
&F_{f(N)}(p') \vee \tilde{\delta}_3'^*((p', T^t(p'), I^t(p'), F^t(p')), a, q') \\
&\quad = (\bigwedge\{F_N(p)|p \in Q_1, f(p) = p'\}) \vee \tilde{\delta}_3'^*((p', T^t(p'), I^t(p'), F^t(p')), a, q') \\
&\quad = \bigwedge\{F_N(p) \vee \tilde{\delta}_3'^*((p', T^t(p'), I^t(p'), F^t(p')), a, q')|p \in Q_1, f(p) = p'\} \\
&\quad = \bigwedge\{F_N(p) \vee \tilde{\delta}_3'^*((f(p), T^t(f(p)), I^t(f(p)), F^t(f(p))), a, f(q))|p \in Q_1, f(p) = p'\} \\
&\quad = \bigwedge\{F_N(p) \vee \bigwedge\{\tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), a, r)|p, r \in Q_1, f(p) = p', f(r) = f(q)\}\} \\
&\quad = \bigwedge\{\bigwedge\{F_N(p) \vee \tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), a, r)|r \in Q_1, f(r) = f(q)\}|p \in Q_1, f(p) = p'\} \\
&\quad \geq \bigwedge\{\bigwedge\{F_N(r)|r \in Q_1, f(r) = f(q)\}|p \in Q_1, f(p) = p'\} \\
&\quad = \bigwedge\{F_{f(N)}(q')|p \in Q_1, f(p) = p'\} \\
&\quad = F_{f(N)}(q').
\end{aligned}$$

Hence, $\mathcal{M}_2'$ is a SVNSGM of $\mathcal{M}_2$. □

Now, in the following example we show that Theorem 3.18 is not true if f is not onto.

Example 3.19. Let $Q_1 = Q_2 = \{p_1, p_2\}, \Sigma = \{a\}$ and $\tilde{R} = \tilde{R}' = \{(p_1, 1, 1, 0), (p_2, 1, 1, 0)\}$ and $\delta(p, a, q) = (1, 1, 0) = \delta'(p, a, q)$, for every $p, q \in Q_1$. Then $\mathcal{M} = (Q_1, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a SVNGM. Let $f : Q_1 \to Q_2$ be a mapping such that $f(p_1) = f(p_2) = p_1$. Then f is not onto. It is clear that, f is a strong homomorphism. Let N be a SVNS of Q such that $N(p_1) = N(p_2) = (\frac{1}{2}, \frac{1}{2}, 0)$. Then

$$T_N(p) = \frac{1}{2} = T_N(q) \wedge \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), a, p),$$
$$I_N(p) = \frac{1}{2} = I_N(q) \wedge \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), a, p),$$
$$F_N(p) = 0 = F_N(q) \vee \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), a, p),$$

for every $p, q \in Q$. Then $\mathcal{M}' = (Q_1, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a SVNSGM of $\mathcal{M}_1$. Now, we have

$$T_{f(N)}(p_2) = 0 \leq \frac{1}{2} = T_{f(N)}(p_1) \wedge \tilde{\delta}_1^*((p_1, T^t(p_1), I^t(p_1), F^t(p_1)), a, p_2),$$
$$I_{f(N)}(p_2) = 0 \leq \frac{1}{2} = I_{f(N)}(p_1) \wedge \tilde{\delta}_2^*((p_1, T^t(p_1), I^t(p_1), F^t(p_1)), a, p_2),$$
$$F_{f(N)}(p_2) = 1 \geq \frac{1}{2} = F_N(p_1) \vee \tilde{\delta}_3^*((p_1, T^t(p_1), I^t(p_1), F^t(p_1)), a, p_2).$$

So, $\mathcal{M}_2' = (Q_2, f(N), \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is not a SVNSGM of $\mathcal{M}_2 = (Q_2, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$.

4. Single-Valued Neutrosophic Strong Sub-General Machine

Definition 4.1. Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM and N be a SVNS of Q. Then we say that $\mathcal{M}' = (Q, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a single-valued neutrosophic strong sub-general machine (SVNSSGM) of $\mathcal{M}$ if and only if for every $p, q \in Q$ if there exists $a \in \Sigma$ such that

$$\tilde{\delta}_1((p, T^t(p), I^t(p), F^t(p)), a, q) > 0,$$
$$\tilde{\delta}_2((p, T^t(p), I^t(p), F^t(p)), a, q) > 0,$$
$$\tilde{\delta}_3((p, T^t(p), I^t(p), F^t(p)), a, q) < 1,$$

then $T_N(q) \geq T_N(p), I_N(q) \geq I_N(p), F_N(q) \leq F_N(p)$.

Theorem 4.2. *Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM and N be a SVNS of Q. Then $\mathcal{M}' = (Q, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a SVNSSGM of $\mathcal{M}$ if and only if for every $p, q \in Q$ if there exists $x \in \Sigma^*$ such that*

$$\tilde{\delta}_1^*((p, T^t(p), I^t(p), F^t(p)), a, q) > 0,$$
$$\tilde{\delta}_2^*((p, T^t(p), I^t(p), F^t(p)), a, q) > 0,$$
$$\tilde{\delta}_3^*((p, T^t(p), I^t(p), F^t(p)), a, q) < 1,$$

then $T_N(q) \geq T_N(p), I_N(q) \geq I_N(p), F_N(q) \leq F_N(p)$.

Proof. Let $\mathcal{M}' = (Q, N, \Sigma, \delta, R, E_1, E_2)$ be a SVNSSGM of $\mathcal{M}$. We prove the claim by induction on $|x| = n$. Let $n = 0$. Then $x = \Lambda$. If $p = q$, then

$$\tilde{\delta}_1((p, T^t(p), I^t(p), F^t(p)), \Lambda, q) = 1 > 0,$$
$$\tilde{\delta}_2((p, T^t(p), I^t(p), F^t(p)), \Lambda, q) = 1 > 0,$$
$$\tilde{\delta}_3((p, T^t(p), I^t(p), F^t(p)), \Lambda, q) = 0 < 1,$$

so, $T_N(q) = T_N(p), I_N(q) = I_N(p), F_N(q) = F_N(p)$. Now, let $p \neq q$. Then

$$\tilde{\delta}_1((p, T^t(p), I^t(p), F^t(p)), \Lambda, q) = 0,$$
$$\tilde{\delta}_2((p, T^t(p), I^t(p), F^t(p)), \Lambda, q) = 0,$$
$$\tilde{\delta}_3((p, T^t(p), I^t(p), F^t(p)), \Lambda, q) = 1,$$

so, the claim is true for $n = 0$. Now, let the result is true for every $y \in \Sigma^*$ such that $|y| = n - 1$, $n \geq 1$. Suppose that $x = ya, y \in \Sigma^*, a \in \Sigma$ and $|y| = n - 1$. Let $\tilde{\delta}_1((p, T^t(p), I^t(p), F^t(p)), x, q) = \tilde{\delta}_1((p, T^t(p), I^t(p), F^t(p)), ya, q)$. Then there exists $r \in Q$ such that

$$\tilde{\delta}_1((p, T^t(p), I^t(p), F^t(p)), y, r) \wedge \tilde{\delta}_1((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) > 0.$$

So, $\tilde{\delta}_1((p, T^t(p), I^t(p), F^t(p)), y, r) > 0$ and $\tilde{\delta}_1((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) > 0$. Therefore, $T_N(r) \geq T_N(p)$ and $T_N(q) \geq T_N(r)$. So, $T_N(q) \geq T_N(p)$. Also, let $\tilde{\delta}_2((p, T^t(p), I^t(p), F^t(p)), x, q) = \tilde{\delta}_2((p, T^t(p), I^t(p), F^t(p)), ya, q) > 0$. Then there exists $r \in Q$ such that $\tilde{\delta}_2((p, T^t(p), I^t(p), F^t(p)), y, r) > 0$ and $\tilde{\delta}_2((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) > 0$. So, $I_N(r) \geq I_N(p)$ and $I_N(q) \geq I_N(r)$. So, $I_N(q) \geq I_N(p)$. Moreover, let $\tilde{\delta}_3((p, T^t(p), I^t(p), F^t(p)), x, q) < 1$. Then

$$\tilde{\delta}_3((p, T^t(p), I^t(p), F^t(p)), ya, q) = \bigwedge_{r \in Q} \tilde{\delta}_3((p, T^t(p), I^t(p), F^t(p)), y, r) \vee \tilde{\delta}_3((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) < 1.$$

Therefore, there exists $r \in Q$ such that $\tilde{\delta}_3((p, T^t(p), I^t(p), F^t(p)), y, r) < 1$ and

$$\tilde{\delta}_3((r, T^{t+n-1}(r), I^{t+n-1}(r), F^{t+n-1}(r)), a, q) < 1.$$

So, $F_N(r) \leq F_N(p)$ and $F_N(q) \leq F_N(r)$. Therefore, $F_N(q) \leq F_N(p)$. Then the claim holds. The converse is clear. □

Theorem 4.3. *Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM and N be a SVNS of Q. If $\mathcal{M}' = (Q, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ is a SVNSSGM of $\mathcal{M}$, then $\mathcal{M}'$ is a SVNSGM of $\mathcal{M}$.*

Proof. Let $\mathcal{M}'$ be a SVNSSGM of $\mathcal{M}$. Let

$$\tilde{\delta}_1((p, T^t(p), I^t(p), F^t(p)), a, q) > 0,$$
$$\tilde{\delta}_2((p, T^t(p), I^t(p), F^t(p)), a, q) > 0,$$
$$\tilde{\delta}_3((p, T^t(p), I^t(p), F^t(p)), a, q) < 1.$$

Then $T_N(q) \geq T_N(p), I_N(q) \geq I_N(p)$ and $F_N(q) \leq F_N(p)$. So,

$$T_N(q) \geq T_N(p) \wedge \tilde{\delta}_1((p, T^t(p), I^t(p), F^t(p)), a, q),$$
$$I_N(q) \geq I_N(p) \wedge \tilde{\delta}_2((p, T^t(p), I^t(p), F^t(p)), a, q) > 0,$$
$$F_N(q) \leq F_N(p) \vee \tilde{\delta}_3((p, T^t(p), I^t(p), F^t(p)), a, q) < 1.$$

Hence, $\mathcal{M}'$ is a SVNSGM of $\mathcal{M}$. □

In the next example, we show that the reverse of the Theorem 4.3, is incorrect.

Example 4.4. Let SVNGM $\mathcal{M}$ be as defined in Example 3.6. $\mathcal{M}'$ is a SVNGSM of $\mathcal{M}$.

$$\tilde{\delta}_1((p_1, T^t(p_1), I^t(p_1), F^t(p_1)), a, p_2) = 0.5 > 0,$$

but $T_N(p_2) = 0.5 < T_N(p_1) = 0.6$. So, $\mathcal{M}'$ is not a SVNSSGM of $\mathcal{M}$.

Theorem 4.5. *Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNGM. Let $\mathcal{M}_1 = (Q, N_1, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ and $\mathcal{M}_2 = (Q, N_2, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be two SVNSSGM of $\mathcal{M}$. Then the following hold:*

1. *$\mathcal{M}_1 \cap \mathcal{M}_2$ is a SVNSSGM of $\mathcal{M}$.*
2. *$\mathcal{M}_1 \cup \mathcal{M}_2$ is a SVNSSGM of $\mathcal{M}$.*

Proof. The proofs 1 and 2 are clear. □

Theorem 4.6. *Let $\mathcal{M} = (Q, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ and $\mathcal{M}' = (Q', \Sigma, \tilde{\delta}', \tilde{R}', E_1, E_2)$ be two SVNGM. Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be an onto strong homomorphism. Let $\mathcal{M}_1 = (Q, N, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a SVNSSGM of $\mathcal{M}$. Then $\mathcal{M}'_1 = (Q', f(N), \Sigma, \tilde{\delta}', \tilde{R}', E_1, E_2)$ is a SVNSSGM of $\mathcal{M}'$.*

Proof. Let $p_1, p_2 \in Q_1$ and $a \in \Sigma$ be such that

$$\tilde{\delta}'_1((f(p_1), T^t(f(p_1)), I^t(f(p_1)), F^t(f(p_1))), a, f(p_2)) > 0,$$
$$\tilde{\delta}'_2((f(p_1), T^t(f(p_1)), I^t(f(p_1)), F^t(f(p_1))), a, f(p_2)) > 0,$$
$$\tilde{\delta}_3((f(p_1), T^t(f(p_1)), I^t(f(p_1)), F^t(f(p_1))), a, f(p_2)) < 1.$$

Also,

$$T_{f(N)}(f(p_1)) = \vee\{T_N(q_1)\big|f(p_1) = f(q_1), q_1 \in Q_1\},$$
$$I_{f(N)}(f(p_1)) = \vee\{I_N(q_1)\big|f(p_1) = f(q_1), q_1 \in Q_1\},$$
$$F_{f(N)}(f(p_1)) = \wedge\{F_N(q_1)\big|f(p_1) = f(q_1), q_1 \in Q_1\},$$

and

$$T_{f(N)}(f(p_2)) = \vee\{T_N(q_2)\big|f(p_2) = f(q_2), q_2 \in Q_1\},$$
$$I_{f(N)}(f(p_2)) = \vee\{I_N(q_2)\big|f(p_2) = f(q_2), q_2 \in Q_1\},$$
$$F_{f(N)}(f(p_2)) = \wedge\{F_N(q_2)\big|f(p_2) = f(q_2), q_2 \in Q_1\}.$$

Now, let $r \in Q_1$ and $T_N(r) > 0$ and $f(r) = f(p_1)$. Let

$$\tilde{\delta}'_1((f(r), T^t(f(r)), I^t(f(r)), F^t(f(r))), a, f(q)) = \tilde{\delta}'_1((f(p_1), T^t(f(p_1)), I^t(f(p_1)), F^t(f(p_1))), a, f(p_2)) > 0,$$
$$\tilde{\delta}'_2((f(r), T^t(f(r)), I^t(f(r)), F^t(f(r))), a, f(q)) = \tilde{\delta}'_2((f(p_1), T^t(f(p_1)), I^t(f(p_1)), F^t(f(p_1))), a, f(p_2)) > 0,$$
$$\tilde{\delta}'_3((f(r), T^t(f(r)), I^t(f(r)), F^t(f(r))), a, f(q)) = \tilde{\delta}'_3((f(p_1), T^t(f(p_1)), I^t(f(p_1)), F^t(f(p_1))), a, f(p_2)) < 1.$$

Then

$$\vee\{\tilde{\delta}_1((r, T^t(r), I^t(r), F^t(r)), a, s)\big|s \in Q_1, f(s) = f(q)\} > 0,$$
$$\vee\{\tilde{\delta}_2((r, T^t(r), I^t(r), F^t(r)), a, s)\big|s \in Q_1, f(s) = f(q)\} > 0,$$
$$\wedge\{\tilde{\delta}_3((r, T^t(r), I^t(r), F^t(r)), a, s)\big|s \in Q_1, f(s) = f(q)\} < 1.$$

Then there exists $q \in Q_1$ such that $f(q) = f(s)$ and

$$\tilde{\delta}_1((r, T^t(r), I^t(r), F^t(r)), a, q) > 0,$$
$$\tilde{\delta}_2((r, T^t(r), I^t(r), F^t(r)), a, q) > 0,$$
$$\tilde{\delta}_3((r, T^t(r), I^t(r), F^t(r)), a, q) < 1.$$

Then $T_N(q) \geq T_N(r), I_N(q) \geq I_N(r), F_N(q) \leq F_N(r)$. So, $T_{f(N)}(f(q)) \geq T_N(r), I_{f(N)}(f(q)) \geq I_N(r), F_{f(N)}(f(q)) \leq F_N(r)$. Therefore, $T_{f(N)}(f(q)) \geq T_{f(N)}(f(p_1)), I_{f(N)}(f(q)) \geq I_{f(N)}(f(p_1)), F_{f(N)}(f(q)) \leq F_{f(N)}(f(p_1))$. Hence, $\mathcal{M}'_1$ is a SVNSSGM of $\mathcal{M}_1$. □

5. Conclusion

In this study, for a given SVNGM $\mathcal{M}$ the notion of single-valued neutrosophic sub-general machine of $\mathcal{M}$ has been introduced and examined in details. Accordingly, the research has shown that the operators have some interesting properties under homomorphism. Moreover, the notion of single-valued neutrosophic strong sub-general machine has been presented. In addition, it has been shown that for a given SVNGM $\mathcal{M}$ if $\mathcal{M}'$ is a SVNSSGM of $\mathcal{M}$, then $\mathcal{M}'$ is a SVNSGM of $\mathcal{M}$, but the converse is not held.

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Properties of SuperHyperGraph and Neutrosophic SuperHyperGraph

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Abstract. New setting is introduced to study dominating, resolving, coloring, Eulerian(Hamiltonian) neutrosophic path, n-Eulerian(Hamiltonian) neutrosophic path, zero forcing number, zero forcing neutrosophic-number, independent number, independent neutrosophic-number, clique number, clique neutrosophic-number, matching number, matching neutrosophic-number, girth, neutrosophic girth, 1-zero-forcing number, 1-zero-forcing neutrosophic-number, failed 1-zero-forcing number, failed 1-zero-forcing neutrosophic-number, global-offensive alliance, t-offensive alliance, t-defensive alliance, t-powerful alliance, and global-powerful alliance in SuperHyperGraph and Neutrosophic SuperHyperGraph. Some Classes of SuperHyperGraph and Neutrosophic SuperHyperGraph are cases of study. Some results are applied in family of SuperHyperGraph and Neutrosophic SuperHyperGraph.

Keywords: SuperHyperGraph; Neutrosophic SuperHyperGraph; Classes; Families

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1. Introduction

Fuzzy set in [11], neutrosophic set in [2], related definitions of other sets in [2, 8, 10], hypergraphs and new notions on them in [6], neutrosophic graphs in [3], studies on neutrosophic graphs in [1], relevant definitions of other graphs based on fuzzy graphs in [7], are proposed. Also, some studies and researches about neutrosophic graphs, are proposed as a book in [5].

2. SuperHyperGraph

Definition 2.1. (Smarandache in 2019 and 2020, [9]).
An ordered pair $(G \subseteq P(V), E \subseteq P(V))$ is called by **SuperHyperGraph** and it's denoted by SHG.

Definition 2.2. (Smarandache in 2019 and 2020, [9]).
An ordered pair $(G_n \subseteq P^n(V), E_n \subseteq P^n(V))$ is called by **n-SuperHyperGraph** and it's denoted by n-SHG.

Definition 2.3. (Dominating, Resolving and Coloring).
Assume SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$.

(a) : SuperHyper-dominating set and number are defined as follows.

- (i) : A SuperVertex X_n **SuperHyper-dominates** a SuperVertex Y_n if there's at least one SuperHyperEdge which have them.
- (ii) : A set S is called **SuperHyper-dominating set** if for every $Y_n \in G_n \setminus S$, there's at least one SuperVertex X_n which SuperHyper-dominates SuperVertex Y_n.
- (iii) : If $\mathcal{S}$ is set of all sets of SuperHyper-dominating sets, then

$$|X| = \min_{S \in \mathcal{S}} |\{\cup X_n | X_n \in S\}|$$

is called **optimal-SuperHyper-dominating number** and X is called **optimal-SuperHyper-dominating set.**

(b) : SuperHyper-resolving set and number are defined as follows.

- (i) : A SuperVertex x **SuperHyper-resolves** SuperVertices y, w if

$$d(x, y) \neq d(x, w).$$

.

- (ii) : A set S is called **SuperHyper-resolving set** if for every $Y_n \in G_n \setminus S$, there's at least one SuperVertex X_n which SuperHyper-resolves SuperVertices Y_n, W_n.
- (iii) : If $\mathcal{S}$ is set of all sets of SuperHyper-resolving sets, then

$$|X| = \min_{S \in \mathcal{S}} |\{\cup X_n | X_n \in S\}|$$

is called **optimal-SuperHyper-resolving number** and X is called **optimal-SuperHyper-resolving set.**

(c) : SuperHyper-coloring set and number are defined as follows.

- (i) : A SuperVertex X_n **SuperHyper-colors** a SuperVertex Y_n differently with itself if there's at least one SuperHyperEdge which is incident to them.
- (ii) : A set S_n is called **SuperHyper-coloring set** if for every $Y_n \in G_n \setminus S_n$, there's at least one SuperVertex X_n which SuperHyper-colors SuperVertex Y_n.
- (iii) : If $\mathcal{S}_n$ is set of all sets of SuperHyper-coloring sets, then

$$|X| = \min_{S_n \in \mathcal{S}_n} |\{\cup X_n | X_n \in S_n\}|$$

is called **optimal-SuperHyper-coloring number** and X is called **optimal-SuperHyper-coloring set.**

Proposition 2.4. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. S *is maximum set of SuperVertices which form a SuperHyperEdge. Then optimal-SuperHyper-coloring set has as cardinality as* S *has.*

Proposition 2.5. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *If optimal-SuperHyper-coloring number is* $|V|$, *then for every SuperVertex there's at least one SuperHyperEdge which contains has all members of* V.

Proposition 2.6. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *If there's at least one SuperHyperEdge which has all members of* V, *then optimal-SuperHyper-coloring number is* $|V|$.

Proposition 2.7. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *If optimal-SuperHyper-dominating number is* $|V|$, *then there's one member of* V, *is contained in, at least one SuperVertex which doesn't have incident to any SuperHyperEdge.*

Proposition 2.8. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *Then optimal-SuperHyper-dominating number is* $< |V|$.

Proposition 2.9. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *If optimal-SuperHyper-resolving number is* $|V|$, *then every given SuperVertex doesn't have incident to any SuperHyperEdge.*

Proposition 2.10. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *Then optimal-SuperHyper-resolving number is* $< |V|$.

Proposition 2.11. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *If optimal-SuperHyper-coloring number is* $|V|$, *then all SuperVertices which have incident to at least one SuperHyperEdge.*

Proposition 2.12. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *Then optimal-SuperHyper-coloring number isn't* $< |V|$.

Proposition 2.13. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *Then optimal-SuperHyper-dominating set has cardinality which is greater than* $n - 1$ *where* n *is the cardinality of the set* V.

Proposition 2.14. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. S *is maximum set of SuperVertices which form a SuperHyperEdge. Then* S *is optimal-SuperHyper-coloring set and* $|\{\cup X_n \mid X_n \in S\}|$ *is optimal-SuperHyper-coloring number.*

Proposition 2.15. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *If* S *is SuperHyper-dominating set, then* D *contains* S *is SuperHyper-dominating set.*

Proposition 2.16. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *If* S *is SuperHyper-resolving set, then* D *contains* S *is SuperHyper-resolving set.*

Proposition 2.17. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *If* S *is SuperHyper-coloring set, then* D *contains* S *is SuperHyper-coloring set.*

Proposition 2.18. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *Then* G_n *is SuperHyper-dominating set.*

Proposition 2.19. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$.*Then* G_n *is SuperHyper-resolving set.*

Proposition 2.20. *Assume SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$. *Then* G_n *is SuperHyper-coloring set.*

Proposition 2.21. *Assume* $\mathcal{G}$ *is a family of SuperHyperGraph. Then* G_n *is SuperHyper-dominating set for all members of* $\mathcal{G}$, *simultaneously.*

Proposition 2.22. *Assume* $\mathcal{G}$ *is a family of SuperHyperGraph. Then* G_n *is SuperHyper-resolving set for all members of* $\mathcal{G}$, *simultaneously.*

Proposition 2.23. *Assume* $\mathcal{G}$ *is a family of SuperHyperGraph. Then* G_n *is SuperHyper-coloring set for all members of* $\mathcal{G}$, *simultaneously.*

Proposition 2.24. *Assume* $\mathcal{G}$ *is a family of SuperHyperGraph. Then* $G_n \setminus \{X_n\}$ *is SuperHyper-dominating set for all members of* $\mathcal{G}$, *simultaneously.*

Proposition 2.25. *Assume* $\mathcal{G}$ *is a family of SuperHyperGraph. Then* $G_n \setminus \{X_n\}$ *is SuperHyper-resolving set for all members of* $\mathcal{G}$, *simultaneously.*

Proposition 2.26. *Assume* $\mathcal{G}$ *is a family of SuperHyperGraph. Then* $G_n \setminus \{X_n\}$ *isn't SuperHyper-coloring set for all members of* $\mathcal{G}$, *simultaneously.*

Proposition 2.27. *Assume* $\mathcal{G}$ *is a family of SuperHyperGraph. Then union of SuperHyper-dominating sets from each member of* $\mathcal{G}$ *is SuperHyper-dominating set for all members of* $\mathcal{G}$, *simultaneously.*

Proposition 2.28. *Assume* $\mathcal{G}$ *is a family of SuperHyperGraph. Then union of SuperHyper-resolving sets from each member of* $\mathcal{G}$ *is SuperHyper-resolving set for all members of* $\mathcal{G}$, *simultaneously.*

Proposition 2.29. *Assume* $\mathcal{G}$ *is a family of SuperHyperGraph. Then union of SuperHyper-coloring sets from each member of* $\mathcal{G}$ *is SuperHyper-coloring set for all members of* $\mathcal{G}$, *simultaneously.*

Proposition 2.30. *Assume $\mathcal{G}$ is a family of SuperHyperGraph. For every given SuperVertex, there's one SuperHyperGraph such that the SuperVertex has another SuperVertex which are incident to a SuperHyperEdge. If for given SuperVertex, all SuperVertices have a common SuperHyperEdge in this way, then $G_n \setminus \{X_n\}$ is optimal-SuperHyper-dominating set for all members of $\mathcal{G}$, simultaneously.*

Proposition 2.31. *Assume $\mathcal{G}$ is a family of SuperHyperGraph. For every given SuperVertex, there's one SuperHyperGraph such that the SuperVertex has another SuperVertex which are incident to a SuperHyperEdge. If for given SuperVertex, all SuperVertices have a common SuperHyperEdge in this way, then $G_n \setminus \{X_n\}$ is optimal-SuperHyper-resolving set for all members of $\mathcal{G}$, simultaneously.*

Proposition 2.32. *Assume $\mathcal{G}$ is a family of SuperHyperGraph. For every given SuperVertex, there's one SuperHyperGraph such that the SuperVertex has another SuperVertex which are incident to a SuperHyperEdge. If for given SuperVertex, all SuperVertices have a common SuperHyperEdge in this way, then G_n is optimal-SuperHyper-coloring set for all members of $\mathcal{G}$, simultaneously.*

Proposition 2.33. *Let SHG be a SuperHyperGraph. An $(k-1)$-set from an k-set of twin SuperVertices is subset of a SuperHyper-resolving set.*

Corollary 2.34. *Let SHG be a SuperHyperGraph. The number of twin SuperVertices is $n-1$. Then SuperHyper-resolving number is $n-2$.*

Corollary 2.35. *Let SHG be SuperHyperGraph. The number of twin SuperVertices is $n-1$. Then SuperHyper-resolving number is $n-2$. Every $(n-2)$-set including twin SuperVertices is SuperHyper-resolving set.*

Proposition 2.36. *Let SHG be SuperHyperGraph such that it's complete. Then SuperHyper-resolving number is $n-1$. Every $(n-1)$-set is SuperHyper-resolving set.*

Proposition 2.37. *Let $\mathcal{G}$ be a family of SuperHyperGraphs with common super vertex set G_n. Then simultaneously SuperHyper-resolving number of $\mathcal{G}$ is $|V|-1$*

Proposition 2.38. *Let $\mathcal{G}$ be a family of SuperHyperGraphs with common SuperVertex set G_n. Then simultaneously SuperHyper-resolving number of $\mathcal{G}$ is greater than the maximum SuperHyper-resolving number of n-$SHG \in \mathcal{G}$.*

Proposition 2.39. *Let $\mathcal{G}$ be a family of SuperHyperGraphs with common SuperVertex set G_n. Then simultaneously SuperHyper-resolving number of $\mathcal{G}$ is greater than simultaneously SuperHyper-resolving number of $\mathcal{H} \subseteq \mathcal{G}$.*

Theorem 2.40. *Twin SuperVertices aren't SuperHyper-resolved in any given SuperHyperGraph.*

Proposition 2.41. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyperGraph. If SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is complete, then every couple of SuperVertices are twin SuperVertices.*

Theorem 2.42. *Let $\mathcal{G}$ be a family of SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ with SuperVertex set G_n and n-$SHG \in \mathcal{G}$ is complete. Then simultaneously SuperHyper-resolving number is $|V| - 1$. Every $(n-1)$-set is simultaneously SuperHyper-resolving set for $\mathcal{G}$.*

Corollary 2.43. *Let $\mathcal{G}$ be a family of SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ with SuperVertex set G_n and n-$SHG \in \mathcal{G}$ is complete. Then simultaneously SuperHyper-resolving number is $|V| - 1$. Every $(|V|-1)$-set is simultaneously SuperHyper-resolving set for $\mathcal{G}$.*

Theorem 2.44. *Let $\mathcal{G}$ be a family of SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ with SuperVertex set G_n and for every given couple of SuperVertices, there's a n-$SHG \in \mathcal{G}$ such that in that, they're twin SuperVertices. Then simultaneously SuperHyper-resolving number is $|V| - 1$. Every $(|V|-1)$-set is simultaneously SuperHyper-resolving set for $\mathcal{G}$.*

Theorem 2.45. *Let $\mathcal{G}$ be a family of SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ with SuperVertex set G_n. If $\mathcal{G}$ contains three SuperHyper-stars with different SuperHyper-centers, then simultaneously SuperHyper-resolving number is $|V| - 2$. Every $(|V|-2)$-set is simultaneously SuperHyper-resolving set for $\mathcal{G}$.*

Corollary 2.46. *Let $\mathcal{G}$ be a family of SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ with SuperVertex set G_n. If $\mathcal{G}$ contains three SuperHyper-stars with different SuperHyper-centers, then simultaneously SuperHyper-resolving number is $|V| - 2$. Every $(|V|-2)$-set is simultaneously SuperHyper-resolving set for $\mathcal{G}$.*

Proposition 2.47. *Consider two antipodal SuperVertices X_n and Y_n in any given even SuperHyper-cycle. Let U_n and V_n be given SuperVertices. Then $d(X_n, U_n) \neq d(X_n, V_n)$ if and only if $d(Y_n, U_n) \neq d(Y_n, V_n)$.*

Proposition 2.48. *Consider two antipodal SuperVertices X_n and Y_n in any given even cycle. Let U_n and V_n be given SuperVertices. Then $d(X_n, U_n) = d(X_n, V_n)$ if and only if $d(Y_n, U_n) = d(Y_n, V_n)$.*

Proposition 2.49. *The set contains two antipodal SuperVertices, isn't SuperHyper-resolving set in any given even SuperHyper-cycle.*

Proposition 2.50. *Consider two antipodal SuperVertices* X_n *and* Y_n *in any given even SuperHyper-cycle.* X_n *SuperHyper-resolves a given couple of SuperVertices,* Z_n *and* Z'_n*, if and only if* Y_n *does.*

Proposition 2.51. *There are two antipodal SuperVertices aren't SuperHyper-resolved by other two antipodal SuperVertices in any given even SuperHyper-cycle.*

Proposition 2.52. *For any two antipodal SuperVertices in any given even SuperHyper-cycle, there are only two antipodal SuperVertices don't SuperHyper-resolve them.*

Proposition 2.53. *In any given even SuperHyper-cycle, for any SuperVertex, there's only one SuperVertex such that they're antipodal SuperVertices.*

Proposition 2.54. *Let SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an even SuperHyper-cycle. Then every couple of SuperVertices are SuperHyper-resolving set if and only if they aren't antipodal SuperVertices.*

Corollary 2.55. *Let SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an even SuperHyper-cycle. Then SuperHyper-resolving number is two.*

Corollary 2.56. *Let SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an even SuperHyper-cycle. Then SuperHyper-resolving set contains couple of SuperVertices such that they aren't antipodal SuperVertices.*

Corollary 2.57. *Let* $\mathcal{G}$ *be a family SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an odd SuperHyper-cycle with common SuperVertex set* G_n*. Then simultaneously SuperHyper-resolving set contains couple of SuperVertices such that they aren't antipodal SuperVertices and SuperHyper-resolving number is two.*

Proposition 2.58. *In any given SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *which is odd SuperHyper-cycle, for any SuperVertex, there's no SuperVertex such that they're antipodal SuperVertices.*

Proposition 2.59. *Let SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an odd SuperHyper-cycle. Then every couple of SuperVertices are SuperHyper-resolving set.*

Proposition 2.60. *Let SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an odd cycle. Then SuperHyper-resolving number is two.*

Corollary 2.61. *Let SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an odd cycle. Then SuperHyper-resolving set contains couple of SuperVertices.*

Corollary 2.62. *Let $\mathcal{G}$ be a family of SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ which are odd SuperHyper-cycles with common SuperVertex set G_n. Then simultaneously SuperHyper-resolving set contains couple of SuperVertices and SuperHyper-resolving number is two.*

Proposition 2.63. *Let SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyper-path. Then every SuperHyper-leaf forms SuperHyper-resolving set.*

Proposition 2.64. *Let SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyper-path. Then a set including every couple of SuperVertices is SuperHyper-resolving set.*

Proposition 2.65. *Let SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyper-path. Then an 1-set contains leaf is SuperHyper-resolving set and SuperHyper-resolving number is one.*

Corollary 2.66. *Let $\mathcal{G}$ be a family of SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ are SuperHyper-paths with common SuperVertex set G_n such that they've a common SuperHyper-leaf. Then simultaneously SuperHyper-resolving number is 1, 1-set contains common leaf, is simultaneously SuperHyper-resolving set for $\mathcal{G}$.*

Proposition 2.67. *Let $\mathcal{G}$ be a family of SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ are SuperHyper-paths with common SuperVertex set G_n such that for every SuperHyper-leaf L_n from n-SHG, there's another n-$SHG \in \mathcal{G}$ such that L_n isn't SuperHyper-leaf. Then an 2-set contains every couple of SuperVertices, is SuperHyper-resolving set. An 2-set contains every couple of SuperVertices, is optimal-SuperHyper-resolving set. Optimal-SuperHyper-resolving number is two.*

Corollary 2.68. *Let $\mathcal{G}$ be a family of SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ are SuperHyper-paths with common SuperVertex set G_n such that they've no common SuperHyper-leaf. Then an 2-set is simultaneously optimal-SuperHyper-resolving set and simultaneously optimal-SuperHyper-resolving number is 2.*

Proposition 2.69. *Let SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyper-t-partite. Then every set excluding couple of SuperVertices in different parts whose cardinalities of them are strictly greater than one, is optimal-SuperHyper-resolving set.*

Corollary 2.70. *Let SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyper-t-partite. Let $|V| \geq 3$. Then every $(|V| - 2)$-set excludes two SuperVertices from different parts whose cardinalities of them are strictly greater than one, is optimal-SuperHyper-resolving set and optimal-SuperHyper-resolving number is $|V| - 2$.*

Corollary 2.71. *Let SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a SuperHyper-bipartite. Let* $|V| \geq 3$*. Then every* $(|V|-2)$*-set excludes two SuperVertices from different parts, is optimal-SuperHyper-resolving set and optimal-SuperHyper-resolving number is* $|V|-2$*.*

Corollary 2.72. *Let SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a SuperHyper-star. Then every* $(|V|-2)$*-set excludes SuperHyper-center and a given SuperVertex, is optimal-SuperHyper-resolving set and optimal-SuperHyper-resolving number is* $(|V|-2)$*.*

Corollary 2.73. *Let SuperHyperGraph* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a SuperHyper-wheel. Let* $|V| \geq 3$*. Then every* $(|V|-2)$*-set excludes SuperHyper-center and a given SuperVertex, is optimal-SuperHyper-resolving set and optimal-SuperHyper-resolving number is* $|V|-2$*.*

Corollary 2.74. *Let* $\mathcal{G}$ *be a family of SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *which are SuperHyper-t-partite with common SuperVertex set* G_n*. Let* $|V| \geq 3$*. Then simultaneously optimal-SuperHyper-resolving number is* $|V|-2$ *and every* $(|V|-2)$*-set excludes two SuperVertices from different parts, is simultaneously optimal-SuperHyper-resolving set for* $\mathcal{G}$*.*

Corollary 2.75. *Let* $\mathcal{G}$ *be a family of SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *which are SuperHyper-bipartite with common SuperVertex set* G_n*. Let* $|V| \geq 3$*. Then simultaneously optimal-SuperHyper-resolving number is* $|V|-2$ *and every* $(|V|-2)$*-set excludes two SuperVertices from different parts, is simultaneously optimal-SuperHyper-resolving set for* $\mathcal{G}$*.*

Corollary 2.76. *Let* $\mathcal{G}$ *be a family of SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *which are SuperHyper-star with common SuperVertex set* G_n*. Let* $|V| \geq 3$*. Then simultaneously optimal-SuperHyper-resolving number is* $|V|-2$ *and every* $(|V|-2)$*-set excludes SuperHyper-center and a given SuperVertex, is simultaneously optimal-SuperHyper-resolving set for* $\mathcal{G}$*.*

Corollary 2.77. *Let* $\mathcal{G}$ *be a family of SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *which are SuperHyper-wheel with common SuperVertex set* G_n*. Let* $|V| \geq 3$*. Then simultaneously optimal-SuperHyper-resolving number is* $|V|-2$ *and every* $(|V|-2)$*-set excludes SuperHyper-center and a given SuperVertex, is simultaneously optimal-SuperHyper-resolving set for* $\mathcal{G}$*.*

Proposition 2.78. *Let SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a SuperHyper-complete. Then optimal-SuperHyper-coloring number is* $|V|$*.*

Proposition 2.79. *Let SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a SuperHyper-path. Then optimal-SuperHyper-coloring number is two.*

Proposition 2.80. *Let SuperHyperGraphs* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an even SuperHyper-cycle. Then optimal-SuperHyper-coloring number is two.*

Proposition 2.81. *Let SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ be an odd SuperHyper-cycle. Then optimal-SuperHyper-coloring number is three.*

Proposition 2.82. *Let SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyper-star. Then optimal-SuperHyper-coloring number is two.*

Proposition 2.83. *Let SuperHyperGraphs $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyper-wheel such that it has even SuperHyper-cycle. Then optimal-SuperHyper-coloring number is Three.*

Proposition 2.84. *Let SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyper-wheel such that it has odd SuperHyper-cycle. Then optimal-SuperHyper-coloring number is four.*

Proposition 2.85. *Let SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyper-complete and SuperHyper-bipartite. Then optimal-SuperHyper-coloring number is two.*

Proposition 2.86. *Let SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a SuperHyper-complete and SuperHyper-t-partite. Then optimal-SuperHyper-coloring number is t.*

Proposition 2.87. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be SuperHyperGraph. Then optimal-SuperHyper-coloring number is 1 if and only if $SHG = (G \subseteq P(V), E \subseteq P(V))$ is SuperHyper-empty.*

Proposition 2.88. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be SuperHyperGraph. Then optimal-SuperHyper-coloring number is 2 if and only if $SHG = (G \subseteq P(V), E \subseteq P(V))$ is both SuperHyper-complete and SuperHyper-bipartite.*

Proposition 2.89. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be SuperHyperGraph. Then optimal-SuperHyper-coloring number is $|V|$ if and only if $SHG = (G \subseteq P(V), E \subseteq P(V))$ is SuperHyper-complete.*

Proposition 2.90. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be SuperHyperGraph. Then optimal-SuperHyper-coloring number is obtained from the number of SuperVertices which is $|G_n|$ and optimal-SuperHyper-coloring number is at most $|V|$.*

Proposition 2.91. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be SuperHyperGraph. Then optimal-SuperHyper-coloring number is at most $\Delta + 1$ and at least 2.*

Proposition 2.92. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be SuperHyperGraph and SuperHyper-r-regular. Then optimal-SuperHyper-coloring number is at most $r + 1$.*

Definition 2.93. (Eulerian(Hamiltonian) Neutrosophic Path).
Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a neutrosophic SuperHyperGraph. Then

(i) **Eulerian(Hamiltonian) neutrosophic path** $\mathcal{M}_e(SHG)(\mathcal{M}_h(SHG))$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is a sequence of consecutive edges(vertices) $x_1, x_2, \cdots, x_{\mathcal{S}(SHG)}(x_1, x_2, \cdots, x_{\mathcal{O}(SHG)})$ which is neutrosophic path;

(ii) **n-Eulerian(Hamiltonian) neutrosophic path** $\mathcal{N}_e(SHG)(\mathcal{N}_h(SHG))$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is the number of sequences of consecutive edges(vertices) $x_1, x_2, \cdots, x_{\mathcal{S}(SHG)}(x_1, x_2, \cdots, x_{\mathcal{O}(SHG)})$ which is neutrosophic path.

Proposition 2.94. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-neutrosophic SuperHyperGraph with two weakest edges. Then*

$$\mathcal{M}_e(CMT_\sigma) : \textit{Not Existed;}$$

$$\mathcal{M}_h(CMT_\sigma) : v_{\tau(1)}, v_{\tau(2)}, \cdots, v_{\tau(\mathcal{O}(CMT_\sigma)-1)}, v_{\tau(\mathcal{O}(CMT_\sigma))}$$

where τ is a permutation on $\mathcal{O}(CMT_\sigma)$.

$$\mathcal{N}_e(CMT_\sigma) = 0;$$

$$\mathcal{N}_h(CMT_\sigma) = \mathcal{O}(CMT_\sigma)!.$$

Proposition 2.95. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a path-neutrosophic SuperHyperGraph. Then*

$$\mathcal{M}_e(PTH) : v_1, v_2, \cdots, v_{\mathcal{S}(PTH)};$$

$$\mathcal{M}_h(PTH) : v_1, v_2, \cdots, v_{\mathcal{O}(PTH)}.$$

$$\mathcal{N}_e(PTH) = 1;$$

$$\mathcal{N}_h(PTH) = 1.$$

Proposition 2.96. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a cycle-neutrosophic SuperHyperGraph where $\mathcal{O}(CYC) \geq 3$. Then*

$$\mathcal{M}_e(CYC) : \textit{Not Existed};$$

$$\mathcal{M}_h(CYC) : x_i, x_{i+1}, \cdots, x_{\mathcal{O}(CYC)-1}, x_{\mathcal{O}(CYC)}, \cdots, x_{i-1}.$$

$$\mathcal{N}_e(CYC) = 0;$$

$$\mathcal{N}_h(CYC) = \mathcal{O}(CYC).$$

Proposition 2.97. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a star-neutrosophic SuperHyperGraph with center c. Then*

$$\mathcal{M}_e(STR_{1,\sigma_2}) : v_1, v_2$$

$$\mathcal{M}_h(STR_{1,\sigma_2}) : v_1, c, v_2$$

where $\mathcal{O}(STR_{1,\sigma_2}) \leq 2$;

$$\mathcal{M}_e(STR_{1,\sigma_2}) : \textit{Not Existed}$$

$$\mathcal{M}_h(STR_{1,\sigma_2}) : \textit{Not Existed}$$

where $\mathcal{O}(STR_{1,\sigma_2}) \geq 3$.

$$\mathcal{N}_e(STR_{1,\sigma_2}) = 2$$

$$\mathcal{N}_h(STR_{1,\sigma_2}) = 3$$

where $\mathcal{O}(STR_{1,\sigma_2}) \leq 2$;

$$\mathcal{N}_e(STR_{1,\sigma_2}) = 0$$

$$\mathcal{N}_h(STR_{1,\sigma_2}) = 0$$

where $\mathcal{O}(STR_{1,\sigma_2}) \geq 3$.

Proposition 2.98. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-bipartite-neutrosophic SuperHyperGraph. Then*

$$\mathcal{M}_e(CMC_{\sigma_1,\sigma_2}) : \textit{Not Existed}$$

$$\mathcal{M}_h(CMC_{\sigma_1,\sigma_2}) : v_1, v_2, \cdots, v_{\mathcal{O}(CMC_{\sigma_1,\sigma_2})-1}, v_{\mathcal{O}(CMC_{\sigma_1,\sigma_2})}$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2}) \geq 3,\ |V_1| = |V_2|,\ v_{2i+1} \in V_1,\ v_{2i} \in V_2$;

$$\mathcal{M}_e(CMC_{\sigma_1,\sigma_2}) : v_1v_2$$

$$\mathcal{M}_h(CMC_{\sigma_1,\sigma_2}) : v_1, v_2$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2}) = 2$;

$$\mathcal{M}_e(CMC_{\sigma_1,\sigma_2}) : -$$

$$\mathcal{M}_h(CMC_{\sigma_1,\sigma_2}) : v_1$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2}) = 1$.

$$\mathcal{N}_e(CMC_{\sigma_1,\sigma_2}) = 0$$

$$\mathcal{N}_h(CMC_{\sigma_1,\sigma_2}) = c$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2}) \geq 3,\ |V_1| = |V_2|,\ v_{2i+1} \in V_1,\ v_{2i} \in V_2$;

$$\mathcal{N}_e(CMC_{\sigma_1,\sigma_2}) = 2$$

$$\mathcal{N}_h(CMC_{\sigma_1,\sigma_2}) = 2$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2}) = 2$;

$$\mathcal{N}_e(CMC_{\sigma_1,\sigma_2}) = -$$

$$\mathcal{N}_h(CMC_{\sigma_1,\sigma_2}) = 1$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2}) = 1.$

Proposition 2.99. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a complete-t-partite-neutrosophic SuperHyperGraph. Then*

$$\mathcal{M}_e(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) : \textit{Not Existed}$$

$$\mathcal{M}_h(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) : v_1, v_2, \cdots, v_{\mathcal{O}(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t})-1}, v_{\mathcal{O}(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t})}$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) \geq 3,\ |V_i| = |V_j|,\ v_{2i+1} \in V_i,\ v_{2i} \in V_j;$

$$\mathcal{M}_e(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) : v_1v_2$$

$$\mathcal{M}_h(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) : v_1, v_2$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = 2;$

$$\mathcal{M}_e(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) : -$$

$$\mathcal{M}_h(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) : v_1$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = 1.$

$$\mathcal{N}_e(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = 0$$

$$\mathcal{N}_h(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = c$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) \geq 3,\ |V_i| = |V_j|,\ v_{2i+1} \in V_i,\ v_{2i} \in V_j;$

$$\mathcal{N}_e(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = 2$$

$$\mathcal{N}_h(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = 2$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = 2;$

$$\mathcal{N}_e(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = -$$

$$\mathcal{N}_h(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = 1$$

where $\mathcal{O}(CMC_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = 1.$

Proposition 2.100. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a wheel-neutrosophic SuperHyperGraph. Then*

$$\mathcal{M}_h(WHL_{1,\sigma_2}) : x_i, x_{i+1}, \cdots, x_{\mathcal{O}(WHL_{1,\sigma_2})-1}, x_{\mathcal{O}(WHL_{1,\sigma_2})}, x_{i-1}.$$

$$\mathcal{M}_e(WHL_{1,\sigma_2}) : v_1, v_2, v_3$$

where $\mathcal{S}(WHL_{1,\sigma_2}) = 3.$

$$\mathcal{M}_h(WHL_{1,\sigma_2}) : x_i, x_{i+1}, \cdots, x_{\mathcal{O}(WHL_{1,\sigma_2})-1}, x_{\mathcal{O}(WHL_{1,\sigma_2})}, x_{i-1}.$$

$$\mathcal{M}_e(WHL_{1,\sigma_2}) : \textit{Not Existed}$$

where $\mathcal{S}(WHL_{1,\sigma_2}) > 3.$

$$\mathcal{N}_h(WHL_{1,\sigma_2}) = \mathcal{O}(WHL_{1,\sigma_2});$$

$$\mathcal{N}_e(WHL_{1,\sigma_2}) = 3;$$

where $\mathcal{S}(WHL_{1,\sigma_2}) = 3$.

$$\mathcal{N}_h(WHL_{1,\sigma_2}) = \mathcal{O}(WHL_{1,\sigma_2});$$

$$\mathcal{N}_e(WHL_{1,\sigma_2}) = 0;$$

where $\mathcal{S}(WHL_{1,\sigma_2}) > 3$.

3. Neutrosophic SuperHyperGraph

Definition 3.1. (Zero Forcing Number).
Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a neutrosophic SuperHyperGraph. Then

(i) **zero forcing number** $\mathcal{Z}(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is minimum cardinality of a set S of black vertices (whereas vertices in $V(G) \setminus S$ are colored white) such that $V(G)$ is turned black after finitely many applications of "the color-change rule": a white vertex is converted to a black vertex if it is the only white neighbor of a black vertex;

(ii) **zero forcing neutrosophic-number** $\mathcal{Z}_n(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is minimum neutrosophic cardinality of a set S of black vertices (whereas vertices in $V(G) \setminus S$ are colored white) such that $V(G)$ is turned black after finitely many applications of "the color-change rule": a white vertex is converted to a black vertex if it is the only white neighbor of a black vertex.

Definition 3.2. (Independent Number).
Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a neutrosophic SuperHyperGraph. Then

(i) **independent number** $\mathcal{I}(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is maximum cardinality of a set S of vertices such that every two vertices of S aren't endpoints for an edge, simultaneously;

(ii) **independent neutrosophic-number** $\mathcal{I}_n(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is maximum neutrosophic cardinality of a set S of vertices such that every two vertices of S aren't endpoints for an edge, simultaneously.

Definition 3.3. (Clique Number).
Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a neutrosophic SuperHyperGraph. Then

(i) **clique number** $\mathcal{C}(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is maximum cardinality of a set S of vertices such that every two vertices of S are endpoints for an edge, simultaneously;

(*ii*) **clique neutrosophic-number** $\mathcal{C}_n(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is maximum neutrosophic cardinality of a set S of vertices such that every two vertices of S are endpoints for an edge, simultaneously.

Definition 3.4. (Matching Number).
Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a neutrosophic SuperHyperGraph. Then

(*i*) **matching number** $\mathcal{M}(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is maximum cardinality of a set S of edges such that every two edges of S don't have any vertex in common;

(*ii*) **matching neutrosophic-number** $\mathcal{M}_n(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is maximum neutrosophic cardinality of a set S of edges such that every two edges of S don't have any vertex in common.

Definition 3.5. (Girth and Neutrosophic Girth).
Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a neutrosophic SuperHyperGraph. Then

(*i*) **girth** $\mathcal{G}(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is minimum crisp cardinality of vertices forming shortest cycle. If there isn't, then girth is ∞;

(*ii*) **neutrosophic girth** $\mathcal{G}_n(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is minimum neutrosophic cardinality of vertices forming shortest cycle. If there isn't, then girth is ∞.

Proposition 3.6. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-neutrosophic SuperHyperGraph. Then*

(1)
$$\mathcal{Z}(CMT_\sigma) = \mathcal{O}(CMT_\sigma) - 1.$$

(2)
$$\mathcal{I}(SHG) = 1.$$

(3)
$$\mathcal{C}(SHG) = \mathcal{O}(SHG).$$

(4)
$$\mathcal{M}(SHG) = \lfloor \frac{n}{2} \rfloor.$$

(5)
$$\mathcal{G}(SHG) = 3.$$

Proposition 3.7. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a path-neutrosophic SuperHyperGraph. Then*

(1)

$$\mathcal{Z}(PTH_n) = 1.$$

(2)

$$\mathcal{I}(SHG) = \lceil \frac{\mathcal{O}(SHG)}{2} \rceil.$$

(3)

$$\mathcal{C}(SHG) = 2.$$

(4)

$$\mathcal{M}(SHG) = \lfloor \frac{n}{2} \rfloor.$$

(5)

$$\mathcal{G}(SHG) = \infty.$$

Proposition 3.8. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a cycle-neutrosophic SuperHyperGraph where $\mathcal{O}(CYC) \geq 3$. Then*

(1)

$$\mathcal{Z}(CYC_n) = 2.$$

(2)

$$\mathcal{I}(SHG) = \lfloor \frac{\mathcal{O}(SHG)}{2} \rfloor.$$

(3)

$$\mathcal{C}(SHG) = 2.$$

(4)

$$\mathcal{M}(SHG) = \lfloor \frac{n}{2} \rfloor.$$

(5)

$$\mathcal{G}(SHG) = \mathcal{O}(SHG).$$

Proposition 3.9. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a star-neutrosophic SuperHyperGraph with center c. Then*

(1)

$$\mathcal{Z}(STR_{1,\sigma_2}) = \mathcal{O}(STR_{1,\sigma_2}) - 2.$$

(2)

$$\mathcal{I}(SHG) = \mathcal{O}(SHG) - 1.$$

(3)

$$\mathcal{C}(SHG) = 2.$$

(4)

$$\mathcal{M}(SHG) = 1.$$

(5)

$$\mathcal{G}(SHG) = \infty.$$

Proposition 3.10. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-bipartite-neutrosophic SuperHyperGraph. Then*

(1)

$$\mathcal{Z}(CMT_{\sigma_1,\sigma_2}) = \mathcal{O}(CMT_{\sigma_1,\sigma_2}) - 2.$$

(2)

$$\mathcal{I}(SHG) = \max\{|V_1|, |V_2|\}.$$

(3)

$$\mathcal{C}(SHG) = 2.$$

(4)

$$\mathcal{M}(SHG) = \min\{|V_1|, |V_2|\}.$$

(5)

$$\mathcal{G}(SHG) = 4$$

where $\mathcal{O}(SHG) \geq 4$. And

$$\mathcal{G}(SHG) = \infty$$

where $\mathcal{O}(SHG) \leq 3$.

Proposition 3.11. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-t-partite-neutrosophic SuperHyperGraph. Then*

(1)

$$\mathcal{Z}(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = \mathcal{O}(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) - 1.$$

(2)

$$\mathcal{I}(SHG) = \max\{|V_1|, |V_2|, \cdots, |V_t|\}.$$

(3)

$$\mathcal{C}(SHG) = t.$$

(4)

$$\mathcal{M}(SHG) = \min |V_i|_{i=1}^{t}.$$

(5)

$$\mathcal{G}(SHG) = 3$$

where $t \geq 3$.

$$\mathcal{G}(SHG) = 4$$

where $t \leq 2$. And

$$\mathcal{G}(SHG) = \infty$$

where $\mathcal{O}(SHG) \leq 2$.

Proposition 3.12. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a complete-neutrosophic SuperHyperGraph. Then*

(1)
$$\mathcal{Z}_n(CMT_\sigma) = \mathcal{O}_n(CMT_\sigma) - \max\{\Sigma_{i=1}^3 \sigma_i(x)\}_{x \in V}.$$

(2)
$$\mathcal{I}_n(SHG) = \max\{\sum_{i=1}^{3} \sigma_i(x)\}_{x \in V}.$$

(3)
$$\mathcal{C}_n(SHG) = \mathcal{O}_n(SHG).$$

(4)
$$\mathcal{M}_n(SHG) = \max\{\sum_{i=1}^{3} \mu_i(x_0x_1) + \sum_{i=1}^{3} \mu_i(x_1x_2) + \cdots + \sum_{i=1}^{3} \mu_i(x_{j-1}x_j)\}_{j=\lfloor \frac{n}{2} \rfloor}.$$

(5)
$$\mathcal{G}_n(SHG) = \min\{\Sigma_{i=1}^3(\sigma_i(x) + \sigma_i(y) + \sigma_i(z))\}.$$

Proposition 3.13. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a path-neutrosophic SuperHyperGraph. Then*

(1)
$$\mathcal{Z}_n(PTH_n) = \min\{\Sigma_{i=1}^3 \sigma_i(x)\}_{x \text{ is a leaf}}.$$

(2)
$$\mathcal{I}_n(SHG) = \max\{\sum_{i=1}^{3}(\sigma_i(x_1) + \sigma_i(x_3) + \cdots + \sigma_i(x_t)),$$
$$\sum_{i=1}^{3} \sigma_i(x_2) + \sigma_i(x_4) + \cdots + \sigma_i(x'_t))\}_{x_i x_{i+1} \in E}.$$

(3)
$$\mathcal{C}_n(SHG) = \max\{\sum_{i=1}^{3}(\sigma_i(x_j) + \sigma_i(x_{j+1}))\}_{x_j x_{j+1} \in E}.$$

(4)
$$\mathcal{M}_n(SHG) = \max\{\sum_{i=1}^{3} \mu_i(x_0x_1) + \sum_{i=1}^{3} \mu_i(x_2x_3) + \cdots + \sum_{i=1}^{3} \mu_i(x_{j-1}x_j)\}_{|S| = \lfloor \frac{n}{2} \rfloor}.$$

(5)
$$\mathcal{G}_n(SHG) = \infty.$$

Proposition 3.14. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a cycle-neutrosophic SuperHyperGraph where* $\mathcal{O}(CYC) \geq 3$. *Then*

(1)

$$\mathcal{Z}_n(CYC_n) = \min\{\Sigma_{i=1}^3 \sigma_i(x) + \Sigma_{i=1}^3 \sigma_i(y)\}_{xy \in E.}.$$

(2)

$$\mathcal{I}_n(SHG) = \max\{\sum_{i=1}^{3}(\sigma_i(x_1) + \sigma_i(x_3) + \cdots + \sigma_i(x_t)),$$
$$\sum_{i=1}^{3} \sigma_i(x_2) + \sigma_i(x_4) + \cdots + \sigma_i(x'_t))\}_{x_i x_{i+1} \in E}.$$

(3)

$$\mathcal{C}_n(SHG) = \max\{\sum_{i=1}^{3}(\sigma_i(x_j) + \sigma_i(x_{j+1}))\}_{x_j x_{j+1} \in E}.$$

(4)

$$\mathcal{M}_n(SHG) = \max\{\sum_{i=1}^{3} \mu_i(x_0 x_1) + \sum_{i=1}^{3} \mu_i(x_2 x_3) + \cdots + \sum_{i=1}^{3} \mu_i(x_{j-1} x_j)\}_{|S| = \lfloor \frac{n}{2} \rfloor}.$$

(5)

$$\mathcal{G}_n(SHG) = \mathcal{O}_n(SHG).$$

Proposition 3.15. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a star-neutrosophic SuperHyperGraph with center c. Then*

(1)

$$\mathcal{Z}_n(STR_{1,\sigma_2}) = \mathcal{O}_n(STR_{1,\sigma_2}) - \max\{\Sigma_{i=1}^3 \sigma_i(c) + \Sigma_{i=1}^3 \sigma_i(x)\}_{x \in V}.$$

(2)

$$\mathcal{I}_n(SHG) = \mathcal{O}_n(SHG) - \sigma(c) = \sum_{i=1}^{3} \sum_{x_j \neq c} \sigma_i(x_j).$$

(3)

$$\mathcal{C}_n(SHG) = \sum_{i=1}^{3} \sigma_i(c) + \max\{\sum_{i=1}^{3} \sigma_i(x_j)\}.$$

(4)

$$\mathcal{M}_n(SHG) = \max\{\sum_{i=1}^{3} \mu_i(x_{j-1} x_j)\}_{x_{j-1} x_j \in E}.$$

(5)

$$\mathcal{G}_n(SHG) = \infty.$$

Proposition 3.16. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-bipartite-neutrosophic SuperHyperGraph. Then*

(1)

$$\mathcal{Z}_n(CMT_{\sigma_1,\sigma_2}) = \mathcal{O}_n(CMT_{\sigma_1,\sigma_2}) - \max\{\Sigma_{i=1}^3 \sigma_i(x) + \Sigma_{i=1}^3 \sigma_i(x')\}_{x,x' \in V}.$$

(2)

$$\mathcal{I}_n(SHG) = \max\{(\sum_{i=1}^{3}\sum_{x_j\in V_1}\sigma_i(x_j)), (\sum_{i=1}^{3}\sum_{x_j\in V_2}\sigma_i(x_j))\}.$$

(3)

$$\mathcal{C}_n(SHG) = \max\{\sum_{i=1}^{3}(\sigma_i(x_j)+\sigma_i(x_{j'}))\}_{x_j\in V_1,\ x_{j'}\in V_2}.$$

(4)

$$\mathcal{M}_n(SHG) = \max\{\sum_{i=1}^{3}\mu_i(x_0x_1)+\sum_{i=1}^{3}\mu_i(x_2x_3)+\cdots+\sum_{i=1}^{3}\mu_i(x_{j-1}x_j)\}_{|S|=\min\{|V_1|,|V_2|\}}.$$

(5)

$$\mathcal{G}_n(SHG) = \min\{\Sigma_{i=1}^{3}(\sigma_i(x)+\sigma_i(y)+\sigma_i(z)+\sigma_i(w))\}_{x,y\in V_1,\ z,w\in V_2}.$$

where $\mathcal{O}(SHG)\geq 4$ *and* $\min\{|V_1|,|V_2|\}\geq 2$. *Also,*

$$\mathcal{G}_n(SHG) = \infty$$

where $\mathcal{O}(SHG)\leq 3$.

Proposition 3.17. *Let* $SHG = (G\subseteq P(V), E\subseteq P(V))$ *be a complete-t-partite-neutrosophic SuperHyperGraph. Then*

(1)

$$\mathcal{Z}_n(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = \mathcal{O}_n(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) - \max\{\Sigma_{i=1}^{3}\sigma_i(x)\}_{x\in V}.$$

(2)

$$\mathcal{I}_n(SHG) = \max\{(\sum_{i=1}^{3}\sum_{x_j\in V_1}\sigma_i(x_j)), (\sum_{i=1}^{3}\sum_{x_j\in V_2}\sigma_i(x_j)), \cdots,$$
$$(\sum_{i=1}^{3}\sum_{x_j\in V_t}\sigma_i(x_j))\}.$$

(3)

$$\mathcal{C}_n(SHG) = \max\{\sum_{i=1}^{3}(\sigma_i(x_{j_1})+\sigma_i(x_{j_2})+\cdots+\sigma_i(x_{j_t}))\}_{x_{j_1}\in V_1, x_{j_2}\in V_2,\cdots,\ x_{j_t}\in V_t}.$$

(4)

$$\mathcal{M}_n(SHG) = \max\{\sum_{i=1}^{3}\mu_i(x_0x_1)+\sum_{i=1}^{3}\mu_i(x_2x_3)+\cdots+\sum_{i=1}^{3}\mu_i(x_{j-1}x_j)\}_{|S|=\min|V_i|_{i=1}^{t}}.$$

(5)

$$\mathcal{G}_n(SHG) = \min\{\Sigma_{i=1}^{3}(\sigma_i(x)+\sigma_i(y)+\sigma_i(z))\}_{x\in V_1,\ y\in V_2,\ z\in V_3}.$$

where $t\geq 3$.

$$\mathcal{G}_n(SHG) = \min\{\Sigma_{i=1}^{3}(\sigma_i(x)+\sigma_i(y)+\sigma_i(z)+\sigma_i(w))\}_{x,y\in V_1,\ z,w\in V_2}.$$

where $t \leq 2$. *And*

$$\mathcal{G}_n(SHG) = \infty$$

where $\mathcal{O}(SHG) \leq 2$.

3.1. *Setting of Neutrosophic 1-Zero-Forcing Number*

Definition 3.18. (1-Zero-Forcing Number).
Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a neutrosophic SuperHyperGraph. Then

(i) **1-zero-forcing number** $\mathcal{Z}(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is minimum cardinality of a set S of black vertices (whereas vertices in $V(G) \setminus S$ are colored white) such that $V(G)$ is turned black after finitely many applications of "the color-change rule": a white vertex is converted to a black vertex if it is the only white neighbor of a black vertex. The last condition is as follows. For one time, black can change any vertex from white to black.

(ii) **1-zero-forcing neutrosophic-number** $\mathcal{Z}_n(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is minimum neutrosophic cardinality of a set S of black vertices (whereas vertices in $V(G) \setminus S$ are colored white) such that $V(G)$ is turned black after finitely many applications of "the color-change rule": a white vertex is converted to a black vertex if it is the only white neighbor of a black vertex. The last condition is as follows. For one time, black can change any vertex from white to black.

Definition 3.19. (Failed 1-Zero-Forcing Number).
Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a neutrosophic SuperHyperGraph. Then

(i) **failed 1-zero-forcing number** $\mathcal{Z}'(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is maximum cardinality of a set S of black vertices (whereas vertices in $V(G) \setminus S$ are colored white) such that $V(G)$ isn't turned black after finitely many applications of "the color-change rule": a white vertex is converted to a black vertex if it is the only white neighbor of a black vertex. The last condition is as follows. For one time, Black can change any vertex from white to black. The last condition is as follows. For one time, black can change any vertex from white to black;

(ii) **failed 1-zero-forcing neutrosophic-number** $\mathcal{Z}'_n(SHG)$ for a neutrosophic SuperHyperGraph $SHG = (G \subseteq P(V), E \subseteq P(V))$ is maximum neutrosophic cardinality of a set S of black vertices (whereas vertices in $V(G) \setminus S$ are colored white) such that $V(G)$ isn't turned black after finitely many applications of "the color-change rule": a white vertex is converted to a black vertex if it is the only white neighbor of a black vertex. The last condition is as follows. For one time, Black can change any vertex from white to black. The last condition is as follows. For one time, black can change any vertex from white to black.

Proposition 3.20. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}(CMT_{\sigma}) = \mathcal{O}(CMT_{\sigma}) - 2.$$

Proposition 3.21. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a path-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}(PTH_n) = 1.$$

Proposition 3.22. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a cycle-neutrosophic SuperHyperGraph where $\mathcal{O}(CYC) \geq 3$. Then*

$$\mathcal{Z}(CYC_n) = 1.$$

Proposition 3.23. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a star-neutrosophic SuperHyperGraph with center c. Then*

$$\mathcal{Z}(STR_{1,\sigma_2}) = \mathcal{O}(STR_{1,\sigma_2}) - 3.$$

Proposition 3.24. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-bipartite-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}(CMT_{\sigma_1,\sigma_2}) = \mathcal{O}(CMT_{\sigma_1,\sigma_2}) - 3.$$

Proposition 3.25. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-t-partite-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = \mathcal{O}(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) - 2.$$

3.2. *Setting of 1-Zero-Forcing Neutrosophic-Number*

Proposition 3.26. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}_n(CMT_{\sigma}) = \mathcal{O}_n(CMT_{\sigma}) - \max\{\Sigma_{i=1}^{3}\sigma_i(x) + \Sigma_{i=1}^{3}\sigma_i(y)\}_{x,y \in V}.$$

Proposition 3.27. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a path-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}_n(PTH_n) = \min\{\Sigma_{i=1}^{3}\sigma_i(x)\}_{x \text{ is a vertex}}.$$

Proposition 3.28. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a cycle-neutrosophic SuperHyperGraph where $\mathcal{O}(CYC) \geq 3$. Then*

$$\mathcal{Z}_n(CYC_n) = \min\{\Sigma_{i=1}^{3}\sigma_i(x)\}_{x \text{ is a vertex}}.$$

Proposition 3.29. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a star-neutrosophic SuperHyperGraph with center c. Then*

$$\mathcal{Z}_n(STR_{1,\sigma_2}) = \mathcal{O}_n(STR_{1,\sigma_2}) - \max\{\Sigma_{i=1}^3 \sigma_i(c) + \Sigma_{i=1}^3 \sigma_i(x) + \Sigma_{i=1}^3 \sigma_i(y)\}_{x,y \in V}.$$

Proposition 3.30. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-bipartite-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}_n(CMT_{\sigma_1,\sigma_2}) = \mathcal{O}_n(CMT_{\sigma_1,\sigma_2}) - \max\{\Sigma_{i=1}^3 \sigma_i(x) + \Sigma_{i=1}^3 \sigma_i(x') + \Sigma_{i=1}^3 \sigma_i(x'')\}_{x,x',x'' \in V}.$$

Proposition 3.31. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-t-partite-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}_n(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = \mathcal{O}_n(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) - \max\{\Sigma_{i=1}^3 \sigma_i(x) + \Sigma_{i=1}^3 \sigma_i(x')\}_{x,x' \in V}.$$

3.3. *Setting of Neutrosophic Failed 1-Zero-Forcing Number*

Proposition 3.32. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}'(CMT_\sigma) = \mathcal{O}(CMT_\sigma) - 3.$$

Proposition 3.33. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a path-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}'(PTH_n) = 0.$$

Proposition 3.34. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a cycle-neutrosophic SuperHyperGraph where $\mathcal{O}(CYC) \geq 3$.*

$$\mathcal{Z}'(CYC_n) = 0.$$

Proposition 3.35. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a star-neutrosophic SuperHyperGraph with center c. Then*

$$\mathcal{Z}'(STR_{1,\sigma_2}) = \mathcal{O}(STR_{1,\sigma_2}) - 4.$$

Proposition 3.36. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-bipartite-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}'(CMT_{\sigma_1,\sigma_2}) = \mathcal{O}(CMT_{\sigma_1,\sigma_2}) - 4.$$

Proposition 3.37. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-t-partite-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}'(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = \mathcal{O}(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) - 3.$$

3.4. *Setting of Failed 1-Zero-Forcing Neutrosophic-Number*

Proposition 3.38. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}'_n(CMT_{\sigma}) = \mathcal{O}_n(CMT_{\sigma}) -$$

$$\min\{\Sigma_{i=1}^{3}\sigma_i(x) + \Sigma_{i=1}^{3}\sigma_i(y) + \Sigma_{i=1}^{3}\sigma_i(z)\}_{x,y,z\in V}.$$

Proposition 3.39. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a path-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}'_n(PTH_n) = 0.$$

Proposition 3.40. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a cycle-neutrosophic SuperHyperGraph where $\mathcal{O}(CYC) \geq 3$. Then*

$$\mathcal{Z}'_n(CYC_n) = 0$$

Proposition 3.41. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a star-neutrosophic SuperHyperGraph with center c. Then*

$$\mathcal{Z}'_n(STR_{1,\sigma_2}) = \mathcal{O}_n(STR_{1,\sigma_2}) -$$

$$\min\{\Sigma_{i=1}^{3}\sigma_i(c) + \Sigma_{i=1}^{3}\sigma_i(x) + \Sigma_{i=1}^{3}\sigma_i(y) + \Sigma_{i=1}^{3}\sigma_i(z)\}_{x,y,z\in V}.$$

Proposition 3.42. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-bipartite-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}'_n(CMT_{\sigma_1,\sigma_2}) = \mathcal{O}_n(CMT_{\sigma_1,\sigma_2}) -$$

$$\min\{\Sigma_{i=1}^{3}\sigma_i(x) + \Sigma_{i=1}^{3}\sigma_i(x') + \Sigma_{i=1}^{3}\sigma_i(x'') + \Sigma_{i=1}^{3}\sigma_i(x''')\}_{x,x',x'',x'''\in V}.$$

Proposition 3.43. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a complete-t-partite-neutrosophic SuperHyperGraph. Then*

$$\mathcal{Z}'_n(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) = \mathcal{O}_n(CMT_{\sigma_1,\sigma_2,\cdots,\sigma_t}) - \min\{\Sigma_{i=1}^{3}\sigma_i(x) + \Sigma_{i=1}^{3}\sigma_i(x') + \Sigma_{i=1}^{3}\sigma_i(x'')\}_{x,x'\in V}.$$

3.5. *Global Offensive Alliance*

Definition 3.44. Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a neutrosophic SuperHyperGraph. Then

(i) a set S is called **global-offensive alliance** if

$$\forall a \in V \setminus S,\ |N_s(a) \cap S| > |N_s(a) \cap (V \setminus S)|;$$

(ii) $\forall S' \subseteq S$, S is global offensive alliance but S' isn't global offensive alliance. Then S is called **minimal-global-offensive alliance**;

(iii) **minimal-global-offensive-alliance number** of SHG is

$$\bigwedge_{S \text{ is a minimal-global-offensive alliance.}} |S|$$

and it's denoted by Γ;

(iv) **minimal-global-offensive-alliance-neutrosophic number** of SHG is

$$\bigwedge_{S \text{ is a minimal-global-offensive alliance.}} \Sigma_{s\in S}\Sigma_{i=1}^{3}\sigma_i(s)$$

and it's denoted by Γ_s.

Proposition 3.45. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a strong neutrosophic SuperHyperGraph. If S is global-offensive alliance, then $\forall v \in V \setminus S,\ \exists x \in S$ such that*

(i) $v \in N_s(x)$;

(ii) $vx \in E$.

Definition 3.46. Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a strong neutrosophic SuperHyperGraph. Suppose S is a set of vertices. Then

(i) S is called **dominating set** if $\forall v \in V \setminus S,\ \exists s \in S$ such that either $v \in N_s(s)$ or $vs \in E$;

(ii) $|S|$ is called **chromatic number** if $\forall v \in V,\ \exists s \in S$ such that either $v \in N_s(s)$ or $vs \in E$ implies s and v have different colors.

Proposition 3.47. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a strong neutrosophic SuperHyperGraph. If S is global-offensive alliance, then*

(i) *S is dominating set;*

(ii) *there's $S \subseteq S'$ such that $|S'|$ is chromatic number.*

Proposition 3.48. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a strong neutrosophic SuperHyperGraph. Then*

(i) $\Gamma \leq \mathcal{O}$;

(ii) $\Gamma_s \leq \mathcal{O}_n$.

Proposition 3.49. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a strong neutrosophic SuperHyperGraph which is connected. Then*

(i) $\Gamma \leq \mathcal{O} - 1$;

(ii) $\Gamma_s \leq \mathcal{O}_n - \Sigma_{i=1}^{3}\sigma_i(x)$.

Proposition 3.50. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be an odd path. Then*

(i) *the set $S = \{v_2, v_4, \cdots, v_{n-1}\}$ is minimal-global-offensive alliance;*

(ii) $\Gamma = \lfloor \frac{n}{2} \rfloor + 1$ *and corresponded set is $S = \{v_2, v_4, \cdots, v_{n-1}\}$;*

(iii) $\Gamma_s = \min\{\Sigma_{s\in S=\{v_2,v_4,\cdots,v_{n-1}\}}\Sigma_{i=1}^{3}\sigma_i(s), \Sigma_{s\in S=\{v_1,v_3,\cdots,v_{n-1}\}}\Sigma_{i=1}^{3}\sigma_i(s)\}$;

(iv) *the sets* $S_1 = \{v_2, v_4, \cdots, v_{n-1}\}$ *and* $S_2 = \{v_1, v_3, \cdots, v_{n-1}\}$ *are only minimal-global-offensive alliances.*

Proposition 3.51. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an even path. Then*

(i) *the set* $S = \{v_2, v_4, \cdots .v_n\}$ *is minimal-global-offensive alliance;*

(ii) $\Gamma = \lfloor \frac{n}{2} \rfloor$ *and corresponded sets are* $\{v_2, v_4, \cdots .v_n\}$ *and* $\{v_1, v_3, \cdots .v_{n-1}\}$;

(iii) $\Gamma_s = \min\{\Sigma_{s \in S = \{v_2, v_4, \cdots, v_n\}} \Sigma_{i=1}^{3} \sigma_i(s), \Sigma_{s \in S = \{v_1, v_3, \cdots .v_{n-1}\}} \Sigma_{i=1}^{3} \sigma_i(s)\}$;

(iv) *the sets* $S_1 = \{v_2, v_4, \cdots .v_n\}$ *and* $S_2 = \{v_1, v_3, \cdots .v_{n-1}\}$ *are only minimal-global-offensive alliances.*

Proposition 3.52. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an even cycle. Then*

(i) *the set* $S = \{v_2, v_4, \cdots, v_n\}$ *is minimal-global-offensive alliance;*

(ii) $\Gamma = \lfloor \frac{n}{2} \rfloor$ *and corresponded sets are* $\{v_2, v_4, \cdots, v_n\}$ *and* $\{v_1, v_3, \cdots, v_{n-1}\}$;

(iii) $\Gamma_s = \min\{\Sigma_{s \in S = \{v_2, v_4, \cdots, v_n\}} \sigma(s), \Sigma_{s \in S = \{v_1, v_3, \cdots, v_{n-1}\}} \sigma(s)\}$;

(iv) *the sets* $S_1 = \{v_2, v_4, \cdots, v_n\}$ *and* $S_2 = \{v_1, v_3, \cdots, v_{n-1}\}$ *are only minimal-global-offensive alliances.*

Proposition 3.53. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an odd cycle. Then*

(i) *the set* $S = \{v_2, v_4, \cdots, v_{n-1}\}$ *is minimal-global-offensive alliance;*

(ii) $\Gamma = \lfloor \frac{n}{2} \rfloor + 1$ *and corresponded set is* $S = \{v_2, v_4, \cdots, v_{n-1}\}$;

(iii) $\Gamma_s = \min\{\Sigma_{s \in S = \{v_2, v_4, \cdots .v_{n-1}\}} \Sigma_{i=1}^{3} \sigma_i(s), \Sigma_{s \in S = \{v_1, v_3, \cdots .v_{n-1}\}} \Sigma_{i=1}^{3} \sigma_i(s)\}$;

(iv) *the sets* $S_1 = \{v_2, v_4, \cdots .v_{n-1}\}$ *and* $S_2 = \{v_1, v_3, \cdots .v_{n-1}\}$ *are only minimal-global-offensive alliances.*

Proposition 3.54. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be star. Then*

(i) *the set* $S = \{c\}$ *is minimal-global-offensive alliance;*

(ii) $\Gamma = 1$;

(iii) $\Gamma_s = \Sigma_{i=1}^{3} \sigma_i(c)$;

(iv) *the sets* $S = \{c\}$ *and* $S \subset S'$ *are only global-offensive alliances.*

Proposition 3.55. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be wheel. Then*

(i) *the set* $S = \{v_1, v_3\} \cup \{v_6, v_9 \cdots, v_{i+6}, \cdots, v_n\}_{i=1}^{6+3(i-1) \leq n}$ *is minimal-global-offensive alliance;*

(ii) $\Gamma = |\{v_1, v_3\} \cup \{v_6, v_9 \cdots, v_{i+6}, \cdots, v_n\}_{i=1}^{6+3(i-1) \leq n}|$;

(iii) $\Gamma_s = \Sigma_{\{v_1, v_3\} \cup \{v_6, v_9 \cdots, v_{i+6}, \cdots, v_n\}_{i=1}^{6+3(i-1) \leq n}} \Sigma_{i=1}^{3} \sigma_i(s)$;

(iv) *the set* $\{v_1, v_3\} \cup \{v_6, v_9 \cdots, v_{i+6}, \cdots, v_n\}_{i=1}^{6+3(i-1) \leq n}$ *is only minimal-global-offensive alliance.*

Proposition 3.56. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an odd complete. Then*

(i) *the set* $S = \{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1}$ *is minimal-global-offensive alliance;*

(ii) $\Gamma = \lfloor \frac{n}{2} \rfloor + 1$;

(iii) $\Gamma_s = \min\{\Sigma_{s \in S} \Sigma_{i=1}^{3} \sigma_i(s)\}_{S=\{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1}}$;

(iv) *the set* $S = \{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1}$ *is only minimal-global-offensive alliances.*

Proposition 3.57. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be an even complete. Then*

(i) *the set* $S = \{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor}$ *is minimal-global-offensive alliance;*

(ii) $\Gamma = \lfloor \frac{n}{2} \rfloor$;

(iii) $\Gamma_s = \min\{\Sigma_{s \in S} \Sigma_{i=1}^{3} \sigma_i(s)\}_{S=\{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor}}$;

(iv) *the set* $S = \{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor}$ *is only minimal-global-offensive alliances.*

Proposition 3.58. *Let* $\mathcal{G}$ *be a* m*-family of neutrosophic stars with common neutrosophic vertex set. Then*

(i) *the set* $S = \{c_1, c_2, \cdots, c_m\}$ *is minimal-global-offensive alliance for* $\mathcal{G}$;

(ii) $\Gamma = m$ *for* $\mathcal{G}$;

(iii) $\Gamma_s = \Sigma_{i=1}^{m} \Sigma_{j=1}^{3} \sigma_j(c_i)$ *for* $\mathcal{G}$;

(iv) *the sets* $S = \{c_1, c_2, \cdots, c_m\}$ *and* $S \subset S'$ *are only minimal-global-offensive alliances for* $\mathcal{G}$.

Proposition 3.59. *Let* $\mathcal{G}$ *be a* m*-family of odd complete graphs with common neutrosophic vertex set. Then*

(i) *the set* $S = \{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1}$ *is minimal-global-offensive alliance for* $\mathcal{G}$;

(ii) $\Gamma = \lfloor \frac{n}{2} \rfloor + 1$ *for* $\mathcal{G}$;

(iii) $\Gamma_s = \min\{\Sigma_{s \in S} \Sigma_{i=1}^{3} \sigma_i(s)\}_{S=\{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1}}$ *for* $\mathcal{G}$;

(iv) *the sets* $S = \{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1}$ *are only minimal-global-offensive alliances for* $\mathcal{G}$.

Proposition 3.60. *Let* $\mathcal{G}$ *be a* m*-family of even complete graphs with common neutrosophic vertex set. Then*

(i) *the set* $S = \{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor}$ *is minimal-global-offensive alliance for* $\mathcal{G}$;

(ii) $\Gamma = \lfloor \frac{n}{2} \rfloor$ *for* $\mathcal{G}$;

(iii) $\Gamma_s = \min\{\Sigma_{s \in S} \Sigma_{i=1}^{3} \sigma_i(s)\}_{S=\{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor}}$ *for* $\mathcal{G}$;

(iv) *the sets* $S = \{v_i\}_{i=1}^{\lfloor \frac{n}{2} \rfloor}$ *are only minimal-global-offensive alliances for* $\mathcal{G}$.

3.6. *Global Powerful Alliance*

Definition 3.61. Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a neutrosophic SuperHyperGraph. Then

(i) a set S of vertices is called **t-offensive alliance** if

$$\forall a \in V \setminus S,\ |N_s(a) \cap S| - |N_s(a) \cap (V \setminus S)| > t;$$

(ii) a t-offensive alliance is called **global-offensive alliance** if $t = 0$;

(iii) a set S of vertices is called **t-defensive alliance** if

$$\forall a \in S,\ |N_s(a) \cap S| - |N_s(a) \cap (V \setminus S)| < t;$$

(iv) a t-defensive alliance is called **global-defensive alliance** if $t = 0$;

(v) a set S of vertices is called **t-powerful alliance** if it's both t-offensive alliance and (t-2)-defensive alliance;

(vi) a t-powerful alliance is called **global-powerful alliance** if $t = 0$;

(vii) $\forall S' \subseteq S$, S is global-powerful alliance but S' isn't global-powerful alliance. Then S is called **minimal-global-powerful alliance**;

(viii) **minimal-global-powerful-alliance number** of SHG is

$$\bigwedge_{S \text{ is a minimal-global-powerful alliance.}} |S|$$

and it's denoted by Γ;

(ix) **minimal-global-powerful-alliance-neutrosophic number** of SHG is

$$\bigwedge_{S \text{ is a minimal-global-offensive alliance.}} \Sigma_{s \in S} \Sigma_{i=1}^{3} \sigma_i(s)$$

and it's denoted by Γ_s.

Proposition 3.62. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a strong neutrosophic SuperHyperGraph. Then following statements hold;*

(i) *if $s \geq t$ and a set S of vertices is t-defensive alliance, then S is s-defensive alliance;*

(ii) *if $s \leq t$ and a set S of vertices is t-offensive alliance, then S is s-offensive alliance.*

Proposition 3.63. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a strong neutrosophic SuperHyperGraph. Then following statements hold;*

(i) *if $s \geq t + 2$ and a set S of vertices is t-defensive alliance, then S is s-powerful alliance;*

(ii) *if $s \leq t$ and a set S of vertices is t-offensive alliance, then S is t-powerful alliance.*

Proposition 3.64. *Let $SHG = (G \subseteq P(V), E \subseteq P(V))$ be a r-regular-strong-neutrosophic SuperHyperGraph. Then following statements hold;*

(i) *if $\forall a \in S,\ |N_s(a) \cap S| < \lfloor \frac{r}{2} \rfloor + 1$, then $SHG = (G \subseteq P(V), E \subseteq P(V))$ is 2-defensive alliance;*

(ii) *if $\forall a \in V \setminus S,\ |N_s(a) \cap S| > \lfloor \frac{r}{2} \rfloor + 1$, then $SHG = (G \subseteq P(V), E \subseteq P(V))$ is 2-offensive alliance;*

(iii) *if* $\forall a \in S,\ |N_s(a) \cap V \setminus S| = 0,$ *then* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is r-defensive alliance;*

(iv) *if* $\forall a \in V \setminus S,\ |N_s(a) \cap V \setminus S| = 0,$ *then* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is r-offensive alliance.*

Proposition 3.65. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a r-regular-strong-neutrosophic SuperHyperGraph. Then following statements hold;*

(i) $\forall a \in S,\ |N_s(a) \cap S| < \lfloor \frac{r}{2} \rfloor + 1$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-defensive alliance;*

(ii) $\forall a \in V \setminus S,\ |N_s(a) \cap S| > \lfloor \frac{r}{2} \rfloor + 1$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-offensive alliance;*

(iii) $\forall a \in S,\ |N_s(a) \cap V \setminus S| = 0$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is r-defensive alliance;*

(iv) $\forall a \in V \setminus S,\ |N_s(a) \cap V \setminus S| = 0$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is r-offensive alliance.*

Proposition 3.66. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a r-regular-strong-neutrosophic SuperHyperGraph which is complete. Then following statements hold;*

(i) $\forall a \in S,\ |N_s(a) \cap S| < \lfloor \frac{\mathcal{O}-1}{2} \rfloor + 1$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-defensive alliance;*

(ii) $\forall a \in V \setminus S,\ |N_s(a) \cap S| > \lfloor \frac{\mathcal{O}-1}{2} \rfloor + 1$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-offensive alliance;*

(iii) $\forall a \in S,\ |N_s(a) \cap V \setminus S| = 0$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is* $(\mathcal{O} - 1)$*-defensive alliance;*

(iv) $\forall a \in V \setminus S,\ |N_s(a) \cap V \setminus S| = 0$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is* $(\mathcal{O} - 1)$*-offensive alliance.*

Proposition 3.67. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a r-regular-strong-neutrosophic SuperHyperGraph which is complete. Then following statements hold;*

(i) *if* $\forall a \in S,\ |N_s(a) \cap S| < \lfloor \frac{\mathcal{O}-1}{2} \rfloor + 1,$ *then* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-defensive alliance;*

(ii) *if* $\forall a \in V \setminus S,\ |N_s(a) \cap S| > \lfloor \frac{\mathcal{O}-1}{2} \rfloor + 1,$ *then* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-offensive alliance;*

(iii) *if* $\forall a \in S,\ |N_s(a) \cap V \setminus S| = 0,$ *then* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is* $(\mathcal{O} - 1)$*-defensive alliance;*

(iv) *if* $\forall a \in V \setminus S,\ |N_s(a) \cap V \setminus S| = 0,$ *then* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is* $(\mathcal{O} - 1)$*-offensive alliance.*

Proposition 3.68. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a r-regular-strong-neutrosophic SuperHyperGraph which is cycle. Then following statements hold;*

(i) $\forall a \in S,\ |N_s(a) \cap S| < 2$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-defensive alliance;*

(ii) $\forall a \in V \setminus S,\ |N_s(a) \cap S| > 2$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-offensive alliance;*

(iii) $\forall a \in S,\ |N_s(a) \cap V \setminus S| = 0$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-defensive alliance;*

(iv) $\forall a \in V \setminus S,\ |N_s(a) \cap V \setminus S| = 0$ *if* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-offensive alliance.*

Proposition 3.69. *Let* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *be a r-regular-strong-neutrosophic SuperHyperGraph which is cycle. Then following statements hold;*

(i) *if* $\forall a \in S,\ |N_s(a) \cap S| < 2$, *then* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-defensive alliance;*

(ii) *if* $\forall a \in V \setminus S,\ |N_s(a) \cap S| > 2$, *then* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-offensive alliance;*

(iii) *if* $\forall a \in S,\ |N_s(a) \cap V \setminus S| = 0$, *then* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-defensive alliance;*

(iv) *if* $\forall a \in V \setminus S,\ |N_s(a) \cap V \setminus S| = 0$, *then* $SHG = (G \subseteq P(V), E \subseteq P(V))$ *is 2-offensive alliance.*

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An assessed framework for manufacturing sustainability based on Industry 4.0 under uncertainty

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Abstract: Globalization and the rapid growth of technologies are the main challenges facing the manufacturer and its sustainability and survival. Sustainability for any manufacturing plays an important role in competitive advantage which make the manufacturing firm a sustainable competitor. Sustainability in manufacturing is integrated with Industry 4.0 (I4.0) to achieve benefits of economic, environmental, and social. But it has many criteria and factors and contains incomplete and uncertain information. So, we used the neutrosophic sets to overcome this incomplete information and treat with uncertainty environment. The Single-Valued Neutrosophic Set (SVNS) is used to evaluate these criteria, which include three values (Truth, indeterminacy, and falsity). The SVNS is integrated with Multi-Criteria Decision Making (MCDM) methods. The MCDM concept is used in this paper to deal with many conflicting criteria. A Decision-making trial and evaluation laboratory (DEMATEL) is utilized for determining the relation between five main criteria and fourteen sub-criteria in this study. Analytic Hierarchy Process (AHP) is used to compute the weights of the main and sub-criteria. Our framework is applied to a real case study in Egypt to show the validity of our framework.

Keywords: Sustainability; Industry 4.0; AHP; Single-Valued Neutrosophic Sets; SVNSs; Multi-Criteria Decision Making; MCDM; DEMATEL.

1. Introduction

In the previous centuries, the industrial revolutions continued until advent of the fourth industrial revolution, known as I 4.0. This revolution includes the use of many technologies that help automate and digitalize operations. The manufacturing industry has undergone many radical changes [1].

This new digital industrial transformation has had a positive impact on manufacturing organizations. This made manufacturing more intelligent which led to businesses changing their way of working. I4.0 is an umbrella for various technologies such as big data analytics (BDA), Internet of Things (IoT) and cloud computing, Cyber-Physical systems (CPS), information and communications technology (ICT), Enterprise Architecture (EA), Enterprise Integration (EI) and Blockchain (BC) [2].

The benefits of utilization of I4.0 technologies in manufacturing are (i) it helped in the emergence of so-called smart manufacturing. Smart manufacturing is expressed in [3] as "manufacturing machines are characterized with interconnection through wireless networks according to modern manufacturing paradigm, monitored by sensors, and controlled by advanced computational intelligence to enhance the quality of product, increase productivity, and sustainability with reducing costs." (ii) manufacturing system becomes an integrated and cooperative production system that responds to any changing requirements and conditions in real-time [4]. (iii) high level of digitization through exchanging data, communication among parts, products, machines, and human-machine interaction (HMI). (iv) Optimization through energy and resource consumption. (v) Global competitiveness through productivity and operational efficiency. (vi) Beneficial decisions through tracking products effectively and analyzing the market on an ongoing basis. (vii) The cost is reduced, and profits are increasing by processing effective information are improving the production planning decisions [5, 6, 7]. (viii) Improvement of product development by transforming the traditional production and operations management techniques [6].

Consequently, manufacturing firms are becoming sustainable by applying I4.0 technologies. Despite it being a complicated process, not simple. From the TBL perspective [8] one of the sustainability requirements for the firm is achieving a balance between the economic, environmental, and social pillars. Sustainability of manufacturing according to TBL represents: Environmentally, products are environment-friendly through using resources efficiently. Socially, the production process is based on ethics and sustainability. Economically, manufacturing processes are highly efficient in saving energy, natural resources utilization and achieving a better global market reputation [9].

The sustainability of manufacturing based on I4.0 has many various conflict criteria, so the Multi-Criteria Decision Making (MCDM) is used to overcome this problem. Numerous MCDM techniques offer a huge variety of approaches for solving complex decision-making problems such as TOPSIS, DEMATEL, Analytic Hierarchy Process (AHP)…etc. MCDM is used in assessments containing numerous criteria to support decision-makers (DMs) and experts to make decisions based on their preferences by breaking the problems into smaller portions [13]. These techniques have been increasingly used in manufacturing practices [14]. According to [15] MCDM deal with many types of problems that contain huge and conflict criteria.

Researchers in [16] have introduced techniques to strengthen MCDM through utilizing Fuzzy Set (FS) where its function is to assign a degree of membership ranging between [0-1] for each element. In [17] an improvement of FS, called Intuitionistic Fuzzy Sets (IFS) is introduced. It considers the membership degree, non-membership degree, and hesitation degree. But the FS can't deal efficiently with the incomplete data due to lack of the indeterminacy value concept.

Neutrosophic theory embraces the idea of FS and IFS more comprehensively. It assigns a degree of membership, indeterminacy, and non-membership function for each element [18]. Furthermore, [19,20] proposed many benefits of neutrosophic theory such as: (i) Neutrosophy helps experts to present their opinions about uncertain preferences by using the degree of indeterminacy to present obscure information. (ii) It deals with different conditions of decision-making through applying truthiness, indeterminacy, and falsity. (iii) It expresses odds between DMs and experts. (iv) It can handle uncertainty and various environments.

All of these are strong motivations for consolidating neutrosophic theory with MCDM techniques to rank and select the best solution (alternative) among possible solutions (alternatives) based on calculation weights of criteria through an expert panel [15]. For the maximum benefit, the criteria with the maximum weight is selected.

The focus of modern organizations is not limited to profitability, but it spans to eco-friendly items production, time utilization of challenging tasks, and increased productivity. In short, modern organizations seek sustainability [21].

The research on sustainability of manufacturing based I4.0 is in its early stages of growth [22]. In Section 2 of this work, more details are given via the Web of Science (WoS) database.

In this study, we will adopt the idea of the influence of I4.0 on manufacturing firms to be environmentally, socially, and economically sustainable. This study aims to fulfill the following objectives:

1. Attempting to answer the question (using literature analysis): Can the adoption of I4.0 technologies have a positive impact on promoting sustainability in manufacturing?

2. Identifying I4.0 enablers or criteria and sub-criteria that affect the achievement of manufacturing sustainability using literature.
3. Assessing the impact of determined I4.0 main and sub-criteria on each other to achieve sustainable manufacturing through a questionnaire offered to a committee of decision makers (DM) and experts.
4. Determine degree of influence among main and sub-criteria using the hybrid framework of MCDM with neutrosophic theory (N-DEMATEL).
5. Applying AHP-based neutrosophic for recommending the most positive influential criteria on three pillars of Triple Bottom Line (TBL).
6. Applying the proposed framework on a case study of real manufacturing firms.

This paper is organized as follows: section 2 presents systematic analysis of related articles and the research methodology used in this study, section 3 presents the literature review of I4.0 and sustainability of manufacturing related I4.0 illustrating basic concepts and technologies. Section 4 clarifies the proposed developed framework for criteria interrelations. In section 5, the hybrid framework validation is assessed through real case study. Finally, conclusions are highlighted in section 6.

2. Systematic Analysis and Research Methodology

In this section, systematic analysis is performed on the available published documents on the study topic. The analysis process facilitates knowing current trends of research in the literature related to a specific field [23,24]. Therefore, research papers and articles on "sustainable manufacturing" and "sustainable manufacturing based I4.0" are analyzed. The source of articles is Web of Science (WoS) database from 2015 until 2020. WoS database contains numerous famous publications and articles in different domains. Figure 1 illustrates the steps to be followed in the methodology.

The proposed research methodology consists of four steps as shown in Figure 1 and summarized below:

Step1: Search WoS database: The database is searched using two key concepts; "sustainable manufacturing" and "sustainable manufacturing based I4.0".

Step2: Trend Analysis: Based on the research results, the study focuses on number of publications in the field per year, type of the publication and area of research. These data are summarized and interpreted allowing for further insights. Table 1 shows the summarized search results.

Step3: Trend Analysis Results (potentials): the trend results are categorized into two parts. First part is for extracting the gaps and limitations in the research area. This is followed by highlighting the potential motivations for contributions in the manufacturing sustainability using I4.0 as part two.

Step4: Influence Evaluation Model: a model is developed for assessing the influence of criteria from I4.0 on the manufacturing sustainability.

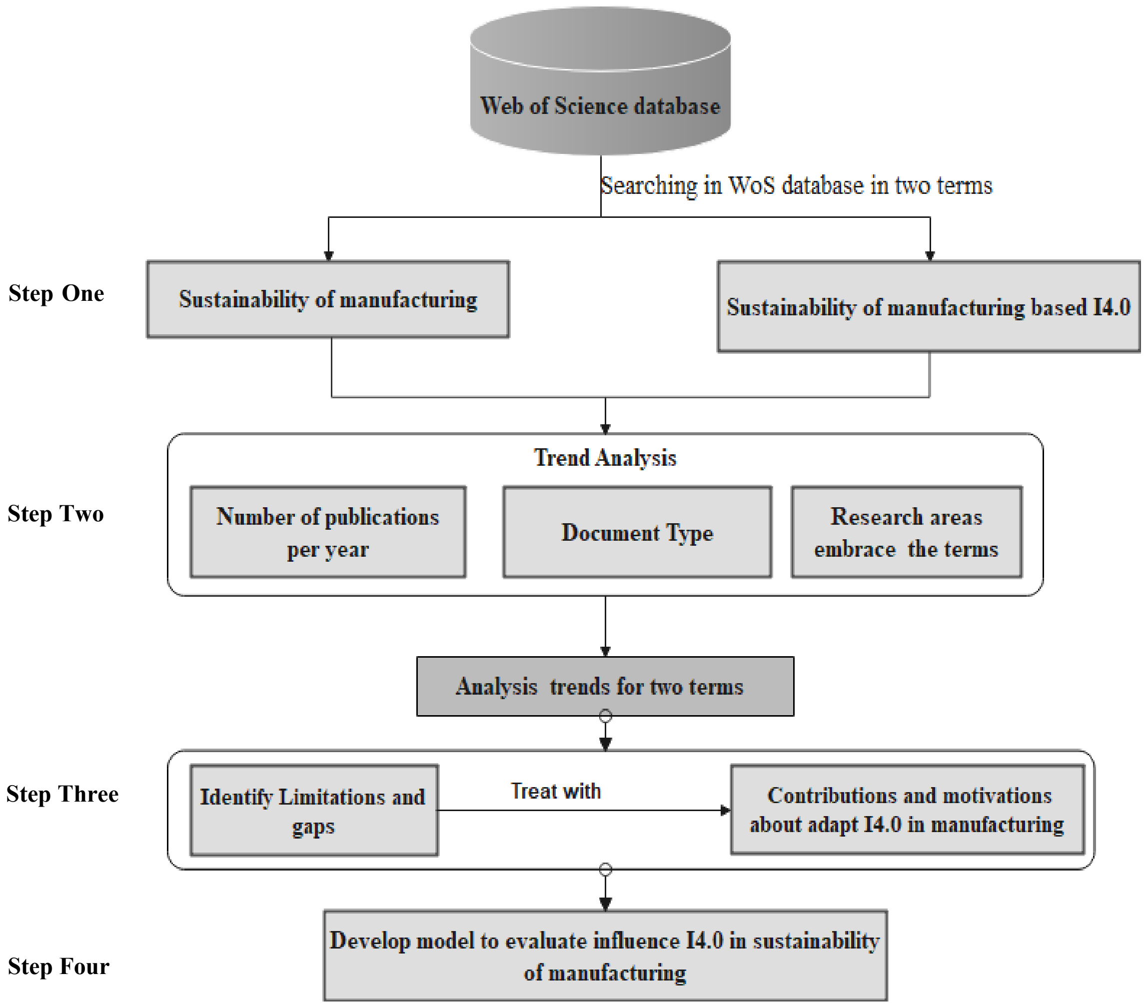

Fig 1. Steps of research methodology.

Table 1. Summary of previous work in sustainability manufacturing and I4.0.

			Sustainability of manufacturing	**Sustainability of manufacturing based I4.0**
No. of Publications		5446	231
Category of publications	Article:	3746	120
	Proceeding papers:	1260	71
	Review:	453	37
	Early Access:	149	11
	Book chapter:	135	0
	Editorial chapter:	35	4
	Books	2	0
Areas and fields	ENGINEERING:	2741	124
	Business Economics:	1562	58
	Computer Science:	301	25
	Telecommunications:	53	5
	Chemistry:	229	4

3. Literature Concepts

3.1 Industry 4.0

In 2011, I4.0 was presented at the Hannover Fair [24]. Later, in 2013, the German government introduced I4.0 [25]. The term "I4.0" is associated with other terms such as smart manufacturing, smart production, or smart factories, due to the use of numerous technologies [26]. For [27], I4.0 includes the connection between physical and digital technologies such as CPS, cloud computing, big data…etc to share information and make intelligent decisions to gain the organization a competitive advantage in the market through fulfilling the needs of clients.

Technologies of I4.0 in [28] are classified into two categories front-end technologies and base technologies as shown in Fig. 2. Other researchers support a different view of base technologies as [29] supposes CPS, IoT, cloud, fog computing, and BDA are yield to base technologies. Reseach in [30] assumes CPS, IoT, ICT, EA, and enterprise integration are base technologies. Moreover, technologies of I4.0 as IoT, CPS, and artificial intelligence (AI) in [33] is a futuristic construct that boosts the development of production systems. That is due, as mentioned in [34] to the capacity of its technologies to enhance the energy, equipment, and use of the human resource. Thus, Organizations are becoming more sustainable and competitive globally.

The goal of I4.0 is to connect intelligent products, manufacturing processes, and machines by developing a network between them [31]. Conforming to that, [32] proposes that organizations are improving their capabilities for data processing through I4.0 which permits each part to interact with each other. Achieving organizational sustainability requires a balance between three pillars of Trible Bottom Line (TBL) economic, environmental, and social perspectives as [35] reported sustainability for industries in Brazil-based three pillars.

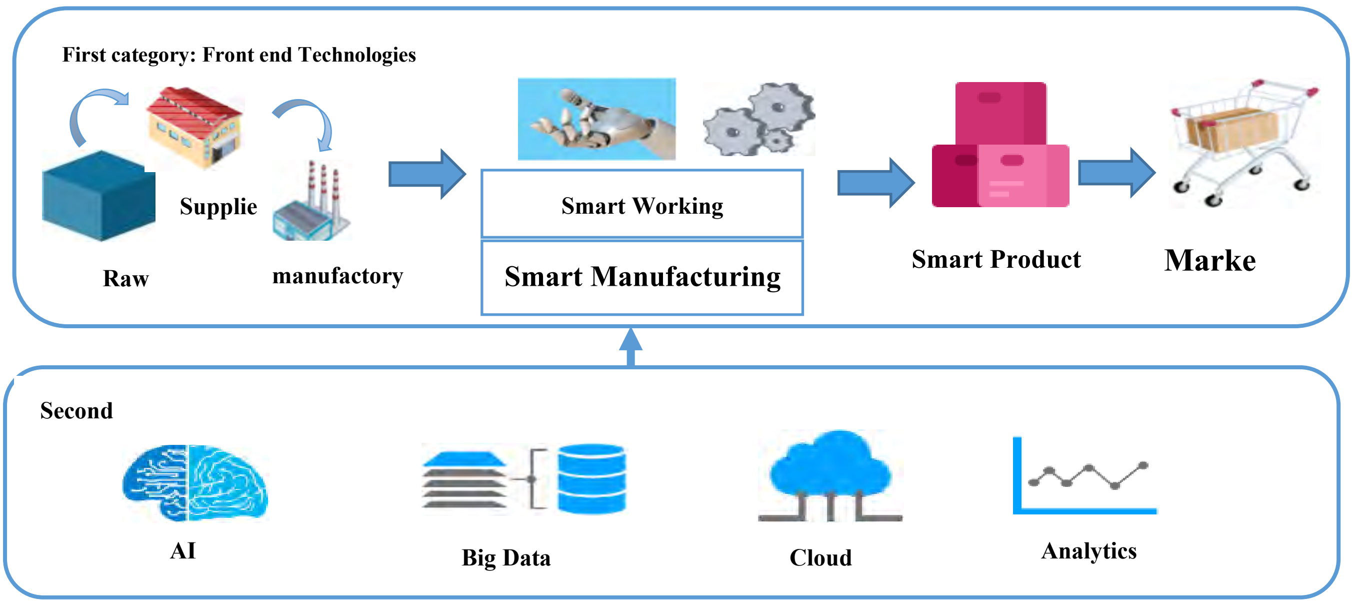

Fig. 2. Classification of I4.0 Technologies adapted from [28]

3.2 Sustainability of Manufacturing Based Industry 4.0

Sustainable Manufacturing is defined in [36] as processes and systems that are merged to use resources such as energy and raw materials wisely for producing a product of high quality, customer satisfaction, and regulatory compliance. Although manufacturing organizations strive to balance three pillars to achieve sustainability, there may be challenges that are threatening their sustainability. The plastics industry in [5] suffers from challenges of three pillars. Addressing such industrial challenges through [37,38] by adopting I4.0 technologies that utilize energy efficiently and effectively and tracking the life cycle of the product from design to delivery. In [39] there are many countries are adopting I4.0 technologies in their manufacturing sector like Australia, China, and Thailand for instance . General Electric Company (GE) is adapted the Predix platform which helps in connectivity, analytics, and machine learning, processing, and analysis big data for adding multiple benefits to its users[40].

CPS [41] is used in many sectors such as automotive, medical, and manufacturing aerospace with a special focus in the United States and the European Research Council. This is due to its ability to acquire and collect data through the sensor and to deal with a large volume of data. This technology is named 5C as for its five levels: Smart Connection, Data-to-Information Conversion, Cyber, Cognition, and Configuration. It consolidates information and machines to enhance the performance of the industry and the decision becomes decentralized [42]. Optimization of production through dynamic models is used in CPS to manage and organize the activities through manufacturing procedures [43]. Its ability to collect and analyze data according to [44] makes it able to increase productivity with higher quality and low cost, promote growth, and increase the efficiency of workers.

IoT supports the manufacturing process and offers advanced methods such as monitoring, managing, and optimizing the operation of manufacturing. International Telecommunication Union (ITU) defined IoT as the ability to connect anytime, anyplace to anyone [45,46]. Also, plays an important role in the observation of energy consumption to save energy thus the energy crisis is reduced [47].

Big Data Analytics are used to obtain information and make an accurate decisions based on analyzing the collected data obtained via IoT technology [9]. The utilization of big data Positively affected the quality of production and monitoring of the damage and work of each machine to facilitate the maintenance of machines and equipment [48].

The manufacturing process can be environmentally friendly by integrating Additive manufacturing to reduce scrap production and facilitates complex designs so, the product becomes flexible and consistent [49]. Applying these new technologies aims to increase efficiency and improve the performance of the entire industrial chain. I4.0technologies have a socially robust impact from the perspective of [44] in transforming operating patterns, design, product services, and production systems to smarter patterns and dispensing with human beings. [50] believes that technologies have a positive impact on the environment through energy consumption is more efficient and safer. Based on [51] I4.0 technologies are adapting to achieve circular economies. The conclusion from the foregoing is that the I4.0 technologies are promoting sustainable development by positively affecting

TBL. Many quantitative and qualitative studies are aimed to analyze and evaluate the impact of the I4.0 on the sustainability of each pillar of TBL's pillars. Robust Best Worst Method (RBWM) is one of the MCDM techniques used to assess the degree of influence of enablers in [10] for I4.0 technologies on the sustainability of manufacturing. Developed frameworks are used Fuzzy Evaluation Method (FEM) for identifying the importance of enablers of I 4.0 as in [52].

Factors affecting sustainability are classified and categorized in [53] into cause and effect. It used DEMATEL as requirements of government (F1), Social responsibility (F2), Green image (F3), and other factors. Grey-based DEMATEL is used in [54] to evaluate the influential strength of drivers for I4.0 to achieve sustainability in Supply Chains (SC). AHP is the most famous technique of MCDM which is used to analyze the drivers in [55] for advanced sustainable manufacturing. A hybrid MCDM techniques-based fuzzy decision-making trial and evaluation laboratory and analytic network process (FDANP with PROMETHEE) in [56] to analyze sustainable risks in the manufacturing of surgical cotton for helping manufacturing organizations avoid unwanted accidents, as well as through early knowledge for sustainable risks.

In this section, the following lierature concepts are introduced; Industry 4.0, sustainability of manufatcuring based I4.0, and related technologies. The proposed framwork is introduced in the following section.

4. Mathematical Model

As mentioned in introduction section, we are identifying I4.0 criteria and suncriteria that achieve sustainability of manufacturing. Assessment process for I4.0's criteria/subcriteria is vital process.

4.1 DMs prespectives based MCDM with neutrosophoic uncertainity method

In this section, we integrated the SVNSs with the MCDM methods to evaluate the criteria I4.0 with sustainability manufacturing. Firstly, the DEMATEL method is applied to show the interrelationships among criteria. The SVNSs are used to scale as [57]. Secondly, the SVNSs AHP is used to compute the weights of the criteria. Fig 3 shows the proposed framework of this paper.

4.2 Determine influencing main/sub criteria Based on N-DEMATEL

Step 1: Select decision-makers and experts who have expertise in this field. The main and sub-criteria of sustainability manufacture based on I4,0 technologies are collected. Then decision-makers offered to evaluate the criteria based on the Single-Valued Neutrosophic Numbers (SVNNs) as in [57].

Step 2: Constructed Pairwise comparison matrices based on relation between criteria by DMs panel.

Step 3: Transformation of pairwise comparison matrices for criteria to deneutrosophic form via Eq. (1).

$$s(a_{ij}) = \frac{(2+T-I-F)}{3} \tag{1}$$

Where T, I, F represent truth, indeterminacy, and falsity, $\mathbf{a_i}$ refers to the value in the comparison matrix and i refers to the number of criteria.

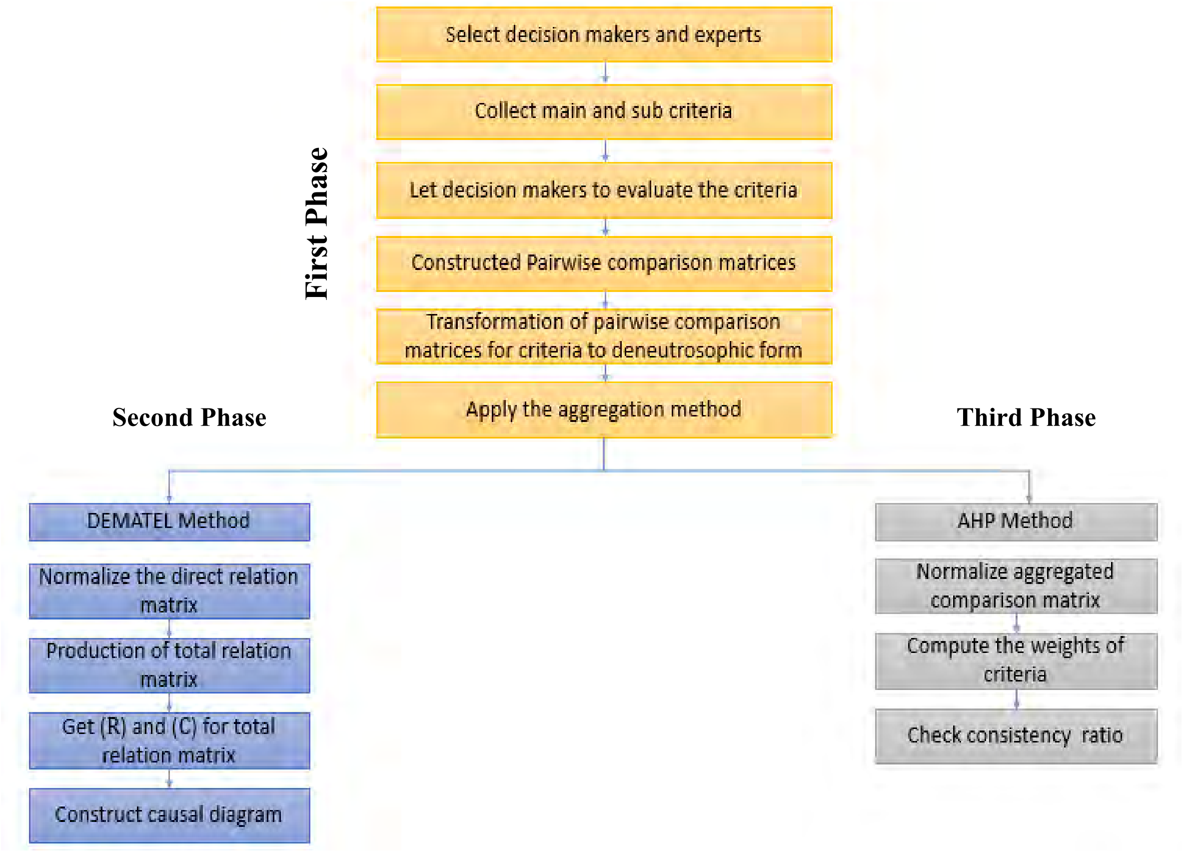

Fig. 3. The proposed Framework

Step 4: Apply the aggregation method to aggregate the opinions of experts into one matrix to obtain the direct relation matrix.

Step 5: Normalize the direct relation matrix as Eqs. (2, 3)

$$S = K * Y \tag{2}$$

where $\boldsymbol{Y}$ refers to the direct relation matrix as in the previous step.

$$K = \frac{1}{max_{1\le i\le n}(\sum_{j=1}^{n} a_{ij})} \quad (i, j = 1,2, \dots, n) \tag{3}$$

Where a_{ij} represent the sum of each raw (i) in matrix Y, $max_{1\le i\le n}(\sum_{j=1}^{n} x_{ij})$ represent the maximum value of a_{ij} and n refers to the number of criteria. a_{ij} refers to the value in the direct relation matrix.

Step 6: Production of total relation matrix

We use the MATLAB software to obtain the total relation matrix as Eq. (4)

$$T = S(I - S)^{-1} \tag{4}$$

Where I refers to the identity matrix.

Step 7: Get (R) and (C) for total relation matrix T.

The Sum of rows (R) and columns (C) are obtained as in Eqs. (5,6).

$\mathrm{T} = [\mathrm{a_{ij}}]_{\mathrm{n\times n}}, \mathrm{i, j} = 1,2,3, \dots \mathrm{n}$

$$\mathrm{R} = \left[\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{a_{ij}}\right]_{1\times n} = [\mathrm{a}_j]_{\mathrm{n}\times 1} \tag{5}$$

$$\mathrm{C} = \left[\sum_{=1}^{\mathrm{n}} \mathrm{a_{ij}}\right]_{1\times n} = [\mathrm{a}_j]_{\mathrm{n}\times 1} \tag{6}$$

Step 8: Construct a causal and effect diagram by the horizontal axis R+C and vertical axis R-C.the values of R-C determine cause and effect criteria/subcriteria. criteria/sub criteria are cause when its values of R-C are positive.

4.3 Neutrosophic AHP Method

Step 1: Repeat steps from 1 to 4 mentioned in section 4.1 to obtain the aggregated pairwise comparison matrix.

Step 2: Normalize aggregated/Average comparison matrix as Eq. (7).

$$\mathrm{Norm_{ij}} = \frac{\mathrm{a_j}}{\sum_{\mathrm{j}=1}^{\mathrm{n}}(\mathrm{a_j})}, \mathrm{j} = 1,2, \dots \dots . \mathrm{n} \tag{7}$$

Where $\sum_{j=1}^{n}(a_j)$ the sum of criteria per column in the aggregate matrix, a_j point to the preference of criterion in aggregated comparison matrix.

Step 3: Compute the weights of criteria by the row average of the previous step.

Step4: Check the consistency ratio (CR) as [58].

$$\mathrm{CR} = \frac{\mathrm{CI}}{\mathrm{RI}} \tag{8}$$

Where, $CI = \frac{\lambda_{max} - n}{n-1}$ (9)

Where n point to number of criteria/sub criteria in this study, RI is consistency ratio where its value determines based on number of criteria/sub criteria are used in the model.

5. **Case Study and Results**

We apply our methodology in a manufacturing enterprise in Egypt. This enterprise is responsible for producing household electrical appliances such as irons, food blenders, ceiling fans, vacuum cleaners, etc. The criteria of sustainable manufacutring based on I4.0 are introduced to the enterprise to increase the performance and achieve sustainability..

5.1 Results of Neutrosophic DEMATEL

Step 1: Table 2. represents demographic information about the experts who evaluated the criteria in this study. We collected five main criteria and fourteen sub-criteria as in Table 3.
Step 2: Four comparison matrices are obtained.
Step 3: Transform these matrices into crisp values-based Eq. (1).
Step 4: Obtain the direct relation matrix by the aggregation method.
Step 5: Obtain the normalized relation matrix based on Eq. (2,3) as Table 4.

Step 6: Obtain the total relation matrix as in Table 5.
Step 7: Obtain the values of R-C and R+C
Step 8: Obtain the causal diagram for the main and sub-criteria. Fig 4. shows the causal diagram. From Fig 4. C_5 is the best criteria and C_1 is the worst criteria.

Table 2. Demographic information about the expert panel

Demographic Information	Gender	Age	Qualifications	Job Title
First member	Male	40	Ph.D.	Executive Manager
Second member	female	35	Bachelor	Financial Consultant
Third member	Male	45	Master	Maintenance Engineer
Fourth member	Male	40	Bachelor	Quality and Safety Manager

Table 3. The main and sub-criteria

Main Criteria	Sub-Criteria
DBA(C1)	Exploration of new customers and opportunities (C_{1-1}). Technologies Upgradation for analyzing(C_{1-2})
Additive Manufacturing(C2)	Green design and environmentally friendly process (C_{2-1}). Ease testing and prototyping (C_{2-2}) Health and safety (C_{2-3}) Reduction cost of operations (C_{2-4})
IoT(C3)	Real time control (C_{3-1}) Efficiency monitoring and traceability(C_{3-2})
Flexible Manufacturing (C4)	Reduction lead time (C_{4-1}) Increase productivity and quality(C_{4-2}) Energy efficient consumption (C_{4-3}) Enhance ethical and sustainable process (C_{4-4})
CPS(C5)	Interactions between human and machine are friendly (C_{5-1}) Automation DM instead human (C_{5-2})

Table 4. Normalized relation matrix

Criteria	C_1	C_2	C_3	C_4	C_5
C_1	0.051368	0.067292	0.067806	0.073456	0.067806
C_2	0.238221	0.051368	0.076538	0.049313	0.0488
C_3	0.170256	0.225449	0.051368	0.084244	0.043663
C_4	0.147062	0.462174	0.125288	0.051368	0.078593
C_5	0.190045	0.305762	0.31727	0.135555	0.051368

Table 5. Total relation matrix

Criteria	C_1	C_2	C_3	C_4	C_5
C_1	0.18232	0.223463	0.15843	0.13358	0.114364
C_2	0.378852	0.218592	0.177966	0.123952	0.108227
C_3	0.368491	0.427165	0.176694	0.171914	0.116716
C_4	0.461954	0.741624	0.310335	0.182239	0.183401
C_5	0.548225	0.686383	0.526992	0.293146	0.177187

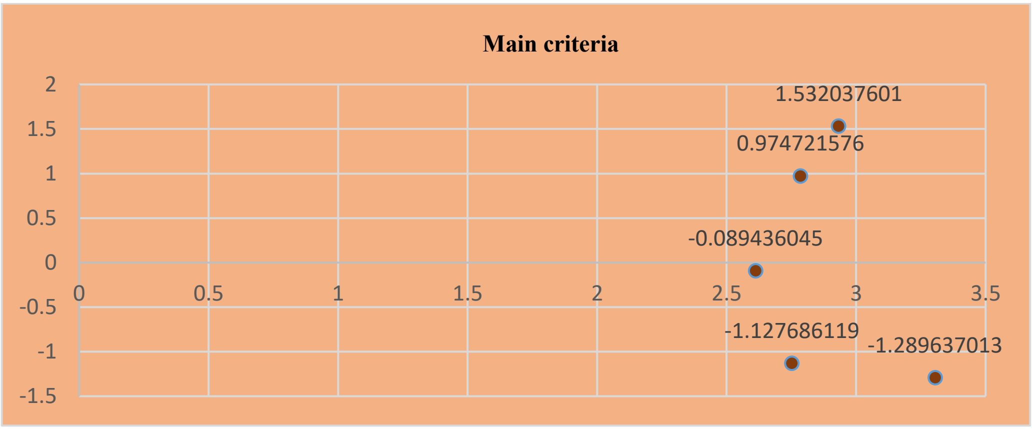

Fig. 4. Causal and effect for main criteria

For sub-criteria, we applied the Neutrosophic DEMATEL method in five sub-criteria. From Fig 5,6,7,8 and 9, we found that $C_{1\text{-}1}$ has the highest impact and $C_{1\text{-}2}$ has the lowest impact. $C_{2\text{-}4}$ has the highest impact and $C_{2\text{-}1}$ has the lowest impact. $C_{3\text{-}2}$ has the highest impact and $C_{3\text{-}1}$ has the lowest impact. $C_{4\text{-}4}$ has the highest impact and $C_{4\text{-}1}$ has the lowest impact. C_{52} has the highest impact and $C_{5\text{-}1}$ has the lowest impact.

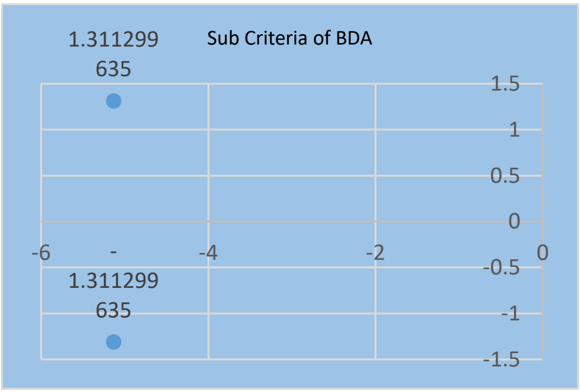

Fig. 5. Causal and effect for BDA sub- criteria

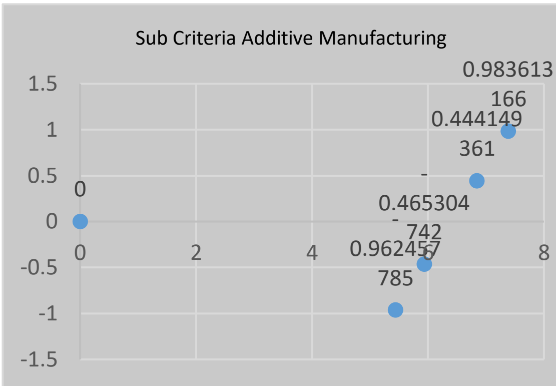

Fig. 6. Causal and effect for Additive Manufacturing sub- criteria

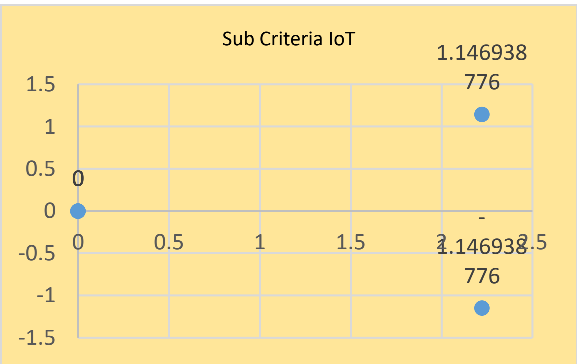

Fig. 7. Causal and effect for IoT sub- criteria

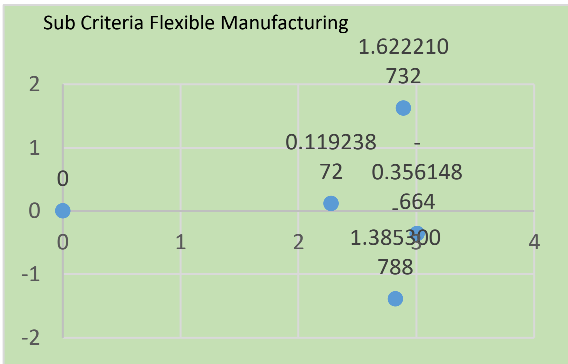

Fig. 8. Causal and effect for Flexible Manufacturing sub- criteria

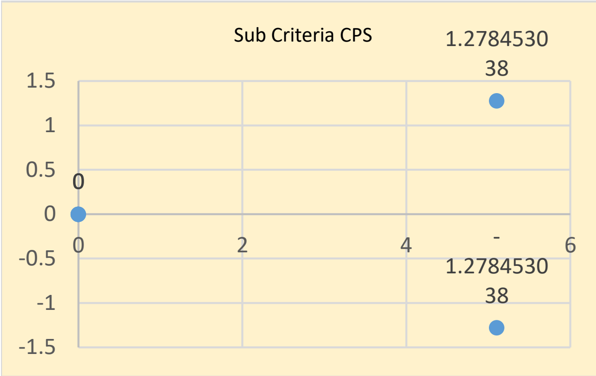

Fig. 9. Causal and effect for CPS sub- criteria

5.2 Results of Neutrosophic AHP Method

Start with the aggregated comparison matrix, then normalized it using Eq. (7) in Table 6. After that, from Table 6. we compute the weights of criteria by the row average in the normalized comparison matrix. The weights of the main criteria are obtained as $W_1 = 0.13026, W_2 = 0.151669, W_3 = 0.172228, W_4 = 0.239525, W_5 = 0.306318$. This means that C_5 has the highest weight and C_2 has the lowest weight. Then we compute the weights of sub-criteria and compute the global weights by multiplying the weights of main criteria by the weights of local criteria. Fig 10. shows the weights of global criteria. From Fig. 10. we deduce that $C_{5\text{-}2}$ has the highest weight and $C_{2\text{-}1}$ has the lowest weight.

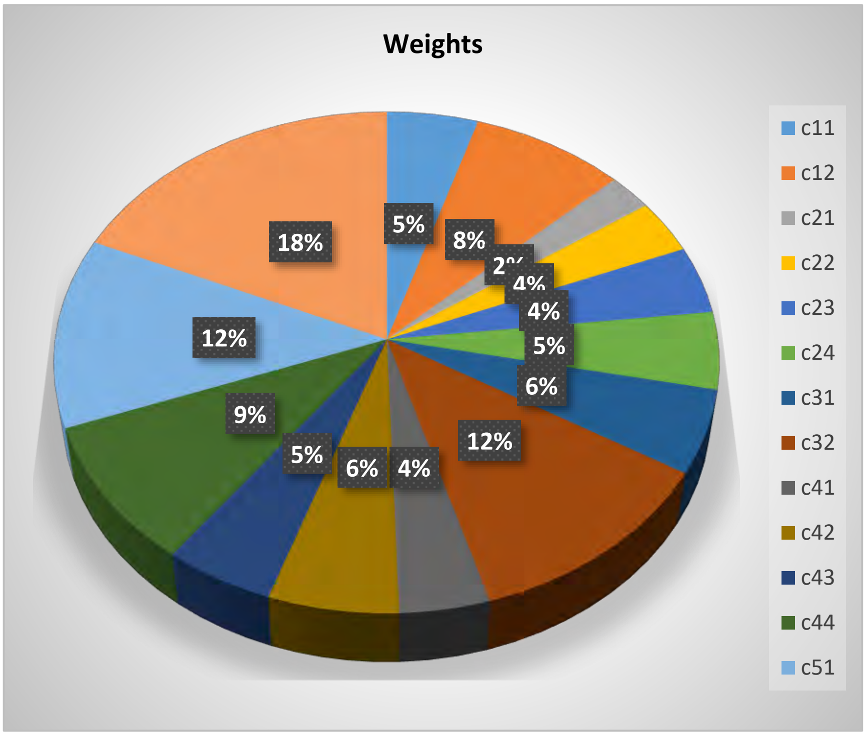

Fig 10. The global weights

Table 6. Normalized aggregated comparison matrix by the AHP method

Criteria	C_1	C_2	C_3	C_4	C_5
C_1	0.064456	0.060512	0.106234	0.186468	0.233628
C_2	0.298915	0.046192	0.119915	0.125181	0.168142
C_3	0.213634	0.202734	0.08048	0.213851	0.150442
C_4	0.18453	0.415607	0.196293	0.130397	0.270796
C_5	0.238465	0.274955	0.497077	0.344103	0.176991

6. Conclusions

Merging I4.0 in the industrial sector contributes to making flexible and efficient processes to produce better quality products with low cost to achieve competitive advantage. I4.0 has a significant impact on digitalizing manufacturing-based technologies as seen earlier.

This study contributes to the understanding of how manufacturing achieves sustainability according to TBL through I4.0 technologies. So, manufacturing firms are encouraged to fully integrate new technologies which have a positive impact on TBL pillars into their practices.

Wherefore**,** we developed a hybrid framework based on MCDM techniques to analyze and evaluate the factors and criteria based on sustainability manufacture related to I4.0. Four decision-makers and experts are selected to evaluate these criteria. Five main and fourteen sub-criteria are collected. The framework has been applied to a real case study in a manufacturing firm in the electrical industry. SVNSs are integrated with the DEMATEL and AHP methods in this work. The DEMATEL method is used to show the relation between the main and sub-criteria while the AHP method is used to compute the weights of the criteria.

Many methods like TOPSIS, VIKOR, and Entropy, can be applied to this problem in future directions. Moreover, the proposed framework can eventually be applied to many MCDM problems with more criteria.

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A note on AntiGeometry and NeutroGeometry and their application to real life

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Abstract: Dealing with NeutroGeometry in true, false, and uncertain regions is becoming of great interested for researchers. Not too many studies have been done on this topic, for that reason, aim of this work is to define a new method to deal with NeutroGeometry in true, false, and neutrogeometry (T,C,I,F). Furthermore, some real-life application examples in 3D computer graphics, Astrophysics, nanostructure, neutrolaw, neutrogender, neutrocitation, neutrohealth-food, neutroenvironment and quantum space are presented.

Keywords: Neutrosophic logic, neutroGeometry, antiGeometry, neutrosophic theory, Non-Euclidian geometry, Euclidian geometry, neutroAlgebra, antiAlgebra.

1. Introduction

Neutrosophy is a new branch of philosophy which was introduced by (Smarandache, 2002) which has been of great interesting of researchers who study different topics (applied or pure science) and it studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra: (B) is an idea, proposition, theory, event, concept or entity; anti (B) is the opposite of (B); and (neut-B) means neither (B) nor anti (B), i.e. neutrality between the two extremes (Bal eta l., 2018). Its fundamental theory states that every idea <B> tends to be neutralized, diminished, balanced by <nonB> ideas (not only <antiB> as Hegel). <noB> = what is not <B>, <antiB> = the opposite of <B>, and <neutB> = what is neither <B> nor <antiB>. In their classical form <B>, <neutB>, <antiB> are disjointed two by two. Smarandache (2002) defined fundamental notion of neutrosophic sets in the following way: Let R be an universe and N be a subset of R. An element y of R is written with respect to the set N as y (T, I, F) and belongs to N as follows: t% of true, i% of indeterminacy (unknown) and f% of false , where t belongs to T, i belongs to I and f belongs to F. Statically T, I, F are subsets, but dynamically T, I, F are functions or operators that depend on many known and unknown parameters. Following the idea of neutrosophic theory, many topics have been developed such that neutrosophic topology, neutrosophic normed spaces, neutrosophic probability, neutrosophic probability, decision making, neutroAlgebra, neutroGeometry and so on.

In our real world, spaces are not homogeneous, but mixed, complex, even ambiguous. The elements that populate them and the rules that act on them are not perfect, uniform or complete but fragmented and disparate, with unclear and contradictory information, and are not applied in the same degree to each element. The perfect, idealistic, exist only in the theoretical sciences. We live in a multi-space endowed with a multi-structure (Smarandache, 2021). Neither the elements of space nor the rules that govern them are egalitarian, all of them are characterized by degrees of diversity and variation. Indeterminate data and procedures (vague, unclear, incomplete, unknown, contradictory, ignorance, etc.) surround us.

While Non-Euclidean Geometries result from the total negation of a single specific axiom (Euclid's Fifth Postulate), AntiGeometry results from the total negation of any axiom and even more axioms of any geometric axiom system (Euclid's Five Postulates). Therefore, NeutroGeometry and AntiGeometry are respectively alternatives and generalizations of Non-Euclidean Geometries.

Smarandache (2021) proposed: Let's consider a classical geometry concept, it forms the following geometric neutrosophic triplet:

Concept(1, 0, 0), NeutroConcept(T, I, F), AntiConcept (0, 0, 1).

Where $(T, I, F) \notin \{(1, 0, 0), (0, 0, 1)\}$.

Concept(1, 0, 0) means that the degree of truth of the concept is $T = 1$, $I = 0$, $F = 0$, or the Concept is 100% true, 0% indeterminate, and 0% false in the given geometric space.

NeutroConcept (T, I, F) means that the concept is T% true, I% indeterminate, and 0% false in the given geometric space, with $(T, I, F) \in [0, 1]$, and $(T, I, F) \notin \{(1, 0, 0), (0, 0, 1)\}$.

AntiConcept (0, 0, 1) means that $T = 0$, $I = 0$, and $F = 1$, or the Concept is 0% true, 0% indeterminate, and 100% false in the given geometric space.

Smarandache (2021) went from the neutrosophic triplet (Algebra, NeutroAlgebra, AntiAlgebra) to a similar neutrosophic triplet (Geometry, NeutroGeometry, AntiGeometry), in the same way. Correspondingly from the algebraic structuires, with respect to the geometries, one has in the classical (Euclidean) Geometry, on a given space, all classical geometric Concepts are 100% true (i.e. true for all elements of the space). While in a NeutroGeometry, on a given space, there is at least one NeutroConcept (and no AntiConcept). In the AntiGeometry, on a given space, there is at least one AntiConcept.

With a view to device a practical tool for inference, Belnap (1977) introduced the notion of a four-valued logic. In his work, corresponding to a certain information he considered four possibilities namely T: True, F: False, none: neither true nor false, and both: both true and false. He symbolized these four truth values as {T, F, both, none}, for more notions derived from this paper, we refer the reader to (Das, et al., 2021; Mohanasundari and Mohana, 2020).

Later on (Smarandache, 2013) Smarandache has generalized Belnap's Logic (True, False, Unknown, and Contradiction), Lukasiewicz' Logic (True, False, and Possible), and Kleene's Logic (True, False, Unknown (or Undefined)) to Refined Neutrosophic Set having any $n \geq 2$ components, where the Truth T was split into Sub-Truths T_1, T_2, …, T_p, the Indeterminacy I was split into Sub-Indeterminacies, I_1, I_2, …, I_r, and the Falsehood F was split into Sub-Falsehoods F_1, F_2, …, F_s, where p, r, s ≥ 0, are positive integers and at least one of them is ≥ 2, with p+r+s = n.

Therefore, he also extended the Fuzzy Set to Refined Fuzzy Set, Intuitioinoistic Fuzzy Set to Refined Intuitionoistic Fuzzy Set, and similarly for other fuzzy extension sets.

In the case of Belnap's Logic, the Indeterminacy was split into two sub-indeterminacies: I_1 = Unknown, and I_2 = Contradiction.

In this work, we use the notions presented by (Singh, 2022) and (Smarandache, 2021), to carry out an exhaustive analysis of the NeutroGeometry in the cases in which the indeterminacy is divided into two categories, ignorance and contradiction as is was proposed by Smarandache (2013) in n-refined neutrosophic logic; given that these cases can occur in real life and are first line, in this way, several application examples are presented where this sort of act can occur. Additionally, a method for dealing with NeutroGeometry in true, false and neutrality (or indeterminacy) (T,C,I,F) is presented, where T is true, C is contradiction, I is ignorance and F is false. Throughout the development of this work, neutrogeometry (T,C,I,F) will also be written as neutrogeometry.

2. BACKGROUND

It is well-known that Euclidian-geometry is one of the oldest disciplines of mathematics. Its origin etymological, gives us a clear idea of the activities to which it appears related in its beginnings. This is how many historians find the roots of this science in ancient Egypt, where the foundations geometry solids by simply introducing the measurement of land with the surveyors, who after the annual floods of the Nile, had the task of rebuilding the boundaries of the lands assigned to the settlers, or also linked to the construction of its famous pyramids According to Salazar (1984) The development of modern geometry was carried out by the mathematician German David Hilbert (1862-1943), who made an analysis of Euclidean-geometry in his work Foundation of geometry (1899), reaching the conclusion that only six primitive concepts (point, line, plane, belongs, congruence and between) and 21 postulates, which lay the solid foundations of geometry, thus becoming a science rational and deductive, of which its components are independent, categorical and enough.

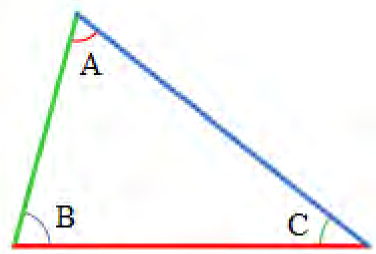

Figure 1: The sum of angles A, B and C in the given triangle is 180° as per Euclidian geometry i.e., A+B+C=180°

On the other hand, it is called non-Euclidean geometry, to any formal system of geometry were different postulates and propositions in some matter from those established by Euclid in his treatise elements.

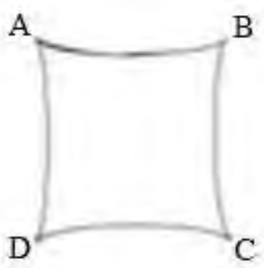

Figure 2: An illustrative example of non-Euclidian geometry

There is not a single system of non-Euclidean geometry, but many, although if the discussion is restricted to homogeneous spaces, in which the curvature of space is the same at each point, in which the points of space are indistinguishable, three formulations of geometries:

I. Euclidean geometry satisfies all five of Euclid's postulates and has zero curvature (i.e., it is assumed to be in flat space so the sum of the three interior angles of a triangle is always 180°).

II. Hyperbolic geometry satisfies only Euclid's first four postulates and has negative curvature (in this geometry, for example, the sum of the three interior angles of a triangle is less than 180°).

III. Elliptic geometry satisfies only Euclid's first four postulates and has positive curvature (in this geometry, for example, the sum of the three interior angles of a triangle is greater than 180°).

IV. Spherical geometry is the geometry of the two-dimensional surface of a sphere. It is an example of non-Euclidean geometry. Spherical geometry is the simplest model of elliptical geometry, in which a line has no parallel lines through a given point. In contrast to hyperbolic geometry, in which a line has two parallels, and an infinite number of ultra-parallels, through a given point.

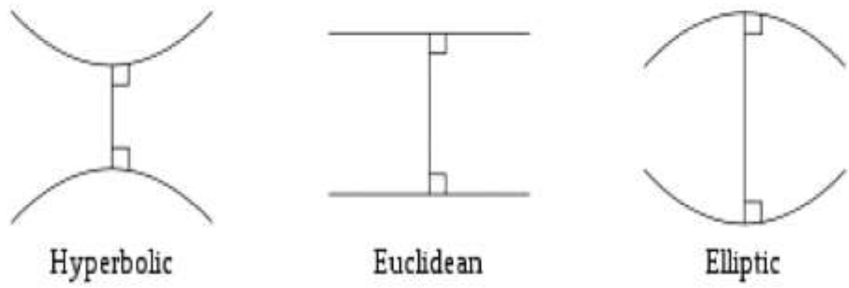

Figure 3: An illustrative example of difference between Euclidian and non-Euclidian geometry

II, III, IV are particular cases of Riemannian geometries, in which the curvature is constant, if the possibility that the intrinsic curvature of the geometry varies from one point to another is allowed, we have a case of general Riemannian geometry, as it happens in the theory of general relativity where gravity causes an inhomogeneous curvature in space-time, the curvature being greater near concentrations of mass, which is perceived as an attractive gravitational field. Smarandache (2021) said that Riemannian geometry, which is called elliptic geometry, is an antigeometry too, since the fifth Euclidean postulate is 100% invalidated in the following antipostulate (second version) place, through a point outside of a line, no parallel can be drawn to that line or (T,I,F)=(0,0,1). Since in this paper indeterminacy factor consists of two divisions namely contradiction (C) and ignorance (I) (Chatterjee et al., 2016), through a point outside of a line, no parallel can be drawn to that line or (T,C,I,F)=(0,0,0,1). This means that for this concept, we form the following geometric neutrosophic triplet:

Concept(1, 0, 0, 0), NeutroConcept(T, C, I, F), AntiConcept (0, 0, 0, 1).

Where (T, C, I, F) $\notin$ {(1, 0, 0, 0), (0, 0, 0, 1)}.

Concept(1, 0, 0, 0) means that the degree of truth of the concept is T = 1, C=0, I = 0, F = 0, or the Concept is 100% true, 0% contradiction, 0% ignorance and 0% false in the given geometric space.

NeutroConcept (T, C, I, F) means that the concept is T% true, C% contradiction, I% ignorance and 0% false in the given geometric space, with (T, C, I, F) ∈ [0, 1], and (T, C, I, F) ∉ {(1, 0, 0,0), (0, 0, 0, 1)}.

AntiConcept (0, 0, 0, 1) means that T = 0, C = 0, I=0 and F = 1, or the Concept is 0% true, 0% contradiction, 0% ignorance and 100% false in the given geometric space.

We go from the algebraic structures, with respect to the geometries, one has in the classical (Euclidean) Geometry, on a given space, and all classical geometric Concepts are 100% true. While in a NeutroGeometry, on a given space, there is at least two NeutroConcept (and no AntiConcept). In the AntiGeometry, on a given space, there is at least one AntiConcept.

How to deal with these sort of phenomenon and characterize them in true, false, or uncertain regions in which these uncertain regions are divided in two parts (contradiction and ignorance) is one of the most crucial tasks. Recently, Singh (2022) presented a method for dealing with one type of indeterminacy, therefore, in the next section, we propose a method to deal with these types of information in neutrogeometry (when indeterminacy is divided in contradiction and ignorance) for multi-decision process and we show some application examples in real life, this method is an extension of the method proposed by (Singh, 2022), but the method proposes in the next section is more general that the method proposed by (Singh, 2022).

3. Method to deal with NeutroGeometry in true, false, and indeterminacy and its application to real life

Step 1. Consider the information with a geometry and its attributes ($\boldsymbol{B}$).

Step 2. Let $\boldsymbol{B}$ be any non-empty set of a given geometrical information.

Step 3. Define the operator as $\boldsymbol{\varphi}: \boldsymbol{B} \times \boldsymbol{B} \to P^m(\boldsymbol{B})$ as (T,C,I,F) ∉ {(1,0,0,0),(0,0,0,1)}.

Step 4. In case any mapping is possible, then it can be characterized by:

I. In case for any elements t, u ∈ $\boldsymbol{B}$, the geometry provides a new element in the geometrical space, i.e., t ○ u ⊆ $\boldsymbol{B}$. It can be considered as true characterization.

II. In case for any elements t, u ∈ $\boldsymbol{B}$, the geometry provides a new element which does not exist in the geometrical space using the given operator as t ○ u ⊈ $\boldsymbol{B}$. It can be considered as false regions.

III. In case for any element of t, u ∈ B, the geometry provides a new element which in saddle space and its quantum state is uncertain and it is divided in two unknown parts. This type of element can be considered in neutrogeometry.

Step 5. It defined a function $\vartheta: T \to U$, which provides three possibilities:

I. In case a well-defined mapping exists among t and u, then it is called as true regions.

II. In case the mapping is outer-defined mapping between t and u, then it is in false regions.

III. It is unknown whether the mapping exists or not and it thinks but not sure what is the value for the mapping among t and u then the element is in neutrogeometry.

Step 6. In this way, the geometrical space and its characterization can be possible.
Step 7. The similarity among the information sets can be found using the geodesic distance.
Step 8. The geodesic distance provides the shortest path among two neutrogeometric spaces rather than its straight line distance of Euclidean geometry as shown in figure 4.
Step 9. The information shading to the defined geodesic distance can be considered as cluster for knowledge processing tasks.

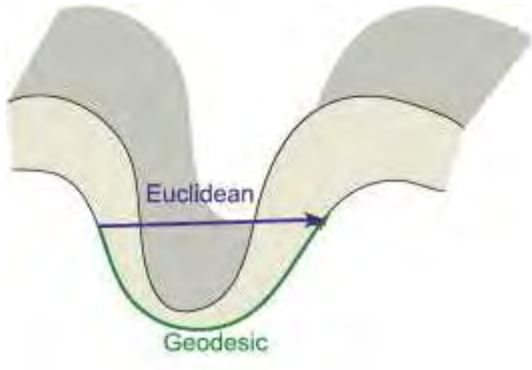

Figure 4: The difference between Euclidean and geodesic distance

Next, we show some applications with non-Euclidian geometry and NeutroGeometry.

Application 1: Consider there are *m* non-Euclidean information sets in a given space. Defining the function will take $O(m^2)$ time complexity among them. The characterization of those information sets in neutrogeometry will take maximum $O(m^2)$ time complexity. In this way, the overall time complexity for characterization of non-Euclidean information in true, false, contradiction, ignorance regions may take maximum $O(m^3)$ time complexity.

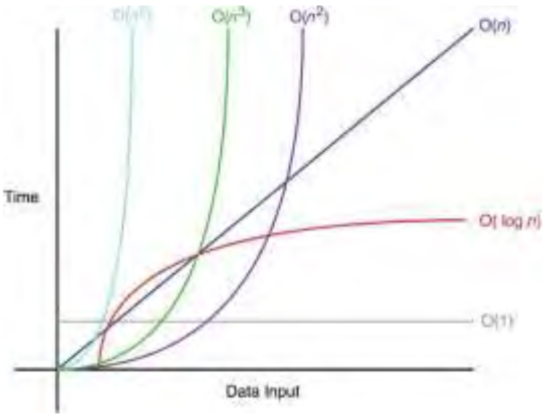

Figure 5: A NeutroGeoemtry time complexity information and its visualization

Application 2: Astrophysics is the development and study of physics applied to astronomy. It studies stars, planets, galaxies, black holes and other astronomical objects as physical bodies, including their composition, structure and evolution. Astrophysics uses physics to explain the properties and phenomena of stellar bodies through their laws, formulas and magnitudes. The beginning of astrophysics was possibly in the 19th century when, thanks to the spectra, the physical composition of the stars could be ascertained. Once it was understood that the celestial bodies are composed of the same ones that make up the Earth and that the same laws of physics and chemistry apply to them, astrophysics was born as an application of physics to the phenomena observed by the Earth astronomy. Astrophysics is therefore based on the assumption that the laws of physics and chemistry are universal, that is, that they are the same throughout the universe (Ginzburg, 1979). In this way, Astrophysics is a branch of information with NeutroGeometry. The representation of astronomy and its pattern is based on spherical geometry and its algebra as shown in figure 6.

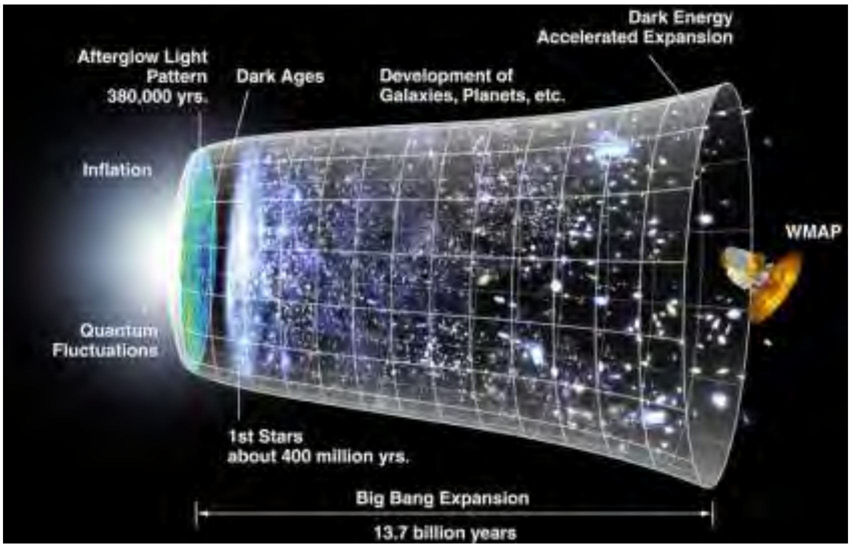

Figure 6: A NeutroGeoemtry astrophysics information and its visualization

Application 3: A nanostructure is a structure with an intermediate size between molecular and microscopic (micron-sized) structures. Here we are talking about the nanoscale. Generally, these structures experience quantum effects that are not as obvious in larger structures and therefore have special physical properties (Farrow et al., 2007). This case can be represented by Riemannian geometry as can be seen in figure 7.

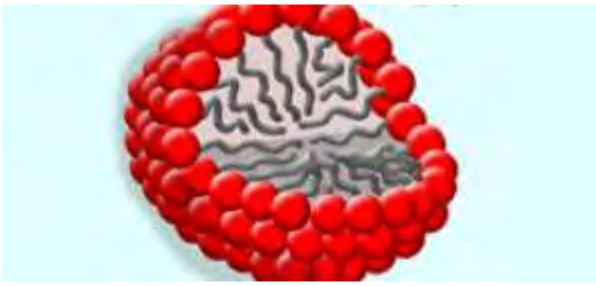

Figure 7: A NeutroGeoemtry nanostructure information and its visualization

Application 4: In 3D computer graphics (abbreviated CG) is a method of improving the quality of a texture on a surface that is viewed from an oblique angle relative to the projection angle of the image. Texture on a surface, like bilinear filtering and trilinear filtering, anisotropic filtering removes aliasing, but it differs from the previous methods in that it reduces blurring and preserves detail at extreme viewing angles. Anisotropy filtering is relatively heavy (mainly because of memory usage and some amount of computational processing) and only became a standard feature on commercial graphics cards in the late 1990s. Computer graphics is now common in modern boards and can be activated and configured both by the user from the driver configuration, or by graphic applications or video games using programming tools as can be seen in figure 8.

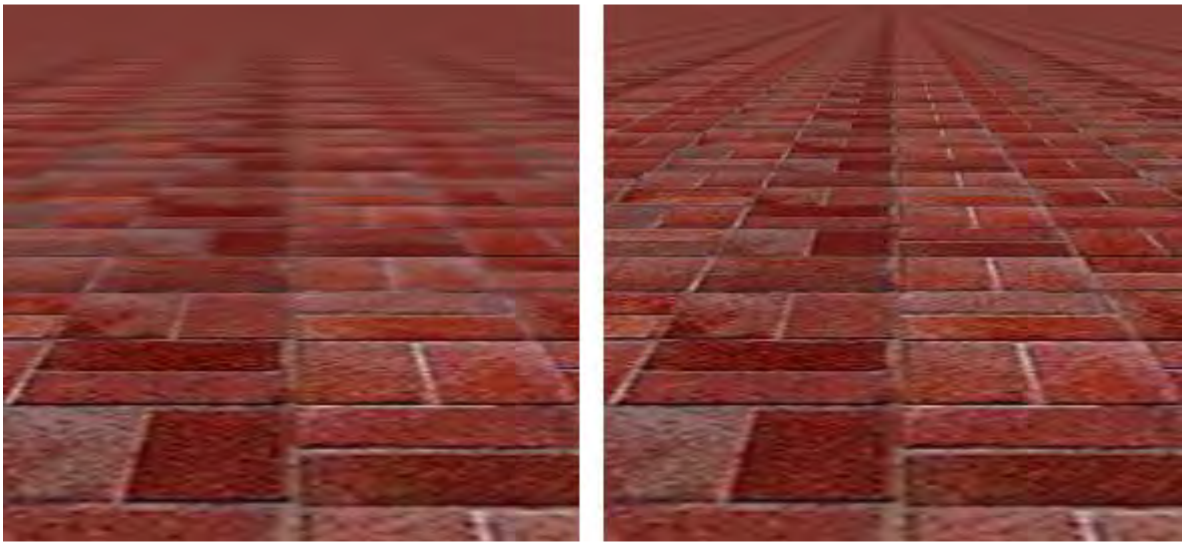

Figure 8: A NeutroGeoemtry computer graphic information and its visualization

This can be done via steeply angled rather than right angled with respect to the given point which required neutrogeometry. This can be characterized as follows:

I. True image (1,0,0,0): In case the true image is made via enhancing the image which can be represented as (1, 0, 0, 0).

II. False image (0,0,0,1): The image does not provide the true image or provide distinct results can be represented as (0, 0, 0, 1).

III. NeutroImage (T, C, I, F): The expert is uncertain about the image and its quality after the enhancement, this means that he/she does not too much about the topic and the does not if the image quality will be as he/she thought.

Application 5: The law in any country is totally uncertain and vague (Singh, 2021; Smarandache, 2021; Kappor and Singh, 2020; Singh, 2022). It depends on hierarchical ordering of citizens and their positional power in the given country which is a neutrogeometric information rather than flat. There are several cases where the same punishment will not be given to each citizen for the same act. This can be defined by:

I. Law (1, 0, 0, 0): In case the given law is fully applied on the particular citizen. in this case, the government or court can be considered as unbiased.

II. AntiLaw (0, 0, 0, 1): in this case, there is no law defined for the particular act. It used to be observed when a politician or business class people never get punishment under the same law.

III. NeutroLaw (T,C,I,F). In this case, the law changes based on person to person, region to region, and, religion to religion. Besides, some laws presented by governments usually contradict what is present in some other decree or do not agree to propose a new law, in turn, there is a high level of ignorance among people, given that they do things out of ignorance and not knowing the law does not exempt them from responsibility. This type of law where partial influence occur by any government or higher authority can be considered as neutro-law. So, the law differs into indeterminacy which is divided in ignorance and contradiction, the hidden pattern in these types of information can be analyzed using neutrogeometry.

Application 6: Neutro-gender law is one of the most suitable examples of neutrogeoemtry information (Singh, 2022) where the law differs based on the gender. This can be characterized as follows:

I. Women law (1, 0, 0, 0): Consider, a woman complains that a man did sexual or mental harassment to her. In this case, the given crime can be accepted immediately without proper proof also.

II. Men Law (0, 0, 0, 1): Consider, a man complains because he suffered sexual abuse by a woman or was psychologically violated. In this case, the given crime cannot be accepted right away with providing several proofs also.

III. NeutroLaw (T,C,I,F): In case a person who belongs to LGBT community reports about sexual or mental abuse, sometimes nobody body listens, sometimes nobody does not what to do, the laws are not clear for these types of people since there is a contradiction if should be care as a man or as a woman . The law differs for them which shows indeterminacy which is divided in ignorance and contradiction. In this case, the entire information can be considered as uncertain and vague.

Application 7: The characterization of a citation for intellectual measurement cannot be done via flat way like Euclidean geometry (Singh, 2022; Smarandache, 2021). It requires neutrogeometry classification which can be characterized as follows:

I. Citation (1, 0, 0, 0): A paper cited by the domain expert, keyword, or methodology matching for the given topic can be considered as relevant citation (1, 0, 0, 0).

II. Anti-Citation (0, 0, 0, 1): A paper cited in irrelevant way, a retracted paper citation, a posthumous authors papers citation, same departmental citations beyond the relevant of topic, host conference citation without relevancy, forced citation, and random citation can be considered as Anti-Citation (0, 0, 0, 1).

III. Neutro-Citation (T,C,I,F): An article that is self-cited, influenced citations, citations added because peer review was required, articles published in predatory journals, etc. It can be considered a Neutro-Citation (T,C,I,F).

Application 8: Quantum field theory in curved space-time is an extension of standard quantum field theory in which the possibility is contemplated that the space-time through which the field propagates is nevertheless not flat (described by the metric of Minkowski). A generic prediction of this theory is that particles can be generated due to time-dependent gravitational fields, or the presence of horizons Quantum field theory in curved space-time may be required as a first approximation of quantum gravity. The next step consists of a semi classical gravity, in which quantum corrections will be taken into account, due to the presence of matter, on space-time as can be seen in figure 9. In this way, the traversal criteria do not matter whether you go x steps right and then you go y steps forward and vice versa (Bresar, 2014). It means the non-commutative geometry cannot be represented precisely which requires fuzzy spherical coordinates.

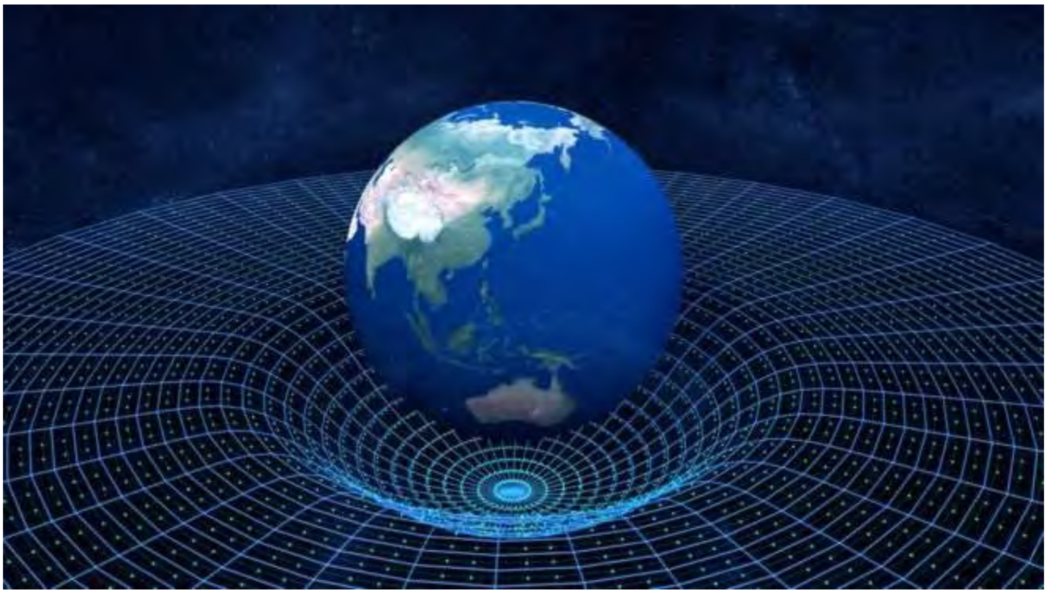

Figure 9: A NeutroGeoemtry Quantum space information and its visualization

Application 9: The way in which we consume perishable foods is something that we do not know and we cannot exactly measure whether they are healthy or not, since the companies that produce, mention that they are made under strict health protocols, while some foundations question these methods, since they mention that these are harmful to health. So, it requires non-Euclidean classification which can be characterized as follows:

I. Health-food (1, 0, 0, 0): People who eat healthy without consuming perishable products for the given topic can be considered as relevant Health-food (1, 0, 0, 0).

II. Anti-Health-food (0, 0, 0, 1): People who eat unhealthy food and/or perishable products, knowing how bad these can be for their health. This can be considered as Anti-Health-food (0, 0, 0, 1).

III. Neutro-Health-food (T,C,I,F): People who claim to take care of themselves but eat unhealthy food and people who are unaware of their health status because they don't have the means or because they don't want to. This can be considered a Neutro-Health-food (T,C,I,F).

Application 10: The conservation of the environment is something that has been of great interest and debate of many researchers and non-researchers. In the last decades, the concern for the conservation of the environment has undergone an amazing growth at all levels, and today it must be considered one of the most relevant matters at a scientific, doctrinal and normative level. Indeed, if less than fifty years ago the relationship between human rights and the environment was ignored, today there are numerous binding normative texts that enshrine both the right to a healthy environment and the so-called rights of environmental action, all of which is now preached as necessary to guarantee that present and future generations can develop in a healthy and beneficial environment for human life (García, 2018). But there have always been some entities that have caused a lot of damage to the environment regardless of the consequences. So, it requires neutrogeometric classification which can be characterized as follows:

I. Environment (1, 0, 0, 0): People who care for the environment and do not consume the products of companies that affect the well-being of the environment, for the given topic can be considered as relevant Environment (1, 0, 0, 0).

II. Anti-Health-food (0, 0, 0, 1): People who do not care for the environment and consume the products of companies that affect the well-being of the environment, even knowing that this can be harmful to themselves. This can be considered as Anti-environment (0, 0, 0, 1).

III. Neutro-environment (T,C,I,F): people who consume the product of said company and talk about conserving the well-being of the environment and people who are unaware of the reality of the environment because they do not read news about it or are not interested in knowing about the subject, since according to them it is not their convenience. This can be considered a Neutro-environment (T,C,I,F).

4. **Conclusions**

In this work, we presented a method to deal with NeutroGeometry of type (T,C,I,F). The analysis of this method is showed together with some applications and illustrative examples. For future works, new applications to this method can be introduced and decision-making applications can be presented for a better study and analysis of this topic.

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6. Conflicts of interest

The author declares that there is no conflict of interest.

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The SuperHyperFunction and the Neutrosophic SuperHyperFunction (revisited again)

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Abstract: In this paper, one recalls the general definition of the SuperHyperAlgebra with its SuperHyperOperations and SuperHyperAxioms [2, 6]. Then one introduces for the first time the SuperHyperTopology and especially the SuperHyperFunction and Neutrosophic SuperHyperFunction. One gives a numerical example of a Neutro-SuperHyperGroup.

Keywords: SuperHyperAlgebra; SuperHyperFunction; Neutrosophic SuperHyperFunction; SuperHyperOperations; SuperHyperAxioms; SuperHyperTopology.

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1. System of Sub-Systems of Sub-Sub-Systems and so on

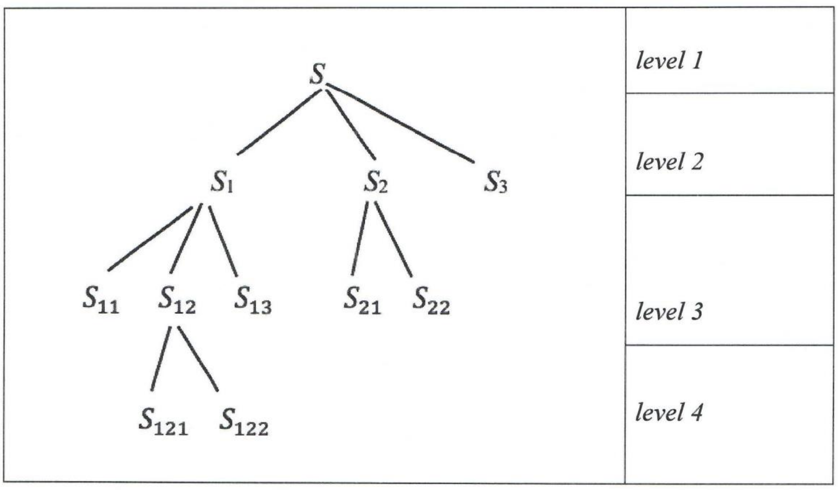
A system may be a set, space, organization, association, team, city, region, country, etc. One considers both: the static and dynamic systems.

With respect to various criteria, such as: political, religious, economic, military, educational, sportive, touristic, industrial, agricultural, etc.,

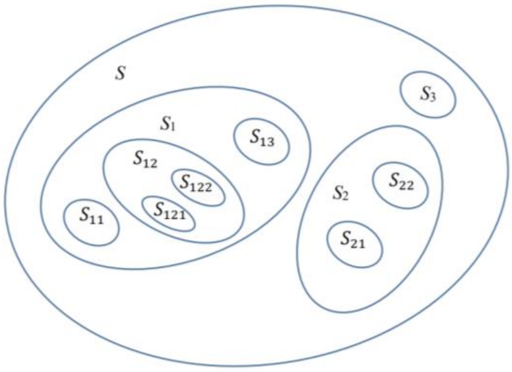
a system S is made up of several sub-systems S_1, S_2, …, S_p, for integer $p \geq 1$; then each sub-system S_i, for $i \in \{1, 2, \ldots, p\}$ is composed of many sub-sub-systems S_{i1}, S_{i2}, …, S_{ip_i}, for integer $p_i \geq 1$; then each sub-sub-system S_{ij}, for $j \in \{1, 2, \ldots, p_i\}$ is composed sub-sub-sub-systems, S_{ij1}, S_{ij2}, …, S_{ijp_j}, for integer $p_j \geq 1$; and so on.

2. Example 1 and Application of Systems made up of Sub-Sub-Sub-Systems (four levels)

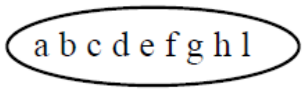
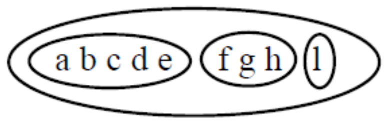
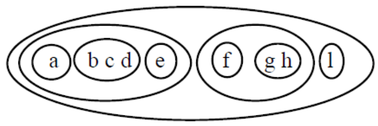
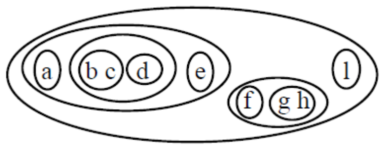
i) Using a *Tree-Graph Representation*, one has:



ii) Using a *Geometric Representation*, one has:



iii) Using an *Algebraic Representation* through pairs of braces { }, one has:

$P^0(S) \stackrel{\text{def}}{=} S = \{a,b,c,d,e,f,g,h,l\}$ 1 level of pairs of braces	 1level of closed curves	*level 1*
$P^1(S) \stackrel{\text{def}}{=} P(S) \ni \{\{a,b,c,d,e\},\{f,g,h\},\{l\}\}$ 2 levels of pairs of braces i.e. a pair of braces { } inside, another pair of braces { }, or { … { … } … }	 2 levels of closed curves	*level 2*
$P^2(S) \stackrel{\text{def}}{=} P(P(S))$ $\ni \{\{\{a\},\{b,c,d\},\{e\}\},\{\{f\},\{g,h\}\},\{l\}\}$ 3 levels of pairs of braces	 3 levels of closed curves	*level 3*
$P^3(S) \stackrel{\text{def}}{=} P(P^2(S))$ $\ni \{\{\{a\},\{b,c\},\{d\},\{e\}\},\{\{f\},\{g,h\}\},\{l\}\}$ 4 levels of pairs of braces	 4 levels of closed curves	*level 4*

where the symbol "∋" means "contain(s)", it is the opposite of the symbol "∈" (belong(s) to), for example $M \ni x$ means the set M contains the element x, which is equivalent to $x \in M$.

Industrial Application.

Let's assume an auto-repair corporation called MotorX Inc. resides in the United States (system S). MotorX has three branches, one in each of the states: California, Arizona, and New Mexico (sub-systems S_1, S_2, and S_3 respectively).

In California, MotorX has branches in three cities: San Francisco, Los Angeles, and San Diego (sub-sub-systems S_{11}, S_{12}, and S_{13} respectively), and in Arizona in two cities: Phoenix and Tucson (sub-sub-systems S_{21}, and S_{22} respectively).

In the city of Los Angeles, MotorX has branches in two of the city's districts or neighborhoods, such as Fairfax and Northridge (sub-sub-sub-systems S_{121}, and S_{122} respectively).

2.1 Remark 1

The pairs of braces { } make a difference on the structure of a set. For example, let's see the distinction between the sets A and B, defined as bellow:

A = $\{a, b, c, d\}$ represents a system (organization) made up of four elements,

while B = $\{\{a, b\}, \{c, d\}\}$ represents a system (organization) made up of two sub-systems and each sub-system made up of two elements. Therefore B has a richer structure, it is a refinement of A.

3. Definition of n^{th}-Power of a Set

The n^{th}-Power of a Set (2016) was introduced by Smarandache in the following way:

$P^n(S)$, as the n^{th}-PowerSet of the Set S, for integer $n \geq 1$, is recursively defined as:

$P^2(S) = P(P(S))$, $P^3(S) = P(P^2(S)) = P\left(P(P(S))\right)$, …,

$P^n(S) = P(P^{n-1}(S))$, where $P^0(S) \stackrel{\text{def}}{=} S$, and $P^1(S) \stackrel{\text{def}}{=} P(S)$. (For any subset A, we identify {A} with A.)

The n^{th}-Power of a Set better reflects our complex reality, since a set S (that may represent a group, a society, a country, a continent, etc.) of elements (such as: people, objects, and in general aany items) is organized onto subsets $P(S)$, which on their turns are also organized onto subsets of subsets, and so on. That is our world.

In the classical HyperOperation and Classical HyperStructures, the empty set $\emptyset$ does not belong to the power set, or $P_*(S) = P(S) \setminus \{\emptyset\}$.

However, in the real world we encounter many situations when a HyperOperation $\circ$ is:

- *indeterminate*, for example $a \circ b = \emptyset$ (unknown, or undefined);
- or *partially indeterminate*, for example $a \circ b = \{[0.2, 0.3], \emptyset\}$.

In our everyday life, there are many more operations and laws that have some degrees of indeterminacy (vagueness, unclearness, unknowingness, contradiction, etc.), than those that are totally determinate.

That is why is 2016 we have extended the classical HyperOperation to the Neutrosophic HyperOperation, by taking the whole power $P(S)$ (that includes the empty-set $\emptyset$ as well), instead $P_*(S)$ (that does not include the empty-set $\emptyset$), as follows.

3.1 Remark 2

Throughout this paper the definitions, theorems, remarks, examples and applications work for both classical-type and Neutrosophic-type SuperHyper-Algebra and SuperHyper Function.

3.2 Theorem 1

Let S be a discrete finite set of 2 or more elements, and $n \geq 1$ an integer.

Then: $P^0(S) \subset P^1(S) \subset P^2(S) \subset \cdots \subset P^{n-1}(S) \subset P^n(S)$.

Proof

For a discrete finite set $S = \{a_1, a_2, \dots, a_m\}$ for integer $m \geq 2$ one has:

$P^0(S) \equiv S = \{a_1, a_2, \dots, a_m\}$.

$P^1(S) = P(S) = \{a_1, a_2, \dots, a_m; \{a_1, a_2\}, \{a_1, a_2, a_3\}, \dots, \{a_1, a_2, \dots, a_m\}\}$,

and cardinal of P(S) is $Card(P(S)) = C_m^1 + C_m^2 + \cdots + C_m^n = 2^m - 1$,

where C_m^i, $1 \leq i \leq m$, means combinations of m elements taken in groups of i elements.

It is clear that $P^0(S) \subset P^1(S)$.

In general, one computes the set of $P^{k+1}(S)$ by taking the set of the previous $P^k(S) = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$, where $r = Card(P^k(S))$, and making all possible combination of its r elements; but, at the beginning, when one takes the elements only by one, we get just $P^k(S)$, afterwards one takes the elements in group of two, then in groups of three, and so on, and finally all r elements together as a single group.

4. Definition of SuperHyperOperations

We recall our 2016 concepts of SuperHyperOperation, SuperHyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation Neutrosophic SuperHyperAxiom and Neutrosophic SuperHyperAlgebra [2].

Let $P_*^n(H)$ be the n[th]-powerset of the set *H* such that none of *P(H), P²(H), …, Pⁿ(H)* contain the empty set ϕ.

Also, let $P_n(H)$ be the n[th]-powerset of the set *H* such that at least one of the *P(H), P²(H), …, Pⁿ(H)* contain the empty set ϕ. For any subset A, we identify {A} with A.

The SuperHyperOperations are operations whose codomain is either $P_*^n(H)$ and in this case one has **classical-type SuperHyperOperations**, or $P^n(H)$ and in this case one has **Neutrosophic SuperHyperOperations**, for integer $n \geq 2$.

4.1 Classical-type m-ary SuperHyperOperation {or more accurate denomination (m, n)-SuperHyperOperation}

Let U be a universe of discourse and a non-empty set $H,\ H \subset U$. Then:

$$\circ_{(m,n)}^{*} : H^m \to P_*^n(H)$$

where the integers $m, n \geq 1$,

$$H^m = \underbrace{H \times H \times \ldots \times H}_{m\ times},$$

and $P_*^n(H)$ is the n[th]-powerset of the set H that includes the empty-set.

This SuperHyperOperation is a m-ary operation defined from the set H to the n[th]-powerset of the set H.

4.2 Neutrosophic m-ary SuperHyperOperation {or more accurate denomination Neutrosophic (m, n)-SuperHyperOperation}

Let U be a universe of discourse and a non-empty set $H,\ H \subset U$. Then:

$$\circ_{(m,n)} : H^m \to P^n(H)$$

where the integers $m, n \geq 1; P^n(H)$ - the n-th powerset of the set H that includes the empty-set.

5. SuperHyperAxiom

A **classical-type SuperHyperAxiom** or more accurately a ***(m, n)*-SuperHyperAxiom** is an axiom based on classical-type SuperHyperOperations.

Similarly, a **Neutrosophic SuperHyperAxiom** {or Neutrosphic (m, n)-SuperHyperAxiom} is an axiom based on Neutrosophic SuperHyperOperations.

There are:

- **Strong SuperHyperAxioms**, when the left-hand side is equal to the right-hand side as in non-hyper axioms,
- and **Week SuperHyperAxioms**, when the intersection between the left-hand side and the right-hand side is non-empty.

For examples, one has:

- **Strong SuperHyperAssociativity**, for any $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_{m-1} \in H$, one has
$$\circ_{(m,n)}(\circ_{(m,n)}(x_1, x_2, \ldots, x_m), y_1, y_2, \ldots, y_{m-1}) = \circ_{(m,n)}(x_1, \circ_{(m,n)}(x_2, x_3, \ldots, x_m, y_1), y_2, y_3, \ldots, y_{m-1}) = \ldots$$
$$\ldots = \circ_{(m,n)}(x_1, x_2, x_3, \ldots, x_{m-1}, \circ_{(m,n)}(x_m, y_1, y_2 \ldots, y_{m-1})).$$
and **Week SuperHyperAssociativity**, for any $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_{m-1} \in H$ one has
$$\circ_{(m,n)}(\circ_{(m,n)}(x_1, x_2, \ldots, x_m), y_1, y_2, \ldots, y_{m-1}) \cap \circ_{(m,n)}(x_1, \circ_{(m,n)}(x_2, x_3, \ldots, x_m, y_1), y_2, y_3, \ldots, y_{m-1}) \cap \ldots$$
$$\ldots \cap \circ_{(m,n)}(x_1, x_2, x_3, \ldots, x_{m-1}, \circ_{(m,n)}(x_m, y_1, y_2 \ldots, y_{m-1})) \neq \phi.$$

6. SuperHyperAlgebra and SuperHyperStructure

A **SuperHyperAlgebra** or more accurately ***(m-n)*-SuperHyperAlgebra** is an algebra dealing with SuperHyperOperations and SuperHyperAxioms.

Again, a **Neutrosophic SuperHyperAlgebra** {or Neutrosphic (m, n)-SuperHyperAlgebra} is an algebra dealing with Neutrosophic SuperHyperOperations and Neutrosophic SuperHyperOperations.

In general, we have **SuperHyperStructures** {or *(m-n)*-SuperHyperStructures}, and corresponding **Neutrosophic SuperHyperStructures**.

For example, there are SuperHyperGrupoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, SuperHyperVectorSpace, etc.

7. Distinction between SuperHyperAlgebra vs. Neutrosophic SuperHyperAlgebra

i. If none of the power sets $P^k(H)$, $1 \le k \le n$, do not include the empty set ϕ, then one has a classical-type SuperHyperAlgebra;

ii. If at least one power set, $P^k(H)$, $1 \le k \le n$, includes the empty set ϕ, then one has a Neutrosophic SuperHyperAlgebra.

8. Example 2 of SuperHyperGroup

The below $(P^2(S), \#)$ Table represents a Commutative Neutro-*SuperHyperGroup*.

$\#$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{\{a\},\{a,b\}\}$	$\{\{b\},\{a,b\}\}$	$\{\{a\},\{b\},\{a,b\}\}$
$\{a\}$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{\{a\},\{a,b\}\}$	$\{\{b\},\{a,b\}\}$	$\{\{a\},\{b\},\{a,b\}\}$
$\{b\}$	$\{b\}$	$\{a\}$	$\{\{a\},\{a,b\}\}$	$\{\{b\},\{a,b\}\}$	$\{\{a\},\{b\},\{a,b\}\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$	$\{\{a\},\{a,b\}\}$	$\{a\}$	$\{b\}$	$\{a\}$	$\{\{b\},\{a,b\}\}$
$\{\{a\},\{a,b\}\}$	$\{\{a\},\{a,b\}\}$	$\{\{b\},\{a,b\}\}$	$\{b\}$	$\{a\}$	$\{a,b\}$	$\{\{a\},\{a,b\}\}$
$\{\{b\},\{a,b\}\}$	$\{\{b\},\{a,b\}\}$	$\{\{a\},\{b\},\{a,b\}\}$	$\{a\}$	$\{a,b\}$	{a}	$\{b\}$
$\{\{a\},\{b\},\{a,b\}\}$	$\{\{a\},\{b\},\{a,b\}\}$	$\{a,b\}$	$\{\{b\},\{a,b\}\}$	$\{\{a\},\{a,b\}\}$	$\{b\}$	$\{a\}$

The *SuperHyperLaw* # is clearly well-defined, according to the above Table. This law is commutative since Table's matrix is symmetric with respect to the main diagonal.

$\{a\}$ is the *SuperHyperNeutral*.

And the *SuperHyperInverse* of an element $x \in P^2(S)$ is itself: $x^{-1} = x$.

8.1 Theorem 2

The above algebraic structure is a Commutative Neutro-SuperHyperGroup.

Proof

The axiom of associativity, with respect to the law #, is a NeutroAxiom, since:

there are three elements *{a}*, *{b}*, and *{a, b}* from the set $P^2(S)$

such that: *{a} # ({b} # {a, b}) = {a} # {{a}, {a, b}} = {{a}, {a, b}}*

and *({a} # {b}) # {a, b} = {b} # {a, b} = {{a}, {a, b}}*,

therefore, it has some degree of truth (*T > 0*);

and there are three elements *{a, b}*, *{{a}, {a, b}}*, and *{{b}, {a, b}}* from the set $P^2(S)$ such that:

({a, b} #{{a}, {a, b}}) # {{b}, {a, b}} = {b} # {{b}, {a, b}} = {{a}, {b}, {a, b}}

and {a, b} # ({{a}, {a, b}} # {{b}, {a,b}}) = {a, b} # {a, b} = {a}, which is different from {{a}, {b}, {a, b}}, therefore, it has some degree of falsehood (F > 0). While the other four axioms (well-defined, commutativity, unit element, and inverse element are classical (100% true, or T = 1, degree of indeterminacy I = 0, and F = 0).

9. SuperHyperTopology and Neutrosophic SuperHyperTopology

A topology defined on a SuperHyperAlgebra $(P_*^n(S), \#)$, for integer $n \geq 2$, is called a SuperHyperTopology, and it is formed from SuperHyperSubsets. Similarly for Neutrosophic SuperHyper of $P_*^n(S)$ Topology, where $P_*^n(S)$ is replaced by $P_n(S)$, that includes the empty-set as well.

10. Definition of classical-type Unary HyperFunction (f_H)

Let S be a non-empty set included in a universe of discourse U.

$$f_H: S \rightarrow P_*(S)$$

11. Definition of classical-type m-ary HyperFunction (f_H^m)

$$f_H^m: S^m \rightarrow P_*(S)$$

where m is an integer ≥ 2, and $P_*(S)$ is the classical powerset of S.

12. Definition of Unary SuperHyperFunction (f_H)

We now introduce for the first time the concept of **SuperHyperFunction** (f_{SH}).

$f_{SH}: S \rightarrow P_*^n(S)$, for integer $n \geq 1$, where $P^n(S)$ is the n-th powerset of the set S.

13. Definition of m-ary SuperHyperFunction (f_{SH}^m)

$f_{SH}^m: S^m \rightarrow P_*^n((S)$, for integer $m \geq 2$.

14. General Definition of SuperHyperFunction

$f_{SH}^{SH}: P_*^r(S) \rightarrow P_*^n(S)$, for integers $r, n \geq 0$.

$f_{SH}: S \rightarrow P^n(S)$

$f_{SH}^m: S^m \longrightarrow P^n(S)$

$f_{SH}^{SH}: P_*^r(S) \rightarrow P^n(S)$

15. Example 3 and Application of SuperHyperFunctions

Let $S = \{a, b\}$ be a discrete set. The first and second powersets of the set S are:

$$P(S) = \{\{a\}, \{b\}, \{a, b\}\}$$

$$P^2(S) = \left\{ \begin{matrix} \{a\}, \{b\}, \{a, b\} \\ \{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\} \\ \{\{a\}, \{b\}, \{a, b\}\} \end{matrix} \right\}$$

Let's define the SuperHyperFunction f_{SH} as follows:

$f_{SH}: S \rightarrow P^2(S)$

$f_{SH}(x)$ = the system (organization) or { } set that x best belongs to

$f_{SH}(a) = \{a, b\}$

$f_{SH}(b) = \{\{b\}, \{a, b\}\}$

For example, the system $\{b\}$ means that person b is a strong personality and himself alone makes a system.

16. Example 4 and Application of Neutrosophic SuperHyperFunctions

$S = [0, 5]$, a continuous set.

$P_o(S) = \{A, A \text{ is a subset}, A \subseteq [0,5]\}$

$P_o^2(S) = \{A_1, \{A_1, A_2\}, \{A_1, A_2, A_3\}, \dots\}$,

where all A_k are subsets of $[0, 5]$ with index $k \in [0,5]$, therefore one has an uncountable infinite set of subsets of [0, 5].

$f_{SH}: [0, 5] \to P_o^2([0, 5])$

$$f_{SH}(x) = \{[x-1, x] \cap [0, 5], [x+1, x+2] \cap [0, 5]\}$$

For example:

$$f_{SH}(2) = \{[1, 2], [3, 4]\}.$$

$$f_{SH}(3.4) = \{[2.4, 3.4], [4.4, 5]\}$$

since $[4.4, 5.4] \cap [0, 5] = [4.4, 5]$.

$$f_{SH}(0) = \{[0, 0], [1, 2]\} = \{0, [1, 2]\}$$

since $[-1, 0] \cap [0, 5] = [0, 0] = 0$.

$$f_{SH}(5) = \{[4,5], \emptyset\}$$

since $[6, 7] \cap [0, 5] = \emptyset$.

Conclusion

In this paper we recalled the concepts of SuperHyperAlgebra and Neutrosophic HyperSuperAlgebra, and presented an example of Neutro-SuperHyperGroup. Then, for the first time we introduced and gave examples of SuperHyperFunction and Neutrosophic SuperHyperFunction.

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