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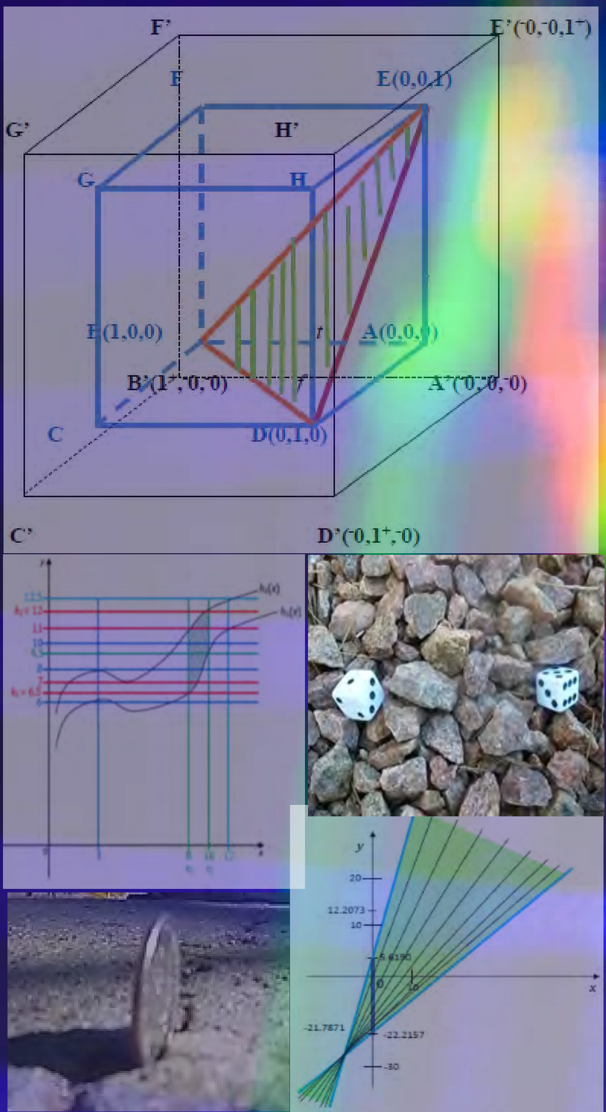

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Fixed Point Theorem of Weak Compatible Maps of Type (γ) in Neutrosophic Metric Space

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Abstract: In this paper, we give definitions of compatible mappings of type (γ) in neutrosophic metric space, and obtain a common fixed point theorem under the conditions of weakly compatible mappings of type (γ) in complete neutrosophic metric spaces. Our research generalizes, extends and improves the results given by Sedghi et al.[19].

Keywords: Fixed point, Neutrosophic metric Space, Compatible Mappings, Weak Compatible Mappings of Type (γ).

1. Introduction :

Fuzzy set was presented by Zadeh [30] as a class of elements with a grade of membership. Kramosil and Michalek [9] defined new notion called Fuzzy Metric Space (FMS). Later on, many authors have examined the concept of fuzzy metric in various aspects. Since then, many authors have obtained fixed point results in fuzzy metric space using these compatible notions. Also, Kutukcu et al.[11] obtained the common fixed points of compatible maps of type(β) on fuzzy metric spaces, and Sedghi et.al.[19] studied the common fixed point of compatible maps of type (γ) in complete fuzzy metric spaces.

Atanassov [1] introduced and studied the notion of intuitionistic fuzzy set by generalizing the notion of fuzzy set. Recently, Park[14] and Park et al. [17] defined the intuitionistic fuzzy metric space. Many authors [15, 16, 17] obtained a fixed point theorems in this space. Also, Park et al. [17] introduced the concept of compatible mappings of type(α) and type(β), and obtained common fixed point theorems in intuitionistic fuzzy metric space.

In 1998, Smarandache [20, 21, 22] characterized the new concept called neutrosophic logic and neutrosophic set and explored many results in it. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. Basset et al. [3] Explored the neutrosophic applications in dif and only iferent fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making. In 2020, Kirisci et al [10] defined NMS as a generalization of IFMS and brings about fixed point theorems in complete NMS. In 2020, Sowndrarajan et al. [23] proved some fixed point results for contraction theorems in neutrosophic metric spaces.

In this paper, we give definitions of compatible mappings of type (γ) in neutrosophic metric space and obtain common fixed point theorem under the conditions of weak compatible mappings of type (γ) in complete neutrosophic metric space.

2. Some Relevent Results:

Definition: 2.1.[18]

A binary operation * : [0, 1] x [0, 1] → [0, 1] is a continuous t-norm [CTN] if it satisfies the following conditions :

1. * is commutative and associative,
2. * is continuous,
3. $\varepsilon_1 * 1 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,
4. $\varepsilon_1 * \varepsilon_2 \le \varepsilon_3 * \varepsilon_4$ whenever $\varepsilon_1 \le \varepsilon_3$ and $\varepsilon_2 \le \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, 1]$.

Definition: 2.2.[18]

A binary operation ⋄ : [0, 1] x [0, 1] → [0, 1] is a continuous t-conorm [CTC] if it satisfies the following conditions:

1. ⋄ is commutative and associative,
2. ⋄ is continuous,
3. $\varepsilon_1 \diamond 0 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,
4. $\varepsilon_1 \diamond \varepsilon_2 \le \varepsilon_3 \diamond \varepsilon_4$ whenever $\varepsilon_1 \le \varepsilon_3$ and $\varepsilon_2 \le \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4 \in [0, 1]$.

Definition: 2.3.[23]

A 6-tuple $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is said to be an Neutrosophic Metric Space (shortly NMS), if Σ is an arbitrary non empty set, $*$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC and Ξ, Θ *and* Υ are neutrosophic on $\Sigma^2 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \delta, \omega \in \Sigma$, $\lambda \in \mathbb{R}^+$.

1. $0 \le \Xi(\zeta, \eta, \lambda) \le 1$; $0 \le \Theta(\zeta, \eta, \lambda) \le 1$; $0 \le \Upsilon(\zeta, \eta, \lambda) \le 1$;
2. $\Xi(\zeta, \eta, \lambda) + \Theta(\zeta, \eta, \lambda) + \Upsilon(\zeta, \eta, \lambda) \le 3$;
3. $\Xi(\zeta, \eta, \lambda) = 1$ if and only if $\zeta = \eta$;
4. $\Xi(\zeta, \eta, \lambda) = \Xi(\eta, \zeta, \lambda)$,
5. $\Xi(\zeta, \eta, \lambda) * \Xi(\eta, \delta, \mu) \le \Xi(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
6. $\Xi(\zeta, \eta, .) : (0, \infty) \to (0, 1]$ is neutrosophic continuous ;
7. $\lim_{\lambda \to \infty} \Xi(\zeta, \eta, \lambda) = 1$ for all $\lambda > 0$;
8. $\Theta(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;
9. $\Theta(\zeta, \eta, \lambda) = \Theta(\eta, \zeta, \lambda)$;
10. $\Theta(\zeta, \eta, \lambda) \diamond \Theta(\eta, \delta, \mu) \ge \Theta(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;

11. $\Theta(\zeta, \eta, .) : (0, \infty) \rightarrow (0, 1]$ is neutrosophic continuous;
12. $\lim_{\lambda\to\infty} \Theta(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;
13. $\Upsilon(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;
14. $\Upsilon(\zeta, \eta, \lambda) = \Upsilon(\eta, \zeta, \lambda)$;
15. $\Upsilon(\zeta, \eta, \lambda) \diamond \Upsilon(\eta, \delta, \mu) \geq \Upsilon(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
16. $\Upsilon(\zeta, \eta, .) : (0, \infty) \rightarrow (0, 1]$ is neutrosophic continuous;
17. $\lim_{\lambda\to\infty} \Upsilon(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;
18. If $\lambda \leq 0$ then $\Xi(\zeta, \eta, \delta, \lambda) = 0$; $\Theta(\zeta, \eta, \delta, \lambda) = 1$; $\Upsilon(\zeta, \eta, \delta, \lambda) = 1$.

Then, (Ξ, Θ, Υ) is called an NMS on Σ. The functions Ξ, Θ and Υ denote degree of closedness, neturalness and non-closedness between ζ and η with respect to λ respectively.

Example: 2.4.[23]

Let (Σ, d) be a metric space. Define $\omega * \tau = \min\{\omega, \tau\}$ and $\omega \diamond \tau = \max\{\omega, \tau\}$ and $\Xi, \Theta, \Upsilon : \Sigma^2 \times \mathbb{R}^+ \rightarrow [0, 1]$ defined by,we define $\Xi(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}$; $\Theta(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda + d(\zeta,\eta)}$; $\Upsilon(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda}$, for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is called NMS induced by a metric d the standard neutrosophic metric.

Definition 2.5.[23]

Let Σ be an NMS. Then Ξ, Θ are said to be continuous on $\Sigma^2 \times \mathbb{R}^+$ if $\lim_{n\to\infty} \Xi(\zeta_n, \eta_n, \lambda_n) = \Xi(\zeta, \eta, \lambda)$; $\lim_{n\to\infty} \Theta(\zeta_n, \eta_n, \lambda_n) = \Theta(\zeta, \eta, \lambda)$; $\lim_{n\to\infty} \Upsilon(\zeta_n, \eta_n, \lambda_n) = \Upsilon(\zeta, \eta, \lambda)$,

Whenever a sequence $\{(\zeta_n, \eta_n, \lambda_n)\} \subset \Sigma^2 \times \mathbb{R}^+$ converges to a point $(\zeta, \eta, \lambda) \in \Sigma^2 \times \mathbb{R}^+$.

Definition 2.6.

Let Γ and Ω be mappings from an NMS Σ into itself. Then the mappings are said to be compatible if $\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1$, $\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$, $\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$, $\forall\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta \in \Sigma$.

Example 2.7.

Let Σ be an NMS, where $\Sigma = [0, 2]$, $*$, $\lozenge$ defined $a * b = \min\{a, b\}$, $a \lozenge b = \max\{a, b\}$ for all $a, b \in [0, 1]$ and $\Xi(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}$; $\Theta(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda + d(\zeta,\eta)}$; $\Upsilon(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda}$, for all $\lambda > 0$ and $\zeta, \eta \in \Sigma$. Define self maps Γ and Ω on Σ as follows:

$$\Gamma\zeta = \begin{cases} 2 \; if \; 0 \leq \zeta \leq 1 \\ \frac{\zeta}{2} \; if \; 1 < \zeta \leq 2 \end{cases} \quad \text{and} \quad \Omega\zeta = \begin{cases} 2 \; if \; \zeta = 1 \\ \frac{\zeta+3}{3} \; otherewise \end{cases}$$

and $\zeta_n = 2 - \frac{1}{2n}$. Then we have $\Omega 1 = 2 = \Gamma 1$ and $\Omega 2 = 1 = \Gamma 2$.

Also, $\Omega\Gamma 2 = \Omega 1 = 2$, $\Gamma\Omega 2 = \Gamma 1 = 2(\Omega\Gamma 2 = \Gamma\Omega 2 = 2)$, thus Γ and Ω are weak compatible.

Also, since $\Gamma\zeta_n = \frac{1}{2}(2 - \frac{1}{2n}) = 1 - \frac{1}{2n}$, $\Omega\zeta_n = \frac{1}{2}(2 - \frac{1}{2n} + 3) = 1 - \frac{1}{10n}$. Thus $\lim_{n\to\infty} \Gamma\zeta_n = 1 = \lim_{n\to\infty} \Omega\zeta_n$.

Furthermore, $\Omega\Gamma\zeta_n = \Omega(1 - \frac{1}{4n}) = \frac{1}{5}(1 - \frac{3}{4n} + 3) = \frac{4}{5} - \frac{1}{20n}$, $\Gamma\Omega\zeta_n = \Gamma(1 - \frac{1}{10n}) = 2$.

Now,

$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = \lim_{n\to\infty} \Xi(2, \frac{4}{5} - \frac{1}{20n}, \lambda) = \frac{5t}{5t+6}$,

$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = \lim_{n\to\infty} \Theta(2, \frac{4}{5} - \frac{1}{20n}, \lambda) = \frac{6}{5t+6}$,

$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = \lim_{n\to\infty} \Upsilon(2, \frac{4}{5} - \frac{1}{20n}, \lambda) = \frac{6}{5t}$, Hence Γ and Ω is not compatible.

3. Weak Compatible mappings of type (γ):

Definition 3.1.

Let Γ and Ω be mappings from an NMS Σ into itself. Then the mappings Γ and Ω are said to be compatible maps of type (γ) if satisfying:

1. Γ and Ω are compatible, that is, $\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1$, $\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$, $\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0, \forall\lambda > 0$.
   Whenever $\{\zeta_n\} \subset \Sigma$ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta \in \Sigma$.
2. They are continuous at ζ. On the other hand, we have
   $\Gamma\zeta = \Gamma(\lim_{n\to\infty} \Gamma\zeta_n) = \Gamma(\lim_{n\to\infty} \Omega\zeta_n) = (\lim_{n\to\infty} \Omega\Gamma\zeta_n) = \Omega(\lim_{n\to\infty} \Gamma\zeta_n) = \Omega\zeta$.

Definition 3.2.

Let Γ and Ω be mappings from an NMS Σ into itself. The mappings Γ and Ω are said to be weak - compatible of type (γ) if $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$ implies that $\Gamma\zeta = \Omega\zeta$.

Remark 3.3.

If self maps Γ and Ω of an NMS Σ are compatible of type(γ), then they are weak compatible type(γ). But the converse is not true.

Lemma 3.5.

Let Σ be an NMS,

1. If we define $E_\alpha : \Sigma^2 \times \mathbb{R}^+$ by
   $E_\alpha(\zeta, \eta) = \inf\{\lambda > 0 ; \Xi(\zeta, \eta, \lambda) > 1-\lambda, \Theta(\zeta, \eta, \lambda) < \lambda \text{ and } \Upsilon(\zeta, \eta, \lambda) < \lambda\}$ for each $\mu \in (0,1)$ there exists $\alpha \in (0,1)$ such that $E_\alpha(\zeta_1, \zeta_n) \leq E_\alpha(\zeta_1, \zeta_2) + E_\alpha(\zeta_2, \zeta_3) + \ldots E_\alpha(\zeta_{n-1}, \zeta_n)$ for any $\zeta_1, \zeta_2 \ldots \zeta_n \in \Sigma$.
2. The sequence $\{\zeta_n\}_{n\in N}$ is convergent in NMS Σ if and only if $E_\alpha(\zeta_n, \zeta) \to 0$.
   Also, the sequence $\{\zeta_n\}_{n\in N}$ is Cauchy sequence if and only if it is Cauchy sequence with E_α.

Lemma 3.6.

Let Σ be an NMS. $\Xi(\zeta_n, \zeta_{n+1}, \lambda) \geq \Xi(\zeta_0, \zeta_1, k^n\lambda)$, $\Theta(\zeta_n, \zeta_{n+1}, \lambda) \leq \Theta(\zeta_0, \zeta_1, k^n\lambda)$ and $\Upsilon(\zeta_n, \zeta_{n+1}, \lambda) \leq \Upsilon(\zeta_0, \zeta_1, k^n\lambda)$ for some $k > 1$ and for every $n \in N$. Then sequence $\{\zeta_n\}$ is a Cauchy sequence.

Lemma 3.7.

Let Σ be an NMS. If there exists a number $k \in (0, 1)$ such that for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. $\Xi(\zeta, \eta, k\lambda) \geq \Xi(\zeta, \eta, \lambda)$, $\Theta(\zeta, \eta, k\lambda) \leq \Theta(\zeta, \eta, \lambda)$ and $\Upsilon(\zeta, \eta, k\lambda) \leq \Upsilon(\zeta, \eta, \lambda)$ then $\zeta = \eta$.

4. Main Results

Lemma 4.1.

Let Γ and Ω are self – mappings of a complete NMS Σ satisfying:
There exists a constant $k \in (0, 1)$ such that

$\Xi^2(\Gamma\zeta, \Omega\eta, k\lambda) * [\Xi(\zeta, \Gamma\zeta, k\lambda)\, \Xi(\eta, \Omega\eta, k\lambda)] * \Xi^2(\eta, \Omega\eta, k\lambda) + a\, \Xi(\eta, \Omega\eta, k\lambda)$
$\Xi(\zeta, \Omega\eta, 2k\lambda) \geq [p\, \Xi(\zeta, \Gamma\zeta, \lambda) + q\, \Xi(\zeta, \eta, \lambda)]\, \Xi(\zeta, \Omega\eta, 2k\lambda)$ (4.1.1)
$\Theta^2(\Gamma\zeta, \Omega\eta, k\lambda) \lozenge [\Theta(\zeta, \Gamma\zeta, k\lambda)\, \Theta(\eta, \Omega\eta, k\lambda)] \lozenge \Theta^2(\eta, \Omega\eta, k\lambda) + a\, \Theta(\eta, \Omega\eta, k\lambda)$
$\Theta(\zeta, \Omega\eta, 2k\lambda) \leq [p\, \Theta(\zeta, \Gamma\zeta, \lambda) + q\, \Theta(\zeta, \eta, \lambda)]\, \Theta(\zeta, \Omega\eta, 2k\lambda)$ (4.1.2)
$\Upsilon^2(\Gamma\zeta, \Omega\eta, k\lambda) \lozenge [\Upsilon(\zeta, \Gamma\zeta, k\lambda)\, \Upsilon(\eta, \Omega\eta, k\lambda)] \lozenge \Upsilon^2(\eta, \Omega\eta, k\lambda) + a\, \Upsilon(\eta, \Omega\eta, k\lambda)$
$\Upsilon(\zeta, \Omega\eta, 2k\lambda) \leq [p\, \Upsilon(\zeta, \Gamma\zeta, \lambda) + q\, \Upsilon(\zeta, \eta, \lambda)]\, \Upsilon(\zeta, \Omega\eta, 2k\lambda)$ (4.1.3)

for every $\zeta, \eta \in \Sigma$ and $\lambda > 0$, where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$. Then Γ and Ω have a unique common fixed point in Σ.

Proof: Let $\zeta_0 \in \Sigma$ be an arbitrary point, there exist $\zeta_1 \in \Sigma$ such that $\Gamma\zeta_0 = \zeta_1$, $\Omega\zeta_1 = \zeta_2$. Inductively, construct the sequences $\{\zeta_n\} \subset \Sigma$ such that $\zeta_{2n+1} = \Gamma\zeta_{2n}$, $\zeta_{2n+2} = \Omega\zeta_{2n+1}$
for n = 0, 1, 2, ... Then we prove that $\{\zeta_n\}$ is a Cauchy sequence.

For $\zeta = \zeta_{2n}$, $\eta = \zeta_{2n+1}$ by we have

$\Xi^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) *[\Xi(\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Xi(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)] * \Xi^2(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)$
$+ a\, \Xi(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Xi(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \geq [p\, \Xi(\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\, \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Xi(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda)$ and
$\Xi^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) *[\Xi(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] * \Xi^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$+ a\, \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)\, \Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \geq [p\, \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda) + q\, \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$,
$\Theta^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \lozenge[\Theta(\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Theta(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)] \lozenge \Theta^2(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)$
$+ a\, \Theta(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Theta(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \leq [p\, \Theta(\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\, \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Theta(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda)$ and
$\Theta^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \lozenge [\Theta(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] \lozenge \Theta^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$+ a\, \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)\, \Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq [p\, \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda) + q\, \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$.
$\Upsilon^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \lozenge[\Upsilon(\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Upsilon(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)] \lozenge \Upsilon^2(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)$
$+ a\, \Upsilon(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Upsilon(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \leq [p\, \Upsilon(\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\, \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Upsilon(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda)$ and
$\Upsilon^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \lozenge [\Upsilon(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] \lozenge \Upsilon^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$+ a\, \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)\, \Upsilon(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq [p\, \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda) + q\, \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Upsilon(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$.

Then

$\Xi^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) *[\Xi(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] + a\, \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$\Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \geq (p + q)\, \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)\, \Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$,
$\Theta^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \lozenge[\Theta(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] + a\, \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$\Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq (p + q)\, \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)\, \Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$,
$\Upsilon^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \lozenge[\Upsilon(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] + a\, \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$\Upsilon(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq (p + q)\, \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda)\, \Upsilon(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$.

So,

$\Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) + a\, \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \geq (p + q)\, \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)$
$\Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) + a\, \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq (p + q)\, \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)$
$\Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) + a\, \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq (p + q)\, \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda)$.

Therefore

$\Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \geq \Xi(\zeta_{2n}\, \zeta_{2n+1}, \lambda)$, $\Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)$ and
$\Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda)$.

Similarly, we also have

$\Xi(\zeta_{2n+2}, \zeta_{2n+3}, k\lambda) \geq \Xi(\zeta_{2n+1}, \zeta_{2n+2}, \lambda)$, $\Theta(\zeta_{2n+2}, \zeta_{2n+3}, k\lambda) \leq \Theta(\zeta_{2n+1}, \zeta_{2n+2}, \lambda)$,

$\Upsilon(\zeta_{2n+2}, \zeta_{2n+3}, k\lambda) \leq \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, \lambda)$.

For $k \in (0, 1)$ if $k_1 = \frac{1}{k} > 1$ and $\lambda = k_1 \lambda_1$, then we have

$\Xi(\zeta_n, \zeta_{n+1}, \lambda) \geq \Xi(\zeta_0, \zeta_1, k_1^n \lambda_1)$, $\Theta(\zeta_n, \zeta_{n+1}, \lambda) \leq \Theta(\zeta_0, \zeta_1, k_1^n \lambda_1)$and

$\Upsilon(\zeta_n, \zeta_{n+1}, \lambda) \leq \Upsilon(\zeta_0, \zeta_1, k_1^n \lambda_1)$.

By Lemma 3.6, since $\{\zeta_n\}$ is a Cauchy sequence in Σ which is complete, $\{\zeta_n\}$ converges to ω in Σ. Hence $\lim_{n\to\infty} \Gamma\zeta_{2n} = \lim_{n\to\infty} \zeta_{2n+1} = \lim_{n\to\infty} \zeta_{2n+2} = \lim_{n\to\infty} \Omega\zeta_{2n+1} = \omega$.

Now, taking $\zeta = \omega$ and $\eta = \zeta_{2n+1}$ in (i), we have as $n\to\infty$,

$$\Xi^2(\Gamma\omega, \omega, k\lambda) * [\Xi(\omega, \Gamma\omega, k\lambda)\, \Xi(\omega, \omega, k\lambda)] * \Xi^2(\omega, \omega, k\lambda) + a\, \Xi(\omega, \omega, k\lambda)\, \Xi(\omega, \omega, 2k\lambda)$$
$$\geq [p\Xi(\omega, \Gamma\omega, \lambda) + q\Xi(\omega, \omega, \lambda)]\, \Xi(\omega, \omega, 2k\lambda),$$

$$\Theta^2(\Gamma\omega, \omega, k\lambda) * [\Theta(\omega, \Gamma\omega, k\lambda)\, \Theta(\omega, \omega, k\lambda)] \lozenge \Theta^2(\omega, \omega, k\lambda) + a\, \Theta(\omega, \omega, k\lambda)\, \Theta(\omega,\omega, 2k\lambda)$$
$$\leq [p\, \Theta(\omega, \Gamma\omega, \lambda) + q\, \Theta(\omega, \omega, \lambda)]\, \Theta(\omega, \omega, 2k\lambda),$$

$$\Upsilon^2(\Gamma\omega, \omega, k\lambda) * [\Upsilon(\omega, \Gamma\omega, k\lambda)\, \Upsilon(\omega, \omega, k\lambda)] \lozenge \Upsilon^2(\omega, \omega, k\lambda) + a\, \Upsilon(\omega, \omega, k\lambda)\, \Theta(\omega,\omega, 2k\lambda)$$
$$\leq [p\, \Upsilon(\omega, \Gamma\omega, \lambda) + q\, \Upsilon(\omega, \omega, \lambda)]\, \Upsilon(\omega, \omega, 2k\lambda).$$

Therefore

$\Xi(\Gamma\omega, \omega, k\lambda) + a \geq p\Xi(\omega, \Gamma\omega, \lambda) + q$, $\Theta(\Gamma\omega, \omega, k\lambda) \leq 0$, $\Upsilon(\Gamma\omega, \omega, k\lambda) \leq 0$,

for all $\lambda > 0$, so $\Gamma\omega = \omega$. Taking $\zeta = \zeta_{2n}$ and $\eta = \omega$ in (i), we have as $n \to \infty$,

$\Xi(\omega, \Omega\omega, \lambda) + a \geq p + q$, $\Theta(\omega, \Omega\omega, \lambda) + a\, \Theta(\omega, \Omega\omega, \lambda) \leq 0$ and $\Upsilon(\omega, \Omega\omega, \lambda) + a\, \Upsilon(\omega, \Omega\omega, \lambda) \leq 0$,

for all $\lambda > 0$, so $\Omega\omega = \omega$. Thus ω is a common fixed point of Γ and Ω,

Let β be another common fixed point of Γ and Ω. Then using (i), we have

$\Xi^2(\omega, \beta, k\lambda) + a\, \Xi(\omega, \beta, 2k\lambda) \geq [p + q\, \Xi(\omega, \beta, \lambda)]\, \Xi(\omega, \beta, 2k\lambda)$,

$\Theta^2(\omega, \beta, k\lambda) \leq q\, \Theta(\omega, \beta, \lambda)\, \Theta(\omega, \beta, 2k\lambda)$ and $\Upsilon^2(\omega, \beta, k\lambda) \leq q\, \Upsilon(\omega, \beta, \lambda)\, \Upsilon(\omega, \beta, 2k\lambda)$

and

$\Xi(\omega, \beta, \lambda)\, \Xi(\omega, \beta, 2k\lambda) + a\Xi(\omega, \beta, 2k\lambda) \geq [p + q\, \Xi(\omega, \beta, \lambda)]\Xi(\omega, \beta, 2k\lambda)$,

$\Theta(\omega, \beta, \lambda)\, \Theta(\omega, \beta, 2k\lambda) \leq q\, \Theta(\omega, \beta, \lambda)\, \Theta(\omega, \beta, 2k\lambda)$,

$\Upsilon(\omega, \beta, \lambda)\, \Upsilon(\omega, \beta, 2k\lambda) \leq q\, \Upsilon(\omega, \beta, \lambda)\, \Upsilon(\omega, \beta, 2k\lambda)$.

Thus, it follows that

$\Xi(\omega, \beta, \lambda) \geq \frac{p-a}{1-q} = 1$, $\Theta(n, \beta, \lambda) \leq 0$, $\Upsilon(n, \beta, \lambda) \leq 0$,

for all $\lambda > 0$, so $\omega = \beta$. Hence Γ and Ω have a unique common fixed point in Σ.

Theorem 4.2.

Let Γ, Ω, Λ and $\vee$ be self mappings of a complete NMS Σ satisfying

1. $\Gamma(\Sigma) \subseteq \vee(\Sigma)$, $\Omega(\Sigma) \subseteq \Lambda(\Sigma)$,
2. There exists a constant $k \in (0, 1)$ such that

$$\Xi^2(\Gamma\zeta, \Omega\eta, k\lambda) * [\Xi(\Lambda\zeta, \Gamma\zeta, k\lambda)\, \Xi(\vee\eta, \Omega\eta, k\lambda)] * \Xi^2(\vee\eta, \Omega\eta, k\lambda) + a\Xi(\vee\eta, \Omega\eta, k\lambda)$$
$$\Xi(\Lambda\zeta, \Omega\eta, 2k\lambda) \geq [p\Xi(\Lambda\zeta, \Gamma\zeta, \lambda) + q\Xi(\Lambda\zeta, \vee\eta, \lambda)]\, \Xi(\Lambda\zeta, \Omega\eta, 2k\lambda) \quad (4.2.1)$$

$$\Theta^2(\Gamma\zeta, \Omega\eta, k\lambda) \lozenge [\Theta(\Lambda\zeta, \Gamma\zeta, k\lambda)\, \Theta(\vee\eta, \Omega\eta, k\lambda)] \lozenge \Theta^2(\vee\eta, \Omega\eta, k\lambda) + a\, \Theta(\vee\eta, \Omega\eta, k\lambda)$$
$$\Theta(\Lambda\zeta, \Omega\eta, 2k\lambda) \leq [p\, \Theta(\Lambda\zeta, \Gamma\zeta, \lambda) + q\, \Theta(\Lambda\zeta, \vee\eta, \lambda)]\, \Theta(\Lambda\zeta, \Omega\eta, 2k\lambda) \quad (4.2.2)$$

$$\Upsilon^2(\Gamma\zeta, \Omega\eta, k\lambda) \lozenge [\Upsilon(\Lambda\zeta, \Gamma\zeta, k\lambda)\, \Upsilon(\vee\eta, \Omega\eta, k\lambda)] \lozenge \Upsilon^2(\vee\eta, \Omega\eta, k\lambda) + a\, \Upsilon(\vee\eta, \Omega\eta, k\lambda)$$
$$\Upsilon(\Lambda\zeta, \Omega\eta, 2k\lambda) \leq [p\, \Upsilon(\Lambda\zeta, \Gamma\zeta, \lambda) + q\, \Upsilon(\Lambda\zeta, \vee\eta, \lambda)]\, \Upsilon(\Lambda\zeta, \Omega\eta, 2k\lambda) \quad (4.2.3)$$

   for every $\zeta, \eta \in \Sigma$ and $\lambda > 0$, where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$,
3. The pairs (Γ, Λ) and $(\Omega, \vee)$ are weak compatible of type (γ).

Then Γ, Ω, Λ and $\vee$ have a unique common fixed point in Σ.

Proof: Let $\zeta_0 \in \Sigma$ be an arbitrary point. Since $\Gamma(\Sigma) \subseteq V(\Sigma)$ and $\Omega(\Sigma) \subseteq \Lambda(\Sigma)$, there exists $\zeta_1, \zeta_2 \in \Sigma$ such that $\Gamma\zeta_0 = V\zeta_1 = \eta_1$, $\Omega\zeta_1 = \Lambda\zeta_2 = \eta_2$. Because we can construct the sequences $\{\zeta_n\}$, $\{\eta_n\} \subset \Sigma$ such that $\eta_{2n+1} = \Gamma\zeta_{2n} = V\zeta_{2n+1}$, $\eta_{2n+2} = \Omega\zeta_{2n+1} = \Lambda\zeta_{2n+2}$, for n = 0, 1, 2, . . . , we prove $\{\eta_n\}$ is Cauchy sequence.

For $\zeta = \zeta_{2n}$, $\eta = \zeta_{2n+1}$ by (ii), we have

$$\Xi^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) * [\Xi(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Xi(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) * \Xi^2(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) + a\, \Xi(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Xi(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \geq [p\Xi(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\Xi(\Lambda\zeta_{2n}, V\zeta_{2n+1}, \lambda)] \times \Xi(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda),$$

$$\Theta^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \Diamond [\Theta(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Theta(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \Diamond \Theta^2(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) + a\, \Theta(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Theta(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \leq [p\,\Theta(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\,\Theta(\Lambda\zeta_{2n}, V\zeta_{2n+1}, \lambda)] \times \Theta(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda),$$

$$\Upsilon^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \Diamond [\Upsilon(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Upsilon(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \Diamond \Upsilon^2(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) + a\, \Upsilon(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Upsilon(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \leq [p\,\Upsilon(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\,\Upsilon(\Lambda\zeta_{2n}, V\zeta_{2n+1}, \lambda)] \times \Upsilon(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda).$$

Hence

$$\Xi(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Xi(\eta_{2n}, \eta_{2n+2}, 2k\lambda) + a\Xi(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Xi(\eta_{2n}, \eta_{2n+2}, 2k\lambda) \geq (p + q)\, \Xi(\eta_{2n}, \eta_{2n+1}, \lambda)\, \Xi(\eta_{2n}, \eta_{2n+2}, 2k\lambda),$$

$$\Theta(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Theta(\eta_{2n}, \eta_{2n+2}, 2k\lambda) + a\, \Theta(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Theta(\eta_{2n}, \eta_{2n+2}, 2k\lambda) \leq (p + q)\, \Theta(\eta_{2n}, \eta_{2n+1}, \lambda)\, \Theta(\eta_{2n}, \eta_{2n+2}, 2k\lambda),$$

$$\Upsilon(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Upsilon(\eta_{2n}, \eta_{2n+2}, 2k\lambda) + a\, \Upsilon(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Upsilon(\eta_{2n}, \eta_{2n+2}, 2k\lambda) \leq (p + q)\, \Upsilon(\eta_{2n}, \eta_{2n+1}, \lambda)\, \Upsilon(\eta_{2n}, \eta_{2n+2}, 2k\lambda).$$

So, we have

$\Xi(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \geq \Xi(\eta_{2n}, \eta_{2n+1}, \lambda)$, $\Theta(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \leq \Theta(\eta_{2n}, \eta_{2n+1}, \lambda)$ and $\Upsilon(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \leq \Upsilon(\eta_{2n}, \eta_{2n+1}, \lambda)$.

Similarly, also we have

$\Xi(\eta_{2n+2}, \eta_{2n+3}, k\lambda) \geq \Xi(\eta_{2n+1}, \eta_{2n+2}, \lambda)$, $\Theta(\eta_{2n+2}, \eta_{2n+3}, k\lambda) \leq \Theta(\eta_{2n+1}, \eta_{2n+2}, \lambda)$,

$\Upsilon(\eta_{2n+2}, \eta_{2n+3}, k\lambda) \leq \Upsilon(\eta_{2n+1}, \eta_{2n+2}, \lambda)$, for $k \in (0, 1)$, if $k_1 = \frac{1}{k} > 1$ and $\lambda = k_1 \lambda_1$, then

$\Xi(\eta_n, \eta_{n+1}, \lambda) \geq \Xi(\eta_{n-1}, \eta_n, k_1 \lambda_1) \geq \ldots \geq \Xi(\eta_0, \eta_1, k_1^n \lambda_1)$,

$\Theta(\eta_n, \eta_{n+1}, \lambda) \leq \Theta(\eta_{n-1}, \eta_n, k_1, \lambda_1) \leq \ldots \leq \Theta(\eta_0, \eta_1, k_1^n \lambda_1)$,

$\Upsilon(\eta_n, \eta_{n+1}, \lambda) \leq \Upsilon(\eta_{n-1}, \eta_n, k_1, \lambda_1) \leq \ldots \leq \Upsilon(\eta_0, \eta_1, k_1^n \lambda_1)$.

Thus $\{\eta_n\}$ is a Cauchy sequence and completeness of Σ, $\{\eta_n\}$ converges to $\omega \in \Sigma$.

Hence

$$\lim_{n\to\infty} \Gamma\zeta_{2n} = \lim_{n\to\infty} \eta_{2n+1} = \lim_{n\to\infty} V\zeta_{2n+1} = \lim_{n\to\infty} \eta_{2n+2} = \lim_{n\to\infty} \Omega\zeta_{2n+1} = \lim_{n\to\infty} \Lambda\zeta_{2n+2} = \lim_{n\to\infty} \Lambda\zeta_{2n} = \omega.$$

Since Γ, Λ are weak compatible of type (γ), $A\omega = \Lambda\omega$.

Now, taking $\zeta = \omega$ and $\eta = \zeta_{2n+1}$ in (ii), we have as $n \to \infty$.

$$\Xi^2(\Gamma\omega, \omega, k\lambda) * [\Xi(\Lambda\omega, \Gamma\omega, k\lambda)\, \Xi(\omega, \omega, k\lambda)] * \Xi^2(\omega, \omega, k\lambda) + a\Xi(\omega, \omega, k\lambda)\, \Xi(\Lambda\omega, \omega, 2k\lambda) \geq [p\Xi(\Lambda\omega, \Gamma\omega, \lambda) + q\Xi(\Lambda\omega, \omega, \lambda)]\Xi(\Lambda\omega, \omega, 2k\lambda),$$

$$\Theta^2(\Gamma\omega, \omega, k\lambda) \Diamond [\Theta(\Lambda\omega, \Gamma\omega, k\lambda)\, \Theta(\omega, \omega, k\lambda)] \Diamond \Theta^2(\omega, \omega, k\lambda) + a\, \Theta(\omega, \omega, k\lambda)\, \Theta(\Lambda\omega, \omega, 2k\lambda) \leq [p\, \Theta(\Lambda\omega, \Gamma\omega, \lambda) + q\, \Theta(\Lambda\omega, \omega, \lambda)]\, \Theta(\Lambda\omega, \omega, 2k\lambda),$$

$$\Upsilon^2(\Gamma\omega, \omega, k\lambda) \Diamond [\Upsilon(\Lambda\omega, \Gamma\omega, k\lambda)\, \Upsilon(\omega, \omega, k\lambda)] \Diamond \Upsilon^2(\omega, \omega, k\lambda) + a\, \Upsilon(\omega, \omega, k\lambda)\, \Upsilon(\Lambda\omega, \omega, 2k\lambda) \leq [p\, \Upsilon(\Lambda\omega, \Gamma\omega, \lambda) + q\, \Upsilon(\Lambda\omega, \omega, \lambda)]\, \Upsilon(\Lambda\omega, \omega, 2k\lambda).$$

It follows that

$\Xi^2(\Gamma\omega, \omega, k\lambda) + a\Xi(\Gamma\omega, \omega, 2k\lambda) \geq [p + q\Xi(\Gamma\omega, \omega, \lambda)]\Xi(\Gamma\omega, \omega, 2k\lambda)$,

$\Theta^2(\Gamma\omega, \omega, k\lambda) \leq q\, \Theta(\Gamma\omega, \omega, \lambda)\, \Theta(\Gamma\omega, \omega, 2k\lambda)$, $\Upsilon^2(\Gamma\omega, \omega, k\lambda) \leq q\, \Upsilon(\Gamma\omega, \omega, \lambda)\, \Upsilon(\Gamma\omega, \omega, 2k\lambda)$.

Since Ξ(ζ, η, .) is nondecreasing , Θ (ζ ,η, .) is nonincreasing and ϒ (ζ, η, .) is nonincreasing for all ζ , η ∈Σ, we have

$$\Xi(\Gamma\omega, \omega, \lambda) \geq \frac{p-a}{1-q} = 1,\ \Theta(\Gamma\omega, \omega, \lambda) \leq \frac{0}{1-q} = 0,\ \Upsilon(\Gamma\omega, \omega, \lambda) \leq \frac{0}{1-q} = 0.$$

for all λ > 0, So Γω = ω. Hence Γω = ∧ω = ω.

Similarly, since Ω, V are weak compatible of type (γ), we get Ωω = Vω.

For taking $\zeta = \zeta_{2n}$ and η = ω in (ii), we have as n → ∞,

$$\Xi^2(\omega, \Omega\omega, k\lambda) * [\Xi(\omega, \omega, k\lambda)\, \Xi(V\omega, \Omega\omega, k\lambda)] * \Xi^2(V\omega, \Omega\omega, k\lambda) + a\Xi(V\omega, \Omega\omega, k\lambda)$$
$$\Xi(\omega, \Omega\omega, 2k\lambda) \geq [p\Xi(\omega, \omega, \lambda) + q\Xi(\omega, V\omega, \lambda)]\, \Xi(\omega, \Omega\omega, 2k\lambda),$$

$$\Theta^2(\omega, \Omega\omega, k\lambda) \diamond [\Theta(\omega, \omega, k\lambda)\, \Theta(V\omega, \Omega\omega, k\lambda)] \diamond \Theta^2(V\omega, \Omega\omega, k\lambda) + a\,\Theta(V\omega, \Omega\omega, k\lambda)$$
$$\Theta(\omega, \Omega\omega, 2k\lambda) \leq [p\,\Theta(\omega, \omega, \lambda) + q\,\Theta(\omega, V\omega, \lambda)]\, \Theta(\omega, \Omega\omega, 2k\lambda),$$

$$\Upsilon^2(\omega, \Omega\omega, k\lambda) \diamond [\Upsilon(\omega, \omega, k\lambda)\, \Upsilon(V\omega, \Omega\omega, k\lambda)] \diamond \Upsilon^2(V\omega, \Omega\omega, k\lambda) + a\,\Upsilon(V\omega, \Omega\omega, k\lambda)$$
$$\Upsilon(\omega, \Omega\omega, 2k\lambda) \leq [p\,\Upsilon(\omega, \omega, \lambda) + q\,\Upsilon(\omega, V\omega, \lambda)]\, \Upsilon(\omega, \Omega\omega, 2k\lambda).$$

Then,

$$\Xi^2(\omega, \Omega\omega, k\lambda) + a\Xi(\omega, \Omega\omega, 2k\lambda) \geq [p+q\Xi(\omega, V\omega, \lambda)]\, \Xi(\omega, \Omega\omega, 2k\lambda),$$
$$\Theta^2(\omega, \Omega\omega, k\lambda) \leq q\,\Theta(\omega, V\omega, \lambda)\, \Theta(\omega, \Omega\omega, 2k\lambda),\ \Upsilon^2(\omega, \Omega\omega, k\lambda) \leq q\,\Upsilon(\omega, V\omega, \lambda)\, \Upsilon(\omega, \Omega\omega, 2k\lambda).$$

Thus it follows that

$$\Xi(\omega, \Omega\omega, \lambda) \geq \frac{p-a}{1-q} = 1,\ \Theta(\omega, \Omega\omega, \lambda) \leq \frac{0}{1-q} = 0 \text{ and } \Upsilon(\omega, \Omega\omega, \lambda) \leq 0, \text{ for all } \lambda > 0, \text{ so } \Omega\omega = \omega.$$

Hence Ωω = Vω = ω.

Therefore ω is a common fixed point of Γ, Ω, ∧ and V.

Let β be another common fixed point of Γ, Ω, ∧ and V. Then we have

$$\Xi^2(\Gamma\omega, \Omega\beta, k\lambda) * [\Xi(\wedge\omega, \Gamma\omega, k\lambda)\, \Xi(V\beta, \Omega\beta, k\lambda)] * \Xi^2(V\beta, \Omega\beta, k\lambda)$$
$$+ a\Xi(V\beta, \Omega\beta, k\lambda)\, \Xi(\wedge\beta, \Omega\beta, 2k\lambda) \geq [p\Xi(\wedge\omega, \Gamma\omega, \lambda) + q\Xi(\wedge\omega, V\beta, \lambda)]\Xi(\wedge\omega, \Omega\beta, 2k\lambda),$$

$$\Theta^2(\Gamma\omega, \Omega\beta, k\lambda) \diamond [\Theta(\wedge\omega, \Gamma\omega, k\lambda)\, \Theta(V\beta, \Omega\beta, k\lambda)] \diamond \Theta^2(V\beta, \Omega\beta, k\lambda)$$
$$+ a\,\Theta(V\beta, \Omega\beta, k\lambda)\, \Theta(\wedge\beta, \Omega\beta, 2k\lambda) \leq [p\,\Theta(\wedge\omega, \Gamma\omega, \lambda) + q\,\Theta(\wedge\omega, V\beta, \lambda)]\, \Theta(\wedge\omega, \Omega\beta, 2k\lambda),$$

$$\Upsilon^2(\Gamma\omega, \Omega\beta, k\lambda) \diamond [\Upsilon(\wedge\omega, \Gamma\omega, k\lambda)\, \Upsilon(V\beta, \Omega\beta, k\lambda)] \diamond \Upsilon^2(V\beta, \Omega\beta, k\lambda)$$
$$+ a\,\Upsilon(V\beta, \Omega\beta, k\lambda)\, \Upsilon(\wedge\beta, \Omega\beta, 2k\lambda) \leq [p\,\Upsilon(\wedge\omega, \Gamma\omega, \lambda) + q\,\Upsilon(\wedge\omega, V\beta, \lambda)]\, \Upsilon(\wedge\omega, \Omega\beta, 2k\lambda),$$

So,

$$\Xi^2(\omega, \beta, k\lambda) + a\Xi(\omega, \beta, 2k\lambda) \geq [p + q\,\Xi(\omega, \beta, \lambda)]\Xi(\omega, \beta, 2k\lambda),$$
$$\Theta^2(\omega, \beta, k\lambda) \leq q\,\Theta(\omega, \beta, \lambda)\, \Theta(\omega, \beta, 2k\lambda) \text{ and } \Upsilon^2(\omega, \beta, k\lambda) \leq q\,\Upsilon(\omega, \beta, \lambda)\, \Upsilon(\omega, \beta, 2k\lambda).$$

Therefore

$$\Xi(\omega, \beta, \lambda) \geq \frac{p-a}{1-q} = 1,\ \Theta(\omega, \beta, \lambda) \leq \frac{0}{1-q} = 0,\ \Upsilon(\omega, \beta, \lambda) \leq 0,$$

for all λ > 0, so ω = β , hence Γ, Ω, ∧ and V have unique common fixed point on Σ.

Example 4.3.

Let (Σ, d) be a metric space with Σ = [0,1]. Denote $\omega * \tau = \min\{\omega, \tau\}$ and $\omega \diamond \tau = \max\{\omega, \tau\}$ for $\omega, \tau \in [0,1]$ and let Ξ_d, Θ_d, Υ_d be neutroshopic sets on $\Sigma^2 \times [0, \infty]$ defined as follows;

$$\Xi_d(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)};\ \Theta_d(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda + d(\zeta,\eta)};\ \Upsilon_d(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda}.$$

Then $(\Xi_d, \Theta_d, \Upsilon_d)$ is an NMS on Σ and $(\Sigma, \Xi_d, \Theta_d, \Upsilon_d, *, \diamond)$ is an NMS.

Define self mappings Γ, Ω, ∧ and V by

$$\Gamma(\Sigma) = 1;\ \Omega(\Sigma) = 1;\ \wedge(\Sigma) = \begin{cases} 1 \text{ if } \zeta \text{ is rational} \\ 0 \text{ if } \zeta \text{ is irrational} \end{cases};\ V(\Sigma) = \frac{\zeta+1}{2}$$

If we define $\{\zeta_n\} \subset \Sigma$ by $\zeta_n = 1 - \frac{1}{n}$, then we have for $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 1$ and Γ1 = 1 = ∧1.

$\lim_{n\to\infty} \Xi (\wedge \Gamma\zeta_n, 1, \lambda) \leq \Xi(\Gamma 1, 1, \lambda) = 1$; $\lim_{n\to\infty} \Theta (\wedge \Gamma\zeta_n, 1, \lambda) \geq \Theta(\Gamma 1, 1, \lambda) = 0$;
$\lim_{n\to\infty} \Upsilon (\wedge \Gamma\zeta_n, 1, \lambda) \geq \Upsilon(\Gamma 1, 1, \lambda) = 0$.

Also, for $\lim_{n\to\infty} \Omega\zeta_n = \lim_{n\to\infty} \vee \zeta_n = 1$ and $\Omega 1 = 1 = \vee 1$.

$\lim_{n\to\infty} \Xi (\vee\Omega\zeta_n, 1, \lambda) \leq \Xi(\Omega 1, 1, \lambda) = 1$; $\lim_{n\to\infty} \Theta (\vee\Omega\zeta_n, 1, \lambda) \geq \Theta(\Omega 1, 1, \lambda) = 0$;
$\lim_{n\to\infty} \Upsilon (\vee\Omega\zeta_n, 1, \lambda) \geq \Upsilon(\Omega 1, 1, \lambda) = 0$.

Therefore, (Γ, $\wedge$) and (Ω, $\vee$) are weak compatible of type (γ). Then all the conditions of Theorem 4.2. are satisfied and 1 is a unique common fixed point of Γ, Ω, $\wedge$ and $\vee$ on Σ.

Conclusion: In this study, we have made common fixed point results for weak compatiple maps of type in neutrosophic metric Space. There is a degree to set up many fixed point brings about the spaces like fuzzy metric, generalized fuzzy metric, bipolar and partial fuzzy metric spaces by utilizing the idea of Neutrosophic Set**.**

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An Intelligent Traffic Control System Using Neutrosophic Sets, Rough sets, Graph Theory, Fuzzy sets and its Extended Approach: A Literature Review

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Abstract: Recently, the intelligent traffic control system and its uncertainty analysis are considered one of the hot spots for utilizing the available techniques. It became more essential when the automatic car, electric vehicle, and other smart cars have introduced the transportation. To control the traffic accident and smooth road services an intelligent traffic control system required. It will be also useful in decreasing the time, reaction time, and efficiency of traffic. However, the problem arises while characterization of true, false or uncertain regions of traffic flow and its future approximation. To deal with this issue some available mathematical technique for traffic flow using rough set, fuzzy rough set, and its extension with the neutrosophic set is discussed in this paper. Some of the papers related to graphical visualization of traffic flow is also discussed for further improvement. The rough set theory can be useful for dealing the uncertain, incomplete, and indeterminate data set. Hence, the hybridization of the neutrosophic set and rough can be considered one of the efficient tools for intelligent traffic control and its approximation via automatic red, green and yellow lights. This paper tried to provide an overview of each available technique to solve the traffic problem. It is hoped that the proposed study will be helpful for several researchers working on traffic flow, traffic accident diagnosis, and its hybridization as future research.

Keywords: Neutrosophic Set ; Rough set ; Fuzzy Set ; Graph theory ; Intelligent Transportation System ; Uncertainty; Urban traffic ; Traffic Flow

1. Introduction

Recently, the urban traffic control system and its analysis have attracted the attention of various researchers. The reason is many electric, automatic, and other smart vehicles are proposed for transportation services. To control the urban traffic beyond red, green, and yellow light signals many traffic management systems have been developed over time. One of the reasons is in the case of the human driver the traffic control is based on Human Turiyam cognition rather than red, green, or yellow light as discussed by Singh (2021). The problem arises when the automatic car needs to be aware of when to stop, when to start and when to slow as the car does not has awareness. It becomes more crucial in the case of large towns and cities. The computerized traffic signal controls, which are known as Urban Traffic Control (UTC), also have some limitations. Hence, the fuzzy set theory, Rough Set Theory, fuzzy-rough set theory, and neutrosophic set theory, and others can be used to synchronize traffic control in crowded metropolitan networks. The reason is these types of data may contain a heteroclinic pattern as discussed by Singh (2022). Hence maximizing vehicle throughput is no longer the only goal of traffic control. The balance demand and flow with extra consideration namely lane assignment, parking limits, turning bans, one-way street systems, and tidal flow schemes. They can be constructed to provide deliberate traffic constraints, such as by prioritizing buses over other vehicles or implementing queue management procedures and deliberate area entry control. These advancements provide traffic engineers and network controllers with the tools they need to implement a highly adaptive type of urban traffic management - one that responds to transportation policies and management priorities, as well as the public's and local politicians' acceptance of them (Wang 2013, Chen et 2014). It is one of the major issues as the public and local politicians acceptation and Turiyam cognition contains lots of uncertainty in the word. Computing with these types of words for precise management of traffic control is one of the major tasks for data science researchers.

The mathematical computation of uncertainty and its analysis is one of the crucial tasks for data science researchers. To achieve this goal, Prof. Zadeh proposed fuzzy set theory in 1965 as an expansion of the classical notion of a set (Zadeh, 1965) as an alternative of probability. With the proposed methodology, Zadeh established a mathematical framework that allows for decision-making based on fuzzy representations of some data. Uncertainty, subjectivity, imprecision, and ambiguity can be found in à wide range of traffic and transportation factors. As a result, mathematical approaches that can deal successfully with uncertainty, ambiguity, and subjectivity must be utilized in the mathematical modeling phase of traffic and transportation processes whose individual parameters are unclear, ambiguous, or subjectively evaluated. Fuzzy set theory is a useful mathematical tool for dealing with indeterminacy, subjectivity, and ambiguity. A fuzzy set is a collection of elements that fits the membership degree of a set. For example, suppose there are two fuzzy sets that represent two categories of people: old and young. As a result, the higher a person's age, the higher his or her membership degree among the elderly, and the lower his or her membership degree among the youth. Calculating the gradual indiscernibility connection in large datasets with many items is difficult in terms of memory and runtime. RST is a revolutionary mathematics technique for dealing with uncertain and inexact knowledge in a variety of real-world applications such as data mining, medicine, and information analysis. Rough set theory is used to analyses and handle data (Wang et al. 2009). Z. Pawlak, a Polish mathematician, initially presented it in 1982 to find the underlying laws of data. It's very useful for dealing with irregularities in information systems. In order to manage data with continuous qualities and find inconsistencies in the data, fuzzy rough set theory can be used with rough set theory. The fuzzy-rough set model has shown to be beneficial in a variety of applications because it is a potent tool for analyzing inconsistent and ambiguous data. RST is concerned with data that is inconsistent, such as two patients with the same symptoms but different diagnoses. Data is intended to be ambiguous in the rough set analysis. As a result, it's necessary to discretize a continuous numerical property. The fuzzy-rough set theory (FRST) is a continuous numerical attribute extension of the rough set theory. It can handle both numerical and discrete data and can address the same problems that a rough set can. The value of FRST can be observed in a variety of applications. The FRST is built on the foundations of two theories

: rough set theory and fuzzy set theory. The two important and mutually orthogonal aspects of faulty data and knowledge are addressed by fuzzy sets and rough sets. While the former allows things to belong to a collection or relation to a certain extent, the latter provides approximations of concepts in the face of incomplete data. The primary goal of fuzzy-rough sets is to define lower and higher approximations of the set when the universe of a fuzzy set turns rough due to equivalence or to transfer the equivalence relation to a similar fuzzy relation. Fuzzy approximations of a fuzzy set in a crisp approximation space are called rough fuzzy sets and fuzzy approximations of a crisp set in a fuzzy approximation space are called fuzzy rough sets (Shao 2015) and their applications (Weng et 2007 ; Chai 2015). The problem arises when data sets contain hesitant parts as an independent value. To represent this type of indeterminacy 3D-Neutrosophic set is introduced by the Smarandache (2010, 2021), with each dimension representing the statement's truth (T), indeterminacy (I), and falsity (F). These functions are unrelated, and the total of their parts does not equal one. It should add up to 3 in the meantime. To deal with ambiguity, many approaches have been devised. Starting with Fuzzy logic (Xiong et al. 2021), which depicts the concept of "partial truth" as the true value ranges between 0 and 1, depending on whether it is wholly false or completely true.

Meanwhile, the researchers proposed interval-valued sets to allow interval membership values within the same set because fuzzy logic had several downsides. An intuitionistic fuzzy set was then created as a generalization of traditional fuzzy sets. Each element has a degree of membership and even non-membership in an intuitionistic fuzzy set (Thakur, 2014). Meanwhile, it had flaws, prompting some scholars to suggest a neutrosophic set of rules. Information is often ambiguous and imprecise in the fields of safety, reliability, risk analysis, and management.

When some barriers against accidents fail to achieve their aim, the "severe occurrence" is frequently an extremely deadly event. They are invaluable resources for information on air transportation safety assurance systems. With such research at hand, it is possible to determine if current safety measures are adequate or whether they need to be improved. Estimation of safety barrier reliability must be carried out in order to evaluate this likelihood.

Unfortunately, most of the time there isn't enough evidence to make statistical inferences about the frequency of events in an accident scenario. Regrettably, finding that information is extremely improbable. The condition is caused by two factors. The first is that some of these events occur seldom, and in the past, events with minor implications were not routinely reported. The second aspect is the human factor, which includes difficult-to-quantify indicators like differing reaction probabilities and mistake activity probability. Uncertainty and subjective judgments are present in such metrics. Expert estimations are the only way to get such information. These estimates aren't exact enough to be used in probabilistic analysis. Information is frequently ambiguous and imprecise in the areas of safety, reliability, risk analysis, and management.

Recently, uncertainty and its characterization is considered one of the major issues. To deal with this issue neutrosophic set and its metric is used for the characterization of data beyond acceptation, rejection, and uncertain part, independently. A parallel rough set also gives away to characterize the uncertainty in positive, negative, and boundary regions. These two methods are applied in several areas for knowledge processing tasks.

In this paper, we tried to focus on dealing with the traffic flow and its approximation. The traffic flow is a complex, changeful, nonlinear, unstructured, space time-varying and random system. With the foundation operationating of the intelligent traffic system, it is imperative to search for a traffic state estimation model, which is suitable for mixed-traffic in China. On the basis of analysis of the multidimensional state characteristics of mixed traffic, using the rough set theory, the four-dimensional state estimation model is founded. By data discretization and attribute reduction, the two-dimensional decision table is gained, and the rules of multidimensional state estimation in urban traffic systems are presented. A case is given and it indicates that this method can eliminate the redundancy information of the system effectively and improve the precision of rule mining. Rough set theory (RST) is a new mathematical tool to deal with vagueness and uncertainty. The main objective of using RST is to combine approximation of concepts from the collected data. This set can easily integrate community opinion and experience without having a precise mathematical model and hence, it is pertinent for applications in traffic prediction and control. Uncertainty in the rough set approach is expressed by a bounded region of a set, not by partial membership like in fuzzy set theory and it is defined by means of topological operations, interior and closure called approximations.

Other parts of the paper are organized as follows: Section 2 provides some literature on road traffic control using a Neutrosophic set and its hybrid method. Section 3 discussed the method for Rough set for traffic control. Whereas Section 4 provides some recent methods for utilization of different graphs for traffic control. Section 5 provides methods for a fuzzy set for traffic control followed by conclusions, acknowledgments, and references

2. Road traffic control management based on neutrosophic approaches:

This section contains some of the available methods using Neutrosophic set for Road Traffic control.

Table 1 summarizes some of the neutrosophic techniques dealing the road traffic control.

Table 1: The neutrosophic approaches for dealing with the Traffic flow

Reference	year	Techniques used	Solve problem
[1]	2017	neutrosophic linear equations	Traffic flow
[2]	2018	Neutrosophic C-means	Road safety
[3]	2019	Type 2 fuzzy and interval neutrosophic	operational laws, and aggregation. operators have been proposed under triangular interval type-2 fuzzy and interval neutrosophic environments. The validity of the proposed concepts has been verified using a numerical example.
[8]	2019	Gauss Jordon	Traffic control in a neutrosophic environment
[9]	2019	Dombi interval neutrosophic	Traffic control in Dombi interval
[4]	2019	Jordan method	Traffic control in a neutrosophic environment
[6]	2019	Neutrosophic set	transport sustainability assessment
[11]	2020	Single valued neutrosophic sets	Emergency Transportation Problem

[17]	2020	Interval-valued neutrosophic soft set	Control traffic signals
[19]	2019	on neutrosophic Markov chain	Crowed management
[5]	2019	Neutrosophic Cognitive maps	Crowded junction in Chennai
[10]	2020	Developed Plithogenic fuzzy hypersoft set based TOPSIS under neutrosophic environment	Developed Plithogenic fuzzy hypersoft set based TOPSIS under a neutrosophic environment to solve a parking problem and validated the findings by taking two different sets of choices compared with fuzzy TOPSIS
[14]	2020	neutrosophic exponentially weighted moving average	Monitoring the road traffic crashes
[13]	2021	Type-2 neutrosophic sets based CRITIC and MABAC	Public transportation pricing system selection
[12]	2021	Fuzzy FUCOM and neutrosophic fuzzy MARCOS	Assessment of alternative fuel vehicles for sustainable road transportation
[15]	2021	Neutrosophic statistical approach	Reducing and identifying the reasons for road accidents and road injuries
[16]	2021	AHP, MABAC, and PROMETHEE II with single-valued neutrosophic sets	Risk Management in Autonomous Vehicles
[18]	2021	multi-valued neutrosophic MULTIMOORA method	Traffic flow and its application in a multi-valued way
[19]	2021	Neutrosophic exponentially weighted Moving Average Statistics	Monitoring road accidents and road injuries
[20]	2022	Neutrosophic weighted Sensors Data Fusion	Occupancy detection system

Prof. Abdel-Basset et al. (2021) propose an opinion that autonomous vehicles play a key part in an intelligent transportation system, however, there are a number of dangers associated with these vehicles. As a result, a new hybrid model for identifying these risks is introduced. Uncertainty and hazy data are present in this procedure. To deal with the uncertainty, the neutrosophic hypothesis is employed. True, indeterminacy and false are the three membership functions provided by the neutrosophic theory (T, I, F). The notion of Multi-Criteria Decision Making (MCDM) is employed with the neutrosophic theory in this research since autonomous cars have various and conflicting criteria. The Analytic Hierarchy Process determines the weights of criteria in the first stage (AHP). Second, methodologies such as Multi-Attributive Border Approximation Area Comparison (MABAC) and Preference Ranking Organization Method for Enrichment Evaluations II are used to rate the risks of autonomous vehicles (PROMETHEE II). Ten different options were used in the case study. To demonstrate the robustness of the proposed model, a sensitivity analysis and a comparative study with a fuzzy environment are presented.

Bendadi (2018) proposed two clustering techniques for road traffic control. The first is Credal C-Means clustering (CCM), and the other is Neutrosophic C-Means clustering (NCM) (NCM). When it comes to overlapping items, both proposed methods have a similar tendency to form a new cluster that decides the imprecision object. Both techniques have different interpretations of the indeterminacy cluster. The number of meta-clusters formed by the CCM algorithm are proportional to the number of singleton clusters, whereas, with the NCM technique, all indeterminate objects are represented by a single indeterminacy cluster.

The application of CCM and NCM approaches to real-world data in the field of road safety, as represented by trajectories gathered in a bend, provides four clusters that represent the behavior of four different types of drivers based on their Turiyam consciousness (Singh 2021):

- The first cluster depicts the family of the slowest safe driving trajectories.
- The second cluster consists of the family of fast trajectories with safe driving.
- The third cluster is the family of sport driving's slowest trajectories.
- The fourth cluster is the family of sport driving's fastest trajectories.

Pamucar et al. (2021) suggested a hybrid model for evaluating alternative fuel cars for sustainable road transportation in the United States that included fuzzy FUCOM and neutrosophic fuzzy MARCOS. For public transportation pricing system selection, Simic et al. (2021) extended the CRITIC and MABAC techniques to type-2 neutrosophic sets.

Rayees et al. (2020) propose four possible categories of Plithogenic hypersoft sets in this study, based on the number of characteristics chosen for the application, the type of alternatives, or the degree of attribute value appurtenance. The fuzzy and neutrosophic scenarios that potentially have neutrosophic applications in symmetry are covered by these four PHSS classes. Then, as an extension of the methodology for order preference by the resemblance to an ideal solution, they introduced a novel multi-criteria decision making (MCDM) method, which is based on PHSS (TOPSIS). A number of real-world MCDM situations are compounded by uncertainty, which necessitates subdividing each selection criteria or attributes into attribute values and evaluating all alternatives independently against each attribute value. The suggested PHSS-based TOPSIS can be utilized to solve real MCDM problems that are precisely characterized by the PHSS concept, in which each attribute value has a neutrosophic degree of appurtenance matching to each alternative under examination, in light of some supplied criteria. In a real-world application, the suggested PHSS-based TOPSIS solves a parking place selection problem in a fuzzy neutrosophic environment, and it is validated by comparing it to fuzzy TOPSIS.

Aslam (2020) developed a control chart for neutrosophic exponentially weighted moving average (NEWMA) employing recurrent sampling under neutrosophic statistics. The author used a NEWMA chart to track traffic collisions on the highway (RTC). According to a simulated analysis and a real-world example, the suggested NEWMA chart outperforms existing control charts for monitoring the RTC. According to the comparative analysis, It is indicated that the proposed NEWMA chart may be successfully used to control RTC. In this way, it built a new S2 N NEWMA control chart for road accident monitoring employing a repeating sample strategy in another study by the same author. The new chart will help notice shifts in accidents and injuries more quickly than existing charts.

Lin et al. (2020) developed a novel emergency transport model that simulates emergency transport from the logistics center to each disaster location as well as between disaster sites. In ambiguous and uncertain contexts, the single-valued neutrosophic set (SVNS) idea was used to convert the emergency transshipment problem into a multi-attribute decision-making problem. To rank and optimise alternate transportation routes, the proposed method was used to an emergency operation scenario.

Enalkachew Teshome Ayele et al. (2020) in developing countries, For controlling traffic flow at traffic intersections, a fixed time traffic signal control method is used. if there are high traffic conditions at the junction because it is unable to identify the level of traffic at the junction and enable vehicles waiting to cross the junction. To address these challenges, operators must formulate their judgment and design an automatic decision-making system to take their place. To make use of fuzziness in traffic flow and find efficient and effective timings for optimal phase changes, the operator's decision process could be examined using the method of interval-valued neutrosophic soft set theory. The proposed interval valued neutrosophic soft sets (IVNSS) traffic control system can improve traffic congestion management. It analyses variable phases and time lengths for the green light time duration depending on the present traffic density at the intersection instead of a constant time duration

Wang et al. (2020) proposed a travel time prediction model based on exclusive disjunctive soft set theory is developed to address the prediction problem of expressway trip time. The key impact factors are retrieved using the soft set parameter reduction theory, and the mapping relationship between the influence factors and the travel time is generated using the exclusive disjunctive soft set decision system. The soft set theory is used to create the journey time model, and the travel time is estimated using the mapping relationship. The experimental results reveal that, when compared to the BPR function model, the trip time model based on exclusive disjunctive soft set theory reduces prediction error and improves performance significantly.

Xiao et al. (2021) proposed a method based on prospect theory, this method improves the multi-valued neutrosophic MULTIMOORA method. The proposed method is used to choose a subway building scheme that is appropriate. Sujatha et al (2019) demonstrated how to use Fuzzy Cognitive Map and Induced Fuzzy Cognitive Map to assess the traffic flow pattern at a busy crossroads in Chennai, India's largest city. Nagarajan et al(2020) developed a decision-making mechanism based on a neutrosophic Markov chain to anticipate the traffic volume.

Fayed et al. (2022) proposed a comprehensive occupancy detection system based on a new fusion technique for fusing heterogeneous sensor data that greatly enhances occupancy detection efficiency. The proposed algorithm can be used in a traffic control system for roads. This study motivated to use its graphical visualization for precise analysis of Traffic Road management. In the next section, some of the available methods related to the traffic control system using graph theory is discussed.

If the Markov Chain (MC) has 'n ' states, The position of the state vector is tracked using the state vector (Fort and colleagues, 2008). Olaleye and colleagues (2009) For the dynamics of the system, the Markov technique was applied to automobile traffic. Ning (2013) investigated traffic flow disruption along a highway length. The traffic bottleneck caused by big trucks was discussed by Rui et al.(2017). Syed Imran Hussain Shah et al (2020) conducted a case study on modern urban transportation sustainability assessment. Uncertain or insufficient data must be dealt with when dealing with traffic flow issues. Partially indeterminacy and/or partial determinacy are common in real-time decision-making challenges. Due to a lack of knowledge or other factors, this is the case. Although fuzzy sets, as proposed by Zadeh (1965), may handle uncertain information and have been widely employed (Koukol et al. (2015). fuzzy numbers cannot represent data with both determinate and indeterminate information. For addressing unclear information, biassed possibilities can often be utilised instead of biassed probabilities to define MC in a neutrosophic environment (Smarandache, 2013). Markov chains are commonly used in vehicle control systems, traffic regulation, currency exchange rates, and queuing systems. Indeterminacy is distinguished from randomness by the fact that the objects in the space are both true and untrue at the same time.

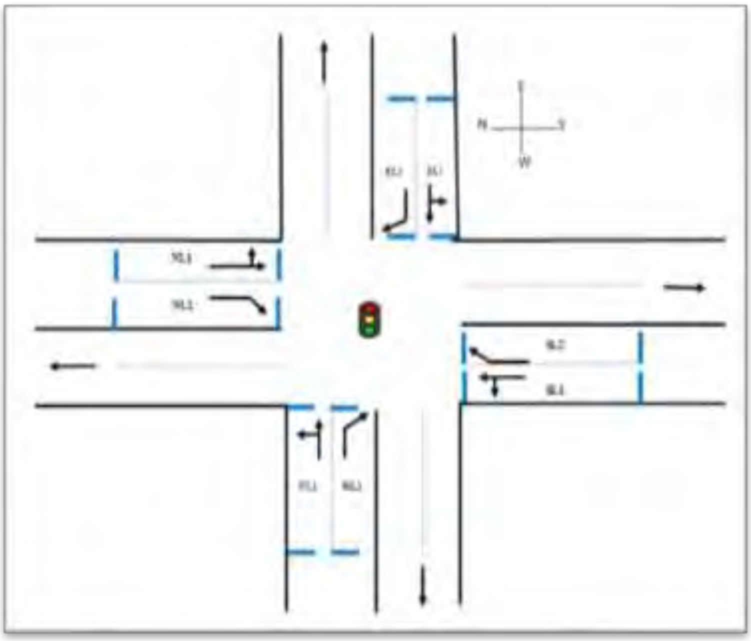

Figure 1 : The understanding of Traffic flow using time based and directions

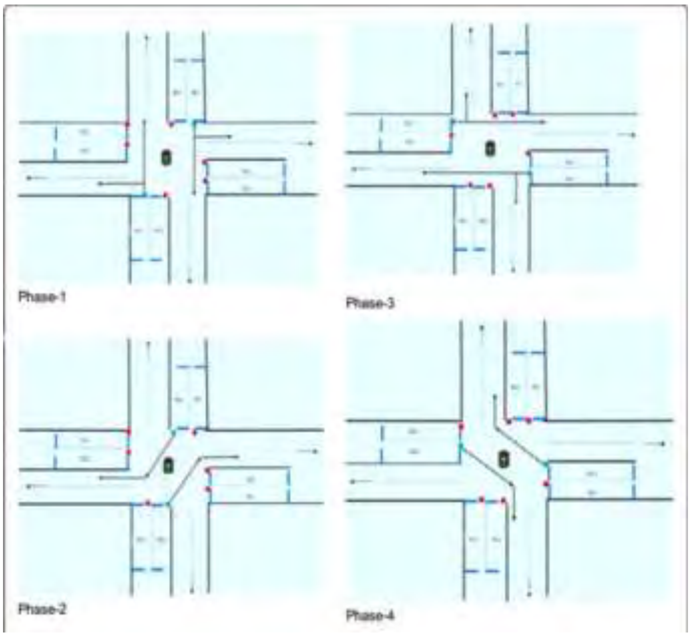

Figure 2: The phases of light and its connection with traffic flow

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3. Road Traffic Control Management Based on Rough Set based Approaches

This section provides some prominent methods for dealing the traffic control using the rough set as shown in Table 2. Table 3 providese some methods for dealing the traffic control using fuzzy rough sets whereas Table 4 contains hybrid methods for rough sets. In addition, this section clearly demonstrates the function of rough set approaches in traffic control management from many angles.

Table 2: Some available methods for dealing the Traffic flow using a Rough set

Author	Year	Country	Techniques used	Solved Problem
[55]	2005	Singapore	Rough set and neural network	Highway traffic flow prediction
[56]	2007	China	Rough set approach	Accident chains exploration
[57]	2007	china	Rough set approach	Determine the most important inducement of black-spot and repair its effect to reduce traffic accident frequency.
[58]	2007	China	Rough set approximation	Multidimensional state estimation rules in the urban traffic system
[59]	2008	China	Rough set	To Identify Causal Factors of Accident Severity
[60]	2009	China	Rough set	Prediction Model of Traffic Flow
[61]	2009	China	Rough set	Analyze the cause of road black-spots
[62]	2009	China	Rough set	Traffic accident diagnosis, Traffic Accident Discrimination
[63]	2009	China	Random Forest Rough Set Theory	To identify the factors Significantly Influencing single Vehicle crash
[64]	2009	Australia	Data Mining	Assess Crash Risk on Curves
[65]	2010	China	rough sets and association rules data mining	Traffic rule and its flow
[66]	2010	China	rough set and neural network	Traffic flow forecasting
[67]	2010	China	Back	Forecasting the railway passenger traffic demand.

			propagation neural network with a rough set	
[68]	2010	China	Rough Set with Support Vector Machine	Travel Time Prediction on Urban Networks
[69]	2010	China	Rough set	Road Traffic Accidents Causes Analysis Based on Data Mining.
[70]	2010	China	Rough set	Accident cause analysis

Table 3 : Some available methods for dealing with the Traffic flow using the Fuzzy Rough set

Author	**Year**	**Country**	**Techniques used**	**Solved Problem**
[71]	2011	China	Rough Set and RBF Neural Network	Traffic Safety Evaluation of Expressway
[72]	2011	India	Tabu Search and rough set	Optimizing parking space
[73]	2011	China	Neural Networks Algorithm and Rough Set Theory	A Traffic Accident Predictive Model
[74]	2012	China	Rough sets	Analyzing traffic accidents
[75]	2012	India	Rough set	Traffic Discretization
[76]	2012	Poland	Reducts Set	Traffic intensity prediction, for junctions of the network graph's arches description
[77]	2013	China	Evidence theory combined with the fuzzy rough set.	Traffic flow
[78]	2013	China	Rough sets+fuzzy set	Trafic prediction
[79]	2013	China	Rough sets+fuzzy set	A Knowledge-Based Fast Recognition Method of Urban Traffic Flow States

[80]	2013	China	Evidence Theory Combining with Fuzzy Rough Sets	Urban Traffic Flow.
[81]	2013	India	Rough set and its extension	Short term traffic prediction
[82]	2014	China	Rough set and granulation	Traffic congestion
[83]	2014	China	Rough sets	Study on Traffic Control of Single Intersection
[84]	2014	Australia	Rough set	Assessing Road-Curve Crash Severity
[85]	2015	China	rough sets+ fuzzy sets	Emergency plan matching highway traffic
[86]	2015	Italy	Dominance-Based Rough Set Approach	Setting Speed Limits for Vehicles in Speed Controlled Zones

Table 4: Recent methods for dealing the Traffic flow using Rough set and it's Hybrid

Author	Year	Country	Techniques used	Solved Problem
[87]	2015	China	Degrees of Attribute Importance of Rough Set	Selecting scientific and reasonable indexes for the prediction model of road traffic accident morphologies
[88]	2015	China	fuzzy rough set	Predicting Urban Traffic Congestion
[89]	2015	China	Rough set tree	Accident morphology diagnoses
[90]	2015	Thailand	Rough set	highway traffic data
[91]	2016	Turkey	Rough set	Accident factor analysis
[92]	2016	China	Rough set decision tree	Accident morphology analysis
[93]	2016	China	grey relational analysis+rough set	To judge the traffic congestion state
[94]	2017	China	fuzzy rough set theory+SVM classifier	Predict city traffic flow breakdown
[95]	2017	Iran	rough sets	Solving Road Pavement Management Problems

[96]	2018	China	Rough set	Data-driven car-following model
[97]	2018	India	Rough sets	Predict the causes of traffic accidents
[98]	2018	Egypt	Rough set	Intelligent Traffic System
[99]	2018	China	Rough Sets (RS) and Bayesian Networks (BN)	Predict accident type.
[100]	2019	India	Combination of Support Vector Machine and Rough Set,	Traffic Flow Prediction using
[101]	2019	China	Rough set	Traffic Network Modeling and Extended Max-Pressure Traffic Control Strategy
[102]	2019	China	Rough sets based on classification	A classification and recognition model for the severity of road traffic accidents.
[103]	2019	Poland	rough sets	Reduce congestion in the city by predicting the intensity of the traffic
[104]	2020	China	fuzzy neural network and rough set theory	Data imputation for traffic flow
[105]	2020	India	Neuro-Fuzzy	Traffic flow
[106]	2020	Egypt	Rough interval	Transportation problem
[107]	2020	Thailand	Rough Set and Decision Tree Classification algorithm	Predict the accident damage magnitude
[108]	2021	China	Rough set	Analyzing Road Users' Precrash Behaviors and Implications for Road Safet

Table 3 shows the hybridization of a rough set with other set theories for handling traffic flow. Motivated by Table 4 Prof. Ang, K. K. (2005) proposes a new rough set–based pseudo-outer-product RSPOP) the algorithm that combines the RSPOP technique with the sound concept of knowledge reduction from rough set theory. The suggested algorithm not only accomplishes feature selection by reducing characteristics but also extends the reduction to rules that are devoid of redundant attributes.

Wong and Chung (2008) used a comparison of methodology approaches to identify causal factors of accident severity. Accident data were first analyzed with a rough set of theories to determine whether they included complete information about the circumstances of their occurrence according to an accident database. Derived circumstances were then compared. For those remaining accidents without sufficient information, logistic regression models were employed to investigate possible associations. Adopting the 2005 Taiwan single auto vehicle accident data set, the results indicated that accident fatality resulted from a combination of unfavorable factors, rather than from a single factor. Moreover, accidents related to rules with high or low support showed distinct features. Li, (2011) developed an enhanced rough set theory algorithm to investigate the cause of roadblock spots in order to confirm the most relevant inducements in road traffic accidents. Pang et al. (2010), proposed traffic flow forecasting based on a rough set and neural network. The forecasting data provided by the neural network-forecasting model is adjusted by rough set theory to improve the traffic flow forecasting accuracy in the proposed traffic flow-forecasting theme. The simulation results suggest that using the proposed traffic flow technique can greatly enhance forecasting accuracy.

Deng (2010) proposed a hybrid intelligent forecasting model combining back propagation neural networks with a rough set to forecast railway passenger traffic demand with pre-treated forecasting data. The experiment used data from China's railway passenger traffic from 1991 to 2008 as learning and testing samples, and the efficiency of this method was established in comparison to the linear recursive method. Chen et al. (2010) put forward a new prediction model that combined a rough set with a support vector machine. The concept of Rough set is used to pre-process the traffic data that is noisy, missing, and inconsistent then deduce some rules for framing the support vector machine (SVM) model. When comparing the committee model to the single SVM predictions utilizing real traffic data collected in Chengdu, the authors concluded that the integration of the two models leads to predicting travel time effectively

Banerjee & Al-Qaheri (2011) developed a revolutionary software interface to guide and help drivers in making better parking spot decisions and dealing with unpredictable traffic situations on the road. The interface is based on an intelligent hybrid strategy for parking space optimization that combines a Tabu metaphor with a rough set. The interface might be tested as an off-line decision support system before being integrated into an online traffic network, with instruction delivered via mobile phone-based voice instructions (Fan, 2013), both traffic prediction and control have been done using a rough set theory. In general, the transport system is a non-linear, time-varying, and delaying large-scale system, whereas the traffic system is a complex, time-varying, high ambiguous, and non-linear large system with human assistance and hence faces the greatest challenge to the transportation system. The fundamental principle of predicting traffic flow is predicting the number of vehicles at the k+1th moment in accordance with the previous moments. Once this prediction is done, then the controller starts controlling accordingly and it can be observed that the prediction of short-term traffic flow is very important in real-time intelligent traffic control.

The objective of signal control is to minimize the average delay time or a number of stops for vehicles passing through the junction. In a cycle, various traffic flows will take the right of passing in an intersection called phase. There are four phases in a normal four-direction intersection. In China, the right turning movement has a special passing rule. Therefore, the four phases movements namely straight going in south-north left turning in south-north, straight-going in east-west, and left-turning in east-west. Cars can pass through only two directions in one phase at the same time. According to the given cycle, the rate of green light is independently adjusted to track the immediate traffic flow. In this work, there are three control variables namely signal period (T), the rate of green light (⅄), and phase difference tp. When the flow of vehicles is infrequent, the signal period may be short but not smaller than 30 seconds. In such a manner, this can prevent the green light time of assured direction smaller than 15 seconds, and vehicles do not have sufficient time through the direction, which affects the safety of traffic. When the flow of vehicles is heavy, the signal period should belong, but cannot exceed 200 seconds. Else ways, the red light time of a certain phase is too long and the drivers cannot suffer from behaviorism. Here, it has been dealt with a multiphase fuzzy control algorithm where the vehicle queues have been characterized by the number of vehicles between two detectors. The distance of detectors is normally from 80 to 100 meters and lies in front of the stopping line in the intersection. In each phase, the basic green light time is 10 seconds and the time of directing is 15 seconds.

By considering the number of vehicles in the controlled phase future in 10 seconds and the vehicle queues in other red phases, the system provides the delaying time and makes the rough set rule judgment. The range of delaying time is 4 to 26 seconds. Using simulation to the general control method and the rough set predicting control algorithm, the delaying time of green light in four phases and eight periods. If the greatest green light time of directing is 70 seconds then turning left and right is 40 seconds. In each period, the loss of green light time is 15 seconds. The signal period and green light time of all the phases can be adjusted accordingly in addition to the variation of traffic flows and mitigates traffic difficulty and the waste of green light resources.

Predicting the short-term traffic flow is expedient using a rough set. The average time delay may be minimized using a rough set than with fixed timing in signal control of the unusual intersection. Here, six state variables have been taken into account for the signal control in a single intersection at the same time and it is found that the present system is highly reliable, compatible, and surpasses the traditional time control during great traffic change. Minal and Bajaj (2013) uses some data mining tools were used to develop a prediction system. The approach helped to advance rough set theory, evolutionary algorithms, and wavelet neural networks. There were three stages to the modeling process: discretion, attribute reduction, and training. To begin, the upgraded genetic algorithm was applied to discrete-continuous qualities with the fewest broken points to keep the discernable ability of the judicial system.

Decretive data was then reduced using rough set theory in order to improve prediction speed and simplify network construction. Finally, nonlinear wavelet neural networks were used to process the reduced data. Through comparative testing, improved precision and speed were gained using the data mining approach, which provided a novel concept for short-term traffic flow prediction.

A paper by Chen et al. (2014) proposed a generalized model based on granular computing to recognize and analyze the traffic congestion of urban road networks. Cheng, (2014) the authors described the experience and principle of traffic control as knowledge. The complete state of the intersection is determined by the classified arrival car number. In the space of intersection state, the knowledge face to the controlling of isolated intersection is applied. After that, a traffic signal control model based on a rough set was created. In Rakotonirainy et al. (2014) the authors utilized Text mining methods such as rough set theory and the Ward clustering algorithm to improve knowledge related to risk and contributing factors to road-curve crash severity. In this study, the authors proved that the proposed techniques could be applied within other safety domains and may reveal heretofore unrealized contributors to incidents and accidents. Shao(2015) the authors applied the concept of the soft fuzzy rough set theory to predict urban traffic congestion. For this purpose, they present a practical example predicting urban traffic congestion based on the soft fuzzy rough set. In Gang (2015) proposed a traffic accident morphology diagnostic model based on a rough set decision tree. The advantage of this model it can be used by road traffic managers to identify the potential accident morphology realized the prediction for potential traffic accidents and formulated targeted accident solutions. Zhang (2016) the authors analyzed urban road traffic information using grey relational clustering and combined the results with rough set theory to establish a decision table system. To evaluate the degree of urban traffic congestion (jam), the authors used three properties of traffic flows (traffic flow velocity, traffic flow density, and traffic volume). They judged which road was allowed smooth traffic flows, which was suffering from a light traffic jam, which was suffered from a traffic jam, and which was suffered from a heavy traffic jam state. Finally, the authors found their method can be more conducive to dynamic traffic warnings. Yang(2017) introduced the fuzzy rough set theory to solve the task of attribute reduction, and then utilized an SVM classifier to forecast city traffic flow breakdown. Particularly, in this paper three definitions to describe city traffic flow more accurately are given that is, 1) Pre-breakdown flow rate, 2) Rate, density, and speed of the traffic flow breakdown, and 3) Duration of the traffic flow breakdown. In another study, Nithya et al. (2018)

described the Rough set approaches for detecting and analyzing the causes of an accident. In this work, they conclude that Driver Fault is the major cause of traffic accidents.

Xiong et al. (2018) applied the rough set-based Bayesian networks as a complementary tool for roadway traffic accident analysis based on Naturalistic driving data (NDD). The proposed framework was demonstrated using the the100-car naturalistic driving data from Virginia Tech Transportation Institute to predict accident type. The authors employed Rough Set Theory to reconstruct and simplify the components that influence the severity of a traffic collision in this research.

The importance of qualities in people, cars, roads, environments, and accidents was calculated using rough set theory. Marek and Anna (2019) utilized using rough set theory, data mining of traffic vehicles and decision rules for the number of traffic vehicles that have been constructed at specific locations around the city (RST). As part of the Green and Sustainable Freight Transport Systems (GRASS) in Cities project, RST was used to extract knowledge from empirical data collected during a study of traffic intensity in favored areas in Szczecin.

In this paper, vehicle traffic volume was investigated using RST with 120 objects, 16 well-defined rules, 9 useful advantageous vague rules, three condition characteristics (vehicle type, experiment location, and vehicle speed), and one decision attribute (number of vehicles). And it was discovered that 65 percent of the examined examples admit to generating specific rules, according to the estimated signal of the quality of approximation of the condition attributes. Furthermore, because RST's knowledge extraction ratio is 4.8, the average of five objects has been characterized by one helpful rule and the connotation of conditional attribution has been checked. Zaher et al. (2020) presented a new rough interval max algebra approach (RIMAA) for solving the traffic problem with rough interval data. It motivated to deal with traffic flow using interval-valued rough set and its hybridization. In the next section, some of the available approaches are interval-set, vague set, and another set.

4. Road Traffic Control Management based on graph approaches:

In this section, some graph theory-based approaches for resolving road transport networks or studying traffic flow across road networks are provided in Table 5 under classical, fuzzy, intuitionistic fuzzy, and neutrosophic environments. One of the reasons for this is that, as illustrated in Figure 3, traffic flow can be represented by the vertex and edges of any defined graph.

Table 5. Summary of the available multi-criteria decision-making (graphs) approaches for the traffic management system.

Reference	Year	Techniques used	Solve problem
[39]	2012	m-polar fuzzy graph	Traffic-accidental zones in a road network.
[23]	2013	Fuzzy graph	Minimize the waiting time of the vehicle using vertex coloring function
[24]	2013	Fuzzy graph	Classify the accidental zone of a traffic flows.
[25]	2014	Interval-valued fuzzy planar graphs	Minimize the crossing between roads
[28]	2018	Neutrosophic bipolar planar graph	To monitor traffic
[29]	2019	product bipolar fuzzy graphs	PBFPG of a road network
[31]	2019	Hesitancy fuzzy magic labeling	Smooth the network traffic and contribute the uniformity of the traffic distribution using fuzzy magic labeling graphs
[32]	2020	Fuzzy graphs+ MatLab program	Using a MATLAB program based on fuzzy graph-FCN-FIS, minimize traffic light cycle time at crossings.
[34]	2020	cubic graphs	Get the least time to reach the destination
[35]	2020	Multigraph with Picture Fuzzy Information	Minimize the crossing between roads
[36]	2020	Fuzzy graph Structures	Detection of the road crimes
[37]	2020	Intuitionistic fuzzy soft digraph	Road Safety Measures
[38]	2020	Edge coloring of fuzzy graphs	Determined the present condition of the traffic in the traffic light system using color density with a percentage

[40]	2021	Cyclic connectivity index of fuzzy incidence graphs	Minimize road accidents
[41]	2021	fuzzy graph	Reduce the traffic congestion in accidental prone zone
[42]	2021	Application of genus graphs under picture fuzzy environment	To control traffic jam on road network
[43]	2021	Fuzzy incidence coloring techniques	Reduce the frequency of accidents and vehicle waiting times in traffic flow scenarios,

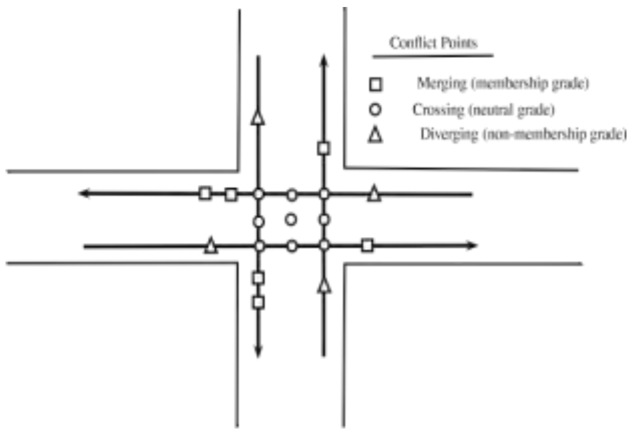

Figure 3. The graphical visualization of Traffic can be possible using vertex and edges [116]

Akram et al. (2012) described how to discover traffic-accidental zones in a road network using various sorts of m-polar fuzzy edges. Dey and Pal (2013) traffic congestion has become a major issue in cities as the number of vehicles on the road grows rapidly. The goal of the traffic light setting problem is to figure out how to set the traffic lights such that the total time vehicles spend on the road is as short as possible. To depict the traffic network in this paper, we employ a fuzzy graph model. The traffic light problem is solved using the vertex coloring function (crisp mode) of a fuzzy graph. Cut of graph G=(V, E), the cuts of fuzzy graph G, is the basis for the function. Using this method, the traffic light issue is investigated. The authors solved the problem discussed in Dey and Pal (2013) by utilizing a fuzzy network to encode the vertex membership value for traffic signal length based on vehicle number. In this scenario, because the route had the most vehicles, the time spent waiting was the longest. When a track has a large number of vehicles, it must be protected in order to avoid accidents, in which all of the vehicles on the track must wait. Moreover, there is a maximum amount of time to wait.

Pramanik et al.(2014) developed a model for designing the road map as an interval valued fuzzy planar graph with membership values of vertices and edges taken as an interval number, and then estimated the degree of planarity of interval valued fuzzy graphs to minimize road crossings (IVF graph). The measurement of congestions in the paper was done as an interval valued fuzzy (IVF) number.

Akram (2018) developed a traffic-monitoring road network model using the concept of bipolar neutrosophic planar graph. The notion of bipolar neutrosophic planar graphs was utilized to build road networks. The proposed method can be used to calculate and track the annual proportion of accidents. By monitoring and implementing extra security steps, the total number of accidents can also be minimized.

Sumera et al. (2019) explained the notion of planarity product bipolar fuzzy graphs was used to solve the problem of crossing roads in a road network modelled by product bipolar fuzzy graphs. In the paper of Fathalian et al. (2019) the authors demonstrate whether any simple graph is hesitancy fuzzy magic labeling in this work by studying the concept of hesitancy fuzzy magic labelling of a graph. We show that any finite path graph, cyclic graph, star graph, and any complete graph derived from these, as well as any connected graph, have hesitancy fuzzy magic labelling. Finally, we discuss various plumbing and traffic flow applications for hesitancy fuzzy magic labelling graphs.

Rosyida et al. (2020) propose a phase scheduling that considers traffic intensities using fuzzy graph and fuzzy chromatic number (FCN) of the fuzzy graph. In this paper, two algorithms are constructed. The first is an algorithm to model a traffic light system on an intersection using fuzzy graph and determine phase scheduling using FCN of the fuzzy graph. The second is an algorithm to determine duration of green lights of the phases in the first algorithm using Mamdani-FIS. In addition they created Matlab codes of the above two algorithms.

The authors evaluated the algorithms through case studies on two intersections with 4 approaches in Semarang City, Indonesia, namely "Kaligarang" intersection and “Lamper Gadjah” intersection. The results show that the combination of FCN of fuzzy graph and the Mamdani-FIS gives some options of phase scheduling with different cycle times. In addition, the approach with high traffic volume gets a longer green time. The phase scheduling proposed in this research increases performances of intersections under study in that the cycle times of the proposed scheduling are shorter than that of the existing systems. It means that it is superior in reducing the average time a driver spends his/her time on the intersection.

Muhiuddin (2020) applied the notion of cubic graphs in traffic flows to arrive at the shortest time possible. They used fuzzy variables and interval-valued fuzzy variables to represent two primary parameters in their study: traffic volume and distance between two intersections. Each intersection is represented by a single vertex, and each highway between two intersections is represented by a graph edge. The authors of Koam (2020) adapted the concept of fuzzy network structures to decision-making in the detection of marine and road crimes, and provided an algorithm to solve these two problems. The authors investigated whether road connecting any two cities is the most important for a certain crime, using the notion of fuzzy graph structure. Singh (2021) tried to provide the threshold for which cubic graph can be approximated for given traffic and its density. It can be used to control the traffic speed based on human turiyam cognition (Singh 2021) rather than red, green or yellow light. It is totally based on human Turiyam cognition that red light need to stop, green light need to go and yellow light means slow. It will be helpful in finding heterocolinic pattern on the traffic in case of Neutrotraffic (Singh 2022).

Sarala and Tharani (2020) tried to minimize the human loss during accidents and reduced the waiting time of vehicles in lane at traffic flow from existing traffic system, Yamuna et al.(2021) proposed a new methodology based on Fuzzy incidence coloring numbers to identify a solution to traffic flow problem. The real-time traffic flow problem was modeled by fuzzy graph including eight vertices. Nazeer et al. (2021) provided real-life applications of cyclic connectivity index of fuzzy incidence graphs in a highway system of different cities to minimize road accidents. In the planning of road maps the crossing between congested (strong) road and non-congested (weak) road may be accepted with certain amount of protection as this crossing is low risky as comparison to the crossing between two congested (strong) roads.

Das et al. (2021) considered the rate of congestion as picture fuzzy set (PFS) and modeled up the design of road map as PFPG. They defined a very important notion of PFG theory called degree of planarity. The concept of degree of planarity (DP) determine the nature of planarity (NP) of a PFPG .If the DP of a PFG is (1, 1, 1), then there is no crossing between two edges on DP. The congestions of roads is a fuzzy quantity as rate of congestions depends on decision makers attitude, practices, behavior, etc. The measurement of congestions as a point is not easy for decision maker.

Mahapatra et al. (2020) discussed the degree of capacity of vehicles of a city is defined in terms of its positive membership and negative membership. Positive membership degree can be depicted as how much capacity, vehicles of a city posses and negative membership can be depicted as how much capacity is lost by the vehicles of a city. The membership values of edges of this graph show the capacity of vehicles on the road joining any two cities. The positive and negative membership degree of edges can be interpreted as the percentage of increasing and decreasing capacity of vehicles on the road between any two cities

The authors claimed that the concept of Fuzzy incidence coloring might be applied to other modes of transportation, such as air, rail, and marine, to reduce human loss. It can be observed that the positive, negative and uncertain regions of traffic flow can be approximated via rough neutrosophic theory and its graph visualization, which will be the future scope of the paper.

5. Traffic management systems based on other novel fuzzy sets approaches

This section will discuss a few applications of fuzzy set extensions on road traffic networks, such as intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, vague sets, and hesitant fuzzy sets. Table 6 contains some of the methods for resolving problems involving road traffic networks.

Table 6: Some new methodology to solve Traffic technique using an extension of Fuzzy set

Reference	year	Techniques used	Solve problem
[44]	2008	Vague set	Route Choice Approach to Transit Travel
[45]	2010	Vague set theory	road safety evaluation
[33]	2014	Linguistic variable in interval type-2 fuzzy entropy weight	Ranking of causes lead to road accidents
[47]	2017	The hesitant distance set on hesitant fuzzy sets	urban road traffic state identification
[48]	2017	Dual hesitant fuzzy rough pattern recognition approach	Urban traffic modes recognition
[46]	2018	Interval-valued intuitionistic fuzzy sets	Prediction of traffic emission
[50]	2018	Entropy Analysis on Intuitionistic Fuzzy Sets And Interval-Valued Intuitionistic Fuzzy Sets	Mode assessment of open communities on surrounding traffic
[51]	2019	Double hierarchy hesitant fuzzy linguistic -ORESTE method	Assessment of traffic congestion
[53]	2020	Euclidean distance intuitionistic fuzzy value with TOPSIS ranking method	Measuring drivers incapability
[54]	2020	Interval-valued intuitionistic fuzzy environment	Public bus route selection
[50]	2021	IVIF-VIKOR method	To assess urban road traffic safety.
[116]	2021	Complex Spherical Set	CSF information could be used to monitor the day and night traffic clashes on four-way road junctions.
[52]	2021	IF-MABAC	Evaluating the intelligent transportation system

[49]	2021	Interval-valued spherical fuzzy analytic hierarchy process method	Evaluate public transportation development

Here's a quick rundown of some of the possible findings: Tan (2008) connected the vague rough set for road section traffic state identification utilizing ambiguous sets was presented to identify the traffic condition of road sections and give decision support for traffic management. They have set up a decision matrix for traffic conditions. The steps for determining the traffic state of a road stretch were supplied by the authors. They presented a vague set and group decision-making-based method for acquiring knowledge about regional road network traffic situations followed by Wei and MA(2020).

The traffic state identification methods could meet the current demand for real-time traffic control and guidance, while the traffic state knowledge acquisition methods might give a mechanism for analyzing the time-space traffic flow evolution pattern of road networks. The hazy aggregation value, the weighted sums, and the scoring value are discovered and used to determine the substantially worst traffic status link called the regional road network's bottleneck link.

Fangwei et al. (2017) proposed a fuzzy traffic state identification method in which the three attributes f 1 (saturation degrees of the traffic flow), f 2 (vehicle queue length), and f 3 (average delay time of vehicles) are described by the Hesitant Fuzzy Sets concept for the four congestion levels E 1 (unobstructed traffic), E 2 (slight congestion traffic), E 3 (congestion traffic), and E_4 (extreme congestion traffic).A another author from China, Tian et al.(2018) offered a novel multiple attribute decision makings strategy for handling the problem of mode assessment of open communities on surrounding traffic in an intuitionistic fuzzy environment under an intuitionistic fuzzy environment. Taking into account road capacity, safety, and other factors. Also, The Chinese authors looked at four aspects of mode assessment of open communities which is based on human Turiyam as discussed by Singh (2021). These attributes are denoted as F = { f_1,f_2, f_3, f_4},where f_1 represents the average delay time at the community; f_2 represents the-safety-level of the community(number of vehicles collisions at the community intersection); f_3 represents the average speed of vehicles;, and df_4 represents the average driving path length of vehicles. Wang (2019) developed the DHHFL-ORESTE method (double hierarchy hesitant fuzzy linguistic ORESTE method) to evaluate traffic congestion and identify the most congested city in new first-tier cities in the article. Akram et al. (2021) created a new concept known as a complicated spherical fuzzy set in their research (CFS). The CSF data can be used to track traffic congestion on four-way intersections during the day and at night. Merging, diverging, and crossing are three common forms of traffic collisions to expect. Figure 2 depicts a clear picture of the clashing spots on a four-way intersection, which include six merging clashes, nine crossing clashes, and four diverging clashes.

The day and night check on traffic collisions may be done with complete information about prospective collisions, which can be demonstrated using CFS data by Akram et al. (2021). The daytime merging, crossing, and diverging clashes are represented by the amplitude term of membership, neutral, and non-membership grades, respectively, whereas the nighttime merging, crossing, and diverging clashes are represented by the phase term of membership, neutral, and non-membership grades, respectively and can be assigned 0.1, 0.7, and 0.3 as the membership, neutral and non-membership respectively, for a stray four-way junction with one merging, five crossing, and two diverging collisions during the day. If there are 12 traffic disputes during the night, three for merging, four for crossing, and five for diverging, the membership, neutral, and non-membership grades might be assigned phase terms of 0.2, 1.2, and 1.4, respectively. These data may be used to create a CSFN that describes information regarding traffic jams at a four-way intersection. Furthermore, the CSF data allows for the investigation of traffic collisions at all types of road crossings, as well as the characterization of traffic flow over a certain time period.

If they utilize a spherical fuzzy set here, it will only gather data during daytime traffic jams because it can't store two-dimensional data. The use of a complex Pythagorean fuzzy set, on the other hand, epitomizes two-dimensional information and only comprises data for merging and diverging traffic confrontations. It does not, however, constitute a crossing clash at any time of day or night. These facts raise CSFS requirements within the existing model by improving the information on day and night traffic collisions, as well as merging, crossing, and diverging collisions.

Yanping (2021) proposes a unique intuitive distance-based IF-MABAC approach to evaluate the performance of financial management, based on the standard multi-attribute border approximation area comparison (MABAC) method and intuitionistic fuzzy sets (IFSs). First, a literature review is carried out on the subject. In addition, certain key IFS theories are briefly discussed. Furthermore, because subjective randomness is common while calculating criteria weights, the maximizing deviation approach is used to determine objectively the weights of criteria. After that, the traditional MABAC approach is extended to the IFSs using innovative distance measurements between intuitionistic fuzzy numbers (IFNs). As a result, all businesses may be ranked, and the one with the best environmental practices and awareness can be found.

Duleba et al. (2021) presented Interval-valued Spherical Fuzzy Analytic Hierarchy Process as a methodological approach presented with the goal of handling both types of problems at the same time, taking into account hesitant scoring and synthesizing different stakeholder group opinions through a mathematical procedure. The additional extensions with a more flexible characterization of membership function are preferable to interval-valued spherical fuzzy sets. Decision makers' judgments regarding the membership functions of a fuzzy set are incorporated into the model using interval-valued spherical fuzzy sets instead of a single point. Also, solved public transportation problems using an interval-valued spherical fuzzy AHP approach. Due to the inclusion of three traditionally antagonistic stakeholder groups, public transportation development is an appropriate case study to explain the new model and analyze the outcomes. This motivated me to utilize neutrosophic set for dealing the Road traffic. In the next section, some of the available methods for road control using a neutrosophic set is discussed.

6. Conclusions

This paper provide a survey on available mathematical model for traffic flow using neutrosophic set, rough set, fuzzy set, and its extensions in Table 1 to 4 and Table 5. The graph based traffic flow methods also discussed in Table 5. It can be observed that neutrosophic rough set the hybrid set structures where computational techniques based on just one of these structures will not always produce the best results. The hybrid of two or more methods can frequently produce better results, which can be considered as one of the efficient method for measuring uncertainty in traffic flow as positive, negative and uncertain region to control the accidents.

In near future the author will focus on neutrosophic rough set based traffic flow and its graphical visualization. As can be seen, even with certain norms and laws in place, passengers and drivers disregard the traffic system, resulting in a variety of large and little incidents, how do we govern, manage, and maintain road discipline? Is the car-sharing system a viable option for eco-friendly and urban mobility?

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Characterization of γ-Single Valued Neutrosophic Rings and Ideals

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Abstract. In this paper, we investigate the notion of γ-single valued neutrosophic subrings and ideals. Also, several properties related to the algebraic structure of rings and ideals are discussed. Moreover, many characterizations are proposed on γ-single valued neutrosophic subrings and ideals.

Keywords: γ-single valued neutrosophic subrings; γ-single valued neutrosophic normal subrings; γ-single valued neutrosophic ideals.

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1. **Introduction**

Generally, the inconvenience of previously established strategies and designs is overcome by recently established fuzzy algebraic structures. Routine mathematics cannot always be used because of unclear and missing knowledge in certain regular structures. Various methodologies were seen as alternative groups to deal with these issues and avoid vulnerabilities, like probability, rough set, anda fuzzy set hypothesis. Unfortunately, each of these alternate mathematics has a side and inconveniences such as the majority of words like real, beautiful, famous that are not clearly observed or indeed vague. Henceforth, the rules for such terms vary from person to person.

Zadeh [1], proposed the idea of the fuzzy set which is focussed on the possibility of the support highlight doling out an enrollment grade in $[0, 1]$ to deal with such sort of vague and questionable data. Taking into account the possibility of enrolment and non-investment, Atanassov [2, 3] proposed an intuitionistic fuzzy set which is an augmentation of a fuzzy set. As an extension of intuitionistic fuzzy set, Smarandache's [4, 5] introduced neutrosophic logic and sets. A neutrosophic set is based on three degrees: the level of participation, indeterminacy, and non-enrollment degree. The notion of a soft set is introduced in [6] by Molodtsov. Several operations were added by Ali et al. in soft set in [7]. In [8]- [10], Yager has executed the idea of the Pythagorean fuzzy set. Peng et al. presented several findings in [11, 12] on the measurements of the Pythagorean fuzzy and soft sets. Moreover, several new models have been investigated in [13]- [16].

In 1971, the concept of a fuzzy subgroup was proposed by Rosenfeld [17] and the investigation of fuzzy subgroups began. Later on, many algebraic structures; like groups, rings, fields, graphs, and modules, etc. have been developed in [18]- [38]. In this piece of work, we investigate the notion of γ-single valued neutrosophic rings, ideals, and sum and product of γ-single valued neutrosophic ideals. The proposed work is the generalization of many existing algebraic structures on fuzzy set, intuitionistic fuzzy set, (α, β)-intuitionistic fuzzy set etc.

The paper is structured as follows: we provide some basic concepts relating to γ-single valued neutrosophic rings and ideals in Section 3. We give an overview of the sum and product of γ-single valued neutrosophic ideals, also suggested several suggested several characterizations in Section 4.

2. Preliminaries

In this section neutrosophic subrings, neutrosophic normal subrings, and neutrosophic ideals are defined.

Definition 2.1. [18] A single valued neutrosophic set U on the universe of discourse R is defined as:

$$U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in R\},$$

where $i, t, f : R \to [0, 1]$ and $0 \le i_U(u) + t_U(u) + f_U(u) \le 3$. Here, $i_U(u), t_U(u)$ and $f_U(u)$ are called membership function, hesitancy function and non-membership function respectively.

Definition 2.2. [18] Let U & V be two SVNS on R. Then

(1) $U \subseteq V, \Leftrightarrow U(u) \le V(u)$. i.e. $i_U(u) \le i_V(u),\ t_U(u) \le t_V(u)$ and $f_U(u) \ge f_V(u)$. Also $U = V \Leftrightarrow U \subseteq V$ and $V \subseteq U$.

(2) $W = U \cup V$ such that $W(u) = U(u) \vee V(u)$ where $U(u) \vee V(u) = (i_U(u) \vee i_V(u), t_U(u) \vee t_V(u), f_U(u) \wedge f_V(u))$, for each $u \in R$. i.e. $i_W(u) = \max\{i_U(u), i_V(u)\}, t_W(u) = \max\{t_U(u), t_V(u)\}$ and $f_W(u) = \min\{f_U(u), f_V(u)\}$.

(3) $W = U \cap V$ such that $W(u) = U(u) \wedge V(u)$ where
$U(u) \wedge V(u) = (i_U(u) \wedge i_V(u),\ t_U(u) \wedge t_V(u),\ f_U(u) \vee f_V(u))$, for each $u \in R$.
i.e. $i_W(u) = \min\{i_U(u), i_V(u)\},\ t_W(u) = \min\{t_U(u), t_V(u)\}$ and
$f_W(u) = \max\{f_U(u), f_V(u)\}$.

(4) $U^c(u) = (f_U(u), 1 - t_U(u), i_U(u))$, for each $u \in R$. Here $(U^c)^c = U$.

Definition 2.3. [39] A single valued neutrosophic set $(SVNS)$ $U = (i_U, t_U, f_U)$ of a ring R is said to be an single valued neutrosophic subring $(SVNSR)$ if

(1) $i_U(u - v) \geq \wedge\{i_U(u),\ i_U(v)\}$.
(2) $t_U(u - v) \geq \wedge\{t_U(u),\ t_U\}$.
(3) $f_U(u - v) \leq \vee\{f_U(u),\ f_U(v)\}$.
(4) $i_U(uv) \geq \wedge\{i_U(u),\ i_U(v)\}$.
(5) $t_U(uv) \geq \wedge\{t_U(u),\ t_U(v)\}$.
(6) $f_U(uv) \leq \vee\{f_U(u),\ f_U(v)\},\ \forall\ u, v \in R$.

Definition 2.4. [39] A subset $U = (i_U, t_U, f_U)$ of a ring R is said to be an single valued neutrosophic normal subring $(SVNNSR)$ of R if

(1) $i_U(uv) = i_U(vu)$.
(2) $t_U(uv) = t_U(vu)$.
(3) $f_U(uv) = f_U(vu),\ \forall\ u, v \in R$.

Definition 2.5. [39] A single valued neutrosophic set $U = (i_U, t_U, f_U)$ a ring R is said to be an single valued neutrosophic left ideal $(SVNLI)$ if

(1) $i_U(u - v) \geq \wedge\{i_U(u),\ i_U(v)\}$.
(2) $i_U(uv) \geq i_U(v)$.
(3) $t_U(u - v) \geq \wedge\{t_U(u),\ t_U(v)\}$.
(4) $t_U(uv) \geq t_U(v)$.
(5) $f_U(u - v) \leq \vee\{f_U(u),\ f_U(v)\}$.
(6) $f_U(uv) \leq f_U(v),\ \forall\ u, v \in R$.

Definition 2.6. [39] A single valued neutrosophic set $U = (i_U, t_U, f_U)$ a ring R is said to be an single valued neutrosophic right ideal $(SVNRI)$ if

(1) $i_U(u - v) \geq \wedge\{i_U(u),\ i_U(v)\}$.
(2) $i_U(uv) \geq i_U(u)$.
(3) $t_U(u - v) \geq \wedge\{t_U(u),\ t_U(v)\}$.
(4) $t_U(uv) \geq t_U(u)$.
(5) $f_U(u - v) \leq \vee\{f_U(u),\ f_U(v)\}$.

(6) $f_U(uv) \leq f_U(u), \ \forall \ u, v \in R$.

Definition 2.7. [39] A single valued neutrosophic set $U = (i_U, t_U, f_U)$ a ring R is said to be an single valued neutrosophic ideal ($SVNI$) if

(1) $i_U(u - v) \geq \wedge\{i_U(u), \ i_U(v)\}$.
(2) $t_U(u - v) \geq \wedge\{t_U(u), \ t_U(v)\}$.
(3) $f_U(u - v) \leq \vee\{f_U(u), \ f_U(v)\}$.
(4) $i_U(uv) \geq \vee\{i_U(u), \ i_U(v)\}$.
(5) $t_U(uv) \geq \vee\{t_U(u), \ t_U(v)\}$.
(6) $f_U(uv) \leq \wedge\{f_U(u), \ f_U(v)\}, \ \forall \ u, v \in R$.

3. γ-Single Valued Neutrosophic Subrings and Ideals

This section discusses some basic concepts and results related to γ-single valued neutrosophic subrings and ideals.

Definition 3.1. If U be a single valued neutrosophic subset of ring R then γ-single valued neutrosophic subset U is described as,

$$U^\gamma = \Big\{\langle u, i^\gamma(u), t^\gamma(u), f^\gamma(u)\rangle \mid i^\gamma(u) = \wedge\{i_U(u), \ \gamma\}, \ t^\gamma(u) = \wedge\{t_U(u), \gamma\}, \ f^\gamma(u) = \vee\{f_U(u), \gamma\}, \ u \in R\Big\},$$

where $\gamma \in [0, 1]$.

Definition 3.2. Let U & V be two γ-SVNS on R. Then

(1) $U^\gamma \subseteq V^\gamma, \Leftrightarrow U^\gamma(u) \leq V^\gamma(u)$. i.e. $i_{U^\gamma}(u) \leq i_{V^\gamma}(u), \ t_{U^\gamma}(u) \leq t_{V^\gamma}(u)$ and $f_{U^\gamma}(u) \geq f_{V^\gamma}(u)$. Also $U^\gamma = V^\gamma \Leftrightarrow U^\gamma \subseteq V^\gamma$ and $V^\gamma \subseteq U^\gamma$.
(2) $W^\gamma = U^\gamma \cup V^\gamma$ such that $W^\gamma(u) = U^\gamma(u) \vee V^\gamma(u)$ where $U^\gamma(u) \vee V^\gamma(u) = (i_{U^\gamma}(u) \vee i_{V^\gamma}(u), t_{U^\gamma}(u) \vee t_{V^\gamma}(u), f_{U^\gamma}(u) \wedge f_{V^\gamma}(u))$, for each $u \in R$. i.e. $i_{W^\gamma}(u) = \max\{i_{U^\gamma}(u), i_{V^\gamma}(u)\}, \ t_{W^\gamma}(u) = \max\{t_{U^\gamma}(u), t_{V^\gamma}(u)\}$ and $f_{W^\gamma}(u) = \min\{f_{U^\gamma}(u), f_{V^\gamma}(u)\}$.
(3) $W^\gamma = U^\gamma \cap V^\gamma$ such that $W^\gamma(u) = U^\gamma(u) \wedge V^\gamma(u)$ where $U^\gamma(u) \wedge V^\gamma(u) = (i_{U^\gamma}(u) \wedge i_{V^\gamma}(u), t_{U^\gamma}(u) \wedge t_{V^\gamma}(u), f_{U^\gamma}(u) \vee f_{V^\gamma}(u))$, for each $u \in R$. i.e. $i_{W^\gamma}(u) = \min\{i_{U^\gamma}(u), i_{V^\gamma}(u)\}, \ t_{W^\gamma}(u) = \min\{t_{U^\gamma}(u), t_{V^\gamma}(u)\}$ and $f_{W^\gamma}(u) = \max\{f_{U^\gamma}(u), f_{V^\gamma}(u)\}$.
(4) $U^{\gamma c}(u) = (f_{U^\gamma}(u), 1 - t_{U^\gamma}(u), i_{U^\gamma}(u))$, for each $u \in R$. Here $(U^{\gamma c})^c = U^\gamma$.

Definition 3.3. A γ-single valued neutrosophic set (γ-$SVNS$) $U^\gamma = (i_U^\gamma, t_U^\gamma, f_U^\gamma)$ of a ring R is said to be an γ-single valued neutrosophic subring (γ-$SVNSR$) if

(1) $i_U^\gamma(u - v) \geq \wedge\{i_U^\gamma(u), \ i_U^\gamma(v)\}$.
(2) $t_U^\gamma(u - v) \geq \wedge\{t_U^\gamma(u), \ t_U^\gamma(v)\}$.

(3) $f_U^\gamma(u-v) \leq \vee\{f_U^\gamma(u),\ f_U^\gamma(v)\}$.
(4) $i_U^\gamma(uv) \geq \wedge\{i_U^\gamma(u),\ i_U^\gamma(v)\}$.
(5) $t_U^\gamma(uv) \geq \wedge\{t_U^\gamma(u),\ t_U^\gamma(v)\}$.
(6) $f_U^\gamma(uv) \leq \vee\{f_U^\gamma(u),\ f_U^\gamma(v)\},\ \forall\ u, v \in R$.

Example 3.4. Let us consider the ring $(Z_2, +_2, *_2)$ where $Z_2 = \{0, 1\}$.
Let we define $U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in Z_2\}$ such that
$i_U(0) = 0.8,\ i_U(1) = 0.4,\ t_U(0) = 0.4,\ t_U(1) = 0.3$ and $f_U(0) = 0.3,\ f_U(1) = 0.6$.
Consider $\gamma = 0.5$, then $U^\gamma = \{\langle u, i_U^\gamma(u), t_U^\gamma(u), f_U^\gamma(u)\rangle \mid u \in Z_2\}$ where
$i_U^\gamma(0) = 0.5,\ i_U^\gamma(1) = 0.4,\ t_U^\gamma(0) = 0.4,\ t_U^\gamma(1) = 0.3$ and $f_U^\gamma(0) = 0.5,\ f_U^\gamma(1) = 0.6$,
$\Rightarrow SVNS\ U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in Z_2\}$ is an 0.5-$SVNSR$ of Z_2.

Proposition 3.5. *If U and V be two γ-single-valued neutrosophic subset of ring R then $(U \cap V)^\gamma = U^\gamma \cap V^\gamma$.*

Proof. Assume that U and V are two γ-single-valued neutrosophic subset of ring R.

$$(U \cap V)^\gamma(u) = \Big\{ \min\{\min\{i_U(u), i_V(u)\}, \gamma\}, \min\{\min\{t_U(u),\ t_V(u)\}, \gamma\}, \max\{\max\{f_U(u), f_V(u)\}, \gamma\} \Big\}$$
$$= \Big\{ \min\{\min\{i_U(u), \gamma\}, \min\{i_V(u), \gamma\}\}, \min\{\min\{t_U(u), \gamma\}, \min\{t_V(u), \gamma\}\}, \max\{\max\{f_U(u), \gamma\}, \max\{f_V(u), \gamma\}\} \Big\}$$
$$= \Big\{ \min(\{i_U^\gamma(u)\}, \{i_V^\gamma(u)\}), \min(\{t_U^\gamma(u)\}, \{t_V^\gamma(u)\}), \max(\{f_U^\gamma(u)\}, \{f_V^\gamma(u)\}) \Big\} = U^\gamma(u) \cap V^\gamma(u),\ \forall\, u \in R.$$

□

Theorem 3.6. *Let U and V be two γ-$SVNSRs$ of a ring R. Then $U \cap V$ is also an γ-$SVNSR$ of R.*

Proof. Let $U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in R\}$ and $V = \{\langle u, i_V(u), t_V(u), f_V(u)\rangle \mid u \in R\}$ be any two γ-$SVNRs$ of a ring R.

$$\Rightarrow\ U^\gamma = \{\langle u, i_U^\gamma(u), t_U^\gamma(u), f_U^\gamma(u)\rangle \mid u \in R\} \text{ and } V^\gamma = \{\langle u, i_V^\gamma(u), t_V^\gamma(u), f_V^\gamma(u)\rangle \mid u \in R\}.$$

Then by using Proposition 3.5

$$(U \cap V)^\gamma = U^\gamma \cap V^\gamma = \{\langle u, (i_U^\gamma \wedge i_V^\gamma)(u), (t_U^\gamma \wedge t_V^\gamma)(u), (f_U^\gamma \vee f_V^\gamma)(u)\rangle \mid u \in R\}.$$

Now for any $u, v \in R$, we have
(i) $(i_U^\gamma \wedge i_V^\gamma)(u-v) = \wedge\{i_U^\gamma(u-v), i_V^\gamma(u-v)\}$
$\geq \wedge\{\wedge\{i_U^\gamma(u), i_U^\gamma(v)\}, \wedge\{i_V^\gamma(u), i_V^\gamma(v)\}\}$
$= \wedge\{\wedge\{i_U^\gamma(u), i_V^\gamma(u)\}, \wedge\{i_U^\gamma(v), i_U^\gamma(v)\}\}$
$= \wedge\{(i_U^\gamma \wedge i_V^\gamma)(u), (i_U^\gamma \wedge i_V^\gamma)(v)\}$.
(ii) $(i_U^\gamma \wedge i_V^\gamma)(uv) = \wedge\{i_U^\gamma(uv), i_V^\gamma(uv)\}$
$\geq \wedge\{\wedge\{i_U^\gamma(u), i_U^\gamma(v)\}, \wedge\{i_V^\gamma(u), i_V^\gamma(v)\}\}$

$= \wedge\{\wedge\{i_U^\gamma(u), i_V^\gamma(u)\}, \wedge\{i_U^\gamma(v), i_U^\gamma(v)\}\}$
$= \wedge\{(i_U^\gamma \wedge i_V^\gamma)(u), (i_U^\gamma \wedge i_V^\gamma)(v)\}.$
(iii) $(t_U^\gamma \wedge t_V^\gamma)(u-v) = \wedge\{t_U^\gamma(u-v), t_V^\gamma(u-v)\}$
$\geq \wedge\{\wedge\{t_U^\gamma(u), t_U^\gamma(v)\}, \wedge\{t_V^\gamma(u), t_V^\gamma(v)\}\}$
$= \wedge\{\wedge\{t_U^\gamma(u), t_V^\gamma(u)\}, \wedge\{t_U^\gamma(v), t_U^\gamma(v)\}\}$
$= \wedge\{(t_U^\gamma \wedge t_V^\gamma)(u), (t_U^\gamma \wedge t_V^\gamma)(v)\}.$
(iv) $(t_U^\gamma \wedge t_V^\gamma)(uv) = \wedge\{t_U^\gamma(uv), t_V^\gamma(uv)\}$
$\geq \wedge\{\wedge\{t_U^\gamma(u), t_U^\gamma(v)\}, \wedge\{T_B^\gamma(u), t_V^\gamma(v)\}\}$
$= \wedge\{\wedge\{t_U^\gamma(u), T_B^\gamma(u)\}, \wedge\{t_U^\gamma(v), t_U^\gamma(v)\}\}$
$= \wedge\{(t_U^\gamma \wedge t_V^\gamma)(u), (t_U^\gamma \wedge t_V^\gamma)(v)\}.$
(v) $(f_U^\gamma \vee f_V^\gamma)(u-v) = \vee\{f_U^\gamma(u-v), f_V^\gamma(u-v)\}$
$\leq \vee\{\vee\{f_U^\gamma(u), f_U^\gamma(v)\}, \vee\{f_V^\gamma(u), f_V^\gamma(v)\}\}$
$= \vee\{\vee\{f_U^\gamma(u), f_V^\gamma(u)\}, \vee\{f_U^\gamma(v), f_U^\gamma(v)\}\}$
$= \vee\{(f_U^\gamma \vee f_V^\gamma)(u), (f_U^\gamma \vee f_V^\gamma)(v)\}.$
(vi) $(f_U^\gamma \vee f_V^\gamma)(uv) = \vee\{f_U^\gamma(uv), f_V^\gamma(uv)\}$
$\leq \vee\{\vee\{f_U^\gamma(u), f_U^\gamma(v)\}, \vee\{f_V^\gamma(u), f_V^\gamma(v)\}\}$
$= \vee\{\vee\{f_U^\gamma(u), f_V^\gamma(u)\}, \vee\{f_U^\gamma(v), f_U^\gamma(v)\}\}$
$= \vee\{(f_U^\gamma \vee f_V^\gamma)(u), (f_U^\gamma \vee f_V^\gamma)(v)\}.$
Therefore $(U \cap V)$ is an γ-$SVNSR$ of R. □

Remark 3.7. However, the union of two γ-$SVNSRs$ is not an γ-$SVNSR$. For example, consider the set $R = \{0, a, b, a+b\}$, where $a+a = 0 = b+b$ and $a+b = b+a$ and $u.v = 0$ for every $u, v \in R$. Then $(R, +, .)$ is a ring.
Let $U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in R\}$ and $V = \{\langle u, i_V(u), t_V(u), f_V(u)\rangle \mid u \in R\}$, where
$i_U(0) = 0.8,\ i_U(a) = 0.5,\ i_U(b) = 0.4 = i_U(a+b).$
$t_U(0) = 0.7,\ t_U(a) = 0.3,\ t_U(b) = 0.2 = t_U(a+b).$
$f_U(0) = 0.4,\ f_U(a) = 0.7,\ f_U(b) = 0.8 = f_U(a+b).$
$i_V(0) = 0.6,\ i_V(a) = 0.1,\ i_V(b) = 0.5, i_V(a+b) = 0.1.$
$t_V(0) = 0.7,\ t_V(a) = 0.1,\ t_V(b) = 0.3, t_V(a+b) = 0.1.$
$f_V(0) = 0.1,\ f_V(a) = 0.2,\ f_V(b) = 0.2, f_V(a+b) = 0.2.$
Consider $\gamma = 0.6$ then $U^\gamma = \{\langle u, i_U^\gamma(u), t_U^\gamma(u), f_U^\gamma(u)\rangle \mid u \in R\}$ and
$V^\gamma = \{\langle u, i_V^\gamma(u), t_V^\gamma(u), f_V^\gamma(u)\rangle \mid u \in R\}$, where
$i_U^\gamma(0) = 0.6,\ i_U^\gamma(a) = 0.5,\ i_U^\gamma(b) = 0.4 = i_U^\gamma(a+b).$
$t_U^\gamma(0) = 0.6,\ t_U^\gamma(a) = 0.3,\ t_U^\gamma(b) = 0.2 = t_U^\gamma(a+b).$
$f_U^\gamma(0) = 0.6,\ f_U^\gamma(a) = 0.7,\ f_U^\gamma(b) = 0.8 = f_U^\gamma(a+b).$
$i_V^\gamma(0) = 0.6,\ i_V^\gamma(a) = 0.1,\ i_V^\gamma(b) = 0.5, i_V^\gamma(a+b) = 0.1.$

$t_V^\gamma(0) = 0.6,\ t_V^\gamma(a) = 0.1,\ t_V^\gamma(b) = 0.3, t_V^\gamma(a+b) = 0.1.$
$f_V^\gamma(0) = 0.6,\ f_V^\gamma(a) = 0.6,\ f_V^\gamma(b) = 0.6, f_V^\gamma(a+b) = 0.6.$
Then U and V are γ-$SVNSRs$ of R. Now
$(U \cup V)^\gamma == \{\langle u, (i_U \vee i_V)^\gamma)(u), (t_U \vee t_V)^\gamma(u), (f_U^\gamma \wedge f_V)^\gamma(u), (u)\rangle \mid u \in R\},$
Here $(i_U \vee i_V)^\gamma(0) = 0.8,\ (i_U \vee i_V)^\gamma(a) = 0.5,\ (i_U \vee i_V)^\gamma(b) = 0.5,\ (i_U \vee i_V)^\gamma(a+b) = 0.4;$
$(t_U \vee t_V)^\gamma(0) = 0.7,\ (t_U \vee t_V)^\gamma(a) = 0.3,\ (t_U \vee t_V)^\gamma(b) = 0.3,\ (t_U \vee t_V)^\gamma(a+b) = 0.2;$
$(f_U^\gamma \wedge f_V)^\gamma(0) = 0.1,\ (f_U^\gamma \wedge f_V)^\gamma(a) = 0.2,\ (f_U^\gamma \wedge f_V)^\gamma(b) = 0.2,\ (f_U^\gamma \wedge f_V)^\gamma(a+b) = 0.2.$
Now
$(i_U \vee i_V)^\gamma(a+b) = 0.4 < \wedge\{((i_U^\gamma \vee i_V)^\gamma(a), (i_U \vee i_V)^\gamma)(b))\} = 0.5$
Therefore $(U \cup V)^\gamma$ is not an γ-$SVNSR$ of R.

Definition 3.8. A $U^\gamma = (i_U^\gamma, t_U^\gamma, f_U^\gamma)$ of a ring R is said to be an γ-single valued neutrosophic normal subring γ-$SVNNSR$ of R if

(1) $i_U^\gamma(uv) = i_U^\gamma(vu)$.
(2) $t_U^\gamma(uv) = t_U^\gamma(vu)$.
(3) $f_U^\gamma(uv) = f_U^\gamma(vu),\ \forall\ u, v \in R$.

Definition 3.9. A γ-single valued neutrosophic set $U^\gamma = (i_U^\gamma, t_U^\gamma, f_U^\gamma)$ a ring R is said to be an γ-single valued neutrosophic left ideal (γ-$SVNLI$) if

(1) $i_U^\gamma(u-v) \geq \wedge\{i_U^\gamma(u),\ i_U^\gamma(v)\}$.
(2) $i_U^\gamma(uv) \geq i_U^\gamma(v)$.
(3) $t_U^\gamma(u-v) \geq \wedge\{t_U^\gamma(u),\ t_U^\gamma(v)\}$.
(4) $t_U^\gamma(uv) \geq t_U^\gamma(v)$.
(5) $f_U^\gamma(u-v) \leq \vee\{f_U^\gamma(u),\ f_U^\gamma(v)\}$.
(6) $f_U^\gamma(uv) \leq f_U^\gamma(v),\ \forall\ u, v \in R$.

Definition 3.10. A γ-single valued neutrosophic set $U^\gamma = (i_U^\gamma, t_U^\gamma, f_U^\gamma)$ a ring R is said to be an γ-single valued neutrosophic right ideal (γ-$SVNRI$) if

(1) $i_U^\gamma(u-v) \geq \wedge\{i_U^\gamma(u),\ i_U^\gamma(v)\}$.
(2) $i_U^\gamma(uv) \geq i_U^\gamma(u)$.
(3) $t_U^\gamma(u-v) \geq \wedge\{t_U^\gamma(u),\ t_U^\gamma(v)\}$.
(4) $t_U^\gamma(uv) \geq t_U^\gamma(u)$.
(5) $f_U^\gamma(u-v) \leq \vee\{f_U^\gamma(u),\ f_U^\gamma(v)\}$.
(6) $f_U^\gamma(uv) \leq f_U^\gamma(u),\ \forall\ u, v \in R$.

Definition 3.11. A γ-single valued neutrosophic set $U^\gamma = (i_U^\gamma, t_U^\gamma, f_U^\gamma)$ a ring R is said to be an γ-single valued neutrosophic ideal (γ-SVNI) if

(1) $i_U^\gamma(u-v) \geq \wedge\{i_U^\gamma(u),\ i_U^\gamma(v)\}$.

(2) $t_U^\gamma(u-v) \geq \wedge\{t_U^\gamma(u),\ t_U^\gamma(v)\}$.
(3) $f_U^\gamma(u-v) \leq \vee\{f_U^\gamma(u),\ f_U^\gamma(v)\}$.
(4) $i_U^\gamma(uv) \geq \vee\{i_U^\gamma(u),\ i_U^\gamma(v)\}$.
(5) $t_U^\gamma(xv) \geq \vee\{t_U^\gamma(u),\ t_U^\gamma(v)\}$.
(6) $f_U^\gamma(uv) \leq \wedge\{f_U^\gamma(u),\ f_U^\gamma(v)\},\ \forall\ u, v \in R$.

Example 3.12. Let us consider a ring $(Z_4, +_4, \times_4)$ where $Z_4 = \{0, 1, 2, 3\}$ and
Consider $U = \{\langle i_U, t_U, f_U\rangle \mid u \in Z_4\}$ be a single valued neutrosophic subset of Z_4, where
$i_U(0) = 0.4,\ i_U(1) = 0.3 = i_U(3),\ i_U(2) = 0.5.$
$t_U(0) = 0.3,\ t_U(1) = 0.2 = t_U(3),\ t_U(2) = 0.6.$ and
$f_U(0) = 0.2,\ f_U(1) = 0.7 = f_U(3),\ f_U(2) = 0.6.$
Suppose $\gamma = 0.5$ then $U^\gamma = \{\langle i_U^\gamma, t_U^\gamma, f_U^\gamma\rangle \mid u \in Z_4\}$ be an γ-single valued neutrosophic subset of Z_4, where
$i_U^\gamma(0) = 0.4,\ i_U^\gamma(1) = 0.3 = i_U^\gamma(3),\ i_U^\gamma(2) = 0.5.$
$t_U^\gamma(0) = 0.3,\ t_U^\gamma(1) = 0.2 = t_U^\gamma(3),\ t_U^\gamma(2) = 0.5.$ and
$f_U^\gamma(0) = 0.5,\ f_U^\gamma(1) = 0.7 = f_U^\gamma(3),\ f_U^\gamma(2) = 0.6.$
$\Rightarrow U$ is an γ-$SVNI$ of Z_4.

Theorem 3.13. *If $U^\gamma = \{\langle i_U^\gamma, t_U^\gamma, f_U^\gamma\rangle \mid u \in R\}$ is a γ-$SVNI$ of a ring R, then*
$i_U^\gamma(0) \geq i_U^\gamma(u),\ t_U^\gamma(0) \geq t_U^\gamma(u),\ f_U^\gamma(0) \leq f_U^\gamma(u)$
and $i_U^\gamma(-u) = i_U^\gamma(u),\ t_U^\gamma(-u) = t_U^\gamma(u),\ f_U^\gamma(-u) = F_U^\gamma(u),\ \forall\ u \in R.$

Proof. Let $i_U^\gamma(0) = i_U^\gamma(u-u) \geq \wedge\{i_U^\gamma(u), i_U^\gamma(u)\} = i_U^\gamma(u)$.
$t_U^\gamma(0) = t_U^\gamma(u-u) \geq \wedge\{t_U^\gamma(u), t_U^\gamma(u)\} = t_U^\gamma(u)$.
Similarly $f_U^\gamma(0) = f_U^\gamma(u-u) \leq \vee\{f_U^\gamma(u), f_U^\gamma(u)\} = f_U^\gamma(u)$.
Next $i_U^\gamma(-x) = i_U^\gamma(0-u) \geq \wedge\{i_U^\gamma(0), i_U^\gamma(u)\} = i_U^\gamma(u)$.
Also $i_U^\gamma(u) = i_U^\gamma\{0-(-u)\} \geq \wedge\{i_U^\gamma(0), i_U^\gamma(-u)\} = i_U^\gamma(-u)$.
Therefore $i_U^\gamma(-u) = i_U^\gamma(u)$.
So $t_U^\gamma(-u) = t_U^\gamma(0-u) \geq \wedge\{t_U^\gamma(0), t_U^\gamma(u)\} = t_U^\gamma(u)$.
Also $t_U^\gamma(u) = t_U^\gamma\{0-(-u)\} \geq \wedge\{t_U^\gamma(0), t_U^\gamma(-u)\} = t_U^\gamma(-u)$.
Therefore $t_U^\gamma(-u) = t_U^\gamma(u)$.
Finally $f_U^\gamma(-u) = f_U^\gamma(0-u) \leq \vee\{f_U^\gamma(0), f_U^\gamma(u)\} = f_U^\gamma(u)$.
Also $f_U^\gamma(u) = f_U^\gamma\{0-(-u)\} \geq \vee\{f_U^\gamma(-u), f_U^\gamma(0)\} = f_U^\gamma(-u)$.
Therefore $f_U^\gamma(-u) = f_U^\gamma(u)$. □

Remark 3.14. Every γ-$SVNI$ of a ring R is an γ-$SVNSR$ of R. However the converse is not true.

For example, let $(R, +, .)$ be the ring of real numbers.
Define, $U = \{\langle u, i_U(u), t_U(u), i_U(u)\rangle | \ u \in R\}$ such that
$i_U(u) = 0.5$ if u is rational, $t_U(u) = 0.8$ if u is rational, $f_U(u) = 0.1$ if u is rational.
$i_U(u) = 0.4$ if u is irrational, $t_U(u) = 0.3$ if u is irrational, $f_U(u) == 0.7$ if u is irrational.
Consider $\gamma = 0.6$, now define $U^\gamma = \{\langle u, i_U^\gamma(u), t_U^\gamma(u), i_U^\gamma(u)\rangle | \ u \in R\}$ then
$i_U^\gamma(u) = 0.5$ if u is rational, $t_U^\gamma(u) = 0.6$ if u is rational, $f_U^\gamma(u) = 0.6$ if u is rational.
$i_U^\gamma(u) = 0.4$ if u is irrational, $t_U^\gamma(u) = 0.3$ if u is irrational, $f_U^\gamma(u) = 0.7$ if u is irrational.
Then U is an γ-$SVNSR$ of R.
But U is not an γ-$SVNI$ of R, since $i_U^\gamma(2\sqrt{2}) = 0.4 < \vee\{i_U^\gamma(2), i_U^\gamma(\sqrt{2})\}$.

Theorem 3.15. *Let U and V be two γ-$SVNIs$ of a ring R. Then $U \cap V$ is also a γ-$SVNI$ of R.*

Proof. Let $U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in R\}$ and $V = \{\langle u, i_V(u), t_V(u), f_V(u)\rangle \mid u \in R\}$ be any two γ-$SVNIs$ of a ring R. Then,
$U^\gamma = \{\langle u, i_U^\gamma(u), t_U^\gamma(u), f_U^\gamma(u)\rangle \mid u \in R\}$ and $V^\gamma = \{\langle u, i_V^\gamma(u), t_V^\gamma(u), f_V^\gamma(u)\rangle \mid u \in R\}$, then by using Proposition 3.5
$(U \cap V)^\gamma = U^\gamma \cap V^\gamma = \{\langle u, (i_U^\gamma \wedge i_V^\gamma)(u), (t_U^\gamma \wedge t_V^\gamma)(u), (f_U^\gamma \vee f_V^\gamma)(u)\rangle \mid u \in R\}$.
Now for any $u, v \in R$, we have
(i) $(i_U^\gamma \wedge i_V^\gamma)(u - v) = \wedge\{i_U^\gamma(u - v), i_V^\gamma(u - v)\}$
$\geq \wedge\{\wedge\{i_U^\gamma(u), i_U^\gamma(v)\}, \wedge\{i_V^\gamma(u), i_V^\gamma(v)\}\}$
$= \wedge\{\wedge\{i_U^\gamma(u), i_V^\gamma(u)\}, \wedge\{i_U^\gamma(v), i_U^\gamma(v)\}\}$
$= \wedge\{(i_U^\gamma \wedge i_V^\gamma)(u), (i_U^\gamma \wedge i_V^\gamma)(v)\}$.
(ii) $(i_U^\gamma \wedge i_V^\gamma)(uv) = \wedge\{i_U^\gamma(uv), i_V^\gamma(xv)\}$
$\geq \wedge\{\vee\{i_U^\gamma(u), i_U^\gamma(v)\}, \vee\{i_V^\gamma(u), i_V^\gamma(v)\}\}$
$\geq \vee\{\wedge\{i_U^\gamma(u), i_V^\gamma(u)\}, \wedge\{i_U^\gamma(v), i_U^\gamma(v)\}\}$
$= \vee\{(i_U^\gamma \wedge i_V^\gamma)(x), (i_U^\gamma \wedge i_V^\gamma)(v)\}$.
(iii) $(t_U^\gamma \wedge t_V^\gamma)(u - v) = \wedge\{t_U^\gamma(u - v), t_V^\gamma(u - v)\}$
$\geq \wedge\{\wedge\{t_U^\gamma(u), t_U^\gamma(v)\}, \wedge\{t_V^\gamma(u), t_V^\gamma(v)\}\}$
$= \wedge\{\wedge\{t_U^\gamma(u), t_V^\gamma(u)\}, \wedge\{t_U^\gamma(v), t_U^\gamma(v)\}\}$
$= \wedge\{(t_U^\gamma \wedge t_V^\gamma)(u), (t_U^\gamma \wedge t_V^\gamma)(v)\}$.
(iv) $(t_U^\gamma \wedge t_V^\gamma)(uv) = \wedge\{t_U^\gamma(uv), t_V^\gamma(uv)\}$
$\geq \wedge\{\vee\{t_U^\gamma(u), t_U^\gamma(v)\}, \vee\{t_V^\gamma(u), t_V^\gamma(v)\}\}$
$\geq \vee\{\wedge\{t_U^\gamma(u), t_V^\gamma(u)\}, \wedge\{t_U^\gamma(v), t_U^\gamma(v)\}\}$
$= \vee\{(t_U^\gamma \wedge t_V^\gamma)(u), (t_U^\gamma \wedge t_V^\gamma)(v)\}$.
(v) $(f_U^\gamma \vee f_V^\gamma)(u - v) = \vee\{f_U^\gamma(u - v), f_V^\gamma(u - v)\}$
$\leq \vee\{\vee\{f_U^\gamma(u), f_U^\gamma(v)\}, \vee\{f_V^\gamma(u), f_V^\gamma(v)\}\}$
$= \vee\{\vee\{f_U^\gamma(u), f_V^\gamma(u)\}, \vee\{f_U^\gamma(v), f_U^\gamma(v)\}\}$

$= \vee\{(f_U^\gamma \vee f_V^\gamma)(u), (f_U^\gamma \vee f_V^\gamma)(v)\}$.
(vi) $(f_U^\gamma \vee f_V^\gamma)(uv) = \vee\{f_U^\gamma(uv), f_V^\gamma(uv)\}$
$\leq \vee\{\wedge\{f_U^\gamma(u), f_U^\gamma(v)\}, \wedge\{f_V^\gamma(u), f_V^\gamma(v)\}\}$
$\leq \wedge\{\vee\{f_U^\gamma(u), f_V^\gamma(u)\}, \vee\{f_U^\gamma(v), f_U^\gamma(v)\}\}$
$= \wedge\{(f_U^\gamma \vee f_V^\gamma)(u), (f_U^\gamma \vee f_V^\gamma)(v)\}$.
Therefore $U \cap V$ is an γ-SVNI of R. □

Remark 3.16. Union of two γ-$SVNIs$ of R need not to be γ-$SVNI$ of R.

Remark 3.17. If U is an γ-$SVNSR$ and V is an γ-$SVNI$ of a ring R then $U \cap V$ is an γ-$SVNSR$ of R but not an γ-$SVNI$ of R. For example, consider the ring $(R, +, .)$ of real numbers and define,
$U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in R\}$ such that
$i_U(u) = 0.7$ if u is rational, $t_U(u) = 0.6$ if u is rational, $f_U(u) = 0.1$ if u is rational.
$i_U(u) = 0.2$ if u is irrational, $i_U(u) = 0.1$ if u is irrational, $f_U(u) = 0.8$ if u is irrational.
Also define $V = \{\langle u, i_V(u), t_V(u), f_V(u)\rangle | u \in R\}$ such that
$i_V(u) = 0.5$, $t_V(u) = 0.4$ and $f_V(u) = 0.6$ $\forall$ $u \in R$. Consider $\gamma = 0.5$ then
$i_U^\gamma(u) = 0.5$ if u is rational, $t_U^\gamma(u) = 0.5$ if u is rational, $f_U^\gamma(u) = 0.5$ if u is rational.
$i_U^\gamma(u) = 0.2$ if u is irrational, $i_U^\gamma(u) = 0.1$ if u is irrational, $f_U^\gamma(u) = 0.8$ if u is irrational.
Then $U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in R\}$ is an γ-$SVNSR$ of R.
Similarly, $i_V^\gamma(u) = 0.5$, $t_V^\gamma(u) = 0.4$ and $i_V^\gamma(u) = 0.6$ $\forall$ $u \in R$.
Then $V = \{\langle u, i_V(u), t_V(u), f_V(u)\rangle | u \in R\}$ is an γ-$SVNI$ of R.
Then by using Proposition 3.5
$(U \cap V)^\gamma = U^\gamma \cap V^\gamma = \{\langle u, (i_U^\gamma \wedge i_V^\gamma)(u), (t_U^\gamma \wedge t_V^\gamma)(u), (f_U^\gamma \vee f_V^\gamma)(u)\rangle \mid u \in R\}$ is not an γ-$SVNI$ of R, because $(i_U^\gamma \wedge i_V^\gamma)(2\sqrt{2}) < \vee\{(i_U^\gamma \wedge i_V^\gamma)(2), (i_U^\gamma \wedge i_V^\gamma)(\sqrt{2})\}$.

4. Sum and Product of γ-Single Valued Neutrosophic Ideal (γ-SVNI)

In this section, we elaborate some fundamental principles and results related to the sum and product of the γ-single valued neutrosophic ideal.

Definition 4.1. Let U and V be two γ-$SVNIs$ of a ring R then their sum $(U+V)^\gamma$ is defined as $(U+V)^\gamma = \{\langle u, (i_U^\gamma + i_V^\gamma)(u), (t_U^\gamma + t_V^\gamma)(u), (f_U^\gamma + f_V^\gamma)(u)\rangle \mid u \in R\}$, where
$(i_U^\gamma + i_V^\gamma)(u) = \sup\limits_{u=a+b}\{\wedge \; \{i_U^\gamma(a), i_U^\gamma(b)\}\}$,
$(t_U^\gamma + t_V^\gamma)(u) = \sup\limits_{u=a+b}\{\wedge \; \{t_U^\gamma(a), t_U^\gamma(b)\}\}$, and
$(f_U^\gamma + f_V^\gamma)(u) = \inf\limits_{u=a+b}\{\vee \; \{f_U^\gamma(a), f_U^\gamma(b)\}\}$.

Definition 4.2. Let U and V be two γ-$SVNIs$ of a ring R then their product $(UV)^\gamma$ is defined as $(UV)^\gamma = \{\langle u, (i_U^\gamma i_V^\gamma)(u), (t_U^\gamma t_V^\gamma)(u), (f_U^\gamma f_V^\gamma)(u)\rangle \mid u \in R\}$, where

$(i_U^\gamma i_V^\gamma)(u) = \sup_{\substack{u=\sum a_i b_i \\ i<\infty}} \{\wedge \{\wedge\{i_U^\gamma(a_i), i_U^\gamma(b_i)\}\}\}$,
$(t_U^\gamma t_V^\gamma)(u) = \sup_{\substack{u=\sum a_i b_i \\ i<\infty}} \{\wedge \{\wedge\{t_U^\gamma(a_i), t_U^\gamma(b_i)\}\}\}$, and
$(f_U^\gamma f_V^\gamma)(u) = \inf_{\substack{u=\sum a_i b_i \\ i<\infty}} \{\vee \{\vee\{f_U^\gamma(a_i), f_U^\gamma(b_i)\}\}\}$.

Theorem 4.3. *If U and V are two γ-$SVNIs$ of a ring R, then $U + V$ is also an γ-$SVNI$ of R.*

Proof. Let $U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in R\}$ and $V = \{\langle u, i_V(u), t_V(u), f_V(u) \mid u \in R\rangle\}$ be two γ-$SVNIs$ of a ring R, so $U^\gamma = \{\langle u, i_U^\gamma(u), t_U^\gamma(u), f_U^\gamma(u)\rangle \mid u \in R\}$ and
$V^\gamma = \{\langle u, i_V^\gamma(u), t_V^\gamma(u), f_V^\gamma(u) \mid u \in R\rangle\}$, then their sum $(U + V)^\gamma$ is given by
$(U + V)^\gamma = \{\langle u, (i_U^\gamma + i_V^\gamma)(u), (t_U^\gamma + t_V^\gamma)(u), (f_U^\gamma + f_V^\gamma)(u)\rangle \mid u \in R\}$.
Let $u, v \in R$ and let $\wedge\{(i_U^\gamma i_V^\gamma)(u),\ (i_U^\gamma i_V^\gamma)(v)\} = l$. Then for any $\epsilon > 0$,
$l - \epsilon < (i_U^\gamma + i_V^\gamma)(u) = \sup_{u=a+b} \{\wedge \{i_U^\gamma(a), i_U^\gamma(b)\}\}$,
$l - \epsilon < (i_U^\gamma + i_V^\gamma)(v) = \sup_{v=c+d} \{\wedge \{i_U^\gamma(c), i_U^\gamma(d)\}\}$.
So there exist representations $u = a + b,\ v = c + d$, where $a, b, c, d \in R$ such that
$l - \epsilon < \wedge\{i_U^\gamma(a), i_V^\gamma(b)\}$ and $l - \epsilon < \wedge\{i_U^\gamma(c), i_U^\gamma(d)\}$.
$\Rightarrow\ l - \epsilon < i_U^\gamma(a), i_V^\gamma(b)$ and $l - \epsilon < i_U^\gamma(c), i_U^\gamma(d)$.
$\Rightarrow\ l - \epsilon < \wedge\{i_U^\gamma(a), i_U^\gamma(c)\} \leq i_U^\gamma(a + c)$ and $l - \epsilon < \wedge\{i_V^\gamma(b), i_V^\gamma(d)\} \leq i_V^\gamma(b + d)$.
Thus we get $u + v = (a + b) + (c + d) = (a + c) + (b + d)$ such that
$l - \epsilon < \wedge\{i_U^\gamma(a + c), i_V^\gamma(b + d\}$.
$\Rightarrow\ l - \epsilon < \sup_{u+v=(a+c)+(b+d)} \{\wedge i_U^\gamma(a + c), i_V^\gamma(b + d)\} = (i_U^\gamma + i_V^\gamma)(u + v)$.
Since ϵ is arbitrary, it follows that,
$(i_U^\gamma + i_V^\gamma)(u + v) \geq l = \wedge\{(i_U^\gamma + i_V^\gamma)(u), (i_U^\gamma + i_V^\gamma)(v)\}$.
Next, let $m = \vee\{(i_U^\gamma + i_V^\gamma)(u), (i_U^\gamma + i_V^\gamma)(v)\} = (i_U^\gamma + i_V^\gamma)(u)$ (say) and $\epsilon > 0$.
Then $m - \epsilon < (i_U^\gamma + i_V^\gamma)(u) = \sup_{u=a+b} \{\wedge i_U^\gamma(a), i_V^\gamma(b)\}$.
So there exists a representation $u = a + b$ such that
$m - \epsilon < \wedge\{i_U^\gamma(a), i_V^\gamma(b)\}$.
$\Rightarrow\ m - \epsilon < i_U^\gamma(a), i_V^\gamma(b)$.
$m - \epsilon < \vee\{i_U^\gamma(a), i_U^\gamma(c + d)\} = i_U^\gamma(a(c + d))$, where $v = c + d$,
and $m - \epsilon < \vee\{i_V^\gamma(b), i_V^\gamma(c + d)\} = i_V^\gamma(b(c + d))$.
$\Rightarrow\ m - \epsilon < \wedge\{i_U^\gamma(a(c + d)), i_V^\gamma(b(c + d))\}$.
So we get, $uv = (a + b)(c + d) = a(c + d) + b(c + d)$, such that
$m - \epsilon < \wedge\{i_U^\gamma(a(c + d)), i_V^\gamma(b(c + d))\}$.
$\Rightarrow\ m - \epsilon < \sup_{uv=a(c+d)+b(c+d)} \{\wedge\{i_U^\gamma(a(c + d)), i_V^\gamma(b(c + d))\}\} = (i_U^\gamma + i_V^\gamma)(uv)$.
Since ϵ is arbitrary,

$(i^{\gamma}_U + i^{\gamma}_V)(uv) \geq m = \vee\{(i^{\gamma}_U + i^{\gamma}_V)(u), (i^{\gamma}_U + i^{\gamma}_V)(v)\}$.

Similarly we can show that

$(t^{\gamma}_U + t^{\gamma}_V)(uv) \geq s = \vee\{(t^{\gamma}_U + t^{\gamma}_V)(u), (t^{\gamma}_U + t^{\gamma}_V)(v)\}$.

Next let $\vee\{(f^{\gamma}_U + f^{\gamma}_V)(u), (f^{\gamma}_U + f^{\gamma}_V)(v)\} = n$ and $\epsilon > 0$.

Then $n + \epsilon > (f^{\gamma}_U + f^{\gamma}_V)(u) = \inf_{u=a+b}\{\vee\{f^{\gamma}_U(a), f^{\gamma}_U(b)\}\}$,

and $n + \epsilon > (f^{\gamma}_U + f^{\gamma}_V)(v) = \inf_{v=c+d}\{\vee\{f^{\gamma}_U(c), f^{\gamma}_U(d)\}\}$.

So, there exist representations $u = a + b$ and $v = c + d$, for some $a, b, c, d \in R$ such that

$n + \epsilon > \vee\{f^{\gamma}_U(a), f^{\gamma}_V(b)\}$ and $n + \epsilon > \vee\{f^{\gamma}_U(c), f^{\gamma}_V(d)\}$.

$\Rightarrow n + \epsilon > f^{\gamma}_U(a), f^{\gamma}_V(b)$ and $n + \epsilon > f^{\gamma}_U(c), f^{\gamma}_V(d)$.

$\Rightarrow n + \epsilon > \vee\{f^{\gamma}_U(a), f^{\gamma}_U(c)\} = f^{\gamma}_U(a + c)$, and $n + \epsilon > \vee\{f^{\gamma}_U(b), f^{\gamma}_U(d)\} \geq f^{\gamma}_U(b + d)$.

Thus we get, $u + v = (a + b) + (c + d) = (a + c) + (b + d)$, such that

$n + \epsilon > \vee\{f^{\gamma}_U(a + c), f^{\gamma}_V(b + d)\}$.

$\Rightarrow\ n + \epsilon < \inf_{u+v=(a+c)+(b+d)}\{\vee\{f^{\gamma}_U(a + c), f^{\gamma}_V(b + d)\}\} = (f^{\gamma}_U + f^{\gamma}_V)(u + v)$.

Since ϵ is arbitrary,

$(f^{\gamma}_U + f^{\gamma}_V)(u + v) \leq n = \vee\{(f^{\gamma}_U + f^{\gamma}_V)(u), (f^{\gamma}_U + f^{\gamma}_V)(v)\}$.

Finally, if $w = \wedge\{(f^{\gamma}_U + f^{\gamma}_V)(u), (f^{\gamma}_U + f^{\gamma}_V)(v)\} = (f^{\gamma}_U + f^{\gamma}_V)(u)$ (say), and $\epsilon > 0$,

then $w + \epsilon > (f^{\gamma}_U + f^{\gamma}_V)(u) = \inf_{u=a+b}\{\vee f^{\gamma}_U(a), f^{\gamma}_U(b)\}$.

So there exists a representation $u = a + b$ such that $w + \epsilon > \vee\{f^{\gamma}_U(a), f^{\gamma}_V(b)\}$.

$\Rightarrow w + \epsilon > f^{\gamma}_U(a)$ and $w + \epsilon > f^{\gamma}_V(b)$.

$\Rightarrow w + \epsilon > \wedge\{f^{\gamma}_U(a), f^{\gamma}_U(c + d)\} = f^{\gamma}_U(a(c + d))$, and

$w + \epsilon > \wedge\{f^{\gamma}_V(b), f^{\gamma}_V(c + d)\} = f^{\gamma}_V(b(c + d))$, where $v = c + d$.

So, we get $uv = (a + b)(c + d) = a(c + d) + b(c + d)$ such that

$w + \epsilon > \vee\{f^{\gamma}_U(a(c + d)), f^{\gamma}_V(b(c + d))\}$.

$\Rightarrow w + \epsilon > \inf_{uv=a(c+d)+b(c+d)}\{\vee(f^{\gamma}_U(a(c + d)), f^{\gamma}_V(b(c + d)))\} = (f^{\gamma}_U + f^{\gamma}_V)(uv)$.

Since ϵ is arbitrary,

$(f^{\gamma}_U + f^{\gamma}_V)(uv) \leq w = \wedge\{(f^{\gamma}_U + f^{\gamma}_V)(u), (f^{\gamma}_U + f^{\gamma}_V)(v)\}$.

Hence $U + V$ is an γ-$SVNI$ of R. □

Theorem 4.4. *If U and V are two γ-$SVNIs$ of a ring R, then UV is also an γ-$SVNI$ of R.*

Proof. Let $U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in R\}$ and $V = \{\langle u, i_V(u), t_V(u), f_V(u) \mid u \in R\rangle\}$ be two γ-$SVNIs$ of a ring R, so

$U^{\gamma} = \{\langle u, i^{\gamma}_U(u), t^{\gamma}_U(u), f^{\gamma}_U(u)\rangle \mid u \in R\}$ and $V^{\gamma} = \{\langle u, i^{\gamma}_V(u), t^{\gamma}_V(u), f^{\gamma}_V(u) \mid u \in R\rangle\}$.

Then $(UV)^{\gamma} = \{\langle u, (i^{\gamma}_U i^{\gamma}_V)(u), (t^{\gamma}_U t^{\gamma}_V)(u), (f^{\gamma}_U f^{\gamma}_V)(u)\rangle \mid u \in R\}$.

Let $u, v \in R$ and let $\wedge\{(i^{\gamma}_U i^{\gamma}_V)(u), (i^{\gamma}_U i^{\gamma}_V)(v)\} = \varsigma$.

Then for any $\epsilon > 0$,

$$\varsigma - \epsilon < (i_U^\gamma i_V^\gamma)(u) = \sup_{\substack{u=\sum a_i b_i \\ i<\infty}} \{\wedge \{\wedge\{i_U^\gamma(a_i), i_U^\gamma(b_i)\}\}\}, \text{ and}$$
$$\varsigma - \epsilon < (i_U^\gamma i_V^\gamma)(v) = \sup_{\substack{u=\sum m_i n_i \\ i<\infty}} \{\wedge \{\wedge\{i_U^\gamma(m_i), i_U^\gamma(n_i)\}\}\}.$$

So we get representations $u = \sum_{i<\infty} a_i b_i$ and $v = \sum_{i<\infty} m_i n_i$ such that

$\varsigma - \epsilon < \{\wedge \{\wedge\{i_U^\gamma(a_i), i_U^\gamma(b_i)\}\}\}$, and $\varsigma - \epsilon < \{\wedge \{\wedge\{i_U^\gamma(m_i), i_U^\gamma(n_i)\}\}\}$,

$\Rightarrow\ \varsigma - \epsilon < \wedge\{i_U^\gamma(a_i), i_U^\gamma(b_i)\}$, and $\varsigma - \epsilon < \wedge\{i_U^\gamma(m_i), i_U^\gamma(n_i)\}\ \forall\ i$,

$\Rightarrow\ \varsigma - \epsilon < i_U^\gamma(a_i), i_U^\gamma(b_i)$, and $\varsigma - \epsilon < i_U^\gamma(m_i), i_U^\gamma(n_i)\ \forall\ i$,

$\Rightarrow\ \varsigma - \epsilon < \wedge\{i_U^\gamma(a_i), i_U^\gamma(b_i)\} \leq i_U^\gamma(a_i + m_i)$, and $\varsigma - \epsilon < \wedge\{i_U^\gamma(m_i), i_U^\gamma(n_i)\} \leq i_V^\gamma(b_i + n_i)\ \forall\ i$.

Thus, we get $u + v = \sum_{i<\infty} (a_i b_i + m_i n_i)$, where $a_i, b_i, m_i, n_i \in R$, such that

$\varsigma - \epsilon < \{\wedge\{i_U^\gamma(a_i + m_i), i_V^\gamma(b_i + n_i)\}\},\ \forall\ i$,

$\Rightarrow\ \varsigma - \epsilon < \bigwedge_i \{\wedge\{i_U^\gamma(a_i + m_i), i_V^\gamma(b_i + n_i)\}\}$,

$$\varsigma - \epsilon < \sup_{\substack{u=\sum(a_i b_i + m_i n_i) \\ i<\infty}} \{\bigwedge_i \{\wedge\{i_U^\gamma(a_i + m_i), i_U^\gamma(b_i + n_i)\}\}\} = (i_U^\gamma i_V^\gamma)(u + v).$$

Since ϵ is arbitrary, so we have,

$(i_U^\gamma i_V^\gamma)(u + v) \geq \varsigma = \wedge\{(i_U^\gamma i_V^\gamma)(u), (i_U^\gamma i_V^\gamma)(v)\}$.

Next let $g = \vee\{(i_U^\gamma i_V^\gamma)(u), (i_U^\gamma i_V^\gamma)(v)\} = (i_U^\gamma i_V^\gamma)(u)$ (say) and let $\epsilon > 0$, then

$$g - \epsilon < (i_U^\gamma i_V^\gamma)(u) = \sup_{\substack{u=\sum a_i b_i \\ i<\infty}} \{\bigwedge_i \{\wedge\{i_U^\gamma(a_i), \{i_V^\gamma b_i)\}\}\}.$$

So there exists a representation $u = \sum_{i<\infty} a_i b_i$ such that

$g - \epsilon < \bigwedge_i \{\wedge\{i_U^\gamma(a_i), \{i_V^\gamma b_i)\}\}\ \Rightarrow \wedge\{i_U^\gamma(a_i), \{i_V^\gamma b_i\},\ \forall\ i$.

$\Rightarrow\ g - \epsilon <\ i_U^\gamma(a_i), i_V^\gamma(b_i),\ \forall\ i$.

If $v = \sum_{i<\infty} m_i n_i$ then

$g - \epsilon < \vee\{i_U^\gamma(a_i), i_U^\gamma(m_i)\} = i_U^\gamma(a_i m_i)\ \forall\ i$,

and $g - \epsilon < \vee\{i_V^\gamma(b_i), i_V^\gamma(n_i)\} = i_V^\gamma(b_i n_i),\ \forall\ i$.

Thus, we get $uv = \sum_{i<\infty} (a_i b_i)(m_i n_i) = \sum_{i<\infty} (a_i m_i)(b_i n_i)$

such that $g - \epsilon < \wedge\{i_U^\gamma(a_i m_i), i_V^\gamma(b_i n_i)\},\ \forall\ i$.

$\Rightarrow\ g - \epsilon < \bigwedge_i \{\wedge\{i_U^\gamma(a_i m_i), i_V^\gamma(b_i n_i)\}\}$.

$$\Rightarrow\ g - \epsilon < \sup_{\substack{uv=\sum(a_i m_i)(b_i n_i) \\ i<\infty}} \{\wedge\{i_U^\gamma(a_i m_i), i_V^\gamma(b_i n_i)\}\} = (i_U^\gamma i_V^\gamma)(uv).$$

Since ϵ is arbitrary

$(i_U^\gamma i_V^\gamma)(uv) \geq g = \vee\{(i_U^\gamma i_V^\gamma)(u), (i_U^\gamma i_V^\gamma)(v)\}$.

Similarly, we can show that

$(t_U^\gamma t_V^\gamma)(u + v) \geq j = \wedge\{(t_U^\gamma t_V^\gamma)(u), (t_U^\gamma i_V^\gamma)(v)\}$.

$(t_U^\gamma t_V^\gamma)(uv) \geq \delta = \vee\{(t_U^\gamma t_V^\gamma)(u), (t_U^\gamma t_V^\gamma)(v)\}$.

Next, let $l = \vee\{(f_U^\gamma f_V^\gamma)(u), (f_U^\gamma f_V^\gamma)(v)\}$ and $\epsilon > 0$, then

$\Rightarrow\ l+\epsilon > (f_U^\gamma f_V^\gamma)(u) = \inf\limits_{\substack{u=\sum a_ib_i \\ i<\infty}} \{\vee\{\bigvee\limits_i \{f_U^\gamma(a_i), f_V^\gamma(b_i)\}\}\}$,

$\Rightarrow\ l+\epsilon > (f_U^\gamma f_V^\gamma)(v) = \inf\limits_{\substack{u=\sum m_in_i \\ i<\infty}} \{\vee\{\bigvee\limits_i \{f_U^\gamma(m_i), f_V^\gamma(n_i)\}\}\}$.

So, we get representations $u = \sum\limits_{i<\infty} a_ib_i$ and $v = \sum\limits_{i<\infty} m_in_i$, where $a_i, b_i, m_i, n_i \in R$, such that

$l+\epsilon > \bigvee\limits_i \{\vee\{f_U^\gamma(a_i), f_V^\gamma(b_i)\}$ and $l+\epsilon > \bigvee\limits_i \{\vee\{f_U^\gamma(m_i), f_V^\gamma(n_i)\}\}$.

$\Rightarrow\ l+\epsilon > \vee\{f_U^\gamma(a_i), f_V^\gamma(b_i)\}$ and $l+\epsilon > \vee\{f_U^\gamma(m_i), f_V^\gamma(n_i)\},\ \forall\ i.$

$\Rightarrow\ l+\epsilon > f_U^\gamma(a_i), f_V^\gamma(b_i)$ and $l+\epsilon > f_U^\gamma(m_i), f_V^\gamma(n_i),\ \forall\ i.$

$\Rightarrow\ l+\epsilon > \vee\{f_U^\gamma(a_i), f_V^\gamma(m_i)\} \geq f_U^\gamma(a_i+m_i)$ and $l+\epsilon > \vee\{f_U^\gamma(b_i), f_V^\gamma(n_i)\} \geq f_U^\gamma(b_i+n_i),\ \forall\ i.$

Thus, we get $u+v = \sum\limits_{i<\infty} (a_ib_i + m_in_i)$, where $a_i, b_i, m_i, n_i \in R$, such that

$l+\epsilon > \vee\{f_U^\gamma(a_i+m_i), f_V^\gamma(b_i+n_i)\},\ \forall\ i.$

$\Rightarrow\ l+\epsilon > \bigvee\limits_i \{\vee\{f_U^\gamma(a_i+m_i), f_V^\gamma(b_i+n_i)\}\}$,

$l+\epsilon > \sup\limits_{\substack{u=\sum(a_ib_i+m_in_i) \\ i<\infty}} \{\bigvee\limits_i \{\vee\{f_U^\gamma(a_i+m_i), f_U^\gamma(b_i+n_i)\}\}\} = (f_U^\gamma f_V^\gamma)(u+v).$

Since ϵ is arbitrary, so we have,

$(f_U^\gamma f_V^\gamma)(u+v) \leq o = \vee\{(f_U^\gamma f_V^\gamma)(u), (f_U^\gamma f_V^\gamma)(v)\}$.

Finally, let $o = \wedge\{(f_U^\gamma f_V^\gamma)(u), (f_U^\gamma f_V^\gamma)(v)\} = (f_U^\gamma f_V^\gamma)(u)$ (say) and let $\epsilon > 0$, then

$o+\epsilon > (f_U^\gamma f_V^\gamma)(u) = \inf\limits_{\substack{u=\sum a_ib_i \\ i<\infty}} \{\bigvee\limits_i \{\vee\{f_U^\gamma(a_i), \{f_V^\gamma b_i)\}\}\}$.

So there exists a representation $u = \sum\limits_{i<\infty} a_ib_i$ such that

$o+\epsilon > \bigvee\limits_i \{\vee\{f_U^\gamma(a_i), \{f_V^\gamma b_i)\}\}\ \Rightarrow \vee\{f_U^\gamma(a_i), \{f_V^\gamma b_i\},\ \forall\ i.$

$\Rightarrow\ r+\epsilon >\ f_U^\gamma(a_i), f_V^\gamma(b_i),\ \forall\ i.$

If $v = \sum\limits_{i<\infty} m_in_i$ then

$o+\epsilon > \vee\{f_U^\gamma(a_i), f_U^\gamma(m_i)\} \geq f_U^\gamma(a_im_i)\ \forall\ i,$

and $o+\epsilon > \vee\{f_V^\gamma(b_i), f_V^\gamma(n_i)\} \geq f_V^\gamma(b_in_i),\ \forall\ i.$

Thus, we get $uv = \sum\limits_{i<\infty} (a_ib_i)(m_in_i) = \sum\limits_{i<\infty} (a_im_i)(b_in_i)$

such that $o+\epsilon > \vee\{f_U^\gamma(a_im_i), f_V^\gamma(b_in_i)\},\ \forall\ i.$

$\Rightarrow\ o+\epsilon > \bigvee\limits_i \{\vee\{f_U^\gamma(a_im_i), f_V^\gamma(b_in_i)\}\}$.

$\Rightarrow\ o+\epsilon > \inf\limits_{\substack{uv=\sum(a_im_i)(b_in_i) \\ i<\infty}} \{\vee\{f_U^\gamma(a_im_i), f_V^\gamma(b_in_i)\}\} = (f_U^\gamma f_V^\gamma)(uv).$

Since ϵ is arbitrary

$(f_U^\gamma f_V^\gamma)(uv) \leq o = \wedge\{(f_U^\gamma f_V^\gamma)(u), (f_U^\gamma f_V^\gamma)(v)\}$.

Hence UV is an γ-$SVNI$ of R. □

Remark 4.5. According to the definition given by Atanassov [1] the sum and product of two γ-$SVNIs$ of a ring R is not necessarily an γ-$SVNI$ of R as shown by the following example: Consider the ring $R = \{0, a, b, a+b\}$ where $a+a = 0 = b+b, a+b = b+a$ and $uv = 0$

$\forall\ u,\ v \in R$. We define,
$i_U^\gamma(0) = 0.9 = i_U^\gamma(a), i_U^\gamma(b) = 0.4 = i_U^\gamma(a+b)$;
$t_U^\gamma(0) = 0.9 = t_U^\gamma(a), t_U^\gamma(b) = 0.4 = t_U^\gamma(a+b)$;
$f_U^\gamma(0) = 0.1 = f_U^\gamma(a), f_U^\gamma(b) = 0.4 = f_U^\gamma(a+b)$.
And $i_V^\gamma(0) = 0.7, i_V^\gamma(a) = 0.3 = i_V^\gamma(a+b), i_V^\gamma(b) = 0.5$;
$i_V^\gamma(0) = 0.7, i_V^\gamma(a) = 0.3 = i_V^\gamma(a+b), i_V^\gamma(b) = 0.5$;
$f_V^\gamma(0) = 0.2, f_V^\gamma(a) = 0.6 = f_V^\gamma(a+b), f_V^\gamma(b) = 0.5$.
Then $U = \{\langle u, i_U(u), t_U(u), f_U(u)\rangle \mid u \in R\}$ and $V = \{\langle u, i_V(u), t_V(u), f_V(u)\rangle \mid u \in R\}$ are γ-$SVNIs$ of R. According to Atanassov [1],
$(U+V)^\gamma = \{\langle u, i_U^\gamma(u)+i_V^\gamma(u)-i_U^\gamma(u)i_V^\gamma(u), t_U^\gamma(u)+t_V^\gamma(u)-t_U^\gamma(u)t_V^\gamma(u), f_U^\gamma(u)f_U^\gamma(u)\rangle \mid u \in R\}$.
And $(UV)^\gamma = \{\langle u, i_U^\gamma(u)i_V^\gamma(u), t_U^\gamma(u)t_U^\gamma(u), f_U^\gamma(u)+f_V^\gamma(u)-f_U^\gamma(u)f_U^\gamma(u)\rangle \mid uinR\}$.
Now $i_U^\gamma(a-b)+i_V^\gamma(a-b)-i_U^\gamma(a-b)i_V^\gamma(a-b) = 0.4+0.3-0.12 = 0.58$,
$i_U^\gamma(a)+i_V^\gamma(a)-i_U^\gamma(a)i_V^\gamma(a) = 0.9+0.3-0.27 = 0.93$,
and $i_U^\gamma(b)+i_V^\gamma(b)-i_U^\gamma(b)i_V^\gamma(b) = 0.4+0.5-0.2 = 0.7$.
Therefore,
$i_U^\gamma(a-b)+i_V^\gamma(a-b)-i_U^\gamma(a-b)i_V^\gamma(a-b) < \wedge\{i_U^\gamma(a)+i_V^\gamma(a)-i_U^\gamma(a)i_V^\gamma(a), i_U^\gamma(b)+i_V^\gamma(b)-i_U^\gamma(b)i_V^\gamma(b)\}$.
Hence $U+V$ is not an γ-$SVNI$ of R. Again for the product, we see that

$$f_U^\gamma(a-b)+f_V^\gamma(a-b)-f_U^\gamma(a-b)f_V^\gamma(a-b) = 0.76,$$
$$f_U^\gamma(a)+f_V^\gamma(a)-f_U^\gamma(a)f_V^\gamma(a) = 0.64,$$
$$and\ f_U^\gamma(b)+f_V^\gamma(b)-f_U^\gamma(b)f_V^\gamma(b) = 0.7.$$

Therefore

$$f_U^\gamma(a-b)+f_V^\gamma(a-b)-f_U^\gamma(a-b)f_V^\gamma(a-b) > \vee\{f_U^\gamma(a)+f_V^\gamma(a)-f_U^\gamma(a)f_V^\gamma(a), f_U^\gamma(b)+f_V^\gamma(b)-f_U^\gamma(b)f_V^\gamma(b)\}.$$

Hence UV is not an γ-$SVNI$ of R.

5. **Conclusions**

A γ-single valued neutrosophic set is a type of SVNS that can be used to tackle real-world challenges for research and engineering. In this work, we introduce the notion of γ-single valued neutrosophic subrings, γ-single valued neutrosophic ideals also the sum and product of γ-single valued neutrosophic ideals. On γ-single valued neutrosophic subrings and ideals, a variety of characterizations have been proposed. Therefore, it is important for researchers to examine γ-single valued neutrosophic subrings and ideals and their characteristics in applications and to understand the basics of uncertainty. We agreed to include the concept of a γ-SVNSR & γ-SVNI in research also examine its key feature. As a consequence of this research, various principles are to be applied to achieve some adequate research value results of γ-SVNSR &

γ-SVNI. In further work, researchers can extend this idea in topological spaces, modules, and fields.

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An Investigation in the Initial Solution for Neutrosophic Transportation Problems (NTP)

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Abstract.

Transportation issues arise often in everyday life. To ensure that the regions' needs for transported material are met at the lowest feasible cost, materials must be carried from production centers to consuming centers as quickly as possible. Operations research approaches, notably mathematical programming, are utilized to solve these recurring and daily challenges. The problem's data is transformed into a mathematical model, and then the best solution is discovered using the proper procedures. When dealing with transportation issues, we arrive at a linear mathematical model, which can be solved using the direct simplex method and its modifications. However, because of the clarity and specificity of the transportation model, scholars and researchers were able to find other methods that were easier than the simplex method.

Whatever method is used, the goal is to determine the number of units transferred for any material from the production centers to the consumption centers in order to minimize transportation costs, keeping in mind that each export center has its own capacity and cannot supply quantities of the material greater than that capacity. Furthermore, each import center has a certain requirement for which it makes a request and for which it is unable to consume further quantities. In this manuscript, the researchers will use the North-West Corner approach, the Least-Cost method, and Vogel's approximation method to discover an initial solution to the balanced neutrosophic transport problems.

The term "neutrosophic transportation problems (i.e. NTP)" refers to the transportation problems in which the required and available quantities have neutrosophic values of the form $Na_i =$

$a_i + \varepsilon_i$, where ε_i is the indeterminacy in the produced quantities, and it is either $\varepsilon_i = [\lambda_{i1}, \lambda_{i2}]$ or $\varepsilon_i = \{\lambda_{i1}, \lambda_{i2}\}$. While the required quantities are also neutrosophic values of the form $Nb_j = b_j + \delta_j$, here δ_j is the indeterminacy on the required quantity, and it is either $\delta_j = [\mu_{j1}, \mu_{j2}]$ or $\delta_j = \{\mu_{j1}, \mu_{j2}\}$. It is worthy to mention that when the problem is unbalanced (i.e. the summation of the required quantities is not equal to the produced quantities), then firstly will convert the problem to a balanced one.

In this article, the authors assumed the representation of the neutrosophic numbers as intervals such as $\varepsilon_i = [\lambda_{i1}, \lambda_{i2}]$, $\delta_j = [\mu_{j1}, \mu_{j2}]$. It is important to notice that the authors did not adopt (trapezoidal numbers, pentagonal numbers, or any other neutrosophic numbers which need to specify using the membership functions, this kind of neutrosophic numbers or parameters represented by intervals have been firstly introduced by Smarandache F. in his main published books [32,33].

The sections of this manuscript has been organized as follow: the introduction is the inception of this article, section one has been dedicated to the north-west corner method containing three subsections for three case studies depending upon the existence of the indeterminacy in the problem, the least- cost method represents section two, while section three has been devoted to Vogel's approximation method, section four is conclusion and results.

Keywords: Transportation Problem; Neutrosophic Transportation Problem (NTP); Initial Solution; Consumption Centers (CC); Production Centers (PC); Available Quantities (AQ); Required Quantities (RQ); North-West Corner Method; Least-Cost Method; Vogel's Method.

Introduction

Transportation problems are among the most prevalent linear programming problems encountered in everyday life. These problems study the transfer of materials from production centers to consumption centers in the shortest period of time or at the lowest cost, or the distribution of transportation modes (such as buses, planes and ships etc.) on the imposed transportation lines in which the requests can meet by the least cost, and since the mathematical models we obtain are linear models, so the simplex method and its modifications can be used to obtain the optimal solution, but the special nature of the transportation problems enabled scientists to find special ways to solve these models that dependent on finding initial solution and then using other ways to improvement this initial solution using heuristics algorithms to find optimal solution [1-5], there are previous studies for transport models at the lowest cost using neutrosophic environments, which is the new vision of modelling and is designed to effectively address the uncertainties inherent in the real world, as it came to replace the binary logic that define merely the truthiness status and falseness status, by introducing a third, neutral state which can be interpreted as undetermined or uncertain .

This neutrosophic logic has been established in 1995 by the philosopher and mathematician Florentin Smarandache [7,9,10,11,13] introduced as a generalization to both: fuzzy logic presented by L. Zadeh in 1965 [6], and intuitionistic fuzzy logic introduced by K. Atanassov in 1983 [8]. In addition, A. A. Salama presented the theory of classical neutrosophic sets as a generalization of the classical sets. Theory [12,20], he developed, introduced and formulated new concepts in the fields of topology, statistics, computer science... etc. through neutrosophic theory [15,17-19, 22,28,29].

The neutrosophic theory has grown significantly in recent years through its application in measurement theory, group theory, graph theory and many scientific and practical fields [8,14-16, 21, 23-27,30-35], this research sheds the light on the modified that same methods used to find the initial solution for the classic transport problems in finding a preliminary solution to the neutrosophic transport problems in its three forms, the first form is that form when the cost is neutrosophic values, while the second form occurs when the demanded quantities and the supplied quantities are neutrosophic values, finally, the third form occurs when the cost of transport and the demanded quantities and the supplied quantities are all neutrosophic values.

Discussion:

It is popular that there are several ways to find a basic (initial) solution to the transfer problem, given the condition that the number of the basic variables in this initial solution must equal the number of the linear conditions (i.e. m+n-1). An initial solution can be found in several ways. In this research, we will use three methods, the North-West corner method, the least-cost method, and Vogel's approximate method, to find an initial solution to the neutrosophic transport model, and we will study the three forms in each of the methods. [1-5].

1. The North-West Corner Method

The North-West Corner Rule is a technique for calculating an initial basic feasible solution to a transportation problem. The method is called North-West Corner because the basic variables are chosen from the extreme left corner. the following three steps gives the initial basic feasible solution:

1. Find the north west corner cell of the transportation tableau. Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.
2. Cross out the row or column with 0 supply or demand. If both a row and a column have a 0, cross out randomly row or column.
3. If one cell is left uncrossed, cross out the cell and stop. Otherwise, go to step 1.

We should not forget that the final allocated cells (nonzero cells) must equal to the value (m+n-1), where m determine how many production center that the problem have, while n refers to how many consumption center exist.

1.1 First Case Study for Neutrosophic Transportation Problem in which the Indeterminacy is in Nc_{ij}

In this case the cost of transportation will be neutrosophic values, that's mean the monetary value of transfer one unit from the production center i to the consumption center j is $Nc_{ij} = c_{ij} \pm \varepsilon$, where ε is the indeterminate value and equal to $\varepsilon = [\lambda_1, \lambda_2]$, so the payment matrix will be $Nc_{ij} = [\, c_{ij} \pm \varepsilon]$. Although all transportation costs have been given the same indeterminate value, it is feasible to assign a different indeterminate to each cost and use the same case study.
The Text of the Problem
A certain amount of oil is to be transported from three stations A_1, A_2, A_3 to four cities B_1, B_2, B_3, B_4.The following table shows the quantities available at each station, the demand quantities in each city,and the transportation costs in each direction:

CC / PC	$B1$	$B2$	$B3$	$B4$	AQ
$A1$	$7+\varepsilon$ $x11$	$4+\varepsilon$ $x12$	$15+\varepsilon$ $x13$	$9+\varepsilon$ $x14$	120
$A2$	$11+\varepsilon$ $x21$	$0+\varepsilon$ $x22$	$7+\varepsilon$ $x23$	$3+\varepsilon$ $x24$	80
$A3$	$4+\varepsilon$ $x31$	$5+\varepsilon$ $x32$	$2+\varepsilon$ $x33$	$8+\varepsilon$ $x34$	100
RQ	85	65	90	60	

In this example, the indeterminate value of ε = [0,2], based on the problem's data we have, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = 300$, this signifies that the issue is balanced. Substituting ε = [0,2] into the preceding tableau yielded the following one:

PC \ CC	B_1	B_2	B_3	B_4	AQ
A_1	[7,9] x_{11}	[4,6] x_{12}	[15,17] x_{13}	[9,11] x_{14}	120
A_2	[11,13] x_{21}	[0,2] x_{22}	[7,9] x_{23}	[3,5] x_{24}	80
A_3	[4,6] x_{31}	[5,7] x_{32}	[2,4] x_{33}	[8,10] x_{34}	100
RQ	85	65	90	60	300 \ 300

The entire calculations to find the initial solution using the North-West method have been summarized as follow:

Start with the cell located in the north-west corner of the table, i.e. that cell corresponds to the first production center crossing with the first consumption center, this cell will carry the value $Min\ \{85,93\} = 85$, hence the first consumption center B_1 has the need been fulfilling of its requirement from the first production center A_1, the remaining amount in $A_1$is $120 - 85 = 35$. Move to the right cell positioned in the crossing of first row with second column and put in it the value $Min\ \{65,35\} = 35$, so the available quantity in A_1, and the required quantity in B_1 both are being zero, but the second consumption center B_2 requires to $65 - 35 = 30$. Go down to the cell of position in the cross of second row with second column (x_{22}) and put the value $Min\ \{30,80\} = 30$, consequently, the second consumption center B_2 has its need been fulfilling, and the remaining value in the second production center A_2 is $80 - 30 = 50$, keep going in the same above technique till all production centers are emptied, and all consumption centers have been fulfilling, finally the following table was gotten:

CC / PC	B_1	B_2	B_3	B_4	AQ
A_1	[7,9] 85	[4,6] 35	[15,17]	[9,11]	120
A_2	[11,13]	[0,2] 30	[7,9] 50	[3,5]	80
A_3	[4,6]	[5,7]	[2,4] 40	[8,10] 60	100
RQ	85	65	90	60	300 / 300

The last table illustrates: $x_{11} = 85, x_{12} = 35, x_{22} = 30, x_{23} = 50, x_{33} = 40, x_{34} = 60, x_{13} = x_{14} = x_{21} = x_{24} = x_{31} = x_{32} = 0$, we have $n = 4, m = 3, m + n - 1 = 6$, meaning that the initial conduction satisfied the necessary condition. Calculate the total cost for this initial solution by substitution the $x's$ values in the cost function:

$$NC = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23} + c_{24}x_{24} + c_{31}x_{31} + c_{32}x_{32} + c_{33}x_{33} + c_{34}x_{34}$$

$$NC = [7,9] * 85 + [4,6] * 35 + [15,17] * 0 + [9,11] * 0 + [11,13] * 0 + [0,2] * 30 + [7,9] * 50 + [3,5] * 0 + [4,6] * 0 + [5,7] * 0 + [2,4] * 40 + [8,10] * 60 = [1645,2245]$$

Which is the cost versus initial solution.

1.2 Second Case Study in which the indeterminacy is in both production center and consumption center

Problem Text

A quantity of fuel is intended to be shipped from three stations to four cities. The available quantities at each station, the demand quantities in each city, and the transportation costs in each direction are demonstrated in the following table:

PC \ CC	$B1$	$B2$	$B3$	$B4$	**AQ**
$A1$	7	4	15	9	$120+\varepsilon 1$
$A2$	11	0	7	3	$80+\varepsilon 2$
$A3$	4	5	2	8	$100+\varepsilon 3$
RQ	$85+\delta 1$	$65+\delta 2$	$90+\delta 3$	$60+\delta 4$	

It is worthy to mention that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ represent the indeterminacies that exist in available quantitiesin the fuel stations, and it can be took as intervals $[\lambda_{i1}, \lambda_{i2}]$ or as sets $\{\lambda_{i1}, \lambda_{i2}\}$ …etc.

For this case study, the following values have been picked: $\varepsilon_1 = [0,11], \varepsilon_2 = [0,9], \varepsilon_3 = [0,15]$

While the values δ_1, δ_2, δ_3, δ_4 are the indeterminacies in the required quantities in the four cities,also these neutrosophic values can be regarded as intervals $[\mu_{i1}, \mu_{i2}]$ or as sets $\{\mu_{i1}, \mu_{i2}\}$ …etc.
For this example, the following neutrosophic values have been took: $\delta_1 = [0,8], \delta_2 = [0,12], \delta_3 = [0,9]$, $\delta_4 = [0,6]$. So the above table becomes:

PC \ CC	$B1$	$B2$	$B3$	$B4$	**A**
$A1$	7	4	15	9	[120,131]
$A2$	11	0	7	3	[80,89]
$A3$	4	5	2	8	[100,115]
RQ	[85,93]	[65,77]	[90,99]	[60,66]	[300,335] / [300,335]

It is clear that the problem is balanced as $\sum_{i=1}^{3} Na_i = \sum_{j=1}^{4} Nb_j = [300,335]$.

Start with the cell located in the north-west corner of the table, i.e. that cell corresponds to the first production center crossing with the first consumption center, this cell will carry the value Min {[85,93], [120,131]} = [85,93], hence the first consumption center $B1$ has the need been fulfilling of its requirement from the first production center $A1$, the remaining amount in $A1$ is [120,131] − [85,93] = [35,38]. Move to the right cell positioned in the crossing of first row with second column and put in it the value Min {[65,77], [35,38]} = [35,38], so the available quantity in $A1$, and the required quantity in $B1$ both are being zero, but the second consumption center $B2$ requires to [65,77] − [35,38] = [30,39]. Go down to the cell of position in the cross of second row with second column ($x22$) and put the value Min {[30,39], [80,89]} = [30,39], consequently, the second consumption center $B2$ has its need been fulfilling, and the remaining value in the second production center $A2$ is [80,89] − [30,39] = [50,50], keep going in the same above technique till all production centers are emptied, and all consumption centers have been fulfilling, finally the following table was gotten:

This table contains the following neutrosophic values:

$Nx_{11} = [85,93], Nx_{12} = [35,38], Nx_{22} = [30,39], Nx_{23} = [50,50], Nx_{33} = [40,49], Nx_{34} = [60,66], Nx_{13} = Nx_{14} =$

PC \ CC	B_1	B_2	B_3	B_4	AQ
A_1	7 [85,93]	4 [35,38]	15 0	9 0	[120,131]
A_2	11 0	0 [30,39]	7 [50,50]	3 0	[80,89]
A_3	4 0	5 0	2 [40,49]	8 [60,66]	[100,115]
RQ	[85,93]	[65,77]	[90,99]	[60,66]	[300,335] [300,335]

$Nx_{21} = Nx_{24} = Nx_{31} = Nx_{32} = 0$. In this problem similar to the previous problem in section (1.1), $n = 4, m = 3 \Rightarrow m + n - 1 = 6$, meaning that the initial condition satisfied the necessary condition. Calculate the total cost for this initial solution by substitution the x's values in the cost function:

$NC = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23} + c_{24}x_{24} + c_{31}x_{31} + c_{32}x_{32} + c_{33}x_{33} + c_{34}x_{34}$

$NC = 7 * [85,93] + 4 * [35,38] + 15 * 0 + 9 * 0 + 11 * 0 + 0 * [30,39] + 7 * [50,50] + 3 * 0 + 4 * 0 + 5 * 0 + 2 * [40,49] + 8 * [60,66] = [1645,1779]$

Which is the cost versus to the initial solution.

1.3 Third Case Study in Which the Transportation Cost, the Available Quantities in the Production Centers, and the Demand Quantities in the Consumption Centers are all Neutrosophic Values

By taking the same context of the studied cases in the previous subsections (1.1, 1.2) subject to the following table has been considered as new example:

PC \ CC	B_1	B_2	B_3	B_4	A
A_1	$7 + \varepsilon$ Nx_{11}	$4 + \varepsilon$ Nx_{12}	$15 + \varepsilon$ Nx_{13}	$9 + \varepsilon$ Nx_{14}	$120 + \varepsilon_1$
A_2	$11 + \varepsilon$ Nx_{21}	$0 + \varepsilon$ Nx_{22}	$7 + \varepsilon$ Nx_{23}	$3 + \varepsilon$ Nx_{24}	$80 + \varepsilon_2$
A_3	$4 + \varepsilon$ Nx_{31}	$5 + \varepsilon$ Nx_{32}	$2 + \varepsilon$ Nx_{33}	$8 + \varepsilon$ Nx_{34}	$100 + \varepsilon_3$
RQ	$85 + \delta_1$	$65 + \delta_2$	$90 + \delta_3$	$60 + \delta_4$	

By assuming $\varepsilon = [0,2], \varepsilon_1 = [0,11], \varepsilon_2 = [0,9], \varepsilon_3 = [0,15], \delta_1 = [0,8], \delta_2 = [0,12], \delta_3 = [0,9], \delta_4 =$

[0,6], the above table can be rewritten as:

CC / PC	B_1	B_2	B_3	B_4	AQ
A_1	[7,9]	[4,6]	[15,17]	[9,11]	[120,131]
A_2	[11,13]	[0,2]	[7,9]	[3,5]	[80,89]
A_3	[4,6]	[5,7]	[2,4]	[8,10]	[100,115]
RQ	[85,93]	[65,77]	[90,99]	[60,66]	[300,335] / [300,335]

Obviously, the problem is balanced because $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = [300,335]$. The same north-west strategy has been implemented to get the following table:

CC / PC	B_1	B_2	B_3	B_4	AQ
A_1	[7,9] [85,93]	[4,6] [35,38]	[15,17] 0	[9,11] 0	[120,131]
A_2	[11,13] 0	[0,2] [30,39]	[7,9] [50,50]	[3,5] 0	[80,89]
A_3	[4,6] 0	[5,7] 0	[2,4] [40,49]	[8,10] [60,66]	[100,115]
RQ	[85,93]	[65,77]	[90,99]	[60,66]	[300,335] / [300,335]

Same as the previous examples we have, $Nx_{11} = [85,93], Nx_{12} = [35,38], Nx_{22} = [30,39], Nx_{23} = [50,50], Nx_{33} = [40,49], Nx_{34} = [60,66], Nx_{13} = Nx_{14} = Nx_{21} = Nx_{24} = Nx_{31} = Nx_{32} = 0$. In this problem similar to the previous problems (1.1 &1.2), n = 4, m = 3 $\Longrightarrow m + n - 1 = 6$, meaning that the initial condition satisfied the necessary condition. Calculate the total cost for this initial solution by substitution the x's values in the cost function:

$$NC = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23} + c_{24}x_{24} + c_{31}x_{31} + c_{32}x_{32} + c_{33}x_{33} + c_{34}x_{34}$$

NC = [7,9] ∗ [85,93] + [4,6] ∗ [35,38] + [9,11] ∗ 0 + [7,9] ∗ 0 + [11,13] ∗ 0 + [0,2] ∗ [30,39] + [7,9]* [50,50] + [4,6] ∗ 0 + [5,7] ∗ 0 + [2,4] ∗ [40,49] + [8,10] ∗ [60,66] = [1645,2449]

Which is the cost versus to the initial solution.

2. The Least- Cost Method

The least cost is another method used to obtain the initial feasible solution for the transportation problem. Here, the allocation begins with the cell which has the minimum cost. The lower cost cells are chosen over the higher-cost cell with the objective to have the least cost of transportation. The Least Cost Method is considered to produce more optimal results than the north-west corner because it considers the shipping cost while making the allocation, whereas the North-West corner method only considers the availability and supply requirement and allocation begin with the extreme left corner, irrespective of the shipping cost [1-5]. We will discuss the existence types of the indeterminacies either in transportation costs, or the indeterminacy exists in both the available quantity in the production centers and in the demand quantities in the consumption centers, or in all of them by the following case studies:

2.1 Case Study Has the Indeterminacy in its Transportation Cost

The same context of example (1.1) has been resolved using least- cost method, where the least cost cell is [0,2] which is located in the position resulting from the intersection of the second row with the second column, and put the value $min\ \{65,80\} = 65$ in it. Thus, we have met the need of the second consumption center B_2 from the second production center A_2, and the remaining quantity in A_2 is $80 - 65 = 15$. Move to the next least cost value among the remaining costs is [2,4], which is located in the cell resulting from the intersection of the third row with the third column, and we put the value $min\ \{90,100\} = 90$ in it. Thus, we have met the need of the third consumption center B_3 from the third productive center A_3, and the remaining value in the third productive center is $100 - 90 = 10$. Continuing by the same strategy until all consumption centers have been saturated and all production centers have been emptied. Consequently, the following table yield:

PC \ CC	B_1	B_2	B_3	B_4	AQ
A_1	[7,9] 75	[4,6]	[15,17]	[9,11] 45	120
A_2	[11,13]	[0,2] 65	[7,9]	[3,5] 15	80
A_3	[4,6] 10	[5,7]	[2,4] 90	[8,10]	100
RQ	85	65	90	60	300 \ 300

Here, $x_{11} = 75,\ x_{14} = 45, x_{22} = 65, x_{24} = 15,\ x_{31} = 10, x_{33} = 90,\ x_{12} = x_{13} = x_{21} = x_{23} = x_{32} = x_{34} = 0$.
NL = [7,9] ∗ 75 + [4,6] ∗ 0 + [15,17] ∗ 0 + [9,11] ∗ 45 + [11,13] ∗ 0 + [0,2] ∗ 65 + [7,9] ∗ 0 + [3,5] ∗ 15 + [4,6] ∗ 10 + [5,7] ∗ 0 + [2,4] ∗ 90 + [8,10] ∗ 0 = [1195,1795]. As usual, it represents the cost versus to the initial solution.

2.2 Case Study in Which the Available Quantities of the Production Centers and the Demanded Quantities of the Consumption Centers are Neutrosophic Values

For the comparison purposes, the same data and problem text that used in the case study (1.2) has been considered here, so the first three tables are the same. By applying the least cost method, it seemsthe cell of the zero value is the required cell which exactly located in the intersection of the second row with the second column, so we put in this cell the value min {[65,77], [80,89]} = [65,77], by moving to the next least cost cell that located in the intersection of the third row with the third columnof 2 value, burden this cell with the value min {[90,99], [100,115]} = [90,99], go on with the same strategy till all consumption centers saturation at the same time all productions centers have been emptied, hence, the following table yielding:

CC / PC	B_1	B_2	B_3	B_4	AQ
A_1	7 [75,77]	4	15	9 [45,54]	[120,131]
A_2	11	0 [65,77]	7	3 [15,12]	[80,89]
A_3	4 [10,16]	5	2 [90,99]	8	[100,115]
RQ	[85,93]	[65,77]	[90,99]	[60,66]	[300,335] / [300,335]

Hence, Nx_{11} = [75,77], Nx_{14} = [45,54], Nx_{22} = [65,77], Nx_{24} = [15,12], Nx_{31} = [10,16], Nx_{33} = [90,99], $Nx_{12} = N_{13} = Nx_{21} = Nx_{23} = Nx_{32} = Nx_{34} = 0$, we should not forget the problem satisfiesthe balancing condition since $m + n - 1 = 6$.

$NL = 7 * [75,77] + 4 * 0 + 15 * 0 + 9 * [45,54] + 11 * 0 + 0 * [65,77] + 7 * 0 + 3 * [15,12] + 4* [10,16] + 5 * 0 + 2 * [90,99] + 8 * 0 = [1259,1323]$

Obviously, it represents the cost versus to the initial solution.

2.3 Case Study in Which the Transportation Cost, the Available Quantities of the Production Centers, and the Demanded Quantities of the Consumption Centers are all Neutrosophic Values

In this section, the text problem and the types of the indeterminacies are same as in the case study ofthe section (1.3), but the values of $\delta_i's,\ \varepsilon_i's$ are assumed to be:
$\varepsilon = [0,2], \varepsilon_1 = [0,35], \varepsilon_2 = [0,10], \varepsilon_3 = [0,15], \delta_1 = [0,7], \delta_2 = [0,18], \delta_3 = [0,25], \delta_4 = [0,10]$.

CC / PC	B_1	B_2	B_3	B_4	AQ
A_1	[7,9] Nx_{11}	[4,6] Nx_{12}	[15,17] Nx_{13}	[9,11] Nx_{14}	[120,155]
A_2	[11,13] Nx_{21}	[0,2] Nx_{22}	[7,9] Nx_{23}	[3,5] Nx_{24}	[80,90]
A_3	[4,6] Nx_{31}	[5,7] Nx_{32}	[2,4] Nx_{33}	[8,10] Nx_{34}	[100,115]
RQ	[85,92]	[65,83]	[90,115]	[60,70]	[300,360] / [300,360]

Again, by using the least cost strategy, the least cost is the cell [0,2] located in the cell intersect the second row with second column, so we will put the value min{65,80} = 65 in it, hence the needy of the second consumption center $B2$ has been met from the second production center A_2 , the remaining quantity in A_2 is 80 − 65 = 15 , by moving to the next least cost cell which is [2,4] represents the cell allocated in the intersection of the third row with the third column, burden this cell with the value min{90,100} = 90 . Thus, we have met the need of the third consumption center B_3 from the third productive center A_3, and the remaining value in the third productive center is 100 − 90 = 10. Go on with the same strategy till all consumption centers saturation at the same time all productions centers have been emptied, hence, the following table yielding:

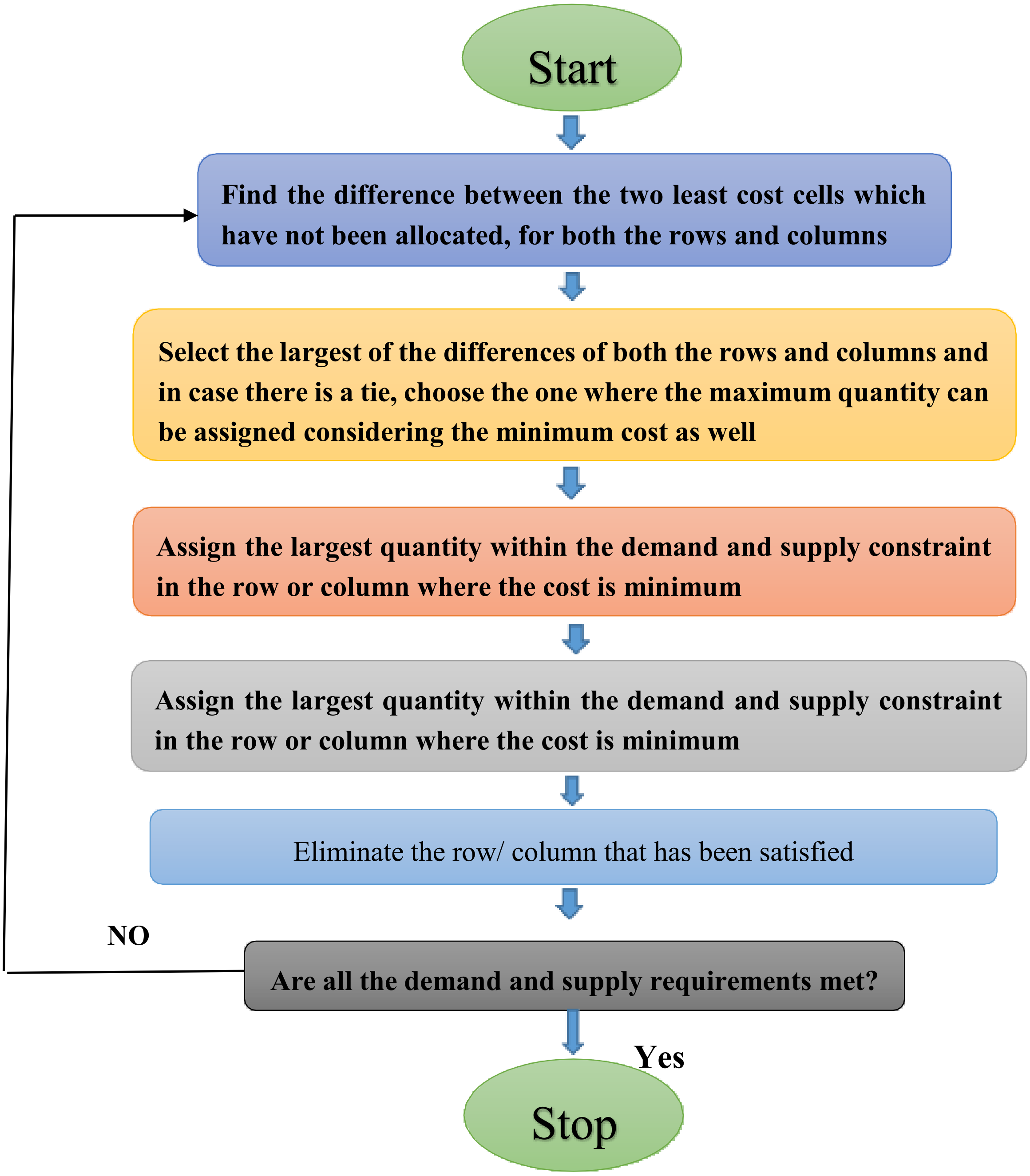
CC / PC	B_1	B_2	B_3	B_4	AQ
A_1	[7,9] [75,77]	[4,6]	[15,17]	[9,11] [45,54]	[120,131]
A_2	[11,13]	[0,2] [65,77]	[7,9]	[3,5] [15,12]	[80,89]
A_3	[4,6] [10,16]	[5,7]	[2,4] [90,99]	[8,10]	[100,115]
RQ	[85,93]	[65,77]	[90,99]	[60,66]	[300,335] / [300,335]

Hence, Nx_{11} = [75,77], Nx_{14} = [45,54], Nx_{22} = [65,77], Nx_{24} = [15,12], Nx_{31} = [10,16], Nx_{33} = [90,99], $Nx_{12} = Nx_{13} = Nx_{21} = Nx_{23} = Nx_{32} = Nx_{34} = 0$, we should not forget the problem satisfies the balancing condition since $m + n - 1 = 6$. The following cost represents the initial feasible solution.

NL = [7,9] ∗ [75,77] + [4,6] ∗ 0 + [15,17] ∗ 0 + [9,11] ∗ [45,54] + [11,13] ∗ 0 + [0,2] ∗ [65,77] + [7,9] ∗0 + [3,5] ∗ [15,12] + [4,6] ∗ [10,16] + [5,7] ∗ 0 + [2,4] ∗ [90,99] + [8,10] ∗ 0 = [1195,1993].

3- Vogel's Approximation Method :

Definition: The Vogel's Approximation Method or VAM is an iterative procedure calculated to findout the initial feasible solution of the transportation problem. Like Least cost Method, here also the shipping cost is taken into consideration, but in a relative sense. The following is the flow chart showing the steps involved in solving the transportation problem using the Vogel's Approximation method.



The same cases studies that have discussed in the all previous sections (1.1,1.2,1.3,2.1,2.2,2.3) can be presented here with the same problems texts, with same data values, to be resolved in the Vogel's iterative method, this will give us a good opportunity to analyze the results which enables us to make a good comparison between (North-West Corner method, Least-Cost Method, and Vogel's Method).

3.1 Case Study Has the Indeterminacy in its Transportation Cost

CC / PC	B_1	B_2	B_3	B_4	**AQ**	Δ_1	Δ_2	Δ_3
A_1	[7,9] 75	[4,6] 45	[15,17]	[9,11]	120	3	3	3
A_2	[11,13]	[0,2] 20	[7,9]	[3,5] 60	80	3	3	11
A_3	[4,6] 10	[5,7]	[2,4] 90	[8,10]	100	2	2	1
RQ	85	65	90	60	300 / 300			
Δ_1'	3	4	5	5				
Δ_2'	3	4	5	_				
Δ_3'	3	4	_	_				

The symbols $\Delta_1', \Delta_2', \Delta_3'$ mean the subtractions between the columns respectively, while the symbols $\Delta_1, \Delta_2, \Delta_3$ mean the subtractions between the rows respectively.

From the above table, we have $x_{11} = 75, x_{12} = 45, x_{22} = 20, x_{24} = 60, x_{31} = 10, x_{33} = 90, x_{13} = x_{14} = x_{21} = x_{23} = x_{32} = x_{34} = 0$,

$$NL = [7,9] * 75 + [4,6] * 45 + [15,17] * 0 + [9,11] * 0 + [11,13] * 0 + [0,2] * 20 + [7,9] * 0 + [3,5] * 60 + [4,6] * 10 + [5,7] * 0 + [2,4] * 90 + [8,10] * 0 = [1105,1705]$$

3.2 Case Study in Which the Available Quantities of the Production Centers and the Demanded Quantities of the Consumption Centers are Neutrosophic Values

CC / PC	B_1	B_2	B_3	B_4	AQ	Δ_1	Δ_2	Δ_3
A_1	7 [75,77]	4 [45,54]	15	9	[120,131]	3	3	3
A_2	11	0 [20,23]	7	3 [60,66]	[80,89]	3	7	–
A_3	4 [10,16]	5	2 [90,99]	8	[100,115]	2	2	2
RQ	[85,93]	[65,77]	[90,99]	[60,66]	[300,335] [300,335]			
Δ'_1	3	4	5	5				
Δ'_2	3	4	5	–				
Δ'_3	3	1	13	–				

Recall the same notations $Nx_{ij}'s$ with their new values,

$Nx_{11} = [75,77], Nx_{12} = [45,54], Nx_{22} = [20,23], Nx_{24} = [60,66], Nx_{31} = [10,16], \quad Nx_{33} = [90,99],$

$$Nx_{13} = N_{14} = Nx_{21} = Nx_{23} = Nx_{32} = Nx_{34} = 0$$

The initial feasible solution is:

$$NL = 7 * [75,77] + 4 * [45,54] + 15 * 0 + 9 * 0 + 11 * 0 + 0 * [20,23] + 7 * 0 + 3 * [60,66] + 4 * [10,16] + 5 * 0 + 2 * [90,99] + 8 * 0 = [1105,1215]$$

3.3 Case Study in Which the Transportation cost, the Available Quantities of the Production Centers, and the Demanded Quantities of the Consumption Centers are all Neutrosophic Values

Recall the same text problem in section (1.3) with respect to resolving it using Vogel's iterative procedure to conclude the following table:

CC / PC	B_1	B_2	B_3	B_4	AQ	Δ_1	Δ_2	Δ_3
A_1	[7,9] [75,77]	[4,6] [45,54]	[15,17]	[9,11]	[120,131]	3	3	3
A_2	[11,13]	[0,2] [20,23]	[7,9]	[3,5] [60,66]	[80,89]	3	7	−
A_3	[4,6] [10,16]	[5,7]	[2,4] [90,99]	[8,10]	[100,115]	2	2	2
RQ	[85,93]	[65,77]	[90,99]	[60,66]	[300,335] [300,335]			
Δ'_1	3	4	5	5				
Δ'_2	3	4	5	_				
Δ'_3	3	1	13	_				

So the initial feasible solution is:

$$NL = [7,9] * [75,77] + [4,6] * [45,54] + [15,17] * 0 + [9,11] * 0 + [11,13] * 0 + [0,2] * [20,23] + [7,9] * 0 + [3,5] * [60,66] + [4,6] * [10,16] + [5,7] * 0 + [2,4] * [90,99] + [8,10] * 0 = [1105,1885]$$

4. Conclusion and Results

This paper sheds the light on a new vision for solving Transportation problems by taking into consideration the existence of indeterminacy in many joints of the problems that have been solve nine times, each time in different method and different aspect of indeterminacy existence. With somedeep insights the reader can notice that the Vogel's iterative procedure yields minimum cost than both the costs that produced by applying North-west method, and Least-Cost method. However, themethods that used is still gives the results regarded as the initial feasible solution not the optimal solution, which mean that these methods are still need to improve to get the optimal solution.
Below table summarizes all previous solutions in comparison strategy:

The Method / Types of the problem	North-West Method	Least- Cost Method	Vogel's Method
First type indeterminacy	[1645,2245]	[1195,1795]	[1105,1705]
Second type indeterminacy	[1645,1779]	[1259,1323]	[1105,1215]
Third type indeterminacy	[1645,2449]	[1195,1993]	[1105,1885]

Looking forward to the further upcoming studies dedicated to implement improvements on theinitial solutions to get an optimal solution in the neutrosophic transportation problems.

As it is well known in the transportation problems, that the (North- West Corner, Least Cost Method, and Vogel's Method), all these methods are for finding the initial solutions (either these initial solutions are suffering from weak accurate by applying the North - West Corner, or having more accurate by applying Vogel's method), it still needs to investigate the optimal solutions in the transportations problems, which will be by intending to publish forthcoming papers.

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A Study of Neutrosophic Cubic Hesitant Fuzzy Hybrid Geometric Aggregation Operators and its Application to Multi Expert Decision Making System

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Abstract: Hesitancy is an imperative part of belief system. In order to counter the hesitancy in neutrosophic cubic set (NCS), the notion of neutrosophic cubic hesitant fuzzy set (NCHFS) is presented. NCHFS couple NCS with hesitant fuzzy set (HFS). Operational laws in NCHFS are developed with examples. To meet the challenges of decision making problems, neutrosophic cubic hesitant fuzzy geometric (NCHFG) aggregation operators, neutrosophic cubic hesitant fuzzy Einstein geometric (NCHFEG) aggregation operators, neutrosophic cubic hesitant fuzzy hybrid geometric (NCHFHEG) aggregation operators are developed in the current study. At the end a multi expert decision making (MEDM) process is proposed and furnished upon numerical data of a company as applications.

Keywords: Neutrosophic Cubic Fuzzy Hesitant Set (NCHFS), Neutrosophic Cubic Hesitant Fuzzy Weighted geometric (NCHFWG) operator, Neutrosophic Cubic Hesitant Fuzzy Einstein Geometric (NCHFEG) operator, Neutrosophic Cubic Hesitant Fuzzy Einstein Hybrid Geometric (NCHFEHG) Operator. multi expert decision making (MEDM)

1. Introduction

We are in different mental states of acceptance, hesitancy and refusal while taking decisions in life. Many methods in MADM ignore the uncertainty and hence yields the results which are unreliable. The role of expert in decision making (DM) is vital. The participation of more than one expert in a DM process reduce the uncertainty. Zadeh proposed the notion of fuzzy set (FS) [1] as a function from a given set of objects to [0,1] called membership. Later Zadeh extended the idea to interval valued fuzzy set (IVFS) [2]. An IVFS a function from a given set of objects to the subintervals of [0,1]. The FS theory has many applications in artificial intelligence, robotics, computer networks, engineering and DM [3,4]. Different researchers [5-8] established similarity measures and other important concepts and successfully apply their models to medical diagnosis and selection criteria. R.A. Krohling and V.C. Campanharo, M. Xia and Z. Xu, M.K. Mehlawat and P.A. Guptal established different useful techniques to sort out MADM problems [9-11]. K. Atanassov introduced non-membership degree and presented the idea of intuitionistic fuzzy set (IFS) [12] which consist of both membership and non-membership degree within [0,1]. An extension of IFS was proposed and named as interval value intuitionistic fuzzy set (IVIFS) [13]. IVIFS contains membership and non-membership in the form of subintervals of [0,1]. This characteristic of intuitionistic fuzzy set made it more applicable then previous versions and attracted researchers [14-16] to apply it to the fields of science, engineering and daily life problems. Jun et al., combined IVFS and FS and proposed cubic set (CS). The CS is the generalization of IFS and IVIFS. CS become vital tool to deal the vague data. Several researchers explored algebraic aspects and apparently define ideal theory in CS [17-20]. F. Smarandache initiated the concept of indeterminacy and describe the notion of neutrosophic set (NS) [21]. NS consist of three components truth, indeterminacy and falsehood and all are independent. This characteristic of neutrosophic set enabled researchers to work with inconsistent and vague data more effictively. Wang *et al* proposed single valued neutrosophic set (SNVS) [22] by restricting components of NS to [0,1]. The NS was further extended to interval neutrosophic set (INS) [23]. After the appearance of NS, researchers put their contributions in theoretical as well as technological developments of the set [24-27]. Several researchers use neutrosophic and interval valued neutrosophic environments to construct MADM methods [28-32]. Zhan *et al.,*define aggregation operators and furnished some applications in MADM [33]. Torra define hesitant fuzzy set (HFS) [34] in contrast of FS. HFS on X is a function that maps every object of X into a subset of [0, 1]. Jun *et al.,* presented the notion of NCS [35] which consist of both INS and NS. These characteristics of NCS make it a powerful tool to deal the vague and inconsistent data more efficiently. Soon after its exploration it attracted the researcher to work in many fields like medicine, algebra, engineering and DM. Later the idea of cubic hesitant fuzzy set was introduced by Tahir *et al.,*[36]. Ye [37] establish similarity measure in neutrosophic hesitant fuzzy sets (NHFS) and established MADM method using these measures. Liu et. Al [38] proposed hybrid geometric aggregation operators in interval neutrosophic hesitant fuzzy sets (INHFS)and discuss its applications in MADM. Zhu et al. [39] proposed the method of β-normalization to enlarge a HFE, which is a useful technique in case of different cardinalities.

The remaining of the paper is formulated as follows. In section 2, we reviewed some basic definitions used later on. Section 3 deals with NCHFS, algebraic and Einstein operational laws in NCHFS. In section 4 we introduced aggregation operators in NCHFS. Section 5 concern with establishing a MEDM method based on NCHFG operators and use this method in MEDM problem.

2. Preliminaries

Definition 2.1: [1] A fuzzy set (FS)on a nonempty set W is a mapping $\Gamma: W \to [0,1]$.

Definition 2.2: [12] The cubic set (CS) on a nonempty set Z is defined by $\mu = \langle x; I(x), \delta(x)/x \in X \rangle$, where $I(x)$ is an IVFS on Z and $\delta(x)$ is an FS on Z.

Definition 2.3: [22] A neutrosophic set associated with a crisp set S, is a set of the form $\mu = \langle e; \xi_T(e), \xi_I(e), \xi_F(e)/e \in S \rangle$ where $\xi_T, \xi_I, \xi_F: S \to [0,1]$ respectively called a truth membership function, a non-membership function and a false membership function.

Definition 2.4: [34] A hesitant fuzzy set on a crisp set W is a mapping which assigns a set of values in [0,1], to each element of W.

Definition 2.5: [35] A neutrosophic cubic set in a nonempty set E is defined as a pair (B, μ) where $B = \langle x; B_T(e), B_I(e), B_F(e)/e \in E \rangle$ is an INS and $\mu = \langle x; \mu_T(e), \mu_I(e), \mu_F(e)/e \in X \rangle$ is a NS.

Definition 2.6: [39] A neutrosophic hesitant fuzzy set a nonempty set E is described as $\mu = \langle x; \mu_T(e), \mu_I(e), \mu_F(e)/e \in E \rangle$where $\mu_T(e), \mu_I(e), \mu_F(e)$ are three HFSs such that $\mu_T(e) + \mu_I(e) + \mu_F(e) \leq 3$.

Definition2.7: [40] The object $\zeta = \langle x; \xi_T(x), \xi_I(x), \xi_F(x)/x \in X\rangle$, s called an INHFS on *X*, where$\xi_T(x), \xi_I(x), \xi_F(x)$ are IHFSs.

Zhu et al. proposed the following β-normalization method to enlarge a hesitant fuzzy element, which is a useful technique in case of different cardinalities.

Definition 2.8:[41] Let m^+ and m^- be the maximum and minimum elements of an hesitant fuzzy set H and $\zeta(0 \leq \zeta \leq 1)$ an optimized parameter. We call $m = \zeta m^+ + (1-\zeta)m^-$ an added element.

3. NCHFS and operational Laws in NCHFS

Definition3.1: Let X be a nonempty set. A neutrosophic cubic hesitant fuzzy set in X is a pair $\alpha = \langle A, \lambda\rangle$ where $A = \langle x; A_T(x), A_I(x), A_F(x)/x \in X\rangle$ is an interval-valued neutrosophic hesitant set in X and $\lambda = \langle x; \lambda_T(x), \lambda_I(x), \lambda_F(x)/x \in X\rangle$ is a neutrosophic hesitant set in X. Furthermore $A_T = \{[A^L_{j_T}, A^U_{j_T}]; j = 1, \ldots, l\}, A_I = \{[A^L_{j_I}, A^U_{j_I}]; j = 1, \ldots, m\}, A_F = \{[A^L_{j_F}, A^U_{j_F}]; j = 1, \ldots, n\}$ are some interval values in [0,1] and $\lambda_T = \{\lambda_{j_T}; j = 1, \ldots, r\}, \lambda_I = \{\lambda_{j_I}; j = 1, \ldots, s\}, \lambda_F = \{\lambda_{j_F}; j = 1, \ldots, t\}$ are some values in [0,1].

Example 3.2: Let$X = \{u, v, w\}$ The pair $\alpha = \langle A, \lambda\rangle$ with

$$A_T(u) = \{[0.1,0.5],[0.2,0.7]\}, \lambda_T(u) = \{0.3,0.5,0.7\}, A_I(u) = \{[0.2,0.4],[0.3,0.6]\}, \lambda_T(u) = \{0.1,0.4,0.7\}, A_F(u) = \{[0.1,0.4],[0,0.3],[0.6,0.8]\}, \lambda_F(u) = \{0.4,0.6\}$$

$$A_T(v) = \{[0.1,0.5],[0.2,0.7]\}, \lambda_T(v) = \{0.3,0.5\}, A_I(v) = \{[0.2,0.3],[0.1,0.6]\}, \lambda_I(v) = \{0.7,0.8\}, A_F(v) = \{[0.1,0.4],[0,0.3]\}, \lambda_F(v) = \{0.4,0.6\}$$

$$A_T(w) = \{[0.1,0.5],[0.2,0.7]\}, \lambda_T(w) = \{0.3,0.5\}, A_I(w) = \{[0.2,0.3],[0.1,0.6]\}, \lambda_I(w) = \{0.7,0.8\}, A_F(z) = \{[0.1,0.4],[0,0.3]\}, \lambda_F(w) = \{0.4,0.6\}$$

is a NCHFS.

Definition 3.3: The sum of two NCHFSs $\alpha = \langle A, \lambda\rangle, \beta = \langle B, \mu\rangle$ is defined as

$$\alpha \oplus \beta = \left\langle \begin{array}{c} x, \left\{\left[A^L_{j_T} + B^L_{j_T} - A^L_{j_T}B^L_{j_T}, A^U_{j_T} + B^U_{j_T} - A^U_{j_T}B^U_{j_T}\right]\right\}, \left\{\left[A^L_{j_I} + B^L_{j_I} - A^L_{j_I}B^L_{j_I}, A^U_{j_I} + B^U_{j_I} - A^U_{j_I}B^U_{j_I}\right]\right\}, \left\{[A^L_{j_F}B^L_{j_F}, A^U_{j_F}B^U_{j_F}]\right\}, \\ \left\{\lambda_{j_T} + \mu_{j_T} - \lambda_{j_T}\mu_{j_T}\right\}, \left\{\lambda_{j_I} + \mu_{j_I} - \lambda_{j_I}\mu_{j_I}\right\}, \left\{\lambda_{j_F}\right\} \end{array} \right\rangle,$$

Moreover the β-normalization is used in case of different cardinalities.

Example3.4: If

$$\alpha = \langle \{[0.1,0.5],[0.2,0.7]\}, \{[0.2,0.3],[0.1,0.6]\}, \{[0.1,0.4],[0.0,0.3]\}, \{0.1,0.2\}, \{0.3,0.5,0.7\}, \{0.4,0.8\}\rangle,$$

and $\beta = \langle \{[0.4,0.5],[0.3,0.4]\}, \{[0.1,0.3],[0.2,0.5]\}, \{[0.1,0.4],[0.7,0.8]\}, \{0.3,0.4,0.5\}, \{0.7,0.8\}, \{0.4,0.6\}\rangle$,

then using above definition and β-normalization with parameter $\xi = 0.5$ we have

$$\alpha \oplus \beta = \langle \{[0.46,0.75],[0.44,0.82]\}, \{[0.28,0.51],[0.28,0.8]\}, \{[0.01,0.16],[0,0.24]\}, \{0.03,0.06,0.1\}, \{0.21,0.375,0.56\}, \{0.64,0.92\}\rangle.$$

Definition 3.5: The Product of two NCHFSs $\alpha = \langle A, \lambda\rangle, \beta = \langle B, \mu\rangle$ is defined by

$$\alpha \otimes \beta = \left\langle \begin{array}{c} x, \left\{[A^L_{j_T}B^L_{j_T}, A^U_{j_T}B^U_{j_T}]\right\}, \left\{[A^L_{j_I}B^L_{j_I}, A^U_{j_I}B^U_{j_I}]\right\}, \left\{[A^L_{j_F} + B^L_{j_F} - A^L_{j_F}B^L_{j_F}, A^U_{j_F} + B^U_{j_F} - A^U_{j_F}B^U_{j_F}]\right\}, \\ \left\{\lambda_{j_T}\mu_{j_T}\right\}, \left\{\lambda_{j_I}\mu_{j_I}\right\}, \left\{\lambda_{j_F} + \mu_{j_F} - \lambda_{j_F}\mu_{j_F}\right\} \end{array} \right\rangle.$$

Moreover the β-normalization is used in case of different cardinalities.

Example3.6: If

$\alpha=\left\langle\{[0.1,0.5],[0.2,0.7]\},\{[0.2,0.3],[0.1,0.6]\},\{[0.1,0.4],[0.0,0.3]\},\{0.1,0.2\},\{0.3,0.5,0.7\},\{0.4,0.8\}\right\rangle$, and

$\beta=\left\langle\{[0.4,0.5],[0.3,0.4]\},\{[0.1,0.3],[0.2,0.5]\},\{[0.1,0.4],[0.7,0.8]\},\{0.3,0.4,0.5\},\{0.7,0.8\},\{0.4,0.6\}\right\rangle$, then using above definition and β-normalization with parameter $\xi = 0.5$ we have

$$\alpha\otimes\beta=\left\langle\begin{array}{c}\{[0.04,0.25],[0.06,0.28]\},\{[0.02,0.09],[0.02,0.3]\},\{[0.19,0.64],[0.7,0.86]\},\\ \{0.37,0.49,0.6\},\{0.79,0.875,0.94\},\{0.16,0.48\}\end{array}\right\rangle.$$

Definition 3.7: The scalar multiplication of a scalar q with a NCHFS $\alpha = \langle A, \lambda\rangle$ is defined by

$$q\alpha=\left\langle\begin{array}{c}\left\{\left[1-\left(1-A^{L}_{j_T}\right)^{q},1-\left(1-A^{U}_{j_T}\right)^{q}\right]\right\},\left\{\left[1-\left(1-A^{L}_{j_I}\right)^{q},1-\left(1-A^{U}_{j_I}\right)^{q}\right]\right\},\left\{\left[\left(A^{L}_{j_F}\right)^{q},\left(A^{U}_{j_F}\right)^{q}\right]\right\},\\ \left\{1-\left(1-\lambda_{j_T}\right)^{q}\right\},\left\{1-\left(1-\lambda_{j_I}\right)^{q}\right\},\left\{\left(\lambda_{j_F}\right)^{q}\right\}\end{array}\right\rangle.$$

Example3.8: If

$\alpha=\left\langle\{[0.1,0.5],[0.2,0.7]\},\{[0.2,0.3],[0.1,0.6]\},\{[0.1,0.4],[0.0,0.3]\},\{0.1,0.2\},\{0.3,0.5\},\{0.4,0.8\}\right\rangle$,

then using above definition with q=3 we have

$$3\alpha=\left\langle\begin{array}{c}\{[0.271,0.875],[0.488,0.973]\},\{[0.488,0.657],[0.271,0.936]\},\{[0.001,0.64],[0,0.27]\},\\ \{0.001,0.008\},\{0.27,0.125\},\{0.784,0.992\}\end{array}\right\rangle.$$

Definition 3.9: For NCHFS $\alpha = \langle A, \lambda\rangle$ and a scalar q

$$\alpha^{q}=\left\langle\begin{array}{c}x,\left\{\left[\left(A^{L}_{p_T}\right)^{q},\left(A^{U}_{p_T}\right)^{q}\right]\right\},\left\{\left[\left(A^{L}_{p_I}\right)^{q},\left(A^{U}_{p_I}\right)\right]^{q}\right\},\left\{\left[1-\left(1-A^{L}_{p_F}\right)^{q},1-\left(1-A^{U}_{p_F}\right)^{q}\right]\right\},\\ \left\{\left(\lambda_{p_T}\right)^{q}\right\},\left\{\left(\lambda_{p_I}\right)^{q}\right\},\left\{1-\left(1-\lambda_{p_F}\right)^{q}\right\}\end{array}\right\rangle$$

where $\alpha^q = \alpha \otimes \alpha\otimes \ldots \otimes\alpha(q - times)$ moreover α^q is a NCHF value for every $q > 0$.

Definition 3.10: The Einstein sum of two NCHFSs $\alpha = \langle A, \lambda\rangle, \beta = \langle B, \mu\rangle$ is defined by

$$\alpha\oplus_{E}\beta=\left\langle\begin{array}{c}\left\{\left[\frac{A^{L}_{j_T}+B^{L}_{j_T}}{1+A^{L}_{j_T}B^{L}_{j_T}},\frac{A^{U}_{j_T}+B^{U}_{j_T}}{1+A^{U}_{j_T}B^{U}_{j_T}}\right]\right\},\left\{\left[\frac{A^{L}_{j_I}+B^{L}_{j_I}}{1+A^{L}_{j_I}B^{L}_{j_I}},\frac{A^{U}_{j_I}+B^{U}_{j_I}}{1+A^{U}_{j_I}B^{U}_{j_I}}\right]\right\},\left\{\left[\frac{A^{L}_{j_F}B^{L}_{j_F}}{1+\left(1-A^{L}_{j_F}\right)\left(1-B^{L}_{j_F}\right)},\frac{A^{U}_{j_F}B^{U}_{j_F}}{1+\left(1-A^{U}_{j_F}\right)\left(1-B^{U}_{j_F}\right)}\right]\right\},\\ \left\{\frac{\lambda_{j_T}+\mu_{j_T}}{1+\lambda_{j_T}\mu_{j_T}}\right\},\left\{\frac{\lambda_{j_I}+\mu_{j_I}}{1+\lambda_{j_I}\mu_{j_I}}\right\},\left\{\frac{\lambda_{j_F}\mu_{j_F}}{1+\left(1-\lambda_{j_F}\right)\left(1-\mu_{j_F}\right)}\right\}\end{array}\right\rangle.$$

Moreover the β-normalization is used in case of different cardinalities.

Definition 3.11: The Einstein product of two NCHFSs $\alpha = \langle A, \lambda\rangle, \beta = \langle B, \mu\rangle$ is defined by

$$\alpha\otimes_{E}\beta=\left\langle\begin{array}{c}\left\{\left[\frac{A^{L}_{j_T}B^{L}_{j_T}}{1+\left(1-A^{L}_{j_T}\right)\left(1-B^{L}_{j_T}\right)},\frac{A^{U}_{j_T}B^{U}_{j_T}}{1+\left(1-A^{U}_{j_T}\right)\left(1-B^{U}_{j_T}\right)}\right]\right\},\left\{\left[\frac{A^{L}_{j_I}B^{L}_{j_I}}{1+\left(1-A^{L}_{j_I}\right)\left(1-B^{L}_{j_I}\right)},\frac{A^{U}_{j_I}B^{U}_{j_I}}{1+\left(1-A^{U}_{j_I}\right)\left(1-B^{U}_{j_I}\right)}\right]\right\},\\ \left\{\left[\frac{A^{L}_{j_F}+B^{L}_{j_F}}{1+A^{L}_{j_F}B^{L}_{j_F}},\frac{A^{U}_{j_F}+B^{U}_{j_F}}{1+A^{U}_{j_F}B^{U}_{j_F}}\right]\right\},\left\{\frac{\lambda_{j_T}\mu_{j_T}}{1+\left(1-\lambda_{j_T}\right)\left(1-\mu_{j_T}\right)}\right\},\left\{\frac{\lambda_{j_I}\mu_{j_I}}{1+\left(1-\lambda_{j_I}\right)\left(1-\mu_{j_I}\right)}\right\},\left\{\frac{\lambda_{j_F}+\mu_{j_F}}{1+\lambda_{j_F}\mu_{j_F}}\right\}\end{array}\right\rangle.$$

Moreover the β-normalization is used in case of different cardinalities.

Definition 3.12: The Einstein scalar multiplication of a scalar q with a NCHFS $\alpha = \langle A, \lambda \rangle$ is defined by

$$q_{_E}\alpha = \left\langle \begin{array}{c} \left\{\left[\frac{\left(1+A^L_{j_T}\right)^q-\left(1-A^L_{j_T}\right)^q}{\left(1+A^L_{j_T}\right)^q+\left(1-A^L_{j_T}\right)^q}, \frac{\left(1+A^U_{j_T}\right)^q-\left(1-A^U_{j_T}\right)^q}{\left(1+A^U_{j_T}\right)^q+\left(1-A^U_{j_T}\right)^q}\right]\right\}, \left\{\left[\frac{\left(1+A^L_{j_I}\right)^q-\left(1-A^L_{j_I}\right)^q}{\left(1+A^L_{j_I}\right)^q+\left(1-A^L_{j_I}\right)^q}, \frac{\left(1+A^U_{j_I}\right)^q-\left(1-A^U_{j_I}\right)^q}{\left(1+A^U_{j_I}\right)^q+\left(1-A^U_{j_I}\right)^q}\right]\right\}, \\ \left\{\left[\frac{2\left(A^L_{j_F}\right)^q}{\left(2-A^L_{j_F}\right)^q+\left(A^L_{j_F}\right)^q}, \frac{2\left(A^U_{j_F}\right)^q}{\left(2-A^U_{j_F}\right)^q+\left(A^U_{j_F}\right)^q}\right]\right\}, \\ \left\{\frac{\left(1+\lambda_{j_T}\right)^q-\left(1-\lambda_{j_T}\right)^q}{\left(1+\lambda_{j_T}\right)^q+\left(1-\lambda_{j_T}\right)^q}\right\}, \left\{\frac{\left(1+\lambda_{j_I}\right)^q-\left(1-\lambda_{j_I}\right)^q}{\left(1+\lambda_{j_I}\right)^q+\left(1-\lambda_{j_I}\right)^q}\right\}, \left\{\frac{2\left(\lambda_{j_F}\right)^q}{\left(2-\lambda_{j_F}\right)^q+\left(\lambda_{j_F}\right)^q}\right\} \end{array} \right\rangle.$$

Theorem 3.13: For a scalar q and a *NCHFS* $\alpha = \langle A, \lambda \rangle$ we have

$$\alpha^{E^q} = \left\langle \begin{array}{c} \left\{\left[\frac{2\left(A^L_{j_T}\right)^q}{\left(2-A^L_{j_T}\right)^q+\left(A^L_{j_T}\right)^q}, \frac{2\left(A^U_{j_T}\right)^q}{\left(2-A^U_{j_T}\right)^q+\left(A^U_{j_T}\right)^q}\right]\right\}, \left\{\left[\frac{2\left(A^L_{j_I}\right)^q}{\left(2-A^L_{j_I}\right)^q+\left(A^L_{j_I}\right)^q}, \frac{2\left(A^U_{j_I}\right)^q}{\left(2-A^U_{j_I}\right)^q+\left(A^U_{j_I}\right)^q}\right]\right\} \\ \left\{\left[\frac{\left(1+A^L_{j_F}\right)^q-\left(1-A^L_{j_T}\right)^q}{\left(1+A^L_{j_F}\right)^q+\left(1-A^L_{j_F}\right)^q}, \frac{\left(1+A^U_{j_F}\right)^q-\left(1-A^U_{j_F}\right)^q}{\left(1+A^U_{j_F}\right)^q+\left(1-A^U_{j_F}\right)^q}\right]\right\}, \left\{\frac{2\left(\lambda_{j_T}\right)^q}{\left(2-\lambda_{j_T}\right)^q+\left(\lambda_{j_T}\right)^q}\right\}, \left\{\frac{2\left(\lambda_{j_I}\right)^q}{\left(2-\lambda_{j_I}\right)^q+\left(\lambda_{j_I}\right)^q}\right\}, \\ \left\{\frac{\left(1+\lambda_{j_F}\right)^q-\left(1-\lambda_{j_F}\right)^q}{\left(1+\lambda_{j_F}\right)^q+\left(1-\lambda_{j_F}\right)^q}\right\} \end{array} \right\rangle.$$

where $\alpha^{E^q} = \alpha \otimes_E \alpha \otimes_E \ldots \otimes_E \alpha(q - times)$ moreover α^{E^q} is a NCHF value for every $q > 0$.

Proof: Using induction on q. for *q=1* we have

$$\alpha^{E^1} = \left\langle \begin{array}{c} \left\{\left[\frac{2\left(A^L_{j_T}\right)}{\left(2-A^L_{j_T}\right)+\left(A^L_{j_T}\right)}, \frac{2\left(A^U_{j_T}\right)}{\left(2-A^U_{j_T}\right)^q+\left(A^U_{j_T}\right)}\right]\right\}, \left\{\left[\frac{2\left(A^L_{j_I}\right)}{\left(2-A^L_{j_I}\right)+\left(A^L_{j_I}\right)}, \frac{2\left(A^U_{j_I}\right)}{\left(2-A^U_{j_I}\right)+\left(A^U_{j_I}\right)}\right]\right\} \\ \left\{\left[\frac{\left(1+A^L_{j_F}\right)-\left(1-A^L_{j_T}\right)}{\left(1+A^L_{j_F}\right)+\left(1-A^L_{j_F}\right)}, \frac{\left(1+A^U_{j_F}\right)-\left(1-A^U_{j_F}\right)}{\left(1+A^U_{j_F}\right)+\left(1-A^U_{j_F}\right)}\right]\right\}, \left\{\frac{2\left(\lambda_{j_T}\right)}{\left(2-\lambda_{j_T}\right)+\left(\lambda_{j_T}\right)}\right\}, \left\{\frac{2\left(\lambda_{j_I}\right)}{\left(2-\lambda_{j_I}\right)+\left(\lambda_{j_I}\right)}\right\}, \\ \left\{\frac{\left(1+\lambda_{j_F}\right)-\left(1-\lambda_{j_F}\right)}{\left(1+\lambda_{j_F}\right)+\left(1-\lambda_{j_F}\right)}\right\} \end{array} \right\rangle$$

Assuming that result is true for *q=m*.

$$\alpha^{E^m} = \left\langle \begin{array}{c} \left\{\left[\frac{2\left(A^L_{j_T}\right)^m}{\left(2-A^L_{j_T}\right)^m+\left(A^L_{j_T}\right)^m}, \frac{2\left(A^U_{j_T}\right)^m}{\left(2-A^U_{j_T}\right)^m+\left(A^U_{j_T}\right)^m}\right]\right\}, \left\{\left[\frac{2\left(A^L_{j_I}\right)^m}{\left(2-A^L_{j_I}\right)^m+\left(A^L_{j_I}\right)^m}, \frac{2\left(A^U_{j_I}\right)^m}{\left(2-A^U_{j_I}\right)^m+\left(A^U_{j_I}\right)^m}\right]\right\} \\ \left\{\left[\frac{\left(1+A^L_{j_F}\right)^m-\left(1-A^L_{j_T}\right)^m}{\left(1+A^L_{j_F}\right)^m+\left(1-A^L_{j_F}\right)^m}, \frac{\left(1+A^U_{j_F}\right)^m-\left(1-A^U_{j_F}\right)^m}{\left(1+A^U_{j_F}\right)^m+\left(1-A^U_{j_F}\right)^m}\right]\right\}, \left\{\frac{2\left(\lambda_{j_T}\right)^m}{\left(2-\lambda_{j_T}\right)^m+\left(\lambda_{j_T}\right)^m}\right\}, \left\{\frac{2\left(\lambda_{j_I}\right)^m}{\left(2-\lambda_{j_I}\right)^m+\left(\lambda_{j_I}\right)^m}\right\}, \\ \left\{\frac{\left(1+\lambda_{j_F}\right)^m-\left(1-\lambda_{j_F}\right)^m}{\left(1+\lambda_{j_F}\right)^m+\left(1-\lambda_{j_F}\right)^m}\right\} \end{array} \right\rangle$$

α^{E^m} is neutrosophic cubic hesitant fuzzy value. Using assumption, we have

$$\alpha^{E^{m}}\otimes\alpha^{E^{1}}=\left\langle\begin{array}{c}\left\{\left[\dfrac{2\left(A_{j_T}^{L}\right)^{m}}{\left(2-A_{j_T}^{L}\right)^{m}+\left(A_{j_T}^{L}\right)^{m}},\dfrac{2\left(A_{j_T}^{U}\right)^{m}}{\left(2-A_{j_T}^{U}\right)^{m}+\left(A_{j_T}^{U}\right)^{m}}\right]\right\},\\ \left\{\left[\dfrac{2\left(A_{j_I}^{L}\right)^{m}}{\left(2-A_{j_I}^{L}\right)^{m}+\left(A_{j_I}^{L}\right)^{m}},\dfrac{2\left(A_{j_I}^{U}\right)^{m}}{\left(2-A_{j_I}^{U}\right)^{m}+\left(A_{j_I}^{U}\right)^{m}}\right]\right\}\\ \left\{\left[\dfrac{\left(1+A_{j_F}^{L}\right)^{m}-\left(1-A_{j_T}^{L}\right)^{m}}{\left(1+A_{j_F}^{L}\right)^{m}+\left(1-A_{j_F}^{L}\right)^{m}},\dfrac{\left(1+A_{j_F}^{U}\right)^{m}-\left(1-A_{j_F}^{U}\right)^{m}}{\left(1+A_{j_F}^{U}\right)^{m}+\left(1-A_{j_F}^{U}\right)^{m}}\right]\right\},\\ \left\{\dfrac{2\left(\lambda_{j_T}\right)^{m}}{\left(2-\lambda_{j_T}\right)^{m}+\left(\lambda_{j_T}\right)^{m}}\right\},\left\{\dfrac{2\left(\lambda_{j_I}\right)^{m}}{\left(2-\lambda_{j_I}\right)^{m}+\left(\lambda_{j_I}\right)^{m}}\right\},\\ \left\{\dfrac{\left(1+\lambda_{j_F}\right)^{m}-\left(1-\lambda_{j_F}\right)^{m}}{\left(1+\lambda_{j_F}\right)^{m}+\left(1-\lambda_{j_F}\right)^{m}}\right\}\end{array}\right\rangle\otimes_{E}\left\langle\begin{array}{c}\left\{\left[\dfrac{2\left(A_{j_T}^{L}\right)}{\left(2-A_{j_T}^{L}\right)+\left(A_{j_T}^{L}\right)},\dfrac{2\left(A_{j_T}^{U}\right)}{\left(2-A_{j_T}^{U}\right)^{q}+\left(A_{j_T}^{U}\right)}\right]\right\},\\ \left\{\left[\dfrac{2\left(A_{j_I}^{L}\right)}{\left(2-A_{j_I}^{L}\right)+\left(A_{j_I}^{L}\right)},\dfrac{2\left(A_{j_I}^{U}\right)}{\left(2-A_{j_I}^{U}\right)+\left(A_{j_I}^{U}\right)}\right]\right\}\\ \left\{\left[\dfrac{\left(1+A_{j_F}^{L}\right)-\left(1-A_{j_T}^{L}\right)}{\left(1+A_{j_F}^{L}\right)+\left(1-A_{j_F}^{L}\right)},\dfrac{\left(1+A_{j_F}^{U}\right)-\left(1-A_{j_F}^{U}\right)}{\left(1+A_{j_F}^{U}\right)+\left(1-A_{j_F}^{U}\right)}\right]\right\},\\ \left\{\dfrac{2\left(\lambda_{j_T}\right)}{\left(2-\lambda_{j_T}\right)+\left(\lambda_{j_T}\right)}\right\},\left\{\dfrac{2\left(\lambda_{j_I}\right)}{\left(2-\lambda_{j_I}\right)+\left(\lambda_{j_I}\right)}\right\},\\ \left\{\dfrac{\left(1+\lambda_{j_F}\right)-\left(1-\lambda_{j_F}\right)}{\left(1+\lambda_{j_F}\right)+\left(1-\lambda_{j_F}\right)}\right\}\end{array}\right\rangle$$

$$=\left\langle\begin{array}{c}\left\{\left[\dfrac{\dfrac{4(A_{j_T}^{L})^{m+1}}{\left((2-A_{j_T}^{L})^{m}+(A_{jT}^{L})^{m}\right)\left((2-A_{j_T}^{L})^{1}+(A_{jT}^{L})^{1}\right)}}{1+\left(1-\dfrac{2(A_{jT}^{L})^{m}}{\left(2-A_{jT}^{L}\right)^{m}+(A_{jT}^{L})^{m}}\right)\left(1-\dfrac{2(A_{jT}^{L})^{1}}{\left(2-A_{jT}^{L}\right)^{1}+(A_{jT}^{L})^{1}}\right)},\dfrac{\dfrac{4(A_{j_T}^{U})^{m+1}}{\left((2-A_{j_T}^{U})^{m}+(A_{jT}^{U})^{m}\right)\left((2-A_{j_T}^{U})^{1}+(A_{jT}^{U})^{1}\right)}}{1+\left(1-\dfrac{2(A_{jT}^{U})^{m}}{\left(2-A_{jT}^{U}\right)^{m}+(A_{jT}^{U})^{m}}\right)\left(1-\dfrac{2(A_{jT}^{U})^{1}}{\left(2-A_{jT}^{U}\right)^{1}+(A_{jT}^{U})^{1}}\right)}\right]\right\},\\ \left\{\left[\dfrac{\dfrac{4(A_{j_I}^{L})^{m+1}}{\left((2-A_{j_I}^{L})^{m}+(A_{jI}^{L})^{m}\right)\left((2-A_{j_I}^{L})^{1}+(A_{jI}^{L})^{1}\right)}}{1+\left(1-\dfrac{2(A_{jI}^{L})^{m}}{\left(2-A_{jI}^{L}\right)^{m}+(A_{jI}^{L})^{m}}\right)\left(1-\dfrac{2(A_{jI}^{L})^{1}}{\left(2-A_{jI}^{L}\right)^{1}+(A_{jI}^{L})^{1}}\right)},\dfrac{\dfrac{4(A_{j_I}^{U})^{m+1}}{\left((2-A_{j_I}^{U})^{m}+(A_{jI}^{U})^{m}\right)\left((2-A_{j_I}^{U})^{1}+(A_{jI}^{U})^{1}\right)}}{1+\left(1-\dfrac{2(A_{jI}^{U})^{m}}{\left(2-A_{jI}^{U}\right)^{m}+(A_{jI}^{U})^{m}}\right)\left(1-\dfrac{2(A_{jI}^{U})^{1}}{\left(2-A_{jI}^{U}\right)^{1}+(A_{jI}^{U})^{1}}\right)}\right]\right\},\\ \left\{\left[\dfrac{\left(\dfrac{(1+A_{jF}^{L})^{m}-(1-A_{jF}^{L})^{m}}{(1+A_{jF}^{L})^{m}+(1-A_{jF}^{L})^{m}}\right)+\left(\dfrac{(1+A_{jF}^{L})^{1}-(1-A_{jF}^{L})^{1}}{(1+A_{jF}^{L})^{1}+(1-A_{jF}^{L})^{1}}\right)}{1+\left(\dfrac{(1+A_{jF}^{L})^{m}-(1-A_{jF}^{L})^{m}}{(1+A_{jF}^{L})^{m}+(1-A_{jF}^{L})^{m}}\right)\left(\dfrac{(1+A_{jF}^{L})^{1}-(1-A_{jF}^{L})^{1}}{(1+A_{jF}^{L})^{1}+(1-A_{jF}^{L})^{1}}\right)},\dfrac{\left(\dfrac{(1+A_{jF}^{U})^{m}-(1-A_{jF}^{U})^{m}}{(1+A_{jF}^{U})^{m}+(1-A_{jF}^{U})^{m}}\right)+\left(\dfrac{(1+A_{jF}^{U})^{1}-(1-A_{jF}^{U})^{1}}{(1+A_{jF}^{U})^{1}+(1-A_{jF}^{U})^{1}}\right)}{1+\left(\dfrac{(1+A_{jF}^{U})^{m}-(1-A_{jF}^{U})^{m}}{(1+A_{jF}^{U})^{m}+(1-A_{jF}^{U})^{m}}\right)\left(\dfrac{(1+A_{jF}^{U})^{1}-(1-A_{jF}^{U})^{1}}{(1+A_{jF}^{U})^{1}+(1-A_{jF}^{U})^{1}}\right)}\right]\right\},\\ \left\{\dfrac{\dfrac{4(\lambda_{j_T})^{m+1}}{\left((2-\lambda_{j_T})^{m}+(\lambda_{j_T})^{m}\right)\left((2-\lambda_{j_T})^{1}+(\lambda_{jF})^{1}\right)}}{1+\left(1-\dfrac{2(\lambda_{j_T})^{m}}{\left(2-\lambda_{j_T}\right)^{m}+(\lambda_{j_T})^{m}}\right)\left(1-\dfrac{2(\lambda_{j_T})^{1}}{\left(2-\lambda_{j_T}\right)^{1}+(\lambda_{j_T})^{1}}\right)}\right\},\left\{\dfrac{\dfrac{4(\lambda_{j_I})^{m+1}}{\left((2-\lambda_{j_I})^{m}+(\lambda_{j_I})^{m}\right)\left((2-\lambda_{j_I})^{1}+(\lambda_{j_I})^{1}\right)}}{1+\left(1-\dfrac{2(\lambda_{j_I})^{m}}{\left(2-\lambda_{j_I}\right)^{m}+(\lambda_{j_I})^{m}}\right)\left(1-\dfrac{2(\lambda_{j_I})^{1}}{\left(2-\lambda_{j_I}\right)^{1}+(\lambda_{j_I})^{1}}\right)}\right\},\\ \left\{\dfrac{\dfrac{(1+\lambda_{j_F})^{m}-(1-\lambda_{j_F})^{m}}{(1+\lambda_{j_F})^{m}+(1-\lambda_{j_F})^{m}}+\dfrac{(1+\lambda_{j_F})^{1}-(1-\lambda_{j_F})^{1}}{(1+\lambda_{j_F})^{1}+(1-\lambda_{j_F})^{1}}}{1+\left(\dfrac{(1+\lambda_{j_F})^{m}-(1-\lambda_{j_F})^{m}}{(1+\lambda_{j_F})^{m}+(1-\lambda_{j_F})^{m}}\right)\left(\dfrac{(1+\lambda_{j_F})^{1}-(1-\lambda_{j_F})^{1}}{(1+\lambda_{j_F})^{1}+(1-\lambda_{j_F})^{1}}\right)}\right\}\end{array}\right\rangle$$

$$
=\left\langle
\begin{array}{c}
\left\{\left[\frac{\dfrac{4(A^{L}_{j_T})^{m+1}}{\left((2-A^{L}_{j_T})^{m}+(A^{L}_{jT})^{m}\right)\left((2-A^{L}_{j_T})^{1}+(A^{L}_{jT})^{1}\right)}}{1+\left(\dfrac{\left(2-A^{L}_{jT}\right)^{m}+(A^{L}_{jT})^{m}-2(A^{L}_{jT})^{m}}{\left(2-A^{L}_{jT}\right)^{m}+(A^{L}_{jT})^{m}}\right)\left(\dfrac{\left(2-A^{L}_{jT}\right)^{1}+(A^{L}_{jT})^{1}-2(A^{L}_{jT})^{1}}{\left(2-A^{L}_{jT}\right)^{1}+(A^{L}_{jT})^{1}}\right)},\frac{\dfrac{4(A^{U}_{j_T})^{m+1}}{\left((2-A^{U}_{j_T})^{m}+(A^{U}_{jT})^{m}\right)\left((2-A^{U}_{j_T})^{1}+(A^{U}_{jT})^{1}\right)}}{1+\left(\dfrac{\left(2-A^{U}_{jT}\right)^{m}+(A^{U}_{jT})^{m}-2(A^{U}_{jT})^{m}}{\left(2-A^{U}_{jT}\right)^{m}+(A^{U}_{jT})^{m}}\right)\left(\dfrac{\left(2-A^{U}_{jT}\right)^{1}+(A^{U}_{jT})^{1}-2(A^{U}_{jT})^{1}}{\left(2-A^{U}_{jT}\right)^{1}+(A^{U}_{jT})^{1}}\right)}\right]\right\}, \\
\left\{\left[\frac{\dfrac{4(A^{L}_{j_I})^{m+1}}{\left((2-A^{L}_{j_I})^{m}+(A^{L}_{jI})^{m}\right)\left((2-A^{L}_{j_I})^{1}+(A^{L}_{jI})^{1}\right)}}{1+\left(\dfrac{\left(2-A^{L}_{jI}\right)^{m}+(A^{L}_{jI})^{m}-2(A^{L}_{jI})^{m}}{\left(2-A^{L}_{jI}\right)^{m}+(A^{L}_{jI})^{m}}\right)\left(\dfrac{\left(2-A^{L}_{jI}\right)^{1}+(A^{L}_{jI})^{1}-2(A^{L}_{jI})^{1}}{\left(2-A^{L}_{jI}\right)^{1}+(A^{L}_{jI})^{1}}\right)},\frac{\dfrac{4(A^{U}_{j_I})^{m+1}}{\left((2-A^{U}_{j_I})^{m}+(A^{U}_{jI})^{m}\right)\left((2-A^{U}_{j_I})^{1}+(A^{U}_{jI})^{1}\right)}}{1+\left(\dfrac{\left(2-A^{U}_{jI}\right)^{m}+(A^{U}_{jI})^{m}-2(A^{U}_{jI})^{m}}{\left(2-A^{U}_{jI}\right)^{m}+(A^{U}_{jI})^{m}}\right)\left(\dfrac{\left(2-A^{U}_{jI}\right)^{1}+(A^{U}_{jI})^{1}-2(A^{U}_{jI})^{1}}{\left(2-A^{U}_{jI}\right)^{1}+(A^{U}_{jI})^{1}}\right)}\right]\right\}, \\
\left\{\left[\begin{array}{c}
\frac{\dfrac{\left((1+A^{L}_{jF})^{m}-(1-A^{L}_{jF})^{m}\right)\left((1+A^{L}_{jF})^{1}+(1-A^{L}_{jF})^{1}\right)+\left((1+A^{L}_{jF})^{1}-(1-A^{L}_{jF})^{1}\right)\left((1+A^{L}_{jF})^{m}+(1-A^{L}_{jF})^{m}\right)}{\left((1+A^{L}_{jF})^{m}+(1-A^{L}_{jF})^{m}\right)\left((1+A^{L}_{jF})^{1}+(1-A^{L}_{jF})^{1}\right)}}{\dfrac{\left((1+A^{L}_{jF})^{m}+(1-A^{L}_{jF})^{m}\right)\left((1+A^{L}_{jF})^{1}+(1-A^{L}_{jF})^{1}\right)+\left((1+A^{L}_{jF})^{1}+(1-A^{L}_{jF})^{1}\right)\left((1+A^{L}_{jF})^{m}-(1-A^{L}_{jF})^{m}\right)}{\left((1+A^{L}_{jF})^{m}+(1-A^{L}_{jF})^{m}\right)\left((1+A^{L}_{jF})^{1}+(1-A^{L}_{jF})^{1}\right)}}, \\
\frac{\dfrac{\left((1+A^{U}_{jF})^{m}-(1-A^{U}_{jF})^{m}\right)\left((1+A^{U}_{jF})^{1}+(1-A^{U}_{jF})^{1}\right)+\left((1+A^{U}_{jF})^{1}-(1-A^{U}_{jF})^{1}\right)\left((1+A^{U}_{jF})^{m}+(1-A^{U}_{jF})^{m}\right)}{\left((1+A^{U}_{jF})^{m}+(1-A^{U}_{jF})^{m}\right)\left((1+A^{U}_{jF})^{1}+(1-A^{U}_{jF})^{1}\right)}}{\dfrac{\left((1+A^{U}_{jF})^{m}+(1-A^{U}_{jF})^{m}\right)\left((1+A^{U}_{jF})^{1}+(1-A^{U}_{jF})^{1}\right)+\left((1+A^{U}_{jF})^{1}+(1-A^{U}_{jF})^{1}\right)\left((1+A^{U}_{jF})^{m}-(1-A^{U}_{jF})^{m}\right)}{\left((1+A^{U}_{jF})^{m}+(1-A^{U}_{jF})^{m}\right)\left((1+A^{U}_{jF})^{1}+(1-A^{U}_{jF})^{1}\right)}}
\end{array}\right]\right\}, \\
\left\{\frac{\dfrac{4(\lambda_{jT})^{m+1}}{\left((2-\lambda_{jT})^{m}+(\lambda_{jT})^{m}\right)\left((2-\lambda_{jF})^{1}+(\lambda_{jT})^{1}\right)}}{1+\left(\dfrac{\left(2-\lambda_{jT}\right)^{m}+(\lambda_{jT})^{m}-2(\lambda_{jT})^{m}}{\left(2-\lambda_{jT}\right)^{m}+(\lambda_{jT})^{m}}\right)\left(\dfrac{\left(2-\lambda_{jT}\right)^{1}+(\lambda_{jT})^{1}-2(\lambda_{jT})^{1}}{\left(2-\lambda_{jT}\right)^{1}+(\lambda_{jT})^{1}}\right)}\right\}, \\
\left\{\frac{\dfrac{4(\lambda_{jI})^{m+1}}{\left((2-\lambda_{jI})^{m}+(\lambda_{jI})^{m}\right)\left((2-\lambda_{jI})^{1}+(\lambda_{jI})^{1}\right)}}{1+\left(\dfrac{\left(2-\lambda_{jI}\right)^{m}+(\lambda_{jI})^{m}-2(\lambda_{jI})^{m}}{\left(2-\lambda_{jI}\right)^{m}+(\lambda_{jI})^{m}}\right)\left(\dfrac{\left(2-\lambda_{jI}\right)^{1}+(\lambda_{jI})^{1}-2(\lambda_{jI})^{1}}{\left(2-\lambda_{jI}\right)^{1}+(\lambda_{jI})^{1}}\right)}\right\}, \\
\left\{\frac{\dfrac{\left((1+\lambda_{jF})^{m}-(1-\lambda_{jF})^{m}\right)\left((1+\lambda_{jF})^{1}+(1-\lambda_{jF})^{1}\right)+\left((1+\lambda_{jF})^{m}+(1-\lambda_{jF})^{m}\right)\left((1+\lambda_{jF})^{1}-(1-\lambda_{jF})^{1}\right)}{\left((1+\lambda_{jF})^{m}+(1-\lambda_{jF})^{m}\right)\left((1+\lambda_{jF})^{1}+(1-\lambda_{jF})^{1}\right)}}{\dfrac{\left((1+\lambda_{jF})^{m}+(1-\lambda_{jF})^{m}\right)\left((1+\lambda_{jF})^{1}+(1-\lambda_{jF})^{1}\right)+\left((1+\lambda_{jF})^{m}-(1-\lambda_{jF})^{m}\right)\left((1+\lambda_{jF})^{1}-(1-\lambda_{jF})^{1}\right)}{\left((1+\lambda_{jF})^{m}+(1-\lambda_{jF})^{m}\right)\left((1+\lambda_{jF})^{1}+(1-\lambda_{jF})^{1}\right)}}\right\}
\end{array}
\right\rangle
$$

$$= \left\langle \begin{array}{c} \left\{\left[\frac{4\left(A^L_{j_T}\right)^{m+1}}{2\left(\left(2-A^L_{j_T}\right)^{m+1}+\left(A^L_{j_T}\right)^{m+1}\right)}, \frac{4\left(A^U_{j_T}\right)^{m+1}}{2\left(\left(2-A^U_{j_T}\right)^{m+1}+\left(A^U_{j_T}\right)^{m+1}\right)}\right]\right\}, \\ \left\{\left[\frac{4\left(A^L_{j_I}\right)^{m+1}}{2\left(\left(2-A^L_{j_I}\right)^{m+1}+\left(A^L_{j_I}\right)^{m+1}\right)}, \frac{4\left(A^U_{j_I}\right)^{m+1}}{2\left(\left(2-A^U_{j_I}\right)^{m+1}+\left(A^U_{j_I}\right)^{m+1}\right)}\right]\right\}, \\ \left\{\left[\frac{\left(1+A^L_{j_F}\right)^{m+1}-\left(1-A^L_{j_T}\right)^{m+1}}{\left(1+A^L_{j_F}\right)^{m+1}+\left(1-A^L_{j_F}\right)^{m+1}}, \frac{\left(1+A^U_{j_F}\right)^{m+1}-\left(1-A^U_{j_F}\right)^{m+1}}{\left(1+A^U_{j_F}\right)^{m+1}+\left(1-A^U_{j_F}\right)^{m+1}}\right]\right\}, \\ \left\{\frac{4\left(\lambda_{j_T}\right)^{m+1}}{2\left(\left(2-\lambda_{j_T}\right)^{m+1}+\left(\lambda_{j_T}\right)^{m+1}\right)}\right\}, \left\{\frac{4\left(\lambda_{j_I}\right)^{m+1}}{2\left(\left(2-\lambda_{j_I}\right)^{m+1}+\left(\lambda_{j_I}\right)^{m+1}\right)}\right\}, \\ \left\{\frac{\left(1+\lambda_{j_F}\right)^{m+1}-\left(1-\lambda_{j_F}\right)^{m+1}}{\left(1+\lambda_{j_F}\right)^{m+1}+\left(1-\lambda_{j_F}\right)^{m+1}}\right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \left\{\left[\frac{2\left(A^L_{j_T}\right)^{m+1}}{\left(2-A^L_{j_T}\right)^{m+1}+\left(A^L_{j_T}\right)^{m+1}}, \frac{2\left(A^U_{j_T}\right)^{m+1}}{\left(2-A^U_{j_T}\right)^{m+1}+\left(A^U_{j_T}\right)^{m+1}}\right]\right\}, \left\{\left[\frac{2\left(A^L_{j_I}\right)^{m+1}}{\left(2-A^L_{j_I}\right)^{m+1}+\left(A^L_{j_I}\right)^{m+1}}, \frac{2\left(A^U_{j_I}\right)^{m+1}}{\left(2-A^U_{j_I}\right)^{m+1}+\left(A^U_{j_I}\right)^{m+1}}\right]\right\} \\ \left\{\left[\frac{\left(1+A^L_{j_F}\right)^{m+1}-\left(1-A^L_{j_T}\right)^{m+1}}{\left(1+A^L_{j_F}\right)^{m+1}+\left(1-A^L_{j_F}\right)^{m+1}}, \frac{\left(1+A^U_{j_F}\right)^{m+1}-\left(1-A^U_{j_F}\right)^{m+1}}{\left(1+A^U_{j_F}\right)^{m+1}+\left(1-A^U_{j_F}\right)^{m+1}}\right]\right\}, \left\{\frac{2\left(\lambda_{j_T}\right)^{m+1}}{\left(2-\lambda_{j_T}\right)^{m+1}+\left(\lambda_{j_T}\right)^{m+1}}\right\}, \left\{\frac{2\left(\lambda_{j_I}\right)^{m+1}}{\left(2-\lambda_{j_I}\right)^{m+1}+\left(\lambda_{j_I}\right)^{m+1}}\right\}, \\ \left\{\frac{\left(1+\lambda_{j_F}\right)^{m+1}-\left(1-\lambda_{j_F}\right)^{m+1}}{\left(1+\lambda_{j_F}\right)^{m+1}+\left(1-\lambda_{j_F}\right)^{m+1}}\right\} \end{array} \right\rangle$$

Definition 3.14: The Score, accuracy and certainty of a *NCHFS* $\alpha = \langle A, \lambda \rangle$ where $A = \langle A_T, A_I, A_F \rangle, A_T = \{[A^L_{j_T}, A^U_{j_T}]; j = 1, \dots, l\}, A_I = \{[A^L_{j_I}, A^U_{j_I}]; j = 1, \dots, m\}, A_F = \{[A^L_{j_F}, A^U_{j_F}]; i = 1, \dots, n\}$ and $\lambda = \langle \lambda_T, \lambda_I, \lambda_F \rangle, \lambda_T = \{\lambda_{j_T}; j = 1, \dots, r\}, \lambda_I = \{\lambda_{j_I}; j = 1, \dots, s\}, \lambda_F = \{\lambda_{j_F}; j = 1, \dots, t\}$ are defined as:

$$S(\alpha) = \frac{1}{2}\left\{ \begin{array}{c} \frac{1}{6}\left(\frac{1}{l}\sum_{j=1}^{l}\left(A^L_{j_T}+A^U_{j_F}\right)+\frac{1}{m}\sum_{j=1}^{m}\left(A^L_{j_I}+A^U_{j_I}\right)+\frac{1}{n}\sum_{j=1}^{n}\left(2-\left(A^L_{j_T}+A^U_{j_F}\right)\right)\right) \\ +\frac{1}{3}\left(\frac{1}{r}\sum_{j=1}^{r}\lambda_{j_T}+\frac{1}{s}\sum_{j=1}^{s}\lambda_{j_I}+\frac{1}{t}\sum_{j=1}^{t}\left(1-\lambda_{j_F}\right)\right) \end{array} \right\},$$

$$H(\alpha) = \tfrac{1}{9}\left\{ \begin{array}{c} \tfrac{1}{l}\sum_{j=1}^{l}\left(A^L_{j_T}+A^U_{j_T}\right)+\tfrac{1}{m}\sum_{j=1}^{m}\left(A^L_{j_I}+A^U_{j_I}\right)+\tfrac{1}{n}\sum_{j=1}^{n}\left(A^U_{j_F}+A^L_{j_F}\right) \\ +\tfrac{1}{r}\sum_{j=1}^{r}\lambda_{j_T}+\tfrac{1}{s}\sum_{j=1}^{s}\lambda_{j_I}+\tfrac{1}{t}\sum_{j=1}^{t}\lambda_{j_F} \end{array} \right\},$$

$$C(\alpha) = \tfrac{1}{3}\left\{\tfrac{1}{l}\sum_{j=1}^{l}\left(A^L_{j_T}+A^U_{j_T}\right)+\tfrac{1}{r}\sum_{j=1}^{r}\lambda_{j_T}\right\}.$$

If $\alpha = \left\langle \{[0.1,0.5],[0.2,0.7]\}, \{[0.2,0.3],[0.1,0.6]\}, \{[0.1,0.4],[0,0.3]\}, \{0.1,0.2\}, \{0.3,0.5\}, \{0.4,0.8\} \right\rangle$, and

$\beta = \langle \{[0.4,0.5],[0.3,0.4]\}, \{[0.1,0.3],[0.2,0.5]\}, \{[0.1,0.4],[0.7,0.8]\}, \{0.3,0.5\}, \{0.7,0.8\}, \{0.4,0.6\} \rangle$, then $S(\alpha) = 0.404167, S(\beta) = 0.470833, H(\alpha) = 0.3222, H(\beta) = 0.4444, C(\alpha) = 0.15, C(\beta) = 0.4.$

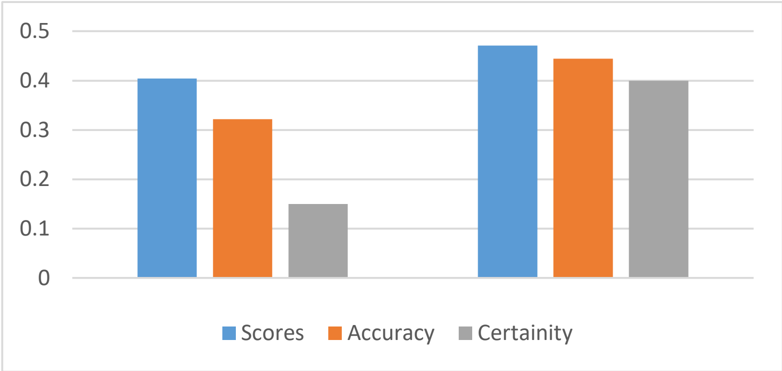

Figure 1: Scores, Accuracy and Certainty of above NHFSs

Definition 3.15: Let $\alpha = \langle A, \lambda \rangle, \beta = \langle B, \mu \rangle$ are two NCHFSs. We say that $\alpha > \beta$ if $S(\alpha) > S(\beta)$. If $S(\alpha) = S(\beta)$, then $\alpha > \beta$ if $A(\alpha) > A(\beta)$. If $A(\alpha) = A(\beta)$, then $\alpha > \beta$ if $C(\alpha) > C(\beta)$. If $S(\alpha) = S(\beta), A(\alpha) = A(\beta), C(\alpha) > C(\beta)$, then $\alpha = \beta$.

In the next section we define aggregation operators on neutrosophic cubic hesitant fuzzy set and prove some elegant results.

4. Aggregation Operators

Definition 4.1: The Neutrosophic cubic hesitant fuzzy weighted geometric operator is defined as

$NCHWG(\alpha_1, ..., \alpha_n) = \bigotimes_{j=1}^{n} \alpha_j^{w_j}$, where $\alpha_{j_{(k)}}$ are neutrosophic cubic hesitant fuzzy values taken in descending order with corresponding weight vector $w = (w_1, ..., w_n)^t$.

Definition 4.2: Neutrosophic cubic hesitant fuzzy order weighted geometric operator is defined as:

$NCHFOWG(\alpha_1, ..., \alpha_n) = \bigotimes_{j=1}^{n} (\alpha_{(k)_j})^{w_j}$ where $\alpha_{j_{(k)}}$ are neutrosophic cubic hesitant fuzzy values taken in descending order with corresponding weight vector $w = (w_1, ..., w_n)^t$.

Definition 4.3: The Neutrosophic cubic hesitant fuzzy Einstein weighted geometric operator is defined as:

$NCEHWG(\alpha_1, ..., \alpha_n) = \bigotimes_{j=1}^{n} (\alpha_j^E)^{w_j}$ where $\alpha_{j_{(k)}}$ are neutrosophic cubic hesitant fuzzy values taken in descending order with corresponding weight vector $w = (w_1, ..., w_n)^t$

Definition 4.4: Neutrosophic cubic hesitant fuzzy Einstein ordered weighted geometric operator is defined as:

$NCHEOWG(\alpha_1, ..., \alpha_n) = \bigotimes_{j=1}^{n} \left(\alpha_j^E\right)^{w_j}$ where $\alpha_{j_{(k)}}$ are neutrosophic cubic hesitant fuzzy values taken in descending order with corresponding weight vector $w = (w_1, ..., w_n)^t$

Theorem 4.5: Let $\left\{\alpha_{(k)}=\left\langle A_{(k)},\lambda_{(k)}\right\rangle\right\}$ the set of neutrosophic cubic hesitant with corresponding weight vector $w=(w_1,\ldots)^t$ fuzzy values, then

$$NCHFWG(\alpha_1,\ldots,\alpha_m)=\left\langle \begin{array}{c} \left\{\left[\prod_{k=1}^{m}\left(A^{L}_{p_{T(k)}}\right)^{w_k},\prod_{k=1}^{m}\left(A^{U}_{p_{T(k)}}\right)^{w_k}\right]\right\},\left\{\left[\prod_{k=1}^{m}\left(A^{L}_{p_{I(k)}}\right)^{w_k},\prod_{k=1}^{m}\left(A^{U}_{p_{I(k)}}\right)^{w_k}\right]\right\}, \\ \left\{\left[1-\prod_{k=1}^{m}\left(1-A^{L}_{p_{F(k)}}\right)^{w_k},1-\prod_{k=1}^{m}\left(1-A^{U}_{p_{F(k)}}\right)^{w_k}\right]\right\},\left\{\prod_{k=1}^{m}\left(\lambda_{p_{T(k)}}\right)^{w_k}\right\}, \\ \left\{\prod_{k=1}^{m}\left(\lambda_{p_{I(k)}}\right)^{w_k}\right\},\left\{1-\prod_{k=1}^{m}\left(1-\lambda_{p_{F(k)}}\right)^{w_k}\right\} \end{array}\right\rangle$$

Proof: Using induction for m=2

$$NCHFWG(\alpha_1,\alpha_2)=\bigotimes_{k=1}^{2}\alpha_k^{w_k}$$

$$=\left\langle \begin{array}{c} \left\{\left[\prod_{k=1}^{2}\left(A^{L}_{p_{T(k)}}\right)^{w_k},\prod_{k=1}^{2}\left(A^{U}_{p_{T(k)}}\right)^{w_k}\right]\right\},\left\{\left[\prod_{k=1}^{2}\left(A^{L}_{p_{I(k)}}\right)^{w_k},\prod_{k=1}^{2}\left(A^{U}_{p_{I(k)}}\right)^{w_k}\right]\right\}, \\ \left\{\left[1-\prod_{k=1}^{2}\left(1-A^{L}_{p_{F(k)}}\right)^{w_k},1-\prod_{k=1}^{2}\left(1-A^{U}_{p_{F(k)}}\right)^{w_k}\right]\right\},\left\{\prod_{k=1}^{2}\left(\lambda_{p_{T(k)}}\right)^{w_k}\right\}, \\ \left\{\prod_{k=1}^{2}\left(\lambda_{p_{I(k)}}\right)^{w_k}\right\},\left\{1-\prod_{k=1}^{2}\left(1-\lambda_{p_{F(k)}}\right)^{w_k}\right\} \end{array}\right\rangle$$

For $m=q$ we have

$$NCHFWG\left(\alpha_1,\ldots,\alpha_q\right)=\left\langle \begin{array}{c} \left\{\left[\prod_{k=1}^{q}\left(A^{L}_{p_{I(k)}}\right)^{w_k},\prod_{k=1}^{q}\left(A^{U}_{p_{I(k)}}\right)^{w_k}\right]\right\},\left\{\left[\prod_{k=1}^{q}\left(A^{L}_{p_{T(k)}}\right)^{w_k},\prod_{k=1}^{q}\left(A^{U}_{p_{T(k)}}\right)^{w_k}\right]\right\}, \\ \left\{\left[1-\prod_{k=1}^{q}\left(1-A^{L}_{p_{F(k)}}\right)^{w_k},1-\prod_{k=1}^{q}\left(1-A^{U}_{p_{F(k)}}\right)^{w_k}\right]\right\}, \\ \left\{\prod_{k=1}^{q}\left(\lambda^{L}_{p_{T(k)}}\right)^{w_k}\right\},\left\{\prod_{k=1}^{q}\left(\lambda^{L}_{p_{I(k)}}\right)^{w_k}\right\},\left\{1-\prod_{k=1}^{q}\left(1-\lambda^{L}_{p_{F(k)}}\right)^{w_k}\right\} \end{array}\right\rangle$$

we prove for m=q+1

$$\bigotimes_{k=1}^{q}\left(\alpha_k\right)^{w_k}\otimes\left(\alpha_{q+1}\right)^{w_{q+1}}=\left\langle \begin{array}{c} \left\{\left[\prod_{k=1}^{q}\left(A^{L}_{p_{T(k)}}\right)^{w_k},\prod_{k=1}^{q}\left(A^{U}_{p_{T(k)}}\right)^{w_k}\right]\right\}, \\ \left\{\left[\prod_{k=1}^{q}\left(A^{L}_{p_{I(k)}}\right)^{w_k},\prod_{k=1}^{q}\left(A^{U}_{p_{I(k)}}\right)^{w_k}\right]\right\}, \\ \left\{\left[1-\prod_{k=1}^{q}\left(1-A^{L}_{p_{F(k)}}\right)^{w_k},1-\prod_{k=1}^{q}\left(1-A^{U}_{p_{F(k)}}\right)^{w_k}\right]\right\}, \\ \left\{\prod_{k=1}^{q}\left(\lambda_{p_{T(k)}}\right)^{w_k}\right\},\left\{\prod_{k=1}^{q}\left(\lambda_{p_{I(k)}}\right)^{w_k}\right\}, \\ \left\{1-\prod_{k=1}^{q}\left(1-\lambda_{p_{F(k)}}\right)^{w_k}\right\} \end{array}\right\rangle\otimes\left\langle \begin{array}{c} \left\{\left[\left(A^{L}_{p_{T(q+1)}}\right)^{w_{q+1}},\left(A^{U}_{p_{T(k)}}\right)^{w_{q+1}}\right]\right\}, \\ \left\{\left[\left(A^{L}_{p_{I(q+1)}}\right)^{w_{q+1}},\left(A^{U}_{p_{I(q+1)}}\right)^{w_{q+1}}\right]\right\}, \\ \left\{\left[1-\left(1-A^{L}_{p_{F(q+1)}}\right)^{w_{q+1}},1-\left(1-A^{U}_{p_{F(q+1)}}\right)^{w_{q+1}}\right]\right\}, \\ \left\{\left(\lambda_{p_{T(q+1)}}\right)^{w_{q+1}}\right\}, \\ \left\{\left(\lambda_{p_{I(q+1)}}\right)^{w_{q+1}}\right\},\left\{1-\left(1-\lambda_{p_{F(q+1)}}\right)^{w_{q+1}}\right\} \end{array}\right\rangle$$

$$
= \left\langle
\begin{array}{c}
\left\{\left[\prod_{k=1}^{q}\left(A^{L}_{p_{T_{(k)}}}\right)^{w_k}\left(A^{L}_{p_{T_{(q+1)}}}\right)^{w_{q+1}}, \prod_{k=1}^{q}\left(A^{U}_{p_{T_{(k)}}}\right)^{w_k}\left(A^{U}_{p_{T_{(q+1)}}}\right)^{w_{q+1}}\right]\right\}, \left\{\left[\prod_{k=1}^{q}\left(A^{L}_{p_{I_{(k)}}}\right)^{w_k}\left(A^{L}_{p_{I_{(q+1)}}}\right)^{w_{q+1}}, \prod_{k=1}^{q}\left(A^{U}_{p_{I_{(k)}}}\right)^{w_k}\left(A^{U}_{p_{I_{(q+1)}}}\right)^{w_{q+1}}\right]\right\}, \\
\left\{\left[\begin{array}{l}
1-\prod_{k=1}^{q}\left(1-A^{L}_{p_{F_{(k)}}}\right)^{w_k}+1-\left(1-A^{L}_{p_{F_{(q+1)}}}\right)^{w_{q+1}}-\left(\left(1-\prod_{k=1}^{q}\left(1-A^{L}_{p_{F_{(k)}}}\right)^{w_k}\right)\left(1-\left(1-A^{L}_{p_{F_{(q+1)}}}\right)^{w_{q+1}}\right)\right), \\
1-\prod_{k=1}^{q}\left(1-A^{U}_{p_{F_{(k)}}}\right)^{w_k}+1-\left(1-A^{U}_{p_{F_{(q+1)}}}\right)^{w_{q+1}}-\left(\left(1-\prod_{k=1}^{q}\left(1-A^{U}_{p_{F_{(k)}}}\right)^{w_k}\right)\left(1-\left(1-A^{U}_{p_{F_{(q+1)}}}\right)^{w_{q+1}}\right)\right)
\end{array}\right]\right\}, \\
\left\{\prod_{k=1}^{q}\left(\lambda_{p_{T_{(k)}}}\right)^{w_k}\left(\lambda_{p_{T_{(q+1)}}}\right)^{w_{q+1}}\right\}, \left\{\prod_{k=1}^{q}\left(\lambda_{p_{I_{(k)}}}\right)^{w_k}\left(\lambda_{p_{I_{(q+1)}}}\right)^{w_{q+1}}\right\}, \\
\left\{1-\prod_{k=1}^{q}\left(1-\lambda_{p_{F_{(k)}}}\right)^{w_k}+1-\left(1-\lambda_{p_{F_{(q+1)}}}\right)^{w_{q+1}}-\left(\left(1-\prod_{k=1}^{q}\left(1-\lambda_{p_{F_{(k)}}}\right)^{w_k}\right)\left(1-\left(1-\lambda_{p_{F_{(q+1)}}}\right)^{w_{q+1}}\right)\right)\right\}
\end{array}
\right\rangle
$$

$$
= \left\langle
\begin{array}{c}
\left\{\left[\prod_{k=1}^{q+1}\left(A^{L}_{p_{T_{(k)}}}\right)^{w_k}, \prod_{k=1}^{q+1}\left(A^{U}_{p_{T_{(k)}}}\right)^{w_k}\right]\right\}, \left\{\left[\prod_{k=1}^{q+1}\left(A^{L}_{p_{I_{(k)}}}\right)^{w_k}, \prod_{k=1}^{q+1}\left(A^{U}_{p_{I_{(k)}}}\right)^{w_k}\right]\right\}, \\
\left\{\left[\begin{array}{l}
2-\prod_{k=1}^{q+1}\left(1-A^{L}_{p_{F_{(k)}}}\right)^{w_k}-1+\prod_{k=1}^{q}\left(1-A^{L}_{p_{F_{(k)}}}\right)+\left(1-A^{L}_{p_{F_{(q+1)}}}\right)^{w_{q+1}}-\left(\prod_{k=1}^{q}\left(1-A^{L}_{p_{F_{(k)}}}\right)^{w_k}\right)\left(1-A^{L}_{p_{F_{(q+1)}}}\right)^{w_{q+1}}, \\
2-\prod_{k=1}^{q+1}\left(1-A^{U}_{p_{F_{(k)}}}\right)^{w_k}-1+\prod_{k=1}^{q}\left(1-A^{U}_{p_{F_{(k)}}}\right)+\left(1-A^{U}_{p_{F_{(q+1)}}}\right)^{w_{q+1}}-\left(\prod_{k=1}^{q}\left(1-A^{U}_{p_{F_{(k)}}}\right)^{w_k}\right)\left(1-A^{U}_{p_{F_{(q+1)}}}\right)^{w_{q+1}}
\end{array}\right]\right\}, \\
\left\{\prod_{k=1}^{q+1}\left(\lambda_{p_{T_{(k)}}}\right)^{w_k}\right\}, \left\{\prod_{k=1}^{q+1}\left(\lambda_{p_{I_{(k)}}}\right)^{w_k}\right\}, \\
\left\{\begin{array}{l}
2-\prod_{k=1}^{q+1}\left(1-\lambda_{p_{F_{(k)}}}\right)^{w_k}-1+\prod_{k=1}^{q}\left(1-\lambda_{p_{F_{(k)}}}\right)+\left(1-\lambda_{p_{F_{(k)}}}\right)^{w_{q+1}} \\
-\left(\prod_{k=1}^{q}\left(1-\lambda_{p_{F_{(k)}}}\right)^{w_k}\right)\left(1-\lambda_{p_{F_{(k)}}}\right)^{w_{q+1}}
\end{array}\right\}
\end{array}
\right\rangle
$$

$$=\left\langle \begin{array}{c} \left\{\left[\prod_{k=1}^{q+1}\left(A^{L}_{p_{T_{(k)}}}\right)^{w_k}, \prod_{k=1}^{q+1}\left(A^{U}_{p_{T_{(k)}}}\right)^{w_k}\right]\right\}, \left\{\left[\prod_{k=1}^{q+1}\left(A^{L}_{p_{I_{(k)}}}\right)^{w_k}, \prod_{k=1}^{q+1}\left(A^{U}_{p_{I_{(k)}}}\right)^{w_k}\right]\right\}, \\ \left\{\left[1-\prod_{k=1}^{q+1}\left(1-A^{L}_{p_{F_{(k)}}}\right)^{w_k}, 1-\prod_{k=1}^{q+1}\left(1-A^{U}_{p_{F_{(k)}}}\right)^{w_k}\right]\right\} \left\{\prod_{k=1}^{q+1}\left(\lambda_{p_{T_{(k)}}}\right)^{w_k}\right\}, \\ \left\{\prod_{k=1}^{q+1}\left(\lambda_{p_{I_{(k)}}}\right)^{w_k}\right\}, \left\{1-\prod_{k=1}^{q+1}\left(1-\lambda_{p_{F_{(k)}}}\right)^{w_k}\right\} \end{array} \right\rangle$$

Theorem 4.6: Let $\left\{\alpha_{(k)} = \left\langle A_{(k)}, \lambda_{(k)} \right\rangle\right\}$ the set of neutrosophic cubic hesitant fuzzy values, then

i) Idempotency: If $\alpha_k = \alpha, k = 1,...m$ then $NCHFWG(\alpha_1,...,\alpha_m) = \alpha.$

ii) Monotonicity: If $S(\alpha_q) \geq S(\alpha_l)$ then $NCHFWG(\alpha_q) \leq NCHFWG(\alpha_l)$.

Theorem 4.7: Let $\left\{\alpha_{(k)} = \left\langle A_{(k)}, \lambda_{(k)} \right\rangle\right\}$ the set of neutrosophic cubic hesitant with corresponding weight vector $w = (w_1, \dots)^t$ fuzzy values, then

$$NCHFEWG\left(\alpha_1,...,\alpha_m\right)=\left\langle\begin{array}{c}\left\{\left[\frac{2\prod\limits_{k=1}^{m}\left(A^L_{p_{T(k)}}\right)^{w_k}}{\prod\limits_{k=1}^{m}\left(2-A^L_{p_{T(k)}}\right)^{w_k}+\prod\limits_{k=1}^{m}\left(A^L_{p_{T(k)}}\right)^{w_k}},\frac{2\prod\limits_{k=1}^{m}\left(A^U_{p_{T(k)}}\right)^{w_k}}{\prod\limits_{k=1}^{m}\left(2-A^U_{p_{T(k)}}\right)^{w_k}+\prod\limits_{k=1}^{m}\left(A^U_{p_{T(k)}}\right)^{w_k}}\right]\right\},\\ \left\{\left[\frac{2\prod\limits_{k=1}^{m}\left(A^L_{p_{I(k)}}\right)^{w_k}}{\prod\limits_{k=1}^{m}\left(2-A^L_{p_{I(k)}}\right)^{w_k}+\prod\limits_{k=1}^{m}\left(A^L_{p_{T(k)}}\right)^{w_k}},\frac{2\prod\limits_{k=1}^{m}\left(A^U_{p_{I(k)}}\right)^{w_k}}{\prod\limits_{k=1}^{m}\left(2-A^U_{p_{I(k)}}\right)^{w_k}+\prod\limits_{k=1}^{m}\left(A^U_{p_{I(k)}}\right)^{w_k}}\right]\right\},\\ \left\{\left[\frac{\prod\limits_{k=1}^{m}\left(1+A^L_{p_{F(k)}}\right)^{w_k}-\prod\limits_{k=1}^{m}\left(1-A^L_{p_{F(k)}}\right)^{w_k}}{\prod\limits_{k=1}^{m}\left(1+A^L_{p_{F(k)}}\right)^{w_k}+\prod\limits_{k=1}^{m}\left(1-A^L_{p_{F(k)}}\right)^{w_k}},\frac{\prod\limits_{k=1}^{m}\left(1+A^U_{p_{F(k)}}\right)^{w_k}-\prod\limits_{k=1}^{m}\left(1-A^U_{p_{F(k)}}\right)^{w_k}}{\prod\limits_{k=1}^{m}\left(1+A^U_{p_{F(k)}}\right)^{w_k}+\prod\limits_{k=1}^{m}\left(1-A^U_{p_{F(k)}}\right)^{w_k}}\right]\right\},\\ \left\{\frac{2\prod\limits_{k=1}^{m}\left(\lambda_{p_{T(k)}}\right)^{w_k}}{\prod\limits_{k=1}^{m}\left(2-\lambda_{p_{T(k)}}\right)^{w_k}+\prod\limits_{k=1}^{m}\left(\lambda_{p_{T(k)}}\right)^{w_k}}\right\},\left\{\frac{2\prod\limits_{k=1}^{m}\left(\lambda_{p_{I(k)}}\right)^{w_k}}{\prod\limits_{k=1}^{m}\left(2-\lambda_{p_{I(k)}}\right)^{w_k}+\prod\limits_{k=1}^{m}\left(\lambda_{p_{I(k)}}\right)^{w_k}}\right\},\\ \left\{\frac{\prod\limits_{k=1}^{m}\left(1+\lambda_{p_{F(k)}}\right)^{w_k}-\prod\limits_{k=1}^{m}\left(1-\lambda_{p_{F(k)}}\right)^{w_k}}{\prod\limits_{k=1}^{m}\left(1+\lambda_{p_{F(k)}}\right)^{w_k}+\prod\limits_{k=1}^{m}\left(1-\lambda_{p_{F(k)}}\right)^{w_k}}\right\}\end{array}\right\rangle$$

Proof: we use induction.

$$\alpha^{E^{w_1}}=\left\langle\begin{array}{c}\left\{\left[\frac{2\left(A^L_{j_T}\right)^{w_1}}{\left(2-A^L_{j_T}\right)^{w_1}+\left(A^L_{j_T}\right)^{w_1}},\frac{2\left(A^U_{j_T}\right)^{w_1}}{\left(2-A^U_{j_T}\right)^{w_1}+\left(A^U_{j_T}\right)^{w_1}}\right]\right\},\left\{\left[\frac{2\left(A^L_{j_I}\right)^{w_1}}{\left(2-A^L_{j_I}\right)^{w_1}+\left(A^L_{j_I}\right)^{w_1}},\frac{2\left(A^U_{j_I}\right)^{w_1}}{\left(2-A^U_{j_I}\right)^{w_1}+\left(A^U_{j_I}\right)^{w_1}}\right]\right\}\\ \left\{\left[\frac{\left(1+A^L_{j_F}\right)^{w_1}-\left(1-A^L_{j_T}\right)^{w_1}}{\left(1+A^L_{j_F}\right)^{w_1}+\left(1-A^L_{j_F}\right)^{w_1}},\frac{\left(1+A^U_{j_F}\right)^{w_1}-\left(1-A^U_{j_F}\right)^{w_1}}{\left(1+A^U_{j_F}\right)^{w_1}+\left(1-A^U_{j_F}\right)^{w_1}}\right]\right\},\left\{\frac{2\left(\lambda_{j_T}\right)^{w_1}}{\left(2-\lambda_{j_T}\right)^{w_1}+\left(\lambda_{j_T}\right)^{w_1}}\right\},\left\{\frac{2\left(\lambda_{j_I}\right)^{w_1}}{\left(2-\lambda_{j_I}\right)^{w_1}+\left(\lambda_{j_I}\right)^{w_1}}\right\},\\ \left\{\frac{\left(1+\lambda_{j_F}\right)^{w_1}-\left(1-\lambda_{j_F}\right)^{w_1}}{\left(1+\lambda_{j_F}\right)^{w_1}+\left(1-\lambda_{j_F}\right)^{w_1}}\right\}\end{array}\right\rangle$$

and $$\alpha^{E^{w2}}=\left\langle \begin{array}{c} \left\{\left[\frac{2\left(A^L_{j_T}\right)^{w2}}{\left(2-A^L_{j_T}\right)^{w2}+\left(A^L_{j_T}\right)^{w2}},\frac{2\left(A^U_{j_T}\right)^{w2}}{\left(2-A^U_{j_T}\right)^{w2}+\left(A^U_{j_T}\right)^{w2}}\right]\right\},\left\{\left[\frac{2\left(A^L_{j_I}\right)^{w2}}{\left(2-A^L_{j_I}\right)^{w2}+\left(A^L_{j_I}\right)^{w2}},\frac{2\left(A^U_{j_I}\right)^{w2}}{\left(2-A^U_{j_I}\right)^{w2}+\left(A^U_{j_I}\right)^{w2}}\right]\right\} \\ \left\{\left[\frac{\left(1+A^L_{j_F}\right)^{w2}-\left(1-A^L_{j_T}\right)^{w2}}{\left(1+A^L_{j_F}\right)^{w2}+\left(1-A^L_{j_F}\right)^{w2}},\frac{\left(1+A^U_{j_F}\right)^{w2}-\left(1-A^U_{j_F}\right)^{w2}}{\left(1+A^U_{j_F}\right)^{w2}+\left(1-A^U_{j_F}\right)^{w1}}\right]\right\},\left\{\frac{2\left(\lambda_{j_T}\right)^{w2}}{\left(2-\lambda_{j_T}\right)^{w2}+\left(\lambda_{j_T}\right)^{w2}}\right\},\left\{\frac{2\left(\lambda_{j_I}\right)^{w2}}{\left(2-\lambda_{j_I}\right)^{w2}+\left(\lambda_{j_I}\right)^{w2}}\right\}, \\ \left\{\frac{\left(1+\lambda_{j_F}\right)^{w2}-\left(1-\lambda_{j_F}\right)^{w2}}{\left(1+\lambda_{j_F}\right)^{w2}+\left(1-\lambda_{j_F}\right)^{w2}}\right\} \end{array}\right\rangle$$

$$NCHFEWG(\alpha_1,\alpha_2)=\bigotimes_{k=1}^{2}\alpha_k^{E^{w_k}}$$

$$=\left\langle \begin{array}{c} \left\{\left[\frac{2\prod_{k=1}^{2}\left(A^L_{p_{T(k)}}\right)^{w_k}}{\prod_{k=1}^{2}\left(2-A^L_{p_{T(k)}}\right)^{w_k}+\prod_{k=1}^{2}\left(A^L_{p_{T(k)}}\right)^{w_k}},\frac{2\prod_{k=1}^{2}\left(A^U_{p_{T(k)}}\right)^{w_k}}{\prod_{k=1}^{2}\left(2-A^U_{p_{T(k)}}\right)^{w_k}+\prod_{k=1}^{2}\left(A^U_{p_{T(k)}}\right)^{w_k}}\right]\right\},\left\{\left[\frac{2\prod_{k=1}^{2}\left(A^L_{p_{I(k)}}\right)^{w_k}}{\prod_{k=1}^{2}\left(2-A^L_{p_{I(k)}}\right)^{w_k}+\prod_{k=1}^{2}\left(A^L_{p_{T(k)}}\right)^{w_k}},\frac{2\prod_{k=1}^{2}\left(A^U_{p_{I(k)}}\right)^{w_k}}{\prod_{k=1}^{2}\left(2-A^U_{p_{I(k)}}\right)^{w_k}+\prod_{k=1}^{2}\left(A^U_{p_{I(k)}}\right)^{w_k}}\right]\right\}, \\ \left\{\left[\frac{\prod_{k=1}^{2}\left(1+A^L_{p_{F(k)}}\right)^{w_k}-\prod_{k=1}^{2}\left(1-A^L_{p_{F(k)}}\right)^{w_k}}{\prod_{k=1}^{2}\left(1+A^L_{p_{F(k)}}\right)^{w_k}+\prod_{k=1}^{2}\left(1-A^L_{p_{F(k)}}\right)^{w_k}},\frac{\prod_{k=1}^{2}\left(1+A^U_{p_{F(k)}}\right)^{w_k}-\prod_{k=1}^{2}\left(1-A^U_{p_{F(k)}}\right)^{w_k}}{\prod_{k=1}^{2}\left(1+A^U_{p_{F(k)}}\right)^{w_k}+\prod_{k=1}^{2}\left(1-A^U_{p_{F(k)}}\right)^{w_k}}\right]\right\}, \\ \left\{\frac{2\prod_{k=1}^{2}\left(\lambda_{pT(k)}\right)^{w_k}}{\prod_{k=1}^{2}\left(2-\lambda_{pT(k)}\right)^{w_k}+\prod_{k=1}^{2}\left(\lambda_{pT(k)}\right)^{w_k}}\right\},\left\{\frac{2\prod_{k=1}^{2}\left(\lambda_{pI(k)}\right)^{w_k}}{\prod_{k=1}^{2}\left(2-\lambda_{pI(k)}\right)^{w_k}+\prod_{k=1}^{2}\left(\lambda_{pI(k)}\right)^{w_k}}\right\},\left\{\frac{\prod_{k=1}^{2}\left(1+\lambda_{pF(k)}\right)^{w_k}-\prod_{k=1}^{2}\left(1-\lambda_{pF(k)}\right)^{w_k}}{\prod_{k=1}^{2}\left(1+\lambda_{pF(k)}\right)^{w_k}+\prod_{k=1}^{2}\left(1-\lambda_{pF(k)}\right)^{w_k}}\right\} \end{array}\right\rangle$$

For m=q we have

$$NCHFEWG\left(\alpha_1,...,\alpha_q\right)=\left\langle \begin{array}{c} \left\{\left[\frac{2\prod_{k=1}^{q}\left(A^L_{p_{T(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-A^L_{p_{T(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(A^L_{p_{T(k)}}\right)^{w_k}},\frac{2\prod_{k=1}^{q}\left(A^U_{p_{T(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-A^U_{p_{T(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(A^U_{p_{T(k)}}\right)^{w_k}}\right]\right\}, \\ \left\{\left[\frac{2\prod_{k=1}^{q}\left(A^L_{p_{I(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-A^L_{p_{I(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(A^L_{p_{T(k)}}\right)^{w_k}},\frac{2\prod_{k=1}^{q}\left(A^U_{p_{I(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-A^U_{p_{I(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(A^U_{p_{I(k)}}\right)^{w_k}}\right]\right\} \\ \left\{\left[\frac{\prod_{k=1}^{q}\left(1+A^L_{p_{F(k)}}\right)^{w_k}-\prod_{k=1}^{q}\left(1-A^L_{p_{F(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(1+A^L_{p_{F(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(1-A^L_{p_{F(k)}}\right)^{w_k}},\frac{\prod_{k=1}^{q}\left(1+A^U_{p_{F(k)}}\right)^{w_k}-\prod_{k=1}^{q}\left(1-A^U_{p_{F(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(1+A^U_{p_{F(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(1-A^U_{p_{F(k)}}\right)^{w_k}}\right]\right\}, \\ \left\{\frac{2\prod_{k=1}^{q}\left(\lambda_{pT(k)}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-\lambda_{pT(k)}\right)^{w_k}+\prod_{k=1}^{q}\left(\lambda_{pT(k)}\right)^{w_k}}\right\},\left\{\frac{2\prod_{k=1}^{q}\left(\lambda_{pI(k)}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-\lambda_{pI(k)}\right)^{w_k}+\prod_{k=1}^{q}\left(\lambda_{pI(k)}\right)^{w_k}}\right\},\left\{\frac{\prod_{k=1}^{q}\left(1+\lambda_{pF(k)}\right)^{w_k}-\prod_{k=1}^{q}\left(1-\lambda_{pF(k)}\right)^{w_k}}{\prod_{k=1}^{q}\left(1+\lambda_{pF(k)}\right)^{w_k}+\prod_{k=1}^{q}\left(\lambda_{pF(k)}\right)^{w_k}}\right\} \end{array}\right\rangle$$

Using assumption, we have

$$NCHFEWG\left(\alpha_1,\ldots,\alpha_{q+1}\right)=\underset{k=1}{\overset{q}{\otimes_E}}\left(\alpha_k^E\right)^{w_k}\otimes_E\left(\alpha_{q+1}^E\right)^{w_{q+1}}$$

$$=\left\langle\begin{array}{c}
\left\{\left[\dfrac{2\prod_{k=1}^{q}\left(A^L_{p_{T(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-A^L_{p_{T(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(A^L_{p_{T(k)}}\right)^{w_k}},\dfrac{2\prod_{k=1}^{q}\left(A^U_{p_{T(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-A^U_{p_{T(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(A^U_{p_{T(k)}}\right)^{w_k}}\right]\right\},\\
\left\{\left[\dfrac{2\prod_{k=1}^{q}\left(A^L_{p_{I(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-A^L_{p_{I(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(A^L_{p_{T(k)}}\right)^{w_k}},\dfrac{2\prod_{k=1}^{q}\left(A^U_{p_{I(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-A^U_{p_{I(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(A^U_{p_{I(k)}}\right)^{w_k}}\right]\right\}\\
\left\{\left[\dfrac{\prod_{k=1}^{q}\left(1+A^L_{p_{F(k)}}\right)^{w_k}-\prod_{k=1}^{q}\left(1-A^L_{p_{F(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(1+A^L_{p_{F(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(1-A^L_{p_{F(k)}}\right)^{w_k}},\dfrac{\prod_{k=1}^{q}\left(1+A^U_{p_{F(k)}}\right)^{w_k}-\prod_{k=1}^{q}\left(1-A^U_{p_{F(k)}}\right)^{w_k}}{\prod_{k=1}^{q}\left(1+A^U_{p_{F(k)}}\right)^{w_k}+\prod_{k=1}^{q}\left(1-A^U_{p_{F(k)}}\right)^{w_k}}\right]\right\},\\
\left\{\dfrac{2\prod_{k=1}^{q}\left(\lambda_{pT(k)}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-\lambda_{pT(k)}\right)^{w_k}+\prod_{k=1}^{q}\left(\lambda_{pT(k)}\right)^{w_k}}\right\},\\
\left\{\dfrac{2\prod_{k=1}^{q}\left(\lambda_{pI(k)}\right)^{w_k}}{\prod_{k=1}^{q}\left(2-\lambda_{pI(k)}\right)^{w_k}+\prod_{k=1}^{q}\left(\lambda_{pI(k)}\right)^{w_k}}\right\},\\
\left\{\dfrac{\prod_{k=1}^{q}\left(1+\lambda_{pF(k)}\right)^{w_k}-\prod_{k=1}^{q}\left(1-\lambda_{pF(k)}\right)^{w_k}}{\prod_{k=1}^{q}\left(1+\lambda_{pF(k)}\right)^{w_k}+\prod_{k=1}^{q}\left(\lambda_{pF(k)}\right)^{w_k}}\right\}
\end{array}\right\rangle
\otimes_E
\left\langle\begin{array}{c}
\left\{\left[\begin{array}{c}\dfrac{2(A^L_{p_T})^{w_{q+1}}}{(2-A^L_{p_T})^{w_{q+1}}+(A^L_{p_T})^{w_{q+1}}},\\ \dfrac{2(A^U_{p_T})^{w_{q+1}}}{(2-A^U_{p_T})^{w_{q+1}}+(A^U_{p_T})^{w_{q+1}}}\end{array}\right]\right\},\\
\left\{\left[\begin{array}{c}\dfrac{2(A^L_{p_I})^{w_{q+1}}}{(2-A^L_{p_I})^{w_{q+1}}+(A^L_{p_I})^{w_{q+1}}},\\ \dfrac{2(A^U_{p_I})^{w_{q+1}}}{(2-A^U_{p_I})^{w_{q+1}}+(A^U_{p_I})^{w_{q+1}}}\end{array}\right]\right\},\\
\left\{\left[\begin{array}{c}\dfrac{(1+A^L_{p_F})^{w_{q+1}}-(1-A^L_{p_F})^{w_{q+1}}}{(1+A^L_{p_F})^{w_{q+1}}+(1-A^L_{p_F})^{w_{q+1}}},\\ \dfrac{(1+A^U_{p_F})^{w_{q+1}}-(1-A^U_{p_F})^{w_{q+1}}}{(1+A^U_{p_F})^{w_{q+1}}+(1-A^U_{p_F})^{w_{q+1}}}\end{array}\right]\right\},\\
\left\{\dfrac{2(\lambda_{p_T})^{w_{q+1}}}{(2-\lambda_{p_T})^{w_{q+1}}+(\lambda_{p_T})^{w_{q+1}}}\right\},\\
\left\{\dfrac{2(\lambda_{p_I})^{w_{q+1}}}{(2-\lambda_{p_I})^{w_{q+1}}+(\lambda_{p_I})^{w_{q+1}}}\right\},\\
\left\{\dfrac{(1+\lambda_{p_F})^{w_{q+1}}-(1-\lambda_{p_F})^{w_{q+1}}}{(1+\lambda_{p_F})^{w_{q+1}}+(1-\lambda_{p_F})^{w_{q+1}}}\right\}
\end{array}\right\rangle$$

$$
= \left\langle \begin{array}{c}
\left\{\left[\dfrac{2\prod_{k=1}^{q+1}\left(A^{L}_{p_{T(k)}}\right)^{w_k}}{\prod_{k=1}^{q+1}\left(2-A^{L}_{p_{T(k)}}\right)^{w_k}+\prod_{k=1}^{q+1}\left(A^{L}_{p_{T(k)}}\right)^{w_k}}, \dfrac{2\prod_{k=1}^{q+1}\left(A^{U}_{p_{T(k)}}\right)^{w_k}}{\prod_{k=1}^{q+1}\left(2-A^{U}_{p_{T(k)}}\right)^{w_k}+\prod_{k=1}^{q+1}\left(A^{U}_{p_{T(k)}}\right)^{w_k}}\right]\right\}, \\
\left\{\left[\dfrac{2\prod_{k=1}^{q+1}\left(A^{L}_{p_{I(k)}}\right)^{w_k}}{\prod_{k=1}^{q+1}\left(2-A^{L}_{p_{I(k)}}\right)^{w_k}+\prod_{k=1}^{q+1}\left(A^{L}_{p_{T(k)}}\right)^{w_k}}, \dfrac{2\prod_{k=1}^{q+1}\left(A^{U}_{p_{I(k)}}\right)^{w_k}}{\prod_{k=1}^{q+1}\left(2-A^{U}_{p_{I(k)}}\right)^{w_k}+\prod_{k=1}^{q+1}\left(A^{U}_{p_{I(k)}}\right)^{w_k}}\right]\right\} \\
\left\{\left[\dfrac{\prod_{k=1}^{q+1}\left(1+A^{L}_{p_{F(k)}}\right)^{w_k}-\prod_{k=1}^{q+1}\left(1-A^{L}_{p_{F(k)}}\right)^{w_k}}{\prod_{k=1}^{q+1}\left(1+A^{L}_{p_{F(k)}}\right)^{w_k}+\prod_{k=1}^{q+1}\left(1-A^{L}_{p_{F(k)}}\right)^{w_k}}, \dfrac{\prod_{k=1}^{q+1}\left(1+A^{U}_{p_{F(k)}}\right)^{w_k}-\prod_{k=1}^{q+1}\left(1-A^{U}_{p_{F(k)}}\right)^{w_k}}{\prod_{k=1}^{q+1}\left(1+A^{U}_{p_{F(k)}}\right)^{w_k}+\prod_{k=1}^{q+1}\left(1-A^{U}_{p_{F(k)}}\right)^{w_k}}\right]\right\}, \\
\left\{\dfrac{2\prod_{k=1}^{q+1}\left(\lambda_{pT_{(k)}}\right)^{w_k}}{\prod_{k=1}^{q+1}\left(2-\lambda_{pT_{(k)}}\right)^{w_k}+\prod_{k=1}^{q+1}\left(\lambda_{pT_{(k)}}\right)^{w_k}}\right\}, \\
\left\{\dfrac{2\prod_{k=1}^{q+1}\left(\lambda_{pI_{(k)}}\right)^{w_k}}{\prod_{k=1}^{q+1}\left(2-\lambda_{pI_{(k)}}\right)^{w_k}+\prod_{k=1}^{q+1}\left(\lambda_{pI_{(k)}}\right)^{w_k}}\right\}, \\
\left\{\dfrac{\prod_{k=1}^{q+1}\left(1+\lambda_{pF_{(k)}}\right)^{w_k}-\prod_{k=1}^{q+1}\left(1-\lambda_{pF_{(k)}}\right)^{w_k}}{\prod_{k=1}^{q+1}\left(1+\lambda_{pF_{(k)}}\right)^{w_k}+\prod_{k=1}^{q+1}\left(\lambda_{pF_{(k)}}\right)^{w_k}}\right\}
\end{array} \right\rangle
$$

Definition4.8: Neutrosophic cubic hesitant fuzzy hybrid geometric operator (NCHFHG) is a mapping defined as $NCHFHG(\alpha_1,...,\alpha_m)=\bigotimes_{j=1}^{m}\left(\alpha_{\sigma(j)}\right)^{w_j}$, where $\alpha_{\sigma(j)}=\left(\alpha_j\right)^{mw_j}$ is the *jth* largest value, *m* is the balancing coefficient and $w=(w_1,\dots,w_m)^t$ is the weighting vector.

Theorem 4.9: Let $\left\{\alpha_{(k)}=\left\langle A_{(k)},\lambda_{(k)}\right\rangle\right\}$ the set of neutrosophic cubic hesitant fuzzy values with corresponding weight vector $w=(w_1,\dots)^t$, then

$$
NCHFHG(\alpha_1,...,\alpha_m)=\left\langle \begin{array}{c}
\left\{\left[\bigotimes_{k=1}^{m}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k},\bigotimes_{k=1}^{m}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}\right]\right\},\left\{\left[\bigotimes_{k=1}^{m}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k},\bigotimes_{k=1}^{m}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \\
\left\{\left[1-\bigotimes_{k=1}^{m}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k},1-\bigotimes_{k=1}^{m}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}\right]\right\},\left\{\bigotimes_{k=1}^{m}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}\right\}, \\
\left\{\bigotimes_{k=1}^{m}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}\right\},\left\{1-\bigotimes_{k=1}^{m}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}\right\}
\end{array}\right\rangle
$$

Furthermore $NCHFHG(\alpha_1,\alpha_2,...,\alpha_m)$ is also a neutrosophic cubic hesitant fuzzy value.

Proof: Using induction for m=2

$$
NCHFHG(\alpha_1,\alpha_2)=\bigotimes_{k=1}^{2}\left(\alpha_{\sigma(k)}\right)^{w_k}
$$

$$= \left\langle \begin{array}{c} \left\{\left[\left(A^{L}_{j_{T_{\sigma(1)}}}\right)^{w_1}, \left(A^{U}_{j_{T_{\sigma(1)}}}\right)^{w_1}\right]\right\}, \left\{\left[\left(A^{L}_{j_{I_{\sigma(1)}}}\right)^{w_1}, \left(A^{U}_{j_{I_{\sigma(1)}}}\right)^{w_1}\right]\right\}, \\ \left\{\left[1-\left(1-A^{L}_{j_{F_{\sigma(1)}}}\right)^{w_1}, 1-\left(1-A^{U}_{j_{F_{\sigma(1)}}}\right)^{w_1}\right]\right\}, \left\{\left(\lambda_{j_{T_{\sigma(1)}}}\right)^{w_1}\right\}, \\ \left\{\left(\lambda_{j_{I_{\sigma(1)}}}\right)^{w_1}\right\}, \left\{1-\left(1-\lambda_{j_{F_{\sigma(1)}}}\right)^{w_1}\right\} \end{array} \right\rangle \otimes \left\langle \begin{array}{c} \left\{\left[\left(A^{L}_{j_{T_{\sigma(2)}}}\right)^{w_2}, \left(A^{U}_{j_{T_{\sigma(2)}}}\right)^{w_2}\right]\right\}, \left\{\left[\left(A^{L}_{j_{I_{\sigma(2)}}}\right)^{w_2}, \left(A^{U}_{j_{I_{\sigma(2)}}}\right)^{w_2}\right]\right\}, \\ \left\{\left[1-\left(1-A^{L}_{j_{F_{\sigma(2)}}}\right)^{w_2}, 1-\left(1-A^{U}_{j_{F_{\sigma(2)}}}\right)^{w_2}\right]\right\}, \left\{\left(\lambda_{j_{T_{\sigma(2)}}}\right)^{w_2}\right\}, \\ \left\{\left(\lambda_{j_{I_{\sigma(2)}}}\right)^{w_2}\right\}, \left\{1-\left(1-\lambda_{j_{F_{\sigma(2)}}}\right)^{w_2}\right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \left\{\left[\bigotimes_{k=1}^{2}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}, \bigotimes_{k=1}^{2}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \left\{\left[\bigotimes_{k=1}^{2}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}, \bigotimes_{k=1}^{2}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \\ \left\{\left[1-\bigotimes_{k=1}^{2}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}, 1-\bigotimes_{k=1}^{2}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \left\{\bigotimes_{k=1}^{2}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}\right\}, \\ \left\{\bigotimes_{k=1}^{2}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}\right\}, \left\{1-\bigotimes_{k=1}^{2}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}\right\} \end{array} \right\rangle$$

For $m = q$ we have

$$NCHFHG(\alpha_1, ..., \alpha_q) = \left\langle \begin{array}{c} \left\{\left[\bigotimes_{k=1}^{q}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}, \bigotimes_{k=1}^{q}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \left\{\left[\bigotimes_{k=1}^{q}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}, \bigotimes_{k=1}^{q}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \\ \left\{\left[1-\bigotimes_{k=1}^{q}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}, 1-\bigotimes_{k=1}^{q}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \left\{\bigotimes_{k=1}^{q}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}\right\}, \\ \left\{\bigotimes_{k=1}^{q}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}\right\}, \left\{1-\bigotimes_{k=1}^{q}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}\right\} \end{array} \right\rangle$$

we prove for $m = q + 1$

$$\bigotimes_{k=1}^{q}\left(\alpha_{\sigma(k)}\right)^{w_k} \otimes \left(\alpha_{\sigma(q+1)}\right)^{w_{q+1}} = \left\langle \begin{array}{c} \left\{\left[\bigotimes_{k=1}^{q}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}, \bigotimes_{k=1}^{q}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \left\{\left[\bigotimes_{k=1}^{q}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}, \bigotimes_{k=1}^{q}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \\ \left\{\left[1-\bigotimes_{k=1}^{q}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}, 1-\bigotimes_{k=1}^{q}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \left\{\bigotimes_{k=1}^{q}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}\right\}, \\ \left\{\bigotimes_{k=1}^{q}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}\right\}, \left\{1-\bigotimes_{k=1}^{q}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}\right\} \end{array} \right\rangle \otimes \left\langle \begin{array}{c} \left\{\left[\left(A^{L}_{j_{T_{\sigma(q+1)}}}\right)^{w_{q+1}}, \left(A^{U}_{j_{T_{\sigma(q+1)}}}\right)^{w_{q+1}}\right]\right\}, \left\{\left[\left(A^{L}_{j_{I_{\sigma(q+1)}}}\right)^{w_{q+1}}, \left(A^{U}_{j_{I_{\sigma(q+1)}}}\right)^{w_{q+1}}\right]\right\}, \\ \left\{\left[1-\left(1-A^{L}_{j_{F_{\sigma(q+1)}}}\right)^{w_{q+1}}, 1-\left(1-A^{U}_{j_{F_{\sigma(q+1)}}}\right)^{w_{q+1}}\right]\right\}, \left\{\left(\lambda_{j_{T_{\sigma(q+1)}}}\right)^{w_{q+1}}\right\}, \\ \left\{\left(\lambda_{j_{I_{\sigma(q+1)}}}\right)^{w_{q+1}}\right\}, \left\{1-\left(1-\lambda_{j_{F_{\sigma(q+1)}}}\right)^{w_{q+1}}\right\} \end{array} \right\rangle$$

$$=\left\langle \begin{array}{c} \left\{\left[\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}, \overset{q+1}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)\right]\right\}, \left\{\left[\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{I_{\sigma(k)}}}\right), \overset{q+1}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \\ \left\{\left[\begin{array}{l} \left[2-\overset{q+1}{\underset{k=1}{\otimes}}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}-1+\overset{q}{\underset{k=1}{\otimes}}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}+\left(1-A^{L}_{j_{F_{\sigma(q+1)}}}\right)^{w_{q+1}}-\left(\overset{q}{\underset{k=1}{\otimes}}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}\right)\left(1-A^{L}_{j_{F_{\sigma(q+1)}}}\right)^{w_{q+1}}, \right. \\ \left. 2-\overset{q+1}{\underset{k=1}{\otimes}}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}-1+\overset{q}{\underset{k=1}{\otimes}}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}+\left(1-A^{U}_{j_{F_{\sigma(q+1)}}}\right)^{w_{q+1}}-\left(\overset{q}{\underset{k=1}{\otimes}}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}\right)\left(1-A^{U}_{j_{F_{\sigma(q+1)}}}\right)^{w_{q+1}}\right] \end{array}\right]\right\}, \\ \left\{\overset{q+1}{\underset{k=1}{\otimes}}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}\right\}, \left\{\overset{q+1}{\underset{k=1}{\otimes}}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}\right\}, \\ \left\{2-\overset{q+1}{\underset{k=1}{\otimes}}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}-1+\overset{q}{\underset{k=1}{\otimes}}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}+\left(1-\lambda_{j_{F_{\sigma(q+1)}}}\right)^{w_{q+1}}-\left(\overset{q}{\underset{k=1}{\otimes}}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}\right)\left(1-\lambda_{j_{F_{\sigma(q+1)}}}\right)^{w_{q+1}}\right\} \end{array}\right\rangle$$

$$=\left\langle \begin{array}{c} \left\{\left[\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}, \overset{q+1}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \left\{\left[\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}, \overset{q+1}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \\ \left\{\left[1-\overset{q+1}{\underset{k=1}{\otimes}}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}, 1-\overset{q+1}{\underset{k=1}{\otimes}}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}\right]\right\}, \left\{\overset{q+1}{\underset{k=1}{\otimes}}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}\right\}, \\ \left\{\overset{q+1}{\underset{k=1}{\otimes}}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}\right\}, \left\{1-\overset{q+1}{\underset{k=1}{\otimes}}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}\right\} \end{array}\right\rangle$$

Theorem 4.10: With $w_j = \frac{1}{m}$, *NCHFHG* becomes *NCHFWG*.

Proof: $$NCHFHG(\alpha_1,...,\alpha_m) = \left(\alpha_{\sigma(1)}\right)^{w_1} \otimes ... \otimes \left(\alpha_{\sigma(m)}\right)^{w_m}$$

$$= \left(\alpha_1\right)^{w_1} \otimes ... \otimes \left(\alpha_m\right)^{w_m}$$

$$= NCHFWG(\alpha_1,...,\alpha_m)$$

Definition 4.11: Neutrosophic cubic hesitant fuzzy Einstein hybrid geometric operator(NCHFEHG) is a mapping defined as $NCHFEHG(\alpha_1,...,\alpha_m) = \underset{j=1}{\overset{m}{\otimes_E}} \left(\alpha_{\sigma(j)}\right)^{w_j}$ where $\alpha_{\sigma(j)} = \left(\alpha_j\right)^{mw_j}$ is the *jth* largest value, *m* is the balancing coefficient and $w = (w_1, \dots, w_m)^t$ is the weighting vector.

Theorem 4.12: Let $\left\{\alpha_{(k)} = \left\langle A_{(k)}, \lambda_{(k)} \right\rangle\right\}$ the set of neutrosophic cubic hesitant fuzzy values with corresponding weight vector $w = (w_1, \dots)^t$ then

$$NCHFEHG(\alpha_1,\alpha_2,...,\alpha_m) = \left\langle \begin{array}{c} \left\{ \left[\dfrac{2 \overset{m}{\underset{k=1}{\otimes}} \left(A^L_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\overset{m}{\underset{k=1}{\otimes}} \left(2 - A^L_{j_{T_{\sigma(k)}}}\right)^{w_k} + \overset{m}{\underset{k=1}{\otimes}} \left(A^L_{j_{T_{\sigma(k)}}}\right)^{w_k}}, \dfrac{2 \overset{m}{\underset{k=1}{\otimes}} \left(A^U_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\overset{m}{\underset{k=1}{\otimes}} \left(2 - A^U_{j_{T_{\sigma(k)}}}\right)^{w_k} + \overset{m}{\underset{k=1}{\otimes}} \left(A^U_{j_{T_{\sigma(k)}}}\right)^{w_k}} \right] \right\}, \\ \left\{ \left[\dfrac{2 \overset{m}{\underset{k=1}{\otimes}} \left(A^L_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\overset{m}{\underset{k=1}{\otimes}} \left(2 - A^L_{j_{I_{\sigma(k)}}}\right)^{w_k} + \overset{m}{\underset{k=1}{\otimes}} \left(A^L_{j_{I_{\sigma(k)}}}\right)^{w_k}}, \dfrac{2 \overset{m}{\underset{k=1}{\otimes}} \left(A^U_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\overset{m}{\underset{k=1}{\otimes}} \left(2 - A^U_{j_{I_{\sigma(k)}}}\right)^{w_k} + \overset{m}{\underset{k=1}{\otimes}} \left(A^U_{j_{I_{\sigma(k)}}}\right)^{w_k}} \right] \right\}, \\ \left\{ \left[\dfrac{\overset{m}{\underset{k=1}{\otimes}} \left(1 + A^L_{j_{F_{\sigma(k)}}}\right)^{w_k} - \overset{m}{\underset{k=1}{\otimes}} \left(1 - A^L_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\overset{m}{\underset{k=1}{\otimes}} \left(1 + A^L_{j_{F_{\sigma(k)}}}\right)^{w_k} + \overset{m}{\underset{k=1}{\otimes}} \left(1 - A^L_{j_{F_{\sigma(k)}}}\right)^{w_k}}, \dfrac{\overset{m}{\underset{k=1}{\otimes}} \left(1 + A^U_{j_{F_{\sigma(k)}}}\right)^{w_k} - \overset{m}{\underset{k=1}{\otimes}} \left(1 - A^U_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\overset{m}{\underset{k=1}{\otimes}} \left(1 + A^U_{j_{F_{\sigma(k)}}}\right)^{w_k} + \overset{m}{\underset{k=1}{\otimes}} \left(1 - A^U_{j_{F_{\sigma(k)}}}\right)^{w_k}} \right] \right\}, \\ \left\{ \dfrac{2 \overset{m}{\underset{k=1}{\otimes}} \left(\left(1 + \lambda_{j_{T_{\sigma(k)}}}\right)\right)^{w_k}}{\overset{m}{\underset{k=1}{\otimes}} \left(\left(2 - \lambda_{j_{T_{\sigma(k)}}}\right)\right)^{w_k} + \overset{m}{\underset{k=1}{\otimes}} \left(\left(1 + \lambda_{j_{T_{\sigma(k)}}}\right)\right)^{w_k}} \right\}, \left\{ \dfrac{2 \overset{m}{\underset{k=1}{\otimes}} \left(\left(1 + \lambda_{j_{I_{\sigma(k)}}}\right)\right)^{w_k}}{\overset{m}{\underset{k=1}{\otimes}} \left(\left(2 - \lambda_{j_{I_{\sigma(k)}}}\right)\right)^{w_k} + \overset{m}{\underset{k=1}{\otimes}} \left(\left(1 + \lambda_{j_{I_{\sigma(k)}}}\right)\right)^{w_k}} \right\}, \\ \left\{ \dfrac{\overset{m}{\underset{k=1}{\otimes}} \left(1 + \lambda_{j_{F_{\sigma(k)}}}\right)^{w_k} - \overset{m}{\underset{k=1}{\otimes}} \left(1 - \lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\overset{m}{\underset{k=1}{\otimes}} \left(1 + \lambda_{j_{F_{\sigma(k)}}}\right)^{w_k} + \overset{m}{\underset{k=1}{\otimes}} \left(1 - \lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}} \right\} \end{array} \right\rangle$$

Proof: Using induction and from Theorem 3.12

$$\alpha^{E^{w_1}} = \left\langle \begin{array}{c} \left\{\left[\dfrac{2(A^L_{j_T})^{w_1}}{(2-A^L_{j_T})^{w_1}+(A^L_{j_T})^{w_1}}, \dfrac{2(A^U_{j_T})^{w_1}}{(2-A^U_{j_T})^{w_1}+(A^U_{j_T})^{w_1}}\right]\right\}, \left\{\left[\dfrac{2(A^L_{j_I})^{w_1}}{(2-A^L_{j_I})^{w_1}+(A^L_{j_I})^{w_1}}, \dfrac{2(A^U_{j_I})^{w_1}}{(2-A^U_{j_I})^{w_1}+(A^U_{j_I})^{w_1}}\right]\right\}, \\ \left\{\left[\dfrac{(1+A^L_{j_F})^{w_1}-(1-A^L_{j_F})^{w_1}}{(1+A^L_{j_F})^{w_1}+(1-A^L_{j_F})^{w_1}}, \dfrac{(1+A^U_{j_F})^{w_1}-(1-A^U_{j_F})^{w_1}}{(1+A^U_{j_F})^{w_1}+(1-A^U_{j_F})^{w_1}}\right]\right\}, \left\{\dfrac{2(\lambda_{j_T})^{w_1}}{(2-\lambda_{j_T})^{w_1}+(\lambda_{j_T})^{w_1}}\right\}, \\ \left\{\dfrac{2(\lambda_{j_I})^{w_1}}{(2-\lambda_{j_I})^{w_1}+(\lambda_{j_I})^{w_1}}\right\}, \left\{\dfrac{(1+\lambda_{j_F})^{w_1}-(1-\lambda_{j_F})^{w_1}}{(1+\lambda_{j_F})^{w_1}+(1-\lambda_{j_F})^{w_1}}\right\} \end{array} \right\rangle$$

and
$$\alpha^{E^{w_2}} = \left\langle \begin{array}{c} \left\{\left[\dfrac{2(A^L_{j_T})^{w_2}}{(2-A^L_{j_T})^{w_2}+(A^L_{j_T})^{w_2}}, \dfrac{2(A^U_{j_T})^{w_2}}{(2-A^U_{j_T})^{w_2}+(A^U_{j_T})^{w_2}}\right]\right\}, \left\{\left[\dfrac{2(A^L_{j_I})^{w_2}}{(2-A^L_{j_I})^{w_2}+(A^L_{j_I})^{w_2}}, \dfrac{2(A^U_{j_I})^{w_2}}{(2-A^U_{j_I})^{w_2}+(A^U_{j_I})^{w_2}}\right]\right\}, \\ \left\{\left[\dfrac{(1+A^L_{j_F})^{w_2}-(1-A^L_{j_F})^{w_2}}{(1+A^L_{j_F})^{w_2}+(1-A^L_{j_F})^{w_2}}, \dfrac{(1+A^U_{j_F})^{w_2}-(1-A^U_{j_F})^{w_2}}{(1+A^U_{j_F})^{w_2}+(1-A^U_{j_F})^{w_2}}\right]\right\}, \left\{\dfrac{2(\lambda_{j_T})^{w_2}}{(2-\lambda_{j_T})^{w_2}+(\lambda_{j_T})^{w_2}}\right\}, \\ \left\{\dfrac{2(\lambda_{j_I})^{w_2}}{(2-\lambda_{j_I})^{w_2}+(\lambda_{j_I})^{w_2}}\right\}, \left\{\dfrac{(1+\lambda_{j_F})^{w_2}-(1-\lambda_{j_F})^{w_2}}{(1+\lambda_{j_F})^{w_2}+(1-\lambda_{j_F})^{w_2}}\right\} \end{array} \right\rangle$$

$$NCHFEWG(\alpha_1, \alpha_2) = \bigotimes_{k=1}^{2} \alpha_k^{E^{w_k}}$$

$$= \left\langle \begin{array}{c} \left\{\left[\dfrac{2\bigotimes_{k=1}^{2}\left(A^L_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{2}\left(2-A^L_{j_{T_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{2}\left(A^L_{j_{T_{\sigma(k)}}}\right)^{w_k}}, \dfrac{2\bigotimes_{k=1}^{2}\left(A^U_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{2}\left(2-A^U_{j_{T_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{2}\left(A^U_{j_{T_{\sigma(k)}}}\right)^{w_k}}\right]\right\}, \\ \left\{\left[\dfrac{2\bigotimes_{k=1}^{2}\left(A^L_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{2}\left(2-A^L_{j_{I_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{2}\left(A^L_{j_{I_{\sigma(k)}}}\right)^{w_k}}, \dfrac{2\bigotimes_{k=1}^{m}\left(A^U_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{2}\left(2-A^U_{j_{I_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{2}\left(A^U_{j_{I_{\sigma(k)}}}\right)^{w_k}}\right]\right\}, \\ \left\{\left[\dfrac{\bigotimes_{k=1}^{2}\left(1+A^L_{j_{F_{\sigma(k)}}}\right)^{w_k}-\bigotimes_{k=1}^{2}\left(1-A^L_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{2}\left(1+A^L_{j_{F_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{2}\left(1-A^L_{j_{F_{\sigma(k)}}}\right)^{w_k}}, \dfrac{\bigotimes_{k=1}^{2}\left(1+A^U_{j_{F_{\sigma(k)}}}\right)^{w_k}-\bigotimes_{k=1}^{2}\left(1-A^U_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{2}\left(1+A^U_{j_{F_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{2}\left(1-A^U_{j_{F_{\sigma(k)}}}\right)^{w_k}}\right]\right\}, \\ \left\{\dfrac{2\bigotimes_{k=1}^{2}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{2}\left(2-\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{2}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}}\right\}, \left\{\dfrac{2\bigotimes_{k=1}^{2}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{2}\left(2-\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{2}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}}\right\}, \\ \left\{\dfrac{\bigotimes_{k=1}^{2}\left(1+\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}-\bigotimes_{k=1}^{2}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{2}\left(1+\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{2}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}}\right\} \end{array} \right\rangle$$

For $m=q$ we have

$$NCHFEHG(\alpha_1,\alpha_2,...,\alpha_q)=\left\langle \begin{array}{c} \left\{\left[\dfrac{2\bigotimes_{k=1}^{q}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{q}\left(2-A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{q}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}},\dfrac{2\bigotimes_{k=1}^{q}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{q}\left(2-A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{q}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}}\right]\right\} \\ ,\left\{\left[\dfrac{2\bigotimes_{k=1}^{q}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{q}\left(2-A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{q}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}},\dfrac{2\bigotimes_{k=1}^{m}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{q}\left(2-A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{q}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}}\right]\right\}, \\ \left\{\left[\dfrac{\bigotimes_{k=1}^{q}\left(1+A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}-\bigotimes_{k=1}^{q}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{q}\left(1+A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{q}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}},\dfrac{\bigotimes_{k=1}^{q}\left(1+A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}-\bigotimes_{k=1}^{q}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{q}\left(1+A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{q}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}}\right]\right\}, \\ \left\{\dfrac{2\bigotimes_{k=1}^{q}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{q}\left(2-\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{q}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}}\right\},\left\{\dfrac{2\bigotimes_{k=1}^{q}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{q}\left(2-\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{q}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}}\right\}, \\ \left\{\dfrac{\bigotimes_{k=1}^{q}\left(1+\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}-\bigotimes_{k=1}^{q}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\bigotimes_{k=1}^{q}\left(1+\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}+\bigotimes_{k=1}^{q}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}}\right\} \end{array}\right\rangle$$

Using assumption, we have

$$\left(\underset{k=1}{\overset{q}{\otimes_E}}\left(\alpha^{E}_{\sigma(k)}\right)^{w_k}\right)\otimes_E\left(\alpha^{E}_{q+1}\right)^{w_{q+1}}=\left\langle\begin{matrix}\left\{\left[\dfrac{2\overset{q}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\overset{q}{\underset{k=1}{\otimes}}\left(2-A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}+\overset{q}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}},\dfrac{2\overset{q}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\overset{q}{\underset{k=1}{\otimes}}\left(2-A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}+\overset{q}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}}\right]\right\},\\ \left\{\left[\dfrac{2\overset{q}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\overset{q}{\underset{k=1}{\otimes}}\left(2-A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}+\overset{q}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}},\dfrac{2\overset{m}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\overset{q}{\underset{k=1}{\otimes}}\left(2-A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}+\overset{q}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}}\right]\right\},\\ \left\{\left[\dfrac{\overset{q}{\underset{k=1}{\otimes}}\left(1+A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}-\overset{q}{\underset{k=1}{\otimes}}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\overset{q}{\underset{k=1}{\otimes}}\left(1+A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}+\overset{q}{\underset{k=1}{\otimes}}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}},\dfrac{\overset{q}{\underset{k=1}{\otimes}}\left(1+A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}-\overset{q}{\underset{k=1}{\otimes}}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\overset{q}{\underset{k=1}{\otimes}}\left(1+A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}+\overset{q}{\underset{k=1}{\otimes}}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}}\right]\right\},\\ \left\{\dfrac{2\overset{q}{\underset{k=1}{\otimes}}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\overset{q}{\underset{k=1}{\otimes}}\left(2-\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}+\overset{q}{\underset{k=1}{\otimes}}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}}\right\},\\ \left\{\dfrac{2\overset{q}{\underset{k=1}{\otimes}}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\overset{q}{\underset{k=1}{\otimes}}\left(2-\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}+\overset{q}{\underset{k=1}{\otimes}}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}}\right\},\\ \left\{\dfrac{\overset{q}{\underset{k=1}{\otimes}}\left(1+\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}-\overset{q}{\underset{k=1}{\otimes}}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\overset{q}{\underset{k=1}{\otimes}}\left(1+\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}+\overset{q}{\underset{k=1}{\otimes}}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}}\right\}\end{matrix}\right\rangle\otimes_E\left\langle\begin{matrix}\left\{\left[\dfrac{2(A^{L}_{j_T})^{w_{q+1}}}{(2-A^{L}_{j_T})^{w_{q+1}}+(A^{L}_{j_T})^{w_{q+1}}},\dfrac{2(A^{U}_{j_T})^{w_{q+1}}}{(2-A^{U}_{j_T})^{w_{q+1}}+(A^{U}_{j_T})^{w_{q+1}}}\right]\right\},\\ \left\{\left[\dfrac{2(A^{L}_{j_I})^{w_{q+1}}}{(2-A^{L}_{j_I})^{w_{q+1}}+(A^{L}_{j_I})^{w_{q+1}}},\dfrac{2(A^{U}_{j_I})^{w_{q+1}}}{(2-A^{U}_{j_I})^{w_{q+1}}+(A^{U}_{j_I})^{w_{q+1}}}\right]\right\},\\ \left\{\left[\dfrac{(1+A^{L}_{j_F})^{w_{q+1}}-(1-A^{L}_{j_F})^{w_{q+1}}}{(1+A^{L}_{j_F})^{w_{q+1}}+(1-A^{L}_{j_F})^{w_{q+1}}},\dfrac{(1+A^{U}_{j_F})^{w_{q+1}}-(1-A^{U}_{j_F})^{w_{q+1}}}{(1+A^{U}_{j_F})^{w_{q+1}}+(1-A^{U}_{j_F})^{w_{q+1}}}\right]\right\},\\ \left\{\dfrac{2(\lambda_{j_T})^{w_{q+1}}}{(2-\lambda_{j_T})^{w_{q+1}}+(\lambda_{j_T})^{w_{q+1}}}\right\},\\ \left\{\dfrac{2(\lambda_{j_I})^{w_{q+1}}}{(2-\lambda_{j_I})^{w_{q+1}}+(\lambda_{j_I})^{w_{q+1}}}\right\},\\ \left\{\dfrac{(1+\lambda_{j_F})^{w_{q+1}}-(1-\lambda_{j_F})^{w_{q+1}}}{(1+\lambda_{j_F})^{w_{q+1}}+(1-\lambda_{j_F})^{w_{q+1}}}\right\}\end{matrix}\right\rangle$$

$$
= \left\langle \begin{array}{c}
\left\{\left[\dfrac{2\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\overset{q+1}{\underset{k=1}{\otimes}}\left(2-A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}+\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{T_{\sigma(k)}}}\right)^{w_k}}, \dfrac{2\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\overset{q+1}{\underset{k=1}{\otimes}}\left(2-A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}+\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{T_{\sigma(k)}}}\right)^{w_k}}\right]\right\}, \left\{\left[\dfrac{2\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\overset{q+1}{\underset{k=1}{\otimes}}\left(2-A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}+\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{L}_{j_{I_{\sigma(k)}}}\right)^{w_k}}, \dfrac{2\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\overset{q+1}{\underset{k=1}{\otimes}}\left(2-A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}+\overset{q+1}{\underset{k=1}{\otimes}}\left(A^{U}_{j_{I_{\sigma(k)}}}\right)^{w_k}}\right]\right\}, \\
\left\{\left[\dfrac{\overset{q+1}{\underset{k=1}{\otimes}}\left(1+A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}-\overset{q+1}{\underset{k=1}{\otimes}}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\overset{q+1}{\underset{k=1}{\otimes}}\left(1+A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}+\overset{q+1}{\underset{k=1}{\otimes}}\left(1-A^{L}_{j_{F_{\sigma(k)}}}\right)^{w_k}}, \dfrac{\overset{q}{\underset{k=1}{\otimes}}\left(1+A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}-\overset{q}{\underset{k=1}{\otimes}}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\overset{q+1}{\underset{k=1}{\otimes}}\left(1+A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}+\overset{q+1}{\underset{k=1}{\otimes}}\left(1-A^{U}_{j_{F_{\sigma(k)}}}\right)^{w_k}}\right]\right\}, \left\{\dfrac{2\overset{q+1}{\underset{k=1}{\otimes}}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}}{\overset{q+1}{\underset{k=1}{\otimes}}\left(2-\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}+\overset{q+1}{\underset{k=1}{\otimes}}\left(\lambda_{j_{T_{\sigma(k)}}}\right)^{w_k}}\right\}, \\
\left\{\dfrac{2\overset{q+1}{\underset{k=1}{\otimes}}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}}{\overset{q+1}{\underset{k=1}{\otimes}}\left(2-\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}+\overset{q+1}{\underset{k=1}{\otimes}}\left(\lambda_{j_{I_{\sigma(k)}}}\right)^{w_k}}\right\}, \left\{\dfrac{\overset{q+1}{\underset{k=1}{\otimes}}\left(1+\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}-\overset{q+1}{\underset{k=1}{\otimes}}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}}{\overset{q+1}{\underset{k=1}{\otimes}}\left(1+\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}+\overset{q+1}{\underset{k=1}{\otimes}}\left(1-\lambda_{j_{F_{\sigma(k)}}}\right)^{w_k}}\right\}
\end{array} \right\rangle
$$

Theorem 4.13: With $w_j = \frac{1}{m}$, *NCHFEG* becomes *NCHFEWG*.

Proof: $NCHFEHG(\alpha_1,...,\alpha_m) = \left(\alpha_{\sigma(1)}\right)^{w_1} \otimes_E ... \otimes\left(\alpha_{\sigma(m)}\right)^{w_m}$

$$= \left(\alpha_1\right)^{w_1} \otimes_E ... \otimes_E \left(\alpha_m\right)^{w_m}$$

$$= \left(\alpha_1\right)^{\frac{1}{m}} \otimes_E ... \otimes_E \left(\alpha_m\right)^{\frac{1}{m}}$$

$$= NCHFEWG(\alpha_1,...,\alpha_m)$$

5. An Application of NCHFG Aggregation Operator to MEDM Problems

This section concern with constructing algorithms using the NCHFWG for MEDM problems.

5.1. Algorithm

Step 1: Allocation of expert corresponding to their weight, identification of alternative and attributes.

Let $\{F_1, F_2, ..., F_r\}$ be the set of *r* alternatives, $\{K_1, K_2, ..., K_s\}$ be *s* attributes with corresponding weight vector $[w_1, w_2, ..., w_s]^T$ such that $w_j \in [0,1]$, $\sum w_j = 1$ $\{M_1, M_2, ..., M_p\}$ be decision experts with corresponding weight vector $\left[w_1, w_2, ..., w_p\right]^T$ such that $w_j \in [0,1]$, $\sum w_j = 1$. we construct decision matrices $D^{(k)} = (d_{ij})_{r\times s}$, with entries as neutrosophic cubic hesitant fuzzy values.

Step 2. Transformation of decision matrices to aggregated decision matrix.

A single matrix consisting of *s* attributes is constructed by aggregating all decision matrices using NCHFWG operators with corresponding weight vector of decision makers.

Step 3. Transformation of aggregated matrix to decision vector.

An $r \times 1$ vector is obtained by aggregating the decision matrix using NCHFWG operators.

Step 4: Ranking alternatives.

The most desirable alternative with highest score by ranking them in descending order of scores.

Example 5.2: Using above algorithm, we have to choose the most desirable alternative among the alternatives (Electronics companies) $F_p (p = 1,2,3)$ on the basis of three attributes A_1 (price), A_2 (Electricity consumption), A_3 (design).

Step 1: Decision matrix for first expert

$$D^{(1)} = \begin{pmatrix} \left\langle \begin{array}{c} \{[0.1,0.5],[0.2,0.7]\}, \\ \{[0.2,0.3],[0.1,0.6]\}, \\ \{[0.1,0.5],[0.2,0.3]\}, \\ \{0.4,0.6\},\{0.3,0.5\},\{0.3,0.4\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.3,0.4],[0.2,0.9]\}, \\ \{[0.2,0.6],[0.3,0.6]\}, \\ \{[0.3,0.4],[0,0.1]\}, \\ \{0.5,0.7\},\{0.3,0.4\},\{0.4,0.5\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.3,0.7],[0.2,0.4]\}, \\ \{[0.2,0.5],[0.1,0.6]\}, \\ \{[0.2,0.4],[0,0.1]\}, \\ \{0.5,0.6\},\{0.2,0.4\},\{0.2,0.3\} \end{array} \right\rangle \\ \left\langle \begin{array}{c} \{[0.2,0.5],[0.3,0.7]\}, \\ \{[0.3,0.4],[0.1,0.5]\}, \\ \{[0.1,0.3],[0,0.2]\}, \\ \{0.7,0.9\},\{0.3,0.4\},\{0.1,0.2\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.1,0.3],[0.2,0.5]\}, \\ \{[0.2,0.3],[0.1,0.6]\}, \\ \{[0.1,0.4],[0,0.3]\}, \\ \{0.4,0.5\},\{0.3,0.7\},\{0.3,0.5\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.1,0.4],[0.2,0.6]\}, \\ \{[0.1,0.3],[0,0.2]\}, \\ \{[0.2,0.4],[0,0.3]\}, \\ \{0.4,0.5\},\{0.3,0.6\},\{0.1,0.3\} \end{array} \right\rangle \\ \left\langle \begin{array}{c} \{[0.4,0.5],[0.3,0.4]\}, \\ \{[0.1,0.3],[0.2,0.5]\}, \\ \{[0.1,0.4],[0.7,0.8]\}, \\ \{0.4,0.5\},\{0.3,0.4\},\{0.2,0.4\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.3,0.5],[0.1,0.4]\}, \\ \{[0.2,0.3],[0.2,0.6]\}, \\ \{[0.1,0.5],[0.6,0.8]\}, \\ \{0.5,0.6\},\{0.1,0.4\},\{0.2,0.3\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.2,0.5],[0.1,0.4]\}, \\ \{[0.2,0.5],[0.2,0.6]\}, \\ \{[0.1,0.4],[0.6,0.7]\}, \\ \{0.4,0.6\},\{0.3,0.4\},\{0.4,0.5\} \end{array} \right\rangle \end{pmatrix}$$

Explanation of decision matrix entries:

In case of $d_{11}^{(1)}$, {[0.1,0.5],[0.2,0.7]} is interval hesitant degree of preference to attribute F_1 corresponding to attribute K_1, {[0.2,0.3],[0.1,0.6]} is interval hesitant degree of indeterminacy (preference/ non-preference) for attributeF_1 corresponding to attribute K_1, {[0.1,0.5],[0.2,0.3]} is interval hesitant degree of non-preference for attribute F_1 corresponding to attribute K_1, {0.4,0.6}} is hesitant degree of preference for attribute F_1 corresponding to attribute K_1, {0.3,0.5} is hesitant degree of indeterminacy (preference/ non-preference) for attribute F_1 corresponding to attribute K_1, {0.3,0.4} is hesitant degree of non-preference for attribute F_1 corresponding to attribute K_1, given by the first expert.

Decision Matrix for second expert

$$D^{(2)} = \begin{pmatrix} \left\langle \begin{array}{c} \{[0.1,0.3],[0.2,0.5]\}, \\ \{[0.2,0.4],[0.1,0.5]\}, \\ \{[0.1,0.4],[0,0.3]\}, \\ \{0.4,0.8\},\{0.3,0.4\},\{0.3,0.4\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.2,0.4],[0.2,0.7]\}, \\ \{[0.1,0.3],[0,0.2]\}, \\ \{[0.3,0.5],[0,0.3]\}, \\ \{0.7,0.8\},\{0.3,0.6\},\{0.2,0.3\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.2,0.5],[0.2,0.7]\}, \\ \{[0,0.4],[0.1,0.3]\}, \\ \{[0.3,0.5],[0,0.4]\}, \\ \{0.6,0.7\},\{0.2,0.5\},\{0.4,0.5\} \end{array} \right\rangle \\ \left\langle \begin{array}{c} \{[0.1,0.5],[0.3,0.7]\}, \\ \{[0.1,0.3],[0.1,0.2]\}, \\ \{[0.2,0.4],[0,0.3]\}, \\ \{0.6,0.7\},\{0.3,0.6\},\{0.4,0.5\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.3,0.6],[0.2,0.7]\}, \\ \{[0.1,0.2],[0.2,0.5]\}, \\ \{[0,0.3],[0.1,0.4]\}, \\ \{0.7,0.8\},\{0.3,0.4\},\{0.2,0.3\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.4,0.5],[0.4,0.7]\}, \\ \{[0.2,0.3],[0.1,0.6]\}, \\ \{[0.3,0.4],[0,0.3]\}, \\ \{0.5,0.6\},\{0.3,0.5\},\{0.3,0.4\} \end{array} \right\rangle \\ \left\langle \begin{array}{c} \{[0.2,0.5],[0.1,0.4]\}, \\ \{[0.2,0.4],[0.2,0.5]\}, \\ \{[0.1,0.3],[0.8,0.9]\}, \\ \{0.7,0.8\},\{0.6,0.8\},\{0.3,0.4\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.2,0.3],[0.4,0.5]\}, \\ \{[0.1,0.4],[0.2,0.5]\}, \\ \{[0.7,0.8],[0.1,0.2]\}, \\ \{0.7,0.9\},\{0.7,0.8\},\{0.2,0.5\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.5,0.7],[0.3,0.4]\}, \\ \{[0.1,0.3],[0.2,0.5]\}, \\ \{[0.1,0.4],[0.2,0.6]\}, \\ \{0.6,0.8\},\{0.1,0.4\},\{0.2,0.3\} \end{array} \right\rangle \end{pmatrix}$$

Step 2

using weight vector (0.5,0.5) for decision makers, the aggregated matrix is

$$D = \begin{pmatrix} \left\langle \begin{array}{c} \{[0.1,0.387298],[0.2,0.591608]\}, \\ \{[0.2,0.34641],[0.1,0.547723]\}, \\ \{[0.1,0.452277],[0.105573,0.3]\}, \\ \{0.4,0.69282\}, \\ \{0.3,0.447214\}, \\ \{0.3,0.4\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.244949,0.4],[0.2,0.793725]\}, \\ \{[0.141421,0.424264],[0,0.34641]\}, \\ \{[0.3,0.452277],[0,0.206275]\}, \\ \{0.591608,0.748331\}, \\ \{0.3,0.489898\}, \\ \{0.30718,0.408392\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.244949,0.591608],[0.2,0.52915]\}, \\ \{[0,0.447214],[0.1,0.424264]\}, \\ \{[0.251669,0.452277],[0,0.265153]\}, \\ \{0.547723,0.648074\}, \\ \{0.2,0.4\}, \\ \{0.30718,0.408392\} \end{array} \right\rangle \\ \left\langle \begin{array}{c} \{[0.141421,0.5],[0.3,0.7]\}, \\ \{[0.173205,0.34641],[0.1,0.316228]\}, \\ \{[0.151472,0.351926],[0,0.251669]\}, \\ \{0.648074,0.793725\}, \\ \{0.3,0.489898\}, \\ \{0.265153,0.367544\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.173205,0.424264],[0.2,0.591608]\}, \\ \{[0.141421,0.244949],[0.141421,0.547723]\}, \\ \{[0.051317,0.351926],[0.051317,0.351926]\}, \\ \{0.52915,0.632456\}, \\ \{0.3,0.52915\}, \\ \{0.251669,0.408392\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.2,0.447214],[0.282843,0.648074]\}, \\ \{[0.141421,0.3],[0,0.34641]\}, \\ \{[0.251669,0.4],[0,0.3]\}, \\ \{0.447214,0.547723\}, \\ \{0.3,0.547723\}, \\ \{0.206275,0.351926\} \end{array} \right\rangle \\ \left\langle \begin{array}{c} \{[0.282843,0.5],[0.173205,0.4]\}, \\ \{[0.141421,0.34641],[0.2,0.5]\}, \\ \{[0.1,0.351926],[0.755051,0.858579]\}, \\ \{0.52915,0.632456\}, \\ \{0.424264,0.565685\}, \\ \{0.251669,0.4\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.244949,0.387298],[0.2,0.447214]\}, \\ \{[0.141421,0.34641],[0.2,0.547723]\}, \\ \{[0.480385,0.683772],[0.4,0.6]\}, \\ \{0.591608,0.734847\}, \\ \{0.264575,0.565685\}, \\ \{0.2,0.408392\} \end{array} \right\rangle & \left\langle \begin{array}{c} \{[0.316228,0.591608],[0.173205,0.4]\}, \\ \{[0.141421,0.387298],[0.2,0.547723]\}, \\ \{[0.1,0.4],[0.434315,0.65359]\}, \\ \{0.489898,0.69282\}, \\ \{0.173205,0.4\}, \\ \{0.30718,0.408392\} \end{array} \right\rangle \end{pmatrix}$$

Step 3

using weight vector (0.3,0.4,0.3) for attributes and NCHWG we have the following decision vector

$$d = \begin{pmatrix} \left\langle \begin{array}{c} \{[0.18722,0.4455],[0.2,0.643505]\}, \\ \{[0,0.405589],[0,0.422372]\}, \\ \{[0.229912,0.452277],[0.032918,0.25311]\}, \\ \{0.514043,0.700344\},\{0.26564,0.463822\},\{0.305034,0.405887\} \end{array} \right\rangle \\ \left\langle \begin{array}{c} \{[0.170172,0.452793],[0.250618,0.639485]\}, \\ \{[0.15029,0.288831],[0,0.404857]\}, \\ \{[0.14557,0.366739],[0.020852,0.30752]\}, \\ \{0.534655,0.648457\},\{0.3,0.522434\},\{0.242474,0.379688\} \end{array} \right\rangle \\ \left\langle \begin{array}{c} \{[0.276117,0.47481],[0.183463,0.418256]\}, \\ \{[0.141421,0.358201],[0.2,0.532946]\}, \\ \{[0.277533,0.524738],[0.549437,0.71955]\}, \\ \{0.540653,0.690198\},\{0.268459,0.509824\},\{0.248982,0.405887\} \end{array} \right\rangle \end{pmatrix}$$

Step 4

Using Score function, we rank the alternatives as $S(F_1) = 0.491743, S(F_2) = 0.511797, S(F_3) = 0.467604$. Hence most desirable alternative is F_2.

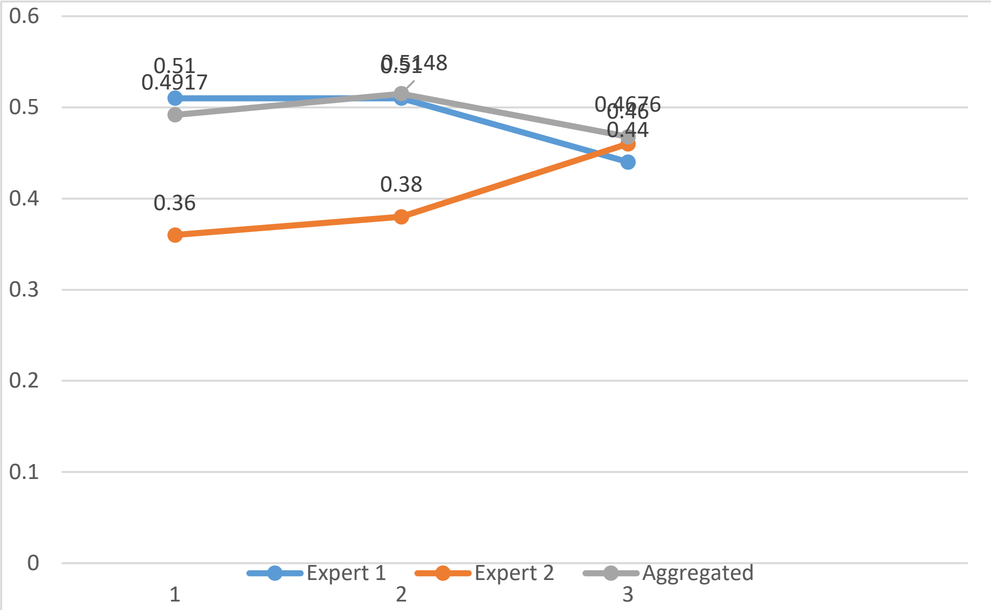

Figure 2: Ranking based on score function

Concluding Remarks

Decision making is one of the crucial problems in real life. For decision making different tools has been established. Torra's hesitant fuzzy set has been used in many practical problems due to flexibility of choosing membership grades. On the other side Jun's neutrosophic cubic set is capable of dealing truth, falsity and indeterminacy membership grades, but the element of hesitancy is missing in truth and falsity membership grades of neutrosophic cubic set. We have discussed the role of hesitancy in truth and falsity membership grades of neutrosophic cubic set. We define neutrosophic cubic hesitant fuzzy set and some basic operations like addition, multiplication, Einstein addition and multiplication in neutrosophic cubic hesitant fuzzy sets. Then we prove some elegant results. In section 4 geometric aggregation operators are defined. Using these aggregation operators an example is constructed.

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A Study on Neutrosophic Algebra

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Abstract: The notion of neutrosophic algebra, ideal of neutrosophic algebra, kernel and neutrosophic quotient algebra have been proposed in this paper. We characterize some properties of neutrosophic algebra and proved that every quotient neutrosophic algebra is quotient algebra. Also we proved that every neutrosophic algebra is algebra and direct product of neutrosophic algebras over a neutrosophic field is algebra.

Key words: Neutrosophic set, neutrosophic algebra, neutrosophic algebra isomorphism, ideals of neutrosophic algebra, quotient neutrosophic algebra, neutrosophic subalgebra.

1. Introduction

A fuzzy set A in X is characterized by a membership function which is associated with each element in X to a real number in the unit interval [0, 1]. In 1965 L. A. Zadeh [24] introduced the concept of fuzzy set theory. This novel concept is used successfully in modeling uncertainty in many fields of real life. Fuzzy sets and its applications have been extensively studied in different aspects. In 1998, Neutrosophic set was introduced by Florentin Smarandache [17, 18], where each element associated with three defining functions, namely the membership function (T), the non-membership function (F) and the indeterminacy function (I) defined on the universe of discourse X, the three functions are completely independent. Relative to the natural problems sometimes one may not be able to decide.

In 2004 W. B. Vasantha Kandaswamy and Florentin Smarandache [**23**] introduced a neutrosophic structure based on indeterminacy I only, which they called I-neutrosophic algebraic structures. Algebraic structure based sets of neutrosophic numbers of the form $a+bI$ where a,b are real (or complex) and Indeterminacy I with $I^2=I$. This I is different from the imaginary $i=\sqrt{-1}$. After the development of the Neutrosophic set theory, one can easily take decision and indeterminacy function of the set is the nondeterministic part of the situation. The applications of the theory have been found in various fields for dealing with indeterminate and consistent information in real world. The neutrosophic set generalizes the concept of classical fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set and so on. Broumi Said et al. [**6, 7**] studied the notion of intuitionistic Neutrosophic Soft Set and Rough neutrosophic sets. The branch of neutrosophic theory is the theory of neutrosophic algebraic structures. Abobala [4, 5] introduced Some Special Substructures of Neutrosophic Rings and AH-Substructures in n-Refined Neutrosophic Vector Spaces**.** The authors in [1, 2] studied the notion of Neutrosophic vector spaces and Mamouni Dhar, Said Broumi and Florentin Smarandache [10] introduced Square Neutrosophic Fuzzy Matrices. In [**3**] Abdel Nasser Hussian, Mai Mohamed, Mohamed Abdel-Baset and Florentin smarandache studied the Neutrosophic Linear Programming Problems. W. B. Vasanntha Kandasamy and F. Smarandache [20, 21] introduced the concept of neutrosophic algebraic structure and neutrosophic N-algebraic structures. P. Narasimha Swamy et al. [14, 15] studied the notion of Fuzzy quasi-ideals of near algebra and Anti Fuzzy Gamma Near-Algebras. T. Nagaiah et al. [11, 12, 13, 16] introduced Partially ordered Gamma semi groups, near-rings, direct product and strongest interval valued anti fuzzy ideals of Gamma near-rings and special class of ring structure. T. Srinivas, T. Nagaiah and P.Narasimha Swamy [**19**] initiated the concept of Anti Fuzzy Ideals of Gamma Near-rings.Hatip and Abobala [9] studied the notion of AH-substructuresin strong refined models. Bijan Davvaz [8] introduced Neutrosophic ideals of Neutrosophic KU-algebra. Since then several researcher have been study the concept of neutrosophic theory and its application in varies branches. In this paper our main aim is to introduce the concept of neutrosophic algebra and its application in varies branches of Mathematics. Also we proved that every neutrosophic algebra is algebra and direct product of neutrosophic algebras over a neutrosophic field is algebra.

This paper is organized into four sections. The first section is introductory. The second section presents the basic concepts needed to make this paper a self-contained one. Section three discusses and describes the neutrosophic algebra and its examples. Final section gives ideal of neutrosophic algebra, neutrosophic quotient algebra and studied their properties.

2. Preliminaries

In this section we recall some basic concepts of neutrosophic set, proposed by W.B. Vasantha Kandasamy and University of New Mexico professor F. Smarandache in their monograph [21, 22].

Definition 2.1 Let U be an universe of discourse then the neutrosophic set A is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$ >, x ∈ U} where the functions T, I, F: U → $]^-0, 1^+[$ define respectively the degree of membership (or truthness), the degree of indeterminacy, and the degree of non-membership (or falsehood)of the element x in U to the set A with the condition. $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$. From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. So instead of $]^-0, 1^+[$ we need to take the interval [0, 1] for technical applications, because $]^-0, 1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems. Let X be a non-empty set. A set $X(I)=\langle X,I\rangle$ generated by X and I is called a neutrosophic set. The elements of X (I) are of the form (a,bI), where a,b are elements of X .

Definition 2.2: Algebra Y over a field X is a linear space Y over a field X on whicha multiplication is defined such that

i. Y forms a semi group under multiplication,

ii. multiplication is distributive over addition

iii. $\lambda (x y) = (\lambda x) y = x (\lambda y)$, for all $x, y \in Y$ and $\lambda \in X$.

3. Neutrosophic algebra

In this section we define the neutrosophic algebra and provide examples. Also define neutrosophic subalgebra and characterize their properties. Excepted otherwise stated, all strong neutrosophic algebras in this paper will be considering neutrosophic algebra.

Definition 3.1: Let Y be algebra over a field X . The set generated by Y and I is denoted by $\langle Y \cup I\rangle$ = Y(I) = $\{a + bI : a , b\epsilon Y\}$ is called a weak neutrosophic algebra over a field X . If Y(I) is a ~~neutrosophic algebra over a neutrosophic field $X(I)$ then~~

Y(I) is called a strong neutrosophic algebra. The elements of Y(I) are calledneutrosophic vectors and the elements of $X(I)$ are called neutrosophic scalars.

Examples 3.2:

i. $\mathbb{R}(I)$ is a weak neutrosophic algebra over a field $\mathbb{Q}$ and it is strongneutrosophic algebra over a neutrosophic field $\mathbb{Q}(I)$, where $\mathbb{Q}(I) = \{a+bI : a,b \in Q\}$

ii. $\mathbb{C}(I)$ is a weak neutrosophic algebra over a field $\mathbb{R}$ and it is a strongneutrosophic algebra over a neutrosophic field $\mathbb{R}(I)$

iii. $\mathbb{R}^n(I)$ is a weak neutrosophic algebra over a field $\mathbb{R}$ and it is a strong neutrosophic algebra over a neutrosophic field $\mathbb{R}(I)$

iv. $M_{m\times n}(I) = \{[a_{ij} + b_{ij}I] ; a_{ij}, b_{ij} \epsilon \mathbb{Q}\}$ is strong neutrosophic algebra over a neutrosohic field $\mathbb{Q}(I)$ and it is weak neutrosophic algebra over a field $\mathbb{Q}$.

Definition 3.3: Let Y(I) be neutrosophic algebra over a neutrosophic field $X(I)$.

The non-empty subset W(I) of Y(I) is called a neutrosophic subalgebra over a field $X(I)$, if $W(I)$ is itself a neutrosophic algebra over a neutrosophic field $X(I)$.

Theorem 3.4: Every strong neutrosophic algebra is weak neutrosophic algebra.

Proof: Suppose $Y(I)$ is strong neutrosophic algebra over a neutrosophic field$X(I)$. Since $X \subseteq X(I)$, so that $Y(I)$ is weak neutrosophic algebra over a field X. Hence every strong neutrosophic algebra is weak neutrosophic algebra.

Theorem 3.5: Every strong (weak) neutrosophic algebra is neutrosophic vector space.

Proof:- Suppose $Y(I)$ is a strong neutrosophic algebra over a neutrosophic field $X(I)$.
This implies that Y is algebra over a field X. So that Y is a vector space over a field X.
From Theorem 3.4, this show $Y(I)$ is a neutrosophic vector space over a field X. Similarly we can prove, if $Y(I)$ is a weak neutrosophic algebra over a field X, then Y is an algebra over a field X and hence Y is a vector space over a field X. Hence $Y(I)$ is a neutrosophic vector space.

Theorem 3.6: Every neutrosophic algebra is algebra.

Proof: Let $Y(I)$ be the neutrosophic algebra over a neutrosophic field $X(I)$. By Theorem 3.5,$Y(I)$ is a neutrosophic vector space over a field $X(I)$.
This implies $Y(I)$ is a vector space over a field $X(I)$. It is easy to verify that all the algebra properties of $Y(I)$ over a field $X(I)$.i.e., (i) $x(yz) = (xy)z$ (ii)$x \cdot (y + z) = x \cdot y + x \cdot z, (x + y) \cdot z = x \cdot z + y \cdot z$ and $\lambda(xy) = (\lambda x)y = x(\lambda y), \forall x, y, z \epsilon Y(I)$ and $\lambda \epsilon X(I)$.
We can easy to see that Y (I) is a neutrosophic algebra over a field X (I).

Theorem 3.7: Let $M_1(I)$ and $M_2(I)$ be neutrosophic algebras over a neutrosophic field $X(I)$. Then the direct product
$M_1(I) \times M_2(I) = \{(u_1, u_2): u_1 \in M_1(I), u_2 \in M_2(I)\}$ is algebra over a neutrosophic field $X(I)$, where addition, multiplication and scalar multiplication is defined by

(i) $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$

(ii) $(u_1, u_2) \cdot (v_1, v_2) = (u_1v_1, u_2v_2)$

(iii) $\alpha(u_1, u_2) = (\alpha u_1, \alpha u_2), \forall \alpha \epsilon X(I)$ and $(u_1, u_2), (v_1, v_2) \epsilon M_1(I) \times M_2(I)$.

Proof: Let $M_1(I)$ and $M_2(I)$ be two neutrosophic algebras over a neutrosophic field $X(I)$. In view of Theorem 3.5, $M_1(I)$ and $M_2(I)$ are linear spaces over a field $X(I)$.
This implies $M_1(I) \times M_2(I)$ is a linear space over a field $X(I)$.
Let $x = (u_1, u_2), y = (v_1, v_2), z = (w_1, w_2) \epsilon M_1(I) \times M_2(I)$.

Consider $x(yz) = (u_1, u_2)\big((v_1, v_2)(w_1, w_2)\big)$
$= (u_1, u_2)(v_1w_1, v_2w_2)$
$= \big(u_1(v_1w_1), u_2(v_2w_2)\big)$

$= ((u_1v_1)w_1, (u_2v_2)w_2)$
$= (u_1v_1, u_2v_2)(w_1, w_2)$
$= ((u_1v_1, u_2v_2))(w_1, w_2) = (xy)z$

This show $M_1(I) \times M_2(I)$ is a semi group under multiplication.

Now $x(y+z) = (u_1, u_2) \cdot [(v_1, v_2) + (w_1, w_2)]$
$= (u_1, u_2) \cdot [(v_1 + w_1, v_2 + w_2)]$
$= (u_1 \cdot (v_1 + w_1),\ u_2 \cdot (v_2 + w_2))$
$= (u_1 \cdot v_1 + u_1 \cdot w_1,\ u_2 \cdot v_2 + u_2 \cdot w_2)$
$= (u_1 \cdot v_1,\ u_2 \cdot v_2) + (u_1 \cdot w_1,\ u_2 \cdot w_2)$
$= (u_1, u_2) \cdot (v_1, v_2) + (u_1, u_2) \cdot (w_1, w_2)$
$= x \cdot y + x \cdot z$

Also $(x+y)z = (u_1 + v_1, u_2 + v_2) \cdot (w_1, w_2)$
$= ((u_1 + v_1) \cdot w_1, (u_2 + v_2) \cdot w_2)$
$= (u_1 \cdot w_1 + v_1 \cdot w_1,\ u_2 \cdot w_2 + v_2 \cdot w_2)$
$= (u_1 \cdot w_1,\ u_2 \cdot w_2) + (v_1 \cdot w_1, v_2 \cdot w_2)$
$= (u_1, u_2) \cdot (w_1, w_2) + (v_1, v_2) \cdot (w_1, w_2)$
$= x\,z + y\,z.$

Let $\alpha \epsilon X(I)$ and $x, y \epsilon M_1(I) \times M_2(I)$.

Consider $\alpha(xy) = \alpha[(u_1, u_2) \cdot (v_1, v_2)]$
$= \alpha(u_1 \cdot v_1,\quad u_2 \cdot v_2)$
$= (\alpha(u_1 \cdot v_1), \alpha(u_2 \cdot v_2))$
$= ((\alpha u_1) \cdot v_1, (\alpha u_2) \cdot v_2) = (\alpha u_1, \alpha u_2) \cdot (v_1, v_2)$
$= (\alpha(u_1, u_2)) \cdot (v_1, v_2) = (\alpha x)y$

Also we prove that $\alpha(xy) = x(\alpha y)$, for all $x, y \epsilon M_1(I) \times M_2(I)$, $\alpha \epsilon X(I)$.

Hence $M_1(I) \times M_2(I)$ is algebra over a neutrosophic field $X(I)$.

Theorem 3.8:- Let $W_1(I)$, $W_2(I)$, …………$W_n(I)$ be a neutrosophic algebra over a neutrosophic field $X(I)$. Then $W_1(I) \times W_2(I) \times \dots \dots W_n(I) = \{(u_1, u_2, \dots \dots u_n) : u_i \in W_i, for\ 1 \le i \le n\}$ is algebra over a neutrosophic field $X(I)$, where addition, multiplication and scalar multiplication defined as follows:

(i) $(u_1, u_2, \dots .. u_n) + (v_1, v_2, \dots .. v_n) = (u_1 + v_1, u_2 + v_2, \dots . u_n + v_n)$
(ii) $(u_1, u_2, \dots .. u_n)(v_1, v_2, \dots .. v_n) = (u_1v_1,\ u_2v_2, \dots\ \dots . u_nv_n)$
(iii) $\alpha(u_1, u_2, \dots .. u_n) = (\alpha u_1, \alpha u_2, \dots .. \alpha u_n)$.

Proof: Proof of this theorem is similar to theorem 3.7.

4. Ideal of neutrosophic algebra and neutrosophic quotient algebra

In this section we define the ideal of neutrosophic algebra and neutrosophic quotient algebra and studied their algebraic properties. Also we define Neutrosophic algebra homomorphism, Neutrosophic algebra isomorphism and kernel of neutrosophic algebra. We proved that every quotient neutrosophic algebra is quotient algebra.

Definition 4.1: A non-empty subset $W(I)$ of a neutrosophic algebra $Y(I)$ over a neutrosophic field $X(I)$ is an ideal of a neutrosophic algebra $Y(I)$ if

i. $W(I)$ is a subspace of a vector space $Y(I)$
ii. $\alpha u \epsilon W(I)$ for every $u \epsilon W(I), \alpha \epsilon X(I)$ and
iii. $v(u+\alpha) - vu \epsilon W(I)$ for every $u, v \epsilon W(I), \alpha \epsilon X(I)$.

If $W(I)$ satisfies (i) and (ii) then $W(I)$ is called a right ideal of neutrosophic algebra and if $W(I)$ satisfies (i) and (iii) then $W(I)$ is called a left ideal of neutrosophic algebra over a neutrosophic field$X(I)$.

Definition 4.2: Let $M_1(I)$ *and* $M_2(I)$ be two neutrosophic algebras over a neutrosophic field X(I). A mapping $\varphi: M_1(I) \rightarrow M_2(I)$ is called neutrosophic algebra homomorphism if the following conditions hold:

i. $\varphi(u+v) = \varphi(u) + \varphi(v)$
ii. $\varphi(uv) = \varphi(u)\,\varphi(v)$
iii. $\varphi(\alpha\, u) = \alpha\, \varphi(u)$
iv. $\varphi(I) = I$, for all u, v in $M_1(I)$, α in X(I) and I is a neutrosophic element of $M_1(I)$.

A neutrosophic algebra homomorphism φ is said to be neutrosophic algebra monomorphism if φ is injective.
A neutrosophic algebra homomorphism φ is said to be neutrosophic algebra epimorphism if φ is surjective.
A neutrosophic algebra homomorphism φ is said to be neutrosophic algebra isomorphism if φ is bijection. A bijective neutrosophic algebra homomorphism from $M_1(I)$ onto $M_1(I)$ is called a neutrosophic algebra automorphism.

Definition 4.3: Let $M_1(I)$and $M_2(I)$ be two neutrosophic algebras over a field $X(I)$. Let $\varphi: M_1(I) \to M_2(I)$ be a neutrosophic algebra homomorphism. Then the kernel of φ is denoted by Kerφ and is defined by $Ker\ \varphi = \{u \in M_1(I);\ \varphi(u) = 0'\}$, where $0' = 0 + 0I \epsilon M_2(I)$.

Definition 4.4: Let $M(I)$ be an ideal of neutosophic algebra $Y(I)$ over a field $X(I)$. Then the set of all co-sets of $M(I)$ in $Y(I)$ is denoted by $Y(I)\ /\ M(I)$ and defined by $Y(I)/M(I) = \{u + M(I):\ for\ every\ \ u\epsilon\, Y(I)\}$.

Addition, multiplication and scalar multiplication on $Y(I)\ /\ M(I)$ defined as

$$\big(a + M(I)\big) + \big(b + M(I)\big) = (a+b) + M(I)$$
$$\big(a + M(I)\big)\big(b + M(I)\big) = ab + M(I)$$

And $\alpha\big(a + M(I)\big) = \alpha a + M(I)$, $\forall \alpha \epsilon X(I), \big(a + M(I)\big), \big(b + M(I)\big) \epsilon Y(I)/M(I)$.
The set $Y(I)/M(I)$ form neutrosophic algebra over a neutrosophic field $X(I)$.
This neutrosophic algebra is called quotient neutrosophic algebra.

Theorem 4.5: Let $Y(I)$ be neutrosophic algebra over a neutrosophic field $X(I)$. The intersection of any collection of right ideals of neutrosophic algebra $Y(I)$ over a neutrosophic field $X(I)$ is a right ideal of $Y(I)$.

Proof: Let $\{W_\alpha(I)\}$ be the collection of right ideals of neutrosophic algebras $Y(I)$ over a neutrosophic field $X(I)$. Let $W(I) = \bigcap_\alpha W_\alpha(I)$ be their intersection. As $\bigcap_\alpha W_\alpha(I)$ is the collection of right ideals of $Y(I)$ over a neutrosophic field $X(I)$.This implies each $W_\alpha(I)$, for each α is a right ideal of $Y(I)$ over a field $X(I)$. For each α, $W_\alpha(I)$ is a subspace of a vector space $Y(I)$. $This\ implies\ that$ $\bigcap_\alpha W_\alpha(I)$ is a subspace of a vector space$Y(I)$.
Therefore $W(I)$ is sub-space of $Y(I)$.
Let $\alpha\epsilon X(I)$ and any $u\epsilon \bigcap_\alpha W_\alpha(I)$.

$\Rightarrow u\epsilon W_\alpha(I)$ for each α.

$\Rightarrow \alpha u\epsilon W_\alpha(I)$, for each $\alpha \Rightarrow \alpha u\ \epsilon\ \bigcap_\alpha W_\alpha(I)$.

Hence $\bigcap_\alpha W_\alpha(I)$ is a right ideal of $Y(I)$ over a neutrosophic field $X(I)$.

Theorem 4.6: Every quotient neutrosophic algebra over a neutrosophic field is quotient algebra.

Proof: Let $M(I)/U(I)$ be quotient algebra over a neutrosophic field $X(I)$.
For proving of $M(I)/U(I)$ is algebra, it is enough to show that the following.

(i) $M(I)/U(I)$ is a vector space over a field $X(I)$.
(ii) $M(I)/U(I)$ form a semigroup under multiplication
(iii) $x\cdot(y+z) = x\cdot y + x\cdot z$ and
$(x+y)\cdot z = x\cdot z + y\cdot z, \forall\, x, y \epsilon\, M(I)/U(I)$
(iv) $\alpha(xy) = (\alpha x)y = x(\alpha y)$, $\forall\ \alpha\epsilon\ X(I)\ and\ \ x, y\epsilon\, M(I)/U(I)$.

We have $M(I)/U(I) = \{m + U(I)/for\ every\ m\ \epsilon\ M(I)\}$. Since $M(I)/U(I)$ is quotient neutrosophic algebra then $U(I)$ is an ideal of a neutrosophic algebra $M(I)$. From the definition of ideal, it is clear that $U(I)$ is a subspace of a vector space $M(I)$ over $X(I)$. As we know every neutrosophic algebra is a vector space, so that $U(I)$ and $M(I)$ are vector spaces with $U(I)\ is\ a\ subset\ of\ M(I)$. Therefore $M(I)/U(I)$ is a vector space over a neutrosophic field $X(I)$.

Let $x = m_1 + U(I), y = m_2 + U(I), z = m_3 + U(I)$ are in $M(I)/U(I)$, where $m_1, m_2, m_3 \epsilon\, M(I)$.

Consider $x(yz) = (m_1 + U(I))(m_2 + U(I) \cdot m_3 + U(I))$

$= (m_1 + U(I))(m_2\, m_3 + U(I)) = m_1(m_2\, m_3) + U(I)$

$= (m_1\, m_2)m_3 + U(I)$ ($\because M(I)$ is a neutrosophic algebra)

$= (m_1\, m_2 + U(I)) \cdot (m_3 + U(I))$

$= ((m_1 + U(I))(m_2 + U(I)))(m_3 + U(I))$

$= (xy)z.$

Therefore $M(I)/U(I)$ form semigroup under multiplication.

Again consider $(x + y)z = [(m_1 + U(I)) + (m_2 + U(I))] \cdot (m_3 + U(I)).$

$= ((m_1 + m_2) + U(I)) \cdot (m_3 + U(I))$

$= (m_1 + m_2) \cdot m_3 + U(I)$

$= (m_1 \cdot m_3 + m_2 \cdot m_3) + U(I)$

$= (m_1 \cdot m_3 + U(I)) + (m_1 \cdot m_3 + U(I))$

$= (m_1 + U(I) \cdot m_3 + U(I)) + (m_2 + U(I) \cdot m_3 + U(I))$

$= x\, z + y\, z.$

Also let $X(I), x, y \epsilon\, M(I)/U(I)$.

Now $\alpha(xy) = \alpha(m_1 + U(I) \cdot m_2 + U(I))$

$= \alpha(m_1\, m_2 + U(I)) = \alpha(m_1\, m_2) + U(I)$

$= (\alpha m_1)\, m_2 + U(I)$

$= (\alpha m_1 + U(I))(m_2 + U(I))$

$= [\alpha(m_1 + U(I))](m_2 + U(I))$

$= (\alpha x)y.$

Also $x(\alpha y) = (m_1 + U(I))\left(\alpha(m_2 + U(I))\right)$

$= (m_1 + U(I))(\alpha m_2 + U(I))$

$= m_1(\alpha m_2) + U(I)$

$= \alpha(m_1\, m_2) + U(I)$

$= \alpha(m_1\, m_2 + U(I))$

$= \alpha[(m_1 + U(I))(m_2 + U(I))] = \alpha(xy)$

Hence complete the proof.

5. Conclusions

In this paper, we have introduced the notion of neutrosophic algebra, neutrosophic subalgebra, quotient neutrosophic algebra, homomorphism and isomorphism of neutrosophic algebra and ideals of neutrosophic algebra. We characterize some properties of neutrosophic algebra and proved that every quotient neutrosophic algebra is quotient algebra. Also we have proved that every neutrosophic algebra is algebra and direct product of neutrosophic algebras over a neutrosophic field is algebra. Several results and examples related to the neutrosophic algebra have been introduced. The concept of neutrosophic theory can be extend to near-algebra, Banach Algebra, C-algebra, Gamma near-algebra and near-modules.

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An MCDM Technique Using Cosine and Set-Theoretic Similarity Measures for Neutrosophic hypersoft set

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Abstract:

A similarity measure is used to tackle many issues that include indistinct and blurred information, excluding is not able to deal with the general fuzziness and obscurity of the problems that have various information. The neutrosophic hypersoft set is the most generalized and advanced extension of neutrosophic sets, which deals with the multi sub-attributes of the considered parameters. In this paper, we study some basic concepts which are helpful to build the structure of the article, such as soft set, neutrosophic soft set, hypersoft set, and neutrosophic hypersoft set, etc. The main objective of the present research is to develop a cosine similarity measure and set-theoretic similarity measure for an NHSS with their necessary properties. A decision-making approach has been established by using cosine and set-theoretic similarity measures. Furthermore, we used to develop a technique to solve multi-criteria decision-making problems. Finally, the advantages, effectiveness, flexibility, and comparative analysis of the algorithms are given with prevailing methods.

Keywords: Neutrosophic set; hypersoft set; neutrosophic hypersoft set; similarity measures

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1. Introduction

Decision-making is an interesting concern to pick the perfect alternate for any specific persistence. Firstly, it is pretended that details about alternatives are accumulated in crisp numbers, but in real-life situations, collective farm information always conquers wrong and inaccurate information. Fuzzy sets are like sets having an element of membership (Mem) degree. In classical set theory, the Mem degree of the elements in a set is examined in binary form to see that the element is not entirely concomitant. In contrast, the fuzzy set theory enables advanced Mem categorization of the components in the set. The Mem function portrays it, and also the multipurpose unit interval of the Mem function is [0, 1]. In some circumstances, decision-makers consider objects' Mem and nonmember-ship (Nmem) values. Zadeh's FS cannot handle imprecise and vague information in such cases. Atanassov [2] developed the notion of intuitionistic fuzzy sets (IFS) to deal above mentioned difficulties. The IFS accommodates the imprecise and inaccurate information using Mem and Nmem values.

Atanassov IFS was unable to solve those problems in which decision-makers considered the membership degree (MD) and nonmembership degrees (NMD) such as MD = 0.5 and NDM = 0.8, then $0.5 + 0.8 \nleq 1$. Yager [3, 4] extended the notion of IFS to Pythagorean fuzzy sets (PFSs) to overcome above-discussed complications by modifying $MD + NMD \leq 1$ to $MD^2 + NMD^2 \leq 1$. After developing PFSs, Zhang and Xu [5] proposed operational laws for PFSs and established a DM approach to resolving the MCDM problem. Wei and Lu [6] planned some power aggregation operators (AOs) and proposed a DM technique to solve multi-attribute decision-making (MADM) issues under the Pythagorean fuzzy environment. Wang and Li [7] presented power Bonferroni mean operators for PFSs with their basic properties using interaction. Gao et al. [8] presented several aggregation operators by considering the interaction and proposed a DM approach to solving MADM difficulties utilizing the developed operators. Wei [9] developed the interaction operational laws for Pythagorean fuzzy numbers (PFNs) by considering interaction and established interaction aggregation operators by using the developed interaction operations. Zhang [10] developed the accuracy function and presented a DM approach to solving multiple criteria group decision-making (MCGDM) problems using PFNs. Wang et al. [11] extended the PFSs and introduced an interactive Hamacher operation with some novel AOs. They also established a DM method to solve MADM problems using their proposed operators. Wang and Li [12] developed some interval-valued PFSs and utilized their operators to resolve multi-attribute group decision-making (MAGDM) issues. Peng and Yuan [13] established novel operators such as Pythagorean fuzzy point operators and developed a DM technique using their proposed operators. Peng and Yang [14] introduced fundamental operations and their necessary possessions under PFSs and planned DM methodology. Garg [15] developed the logarithmic operational laws for PFSs and proposed some AOs. Arora and Garg [16] presented the operational laws for linguistic IFS and developed prioritized AOs. Ma and Xu [17] presented some innovative AOs for PFSs and proposed the score and accuracy functions for PFNs.

Above mentioned theories and their DM methodologies have been used in several fields of life. But, these theories cannot deal with the parametrization of the alternatives. Molodtsov [18] developed soft sets (SS) to overcome the complications above. Molodtsov's SS competently deals with imprecise, vague, and unclear objects considering their parametrization. Maji et al. [19] prolonged the notion of SS and introduced some necessary operators with their properties. Maji et al. [20] established a DM technique using their developed operations for SS. They also merged two well-known theories, such as FS and SS, and established the concept of fuzzy soft sets (FSS) [21]. They also proposed an intuitionistic fuzzy soft set (IFSS) [22] and discussed their basic operations. Garg and Arora [23] extended the idea of IFSS and presented a generalized form of IFSS with AOs. They also planned a DM technique to resolve undefined and inaccurate information under IFSS information. Garg and Arora [24] presented the correlation and weighted correlation coefficients for IFSS and developed the TOPSIS approach utilizing established correlation procedures. Zulqarnain et al. [25] introduced some AOs and correlation coefficients for interval-valued IFSS. They also extended the TOPSIS technique using their developed correlation measures to solve the MADM problem. Peng et al. [26] proposed the Pythagorean fuzzy soft sets (PFSSs) and presented fundamental operations of PFSSs with their desirable properties by merging PFS and SS. Zulqarnain et al. [27-28] proposed the Einstein weighted ordered average and geometric operators for PFSSs. Zulqarnain et al. [29] introduced operational laws for Pythagorean fuzzy soft numbers (PFSNs) and developed AOs utilizing defined operational laws for PFSNs. They also planned a DM approach to solve MADM problems with the help of presented operators. Riaz et al. [30] prolonged the idea of PFSSs and developed the m polar PFSSs. They also established the TOPSIS method under the considered hybrid structure and proposed a DM methodology to solve the MCGDM problem. Siddique et al. [31] introduced the score matrix for PFSS and established a DM approach using their developed concept. Zulqarnain et al. [32-34] planned the TOPSIS methodology in the PFSS environment based on the correlation coefficient. They also proposed some AOs and interaction AOs for PFSS.

All the above studies only deal the inadequate information because of membership and non-membership values. However, these theories cannot handle the overall incompatible and imprecise data. To address such inconsistent and inaccurate records, the idea of the neutrosophic set (NS) was developed by Smarandache [35]. Maji [36] offered the perception of a neutrosophic soft set (NSS) with necessary operations. Broumi [37] developed the generalized NSS with some operations and properties and used the projected concept for DM. Deli and Subas [38] developed the single-valued Neutrosophic numbers (SVNNs) to solve MCDM problems. They also established the cut sets for SVNNs. Wang et al. [39] proposed the correlation coefficient (CC) for SVNSs. Ye [40] introduced the simplified NSs with operational laws and AOs. Also, he presented an MCDM technique utilizing his planned AOs. Zulqarnain et al. [41-42] offered the generalized neutrosophic TOPSIS and an integrated model for neutrosophic TOPSIS. They used their developed techniques for supplier selection and MCDM problems.

All the above studies have some limitations. When any attribute from a set of attributes contains further sub-attributes, then the above-presented theories fail to solve such problems. To overcome the limitations mentioned above, Smarandache [43] protracted the idea of SS to hypersoft sets (HSS) by substituting the one-parameter function f to a multi-parameter (sub-attribute) function. Samarandache claimed that the established HSS competently deals with uncertain objects compared to SS. Several researchers explored the HSS and presented a lot of extensions for HSS [44, 45]. Zulqarnain et al. [46] presented the IFHSS, the generalized version of IFSS. They established the TOPSIS method utilizing the developed correlation coefficient. Zulqarnain et al. [47] proposed the Pythagorean fuzzy hypersoft sets with AOs and correlation coefficients. They also established the TOPSIS technique using their developed correlation coefficient and utilized the presented approach to select appropriate anti-virus face masks. Zulqarnain et al. [48, 49] presented some fundamental operations with their properties for interval-valued NHSS. Also, they proposed the CC and WCC for interval-valued NHSS and established a decision-making approach utilizing their developed CC. Several researchers extended the notion of HSS and introduced different extensions of HSS with their DM methodologies [50-58]. However, all existing studies only deal with the scenario by using MD and NMD of sub-attributes of the considered attributes. If any decision-maker considers the MD = 0.7 and NDM = 0.6, then $0.7 + 0.6 \leq 1$ of any sub-attribute of the alternatives. We can observe that the theories mentioned above cannot handle it. To overwhelm the above boundaries, we proposed some AOs for PFHSS such as PFHSWA and PFHSWG operators by modifying the condition $\mathcal{T}_{\mathcal{F}(\check{d})}(\boldsymbol{\delta})$ + $\mathcal{J}_{\mathcal{F}(\check{d})}(\boldsymbol{\delta}) \leq 1$ to $\left(\mathcal{T}_{\mathcal{F}(\check{d})}(\boldsymbol{\delta})\right)^2 + \left(\mathcal{J}_{\mathcal{F}(\check{d})}(\boldsymbol{\delta})\right)^2 \leq 1$.. The essential objective of the following scientific research is to grow novel AOs for the PFHSS environment and processing mechanism, which can also follow the assumptions of PFHSNs. Furthermore, I developed an algorithm to explain the MCGDM problem and presented a numerical illustration to justify the effectiveness of the proposed approach under the PFHSS environment.

The following research is organized: In section 2, we recollected some basic definitions used in the subsequent sequel, such as NS, SS, NSS, HSS, and NHSS. Section 3 proposes the similarity measures such as cosine and set-theoretic for NHSS with its properties. We also introduced some operational laws for NHSS in the same section and established a decision-making technique to solve decision-making complications utilizing our developed similarity measures. In section 4, we use the proposed similarity measures for decision-making. A brief comparative analysis has been conducted between proposed techniques with existing methodologies in section 5. Finally, the conclusion and future directions are presented in section 6.

2. Preliminaries

The following section recalled fundamental concepts that helped us develop the current article's structure, such as SS, NS, NSS, HSS, FHSS, and NHSS.

Definition 2.1 [18]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathsf{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathsf{A})$ is called a soft set over $\mathcal{U}$, and its mapping is given as

$$\mathcal{F}:\mathsf{A} \to \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F},\mathsf{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \; if \; e \notin \mathsf{A}\}$$

Definition 2.2 [21] $\mathcal{U}$ and $\mathcal{E}$ be a universe of discourse and set of attributes respectively and $\mathcal{F}(\mathcal{U})$ be a power set of $\mathcal{U}$. Let $\mathcal{A} \subseteq \mathcal{E}$, then $(\mathcal{F}, \mathcal{A})$ is an FSS over $\mathcal{U}$, its mapping can be stated as follows: $\mathcal{F}$: $\mathcal{A} \to F(\mathcal{U})$

Definition 2.3 [35] Let $\boldsymbol{\mathcal{U}}$ be a universe and $\boldsymbol{\mathcal{A}}$ be an NS on $\boldsymbol{\mathcal{U}}$ is defined as $\boldsymbol{\mathcal{A}} = \{\boldsymbol{\delta}, (\boldsymbol{\sigma_{\mathcal{F}}(\delta)}, \boldsymbol{\tau_{\mathcal{F}}(\delta)}, \boldsymbol{\gamma_{\mathcal{F}}(\delta)}): \boldsymbol{\delta} \in \boldsymbol{\mathcal{U}}\}$, where $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, $\boldsymbol{\gamma}$: $\boldsymbol{\mathcal{U}} \to\ ]\mathbf{0}^-, \mathbf{1}^+[$ and $\mathbf{0}^- \leq \boldsymbol{\sigma_{\mathcal{F}}(\delta)} + \boldsymbol{\tau_{\mathcal{F}}(\delta)} + \boldsymbol{\gamma_{\mathcal{F}}(\delta)} \leq \mathbf{3}^+$.

Definition 2.4 [36] Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the Neutrosophic values of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a Neutrosophic soft set over $\mathcal{U}$ and its mapping is given as

$$\mathcal{F}: \mathcal{A} \to \mathcal{P}(\mathcal{U})$$

Definition 2.5 [43]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, ..., k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of multi-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}})$ is said to be HSS over $\mathcal{U}$, and its mapping is defined as

$\mathcal{F}$: $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \to \mathcal{P}(\mathcal{U})$.

It is also defined as

$(\mathcal{F}, \ddot{\mathsf{A}}) = \{\breve{a}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}): \breve{a} \in \ddot{\mathsf{A}}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) \in \mathcal{P}(\mathcal{U})\}$

Definition 2.6 [43] $\mathcal{U}$ be a universal set and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, ..., k_n\}$, $(n \geq 1)$ and K_i denoted the set of attributes and their corresponding sub-attributes like $K_i \cap K_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3 \ldots n\}$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, $\gamma \in \mathbb{N}$. Where $IFS^{\mathcal{U}}$ represents the intuitionistic fuzzy subsets of $\mathcal{U}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = (\mathcal{F}, \ddot{\mathsf{A}})$ is known as IFHSS defined as follows:

$$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \to IFS^{\mathcal{U}}.$$

It is also defined as

$(\mathcal{F}, \ddot{\mathsf{A}}) = \{(\breve{a}, \mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a})): \breve{a} \in \ddot{\mathsf{A}}, \mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a}) \in IFS^{\mathcal{U}} \in [0,1]\}$, where $\mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a}) = \{\langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle: \delta \in \mathcal{U}\}$, where $\sigma_{\mathcal{F}(\breve{a})}(\delta)$ and $\tau_{\mathcal{F}(\breve{a})}(\delta)$ signifies the Mem and NMem values of the attributes: $\sigma_{\mathcal{F}(\breve{a})}(\delta)$, $\tau_{\mathcal{F}(\breve{a})}(\delta) \in [0,1]$, and $0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta) + \tau_{\mathcal{F}(\breve{a})}(\delta) \leq 1$.

Definition 2.7 [47] Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, ..., k_n\}$, $(n \geq 1)$ and K_i represented the set of attributes and their corresponding sub-attributes such as $K_i \cap K_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3 \ldots n\}$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} =$

$\{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha, 1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in \mathbb{N}$. and $PFS^{\mathcal{U}}$ be a collection of all fuzzy subsets over $\mathcal{U}$. Then the pair ($\mathcal{F}$, $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ = ($\mathcal{F}$, Ä) is known as PFHSS defined as follows:

$\mathcal{F}$: $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ = Ä $\rightarrow PFS^{\mathcal{U}}$.

It is also defined as

$(\mathcal{F}, \ddot{A}) = \{(\breve{a}, \mathcal{F}_{\ddot{A}}(\breve{a})): \breve{a} \in \ddot{A}, \mathcal{F}_{\ddot{A}}(\breve{a}) \in PFS^{\mathcal{U}} \in [0,1]\}$, where $\mathcal{F}_{\ddot{A}}(\breve{a}) = \{\langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle: \delta \in \mathcal{U}\}$, where $\sigma_{\mathcal{F}(\breve{a})}(\delta)$ and $\tau_{\mathcal{F}(\breve{a})}(\delta)$ signifies the Mem and NMem values of the attributes:

$\sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta) \in [0,1]$, and $0 \leq \left(\sigma_{\mathcal{F}(\breve{a})}(\delta)\right)^2 + \left(\tau_{\mathcal{F}(\breve{a})}(\delta)\right)^2 \leq 1$.

A Pythagorean fuzzy hypersoft number (PFHSN) can be stated as $\mathcal{F} = \left\{\left(\sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right)\right\}$, where $0 \leq \left(\sigma_{\mathcal{F}(\breve{a})}(\delta)\right)^2 + \left(\tau_{\mathcal{F}(\breve{a})}(\delta)\right)^2 \leq 1$.

Definition 2.8 [43]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}$, (n ≥ 1) be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ = Ä = $\{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $NS^{\mathcal{U}}$ be a collection of all neutrosophic subsets over $\mathcal{U}$. Then the pair ($\mathcal{F}$, $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ = Ä) is said to be NHSS over $\mathcal{U}$, and its mapping is defined as

$\mathcal{F}$: $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ = Ä $\rightarrow NS^{\mathcal{U}}$.

It is also defined as

$(\mathcal{F}, \ddot{A}) = \{(\breve{a}, \mathcal{F}_{\ddot{A}}(\breve{a})): \breve{a} \in \ddot{A}, \mathcal{F}_{\ddot{A}}(\breve{a}) \in NS^{\mathcal{U}}\}$, where $\mathcal{F}_{\ddot{A}}(\breve{a}) = \{\langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta), \gamma_{\mathcal{F}(\breve{a})}(\delta)\rangle: \delta \in \mathcal{U}\}$, where $\sigma_{\mathcal{F}(\breve{a})}(\delta)$, $\tau_{\mathcal{F}(\breve{a})}(\delta)$, and $\gamma_{\mathcal{F}(\breve{a})}(\delta)$ represent the truth, indeterminacy, and falsity grades of the attributes such as $\sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta), \gamma_{\mathcal{F}(\breve{a})}(\delta) \in [0,1]$, and $0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta) + \tau_{\mathcal{F}(\breve{a})}(\delta) + \gamma_{\mathcal{F}(\breve{a})}(\delta) \leq 3$. Simply a neutrosophic hypersoft number (NHSN) can be expressed as $\mathcal{F} = \left\{\left(\sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta), \gamma_{\mathcal{F}(\breve{a})}(\delta)\right)\right\}$, where $0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta) + \tau_{\mathcal{F}(\breve{a})}(\delta) + \gamma_{\mathcal{F}(\breve{a})}(\delta) \leq 3$.

Example 2.7

Consider the universe of discourse $\mathcal{U} = \{\delta_1, \delta_2\}$ and $\mathfrak{L} = \{\ell_1 = Teaching\ methdology, \ell_2 = Subjects, \ell_3 = Classes\}$ be a collection of attributes with following their corresponding attribute values are given as teaching methodology = $L_1 = \{a_{11} = project\ base, a_{12} = class\ discussion\}$, Subjects = $L_2 = \{a_{21} = Mathematics, a_{22} = Computer\ Science, a_{23} = Statistics\}$, and Classes = $L_3 = \{a_{31} = Masters, a_{32} = Doctorol\}$. Let Ä = $L_1 \times L_2 \times L_3$ be a set of attributes

$\ddot{A} = L_1 \times L_2 \times L_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}, a_{23}\} \times \{a_{31}, a_{32}\}$

$$= \begin{Bmatrix} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}), (a_{11}, a_{23}, a_{31}), (a_{11}, a_{23}, a_{32}), \\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{21}, a_{32}), (a_{12}, a_{22}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{12}, a_{23}, a_{31}), (a_{12}, a_{23}, a_{32}), \end{Bmatrix}$$

$\ddot{A} = \{\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5, \breve{a}_6, \breve{a}_7, \breve{a}_8, \breve{a}_9, \breve{a}_{10}, \breve{a}_{11}, \breve{a}_{12}\}$

Then the NHSS over $\mathcal{U}$ is given as follows

$(\mathcal{F}, \ddot{A})$ =

$$\begin{Bmatrix} (\breve{a}_1, (\delta_1, (.6, .3, .8)), (\delta_2, (.9, .3, .5))), (\breve{a}_2, (\delta_1, (.5, .2, .7)), (\delta_2, (.7, .1, .5))), (\breve{a}_3, (\delta_1, (.5, .2, .8)), (\delta_2, (.4, .3, .4))), \\ (\breve{a}_4, (\delta_1, (.2, .5, .6)), (\delta_2, (.5, .1, .6))), (\breve{a}_5, (\delta_1, (.8, .4, .3)), (\delta_2, (.2, .3, .5))), (\breve{a}_6, (\delta_1, (.9, .6, .4)), (\delta_2, (.7, .6, .8))), \\ (\breve{a}_7, (\delta_1, (.6. .5, .3)), (\delta_2, (.4, .2, .8))), (\breve{a}_8, (\delta_1, (.8, .2, .5)), (\delta_2, (.6, .8, .4))), (\breve{a}_9, (\delta_1, (.7, .4, .9)), (\delta_2, (.7. .3, .5))), \\ (\breve{a}_{10}, (\delta_1, (.8, .4, .6)), (\delta_2, (.7, .2, .9))), (\breve{a}_{11}, (\delta_1, (.8, .4, .5)), (\delta_2, (.4, .2, .5))), (\breve{a}_5, (\delta_1, (.7, .5, .8)), (\delta_2, (.7, .5, .9))) \end{Bmatrix}$$

3. Similarity Measures and Their Decision-Making Approaches

Many mathematicians developed various methodologies to solve MCDM problems in the past few years, such as aggregation operators for different hybrid structures, CC, similarity measures, and decision-making applications. The following section develops the cosine and set-theoretic similarity measure for NHSS.

Definition 3.1

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs defined over a universe of discourse $\mathcal{U}$. Then, the then cosine similarity measure of $(\mathcal{F},\ddot{A})$ and $(\mathcal{G},\ddot{\mathbb{A}})$ can be described as follows:

$\mathcal{S}^1_{NHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{\mathbb{A}})) =$

$$\frac{1}{mn}\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)\right)}{\left(\sqrt{\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}\sqrt{\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)}\right)}$$

Proposition 3.2

Let $(\mathcal{F},\ddot{A})$, $(\mathcal{G},\ddot{\mathbb{A}})$, and $(\mathcal{H},\breve{C}) \in$ NHSS, then the following properties hold

1. $0 \leq \mathcal{S}^1_{NHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{\mathbb{A}})) \leq 1$
2. $\mathcal{S}^1_{NHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{\mathbb{A}})) = \mathcal{S}^1_{NHSS}((\mathcal{G},\ddot{\mathbb{A}}), (\mathcal{F},\ddot{A}))$
3. If $(\mathcal{F},\ddot{A}) \subseteq (\mathcal{G},\ddot{\mathbb{A}}) \subseteq (\mathcal{H},\breve{C})$, then $\mathcal{S}^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{H},\breve{C})) \leq \mathcal{S}^1_{NHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{\mathbb{A}}))$ and $\mathcal{S}^1_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{H},\breve{C})) \leq \mathcal{S}^1_{NHSS}((\mathcal{G},\ddot{\mathbb{A}}),(\mathcal{H},\breve{C}))$.

Proof: Using the above definition, the proof of these properties can be done easily.

Definition 3.3

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSSs defined over a universe of discourse $\mathcal{U}$. Then, the then set-theoretic similarity measure of $(\mathcal{F},\ddot{A})$ and $(\mathcal{G},\ddot{\mathbb{A}})$ can be described as follows:

$\mathcal{S}^2_{NHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{\mathbb{A}})) =$

$$\frac{1}{mn}\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)\right)}{max\left\{\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right),\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\right\}}$$

Proposition 3.4

Let $(\mathcal{F},\ddot{A})$, $(\mathcal{G},\ddot{\mathbb{A}})$, and $(\mathcal{H},\breve{C}) \in$ NHSS, then the following properties hold

1. $0 \leq \mathcal{S}^2_{NHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{\mathbb{A}})) \leq 1$
2. $\mathcal{S}^2_{NHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{\mathbb{A}})) = \mathcal{S}^2_{NHSS}((\mathcal{G},\ddot{\mathbb{A}}), (\mathcal{F},\ddot{A}))$
3. If $(\mathcal{F},\ddot{A}) \subseteq (\mathcal{G},\ddot{\mathbb{A}}) \subseteq (\mathcal{H},\breve{C})$, then $\mathcal{S}^2_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{H},\breve{C})) \leq \mathcal{S}^2_{NHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{\mathbb{A}}))$ and $\mathcal{S}^2_{NHSS}((\mathcal{F},\ddot{A}),(\mathcal{H},\breve{C})) \leq \mathcal{S}^2_{NHSS}((\mathcal{G},\ddot{\mathbb{A}}),(\mathcal{H},\breve{C}))$.

Proof: Using the above definition, the proof of these properties can be done easily.

3.5 Algorithm 1 for Similarity Measures of NHSS

Step 1. Pick out the set containing parameters.

Step 2. Construct the NHSS according to experts.

Step 3. Compute the cosine similarity measure by using definition 3.1.

Step 4. Compute the set-theoretic similarity measure for NHSS by utilizing definition 3.3.

Step 5. An alternative with a maximum value with cosine similarity measure has the maximum rank according to considered numerical illustration.

Step 6. An alternative with a maximum value with a set-theoretic similarity measure has the maximum rank according to considered numerical illustration.

Step 7. Analyze the ranking.

Definition 3.6

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$, $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$, and $(\mathcal{H},\check{C}) = \left\{\left(\delta_i, \sigma_{\mathcal{H}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{H}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{H}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be three NHSSs defined over a universe of discourse $\mathcal{U}$ when $\delta > 0$, then the following laws hold.

$$(\mathcal{F},\ddot{A}) \oplus (\mathcal{G},\ddot{\mathbb{A}}) = \left\langle \begin{matrix} \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i) + \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i) - \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \\ \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i) \end{matrix} \right\rangle$$

$$(\mathcal{F},\ddot{A}) \otimes (\mathcal{G},\ddot{\mathbb{A}}) = \left\langle \begin{matrix} \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i) + \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i) - \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \\ \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i) + \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i) - \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i) \end{matrix} \right\rangle$$

$$\delta(\mathcal{F},\ddot{A}) = \left\langle 1 - \left(1 - \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^{\delta}, \left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^{\delta}, \left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^{\delta} \right\rangle$$

$$((\boldsymbol{\mathcal{F}},\ddot{A}))^{\boldsymbol{\delta}} = \left\langle \left(\boldsymbol{\sigma}_{\boldsymbol{\mathcal{F}}(\breve{a}_k)}(\boldsymbol{\delta}_i)\right)^{\boldsymbol{\delta}}, \mathbf{1} - \left(\mathbf{1} - \boldsymbol{\tau}_{\boldsymbol{\mathcal{F}}(\breve{a}_k)}(\boldsymbol{\delta}_i)\right)^{\boldsymbol{\delta}}, \mathbf{1} - \left(\mathbf{1} - \boldsymbol{\gamma}_{\boldsymbol{\mathcal{F}}(\breve{a}_k)}(\boldsymbol{\delta}_i)\right)^{\boldsymbol{\delta}} \right\rangle.$$

Proposition 3.7

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$, $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$, and $(\mathcal{H},\check{C}) = \left\{\left(\delta_i, \sigma_{\mathcal{H}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{H}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{H}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be three NHSSs defined over a universe of discourse $\mathcal{U}$ and $\delta,\ \delta_1,\ \delta_2 > 0$, then the following laws hold

1. $(\mathcal{F},\ddot{A}) \oplus (\mathcal{G},\ddot{\mathbb{A}}) = (\mathcal{G},\ddot{\mathbb{A}}) \oplus (\mathcal{F},\ddot{A})$
2. $(\mathcal{F},\ddot{A}) \otimes (\mathcal{G},\ddot{\mathbb{A}}) = (\mathcal{G},\ddot{\mathbb{A}}) \otimes (\mathcal{F},\ddot{A})$
3. $\delta((\mathcal{F},\ddot{A}) \oplus (\mathcal{G},\ddot{\mathbb{A}})) = \delta(\mathcal{G},\ddot{\mathbb{A}}) \oplus \delta(\mathcal{F},\ddot{A})$
4. $((\mathcal{F},\ddot{A}) \otimes (\mathcal{G},\ddot{\mathbb{A}}))^{\delta} = ((\mathcal{F},\ddot{A}))^{\delta} \otimes ((\mathcal{G},\ddot{\mathbb{A}}))^{\delta}$
5. $\delta_1(\mathcal{F},\ddot{A}) \oplus \delta_2(\mathcal{F},\ddot{A}) = (\delta_1 \oplus \delta_2)(\mathcal{F},\ddot{A})$
6. $((\mathcal{F},\ddot{A}))^{\delta_1} \otimes ((\mathcal{F},\ddot{A}))^{\delta_2} = ((\mathcal{F},\ddot{A}))^{\delta_1+\delta_2}$
7. $((\mathcal{F},\ddot{A}) \oplus (\mathcal{G},\ddot{\mathbb{A}})) \oplus (\mathcal{H},\check{C}) = (\mathcal{F},\ddot{A}) \oplus \left((\mathcal{G},\ddot{\mathbb{A}}) \oplus (\mathcal{H},\check{C})\right)$
8. $((\mathcal{F},\ddot{A}) \otimes (\mathcal{G},\ddot{\mathbb{A}})) \otimes (\mathcal{H},\check{C}) = (\mathcal{F},\ddot{A}) \otimes \left((\mathcal{G},\ddot{\mathbb{A}}) \otimes (\mathcal{H},\check{C})\right)$

Proof. The proof of the above laws is straightforward by using definition 4.6.

Definition 3.8

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be a collection of NHSNs, Ω_i and γ_k are weight vector for expert's and parameters respectively with given conditions $\Omega_i > 0$, $\sum_{i=1}^{n} \Omega_i = 1$, $\gamma_k > 0$, $\sum_{k=1}^{m} \gamma_k = 1$, where $(i = 1, 2, \dots, n, and\ k = 1, 2, \dots, m)$. Then NHSWA operator defined as NHSWA: $\Delta^n \rightarrow \Delta$ defined as follows

$$NHSWA\left(\mathcal{F}_{\ddot{A}}(\delta_{11}), \mathcal{F}_{\ddot{A}}(\delta_{12}), \dots, \mathcal{F}_{\ddot{A}}(\delta_{nm})\right) = \oplus_{k=1}^{m} \gamma_j (\oplus_{i=1}^{n} \Omega_i \mathcal{F}_{\ddot{A}}(\delta_i)).$$

Proposition 3.9

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be a collection of NHSNs, the aggregated value is also an NHSN, such as

$$NHSWA\left(\mathcal{F}_{\ddot{A}}(\delta_{11}), \mathcal{F}_{\ddot{A}}(\delta_{12}), \dots, \mathcal{F}_{\ddot{A}}(\delta_{nm})\right)$$

$$= \left\langle \begin{matrix} \mathbf{1} - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\mathbf{1} - \boldsymbol{\sigma}_{\boldsymbol{\mathcal{F}}(\breve{a}_k)}(\boldsymbol{\delta}_i)\right)^{\Omega_i}\right)^{\gamma_k}, \mathbf{1} - \left(\mathbf{1} - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\mathbf{1} - \boldsymbol{\tau}_{\boldsymbol{\mathcal{F}}(\breve{a}_k)}(\boldsymbol{\delta}_i)\right)^{\Omega_i}\right)^{\gamma_k}\right), \\ \mathbf{1} - \left(\mathbf{1} - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\mathbf{1} - \boldsymbol{\gamma}_{\boldsymbol{\mathcal{F}}(\breve{a}_k)}(\boldsymbol{\delta}_i)\right)^{\Omega_i}\right)^{\gamma_k}\right) \end{matrix} \right\rangle$$

Definition 3.10

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be an NHSNs, then the score, accuracy, and certainty functions for NHSN respectively defined as follows:

$$\mathbb{S}((\mathcal{F},\ddot{A})) = \frac{1}{6m}\sum_{\alpha=1}^{m}\left(6 + \sigma^{\alpha}_{\mathcal{F}(\breve{a}_k)}(\delta_i) - \tau^{\alpha}_{\mathcal{F}(\breve{a}_k)}(\delta_i) - \gamma^{\alpha}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)$$

$$\mathbb{A}((\mathcal{F},\ddot{A})) = \frac{1}{4m}\left(4 + \sigma^{\alpha}_{\mathcal{F}(\breve{a}_k)}(\delta_i) - \gamma^{\alpha}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)$$

$$\mathbb{C}((\mathcal{F},\ddot{A})) = \frac{1}{2m}\left(2 + \sigma^{\alpha}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)$$

where $\alpha = 1, 2, \cdots, m$.

Definition 3.11

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$, and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two NHSNs. The comparison approach is present as follows:

1. If $\mathbb{S}(\mathcal{F},\ddot{A}) > \mathbb{S}(\mathcal{G},\ddot{\mathbb{A}})$, then $(\mathcal{F},\ddot{A})$ is superior to $(\mathcal{G},\ddot{\mathbb{A}})$.
2. If $\mathbb{S}(\mathcal{F},\ddot{A}) = \mathbb{S}(\mathcal{G},\ddot{\mathbb{A}})$ and $\mathbb{A}(\mathcal{F},\ddot{A}) > \mathbb{A}(\mathcal{G},\ddot{\mathbb{A}})$, then $(\mathcal{F},\ddot{A})$ is superior to $(\mathcal{G},\ddot{\mathbb{A}})$.
3. If $\mathbb{S}(\mathcal{F},\ddot{A}) = \mathbb{S}(\mathcal{G},\ddot{\mathbb{A}})$, $\mathbb{A}(\mathcal{F},\ddot{A}) = \mathbb{A}(\mathcal{G},\ddot{\mathbb{A}})$, and $\mathbb{C}(\mathcal{F},\ddot{A}) > \mathbb{C}(\mathcal{G},\ddot{\mathbb{A}})$, then $(\mathcal{F},\ddot{A})$ is superior to $(\mathcal{G},\ddot{\mathbb{A}})$.
4. If $\mathbb{S}(\mathcal{F},\ddot{A}) = \mathbb{S}(\mathcal{G},\ddot{\mathbb{A}})$, $\mathbb{A}(\mathcal{F},\ddot{A}) > \mathbb{A}(\mathcal{G},\ddot{\mathbb{A}})$, and $\mathbb{C}(\mathcal{F},\ddot{A}) = \mathbb{C}(\mathcal{G},\ddot{\mathbb{A}})$, then $(\mathcal{F},\ddot{A})$ is indifferent to $(\mathcal{G},\ddot{\mathbb{A}})$, can be denoted as $(\mathcal{F},\ddot{A}) \sim (\mathcal{G},\ddot{\mathbb{A}})$.

4. Application of Similarity Measures in Decision Making

In this section, we proposed the algorithm for NHSS by using developed similarity measures. We also used the proposed methods for decision-making in real-life problems.

4.1. Problem Formulation and Application of NHSS For Decision Making

A construction company calls for the appointment of a civil engineer to supervise the workers. Several engineers apply for the civil engineer post, simply four engineers call for an interview based on experience for undervaluation such as $S = \{S_1, S_2, S_3, S_4\}$ be a set of selected engineers call for the interview. The managing director hires a committee of four experts $\mathcal{U} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}\}$ for the selection of civil engineer. The group of experts chooses the set of attributes for the selection of an appropriate civil engineer such as $\mathfrak{L} = \{\ell_1 = personality, \ell_2 = communication\ skills, \ell_3 = qualification\}$ with their corresponding sub-attribute: personality $= \ell_1 = \{d_{11} = \text{attractive}\}$, communication skills $= \ell_2 = \{d_{21} = \text{normal}, d_{22} = \text{excellent}\}$, and qualification $= \ell_3 = \{d_{31} = \text{masters}, d_{32} = \text{doctor}\}$. The experts evaluate the applicants under defined parameters and forward the evaluation performa to the company's managing director. Finally, the director scrutinizes the best applicant based on the expert's evaluation report.

4.1.1. Application of NHSS For Decision Making

Let $S = \{S_1, S_2, S_3, S_4\}$ be a set of civil engineers who are shortlisted for interviews (alternatives) such as. The managing director hires a team of four experts such as $\mathcal{U} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$. The group of experts chooses the set of attributes for the selection of an appropriate civil engineer such as $\mathfrak{L}$ = $\{\ell_1 = personality, \ell_2 = communication\ skills, \ell_3 = qualification\}$ with their corresponding sub-attribute: personality = ℓ_1 = $\{d_{11} = \text{attractive}\}$, communication skills = ℓ_2 = $\{d_{21} = \text{normal}, d_{22} = \text{excellent}\}$, and qualification = ℓ_3 = $\{d_{31} = \text{masters}, d_{32} = \text{doctor}\}$. Let $\mathfrak{L}'$ = $\ell_1 \times \ell_2 \times \ell_3$ shows the multi sub-attributes

$\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3 = \{d_{11}\} \times \{d_{21}, d_{22}\} \times \{d_{31}, d_{32}\}$
$= \{(d_{11}, d_{21}, d_{31}),\ (d_{11}, d_{22}, d_{31}), (d_{11}, d_{21}, d_{32}), (d_{11}, d_{22}, d_{32})\}$,

$\mathfrak{L}' = \{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4\}$ with weights $(0.2, 0.1, 0.4, 0.3)^T$. Experts' opinion in the form of NHSNs following multi sub-attributes of considered attributes.

Step 2. Construct the NHSS according to experts.

Table 1. Construction of NHSS of all Applicants According to Company Requirement

S	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
κ_1	(0.7,0.2,0.4)	(0.4, 0.3, 0.7)	(0.9, 0.7, 0.4)	(0.6,0.3,0.7)
κ_2	(0.5, 0.6,0.2)	(0.8,0.5, 0.6)	(0.8, 0.2, 0.5)	(0.7, 0.5, 0.9)
κ_3	(0.6,0.6,0.2)	(0.5,0.8,0.3)	(0.4, 0.7, 0.3)	(0.6, 0.7, 0.4)
κ_4	(0.8, 0.7, 0.5)	(0.2,0.4,0.9)	(0.7, 0.5, 0.1)	(0.6,0.8,0.2)

Now we will construct the NHSS S_t according to four experts, where t = 1, 2, 3, 4.

Table 2. Decision Matrix for alternative S_1

S_1	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
κ_1	(0.9,0.2,0.1)	(0.3, 0.3, 0.7)	(0.6, 0.4, 0.2)	(0.7,0.1,0.3)
κ_2	(0.8, 0.3,0.2)	(0.6,0.2, 0.6)	(0.8, 0.3, 0.1)	(0.2, 0.6, 0.8)
κ_3	(0.6,0.1,0.3)	(0.6,0.1,0.3)	(0.8, 0.2, 0.1)	(0.6, 0.3, 0.4)
κ_4	(0.9, 0.1, 0.1)	(0.9,0.1,0.1)	(0.8, 0.1, 0.1)	(0.9,0.1,0.2)

Table 3. Decision Matrix for alternative S_2

S_2	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
κ_1	(0.3,0.3,0.7)	(0.9, 0.2, 0.1)	(0.6, 0.1, 0.3)	(0.3,0.6,0.2)
κ_2	(0.8, 0.2,0.1)	(0.8,0.3, 0.2)	(0.9, 0.1, 0.1)	(0.8, 0.3, 0.1)
κ_3	(0.6,0.3,0.4)	(0.8,0.1,0.2)	(0.9, 0.1, 0.1)	(0.2, 0.3, 0.8)
κ_4	(0.9, 0.1, 0.2)	(0.8,0.1,0.1)	(0.7, 0.1, 0.3)	(0.6,0.3,0.4)

Table 4. Decision Matrix for alternative S_3

S_3	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
κ_1	(0.6,0.3,0.4)	(0.2, 0.3, 0.8)	(0.3, 0.6, 0.2)	(0.3,0.6,0.2)
κ_2	(0.9, 0.1,0.1)	(0.9,0.1, 0.1)	(0.9, 0.1, 0.1)	(0.8, 0.3, 0.2)
κ_3	(0.8,0.3,0.2)	(0.9,0.2,0.1)	(0.9, 0.1, 0.1)	(0.2, 0.3, 0.8)
κ_4	(0.3, 0.3, 0.7)	(0.9,0.1,0.2)	(0.7, 0.1, 0.3)	(0.6,0.3,0.4)

Table 5. Decision Matrix for alternative S_4

S_4	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
κ_1	(0.9,0.1,0.1)	(0.9, 0.1, 0.2)	(0.8, 0.2, 0.1)	(0.3,0.6,0.2)
κ_2	(0.8, 0.2,0.1)	(0.8,0.2, 0.1)	(0.6, 0.3, 0.4)	(0.8, 0.3, 0.2)
κ_3	(0.8,0.1,0.1)	(0.8,0.1,0.2)	(0.9, 0.1, 0.1)	(0.3, 0.6, 0.2)
κ_4	(0.9, 0.1, 0.2)	(0.3,0.3,0.7)	(0.8, 0.3, 0.2)	(0.9,0.1,0.1)

Step 3. Compute the cosine similarity measure by using definition 3.1.

By using Tables 1-5, compute the cosine similarity measure between $\mathcal{S}^1_{NHSS}(S,S_1)$, $\mathcal{S}^1_{NHSS}(S,S_2)$, $\mathcal{S}^1_{NHSS}(S,S_3)$, and $\mathcal{S}^1_{NHSS}(S,S_t)$ by using equation 3.1, such as

$$\mathcal{S}^1_{NHSS}(S, S_1) = \frac{1}{3\times 4}\left\{\frac{(.8)(.3)+(.5)(.5)+(.6)(.2)}{\sqrt{(.8)^2+(.5)^2+(.6)^2}\sqrt{(.3)^2+(.5)^2+(.2)^2}} + \frac{(.5)(.8)+(.4)(.7)+(.2)(.3)}{\sqrt{(.5)^2+(.4)^2+(.2)^2}\sqrt{(.8)^2+(.7)^2+(.3)^2}} + \cdots + \frac{(.4)(.7)+(.7)(.7)+(.6)(.9)}{\sqrt{(.4)^2+(.7)^2+(.6)^2}\sqrt{(.7)^2+(.7)^2+(.9)^2}}\right\} = \frac{1}{12}\left(\frac{28.99}{34.4799}\right) = 0.07007.$$

Similarly, we can find the cosine similarity measure between $\mathcal{S}^1_{NHSS}(S,S_2)$, $\mathcal{S}^1_{NHSS}(S,S_3)$, and $\mathcal{S}^1_{NHSS}(S,S_4)$ given as

$\mathcal{S}^1_{NHSS}(S,S_2) = \frac{1}{12}\left(\frac{26.32}{32.3767}\right) = 0.06771$, $\mathcal{S}^1_{NHSS}(S,S_3) = \frac{1}{12}\left(\frac{25.4}{29.4056}\right) = 0.06943$, and $\mathcal{S}^1_{NHSS}(S,S_4) = \frac{1}{12}\left(\frac{25.48}{30.88764}\right) = 0.06874$. This shows that $\mathcal{S}^1_{NHSS}(S,S_1) > \mathcal{S}^1_{NHSS}(S,S_3) > \mathcal{S}^1_{NHSS}(S,S_4) > \mathcal{S}^1_{NHSS}(S,S_2)$. It can be seen from this ranking alternative S_1 is most relevant and similar to S. Therefore S_1 is the best alternative for the civil engineer, the ranking of other alternatives given as $S_1 > S_3 > S_4 > S_2$.

Now we compute the set-theoretic similarity measure by using Definition 4.3 between $\mathcal{S}^2_{NHSS}(S,S_1)$, $\mathcal{S}^2_{NHSS}(S,S_2)$, $\mathcal{S}^2_{NHSS}(S,S_3)$, and $\mathcal{S}^2_{NHSS}(S,S_4)$ given as From Tables 1-5, we can find the set-theoretic similarity measure for each alternative by using definition 4.3 given as $\mathcal{S}^2_{NHSS}(S,S_1) = 0.06986$, $\mathcal{S}^2_{NHSS}(S,S_2) = 0.06379$, $\mathcal{S}^2_{NHSS}(S,S_3) = 0.06157$, and $\mathcal{S}^2_{NHSS}(S,S_4) = 0.06176$. $\mathcal{S}^1_{NHSS}(S,S_1) > \mathcal{S}^1_{NHSS}(S,S_2) > \mathcal{S}^1_{NHSS}(S,S_4) > \mathcal{S}^1_{NHSS}(S,S_3)$. It can be seen from this ranking alternative S_1 is most relevant and similar to S. Therefore S_1 is the best alternative for the civil engineer, the ranking of other alternatives given as $S_1 > S_2 > S_4 > S_3$.

5. Discussion and Comparative Analysis

In the subsequent section, we will talk over the usefulness, easiness, manageability, and assistance of the planned method. We also performed an ephemeral evaluation of the undermentioned: the planned technique along with some prevailing methodologies.

5.1. Superiority of the Proposed Approach

Through this study and comparison, it could be determined that the consequences acquired by the suggested approach have been more common than either available method. Overall, the DM procedure associated with the prevailing DM methods accommodates extra information to address hesitation. Also, FS's various hybrid structures are becoming a particular feature of NHSS, along with some appropriate circumstances added. The general info associated with the object could be stated precisely and analytically, see Table 6. Therefore, it is a suitable technique to syndicate inaccurate and ambiguous information in the DM process. Hence, the suggested approach is practical, modest, and in advance of fuzzy sets' distinctive hybrid structures.

Table 6. Comparison between NHSS and some existing techniques

	Set	Truthiness	Indeterminacy	Falsity	Parametrization	Attributes	Sub-attributes
Zadeh [1]	FS	✓	×	×	×	✓	×
Atanassov [2]	IFS	✓	×	✓	×	✓	×
Smarandache [35]	NS	✓	✓	✓	×	✓	×
Maji et al. [21]	FSS	✓	×	×	✓	✓	×
Maji et al. [22]	IFSS	✓	×	✓	✓	✓	×
Peng et al. [26]	PFSS	✓	×	✓	✓	✓	×
Maji [36]	NSS	✓	✓	✓	✓	✓	×
Zulqarnain et al. [46]	IFHSS	✓	×	✓	✓	✓	✓

Zulqarnain et al. [47]	PFHSS	✓	×	✓	✓	✓	✓
Proposed approach	NHSS	✓	✓	✓	✓	✓	✓

It turns out that this is a contemporary issue. Why do we have to embody novel algorithms based on the proposed novel structure? Many indications compared with other existing methods; the recommended method may be an exception. We remember the following fact: the mixed structure limits IFS, picture fuzzy sets, FS, hesitation fuzzy sets, NS, and other fuzzy sets and cannot provide complete information about the situation. But our m-polar model GmPNSS can deal with truthiness, indeterminacy, and falsity, so it is most suitable for MCDM. Due to the exaggerated multipolar neutrosophy, these three degrees are independent and provide a lot of information about alternative norms. Other similarity measures of available hybrid structures are converted into exceptional cases of GmPNSS. A comparative analysis of some already existing techniques is listed above in Table 6. Therefore, this model has more versatility and can efficiently resolve complications than intuitionistic, neutrosophic, hesitant, image, and ambiguity substitution. The similarity measure established for GmPNSS becomes better than the existing similarity measure for MCDM.

5.2. Comparative Analysis

In the following section, we recommend another algorithmic rule under NHSS by utilizing the progressed cosine similarity measure and set-theoretic similarity measure. Subsequently, we use the suggested algorithm to a realistic problem, namely the appropriate civil engineer in a company. The overall outcomes prove that the algorithmic rule is valuable and practical. It can be observed that S_1 is the most acceptable alternative for the civil engineer position. The recommended approach may be compared to other available methods. From the research findings, it has been concluded that the outcomes acquired by the planned approach exceed the consequences of the prevailing ideas. Therefore, compared to existing techniques, the established similarity measures handled the uncertain and ambiguous information competently. However, under current DM strategies, the core advantage of the planned method is that it can accommodate extra info in data comparative to existing techniques. It is also a beneficial tool to solve inaccurate and imprecise information in DM procedures. The benefit of the planned approach and related measures over present methods is evading conclusions grounded on adverse reasons.

5.3. Discussion

Zadeh's [1] FS handled the inaccurate and imprecise information using MD of sub-attributes of considered attributes for each alternative. But the FS has no evidence around the NMD of the considered parameters. Atanassov's [2] IFS accommodates the unclear and undefined objects using MD and NMD. However, IFS cannot handle the circumstances when MD + NMD ≥ 1, conversely, is presented notion competently deals with such difficulties. Meanwhile, these theories have no information about the indeterminacy of the attributes. To overcome such problems, Smarandache [35] proposed the idea of NS. Maji et al. [21] presented the notion of FSS to deal with the parametrization of the objects, which contains uncertainty by considering the MD of the attributes. But, the presented FSS provides no information about the NMD of the object. To overcome the presented drawback, Maji et al. [22] offered the concept of IFSS. The proposed notion handles the uncertain object more accurately by using the MD and NMD of the attributes with their parametrization. The sum of MD and NMD does not exceed 1. To handle this scenario, Peng et al. [26] proposed the notion of PFSS by modifying the condition $MD + NMD \leq 1$ to $MD^2 + NMD^2 \leq 1$ with their parametrization. The PFSS is unable to deal with the indeterminacy of the attributes. Maji

[36] introduced the concept of NSS, in which decision-makers competently solve the DM problems comparative to the above-studied theories using truthiness, falsity, and indeterminacy of the object. But all the studies mentioned above have no information about the sub-attributes of the considered attributes. So the theories discussed above cannot handle the scenario when attributes have their corresponding sub-attributes. Utilizing the MD and NMD, Zulqarnain et al. [46] extended the IFSS to IFHSS and proposed the CC and WCC for IFHSS in which $MD + NMD \leq 1$ for each sub-attribute. But IFHSS cannot provide any information on the NMem values of the sub-attribute of the considered attribute. Zulqarnain et al. [47] proposed the more generalized notion of PFHSS comparative to IFHSS. The PFHSS accommodates more uncertainty compared to IFHSS by updating the condition $MD + NMD \leq 1$ to $\left(\sigma_{\mathcal{F}(\breve{\alpha})}(\delta)\right)^2 + \left(\tau_{\mathcal{F}(\breve{\alpha})}(\delta)\right)^2 \leq 1$. All existing hybrid structures of FS cannot handle the indeterminacy of sub-attributes of considered n-tuple attributes. On the other hand, developed aggregation operators can accommodate the sub-attributes of considered attributes using truthness, indeterminacy, and falsity objects of sub-attributes with the following condition $0 \leq \sigma_{\mathcal{F}(\breve{\alpha})}(\delta)$, $\tau_{\mathcal{F}(\breve{\alpha})}(\delta)$, $\gamma_{\mathcal{F}(\breve{\alpha})}(\delta) \leq 3$. It may be seen that the best selection of the suggested approach is to resemble the verbalized own method, and that ensures the liableness along with the effectiveness of the recommended approach.

6. Conclusion

This paper studies some basic concepts such as soft set, NSS, HSS, IFHSS, PFHSS, and NHSS. We developed the idea of cosine similarity measure and set-theoretic similarity measure for NHSS and described their desirable properties. Some operational laws have been established for NHSS. The concept of score function, accuracy function, and certainty function is developed to compare NHSNs. Furthermore, a decision-making approach has been developed for NHSS based on the proposed technique. To verify the effectiveness of our developed techniques, we presented an illustration to solve MCDM problems. We presented a comprehensive comparative analysis of proposed techniques with existing methods. In the future, the concept of NHSS will be extended to interval-valued NHSS. It will solve different real-life problems such as medical diagnoses, decision-making, etc.

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Cosine and Set-Theoretic Similarity Measures for Generalized Multi-Polar Neutrosophic Soft Set with Their Application in Decision Making

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Abstract:

A similarity measure and correlation coefficients are used to tackle many issues that include indistinct and blurred information, excluding is not in a position to deal with the general fuzziness and obscurity of the various problems. In this paper, we study some basic concepts which are helpful to build the structure of the article, such as soft set, neutrosophic soft set, and generalized m-polar neutrosophic soft set. The main objective of this paper is to develop the cosine and set-theoretic similarity measures for the generalized multipolar neutrosophic soft set (GmPNSS). We discuss some basic operations with their properties for GmPNSS. A decision-making approach has been established by using cosine and set-theoretic similarity measures. Also, we introduce the multipolar neutrosophic soft weighted average (mPNSWA) operator and develop a decision-making approach based on mPNSWA. Furthermore, we use to develop techniques to solve multi-criteria decision-making problems. Finally, the advantages, effectiveness, flexibility, and comparative analysis of the algorithms are given with prevailing methods.

Keywords: Neutrosophic set; multipolar neutrosophic set; neutrosophic soft set; multipolar neutrosophic soft set

1. Introduction

Uncertainty plays a dynamic role in many areas of life (such as modeling, medicine, engineering, etc.). However, people have raised a general question: how do we express and use the concept of uncertainty in mathematical modeling. Many researchers have proposed and recommended different methods of using uncertainty theory. First of all, Zadeh proposed the concept of fuzzy sets [1] to solve those problems that contain uncertainty and ambiguity. It can be seen that in some cases, fuzzy sets cannot handle the situation. To overcome such situations, Turksen [2] proposed the idea of interval-valued fuzzy sets (IVFS). In some cases, we must consider the appropriate representation of the object under the conditions of uncertainty and uncertainty and regard its unbiased value as the fair value of the proper representation of the object, which these fuzzy sets or IVFS cannot process. To overcome these difficulties, Atanassov proposed the Intuitionistic Fuzzy Set (IFS) [3]. The theory proposed by Atanassov only deals with insufficient data considerations and membership and non-member values. However, the IFS theory cannot deal with overall incompatibility and imprecise information. To solve such incompatible and inaccurate records, Smarandache [4] proposed the idea of the neutrosophic set (NS).

A general mathematical tool was proposed by Molodtsov [5] to deal with indeterminate, fuzzy, and not clearly defined substances known as a soft set (SS). Maji et al. [6] extended the work on SS and described some operations and properties. They also used the SS theory for decision-making [7]. Ali et al. [8] revised the Maji approach to SS and developed new operations with their properties. De Morgan's Law on SS theory was proved [9] by using different operators. Cagman and Enginoglu [10] developed the concept of soft matrices with operations and discussed their properties. They also introduced a decision-making method to resolve those problems which contain uncertainty and revised the operations proposed by Molodtsov's SS [11]. In [12], the author's planned some new operations on soft matrices like soft difference product, soft restricted difference product, soft extended difference product, and soft weak-extended difference product with their properties.

Maji [13] offered the idea of a neutrosophic soft set (NSS) with necessary operations and properties. The concept of the possibility NSS was developed by Karaaslan [14] and introduced a neutrosophic soft decision-making method to solve those problems that contain uncertainty based on And-product. Broumi [15] developed the generalized NSS with some operations and properties and used the proposed decision-making concept. To solve MCDM problems with single-valued Neutrosophic numbers (SVNNs) presented by Deli and Subas in [16], they constructed the concept of cut sets of SVNNs. Based on the correlation of IFS, the term CC of SVNSs [17] was introduced. In [18], simplified NSs introduced with some operational laws and aggregation operators such as weighted arithmetic and weighted geometric average operators. They constructed an MCDM method on the base of proposed aggregation operators. Masooma et al. [19] progressed a new concept by combining the multipolar fuzzy set and neutrosophic set, known as the multipolar neutrosophic set. They also established various characterization and operations with examples.

Zulqarnain et al. [20-21] proposed the Einstein weighted ordered average and geometric operators for PFSSs. Zulqarnain et al. [22] introduced operational laws for Pythagorean fuzzy soft numbers (PFSNs) and developed AOs utilizing defined operational laws for PFSNs. They also planned a DM approach to solve MADM problems with the help of presented operators. Riaz et al. [23] prolonged the idea of PFSSs and developed the m polar PFSSs. They also established the TOPSIS method under the considered hybrid structure and proposed a DM methodology to solve the MCGDM problem. Siddique et al. [24] introduced the score matrix for PFSS and established a DM approach using their developed concept. Zulqarnain et al. [25-27] planned the TOPSIS methodology in the PFSS environment based on the correlation coefficient. They also proposed some AOs and interaction AOs for PFSS. Basset et al. [28] applied TODIM and TOPSIS methods under the best-worst approach to raising the overall efficiency of rating beneath uncertainty according to the NS. They also utilized plithogenic set theory to resolve the unsure info and assess the overall commercial enterprise world premiere of manufacturing industries. They utilized the AHP approach to come across the weight vector of your business enterprise concentrations to gain that destination afterward; they had to use VIKOR and TOPSIS methods to utilize the firm's ranking [29].

The authors established the probability multi-valued neutrosophic set by combining the multi-valued neutrosophic set and probability distribution to solve decision-making issues [30]. Kamal et al. [31] proposed the idea of mPNSS with some significant operations and properties. They also used the developed technique for decision-making. Saeed et al. [32] established the concept of mPNSS with its properties and operators. They also developed the distance-based similarity measures and used the proposed similarity measures for decision-making and medical diagnoses. In [33], the authors established the concept of mPNSS with its properties and operators. They also developed the distance-based similarity measures and used the proposed similarity measures for decision-making and medical diagnoses. Zulqarnain et al. [34-35] offered the generalized neutrosophic TOPSIS and an integrated model for neutrosophic TOPSIS. They used their developed techniques for supplier selection and MCDM problems.

In this epoch, experts consider that real life will be moving toward multi-polarization. Thus, there is no doubt that the multi-polarization of information must have managed to succeed in the prosperity of many science and engineering science fields. In information technology, multi-polar technology can be utilized to manipulate several structures. The motivation for expanding and mixing this research work is gradually given in the entire manuscript. We demonstrate that under any appropriate circumstances, different hybrid structures containing fuzzy sets will be converted into the unique privilege of GmPNSS. The multipolar neutrosophic environment is novel to the concept of neutrosophic soft sets of multipolar values. We discuss the effectiveness, flexibility, quality, and advantages of planning work and algorithms. This research will be the most versatile form and will integrate data with appropriate medicine, engineering, artificial intelligence, agriculture, and other daily complications. Current work may apply to other methods and different types of hybrid structures in the future.

The following research is organized: In section 2, we recollected some basic definitions used in the subsequent sequel, such as NS, SS, NSS, and multipolar neutrosophic set. In section 3, we propose the GmPNSS with its properties and operations. Section 4 establishes two different types of similarity measures such as cosine and set-theoretic similarity with their decision-making approaches and graphically representation. We also introduce some operational laws and mPNSWA operators with its decision-making technique based on GmPNSS. Section 5 uses the developed similarity measures and mPNSWA operator for decision-making. A brief comparative analysis has been conducted between proposed methods with existing methodologies in section 6. Finally, the conclusion and future directions are presented in section 7.

2. Preliminaries

In this section, we recollect some basic concepts such as the neutrosophic set, soft set, neutrosophic soft set, and m-polar neutrosophic soft set used in the following sequel.

Definition 2.1 [4]

Let $\mathcal{U}$ be a universe, and $\mathcal{A}$ be an NS on $\mathcal{U}$ is defined as $\mathcal{A} = \{< u, \mathcal{u}_{\mathcal{A}}(u), \mathcal{v}_{\mathcal{A}}(u), \mathcal{w}_{\mathcal{A}}(u) > : u \in \mathcal{U}\}$, where $\mathcal{u}$, $\mathcal{v}$, $\mathcal{w}$: $\mathcal{U} \rightarrow\ ]0^-,\ 1^+[$ and $0^- \leq \mathcal{u}_{\mathcal{A}}(u) + \mathcal{v}_{\mathcal{A}}(u) + \mathcal{w}_{\mathcal{A}}(u) \leq 3^+$.

Definition 2.2 [5]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a soft set over $\mathcal{U}$, and its mapping is given as

$$\mathcal{F}: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, \mathcal{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset\ if\ e \neq \mathcal{A}\}$$

Definition 2.3 [13]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the Neutrosophic values of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a Neutrosophic soft set over $\mathcal{U}$ and its mapping is given as

$$\mathcal{F}: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{U})$$

Definition 2.4 [32]

Let $\boldsymbol{\mathcal{U}}$ be a universe of discourse and $\boldsymbol{\mathcal{E}}$ be a set of attributes, and m-polar neutrosophic soft set (mPNSS) $\wp_{\mathfrak{R}}$ over $\boldsymbol{\mathcal{U}}$ defined as

$\wp_{\mathfrak{R}} = \{(< \boldsymbol{e}, \{(\boldsymbol{u}, \boldsymbol{u}_{\alpha}(\boldsymbol{u}), \boldsymbol{v}_{\alpha}(\boldsymbol{u}), \boldsymbol{w}_{\alpha}(\boldsymbol{u})): \boldsymbol{u} \in \boldsymbol{\mathcal{U}}, \boldsymbol{\alpha} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \boldsymbol{m}\}) > : \boldsymbol{e} \in \boldsymbol{\mathcal{E}}\}$,

where $\boldsymbol{u}_{\alpha}(\boldsymbol{u})$, $\boldsymbol{v}_{\alpha}(\boldsymbol{u})$, and $\boldsymbol{w}_{\alpha}(\boldsymbol{u})$ represents the truthiness, indeterminacy, and falsity, respectively, $\boldsymbol{u}_{\alpha}(\boldsymbol{u}), \boldsymbol{v}_{\alpha}(\boldsymbol{u}), \boldsymbol{w}_{\alpha}(\boldsymbol{u}) \subseteq [\mathbf{0}, \mathbf{1}]$ and $0 \leq \boldsymbol{u}_{\alpha}(\boldsymbol{u}) + \boldsymbol{v}_{\alpha}(\boldsymbol{u}) + \boldsymbol{w}_{\alpha}(\boldsymbol{u}) \leq 3$, for all $\boldsymbol{\alpha} = 1, 2, 3, \ldots, \boldsymbol{m}$; $\boldsymbol{e} \in \boldsymbol{\mathcal{E}}$ and $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$. Simply an m-polar neutrosophic number (mPNSN) can be expressed as $\wp = \{< \boldsymbol{u}_{\alpha}, \boldsymbol{v}_{\alpha}, \boldsymbol{w}_{\alpha} >\}$, where $0 \leq \boldsymbol{u}_{\alpha} + \boldsymbol{v}_{\alpha} + \boldsymbol{w}_{\alpha} \leq 3$ and $\boldsymbol{\alpha} = 1, 2, 3, \ldots, \boldsymbol{m}$.

3. Generalized Multi-Polar Neutrosophic soft Set (GmPNSS) with Operators and Properties

In this section, we study the concept of GmPNSS and introduce some basic operations and their properties on GmPNSS.

Definition 3.1

Let $\mathcal{U}$ and E are universal and set of attributes respectively, and $\mathcal{A} \subseteq$ E, if there exists a mapping Φ such as

$\Phi: \mathcal{A} \rightarrow GmPNSS^{\mathcal{U}}$

Then (Φ, $\mathcal{A}$) is called GmPNSS over $\mathcal{U}$ defined as follows

$\Upsilon_K = (\Phi, \mathcal{A}) = \left\{\left(u, \Phi_{\mathcal{A}(e)}(u)\right): e \in E, u \in \mathcal{U}\right\}$, where

$\Phi_{\mathcal{A}}(e) = \{e, < u, u^{\alpha}_{\mathcal{A}(e)}(u), v^{\alpha}_{\mathcal{A}(e)}(u), w^{\alpha}_{\mathcal{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\}$, and

$0 \leq u^{\alpha}_{\mathcal{A}(e)}(u) + v^{\alpha}_{\mathcal{A}(e)}(u) + w^{\alpha}_{\mathcal{A}(e)}(u) \leq 3$ for all $\alpha \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathcal{U}$.

Definition 3.2

Let $\Upsilon_{\mathcal{A}}$ and Υ_B are two GmPNSS over $\mathcal{U}$, then $\Upsilon_{\mathcal{A}}$ is called a multi-polar neutrosophic soft subset of Υ_B. If

$u^{\alpha}_{\mathcal{A}(e)}(u) \leq u^{\alpha}_{B(e)}(u)$, $v^{\alpha}_{\mathcal{A}(e)}(u) \leq v^{\alpha}_{B(e)}(u)$ and $w^{\alpha}_{\mathcal{A}(e)}(u) \geq w^{\alpha}_{B(e)}(u)$

for all $\alpha \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathcal{U}$.

Example 1 Assume $\mathcal{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{x_1, x_2, x_3, x_4\}$ be a set of attribuites and $\mathcal{A} = B = \{x_1, x_2\} \subseteq E$. Consider $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{G}_B \in$ G3-PNSS over $\mathcal{U}$ can be represented as follows

$$\mathcal{F}_{\mathcal{A}} = \begin{Bmatrix} (x_1, \{\langle u_1, (.5, .2, .1), (.3, .1, .3), (.4, .3, .8), (u_2, (.2, .3, .2), (.2, .1, .3), (.3, .4, .6))\rangle, \\ (x_2, \{\langle u_1, (.3, .1, .4), (0, .1, .5), (.3, .1, .5)), (u_2, (.2, .2, .5), (.3, .1, .5), (.4, .3, .6))\rangle \end{Bmatrix}$$

and

$$\mathcal{G}_B = \begin{Bmatrix} (x_1, \{\langle u_1, (.6, .4, .1), (.4, .3, .2), (.5, .4, .5), (u_2, (.3, .5, .1), (.3, .2, .1), (.4, .5, .4))\rangle, \\ (x_2, \{\langle u_1, (.4, .3, .3), (0, .2, .3), (.4, .2, .5)), (u_2, (.2, .1, .3), (.6, .3, .1), (.5, .3, .1))\rangle \end{Bmatrix}$$

Thus

$\mathcal{F}_{\mathcal{A}} \subseteq \mathcal{G}_B$.

Definition 3.3

Let $\Upsilon_{\mathcal{A}}$ and Υ_B are two GmPNSS over $\mathcal{U}$, then $\Upsilon_{\mathcal{A}} = \Upsilon_B$, if

$u^{\alpha}_{\mathcal{A}(e)}(u) \leq u^{\alpha}_{B(e)}(u)$, $u^{\alpha}_{B(e)}(u) \leq u^{\alpha}_{\mathcal{A}(e)}(u)$

$v^{\alpha}_{\mathcal{A}(e)}(u) \leq v^{\alpha}_{B(e)}(u)$, $v^{\alpha}_{B(e)}(u) \leq v^{\alpha}_{\mathcal{A}(e)}(u)$

$w^{\alpha}_{\mathcal{A}(e)}(u) \geq w^{\alpha}_{B(e)}(u)$, $w^{\alpha}_{B(e)}(u) \geq w^{\alpha}_{\mathcal{A}(e)}(u)$

for all $i \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathcal{U}$.

Definition 3.4

Let $\mathcal{F}_{\mathcal{A}}$ be a GmPNSS over $\mathcal{U}$; then empty GmPNSS can be represented as $\mathcal{F}_{\breve{0}}$And defined as follows

$\mathcal{F}_{\breve{0}} = \{e, < u, (0, 1, 1),\ (0, 1, 1), \dots, (0, 1, 1) > : e \in E, u \in \mathcal{U}\}$.

Definition 3.5

Let $\mathcal{F}_{\mathcal{A}}$ be a GmPNSS over $\mathcal{U}$; then universal GmPNSS can be represented as $\mathcal{F}_{\breve{E}}$And defined as follows

$\mathcal{F}_{\breve{E}} = \{e, < u, (1, 1, 0),\ (1, 1, 0), \dots, (1, 1, 0) > : e \in E, u \in \mathcal{U}\}$.

Example 2 Assume $\mathcal{U} = \{u_1,\ u_2\}$ be a universe of discourse and $E = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a set of attributes. The tabular representation of $\mathcal{F}_{\breve{0}}$ and $\mathcal{F}_{\breve{E}}$ given as follows in table 1 and table 2, respectively.

Table 1. Tablur representation of GmPNSS $\mathcal{F}_{\breve{0}}$

$\mathcal{U}$	$\mathbf{u_1}$	$\mathbf{u_2}$	$\cdots$	$\mathbf{u_n}$
x_1	$(0, 1, 1)$	$(0, 1, 1)$	$\cdots$	$(0, 1, 1)$
x_2	$(0, 1, 1)$	$(0, 1, 1)$	$\cdots$	$(0, 1, 1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$(0, 1, 1)$	$(0, 1, 1)$	$\cdots$	$(0, 1, 1)$

Table 2. Tablur representation of GmPNSS $\mathcal{F}_{\breve{E}}$

$\mathcal{U}$	$\mathbf{u_1}$	$\mathbf{u_2}$	$\cdots$	$\mathbf{u_n}$
x_1	$(1, 1, 0)$	$(1, 1, 0)$	$\cdots$	$(1, 1, 0)$
x_2	$(1, 1, 0)$	$(1, 1, 0)$	$\cdots$	$(1, 1, 0)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_n	$(1, 1, 0)$	$(1, 1, 0)$	$\cdots$	$(1, 1, 0)$

Definition 3.6

Let $\mathcal{F}_{\mathcal{A}}$ be a GmPNSS over $\mathcal{U}$, then the complement of GmPNSS is defined as follows

$\mathcal{F}^c_{\mathcal{A}}(e) = \left\{< u, w^{\alpha}_{\mathcal{A}(e)}(u),\ (1, 1, \dots, 1) - v^{\alpha}_{\mathcal{A}(e)}(u), u^{\alpha}_{\mathcal{A}(e)}(u) > : u \in \mathcal{U}\right\}$, for all $\alpha \in 1, 2, 3, \dots, m$; $e \in E$ and $u \in \mathcal{U}$.

Example 3. Assume $\mathcal{U} = \{u_1,\ u_2\}$ be a universe of discourse and $E = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a set of attributes and $\mathcal{A} = \{x_1,\ x_2\} \subseteq E$. Consider $\mathcal{F}_{\mathcal{A}} \in$ G3-PNSS over $\mathcal{U}$ can be represented as follows

$$\mathcal{F}_{\mathcal{A}} = \begin{Bmatrix} (x_1, \{\langle u_1, (.6, .4, .1), (.4, .3, .2), (.5, .6, 1), \big(u_2, (.3, .5, .1), (.3, .2, .1), (.4, .5, .4)\big)\rangle, \\ \big(x_2, \{\langle u_1, (.4, .3, .3), (0, .2, .3), (.4, .2, .5)\big), \big(u_2, (.2, .1, .7), (.6, .3, 1), (.5, .3, .1)\big)\rangle \end{Bmatrix}$$

Then,

$$\mathcal{F}^c_{\mathcal{A}}(x) = \begin{Bmatrix} (x_1, \{\langle u_1, (.1, .6, .6),\ (.2, .7, .4), (.1, .4, .5), \big(u_2, (.1, .5, .3), (.1, .8, .3), (.4, .5, .4)\big)\rangle, \\ \big(x_2, \{\langle u_1, (.3, .7, .4), (.3, .8, 0), (.5, .8, .4)\big), \big(u_2, (.7, .9, .2), (1, .7, .6), (.1, .7, .5)\big)\rangle \end{Bmatrix}$$

Proposition 3.7

If $\mathcal{F}_{\mathcal{A}}$ be a GmPNSS, then

1. $(\mathcal{F}^c_{\mathcal{A}})^c = \mathcal{F}_{\mathcal{A}}$
2. $(\mathcal{F}_{\breve{0}})^c = \mathcal{F}_{\breve{E}}$
3. $(\mathcal{F}_{\breve{E}})^c = \mathcal{F}_{\breve{0}}$

Proof 1 Let $\mathcal{F}_{\mathcal{A}}(e) = \{e, < u, u^{\alpha}_{\mathcal{A}(e)}(u), v^{\alpha}_{\mathcal{A}(e)}(u),\ w^{\alpha}_{\mathcal{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\}$. Then by using definition 3.6, we get

$\mathcal{F}^c_{\mathcal{A}}(e) = \left\{e, < u, \left(w^{\alpha}_{\mathcal{A}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\mathcal{A}(e)}(u), u^{\alpha}_{\mathcal{A}(e)}(u)\right), > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$

Again, by using definition 3.6

$(\mathcal{F}^c_{\mathcal{A}}(e))^c =$

$\left\{< u, \left(w^{\alpha}_{\mathcal{A}(e)}(u),\ (1,1,\dots,1) - \left((1,1,\dots,1) - v^{\alpha}_{\mathcal{A}(e)}(u)\right), u^{\alpha}_{\mathcal{A}(e)}(u)\right) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$

$.(\mathcal{F}^c_{\mathcal{A}}(e))^c = \left\{< u, \left(u^{\alpha}_{\mathcal{A}(e)}(u),\ v^{\alpha}_{\mathcal{A}(e)}(u), w^{\alpha}_{\mathcal{A}(e)}(u)\right) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$.

$(\mathcal{F}^c_{\mathcal{A}}(e))^c = \mathcal{F}_{\mathcal{A}}(e)$.

Similarly, we can prove 2 and 3.

Definition 3.8

Let $\mathcal{F}_{\mathcal{A}(e)}$ and $\mathcal{G}_{B(e)}$ are two GmPNSS over $\mathcal{U}$, then

$$\mathcal{F}_{\mathcal{A}(e)} \cup \mathcal{G}_{B(e)} = \left\{ e, < u, \begin{pmatrix} max\{u^{\alpha}_{\mathcal{A}(e)}(u), u^{\alpha}_{B(e)}(u)\}, \\ min\{v^{\alpha}_{\mathcal{A}(e)}(u), v^{\alpha}_{B(e)}(u)\}, \\ min\{w^{\alpha}_{\mathcal{A}(e)}(u), w^{\alpha}_{B(e)}(u)\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Example 4. Assume $\mathcal{U} = \{u_1,\ u_2\}$ be a universe of discourse and $E = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a set of attribuites and $\mathcal{A} = B = \{x_1,\ x_2\} \subseteq E$. Consider $\mathcal{F}_{\mathcal{A}(e)}$ and $\mathcal{G}_{B(e)} \in$ G3-PNSS over $\mathcal{U}$ can be represented as follows

$$\mathcal{F}_{\mathcal{A}(e)} = \begin{Bmatrix} (x_1, \{\langle u_1, (.5,.2,.1), (.3,.1,.3), (.4,.3,.8), (u_2, (.2,.3,.2), (.2,.1,.3), (.3,.4,.6))\rangle, \\ (x_2, \{\langle u_1, (.3,.1,.4), (0,.1,.5), (.3,.1,.5)), (u_2, (.2,.2,.5), (.3,.1,.5), (.4,.3,.6))\rangle \end{Bmatrix}$$

and

$$\mathcal{G}_{B(e)} = \begin{Bmatrix} (x_1, \{\langle u_1, (.6,.4,.1), (.4,.3,.2), (.5,.4,.5), (u_2, (.3,.5,.1), (.3,.2,.1), (.4,.5,.4))\rangle, \\ (x_2, \{\langle u_1, (.4,.3,.3), (0,.2,.3), (.4,.2,.5)), (u_2, (.2,.1,.3), (.6,.3,.1), (.5,.3,.1))\rangle \end{Bmatrix}$$

Then

$$\mathcal{F}_{\mathcal{A}(e)} \cup \mathcal{G}_{B(e)} = \begin{Bmatrix} (x_1, \{\langle u_1, (.6,.2,.1), (.4,.1,.2), (.5,.3,.5)), (u_2, (.3,.3,.1), (.3,.1,.1), (.4,.4,.4))\rangle, \\ (x_2, \{\langle u_1, (.4,.1,.3), (0,.1,.3), (.4,.1,.5)), (u_2, (.2,.1,.3), (.6,.1,.1), (.5,.3,.1))\rangle \end{Bmatrix}$$

Proposition 3.9

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}}$ are GmPNSS over $\mathcal{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}}$
3. $(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cup \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \cup (\mathcal{G}_{\breve{B}} \cup \mathcal{H}_{\breve{C}})$

Proof 1. As we know that

$\mathcal{F}_{\breve{A}}(e) = \left\{e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u),\ w^{\alpha}_{\breve{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$ be a GmPNSS over $\mathcal{U}$. Then by using definition 3.8

$$\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \left\{ e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{A}(e)}(u)\right\}, \\ min\left\{v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u)\right\}, \\ min\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \left\{e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u),\ w^{\alpha}_{\breve{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

$\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$.

Proof 2. As we know that

$\mathcal{F}_{\breve{A}}(e) = \left\{e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u),\ w^{\alpha}_{\breve{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$ and $\mathcal{G}_{\breve{B}}(e) = \left\{e, < u, u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u),\ w^{\alpha}_{\breve{B}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$ are two GmPNSS over $\mathcal{U}$. Then

$$\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \left\{ e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ min\left\{v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ min\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \left\{ e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\breve{B}(e)}(u), u^{\alpha}_{\breve{A}(e)}(u)\right\}, \\ min\left\{v^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u)\right\}, \\ min\left\{ w^{\alpha}_{\breve{B}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}}$

Similarly, we can prove 3.

Definition 3.10

Let $\mathcal{F}_{\mathcal{A}(e)}$ and $\mathcal{G}_{B(e)}$ are GmPNSS over $\mathcal{U}$, then

$$\mathcal{F}_{\mathcal{A}(e)} \cap \mathcal{G}_{B(e)} = \left\{ e, < u, \begin{pmatrix} min\{u^{\alpha}_{\mathcal{A}(e)}(u), u^{\alpha}_{B(e)}(u)\}, \\ max\{v^{\alpha}_{\mathcal{A}(e)}(u), v^{\alpha}_{B(e)}(u)\}, \\ max\{w^{\alpha}_{\mathcal{A}(e)}(u), w^{\alpha}_{B(e)}(u)\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Proposition 3.11

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}}$ are GmPNSS over $\mathcal{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cap \mathcal{F}_{\breve{A}}$
3. $(\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \cap \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \cap (\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}})$

Proof Similar to proposition 3.9, by using definition 3.11, we can prove easily.

Remark 3.12 Generally, if $\mathcal{F}_{\breve{A}} \neq \mathcal{F}_{\breve{0}}$ and $\mathcal{F}_{\breve{A}} \neq \mathcal{F}_{E}$, then the law of contradiction $\mathcal{F}_{\breve{A}} \cap {\mathcal{F}_{\breve{A}}}^{C} = \mathcal{F}_{\breve{0}}$ and the law of the excluded middle $\mathcal{F}_{\breve{A}} \cap {\mathcal{F}_{\breve{A}}}^{C} = \mathcal{F}_{E}$ does not hold in mPIVNSS. But in classical set theory law of contradiction and excluded middle always hold.

Proposition 3.13

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ are GmPNSS over $\mathcal{U}$, then

1. $(\mathcal{F}_{\breve{A}(e)} \cup \mathcal{G}_{\breve{B}(e)})^{C} = {\mathcal{F}_{\breve{A}(e)}}^{C} \cap {\mathcal{G}_{\breve{B}(e)}}^{C}$

2. $(\mathcal{F}_{\breve{A}(e)} \cap G_{B(e)})^C = \mathcal{F}_{\breve{A}(e)}{}^C \cup \mathcal{G}_{\breve{B}(e)}{}^C$

Proof 1 As we know that

$\mathcal{F}_{\breve{A}}(e) = \left\{e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$ and $\mathcal{G}_{\breve{B}}(e) = \left\{e, < u, u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$ are two GmPNSS.

By using definition 3.8, we get

$$\mathcal{F}_{\breve{A}}(e) \cup \mathcal{G}_{\breve{B}(e)} = \left\{ e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ min\left\{v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ min\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Now by using definition 3.6, we get

$$\left(\mathcal{F}_{\breve{A}}(e) \cup \mathcal{G}_{\breve{B}(e)}\right)^c = \left\{ (e, (< u, \begin{pmatrix} min\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ (1,1,\dots,1) - min\left\{v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Now

$$\mathcal{F}_{\breve{A}}(e)^C = \left\{< u, \left(w^{\alpha}_{\breve{A}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{A}(e)}(u)\right) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

$$\mathcal{G}_{\breve{B}(e)}{}^C = \left\{< u, \left(w^{\alpha}_{\breve{B}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\breve{B}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

By using definition 3.10

$$\mathcal{F}_{\breve{A}}(e)^C \cap \mathcal{G}_{\breve{B}(e)}{}^C = \left\{ (e, < u, \begin{pmatrix} min\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ \max\left\{(1,1,\dots,1) - v^{\alpha}_{\breve{A}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\breve{A}}(e)^C \cap \mathcal{G}_{\breve{B}(e)}{}^C = \left\{ (e, (< u, \begin{pmatrix} min\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ (1,1,\dots,1) - min\left\{v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E; \alpha \in 1,2,3,\dots,m \right\}$$

Hence

$(\mathcal{F}_{\breve{A}}(e) \cup \mathcal{G}_{\breve{B}(e)})^C = \mathcal{F}_{\breve{A}}(e)^C \cap \mathcal{G}_{\breve{B}(e)}{}^C$.

Proof 2 By using definition 3.10, we have

$$\mathcal{F}_{\breve{A}}(e) \cap \mathcal{G}_{\breve{B}(e)} = \left\{ e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Now by using definition 3.6, we get

$$\left(\mathcal{F}_{\check{A}}(e) \cap \mathcal{G}_{\check{B}(e)}\right)^{c} = \left\{ \left(e, \left(< u, \begin{pmatrix} max\left\{w^{\alpha}_{\check{A}(e)}(u), w^{\alpha}_{\check{B}(e)}(u)\right\}, \\ (1,1,\dots,1) - max\left\{v^{\alpha}_{\check{A}(e)}(u), v^{\alpha}_{\check{B}(e)}(u)\right\}, \\ min\left\{u^{\alpha}_{\check{A}(e)}(u), u^{\alpha}_{\check{B}(e)}(u)\right\} \end{pmatrix} >: u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Now

$$\mathcal{F}_{\check{A}}(e)^{C} = \left\{< u, \left(w^{\alpha}_{\check{A}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\check{A}(e)}(u), u^{\alpha}_{\check{A}(e)}(u)\right) >: u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

$$\mathcal{G}_{\check{B}(e)}{}^{C} = \left\{< u, \left(w^{\alpha}_{\check{B}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\check{B}(e)}(u), u^{\alpha}_{\check{B}(e)}(u)\right) >: u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

By using definition 3.8

$$\mathcal{F}_{\check{A}}(e)^{C} \cup \mathcal{G}_{\check{B}(e)}{}^{C} = \left\{ \left(e, < u, \begin{pmatrix} max\left\{w^{\alpha}_{\check{A}(e)}(u), w^{\alpha}_{\check{B}(e)}(u)\right\}, \\ min\left\{(1,1,\dots,1) - v^{\alpha}_{\check{A}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\check{B}(e)}(u)\right\}, \\ min\left\{u^{\alpha}_{\check{A}(e)}(u), u^{\alpha}_{\check{B}(e)}(u)\right\} \end{pmatrix} >: u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\check{A}}(e)^{C} \cup \mathcal{G}_{\check{B}(e)}{}^{C} = \left\{ \left(e, \left(< u, \begin{pmatrix} max\left\{w^{\alpha}_{\check{A}(e)}(u), w^{\alpha}_{\check{B}(e)}(u)\right\}, \\ (1,1,\dots,1) - max\left\{v^{\alpha}_{\check{A}(e)}(u), v^{\alpha}_{\check{B}(e)}(u)\right\}, \\ min\left\{u^{\alpha}_{\check{A}(e)}(u), u^{\alpha}_{\check{B}(e)}(u)\right\} \end{pmatrix} >: u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Hence

$(\mathcal{F}_{\check{A}(e)} \cap G_{B(e)})^{C} = \mathcal{F}_{\check{A}(e)}{}^{C} \cup \mathcal{G}_{\check{B}(e)}{}^{C}$.

Proposition 3.14

Let $\mathcal{F}_{\overline{A(e)}}$, $\mathcal{G}_{\overline{B(e)}}$, $\mathcal{H}_{\overline{C(e)}}$ are GmPNSS over $\mathcal{U}$. Then

1. $\mathcal{F}_{\overline{A(e)}} \cup (\mathcal{G}_{\overline{B(e)}} \cap \mathcal{H}_{\overline{C(e)}}) = (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}}) \cap (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{H}_{\overline{C(e)}})$
2. $\mathcal{F}_{\overline{A(e)}} \cap (\mathcal{G}_{\overline{B(e)}} \cup \mathcal{H}_{\overline{C(e)}}) = (\mathcal{F}_{\overline{A(e)}} \cap \mathcal{G}_{\overline{B(e)}}) \cup (\mathcal{F}_{\overline{A(e)}} \cap \mathcal{H}_{\check{C}(e)})$
3. $\mathcal{F}_{\overline{A(e)}} \cup (\mathcal{F}_{\check{A}(e)} \cap \mathcal{G}_{\overline{B(e)}}) = \mathcal{F}_{\overline{A(e)}}$
4. $\mathcal{F}_{\overline{A(e)}} \cap (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}}) = \mathcal{F}_{\overline{A(e)}}$

Proof 1 As we know that

$$\mathcal{G}_{\overline{B(e)}} \cap \mathcal{H}_{\overline{C(e)}} = \left\{ e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\overline{B(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ max\left\{v^{\alpha}_{\overline{B(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ max\left\{w^{\alpha}_{\overline{B(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\} \end{pmatrix} >: u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$\mathcal{F}_{\overline{A(e)}} \cup (\mathcal{G}_{\overline{B(e)}} \cap \mathcal{H}_{\overline{C(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\overline{A(e)}}(u), min\left\{u^{\alpha}_{\overline{B(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ min\left\{v^{\alpha}_{\overline{A(e)}}(u), max\left\{v^{\alpha}_{\overline{B(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}\right\} \\ min\left\{w^{\alpha}_{\overline{B(e)}}(u), max\left\{w^{\alpha}_{\overline{B(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$$\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}} = \left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{B(e)}}(u)\right\}, \\ min\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{B(e)}}(u)\right\}, \\ min\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{B(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$$\mathcal{F}_{\overline{A(e)}} \cup \mathcal{H}_{\overline{C(e)}} = \left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ min\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ min\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$(\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}}) \cap (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{H}_{\overline{C(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} min\left\{max\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{B(e)}}(u)\right\}, max\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{min\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{B(e)}}(u)\right\}, min\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{min\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{B(e)}}(u)\right\}, min\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E; \alpha \in 1, 2, 3, \ldots, m\right\}$$

$(\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}}) \cap (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{H}_{\overline{C(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\overline{A(e)}}(u), max\left\{u^{\alpha}_{\overline{B(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{v^{\alpha}_{\overline{A(e)}}(u), min\left\{v^{\alpha}_{\overline{B(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{w^{\alpha}_{\overline{A(e)}}(u), min\left\{w^{\alpha}_{\overline{B(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E; \alpha \in 1, 2, 3, \ldots, m\right\}$$

Hence

$\mathcal{F}_{\overline{A(e)}} \cup (\mathcal{G}_{\overline{B(e)}} \cap \mathcal{H}_{\overline{C(e)}}) = (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}}) \cap (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{H}_{\overline{C(e)}})$.

Proof 2. As we know that

$$\mathcal{G}_{\overline{B(e)}} \cup \mathcal{H}_{\overline{C(e)}} = \left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\overline{B(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ min\left\{v^{\alpha}_{\overline{B(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ min\left\{w^{\alpha}_{\overline{B(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$\mathcal{F}_{\overline{A(e)}} \cap (\mathcal{G}_{\overline{B(e)}} \cup \mathcal{H}_{\overline{C(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\overline{A(e)}}(u), max\left\{u^{\alpha}_{\overline{B(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{v^{\alpha}_{\overline{A(e)}}(u), min\left\{v^{\alpha}_{\overline{B(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{w^{\alpha}_{\overline{B(e)}}(u), min\left\{w^{\alpha}_{\overline{B(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\}\right\} \end{pmatrix} >: u \in \mathcal{U}, e \in E; \alpha \in 1, 2, 3, \ldots, m\right\}$$

$$\mathcal{F}_{\overline{A(e)}} \cap \mathcal{G}_{\overline{B(e)}} = \left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{B(e)}}(u)\right\}, \\ max\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{B(e)}}(u)\right\}, \\ max\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{B(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$$\mathcal{F}_{\overline{A(e)}} \cap \mathcal{H}_{\overline{C(e)}} = \left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ max\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ max\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$(\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}}) \cup (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{H}_{\widetilde{C(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} max\left\{min\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{B(e)}}(u)\right\}, min\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\}, \\ min\left\{max\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{B(e)}}(u)\right\}, max\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\}, \\ min\left\{max\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{B(e)}}(u)\right\}, max\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

$(\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}}) \cup (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{H}_{\widetilde{C(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\widetilde{A(e)}}(u), min\left\{u^{\alpha}_{\widetilde{B(e)}}(u), u^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\}, \\ min\left\{v^{\alpha}_{\widetilde{A(e)}}(u), max\left\{v^{\alpha}_{\widetilde{B(e)}}(u), v^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\}, \\ min\left\{w^{\alpha}_{\widetilde{A(e)}}(u), max\left\{w^{\alpha}_{\widetilde{B(e)}}(u), w^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

Hence

$\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{G}_{\widetilde{B(e)}} \cup \mathcal{H}_{\widetilde{C(e)}}) = (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}}) \cup (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{H}_{\breve{C}(e)})$.

Proof 3. As

$$\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}} = \left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{B(e)}}(u)\right\}, \\ max\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{B(e)}}(u)\right\}, \\ max\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{B(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

$\mathcal{F}_{\widetilde{A(e)}} \cup (\mathcal{F}_{\breve{A}(e)} \cap \mathcal{G}_{\widetilde{B(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\widetilde{A(e)}}(u), min\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\}, \\ min\left\{v^{\alpha}_{\widetilde{A(e)}}(u), max\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\}, \\ min\left\{w^{\alpha}_{\widetilde{A(e)}}(u), max\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

$$\mathcal{F}_{\widetilde{A(e)}} \cup (\mathcal{F}_{\breve{A}(e)} \cap \mathcal{G}_{\widetilde{B(e)}}) = \left\{e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u),\ w^{\alpha}_{\breve{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

Hence

$\mathcal{F}_{\widetilde{A(e)}} \cup (\mathcal{F}_{\breve{A}(e)} \cap \mathcal{G}_{\widetilde{B(e)}}) = \mathcal{F}_{\widetilde{A(e)}}$.

Proof 4. $\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{F}_{\widetilde{A(e)}} \cup \mathcal{G}_{\widetilde{B(e)}}) = \mathcal{F}_{\widetilde{A(e)}}$

As

$$\mathcal{F}_{\widetilde{A(e)}} \cup \mathcal{G}_{\widetilde{B(e)}} = \left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{B(e)}}(u)\right\}, \\ min\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{B(e)}}(u)\right\}, \\ min\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{B(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

$\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{F}_{\breve{A}(e)} \cup \mathcal{G}_{\widetilde{B(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\widetilde{A(e)}}(u), max\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\}, \\ max\left\{v^{\alpha}_{\widetilde{A(e)}}(u), min\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\}, \\ max\left\{w^{\alpha}_{\widetilde{A(e)}}(u), min\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

$$\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{F}_{\breve{A}(e)} \cup \mathcal{G}_{\widetilde{B(e)}}) = \left\{e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u),\ w^{\alpha}_{\breve{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

Hence

$\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{F}_{\breve{A}(e)} \cup \mathcal{G}_{\widetilde{B(e)}}) = \mathcal{F}_{\widetilde{A(e)}}$.

Definition 3.15

Let $\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}$ are GmPNSS, then their difference defined as follows

$$\mathcal{F}_{\breve{A}} \setminus \mathcal{G}_{\breve{B}} = \begin{Bmatrix} (< u, min\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\}, max\left\{v^{\alpha}_{\breve{A}(e)}(u),\ (1,1,\ldots,1) - v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\} > : u \in \mathcal{U}; \alpha \in 1,2,3,\ldots,m) \end{Bmatrix}$$

Definition 3.16

Let $\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}$ are GmPNSS, then their addition is defined as follows

$$\mathcal{F}_{\breve{A}} + \mathcal{G}_{\breve{B}} = \begin{Bmatrix} (< u, min\left\{u^{\alpha}_{\breve{A}(e)}(u) + u^{\alpha}_{\breve{B}(e)}(u), (1,1,\ldots,1)\right\}, min\left\{v^{\alpha}_{\breve{A}(e)}(u) + v^{\alpha}_{\breve{B}(e)}(u), (1,1,\ldots,1)\right\}, \\ min\left\{w^{\alpha}_{\breve{A}(e)}(u) + w^{\alpha}_{\breve{B}(e)}(u), (1,1,\ldots,1)\right\} > : u \in \mathcal{U};\ i \in 1,2,3,\ldots,m) \end{Bmatrix}$$

Definition 3.17

Let $\mathcal{F}_{\breve{A}}$ be a GmPNSS; then its scalar multiplication is represented as $\mathcal{F}_{\breve{A}}(e).\breve{\alpha}$, where $\breve{\alpha} \in [0, 1]$ and defined as follows

$$\mathcal{F}_{\breve{A}}.\breve{\alpha} = \left\{(< u, min\left\{u^{\alpha}_{\breve{A}(e)}(u).\breve{\alpha}, (1,1,\ldots,1)\right\}, min\left\{v^{\alpha}_{\breve{A}(e)}(u).\breve{\alpha}, (1,1,\ldots,1)\right\}, min\left\{w^{\alpha}_{\breve{A}(e)}(u).\breve{\alpha}, (1,1,\ldots,1)\right\} >: u \in \mathcal{U})\right\}$$

Definition 3.18

Let $\mathcal{F}_{\breve{A}}$ be a GmPNSS; then its scalar division is represented as $\mathcal{F}_{\breve{A}}/\breve{\alpha}$, where $\breve{\alpha} \in [0, 1]$ and defined as follows

$$\mathcal{F}_{\breve{A}}/\breve{\alpha} = \left\{(< u, min\left\{u^{\alpha}_{\breve{A}(e)}(u)/\breve{\alpha}, (1,1,\ldots,1)\right\}, min\left\{v^{\alpha}_{\breve{A}(e)}(u)/\breve{\alpha}, (1,1,\ldots,1)\right\},\ min\left\{w^{\alpha}_{\breve{A}(e)}(u)/\breve{\alpha}, (1,1,\ldots,1)\right\} > : u \in \mathcal{U})\right\}$$

4. Similarity Measures and Their Decision-Making Approaches

Many mathematicians developed various methodologies to solve MCDM problems in the past few years, such as aggregation operators for different hybrid structures, CC, similarity measures, and decision-making applications. Some operational laws and mPNSWA operator with its decision-making approach are established for GmPNSS.

Definition 4.1

Let $\boldsymbol{\mathcal{F}_{\breve{A}}}, \boldsymbol{\mathcal{G}_{\breve{B}}}$ are two GmPNSS over the universe of discourse $\boldsymbol{\mathcal{U}} = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_j}\}$, then cosine similarity measure between $\boldsymbol{\mathcal{F}_{\breve{A}}}$ and $\boldsymbol{\mathcal{G}_{\breve{B}}}$ defined as

$\mathcal{F}_{\breve{A}}(e) = \left\{e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u),\ w^{\alpha}_{\breve{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\ldots,m\right\}$ and $\mathcal{G}_{\breve{B}}(e) = \left\{e, < u, u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u),\ w^{\alpha}_{\breve{B}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\ldots,m\right\}$

$$\mathcal{S}^1_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{1}{mn}\sum_{j=1}^{n}\sum_{\alpha=1}^{m} \frac{\left(\left(u^{\alpha}_{\breve{A}(e)}(u)\right)\left(u^{\alpha}_{\breve{B}(e)}(u)\right) + \left(v^{\alpha}_{\breve{A}(e)}(u)\right)\left(v^{\alpha}_{\breve{B}(e)}(u)\right) + \left(w^{\alpha}_{\breve{A}(e)}(u)\right)\left(w^{\alpha}_{\breve{B}(e)}(u)\right)\right)}{\left(\sqrt{\left(\left(u^{\alpha}_{\breve{A}(e)}(u)\right)^2 + \left(v^{\alpha}_{\breve{A}(e)}(u)\right)^2 + \left(w^{\alpha}_{\breve{A}(e)}(u)\right)^2\right)}\sqrt{\left(\left(u^{\alpha}_{\breve{B}(e)}(u)\right)^2 + \left(v^{\alpha}_{\breve{B}(e)}(u)\right)^2 + \left(w^{\alpha}_{\breve{B}(e)}(u)\right)^2\right)}\right)}$$

Proposition 4.2

Let $\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}$, and $\mathcal{H}_{\breve{C}} \in$ GmPNSS, then the following properties hold

1. $0 \le \mathcal{S}^1_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) \le 1$
2. $\mathcal{S}^1_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \mathcal{S}^1_{GmPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{F}_{\breve{A}})$

3. If $\mathcal{F}_{\breve{A}} \subseteq \mathcal{G}_{\breve{B}} \subseteq \mathcal{H}_{\breve{C}}$, then $\mathcal{S}^1_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{H}_{\breve{C}}) \leq \mathcal{S}^1_{mPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})$ and $\mathcal{S}^1_{mPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{H}_{\breve{C}}) \leq \mathcal{S}^1_{mPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{H}_{\breve{C}})$.

Proof: Using the above definition, the proof of these properties can be done quickly.

Definition 4.3

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$ are two GmPNSS over the universe of discourse $\mathcal{U} = \{u_1, u_2, \dots, u_j\}$, then set-theoretic similarity measure between $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ defined as

$$\mathcal{S}^2_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{1}{mn}\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\frac{\left(\left(u^{\alpha}_{\breve{A}(e)}(u)\right)\left(u^{\alpha}_{\breve{B}(e)}(u)\right)+\left(v^{\alpha}_{\breve{A}(e)}(u)\right)\left(v^{\alpha}_{\breve{B}(e)}(u)\right)+\left(w^{\alpha}_{\breve{A}(e)}(u)\right)\left(w^{\alpha}_{\breve{B}(e)}(u)\right)\right)}{max\left\{\left(\left(u^{\alpha}_{\breve{A}(e)}(u)\right)^2+\left(v^{\alpha}_{\breve{A}(e)}(u)\right)^2+\left(w^{\alpha}_{\breve{A}(e)}(u)\right)^2\right),\left(\left(u^{\alpha}_{\breve{B}(e)}(u)\right)^2+\left(v^{\alpha}_{\breve{B}(e)}(u)\right)^2+\left(w^{\alpha}_{\breve{B}(e)}(u)\right)^2\right)\right\}}$$

Proposition 4.4

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, and $\mathcal{H}_{\breve{C}} \in$ GmPNSS, then the following properties hold

1. $0 \leq \mathcal{S}^2_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) \leq 1$
2. $\mathcal{S}^2_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \mathcal{S}^2_{GmPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{F}_{\breve{A}})$
3. If $\mathcal{F}_{\breve{A}} \subseteq \mathcal{G}_{\breve{B}} \subseteq \mathcal{H}_{\breve{C}}$, then $\mathcal{S}^2_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{H}_{\breve{C}}) \leq \mathcal{S}^2_{mPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})$ and $\mathcal{S}^2_{mPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{H}_{\breve{C}}) \leq \mathcal{S}^2_{mPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{H}_{\breve{C}})$.

Proof: Using the above definition, the proof of these properties can be done quickly.

4.5 Algorithm 1 for Similarity Measures of GmPNSS

Step 1. Pick out the set containing parameters.

Step 2. Construct the GmPNSS according to experts.

Step 3. Compute the cosine similarity measure by using definition 4.1.

Step 4. Compute the set-theoretic similarity measure for GmPNSS by utilizing definition 4.3.

Step 5. An alternative with a maximum value with cosine similarity measure has the maximum rank according to considered numerical illustration.

Step 6. An alternative with a maximum value with a set-theoretic similarity measure has the maximum rank according to considered numerical illustration.

Step 7. Analyze the ranking.

A flowchart of the presented algorithm can see in figure 1.

Definition 4.6

Let $\mathcal{F}_{\breve{A}}(e) = \left\{< u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u) >\right\}$, $\mathcal{G}_{\breve{B}}(e) = \left\{< u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u) >\right\}$, and $\mathcal{H}_{\breve{B}(e)} = \left\{< u^{\alpha}_{\breve{C}(e)}(u), v^{\alpha}_{\breve{C}(e)}(u), w^{\alpha}_{\breve{C}(e)}(u) >\right\}$ are three mPNSNs, the basic operators for mPNSNs are defined as when $\delta > 0$

1. $\mathcal{F}_{\breve{A}}(e) \oplus \mathcal{G}_{\breve{B}}(e) = \left\langle u^{\alpha}_{\breve{A}(e)}(u) + u^{\alpha}_{\breve{B}(e)}(u) - u^{\alpha}_{\breve{A}(e)}(u)u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u) * v^{\alpha}_{\breve{B}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u) * w^{\alpha}_{\breve{B}(e)}(u)\right\rangle$
2. $\mathcal{F}_{\breve{A}}(e) \otimes \mathcal{G}_{\breve{B}}(e) = \left\langle u^{\alpha}_{\breve{A}(e)}(u) * u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u) + v^{\alpha}_{\breve{B}(e)}(u) - v^{\alpha}_{\breve{A}(e)}(u)v^{\alpha}_{\breve{B}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u) + w^{\alpha}_{\breve{B}(e)}(u) - w^{\alpha}_{\breve{A}(e)}(u)w^{\alpha}_{\breve{B}(e)}(u)\right\rangle$
3. $\delta\mathcal{F}_{\breve{A}}(e) = \left\langle 1 - \left(1 - u^{\alpha}_{\breve{A}(e)}(u)\right)^{\delta}, \left(v^{\alpha}_{\breve{A}(e)}(u)\right)^{\delta}, \left(w^{\alpha}_{\breve{A}(e)}(u)\right)^{\delta}\right\rangle$

4. $\left(\boldsymbol{\mathcal{F}}_{\breve{A}}(\boldsymbol{e})\right)^{\delta} = \left\langle \left(\boldsymbol{u}^{\alpha}_{\breve{A}(e)}(\boldsymbol{u})\right)^{\delta}, \mathbf{1}-\left(\mathbf{1}-\boldsymbol{v}^{\alpha}_{\breve{A}(e)}(\boldsymbol{u})\right)^{\delta}, \mathbf{1}-\left(\mathbf{1}-\boldsymbol{w}^{\alpha}_{\breve{A}(e)}(\boldsymbol{u})\right)^{\delta}\right\rangle$.

Proposition 4.7

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, and $\mathcal{H}_{\breve{C}}$ ∈ mPNSNs and $\delta,\ \delta_1,\ \delta_2 > 0$, then the following laws hold

1. $\mathcal{F}_{\breve{A}} \oplus \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \oplus \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \otimes \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \otimes \mathcal{F}_{\breve{A}}$
3. $\delta(\mathcal{F}_{\breve{A}} \oplus \mathcal{G}_{\breve{B}}) = \delta\mathcal{G}_{\breve{B}} \oplus \delta\mathcal{F}_{\breve{A}}$
4. $\left(\mathcal{F}_{\breve{A}} \otimes \mathcal{G}_{\breve{B}}\right)^{\delta} = \left(\mathcal{F}_{\breve{A}}\right)^{\delta} \otimes \left(\mathcal{G}_{\breve{B}}\right)^{\delta}$
5. $\delta_1\mathcal{F}_{\breve{A}} \oplus \delta_2\mathcal{F}_{\breve{A}} = (\delta_1 \oplus \delta_2)\mathcal{F}_{\breve{A}}$
6. $\left(\mathcal{F}_{\breve{A}}\right)^{\delta_1} \otimes \left(\mathcal{F}_{\breve{A}}\right)^{\delta_2} = \left(\mathcal{F}_{\breve{A}}\right)^{\delta_1+\delta_2}$
7. $\left(\mathcal{F}_{\breve{A}} \oplus \mathcal{G}_{\breve{B}}\right) \oplus \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \oplus \left(\mathcal{G}_{\breve{B}} \oplus \mathcal{H}_{\breve{C}}\right)$
8. $\left(\mathcal{F}_{\breve{A}} \otimes \mathcal{G}_{\breve{B}}\right) \otimes \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \otimes \left(\mathcal{G}_{\breve{B}} \otimes \mathcal{H}_{\breve{C}}\right)$

Proof. The proof of the above laws is straightforward by using definition 4.6.

Definition 4.8

Let $\mathcal{F}_{\breve{A}}(e_{ij}) = \left\langle\left\{u^{\alpha}_{\breve{A}(e_{ij})}(u), v^{\alpha}_{\breve{A}(e_{ij})}(u),\ w^{\alpha}_{\breve{A}(e_{ij})}(u)\right\}\right\rangle$ be a collection of mPNSNs, Ω_i and γ_j are weight vector for expert's and parameters respectively with given conditions $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i = 1$, $\gamma_j > 0$, $\sum_{j=1}^{m}\gamma_j = 1$, where $(i = 1,2,\dots,n, and\ j = 1,2,\dots,m)$. Then mPIVNSWA operator defined as mPNSWA: $\Delta^n \rightarrow \Delta$ defined as follows

$mPNSWA\left(\mathcal{F}_{\breve{A}}(e_{11}), \mathcal{F}_{\breve{A}}(e_{12}), \dots, \mathcal{F}_{\breve{A}}(e_{nk})\right) = \oplus_{j=1}^{k}\gamma_j\left(\oplus_{i=1}^{n}\Omega_i\mathcal{F}_{\breve{A}}(e_{ij})\right)$.

Proposition 4.9

Let $\mathcal{F}_{\breve{A}}(e_{ij}) = \left\langle\left\{u^{\alpha}_{\breve{A}(e_{ij})}(u), v^{\alpha}_{\breve{A}(e_{ij})}(u),\ w^{\alpha}_{\breve{A}(e_{ij})}(u)\right\}\right\rangle$ be a collection of mPNSNs, where $(i = 1,2,\dots,n, and\ j = 1,2,\dots,k)$, the aggregated value is also an mPNSNs, such as

$$mPNSWA\left(\mathcal{F}_{\breve{A}}(e_{11}), \mathcal{F}_{\breve{A}}(e_{12}), \dots, \mathcal{F}_{\breve{A}}(e_{nk})\right)$$
$$= \left\langle \mathbf{1}-\prod_{j=}^{m}\left(\prod_{i=1}^{n}\left(\mathbf{1}-\boldsymbol{u}^{\alpha}_{\breve{A}(e_{ij})}(\boldsymbol{u})\right)^{\Omega_i}\right)^{\gamma_j}, \mathbf{1}-\left(\mathbf{1}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathbf{1}-\boldsymbol{v}^{\alpha}_{\breve{A}(e_{ij})}(\boldsymbol{u})\right)^{\Omega_i}\right)^{\gamma_j}\right), \mathbf{1}-\left(\mathbf{1}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathbf{1}-\boldsymbol{w}^{\alpha}_{\breve{A}(e_{ij})}(\boldsymbol{u})\right)^{\Omega_i}\right)^{\gamma_j}\right)\right\rangle$$

Proof. We can prove easily by using IFSWA [32].

Definition 4.10

Let $\mathcal{F}_{\breve{A}}(e) = \left\{< u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u),\ w^{\alpha}_{\breve{A}(e)}(u) >\right\}$ be an mPNSN, then the score, accuracy, and certainty functions for GmPNSN respectively defined as follows

$$\mathbb{S}(\mathcal{F}_{\breve{A}}) = \frac{1}{6m}\sum_{\alpha=1}^{m}\left(6 + u^{\alpha}_{\breve{A}(e)}(u) - v^{\alpha}_{\breve{A}(e)}(u) - w^{\alpha}_{\breve{A}(e)}(u)\right)$$

$$\mathbb{A}(\mathcal{F}_{\breve{A}}) = \frac{1}{4m}\left(4 + u^{\alpha}_{\breve{A}(e)}(u) - w^{\alpha}_{\breve{A}(e)}(u)\right)$$

$$\mathbb{C}(\mathcal{F}_{\breve{A}}) = \frac{1}{2m}\left(2 + u^{\alpha}_{\breve{A}(e)}(u)\right)$$

where $\alpha = 1, 2, \cdots, m$.

Definition 4.11

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}} \in$ mPIVNSS, then comparison approach is present as follows

1. If $\mathbb{S}(\mathcal{F}_{\breve{A}}) > \mathbb{S}(\mathcal{G}_{\breve{B}})$, then $\mathcal{F}_{\breve{A}}$ is superior to $\mathcal{G}_{\breve{B}}$.
2. If $\mathbb{S}(\mathcal{F}_{\breve{A}}) = \mathbb{S}(\mathcal{G}_{\breve{B}})$ and $\mathbb{A}(\mathcal{F}_{\breve{A}}) > \mathbb{A}(\mathcal{G}_{\breve{B}})$, then $\mathcal{F}_{\breve{A}}$ is superior to $\mathcal{G}_{\breve{B}}$.
3. If $\mathbb{S}(\mathcal{F}_{\breve{A}}) = \mathbb{S}(\mathcal{G}_{\breve{B}})$, $\mathbb{A}(\mathcal{F}_{\breve{A}}) = \mathbb{A}(\mathcal{G}_{\breve{B}})$, and $\mathbb{C}(\mathcal{F}_{\breve{A}}) > \mathbb{C}(\wp_{\Re_1})$, then $\mathcal{F}_{\breve{A}}$ is superior to $\mathcal{G}_{\breve{B}}$.
4. If $\mathbb{S}(\mathcal{F}_{\breve{A}}) = \mathbb{S}(\mathcal{G}_{\breve{B}})$, $\mathbb{A}(\mathcal{F}_{\breve{A}}) > \mathbb{A}(\mathcal{G}_{\breve{B}})$, and $\mathbb{C}(\mathcal{F}_{\breve{A}}) = \mathbb{C}(\wp_{\Re_1})$, then $\mathcal{F}_{\breve{A}}$ is indifferent to $\mathcal{G}_{\breve{B}}$, can be denoted as $\mathcal{F}_{\breve{A}} \sim \mathcal{G}_{\breve{B}}$.

4.2 Decision-making approach based mPNSWA for GmPNSS

Assume a set of "s" alternatives such as $\beta = \{\beta^1, \beta^2, \beta^3, \dots, \beta^s\}$ for assessment under the team of experts such as $\mathcal{U} = \{u_1, u_2, u_3, \dots, u_n\}$ with weights $\Omega = (\Omega_1, \Omega_1, \dots, \Omega_n)^T$ such that $\Omega_i > 0$, $\sum_{i=1}^{n} \Omega_i = 1$. Let $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$ be a set of attributes with weights $\gamma = (\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_m)^T$ be a weight vector for parameters such as $\gamma_j > 0$, $\sum_{j=1}^{m} \gamma_j = 1$. The team of experts $\{u_i$: $i = 1, 2, \dots, n\}$ evaluate the alternatives $\{\beta^{(z)}$: $z = 1, 2, \dots, s\}$ under the considered parameters $\{e_j$: $j = 1, 2, \dots, m\}$ given in the form of mPIVNSNs $\mathcal{L}_{ij}^{(z)} = \left(u_{\alpha_{ij}}^{(z)}, v_{\alpha_{ij}}^{(z)}, w_{\alpha_{ij}}^{(z)}\right)$, where $0 \le u_{\alpha_{ij}}^{(z)}, v_{\alpha_{ij}}^{(z)}, w_{\alpha_{ij}}^{(z)} \le 1$ and $0 \le u_{\alpha_{ij}}^{(z)}, + v_{\alpha_{ij}}^{(z)} + w_{\alpha_{ij}}^{(z)} \le 3$. So $\Delta_k = \left(u_{\alpha_{ij}}^{(z)}, v_{\alpha_{ij}}^{(z)}, w_{\alpha_{ij}}^{(z)}\right)$ for all i, j. Experts give their preferences for each alternative in terms of mPNSNs by using the mPNSWA operator in the form of $\Delta_k = \left(u_{\alpha_{ij}}^{(z)}, v_{\alpha_{ij}}^{(z)}, w_{\alpha_{ij}}^{(z)}\right)$. Compute the score values for each alternative and analyze the ranking of the alternatives, the algorithm of the proposed approach is presented in Figure: 1.

4.2.1 Algorithm 2 for mPNSWA Operator

Step 1. Develop the m-polar neutrosophic soft matrix for each alternative.

Step 2. Aggregate the mPNSNs for each alternative into a collective decision matrix Δ_k by using the mPNSWA operator.

Step 3. Compute the score value for each alternative Δ_k by using equation 14, where $k = 1, 2, \cdots, s$.

Step 4. Rank the alternatives $\beta^{(k)}$ and choose the best alternative.

Step 5. End.

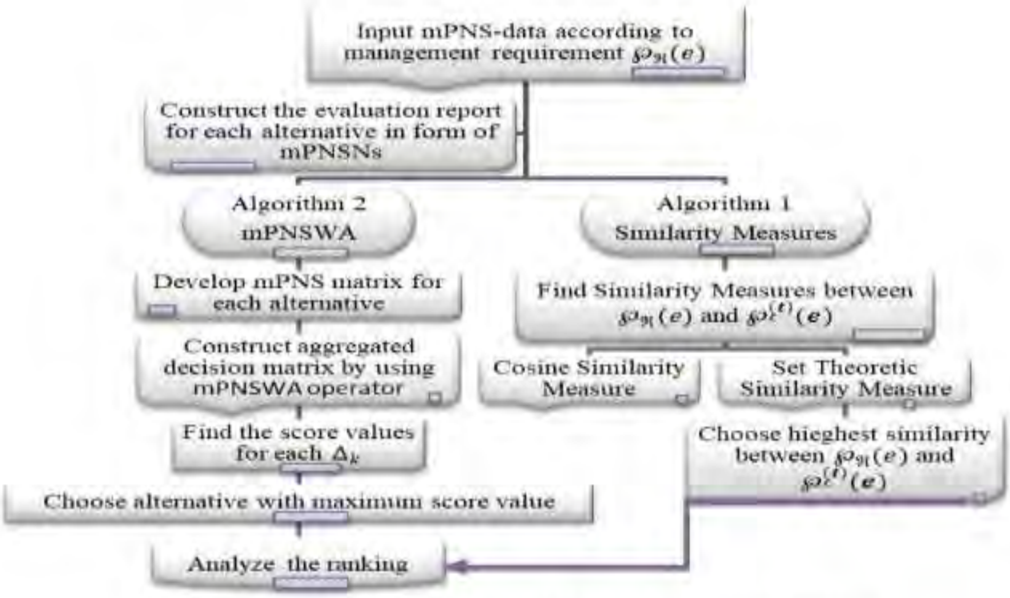


Figure 1: Flowchart of Proposed Algorithm 1 and Algorithm 2

5. Application of Similarity Measures and mPNSWA Operator in Decision Making

In this section, we proposed the algorithm for GmPNSS by using developed similarity measures and the mPNSWA operator. We also used the proposed methods for decision-making in real-life problems.

5.1. Problem Formulation and Application of GmPNSS For Decision Making

A construction company calls for the appointment of a civil engineer to supervise the workers. Several engineers apply for the civil engineer post, simply four engineers call for an interview based on experience for undervaluation such as $S = \{S_1, S_2, S_3, S_4\}$ be a set of selected engineers call for the interview. The managing director of the hires a committee of four experts $X = \{X_1, X_2, X_3, X_4\}$ for the selection of civil engineer. First of all, the committee decides the set of parameters such as $E = \{x_1, x_2, x_3\}$, where x_1, x_2, and x_3 represents the personality, communication skills, and qualifications for the selection of civil engineer. The experts evaluate the applicants under defined parameters and forward the evaluation performa to the company's managing director. Finally, the director scrutinizes the best applicant based on the expert's evaluation report.

5.1.1. Application of GmPNSS For Decision Making

Assume $S = \{S_1, S_2, S_3, S_4\}$ be a set of civil engineers who are shortlisted for interview and $E = \{x_1$ = personality, x_2 = communication skills, x_3 = qualification$\}$ be a set of parameters for the selection of civil engineer. Let $\mathcal{F}$ and $\mathcal{G} \subseteq E$; then we construct the G3-PNSS $\Phi_{\mathcal{F}}(x)$ according to the requirement of the construction company such as follows

Table 3. Construction of G3-PNSS of all Applicants According to Company Requirement

$\Phi_{\mathcal{F}}(x)$	x_1	x_2	x_3
$\mathbf{X_1}$	(.8,.5,.6),(.5,.4,.2),(.4,.3,.6)	(.4,.8,.6),(.7,.6,.5),(.4,.1,.3)	(.7,.8,.5),(.8,.4,.7),(.6,.5,.2)
$\mathbf{X_2}$	(.5,.6,.5),(.9,.5,.8),(.6,.4,.5)	(.7,.5,.8),(.7,.5,.7),(.3,.5,.9)	(.6,.4,.9),(.2,.5,.2),(.9,.4,.6)
$\mathbf{X_3}$	(.2,.5,.4),(.7,.3,.2),(.6,.4,.5)	(.3,.5,.7),(.4,.6,.2),(.6,.7,.9)	(.5,.2,.4),(.7,.5,.9),(.6,.3,.4)

X_4	(.9,.5,.1),(.3,.4,.6),(.6,.5,.2)	(.9,.5,.6),(.3,.4,.3),(.6,.3,.9)	(.9,.5,.7),(.7,.4,.3),(.4,.7,.6)

Now we will construct the G3-PNSS $\varphi_{\mathcal{G}}^{t}$ according to four experts, where t = 1, 2, 3, 4.

Table 4. G3-PNSS Evaluation Report According to Experts of S_1

$\varphi_{\mathcal{G}}^{1}$	x_1	x_2	x_3
X_1	(.3,.5,.2),(.8,.7,.3),(.7,.2,.9)	(.9,.5,.1),(.3,.4,.6),(.1,.5,.2)	(.9,.5,.1),(.7,.4,.3),(.6,.7,.2)
X_2	(.7,.8,.3),(.6,.1,.2),(.2,.4,.6)	(.9,.5,.6),(.7,.2,.3),(.4,.7,.6)	(.7,.2,.4),(.3,.9,.7),(.5,.9,.1)
X_3	(.7,.3,.2),(.2,.1,.2),(.7,.9,.8)	(.7,.2,.1),(.7,.4,.5),(.1,.7,.9)	(.7,.8,.6),(.7,.2,.5),(.7,.3,.2)
X_4	(.3,.2,.7),(.5,.6,.2),(.4,.6,.8)	(.7,.2,.6),(.7,.4,.9),(.8,.6,.9)	(.2,.9,.6),(.7,.4,.2),(.7,.7,.9)

Table 5. G3-PNSS Evaluation Report According to Experts of S_2

$\varphi_{\mathcal{G}}^{2}$	x_1	x_2	x_3
X_1	(.6,.2,.7),(.8,.7,.9),(.7,.5,.6)	(.1,.5,.6),(.3,.4,.6),(.6,.5,.2)	(.9,.5,.1),(.7,.4,.2),(.6,.3,.9)
X_2	(.1,.2,.4),(.1,.2,.2),(.7,.4,.9)	(.3,.5,.7),(.4,.2,.3),(.4,.7,.6)	(.7,.2,.4),(.3,.9,.7),(.3,.5,.1)
X_3	(.2,.6,.7),(.2,.7,.6),(.4,.5,.2)	(.7,.2,.1),(.6,.3,.5),(.1,.7,.4)	(.7,.5,.6),(.7,.2,.5),(.7,.3,.9)
X_4	(.8,.1,.9),(.4,.2,.6),(.2,.7,.1)	(.4,.2,.6),(.7,.4,.3),(.5,.7,.9)	(.2,.9,.1),(.1,.4,.2),(.4,.7,.9)

Table 6. G3-PNSS Evaluation Report According to Experts of S_3

$\varphi_{\mathcal{G}}^{3}$	x_1	x_2	x_3
X_1	(.7,.4,.1),(.7,.3,.1),(.7,.4,.6)	(.4,.9,.6),(.7,.2,.5),(.7,.3,.2)	(.7,.4,.6),(.9,.4,.3),(.1,.4,.5)
X_2	(.6,.2,.3),(.7,.4,.3),(.6,.2,.5)	(.6,.2,.1),(.5,.4,.7),(.3,.5,.1)	(.6,.2,.7),(.5,.4,.3),(.6,.4,.7)
X_3	(.6,.2,.1),(.6,.3,.5),(.4,.7,.9)	(.2,.7,.4),(.3,.6,.2),(.5,.3,.9)	(.4,.2,.6),(.7,.4,.3),(.5,.4,.9)
X_4	(.4,.2,.3),(.4,.1,.3),(.4,.5,.2)	(.1,.6,.5),(.3,.2,.6),(.1,.5,.2)	(.6,.1,.4),(.3,.7,.4),(.4,.3,.2)

Table 7. G3-PNSS Evaluation Report According to Experts of S_4

$\varphi_{\mathcal{G}}^{4}$	x_1	x_2	x_3
X_1	(.2,.1,.2),(.3,.5,.4),(.9,.2,.7)	(.4,.8,.6),(.4,.7,.5),(.4,.5,.3)	(.2,.5,.6),(.5,.6,.2),(.4,.8,.6)
X_2	(.1,.3,.1),(.9,.4,.6),(.3,.3,.8)	(.7,.2,.6),(.7,.4,.2),(.4,.7,.9)	(.5,.6,.5),(.3,.5,.8),(.6,.4,.5)
X_3	(.7,.2,.1),(.6,.3,.5),(.4,.5,.9)	(.7,.2,.1),(.6,.3,.5),(.1,.7,.4)	(.3,.5,.7),(.4,.5,.2),(.6,.3,.9)
X_4	(.4,.1,.7),(.9,.6,.2),(.4,.8,.1)	(.6,.1,.7),(.2,.4,.7),(.4,.5,.2)	(.2,.6,.4),(.3,.1,.6),(.4,.3,.2)

5.1.2 Solution by using Algorithm 1

By using Tables 3-7, compute the cosine similarity measure between $\mathcal{S}_{GmPNSS}^{1}(\Phi_{\mathcal{F}}(x),\varphi_{\mathcal{G}}^{1}(x))$, $\mathcal{S}_{GmPNSS}^{1}(\Phi_{\mathcal{F}}(x),\varphi_{\mathcal{G}}^{2}(x))$, $\mathcal{S}_{GmPNSS}^{1}(\Phi_{\mathcal{F}}(x),\varphi_{\mathcal{G}}^{3}(x))$, and $\mathcal{S}_{GmPNSS}^{1}(\Phi_{\mathcal{F}}(x),\varphi_{\mathcal{G}}^{1}(x))$ by using equation 4.1, such as

$\mathcal{S}^1_{GmPNSS}\,(\,\Phi_{\mathcal{F}}(x)\,,\,\varphi^1_{\mathcal{G}}(x)\,) = \frac{1}{3\times4}\left\{\frac{(.8)(.3)+(.5)(.5)+(.6)(.2)}{\sqrt{(.8)^2+(.5)^2+(.6)^2}\sqrt{(.3)^2+(.5)^2+(.2)^2}} + \frac{(.5)(.8)+(.4)(.7)+(.2)(.3)}{\sqrt{(.5)^2+(.4)^2+(.2)^2}\sqrt{(.8)^2+(.7)^2+(.3)^2}} + \cdots + \frac{(.4)(.7)+(.7)(.7)+(.6)(.9)}{\sqrt{(.4)^2+(.7)^2+(.6)^2}\sqrt{(.7)^2+(.7)^2+(.9)^2}}\right\} = \frac{1}{12}\left(\frac{28.99}{34.4799}\right) = 0.07007.$

Similarly, we can find the cosine similarity measure between $\mathcal{S}^1_{GmPNSS}\,(\,\Phi_{\mathcal{F}}(x)\,,\,\varphi^2_{\mathcal{G}}(x)\,)$, $\mathcal{S}^1_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^3_{\mathcal{G}}(x))$, and $\mathcal{S}^1_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^4_{\mathcal{G}}(x))$ given as $\mathcal{S}^1_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^2_{\mathcal{G}}(x)) = \frac{1}{12}\left(\frac{26.32}{32.3767}\right) = 0.06771$, $\mathcal{S}^1_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^3_{\mathcal{G}}(x)) = \frac{1}{12}\left(\frac{25.4}{29.4056}\right) = 0.06943$, and $\mathcal{S}^1_{GmPNSS}\,(\,\Phi_{\mathcal{F}}(x)\,,\,\varphi^4_{\mathcal{G}}(x)\,) = \frac{1}{12}\left(\frac{25.48}{30.88764}\right) = 0.06874$. This shows that $\mathcal{S}^1_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^1_{\mathcal{G}}(x)) > \mathcal{S}^1_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^3_{\mathcal{G}}(x)) > \mathcal{S}^1_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^4_{\mathcal{G}}(x)) > \mathcal{S}^1_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^2_{\mathcal{G}}(x))$. It can be seen from this ranking alternative $\beta^{(1)}$ is most relevant and similar to $\Phi_{\mathcal{F}}(x)$. Therefore $\beta^{(1)}$ is the best alternative for the vacant position of associate professor, the ranking of other alternatives given as $\beta^{(1)} > \beta^{(3)} > \beta^{(4)} > \beta^{(2)}$.

Now we compute the set-theoretic similarity measure by using Definition 4.3 between $\mathcal{S}^2_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^1_{\mathcal{G}}(x))$, $\mathcal{S}^2_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^2_{\mathcal{G}}(x))$, $\mathcal{S}^2_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^3_{\mathcal{G}}(x))$, and $\mathcal{S}^4_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^1_{\mathcal{G}}(x))$ From Tables 1-5, we can find the set-theoretic similarity measure for each alternative by using definition 4.3 given as $\mathcal{S}^2_{GmPNSS}\,(\,\Phi_{\mathcal{F}}(x)\,,\,\varphi^1_{\mathcal{G}}(x)\,) = 0.06986$, $\mathcal{S}^2_{GmPNSS}\,(\,\Phi_{\mathcal{F}}(x)\,,\,\varphi^2_{\mathcal{G}}(x)\,) = 0.06379$, $\mathcal{S}^2_{GmPNSS}\,(\,\Phi_{\mathcal{F}}(x)\,,\,\varphi^3_{\mathcal{G}}(x)\,) = 0.06157$, and $\mathcal{S}^2_{GmPNSS}\,(\,\Phi_{\mathcal{F}}(x)\,,\,\varphi^4_{\mathcal{G}}(x)\,) = 0.06176$. This shows that $\mathcal{S}^2_{mPNSS}(\wp_{\Re}(e),\wp^{(1)}_{\mathcal{L}}(e)) > \mathcal{S}^2_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^2_{\mathcal{G}}(x)) > \mathcal{S}^2_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^4_{\mathcal{G}}(x)) > \mathcal{S}^2_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^3_{\mathcal{G}}(x))$. Therefore $\beta^{(1)}$ is the best alternative for the vacant position of associate professor by using set-theoretic similarity measure, the ranking of other alternatives given as $\beta^{(1)} > \beta^{(2)} > \beta^{(4)} > \beta^{(3)}$. Graphically representation of results can be seen in Fig. 2.

5.1.3 Solution by using Algorithm 2

Step 1. The experts will evaluate the condition in the case of mPNSNs, and there are just four alternatives; parameters and a summary of their scores given in Tables 4, 5, 6, and 7.

Step 2. Experts' opinions on each alternative are summarized by using proposition 4.9. Therefore, we have

$\Delta_1 = \langle(.3144,.5379,.4259),(.1819,.3711,.4126),(.2129,.3421,.1328)\rangle$,

$\Delta_2 = \langle(.1815,.5420,.3844),\ (.3546,.5937,.2725),(.4526,.5031,.3725)\rangle$,

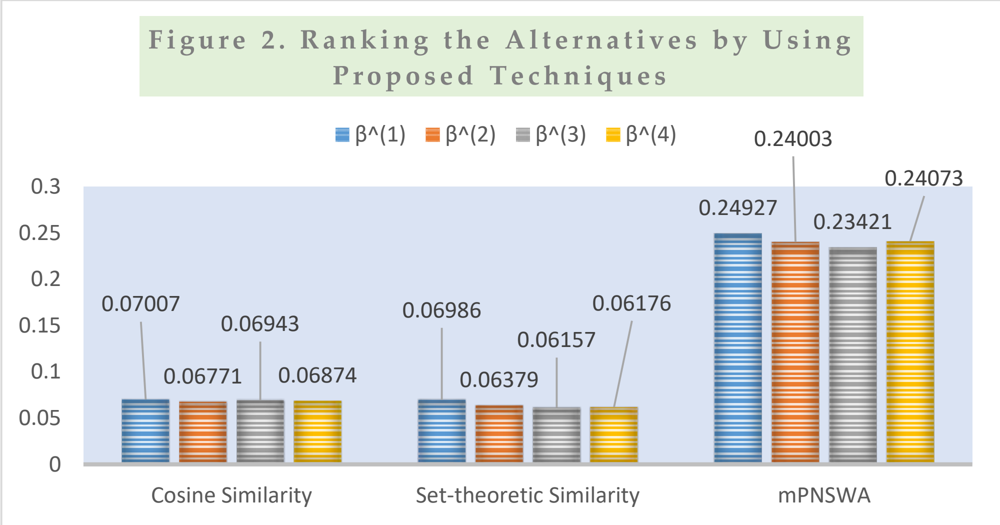
$\Delta_3 = \langle(.2904,.4223,.3755),(.3761,.5547,.4136),(.2516,.4732,.4631)\rangle$, and

$\Delta_4 = \langle(.2713,.5445,.1756),(.3530,.5201,.5641),(.4547,.4153,.5263)\rangle$.

Step 3. Compute the Score values by using definition 4.10.

$\mathbb{S}(\Delta_1) = .24927$, $\mathbb{S}(\Delta_2) = .24003$, $\mathbb{S}(\Delta_3) = .23421$, and $\mathbb{S}(\Delta_4) = .24073$

Step 4. Therefore, the ranking of the alternatives is as follows $\mathbb{S}(\Delta_1) > \mathbb{S}(\Delta_4) > \mathbb{S}(\Delta_2) > \mathbb{S}(\Delta_3)$. So, $\beta^{(1)} > \beta^{(4)} > \beta^{(2)} > \beta^{(3)}$, hence, the alternative $\beta^{(1)}$ is the most suitable alternative for the company.



6. Discussion and Comparative Analysis

The following section will discuss the effectiveness, naivety, flexibility, and advantages of the proposed methods and algorithms. We also conducted a brief comparative analysis of suggested strategies and existing methods.

6.1 Advantages, flexibility, and Superiority of Proposed Approach

The recommended technique is practical and applicable to all forms of input data. We introduce two novel algorithms based on GmPNSS, and one is similarity measures, the other is mPNSWA. This manuscript has established two different types of similarity measures, such as cosine and set-theoretic similarity measures. Both algorithms are practical and can provide the best results in MCDM problems. The recommended algorithms are simple and easy to understand, can deepen their understanding, and apply to many choices and metrics. All algorithms are flexible and easy to change to adapt to different situations, inputs, and outputs. There are subtle differences between the rankings of the suggested methods because different techniques have different ranking methods, so that they can be affordable according to their considerations.

6.2. Results and Discussion

Through this research and comparative analysis, we have concluded that the results obtained by the proposed method are more general than the existing methods. However, in the decision-making process, compared with the current decision-making methods, it contains more information to deal with the uncertainty in the data. Moreover, the hybrid structure of many FSs becomes a particular case of mPNSS, add some suitable conditions. Among them, the information related to the object can be expressed more accurately and empirically, so it is a convenient tool for combining inaccurate and uncertain information in the decision-making process. Therefore, our proposed method is effective, flexible, simple, and superior to other hybrid structures of fuzzy sets.

Table 8: Comparative analysis between some existing techniques and the proposed approach

	Set	Truthiness	Indeterminacy	Falsity	Multi-polarity	Loss of information
Chen et al. [38]	mPFS	✓	×	×	✓	×

Xu et al. [36]	IFS	✓	×	✓	×	×
Zhang et al. [40]	IFS	✓	×	✓	×	✓
Yager [43, 44]	PFS	✓	×	✓	×	×
Naeem et al. [37]	mPFS	✓	×	✓	✓	×
Zhang et al. [31]	INSs	✓	✓	✓	×	×
Ali et al. [39]	BPNSS	✓	✓	✓	×	×
Saeed et al. [45]	mPIVNS	✓	✓	✓	✓	✓
Saqlain et al. [32]	mPNSS	✓	✓	✓	✓	×
Proposed approach	GmPNSS	✓	✓	✓	✓	×

It turns out that this is a contemporary issue. Why do we have to embody novel algorithms based on the proposed novel structure? Many indications compared with other existing methods; the recommended method may be an exception. We remember the following fact: the mixed form limits IFS, picture fuzzy sets, FS, fuzzy hesitation sets, NS, and other fuzzy sets and cannot provide complete information about the situation. But our m-polar model GmPNSS can deal with truthiness, indeterminacy, and falsity, so it is most suitable for MCDM. Due to the exaggerated multipolar neutrosophy, these three degrees are independent and provide a lot of information about alternative norms. Other similarity measures of available hybrid structures are converted into exceptional cases of GmPNSS. A comparative analysis of some already existing techniques is listed in Table 8. Therefore, this model has more versatility and can efficiently resolve complications than intuitionistic, neutrosophic, hesitant, image, and ambiguity substitution. The similarity measure established for GmPNSS becomes better than the existing similarity measure for MCDM.

7. Conclusion

This paper studies some basic concepts such as soft set, NSS, mPNSS, and GmPNSS. We discussed various operations with their properties and numerical examples for GmPNSS. We developed the idea of cosine similarity measure and set-theoretic similarity measure for GmPNSS with some properties in this research. We also presented the introduced multipolar neutrosophic weighted average operator for GmPNSS and established some operational laws for GmPNSS. The concept of score function, accuracy function, and certainty function is developed to compare m-polar neutrosophic numbers. Furthermore, decision-making approaches have been developed for GmPNSS based on proposed techniques. To verify the effectiveness of our developed techniques, we presented an illustration to solve MCDM problems. We gave a comprehensive comparative analysis of proposed techniques with existing methods. In the future, the concept of mPNSS will be extended to interval-valued mPNSS. It will solve real-life problems such as medical diagnoses, decision-making, etc.

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NeutroAlgebra of Idempotents in Group Rings

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Abstract: In this paper, the authors study the new concept of NeutroAlgebra of idempotents in group rings. It is assumed that *RG* is the group ring of a group G over the ring *R*. *R* should be a commutative ring with unit 1. *G* can be a finite or an infinite order group which can be commutative or non-commutative. We obtain conditions under which the idempotents of the group rings *ZG*, Z_nG, and *QG* form a NeutroAlgebra under the operations + or ×. Some collection of idempotents in these group rings form an AntiAlgebra. We propose some open problems which has resulted from this study.

Keywords: Symmetric group; NeutroAlgebra; AntiAlgebra; group ring, NeutrosubAlgebra, Partial Algebra.

1. Introduction

In this paper, we study the NeutroAlgebra of idempotent elements of the group ring RG, where R is a commutative ring with unit 1 (R can be Z or R or Q or Z_n; n a composite or a prime number) and G is a commutative or a non-commutative group of finite order. We only study the NeutroAlgebra of idempotent elements in the group ring under '+' and '×' operations inherited from the group ring *RG*.

The study of neutrosophy was first carried out by [1]. This concept can analyze real-world data's uncertainty, inconsistency, and indeterminacy. The new notion of NeutroAlgebraic structures and AntiAlgebraic structures was first introduced in [2] in 2019. There are several interesting results in this direction, like NeutroAlgebra as a generalization of partial algebra [6-7], Neutro-BE-Algebra and Anti-BE-Algebra, Neutro-BCK Algebra introduced in [8]. [9] has analyzed NeutroAlgebras in the context of number systems Neutrosophic triplets as NeutroAlgebra was carried out in [11-19]. [20] introduces Neutrosophic quadruple vector spaces. Extended Neutrosophic triplets are introduced and analyzed in [21-24]. Various researchers studied other unique properties of Neutrosophic triplets in [25-30]. Application of Neutrosophic theory is carried out in [31- 36], has been extended to the study of neutrosophic vector spaces, and algebraic codes.

This paper is organized into five sections. The first section is introductory. The second section presents the basic concepts needed to make this paper a self-contained one. Section three discusses and describes the NeutroAlgebra of idempotents in the group rings *ZG* and *QG* and the NeutroAlgebra of idempotents in the group ring Z_nG. The final section gives the conclusions based on the study and suggests a few open conjectures which will be taken for future research.

2. Basic Concepts

This section gives a few essential concepts for this paper to be self-contained. First, we recall the concept of the group ring, then recall the definitions and describe a few properties of the NeutroAlgebra and AntiAlgebra by some illustrative examples.

Definition 2.1. *Let R be a commutative ring with unit 1 and G be a multiplicative group. The group RG of the group G over the ring R consists of all finite formal sums of the form $\sum_i \alpha_i g_i$ (i – runs over a finite number) where $\alpha_i \in G$ and $g_i \in G$ satisfy the following conditions.*

i) $\sum_{i=1}^{n} \alpha_i g_i = \sum_{i=1}^{n} \beta_i g_i \Leftrightarrow \alpha_i, \beta_i \in R; \alpha_i = \beta_i$ for $i = 1, 2, \ldots, n$; $g_i \in G$.

ii) $\left(\sum_{i=1}^{n} \alpha_i g_i\right) + \left(\sum_{i=1}^{n} \beta_i g_i\right) = \sum_{i=1}^{n} (\alpha_i + \beta_i)\ g_i,\ g_i \in G;\ \alpha_i, \beta_i \in R$

iii) $\left(\sum_{i=1}^{n} \alpha_i g_i\right)\left(\sum_{i=1}^{n} \beta_i g_i\right) = \sum_k \gamma_k m_k$ where $\gamma_k = \sum \alpha_i \beta_j$ and $g_i = m_k$

iv) $rg = gr$ for all $r \in R$ and $g \in G$

v) $r\sum_{i=1}^{n} r_i g_i = \sum_{i=1}^{n} (rr_i) g_i$ for $r_i, r_i \in R$, $g_i \in G$ and $\sum r_i g_i \in RG$.

RG is a ring with $0 \in R$, which acts as the identity for addition. Since $1 \in R$ and we have 1. $G = G \subseteq G$ and $\mathrm{Re} = R \subseteq G$, where e is the identity of G.

For more about grouprings and their properties refer [3].

Example 2.1. Let Z_4 = *{0, 1, 2, 3}* be the ring of modulo integers. $G = \langle g \mid g^2 = 1 \rangle$ be the cyclic group of order 2. Then the group ring Z_4G = *{1, 0, 2, 3, g, 2g, 3g, 1 + g, 2 + g, 3 + g, 1 + 2g, 1 + 3g, 2 + 2g, 2 + 3g, 3 + 2g, 3 + 3g}.*

We now proceed to recall the definition of support of α in a group ring RG where $\alpha \in RG$. We denote support of α by *supp* α = {all group elements in α with non-zero coefficients from R } and |*supp* α | = {number of group elements in α which has non-zero coefficient}.

Suppose $\alpha = 1 + 3g + 0g + 5g^3 + 0g^4 + 6g^5 \in RG$ where $R = G$ and $G = \langle g \mid g^6 = 1 \rangle$ then *supp* α = $\{1, g, g^3, g^5\}$ of the group ring *RG* of the group *G* over the ring *R*; which is subset of the group *G* and |*supp* α | = *4*.

Now we recall the definition of NeutroAlgebra and describe this concept as in [2].

A NeutroAlgebra is an algebra with at least one Neutro-operation or one Neutroaxiom (axiom that is true for some elements, indeterminate or false for other elements) [2]. A partial algebra has at the minimum one partial operation, and all axioms are classical. [6] has described NeutroAlgebra that are partial algebras.

Similarly, an AntiAlgebra is a non-empty set endowed with at least one anti operation (or anti operations) or at least one anti axiom.

We proceed to give examples of NeutroAlgebra and AntiAlgebra.

Example 2.2. Let Z_{12} be the ring of modulo integers 12. The idempotents of Z_{12} are *{4, 9}* = W ; 0 and 1 in Z_{12} are defined as trivial idempotents of Z_{12}.

The Cayley table of *W* is as follows under +.

Table 1 Cayley table of *{W, +}*

+	4	9
4	od	od
9	od	od

So $\{W,+\}$ is an AntiAlgebra. The Cayley table of W under × is as follows.

Table 2 Cayley table of *{W, ×}*

×	4	9
4	4	od
9	od	9

Clearly if *V = {0, 1, 9, 4}* then the Cayley table of *V* under + is as follows.

Table 3 Cayley table of *{V, +}*

+	0	1	4	9
0	0	1	4	9
1	1	od	od	od
4	4	od	od	1
9	9	od	1	od

$\{V,+\}$ is a NeutroAlgebra of idempotents under +. Clearly $\{V,\times\}$ is a commutative semigroup of order 4.

3. NeutroAlgebra of idempotents in the group ring *ZG(QG)*

This section deals with NeutroAlgebra of idempotents in the group ring RG, where R is the ring of integers Z or the field of rationals Q of characteristic zero. This section finds the NeutroAlgebra of idempotents in the group ring ZG and QG, where G is taken as a commutative or a non-commutative group of finite order.

Example 3.1. Let QG be the group ring of G over Q where $G=\langle g \mid g^2=1\rangle$ is a cyclic group of order 2. A few of the idempotents of G are $\alpha=\frac{1}{2}(1-g)$ that is

$$\alpha^2=\frac{1}{4}(1-2g+g^2) = \frac{1}{4}\times\{2(1-g)\} = \frac{1}{2}(1-g) \text{ (using the fact } g^2=1).$$

If $\beta=\frac{1}{2}(1+g)\in QG$ then

$$\beta^2=\left\{\frac{1}{2}(1+g)\right\}^2 = \frac{1}{4}(1+2g+g^2)=\frac{1}{4}(2+2g)=\frac{1}{2}(1+g) \text{ as } g^2=1.$$

Now $QG=\{\alpha+\beta g \mid \alpha,\beta\in Q,\ g^2=1\}$.

Thus, the only two non-trivial idempotents of QG are $\alpha = \frac{1-g}{2}$ and $\beta = \frac{1+g}{2}$. QG has no other non-trivial idempotents. For if $x+yg$ is a nontrivial idempotent in QG with $x, y \in Q \setminus \{0\}$.

If $x+yg = t$ is an idempotent in QG then $t^2 = (x+yg)^2 = x+yg = t$. This implies

$$t^2 = (x^2 + 2xyg + y^2) = x + yg = t \text{ as } g^2 = 1.$$

$$(x^2 + y^2) + 2xyg = x + yg$$

By equating the like terms.

$$x^2 + y^2 = x \qquad (1)$$

and

$$2xy = y \qquad (2)$$

Since $x, y \in Q \setminus \{0\};\ y \neq 0$ so $y^{-1} \in Q$.

Hence $2xy = y$ implies $(2x-1)y = 0$ as $y \neq 0$. $2x = 1$ or $x = \frac{1}{2}$. Using $x = \frac{1}{2}$ in equation (1) we get

$\left(\frac{1}{2}\right)^2 + y^2 = \frac{1}{2}$ so that $y^2 + \frac{1}{2} - \frac{1}{4}$ or $y = \frac{1}{\pm 2}$.

Thus, the element $x = yg$ is an idempotent if and only if

$$x = y = \frac{1}{2} \text{ or } x = \frac{1}{2} \text{ and } y = -\frac{1}{2}.$$

That is $\alpha = \frac{1}{2}(1+g)$ or $\beta = \frac{1}{2}(1-g)$.

Other possibilities are $x = y = -\frac{1}{2}$ in this case $a = \frac{-1}{2}(1+g)$ but

$$a^2 = \frac{1}{4}(1+2g+g^2) = \frac{2(1+g)}{4} = \frac{(1+g)}{2} \neq a.$$

Hence $a = \frac{-1}{2}(1+g)$ is not an idempotent of QG. So, if $x = y = \frac{-1}{2}$ does not yield an idempotent. Suppose $x = \frac{-1}{2}$ and $y = \frac{1}{2}$ then $b = \frac{-1+g}{2}$. Now

$$b^2 = \frac{1}{4}\left[1+g^2-2g\right] = \frac{1}{4}\left[2-2g\right] = \frac{1}{2}\left[1-g\right] \neq b.$$

So $b = \frac{-1+g}{2}$ too is not an idempotent of QG. Thus $\alpha = \frac{1}{2}(1+g)$ and $\beta = \frac{1}{2}(1-g)$ are the only nontrivial idempotents of QG.

Let $V = \left\{\frac{1}{2}(1+g), \frac{1}{2}(1-g)\right\}$ be the collection of all non-trivial idempotents of QG.

We give the Cayley table of V under +.

Table 4 Cayley table of *{V, +}*.

$+$	$\frac{1}{2}(1+g)$	$\frac{1}{2}(1-g)$
$\frac{1}{2}(1+g)$	*od*	*od*
$\frac{1}{2}(1-g)$	*od*	*od*

So, V under + is an AntiAlgebra of idempotents in QG. (od denotes the term outerdefined).

Now consider V under ×. The Cayley table of V is as follows:

Table 5 Cayley table of *{V, ×}*

$\times$	$\frac{1}{2}(1+g)$	$\frac{1}{2}(1-g)$
$\frac{1}{2}(1+g)$	$\frac{1}{2}(1+g)$	*od*
$\frac{1}{2}(1-g)$	*od*	$\frac{1}{2}(1-g)$

V under × is a NeutroAlgebra of idempotents of G.

Suppose $W = \left\{\frac{1+g}{2}, \frac{1-g}{2}, 0, 1\right\}$; now we find the Cayley table under +.

Table 6 Cayley table of *{W, +}*.

$+$	*1*	*0*	$\frac{1+g}{2}$	$\frac{1-g}{2}$
1	*od*	*1*	*od*	*od*
0	*1*	*0*	$\frac{1+g}{2}$	$\frac{1-g}{2}$
$\frac{1+g}{2}$	*od*	$\frac{1+g}{2}$	*od*	*od*
$\frac{1-g}{2}$	*od*	$\frac{1-g}{2}$	*od*	*od*

Clearly, W under + is a NeutroAlgebra of idempotents in QG under the + operation.

Consider the Cayley table under × of W given in the following:

Table 7 Cayley Table of W under ×

$\times$	*0*	*1*	$\frac{1+g}{2}$	$\frac{1-g}{2}$
0	*0*	*0*	*0*	*0*

1	0	1	$\frac{1+g}{2}$	$\frac{1-g}{2}$
$\frac{1+g}{2}$	0	$\frac{1+g}{2}$	$\frac{1+g}{2}$	0
$\frac{1-g}{2}$	0	$\frac{1-g}{2}$	0	$\frac{1-g}{2}$

Thus $\{W,\times\}$ is a semigroup of idempotents in QG.

Example 3.2. Let QS_3 be the group ring of the symmetric group S_3 over the field of rationals. Here the Cayley table for S_3 is as follows.

$$S_3 = \left\{ e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \ p_1 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \ p_2 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \ p_3 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \right.$$

$$\left. p_4 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ p_5 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$ is the permutation group of degree 3.

The Cayley table of the group S3 under composition '*o*' of maps is as follows:

Table 8. Cayley table of S3 under 'o'.

o	*e*	p_1	p_2	p_3	p_4	p_5
e	*e*	p_1	p_2	p_3	p_4	p_5
p_1	p_1	*e*	p_5	p_4	p_3	p_2
p_2	p_2	p_4	*e*	p_5	p_1	p_3
p_3	p_3	p_5	p_4	*e*	p_2	p_1
p_4	p_4	p_2	p_3	p_1	p_5	*e*
p_5	p_5	p_3	p_1	p_2	*e*	p_4

The nontrivial idempotents of QS_2 are $\alpha_1 = \frac{1}{2}(1+p_1)$, $\alpha_2 = \frac{1}{2}(1 \pm p_2)$, $\alpha_3 = \frac{1}{2}(1 \pm p_3)$, $\alpha_4 = \frac{1}{3}(1+p_4+p_5)$ and $\alpha_5 = \frac{1}{6}(1+p_1+p_2+p_3+p_4+p_5)$.

Let $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be the set of some nontrivial idempotents in QG.

Now we find the Cayley table of *B* under + in the following.

Thus, *B* under

Table 9. Cayley table of *B* under +

+	α_1	α_2	α_3	α_4	α_5
α_1	*od*	*od*	*od*	*od*	*od*
α_2	*od*	*od*	*od*	*od*	*od*
α_3	*od*	*od*	*od*	*od*	*od*
α_4	*od*	*od*	*od*	*od*	*od*

α_5	od	od	od	od	od

Thus, *B* under + is an AntiAlgebra of idempotents in QS_3.

Now we consider the Cayley table of B under ×.

Table 10. Cayley table of B under ×.

×	α_1	α_2	α_3	α_4	α_5
α_1	α_1	od	od	od	α_5
α_2	od	α_2	od	od	α_5
α_3	od	od	α_3	od	α_5
α_4	od	od	od	α_4	α_5
α_5	α_5	α_5	α_5	α_5	α_5

Clearly, *B* under × is NeutroAlgebra of idempotents of QS_3.

We give yet another example of a cyclic group of composite order. Based on these examples, we will proceed onto prove the following results.

Example 3.3. Let $G = \langle g \mid g^{24} = 1 \rangle$ 1 be the cyclic group of order 24. *Q* be the field of rationals. *QG* be the group ring of *G* order *Q*.

The idempotents of *QG* are

$$x_1 = \frac{1}{2}(1+g^{12})$$

$$x_2 = \frac{1}{3}(1+g^8+g^{16})$$

$$x_3 = \frac{1}{4}(1+g^6+g^{12}+g^{18})$$

$$x_4 = \frac{1}{6}(1+g^4+g^6+g^{12}+g^{16}+g^{20})$$

$$x_5 = \frac{1}{8}(1+g^3+g^6+g^9+g^{12}+g^{15}+g^{18}+g^{21})$$

$$x_6 = \frac{1}{12}(1+g^2+g^4+g^6+g^8+g^{14}+g^{10}+g^{12}+g^{16}+g^{18}+g^{20}+g^{22})$$

and $x_7 = \frac{1}{24}(1+g+g^2+\ldots g^{23})$.

Now let $W = (x_1, x_2, x_3, \ldots, x_6, x_7)$ be the collection of some set of idempotents in QG.

We see $y_1 = \frac{1}{2}(1-g^{12})$

$$y_2 = \frac{1}{4}(1-g^6+g^{12}-g^{18})$$

$$y_3 = \frac{1}{6}(1 - g^4 + g^3 - g^{12} + g^{16} - g^{20})$$

$$y_4 = \frac{1}{8}(1 - g^3 + g^6 - g^9 + g^{12} - g^{15} + g^{18} - g^{21})$$

$$y_5 = \frac{1}{12}(1 - g^2 + g^4 - g^6 + g^8 - g^{10} + g^{12} - g^{14} + g^{16} - g^{18} + g^{20} - g^{22})$$

are also idempotents of QG.

Now we find the Cayley tables of W under + and ×.

Let $M = \{y_1, y_3, y_4, y_5\}$ be the set of some idempotents of QG we find the Cayley table of M also under + and × is given in Tables 14 and 15 respectively.

First, the Cayley table of W under + is as follows.

Table 11. Cayley table of W under +.

+	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	od	od	od	od	od	od	od
x_2	od	od	od	od	od	od	od
x_3	od	od	od	od	od	od	od
x_4	od	od	od	od	od	od	od
x_5	od	od	od	od	od	od	od
x_6	od	od	od	od	od	od	od
x_7	od	od	od	od	od	od	od

Clearly, the set W of idempotents of QG is an AntiAlgebra under + as every term is outer defined in W. Now we give the table of W under product.

Table 12. Cayley table of W under ×.

×	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	x_1	x_4	x_3	x_4	x_5	x_6	x_7
x_2	x_4	x_2	x_6	x_4	x_7	x_6	x_7
x_3	x_3	x_6	x_3	x_6	x_5	x_6	x_7
x_4	x_4	x_4	x_6	x_4	x_7	x_6	x_7
x_5	x_5	x_7	x_5	x_7	x_5	x_7	x_7
x_6	x_6	x_6	x_6	x_6	x_7	x_6	x_7
x_7	x_7	x_7	x_7	x_7	x_7	x_7	x_7

Clearly, W under × is a semigroup and is not a NeutroAlgebra or AntiAlgebra.

If, on the other hand, x_7, the whole group sum is deleted as the support of x_7 is G, we will get for the corresponding set $\{W \setminus x_7\}$ the Cayley table under × which is as follows.

Table 13. Cayley table $W \setminus \{x_7\}$ under ×.

×	x_1	x_2	x_3	x_4	x_5	x_6
x_1	x_1	x_4	x_3	x_4	x_5	x_6
x_2	x_4	x_2	x_6	x_4	*od*	x_6
x_3	x_3	x_6	x_3	x_6	x_5	x_6
x_4	x_4	x_4	x_6	x_4	*od*	x_6
x_5	x_5	*od*	x_5	*od*	x_5	*od*
x_6	x_6	x_6	x_6	x_6	*od*	x_6

Thus $\{W \setminus x_7\}$ is a NeutroAlgebra of idempotents in QG.

The Cayley table of M under + is as follows.

Table 14: Cayley table of M under +

+	y_1	y_2	y_3	y_4	y_5
y_1	*od*	*od*	*od*	*od*	*od*
y_2	*od*	*od*	*od*	*od*	*od*
y_3	*od*	*od*	*od*	*od*	*od*
y_4	*od*	*od*	*od*	*od*	*od*
y_5	*od*	*od*	*od*	*od*	*od*

Thus, M under + is an AntiAlgebra of idempotents of QG.

Now we find the Cayley table of M under × which is as follows.

Table 15. Cayley table of M under ×.

×	y_1	y_2	y_3	y_4	y_5
y_1	y_1	*od*	y_3	*od*	*od*
y_2	*od*	y_2	*od*	*od*	y_5
y_3	y_3	*od*	y_3	*od*	*od*
y_4	*od*	*od*	*od*	y_4	*od*
y_6	*od*	y_5	*od*	*od*	y_5

Thus, the set M under × is a NeutroAlgebra of idempotents.

Now we proceed on to prove the following results.

Let G be a cyclic group of order *n, n* a composite number. Q be the field of rationals QG be the group ring of G over Q.

i) All proper idempotents in QG are obtained from the proper subgroups of G.

ii) If p_1 is the order of the subgroup H of G, then $|\text{supp } p_1| < O(G)$ and $p_1 / O(G)$.

iii) The idempotents α formed by the subgroups of G will have a $|\text{supp } \alpha| < O(G)$

iv) If $|\text{supp } \alpha| = n$; $\alpha \in$ QG then this idempotent for all practical situations will be taken as a trivial idempotent. Similarly, $1 \in G$ is an idempotent, which is trivial. Also, $0,1 \in Q$ are trivial idempotents of QG.

These four conditions are strictly adhered to while finding the NeutroAlgebra of idempotents of the group ring QG under the operations + and ×.

v) When n, order of the cyclic group G is a product of odd prime then $n = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$ where p_i's are distinct primes; $1 \le i \le t$ and $\alpha_i \ge 1$; $1 \le i \le t$.

vi) We see all subgroups of G are again an odd prime or a power of a prime or the product of some primes less than n.

vii) Furtherer if $\alpha = \frac{1}{p_i}\left(1 + g^{p_i} + \ldots + g^{p_i^{\alpha_i} - p_i}\right)$ is an idempotent then $\frac{1}{p_i}\left(1 - g^{p_i} + \ldots \mp \frac{1}{p_i^{\alpha_i}} g^{p_i^{\alpha_i - p_i}}\right)$ in general is not an idempotent.

To this effect, we propose an open problem in the section on the conclusion of this paper.

Suppose G is a cyclic group of order n; then G can have subgroups of both even and odd order unless $|G| = 2^n$.

If $|G| = 2^n$ and if $x = 1 + h + \ldots + h^t$ is an idempotent of QG then so is $y = 1 - h + h^2 - \ldots - h^t$ where h is a suitable power of g is the cyclic subgroup of G. In this case x an idempotent of QG with support of $x = \{1, h, \ldots, h^t\}$.

However, product of these two idempotents $x \times y = 0$ is not a proper idempotent of QG, only the trivial idempotent zero.

Theorem 3.1. *Let G be a cyclic group of odd order; QG be the group ring of G over Q.*

i) *QG has only idempotents of the form $\frac{1}{t}(1 + h + h^2 + \ldots + h^{t-1})$ where $h \in G$ and $t < n$, and $\{1, h, h^2, \ldots, h^{t-1}\}$ is subgroup of G of order t.*

ii) *If W = {collection of nontrivial idempotents of G }, then*

a) $\{W, +\}$ is an AntiAlgebra of idempotents of QG and

b) $\{W, \times\}$ is a NeutroAlgebra of idempotents of QG

$\left(0, 1 \text{ and } \frac{1}{n}(1 + g + \ldots + g^{n-1}) \text{ are the trivial idempotents of } QG\right)$.

Proof of (i). Given G is a cyclic group of odd order with $|G| = n$ (n a non-prime). So, G has only subgroups H_t^t of odd order, say t where t / n (t can be prime or non-prime).

Clearly $\alpha = \frac{1}{t}(1 + h + \ldots + h^{t-1})$ is an idempotent of QG, where $h \in G$.

Now 0, 1 and $x = \frac{1}{n}(1 + g + ... + g^{n-1})$ are assumed to be trivial idempotents of QG as $|\text{supp } x| = |G| = n$.

The other type of idempotents can be $\beta = \frac{1}{t}(1 - h + h^2 - ... + h^{t-1})$ but β^2 is not an idempotent easily verified using number theoretic or group theoretic properties.

$$\Big(\beta^2 = \frac{1}{t^2}\Big[1 - h + h^2 - ... - h^{t-2} + h^{t-1} - h + h^2 - h^3 - ... + h^{t-1} - 1$$

$$+ h^2 - h^3 + h^4 - ... - 1 + h - h^3 + h^4 - ... - 1 + h - h^2 + h^{t-1} - 1 + ... + h^{t-2}\Big]$$

$$= \frac{1}{t^2}\Big[-(t-2) + (t-2)h ... + th^{t+1}\Big] \neq \beta .$$

Hence the claim.

Proof of (ii). Given W is the collection of all non-trivial idempotents of QG, so $\frac{1}{n}(1 + g + ... + g^{n-1}) \notin W$. $(W, +)$ is an AntiAlgebra.

For if $\alpha = \frac{1}{t}(1 + h + ... + h^{t-1})$, then $2\alpha = \frac{2}{t}(1 + h + ... + h^{t-1}) \notin W$.

Similarly, if $\beta \in W (\alpha \neq \beta)$ we see $\alpha + \beta \notin W$.

So, under +, every pair is outer defined.

Hence $(W, +)$ is an AntiAlgebra. Thus (a) of (ii) is proved. $(W, \times)$ is a NeutroAlgebra of idempotents of QG.

For if α and $\beta \in W$ such that $|\text{supp}\alpha| = m$ and $|\text{supp}\beta| = p$ such that $pm = n$ then $\alpha\beta = \frac{1}{n}(1 + g + ... + g^{n-1})$ which is a trivial idempotent of RG. As $n = pm$ can be written in a different way we have in the Cayley table of W under × has several od(outer defined) terms. Hence (b) of (ii) is proved.

Corollary 3.1. Let QG be as in the above theorem. If D, the trivial idempotent is taken in W, $(W, +)$ is a NeutroAlgebra of idempotents of QG.

Proof. If $0 \in W$ for every $\alpha \in W$, $\alpha + 0 = \alpha \in W$, so W under + is a NeutroAlgebra as we have some elements to be defined in W. Hence the claim.

Corollary 3.2. Let QG be as in the above theorem.

If the trivial idempotent $\alpha = \frac{1}{n}(1 + g + ... + g^{n-1}) \in W$ that is $|\text{supp}\alpha| = n$ then W under product × is not a NeutroAlgebra is a semigroup under ×.

Proof. Let $x=\frac{1}{t}(1+h+...+h^{t-1})$ and $y=\frac{1}{m}(1+K+...+K^{m-1})$ where h and K are powers of g and is in G. $x,y\in W$ with $|\operatorname{supp} x|=t$ and $|\operatorname{supp} y|=m$ with $mt=n$..

Thus $xy=\frac{1}{n}(1+g+...+g^{n-1})$ as $\operatorname{supp} x$ and $\operatorname{supp} y$ are subgroups of G.

Hence $(W,\times)$ is a semigroup, so W under $\times$ is not a NeutroAlgebra of idempotents.

Now we consider a cyclic group G of even order and obtain analogous results as in theorem for this QG when G is an odd composite number.

Theorem 3.2. *Let G be a cyclic group of even order say m; QG be the group ring of G over Q.*

i) The nontrivial idempotents of QG are of the form $\alpha=\frac{1}{t}(1+h+...+h^{t-1})$ or $\alpha'=\frac{1}{t}(1-h+h^2...-h^{t-1}+h^{t-1})$ where $h\in G$ with $\{1,h,...,h^{t-1}\}$ forming a proper subgroup of G of order t,t an even value (t can be only of even order if α' is to exist if t is of odd order; α' does not exist).

ii) If W = {collection of all idempotents of the form α and α' } then whenever α' is given as in (i) for the α given.

a) $\{W,\times\}$ is a NeutroAlgebra of idempotents of QG.

b) $\{W,+\}$ is an AntiAlgebra of idempotents of QG.

Proof. Given proper subgroups of G say of order t; t even, we have for $\alpha=\frac{1}{t}(1+h+...+h^{t-1})$ and $\alpha'=\frac{1}{t}(1-h+h^2-...-h^{t-2}+h^{t-1})$ are non-trivial idempotents of G.

Taking all even ordered subgroups of G, we have a collection of idempotents of the form α and α'. If the proper subgroup of G is odd-order say m then $\alpha=\frac{1}{m}(1+K+...+K^{m-1})$ are the only idempotents of QG.

If W is the collection of all idempotents of the form α, α' and so on then $(W,+)$ is an AntiAlgebra as no sum is defined.

If on the other hand, we include the trivial idempotent $0=0+0g+0g^2+...+0g^n$ then we see W under + is a NeutroAlgebra of idempotents of QG as $0+\alpha=\alpha$ for all $\alpha\in W$.

Now W under $\times$ is a NeutroAlgebra of idempotents for if α and β are two idempotents in W such that $|\operatorname{supp}\alpha|=K$ and $|\operatorname{supp}\beta|=m$ with $Km=n$ then $\alpha\times\beta=\frac{1}{n}(1+g+...+g^{n-1})$ the trivial idempotent of QG but by definition $|\operatorname{supp}\alpha\beta|=n$, the order of the whole group.

Thus, W under $\times$ is only a NeutroAlgebra, but if we allow the whole group idempotent $\frac{1}{n}(1+g+g^2+...+g^{n-1})$ in W, then W under $\times$ is not a NeutroAlgebra, in fact a semigroup. Hence the theorem.

Next, we proceed to prove the group ring QS_n has some idempotents sets W which forms AntiAlgebra under + and W under $\times$ happens to be a NeutroAlgebra.

We work mainly for this group S_n as every group G has a subgroup H of S_n, which is isomorphic with G [4, 5].

Theorem 3.3. *Let S_n be the symmetric group of degree n (S_n, in particular, be a permutation on (1, 2, 3, …, n)) Q be the field of rationals. QS_n the group ring of the group S_n over Q. QS_n has subsets of nontrivial idempotents, which under ×, is a NeutroAlgebra and under addition + is an AntiAlgebra of idempotents of QS_n.*

Proof. Every subgroup H in S_n for an appropriate n there exists a group G isomorphic with H. Thus, if H be a cyclic group say of some order m, then $G \cong H \subseteq S_n$ for some appropriate cyclic subgroup of order m.

Now, apart from this, S_n has ${}_nC_2$ number of subgroups of order two.

All elements of the form $W = \left\{\frac{1}{2}(1-p_1),\ \frac{1}{2}(1+p_1),\ \frac{1}{2}(1-p_2),\ \frac{1}{2}(1+p_2), \ldots,\ \frac{1}{2}(1-p_n),\ \frac{1}{2}(1+p_n)\right\} \subseteq QS_n$ are idempotents where p_i 's are permutations in S_n such that $p_i.p_i$ = (1, 2, 3, …, n) the identity permutation of S_n.

$\{W,+\}$ can easily be realized as an AntiAlgebra as no element under + in W is in W.

Now similarly $\{W,\times\}$ is a NeutroAlgebra as

$$(1-p_i)\times 1+p_i = 0 \notin W \text{ and } (1-p_i)(1-p_j) = 1-p_i-p_j+p_jp_i \text{ and so on.}$$

Thus $\{W,\times\}$ is only a NeutroAlgebra of idempotents from QS_n.

Hence the theorem.

Based on this study, we propose a few open problems in the last section of this paper.

4. NeutroAlgebra of idempotents in the group ring Z_nG

Next, we study idempotents in the group ring Z_nG where Z_n is the ring of modulo integers and n a prime or a composite number and G a group of finite order. Thus, the group rings in this section are of finite order.

We will first illustrate this situation with some examples.

Example 4.1. Let Z_2 be the field of order two $G = \langle g \mid g^3 = 1 \rangle$ be the cyclic group of order 3.

Z_2G be the group ring of G order Z_2

$\alpha = 1 + g + g^2$ is the only non-trivial idempotent of Z_2G; for $(1+g+g^2)^2 = 1+g+g^2$.

Remark 4.1. *Let Z_p be the field of primes. G be the cyclic group of order $p+1$ (or any other group which has subgroups of order $p+1$), then Z_pG has an idempotent of the form $\alpha = 1 + g + ... + g^p$.*

Proof. For the group ring Z_pG; $\alpha = \left(1 + g + ... + g^p\right)$ is the non-trivial idempotent of Z_pG.

Example 4.2. $Z_{11}G$ be the group ring of G over Z_{11}. $G = \langle g \mid g^{12} = 1 \rangle$ be a cyclic group of order 12.

$\alpha = 1 + g + ... + g^{11} \in Z_{11}G$ is an idempotent of $Z_{11}G$.

$\beta = (6 + 5g^6) \in Z_{11}G$ is also an idempotent of $Z_{11}G$.

$\gamma = (6 + 6g^6)$ is an idempotent of $Z_{11}G$.

Let $W = \{\alpha, \beta, \gamma\}$ be the 3 nontrivial idempotents of $Z_{11}G$.

We give the Cayley table for W under + given by Table 17 in the following.

Table 17: Table of $\{W, +\}$

+	α	β	γ
α	*od*	*od*	*od*
β	*od*	*od*	*od*
γ	*od*	*od*	*od*

$(W, +)$ is an AntiAlgebra of idempotents of the group ring $Z_{11}G$.

The Cayley table of W under $\times$ is as follows.

Table 18: Table of $\{W, \times\}$

$\times$	α	β	γ
α	α	*od*	γ
β	*od*	β	*od*

γ	γ	od	γ

Thus $(W,\times)$ is a NeutroAlgebra of idempotents of the group ring $Z_{11}G$.

Example 4.3. Let Z_7 be the field of prime order 7. S_8 be the permutation group of degree 8. Z_7S_8 be the group ring of S_8 over Z_7.

Let $H=\{1,p_1,p_2,...,p_7\}$ be the cyclic group generated by

$p=\begin{pmatrix}1&2&3&4&5&6&7&8\\8&1&2&3&4&5&6&7\end{pmatrix}$ and H is of order 8.

Clearly $\alpha=(1+p+p^2+...+p^7)\in Z_7S_8$ is a nontrivial idempotent of Z_7S_8

Consider $\beta=4+3p^4\in Z_7S_8$, we have $\beta^2=(4+3p^4)^2$

$= (16+9p^8+24p^4)$ (using $p^8=1$)

$= (25+24p^4)$

$= 4+3p^4=\beta$.

Thus, β is an idempotent of Z_7S_8.

Take $g=\begin{pmatrix}1&2&3&4&\dots&8\\2&1&3&4&\dots&8\end{pmatrix}\in S_8$

$g^2=\begin{pmatrix}1&2&3&4&\dots&8\\1&2&3&4&\dots&8\end{pmatrix}$;

the identity of S_8. Thus, we have ${}_8C_2$ number of such elements of order two in S_8.

Consider m = $4+3g\in Z_7S_8$, we see m² = m is an idempotent of Z_7S_8. In fact, we have ${}_8C_2$ number of such type of idempotents in the group ring Z_7S_8, where $g\in S_8$ is such that $g^2=\begin{pmatrix}1&2&3&4&\dots&8\\1&2&3&4&\dots&8\end{pmatrix}$.

Consider

$$t=4(1+g)\in Z_7S_8$$

$$t^2 = 16(1+g^2+2g)(g^2=1) = 16(2+2g) = 32(1+g) = 4(1+g) \in Z_7S_8$$

is an idempotent of the group ring Z_7S_8.

Thus, by this method also we have at least ${}_8C_2$ number of idempotents in Z_7S_8.

Now, if W is the collection of all idempotents of form $4(1+g)$ and $4+3g$ for varying $g \in S_8$ such that, $g^2 = id$ of S_8.

We see sum of $4+4g+4+3g=1$ is only a trivial idempotent.

$4+4g+4+4g=1+g$ is not an idempotent of this group ring Z_7S_8.

$3g+4+3g+4=6g+1$ is not an idempotent of Z_7S_8.

Thus, if $V = \{$collection of all non-trivial idempotents of the group ring Z_7S_8 of form $4(1+g)$ and $4+3g$ with all $g \in S_8$ such that g^2 is the identity element of $S_8\}$; then $(V,+)$ an AntiAlgebra of idempotents in Z_7S_8.

Also, we consider $4+3g \times 4+4g = 16+12g+16g+12 = 28+28g = 0$ is only a trivial idempotent of V and $0 \notin V$.

Consider $4+3g \times 4+4h$ $(h^2=1)$.

We see $16+12g+12h+12gh \notin V$.

Thus, V under $\times$ is a NeutroAlgebra of idempotents of Z_7S_8 as $(4+3g)^2 = (4+3g)$ and $(4+4h)^2 = (4+4h)$.

Based on all these we have the following results.

Example 4.4. Let Z_{11} be the finite prime field of order 11. S_{12} be the symmetric group of order $12!$. The group ring $Z_{11}S_{12}$ has a collection W of nontrivial idempotents from $Z_{11}S_{12}$ such that W under + is an AntiAlgebra of idempotents of the group ring $Z_{11}S_{12}$ and W under $\times$ is a NeutroAlgebra of idempotents of the group ring.

S_{12} has ${}_{12}C_2$ number of elements of order two. That is if

$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 12 \\ 2 & 1 & 3 & 4 & \dots & 12 \end{pmatrix} \text{ is such that } g_1^2 = \begin{pmatrix} 1 & 2 & 3 & \dots & 12 \\ 1 & 2 & 3 & \dots & 12 \end{pmatrix}$$

= identity permutation.

$$g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 12 \\ 3 & 2 & 1 & 4 & \dots & 12 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 12 \\ 4 & 2 & 3 & 1 & 5 & \dots & 12 \end{pmatrix}$$

and so $g_{11} = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 12 \\ 12 & 2 & 3 & 4 & \dots & 1 \end{pmatrix}$

Now $g_{12} = g_0 = \begin{pmatrix} 1 & 2 & 3 & \dots & 12 \\ 1 & 3 & 2 & \dots & 12 \end{pmatrix}$ identity element of S_{12}. Likewise, in W have ${}_{12}C_2$ number of such elements which are of order two.

Of course, there are other types of elements of order two also.

Our primary purpose is to prove the existence of some set of idempotents W of the group ring $Z_{11}S_{12}$ such that $(W,+)$ is an AntiAlgebra of idempotents and $(W,\times)$ is a NeutroAlgebra of idempotents.

So if we consider $W = \{6+6g_i, 6+5g_i \mid g_i$ is an element of order two in S_{12} described above} then first we show W is a collection of idempotents; then prove $\{W,+\}$ is an AntiAlgebra and {W, ×} is a NeutroAlgebra of idempotents under ×. Consider

$$x = 6+6g_i \text{ in } W,\ x^2 = (6+6g_i)^2 = 36+72g_i+36g_i^2 = = 72+72g_i (g_i^2 = 1) = 6+6g_i = x.$$

On similar lines it can be easily proved

$$y = (6+5g_i), y^2 = (6+5g_i)^2 = 36+60g_i+25g_i^2 = 61+60g_i = 6+5g_i = y.$$

So, W is the collection of idempotents.

Now W under + is not even closed for any pair. So $(W,+)$ is an AntiAlgebra of idempotents.

Further W under × is closed only for $(x \in W,\ x^2 = x)$ and not for any other pair.

So $(W,\times\}$ is a NeutroAlgebra of idempotents of the group ring $Z_{11}S_{12}$. Hence the claim.

However, for general group ring Z_pS_{p+1} (p a prime) we suggest it as an open problem in section 5.

Example 4.3. Let $G = \langle g \mid g^{10} = 1 \rangle$ be a cyclic group of order 10 and Z_{10} be the ring of integers modulo 10. $Z_{10}G$ be the group ring of G over Z_{10}.

Consider $\alpha = 3 + 2g^5 \in Z_{10}G$. We see

$$\alpha^2 = (3 + 2g^5) = 9 + 4 + 12g^5 = 3 + 2g^5 = \alpha$$

is an idempotent. of $Z_{10}G$.

Also $\beta = 3 + 8g$ in $Z_{10}G$ is such that

$$\beta^2 = (3 + 8g^5) = (9 + 64 + 48g^5) = 3 + 8g = \beta.$$

Hence β is an idempotent.

Let $a = 5(1 + g^2 + g^4 + g^6 + g^8) \in Z_{10}G$ where $a^2 = a$ so is an idempotent of $Z_{10}G$.

Take $b = 8 + 2g^5 \in Z_{10}G$; clearly

$$b^2 = (8 + 2g^5) = 64 + 4 + 32g^5 = 8 + 2g^5 = b.$$

Suppose we take the collection of some idempotents W in this group ring $Z_{10}G$; where

$$W = \{8 + 2g^5, 3 + 8g^5, 3 + 2g^5, 5(1 + g^2 + g^4 + g^6 + g^8)\}.$$

The Cayley table of W under + is given below.

Table 19. Cayley table of W under +.

+	α	β	a	b
α	*od*	*od*	*od*	*od*
β	*od*	*od*	*od*	*od*
a	*od*	*od*	*od*	*od*
b	*od*	*od*	*od*	*od*

The Cayley table of W are under × is given below.

Table 20. Cayley table with ×.

×	α	β	a	b
α	*od*	*od*	*od*	*od*
β	*od*	β	*od*	*od*
a	*od*	*od*	a	*od*

b	od	od	od	b

Thus, W under × is a NeutroAlgebra of idempotents of the group ring $Z_{10}G$ under product (×) operation.

We propose some open problems in the following section on conclusions.

5. Conclusions

In this section, we prove in general the a set of all non-trivial idempotents W in a group ring RG of a group G over a ring R have $\{W,+\}$ to be an AntiAlgebra of idempotents under + and $\{W,\times\}$ to be a NeutroAlgebra of idempotents under × for depending on R to be a ring of rationals or modulo integers Z_n (n a prime or a composite number) and G an appropriate finite group in the case of Z_n. Several examples are provided in the earlier for easy understanding.

We suggest some open problems for researchers in this direction, which will be taken by the authors for the future research.

Problem 5.1: Let Z_m be the ring of modulo integers n. S_n be the permutation group of degree n.

Given n and m fixed integers (we can find the solution for both small m and n; but finding for big m and n or a general m and n is challenging). We leave it as an open problem to find a collection of idempotents of the form.

$$W = \left\{(p+qg_i)/p^2+q^2 = p(\text{mod } n) \text{ and } 2pq = q(\text{mod } n) \text{ and } g_i \in S_m \text{ with } g_1^2 = \begin{pmatrix} 1 & 2 & 3 & \ldots & m \\ 1 & 2 & 3 & \ldots & , \end{pmatrix}\right\}$$

i) Further prove or disprove $(W,+)$ is an AntiAlgebra of idempotents of the group ring Z_mS_n.

ii) Prove or disprove $\{W,\times\}$ is a NeutroAlgebra of idempotents of the group ring Z_mS_n.

Can $\left\{\left(\frac{p+1}{2}+\frac{p+1}{2}g_i\right) \text{ and } \left(\frac{p+1}{2}+\frac{p-1}{2}g_i\right) \middle| g_i \in S_n \text{ set of group elements of order two in } S_{p+1}\right\}$, where the group ring is taken as Z_pS_{p+1} ; p is a prime? In the problem 5.1 we are replacing $m = p$(p is a prime) and $n = p + 1$.

Problem 5.2. Can QG and RG have idempotents (nontrivial) other than those mentioned in this paper to form a NeutroAlgebra or AntiAlgebra of idempotents of QG and RG under × or + respectively?

Problem 5.3. Can we have idempotents of the form $a_1 + a_2 g + a_3 g^2 + ... + a_n g^{n-1}$ with $g^n = 1;\ a_i \in Q \setminus \{0,1\};\ 1 \le i \le n$ in the group ring QG where $G = \langle g \mid g^n = 1 \rangle$ is a cyclic group of order n?

Problem 5.4. Let QG be the group ring. Can $\alpha g + \beta h \in QG$ where g and h are some two elements of $G(\alpha, \beta \in Q \setminus \{0,1\})$ be an idempotent for suitable α and β?

Problem 5.5. Let Z_n be the ring of integers modulo n (n a composite number). Prove there exists two integers p and q (p and q need not be prime in Z_n) such that $p^2 + q^2 = p \pmod n$ and $2pq = q (\bmod n)$.

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Neutrosophic Fuzzy X-Sub algebra of Near-Subtraction Semigroups

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Abstract: Neutrosophy fuzzy set is the extended research version of the fuzzy set that deals with imprecise and indeterminate data Neutrosophic deals with the membership, non-membership and indeterminacy function. Neutrosophy have achieved in various fields such as medical diagnosis, decision making problems, image processing etc.,The motivation of the present article is to extend the concept of Neutrosophic fuzzy X-subalgebra in near-subtraction semigroups. We will discuss along with some fundamentals and their algebraic Properties.

Keywords: Near subtraction Semigroup, Fuzzy Sub algebra , Fuzzy X-sub algebra, Neutrosophic Fuzzy Sub algebra, Neutrosophic Fuzzy X-sub algebra

1. Introduction

The Theory of Fuzzy subsets, fuzzy logic found in the research area of L.A. Zadeh[15]. The theory of Intuitionistic fuzzy set is the extension of the fuzzy set that deals with truth and false membership data. From the extension version, the term Neutrosophy was identified in the F. Smarandache [13]. Neutrosophy is a new concept in philosophy. Neutrosophic deals with the membership, non-membership and indeterminacy function. Neutrosophy have achieved in various fields such as medical diagnosis, decision making problems, image processing etc., Neutrosophy became the motivation of our manuscript.

Our present manuscript describes the Neutrosophic Fuzzy X-sub algebra (NFX-SA) of Near-Subtraction Semigroup and has conceptualized some basic algebraic properties.

The results obtained are entirely more beneficial to the researchers. Our aim of this manuscript is given as follows:

(i)To examine the some basic properties and fundamentals.

(ii) Also expand the Intersection, Quotient of the Set.

(iii) We also describe the Complement of the set.

2. Preliminaries

2.1 Definition[8]

Consider X to be define as a non empty along with the operations '-' and '•' is said to be a ***right near-subtraction semigroups*** if for p,q and r in X.

(i)With respect to '-' it defines as a subtraction algebra

(ii) With respect to '•' it defines as a semigroup

(iii)Right Distributive Law follows.

2.2Definition[9]

A fuzzy set μ in X is defined to be a ***fuzzy X-sub algebra*** of X if

(i) $\mu(p-q) \geq \min\{\mu(p), \mu(q)\}$
(ii) $\mu(pq) \geq \mu(q)$
(iii) $\mu(pq) \geq \mu(p)$ for each $p, q \in X$

(i)and (ii) gives μ is called ***fuzzy left X-Sub algebra*** of X and conditions (i) and (iii) gives μ is a ***fuzzy right X-sub algebra*** of X .

2.3Definition[9]

A ***Intuitionistic Fuzzy*** (IF) set $\upsilon = (\mu_\upsilon, \lambda_\upsilon)$ of X is said to be IF X-Sub algebra of X if

(i) $\mu_\upsilon(p-q) \geq \min\{\mu_\upsilon(p), \mu_\upsilon(q)\}$
$\lambda_\upsilon(p-q) \leq \max\{\lambda_\upsilon(p), \lambda_\upsilon(q)\}$
(ii) $\mu_\upsilon(pq) \geq \mu_\upsilon(p)$
$\lambda_\upsilon(pq) \leq \lambda_\upsilon(p)$
(iii) $\mu_\upsilon(pq) \geq \mu_\upsilon(q)$
$\lambda_\upsilon(pq) \leq \lambda_\upsilon(q)$ for each $p,q \in X$

Conditions that satisfy equation (i) and (ii) is called IF right X-sub algebra of X and the conditions that satisfies equation (i) and(iii) is called IF left X-sub algebra of X .

2.4Definition[8]

A ***Neutrosophic Fuzzy*** Set S defines on the universe of discourse X defined by a truth membership $T_S(p)$, indeterminacy function $I_S(p)$, and a false membership function $F_S(p)$ as $S = \{< p, T_S(p), I_S(p), F_S(p) > / p \text{ in } X\}$.Here, T_S, I_S, F_S:X→[0,1] and $0 \leq T_S(p) + I_S(p) + F_S(p) \leq 3$.

2.5Definition[8]

Consider a Neutrosophic fuzzy set V in X is defined to be ***Neutrosphic fuzzy near -subtraction subsemigroup*** of X if for all p,q, in X.

(i) $T_V(p-q) \geq \min\{T_V(p), T_V(q)\}$; $T_V(pq) \geq \min\{T_V(p), T_V(q)\}$

(ii) $I_V(p-q) \leq \max\{I_V(p), I_V(q)\}$; $I_V(pq) \leq \max\{I_V(p), I_V(q)\}$

(iii) $F_V(p-q) \leq \max\{F_V(p), F_V(q)\}$; $F_V(pq) \leq \max\{F_V(p), F_V(q)\}$

3. Neutrosophic Fuzzy X-sub algebra of Near-Subtraction Semigroups

This Section we introduced the basic properties of NFX-SA in Near-Subtraction Semigroup.

3.1Definition

A Neutrosophic fuzzy set S=(T_S, I_S, F_S) in X is said to be **NFX-SA** of X if for each p,q in X.

(i) T_S (p-q)≥min{ T_S (p), T_S(q)};I_S (p-q)≤max{ I_S (p), I_S(q)};F_S (p-q) ≤max{ F_S (p), F_S(q)}

(ii) T_S (pq)≥ T_S (p);I_S (pq) ≤ I_S (p);F_S (pq) ≤ F_S (p)

(iii)T_S (pq)≥ T_S (q);I_S (pq)≤ I_S (q);F_S (pq)≤ F_S (q)

Conditions that satisfies equation (i) and (ii) is called ***Neutrosophic Fuzzy right X-sub algebra*** of X and the conditions that satisfies equation(i) and(iii) is called ***Neutrosophic Fuzzy left X-sub algebra*** of X.

3.2 Example

Define X={0,p,q,r} to be a set defined by binary operations '- ' and '•' is

-	0	p	q	r
0	0	0	0	0
p	p	0	p	0
q	q	q	0	0
r	r	q	P	0

•	0	p	q	r
0	0	0	0	0
p	0	p	0	p
q	0	q	0	q
r	0	r	0	r

Let S:X→[0,1] be a fuzzy subset of X defined by

T_S(0)=.7 T_S(p)=.5 T_S(q)=.4 T_S(r)=.3

I_S(0)=.1 I_S(p)=.2 I_S(q)=.3 I_S(r)=.5

F_S(0)=.02 F_S(p)=.3 F_S(q)=.5 F_S(r)=.7

Hence, S is a NFX-SA of X.

3.3Theorem

If S=(T_S , I_S , F_S) be a NFX-SA of X ,then the set X_S={p in X/ T_S(p)= T_S(0); I_S(p)= I_S(0); F_S(p)= F_S(0)} is a X-sub algebra of X.

Proof:

Choose p,q in X_S. Thus T_S(p)= T_S(0); I_S(p)= I_S(0); F_S(p)= F_S(0) ;T_S(q)= T_S(0); I_S(q)= I_S(0); F_S(q)= F_S(0).

(i)T_S(p-q) ≥min{ T_S(p), T_S(q)}= T_S(0)

I_S(p-q) ≤max{ I_S(p), I_S(q)}= I_S(0).

F_S(p-q) ≤max{ F_S(p), F_S(q)}= F_S(0).

So,p-q∊X_S. Now

(ii)T_S(pq)≥T_S(p)= T_S(0).

I_S(pq)≤I_S(p)= I_S(0).

F_S(pq)≤ F_S(p)= F_S(0).

(iii)$T_S(pq) \geq T_S(q) = T_S(0)$.

$I_S(pq) \leq I_S(q) = I_S(0)$.

$F_S(pq) \leq F_S(q) = F_S(0)$.

So, $pq \in X_S$.

Thus, X_S is a X-sub algebra of X.

3.4 Theorem

The Complement of NFX-SA is again a NFX-SA of X.

Proof:

Assume that $S^c=(T_S^c, I_S^c, F_S^c)$ be the Complement set of the Neutrosophic fuzzy set $S=(T_S , I_S , F_S)$ of X.

Select $p,q,r \in X$.

Then

(i)$T_S^c(p-q)=1- T_S(pq)$

$\leq 1-\min\{ T_S(p), T_S(q)\}$

$=\max\{1- T_S(p),1- T_S(q)\}$

$=\max\{ T_S^c(p), T_S^c(q)\}$

$I_S^c(p-q) =1- I_S(pq)$

$\geq 1-\max\{ I_S(p), I_S(q)\}$

$=\min\{1- I_S(p),1- I_S(q)\}$

$=\min\{ I_S^c(p), I_S^c(q)\}$

$F_S^c(p-q) =1- F_S(pq)$

$\geq 1-\max\{ F_S(p), F_S(q)\}$

$=\min\{1- F_S(p),1- F_S(q)\}$

$=\min\{ F_S^c(p), F_S^c(q)\}$

(ii)$T_S^c(pq)=1- T_S(pq)$

$\leq 1-T_S(p)$

$= T_S^c(p)$

$I_S^c(pq) =1- I_S(pq)$

$\geq 1- I_S(p)$

$=I_S^c(p)$

$F_S^c(pq) =1- F_S(pq)$

$\geq 1- F_S(p)$

$= F_S^c(p)$

(iii)$T_S^c(pq)=1- T_S(pq)$

$\leq 1- T_S(q)$

$= T_S^c(q)$

$I_S^c(pq) =1- I_S(pq)$

$\geq 1-I_S(q)$

$= I_S^c(q)$

$F_S^c(pq) =1- F_S(pq)$

$\geq 1- F_S(q)$

$= F_{S^c}(q)$

Therefore, S^c is a NFX-SA of X.

3.5 Corollary

A Neutrosophic fuzzy set S=(T_S, I_S, F_S) of X is a NFX-SA of X iff (T_S,I_S T_S^c), (F_S^c, I_S, T_S^c), (F_S^c, I_S F_S) are NFX-SA's of X.

3.6 Theorem

If S_j =$\{(T_{S_j}, I_{S_j}, F_{S_j})/j\epsilon\ \delta\}$be a family of NFX-SA on X, then the set $\bigcap_{j\epsilon\delta} T_j$,$\bigcup_{j\epsilon\delta} I_{S_j}$ and $\bigcup_{j\epsilon\delta} F_{S_j}$ are also family of NFX-SA of X, where δ defines an index set.

Proof:

Assume that p,q,r in X.

Also $\bigcap_{j\epsilon\delta} T_{S_j}(p)=\inf_{j\epsilon\delta} T_{S_j}(p)$

$\bigcup_{j\epsilon\delta} I_{S_j}(p) = \sup_{j\epsilon\delta} I_{S_j}(p)$;

$\bigcup_{j\epsilon\delta} F_{S_j}(p) = \sup_{j\epsilon\delta} F_{S_j}(p)$.

Also T_{S_j} , I_{S_j} and F_{S_j} be a family of fuzzy X-sub algebra of X.

Now

(i)$\bigcap_{j\epsilon\delta} T_{S_j}(p-q) = \inf_{j\epsilon\delta} T_{S_j}(p-q) \geq \inf_{j\epsilon\delta} \min\{T_{S_j}(p), T_{S_j}(q)\}$

$=\min\{\inf_{j\epsilon\delta} T_{S_j}(p), \inf_{j\epsilon\delta} T_{S_j}(q)\}$

$=\min\{\bigcap_{j\epsilon\delta} T_{S_j}(p), \bigcap_{j\epsilon\delta} T_{S_j}(q)\}$

$\bigcup_{j\epsilon\delta} I_{S_j}(p-q)= \sup_{j\epsilon\delta} I_{S_j}(p-q) \leq \sup_{j\epsilon\delta} \max\{I_{S_j}(p), I_{S_j}(q)\}$

$=\max\{\sup_{j\epsilon\delta} I_{S_j}(p), \sup_{j\epsilon\delta} I_{S_j}(q)\}$

$=\max\{\bigcup_{j\epsilon\delta} I_{S_j}(p), \bigcup_{j\epsilon\delta} I_{S_j}(q)\}$

$\bigcup_{j\epsilon\delta} F_{S_j}(p-q)= \sup_{j\epsilon\delta} F_{S_j}(p-q) \leq \sup_{j\epsilon\delta} \max\{F_{S_j}(p), F_{S_j}(q)\}$

$=\max\{\sup_{j\epsilon\delta} F_{S_j}(p), \sup_{j\epsilon\delta} F_{S_j}(q)\}$

$=\max\{\bigcup_{j\epsilon\delta} F_{S_j}(p), \bigcup_{j\epsilon\delta} F_{S_j}(q)\}$

(ii)$\bigcap_{j\epsilon\delta} T_{S_j}(pq) = \inf_{j\epsilon\delta} T_{S_j}(pq) \geq \inf_{j\epsilon\delta} T_j(p) = \bigcap_{j\epsilon\delta} T_{S_j}(p)$

$\bigcup_{j\epsilon\delta} I_{S_j}(pq) = \sup_{j\in\delta} I_{S_j}(pq) \leq \sup_{j\in\delta} I_{S_j}(p) = \bigcup_{j\epsilon\delta} I_{S_j}(p)$

$\bigcup_{j\epsilon\delta} F_{S_j}(pq) = \sup_{j\in\delta} F_{S_j}(pq) \leq \sup_{j\in\delta} F_{S_j}(p) = \bigcup_{j\epsilon\delta} F_{S_j}(p)$

(iii) $\bigcap_{j\epsilon\delta} T_{S_j}(pq) = \inf_{j\in\delta} T_{S_j}(pq) \geq \inf_{j\in\delta} T_{S_j}(q) = \bigcap_{j\epsilon\delta} T_{S_j}(q)$

$\bigcup_{j\epsilon\delta} I_{S_j}(pq) = \sup_{j\in\delta} I_{S_j}(pq) \leq \sup_{j\in\delta} I_{S_j}(q) = \bigcup_{j\epsilon\delta} I_{S_j}(q)$

$\bigcup_{i\epsilon\delta} F_{S_j}(pq) = \sup_{j\in\delta} F_{S_j}(pq) \leq \sup_{j\in\delta} F_{S_j}(q) = \bigcup_{j\epsilon\delta} F_{S_j}(q)$

Hence the Proof.

3.7 Theorem

Consider S as a NFX-SA of X, then the fuzzy set S of X/I, where I is an ideal of X defined by

$T_S{}^{o}(p+I) = \sup_{q\in I} T_S(p+q);$

$I_S{}^{o}(p+I) = \inf_{q\in I} I_S(p+q);$

$F_S{}^{o}(p+I) = \inf_{q\in I} F_S(p+q)$

is a NFX-SA of Quotient near-subtraction Semigroup $X/_I$.

Proof:

Choose l,m in X so that l+I=m+I.

Then m=l+q where q in I.

To prove that S is well-defined.

$T_S{}^{o}(m+I) = \sup_{p\in I} T_S(m+p)$

$= \sup_{p\in I} T_S(l+q+p)$

$= \sup_{q+p=u\in I} T_S(l+u)$

$= T_S{}^{o}(l+I)$

$I_S{}^{o}(m+I) = \inf_{p\in I} I_S(m+p)$

$= \inf_{p\in I} I_S(l+q+p)$

$= \inf_{q+p=u\in I} I_S(l+u)$

$= I_S{}^{o}(l+I)$

$F_S{}^{o}(m+I)=\inf_{p\in I} F_S(m+p)$

$=\inf_{p\in I} F_S(l+q+p)$

$= \inf_{q+p=u\in I} F_S(l+u)$

$= F_S{}^{o}(l+I)$

Now

(i) $T_S{}^{o}((p+I)-(q+I))\geq\min\{ T_S{}^{o}(p+I), T_S{}^{o}(q+I)\}$

$I_S{}^{o}((p+I)-(q+I))\leq\max\{ I_S{}^{o}(p+I), I_S{}^{o}(q+I)\}$

$F_S{}^{o}((p+I)-(q+I))\leq\max\{ F_S{}^{o}(p+I), F_S{}^{o}(q+I)\}$

Let p+I,q+I in X/I

(ii) $T_S{}^{o}[(p+I)(q+I)]= T_S{}^{o}(pq+I)=\sup_{l\in I} T_S(pq+l)$

$= \sup_{l=ab\in I} T_S(pq+ab)$

$= \sup_{a,b\in I} T_S[(p+a)(q+b)]$

$\geq\sup_{a\in I} \{T_S(p+a)\}$

$=T_S{}^{o}(p+I)$

$I_S{}^{o}[(p+I)(q+I)]= I_S{}^{o}(pq+I)=\inf_{l\in I} I_S(pq+l)$

$= \inf_{l=ab\in I} I_S(pqr+ab)$

$= \inf_{a,b\in I} I_S[(p+a)(q+b)]$

$\leq\inf_{a,\in I} \{I_S(p+a)\}$

$= I_S{}^{o}(p+I)$

$F_S{}^{o}[(p+I)(q+I)]= F_S{}^{o}(pq+I)=\inf_{l\in I} F_S(pq+l)$

$= \inf_{l=ab\in I} F_S(pq+ab)$

$= \inf_{a,b\in I} F_S[(p+a)(q+b)]$

$\leq \inf_{a,b,c\in I} \{F_S(p+a)\}$

$=F_S{}^{o}(p+I)$

(iii)$T_S{}^{o}[(p+I)(q+I)]= T_S{}^{o}(pq+I)=\sup_{l\in I} T_S(pq+l)$

$= \sup_{l=ab\in I} T_S(pq+ab)$

$= \sup_{a,b\in I} T_S[(p+a)(q+b)]$

$\geq \sup_{a\in I} \{T_S(q+b)\}$

$=T_S{}^{o}(q+I)$

$I_S{}^{o}[(p+I)(q+I)]= I_S{}^{o}(pq+I)$ $=\inf_{l\in I} I_S(pq+l)$

$= \inf_{l=ab\in I} I_S(pqr+ab)$

$= \inf_{a,b\in I} I_S[(p+a)(q+b)]$

$\leq \inf_{a,\in I} \{I_S(q+b)\}$

$= I_S{}^{o}(q+I)$

$F_S{}^{o}[(p+I)(q+I)]= F_S{}^{o}(pq+I)$ $=\inf_{l\in I} F_S(pq+l)$

$= \inf_{l=ab\in I} F_S(pq+ab)$

$= \inf_{a,b\in I} F_S[(p+a)(q+b)]$

$\leq \inf_{a,b,c\in I} \{F_S(q+b)\}$

$=F_S{}^{o}(q+I)$

Hence the Proof.

4. Conclusion

In the present manuscript, we have defined the Intersection, Complement set, Quotient Set of NFX-SA in Near subtraction Semi group. This research work can be extended to other types of ideals and other algebraic structures of Near Subtraction Semi groups.

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Neutrosophic Goal Programming Approach for the Dash Diet Model

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Abstract: This paper deals with the modelling and optimization of health care management with particular reference to the Dietary Approaches to Stop Hypertension (DASH) diet problem in a neutrosophic environment. We have considered the degree of acceptance, indeterminacy, and rejection of objectives to express the DASH diet problem's vulnerability in modelling. Further, neutrosophic goal programming (NGP) by considering three different types of membership functions have been used to minimize the sum of deviation from the set goal. A case study has been discussed to determine an appropriate DASH diet based on cost and user preferences. The results indicated that goal programming (GP) and fuzzy goal programming (FGP) approach failed to provide the value of all the concerned decision variables related to different types of food. However, we can get all the concerned decision variables valuable for diet problems through NGP. The application developed in this study is a problem of optimization that provides users with a daily diet that contains all the necessary amounts of supplements with less expense. The fundamental fact of the DASH diet is not only to shed blood pressure however to decrease the circulatory strain of the body, and so that it can likewise enable the individuals who need to get in shape, lessen Cholesterol, and counteract diabetes.

Keywords: Health care Management; DASH Diet; Neutrosophic Goal Programming.

1. Introduction

The heightened pressure (and boredom) will lead people to neglect their safe eating plans and binge on whatever is around. A substantial number of people globally have diabetes and other forms of infectious diseases, and a significant amount of money is spent on this chronic disease. It is essential to monitor the healthy plan with a suitable diet for patients suffering from lifestyle-related diseases. Carbohydrates usually involve foods like Bread, roti, rice, vegetable, and other food grains; throughout, consumption of these has been curtailed. Individuals work from home and might have neglected their regular schedule, so they have found that their dietary patterns go for a flip. His/Her may contribute to two primary factors– One, they might end up eating and drinking healthier foods during the day and having too much unhealthier food when they feel depressed or anxious. A nutritious and balanced diet and physical activity are the most common and effective means to maintain a healthier body. There are different diets with a broad array of targets. A diet may be used to promote weight loss, ensure the maintenance of muscle mass, reduce premature weight gain during breastfeeding or regulate many chronic diseases such as cardiovascular disease, hypertension and kidney disease. The "Diet Problem" (the quest for a low-cost diet that can satisfy a soldier's dietary needs) is distinguished by a lengthy background, while in 2000 or later, most approaches to similar diet problems were created, after which computers with massive computation capacities were readily accessible, and linear programming (L.P.) techniques were established.

Operations Research approaches provide an essential and efficient resource for grappling with many healthcare issues. Bailey [1] wrote an article related to a queuing methodology to examine waiting times and appointments in hospital emergency services, which is believed to be the first optimization research to be applied to health care management. One of Operations Research's most effective techniques is the L.P. technique that can be extended to many nutritional problems relevant to food relief, regional food services, and specific dietary recommendations. Most researchers have used dietary limitations and cost limitations to evaluate dietary challenges and alternatives, although these work revealed vulnerabilities in the circumstances with a limited number of food products and nutritional restrictions. Effective strategies were obtained for diet problems using L.P. techniques (Smith [2]; Dantzig [3]; Fletcher [4]). However, while this strategy gives the greatest solution to the problem (the cheapest diet), the resulting diet looks to be both distasteful (requiring the consumption of the same items every day) and impracticable (specifying excessive quantities of the food types chosen). Dantzig [5] utilized L.P. to simulate optimal meal patterns under a number of limitations. Khoshbakht [6] wanted to see how a DASH diet affected youngsters with attention deficit hyperactivity disorder (ages 6–12 years). Paidipati et al. [7] provided some dependable approaches for determining optimal menu planning utilizing GP and reduced the variations of over and under accomplishment for suitable meal menu selection with varied energy (calorie) levels. A neutrosophic logic set was invented by Smarandache [8]. Brouni and Smarandache [9] investigated interval neutrosophic set (N.S.) and proposed a new operation on interval neutrosophic numbers. In a real-world example, Abdel-Basset et al. [10] proposed a strategy for addressing the L.P. issue in which N.S. theory plays a critical role. Their parameters were represented by a trapezoidal neutrosophic number, and a neutrosophic L.P. model approach was proposed. To cope with multi-production planning difficulties, Khan et al. [11] suggested an unique multi-objective model operating in an intuitionistic and Neutrosophic context.

This research aims to develop a new mathematical model that generates hypocaloric diets with high protein content. The mathematical model has two goals: the one is to reduce the diet's calorie count, and the other is to minimize the diet's expense along with some restrictions in the form of constraints, *i.e.*, amount of the Fat, amount of the Sodium, amount of the Cholesterol, amount of the saturated Fat, amount of the Calcium, amount of the Magnesium, amount of the Fibre, amount of the Potassium in the food. The model has been formulated in an uncertain environment and solved using a NGP approach. The results have also been compared with the GP approach and FGP approach.

The following is how the rest of the paper is structured: Section 2 is an overview of the literature on health care administration, diet management, NGP; Section 3 deals with the model formulation of the concerned problem along with preliminaries related to N.S. theory; computational experiment is performed in Section 4; and finally, in the last segment closing remarks are made.

2. Literature Review

There is comprehensive literature available on management strategies for managing health care services. One of the most often discussed topics is hospital resource management, focusing on staff workforce planning and correct nutritional capital distribution. Healthcare management's complexity and significance cannot be overstated, and optimization techniques have become a commonly employed healthcare management technique. Guo et al. [12] proposed a bi-objective distribution model for Community-based health resources assessment. The model explores a cost-price trade-off, where the price is represented as the overall number of demand nodes providing service over a defined distance threshold. Harris [13] used a non-linear modelling model to assess resource distribution in a multi-site needle exchange network to accomplish the highest potential decrease at reduced risk of new HIV infections. Benneyan et al. [14] implemented a destination-allocation model for long-term decision-making in the Veterans Health Management market. The objective feature is a weighted total of competing factors, including travel time, unoccupied ability and obscured demands. Günes et al. [15] proposed an allocation based model for implementing a primary health network. Three parameters are listed individually as usability factors, including the maximization of reach, attendance, and overall travel distance. M'Hallah and Alkhabbaz [16] examine the usage of the operational techniques in scheduling a Kuwaiti hospital recognizing specific restrictions on ethnicity, class, and nationality. They proposed a mixed integer L.P. model to reduce the number of nurses outsourced. Turgay and Taskın [17] presented a FGP model using exponential membership function to solve the healthcare model for efficient management solution and explained with data produced by a medical facility in Turkey-Sakarya. Jafari and Salmasi [18] established a mathematical programming metaheuristic method to optimize nursing priorities by analyzing patient and local policies and nurses' roles in Iran government hospitals. Because of the fluctuation of demands, Singh and Goh [19] proposed a pharmaceutical supply chain model comprising several raw material manufacturers, producers, and service centres of multiple hospitals. The developed model combined supply chain planning approaches from raw material sourcing to optimal drugs to hospital-level delivery plans. Yazdani et al. [20] addressed the control of healthcare waste disposal, which can create severe healthcare staff, patients, and the general population and suggested a novel best-worst approach of approximate interval figures due to the shortage of accurate information.

The importance of diet planning is not hidden to anyone, and in the past author used optimization techniques to get the desired amount of diet required for a healthier body. Eghbali et al. [21] addressed the human diet concern in a fuzzy context by considering nutritional diet variables-including taste and size, the volume of nutrients and their dietary intake. Mamat et al. [22] built a Fuzzy L.P. model for balanced diet planning that carries various nutrients a few times a day for each person. Eghbali et al. [23] addressed the application of fuzzy L.P. in diet meal preparation for eating disorders and lifestyle linked with illness. The formula is used to measure the volume of nutrients in the day to day routine. Fourer et al. [24] created an L.P. model to serve a week of fixed nutritious material from the mix of economic foods such as meat, macaroni, spaghetti and others. Another approach was generalized to produce the problem formulation of fish feeds, which would improve fish productivity (Nath and Talukdar [25]). Ali et al. [26] developed a quantitative diet planning model that satisfies the high school student has required nutritional consumption and minimizes a budget. Using an optimization approach coupled with 0-1 Integer Programming, the problem was solved. Ducrot et al. [27] studied the relation between meal preparation and diet consistency, including adherence to dietary recommendations and various foods and weight status. Eghbali-Zarch et al. [28] built a novel multi-objective mixed integer L.P. model to structure the diet plans for patients who have diabetes. The model's goals are to reduce the overall volume of saturated Fat, caffeine, Cholesterol, and the overall diet plan costs. The model's restrictions satisfy the body's nutritional needs and the complex regulation of each individual's diet. Sheng and Sufahani [29] addressed using integer programming to construct the quantitative diet planning design for eczema patients to cut diet costs by achieving the required amounts of nutrients, preventing food allergens and bringing other items into the diet that relieve eczema. Ghorabi et al. [30] reported their findings on the relationship between adherence to the dietary methods to stop hypertension (DASH) diet and metabolic syndrome and its components. Rodriguez et al. [31] investigated the effects of a Transtheoretical model-based personalized behavioral intervention, a non-tailored intervention, and usual care on the DASH eating pattern. According to Farhadnejad et al. [32], following the DASH diet may be beneficial in reducing metabolic abnormalities in overweight and obese people. Pirozeh et al. [33] described the DASH diet, which contains several antioxidants and helps to reduce oxidative stress. Soltani et al. [34] conducted a comprehensive review and meta-analysis to investigate the linear and non-linear dose-response relationship between DASH diet adherence and the causes of particular mortality. Khan *et al.* [35] discussed a daily diet model and minimized the cost of diet, Saturated Fat and carbohydrate. The diet model was solved by fuzzy multi-objective GP to satisfy daily nutrients and compared different methods. Kim *et al.* [36] investigated the similarity of metabolic urine maker and Serum metabolomic markers of the Dietary Approaches to Stop Hypertension (DASH) diet was reported. Ahmed et al. [37] presented a bipolar single-valued neutrosophic issue and used the score function to convert the fuzzy set into a crisp L.P. problem. Ahmed [38] defined the ranking function for transforming LR-type single-valued neutrosophic numbers and proposed a method for solving the LR-type single-valued neutrosophic L.P. problem using the transformation methodology. Das et al. [39] proposed the notion of single-valued neutrosophic numbers and a computer approach for solving the trapezoidal neutrosophic L.P. problem using the ranking function. Das and Edalatpanah [40] examined the diet issue using the Pythagorean fuzzy idea and used the score function to convert proportionate crisp L.P. issues; and proposed a unique technique for addressing the L.P. problem using Pythagorean fuzzy numbers. Das et al. [41] presented a theoretical study of completely fuzzy L.P. and solved it using the lexicographic ordering approach.

3. Model Formulation

This paper has considered one of the essential healthcare management applications, *i.e.,* the balanced diet problem. A healthy or balanced diet gives the body essential nutrients to function adequately. We eat much of the daily calories in fresh fruits, fresh herbs, rice, legumes, nuts, and lean proteins to get the diet's best nutrients. The calorie count of a meal is a calculation of the amount of energy contained in that product. In walking, speaking, swallowing and other essential tasks, the body utilizes calories from food. To preserve well-being, the average individual requires to consume around 2,000 calories per day.

Nevertheless, the same daily intake of calories may differ based on age, gender and degree of physical activity. People require more calories than women in general, and people who work out need to get more calories than people who do not. It is necessary to have a healthy diet since our organs and tissues need proper nutrition to function effectively. The body is more vulnerable to illness, exhaustion and reduced results without adequate nutrition. Children with a low diet run the risk of rising and developing problems, poor academic results, and bad eating habits that last for the rest of their lives. Keeping this thing in mind, we have considered the DASH diet problem for our model formulation. The DASH diet demonstrates the appropriate portion sizes, food diversity and nutrients and finds out how to improve health and reduce blood pressure. The DASH diet emphasizes veggies, fruits and low-fat dairy products with reasonable amounts of whole grains, meats, poultry and nuts. The diet is influencing the body in many respects:

- With the help of the DASH diet, healthy people and high blood pressure can reduce blood pressure.
- People cut out lots of high-fat with the DASH diet aid and may notice that they effectively reduce calorie intake and assist in weight reduction.
- There is a reduced chance of certain tumours with the DASH diet, including colorectal.
- The DASH diet decreases cardiovascular disease risk by as much as 81%.
- The DASH diet helps in reducing type 2 diabetes.

The following decision variables and parameters are used for the model formulation:

Nomenclature

***Decision Variable*:**

x_j Optimal quantity of food items

Parameters:

C_{Dj} Diet Cost of the j^{th} Food Item

c_{lj} Calorie in the j^{th} Food Item

f_{tj} Content/amount of the Fat in the j^{th} Food Item

F_t Tolerable maximum input level of the Total Fat

s_{dj} Content/amount of the Sodium in the j^{th} Food Item

S_d Tolerable maximum input level of the Sodium

c_{hj} Content/amount of the Cholesterol in the j^{th} Food Item

C_h Tolerable maximum input level of the Cholesterol

s_{fj} Content/amount of the Saturated Fat in the j^{th} Food Item

S_f Tolerable maximum input level of the total Saturated Fat

c_{aj} Content/amount of the Calcium in the j^{th} Food Item

C_a Tolerable minimum input level of the Calcium

m_{gj} Content/amount of the Magnesium in the j^{th} Food Item

M_g Tolerable minimum input level of the Magnesium

f_{bj} Content/amount of the Fibre in the j^{th} Food Item

F_b Tolerable minimum input level of the Fibre

p_{tj} Content/amount of the Potassium in the j^{th} Food Item

P_t Tolerable minimum input level of the Potassium

w_{fj} Weight of the j^{th} Food Item

W_f Maximum amount of all food

S_{Ljc} Estimated minimum number of daily servings of the j^{th} food item for calorie level c

S_{Ujc} Estimated maximum number of daily servings of the j^{th} food item for calorie level

With all these above define parameters and decision variables, the problem has been formulated as follows:

In our considered DASH diet model, let $x_j \, (j = 1, 2, ..., n)$ be the different types of food items required for the proper diet, and it works as a decision variable for us. The cost of serving and calories of each food item is $C_{Dj}, c_{lj} \, (j = 1, 2, ..., n)$. Then the objective function will be formulated as:

$$Min\, Z_1 = \sum_{j=1}^{n} C_{Dj} x_j \tag{1}$$

$$Min\, Z_2 = \sum_{j=1}^{n} c_{lj} x_j \tag{2}$$

The non-negative constraints of the model satisfy the nutrients requirements of the diet. The left-hand side of the non-negative constraints is the food items' nutrient contents based on the DASH concerning nutrients. The DASH research suggested that Sodium, Saturated Fat, Total Fat and Cholesterol be taken less and Magnesium, Potassium, Calcium, and Fibre be taken more to reduce the human body's high blood pressure.

Then the non-negative constraints are as follow:

$$\sum_{j=1}^{n} f_{tj} x_j \le F_t, \text{ Constraint for Total Fat} \tag{3}$$

$$\sum_{j=1}^{n} s_{dj} x_j \le S_d, \text{ Constraint for Sodium} \tag{4}$$

$$\sum_{j=1}^{n} c_{hj} x_j \le C_h, \quad \text{Constraint for Cholesterol} \tag{5}$$

$$\sum_{j=1}^{n} s_{fj} x_j \le S_f, \quad \text{Constraint for Saturated Fat} \tag{6}$$

$$\sum_{j=1}^{n} c_{aj} x_j \ge C_a, \quad \text{Constraint for Calcium} \tag{7}$$

$$\sum_{j=1}^{n} m_{gj} x_j \ge M_g, \text{ Constraint for Magnesium} \tag{8}$$

$$\sum_{j=1}^{n} f_{bj} x_j \ge F_b, \quad \text{Constraint for Fibre} \tag{9}$$

$$\sum_{j=1}^{n} p_{tj} x_j \ge P_t, \text{ Constraint on Potassium} \tag{10}$$

$$\sum_{j=1}^{n} w_{fj} x_j \ge W_f, \quad \text{Constraint for Food Weight} \tag{11}$$

$$S_{Ljc} \le x_j \le S_{Hjc}, \text{ Upper and lower limit of the daily serving} \tag{12}$$

The conceptual frameworks for analyzing health care management difficulties are frequently used to examine various success measures, which may be further subdivided into economic performance metrics and quality of service metrics. The existence of contradictory objective functions necessitates an ideal universal method to considering a viable response. In contrast, in recent years, modifications or generalizations of fuzzy set and intuitionistic fuzzy set (IFS) have been confronted with the concept that there is a degree of determinism in real existence, and as a result, a set known as N.S. has emerged. Smarandache suggested the concept of N.S. [42]. The term "neutosophic" is a mixture of two words: the French word "Neutre" means "neutral," and the Greek word "Sophia" means "talent." The concept of indeterminacy in N.S. helps to the possible scope of study in this area. The NGP technique was created based on the N.S. principle to find the optimum compromise solution for the multi-objective optimization issue.

The NGP involves three membership characteristics: maximizing reality "belongingness," indeterminacy "belongingness to some extent," and eliminating falsehood "non-belongingness." Abdel-Basset et al. [10] proposed and developed an effective approach for solving completely neutrosophic L.P. in production planning. Liu and Teng [43] proposed certain standard neutrosophic operators based on particular neutrosophic numbers and constructed a multi-criteria decision-making model based on the generic weighted power mean neutrosophic number operator. Rizk-Allah et al. [44] posed the transportation issue in a neutrosophic setting and enhanced the results achieved with existing approaches by computing the classification degree with the TOPSIS method. For example, if 0.6 is the chance that the diet is healthy, 0.3 is the diet that is not healthy, and 0.1 is the diet about which we are unsure. In this scenario, this type of linguistic ambiguity or inaccuracy extends beyond the bounds of a fuzzy and IFS in order to make the correct judgment. As a result, the neutrosophic decision-based optimization strategy is more applicable to real-world optimization problems than other well-known approaches since it works with three membership functions, namely truth, indeterminacy, and a false membership function. The indeterminacy membership functionality, on the other hand, cannot be accepted by the fuzzy and intuitive fuzzy decision set. Some of the necessary preliminaries belong to N.S. has been taken from (Ali et al. [45]; Abdel-Baset et al. [46]; Haq et al. [47]; Gupta et al. [48]) and are given below:

Definition 1: A real fuzzy number $\tilde{x}$ is a continuous fuzzy subset from the real line $\Re$ whose triangular membership function $\alpha_{\tilde{x}}(x)$ is defined by a continuous mapping from $\Re$ in the closed interval [0,1], where

1. $\alpha_{\tilde{x}}(x) = 0 \; \forall \; x \in (-\infty, x_1]$,
2. $\alpha_{\tilde{x}}(x)$ to be strictly increasing on $x \in [x_1, m]$,
3. $\alpha_{\tilde{x}}(x) = 1$ for $x = m$,
4. $\alpha_{\tilde{x}}(x)$ to be strictly decreasing on $x \in [m, x_2]$,
5. $\alpha_{\tilde{x}}(x) = 0 \; \forall \; x \in (x_2, +\infty]$

It is elicited by:

$$\alpha_{\tilde{x}}(x) = \begin{cases} \dfrac{x - x_1}{m - x_1}, & x_1 \le x \le m, \\ \dfrac{x_2 - x}{x_2 - m}, & m \le x \le x_2, \\ 0, & \text{otherwise} \end{cases}$$

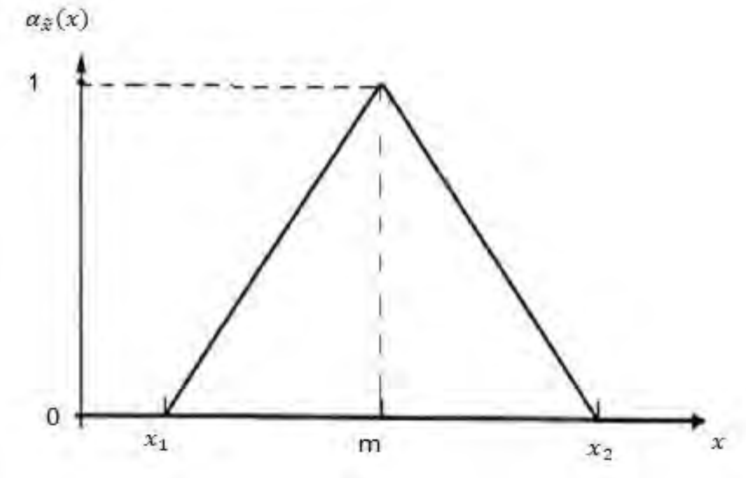


Fig. 1: Membership Function $\tilde{x}$

Where m is a targeted value and x_1 and x_2 denote the value of the lower and upper bound. In this case, we obtain

$$\alpha(x\,; x_1, m, x_2) = \text{Max}\left\{\text{Min}\left[\frac{x - x_1}{m - x_1}, \frac{x_2 - x}{x_2 - m}\right], 0\right\}$$

Definition 2: Let $T = \{t_1, t_2, \ldots, t_n\}$ is a fixed non-empty universe, an IFS X in T is defined as

$$X = \{\langle t, \alpha_X(t), \gamma_X(t)\rangle | t \in T\},$$

which is characterized by a membership function $\alpha_X : T \to [0,1]$ and a non-membership function $\gamma_X : T \to [0,1]$ with the condition $0 \le \alpha_X(t) + \gamma_X(t) \le 1 \,\forall\, t \in T$ where $\alpha_X(t)$ and $\gamma_X(t)$ represent, respectively, the degree of membership and non-membership of the element t to the set X. Also, for each IFS X in $T, \pi_X(t) = 1 - \alpha_X(t) + \gamma_X(t) \,\forall\, t \in T$ is called the degree of hesitation of the element t to the set X. Significantly if $\pi_X(t) = 0$, then the IFS X is degraded to a fuzzy set.

Definition 3: The α-level set of the fuzzy parameters $\tilde{x}$ in definition (1) is defined as the ordinary set $L_\alpha(\tilde{x})$ for which the degree of membership function exceeds the level, $\alpha, \alpha \in [0,1]$, where

$$L_\alpha(\tilde{x}) = \{x \in \Re | \alpha_{\tilde{x}}(x) \ge \alpha\},$$

for specific values α_x^* to be in the unit interval.

Definition 4: Let T is an object and $t \in T$. A N.S. X in T is defined by a truth membership function (t), an indeterminacy membership function (t) and a falsity membership function (t). It has shown in figure 2. Truth-membership function (t), indeterminacy membership function (t) and falsity-membership function (t) are real standard or real nonstandard subsets of $]0^-,1^+[$. That is $T_X(t): T \to]0^-,1^+[$, $I_X(t): T \to]0^-,1^+[$ and $F_X(t): T \to]0^-,1^+[$. There are no restrictions on the sum of truth-membership function (t), indeterminacy membership function (t) and falsity-membership function (t), $0^- \le \sup T_X(t) \le \sup I_X(t) \le F_X(t) \le 3^+$.

In the following, we adopt the notations $\alpha_X(x), \beta_X(x)$ and $\gamma_X(x)$ instead of $T_X(t),\ I_X(t)$ and $F_X(t)$ respectively.

Definition 5: Let T is a universe of discourse. A single-valued neutrosophic (SVN) set X over T is an object having the form

$$\mathrm{X} = \{\langle \alpha_X(t), \beta_X(t), \gamma_X(t)\rangle : t \in T\},$$

where $\alpha_X(t): T \to [0,1]$, $\beta_X(t): T \to [0,1]$ and $\gamma_X(t): T \to [0,1]$ with $0 \le \alpha_X(t) + \beta_X(t) + \gamma_X(t) \le 3\ \forall\, t \in T$. The intervals $\alpha_X(t), \beta_X(t)$ and $\gamma_X(t)$ denote the truth membership degree, the indeterminacy-membership degree and the falsity membership degree of t to X, respect.

For convenience, an SVN number is denoted by $X = (a,b,c)$, where $a,b,c \in [0,1]$ and $a+b+c \le 3$.

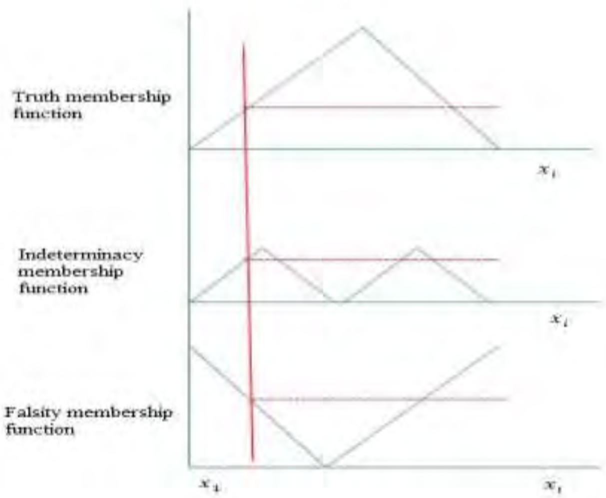


Fig. 2: Neutrosophication Process

Definition 6: Let $\tilde{x}$ is a neutrosophic number in the set of real numbers $\Re$, then its truth-membership function is

$$\alpha_{\tilde{x}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ 1+\dfrac{a_2-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & \text{otherwise} \end{cases}$$

its indeterminacy-membership function is

$$\beta_{\tilde{x}}(x) = \begin{cases} \dfrac{x-b_1}{b_2-b_1}, & b_1 \le x \le b_2 \\ 1+\dfrac{b_2-x}{b_3-b_2}, & b_2 \le x \le b_3 \\ 0, & \text{otherwise} \end{cases}$$

and its falsity-membership function is

$$\gamma_{\tilde{x}}(x) = \begin{cases} 1-\dfrac{x-c_1}{c_2-c_1}, & c_1 \le x \le c_2 \\ 1-\dfrac{c_3-x}{c_3-c_2}, & c_2 \le x \le c_3 \\ 0, & \text{otherwise} \end{cases}$$

Let $Z_k, k=1,2$ be the objective function with the target value T_k, acceptance tolerance limit A_k, Indeterminacy tolerance limit I_k, rejection tolerance limit R_k. Then, the Truth Membership, Indeterminacy Membership and Falsity membership Functions will be defined as follows:

Truth membership function

$$\alpha_k^T(Z_k) = \begin{cases} 1, & \text{if } Z_k \le T_k \\ 1-\dfrac{(Z_k-T_k)}{A_k}, & \text{if } T_k \le Z_k \le T_k + A_k \\ 0, & \text{if } Z_k \ge T_k + A_k \end{cases} \quad (13)$$

Indeterminacy membership function

$$\beta_k^I(Z_k) = \begin{cases} 0, & \text{if } Z_k \le T_k \\ \dfrac{Z_k-T_k}{I_i}, & \text{if } T_k \le Z_k \le T_k + I_k \\ 1-\dfrac{(Z_k-(T_k+I_k))}{A_k-I_k}, & \text{if } T_k + I_k \le Z_k \le T_k + A_k \\ 0, & \text{if } Z_k \ge T_k + A_k \end{cases} \quad (14)$$

Falsity membership function

$$\gamma_k^F(Z_k) = \begin{cases} 0, & \text{if } Z_k \le T_k \\ \dfrac{(Z_k-T_k)}{R_k}, & \text{if } T_k \le Z_k \le T_k + R_k \\ 1, & \text{if } Z_k \ge T_k + R_k \end{cases} \quad (15)$$

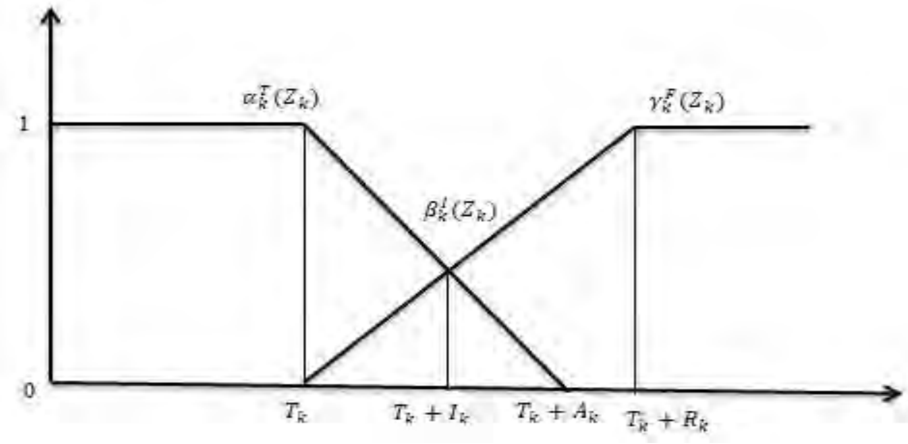


Fig. 3: Truth, Indeterminacy and Falsity Membership Functions for Z_k

To solve the above formulated multi-objective programming problem of the DASH diet, we have used the NGP approach by using the Truth, Indeterminacy and Falsity Membership Functions, and therefore further, the problem can be re-written as:

$$
\begin{aligned}
& Min\,(1\text{-}\alpha_k^T)(1-\beta_k^I).\gamma_k^F \\
& \alpha_k^T(Z_k)=\alpha_k,\ \ k=1,2,\ldots,K \\
& \beta_k^I(Z_k)=\beta_k,\ \ k=1,2,\ldots,K \\
& \gamma_k^F(Z_k)=\alpha_k,\ \ k=1,2,\ldots,K \\
& \sum_{j=1}^{n} f_{tj}x_j \le F_t\,,\quad \sum_{j=1}^{n} s_{dj}x_j \le S_d\,, \\
& \sum_{j=1}^{n} c_{hj}x_j \le C_h\,,\quad \sum_{j=1}^{n} s_{fj}x_j \le S_f\,, \\
& \sum_{j=1}^{n} c_{aj}x_j \ge C_a\,,\quad \sum_{j=1}^{n} m_{gj}x_j \ge M_g\,, \\
& \sum_{j=1}^{n} f_{bj}x_j \ge F_b\,,\quad \sum_{j=1}^{n} p_{tj}x_j \ge P_t\,, \\
& \sum_{j=1}^{n} w_{fj}x_j \ge W_f\,, \\
& x_j \ge S_{Ljc}\,,\quad j=1,2,\ldots,n \\
& x_j \le S_{Hjc}\,,\quad j=1,2,\ldots,n \\
& 0 \le \alpha_k+\beta_k+\gamma_k \le 3 \\
& \alpha_k,\gamma_k \ge 0, \beta_k \le 1,\ \ \alpha_k,\beta_k,\gamma_k \in [0,1]
\end{aligned}
$$

where $Max\,\alpha_k^T, Max\,\beta_k^I$ are equivalent to $Min\,(1-\alpha_k^T), Min\,(1\text{-}\beta_k^I)$ respectively for all

$0 \le \alpha_k^T, \beta_k^I \le 1$. If we take $(1-\alpha_k^T)(1-\beta_k^I)\gamma_k^F = v_k$ then the problem further reduces to:

$$
\begin{aligned}
& Min \;\; v_k \\
& (1-\alpha_k)(1-\beta_k)\gamma_k = v_k \\
& \sum_{j=1}^{n} f_{tj} x_j \le F_t \;, \quad \sum_{j=1}^{n} s_{dj} x_j \le S_d \;, \quad \sum_{j=1}^{n} c_{hj} x_j \le C_h, \quad \sum_{j=1}^{n} s_{fj} x_j \le S_f, \\
& \sum_{j=1}^{n} c_{aj} x_j \ge C_a, \quad \sum_{j=1}^{n} m_{gj} x_j \ge M_g, \quad \sum_{j=1}^{n} f_{bj} x_j \ge F_b \quad , \quad \sum_{j=1}^{n} p_{tj} x_j \ge P_t, \\
& \sum_{j=1}^{n} w_{fj} x_j \ge W_f, \quad x_j \ge S_{Ljc}, \quad x_j \le S_{Hjc} \;, \;\; j = 1,2,\ldots,n \\
& \alpha_k^T (Z_k) = \alpha_k, \quad \beta_k^I (Z_k) = \beta_k, \quad \gamma_k^F (Z_k) = \alpha_k, \quad k = 1,2 \\
& 0 \le \alpha_k + \beta_k + \gamma_k \le 3, \;\; \alpha_k, \gamma_k \ge 0, \beta_k \le 1, \;\; \alpha_k, \beta_k, \gamma_k \in [0,1]
\end{aligned}
$$

4. Case Study

The developed diet model is explained using a collection of the real data set (Iwuji et al. [49]; Iwuji and Agwu [50]). Here we assessed the situation where a person wanted the best DASH Diet plan with 2000 calories. Table 1 indicates the required calories needed by male and female individuals in some age groups and activity rates. Here we offer a random set of 8 sample foods from the different food groups (i.e. wheat, beans, fruits, low-fat milk items, Fish and nuts) for the DASH diet and the maximum and minimum intake level of the nutrients. The foods packages with their nutrient composition, weight (in grams), requirements, availability and cost, are shown in Tables 2 and 3.

Table 1: DASH daily calorie chart for the different levels of activities

Gender	Age	Calorie needed for each activity level		
		Sedentary	Moderately Active	Active
Male	19-30	2400	2600-2800	3000
	31-50	2200	2400-2600	2800-3000
	51+	2000	2200-2400	2400-2800
Female	19-30	2000	2000-2200	2400
	31-50	1800	2000	2200
	51+	1600	1800	2000-2200

Table 2: Foods with their nutrient composition, weight (in grams), requirements

Nutrients	Foods								Max./Min. requirement
	Carrot	Ground Nut	Bread (Whole Wheat)	Sweet Potato (Boiled)	Milk (Low Fat)	Orange	Water Melon	Fish (Grilled)	
Total Fat	0.24	11.48	0.58	0.30	0.10	0.48	0.16	4.10	≤68
Sodium (mg)	33.60	1.50	124.80	15.00	8.10	3.20	2.40	73.00	≤1500
Cholesterol (mg)	0	0	0	0	3.00	0	0	0.29	≤129

Saturated Fat (g)	0	1.55	0.20	0	0.60	0	0	34.00	≤ 16
Calcium (mg)	28	4.25	12.25	24	25	49.6	5.6	40	≥ 1334
Magnesium (mg)	9.6	47.75	13.25	14	2.4	17.6	8	43	≥ 542
Fiber (g)	2.48	2.33	1.55	3	0	2.72	0.29	0	≥ 34
Potassium (mg)	212.8	181.75	56.5	264	31	262.6	87.2	397	≥ 4721
Weight per serving of Foods (g)	80	25	25	100	2	160	80	100	≤ 4000

Table 3: Minimum and maximum Availability of Foods with cost and Calorie

	Carrot	**Ground Nut**	**Bread (Whole Wheat)**	**Sweet Potato (Boiled)**	**Milk (Low Fat)**	**Orange**	**Water Melon**	**Fish (Grilled)**
Minimum Availability	4	0	3	4	6	4	4	0
Maximum Availability	20	1	8	6	9	8	9	6
Cost of per serving of Food ($)	15	20	15	15	30	15	15	50
Calorie	28	144.5	58.5	90	7	72	23.2	151

Cost per serving of the foods and their nutrient information with the maximum and minimum intake level; the estimated minimum and the maximum number of servings of foods into the above-formulated DASH diet model. The above-given table values, with the Target value (T_k), acceptance tolerance limit (A_k), Indeterminacy tolerance limit (I_k), Rejection tolerance limit (R_k), for k^{th} objectives are shown in Table 4.

Table 4: Target value, Acceptance, Indeterminacy and Rejection Tolerance Limit

k	**Target Value** T_k	**Acceptance Tolerance Limit** A_k	**Indeterminacy Tolerance Limit** I_k	**Rejection Tolerance Limit** R_k
1.	825	150	100	200
2.	2000	600	400	700

Using the table (3) value; Truth, Indeterminacy and Falsity membership Functions have been constructed as:

Truth Membership Function

$$\alpha_1^T(Z_1)=\begin{cases}1, & \text{if } Z_1 \le 825\\ 1-\dfrac{(Z_1-825)}{150}, & \text{if } 825 \le Z_1 \le 975\\ 0, & \text{if } Z_1 \ge 975\end{cases}, \quad \alpha_2^T(Z_2)=\begin{cases}1, & \text{if } Z_2 \le 2000\\ 1-\dfrac{(Z_2-2000)}{600}, & \text{if } 2000 \le Z_2 \le 2600\\ 0, & \text{if } Z_2 \ge 2600\end{cases}$$

Indeterminacy Membership Function

$$\beta_1^i(Z_1)=\begin{cases}0, & \text{if } Z_1 \le 825\\ \dfrac{Z_1-825}{100}, & \text{if } 825 \le Z_1 \le 925\\ 1-\dfrac{(Z_1-925)}{50}, & \text{if } 925 \le Z_1 \le 975\\ 0, & \text{if } Z_1 \ge 975\end{cases}, \quad \beta_2^i(Z_2)=\begin{cases}0, & \text{if } Z_2 \le 2000\\ \dfrac{Z_2-2000}{400}, & \text{if } 2000 \le Z_2 \le 2400\\ 1-\dfrac{(Z_2-2400)}{200}, & \text{if } 2400 \le Z_2 \le 2600\\ 0, & \text{if } Z_2 \ge 2600\end{cases}$$

Falsity Membership Function

$$\gamma_1^F(Z_1)=\begin{cases}0, & \text{if } Z_1 \le 825\\ \dfrac{(Z_1-825)}{200}, & \text{if } 825 \le Z_1 \le 1025\\ 1, & \text{if } Z_1 \ge 1025\end{cases} \quad \gamma_2^F(Z_2)=\begin{cases}0, & \text{if } Z_2 \le 2000\\ \dfrac{(Z_2-2000)}{700}, & \text{if } 2000 \le Z_2 \le 2700\\ 1, & \text{if } Z_2 \ge 2700\end{cases}$$

Using all the membership functions in the NCG model, finally, it has been solved using the optimizing software LINGO 16.0. The optimal compromise solution for the proposed model is summarized in Table 5.

Table 5: Optimal Compromise Daily Diet Plan

Foods	Daily Serving Sizes	Cost of Servings ($)	Calorie available in the foods
Carrot (cut up)	20	300.00	560.00
Groundnut (boiled, without salt)	0.7158484	14.31	103.44
Bread (whole wheat)	3	45.00	175.50
Sweet potato (boiled, without salt)	5.427117	81.40	488.44
Milk (low fat, skimmed) (' 00gm)	7	210.00	49.00
Orange	8	120.00	576.00
Watermelon	4	60.00	92.80
Fish (grilled, without salt)	0.2439209	12.20	36.83
Optimal daily diet cost		**842.91**	
Total calories			**2082.01**

The proposed healthy diet is composed of 20 servings of carrots, 0.7158484mg of Groundnut (boiled, without salt), three servings of Bread (whole wheat), around five servings of Sweet potato (boiled, without salt), around 700gm serving of Milk (low Fat, skimmed), around eight serving of Orange, around four serving of Watermelon, and around 0.2439209mg of Fish (grilled, without salt). The graphical representation of the compromise solution and the proposed model's membership values is shown in **Fig. 4** and **Fig. 5,** respectively.

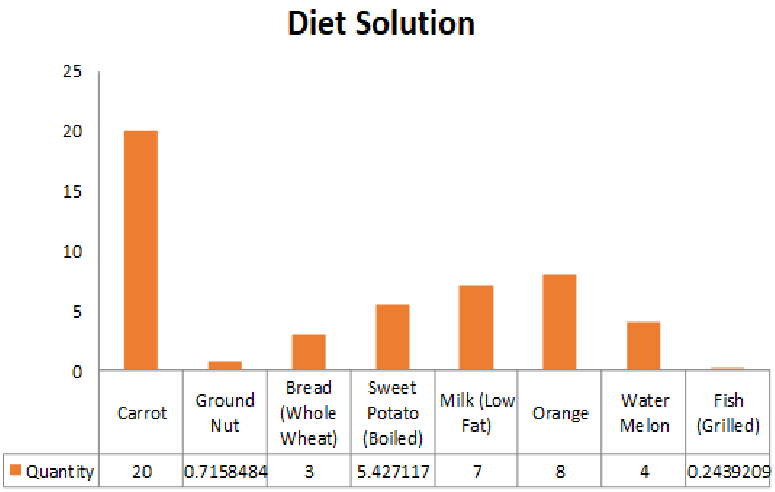


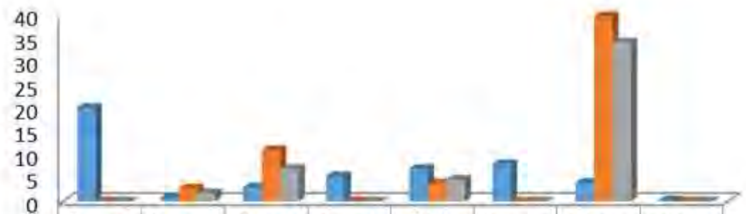
Fig. 4: The Optimal Compromise Solution

Fig. 5: Membership values of the proposed model

Table 6 compares the proposed approach with the famous GP approach and the FGP approach.

Table 6: Comparison of Results

Food	**Carrot (cut up)**	**Groundnut (boiled, without salt)**	**Bread (whole wheat)**	**Sweet potato (boiled, without salt)**	**Milk (low Fat, skimmed)**	**Orange**	**Water melon**	**Fish (grilled, without salt)**
Proposed Approach	20	0.7158484	3	5.427117	7	8	4	0.2439209
GP	0	2.8	11	0	3.9	0	39.7	0.02
FGP	0	1.8	7	0	4.7	0	34.1	0.02



	Carrot (cut up)	Ground nut (boiled, without salt)	Bread (whole wheat)	Sweet potato (boiled, without salt)	Milk (low fat, skimmed)	Orange	Water melon	Fish (grilled, without salt)
Proposed Approach	20	0.7158484	3	5.427117	7	8	4	0.2439209
Goal Programming	0	2.8	11	0	3.9	0	39.7	0.02
Fuzzy Programming	0	1.8	7	0	4.7	0	34.1	0.02

Fig. 6: Comparison of Result

Table 5 and Fig. 6 show our proposed method's supremacy over GP and FGP approach. The results indicated that GP and FGP approach failed to provide the value of all the concerned decision variables, but through the NGP, we can get all the concerned decision variables that are very important for diet problems. There are other behaviours that people can follow to improve their good well-being and the general standard of wellness. Making sure that one consumes a well-planned diet is essential. Nutrients, carbohydrates and minerals can be incorporated into a daily diet. They will also concentrate on including nuts, beans, fruits and vegetables in their diet, alongside garlic and garlic. It is therefore essential to take caution to reduce processed carbs and turn to foods containing natural carbs.

The right amount of nutrition is much more important in situations where the immune system may need to strike back to protect from infections. Eating a diverse, nutritious diet is the easiest way to achieve a full range of nutrients like micronutrients and vitamins. This will also reduce the chances of various severe health problems, including obesity, type 2 diabetes and cardiac failure. There is growing proof that vitamin D can have intestinal safety benefits. Recently, a report has suggested that vitamin d consumption are linked to higher death rates. Dietary Fibre lets you shed weight and reduce belly fat, lowering the chances of diabetes and cardiac failure. This encourages better gut microbes and leads to a balanced immune system. A tasty and healthy diet — comprising of lots of fresh fruits, leafy green vegetables — together with physical exercise can improve our immunity and keep us fit.

6. Motivation and Contribution

This work is prompted by a research topic in NGP that has the potential to capture decision-makers. The following are the study's contributions:

i. It contributes to the existing literature on the DASH Diet issue.

ii. ii. Solution strategies for multi-objective multi-product problem formulation are described in a case study.

iii. For the DASH Diet, a novel technique based on neutrosophic was used in this study.

The technique is compared to GP and FGP, with the conclusion demonstrating that the proposed work is superior.

6. Conclusion

The human body needs foods with low sodium content, saturated Fat, total Fat, and Cholesterol, although high in Potassium, Magnesium, Calcium, and Fibre. The DASH diet has difficulty making regular dietary schedules that fulfil the tolerable consumption rates of nutrients of the diets at a defined expenditure dependent on the required daily calorie and sodium amounts by the persons involved to minimize elevated blood pressure and other diseases. Here, to find out the optimum solution of the formulated multi-objective DASH diet optimization model, we implemented a neutrosophic optimization approach by combining three different types of membership functions, *i.e.,* Truth, Indeterminacy and Falsity. In the formulated model, our main aim is to concurrently optimize calorie consumption and diet cost by minimizing the deviation from the set goal. The formulated DASH diet NGP model has turned into a crisp type model by utilizing truth, indeterminacy, and falsity membership functions. The GP and FGP approach failed to provide the value of all the concerned decision variables, but through the NGP, we can get all the concerned decision variables that are very important for diet problems. The finding obtained in the neutrosophic optimization approach contrasts with the GP and FGP approaches. It demonstrates that the NGP approach gives a more transparent and accurate solution and is a useful optimization technique than the other current method.

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Multi-objective Production Planning Problem: Interval-valued Trapezoidal Neutrosophic and Multi-Choice Parameters

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Abstract: An intuitionistic multi-objective programming problem with interval-valued trapezoidal neutrosophic (IVTN) and multi-choice interval type has been considered in this paper. The coefficients of objective functions and parameters of the left side of the constraints are in the multi-choice environment, and the right-hand side of the constraints are in the IVTN number type. The formulated problem's multi-choice parameters were transformed into the deterministic form using the binary variable transformation technique. A procedure is defined to change the IVTN number into the deterministic form. Then, intuitionistic fuzzy programming (IFP) with two different scalarization models has been used to achieve each membership goal's highest degree and obtain a satisfactory decision-making solution. Finally, a numerical case study for production planning (PP) is explored to validate the work's efficiency and usefulness.

Keywords: Multi-objective programming, multi-choice, interval-valued trapezoidal neutrosophic number, intuitionistic fuzzy programming.

1. Introduction

In PP, a well-organized approach is used in which raw materials are transformed into an optimal quantity of the final products to maintain the performance and quality of the item. The main aim of PP is to comprehend consumer conditions and demands and improve the product design and other enhancements to meet customers' needs while achieving a desirable profit. The universal character of machinery must be used for a specific resource to determine the number of items for the specific time, product categories, labour character classification, and manufacturing process (cycle). Customers who demand resistance in the PP process and further reflection on comfort service rates and organizational benefit are influenced by the firm's profit and level of service planning. The manufacturing processes are streamlined to win the company struggle in the global marketplace. Nowadays, industry experts and analysts use optimization methods for the PP models to ensure maximum benefit with minimum unit production. Mosadegh et al. [33] addressed four criteria: idle time and overtime, employment size, inventory and scarcity, and currency preservation. Jaggi et al. [20] described the multi-objective PP problem under certain conditions for a lock industry. Ghosh and Mondal [12] discussed a production-distribution planning model and found a suitable solution using the genetic algorithm and a two-echelon supply chain. Gupta et al.(2019) [14] discussed certain and uncertain environments for a two-stage transportation problem and used fuzzy goal programming to find a compromise solution.

The experts or decision-makers fixed the mathematical problem's parameter. In real-life scenarios, the parameters are unknown, and the parameters of the optimization problem have become either random or fuzzy variables. In this study, the problem's parameters are in the form of a multi-choice or IVTN number. The multi-choice programming problem avoids the wastage of resources and chooses the best resource. Such problems arise in finance, health care, manufacturing, agriculture, transportation, engineering, military, and technology.

2. Literature Review

The most critical activity in the manufacturing process is PP. Manufacturing firms establish a development schedule at the start of each fiscal year. The ideal development schedule provides a complete picture of how many products will be manufactured during each period and the demand for each period over the fiscal year. The production schedule may be carried out regularly, monthly, yearly, or even annually, depending on the product's demand. Production scheduling is the process of allocating available production capacity over time to meet certain requirements such as delivery time, cost, supply and demand. Machine capacity planning, production management, transportation, and freight schedules are all examples of production-related issues. Over the last two decades, international competitions, technical advances, and market dynamics have impacted the manufacturing sector. The majority of PP issues are multi-objective. The researchers used the e-constraint approach to reduce a multi-objective problem to a single goal. The problem of organizing the output and distribution functions was explored by Chandra and Fisher [4]. A single plant with multi-commodity, multi-period manufacturing environments produces products processed in the plant before being shipped to consumers. Yan et al. [45] defined a strategic production-distribution model with multiple manufacturers, distribution centres, retailers, and consumers in which multiple goods are produced in a single cycle. The fuzzy multi-objective linear programming model effectively solves real-world PP problems. Nowadays, businesses seek to achieve more than one target goals to improve the PP system's consistency and response. The concept of fuzzy set was developed by Zadeh [46]. Zimmermann [49] applied the fuzzy linear programming approaches to the linear vector maximum problem.

In multi-choice programming problems, the decision-maker can consider many options for a parameter problem, but only one must be chosen to optimize the goal value. Healey [17] introduced the concept of multi-choice and considered a case study on the mixed-integer programming problem. Chang [5,6] formulated the multi-choice programming problem with binary variables and suggested a modified approach for the multi-choice objective programming model. Biswal and Acharya [2] recommended the generalized transformation technique for solving multi-choice linear programming problems in which constraints parameters are bound to certain multi-choices. Haq et al. [15] used fuzzy goal programming to solve the optimal case study's PP problem. Khan et al. [23] discussed the IVTN number and used neutrosophic and intuitionistic fuzzy programming to solve a PP problem. Haq et al. [16] discussed the neutrosophic fuzzy programming for the sustainable development goal's problem. Oluyisola et al. [34] prescribed a methodology for designing and developing a smart PP and control system and discussed PP and control challenges in manufacturing technologies for planning environment characteristics. Lohmer and Lasch [28] studied the multi-factory PP and scheduling problems; and classified the literature according to shop configuration, network structure, objectives, and solution methods. Raza and Hameed [36] worked on maintenance planning and scheduling and provided effective guidelines for future studies in the research area. Some conflicting issues such as growing economic demand, increasing energy supply, shrinking energy resources, changing climate conditions, and tightening environmental requirements pose significant challenges for planning energy systems towards cleaner production and sustainable development. Suo et al. [42] developed the ensemble energy system model for China (CN-EES model), incorporating a computable general equilibrium model and interval-parameter programming method within an energy system optimization framework.

The CN-EES model can predict energy demands under different economic-development scenarios, reflecting uncertainties derived from the long-term (2021–2050) planning period and providing optimal solutions for China's energy system transition and management. Some significant research contributions in PP are summarized in Table 1:

Table 1: Review Summary

Researchers	**Goals**	**Applied Techniques**							**Clarifications**
		GP	FP	LP	SP	GA	BL	IP	
Kanyalkar and Adil (2005) [22]	Multi	✓			✓				Dynamic Allocation, Hierarchical Planning Approach, Stochastic Programming, Goal Programming,
Leung and Chan (2009) [24]	Multi	✓							Goal Programming
Baykasoglu and Gocken (2010) [1]	Multi		✓						Tabu Search, Ranking Method, Metaheuristic Algorithm
Che and Chiang (2010) [7]	Multi					✓			Modified Pareto, Genetic Algorithm
Liu et al. (2011) [27]	Multi					✓			Genetic Algorithm, Aggregate PP
Sillekens et al. (2011) [39]	Single								Linear Approximation, Mixed Integer Linear Programming, Aggregate PP
Ramezanian et al. (2012) [35]	Single					✓			Tabu search Mixed, Two-phase Aggregate PP, Integer Linear Programming Model, Genetic Algorithm
Mortezaei et al. (2013) [32]	Multi		✓			✓			Aggregate PP
Liu and Papageorgiou (2013) [26]	Multi			✓					ε-constraint Method, Mixed-integer Linear Programming, Lexicographic Mini-max Method
Madadi and Wong [29]	Multi		✓						IBM ILOG CPLEX Optimization Studio Software, Multiobjective Fuzzy Aggregate PP
Chen and Huang (2014) [8]	Multi		✓						Aggregate PP, Parametric Programming
da Silva and Marins (2014) [9]	Multi	✓	✓						Agricultural and Logistics Phase, Fuzzy Goal Programming
Singh and Yadav (2015) [40]	Multi							✓	Multi-objective Linear Programming, Interval-valued Intuitionistic
Gholamian et al. (2016) [11]	Multi		✓						GAMS Software, Mixed-integer Non-linear Programming, Supply Chain Planning
Lin et al. (2016) [25]	Multi		✓						Multi-objective Optimization Evolutionary Algorithm, Integrated PP
Meistering and Stadtler (2019) [31]	Multi						✓		Bi-level Programming, Hierarchical PP
Zhao et al. [48]	Multi				✓				Multi-stage Stochastic Programming, Progressive Hedging Algorithm
Goli et al. (2019) [13]	Multi	✓							Goal Programming, Robust Multi-objective Multi-period Aggregate PP
Hu et al. (2020) [18]	Single				✓				Two-stage Stochastic Programming
Discussed Model	Multi							✓	IVTN number, IFP, Multi-choice

In this paper, we propose a multi-objective industrial development planning problem with a multi-choice interval type and IVTN parameter. This paper addresses the situation of multi-choices in the objective and the constraints' left-hand side. We have used general transformation methodology given by Roy and Maity [37] to achieve the crisp form for the multi-choice parameter. Moreover, the right-hand sides of the restrictions are of the IVTN form. The formulated problem's compromise solution is obtained by the IFP and compared for both models.

3. Prerequisites

This section discussed some fundamental definitions regarding the intuitionistic fuzzy (IF) number and neutrosophic fuzzy number.

Definition 3.1: [Zadeh (1965) [46]] $\tilde{A}$ of X is a fuzzy set having the form $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ that represents the membership degree with $\mu_{\tilde{A}} : X \to [0,1]$. $\tilde{A}$ on $\Re$ is convex if and if for each pair of point x_1, x_2 in X, and $\tilde{A}$ satisfies the inequality

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} \quad \forall\, x_1, x_2 \in X, \lambda \in [0,1]$$

Definition 3.2: [Ebrahimnejad and Verdegay, 2018 [10]; Mahajan and Gupta, 2019 [30]] $\tilde{A}^I$ in X is an IF set of ordered triples $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \upsilon_{\tilde{A}^I}(x)) : x \in X\}$, where $\mu_{\tilde{A}^I}(x) : X \to [0,1]$ and $\upsilon_{\tilde{A}^I}(x) : X \to [0,1]$ represent the membership degree and non-membership degree, such that $0 \leq \mu_{\tilde{A}^I}(x) + \upsilon_{\tilde{A}^I}(x) \leq 1, \forall x \in X$.

Definition 3.3: [Smarandache, 1999 [41]; Wang et al., 2010 [44]] A in X is a neutrosophic set characterized by a truth-membership $\mu_A^T(x)$, an indeterminacy membership $\sigma_A^I(x)$, and a falsity-membership $\upsilon_A^F(x)$, $\mu_A^T(x), \sigma_A^I(x), \upsilon_A^F(x) \in (0,1)$ or $[0,1], \forall x \in X$ and $0^+ \leq \sup\, \mu_A^T(x) + \sup\, \sigma_A^I(x) + \sup\, \upsilon_A^F(x) \leq 3^-$ or $0 \leq \sup\, \mu_A^T(x) + \sup\, \sigma_A^I(x) + \sup\, \upsilon_A^F(x) \leq 3$.

Definition 3.4: [Ishibuchi and Tanaka, [19]] An interval on $\Re$ is as $A = [a^L, a^R] = \{a : a^L \leq a \leq a^R, a \in \Re\}$, a^R is the right limit and a^L is left limit of A. Or $A = \langle a_c, a_w \rangle = \{a : a_c - a_w \leq a \leq a_c + a_w, a \in \Re\}$, centre and width of A is $a_c = \frac{1}{2}(a^R + a^L)$ and $a_w = \frac{1}{2}(a^R - a^L)$ respectively.

Definition 3.5: [Broumi and Smarandache, 2015] [3] An interval-valued neutrosophic (IVN) set $\tilde{A}^{IV}$ of X is as follows:

$$\tilde{A}^{IV} = \{\langle x;\ [\mu_k^{TL}, \mu_k^{TU}], [\sigma_k^{IL}, \sigma_k^{IU}], [\upsilon_k^{FL}, \upsilon_k^{FU}] \rangle : x \in X\},$$

where $[\mu_k^{TL}, \mu_k^{TU}], [\sigma_k^{IL}, \sigma_k^{IU}]$ and $[\upsilon_k^{FL}, \upsilon_k^{FU}] \subset [0,1]$ for each $x \in X$

Definition 3.6: [Broumi and Smarandache, 2015] [3] Let
$\tilde{A}^{IV} = \{ \langle x;\ [\mu_k^{TL}, \mu_k^{TU}], [\sigma_k^{IL}, \sigma_k^{IU}], [\upsilon_k^{FL}, \upsilon_k^{FU}] \rangle : x \in X\}$ be IVN set, then

- **(i)** $\tilde{A}^{IV}$ is empty if $\mu_k^{TL} = \mu_k^{TU} = 0,\quad \sigma_k^{IL} = \sigma_k^{IU} = 1,\ \upsilon_k^{FL} = \upsilon_k^{FU} = 1,\ \forall\, x \in \tilde{A}$
- **(ii)** Let $\underline{1} = \langle x;\ 1, 0, 0 \rangle$ **and** $\underline{0} = \langle x;\ 0, 1, 1 \rangle$.

Definition 3.7: (IVTN number) Let $\mu_{\tilde{a}}, \sigma_{\tilde{a}}, \upsilon_{\tilde{a}} \subset [0,1]$, and $a_1, a_2, a_3, a_4 \in \Re$ such that $a_1 \leq a_2 \leq a_3 \leq a_4$. Then an interval-valued trapezoidal fuzzy neutrosophic number,

$$\tilde{a} = \langle (a_1, a_2, a_3, a_4);\ [\mu_{\tilde{a}}^L, \mu_{\tilde{a}}^U], [\sigma_{\tilde{a}}^L, \sigma_{\tilde{a}}^U], [\upsilon_{\tilde{a}}^L, \upsilon_{\tilde{a}}^U] \rangle,$$

membership degrees, indeterminacy degrees and non-membership degrees are

$$\mu_{\tilde{a}}(x)=\begin{cases}\mu_{\tilde{a}}^{U}\left(\dfrac{x-a_1}{a_2-a_1}\right), & x\in[a_1,a_2],\\ \mu_{\tilde{a}}^{U}, & x\in[a_2,a_3],\\ \mu_{\tilde{a}}^{U}\left(\dfrac{x-a_4}{a_3-a_4}\right), & x\in[a_3,a_4],\\ 0, & \text{other}\end{cases}\qquad \upsilon_{\tilde{a}}(x)=\begin{cases}\dfrac{\upsilon_{\tilde{a}}^{L}(x-a_1)+a_2-x}{a_2-a_1}, & x\in[a_1,a_2]\\ \upsilon_{\tilde{a}}^{L}, & x\in[a_2,a_3]\\ \dfrac{\upsilon_{\tilde{a}}^{L}(a_4-x)+x-a_3}{a_4-a_3}, & x\in[a_3,a_4]\\ 1, & \text{other}\end{cases}$$

$$\sigma_{\tilde{a}}(x)=\begin{cases}\dfrac{a_2-x+\sigma_{\tilde{a}}^{L}(x-a_1)}{a_2-a_1}, & x\in[a_1,a_2]\\ \sigma_{\tilde{a}}^{L}, & x\in[a_2,a_3]\\ \dfrac{x-a_3+\sigma_{\tilde{a}}^{L}(a_4-x)}{a_4-x_3}, & x\in[a_3,a_4]\\ 1, & \text{other}\end{cases}$$

Definition 3.8: [Thamaraiselvi and Santhi, 2015 [43]] The score function for the IVN number $\tilde{a}=\left\langle (a_1,a_2,a_3,a_4);\ \left[\mu_{\tilde{a}}^{L},\mu_{\tilde{a}}^{U}\right],\left[\sigma_{\tilde{a}}^{L},\sigma_{\tilde{a}}^{U}\right],\left[\upsilon_{\tilde{a}}^{L},\upsilon_{\tilde{a}}^{U}\right]\right\rangle$ is defined as

$$S(\tilde{a})=\frac{(a_1+a_2+a_3+a_4)}{16}\left[\mu_{\tilde{a}}(x)+\left(1-\upsilon_{\tilde{a}}(x)\right)+\left(1-\sigma_{\tilde{a}}(x)\right)\right]$$

3.1 Transformation Technique for Multi-Choice Parameter [Roy et al., 2017 [38]]

The selection procedure of multi-choice in the problem parameter should help to optimize the problem. The binary variable concepts play a vital role in selecting a choice from the problem's values. Among t numbers of possibilities, p numbers of binary variables are used, where $2^{p-1}<t\le 2^{p}$.

Let $t={}^{p}C_0+{}^{p}C_1+{}^{p}C_2+...+{}^{p}C_d+k$ for some d satisfying $1\le d\le p,\ 0\le k<{}^{p}C_{d+1}$.

If $d=p$, then $k=0$, and $k\ne 0$, then $d<p$ in the selection procedure, ${}^{p}c_i$ numbers of possibilities have value zero for i binary variables among p variables in selecting a single choice from multi-choice parameters.

p binary variables $z_j^1, z_j^2, ..., z_j^p$ are taken to reduce the formula in selecting the t values of $c_j^1, c_j^2, ..., c_j^t$. We further construct p binary variable's function $f_0(z)=(z_j^1\ z_j^2\ ...\ z_j^p)c_j^1$, where $z=(z_j^1\ z_j^2\ ...\ z_j^p)$, when each $z_j^1=1$ for $j=1, 2, ..., p$. where, $f_0(z)=c_j^1$, while $z_j^1+z_j^2+...+z_j^p=p$ and we adopt a function $f_1(z)=(1-z_j^1)z_j^2...z_j^p c_j^2+(1-z_j^2)z_j^1 z_j^3...z_j^p c_j^3+...+(1-z_j^p)z_j^1...z_j^{p-1}c_j^{1+{}^{p}c_1}$.

If $z_j^1 + z_j^2 + ... + z_j^p = p-1,\ f_1(z)$ gives a value among the following parameters of $c_j^1, c_j^2, c_j^3 ..., c_j^{1+{}^pc_1}$. Similarly,

$$
\begin{aligned}
f_2(z) = & (1-z_j^1)(1-z_j^2)\, z_j^3 ... z_j^p\, c_j^{1+{}^pc_1+1} + (1-z_j^1)(1-z_j^3)\, z_j^2\, z_j^4 ... z_j^p\, c_j^{1+{}^pc_1+2} \\
& + ... + (1-z_j^1)(1-z_j^p)\, z_j^2 ... z_j^{p-1}\, c_j^{1+{}^pc_1+(p-2)} + (1-z_j^2)(1-z_j^3)\, z_j^1\, z_j^4 ... z_j^p\, c_j^{1+{}^pc_1+(p-2)+1} \\
& + ... + (1-z_j^{p-1})(1-z_j^p)\, z_j^1 ... z_j^{p-2}\, c_j^{1+{}^pc_1+{}^pc_2}
\end{aligned}
$$

If $z_j^1 + z_j^2 + ... + z_j^p = p-2$, the $f_2(z)$ function gives a value from the parameters $c_j^t:\ c_j^{1+{}^pc_1+1},\ c_j^{1+{}^pc_1+2},\ ..., c_j^{1+{}^pc_1+{}^pc_2}$.

Proceeding similarly, we have

$$
\begin{aligned}
f_d(z) = & (1-z_j^1)(1-z_j^2)...(1-z_j^d)\, z_j^{d+1} ... z_j^p\, c_j^{1+{}^pc_1+{}^pc_2+...+{}^pc_{d-1}+1} \\
& + (1-z_j^1)(1-z_j^2)...(1-z_j^{d-1})(1-z_j^{d+1})\, z_j^d\, z_j^{d+2} ... z_j^p\, c_j^{1+{}^pc_1+{}^pc_2+...+{}^pc_{d-1}+2} \\
& + ... + (1-z_j^{p-d+1})(1-z_j^{p-d+2})(1-z_j^p)\, z_j^1 ... z_j^{p-d}\, c_j^{1+{}^pc_1+{}^pc_2+...+{}^pc_d}
\end{aligned}
$$

If $z_j^1 + z_j^2 + ... + z_j^p = p-d$, the $f_d(z)$ function gives a number from the

$$C_j^{t's}:\ c_j^{1+{}^pc_1+{}^pc_2+...+{}^pc_{d-1}+1},\ c_j^{1+{}^pc_1+{}^pc_2+...+{}^pc_{d-1}+2},\ ...,\ c_j^{1+{}^pc_1+{}^pc_2+...+{}^pc_{d-1}+{}^pc_d}.$$

For $k=0$, then $f(z) = f_0(z) + f_1(z) + ... + f_d(z)$ & $f(z)$ function gives a value from the $c_j^t, \forall z$ that satisfy $p-d \le z_j^1 + z_j^2 + ... + z_j^p \le p$.

If $k \ne 0$ then $k < {}^pc_{d+1}$ and the formulated function

$$
\begin{aligned}
f_{d+1}(z) = & (1-z_j^{i_1})(1-z_j^{i_2})...(1-z_j^{i_d})(1-z_j^{i_{d+1}})\, z_j^{d+2} ... z_j^n\, c_j^{t-k+1} \\
& + (1-z_j^{i_1})(1-z_j^{i_2})...(1-z_j^{i_d})(1-z_j^{i_{d+2}})\, z_j^{d+1}\, z_j^{d+3} ... z_j^p\, c_j^{t-k+2} \\
& + ... + (\text{terms up-to } c_j^t)
\end{aligned}
$$

Whenever $z_j^1 + z_j^2 + ... + z_j^p = p-(d+1),\ f_{d+1}(z)$ gives a value from ${}^pc_{d+1}$ numbers and restrictions ${}^pc_{d+1} - k$ are used to reduce the possible outputs on the k number. The k^{th} term occurred at $i_1 = i_1',\ i_2 = i_2',\ ...,\ i_{d+1} = i_{d+1}'$ so the restrictions will be

$$p-(d+1)\le z_j^{i_1}+z_j^{i_2}+\ldots+z_j^{i_p}\le p;\ z_j^{i_1}+z_j^{i_2}+\ldots+z_j^{i_{d+1}}\ge 1,\forall\ i_1=i_1',\ i_2=i_2',\ \ldots,\ i_d=i_d';i_p\ge i_{d+1}>i_{d+1}';$$
$$z_j^{i_1}+z_j^{i_2}+\ldots+z_j^{i_{d+1}}\ge 1,\ \forall\ i_1=i_1',\ i_2=i_2',\ \ldots,\ i_{d-1}=i_{d-1}',\ i_{p-1}\ge i_d>i_d';$$
$$z_j^{i_1}+z_j^{i_2}+\ldots+z_j^{i_{d+1}}\ge 1,\ \forall\ i_{p-d-1}\ge i_1>i_1'$$

So, $f(z)=f_0(z)+f_1(z)+\ldots+f_d(z)+f_{d+1}(z)$ gives the general function without loss of the generality for the selection of the multi-choice parameters c_j^t, the value of $c_j^t=1$ and used the summation and product multiplication, the formula in selecting the crisp values for the multi-choice parameters are:

$$\prod_{i=1}^{p} z_j^i+\sum_{i_1}^{p}\left[(1-z_j^{i_1})\prod_{i=1,i\ne i_1}^{p} z_j^i\right]+\sum_{\substack{i_2=2\\ i_2>i_1}}^{p}\sum_{i_1=1}^{p}\left[\left[(1-z_j^{i_1})(1-z_j^{i_2})\prod_{i=1,i\ne i_1,i_2}^{p} z_j^i\right]\right]$$
$$+\ldots+\sum_{\substack{i_d=d\\ i_d>i_{(d-1)}}}^{p}\ \sum_{\substack{i_{d-1}=d-1\\ i_{d-1}>i_{d-2}}}^{p}\cdots\sum_{i_1=1}^{p}\left[(1-z_j^{i_1})(1-z_j^{i_2})\ldots(1-z_j^{i_d})\prod_{i=1,i\ne i_1,\ldots,i_d}^{p} z_j^i\right]$$

where, $p-d\le z_j^{i_1}+z_j^{i_2}+\ldots+z_j^{i_p}\le p,\ \forall\ i_1<i_2<\ldots.<i_p$

If $k\ne 0$, the first *k* terms will be added with the above function through the formula:

$$(1-z_j^{i_1})(1-z_j^{i_2})\ldots(1-z_j^{i_d})(1-z_j^{i_{d+1}})\prod_{\substack{i=1,i\ne i_1,\\ \ldots,i_d,i_{d+1}}}^{p} z_j^i+(1-z_j^{i_1})(1-z_j^{i_2})\ldots(1-z_j^{i_d})(1-z_j^{i_{d+2}})\prod_{\substack{i=1,i\ne i_1,\\ \ldots,i_d,i_{d+2}}}^{p} z_j^i$$
$$+\ldots+(1-z_j^{i_1})(1-z_j^{i_2})\ldots(1-z_j^{i_d})(1-z_j^{i_p})\prod_{\substack{i=1,i\ne i_1,\\ \ldots,i_d,i_p}}^{p} z_j^i+(1-z_j^{i_1})(1-z_j^{i_2})\ldots(1-z_j^{i_{d+1}})(1-z_j^{i_{d+2}})\prod_{\substack{i=1,i\ne i_1,\\ \ldots,i_{d+1},i_{d+2}}}^{p} z_j^i$$
$$+(1-z_j^{i_1})(1-z_j^{i_2})\ldots(1-z_j^{i_{d+1}})(1-z_j^{i_{d+3}})\prod_{\substack{i=1,i\ne i_1,\\ \ldots,i_{d+1}}}^{p} z_j^i+\ldots+(1-z_j^{i_{1p-(d+1)}})(1-z_j^{i_{p-(d-1)}})\ldots(1-z_j^{i_{p-1}})(1-z_j^{i_p})\prod_{\substack{i\ne i_{p-d-1},\\ i_{p-d+1},\ldots,i_p}}^{p} z_j^i.$$

If, $i_1<i_2<\ldots.<i_p$ and the k[th] term occurs at $i_1'\ ,i_2'\ ,\ldots,i_{d+1}'$, so the restrictions are

$$p-(d+1)\le z_j^{i_1}+z_j^{i_2}+\ldots+z_j^{i_p}\le p;\ z_j^{i_1}+z_j^{i_2}+\ldots+z_j^{i_{d+1}}\ge 1,\forall\ i_1=i_1',i_2=i_2',\ldots,i_d=i_d',\ i_p\ge i_{d+1}>i_{d+1}';$$
$$z_j^{i_1}+z_j^{i_2}+\ldots+z_j^{i_{d+1}}\ge 1,\ \forall\ i_1=i_1',i_2=i_2',\ldots,i_{d-1}=i_{d+1}',\ i_{p-1}\ge i_d>i_d';$$
$$z_j^{i_1}+z_j^{i_2}+\ldots+z_j^{i_{d+1}}\ge 1,\ \forall i_{p-d-1}\ge i_1>i_1'$$

Let, $\tilde{C}_{ij}'=\sum_{g=1}^{t}(term)^g\left[C_{ij}^{g^l}(1-\lambda^{C_{ij}^g})\right]+C_{ij}^{g^u}\lambda^{C_{ij\alpha_c}^g},\ i=1,2,,\ldots,m\ ;\ j=1,2,\ldots,n$

where, $(\text{term})^g,\ \forall g=1,\ 2,\ \ldots,\ t$ are the *t* numbers in the form of binary variables. Similarly,

$$\tilde{a}'_{ij} = \sum_{g=1}^{p} (term)^g \left[a_{ij}^{g^l} (1-\lambda^{a_{ij}^g}) \right] + a_{ij}^{g^u} \lambda^{a_{ij}^g},\ j=1,2,,...,n,\ (i=1,2,,...,m).$$

and $$\tilde{b}'_{i} = \sum_{g=1}^{q} (term)^g \left[b_{i}^{g^l} (1-\lambda^{b_{i}^g}) \right] + b_{i}^{g^u} \lambda^{b_{i}^g},\ (i=1,2,,...,m).$$

3.2 Conversion process for IVNN

The PP problem is expressed in terms of IVTN numbers. To demonstrate, assume the right-hand side of the multi-objective PP model constraints is IVTN numbers. The following is a step-by-step procedure shown to convert in crisp form.

Step 1: Discuss the problems in terms of IVTN numbers.
Step 2: The score function transforms the IVTN numbers problem into an interval-valued problem.
Step 3: The $\alpha -$cut approach is used to transform an interval-valued in a crisp form. For $\left[a,b\right]$

$$\alpha a + (1-\alpha) b$$

3.3 Methodologies

The multi-objective Model:

Maximize (Minimize) Z(x) $=\left\{Z_1(x), Z_2(x), ..., Z_k(x)\right\}$

Subject to

$g(x) \le 0$

$x \in X$

In solving the multi-objective optimization challenge, each objective function is independently solved to find the best solution, ignoring the other objectives of the Model. The process will be repeated until the optimal solutions for each objective are obtained, and a payoff matrix is generated. The lower and upper bounds for each goal function are identified U_k and L_k $\forall\, k = 1, 2, ..., K$ from the payoff matrix. For the solution of a multi-objective PP problem, we used the IFP approach.

Intuitionistic Fuzzy Programming

Maximized type objective: The membership and non-membership functions are defined as follows:

$$\mu_k^M(Z_k(x)) = \begin{cases} 0, & L_k > Z_k(x) \\ \dfrac{Z_k(x) - L_k}{U_k - L_k}, & Z_k(x) \in [L_k, U_k] \\ 1, & U_k < Z_k(x) \end{cases} \tag{1}$$

$$\upsilon_k^{NM}(Z_k(x)) = \begin{cases} 1, & Z_k(x) < R_k \\ \dfrac{U_k - Z_k(x)}{U_k - R_k}, & Z_k(x) \in [R_k, U_k] \\ 0, & Z_k(x) > U_k \end{cases} \tag{2}$$

where, $R_k < L_k < U_k$

Minimized type objective: The membership and non-membership functions are defined as follows:

$$\mu_k^M(Z_k(x)) = \begin{cases} 1, & L_k > Z_k(x) \\ \dfrac{U_k - Z_k(x)}{U_k - L_k}, & Z_k(x) \in [L_k, U_k] \\ 0, & U_k < Z_k(x) \end{cases} \quad \textbf{(3)}$$

$$\upsilon_k^{NM}(Z_k(x)) = \begin{cases} 0, & Z_k(x) < L_k \\ \dfrac{Z_k(x) - L_k}{W_k - L_k}, & Z_k(x) \in [L_k, W_k] \\ 1, & Z_k(x) > W_k \end{cases} \quad (4)$$

where, $L_k < U_k \le W_k$

The above-defined Eqns [1-4] are used in solving the multi-objective optimization problem. As follows:

$$Max\ \mu_k,\ Min\ \upsilon_k$$

Subject to

for maximization problem

$$\mu_k^M = \frac{Z_k(x) - L_k}{U_k - L_k}, \upsilon_k^{NM} = \frac{U_k - Z_k(x)}{U_k - R_k},\ R_k < L_k < U_k$$

for minimization problem

$$\mu_k^M = \frac{U_k - Z_k(x)}{U_k - L_k}, \upsilon_k^{NM} = \frac{Z_k(x) - L_k}{W_k - L_k},\ L_k < U_k < W_k$$

$$\mu_k^M \le \mu_k,\ \ \upsilon_k^{NM} \ge \upsilon_k,\ \ \mu_k \ge \upsilon_k,\ \ \mu_k + \upsilon_k \le 1,$$

$$\mu_k,\ \upsilon_k \in [0,1],\ \ \forall k = 1,\ 2, \ldots, K$$

$$g(x) \le 0,\ \forall\ x \in X$$

Therefore, we can write

$$Max\ \sum_{k=1}^{K} \left(\mu_k - \upsilon_k\right)$$

Subject to

for maximization problem

$$\mu_k^M = \frac{Z_k(x) - L_k}{U_k - L_k}, \upsilon_k^{NM} = \frac{U_k - Z_k(x)}{U_k - R_k},\ R_k < L_k < U_k$$

for minimization problem

$$\mu_k^M = \frac{U_k - Z_k(x)}{U_k - L_k}, \upsilon_k^{NM} = \frac{Z_k(x) - L_k}{W_k - L_k},\ L_k < U_k < W_k$$

$$\mu_k^M \le \mu_k,\ \ \upsilon_k^{NM} \ge \upsilon_k,\ \mu_k \ge \upsilon_k,\ \ \mu_k + \upsilon_k \le 1,$$

$$\mu_k,\ \upsilon_k \in [0,1],\ \ \forall k = 1,\ 2, \ldots, K$$

$$g(x) \le 0,\ \forall\ x \in X$$

4. Mathematical Model

The PP challenge has selected various machinery types for the manufacturing process: lathe, milling machine, grinder, jigsaw, band saw, and drill press. The main objective of the production industry is to make profit so that the company runs smoothly. It is always advisable for a company to prepare a production plan based on scientific methods to get a clear direction for how the production process should be carried out. The main objective of this study is to optimize the profit, product liability, quality, and satisfaction of workers. The input information such as, the available facilities and resource information, the units of machine available for the manufacturing items, and the number of hours spent using the machine to produce the product, including production machinery is required to formulate the problem,. The PP and control model is shown in Figure 1.

Fig. 1: Model for PP and control

The following principles and drawbacks are essential for an organization's industrial planning model:

- We maximize the industry's profit, productivity, product liability, and worker satisfaction.
- The multi-item output model is taken into account.
- A single unit is running a single task at a time on a machine.
- It is not likely to have a shortage of products in the manufacturing process.
- Final products demand only.
- In any case, it cannot reach the maximum level of the machine timing.

Kamal et al. [21] have discussed the mathematical model of the industrial programming problem as follows:

$$optimize \begin{cases} Max\ \ Z_1 = \sum_{j=1}^{3} P_j x_j & \text{(Profit)} \\ Max\ \ Z_2 = \dfrac{\sum_{j=1}^{3} L_j x_j}{\sum_{j=1}^{3} x_j} & \text{(Overall Product Reliability)} \\ Max\ \ Z_3 = \sum_{j=1}^{3} Q_j x_j & \text{(Quality of the product)} \\ Max\ \ Z_4 = \sum_{j=1}^{3} W_j x_j & \text{(Worker's satisfaction)} \end{cases}$$

$$\text{constraints}\begin{cases} \sum_{j=1}^{3} m_j x_j \le M & \text{(for Milling machine)} \\ \sum_{j=1}^{3} l_j x_j \le L & \text{(for Lathe machine)} \\ \sum_{j=1}^{3} g_j x_j \le G & \text{(for Grinder machine)} \\ \sum_{j=1}^{3} s_j x_j \le S & \text{(for Jig saw machine)} \\ \sum_{j=1}^{3} d_j x_j \le D & \text{(for Drill press machine)} \\ \sum_{j=1}^{3} b_j x_j \le B & \text{(for Band saw machine)} \end{cases}$$

In real-life problems, the parameters of the optimization are commonly unknown. The coefficients of objective functions and constraints are represented in the form of interval multi-choice. There are several options in such a case, and the decision-maker is perplexed on one of the problem's criteria to choose. The multi-choice interval form deals with the problem's complexity in the Model. The right-hand sides of the constraint are in the form of an IVTN number type. Then, the industrial programming problem is as follows:

$$Max\ Z_1 = \tilde{P}_1^{MC} x_1 + \tilde{P}_2^{MC} x_2 + \tilde{P}_3^{MC} x_3, \quad Max\ Z_2 = (\tilde{L}_1^{MC} x_1 + \tilde{L}_2^{MC} x_2 + \tilde{L}_3^{MC} x_3) / (x_1 + x_2 + x_3)$$

$$Max\ Z_3 = \tilde{Q}_1^{MC} x_1 + \tilde{Q}_2^{MC} x_2 + \tilde{Q}_3^{MC} x_3, \quad Max\ Z_4 = \tilde{W}_1^{MC} x_1 + \tilde{W}_2^{MC} x_2 + \tilde{W}_3^{MC} x_3$$

Subject to the constraints

$$\tilde{m}_1^{MC} x_1 + \tilde{m}_2^{MC} x_2 \le \tilde{M}^{IVTN}, \quad \tilde{l}_1^{MC} x_1 + \tilde{l}_2^{MC} x_2 + \tilde{l}_3^{MC} x_3 \le \tilde{L}^{IVTN}$$

$$\tilde{g}_1^{MC} x_1 + \tilde{g}_3^{MC} x_3 \le \tilde{G}^{IVTN}, \quad \tilde{s}_1^{MC} x_1 + \tilde{s}_3^{MC} x_3 \le \tilde{S}^{IVTN}$$

$$\tilde{d}_2^{MC} x_2 + \tilde{d}_3^{MC} x_3 \le \tilde{D}^{IVTN}, \quad \tilde{b}_1^{MC} x_1 + \tilde{b}_2^{MC} x_2 + \tilde{b}_3^{MC} x_3 \le \tilde{B}^{IVTN}$$

$$x_1, x_2, x_3 \ge 0$$

5. Numerical Illustration

Zeleny [47] considered six types of machines, i.e. Lathe, milling machine, jig saw, band saw, drill press, grinder for the PP problem and used the deterministic parameters in the formulation.

The available capacities of each machine are in the form of an IVTN number, i.e. given in Table 2.

Table 2: Capacity of machines

Right Side of the Constraints	
Interval-valued neutrosophic form	***Interval-valued form***
$\tilde{M}^{IVTN}$ *=1200,1350,1500,1600; [0.7,0.9], [0.1,0.3], [0.5,0.7]*	*[1200,1366.67]*
$\tilde{L}^{IVTN}$ *=800,1000,1200,1400; [0.5,0.7], [0.3,0.5], [0.4,0.6]*	*[813.54,1085.71]*
$\tilde{G}^{IVTN}$ *=1650,1800,1950,2050; [0.7,1.0], [0.2,0.3], [0.2,0.5]*	*[1650,1925]*
$\tilde{S}^{IVTN}$ *=1225,1290,1340,1425; [0.3,0.7], [0.1,0.4], [0.3,0.7]*	*[1225,1297.22]*
$\tilde{D}^{IVTN}$ *=700,900,1100,1300; [0.6,0.8], [0.3,0.6], [0.2,0.4]*	*[893.93,1050]*
$\tilde{B}^{IVTN}$ *=1075,1275,1475,1675; [0.5,0.9], [0.1,0.3], [0.3,0.6]*	*[1075,1452.78]*

The coefficients of objective functions are given in Table 3.

Table 3: Coefficients of objective functions

$\tilde{P}_1^{MC}$ =[40,50] or [50,60] or [60,70]	$\tilde{P}_2^{MC}$ =[90,100] or [100,110] or [110,120]	$\tilde{P}_2^{MC}$ =[16.5,17.5] or [17.5,18.5]
$\tilde{L}_1^{MC}$ =[0.70,0.72] or [0.72,0.74] or [0.74,0.76]	$\tilde{L}_2^{MC}$ =[0.81,0.85] or [0.85,0.89]	$\tilde{L}_3^{MC}$ =[0.75,0.78] or [0.78,0.81] or [0.81,0.84] or [0.84,0.87]
$\tilde{Q}_1^{MC}$ =[82,92] or [92,102]	$\tilde{Q}_2^{MC}$ =[65,75] or [75,85] or [85,95] or [95,105]	$\tilde{Q}_3^{MC}$ =[40,50] or [50,60] or [60,70]
$\tilde{W}_1^{MC}$ =[15,25] or [25,35]	$\tilde{W}_2^{MC}$ =[90,100] or [100,110] or [110,120] or [120,130]	$\tilde{W}_3^{MC}$ =[65,75] or [75,85]

The left side's coefficients of the constraints are given in Table 4.

Table 4: Coefficients of left side of the constraints

$\tilde{m}_1^{MC}$ =[10,12] or [12,14]	$\tilde{m}_2^{MC}$ =[15,17] or [17,19] or [19,21]	---
$\tilde{l}_1^{MC}$ =[3,5] or [5,7]	$\tilde{l}_2^{MC}$ =[7,9] or [9,11] or [11,13]	$\tilde{l}_3^{MC}$ =[6,8] or [8,10]
$\tilde{g}_1^{MC}$ =[8,10] or [10,12] or [12,14]	$\tilde{g}_2^{MC}$ =[13,15] or [15,17]	$\tilde{g}_3^{MC}$ =[15,17] or [17,19]
$\tilde{s}_1^{MC}$ =[4,6] or [6,8]	---	$\tilde{s}_3^{MC}$ =[12,14] or [14,16] or [16,18]
---	$\tilde{d}_2^{MC}$ =[10,12] or [12,14]	$\tilde{d}_3^{MC}$ =[5,7] or [7,9] or [9,11]
$\tilde{b}_1^{MC}$ =[9.5,11.5] or [11.5,13.5]	$\tilde{b}_2^{MC}$ =[9.5,11.5] or [11.5,13.5]	$\tilde{b}_3^{MC}$ =[4,6] or [6,8] or [8,10] or [10,12]

The membership and non-membership of the intuitionistic function of the problem are as follows:

The objective membership of the problem will be

$$\mu_1(Z_1)=\begin{cases}0, & Z_1\le 6152.5\\ \dfrac{Z_1-6152.5}{12348-6152.5}, & 6152.5<Z_1\le 12348\\ 1, & Z_1>12348\end{cases}\qquad \mu_2(Z_2)=\begin{cases}0, & Z_2\le 0.72\\ \dfrac{Z_2-0.72}{0.89-0.72}, & 0.72<Z_2\le 0.89\\ 1, & Z_1>0.89\end{cases}$$

$$\mu_3(Z_3)=\begin{cases}0, & Z_3\le 7035\\ \dfrac{Z_3-7035}{16006-7035}, & 7035<Z_3\le 16006\\ 1, & Z_3>16006\end{cases}\qquad \mu_4(Z_4)=\begin{cases}0, & Z_4\le 7330\\ \dfrac{Z_4-7330}{13710-7330}, & 7330<Z_4\le 13710\\ 1, & Z_4>13710\end{cases}$$

The objective non-membership of the problem will be

$$\upsilon_1(Z_1)=\begin{cases}1, & Z_1\le 6772.05\\ \dfrac{12348-Z_1}{12348-6772.05}, & 6772.05<Z_1\le 12348\\ 0, & Z_1>12348\end{cases}\qquad \upsilon_2(Z_2)=\begin{cases}1, & Z_2\le 0.75\\ \dfrac{0.89-Z_2}{0.89-0.75}, & 0.75<Z_2\le 0.89\\ 0, & Z_2>0.89\end{cases}$$

$$\upsilon_3(Z_3)=\begin{cases}1, & Z_3\le 7932.1\\ \dfrac{16006-Z_3}{16006-7932.1}, & 7932.1<Z_3\le 16006\\ 0, & Z_3>16006\end{cases}\qquad \upsilon_4(Z_4)=\begin{cases}1, & Z_4\le 7968\\ \dfrac{13710-Z_4}{13710-7968}, & 7968<Z_4\le 13710\\ 0, & Z_4>13710\end{cases}$$

The intuitionistic formulation of the problem is

$$Max\ \ (\mu-\upsilon)$$

Subject to constraints

$$\mu_1(Z_1)\ge\frac{Z_1-6152.5}{12348-6152.5},\ \ \upsilon_1(Z_1)\le\frac{12348-Z_1}{12348-6772.05},\ \ \mu_2(Z_2)\ge\frac{Z_2-0.72}{0.89-0.72},\ \ \upsilon_2(Z_2)\le\frac{0.89-Z_2}{0.89\text{-}0.75}$$

$$\mu_3(Z_3)\ge\frac{Z_3-7035}{16006-7035},\ \ \upsilon_3(Z_3)\le\frac{16006-Z_3}{16006-7932.1},\ \ \mu_4(Z_4)\ge\frac{Z_4-7330}{13710-7330},\ \ \upsilon_4(Z_4)\le\frac{13710-Z_4}{13710-7968}$$

$$Z_1=\tilde{P}_1^{MC}x_1+\tilde{P}_2^{MC}x_2+\tilde{P}_3^{MC}x_3,\ \ Z_2=(\tilde{L}_1^{MC}x_1+\tilde{L}_2^{MC}x_2+\tilde{L}_3^{MC}x_3)/(x_1+x_2+x_3)$$

$$Z_3=\tilde{Q}_1^{MC}x_1+\tilde{Q}_2^{MC}x_2+\tilde{Q}_3^{MC}x_3,\ \ Z_4=\tilde{W}_1^{MC}x_1+\tilde{W}_2^{MC}x_2+\tilde{W}_3^{MC}x_3$$

$$\tilde{m}_1^{MC}x_1+\tilde{m}_2^{MC}x_2\le\tilde{M}^{IVTN},\ \ \tilde{l}_1^{MC}x_1+\tilde{l}_2^{MC}x_2+\tilde{l}_3^{MC}x_3\le\tilde{L}^{IVTN}$$

$$\tilde{g}_1^{MC}x_1+\tilde{g}_3^{MC}x_3\le\tilde{G}^{IVTN},\ \ \ \tilde{s}_1^{MC}x_1+\tilde{s}_3^{MC}x_3\le\tilde{S}^{IVTN}$$

$$\tilde{d}_2^{MC}x_2+\tilde{d}_3^{MC}x_3\le\tilde{D}^{IVTN},\ \ \ \tilde{b}_1^{MC}x_1+\tilde{b}_2^{MC}x_2+\tilde{b}_3^{MC}x_3\le\tilde{B}^{IVTN}$$

$$x_1,x_2,x_3\ge 0,\ \mu_j(Z_j)+\upsilon_j(Z_j)\le 1,\forall\ \mu_j(Z_j)\ \&\ \upsilon_j(Z_j)\in[0,1]$$

$$\mu\le\mu_j(Z_j),\upsilon\ge\upsilon_j(Z_j),\mu\ge\upsilon,\ \ \mathrm{j}=1,\ 2,\ 3,\ 4.$$

We get the following compromise solution using the IFP technique, which is given below in Table 5.

Table 5: Compromise optimal solution

Result
$F_1=11852, F_2=0.8652414, F_3=12320, F_4=13321.67, x_1=15, x_2=78, x_3=52$

The graphical representations of compromise optimal solution through IFP for both models are shown in Figure 2. The Membership and non-membership values are shown in Figure 3.

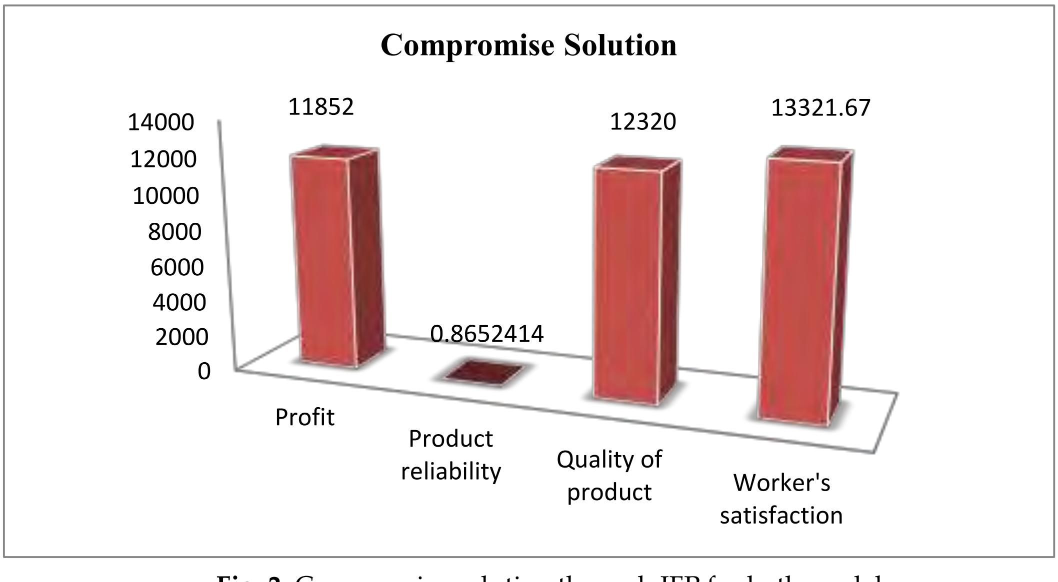

Fig. 2: Compromise solution through IFP for both models

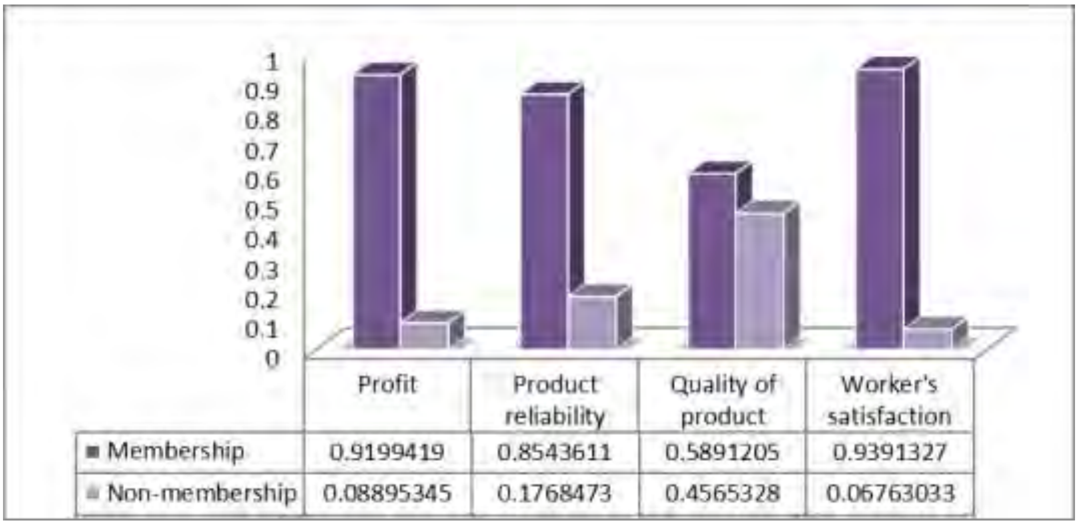

Fig. 3: Membership and non-membership value for the Model I

The comparison of the IFP with fuzzy goal programming (FGP) and bi-level fuzzy goal programming (BL-FGP) are shown in Table 6.

Table 6: Comparison IFP with BL-FGP and FGP

Model	Methods	Profit	Product Reliability	Quality of the Product	Worker's Satisfaction	Decision Variable's Values
Kamal et al. (2019) [21]	BL-FGP	5697.5	0.80734	12307	10000	$x_1 = 71,\ x_2 = 3,\ x_3 = 95$
	FGP	11894	0.812350	10989	11000	$x_1 = 42,\ x_2 = 59,\ x_3 = 54$
Discussed Model	IFP	11852	0.8652414	12320	13321.67	$x_1 = 15, x_2 = 78, x_3 = 52$

From the solutions of Table 6, it can be observed that discussed Model gives the more optimal solution than Kamal et al. [21] Model for profit, product reliability and worker's satisfaction, but for product quality, the BL-FGP Model has the more improved solution.

6. Motivation and Contribution

This study is motivated by an Intuitionistic programming research area with the potential to capture decision-makers. The following are the contributions of the study:

i. It serves as an additional contribution to the literature of PP.

ii. A case study is provided in which solution procedures for multi-objective multi-product problem formulation is reported.

iii. In this study, a new approach based on intuitionistic has been applied.

iv. The approach is compared with BL-FGP and FGP, and the result proves to be better.

v. The applicability of Interval-valued Neutrosophic and multi-choice parameters have also been discussed and reported.

Conclusion

The Model for the PP problem with rational expectations was explored in this article. The optimization problem is represented with multi-choice type parameters in the objective's coefficient and the left-hand side of the constraints, and it is transformed into the deterministic form using the binary variable transformation technique. Some parameters are IVTN number types that are transformed into deterministic forms using the score function. In determining the optimum quantity of products, the industrial PP problem is solved using intuitionistic fuzzy IFP. The comparison of the IFP with BL-FGP and FGP are shown in Table 6.

Furthermore, it can be observed that the discussed model gives a more compromise solution than BL-FGP and FGP for profit, product reliability and worker's satisfaction, but for product quality, the BL-FGP model has the more improved solution. This paper presents a detailed investigation into IFP approaches for solving multi-objective optimization problems in the presence of multi-choice and neutrosophic environments. It will help solve and understand the production-related problems by the IFP approach. The IFP approach will help in solving the other complex production-related problems. The complex PP challenge will be considered with several new solutions based on fuzzy logic.

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Single Valued Pentapartitioned Neutrosophic Graphs

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Abstract

Background:

The notion of single valued pentapartitioned neutrosophic set is the extension of single valued neutrosophic set and quadripartitioned single valued neutrosophic set. The single valued pentapartitioned neutrosophic set is a powerful mathematical tool that comprehensively deals with indeterminacy by splitting it into three independent components, namely, unknown, contradiction, ignorance. We apply the concept of single valued pentapartitioned neutrosophic set to graph theory.

Findings:

We develop the notions of Single-Valued Pentapartitioned Neutrosophic graph (SVPN-graph) as an extension of single valued neutrosophic graph theory. Besides, we introduce the notions of degree, size and order of an SVPN-graph. Further, we furnish a few suitable examples on SVPN-graph. Single valued pentapartitioned neutrosophic set.

Limitations:

Pentapartitioned neutrosophic graph is proposed in this model which is based on pentapartitioned neutrosophic sets. A few studies on pentapartitioned neutrosophic sets are reported in the literature.

Future directions:

In future, the single valued pentapartitioned neutrosophic graph can be extended to regular and irregular single valued pentapartitioned neutrosophic graph, single-valued pentapartitioned neutrosophic intersection graphs, single-valued pentapartitioned neutrosophic hypergraphs, and so on. The single-valued pentapartitioned neutrosophic graph can be employed in modeling the computer networks, expert systems, image processing, social network , and telecommunication.

Keywords: Neutrosophic Set; Pentapartitioned NS; Neutrosophic Graph; SVPN-Graph.

1. Introduction:

Graph theory [1] is generally used as a tool to deal with the combinatorial problems in number theory, geometry, topology, algebra, etc. Euler presented the concept of graph theory [2] in 1736. When there exists uncertainty in the description of a graph, traditional graph theory fails to deal with the problem. To deal with such situation, Rosenfeld [3] developed the Fuzzy Graph (FG) by considering fuzzy relation [4] on Fuzzy Set (FS) [5]. Sunitha and Mathew [6] presented a survey of fuzzy graph in 2013. Shannon and Atanassov [7] developed intuitionistic FG based on Intuitionistic FS (IFS) [8]. Intuitionistic FGs have been further studied in [9-15].

To deal with inconsistency and indeterminacy, Smarandache [16] developed the Neutrosophic Set (NS) in 1998. The Neutrosophic Graphs (NGs) using the NSs were developed by several authors[17-19]. Akram [20] presented the Single Valued Neutrosophic (SVN) planar graph. NGs have been further studied in [21-24]. Broumi et al. [25] presented interval NGs, which have been further studied in [26-27]. NGs have been further studied in different hybrid environment such as neutrosophic soft graph [28], bipolar SVN graphs [29], rough neutrosophic diagraph [30], neutrosophic soft rough graph [31], etc. Recent trends in graph theory have been depicted in [32] in different environments.

Recently, Mallick and Pramanik [33] defined Pentapartitioned Neutrosophic Set (PNS) using the n-valued logic [34]. PNS is a powerful mathematical tool, which is capable of dealing with uncertainty and indeterminacy comprehensively as indeterminacy is divided into three independent components, namely, unknown, contradiction, and ignorance.

In this study, we procure the Single Valued Pentapartitioned Neutrosophic (SVPN) graph and establish some basic its properties.

Research Gap: No investigation on SVPN-graph has been reported in the literature.

Motivation: To fill the research gap, we present the concept of SVPN-graph.

The rest of the article has been organized into four sections:

In section 2, we recall some relevant definitions on PNS those are relevant to the main results of this article. In section 3, we procure the notion of SVPN-graph, and investigate some properties of different types of degree, size and order of an SVPN-graph. Section 4 presents results and discussion section. Section 5 concludes the paper with stating the future scope of research.

2. Some Relevant Definitions:

In this section, we present some existing definitions those are relevant to the main results of this article.

Definition 2.1.[33] Suppose that Ω be a fixed set. Then, a Single Valued Pentapartitioned Neutrosophic Set (SVPN-set) P over Ω is defined by:

$$P=\{(\kappa, T_P(\kappa), C_P(\kappa), R_P(\kappa), U_P(\kappa), F_P(\kappa)) : \kappa\in\Omega\}.$$

Here, T_P, C_P, R_P, U_P and F_P are the truth, contradiction, ignorance, unknown and falsity membership functions respectively from Ω to [0, 1]. So, $0 \leq T_P(\kappa) + C_P(\kappa) + R_P(\kappa) + U_P(\kappa) + F_P(\kappa) \leq 5$, for all $\kappa\in\Omega$.

Definition 2.2.[33] Suppose that $X=\{(\kappa, T_X(\kappa), C_X(\kappa), R_X(\kappa), U_X(\kappa), F_X(\kappa)) : \kappa\in\Omega\}$ and $Y=\{(\kappa, T_Y(\kappa), C_Y(\kappa), R_Y(\kappa), U_Y(\kappa), F_Y(\kappa)) : \kappa\in\Omega\}$ be two SVPN-sets over Ω. Then, an SVPN-set X is said to be a subset of a SVPN-set Y (i.e., $X\subseteq Y$) if and only if $T_X(\kappa) \leq T_Y(\kappa)$, $C_X(\kappa) \leq C_Y(\kappa)$, $R_X(\kappa) \geq R_Y(\kappa)$, $U_X(\kappa) \geq U_Y(\kappa)$, $F_X(\kappa) \geq F_Y(\kappa)$, $\forall\kappa\in\Omega$.

Definition 2.3.[33] Suppose that X={(κ, $T_X(\kappa)$, $C_X(\kappa)$, $R_X(\kappa)$, $U_X(\kappa)$, $F_X(\kappa)$) : κ∈Ω} and Y={(κ, $T_Y(\kappa)$, $C_Y(\kappa)$, $R_Y(\kappa)$, $U_Y(\kappa)$, $F_Y(\kappa)$) : κ∈Ω} be two SVPN-sets over Ω. Then, union of X and Y is defined by X∪Y={(κ, max{$T_X(\kappa)$, $T_Y(\kappa)$}, max{$C_X(\kappa)$, $C_Y(\kappa)$}, min{$R_X(\kappa)$, $R_X(\kappa)$}, min{$U_X(\kappa)$, $U_X(\kappa)$}, min{$F_X(\kappa)$, $F_X(\kappa)$}) : κ∈Ω}.

Definition 2.4.[33] Suppose that X={(κ, $T_X(\kappa)$, $C_X(\kappa)$, $R_X(\kappa)$, $U_X(\kappa)$, $F_X(\kappa)$) : κ∈Ω} and Y={(κ, $T_Y(\kappa)$, $C_Y(\kappa)$, $R_Y(\kappa)$, $U_Y(\kappa)$, $F_Y(\kappa)$) : κ∈Ω} be any two SVPN-sets over Ω. Then, the complement of X is defined by X^c={(κ, $F_X(\kappa)$, $U_X(\kappa)$, 1-$R_X(\kappa)$, $C_X(\kappa)$, $T_X(\kappa)$) : κ∈Ω}.

Definition 2.5.[33] Suppose that X={(κ, $T_X(\kappa)$, $C_X(\kappa)$, $R_X(\kappa)$, $U_X(\kappa)$, $F_X(\kappa)$) : κ∈Ω} and Y={(κ, $T_Y(\kappa)$, $C_Y(\kappa)$, $R_Y(\kappa)$, $U_Y(\kappa)$, $F_Y(\kappa)$) : κ∈Ω} be two SVPN-sets over Ω. Then, intersection of X and Y is defined by X∩Y={(κ, min{$T_X(\kappa)$, $T_Y(\kappa)$}, min{$C_X(\kappa)$, $C_Y(\kappa)$}, max{$R_X(\kappa)$, $R_X(\kappa)$}, max{$U_X(\kappa)$, $U_X(\kappa)$}, max{$F_X(\kappa)$, $F_X(\kappa)$}) : κ∈Ω}.

Definition 2.6.[18] Suppose that V be a fixed set of n vertex. Assume that E be the set of edges between the vertices. Then, Ĝ=(P, Q) is called a single valued neutrosophic graph (in short SVN-graph), where (i) T_P, I_P, F_P : V→[0, 1] denotes the truth, indeterminacy and false membership functions of a vertex κ_i∈V respectively, such that $0 \le T_P(k_i) + I_P(k_i) + F_P(k_i) \le 3$ (∀k_i∈V, i=1, 2, …., n).
(ii) T_Q, I_Q, F_Q : E⊆V×V→[0, 1] defined by $T_Q(k_i, k_j) \le \min\{T_P(k_i), T_P(k_j)\}$, $I_Q(k_i, k_j) \ge \max\{I_P(k_i), I_P(k_j)\}$, $F_Q(k_i, k_j) \ge \max\{F_P(k_i), F_P(k_j)\}$ denotes the truth, indeterminac y and false membership functions of the edge (k_i, k_j)∈E respectively, such that $0 \le T_Q(k_i, k_j) + I_Q(k_i, k_j) + F_Q(k_i, k_j) \le 3$ (∀(k_i, k_j)∈E, i=1, 2, …., n).

Here, P is said to be the SVN vertex set of V and Q is said to be the SVN edge set of E, respectively.

3. Single-Valued Pentapartitioned Neutrosophic-Graph

Here, we introduce the notions of degree, size, and order of SVPN-graph and present few illustrative examples.

Definition 3.1. Suppose that V={k_i: i=1, 2, …, n} be a fixed set of vertices and E={(k_i, k_j): i, j=1, 2, …, n} be the set of edges between the vertices of V. An SVPN-graph of Ĝ*=(V, E) is defined by Ĝ=(P, Q), where (i) T_P: V→[0, 1], C_P: V→[0, 1], R_P: V→[0, 1], U_P: V→[0, 1] and F_P: V→[0, 1] denotes the truth, contradiction, ignorance, unknown and false membership functions of the vertices k_i∈V respectively, such that $0 \le T_P(k_i) + C_P(k_i) + R_P(k_i) + U_P(k_i) + F_P(k_i) \le 5$, ∀$k_i$ ∈V (i=1, 2, …, n);
(ii) T_Q : E⊆V×V→[0, 1], C_Q : E⊆V×V→[0, 1], R_Q : E⊆V×V→[0, 1], U_Q : E⊆V×V→[0, 1] and F_Q : E⊆V×V→[0, 1] defined by $T_Q(k_i, k_j) \le \min\{T_P(k_i), T_P(k_j)\}$, $C_Q(k_i, k_j) \le \min\{C_P(k_i), C_P(k_j)\}$, $R_Q(k_i, k_j) \ge \max\{R_P(k_i), R_P(k_j)\}$, $U_Q(k_i, k_j) \ge \max\{U_P(k_i), U_P(k_j)\}$, and $F_Q(k_i, k_j) \ge \max\{F_P(k_i), F_P(k_j)\}$, indicates the truth, contradiction, ignorance, unknown and false-membership functions from E⊆V×V to [0, 1], respectively, such that $0 \le T_P(k_i) + C_P(k_i) + R_P(k_i) + U_P(k_i) + F_P(k_i) \le 5$, ∀($k_i$, k_j)∈E (i, j = 1, 2, …., n).

Here, P is the SVN vertex set of V and Q is the SVN edge set of E respectively. Therefore, Ĝ=(P, Q) is an SVPN-graph of Ĝ*=(V, E) if and only if $T_Q(k_i, k_j) \le \min\{T_P(k_i), T_P(k_j)\}$; $C_Q(k_i, k_j) \le \min\{C_P(k_i), C_P(k_j)\}$; $R_Q(k_i, k_j) \ge \max\{R_P(k_i), R_P(k_j)\}$; $U_Q(k_i, k_j) \ge \max\{U_P(k_i), U_P(k_j)\}$; and $F_Q(k_i, k_j) \ge \max\{F_P(k_i), F_P(k_j)\}$.

Clearly, both P and Q are the SVPN-set over V and E respectively.

Example 3.1. Assume that Ĝ=(V, E) is a graph, where V={k_1, k_2, k_3, k_4} and E={(k_1, k_2), (k_2, k_3), (k_3, k_4), (k_4, k_1)}. Suppose that P is an SVPN vertex set of V and Q is an SVPN edge set of E defined by the Table 1 and Table 2.:

Table 1. Tabular representation of Example 3.1

	k_1	k_2	k_3	k_4	k_5
T_P	0.4	0.3	0.4	0.5	0.2
C_P	0.5	0.5	0.5	0.5	0.5
R_P	0.3	0.4	0.5	0.3	0.4
U_P	0.4	0.3	0.6	0.6	0.3
F_P	0.4	0.5	0.5	0.4	0.5

Table 2. Tabular representation of Example 3.1

	(k_1, k_2)	(k_2, k_3)	(k_3, k_4)	(k_4, k_5)	(k_5, k_1)
T_P	0.3	0.2	0.2	0.1	0.2
C_P	0.4	0.4	0.3	0.5	0.2
R_P	0.5	0.7	0.8	0.6	0.5
U_P	0.4	0.8	0.9	0.7	0.5
F_P	0.6	0.6	0.9	0.8	0.8

The graph of Example 3.1 is presented in Figure 1.

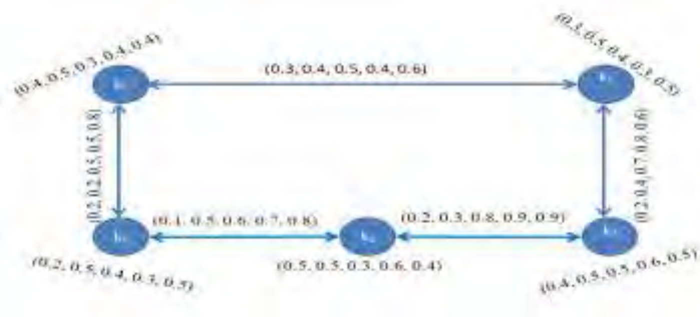

Figure 1: SPVN graph for Example 3.1

Therefore, $\hat{G}$= (P, Q) is an SVPN-graph of $\hat{G}^*$=(V, E).

Remark 3.1. Assume that $\hat{G}$=(P, Q) is an SVPN-graph. Then, the edge (k_i, k_j) is said to be incident at k_i and k_j.

Definition 3.2. Suppose that $\hat{G}$=(P, Q) be an SVPN-graph. Then,

(i) $(k_i, T_P(k_i), C_P(k_i), R_P(k_i), U_P(k_i), F_P(k_i))$ is called a Single Valued Pentapartitioned Neutrosophic (SVPN) vertex (in short SVPN-vertex).

(ii) $((k_i, k_j), T_Q((k_i, k_j)), C_Q((k_i, k_j)), R_Q((k_i, k_j)), U_Q((k_i, k_j)), F_Q((k_i, k_j)))$ is called an SVPN edge (in short SVPN-edge).

Definition 3.3. Suppose that $\hat{G}$=(P, Q) be an SVPN-graph. Then, H=(P′, Q′) is called an SVPN sub-graph (SVPN-sub-graph) of $\hat{G}$=(P, Q) if H=(P′, Q′) is also an SVPN-graph such that:

(i) $P' \subseteq P$ i.e. $T'_{Pi} \leq T_{Pi}$, $C'_{Pi} \leq C_{Pi}$, $R'_{Pi} \geq R_{Pi}$, $U'_{Pi} \geq U_{Pi}$, and $F'_{Pi} \geq F_{Pi}$, $\forall k_i \in V$;

(ii) $Q' \subseteq Q$ i.e. $T'_{Qi} \leq T_{Qi}$, $C'_{Qi} \leq C_{Qi}$, $R'_{Qi} \geq R_{Qi}$, $U'_{Qi} \geq U_{Qi}$, and $F'_{Qi} \geq F_{Qi}$, $\forall (k_i, k_j) \in E$.

Example 3.2. Assume that $\hat{G}$=(P, Q) be an SVPN-graph as shown in Example 3.1. Then, H=(P′, Q′), where V′={k_1, k_2, k_5}, E′={(k_1, k_2), (k_1, k_5)} defined by the Table 3 and Table 4:

Table 3. Tabular representation of Example 3.2

	k_1	k_2	k_5
$T_{P'}$	0.3	0.2	0.2
$C_{P'}$	0.3	0.4	0.2
$R_{P'}$	0.5	0.6	0.5
$U_{P'}$	0.6	0.4	0.4
$F_{P'}$	0.6	0.6	0.8

Table 4. Tabular representation of Example 3.2

	(k_1, k_2)	(k_1, k_5)
$T_{P'}$	0.1	0.1
$C_{P'}$	0.3	0.2
$R_{P'}$	0.8	0.6
$U_{P'}$	0.6	0.8
$F_{P'}$	0.8	0.9

Then, the graph H=(P′, Q′) is represented in Figure 2.

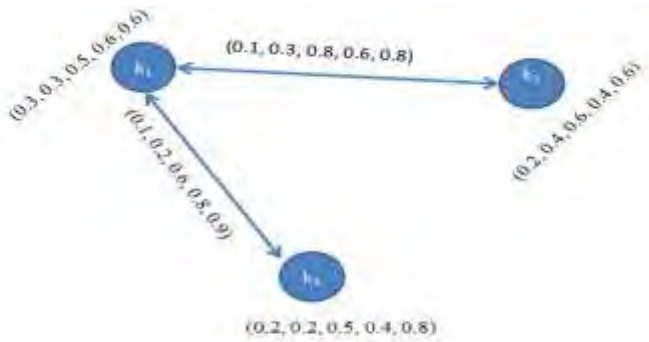

Figure 2: Graph of Example 3.2

Here, H=(P′, Q′) is an SVPN-sub-graph of Ĝ=(P, Q).

Definition 3.4. Suppose that Ĝ=(P, Q) be an SVPN-graph of Ĝ*=(V, E). Then, the complement of Ĝ=(P, Q) is an SVPN-graph $\overline{\hat{G}}$ of Ĝ*=(V, E), where

(ii) $\overline{T}_P(k_i)= T_P(k_i)$, $\overline{C}_P(k_i)=C_P(k_i)$, $\overline{R}_P(k_i)=R_P(k_i)$, $\overline{U}_P(k_i)=U_P(k_i)$, $\overline{F}_P(k_i)=F_P(k_i)$, $\forall k_i \in V$;

(iii) $\overline{T}_Q(k_i, k_j) = \min\{T_P(k_i), T_P(k_j)\} - T_Q(k_i, k_j)$, $\overline{C}_Q(k_i, k_j) = \min\{C_P(k_i), C_P(k_j)\} - C_Q(k_i, k_j)$, $\overline{R}_Q(k_i, k_j) = \max\{R_P(k_i), R_P(k_j)\} - R_Q(k_i, k_j)$, $\overline{U}_Q(k_i, k_j) = \max\{U_P(k_i), U_P(k_j)\} - U_Q(k_i, k_j)$ and $\overline{F}_Q(k_i, k_j) = \max\{F_P(k_i), F_P(k_j)\} - F_Q(k_i, k_j)$, $\forall (k_i, k_j) \in E$.

Definition 3.5. Suppose that Ĝ=(P, Q) be an SVPN-graph. Then, the vertices k_i and k_j are called adjacent in Ĝ=(P, Q) if and only if $T_Q(k_i, k_j) = \min\{T_P(k_i), T_P(k_j)\}$, $C_Q(k_i, k_j) = \min\{C_P(k_i), C_P(k_j)\}$, $R_Q(k_i, k_j) = \max\{R_P(k_i), R_P(k_j)\}$, $U_Q(k_i, k_j) = \max\{U_P(k_i), U_P(k_j)\}$ and $F_Q(k_i, k_j) = \max\{F_P(k_i), F_P(k_j)\}$.

Example 3.3. Assume that Ĝ=(P, Q) be an SVPN-graph, which is defined in Table 5 and Table 6.

Table 5. Tabular representation of Example 3.3

	k_1	k_2	k_3
T_P	0.3	0.2	0.3
C_P	0.3	0.8	0.4
R_P	0.5	0.6	0.6
U_P	0.6	0.5	0.7
F_P	0.6	0.5	0.8

Table 6. Tabular representation of Example 3.3

	(k_1, k_2)	(k_2, k_3)	(k_3, k_1)
T_P	0.2	0.1	0.3
C_P	0.3	0.4	0.3
R_P	0.6	0.8	0.6
U_P	0.6	0.7	0.7
F_P	0.6	0.9	0.8

The representation of the graph of Example 3 is shown in Figure-3.

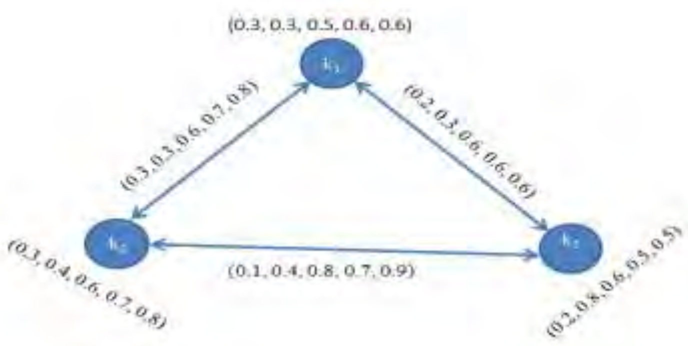

Figure 3: Graph of Example 3.3

Here, the vertices k_1 and k_2 are adjacent in the SVPN-graph Ĝ=(P, Q). Similarly, the vertices k_3 and k_1 are adjacent in the SVPN-graph Ĝ=(P, Q). But, the vertices k_2 and k_3 are not adjacent in the SVPN-graph Ĝ=(P, Q).

Definition 3.6. In an SVPN-graph Ĝ=(P, Q), a vertex $k_j \in V$ is called an isolated vertex if there exists no edge incident at k_j.

Example 3.4. Suppose that Ĝ=(P, Q) be an SVPN-graph, which is defined in Table 7 and Table 8.

Table 7. Tabular representation of Example 3.4

	k_1	k_2	k_3	k_4
T_P	0.3	0.2	0.5	0.3
C_P	0.3	0.8	0.6	0.4
R_P	0.5	0.6	1.0	0.6
U_P	0.6	0.5	0.8	0.7
F_P	0.6	0.5	0.8	0.8

Table 8. Tabular representation of Example 3.4

	(k_1, k_2)	(k_2, k_4)	(k_4, k_1)
T_P	0.3	0.1	0.3
C_P	0.3	0.4	0.3
R_P	0.6	0.7	0.6
U_P	0.7	0.8	0.7
F_P	0.8	0.8	0.8

The graph of Example 3.4 is represented in Figure 4.

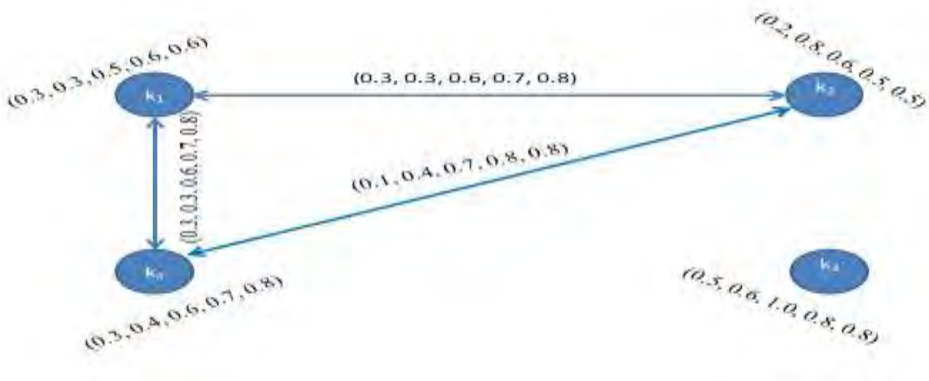

Figure 4: Graph of Example 3.4

In the above SVPN-graph Ĝ=(P, Q), the vertex k_3 is an isolated vertex.

Definition 3.7. Suppose that Ĝ=(P, Q) is an SVPN-graph. Assume that k_0 and k_n be two vertices in Ĝ=(P, Q). Then, an SVPN path $P(k_0, k_n)$ in an SVPN-graph Ĝ=(P, Q) is a sequence of distinct vertices $k_0, k_1, k_2, k_3, \ldots, k_n$ such that $T_Q(k_{i-1}, k_i) > 0$, $C_Q(k_{i-1}, k_i) > 0$, $R_Q(k_{i-1}, k_i) > 0$, $U_Q(k_{i-1}, k_i) > 0$ and $F_Q(k_{i-1}, k_i) > 0$, where $0 \le i \le n$. Here, n (≥ 1) is called the length of the path $P(k_0, k_n)$. The consecutive pairs (k_{i-1}, k_i) $(0 \le i \le n)$ are called the edges of the path $P(k_0, k_n)$. The path $P(k_0, k_n)$ is called a cycle if $k_0 = k_n$, where $n \ge 3$.

Definition 3.8. Suppose that Ĝ=(P, Q) be an SVPN-graph. Then, Ĝ=(P, Q) is said to be an SVPN Connected graph (in short SVPN-C-graph) if there exists at least one SVPN-path between two vertices.

Remark 3.2. If an SVPN-graph $\hat{G}$=(P, Q) is not an SVPN-C-graph, then it is called an SVPN Dis-Connected graph (in short SVPN-DC-graph).

Definition 3.9. Assume that $\hat{G}$=(P, Q) be an SVPN-graph. Then, a vertex having exactly one edge incident on it is called a pendent vertex. If a vertex is not a pendent vertex, then it is called a non-pendent vertex.

Remark 3.3. (i) If an edge is incident with a pendent vertex, then the edge is said to be a pendent edge. Otherwise, it is called a non-pendent edge.

(ii) If a vertex is adjacent to a pendent vertex, then the vertex is said to be a support of that pendent edge.

Example 3.5. Let $\hat{G}$=(P, Q) be an SVPN-graph, which is defined by Table 9 and Table 10.

Table 9. Tabular representation of Example 3.5

	k_1	k_2	k_3	k_4
T_P	0.3	0.2	0.5	0.3
C_P	0.3	0.8	0.6	0.4
R_P	0.5	0.6	1.0	0.6
U_P	0.6	0.5	0.8	0.7
F_P	0.6	0.5	0.8	0.8

Table 10. Tabular representation of Example 3.5

	(k_1, k_2)	(k_2, k_3)	(k_3, k_4)
T_P	0.1	0.2	0.3
C_P	0.2	0.5	0.4
R_P	0.7	1.0	1.0
U_P	0.7	0.9	0.8
F_P	0.7	0.8	0.8

The representation of the graph for Example 3.5 is presented in Figure 5.

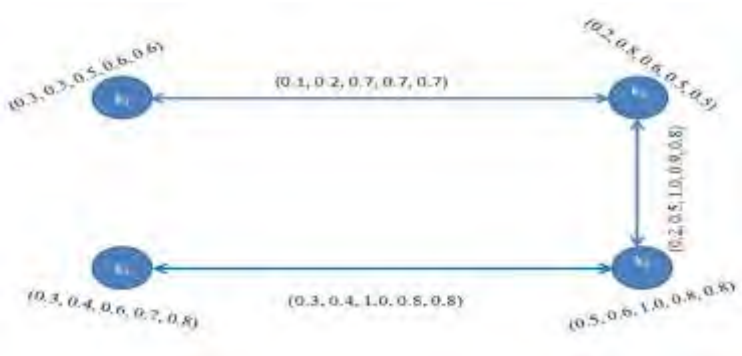

Figure 5: Graph for Example 3.5

In the above SVPN-graph $\hat{G}$=(P, Q), the vertices k_1 and k_4 are the pendent vertices. But the vertices k_2 and k_3 are the non-pendent vertices. Similarly, the edges (k_1, k_2) and (k_3, k_4) are the pendent edges. But the edge (k_2, k_3) is a non-pendent edge. The vertex k_3 is support of the pendent edge (k_3, k_4). But k_2 is not the support of the pendent edge (k_1, k_2).

Definition 3.10. A SVPN-graph $\hat{G}$=(P, Q) of $\hat{G}^*$=(V, E) is said to be a complete SVPN-graph if

$T_Q(k_i, k_j)= \min\{T_P(k_i), T_P(k_j)\}$;

$C_Q(k_i, k_j)= \min\{C_P(k_i), C_P(k_j)\}$;

$R_Q(k_i, k_j)= \max\{R_P(k_i), R_P(k_j)\}$;

$U_Q(k_i, k_j)= \max\{U_P(k_i), U_P(k_j)\}$;

and $F_Q(k_i, k_j)= \max\{F_P(k_i), F_P(k_j)\}$, $\forall k_i, k_j \in V$.

Example 3.6. Assume that $\hat{G}^*$=(V, E) is a graph, where V = {k_1, k_2, k_3} and E = {(k_1, k_2), (k_2, k_3), (k_3, k_1)}. Suppose that $\hat{G}$=(P, Q) is an SVPN-graph defined by Table 11 and Table 12.

Table 11. Tabular representation of Example 3.6

	k_1	k_2	k_3
T_P	0.4	0.3	0.4
C_P	0.5	0.5	0.5
R_P	0.3	0.4	0.5
U_P	0.4	0.3	0.6
F_P	0.4	0.5	0.5

Table 12. Tabular representation of Example 3.6

	(k_1, k_2)	(k_2, k_3)	(k_3, k_1)
T_P	0.3	0.3	0.4
C_P	0.5	0.5	0.5
R_P	0.4	0.5	0.5
U_P	0.4	0.6	0.6
F_P	0.5	0.5	0.5

The representation of the graph for Example 3.6 is presented in Figure 6.

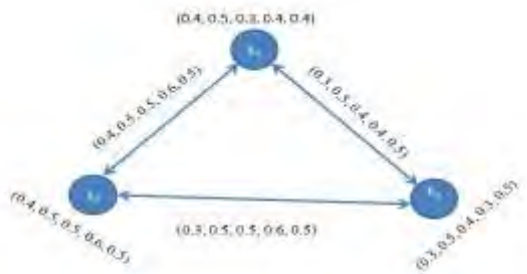

Figure 6: Graph of Example 3.6.

Here, the above SVPN-graph is a complete SVPN-graph.

Definition 3.11. An SVPN-graph $\hat{G}$=(P, Q) of $\hat{G}^*$=(V, E) is called a bipartite SVPN-graph if the graph $\hat{G}^*$=(V, E) is a bipartite graph.

Example 3.7. Assume that $\hat{G}^*$=(V, E) be a graph, where V= {k_1, k_2, k_3, k_4, k_5, k_6} and E={(k_1, k_2), (k_2, k_3), (k_3, k_1)}. Suppose that $\hat{G}$=(P, Q) be an SVPN-graph defined by Table 13 and Table 14.

Table 13. Tabular representation of Example 3.7

	k_1	k_2	k_3	k_4	k_5	k_6
T_P	0.4	0.3	0.4	0.6	0.9	0.8
C_P	0.5	0.5	0.5	0.3	0.8	0.4
R_P	0.3	0.4	0.5	0.5	0.5	0.3
U_P	0.4	0.3	0.6	0.8	0.7	0.6
F_P	0.4	0.5	0.5	0.4	0.8	0.5

Table 14. Tabular representation of Example 3.7

	(k_1, k_2)	(k_1, k_3)	(k_1, k_6)	(k_3, k_5)	(k_2, k_4)	(k_2, k_6)	(k_3, k_6)	(k_4, k_6)	(k_5, k_6)
T_P	0.3	0.3	0.4	0.4	0.3	0.3	0.4	0.6	0.8
C_P	0.5	0.5	0.4	0.5	0.3	0.4	0.4	0.3	0.4
R_P	0.4	0.5	0.3	0.5	0.5	0.4	0.5	0.5	0.5
U_P	0.4	0.6	0.6	0.6	0.8	0.6	0.6	0.8	0.7
F_P	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.8

The representation of the graph of Example 3.7 is presented in Figure 7.

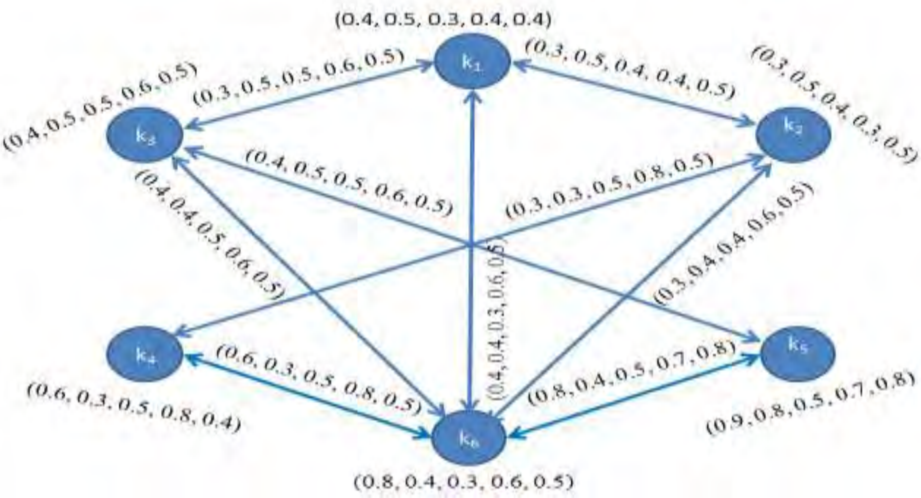

Figure 7: Graph of Example 3.7

Here, the crisp graph $\hat{G}^*$=(V, E) is a bipartite graph and $\hat{G}$=(P, Q) is a SVPN-graph of $\hat{G}^*$=(V, E). Hence $\hat{G}$=(P, Q) is a bipartite SVPN-graph.

Definition 3.12. Suppose that $\hat{G}$=(P, Q) be an SVPN-graph. Then, the degree of the vertex k is defined by d(k)=(d_T(k), d_C(k), d_R(k), d_U(k), d_F(k)),

where, d_T(k) = degree of the truth-membership vertex = sum of the truth-membership of all edges those are incident on the vertex k = $\sum_{\mathbf{u}\neq\mathbf{k}} T_Q(u, k)$;

d_C(k) = degree of the contradiction-membership vertex = sum of the contradiction-membership of all edges those are incident on the vertex k = $\sum_{\mathbf{u}\neq\mathbf{k}} C_Q(u, k)$;

d_R(k) = degree of the ignorance-membership vertex = sum of the ignorance-membership of all edges those are incident on the vertex k = $\sum_{\mathbf{u}\neq\mathbf{k}} R_Q(u, k)$;

d_U(k) = degree of the unknown-membership vertex = sum of the unknown-membership of all edges those are incident on the vertex k = $\sum_{\mathbf{u}\neq\mathbf{k}} U_Q(u, k)$;

d_F(k) = degree of the falsity-membership vertex = sum of the false-membership of all edges those are incident on the vertex k = $\sum_{\mathbf{u}\neq\mathbf{k}} F_Q(u, k)$.

Example 3.8. Assume that $\hat{G}$=(P, Q) be an SVPN-graph of $\hat{G}^*$=(V, E) defined by Table 15, Table 16.

Table 15. Tabular representation of example 3.8

	k_1	k_2	k_3	k_4
T_P	0.3	0.2	0.5	0.3
C_P	0.3	0.8	0.6	0.4
R_P	0.5	0.6	1.0	0.6
U_P	0.6	0.5	0.8	0.7
F_P	0.6	0.5	0.8	0.8

Table 16. Tabular representation of example 3.8

	(k_1, k_2)	(k_2, k_3)	(k_3, k_4)	(k_4, k_1)	(k_2, k_3)	(k_2, k_4)
T_P	0.1	0.2	0.3	0.2	0.1	0.1
C_P	0.2	0.5	0.4	0.3	0.4	0.3
R_P	0.7	1.0	1.0	0.8	1.0	0.7
U_P	0.7	0.9	0.8	0.8	0.9	0.9
F_P	0.7	0.8	0.8	0.9	0.8	0.9

The representation of the graph of example 3.8 is shown in Figure 8.

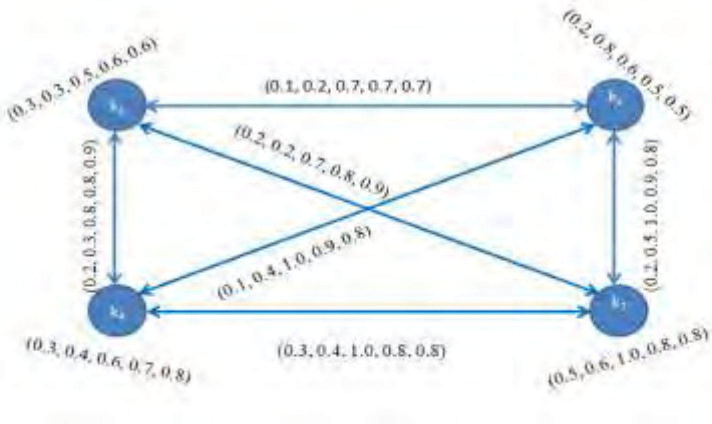

Figure 8: Graph of Example 3.8

Then, $d(k_1) = (0.3, 0.5, 1.5, 1.5, 1.6)$, $d(k_2) = (0.5, 1.4, 3.4, 3.4, 3.2)$, $d(k_3)= (0.6, 1.3, 3.0, 2.6, 2.4)$, and $d(k_4)= (0.6, 1.0, 2.5, 2.5, 2.6)$.

Definition 3.13. Suppose that $\hat{G}=(P, Q)$ is an SVPN-graph of $\hat{G}^*=(V, E)$. Then, $\hat{G}=(P, Q)$ is called a constant SVPN-graph if degree of each vertex is same i.e., $d(k) = (y_1, y_2, y_3, y_4, y_5)$, $\forall k \in V$.

Example 3.9. Assume that $\hat{G}=(P, Q)$ be an SVPN-graph, which is defined by Table 17 and Table 18.

Table 17. Tabular representation of example 3.9

	k_1	k_2	k_3	k_4
T_P	0.4	0.2	0.4	0.3
C_P	0.3	0.4	0.6	0.5
R_P	0.6	0.6	0.7	0.6
U_P	0.7	0.6	0.7	0.7
F_P	0.7	0.4	0.8	0.7

Table 18. Tabular representation of example 3.9

	(k_1, k_2)	(k_2, k_3)	(k_3, k_4)	(k_4, k_1)
T_P	0.2	0.1	0.2	0.1
C_P	0.2	0.3	0.2	0.3
R_P	0.7	0.9	0.7	0.9
U_P	0.8	0.8	0.8	0.8
F_P	0.9	0.9	0.9	0.9

The representation of the graph for Example 3.9 is shown in Figure 9.

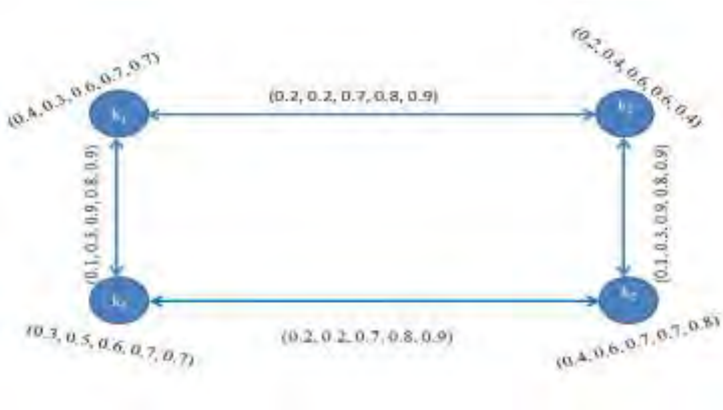

Figure 9: Graph of Example 3.9

In the above SVPN-graph $\hat{G}$= (P, Q), the degree of the vertices k_1, k_2, k_3, and k_4 are $d(k_1)$ = (0.3, 0.5, 1.6, 1.6, 1.8), $d(k_2)$ = (0.3, 0.5, 1.6, 1.6, 1.8), $d(k_3)$ = (0.3, 0.5, 1.6, 1.6, 1.8) and $d(k_4)$ = (0.3, 0.5, 1.6, 1.6, 1.8). Hence, $\hat{G}$= (P, Q) is a constant SVPN-graph.

Definition 3.14. Assume that $\hat{G}$= (P, Q) be a SVPN-graph. Then, the order of $\hat{G}$= (P, Q), denoted by $O(\hat{G})$ is defined by $O(\hat{G})= (O_T(\hat{G}), O_C(\hat{G}), O_R(\hat{G}), O_U(\hat{G}), O_F(\hat{G}))$, where

$O_T(\hat{G})= \sum_{k\in V} T_P$ denotes the T-order of $\hat{G}$= (P, Q);

$O_C(\hat{G})= \sum_{k\in V} C_P$ denotes the C-order of $\hat{G}$= (P, Q)

$O_R(\hat{G})= \sum_{k\in V} R_P$ denotes the R-order of $\hat{G}$= (P, Q);

$O_U(\hat{G})= \sum_{k\in V} U_P$ denotes the U-order of $\hat{G}$= (P, Q);

$O_F(\hat{G})= \sum_{k\in V} F_P$ denotes the F-order of $\hat{G}$= (P, Q).

Example 3.10. Assume that $\hat{G}$= (P, Q) is an SVPN-graph of $\hat{G}^*$= (V, E) as shown in Example 3.6. Then, order of the SVPN-graph $\hat{G}$= (P, Q) is $O(\hat{G})$= (1.3, 2.1, 2.7, 2.6, 2.7).

Definition 3.15. Suppose that $\hat{G}$= (P, Q) is an SVPN-graph. Then, the size of $\hat{G}$= (P, Q), denoted by $S(\hat{G})$ is defined by $S(\hat{G})= (S_T(\hat{G}), S_C(\hat{G}), S_R(\hat{G}), S_U(\hat{G}), S_F(\hat{G}))$, where

$S_T(\hat{G})=\sum_{u\neq k} T_Q(u, k)$ denotes the T-size of $\hat{G}$= (P, Q);

$S_C(\hat{G})=\sum_{u\neq k} C_Q(u, k)$ denotes the C-size of $\hat{G}$= (P, Q);

$S_R(\hat{G})=\sum_{u\neq k} R_Q(u, k)$ denotes the R-size of $\hat{G}$= (P, Q);

$S_U(\hat{G})=\sum_{u\neq k} U_Q(u, k)$ denotes the U-size of $\hat{G}$= (P, Q);

$S_F(\hat{G})= \sum_{u\neq k} F_Q(u, k)$ denotes the F-size of $\hat{G}$= (P, Q).

Example 3.11. Assume that $\hat{G}$= (P, Q) is an SVPN-graph of $\hat{G}^*$= (V, E) as shown in Example 3.6. Then, size of the SVPN-graph $\hat{G}$= (P, Q) is $S(\hat{G})$ = (1.0, 2.1, 5.2, 5, 4.9).

4. Result and discussion

Graph theory is utilized to deal with many real- problems in operations research. In real-life situation, however, indeterminacy and uncertainty may exist in almost every graph theoretic problem. SVPN-graph is a useful graph theory to model uncertainty and indeterminacy in convincing way based on pentapartitioned neutrosophic set which is an extension of neutrosophic set. So, there is a possibility that SVPN-graph will be more successful in dealing with graph theoretic problems having indeterminacy in the form of three independent components, namely, unknown, contradiction, and ignorance.

5. Conclusions

In this article, we have presented the notions of SVPN-graph. Also, we have defined the degree, order, size of a SVPN-graph and investigated some properties of them. By defining degree, order, size of SVPN-graphs, we formulate some results on SVPN-graphs. Further, we give few examples to justify the definitions and results. We hope that the approach presented in this paper will open up new avenues of research on SVPN-graph for its application in real life problems in the current neutrosophic area.

In future study, the single valued pentapartitioned neutrosophic graph can be extended to regular and irregular single valued pentapartitioned neutrosophic graph. The proposed single valued pentapartitioned graph can be extended to single-valued pentapartitioned neutrosophic intersection graphs, single-valued pentapartitioned neutrosophic hypergraphs, and so on. The single-valued pentapartitioned neutrosophic graph can be employed to model the computer networks, expert systems, image processing, social network , and telecommunication.

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On New Types of Weakly Neutrosophic Crisp Closed Functions

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Abstract. This study utilizes some ideas of α-closed and semi-α-closed sets in neutrosophic crisp topological space to state roughly innovative categories of weakly neutrosophic crisp closed functions such as; neutrosophic crisp α*-closed functions, neutrosophic crisp α**-closed functions, neutrosophic crisp semi-α-closed functions, neutrosophic crisp semi-α*-closed and neutrosophic crisp semi-α**-closed functions. Moreover, the interactions among these kinds of feebly neutrosophic crisp closed functions and the suggestions for crisp closed functions are described. Furthermore, some theorems, properties, and remarks are debated.

Keywords: Neutrosophic crisp α*-closed, neutrosophic crisp α**-closed, neutrosophic crisp semi-α-closed, neutrosophic crisp semi-α*-closed and neutrosophic crisp semi-α**-closed functions.

1. Introduction

Smarandache [1,2] extended the view of sets by defending neutrosophic sets as a generality of Zadeh's fuzzy set concept, which states there is no accurate meaning for the set [3]. Soon after, the intuitionistic fuzzy set theory was submitted by Atanassov, such that he suggested that some elements have the degree of non-membership in the set [4]. The recently exhibited notions fascinated numerous scholars of conventional mathematics. Perhaps, fuzzy topology was set up by Chang [5] and Lowen [6] by redirecting the constructs from fuzzy sets to the traditional topological spaces. Additionally, the extraction of neutrosophic crisp topological space (shortly, NCTS) was announced by A. A. Salama et al. [7]. M. Abdel-Basset et al. [8-13] provided a new neutrosophic technique. The interpretation of neutrosophic crisp semi-α-closed sets was tendered by R. K. Al-Hamido et al. [14]. Some views of αgs continuity and αgs irresolute functions was examined by V. Banupriya et al. [15]. Some principles of neutrosophic α^m-continuity was demonstrated by R. Dhavaseelan et al. [16]. The gb-closed sets then gb-continuity was directed by C. Maheswari et al. [17]. The homeomorphism in neutrosophic topological spaces was generalized by M. H. PAGE et al. [18]. A weakly neutrosophic crisp continuity was established by Q. H. Imran et al. [19,20]. Recently, new types of open mappings in weakly neutrosophic crisp topology was defined by Al-Obaidi et al. [21].

The target of the study is to submit different categories of neutrosophic crisp closed functions in weakly forms, such as; neutrosophic crisp α*-closed, neutrosophic crisp α**-closed, neutrosophic crisp semi-α-closed, neutrosophic crisp semi-α*-closed and neutrosophic crisp semi-α**-closed functions. Additionally, the connections concerning these kinds of weakly neutrosophic crisp closed functions are illuminated, corresponding to the thoughts of neutrosophic crisp closed functions. As Well, some theorems, properties and remarks are demonstrated.

2. Preliminaries

Through this paper, $(\mathcal{I},\zeta),(\mathcal{J},\eta)$ and $(\mathcal{K},\gamma)$ (or in short $\mathcal{I},\mathcal{J}$ and $\mathcal{K}$) constantly imply NCTSs. Assume that $\mathcal{L}$ be a neutrosophic crisp set in a NCTS $\mathcal{I}$, then

- $\mathcal{L}^c = \mathcal{I} - \mathcal{L}$ signifies the neutrosophic crisp complement of $\mathcal{L}$.
- $NC\text{-}cl(\mathcal{L})$ refers to the neutrosophic crisp closure of $\mathcal{L}$.
- $NC\text{-}int(\mathcal{L})$ speaks of the neutrosophic crisp interior of $\mathcal{L}$.

Definition 2.1 [7]: Assume non-empty fixed set $\mathcal{L}$ is sample space. The object with form $\mathcal{N} = \langle \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \rangle$ is called a neutrosophic crisp set, for short NC-set such that $\mathcal{N}_1, \mathcal{N}_2$ and $\mathcal{N}_3$ are subsets of $\mathcal{L}$ with the mutually disjoint property.

Definition 2.2: Let $\mathcal{L}$ be a NC-subset of a NCTS $\mathcal{I}$, then we have the following

i. NCα-CS is denoted as a neutrosophic crisp α-closed set [18] if $NC\text{-}cl(NC\text{-}int(NC\text{-}cl(\mathcal{L}))) \subseteq \mathcal{L}$.
ii. NCα-OSs is signified as a neutrosophic crisp α-open set (the complement of a NCα-CS) in $\mathcal{I}$.
iii. $\mathrm{NC\alpha C}(\mathcal{I})$ (resp. $\mathrm{NC\alpha O}(\mathcal{I})$) is represented as the collection of each NCα-CSs (resp. NCα-OSs) of $\mathcal{I}$.
iv. NCSα-CS is indicated as a neutrosophic crisp semi-α-closed set [14] if

$$NC\text{-}int(NC\text{-}cl(NC\text{-}int(NC\text{-}cl(\mathcal{L})))) \subseteq \mathcal{L}$$

or regularly if there exists a NCα-CS $\mathcal{D}$ in $\mathcal{I}$ such that $NC\text{-}int(\mathcal{D}) \subseteq \mathcal{L} \subseteq \mathcal{D}$.
v. NCSα-OS is designated as a neutrosophic crisp semi-α-open set (the complement of a NCSα-CS) in $\mathcal{I}$.
vi. NCSαC($\mathcal{I}$) (resp. NCSαO($\mathcal{I}$)) is shown as the collection of each NCSα-CSs (resp. NCSα-OSs) of $\mathcal{I}$.

Remark 2.3 [14,20]: In a NCTS $\mathcal{I}$, the resulting declarations hang on, and the reverse of each declaration is a fallacy:

i. Each NC-CS is a NCα-CS and NCSα-CS.
ii. Each NCα-CS is a NCSα-CS.

Theorem 2.4 [18]: For any NC-subset $\mathcal{L}$ of a NCTS $\mathcal{I}$, $\mathcal{L} \in \mathrm{NC\alpha C}(\mathcal{I})$ iff there exists a NC-CS $\mathcal{D}$ such that $NC\text{-}cl(NC\text{-}int(\mathcal{D})) \subseteq \mathcal{L} \subseteq \mathcal{D}$.

Definition 2.5: A function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is called:

i. Neutrosophic crisp closed (briefly NC-closed) [7] iff for each NC-CS $\mathcal{L}$ in $\mathcal{I}$, then $\psi(\mathcal{L})$ is a NC-CS in $\mathcal{J}$.
ii. Neutrosophic crisp α-closed (briefly NCα-closed) [19] iff for each NC-CS $\mathcal{L}$ in $\mathcal{I}$, then $\psi(\mathcal{L})$ is a NCα-CS in $\mathcal{J}$.

Theorem 2.6 [7]:

i. A function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is NC-closed iff $NC\text{-}cl(\psi(\mathcal{L})) \subseteq \psi(NC\text{-}cl(\mathcal{L}))$, for every $\mathcal{L} \subseteq \mathcal{I}$.
ii. A function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is neutrosophic crisp continuous (shortly NC-continuous) iff for each NC-CS $\mathcal{L}$ in $\mathcal{J}$, then $\psi^{-1}(\mathcal{L})$ is a NC-CS in $\mathcal{I}$.
iii. A function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is NC-continuous iff $NC\text{-}int(\psi(\mathcal{L})) \subseteq \psi(NC\text{-}int(\mathcal{L}))$, for every $\mathcal{L} \subseteq \mathcal{I}$.

3. Weakly Neutrosophic Crisp Closed Functions

Definition 3.1: Assume $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is a function, then ψ is named as the following:

i. Neutrosophic crisp α*-closed (briefly NCα*-closed) iff for each NCα-CS $\mathcal{L}$ in $\mathcal{I}$, then $\psi(\mathcal{L})$ is a NCα-CS in $\mathcal{J}$.

ii. Neutrosophic crisp α**-closed (briefly NCα**-closed) iff for each $\mathcal{L}$ NCα-CS in $\mathcal{I}$, then $\psi(\mathcal{L})$ is a NC-CS in $\mathcal{J}$.

Definition 3.2: A function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is called:

i. Neutrosophic crisp semi-α-closed (briefly NCSα-closed) iff for each $\mathcal{L}$ NC-CS in $\mathcal{I}$, then $\psi(\mathcal{L})$ is a NCSα-CS in $\mathcal{J}$.

ii. Neutrosophic crisp semi-α*-closed (briefly NCSα*-closed) iff for each $\mathcal{L}$ NCSα-CS in $\mathcal{I}$, then $\psi(\mathcal{L})$ is a NCSα-CS in $\mathcal{J}$.

iii. Neutrosophic crisp semi-α**-closed (briefly NCSα**-closed) iff for each $\mathcal{L}$ NCSα-CS in $\mathcal{I}$, then $\psi(\mathcal{L})$ is a NC-CS in $\mathcal{J}$.

Theorem 3.3: A function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is NCSα-closed iff for every $\mathcal{L} \subseteq \mathcal{I}$, $NC\text{-}int(NC\text{-}cl(NC\text{-}int(NC\text{-}cl(\psi(\mathcal{L}))))) \subseteq \psi(NC\text{-}cl(\mathcal{L}))$.

Proof:

Necessity: For any $\mathcal{L} \subseteq \mathcal{I}$, $\psi(NC\text{-}cl(\mathcal{L}))$ is NCSα-CS in $\mathcal{J}$ this implies that

$$NC\text{-}int(NC\text{-}cl(NC\text{-}int(\psi(NC\text{-}cl(\mathcal{L}))))) \subseteq \psi(NC\text{-}cl(\mathcal{L})).$$

Hence, we have

$$NC\text{-}int(NC\text{-}cl(NC\text{-}int(NC\text{-}cl(\psi(\mathcal{L}))))) \subseteq \psi(NC\text{-}cl(\mathcal{L})).$$

Sufficiency: For any $\mathcal{L} \subseteq \mathcal{I}$, we have by hypothesis

$$NC\text{-}int(NC\text{-}cl(NC\text{-}int(NC\text{-}cl(\psi(\mathcal{L}))))) \subseteq \psi(NC\text{-}cl(\mathcal{L})).$$

So $\psi(\mathcal{L})$ is NCSα-CS in $\mathcal{J}$ and then we get that the function ψ is a NCSα-closed. ■

Theorem 3.4:

i. Any function NC-closed is NCα-closed, then it is NCSα-closed. Nonetheless, the inverse is generally a fallacy.

ii. Any function NCα-closed is NCSα-closed. Nonetheless, the inverse is generally a fallacy.

Proof:

i. Assume $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is a NC-closed function, and $\mathcal{L}$ is a NC-CS in $\mathcal{I}$. Then $\psi(\mathcal{L})$ is a NC-CS in $\mathcal{J}$. Because NC-CS is NCα-CS (NCSα-CS), $\psi(\mathcal{L})$ is a NCα-CS (NCSα-CS) in $\mathcal{J}$. Thus, the function ψ is NCα-closed (NCSα-closed).

ii. Assume $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is a NCα-closed function and $\mathcal{L}$ is a NC-CS in $\mathcal{I}$. Then $\psi(\mathcal{L})$ is a NCα-CS in $\mathcal{J}$. Because NCα-CS is NCSα-CS, $\psi(\mathcal{L})$ is a NCSα-CS in $\mathcal{J}$. Thus, the function ψ is NCSα-closed. ■

Example 3.5: Assume $\mathcal{I} = \{p_1, p_2, p_3, p_4\}$. Then

$$\zeta_{\mathcal{I}} = \{\phi_N, \langle\{p_3\}, \phi, \phi\rangle, \langle\{p_1, p_3\}, \phi, \phi\rangle, \langle\{p_1, p_2, p_3\}, \phi, \phi\rangle, \mathcal{I}_N\}$$

is a NCTS. The collection of every NC-CSs of $\mathcal{I}$ is:

$$\text{NC-C}(\mathcal{I}) = \{\mathcal{I}_N, \langle\{p_1, p_2, p_4\}, \phi, \phi\rangle, \langle\{p_2, p_4\}, \phi, \phi\rangle, \langle\{p_4\}, \phi, \phi\rangle, \phi_N\}.$$

The collection of every NCα-CSs (NCSα-CSs) of $\mathcal{I}$ is:

$$\text{NCS}\alpha\text{C}(\mathcal{I}) = \text{NC}\alpha\text{C}(\mathcal{I}) = \text{NC-C}(\mathcal{I}) \cup \{\langle\{p_1, p_4\}, \phi, \phi\rangle, \langle\{p_1, p_2\}, \phi, \phi\rangle, \langle\{p_2\}, \phi, \phi\rangle, \langle\{p_1\}, \phi, \phi\rangle\}.$$

Define a function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ by

$$\psi(\langle\{p_1\}, \phi, \phi\rangle) = \langle\{p_1\}, \phi, \phi\rangle, \psi(\langle\{p_2\}, \phi, \phi\rangle) = \langle\{p_4\}, \phi, \phi\rangle,$$

$$\psi(\langle\{p_3\}, \phi, \phi\rangle) = \langle\{p_3\}, \phi, \phi\rangle, \psi(\langle\{p_4\}, \phi, \phi\rangle) = \langle\{p_2\}, \phi, \phi\rangle.$$

We observe ψ is a NCα-closed. It is NCSα-closed; nonetheless, it is not a NC-closed function because of $\langle\{p_4\}, \phi, \phi\rangle$ is NC-CS in $\mathcal{I}$ and $\psi(\langle\{p_4\}, \phi, \phi\rangle) = \langle\{p_2\}, \phi, \phi\rangle$ is not a NC-CS in $\mathcal{I}$.

Example 3.6: Let $\mathcal{J} = \{p_1, p_2, p_3, p_4\}$. Then

$$\zeta_{\mathcal{J}} = \{\phi_N, \langle\{p_1\}, \phi, \phi\rangle, \langle\{p_2\}, \phi, \phi\rangle, \langle\{p_1, p_2\}, \phi, \phi\rangle, \langle\{p_1, p_2, p_3\}, \phi, \phi\rangle, \mathcal{J}_N\}$$

is a NCTS. The collection of every NC-CSs of $\mathcal{J}$ is:

$$\text{NC-C}(\mathcal{J}) = \{\mathcal{J}_N, \langle\{p_2, p_3, p_4\}, \phi, \phi\rangle, \langle\{p_1, p_3, p_4\}, \phi, \phi\rangle, \langle\{p_3, p_4\}, \phi, \phi\rangle, \langle\{p_4\}, \phi, \phi\rangle, \phi_N\}.$$

The collection of every NCα-CSs of $\mathcal{J}$ is:

$$\text{NC}\alpha\text{C}(\mathcal{J}) = \text{NC-C}(\mathcal{J}) \cup \{\langle\{p_3\}, \phi, \phi\rangle\}.$$

The collection of every NCSα-CSs of $\mathcal{J}$ is:

$$\text{NCS}\alpha\text{C}(\mathcal{J}) = \text{NC}\alpha\text{C}(\mathcal{J}) \cup \{\langle\{p_2, p_4\}, \phi, \phi\rangle, \langle\{p_2, p_3\}, \phi, \phi\rangle, \langle\{p_1, p_4\}, \phi, \phi\rangle, \langle\{p_1, p_3\}, \phi, \phi\rangle, \langle\{p_2\}, \phi, \phi\rangle, \langle\{p_1\}, \phi, \phi\rangle\}.$$

Define a function $\psi: \mathcal{J} \longrightarrow \mathcal{J}$ by

$$\psi(\langle\{p_1\}, \phi, \phi\rangle) = \langle\{p_1\}, \phi, \phi\rangle, \psi(\langle\{p_2\}, \phi, \phi\rangle) = \langle\{p_2\}, \phi, \phi\rangle,$$
$$\psi(\langle\{p_3\}, \phi, \phi\rangle) = \psi(\langle\{p_4\}, \phi, \phi\rangle) = \langle\{p_4\}, \phi, \phi\rangle.$$

We observe that the function ψ is a NCSα-closed. Furthermore, it is not NCα-closed function because of $\langle\{p_1, p_3, p_4\}, \phi, \phi\rangle$ is NC-CS in $\mathcal{J}$ and $\psi(\langle\{p_1, p_3, p_4\}, \phi, \phi\rangle) = \langle\{p_1, p_4\}, \phi, \phi\rangle$ is not a NCα-CS in $\mathcal{J}$.

Remark 3.7: There is no relation between the ideas of NC-closed and NCα*-closed functions, as the next two examples are displayed below.

Example 3.8: The function ψ in Example (3.5) is a NCα*-closed. Nonetheless, it is not NC-closed.

Example 3.9: Assume that $\mathcal{J} = \{p_1, p_2, p_3, p_4\}$ is a set. Then

$$\zeta_{\mathcal{J}} = \{\phi_N, \langle\{p_1\}, \phi, \phi\rangle, \langle\{p_2\}, \phi, \phi\rangle, \langle\{p_1, p_2\}, \phi, \phi\rangle, \langle\{p_1, p_2, p_3\}, \phi, \phi\rangle, \mathcal{J}_N\}$$

is a NCTS. The collection of each NC-CSs of $\mathcal{J}$ is:

$$\text{NC-C}(\mathcal{J}) = \{\mathcal{J}_N, \langle\{p_2, p_3, p_4\}, \phi, \phi\rangle, \langle\{p_1, p_3, p_4\}, \phi, \phi\rangle, \langle\{p_3, p_4\}, \phi, \phi\rangle, \langle\{p_4\}, \phi, \phi\rangle, \phi_N\}.$$

The family of all NCα-CSs of $\mathcal{J}$ is: $\text{NC}\alpha\text{C}(\mathcal{J}) = \text{NC-C}(\mathcal{J}) \cup \{\langle\{p_3\}, \phi, \phi\rangle\}$. The collection of each NCSα-CSs of $\mathcal{J}$ is:

$$\text{NCS}\alpha\text{C}(\mathcal{J}) = \text{NC}\alpha\text{C}(\mathcal{J}) \cup \{\langle\{p_2, p_4\}, \phi, \phi\rangle, \langle\{p_2, p_3\}, \phi, \phi\rangle, \langle\{p_1, p_4\}, \phi, \phi\rangle, \langle\{p_1, p_3\}, \phi, \phi\rangle, \langle\{p_2\}, \phi, \phi\rangle, \langle\{p_1\}, \phi, \phi\rangle\}$$

Define a function $\psi: \mathcal{J} \longrightarrow \mathcal{J}$ by

$$\psi(\langle\{p_1\}, \phi, \phi\rangle) = \psi(\langle\{p_2\}, \phi, \phi\rangle) = \langle\{p_1\}, \phi, \phi\rangle,$$
$$\psi(\langle\{p_3\}, \phi, \phi\rangle) = \langle\{p_2\}, \phi, \phi\rangle, \psi(\langle\{p_4\}, \phi, \phi\rangle) = \langle\{p_3\}, \phi, \phi\rangle.$$

We observe that the ψ is a NC-closed. Furthermore, it is not NCα*-closed function because of $\langle\{p_3\}, \phi, \phi\rangle$ is NCα-CS in $\mathcal{J}$ and $\psi(\langle\{p_3\}, \phi, \phi\rangle) = \langle\{p_2\}, \phi, \phi\rangle$ is not a NCα-CS in $\mathcal{J}$.

Proposition 3.10:

i. A function $\psi: \mathcal{J} \longrightarrow \mathcal{J}$ is NC-closed and NC-continuous, then this function is NCα*-closed.
ii. A function $\psi: \mathcal{J} \longrightarrow \mathcal{J}$ is a NCα*-closed iff $\psi: (\mathcal{J}, \text{NC}\alpha\text{O}(\mathcal{J})) \longrightarrow (\mathcal{J}, \text{NC}\alpha\text{O}(\mathcal{J}))$ is NC-closed.

Proof:

i. Assume that a function $\psi: \mathcal{J} \longrightarrow \mathcal{J}$ is NC-closed and NC-continuous. To verify the function ψ is NCα*-closed, we suppose that $\mathcal{L} \in \text{NC}\alpha\text{C}(\mathcal{J})$, then for some sets like NC-CS $\mathcal{N}$ with this fact $NC\text{-}cl(NC\text{-}int(\mathcal{N})) \subseteq \mathcal{L} \subseteq \mathcal{N}$ (by theorem (2.4)). Hence $\psi(NC\text{-}cl(NC\text{-}int(\mathcal{N}))) \subseteq \psi(\mathcal{L}) \subseteq \psi(\mathcal{N})$ but $NC\text{-}cl(\psi(NC\text{-}int(\mathcal{N}))) \subseteq \psi(NC\text{-}cl(NC\text{-}int(\mathcal{N})))$ (because the function ψ is NC-closed). Then $NC\text{-}cl(\psi(NC\text{-}int(\mathcal{N}))) \subseteq \psi(NC\text{-}cl(NC\text{-}int(\mathcal{N}))) \subseteq \psi(\mathcal{L}) \subseteq \psi(\mathcal{N})$. But $NC\text{-}cl(NC\text{-}int(\psi(\mathcal{N}))) \subseteq NC\text{-}cl(\psi(NC\text{-}int(\mathcal{N})))$ (because the function ψ is NC-continuous). Consequently, we get $NC\text{-}cl(NC\text{-}int(\psi(\mathcal{N}))) \subseteq \psi(\mathcal{L}) \subseteq \psi(\mathcal{N})$. However, $\psi(\mathcal{N})$ is a NC-CS in $\mathcal{J}$ (because the function ψ is NC-closed). Hence $\psi(\mathcal{L}) \in \text{NC}\alpha\text{C}(\mathcal{J})$ (it is clear from theorem (2.4)). Thus, the function ψ is NCα*-closed.

ii. Part (ii) is clear for proof. ■

Remark 3.11: It is understood that every function is defined as NCα*-closed, then it is NCα-closed as well as NCSα-closed. Nonetheless, the inverse is generally a fallacy, as the next example is displayed below.

Example 3.12: It is clear to note that the function ψ is a NCα-closed and NCSα-closed in Example (3.12). However, it is not NCα*-closed.

Remark 3.13: There is no relation between the ideas of NC-closed and NCSα*-closed functions, as the next two examples are displayed below.

Example 3.14: The function ψ in Example (3.5) is a NCSα*-closed. However, it is not NC-closed.

Example 3.15: Let $\mathcal{I} = \{p_1, p_2, p_3\}$. Then $\zeta = \{\phi_N, \langle\{p_1\}, \phi, \phi\rangle, \mathcal{I}_N\}$ is a NCTS. The collection of each NC-CSs of $\mathcal{I}$ is $\text{NC-C}(\mathcal{I}) = \{\mathcal{I}_N, \langle\{p_2, p_3\}, \phi, \phi\rangle, \phi_N\}$. The collection of each NCα-CSs (NCSα-CSs) of $\mathcal{I}$ is:

$$\text{NC}\alpha\text{C}(\mathcal{I}) = \text{NCS}\alpha\text{C}(\mathcal{I}) = \text{NC-C}(\mathcal{I}) \cup \{\langle\{p_3\}, \phi, \phi\rangle, \langle\{p_2\}, \phi, \phi\rangle\}.$$

Let $\mathcal{J} = \{q_1, q_2, q_3, q_4\}$. Then $\eta = \{\phi_N, \langle\{q_1\}, \phi, \phi\rangle, \langle\{q_2, q_3\}, \phi, \phi\rangle, \langle\{q_1, q_2, q_3\}, \phi, \phi\rangle, \mathcal{J}_N\}$ is a NCTS. The collection of each NC-CSs of $\mathcal{J}$ is:

$$\text{NC-C}(\mathcal{J}) = \{\mathcal{J}_N, \langle\{q_2, q_3, q_4\}, \phi, \phi\rangle, \langle\{q_1, q_4\}, \phi, \phi\rangle, \langle\{q_4\}, \phi, \phi\rangle, \phi_N\}.$$

The collection of each NCα-CSs of $\mathcal{J}$ is: $\text{NC}\alpha\text{C}(\mathcal{J}) = \text{NC-C}(\mathcal{J})$. The family of all NCSα-CSs of $\mathcal{J}$ is:

$$\text{NCS}\alpha\text{C}(\mathcal{J}) = \text{NC}\alpha\text{C}(\mathcal{J}) \cup \{\langle\{q_1\}, \phi, \phi\rangle\}.$$

Define a function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ by

$$\psi(\langle\{p_1\}, \phi, \phi\rangle) = \langle\{q_1\}, \phi, \phi\rangle, \psi(\langle\{p_2\}, \phi, \phi\rangle) = \langle\{q_2\}, \phi, \phi\rangle, \ \psi(\langle\{p_3\}, \phi, \phi\rangle) = \langle\{q_3\}, \phi, \phi\rangle.$$

Clearly, we can realize that the function ψ is a NC-closed; nonetheless, this function doesn't represent NCSα*-closed because of $\langle\{p_3\}, \phi, \phi\rangle \in \text{NCS}\alpha\text{C}(\mathcal{I})$ and $\psi(\langle\{p_3\}, \phi, \phi\rangle) = \langle\{q_3\}, \phi, \phi\rangle \notin \text{NCS}\alpha\text{C}(\mathcal{J})$.

Proposition 3.16: A function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is a NCSα*-closed iff this function $\psi: (\mathcal{I}, \text{NCS}\alpha\text{O}(\mathcal{I})) \longrightarrow (\mathcal{J}, \text{NCS}\alpha\text{O}(\mathcal{J}))$ is a NC-closed.

Proof: Understandable. ■

Remark 3.17: There is no relation between the ideas of NCα*-closed and NCSα*-closed functions, as the next two examples are displayed below.

Example 3.18: The function ψ in Example (3.9) is a NCSα*-closed. However, it is not NCα*-closed.

Example 3.19: Assume that the set $\mathcal{I} = \{p_1, p_2, p_3, p_4\}$. Then

$$\zeta = \{\phi_N, \langle\{p_1\}, \phi, \phi\rangle, \langle\{p_2, p_4\}, \phi, \phi\rangle, \langle\{p_1, p_2, p_4\}, \phi, \phi\rangle, \mathcal{I}_N\}$$

is a NCTS. The collection of each NC-CSs of $\mathcal{I}$ is:

$$\text{NC-C}(\mathcal{I}) = \{\mathcal{I}_N, \langle\{p_2, p_3, p_4\}, \phi, \phi\rangle, \langle\{p_1, p_3\}, \phi, \phi\rangle, \langle\{p_3\}, \phi, \phi\rangle, \phi_N\}.$$

Assume that the set $\mathcal{J} = \{q_1, q_2, q_3, q_4\}$. Then

$$\eta = \{\phi_N, \langle\{q_1\}, \phi, \phi\rangle, \langle\{q_2, q_4\}, \phi, \phi\rangle, \langle\{q_1, q_2, q_4\}, \phi, \phi\rangle, \mathcal{J}_N\}$$

is a NCTS. The collection of each NC-CSs of $\mathcal{J}$ is

$$\text{NC-C}(\mathcal{J}) = \{\mathcal{J}_N, \langle\{q_2, q_3, q_4\}, \phi, \phi\rangle, \langle\{q_1, q_3\}, \phi, \phi\rangle, \langle\{q_3\}, \phi, \phi\rangle, \phi_N\}.$$

Define a function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ by

$$\psi(\langle\{p_1\}, \phi, \phi\rangle) = \langle\{q_1\}, \phi, \phi\rangle,$$

$$\psi(\langle\{p_2\}, \phi, \phi\rangle) = \psi(\langle\{p_3\}, \phi, \phi\rangle) = \langle\{q_2\}, \phi, \phi\rangle,$$

$$\psi(\langle\{p_4\}, \phi, \phi\rangle) = \langle\{q_4\}, \phi, \phi\rangle.$$

Clearly, we can note that the function ψ is NCα*-closed. However, it is not NCSα*-closed.

Theorem 3.20: If a function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is NCα*-closed and NC-continuous, then it is NCSα*-closed.

Proof: Assume that the function $\psi: \mathcal{I} \longrightarrow \mathcal{J}$ is NCα*-closed as well as NC-continuous. Moreover, suppose $\mathcal{L}$ is NCSα-CS in $\mathcal{I}$. Then for some sets, like NCα-CS $\mathcal{N}$ with this fact $NC\text{-}int(\mathcal{N}) \subseteq \mathcal{L} \subseteq \mathcal{N}$. Consequently, we have

$$NC\text{-}int(\psi(\mathcal{N})) \subseteq \psi(NC\text{-}int(\mathcal{N})) \subseteq \psi(\mathcal{L}) \subseteq \psi(\mathcal{N})$$

(this is because the function ψ is NC-continuous). However, the set $\psi(\mathcal{N}) \in \mathrm{NC\alpha C}(\mathcal{J})$ because the function ψ is NCα*-closed). Thus, the set $NC\text{-}int(\psi(\mathcal{N})) \subseteq \psi(\mathcal{L}) \subseteq \psi(\mathcal{N})$. Therefore, $\psi(\mathcal{L}) \in \mathrm{NCS\alpha C}(\mathcal{J})$. Thus, ψ is a NCSα*-closed function. ■

Theorem 3.21: The two functions $\psi_1: \mathcal{I} \longrightarrow \mathcal{J}$ and $\psi_2: \mathcal{J} \longrightarrow \mathcal{K}$ are satisfying the following

i. If a function ψ_1 is NC-closed and a function ψ_2 is NCα-closed, then a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα-closed.
ii. If a function ψ_1 is NCα-closed and a function ψ_2 is NCα*-closed, then a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα-closed.
iii. If functions ψ_1 and ψ_2 are NCα*-closed, then a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα*-closed.
iv. If functions ψ_1 and ψ_2 are NCSα*-closed, then a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCSα*-closed.
v. If functions ψ_1 and ψ_2 are NCα**-closed, then a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα**-closed.
vi. If functions ψ_1 and ψ_2 are NCSα**-closed, then a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCSα**-closed.
vii. If a function ψ_1 is NCα**-closed and a function ψ_2 is NCα*-closed, then a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα*-closed.
viii. If a function ψ_1 is NCα-closed and a function ψ_2 is NCα**-closed, then a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NC-closed.
ix. If a function ψ_1 is NCα**-closed and a function ψ_2 is NCα-closed, then a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα*-closed.
x. If a function ψ_1 is NCα**-closed and a function ψ_2 is NC-closed, then a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is a NCα**-closed.

Proof:

i. Let $\mathcal{L}$ be a NC-CS in $\mathcal{I}$. Since ψ_1 is a NC-closed function, $\psi_1(\mathcal{L})$ is a NC-CS in $\mathcal{J}$. Since ψ_2 is a Nα-closed function, $\psi_2 \circ \psi_1(\mathcal{L}) = \psi_2(\psi_1(\mathcal{L}))$ is a NCα-CS in $\mathcal{K}$. Thus, a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is Nα-closed.
ii. Let $\mathcal{L}$ be a NC-CS in $\mathcal{I}$. Since ψ_1 is a NCα-closed function, $\psi_1(\mathcal{L})$ is a NCα-CS in $\mathcal{J}$. Since ψ_2 is a NCα*-closed function, $\psi_2 \circ \psi_1(\mathcal{L}) = \psi_2(\psi_1(\mathcal{L}))$ is a NCα-CS in $\mathcal{K}$. Thus, a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα-closed.
iii. Let $\mathcal{L}$ be a NCα-CS in $\mathcal{I}$. Since ψ_1 is a NCα*-closed function, $\psi_1(\mathcal{L})$ is a NCα-CS in $\mathcal{J}$. Since ψ_2 is a NCα*-closed function, $\psi_2 \circ \psi_1(\mathcal{L}) = \psi_2(\psi_1(\mathcal{L}))$ is a NCα-CS in $\mathcal{K}$. Thus, a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα*-closed.
iv. Let $\mathcal{L}$ be a NCSα-CS in $\mathcal{I}$. Since ψ_1 is a NCSα*-closed function, $\psi_1(\mathcal{L})$ is a NCSα-CS in $\mathcal{J}$. Since ψ_2 is a NCSα*-closed function, $\psi_2 \circ \psi_1(\mathcal{L}) = \psi_2(\psi_1(\mathcal{L}))$ is a NCSα-CS in $\mathcal{K}$. Thus, a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCSα*-closed.
v. Let $\mathcal{L}$ be a NCα-CS in $\mathcal{I}$. Since ψ_1 is a NCα**-closed function, $\psi_1(\mathcal{L})$ is a NC-CS in $\mathcal{J}$. Since any NC-CS is NCα-CS, $\psi_1(\mathcal{L})$ is a NCα-CS in $\mathcal{J}$. Since ψ_2 is a NCα**-closed function, $\psi_2 \circ \psi_1(\mathcal{L}) = \psi_2(\psi_1(\mathcal{L}))$ is a NC-CS in $\mathcal{K}$. Thus, a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα**-closed.

vi. Let $\mathcal{L}$ be a NCSα-CS in $\mathcal{I}$. Since ψ_1 is a NCSα**-closed function, $\psi_1(\mathcal{L})$ is a NC-CS in $\mathcal{J}$. Since any NC-CS is NCSα-CS, $\psi_1(\mathcal{L})$ is a NCSα-CS in $\mathcal{J}$. Since ψ_2 is a NCSα**-closed function, $\psi_2 \circ \psi_1(\mathcal{L}) = \psi_2(\psi_1(\mathcal{L}))$ is a NC-CS in $\mathcal{K}$. Thus, a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCSα**-closed.

vii. Let $\mathcal{L}$ be a NCα-CS in $\mathcal{I}$. Since ψ_1 is a NCα**-closed function, $\psi_1(\mathcal{L})$ is a NC-CS in $\mathcal{J}$. Since any NC-CS is NCα-CS, $\psi_1(\mathcal{L})$ is a NCα-CS in $\mathcal{J}$. Since ψ_2 is a NCα*-closed function, $\psi_2 \circ \psi_1(\mathcal{L}) = \psi_2(\psi_1(\mathcal{L}))$ is a NCα-CS in $\mathcal{K}$. Thus, a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα*-closed.

viii. Let $\mathcal{L}$ be a NC-CS in $\mathcal{I}$. Since ψ_1 is a NCα-closed function, $\psi_1(\mathcal{L})$ is a NCα-CS in $\mathcal{J}$. Since ψ_2 is a NCα**-closed function, $\psi_2 \circ \psi_1(\mathcal{L}) = \psi_2(\psi_1(\mathcal{L}))$ is a NC-CS in $\mathcal{J}$. Thus, a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NC-closed.

ix. Let $\mathcal{L}$ be a NCα-CS in $\mathcal{J}$. Since ψ_1 is a NCα**-closed function, $\psi_1(\mathcal{L})$ is a NC-CS in $\mathcal{J}$. Since ψ_2 is a NCα-closed function, $\psi_2 \circ \psi_1(\mathcal{L}) = \psi_2(\psi_1(\mathcal{L}))$ is a NCα-CS in $\mathcal{J}$. Thus, a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is a NCα*-closed.

x. Let $\mathcal{L}$ be a NCα-CS in $\mathcal{J}$. Since ψ_1 is a NCα**-closed function, $\psi_1(\mathcal{L})$ is a NC-CS in $\mathcal{J}$. Since ψ_2 is a NC-closed function, $\psi_2 \circ \psi_1(\mathcal{L}) = \psi_2(\psi_1(\mathcal{L}))$ is a NC-CS in $\mathcal{L}$. Thus, a function $\psi_2 \circ \psi_1: \mathcal{I} \longrightarrow \mathcal{K}$ is NCα**-closed. ■

Remark 3.22: The following diagram on the next page explains the relationship between weakly NC-closed functions.

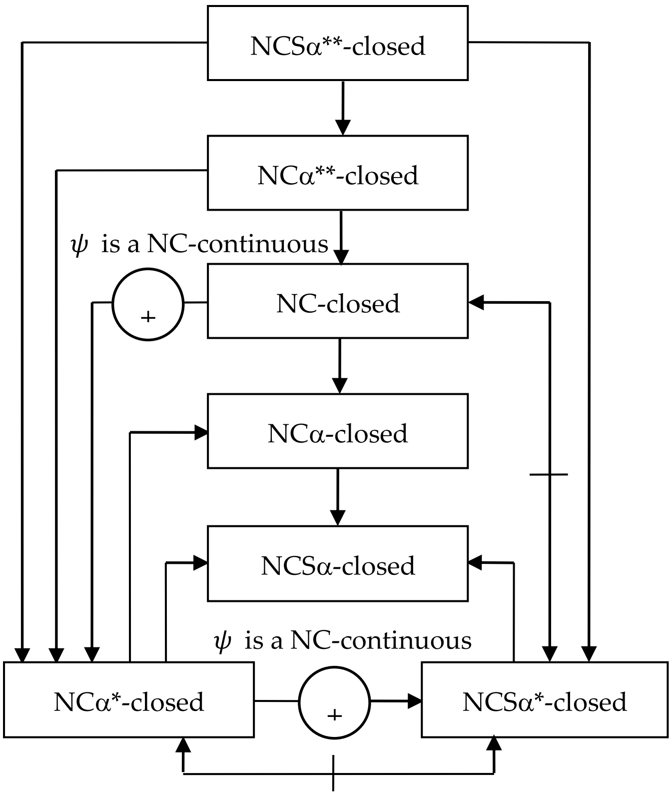

Fig. 3.1

4. Conclusion

We utilize the ideas of NCα-CSs and NCSα-CSs to identify some different kinds of weakly NC-closed functions, for instance; NCα*-closed, NCα**-closed, NCSα-closed, NCSα*-closed and NCSα**-closed functions. The most significant results are that the neutrosophic crisp semi-α**-closed maps are neutrosophic crisp α**-closed, neutrosophic crisp α*-closed and neutrosophic crisp semi-α*-closed. Moreover, neutrosophic crisp α**-closed map is neutrosophic crisp α*-closed and neutrosophic crisp closed. However, the neutrosophic crisp closed map is not neutrosophic crisp α*-closed and the latter is not neutrosophic crisp semi-α*-closed unless they are NC-continuous maps. Furthermore, neutrosophic crisp α*-closed and neutrosophic crisp closed maps are neutrosophic crisp α-closed and neutrosophic crisp semi-α-closed because crisp α-closed map is neutrosophic crisp semi-α-closed. Finally, neutrosophic crisp semi-α*-closed map is neutrosophic crisp semi-α-closed. As future works, the NCα-CSs and NCSα-CSs can be used to derive some neutrosophic crisp separation axioms, and we can generalize our results from the multivalued neutrosophic crisp closed.

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Theory and Applications of Fermatean Neutrosophic Graphs

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Abstract: Yager et. al. defined a q-rung orthopair fuzzy sets as a new general class of Pythagorean fuzzy set in which the sum of the qth power of the support for and support against is bonded by one. Tapan et. al. extended the concept of intuitionistic fuzzy sets as 3-rung orthopair fuzzy sets or Fermatean fuzzy sets (FFSs). Later C. Antony et. al. introduced the concept of Fermatean Neutrosophic Sets. In this work, we define Fermatean neutrosophic graphs and present some operations on Fermatean neutrosophic graphs. Further, we introduce the concepts of regular Fermatean neutrosophic graphs, strong Fermatean neutrosophic graphs, Cartesian, Composition, Lexicographic product of Fermatean neutrosophic graphs. Finally, we give the applications of Fermatean neutrosophic graphs.

Keywords: Pythagorean Fuzzy sets, Fermatean Fuzzy sets, Fermatean Neutrosophic sets, Fermatean Neutrosophic graphs

1. Introduction

Mohamed [1, 2] introduced the concept of strong interval-valued Pythagorean fuzzy graphs and established some algebraic operations. Sangeetha et al. [3] defined the concept of Pythagorean Fuzzy Digraph (PyFDG), and PyFDG's score function in addition they proposed an algorithm for Pythagorean shortest path in package delivery robots. Peng et al. [4] introduced the concept of interval-valued Pythagorean fuzzy sets (IVPFSs) which is a generalization of Pythagorean Fuzzy Set (PFS) and interval-valued intuitionistic fuzzy set. Mohanta et al. [5] introduced the idea of Dombi picture fuzzy graph and develope some dombi picture graph operations. Akram et al. [6] proposed a new generalization of fuzzy graph, called Simplified Interval-Valued Pythagorean Fuzzy Graph (SIVPFGs), to describe uncertain information in graph theory. Then, they developed a series of operations on two SIVPFGs and investigated their properties and introduced new multi-agent decision-making approach based on SIVPFG. By integrating the concepts Pythagorean Neutrosophic

Fuzzy Graph (PNDFG) and Dombi operator, Ajay et al. [7] defined a new concept Pythagorean Neutrosophic Graphs by applying the concepts of Pythagorean Neutrosophic Set to fuzzy graph and defined some of its basic definitions and properties. Ajay et al. [8, 9] proposed Pythagorean Neutrosophic fuzzy graphs using Dombi operator called Pythagorean Neutrosophic Dombi Fuzzy Graphs and solved a decision-making problem involving the selection of the best money-transfer applications. Recently, they developed a new Multi Criteria Decision Making (MCDM) method using the Pythagorean Neutrosophic graphs. Jun et al. [10] introduced Neutrosophic Cubic Sets as the combination of cubic sets with Neutrosophic sets. They also defined different operations of such sets. Muhammad et al. [11] applied Cubic Neutrosophic Set concept on graphs and introduced the notion of Cubic Neutrosophic Graphs.

Senapati et al. [12, 13] proposed a new concept known as the Fermatean fuzzy set, in which the restrictions are that the total of the third powers of the membership grades and non-membership grades be less than one. By expanding the spatial extent of membership and non-membership grade, FFSs have a greater potential to support uncertain information. Later, they develop some Fermatean Fuzzy Sets operations. An extensively study of Fermatean Fuzzy Set and its applications is illustrated in [14 - 30]. Thamizhendhi et al. [31] defined the concept of Fermatean Fuzzy Hyper- Graphs (FFHGs) and developed some definition and properties. Operations on single valued Neutrosophic graphs are studied in [32] Further, the operations on Neutrosophic vague graphs are discussed in [33]. In [34], the authors extensively studied about the concept of single valued Neutrosophic graphs. Moreover, in [35], bipolar single valued Neutrosophic graphs are investigated with its related properties. R. Sundareswaran et. al. introduced and studied the vulnerability parameters in Neutrosophic environment in [36, 37].

Recently, Antony and Jansi [38] proposed a new emerging concept of Fermatean neutrosophic by blending the concept of Neutrosophic sets and Fermatean fuzzy sets. By employing the concept of Fermatean Neutrosophic Sets (FNSs), this paper presents the Fermatean neutrosophic graphs. Motivated by the above-mentioned works, to the best of the authors' knowledge, there is no work reported on the concepts of Fermatean neutrosophic graphs with the application. The major contributions in this work are explained as follows:

1) The notions of Fermatean Neutrosophic Graphs (FNGs) are introduced. This study makes the first attempt in the literature about the concept in Fermatean Neutrosophic graphs.
2) The importance of this new class of graphs and distinguishing this class with other existing classes are studied.
3) In addition, the complete and strong FNG are defined. The operations like a Cartesian product, lexicographic product, composition, union and the join of FNGs with their properties are discussed.
4) The optimum selection of a power plant among various power plants are identified by using FNG is made.

The layout of this article is arranged systematically as follows: Section 2 provides some basic concepts Pythagorean Fuzzy Sets (PFS), Fermatean Fuzzy Set (FFS), Pythagorean Neutrosophic Set (PNS), Fermatean Neutrosophic Set (FNS) and Pythagorean Neutrosophic Fuzzy Graph (PNFG) and we present the geometrical interpretation of Fermatean Neutrosophic Set and illustrated in subsection 2.1. In section 3, we introduce a new class of Neutrosophic graphs called Fermatean Neutrosophic Graphs with an illustration. In Section 4, we present the idea of Size and Types of degrees in Fermatean Neutrosophic Graphs. Finally, we discuss different types of Fermatean Neutrosophic Graphs in Section 5. The conclusion of this research work is summarized in the last Section.

2. Preliminaries

In this section, we provide the basic concepts and definitions in of PFS, PFN, FFS, FNS, FFR, PFR and PNFG. In 1999, Smarandache, F. introduced the following definition for Neutrosophic Sets [NS].

Definition 2.1 [39]

A **fuzzy set** (class) A in X is characterized by a membership (characteristic) function $f_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$, with the value of $f_A(x)$ at x representing the "grade of membership" of x in A.

Definition 2.2 [40]

Let X be a non-empty set. An **intuitionistic fuzzy set** A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A(x), \nu_A(x) : X \to [0,1]$ define respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A and for every element $x \in X, 0 \le \mu_A(x) + \nu_A(x) \le 1$.

Definition 2.3 [41]

Let X be the universe. A **Neutrosophic set** (NS) A in X is characterized by a truth membership function T_A , an indeterminacy membership function I_A, and a falsity membership function F_A where T_A, I_A and F_A are real standard elements of [0,1]. It can be written as $A = \{\langle x, (T_A(x), I_A(x), F_A(x)\rangle : x \in X, T_A, I_A, F_A \in]0^-, 1^+[\ \}$. There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$ and so $0^- \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

Definition 2.4 [42]

A **Pythagorean fuzzy set** (PFS) A on a universe of discourse X is a structure having the form as

$$A = \{\langle x, T_A(x), F_A(x)\rangle |\ x \in X\}$$

where $T_A(x): X \to [0,1]$ indicates the degree of membership and $F_A(x): X \to [0,1]$ indicates the degree of non-membership of every element $x \in X$ to the set A, respectively, with the constraints: $0 \le \left(T_A(x)\right)^2 + \left(F_A(x)\right)^2 \le 1$.

Definition 2.5 [7]

A **Pythagorean neutrosophic set** ($\mathbb{PN} - set$) A on a universe of discourse X is a structure having the form as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle |\ x \in X\}$$

where $T_A(x): X \to [0,1]$ indicates the degree of membership, $I_A(x): X \to [0,1]$ indicates the degree of indeterminacy-membership, and $F_A(x): X \to [0,1]$ indicates the degree of non-membership of every element $x \in X$ to the set A, respectively, with the constraints: $0 \leq (T_A(x))^2 + (F_A(x))^2 \leq 1$ and $0 \leq (I_A(x))^2 \leq 1$ then $0 \leq (T_A(x))^2 + (I_A(x))^2 + (T_A(x))^2 \leq 2$.

Here, $T_A(x)$ and $F_A(x)$ are dependent component and $I(x)$ is independent component.

Definition 2.6 [12, 13]

A **Fermatean fuzzy set** ($\mathbb{FF} - set$) A on a universe of discourse X is a structure having the form as:

$$A = \{\langle x, T_A(x), F_A(x)\rangle | \; x \in X\}$$

where $T_A(x): X \to [0,1]$ indicates the degree of membership, and $F_A(x): X \to [0,1]$ indicates the degree of non-membership of the element $x \in X$ to the set A, respectively, with the constraints :

$$0 \leq (T_A(x))^3 + (F_A(x))^3 \leq 1$$

Antony et al. [36] proposed the concept of Fermatean neutrosophic set considering more possible types of uncertainty including the measure of neutral membership. These are defined below

Definition 2.7 [36]

Fermatean neutrosophic set ($\mathbb{FN} - set$) A on a universe of discourse X is a structure having the form as :

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | \; x \in X\}$$

where $T_A(x): X \to [0,1]$ indicates the degree of membership, $I_A(x): X \to [0,1]$ indicates the degree of indeterminacy-membership, and $F_A(x): X \to [0,1]$ indicates the degree of non-membership of the element $x \in X$ to the set A, respectively, with the constraints : $0 \leq (T_A(x))^3 + (F_A(x))^3 \leq 1$ and $0 \leq (I_A(x))^3 \leq 1$ then $0 \leq (T_A(x))^3 + (I_A(x))^3 + (T_A(x))^3 \leq 2 \quad \forall x \in X$.

Here, $T_A(x)$ and $F_A(x)$ are dependent component and $I_A(x)$ is independent component.

Definition 2.8 [43]

Let $G = (V, E)$ be **a graph which** is an ordered pair a set of vertices (nodes or points) and a set of edges (links or lines), which an edge is associated with two distinct vertices.

Definition 2.9 [44, 45]

Any fuzzy relation $\mu: S \times S \to [0,1]$ can be regarded as defining a weighted graph, or **fuzzy graph**, where the arc $(x, y) \in S \times S$, for all x, y in S has weight $\mu(X, Y) \in [0,1]$.

Definition 2.10 [46]

An **intuitionistic fuzzy graph** is defined as $G = (V, E, \mu, \nu)$, where

(i) $V = \{v_1, v_2, v_3, \dots v_n\}$ (non-empty set) such that $\mu_1 : V \longrightarrow [0,1]$, $\nu_1: V \longrightarrow [0,1]$ denote the degree of membership and non-membership of the element $v_i \in V$ respectively and $0 \leq \mu_1(v_i) + \nu_1(v_i) \leq 1$ for every $v_i \in V, i = 1,2 \dots, n$

(ii) $E \subset V \times V$ where $\mu_2 : V \times V \to [0,1]$and $\nu_2 : V \times V \to [0,1]$ are such that $\mu_2(v_i, v_j) \leq \min\{\mu_1(v_i), \mu_1(v_j)\}$, $\nu_2(v_i, v_j) \leq \max\{\nu_1(v_i), \nu_1(v_j)\}$ and $0 \leq \mu_2(v_i, v_j) + \nu_2(v_i, v_j) \leq 1$, $0 \leq \mu_2(v_i, v_j), \nu_2(v_i v_j), \pi(v_i, v_j) \leq 1$ where $\pi(v_i, v_j) = 1 - \mu_2(v_i, v_j) - \nu_2(v_i, v_j)$ for every $(v_i, v_j) \in E, i = 1,2, \dots, n$

Definition 2.11 [47]

A **Neutrosophic graph** is of the form $G^* = (V, \sigma, \mu)$ where $\sigma = (T_1, I_1, F_1)$ & $\mu = (T_2, I_2, F_2)$

(i) $V = \{v_1, v_2, v_3, \dots v_n\}$ such that $T_1 : V \longrightarrow [0,1]$, $I_1 : V \longrightarrow [0,1]$ and $F_1 : V \to [0,1]$ denote the degree of truth-membership function, indeterminacy –membership function and falsity-membership function of the vertex $v_1 \in V$ respectively and $0 \leq T_i(v) + I_i(v) + F_i(v) \leq 3, \forall\, v_i \in V$ ($i = 1,2, 3, \dots n$).

(ii) $T_2 : V \times V \longrightarrow [0,1]$, $I_2 : V \times V \longrightarrow [0,1]$ and $F_2 : V \times V \to [0,1]$ where $T_2(v_i, v_j), I_2(v_i, v_j)$ and $F_2(v_i, v_j)$ denote the degree of truth-membership function , indeterminacy –membership function and falsity-membership function of the edge (v_i, v_j) respectively such that for every edge (v_i, v_j),

$$T_2(v_i, v_j) \leq \min\{T_1(v_i), T_1(v_j)\},$$
$$I_2(v_i, v_j) \leq \min\{I_1(v_i), I_1(v_j)\},$$
$$F_2(v_i, v_j) \leq \max\{F_1(v_i), F_1(v_j)\},$$

and $T_2(v_i, v_j) + I_2(v_i, v_j) + F_2(v_i, v_j) \leq 3$.

Definition 2. 12 [1, 2]

A **Pythagorean Fuzzy Graph** on a universal set X is a pair $\mathbb{G}$=$(\mathcal{P}, \mathcal{Q})$ where $\mathcal{P}$ is Pythagorean fuzzy set on X and $\mathcal{Q}$ is a pythagorean fuzzy relation on X such that:

$$\begin{cases} T_{\mathcal{Q}}(u, v) \leq \min\{T_{\mathcal{P}}(u), T_{\mathcal{P}}(v)\} \\ F_{\mathcal{Q}}(u, v) \geq \max\{F_{\mathcal{P}}(u), F_{\mathcal{P}}(v)\} \end{cases}$$

and $0 \leq T_{\mathcal{Q}}^2(u, v) + F_{\mathcal{Q}}^2(u, v) \leq 1 \; for\ all\ u, v \in X$, where , $T_{\mathcal{Q}} : X \times X \to [0,1]$. $F_{\mathcal{Q}} : X \times X \to [0,1]$ indicates degree of membership, and degree of non-membership of $\mathcal{Q}$, correspondingly.

Definition 2. 13 [31]

A **Fermatean fuzzy Graph** (FFG) on a universal set X is a pair $\mathbb{G}$=$(\mathcal{P}, \mathcal{Q})$ where $\mathcal{P}$ is Fermatean fuzzy set on X and $\mathcal{Q}$ is a Fermatean fuzzy relation on X such that :

$$\begin{cases} T_{\mathcal{Q}}(u, v) \leq \min\{T_{\mathcal{P}}(u), T_{\mathcal{P}}(v)\} \\ F_{\mathcal{Q}}(u, v) \geq \max\{F_{\mathcal{P}}(u), F_{\mathcal{P}}(v)\} \end{cases}$$

and $0 \leq T_{\mathcal{Q}}^3(u, v) + F_{\mathcal{Q}}^3(u, v) \leq 1 \; for\ all\ u, v \in X$, where , $T_{\mathcal{Q}} : X \times X \to [0,1]$, $F_{\mathcal{Q}} : X \times X \to [0,1]$ indicates degree of membership and degree of non-membership of $\mathcal{Q}$, correspondingly. Here $\mathcal{P}$ is the Fermatean fuzzy vertex set of $\mathbb{G}$ and $\mathcal{Q}$ is the Fermatean fuzzy edge set of $\mathbb{G}$.

Definition 2. 14 [7]

Pythagorean Neutrosophic Fuzzy Graph (PNFG) is of the form $G^* = (V, \sigma, \mu)$ where $\sigma = (T_1, I_1, F_1)$ & $\mu = (T_2, I_2, F_2)$

(i) $V = \{v_1, v_2, v_3, \dots v_n\}$ such that $T_1 : V \longrightarrow [0,1]$, $I_1: V \longrightarrow [0,1]$ and $F_1: V \rightarrow [0,1]$ denote the degree of truth-membership function , indeterminacy –membership function and falsity-membership function of the vertex $v_1 \in V$ respectively and $0 \le T_i(v)^2 + I_i(v)^2 + F_i(v)^2 \le 2, \forall\, v_i \in V$ ($i = 1,2\,,3, \dots n$).

(ii) $T_2 : V \times V \longrightarrow [0,1]$, $I_2: V \times V \longrightarrow [0,1]$ and $F_2 : V \times V \rightarrow [0,1]$ where $T_2(v_i, v_j), I_2(v_i, v_j)$ and $F_2(v_i, v_j)$ denote the degree of truth-membership function , indeterminacy –membership function and falsity-membership function of the edge (v_i, v_j) respectively such that for every edge (v_i, v_j),

$$T_2(v_i, v_j) \le \min\{\, T_1(v_i), T_1(v_j)\},$$
$$I_2(v_i, v_j) \le \min\{\, I_1(v_i), I_1(v_j)\},$$
$$F_2(v_i, v_j) \le \max\{F_1(v_i), F_1(v_j)\},$$

and $T_2(v_i, v_j) + I_2(v_i, v_j) + F_2(v_i, v_j) \le 3$.

2.1 Merits and De-merits of uncertainty sets

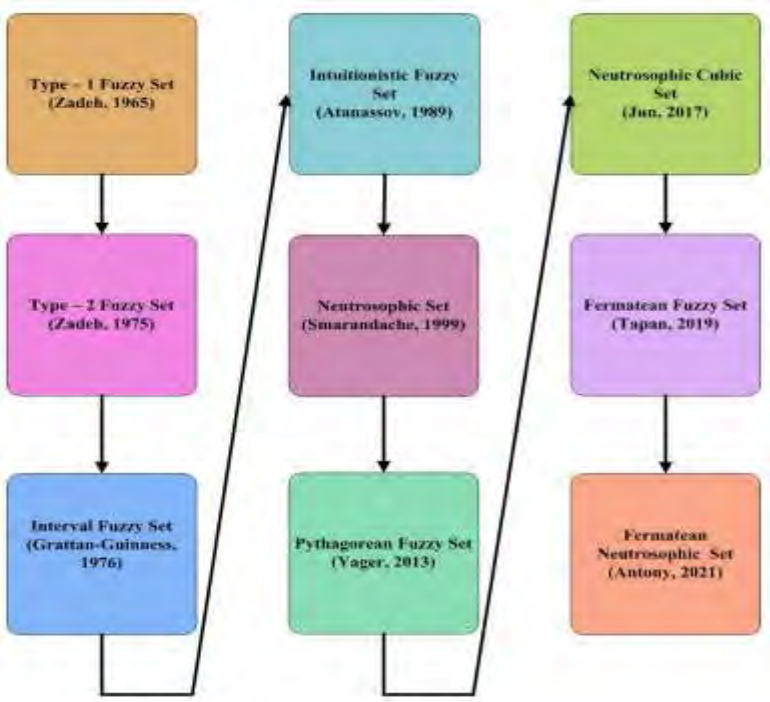
Several researchers have been introduced different kinds of sets based on the uncertainty situations. Each time, a new set is introduced, it gives an information about the limitations and advantages of the new set with a comparison of an existing one. In this section, we have listed out such discussions.

Sets	Advantages	Limitations
Fuzzy - Zadeh (1965)	Problems with uncertainty can be solved by fuzzy sets with membership values.	Decision makers can be used only membership degree $0 \le \mu \le 1$.
Intuitionistic Fuzzy – Atanassov (1986)	The concept of fuzzy sets is inconclusive because the exclusion of non-membership function. The IFS incorporates both membership function, μ and nonmembership function, ν with hesitation margin, π (that is, neither membership nor nonmembership functions), such that $\mu + \nu \le 1$ *and* $\mu + \nu + \pi = 1$.	Intuitionistic fuzzy sets can only handle incomplete information not the indeterminate information and inconsistent information which exists commonly in belief system. For example, when we ask the opinion of an expert about certain statement, he or she may that the possibility that the statement is true is 0.6 and the statement is false is 0.5 and the degree that he or she is not sure is 0.1
Neutrosophic – Smarandache(2019)	In Neutrosophic set, indeterminacy is quantified explicitly and truth-membership, indeterminacy membership and falsity-membership are independent. Neutrosophy was introduced by Smarandache in 1995. "It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra".	A Neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminacy membership function I_A and a falsity-membership function F_A. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of]0- ,1+ [. That is $T_A: X \rightarrow]0-,1[$ $I_A: X \rightarrow]0-,1+[$ $F_A: X \rightarrow]0-,1+[$ There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0- \le sup\, T_A(x) + sup\, I_A(x) + supF_A(x) \le 3+$.

<table>
<tr><td>Single valued Neutrosophic</td><td>The set theoretic operators on an instance of Neutrosophic set is single valued Neutrosophic set (SVNS).</td><td>A Single Valued Neutrosophic Set (SVNS) A in X is characterized by truth-membership function T_A, indeterminacy-membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x), I_A(x), F_A(x) \in [0,1]$.
$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$</td></tr>
<tr><td>Pythagorean fuzzy
-Yager (2014)</td><td>FFS is firstly proposed by Senapati and Yager (2020) as a special case of q-rung orthopair fuzzy sets (q-ROFS). The theory of q-ROFS which is developed by Yager (2017) requires the sum of the qth power of membership (e.g., support for an idea) and non-membership (e.g., support against an idea) degrees should be equal to or smaller than 1. It is obvious that when q increases the space of acceptable orthopairs will increase and this geometric area supplies more independence to users or decision-makers while declaring their preferences, ideas, and claims. By setting q = 2, Yager (2014) rename the q-ROFS as Pythagorean fuzzy sets (PFS) and developed basic operations on them. It deals with vagueness considering the membership grade, μ and nonmembership grade, ν satisfying the conditions $\mu + \nu \le 1$ or $\mu + \nu \ge 1$, and, it follows that $\mu^2 + \nu^2 + \pi^2 = 1$, where π is the Pythagorean fuzzy set index.</td><td rowspan="2">In a voting process, a judgement may give based on a candidate satisfies his expectations with a possibility of 0.80 and this candidate dissatisfies the expectations with a possibility of 0.75. But their sum is 1.55 (>1) and their square sum is 1.20 (>1). the sum of the cubes is equal to 0.93 (<1).</td></tr>
<tr><td>Fermatean Fuzzy
- Sanapati(2019)</td><td>Senapati and Yager (2019) set q = 3 and this novel q-ROFS is called Fermatean fuzzy sets (FFS). Under this new concept, the decision-makers have more freedom since they can specify their ideas about agreeing (membership) and/or disagreeing (non-membership) regarding the state of a subject. It deals with vagueness considering the membership grade, μ and non-membership grade, ν satisfying the conditions $\mu + \nu \le 1$ or $\mu + \nu \ge 1$, and, it follows that $\mu^3 + \nu^3 + \pi^3 = 1$, where π is the Pythagorean fuzzy set index.</td></tr>
<tr><td>Pythagorean Neutrosophic</td><td>Pythagorean fuzzy sets has limitation that their square sum is less than or equal to 1. In neutrosophic set, if truth membership and falsity membership are 100% dependent and indeterminacy is 100% independent, that is $0 \le T_A(x) + I_A(x) + F_A(x) \le 2$. Sometimes in real life, we face many problems which cannot be handled by using neutrosophic when $T_A(x) + I_A(x) + F_A(x) > 2$. In such condition, a neutrosophic set has no ability to obtain any satisfactory result. In Pythagorean neutrosophic set with T and F are dependent neutrosophic components [PNS] of condition is as their square sum does not exceeds 2. Here, T and F are dependent neutrosophic components and we make $T_A(x), F_A(x)$ as Pythagorean, then $(T_A(x))^2 + (F_A(x))^2 \le 1$ with</td><td>In a voting process, a judgement may give based on a candidate satisfies his expectations with a possibility of 0.80 and this candidate dissatisfies the expectations with a possibility of 0.95 and neutrally give 0.85 But their sum is 2.80 (>2) and their square sum is 2.265 (<2). the sum of the cubes is equal to 1.9835 (<2).</td></tr>
</table>

	$T_A(x), F_A(x)$ in [0,1]. If $I_A(x)$ is an Independent from them, then $0 \leq I_A(x) \leq 1$. Then $0 \leq ((T_A(x))^2 + (F_A(x))^2 + (I_A(x))^2 \leq 2$, with $T_A(x), I_A(x), F_A(x)$ in [0,1].	
Fermatean Neutrosophic sets	Fermatean neutrosophic sets, then $(T_A(x))^3 + (F_A(x))^3 \leq 1$ with $T_A(x), F_A(x)$ in [0,1]. If $I_A(x)$ is an Independent from them, then $0 \leq I_A(x) \leq 1$. Then $0 \leq ((T_A(x))^3 + (F_A(x))^3 + (I_A(x))^3 \leq 2, with\ T_A(x), I_A(x), F_A(x)\ in$ [0,1].	

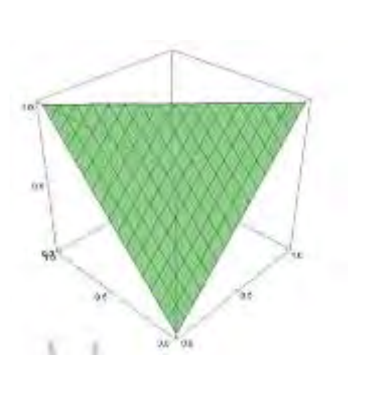
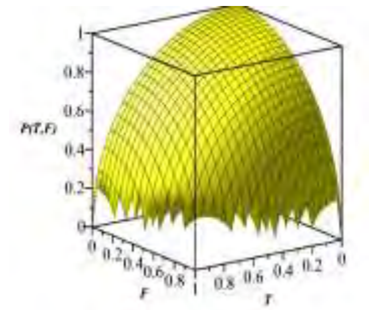
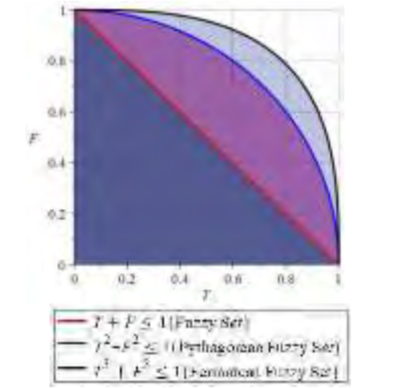
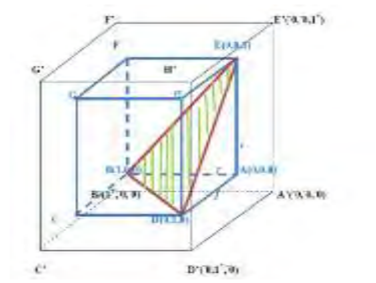
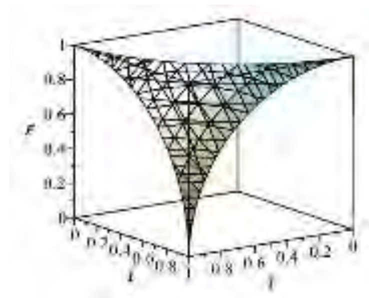
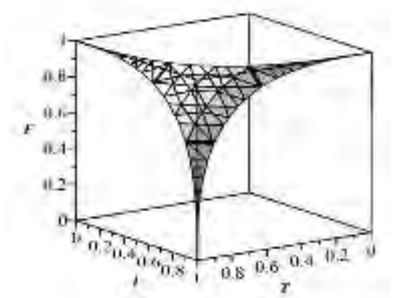
2.2 Flow chart of literature survey of uncertainty sets



2.3 Geometrical interpretation

The Graphical representation of sets which deal with uncertain may be useful to the reader to understand the flow of the T, F and I values. In this section, we give a graphical representation of membership, non-membership, and indeterminacy grades for all fuzzy sets and Neutrosophic sets.

Intuitionistic fuzzy set	Pythagorean fuzzy set	Fermatean fuzzy Set (Benchmark of IFS, PFS, and FFS)

		
Neutrosophic set	**Pythagorean Neutrosophic set**	**Fermatean Neutrosophic set**
		

3. Fermatean neutrosophic graphs

In this section, we propose the new class of graph namely, Fermatean Neutrosophic Graph which is associated with Fermatean Neutrosophic Set (FNS).

Definition 3.1: Let X be a universal set. A mapping $\mathcal{P} = (T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}}) : X \times X \rightarrow [0,1]$ is called a Fermatean Neutrosophic relation on X such that $T_{\mathcal{P}}(u,v)$, $I_{\mathcal{P}}(u,v)$, $F_{\mathcal{P}}(u,v) \in [0,1]$ for all $u, v \in X$.

Definition 3.2: Let $\mathcal{P} = (T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}})$ and $\mathcal{Q} = \left(T_{\mathcal{Q}}, I_{\mathcal{Q}}, F_{\mathcal{Q}}\right)$ be Fermatean Neutrosophic sets on X if $\mathcal{Q}$ is Fermatean Neutrosophic relation on X, then $\mathcal{Q}$ is called a Fermatean Neutrosophic relation on $\mathcal{P}$ if

$$\begin{cases} T_{\mathcal{Q}}(u,v) \leq \min\{T_{\mathcal{P}}(u), T_{\mathcal{P}}(v)\} \\ I_{\mathcal{Q}}(u,v) \geq \max\{I_{\mathcal{P}}(u), I_{\mathcal{P}}(v)\} \\ F_{\mathcal{Q}}(u,v) \geq \max\{F_{\mathcal{P}}(u), F_{\mathcal{P}}(v)\} \end{cases}$$

if $T_{\mathcal{P}}(u,v)$, $I_{\mathcal{P}}(u,v)$, $F_{\mathcal{P}}(u,v) \in [0,1]$ for all $u, v \in X$.

Definition 3.3: A **Fermatean neutrosophic graph** on a universal set X is a pair $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ where $\mathcal{P}$ is Fermatean Neutrosophic set on X and $\mathcal{Q}$ is a Fermatean Neutrosophic relation on X such that:

$$\begin{cases} T_{\mathcal{Q}}(u,v) \leq \min\{T_{\mathcal{P}}(u), T_{\mathcal{P}}(v)\} \\ I_{\mathcal{Q}}(u,v) \geq \max\{I_{\mathcal{P}}(u), I_{\mathcal{P}}(v)\} \\ F_{\mathcal{Q}}(u,v) \geq \max\{F_{\mathcal{P}}(u), F_{\mathcal{P}}(v)\} \end{cases}$$

and $0 \le T_Q^3(u,v) + I_Q^3(u,v) + F_Q^3(u,v) \le 2 \quad for\ all\ u, v \in X$, where , $T_Q: X \times X \to [0,1]$, $I_Q: X \times X \to [0,1]$ and $F_Q: X \times X \to [0,1]$ indicates degree of membership, degree of indeterminacy-membership and degree of non-membership of Q, correspondingly.

Here, $\mathcal{P}$ is the Fermatean Neutrosophic vertex set of $\mathbb{G}$ and Q is the Fermatean Neutrosophic edge set of $\mathbb{G}$.

An example of Fermatean Neutrosophic graph is given below.

Example 3.1 Consider a Fermatean neutrosophic graph $\mathbb{G}=(\mathcal{P}, Q)$ defined on G = (V, E), where $\mathcal{P}$ be a Fermatean Neutrosophic set on V and Q be a Fermatean Neutrosophic relation on V, defined by

$\mathcal{P}$={ $\langle v_1$,(0.6, 1,0.7)$\rangle$,$\langle v_2$,(0.5, 0.8,0.4)$\rangle$,$\langle v_3$,(0.7, 0.5,0.3)$\rangle$}

and

Q={ $\langle v_1v_2$, (0.4, 1,0.8)$\rangle$, $\langle v_2v_3$, (0.4, 0.9,0.6)$\rangle$, $\langle v_1v_3$, (0.5, 1,0.8)$\rangle$}

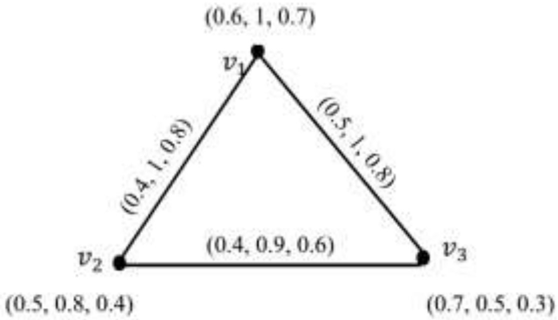


Figure 1. Fermatean Neutrosophic graph

Definition 3.4 Let $\mathbb{G}=(\mathcal{P}, Q)$ be A Fermatean neutrosophic graph $\mathbb{FNG}$ on G=(V, E). The complement of Fermatean Neutrosophic graph is $\mathbb{FNG}$ $\bar{\mathbb{G}}=(\bar{\mathcal{P}}, \bar{Q})$ where $\bar{\mathcal{P}} = (\overline{T_{\mathcal{P}}}, \overline{I_{\mathcal{P}}}, \bar{F}_{\mathcal{P}})$ and $\bar{Q} = \left(\overline{T_Q}, \overline{I_Q}, \overline{F_Q}\right)$, defined by

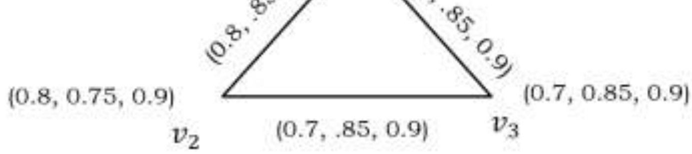
(i) $\mathcal{P} = \bar{\mathcal{P}}$

(ii) $\overline{T_{\mathcal{P}}}(u)= T_{\mathcal{P}}(u)$, $\overline{I_{\mathcal{P}}}(u)= I_{\mathcal{P}}(u)$, $\bar{F}_{\mathcal{P}}(u) = F_{\mathcal{P}}(u)\ \forall\ u \in V$

(iii) $\overline{T_Q}(uv) = \left|T_{\mathcal{P}}(u) \wedge T_{\mathcal{P}}(v) - T_Q(uv)\right|$, $\overline{T_Q}(uv) = \left|I_{\mathcal{P}}(u) \vee I_{\mathcal{P}}(v) - I_Q(uv)\right|$ and

(iv) $\overline{T_Q}(uv) = \left|F_{\mathcal{P}}(u) \vee F_{\mathcal{P}}(v) - F_Q(uv)\right|$, for all $u, v \in V$

Note: In the below example, T, I and F values are very close to 1. This situation will happen in the most of real time problems. But $0 \le T^2 + I^2 + F^2 \nleq 2$. So , we adopt $0 \le T^3 + I^3 + F^3 \le 2$. Hence, we can model this situation by Fermatean Neutrosophic graphs.



4. Size and Types of degrees in Fermatean Neutrosophic graphs

The concept of regularity has been explored by many academics on fuzzy graphs and several of its generalizations. We will now propose a description on regularity of Fermatean Neutrosophic graphs ($\mathbb{FNG}$). First, we introduce few definitions in this context.

Definition 4.1 Let $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ be a Fermatean Neutrosophic graph ($\mathbb{FNG}$) defined on G = (V, E). The order *of* $\mathbb{G}$ is symbolized by O($\mathbb{G}$) and defined as

$$O(\mathbb{G})=\left(\sum_{u\in V} T_{\mathcal{P}}(u), \sum_{u\in V} I_{\mathcal{P}}(u), \sum_{u\in V} F_{\mathcal{P}}(u)\right)$$

Definition 4.2 Let $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ be a Fermatean Neutrosophic graph ($\mathbb{FNG}$) defined on G = (V, E). The size of $\mathbb{G}$ is symbolized by S($\mathbb{G}$) and defined as

$$S(\mathbb{G})=\left(\sum_{uv\in E} T_{\mathcal{Q}}(uv), \sum_{uv\in E} I_{\mathcal{Q}}(uv), \sum_{uv\in E} F_{\mathcal{Q}}(uv)\right)$$

Example 4.1 Consider a Fermatean Neutrosophic graph $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ defined on G = (V, E), where $\mathcal{P}$ be a Fermatean Neutrosophic set on V and $\mathcal{Q}$ be a Fermatean Neutrosophic relation on V, defined by
$\mathcal{P}=\{\langle v_1,(0.6, 1,0.7)\rangle,\langle v_2,(0.5, 0.8,0.4)\rangle,\langle v_3,(0.7, 0.5,0.3)\rangle\}$ and
$\mathcal{Q}=\{\langle v_1v_2, (0.4, 1,0.8)\rangle, \langle v_2v_3, (0.4, 0.9,0.6)\rangle, \langle v_1v_3, (0.5, 1,0.8)\rangle\}$
The order and size of Fermatean Neutrosophic graph displayed in Fig. 1 are
O($\mathbb{G}$) = (1.8, 2.3, 1.4) and S($\mathbb{G}$)= (1.3, 2.9, 2.2), respectively.

Definition 4.3 Let $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ be a Fermatean Neutrosophic graph $\mathbb{FNG}$ defined on G= (V, E). The degree of a vertex u *of* $\mathbb{G}$ is symbolized by $d_{\mathbb{G}}(u) = (\mathrm{d_T(u), d_I(u), d_F(u)})$ and defined as

$$d_{\mathbb{G}}(u)=\left(\sum_{u\neq v} T_{\mathcal{P}}(u), \sum_{u\neq v} I_{\mathcal{P}}(u), \sum_{u\neq v} F_{\mathcal{P}}(u)\right) \text{ for } uv \in E.$$

Definition 4.4 $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ is a Fermatean Neutrosophic graph $\mathbb{FNG}$ defined on G= (V, E). The total degree of a vertex u of $\mathbb{G}$ is symbolized by $td_{\mathbb{G}}(u) = (\mathrm{td_T(u), td_I(u), td_F(u)})$ and defined as

$$td_{\mathbb{G}}(u)=\left(\sum_{u\neq v} T_{\mathcal{Q}}(uv)+T_{\mathcal{P}}(u), \sum_{u\neq v} I_{\mathcal{Q}}(uv)+I_{\mathcal{P}}(u), \sum_{u\neq v} F_{\mathcal{Q}}(uv)+F_{\mathcal{P}}(u)\right) \text{ for } uv \in E.$$

Example 4.2. For the Fermatean Neutrosophic graph $\mathbb{G}$ in Figure 1, the degree and the total degree of the vertices are
$d_{\mathbb{G}}(v_1)$ = (1.2, 1.3, 0.7) and $td_{\mathbb{G}}(v_1)$= (1.5, 2.8, 1.8) ;
$d_{\mathbb{G}}(v_2)$ = (1.3, 1.5, 1.0) and $td_{\mathbb{G}}(v_2)$= (1.8, 2.5, 1.8) ;
$d_{\mathbb{G}}(v_3)$ = (1.1, 1.8, 1.1) and $td_{\mathbb{G}}(v_3)$= (1.5, 2.8, 1.8), respectively.
The following theorem is developed to demonstrate an interesting fact regarding degree of vertices of $\mathbb{FNG}$s.

Theorem 4.1 For any Fermatean Neutrosophic graph $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ defined on V=$\{u_1,u_2,\dots,u_n\}$, the following relation for degree of vertices of $\mathbb{G}$ must holds:

$$\sum_{j=1}^{n} d_{\mathbb{G}}(u_j)= 2\left(\sum_{\substack{j=1\\ i>j}}^{n-1} T_{\mathcal{Q}}(u_ju_i), \sum_{\substack{j=1\\ i>j}}^{n-1} I_{\mathcal{Q}}(u_ju_i), \sum_{\substack{j=1\\ i>j}}^{n-1} F_{\mathcal{Q}}(u_ju_i)\right) \text{ for all } 1\leq i \leq n.$$

Proof : Let V=$\{u_1,u_2,\dots,u_n\}$, and $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ be a Fermatean neutrosophic graph defined on G = (V, E)

$$\sum_{j=1}^{n} d_{\mathbb{G}}(u_j)=\sum_{j=1}^{n}\left(d_T(u_j), d_I(u_j), d_F(u_j)\right)$$

$$= (d_T(u_1), d_I(u_1), d_F(u_1))+ (d_T(u_2), d_I(u_2), d_F(u_2))+\dots+(d_T(u_n), d_I(u_n), d_F(u_n))$$

$$=\left[\left(T_{\mathcal{Q}}(u_1u_2), I_{\mathcal{Q}}(u_1u_2), F_{\mathcal{Q}}(u_1u_2)\right) + \left(T_{\mathcal{Q}}(u_1u_3), I_{\mathcal{Q}}(u_1u_3), F_{\mathcal{Q}}(u_1u_3)\right) + \dots\right.$$

$+\left(T_Q(u_1u_n), I_Q(u_1u_n), F_Q(u_1u_n)\right)]$
$+[\left(T_Q(u_2u_1), I_Q(u_2u_1), F_Q(u_2u_1)\right) + \left(T_Q(u_2u_2), I_Q(u_2u_2), F_Q(u_2u_2)\right) + \ldots$
$+\left(T_Q(u_2u_n), I_Q(u_2u_n), F_Q(u_2u_n)\right)]$
$+[\left(T_Q(u_nu_1), I_Q(u_nu_1), F_Q(u_nu_1)\right) + \left(T_Q(u_nu_2), I_Q(u_nu_2), F_Q(u_nu_2)\right) + \ldots$
$+\left(T_Q(u_nu_{n-1}), I_Q(u_nu_{n-1}), F_Q(u_nu_{n-1})\right)]$
$=2[\left(T_Q(u_1u_2), I_Q(u_1u_2), F_Q(u_1u_2)\right) + \left(T_Q(u_1u_3), I_Q(u_1u_3), F_Q(u_1u_3)\right) + \ldots$
$+\left(T_Q(u_1u_n), I_Q(u_1u_n), F_Q(u_1u_n)\right)]$
$+ 2[\left(T_Q(u_2u_3), I_Q(u_2u_3), F_Q(u_2u_3)\right) + \ldots$
$+T_Q(u_2u_n), I_Q(u_2u_n), F_Q(u_2u_n)] + \ldots$
$+2\left(T_Q(u_{n-1}u_n), I_Q(u_{n-1}u_n), F_Q(u_{n-1}u_n)\right)]$
$=2\left(\sum_{\substack{j=1\\ i>j}}^{n-1} T_Q(u_ju_i), \sum_{\substack{j=1\\ i>j}}^{n-1} I_Q(u_ju_i), \sum_{\substack{j=1\\ i>j}}^{n-1} F_Q(u_ju_i)\right)$

Hence proved.

Theorem 4.2 For any Fermatean Neutrosophic graph $\mathbb{G}=(\mathcal{P},Q)$ defined on V=$\{u_1, u_2, \ldots, u_n\}$, the following relation for total degree of vertices of $\mathbb{G}$ must holds:

$$\sum_{j=1}^{n} td_{\mathbb{G}}(u_j)=\left(2\sum_{\substack{j=1\\ i>j}}^{n-1} T_Q(u_ju_i) + \sum_{j=1}^{n} T_{\mathcal{P}}(u_j), 2\sum_{\substack{j=1\\ i>j}}^{n-1} I_Q(u_ju_i) + \sum_{j=1}^{n} I_{\mathcal{P}}(u_j), 2\sum_{\substack{j=1\\ i>j}}^{n-1} F_Q(u_ju_i) + + \sum_{j=1}^{n} F_{\mathcal{P}}(u_j)\right), \text{ for all } 1 \leq i \leq n.$$

Proof : The proof directly follows from Theorem 4.1 and Definition 4.4.

Definition 4.5. A Fermatean Neutrosophic graph is complete if

$$T_Q(u,v)= \min\{T_{\mathcal{P}}(u), T_{\mathcal{P}}(v)\}$$
$$I_Q(u,v)= \max\{I_{\mathcal{P}}(u), I_{\mathcal{P}}(v)\}$$
$$F_Q(u,v)= \max\{F_{\mathcal{P}}(u), F_{\mathcal{P}}(v)\}$$

We illustrate it by giving an example.

Example 4.3. Let the vertex set $\mathbb{V} = \{v_1, v_2, v_3\}$ and the edge sets $\mathbb{E}=\{v_1v_2, v_2v_3, v_1v_3\}$ in $\mathbb{G}'=(V,E)$. Take the Fermatean Neutrosophic set $\mathcal{P} = (T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}})$ in $\mathbb{V}$ and the Fermatean Neutrosophic edge sets in $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ defined by

$(T_{\mathcal{P}}(v_1), I_{\mathcal{P}}(v_1), F_{\mathcal{P}}(v_1))$ = (0.6, 1, 0.7)
$(T_{\mathcal{P}}(v_2), I_{\mathcal{P}}(v_2), F_{\mathcal{P}}(v_2))$ = (0.5, 0.8, 0.4)
$(T_{\mathcal{P}}(v_3), I_{\mathcal{P}}(v_3), F_{\mathcal{P}}(v_3))$ = (0.7, 0.5, 0.3)
and
$\left(T_Q(v_1v_2), I_Q(v_1v_2), F_Q(v_1v_2)\right)$ = (0.5, 1, 0.7)
$(T_{\mathcal{P}}(v_2v_3), I_{\mathcal{P}}(v_2v_3), F_{\mathcal{P}}(v_2v_3))$ = (0.5, 0.8, 0.4)
$(T_{\mathcal{P}}(v_1v_3), I_{\mathcal{P}}(v_1v_3), F_{\mathcal{P}}(v_1v_3))$ = (0.6, 1, 0.7)

Then, it is a complete $\mathbb{FNG}$.

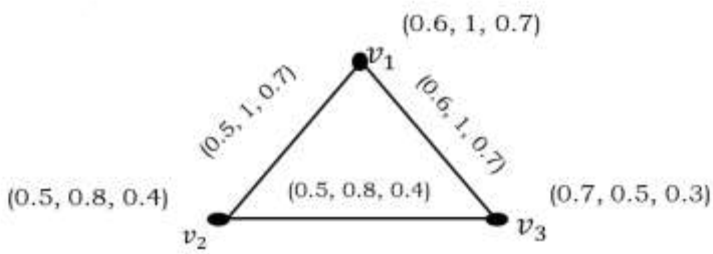


Figure 2. Complete Fermatean Neutrosophic graph

Definition 4.6: The minimum degree of Fermatean Neutrosophic graph $\mathbb{FNG}$, $\mathbb{G}$=$(\mathcal{P}, \mathcal{Q})$ is designated as δ($\mathbb{G}$)=($\delta_T(\mathbb{G})$, $\delta_I(\mathbb{G})$, $\delta_F(\mathbb{G})$) where,

$\delta_T(\mathbb{G})$=min$\{d_T(u)|u \in V\}$; is minimum T-degree of $\mathbb{G}$

$\delta_I(\mathbb{G})$=min$\{d_I(u)|u \in V\}$; is minimum I-degree of $\mathbb{G}$

$\delta_F(\mathbb{G})$=min$\{d_F(u)|u \in V\}$; is minimum F-degree of $\mathbb{G}$

Definition 4.7: The maximum degree of Fermatean Neutrosophic graph $\mathbb{FNG}$, $\mathbb{G}$=$(\mathcal{P}, \mathcal{Q})$ is designated as Δ($\mathbb{G}$)=($\Delta_T(\mathbb{G})$, $\Delta_I(\mathbb{G})$, $\Delta_F(\mathbb{G})$) where,

$\Delta_T(\mathbb{G})$=max$\{d_T(u)|u \in V\}$; is maximum T-degree of $\mathbb{G}$

$\Delta_I(\mathbb{G})$=max$\{d_I(u)|u \in V\}$; is maximum I-degree of $\mathbb{G}$

$\Delta_F(\mathbb{G})$=max$\{d_F(u)|u \in V\}$; is maximum F-degree of $\mathbb{G}$

Example 4.4. Let the vertex set $\mathbb{V} = \{v_1, v_2, v_3, v_4\}$ and the edge sets $\mathbb{E}$=$\{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$ in $\mathbb{G}'$=(V, E). Take the Fermatean Neutrosophic set $\mathcal{P} = (T_\mathcal{P}, I_\mathcal{P}, F_\mathcal{P})$ in $\mathbb{V}$ and the Fermatean Neutrosophic edge sets in $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ defined by

$(T_\mathcal{P}(v_1), I_\mathcal{P}(v_1), F_\mathcal{P}(v_1))$ = (0.3, 0.7, 0.5)

$(T_\mathcal{P}(v_2), I_\mathcal{P}(v_2), F_\mathcal{P}(v_2))$ = (0.6, 0.5, 0.7)

$(T_\mathcal{P}(v_3), I_\mathcal{P}(v_3), F_\mathcal{P}(v_3))$ = (0.8, 0.3, 0.7)

$(T_\mathcal{P}(v_4), I_\mathcal{P}(v_4), F_\mathcal{P}(v_4))$ = (0.7, 0.2, 0.4)

and

$\left(T_\mathcal{Q}(v_1v_2), I_\mathcal{Q}(v_1v_2), F_\mathcal{Q}(v_1v_2)\right)$ = (0.3, 0.7, 0.7)

$(T_\mathcal{P}(v_2v_3), I_\mathcal{P}(v_2v_3), F_\mathcal{P}(v_2v_3))$ = (0.4, 0.6, 0.7)

$(T_\mathcal{P}(v_3v_4), I_\mathcal{P}(v_3v_4), F_\mathcal{P}(v_3v_4))$ = (0.6, 0.5, 0.8)

$(T_\mathcal{P}(v_1v_4), I_\mathcal{P}(v_1v_4), F_\mathcal{P}(v_1v_4))$ = (0.3, 0.8, 0.6)

Then, it is $\mathbb{FNG}$.

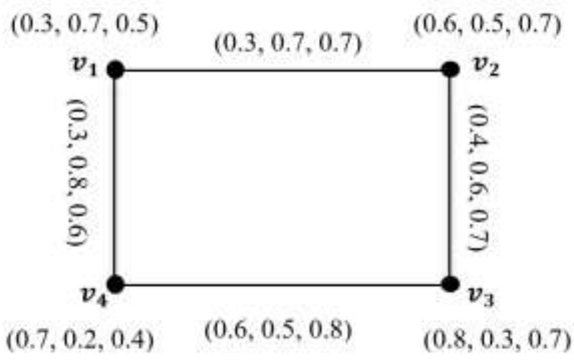


Figure 3. Minimum and maximum degree of a Fermatean Neutrosophic graph
δ(𝔾)=(0.6,1.1,1.3); Δ(𝔾)=(1,1.5,1.5)

Next, the definition of effective edge of 𝔽ℕ𝔾 are

Definition 4.9. The edge $e = (u, v)$ of 𝔾=$(\mathcal{P}, \mathcal{Q})$ be a 𝔽ℕ𝔾 is called an effective edge of 𝔾 is defined as

$$T_{\mathcal{Q}}(u, v) = \min\{T_{\mathcal{P}}(u), T_{\mathcal{P}}(v)\}$$
$$I_{\mathcal{Q}}(u, v) = \max\{I_{\mathcal{P}}(u), I_{\mathcal{P}}(v)\}$$
$$F_{\mathcal{Q}}(u, v) = \max\{F_{\mathcal{P}}(u), F_{\mathcal{P}}(v)\}$$

In Fig. 3, v_1v_2 is an effective edge of 𝔽ℕ𝔾.

Definition 4.10. The effective degree of a vertex u of 𝔽ℕ𝔾, 𝔾=$(\mathcal{P}, \mathcal{Q})$ is defined by $d_{\mathcal{E}}(u)$=($d_{\mathcal{E}_T}(u), d_{\mathcal{E}_I}(u), d_{\mathcal{E}_F}(u)$) $\forall\, u \in \mathcal{E}$; here $d_{\mathcal{E}_T}(u)$ is the sum of the T−values of the effective edges of 𝔽ℕ𝔾 incident with u, $d_{\mathcal{E}_I}(u)$is the sum of the I−values of the effective edges of 𝔽ℕ𝔾 incident with u and $d_{\mathcal{E}_F}(u)$ is the sum of the F−values of the effective edges of 𝔽ℕ𝔾 incident with u.

Definition 4.11. The minimum effective degree of 𝔾=$(\mathcal{P}, \mathcal{Q})$ in 𝔽ℕ𝔾 is designated as
$\delta_{\mathcal{E}}(\mathbb{G})$ =($\delta_{\mathcal{E}_T}(\mathbb{G})$, $\delta_{\mathcal{E}_I}(\mathbb{G})$, $\delta_{\mathcal{E}_F}(\mathbb{G})$) where,

$$\delta_{\mathcal{E}_T}(\mathbb{G}) = \wedge\{d_{\mathcal{E}_T}(u) | u \in \mathcal{P}\};$$
$$\delta_{\mathcal{E}_I}(\mathbb{G}) = \wedge\{d_{\mathcal{E}_I}(u) | u \in \mathcal{P}\};$$
$$\delta_{\mathcal{E}_F}(\mathbb{G}) = \wedge\{d_{\mathcal{E}_F}(u) | u \in \mathcal{P}\}$$

Definition 4.12. The maximum effective degree of 𝔾=$(\mathcal{P}, \mathcal{Q})$ in 𝔽ℕ𝔾 is designated as
$\Delta_{\mathcal{E}}(\mathbb{G})$ =($\Delta_{\mathcal{E}_T}(\mathbb{G})$, $\Delta_{\mathcal{E}_I}(\mathbb{G})$, $\Delta_{\mathcal{E}_F}(\mathbb{G})$) where,

$$\Delta_{\mathcal{E}_T}(\mathbb{G}) = \vee\{d_{\mathcal{E}_T}(u) | u \in \mathcal{P}\};$$
$$\Delta_{\mathcal{E}_I}(\mathbb{G}) = \vee\{d_{\mathcal{E}_I}(u) | u \in \mathcal{P}\};$$
$$\Delta_{\mathcal{E}_F}(\mathbb{G}) = \vee\{d_{\mathcal{E}_F}(u) | u \in \mathcal{P}\}$$

Example 4.6. Let the vertex set $\mathbb{V} = \{v_1, v_2, v_3, v_4\}$ and the edge sets 𝔼=$\{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$ in 𝔾′=(V, E).Take the Fermatean Neutrosophic set $\mathcal{P} = (T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}})$ in 𝕍 and the Fermatean Neutrosophic edge sets in $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ defined by

$(T_{\mathcal{P}}(v_1), I_{\mathcal{P}}(v_1), F_{\mathcal{P}}(v_1)) = (0.3, 0.7, 0.5)$
$(T_{\mathcal{P}}(v_2), I_{\mathcal{P}}(v_2), F_{\mathcal{P}}(v_2)) = (0.6, 0.5, 0.7)$
$(T_{\mathcal{P}}(v_3), I_{\mathcal{P}}(v_3), F_{\mathcal{P}}(v_3)) = (0.8, 0.3, 0.7)$
$(T_{\mathcal{P}}(v_4), I_{\mathcal{P}}(v_4), F_{\mathcal{P}}(v_4)) = (0.7, 0.2, 0.4)$
and
$(T_{\mathcal{Q}}(v_1v_2), I_{\mathcal{Q}}(v_1v_2), F_{\mathcal{Q}}(v_1v_2)) = (0.3, 0.7, 0.7)$
$(T_{\mathcal{P}}(v_2v_3), I_{\mathcal{P}}(v_2v_3), F_{\mathcal{P}}(v_2v_3)) = (0.6, 0.5, 0.7)$
$(T_{\mathcal{P}}(v_3v_4), I_{\mathcal{P}}(v_3v_4), F_{\mathcal{P}}(v_3v_4)) = (0.7, 0.3, 0.7)$
$(T_{\mathcal{P}}(v_1v_4), I_{\mathcal{P}}(v_1v_4), F_{\mathcal{P}}(v_1v_4)) = (0.3, 0.8, 0.6)$
Then, it is 𝔽ℕ𝔾.

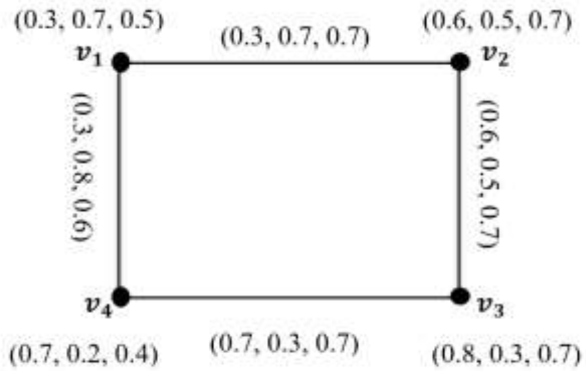


Figure 4. Fermatean Neutrosophic graph

In Fig. 4, v_1v_2, v_2v_3, v_3v_4 are the effective edges of $\mathbb{FNG}$.

$d_\mathcal{E}(v_1)$=(1.3,0.7,1.1)	$\delta_\mathcal{E}(\mathbb{G})$ =(0.8,0.3,0.7)
$d_\mathcal{E}(v_2)$=(1.1,1.0,1.2)	
$d_\mathcal{E}(v_3)$=(1.3,0.7,1.1)	$\Delta_\mathcal{E}(\mathbb{G})$ =(1.3,1.0,1.2)
$d_\mathcal{E}(v_4)$=(0.8,0.3,0.7)	

Definition 4.13. The neighborhood of any vertex u in $\mathbb{G}=(\mathcal{P},\mathcal{Q})$of a $\mathbb{FNG}$ is designated as $\mathcal{N}(u)=(\mathcal{N}_T(u), \mathcal{N}_I(u), \mathcal{N}_F(u))$ where,

$$\mathcal{N}_T(u) = \{v \in \mathcal{P}: T_\mathcal{Q}(u,v) = T_\mathcal{P}(u) \wedge T_\mathcal{P}(v)\};$$
$$\mathcal{N}_I(u) = \{v \in \mathcal{P}: I_\mathcal{Q}(u,v) = I_\mathcal{P}(u) \vee I_\mathcal{P}(v)\};$$
$$\mathcal{N}_F(u) = \{v \in \mathcal{P}: F_\mathcal{Q}(u,v) = F_\mathcal{P}(u) \vee F_\mathcal{P}(v)\}$$

And $\mathcal{N}[u]= \mathcal{N}(u) \cup u$ is called the closed neighbourhood of u.

Definition 4.14. The neighborhood degree of a vertex u in $\mathbb{G}=(\mathcal{P},\mathcal{Q})$of a $\mathbb{FNG}$ is designated as $\mathrm{d}_\mathcal{N}(u)=(\mathrm{d}_{\mathcal{N}_T}(u), \mathrm{d}_{\mathcal{N}_I}(u), \mathrm{d}_{\mathcal{N}_F}(u))$ where,

$$\mathrm{d}_{\mathcal{N}_T}(u) = \sum_{u\in\mathcal{N}(p)} T_\mathcal{P}(u),$$
$$\mathrm{d}_{\mathcal{N}_I}(u) = \sum_{u\in\mathcal{N}(p)} I_\mathcal{P}(u),$$
$$\mathrm{d}_{\mathcal{N}_F}(u) \sum_{u\in\mathcal{N}(p)} F_\mathcal{P}(u)$$

Definition 4.15. The minimum neighborhood degree of $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ in $\mathbb{FNG}$ is designated as $\delta_\mathcal{N}(\mathbb{G})=(\delta_{\mathcal{N}_T}(\mathbb{G}), \delta_{\mathcal{N}_I}(\mathbb{G}), \delta_{\mathcal{N}_F}(\mathbb{G}))$ where,

$$\delta_{\mathcal{N}_T}(\mathbb{G})=\wedge\{d_{\mathcal{N}_T}(u)|u \in \mathcal{P}\};$$
$$\delta_{\mathcal{N}_I}(\mathbb{G})=\wedge\{d_{\mathcal{N}_I}(u)|u \in \mathcal{P}\};$$
$$\delta_{\mathcal{N}_F}(\mathbb{G}) =\wedge\{d_{\mathcal{N}_F}(u)|u \in \mathcal{P}\}$$

Definition 4.16. The maximum neighborhood degree of $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ in $\mathbb{FNG}$ is designated as $\Delta_\mathcal{N}(\mathbb{G})=(\Delta_{\mathcal{N}_T}(\mathbb{G}), \Delta_{\mathcal{N}_I}(\mathbb{G}), \Delta_{\mathcal{N}_F}(\mathbb{G}))$ where,

$$\Delta_{\mathcal{N}_T}(\mathbb{G})=\vee\{d_{\mathcal{N}_T}(u)|u \in \mathcal{P}\};$$
$$\Delta_{\mathcal{N}_I}(\mathbb{G})=\vee\{d_{\mathcal{N}_I}(u)|u \in \mathcal{P}\};$$
$$\Delta_{\mathcal{N}_F}(\mathbb{G}) =\vee\{d_{\mathcal{N}_F}(u)|u \in \mathcal{P}\}$$

Example 4.7.

Let the vertex set $\mathbb{V} = \{v_1, v_2, v_3, v_4\}$ and the edge sets $\mathbb{E}$=$\{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$ in $\mathbb{G}'$=(V, E). Take the Fermatean Neutrosophic set $\mathcal{P} = (T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}})$ in $\mathbb{V}$ and the Fermatean Neutrosophic edge sets in $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ defined by

$(T_{\mathcal{P}}(v_1), I_{\mathcal{P}}(v_1), F_{\mathcal{P}}(v_1))$ = (0.3, 0.7, 0.5)

$(T_{\mathcal{P}}(v_2), I_{\mathcal{P}}(v_2), F_{\mathcal{P}}(v_2))$ = (0.6, 0.5, 0.7)

$(T_{\mathcal{P}}(v_3), I_{\mathcal{P}}(v_3), F_{\mathcal{P}}(v_3))$ = (0.8, 0.3, 0.7)

$(T_{\mathcal{P}}(v_4), I_{\mathcal{P}}(v_4), F_{\mathcal{P}}(v_4))$ = (0.7, 0.2, 0.4)

and

$(T_{\mathcal{Q}}(v_1v_2), I_{\mathcal{Q}}(v_1v_2), F_{\mathcal{Q}}(v_1v_2))$ = (0.2, 0.7, 0.8)

$(T_{\mathcal{P}}(v_2v_3), I_{\mathcal{P}}(v_2v_3), F_{\mathcal{P}}(v_2v_3))$ = (0.6, 0.5, 0.7)

$(T_{\mathcal{P}}(v_3v_4), I_{\mathcal{P}}(v_3v_4), F_{\mathcal{P}}(v_3v_4))$ = (0.7, 0.3, 0.7)

$(T_{\mathcal{P}}(v_1v_4), I_{\mathcal{P}}(v_1v_4), F_{\mathcal{P}}(v_1v_4))$ = (0.3, 0.8, 0.6)

Then, it is $\mathbb{FNG}$.

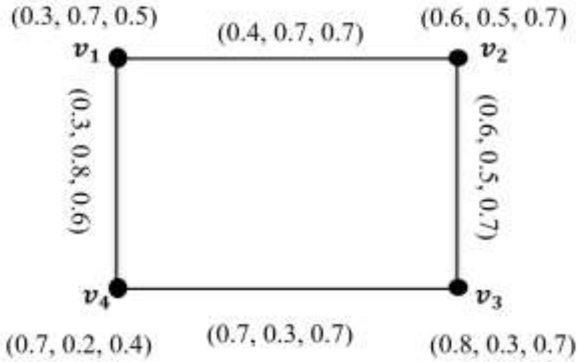


Figure 5. Fermatean Neutrosophic graph

v_1v_2, v_2v_3, v_3v_4 are the effective edges of $\mathbb{FNG}$

$\mathcal{N}(v_1)$=$(\mathcal{N}_T(v_1), \mathcal{N}_I(v_1), \mathcal{N}_F(v_1))$ $\mathcal{N}_T(v_1) = \{v_2\}$; $\mathcal{N}_I(v_1) = \{v_2\}$; $\mathcal{N}_F(v_1) = \{v_2\}$	$d_{\mathcal{N}}(v_1) = (0.6, 0.5, 0.7)$
$\mathcal{N}(v_2)$=$(\mathcal{N}_T(v_2), \mathcal{N}_I(v_2), \mathcal{N}_F(v_2))$ $\mathcal{N}_T(v_2) = \{v_1, v_3\}$; $\mathcal{N}_I(v_2) = \{v_1, v_3\}$; $\mathcal{N}_F(v_2) = \{v_1, v_3\}$	$d_{\mathcal{N}}(v_2) = (1.1,1.0,1.2)$
$\mathcal{N}(v_3)$=$(\mathcal{N}_T(v_3), \mathcal{N}_I(v_3), \mathcal{N}_F(v_3))$ $\mathcal{N}_T(v_3) = \{v_2, v_4\}$; $\mathcal{N}_I(v_3) = \{v_2, v_4\}$;$\mathcal{N}_F(v_3) = \{v_2, v_4\}$	$d_{\mathcal{N}}(v_3) = (1.3,0.7,1.1)$
$\mathcal{N}(v_4)$=$(\mathcal{N}_T(v_4), \mathcal{N}_I(v_4), \mathcal{N}_F(v_4))$ $\mathcal{N}_T(v_4) = \{v_3\}$; $\mathcal{N}_I(v_4) = \{v_3\}$;$\mathcal{N}_F(v_4) = \{v_3\}$	$d_{\mathcal{N}}(v_4) = (0.8,0.3,0.7)$

$$\delta_{\mathcal{N}}(\mathbb{G}) = = (0.6,0.3,0.7);\ \Delta_{\mathcal{N}}(\mathbb{G}) = (1.3,1.0,1.2)$$

Definition 4.17. The closed neighborhood degree of a vertex u of $\mathbb{G}$=$(\mathcal{P}, \mathcal{Q})$in a $\mathbb{FNG}$ is designated as $d_{\mathcal{N}}[u]$=$(d_{\mathcal{N}_T}[u], d_{\mathcal{N}_I}[u], d_{\mathcal{N}_F}[u])$

where,

$$d_{\mathcal{N}_T}[u] = \sum_{v\in\mathcal{N}(p)} T_{\mathcal{P}}(v) + T_{\mathcal{P}}(u),$$

$$d_{\mathcal{N}_I}[u] \sum_{v\in\mathcal{N}(p)} I_{\mathcal{P}}(v) + I_{\mathcal{P}}(u),$$

$$d_{\mathcal{N}_F}[u]\ \sum_{v\in\mathcal{N}(p)} F_{\mathcal{P}}(v) + F_{\mathcal{P}}(u),$$

Definition 4.18. The minimum closed neighborhood degree of $\mathbb{G}$=$(\mathcal{P},\mathcal{Q})$in a $\mathbb{FNG}$ is designated as $\delta_{\mathcal{N}}[\mathbb{G}]$=($\delta_{\mathcal{N}_T}[\mathbb{G}]$, $\delta_{\mathcal{N}_I}[\mathbb{G}]$, $\delta_{\mathcal{N}_F}[\mathbb{G}]$) where,

$$\delta_{\mathcal{N}_T}[\mathbb{G}]=\wedge\{d_{\mathcal{N}_T}(u)|u\in\mathcal{P}\};$$
$$\delta_{\mathcal{N}_I}[\mathbb{G}]=\wedge\{d_{\mathcal{N}_I}(u)|u\in\mathcal{P}\};$$
$$\delta_{\mathcal{N}_F}[\mathbb{G}]=\wedge\{d_{\mathcal{N}_F}(u)|u\in\mathcal{P}\};$$

Definition 4.19. The maximum closed neighborhood degree of $\mathbb{G}$=$(\mathcal{P},\mathcal{Q})$in a $\mathbb{FNG}$ is designated as $\Delta_{\mathcal{N}}[\mathbb{G}]$=($\Delta_{\mathcal{N}_T}[\mathbb{G}]$, $\Delta_{\mathcal{N}_I}[\mathbb{G}]$, $\Delta_{\mathcal{N}_F}[\mathbb{G}]$) where,

$$\Delta_{\mathcal{N}_T}[\mathbb{G}]=\wedge\{d_{\mathcal{N}_T}(u)|u\in\mathcal{P}\};$$
$$\Delta_{\mathcal{N}_I}[\mathbb{G}]=\wedge\{d_{\mathcal{N}_I}(u)|u\in\mathcal{P}\};$$
$$\Delta_{\mathcal{N}_F}[\mathbb{G}]=\wedge\{d_{\mathcal{N}_F}(u)|u\in\mathcal{P}\};$$

Example 4.8.

From Fig. 5,

$\mathcal{N}[v_1]$=($\mathcal{N}_T[v_1]$, $\mathcal{N}_I[v_1]$, $\mathcal{N}_F[v_1]$) $\mathcal{N}_T[v_1]=\{v_1,v_2\}$; $\mathcal{N}_I[v_1]=\{v_1,v_2\}$; ; $\mathcal{N}_F[v_1]=\{v_1,v_2\}$	$d_{\mathcal{N}}[v_1]=(0.9, 1.2, 1.2)$
$\mathcal{N}[v_2]$=($\mathcal{N}_T[v_2]$, $\mathcal{N}_I[v_2]$, $\mathcal{N}_F[v_2]$) $\mathcal{N}_T[v_2]=\{v_1,v_2,v_3\}$; $\mathcal{N}_I[v_2]=\{v_1,v_2,v_3\}$; $\mathcal{N}_F[v_2]=\{v_1,v_2,v_3\}$	$d_{\mathcal{N}}(v_2)=(1.7,1.5,1.9)$
$\mathcal{N}[v_3]$=($\mathcal{N}_T[v_3]$, $\mathcal{N}_I[v_3]$, $\mathcal{N}_F[v_3]$) $\mathcal{N}_T[v_3]=\{v_2,v_3,v_4\}$; $\mathcal{N}_I[v_3]=\{v_2,v_3,v_4\}$; $\mathcal{N}_F[v_3]=\{v_2,v_3,v_4\}$	$d_{\mathcal{N}}(v_3)=(2.1,1.0,1.8)$
$\mathcal{N}[v_4]$=($\mathcal{N}_T[v_4]$, $\mathcal{N}_I[v_4]$, $\mathcal{N}_F[v_4]$) $\mathcal{N}_T[v_4]=\{v_3,v_4\}$; $\mathcal{N}_I[v_4]=\{v_3,v_4\}$; $\mathcal{N}_F[v_4]=\{v_3,v_4\}$	$d_{\mathcal{N}}(v_4)=(1.5,0.5,1.1)$

$$\delta_{\mathcal{N}}[\mathbb{G}]==(0.9,0.5,1.2);\ \Delta_{\mathcal{N}}[\mathbb{G}]=(2.1,1.5,1.8)$$

5. Types of Fermatean neutrosophic graphs

In this section, we introduce different types of Fermatean Neutrosophic graphs based on the degree of each node in FNG such as regular, totally regular and uniform FNGs with suitable examples.

Definition 5.1 Let $\mathbb{G}$=$(\mathcal{P},\mathcal{Q})$ be A Fermatean Neutrosophic graph $\mathbb{FNG}$ defined on G = (V, E). If each vertex of $\mathbb{G}$ has same degree, that is

$$d_{\mathbb{G}}(u)=(l_1, l_2, l_3)\ \ \forall\ u\in V$$

Then $\mathbb{G}$ is called (l_1, l_2, l_3) - regular $\mathbb{FNG}$.

Example 5.2 Consider a Fermatean Neutrosophic graph $\mathbb{G}$=$(\mathcal{P},\mathcal{Q})$ defined on G = (V, E), where $\mathcal{P}$ be a Fermatean Neutrosophic set on V and $\mathcal{Q}$ be a Fermatean Neutrosophic relation on V, defined by

$\mathcal{P}$={ $\langle v_1,(0.6, 1,0.7)\rangle, \langle v_2,(0.5, 0.8,0.4)\rangle, \langle v_3,(0.7, 0.5,0.3)\rangle$ }

And Q={ $\langle v_1v_2, (0.4, 1,0.8)\rangle, \langle v_2v_3, (0.4, 1,0.8)\rangle, \langle v_1v_3, (0.4, 1,0.8)\rangle$ }

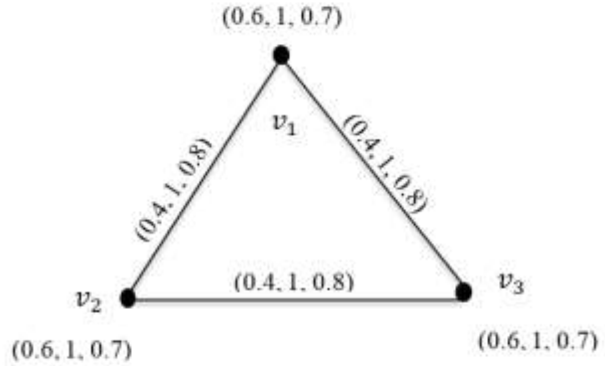


Figure 6. Regular Fermatean Neutrosophic graph

We see that the degree of each vertex in $\mathbb{G}$ is $d_{\mathbb{G}}(v_1)$ = $d_{\mathbb{G}}(v_2)$ = $d_{\mathbb{G}}(v_3)$ = (1.2, 2,1.4). Hence the Fermatean Neutrosophic graph, displayed in Fig. 6, is (1.2, 2,1.4) – regular.

Definition 5.3. A Fermatean Neutrosophic graph $\mathbb{FNG}$ $\mathbb{G}=(\mathcal{P}, Q)$ is called Strong Fermatean Neutrosophic graph if the following conditions are satisfied:

$$T_Q(u, v)= \min\{T_{\mathcal{P}}(u), T_{\mathcal{P}}(v)\}$$
$$I_Q(u, v)= \max\{I_{\mathcal{P}}(u), I_{\mathcal{P}}(v)\}$$
$$F_Q(u, v)= \max\{F_{\mathcal{P}}(u), F_{\mathcal{P}}(v)\} \text{ for all } u, v \in E$$

That is , all the edges in a Fermatean Neutrosophic graph are effective edges.

An example of a Strong Fermatean neutrosophic graph is shown in Figure 7.

Example 5.4

Consider a graph G=(V, E) where the vertex set V=$\{v_1, v_2, v_3, v_4\}$ and the edge se E=$\{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$. Let $\mathbb{G}=(\mathcal{P}, Q)$be a Fermatean Neutrosophic graph on V as shown in Figure 7, defined by $\mathcal{P}$={ $\langle v_1, (0.3, 0.7,0.5)\rangle, \langle v_2, (0.4, 0.6,0.7)\rangle, \langle v_3, (0.8, 0.3,0.7)\rangle, \langle v_4, (0.7, 0.2,0.4)\rangle$ } and Q={ $\langle v_1v_2, (0.3, 0.7,0.7)\rangle, \langle v_2v_3, (0.6, 0.5,0.7)\rangle, \langle v_3v_4, (0.7, 0.3,0.7)\rangle, \langle v_1v_4, (0.3, 0.7,0.5)\rangle$}.

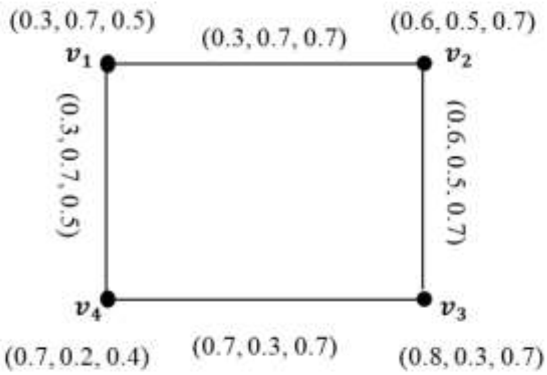


Figure 7. Strong Fermatean Neutrosophic graph

Following and extending the idea of uniform single valued neutrosophic graphs by Broumi et al. [32], we describe the concept of regularity of uniform single valued neutrosophic graphs under Fermatean neutrosophic environment.

Definition 5.5 Let $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ be a Fermatean Neutrosophic graph $\mathbb{FNG}$ defined on G= (V, E), where $\mathcal{P}=(T_{\mathcal{P}},I_{\mathcal{P}},F_{\mathcal{P}})$ is a Fermatean Neutrosophic sets on V and $\mathcal{Q}=(T_{\mathcal{Q}},I_{\mathcal{Q}},F_{\mathcal{Q}})$ is a Fermatean Neutrosophic relation on V. $\mathbb{G}$ is called uniform Fermtean Neutrosophic of level (k_1,k_2,k_3) if $T_{\mathcal{Q}}(u,v)=k_1$, $I_{\mathcal{Q}}(u,v)=k_2$ and $F_{\mathcal{Q}}(u,v)=k_3$, $\forall\ (u,v)\in V\times V$ and $T_{\mathcal{P}}(u)=k_1$, $I_{\mathcal{P}}(u)=k_2$ and $F_{\mathcal{P}}(u)=k_3$ $\forall$ $u\in V$, where, $0<k_1,k_2,k_3\le 1$.

Example 5.5 : The following figure is an uniform Fermatean Neutrosophic graph $\mathbb{G}=(\mathcal{P},\mathcal{Q})$.

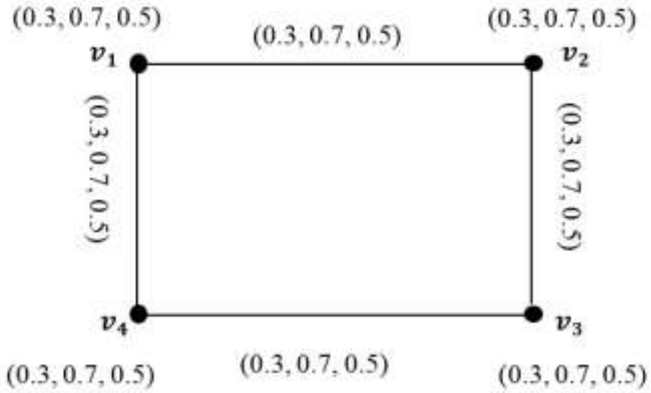


Figure 8. Uniform Fermatean Neutrosophic graph

Theorem 5.6 Every uniform Fermatean Neutrosophic graph is perfectly regular Fermatean Neutrosophic.

Proof. Let $\mathbb{G}=(\mathcal{P},\mathcal{Q})$ be a Fermatean Neutrosophic graph $\mathbb{FNG}$ defined on G= (V, E) with V=$\{u_1,u_2,\dots,u_n\}$, then $T_{\mathcal{Q}}(u,v)=k_1$, $I_{\mathcal{Q}}(u,v)=k_2$ and $F_{\mathcal{Q}}(u,v)=k_3$, $\forall\ (u,v)\in V\times V$ and $T_{\mathcal{P}}(u)=k_1$, $I_{\mathcal{P}}(u)=k_2$ and $F_{\mathcal{P}}(u)=k_3$ $\forall\ (u,v)\in V\times V$, where $0<k_1,k_2,k_3\le 1$.

Then for each u in V,

$$d_{\mathbb{G}}(u)=(\mathrm{d_T}(u),\mathrm{d_I}(u),\mathrm{d_F}(u))$$
$$=\left(\textstyle\sum_{uv\in E}T_{\mathcal{Q}}(uv),\sum_{uv\in E}I_{\mathcal{Q}}(uv),\sum_{uv\in E}F_{\mathcal{Q}}(uv)\right)$$
$$=((n-1)k_1,(n-1)k_2,(n-1)k_3)$$

This shows that $\mathbb{G}$ is $((n-1)k_1,(n-1)k_2,(n-1)k_3)$ regular $\mathbb{FNG}$. Moreover for each vertex u in V,

$$td_{\mathbb{G}}(u)=(\mathrm{td_T}(u),\mathrm{td_I}(u),\mathrm{td_F}(u))$$
$$=\left(\textstyle\sum_{uv\in E}T_{\mathcal{Q}}(uv)+T_{\mathcal{P}}(u),\sum_{uv\in E}I_{\mathcal{Q}}(uv)+I_{\mathcal{P}}(u),\sum_{uv\in E}F_{\mathcal{Q}}(uv)+F_{\mathcal{P}}(u)\right)$$
$$=\left((n-1)k_1+k_1,(n-1)k_2+k_2,(n-1)k_3+k_3\right)$$
$$=(nk_1,nk_2,nk_3)$$

This shows that $\mathbb{G}$ is (nk_1,nk_2,nk_3) totally regular $\mathbb{FNG}$.

Theorem 5.7 If $\mathbb{G}$ is $a\ uniform$ $\mathbb{FNG}$ of level (k_1,k_2,k_3) on G= (V, E), then

a) O($\mathbb{G}$)= (nk_1,nk_2,nk_3) where n=$|V|$.

b) S($\mathbb{G}$)= (mk_1, mk_2, mk_3) where m=$|E|$.

Proof. Let $\mathbb{G}$=$(\mathcal{P}, \mathcal{Q})$ be a uniform Fermatean Neutrosophic graph $\mathbb{FNG}$ defined on G= (V, E) with V=$\{u_1, u_2, \dots, u_n\}$, then $T_Q(u,v)= k_1$, $I_Q(u,v)= k_2$ and $F_Q(u,v)= k_3$, $\forall\ (u,v) \in V \times V$ and $T_{\mathcal{P}}(u)= k_1$, $I_{\mathcal{P}}(u)= k_2$ and $F_{\mathcal{P}}(u)= k_3$ $\forall\ (u,v) \in V \times V$, where $0 < k_1, k_2, k_3 \leq 1$.

a) for each vertex u in V

$$O(\mathbb{G})= (\sum_{u\in V} T_{\mathcal{P}}(u), \sum_{u\in V} I_{\mathcal{P}}(u), \sum_{u\in V} F_{\mathcal{P}}(u))$$
$$=(\sum_{u\in V} k_1, \sum_{u\in V} k_2, \sum_{u\in V} k_3)$$
$$= (nk_1, nk_2, nk_3) \quad \text{where } n=|V|.$$

b) for each edge uv in E

$$S(\mathbb{G})= \left(\sum_{uv\in E} T_Q(uv), \sum_{uv\in E} I_Q(uv), \sum_{uv\in E} F_Q(uv)\right)$$
$$=(\sum_{uv\in E} k_1, \sum_{uv\in E} k_2, \sum_{uv\in E} k_3)$$
$$= (mk_1, mk_2, mk_3) \text{ where } m=|E|.$$

Hence proved.

Remark 5.8 The underlying crisp graph of complement of a Fermatean Neutrosophic graph is always an empty graph.

6. Operations on Fermatean Neutrosophic Graphs

In this section, we propose some important graph-theoretic operations over Fermatean Neutrosophic graphs along with various important results and illustrative examples.

Let $\mathbb{G}_1$=$(\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathbb{G}_2$=$(\mathcal{P}_2, \mathcal{Q}_2)$ be two Fermatean Neutrosophic graphs with references to the graphs G^1=(V_1, E_1) and G^2=(V_2, E_2), correspondingly, where $\mathcal{P}_1$& $\mathcal{P}_2$ are the Fermatean Neutrosophic vertex sets in V_1& V_2 corespondingly, and $\mathcal{Q}_1$& $\mathcal{Q}_2$ are the the Fermatean Neutrosophic edge sets in E_1& E_2, correspondingly.

There are many operations on two graphs G^1=(V_1, E_1) and G^2=(V_2, E_2),which result in a graph whose vertex set is the Cartesian product V_1& V_2.

In the following section, we discuss a few operations on two graphs in the structure of Fermatean Neutrosophic sets theory and investigate their properties.

6.1 Cartesian Product of Fermatean Neutrosophic Graphs

Definition 6.1.1 The Cartesian product of two Fermatean Neutrosophic graphs $\mathbb{G}_1$and $\mathbb{G}_2$, denoted by $\mathbb{G}_1$**Cartesian Product** $\mathbb{G}_2$, is defined as follows:

$\mathbb{G}_1 \times \mathbb{G}_2$=$(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{Q}_1 \times \mathcal{Q}_2)$

where

- $$T_{\mathcal{P}_1\times\mathcal{P}_2}(u_1,u_2)=\min\left(T_{\mathcal{P}_1}(u_1),T_{\mathcal{P}_2}(u_2)\right)$$
$$I_{\mathcal{P}_1\times\mathcal{P}_2}(u_1,u_2)=\max\left(I_{\mathcal{P}_1}(u_1),I_{\mathcal{P}_2}(u_2)\right)$$
$$F_{\mathcal{P}_1\times\mathcal{P}_2}(u_1,u_2)=\max\left(F_{\mathcal{P}_1}(u_1),F_{\mathcal{P}_2}(u_2)\right)\ \forall(u_1,u_2)\in V_1\times V_2$$

The membership value of the edges in $\mathbb{G}_1\times\mathbb{G}_2$ can be computed as

- $$T_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u,u_2),(u,v_2)\big)=\min\left(T_{\mathcal{P}_1}(u),T_{\mathcal{Q}_2}(u_2,v_2)\right)$$
$$I_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u,u_2),(u,v_2)\big)=\max\left(I_{\mathcal{P}_1}(u),I_{\mathcal{Q}_2}(u_2,v_2)\right)$$
$$F_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u,u_2),(u,v_2)\big)=\max\left(F_{\mathcal{P}_1}(u),F_{\mathcal{Q}_2}(u_2,v_2)\right)\ \forall\ u\in V_1,(u_2,v_2)\in E_2$$
- $$T_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big)=\min\left(T_{\mathcal{Q}_1}(u_1,v_1),T_{\mathcal{P}_2}(\gamma)\right),$$
$$I_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big)=\max\left(I_{\mathcal{Q}_1}(u_1,v_1),I_{\mathcal{P}_2}(\gamma)\right)$$
$$F_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big)=\max\left(F_{\mathcal{Q}_1}(u_1,v_1),F_{\mathcal{P}_2}(\gamma)\right)\ \forall\ \gamma\in V_2,(u_1,v_1)\in E_1$$

Theorem 6.1.2 The **Cartesian Product** of two Fermatean Neutrosophic graphs is a Fermatean Neutrosophic graph.

Proof suppose $u\in V_1,(u_2,v_2)\in E_2$. Then,

$$T_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u,u_2),(u,v_2)\big)=min\left(T_{\mathcal{P}_1}(u),T_{\mathcal{Q}_2}(u_2,v_2)\right),$$
$$\leq min\left(T_{\mathcal{P}_1}(u),\min\left(T_{\mathcal{P}_2}(u_2),T_{\mathcal{P}_2}(v_2)\right)\right),$$
$$=min\left(min\left(T_{\mathcal{P}_1}(u),T_{\mathcal{P}_2}(u_2)\right),\min\left(T_{\mathcal{P}_2}(u),T_{\mathcal{P}_2}(v_2)\right)\right),$$
$$=min\big(T_{\mathcal{P}_1\times\mathcal{P}_2}(u,u_2),T_{\mathcal{P}_1\times\mathcal{P}_2}(u,v_2)\big)$$
$$I_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u,u_2),(u,v_2)\big)=max\left(I_{\mathcal{P}_1}(u),I_{\mathcal{Q}_2}(u_2,v_2)\right),$$
$$\geq\ max\left(I_{\mathcal{P}_1}(u),\max\left(I_{\mathcal{P}_2}(u_2),I_{\mathcal{P}_2}(v_2)\right)\right),$$
$$=max\left(max\left(I_{\mathcal{P}_1}(u),I_{\mathcal{P}_2}(u_2)\right),\max\left(I_{\mathcal{P}_2}(u),I_{\mathcal{P}_2}(v_2)\right)\right),$$
$$=\ max\big(I_{\mathcal{P}_1\times\mathcal{P}_2}(u,u_2),I_{\mathcal{P}_1\times\mathcal{P}_2}(u,v_2)\big)$$

and

$$F_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u,u_2),(u,v_2)\big)=max\left(F_{\mathcal{P}_1}(u),F_{\mathcal{Q}_2}(u_2,v_2)\right),$$
$$\geq\ max\left(F_{\mathcal{P}_1}(u),\max\left(F_{\mathcal{P}_2}(u_2),F_{\mathcal{P}_2}(v_2)\right)\right),$$
$$=max\left(max\left(F_{\mathcal{P}_1}(u),F_{\mathcal{P}_2}(u_2)\right),\max\left(F_{\mathcal{P}_2}(u),F_{\mathcal{P}_2}(v_2)\right)\right),$$
$$=max\ \big(F_{\mathcal{P}_1\times\mathcal{P}_2}(u,u_2),F_{\mathcal{P}_1\times\mathcal{P}_2}(u,v_2)\big)$$

Again, let $\forall \gamma \in V_2, (u_1, v_1) \in E_1$, then we have

$$T_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big) = min\left(T_{\mathcal{Q}_1}(u_1,v_1), T_{\mathcal{P}_2}(\gamma)\right),$$

$$\leq min\left(min\left(T_{\mathcal{P}_1}(u_1), T_{\mathcal{P}_1}(v_1), T_{\mathcal{P}_2}(\gamma)\right)\right),$$

$$= min\left(min\left(T_{\mathcal{P}_1}(u_1), T_{\mathcal{P}_2}(\gamma)\right), min\left(T_{\mathcal{P}_1}(v_1), T_{\mathcal{P}_2}(\gamma)\right)\right),$$

$$= min\big(T_{\mathcal{P}_1\times\mathcal{P}_2}(u_1,\gamma), T_{\mathcal{P}_1\times\mathcal{P}_2}(v_1,\gamma)\big).$$

$$I_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big) = max\left(I_{\mathcal{Q}_1}(u_1,v_1), I_{\mathcal{P}_2}(\gamma)\right),$$

$$\geq max\left(max\left(I_{\mathcal{P}_1}(u_1), I_{\mathcal{P}_1}(v_1), I_{\mathcal{P}_2}(\gamma)\right)\right),$$

$$= max\left(max\left(I_{\mathcal{P}_1}(u_1), I_{\mathcal{P}_2}(\gamma)\right), max\left(I_{\mathcal{P}_1}(v_1), I_{\mathcal{P}_2}(\gamma)\right)\right),$$

$$= max\big(I_{\mathcal{P}_1\times\mathcal{P}_2}(u_1,\gamma), I_{\mathcal{P}_1\times\mathcal{P}_2}(v_1,\gamma)\big).$$

and

$$F_{\mathcal{Q}_1\times\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big) = max\left(F_{\mathcal{Q}_1}(u_1,v_1), F_{\mathcal{P}_2}(\gamma)\right),$$

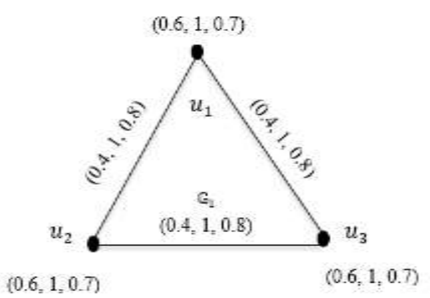
$$\geq max\left(max\left(F_{\mathcal{P}_1}(u_1), F_{\mathcal{P}_1}(v_1), F_{\mathcal{P}_2}(\gamma)\right)\right),$$

$$= max\left(max\left(F_{\mathcal{P}_1}(u_1), F_{\mathcal{P}_2}(\gamma)\right), max\left(F_{\mathcal{P}_1}(v_1), F_{\mathcal{P}_2}(\gamma)\right)\right),$$

$$= max\big(F_{\mathcal{P}_1\times\mathcal{P}_2}(u_1,\gamma), F_{\mathcal{P}_1\times\mathcal{P}_2}(v_1,\gamma)\big).$$

Thus, in view of the definition of the Fermatean Neutrosophic, the result follows. The following example illustrates the above defined graph-theoretic operation.

Example 6.3 Consider two Fermatean Neutrosophic $\mathbb{G}_1$ *and* $\mathbb{G}_2$ as shown in the below Figure 9.



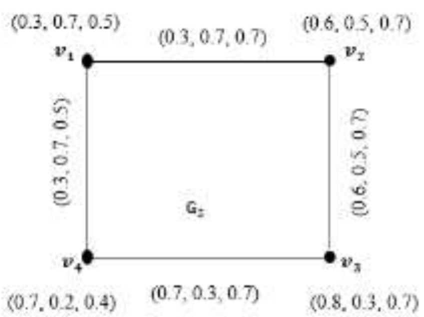


Figure 9. Fermatean Neutrosophic graphs $\mathbb{G}_1$ *and* $\mathbb{G}_2$

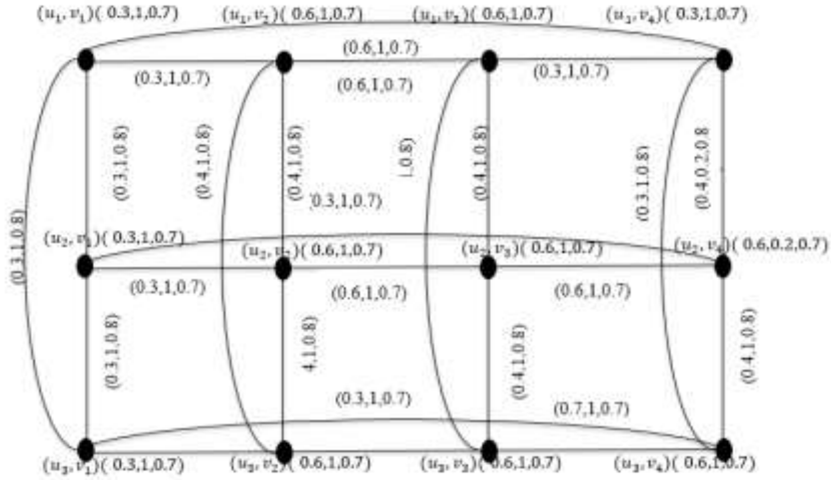


Figure 10. Composition graph $\mathbb{G}_1 \times \mathbb{G}_2$

Then, the graphs $\mathbb{G}_1$, $\mathbb{G}_2$ and their composition graph $\mathbb{G}_1 \times \mathbb{G}_2$ are being graphically presented in the above Figure 10.

6.2 Composition of Fermatean Neutrosophic Graphs

Definition 6.2.1 The composition of two Fermatean Neutrosophic graphs $\mathbb{G}_1$and $\mathbb{G}_2$, denoted by $\mathbb{G}_1 \circ \mathbb{G}_2$, is defined as follows:

$\mathbb{G}_1 \circ \mathbb{G}_2$=$(\mathcal{P}_1 \circ \mathcal{P}_2, \mathcal{Q}_1 \circ \mathcal{Q}_2)$

where

- $$T_{\mathcal{P}_1\times\mathcal{P}_2}(u_1,u_2) = \min\left(T_{\mathcal{P}_1}(u_1), T_{\mathcal{P}_2}(u_2)\right)$$ $$I_{\mathcal{P}_1\times\mathcal{P}_2}(u_1,u_2) = \max\left(I_{\mathcal{P}_1}(u_1), I_{\mathcal{P}_2}(u_2)\right)$$ $$F_{\mathcal{P}_1\times\mathcal{P}_2}(u_1,u_2) = \max\left(F_{\mathcal{P}_1}(u_1), F_{\mathcal{P}_2}(u_2)\right)\ \forall (u_1,u_2) \in V_1 \times V_2$$

- $$T_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((\beta,u_2),(\beta,v_2)\big) = \min\left(T_{\mathcal{P}_1}(\beta), T_{\mathcal{Q}_2}(u_2,v_2)\right)$$ $$I_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((\beta,u_2),(\beta,v_2)\big) = \max\left(I_{\mathcal{P}_1}(\beta), I_{\mathcal{Q}_2}(u_2,v_2)\right)$$ $$F_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((\beta,u_2),(\beta,v_2)\big) = \max\left(F_{\mathcal{P}_1}(\beta), F_{\mathcal{Q}_2}(u_2,v_2)\right)\ \forall\, \beta \in V_1, (u_2,v_2) \in E_2$$

- $$T_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big) = \min\left(T_{\mathcal{Q}_1}(u_1,v_1), T_{\mathcal{P}_2}(\gamma)\right),$$ $$I_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big) = \max\left(I_{\mathcal{Q}_1}(u_1,v_1), I_{\mathcal{P}_2}(\gamma)\right)$$ $$F_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big) = \max\left(F_{\mathcal{Q}_1}(u_1,v_1), F_{\mathcal{P}_2}(\gamma)\right)\ \forall\, \gamma \in V_2, (u_1,v_1) \in E_1$$

- $$T_{Q_1\circ Q_2}\big((u_1,u_2),(v_1,v_2)\big) = \min\Big(T_{\mathcal{P}_2}(u_2), T_{\mathcal{P}_2}(v_2), T_{Q_1}(u_1,v_1)\Big),$$
  $$I_{Q_1\circ Q_2}\big((u_1,u_2),(v_1,v_2)\big) = \max\Big(I_{\mathcal{P}_2}(u_2), I_{\mathcal{P}_2}(v_2), I_{Q_1}(u_1,v_1)\Big)$$
  $$F_{Q_1\circ Q_2}\big((u_1,u_2),(v_1,v_2)\big) = \max\Big(F_{\mathcal{P}_2}(u_2), F_{\mathcal{P}_2}(v_2), F_{Q_1}(u_1,v_1)\Big)$$
  $$\forall\big((u_1,u_2),(v_1,v_2)\big) \in E^\circ, where$$
  $$E^\circ = \{(u_1,u_2),(v_1,v_2) | (u_1,v_1) \in E_1 \ and \ u_2 \neq v_2\}$$

Theorem 6.2.2 The composition of two Fermatean Neutrosophic graphs is a Fermatean Neutrosophic graph.

Proof: Suppose $\beta \in V_1, (u_2,v_2) \in E_2$. Then,

$$T_{Q_1\circ Q_2}\big((\beta,u_2),(\beta,v_2)\big) = min\Big(T_{\mathcal{P}_1}(\beta), T_{Q_2}(u_2,v_2)\Big),$$

$$\leq \ min\Big(T_{\mathcal{P}_1}(\beta), \min\Big(T_{\mathcal{P}_2}(u_2), T_{\mathcal{P}_2}(v_2)\Big)\Big),$$

$$= min\Big(min\Big(T_{\mathcal{P}_1}(\beta), T_{\mathcal{P}_2}(u_2)\Big), \min\Big(T_{\mathcal{P}_2}(\beta), T_{\mathcal{P}_2}(v_2)\Big)\Big),$$

$$= min\big(T_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,u_2), T_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,v_2)\big).$$

$$I_{Q_1\circ Q_2}\big((\beta,u_2),(\beta,v_2)\big) = max\Big(I_{\mathcal{P}_1}(\beta), I_{Q_2}(u_2,v_2)\Big),$$

$$\geq \ max\Big(I_{\mathcal{P}_1}(\beta), \max\Big(I_{\mathcal{P}_2}(u_2), I_{\mathcal{P}_2}(v_2)\Big)\Big),$$

$$= \ max\Big(max\Big(I_{\mathcal{P}_1}(\beta), I_{\mathcal{P}_2}(u_2)\Big), max\Big(I_{\mathcal{P}_2}(\beta), I_{\mathcal{P}_2}(v_2)\Big)\Big),$$

$$= max\big(I_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,u_2), I_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,v_2)\big).$$

and

$$F_{Q_1\circ Q_2}\big((\beta,u_2),(\beta,v_2)\big) = max\Big(F_{\mathcal{P}_1}(\beta), F_{Q_2}(u_2,v_2)\Big),$$

$$\geq \ max\Big(F_{\mathcal{P}_1}(\beta), \max\Big(F_{\mathcal{P}_2}(u_2), F_{\mathcal{P}_2}(v_2)\Big)\Big),$$

$$= \ max\Big(max\Big(F_{\mathcal{P}_1}(\beta), F_{\mathcal{P}_2}(u_2)\Big), \max\Big(F_{\mathcal{P}_2}(\beta), F_{\mathcal{P}_2}(v_2)\Big)\Big),$$

$$= \ max\big(F_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,u_2), F_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,v_2)\big).$$

Again, let $\forall\, \gamma \in V_2, (u_1,v_1) \in E_1$, then we have

$$T_{Q_1\circ Q_2}\big((u_1,\gamma),(v_1,\gamma)\big) = min\Big(T_{Q_1}(u_1,v_1), T_{\mathcal{P}_2}(\gamma)\Big),$$

$$\leq \ min\ \Big(min\Big(T_{\mathcal{P}_1}(u_1), T_{\mathcal{P}_1}(v_1), T_{\mathcal{P}_2}(\gamma)\Big)\Big),$$

$$= \ min\ \Big(min\Big(T_{\mathcal{P}_1}(u_1), T_{\mathcal{P}_2}(\gamma)\Big), min\Big(T_{\mathcal{P}_1}(v_1), T_{\mathcal{P}_2}(\gamma)\Big)\Big),$$

$$= \ min\big(T_{\mathcal{P}_1\circ\mathcal{P}_2}(u_1,\gamma), T_{\mathcal{P}_1\circ\mathcal{P}_2}(v_1,\gamma)\big).$$

$$I_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big) = max\left(I_{\mathcal{Q}_1}(u_1,v_1), I_{\mathcal{P}_2}(\gamma)\right),$$

$$\geq\ max\left(max\left(I_{\mathcal{P}_1}(u_1), I_{\mathcal{P}_1}(v_1), I_{\mathcal{P}_2}(\gamma)\right)\right),$$

$$=\ max\left(max\left(I_{\mathcal{P}_1}(u_1), I_{\mathcal{P}_2}(\gamma)\right), max\left(I_{\mathcal{P}_1}(v_1), I_{\mathcal{P}_2}(\gamma)\right)\right),$$

$$=\ max\big(I_{\mathcal{P}_1\circ\mathcal{P}_2}(u_1,\gamma), I_{\mathcal{P}_1\circ\mathcal{P}_2}(v_1,\gamma)\big).$$

and

$$F_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((u_1,\gamma),(v_1,\gamma)\big) = max\left(F_{\mathcal{Q}_1}(u_1,v_1), F_{\mathcal{P}_2}(\gamma)\right),$$

$$\geq\ max\left(max\left(F_{\mathcal{P}_1}(u_1), F_{\mathcal{P}_1}(v_1), F_{\mathcal{P}_2}(\gamma)\right)\right),$$

$$=\ max\left(max\left(F_{\mathcal{P}_1}(u_1), F_{\mathcal{P}_2}(\gamma)\right), max\left(F_{\mathcal{P}_1}(v_1), F_{\mathcal{P}_2}(\gamma)\right)\right),$$

$$=\ max\big(F_{\mathcal{P}_1\circ\mathcal{P}_2}(u_1,\gamma), F_{\mathcal{P}_1\circ\mathcal{P}_2}(v_1,\gamma)\big).$$

Further, if $\big((u_1,u_2),(v_1,v_2)\big) \in E^{\circ}$, $(u_1,v_1) \in E_1$ *and* $u_2 \neq v_2$, then we have

$$T_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((u_1,u_2),(v_1,v_2)\big) = min\left(T_{\mathcal{P}_2}(u_2), T_{\mathcal{P}_2}(v_2), T_{\mathcal{Q}_1}(u_1,v_1)\right)$$

$$\leq\ min\left(T_{\mathcal{P}_2}(u_2), T_{\mathcal{P}_2}(v_2), min\left(T_{\mathcal{P}_1}(u_1), T_{\mathcal{P}_1}(v_1)\right)\right)$$

$$=\ min\left(T_{\mathcal{P}_2}(u_2), T_{\mathcal{P}_2}(v_2), min\left(T_{\mathcal{P}_1}(u_1), T_{\mathcal{P}_1}(v_1)\right)\right)$$

$$=\ min(T_{\mathcal{P}_1\circ\mathcal{P}_2}(u_1,u_2),\ T_{\mathcal{P}_1\circ\mathcal{P}_2}(v_1,v_2)$$

$$I_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((u_1,u_2),(v_1,v_2)\big) = max\left(I_{\mathcal{P}_2}(u_2), I_{\mathcal{P}_2}(v_2), I_{\mathcal{Q}_1}(u_1,v_1)\right)$$

$$\geq\ max\left(I_{\mathcal{P}_2}(u_2), I_{\mathcal{P}_2}(v_2), max\left(I_{\mathcal{P}_1}(u_1), I_{\mathcal{P}_1}(v_1)\right)\right)$$

$$=\ max\left(I_{\mathcal{P}_2}(u_2), I_{\mathcal{P}_2}(v_2), max\left(I_{\mathcal{P}_1}(u_1), I_{\mathcal{P}_1}(v_1)\right)\right)$$

$$=\ max(I_{\mathcal{P}_1\circ\mathcal{P}_2}(u_1,u_2),\ I_{\mathcal{P}_1\circ\mathcal{P}_2}(v_1,v_2)$$

and

$$F_{\mathcal{Q}_1\circ\mathcal{Q}_2}\big((u_1,u_2),(v_1,v_2)\big) = max\left(F_{\mathcal{P}_2}(u_2), F_{\mathcal{P}_2}(v_2), F_{\mathcal{Q}_1}(u_1,v_1)\right)$$

$$\geq\ max\left(F_{\mathcal{P}_2}(u_2), F_{\mathcal{P}_2}(v_2), max\left(F_{\mathcal{P}_1}(u_1), F_{\mathcal{P}_1}(v_1)\right)\right)$$

$$=\ max\left(F_{\mathcal{P}_2}(u_2), F_{\mathcal{P}_2}(v_2), max\left(F_{\mathcal{P}_1}(u_1), F_{\mathcal{P}_1}(v_1)\right)\right)$$

$$=\ max(F_{\mathcal{P}_1\circ\mathcal{P}_2}(u_1,u_2),\ F_{\mathcal{P}_1\circ\mathcal{P}_2}(v_1,v_2)$$

Thus, in view of the definition of the Fermatean Neutrosophic, the result follows. The following example illustrates the above defined graph-theoretic operation.

Example 6.2.3

Consider two Fermatean Neutrosophic $\mathbb{G}_1 and$ $\mathbb{G}_2$ as shown in the below Figure 11.

Then, the graphs $\mathbb{G}_1$, $\mathbb{G}_2$ and their composition graph $\mathbb{G}_1 \circ \mathbb{G}_2$ are being graphically presented in the below Figure 12.

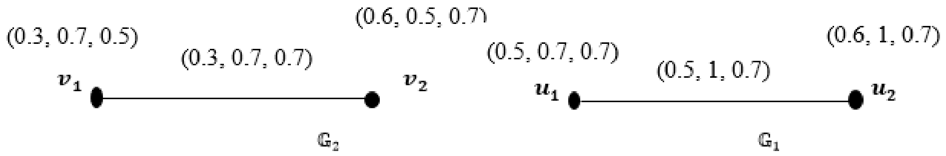


Figure 11. Fermatean Neutrosophic graphs $\mathbb{G}_1 and$ $\mathbb{G}_2$

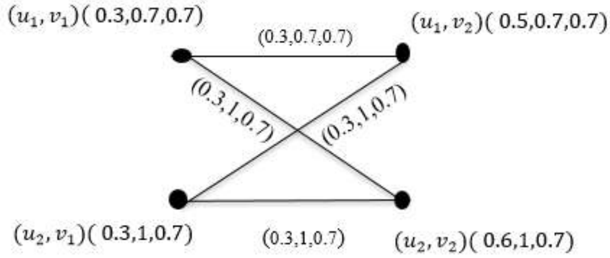


Figure 11. Composition graph $\mathbb{G}_1 \circ \mathbb{G}_2$

6.3 The lexicographic product

Definition 6.3.1 The lexicographic product of two Fermatean Neutrosophic graphs $\mathbb{G}_1$and $\mathbb{G}_2$, denoted by $\mathbb{G}_1 \cdot \mathbb{G}_2$, is defined as follows:

$\mathbb{G}_1 \cdot \mathbb{G}_2$=$(\mathcal{P}_1 \cdot \mathcal{P}_2, \mathcal{Q}_1 \cdot \mathcal{Q}_2)$

- $$T_{\mathcal{P}_1 \cdot \mathcal{P}_2}(u_1, u_2) = min\left(T_{\mathcal{P}_1}(u_1), T_{\mathcal{P}_2}(u_2)\right)$$
$$I_{\mathcal{P}_1 \cdot \mathcal{P}_2}(u_1, u_2) = max\left(I_{\mathcal{P}_1}(u_1), I_{\mathcal{P}_2}(u_2)\right)$$
$$F_{\mathcal{P}_1 \cdot \mathcal{P}_2}(u_1, u_2) = max\left(F_{\mathcal{P}_1}(u_1), F_{\mathcal{P}_2}(u_2)\right) \ \forall (u_1, u_2) \in \mathcal{P}_1 \cdot \mathcal{P}_2$$

- $$T_{\mathcal{Q}_1 \cdot \mathcal{Q}_2}\left((\beta, u_2), (\beta, v_2)\right) = min\left(T_{\mathcal{P}_1}(\beta), T_{\mathcal{Q}_2}(u_2, v_2)\right)$$
$$I_{\mathcal{Q}_1 \cdot \mathcal{Q}_2}\left((\beta, u_2), (\beta, v_2)\right) = max\left(I_{\mathcal{P}_1}(\beta), I_{\mathcal{Q}_2}(u_2, v_2)\right)$$
$$F_{\mathcal{Q}_1 \cdot \mathcal{Q}_2}\left((\beta, u_2), (\beta, v_2)\right) = max\left(F_{\mathcal{P}_1}(\beta), F_{\mathcal{Q}_2}(u_2, v_2)\right) \ \forall \beta \in V_1, (u_2, v_2) \in E_2$$

- $$T_{\mathcal{Q}_1 \cdot \mathcal{Q}_2}\left((u_1, u_2), (v_1, v_2)\right) = min\left(T_{\mathcal{Q}_1}(u_1, v_1), T_{\mathcal{Q}_2}(u_2, v_2)\right),$$
$$I_{\mathcal{Q}_1 \cdot \mathcal{Q}_2}\left((u_1, u_2), (v_1, v_2)\right) = max\left(I_{\mathcal{Q}_1}(u_1, v_1), I_{\mathcal{Q}_2}(u_2, v_2)\right)$$
$$F_{\mathcal{Q}_1 \cdot \mathcal{Q}_2}\left((u_1, u_2), (v_1, v_2)\right) = max\left(F_{\mathcal{Q}_1}(u_1, v_1), F_{\mathcal{Q}_2}(u_2, v_2)\right) \ \forall (u_1, v_1) \in E_1, (u_2, v_2) \in E_2$$

Theorem 6.3.2 The lexicographic product of two Fermatean Neutrosophic graphs is also the Fermatean Neutrosophic graph.

Proof: We have two cases.

Case 1: $\forall \beta \in V_1, (u_2, v_2) \in E_2$. Then,

$$T_{\mathcal{Q}_1\cdot\mathcal{Q}_2}\big((\beta,u_2),(\beta,v_2)\big)=min\left(T_{\mathcal{P}_1}(\beta),T_{\mathcal{Q}_2}(u_2,v_2)\right),$$

$$\leq\ min\left(T_{\mathcal{P}_1}(\beta),\min\left(T_{\mathcal{P}_2}(u_2),T_{\mathcal{P}_2}(v_2)\right)\right),$$

$$=\ min\left(min\left(T_{\mathcal{P}_1}(\beta),T_{\mathcal{P}_2}(u_2)\right),\min\left(T_{\mathcal{P}_2}(\beta),T_{\mathcal{P}_2}(v_2)\right)\right),$$

$$=\ min\big(T_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,u_2),T_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,v_2)\big)\ .$$

$$I_{\mathcal{Q}_1\cdot\mathcal{Q}_2}\big((\beta,u_2),(\beta,v_2)\big)=max\left(I_{\mathcal{P}_1}(\beta),I_{\mathcal{Q}_2}(u_2,v_2)\right),$$

$$\geq\ max\left(I_{\mathcal{P}_1}(\beta),\max\left(I_{\mathcal{P}_2}(u_2),I_{\mathcal{P}_2}(v_2)\right)\right),$$

$$=\ max\left(max\left(I_{\mathcal{P}_1}(\beta),I_{\mathcal{P}_2}(u_2)\right),\max\left(I_{\mathcal{P}_2}(\beta),I_{\mathcal{P}_2}(v_2)\right)\right),$$

$$=\ max\big(I_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,u_2),I_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,v_2)\big).$$

and

$$F_{\mathcal{Q}_1\cdot\mathcal{Q}_2}\big((\beta,u_2),(\beta,v_2)\big)=max\left(F_{\mathcal{P}_1}(\beta),F_{\mathcal{Q}_2}(u_2,v_2)\right),$$

$$\geq\ max\left(F_{\mathcal{P}_1}(\beta),\max\left(F_{\mathcal{P}_2}(u_2),F_{\mathcal{P}_2}(v_2)\right)\right),$$

$$=\ max\left(max\left(F_{\mathcal{P}_1}(\beta),F_{\mathcal{P}_2}(u_2)\right),\max\left(F_{\mathcal{P}_2}(\beta),F_{\mathcal{P}_2}(v_2)\right)\right),$$

$$=\ max\big(F_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,u_2),F_{\mathcal{P}_1\circ\mathcal{P}_2}(\beta,v_2)\big).$$

Case 2 : $\forall\ (u_1,v_1)\in E_1,(u_2,v_2)\in E_2$

$$T_{\mathcal{Q}_1\cdot\mathcal{Q}_2}\big((u_1,u_2),(v_1,v_2)\big)=min\left(T_{\mathcal{Q}_1}(u_1,v_1),T_{\mathcal{Q}_2}(u_2,v_2)\right)$$

$$\leq\ min\ \left(min\left(T_{\mathcal{Q}_1}(u_1),T_{\mathcal{Q}_1}(v_1)\right),min\left(T_{\mathcal{Q}_2}(u_2),T_{\mathcal{Q}_2}(v_2)\right)\right),$$

$$=\ min\ \left(min\left(T_{\mathcal{Q}_1}(u_1),T_{\mathcal{Q}_2}(u_2)\right),min\left(T_{\mathcal{Q}_1}(v_1),T_{\mathcal{Q}_2}(v_2)\right)\right),$$

$$=\ min\big(T_{\mathcal{P}_1\cdot\mathcal{P}_2}(u_1,u_2),T_{\mathcal{P}_1\cdot\mathcal{P}_2}(v_1,v_2)\big)\ .$$

$$I_{\mathcal{Q}_1\cdot\mathcal{Q}_2}\big((u_1,u_2),(v_1,v_2)\big)=max\left(I_{\mathcal{Q}_1}(u_1,v_1),I_{\mathcal{Q}_2}(u_2,v_2)\right)$$

$$\geq\ max\ \left(max\left(I_{\mathcal{Q}_1}(u_1),I_{\mathcal{Q}_1}(v_1)\right),max\left(I_{\mathcal{Q}_2}(u_2),I_{\mathcal{Q}_2}(v_2)\right)\right),$$

$$=\ max\ \left(max\left(I_{\mathcal{Q}_1}(u_1),I_{\mathcal{Q}_2}(u_2)\right),max\left(I_{\mathcal{Q}_1}(v_1),I_{\mathcal{Q}_2}(v_2)\right)\right),$$

$$=\ max\big(I_{\mathcal{P}_1\cdot\mathcal{P}_2}(u_1,u_2),I_{\mathcal{P}_1\cdot\mathcal{P}_2}(v_1,v_2)\big)\ .$$

and

$$F_{\mathcal{Q}_1\cdot\mathcal{Q}_2}\big((u_1,u_2),(v_1,v_2)\big)=max\left(F_{\mathcal{Q}_1}(u_1,v_1),F_{\mathcal{Q}_2}(u_2,v_2)\right)$$

$$\geq max\left(max\left(F_{\mathcal{Q}_1}(u_1), F_{\mathcal{Q}_1}(v_1)\right), max\left(F_{\mathcal{Q}_2}(u_2), F_{\mathcal{Q}_2}(v_2)\right)\right),$$

$$= max\left(max\left(F_{\mathcal{Q}_1}(u_1), F_{\mathcal{Q}_2}(u_2)\right), max\left(F_{\mathcal{Q}_1}(v_1), F_{\mathcal{Q}_2}(v_2)\right)\right),$$

$$= max\left(F_{\mathcal{P}_1 \cdot \mathcal{P}_2}(u_1, u_2), F_{\mathcal{P}_1 \cdot \mathcal{P}_2}(v_1, v_2)\right).$$

Example 6.3.3

Consider two Fermatean neutrosophic $\mathbb{G}_1 and\ \mathbb{G}_2$ as shown in the below Figure 13.

Then, lexicographic product the graphs $\mathbb{G}_1$, $\mathbb{G}_2$ ($\mathbb{G}_1 \circ \mathbb{G}_2$) is graphically presented in the below Figure 14.

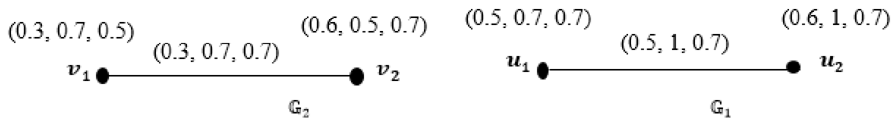


Figure 13. Fermatean neutrosophic graphs $\mathbb{G}_1 and\ \mathbb{G}_2$

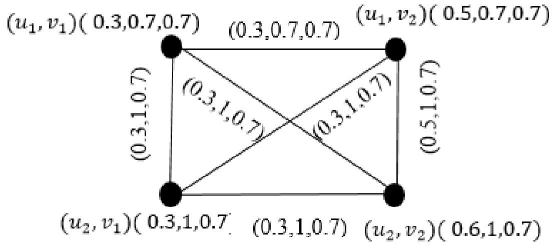


Figure 14. Lexicographic product the graphs $\mathbb{G}_1$, $\mathbb{G}_2$ ($\mathbb{G}_1 \circ \mathbb{G}_2$)

6.4 Union of Fermatean Neutrosophic Graphs

Definition 6.4.1 The union of two Fermatean Neutrosophic graphs $\mathbb{G}_1$and $\mathbb{G}_2$, denoted by $\mathbb{G}_1 \cup \mathbb{G}_2$, is defined as follows:

$\mathbb{G}_1 \cup \mathbb{G}_2$=$(\mathcal{P}_1 \cup \mathcal{P}_2, \mathcal{Q}_1 \cup \mathcal{Q}_2)$

where

- $T_{\mathcal{P}_1 \cup \mathcal{P}_2}(u)=\begin{cases} T_{\mathcal{P}_1}(u) & if\ u \in V_1 - V_2 \\ T_{\mathcal{P}_2}(u) & if\ u \in V_2 - V_1 \\ max\left(T_{\mathcal{P}_1}(u), T_{\mathcal{P}_2}(v)\right) & if\ u \in V_1 \cup V_2 \end{cases}$

  $I_{\mathcal{P}_1 \cup \mathcal{P}_2}(u)=\begin{cases} I_{\mathcal{P}_1}(u) & if\ u \in V_1 - V_2 \\ I_{\mathcal{P}_2}(u) & if\ u \in V_2 - V_1 \\ min\left(I_{\mathcal{P}_1}(u), I_{\mathcal{P}_2}(v)\right) & if\ u \in V_1 \cup V_2 \end{cases}$

  $F_{\mathcal{P}_1 \cup \mathcal{P}_2}(u)=\begin{cases} F_{\mathcal{P}_1}(u) & if\ u \in V_1 - V_2 \\ F_{\mathcal{P}_2}(u) & if\ u \in V_2 - V_1 \\ min\left(F_{\mathcal{P}_1}(u), F_{\mathcal{P}_2}(v)\right) & if\ u \in V_1 \cup V_2 \end{cases}$

- $T_{\mathcal{Q}_1 \cup \mathcal{Q}_2}(u,v)=\begin{cases} T_{\mathcal{Q}_1}(u,v) & if\ (u,v) \in E_1 - E_2 \\ T_{\mathcal{Q}_2}(u,v) & if\ (u,v) \in E_2 - E_1 \\ max\left(T_{\mathcal{Q}_1}(u,v), T_{\mathcal{Q}_2}(u,v)\right) & if\ (u,v) \in E_1 \cup E_2 \end{cases}$

  $I_{\mathcal{Q}_1 \cup \mathcal{Q}_2}(u,v)=\begin{cases} I_{\mathcal{Q}_1}(u,v) & if\ (u,v) \in E_1 - E_2 \\ I_{\mathcal{Q}_2}(u,v) & if\ (u,v) \in E_2 - E_1 \\ min\left(I_{\mathcal{Q}_1}(u,v), I_{\mathcal{Q}_2}(u,v)\right) & if\ (u,v) \in E_1 \cup E_2 \end{cases}$

$$F_{\mathcal{Q}_1 \cup \mathcal{Q}_2}(u,v)=\begin{cases} F_{\mathcal{Q}_1}(u,v) & if\ (u,v) \in E_1 - E_2 \\ F_{\mathcal{Q}_2}(u,v) & if\ (u,v) \in E_2 - E_1 \\ min\left(F_{\mathcal{Q}_1}(u,v), F_{\mathcal{Q}_2}(u,v)\right) & if\ (u,v) \in E_1 \cup E_2 \end{cases}$$

6.5 Join of Fermatean Neutrosophic Graphs

Definition 6.5.1 The join of two Fermatean Neutrosophic graphs $\mathbb{G}_1$and $\mathbb{G}_2$, denoted by $\mathbb{G}_1$+ $\mathbb{G}_2$, is defined as follows:

$\mathbb{G}_1$+ $\mathbb{G}_2$=$(\mathcal{P}_1 + \mathcal{P}_2, \mathcal{Q}_1 + \mathcal{Q}_2)$

where

- $$T_{\mathcal{P}_1+\mathcal{P}_2}(u)=\begin{cases} T_{\mathcal{P}_1}(u) & if\ u \in V_1 - V_2 \\ T_{\mathcal{P}_2}(u) & if\ u \in V_2 - V_1 \\ T_{\mathcal{P}_1 \cup \mathcal{P}_2}(u) & if\ u \in V_1 \cup V_2 \end{cases}$$

  $$I_{\mathcal{P}_1+\mathcal{P}_2}(u)=\begin{cases} I_{\mathcal{P}_1}(u) & if\ u \in V_1 - V_2 \\ I_{\mathcal{P}_2}(u) & if\ u \in V_2 - V_1 \\ I_{\mathcal{P}_1 \cup \mathcal{P}_2}(u) & if\ u \in V_1 \cup V_2 \end{cases}$$

  $$F_{\mathcal{P}_1+\mathcal{P}_2}(u)=\begin{cases} F_{\mathcal{P}_1}(u) & if\ u \in V_1 - V_2 \\ F_{\mathcal{P}_2}(u) & if\ u \in V_2 - V_1 \\ F_{\mathcal{P}_1 \cup \mathcal{P}_2}(u) & if\ u \in V_1 \cup V_2 \end{cases}$$

- $$T_{\mathcal{Q}_1+\mathcal{Q}_2}(u,v)=\begin{cases} T_{\mathcal{Q}_1}(u,v) & if\ (u,v) \in E_1 - E_2 \\ T_{\mathcal{Q}_2}(u,v) & if\ (u,v) \in E_2 - E_1 \\ T_{\mathcal{Q}_1 \cup \mathcal{Q}_2}(u,v) & if\ (u,v) \in E_1 \cup E_2 \end{cases}$$

  $$I_{\mathcal{Q}_1+\mathcal{Q}_2}(u,v)=\begin{cases} I_{\mathcal{Q}_1}(u,v) & if\ (u,v) \in E_1 - E_2 \\ I_{\mathcal{Q}_2}(u,v) & if\ (u,v) \in E_2 - E_1 \\ I_{\mathcal{Q}_1 \cup \mathcal{Q}_2}(u,v) & if\ (u,v) \in E_1 \cup E_2 \end{cases}$$

  $$F_{\mathcal{Q}_1+\mathcal{Q}_2}(u,v)=\begin{cases} F_{\mathcal{Q}_1}(u,v) & if\ (u,v) \in E_1 - E_2 \\ F_{\mathcal{Q}_2}(u,v) & if\ (u,v) \in E_2 - E_1 \\ F_{\mathcal{Q}_1 \cup \mathcal{Q}_2}(u,v) & if\ (u,v) \in E_1 \cup E_2 \end{cases}$$

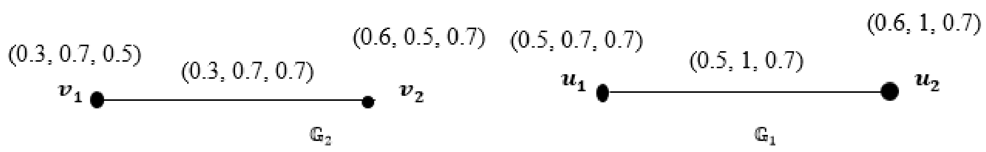
- $$T_{\mathcal{Q}_1+\mathcal{Q}_2}(u,v) = \min\left(T_{\mathcal{P}_1}(u), T_{\mathcal{P}_2}(v)\right) \text{ if } (u,v) \in E'$$
  $$I_{\mathcal{Q}_1+\mathcal{Q}_2}(u,v) = \max\left(I_{\mathcal{P}_1}(u), I_{\mathcal{P}_2}(v)\right) \text{ if } (u,v) \in E'$$
  $$F_{\mathcal{Q}_1+\mathcal{Q}_2}(u,v) = \max\left(F_{\mathcal{P}_1}(u), F_{\mathcal{P}_2}(v)\right) \text{ if } (u,v) \in E'$$

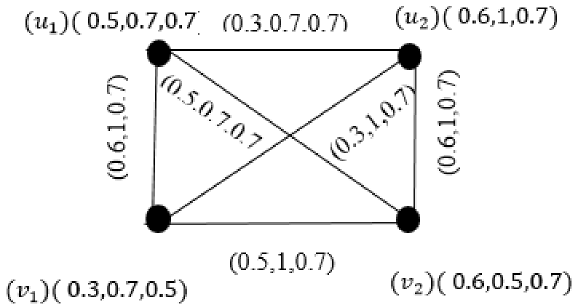
  where E'denotes the set of all the edge joining the nodes of $V_1$and V_2.

Example 6.5.2

Consider two Fermatean Neutrosophic $\mathbb{G}_1 and\ \mathbb{G}_2$ as shown in the below Figure 13.

Then, the join of two Fermatean Neutrosophic graphs $\mathbb{G}_1$and $\mathbb{G}_2$, denoted by $\mathbb{G}_1$+ $\mathbb{G}_2$, is graphically presented in the below Figure 14.





Theorem 6.5.3 The union and join of two Fermatean Neutrosophic graphs are also Fermatean Neutrosophic graphs.

Proof The proof can be outlined similarly as the proof of Theorem 6.3.2.

7. Applications of FNG

Recent days, many researchers who have studied the decision-making problems in different sectors like production, manufacturing, social networking, etc. by using fuzzy, neutrosophic tools [49 – 66]. Sriganesh et. al. [48] investigated the selection of the best power plant among three of the major power plants like hydroelectric power plant, thermal power plant, and nuclear power plant using a graph-theoretic approach. They used digraph characteristic between the factors and cofactors in the selection of the power plant. The interdependency of the factors and their inheritances are identified and they have been represented by using numerical values in their work. Among all these decision-making problems, power plants play a prominent role in for all industry sectors that depend on exergy processes. This section reports the selection of the best power plant among six of the major power plants using Fermatean Neutrosophic graph-theoretic approach. A power plant or power generating station where electric power is generated and distributed on a mass scale. It can be classified into different types based on the fuel used for the generation of electricity. There are many power plants depend on the availability of coal, fuel, wind, and water, etc. We have considered the following six power plants in this case study.

Hydroelectric power plant (P_1): Electricity is produced in a hydroelectric power plant by the flow of water from a height that is used to drive the turbine. The fast-flowing water is converted into mechanical energy when the turbine rotates which is further converted into electric power by the generator.

Thermal power plants (P_2): It converts heat energy into electricity. The heat energy is used to convert fluid into gas which turns the turbine producing mechanical energy which is an intermediate in the process and is converted into electricity in the generators.

A nuclear power plant (P_3): It is similar to a thermal power plant but in nuclear power plants, a nuclear reactor acts as the heat source. In a nuclear reactor, controlled nuclear fission takes place which produces an enormous amount of heat. This heat is dissipated in the water, and it is converted into high-pressure steam which in turn runs the turbine.

Geothermal power plant (P_4): The geothermal power plants are related to other steam turbine thermal power plants. In this heat from the fuel source is used to heat water or any other working fluid. The working fluid is then used to rotate on the turbine of a generator, for producing electricity.
Tidal power plant (P_5): Tidal power or tidal energy is a form of hydropower that converts energy derived from tides primarily into useful forms of electricity. Although not yet generally used, tidal energy has the potential to generate future electricity.
Solar power plant (P_6): A solar power plant is based on the conversion of sunlight into electricity either directly photovoltaics or indirectly using concentrated solar power. Concentrated solar power systems use lenses, mirrors and tracking systems to focus a large area of sunlight into a small beam.

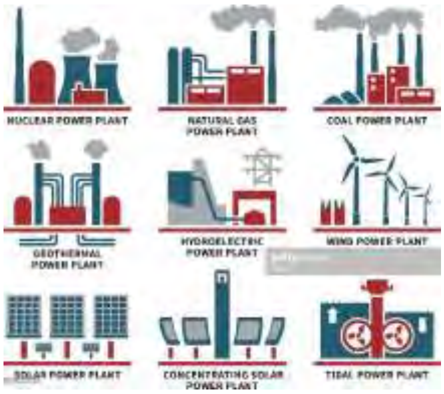


Figure 16. Different power plants

The identification of a site for a power plant selection depends on various factors like land, space, water, cost, transport, fuel, availability of cooling water, nature of the load, etc. Apart from these factors, there are a few sub-factors involving in this process (Figure. 17).

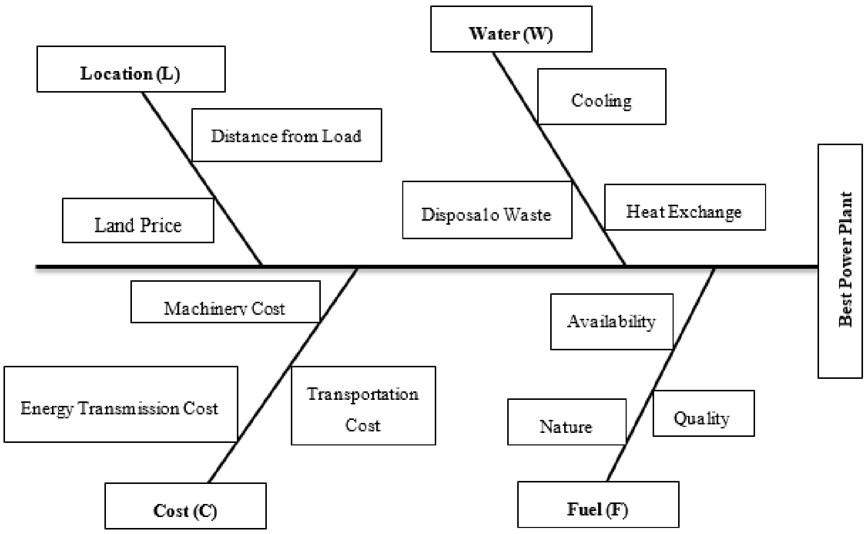


Figure 17. Fishbone diagram representing the necessities for setting up a power plant

In the process of applying FNG in finding the best power plant. FNG can be represented as a matrix whose rows and columns are the sub-factors. Let $V = \{ P_1, P_2, P_3, P_4, P_5, P_6\}$ be the six different power plants under the selection on the basis of wishing parameters or attributes set $A = \{L, W, C, F\}$. The following figures represents the Fermatean Neutrosophic graphs of location, water, cost, and fuel.

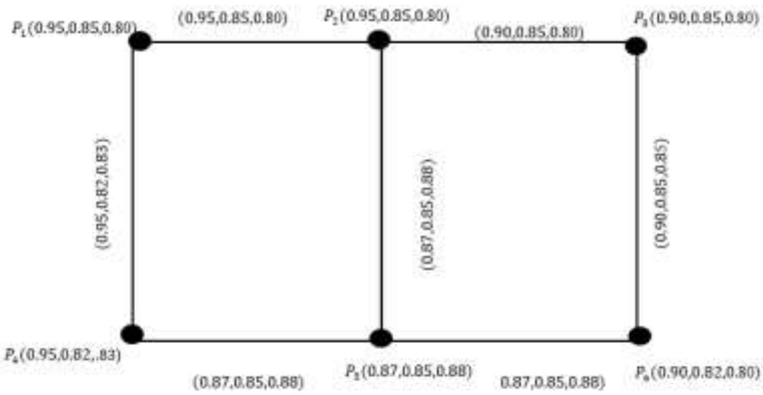


Figure 18. Location based Fermatean Neutrosophic graphs

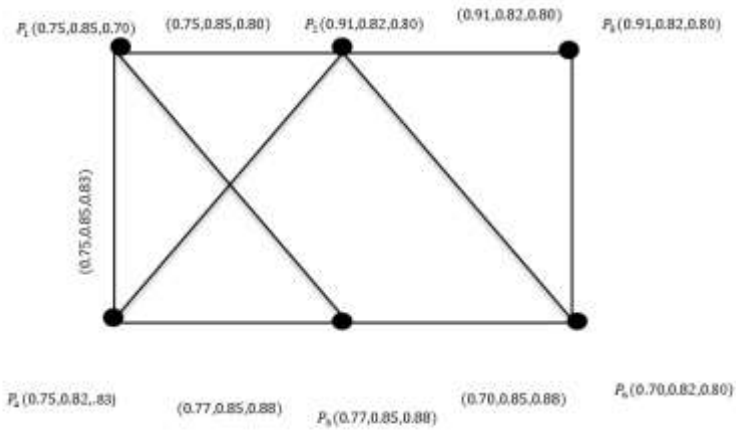


Figure 19. Water based Fermatean Neutrosophic graphs

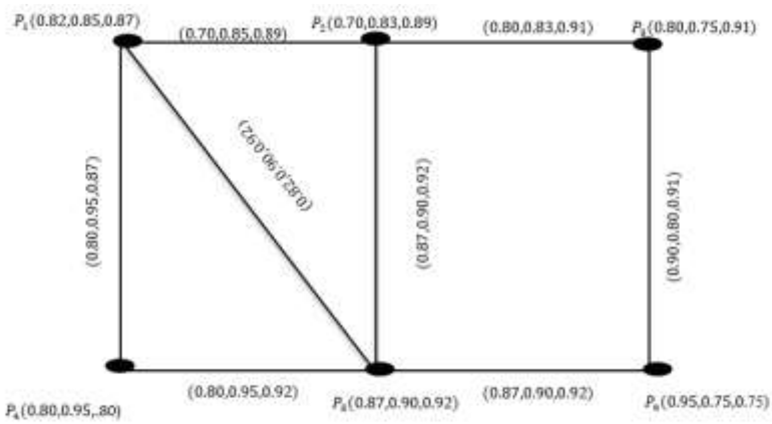


Figure 20. Cost based Fermatean Neutrosophic graphs

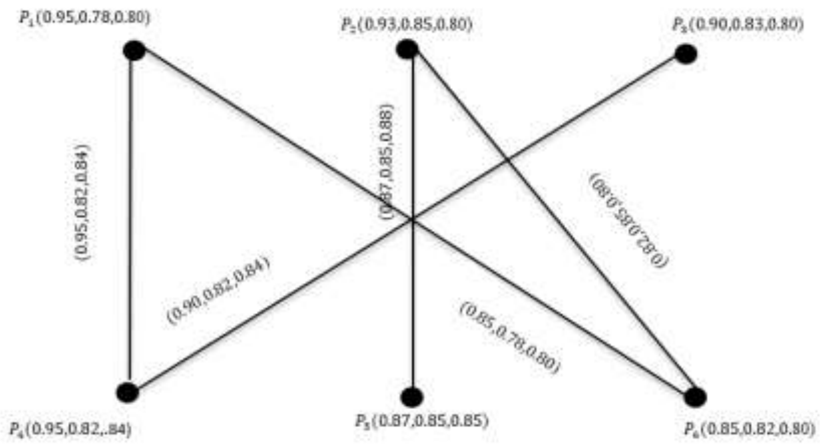


Figure 21. Fuel based Fermatean Neutrosophic graphs

We construct the incidence matrix for $P(L)$, $P(C), P(W), P(F)$ listed below:

$$\boldsymbol{P(L)} = \begin{pmatrix} (0,0,0) & (0.95,0.85,0.80) & (0,0,0) & (0.95,0.82,0.83) & (0,0,0) & (0,0,0) \\ (0.95,0.85,0.80) & (0,0,0) & (0.90,0.85,0.80) & (0,0,0) & (0.87,0.85,0.88) & (0,0,0) \\ (0,0,0) & (0.90,0.85,0.80) & (0,0,0) & (0,0,0) & (0,0,0) & (0.90,0.85,0.85) \\ (0.95,0.82,0.83) & (0,0,0) & (0,0,0) & (0,0,0) & (0.87,0.85,0.8) & (0,0,0) \\ (0,0,0) & (0.87,0.85,0.88) & (0,0,0) & (0.87,0.85,0.8) & (0,0,0) & (0.87,0.85,0.88) \\ (0,0,0) & (0,0,0) & (0.90,0.85,0.85) & (0,0,0) & (0.87,0.85,0.88) & (0,0,0) \end{pmatrix}$$

$$\boldsymbol{P(W)} = \begin{pmatrix} (0,0,0) & (0.75,0.85,0.80) & (0,0,0) & (0.75,0.85,0.83) & (0.75,0.85,0.88) & (0,0,0) \\ (0.91,0.82,0.80) & (0,0,0) & (0.91,0.82,0.80) & (0.75,0.82,0.83) & (0,0,0) & (0.70,0.82,0.80) \\ (0,0,0) & (0.91,0.82,0.80) & (0,0,0) & (0,0,0) & (0,0,0) & (0.70,0.82,0.80) \\ (0.75,0.85,0.83) & (0.75,0.82,0.83) & (0,0,0) & (0,0,0) & (0.95,0.85,0.80) & (0,0,0) \\ (0.75,0.85,0.88) & (0,0,0) & (0,0,0) & (0.77,0.85,0.88) & (0,0,0) & (0.70,0.85,0.88) \\ (0,0,0) & (0.70,0.82,0.80) & (0.95,0.85,0.80) & (0,0,00) & (0.70,0.85,0.88) & (0,0,0) \end{pmatrix}$$

$$\boldsymbol{P(C)} = \begin{pmatrix} (0,0,0) & (0.70,0.85,0.89) & (0,0,0) & (0.80,0.95,0.87) & (0.95,0.85,0.80) & (0,0,0) \\ (0.70,0.85,0.80) & (0,0,0) & (0.80,0.83,0.91) & (0,0,0) & (0.87,0.90,0.92) & (0,0,0) \\ (0,0,0) & (0.80,0.83,0.91) & (0,0,0) & (0,0,0) & (0,0,0) & (0.87,0.90,0.92) \\ (0.80,0.95,0.87) & (0,0,0) & (0,0,0) & (0,0,0) & (0.80,0.95,0.92) & (0,0,0) \\ (0.82,0.90,0.92) & (0.87,0.85,0.80) & (0,0,0) & (0.80,0.95,0.92) & (0,0,0) & (0.87,0.90,0.92) \\ (0,0,0) & (0,0,0) & (0.90,0.80,0.91) & (0,0,0) & (0.87,0.90,0.92) & (0,0,0) \end{pmatrix}$$

$$\boldsymbol{P(F)} = \begin{pmatrix} (0,0,0) & (0,0,0) & (0,0,0) & (0.95,0.85,0.80) & (0,0,0) & (0.85,0.78,0.80) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0.87,0.85,0.88) & (0.82,0.85,0.80) \\ (0.90,0.82,0.84) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0.95,0.82,0.84) & (0,0,0) & (0.90,0.82,0.84) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0.87,0.85,0.88) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0.85,0.78,0.80) & (0.82,0.85,0.80) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \end{pmatrix}$$

The incidence matrix of resultant FNG is obtained from the combination of all attributes for each power plant

	P_1	P_2	P_3	P_4	P_5	P_6	Overall
P_1	0.333333	0.32	0.333333	0.61	0.323333	0.326667	2.246667
P_2	0.35	0.333333	0.333333	0.33	0.326667	0.35	2.023333
P_3	0.326667	0.313333	0.333333	0.333333	0.343333	0.326667	1.976667
P_4	0.61	0.333333	0.326667	0.333333	0.343333	0.333333	2.28
P_5	0.326667	0.323333	0.333333	0.343333	0.333333	0.326667	1.986667
P_6	0.326667	0.313333	0.313333	0.333333	0.326667	0.333333	1.946667

$$\boldsymbol{P(with\ respect\ all\ attributes)} = \begin{pmatrix} (0,0,0) & (0,0.85,0.89) & (0,0,0) & (0.75,0.95,0.87) & (0,0.85,0.88) & (0,0.78,0.80) \\ (0,0.85,0.80) & (0,0,0) & (0,0,0) & (0,0.82,0.83) & (0,0.90,0.92) & (0,0.85,0.80) \\ (0,0.82,0.84) & (0,0.85,0.91) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0.90,0.92) \\ (0.75,0.95,0.87) & (0,0,0) & (0,0.82,0.84) & (0,0,0) & (0,0.95,0.92) & (0,0,0) \\ (0,0.90,0.92) & (0,0.85,0.88) & (0,0,0) & (0,0.95,0.92) & (0,0,0) & (0,0.90,0.92) \\ (0,0.78,0.80) & (0,0.85,0.80) & (0,0.85,0.91) & (0,0,0) & (0,0.90,0.92) & (0,0,0) \end{pmatrix}$$

Tabular representation of score values of incidence matrix of resultant FNG with average score function $S = \frac{T+I+1-F}{3}$.

Clearly, the maximum score value is 2.28, scored by the plant P_4. According the data **Geothermal power plant is the beast choice.**

8. Conclusion

Fuzzy theory plays a vital role in uncertainty situations. The extension of fuzzy sets are the popular Intuitionistic fuzzy sets and then Smarandache introduced the most general concept called the Neutrosophic sets. There are many variants of NS are available in the literature like Pythagorean Neutrosophic, Single Valued Neutrosophic, Bipolar Neutrosophic sets. In the list, we have introduced a new class of set namely, Fermatean Neutrosophic sets in this work. We have discussed various types of Fermatean Neutrosophic graphs and the properties of these graphs in this paper. We also apply this new type of graph in a decision making problem. We are extending our research on this new concept to introduce Fermatean Neutrosophic number and Fermatean triangle and trapezoidal Neutrosophic number and its applications in our future work.

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Neutrosophic Orbit Continuous Mappings

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Abstract: The purpose of this paper is to introduce the new concepts of neutrosophic orbit open set, neutrosophic orbit continuous, almost-neutrosophic orbit continuous, weakly-neutrosophic orbit continuous, neutrosophic orbit* continuous functions and analyze some of their interesting properties.

Keywords: Neutrosophic orbit set; Neutrosophic orbit open set; Neutrosophic orbit continuous.

1. Introduction

Fuzzy concept has invaded almost all branches of Mathematics since its introduction by Zadeh[23]. Fuzzy sets have applications in many fields such as information [21] and control [22]. The theory of fuzzy topological spaces was introduced and developed by Chang[7] and from then various notions in classical topology have been extended to fuzzy topological spaces[4, 5, 6]. Following this concept K.Atanassov[1,2,3] in 1983 devised the idea of intuitionistic fuzzy set on a universe X as a generalization of fuzzy set. Here besides the degree of membership a degree of non-membership for each element is also defined. The topological framework of intuitionistic fuzzy set was initiated by D.Coker[8].

As a generalization of intuitionistic fuzzy sets neutrosophic set was formulated by Smarandache. Smarandache[16,17,18] originally gave the definition of a neutrosophic set and neutrosophic logic. The neutrosophic logic is a formal frame trying to measure the truth, indeterminacy and falsehood. In 2012 Salama and Alblowi[19,21] introduced the concept of neutrosophic topological spaces. Prem Kumar Singh [14,15] introduced the concept of neutrosophic context analysis at distinct multi-granulation using single valued neutrosophic numbers and also graphical representation of lattices by applying interval valued neutrosophic numbers

The orbit in mathematics has an important role in the study of dynamical systems, an orbit is a collection of points associated by the evolution function of the dynamical system. One of the objectives of the modern theory of dynamical systems is using topological methods to understanding the properties of dynamical systems[12]. The concept of the fuzzy orbit set was introduced by R.Malathi and M.K.Uma[13] in 2017, as a generalization to the concept of the orbit point in general metric space[9]. Also, R.Malathi and M.K.Uma[13] introduced the concept of fuzzy orbit open sets and fuzzy orbit continuous mappings.

In this paper various novel concepts of neutrosophic orbit open set, almost-neutrosophic orbit continuous, weakly-neutrosophic orbit continuous, neutrosophic orbit* continuous are created which paves way to discuss. Some of its interesting properties and characterizations. Also neutrosophic orbit* continuous mappings are discussed with necessary examples and counterexamples.

2. Preliminaries

2.1 Definition [13] Let X be a non empty set. A neutrosophic set (NS for short) A is an object having the form A = <x,A^T,A^I,A^F> where A^T,A^I,A^F represent the degree of membership , the degree of indeterminacy and the degree of non-membership respectively of each element x∈X to the set A.

2.2 Definition [13] Let X be a non empty set, A = <x,A^T,A^I,A^F> and B = <x,B^T,B^I,B^F> be neutrosophic sets on X, and let {A_i : i∈ J} be an arbitrary family of neutrosophic sets in X, where A_i = <x,A^T, A^I, A^F>

(i) A $\subseteq$ B if and only if $A^T \leq B^T$, $A^I \geq B^I$ and $A^F \geq B^F$

(ii) A = B if and only if A $\leq$ B and B $\leq$ A.

(iii) $\overline{A}$ = <x,A^F,1-A^I,A^T>

(iv) A∩B=<x,$A^T \wedge B^T$,$A^I \vee B^I$,$A^F \vee B^F$>

(v) A∪B=<x,$A^T \vee B^T$,$A^I \wedge B^I$,$A^F \wedge B^F$>

(vi) $\cup A_i$ = <x, $\vee A_i^T$, $\wedge A_i^I$, $\wedge A_i^F$>

(vii) $\cap A_i$ = <x, $\wedge A_i^T$, $\vee A_i^I$, $\vee A_i^F$>

(viii) A − B = A $\wedge$ $\overline{B}$.

(ix) 0_N = <x,0,1,1>; 1_N = <x,1,0,0>.

2.3 Definition [18] A neutrosophic topology (NT for short) on a nonempty set X is a family τ of neutrosophic set in X satisfying the following axioms:

(i) 0_N, 1_N∈τ.

(ii) $G_1 \wedge G_2$ ∈ τ for any G_1,G_2∈ τ.

(iii) $\vee G_i$∈ τ for any arbitrary family {G_i :i∈J} ⊆ τ.

In this case the pair (X, τ) is called a neutrosophic topological space (NTS for short) and any neutrosophic set in τ is called a neutrosophic open set(NOS for short) in X. The complement A of a neutrosophic open set A is called a neutrosophic closed set (NCS for short) in X.

2.4 Definition [18] Let (X, τ) be a neutrosophic topological space and A = <X,A^T,A^I,A^F> be a set in X. Then the closure and interior of A are defined by

Ncl(A) = $\wedge${K : K is a neutrosophic closed set in X and A $\leq$ K},

Nint(A) = $\vee${G : G is a neutrosophic open set in X and G $\leq$ A}.

It can be also shown that Ncl(A) is a neutrosophic closed set and Nint(A) is a neutrosophic open set in X, and A is a neutrosophic closed set in X iff Ncl(A) = A; and A is a neutrosophic open set in X iff Nint(A) = A.

2.5 Definition [9] Orbit of a point x in X under the mapping f is $O_f(x)=\{x, f(x), f^2(x),...\}$

2.6 Definition [10] A neutrosophic set A=<x, A^T, A^I, A^F> in a neutrosophic topological space (X, τ) is said to be a neutrosophic neighbourhood of a neutrosophic point $x_{r,t,s}$, x∈X, if there exists a neutrosophic open set B=<x, B^T, B^I, B^F> with $x_{r,t,s} \subseteq B \subseteq A$.

2.7 Corollary [11] Let A, A_i(i∈ J) be neutrosophic sets in X, B, B_i(i∈ K) be neutrosophic sets in Y and f: X → Y a function. Then

(a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$,

(b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(A_1) \subseteq f^{-1}(A_2)$,

(c) $A \subseteq f^{-1}(f(A))$ {If f is injective, then $A= f^{-1}(f(A))$},

(d) $f^{-1}(f(B)) \subseteq B$ {If f is surjective, then $f^{-1}(f(B))= B$},

(e) $\overline{f(A)} \subseteq f(\overline{A})$, if f is surjective,

(f) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

3. Properties and characterization of neutrosophic orbit continuous Mappings

3.1 Definition A neutrosophic set A in a neutrosophic topological space (X, τ) is a neighbourhood of a neutrosophic set B, if there exists a neutrosophic open set O such that $B \subseteq O \subseteq A$.

3.2 Definition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. Let f: (X, τ) → (Y, σ) is said to be almost neutrosophic continuous, if for every neutrosophic set α and every neutrosophic open set μ with $f(\alpha) \leq \mu$, there exists a neutrosophic open set σ with $\alpha \leq \sigma$ such that $f(\sigma) \leq int(cl(\mu))$.

3.3 Definition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. Let f: (X, τ) → (Y, σ) is said to be weakly neutrosophic continuous, if for every neutrosophic set α and every neutrosophic open set μ with $f(\alpha) \leq \mu$, there exists a neutrosophic open set σ with $\alpha \leq \sigma$

such that $f(\sigma) \leq cl(\mu)$.

3.4 Definition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. Let f: (X, τ) → (Y, σ) is said to be slightly neutrosophic continuous, if for every neutrosophic set α and every neutrosophic open set μ with $f(\alpha) \leq \mu$, there exists a neutrosophic open set σ with $\alpha \leq \sigma$ such that $f(\sigma) \leq \mu$.

3.5 Definition Let X be a nonempty set and f : X → X be any mapping. Let α be any neutrosophic set in X. The neutrosophic orbit $O_f(\alpha)$ of α under the mapping f is defined as $O_{fT}(\alpha)$ = { $\alpha, f^1(\alpha), f^2(\alpha), \ldots f^n(\alpha)$}, $O_{fI}(\alpha)$ = { $\alpha, f^1(\alpha), f^2(\alpha), \ldots f^n(\alpha)$} , $O_{fF}(\alpha)$ = { $\alpha, f^1(\alpha), f^2(\alpha), \ldots f^n(\alpha)$} for $\alpha \in X$ and $n \in Z^+$.

3.6 Definition Let X be a nonempty set and let f : X→ X be any mapping. The neutrosophic orbit set of α under the mapping f is defined as $NO_f(\alpha) = < \alpha, O_{fT}(\alpha), O_{fI}(\alpha), O_{fF}(\alpha)>$ for $\alpha \in X$, where $O_{fT}(\alpha)$= { $\alpha \wedge f^1(\alpha) \wedge f^2(\alpha) \wedge \ldots \wedge f^n(\alpha)$}, $O_{fI}(\alpha)$= { $\alpha \vee f^1(\alpha) \vee f^2(\alpha) \vee \ldots \vee f^n(\alpha)$}, $O_{fF}(\alpha)$= { $\alpha \vee f^1(\alpha) \vee f^2(\alpha) \vee \ldots \vee f^n(\alpha)$}.

3.7 Example Let X={a, b, c}. Define a neutrosophic set α where $\alpha^T : X \to \left]^-0, 1^+\right[$

$\alpha^I : X \to \left]^-0, 1^+\right[$

$\alpha^F : X \to \left]^-0, 1^+\right[$ as follows

$\alpha^T(a) = 0.5, \alpha^I(a) = 0.4, \alpha^F(a) = 0.5, \alpha^T(b) = 0.6, \alpha^I(b) = 0.5, \alpha^F(b) = 0.4, \alpha^T(c) = 0.7, \alpha^I(c) = 0.6, \alpha^F(c) = 0.3$

Define f : X→ X as f(a)=b, f(b)=c, f(c)=a. The neutrosophic orbit set of α under the mapping f is defined as $NO_f(\alpha) = \alpha \cap f^1(\alpha) \cap f^2(\alpha) \cap \ldots \cap f^n(\alpha)$

$NO_f(\alpha)(a)$=<x, 0.7, 0.6, 0.3>, $NO_f(\alpha)(b)$=<x, 0.5, 0.4, 0.5>, $NO_f(\alpha)(c)$=<x, 0.6, 0.5, 0.4>

3.8 Definition Let (X, τ) be a neutrosophic topological space. Let f : X → X be any mapping. The neutrosophic orbit set under the mapping f which is in neutrosophic topology τ is called neutrosophic orbit open set under the mapping f. Its complement is called a neutrosophic orbit closed set under the mapping f.

3.9 Example Let X={a, b, c}=Y. Define τ = {0_N, 1_N, α, γ }where $\lambda^T, \gamma^T : X \to \left]^-0, 1^+\right[$

$\alpha^I, \gamma^I : X \to \left]^-0, 1^+\right[$

$\alpha^F, \gamma^F : X \to \left]^-0, 1^+\right[$

are defined as

$\alpha^T(a) = 0.3, \alpha^I(a) = 0.5, \alpha^F(a) = 0.6, \alpha^T(b) = 0.4, \alpha^I(b) = 0.6,$

$\alpha(b) = 0.8, \alpha^T(c) = 0.1,$

$\alpha^I(c) = 0.3, \alpha^F(c) = 0.7$

$\gamma^T(a) = 0.3, \gamma^I(a) = 0.5, \gamma^F(a) = 0.6, \gamma^T(b) = 0, \gamma^I(b) = 0, \gamma^F(b) = 1,$

$\gamma^T(c) = 0, \gamma^I(c) = 0, \gamma^F(c) = 1.$

Define f : X→ X as f(a)=a, f(b)=a, f(c)=a. The neutrosophic orbit set of α under the mapping f is defined as $NO_f(\alpha)$ = $\alpha \cap f^1(\alpha) \cap f^2(\alpha) \cap \ldots \cap f^n(\alpha)$, $NO_f(\alpha) = \gamma$. Then γ is a neutrosophic orbit open set under the mapping f.

3.10 Definition Let (X, τ) be a neutrosophic topological space. Let f : X → X be any mapping. The neutrosophic orbit under the mapping f in a neutrosophic topological space (X, T) is said to be neutrosophic orbit clopen set under the mapping f, if it is both neutrosophic orbit open and neutrosophic orbit closed under the mapping f.

3.11 Definition A neutrosophic set α in a neutrosophic topological space (X, τ) is a neutrosophic orbit neighborhood, or NOnbhd for short, of a neutrosophic set μ, if there exists a neutrosophic orbit open set λ such that $\mu \subset \lambda \subset \alpha$.

3.12 Definition Let (X, τ) be a neutrosophic topological space and α = <X, α^1, α^2, α^3 > be a set in

X. Then the closure and interior of α are defined by

Ncl(α) = $\wedge$\{ β : β is a neutrosophic orbit closed set in X and $\alpha \leq \beta$ \},

Nint(α) = $\vee$\{ β : β is a neutrosophic orbit open set in X and $\beta \leq \alpha$ \}.

3.13 Definition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. Let f: X → X be a mapping. A mapping g : (X, τ) → (Y, σ) is said to be neutrosophic orbit continuous, if the inverse image of every neutrosophic open set in (Y, σ) is neutrosophic orbit open set under the mapping f in (X, τ).

3.14 Proposition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. Let g : (X, τ) → (Y,σ) and f_1 : X → X be any two mappings. Then the following are equivalent

(i) g is neutrosophic orbit continuous mapping

(ii) inverse image of every neutrosophic closed set in (Y, σ) is a neutrosophic orbit closed set under the mapping f_1 in (X, τ).

Proof: (i) $\Rightarrow$ (ii): Assume that g is a neutrosophic orbit continuous mapping. Let λ be any neutrosophic closed set in (Y, σ). Then $1-\lambda$ is a neutrosophic open set in in (Y, σ). Thus by assumption, $g^{-1}(1-\lambda)$ is a neutrosophic orbit open set under the mapping f_1 in (X, τ). Now, $g^{-1}(1-\lambda) = 1-g^{-1}(\lambda)$. So, $g^{-1}(\lambda)$ is a neutrosophic orbit closed set under the mapping f_1 in (X, τ).

(ii) $\Rightarrow$ (i): The proof is similar to (i) $\Rightarrow$ (ii).

3.15 Proposition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. Let g : (X, τ) → (Y,σ) and f_1 : X → X and f_2 : Y → Y be any two mappings. Then the following are equivalent

(i) g is neutrosophic orbit continuous mapping

(ii) for each neutrosophic set λ of X and every neutrosophic neighbourhood λ of

$g(\lambda), g^{-1}(\lambda)$ is a neutrosophic orbit neighbourhood of γ

(iii) for each neutrosophic set λ of X and every neutrosophic neighbourhood λ of $g(\lambda)$, there exists a neutrosophic orbit neighbourhood μ of γ such that $g(\mu) \leq \lambda$.

Proof:(i) $\Rightarrow$ (ii): Let γ be a neutrosophic set of X. Let λ be a neutrosophic neighbourhood of $g(\gamma)$. Then there exists a neutrosophic open set μ such that $g(\gamma) \leq \mu \leq \lambda$. Now $g^{-1}(g(\gamma)) \leq g^{-1}(\mu) \leq g^{-1}(\lambda)$. By hypothesis, $g^{-1}(\mu)$ is a neutrosophic orbit open set under the mapping f_1 in (X, τ). But, $\gamma \leq g^{-1}(g(\gamma))$. Thus $g^{-1}(\lambda)$ is a neutrosophic orbit neighbourhood of γ.

(ii) $\Rightarrow$ (iii): Let γ be a neutrosophic set of X. Let λ be a neutrosophic neighborhood of $g(\gamma)$. By hypothesis, $g^{-1}(\lambda)$ is a neutrosophic orbit neighbourhood of γ in (X, τ) such that $g^{-1}(g(\lambda)) \leq \lambda$.

(iii) $\Rightarrow$ (i): Let γ be a neutrosophic set of X such that $\gamma \leq g^{-1}(\lambda)$. Let λ be a neutrosophic orbit open set under the mapping f_2 in (Y, σ). Since every neutrosophic orbit open set is a neutrosophic neighborhood, λ is a neutrosophic neighbourhood of $g(\delta)$ in (Y, σ). Then by hypothesis, $g^{-1}(\lambda)$ is a neutrosophic orbit neighbourhood of γ in (X, τ). Since every neutrosophic orbit neighborhood set is a neutrosophic orbit open set, $g^{-1}(\lambda)$ is a neutrosophic orbit open set under the mapping f_1 in (X, τ). Thus g is neutrosophic orbit continuous.

3.16 Proposition Let (X, τ), (Y, σ) and (Z, η) be any three neutrosophic topological spaces. Let f_1 : X → X be any mappings. Let g : (X, τ) → (Y,σ) be neutrosophic orbit continuous and h : (Y, σ) → (Z, η) be neutrosophic continuous mappings, then their composition h ∘ g is neutrosophic orbit continuous.

Proof: Let λ be a open set of (Z, η). By Definition, $h^{-1}(\lambda)$ is a neutrosophic open set of (Y, σ). Since f is neutrosophic orbit continuous, $g^{-1}(h^{-1}(\lambda))$ is a neutrosophic orbit open set under the mapping f_1 of (X, τ). But $g^{-1}(h^{-1}(\lambda)) = (h \circ g)^{-1}(\lambda)$. Then $h \circ g$ is neutrosophic orbit continuous.

3.17 Definition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. Let $f_1 : X \rightarrow X$ and $f_2 : Y \rightarrow Y$ be any two mappings. A mapping g : (X, τ) → (Y, σ) is said to be neutrosophic orbit* continuous, if for every neutrosophic set α and every neutrosophic orbit open set μ under the mapping f_2 with g(α) $\leq$ μ, there exists a neutrosophic orbit open set λ under the mapping f_1 with $\alpha \leq \lambda$ such that g(λ) $\leq \mu$.

3.18 Example Let X={a, b, c}=Y. Define τ = {0_N, 1_N, λ, λ_1 }and σ = {0_N, 1_N, μ, μ_1 } where

$\lambda^T, {\lambda_1}^T, \mu^T, {\mu_1}^T : X \rightarrow \left]^-0, 1^+\right[$

$\lambda^I, {\lambda_1}^I, \mu^I, {\mu_1}^I : X \rightarrow \left]^-0, 1^+\right[$

$\lambda^F, {\lambda_1}^F, \mu^F, {\mu_1}^F : X \rightarrow \left]^-0, 1^+\right[$

are such that

$\lambda^T(a) = 0, \lambda^I(a) = 1, \lambda^F(a) = 1, \lambda^T(b) = 0, \lambda^I(b) = 1, \lambda^F(b) = 1, \lambda^T(c) = 0.6,$

$\lambda^I(c) = 0.5, \lambda^F(c) = 0.4$

${\lambda_1}^T(a) = 0.5, {\lambda_1}^I(a) = 0.4, {\lambda_1}^F(a) = 0.3, {\lambda_1}^T(b) = 0.7, {\lambda_1}^I(b) = 0.5, {\lambda_1}^F(b) = 0.6,$

${\lambda_1}^T(c) = 0.6, {\lambda_1}^I(c) = 0.5, {\lambda_1}^F(c) = 0.4$

$\mu^T(a) = 0.7, \mu^I(a) = 0.6, \mu^F(a) = 0.5, \mu^T(b) = 0.7, \mu^I(b) = 0.6, \mu^F(b) = 0.5,$

$\mu^T(c) = 0.7, \mu^I(c) = 0.6, \mu^F(c) = 0.5$

${\mu_1}^T(a) = 0.9, {\mu_1}^I(a) = 0.5, {\mu_1}^F(a) = 0.4, {\mu_1}^T(b) = 0.7, {\mu_1}^I(b) = 0.4, {\mu_1}^F(b) = 0.3,$

${\mu_1}^T(c) = 0.8, {\mu_1}^I(c) = 0.4, {\mu_1}^F(c) = 0.2$

Clearly (X, τ) and (Y, σ) are neutrosophic topological spaces.

Define g : (X, τ) → (Y, σ), f_1 : X → X and f_2 : Y→ Y as g(a) = b, g(b) = c, g(c) =a, f_1(a) = c, f_1(b) =c, f_1(c) = c and f_2(a) = b, f_2(b) =c, f_2(c) = a.

Let $\alpha^T : X \to \left]^-0, 1^+\right[$

$\alpha^I : X \to \left]^-0, 1^+\right[$

$\alpha^F : X \to \left]^-0, 1^+\right[$ be any neutrosophic set such that

$$\alpha^T(a) = 0, \alpha^I(a) = 0.8, \alpha^F(a) = 0.9, \alpha^T(b) = 0, \alpha^I(b) = 1, \alpha^F(b) = 1, \alpha^T(c) = 0, \alpha^I(c) = 1, \alpha^F(c) = 1$$

For the neutrosophic orbit open set μ under the mapping f_2 in (Y, σ), g(α) $\leq \mu$. Now, λ is a neutrosophic orbit open set under the mapping f_1 in (X, τ) with $\alpha \leq \lambda$ such that g(λ) $\leq \mu$. Hence g is neutrosophic orbit* continuous.

3.19 Proposition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. Let g : (X, τ) → (Y,σ) be a mappings. Then the following are equivalent

(i) g is neutrosophic orbit* continuous.

(ii) inverse image of every neutrosophic orbit open set of (Y, σ) is neutrosophic orbit open set of (X, τ).

(iii) inverse image of every neutrosophic orbit clopen set of (Y, σ) is neutrosophic orbit open set of (X, τ).

Proof:(i) ⇒ (ii): Let f_1 : X → X and f_2 : Y → Y be any two mappings. Let μ be a neutrosophic orbit open set under the mapping f_2 of (Y, σ) and any neutrosophic set α with g(α) $\leq \mu$. Since g is neutrosophic orbit* continuous, there exists a neutrosophic orbit open set λ under the mapping f_1 of (X, τ) with $\alpha \leq \lambda$ such that g(λ) $\leq \mu$. Hence $g^{-1}(\mu)$ is a neutrosophic orbit open set.

(ii) $\Rightarrow$ (iii): Let μ be a neutrosophic orbit open set under the mapping f_2 of (Y, σ). By (ii) $g^{-1}(\mu)$ is a neutrosophic orbit open set under the mapping f_1 of (X, τ). Now $1-\mu$ is also neutrosophic orbit clopen set. By (ii) $g^{-1}(1-\mu)$ is neutrosophic orbit open set under the mapping f_1 in (X, τ). So $1-g^{-1}(1-\mu)$ is neutrosophic orbit closed set under the mapping f_1 in (X, τ). This implies that $g^{-1}(\mu)$ is neutrosophic orbit closed. Therefore, $g^{-1}(\mu)$ is a neutrosophic orbit open set clopen set in (X, τ).

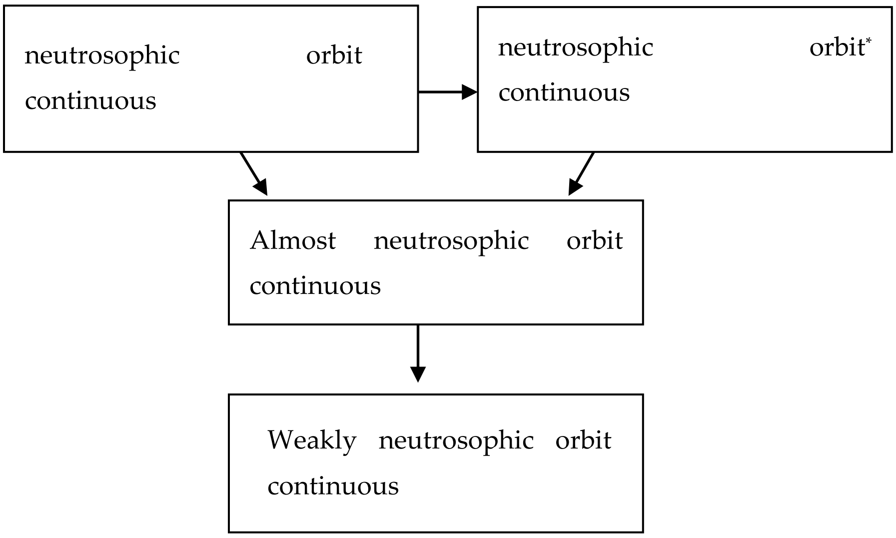
(iii) $\Rightarrow$ (i): Let μ be a neutrosophic orbit clopen set under the mapping f_2 and any fuzzy set α with g(α) $\leq$ μ. Now $g^{-1}(\mu)$ is neutrosophic orbit open set under the mapping f_1 of (X, τ) and $g(g^{-1}(\mu)) \leq \mu$. Hence, g is neutrosophic orbit* continuous.

3.20 Definition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. Let $f_1 : X \to X$ and $f_2 : Y \to Y$ be an y two mappings. A mapping g : (X, τ) → (Y, σ) is said to be almost-neutrosophic orbit continuous, if for every neutrosophic set α and every neutrosophic orbit open set μ under the mapping f_2 with g(α) $\leq$ μ, there exists a neutrosophic orbit open set λ under the mapping f_1 with $\alpha \leq \lambda$ such that g(λ) $\leq$ int (cl(μ)).

3.21 Definition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. Let $f_1 : X \to X$ and $f_2 : Y \to Y$ be any two mappings. A mapping g : (X, τ) → (Y, σ) is said to be weakly-neutrosophic orbit continuous, if for every neutrosophic set α and every neutrosophic orbit open set μ under the mapping f_2 with g(α) $\leq$ μ, there exists a neutrosophic orbit open set λ under the mapping f_1 with $\alpha \leq \lambda$ such that g(λ) $\leq$ cl(μ).

3.22 Remark

Figure 1.



3.23 Proposition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. If g : (X, τ) → (Y,σ) is neutrosophic orbit continuous, then g is almost neutrosophic orbit continuous.

Proof. Let $f_1 : X \to X$ and $f_2 : Y \to Y$ be any two mappings. Let α be any neutrosophic set and μ be any neutrosophic orbit open set under the mapping f_2 with $g(\alpha) \leq \mu$. By Corollary 2.7, $\alpha \leq g^{-1}(g(\alpha)) \leq g^{-1}(\mu)$. Then $\leq g^{-1}(\mu)$. Since g is neutrosophic orbit continuous, $\alpha \leq g^{-1}(\mu) = \lambda, \lambda$ is a neutrosophic orbit open set under the mapping f_1. By Corollary 2.7, $gg^{-1}(\mu) \leq \mu$. Thus g(λ) = $gg^{-1}(\mu) \leq \mu$. Since μ is neutrosophic orbit open, μ is neutrosophic open and hence $\mu \leq$ int(cl(μ)) which implies that g(λ) $\leq$ int(cl(μ)). So g is almost-neutrosophic orbit continuous.

3.24 Remark The converse of the Proposition 3.15 need not be true as shown in the following example.

3.25 Example Let X={a, b, c}=Y. Define τ = {0_N, 1_N, λ, λ_1 }and σ= {0_N, 1_N, μ, μ_1, μ_2, μ_3} where $\lambda^T, {\lambda_1}^T, \mu^T, {\mu_1}^T, {\mu_2}^T, {\mu_3}^T : X \to \left]^-0, 1^+\right[$

$\lambda^I, \lambda_1{}^I, \mu^I, \mu_1{}^I, \mu_2{}^I, \mu_3{}^I : X \to \left]^-0, 1^+\right[$

$\lambda^F, \lambda_1{}^F, \mu^F, \mu_1{}^F, \mu_2{}^F, \mu_3{}^F : X \to \left]^-0, 1^+\right[$

are such that

$\lambda^T(a) = 0, \lambda^I(a) = 1, \lambda^F(a) = 1, \lambda^T(b) = 0, \lambda^I(b) = 1, \lambda^F(b) = 1, \lambda^T(c) = 0.3,$

$\lambda^I(c) = 0.4, \lambda^F(c) = 0.4$

$\lambda_1{}^T(a) = 0.7, \lambda_1{}^I(a) = 0.4, \lambda_1{}^F(a) = 0.5, \lambda_1{}^T(b) = 0.6, \lambda_1{}^I(b) = 0.3, \lambda_1{}^F(b) = 0.5,$

$\lambda_1{}^T(c) = 0.3, \lambda_1{}^I(c) = 0.4, \lambda_1{}^F(c) = 0.4$

$\mu^T(a) = 0.6, \mu^I(a) = 0.5, \mu^F(a) = 0.4, \mu^T(b) = 0.6, \mu^I(b) = 0.5, \mu^F(b) = 0.4,$

$\mu^T(c) = 0.6, \mu^I(c) = 0.5, \mu^F(c) = 0.4$

$\mu_1{}^T(a) = 0.6, \mu_1{}^I(a) = 0.5, \mu_1{}^F(a) = 0.3, \mu_1{}^T(b) = 0.7, \mu_1{}^I(b) = 0.2, \mu_1{}^F(b) = 0.3,$

$\mu_1{}^T(c) = 0.8, \mu_1{}^I(c) = 0.2, \mu_1{}^F(c) = 0.1$

$\mu_2{}^T(a) = 0.3, \mu_2{}^I(a) = 0.5, \mu_2{}^F(a) = 0.4, \mu_2{}^T(b) = 0.3, \mu_2{}^I(b) = 0.5, \mu_2{}^F(b) = 0.4,$

$\mu_2{}^T(c) = 0.3, \mu_2{}^I(c) = 0.5, \mu_2{}^F(c) = 0.4$

$\mu_3{}^T(a) = 0.3, \mu_3{}^I(a) = 0.5, \mu_3{}^F(a) = 0.4, \mu_3{}^T(b) = 0.4, \mu_3{}^I(b) = 0.5, \mu_3{}^F(b) = 0.4,$

$\mu_3{}^T(c) = 0.5, \mu_3{}^I(c) = 0.5, \mu_3{}^F(c) = 0.4$

Clearly (X, τ) and (Y, σ) are neutrosophic topological spaces.

Define $g : (X, \tau) \to (Y, \sigma)$, $f_1 : X \to X$ and $f_2 : Y \to Y$ as g(a) = b, g(b) = c, g(c) =a, f_1(a) = c, f_1(b) =c, f_1(c) = c and f_2(a) = b, f_2(b) =c, f_2(c) = a.

Let $\alpha^T : X \to \left]^-0, 1^+\right[$

$\alpha^I : X \to \left]^-0, 1^+\right[$

$\alpha^F : X \to \left]^-0, 1^+\right[$ be any neutrosophic set such that

$\alpha^T(a) = 0, \alpha^I(a) = 1, \alpha^F(a) = 1, \alpha^T(b) = 0, \alpha^I(b) = 1, \alpha^F(b) = 1, \alpha^T(c) = 0.2, \alpha^I(c) = 0.5, \alpha^F(c) = 0.8$

For the neutrosophic orbit open set μ under the mapping f_2 in (Y, σ) with g(α) $\leq$ μ.

Now the neutrosophic orbit open set λ under the mapping f_1 in (X, τ) with $\alpha \leq \lambda$ such that g(λ) $\leq$ int (cl(μ)). Then g is almost neutrosophic orbit continuous.

Now the neutrosophic open sets μ_1, μ_2 and μ_3 in (Y, σ), but $g^{-1}(\mu_1), g^{-1}(\mu_2)$ and $g^{-1}(\mu_3)$ are not neutrosophic orbit open under the mapping f_1 in (X, τ). Thus g is not neutrosophic orbit continuous.

3.26 Proposition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. If g : (X, τ) $\to$ (Y,σ) is neutrosophic orbit continuous, then g is weakly neutrosophic orbit continuous.

Proof: Let $f_1 : X \to X$ and $f_2 : Y \to Y$ be any two mappings. Let α be any neutrosophic set and μ be any neutrosophic orbit open set under the mapping f_2 with $g(\alpha) \leq \mu$. By Corollary 2.7, $\alpha \leq g^{-1}(g(\alpha)) \leq g^{-1}(\mu)$. Then $\alpha \leq g^{-1}(\mu)$. Since g is neutrosophic orbit continuous, $\alpha \leq g^{-1}(\mu) = \lambda, \lambda$ is a neutrosophic orbit open set under the mapping f_1. By Corollary 2.7, $gg^{-1}(\mu) \leq \mu$. Thus g(λ) = $gg^{-1}(\mu) \leq cl(\mu)$. So g is weakly neutrosophic orbit continuous.

3.27 Remark The converse of the Proposition 3.18 need not be true as shown in the following example.

3.28 Example Let X={a, b, c}=Y. Define τ = {0_N, 1_N, λ, λ_1 }and σ= {0_N, 1_N, μ, μ_1, μ_2, μ_3} where

$\lambda^T, {\lambda_1}^T, \mu^T, {\mu_1}^T, {\mu_2}^T, {\mu_3}^T : X \to \left]^-0, 1^+\right[$

$\lambda^I, {\lambda_1}^I, \mu^I, {\mu_1}^I, {\mu_2}^I, {\mu_3}^I : X \to \left]^-0, 1^+\right[$

$\lambda^F, {\lambda_1}^F, \mu^F, {\mu_1}^F, {\mu_2}^F, {\mu_3}^F : X \to \left]^-0, 1^+\right[$

are such that

$\lambda^T(a) = 0, \lambda^I(a) = 1, \lambda^F(a) = 1, \lambda^T(b) = 0.5, \lambda^I(b) = 0.4, \lambda^F(b) = 0.6, \lambda^T(c) = 0,$

$\lambda^I(c) = 1, \lambda^F(c) = 1$

${\lambda_1}^T(a) = 0.7, {\lambda_1}^I(a) = 0.4, {\lambda_1}^F(a) = 0.3, {\lambda_1}^T(b) = 0.5, {\lambda_1}^I(b) = 0.4, {\lambda_1}^F(b) = 0.6,$

${\lambda_1}^T(c) = 0.4, {\lambda_1}^I(c) = 0.2, {\lambda_1}^F(c) = 0.3$

$\mu^T(a) = 0.3, \mu^I(a) = 0.4, \mu^F(a) = 0.5, \mu^T(b) = 0.3, \mu^I(b) = 0.4, \mu^F(b) = 0.5,$

$\mu^T(c) = 0.3, \mu^I(c) = 0.4, \mu^F(c) = 0.5$

${\mu_1}^T(a) = 0.6, {\mu_1}^I(a) = 0.3, {\mu_1}^F(a) = 0.4, {\mu_1}^T(b) = 0.8, {\mu_1}^I(b) = 0.2, {\mu_1}^F(b) = 0.4,$

${\mu_1}^T(c) = 0.9, {\mu_1}^I(c) = 0.4, {\mu_1}^F(c) = 0.3$

${\mu_2}^T(a) = 0.6, {\mu_2}^I(a) = 0.3, {\mu_2}^F(a) = 0.5, {\mu_2}^T(b) = 0.6, {\mu_2}^I(b) = 0.3, {\mu_2}^F(b) = 0.4,$

${\mu_2}^T(c) = 0.6, {\mu_2}^I(c) = 0.4, {\mu_2}^F(c) = 0.3$

${\mu_3}^T(a) = 0.3, {\mu_3}^I(a) = 0.4, {\mu_3}^F(a) = 0.5, {\mu_3}^T(b) = 0.4, {\mu_3}^I(b) = 0.3, {\mu_3}^F(b) = 0.4,$

${\mu_3}^T(c) = 0.5, {\mu_3}^I(c) = 0.4, {\mu_3}^F(c) = 0.5$

Clearly (X, τ) and (Y, σ) are neutrosophic topological spaces.

Define g : (X, τ) → (Y, σ), f_1 : X → X and f_2 : Y→ Y as g(a) = b, g(b) = c, g(c) =a, f_1(a) = b, f_1(b) =b, f_1(c) = b and f_2(a) = b, f_2(b) =c, f_2(c) = a.

Let $\alpha^T : X \to \left]^-0, 1^+\right[$

$\alpha^I : X \to \left]^-0, 1^+\right[$

$\alpha^F : X \to \left]^-0, 1^+\right[$ be any neutrosophic set such that

$\alpha^T(a) = 0, \alpha^I(a) = 1, \alpha^F(a) = 1, \alpha^T(b) = 0, \alpha^I(b) = 0.6, \alpha^F(b) = 0.8, \alpha^T(c) = 0, \alpha^I(c) = 1, \alpha^F(c) = 1$

For the neutrosophic orbit open set μ under the mapping f_2 in (Y, σ) with g(α) $\leq$ μ.

Now the neutrosophic orbit open set λ under the mapping f_1 in (X, τ) with $\alpha \leq \lambda$ such that g(λ) $\leq$ cl(μ). Then g is weakly neutrosophic orbit continuous.

Now the neutrosophic open sets μ_1, μ_2 and μ_3 in (Y, σ), but $g^{-1}(\mu_1), g^{-1}(\mu_2)$ and $g^{-1}(\mu_3)$ are not neutrosophic orbit open under the mapping f_1 in (X, τ). Thus g is not neutrosophic orbit continuous.

3.29 Proposition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. If g : (X, τ) → (Y,σ) is almost neutrosophic orbit continuous, then g is weakly neutrosophic orbit continuous.

Proof. Let f_1 : X → X and f_2 : Y→ Y be any two mappings. Let α be any neutrosophic set and μ be any neutrosophic orbit open set under the mapping f_2 with $g(\alpha) \leq \mu$. Since g is almost neutrosophic orbit continuous, there exists a neutrosophic orbit open set λ under the mapping f_1 with $\alpha \leq \lambda$ such that g(λ) $\leq$ int(cl(μ)), which implies that g(λ) $\leq$ cl(μ). Then g is weakly neutrosophic orbit continuous.

3.30 Remark The converse of the Proposition 3.21 need not be true as shown in the following example.

3.31 Example Let X={a, b, c}=Y. Define τ = {0_N, 1_N, λ, λ_1 }and σ= {0_N, 1_N, μ, μ_1, μ_2, μ_3} where

$$\lambda^T, {\lambda_1}^T, \mu^T, {\mu_1}^T, {\mu_2}^T, {\mu_3}^T : X \to \left]^-0, 1^+\right[$$

$$\lambda^I, {\lambda_1}^I, \mu^I, {\mu_1}^I, {\mu_2}^I, {\mu_3}^I : X \to \left]^-0, 1^+\right[$$

$$\lambda^F, {\lambda_1}^F, \mu^F, {\mu_1}^F, {\mu_2}^F, {\mu_3}^F : X \to \left]^-0, 1^+\right[$$

are such that

$$\lambda^T(a) = 0, \lambda^I(a) = 1, \lambda^F(a) = 1, \lambda^T(b) = 0.4, \lambda^I(b) = 0.4, \lambda^F(b) = 0.6, \lambda^T(c) = 0,$$

$$\lambda^I(c) = 1, \lambda^F(c) = 1$$

$${\lambda_1}^T(a) = 0.6, {\lambda_1}^I(a) = 0.4, {\lambda_1}^F(a) = 0.3, {\lambda_1}^T(b) = 0.4, {\lambda_1}^I(b) = 0.4, {\lambda_1}^F(b) = 0.6,$$

$${\lambda_1}^T = 0.4, {\lambda_1}^I(c) = 0.6, {\lambda_1}^F(c) = 0.6$$

$\mu^T(a) = 0.3, \mu^I(a) = 0.6, \mu^F(a) = 0.4, \mu^T(b) = 0.3, \mu^I(b) = 0.6, \mu^F(b) = 0.4,$

$\mu^T(c) = 0.3, \mu^I(c) = 0.6, \mu^F(c) = 0.4$

${\mu_1}^T(a) = 0.3, {\mu_1}^I(a) = 0.5, {\mu_1}^F(a) = 0.4, {\mu_1}^T(b) = 0.4, {\mu_1}^I(b) = 0.4, {\mu_1}^F(b) = 0.3,$

${\mu_1}^T(c) = 0.5, {\mu_1}^I(c) = 0.4, {\mu_1}^F(c) = 0.2$

${\mu_2}^T(a) = 0.6, {\mu_2}^I(a) = 0.5, {\mu_2}^F(a) = 0.3, {\mu_2}^T(b) = 0.6, {\mu_2}^I(b) = 0.3, {\mu_2}^F(b) = 0.2,$

${\mu_2}^T(c) = 0.6, {\mu_2}^I(c) = 0.4, {\mu_2}^F(c) = 0.1$

${\mu_3}^T(a) = 0.6, {\mu_3}^I(a) = 0.4, {\mu_3}^F(a) = 0.2, {\mu_3}^T(b) = 0.8, {\mu_3}^I(b) = 0.3, {\mu_3}^F(b) = 0.1,$

${\mu_3}^T(c) = 0.9, {\mu_3}^I(c) = 0.4, {\mu_3}^F(c) = 0.1$

Clearly (X, τ) and (Y, σ) are neutrosophic topological spaces.

Define g : (X, τ) → (Y, σ), f_1 : X → X and f_2 : Y→ Y as g(a) = b, g(b) = c, g(c) =a, f_1(a) = b, f_1(b) =b, f_1(c) = b and f_2(a) = b, f_2(b) =c, f_2(c) = a.

Let $\alpha^T : X \to \left]^-0, 1^+\right[$

$\alpha^I : X \to \left]^-0, 1^+\right[$

$\alpha^F : X \to \left]^-0, 1^+\right[$ be any neutrosophic set such that

$\alpha^T(a) = 0, \alpha^I(a) = 1, \alpha^F(a) = 1, \alpha^T(b) = 0.3, \alpha^I(b) = 0.7, \alpha^F(b) = 0.8, \alpha^T(c) = 0, \alpha^I(c) = 1, \alpha^F(c) = 1$

For the neutrosophic orbit open set μ under the mapping f_2 in (Y, σ) with g(α) $\leq$ μ.

Now the neutrosophic orbit open set λ under the mapping f_1 in (X, τ) with $\alpha \leq \lambda$ such that g(λ) $\leq$ cl(μ). Then g is weakly neutrosophic orbit continuous.

Now, g(α) $\leq$ μ. But there is no neutrosophic orbit open set λ under the mapping f_1 with $\alpha \leq \lambda$ such that g(λ) $\nleq$ int(cl(μ)).Thus g is not almost neutrosophic orbit continuous.

3.32 Proposition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. If g : (X, τ) → (Y,σ) is neutrosophic orbit* continuous, then g is almost neutrosophic orbit continuous.

Proof. Let f_1 : X → X and f_2 : Y→ Y be any two mappings. Let α be any neutrosophic set and μ be any neutrosophic orbit open set under the mapping f_2 with $g(\alpha) \leq \mu$. Since g is neutrosophic orbit* continuous, there exists a neutrosophic orbit open set λ under the mapping f_1 with $\alpha \leq \lambda$ such that g(λ) $\leq \mu$. Since μ is neutrosophic open, which implies that g(λ) $\leq$ int(cl(μ)). Then g is almost neutrosophic orbit continuous.

3.33 Remark The converse of the Proposition 3.24 need not be true as shown in the following example.

3.34 Example Let X={a, b, c}=Y. Define τ = {0_N, 1_N, λ, λ_1 }and σ= {0_N, 1_N, μ, μ_1} where

$$\lambda^T, {\lambda_1}^T, \mu^T, {\mu_1}^T : X \to \left]^-0, 1^+\right[$$

$$\lambda^I, {\lambda_1}^I, \mu^I, {\mu_1}^I : X \to \left]^-0, 1^+\right[$$

$$\lambda^F, {\lambda_1}^F, \mu^F, {\mu_1}^F : X \to \left]^-0, 1^+\right[$$

are such that

$$\lambda^T(a) = 0.4, \lambda^I(a) = 0.5, \lambda^F(a) = 0.5, \lambda^T(b) = 0.4, \lambda^I(b) = 0.5, \lambda^F(b) = 0.5, \lambda^T(c) = 0.4,$$

$$\lambda^I(c) = 0.5, \lambda^F(c) = 0.5$$

$${\lambda_1}^T(a) = 0.4, {\lambda_1}^I(a) = 0.4, {\lambda_1}^F(a) = 0.5, {\lambda_1}^T(b) = 0.5, {\lambda_1}^I(b) = 0.5, {\lambda_1}^F(b) = 0.4,$$

$${\lambda_1}^T = 0.6, {\lambda_1}^I(c) = 0.5, {\lambda_1}^F(c) = 0.4$$

$$\mu^T(a) = 0.3, \mu^I(a) = 0.5, \mu^F(a) = 0.6, \mu^T(b) = 0.3, \mu^I(b) = 0.5, \mu^F(b) = 0.6,$$

$$\mu^T(c) = 0.3, \mu^I(c) = 0.5, \mu^F(c) = 0.6$$

$${\mu_1}^T(a) = 0.4, {\mu_1}^I(a) = 0.5, {\mu_1}^F(a) = 0.5, {\mu_1}^T(b) = 0.4, {\mu_1}^I(b) = 0.5, {\mu_1}^F(b) = 0.5,$$

$${\mu_1}^T(c) = 0.5, {\mu_1}^I(c) = 0.5, {\mu_1}^F(c) = 0.5$$

Clearly (X,τ) and (Y,σ) are neutrosophic topological spaces.

Define g : (X, τ) → (Y, σ), f_1 : X → X and f_2 : Y→ Y as g(a) = b, g(b) = c, g(c) =a, f_1(a) = b, f_1(b) =c, f_1(c) = a and f_2(a) = b, f_2(b) =c, f_2(c) = a.

Let $\alpha^T : X \to \left]^{-}0, 1^{+}\right[$

$\alpha^I : X \to \left]^{-}0, 1^{+}\right[$

$\alpha^F : X \to \left]^{-}0, 1^{+}\right[$ be any neutrosophic set such that

$\alpha^T(a) = 0.2, \alpha^I(a) = 0.6, \alpha^F(a) = 0.8, \alpha^T(b) = 0.2, \alpha^I(b) = 0.6, \alpha^F(b) = 0.6, \alpha^T(c) = 0.2, \alpha^I(c) = 0.6, \alpha^F(c) = 0.7$

For the neutrosophic orbit open set μ under the mapping f_2 in (Y, σ) with g(α) $\leq$ μ.

Now the neutrosophic orbit open set λ under the mapping f_1 in (X, τ) with $\alpha \leq \lambda$ such that g(λ) $\leq$ int(cl(μ)). Then g is almost neutrosophic orbit continuous.

Now, g(α) $\leq$ μ. But there is no neutrosophic orbit open set λ under the mapping f_1 with $\alpha \leq \lambda$ such that g(λ) $\nleq$ μ.Thus g is not neutrosophic orbit* continuous.

3.35 Proposition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. If g : (X, τ) → (Y,σ) is neutrosophic orbit* continuous, then g is weakly neutrosophic orbit continuous.

Proof. Let f_1 : X → X and f_2 : Y→ Y be any two mappings. Let α be any neutrosophic set and μ be any neutrosophic orbit open set under the mapping f_2 with $g(\alpha) \leq \mu$. Since g is neutrosophic orbit* continuous, there exists a neutrosophic orbit open set λ under the mapping f_1 with $\alpha \leq \lambda$ such that g(λ) $\leq$ μ, which implies that g(λ) $\leq$ cl(μ). Then g is weakly neutrosophic orbit continuous.

3.36 Remark The converse of the Proposition 3.27 need not be true as shown in the following example.

3.37 Example Let X={a, b, c}=Y. Define τ = {0_N, 1_N, λ, λ_1 }and σ = {0_N, 1_N, μ, μ_1 } where

$\lambda^T, {\lambda_1}^T, \mu^T, {\mu_1}^T : X \to \left]^-0, 1^+\right[$

$\lambda^I, {\lambda_1}^I, \mu^I, {\mu_1}^I : X \to \left]^-0, 1^+\right[$

$\lambda^F, {\lambda_1}^F, \mu^F, {\mu_1}^F : X \to \left]^-0, 1^+\right[$

are such that

$\lambda^T(a) = 0, \lambda^I(a) = 1, \lambda^F(a) = 1, \lambda^T(b) = 0.7, \lambda^I(b) = 0.7, \lambda^F(b) = 0.7, \lambda^T(c) = 0,$

$\lambda^I(c) = 1, \lambda^F(c) = 1$

${\lambda_1}^T(a) = 0.6, {\lambda_1}^I(a) = 0.3, {\lambda_1}^F(a) = 0.3, {\lambda_1}^T(b) = 0.7, {\lambda_1}^I(b) = 0.7, {\lambda_1}^F(b) = 0.7,$

${\lambda_1}^T = 0.4, {\lambda_1}^I(c) = 0.1, {\lambda_1}^F(c) = 0.4$

$\mu^T(a) = 0.6, \mu^I(a) = 0.5, \mu^F(a) = 0.4, \mu^T(b) = 0.6, \mu^I(b) = 0.5, \mu^F(b) = 0.4,$

$\mu^T(c) = 0.6, \mu^I(c) = 0.5, \mu^F(c) = 0.4$

${\mu_1}^T(a) = 0.6, {\mu_1}^I(a) = 0.4, {\mu_1}^F(a) = 0.3, {\mu_1}^T(b) = 0.7, {\mu_1}^I(b) = 0.5, {\mu_1}^F(b) = 0.2,$

${\mu_1}^T(c) = 0.8, {\mu_1}^I(c) = 0.4, {\mu_1}^F(c) = 0.4$

Clearly (X, τ) and (Y, σ) are neutrosophic topological spaces.

Define g : (X, τ) → (Y, σ), f_1 : X → X and f_2 : Y→ Y as g(a) = b, g(b) = c, g(c) =a, f_1(a) = b, f_1(b) =b, f_1(c) = b and f_2(a) = b, f_2(b) =c, f_2(c) = a.

Let $\alpha^T : X \to \left]^-0, 1^+\right[$

$\alpha^I : X \to \left]^-0, 1^+\right[$

$\alpha^F : X \to \left]^-0, 1^+\right[$ be any neutrosophic set such that

$\alpha^T(a) = 0, \alpha^I(a) = 1, \alpha^F(a) = 1, \alpha^T(b) = 0.3, \alpha^I(b) = 0.8, \alpha^F(b) = 0.8, \alpha^T(c) = 0, \alpha^I(c) = 1, \alpha^F(c) = 1$

For the neutrosophic orbit open set μ under the mapping f_2 in (Y, σ) with g(α) $\leq$ μ.

Now the neutrosophic orbit open set λ under the mapping f_1 in (X, τ) with $\alpha \leq \lambda$ such that g(λ) $\leq$ cl(μ). Then g is weakly neutrosophic orbit continuous.

Now, g(α) $\leq \mu$. But there is no neutrosophic orbit open set λ under the mapping f_1 with $\alpha \leq \lambda$ such that g(λ) $\nleq \mu$.Thus g is not neutrosophic orbit* continuous.

3.38 Proposition Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces. If g : (X, τ) → (Y,σ) is neutrosophic orbit continuous, then g is neutrosophic orbit* continuous.

Proof: Let $f_1 : X \to X$ and $f_2 : Y \to Y$ be any two mappings. Let α be any neutrosophic set and μ be any neutrosophic orbit open set under the mapping f_2 with $g(\alpha) \leq \mu$. By Corollary 2.7, $\alpha \leq g^{-1}(g(\alpha)) \leq g^{-1}(\mu)$. Then $\alpha \leq g^{-1}(\mu)$. Since g is neutrosophic orbit continuous, $\alpha \leq g^{-1}(\mu) = \lambda, \lambda$ is a neutrosophic orbit open set under the mapping f_1. By Corollary 2.7, $gg^{-1}(\mu) \leq \mu$. Therefore g(λ) = $gg^{-1}(\mu) \leq \mu$ which implies that g(λ) $\leq \mu$. Then g is neutrosophic orbit* continuous.

3.39 Remark The converse of the Proposition 3.30 need not be true as shown in the following example.

3.40 Example Let X={a, b, c}=Y. Define τ = {0_N, 1_N, λ, λ_1 }and σ = {0_N, 1_N, μ, μ_1 } where

$\lambda^T, {\lambda_1}^T, \mu^T, {\mu_1}^T : X \to \left]^-0, 1^+\right[$

$\lambda^I, {\lambda_1}^I, \mu^I, {\mu_1}^I : X \to \left]^-0, 1^+\right[$

$\lambda^F, {\lambda_1}^F, \mu^F, {\mu_1}^F : X \to \left]^-0, 1^+\right[$

are such that

$\lambda^T(a) = 0, \lambda^I(a) = 1, \lambda^F(a) = 1, \lambda^T(b) = 0, \lambda^I(b) = 1, \lambda^F(b) = 1, \lambda^T(c) = 0.5,$

$\lambda^I(c) = 0.6, \lambda^F(c) = 0.8$

${\lambda_1}^T(a) = 0.4, {\lambda_1}^I(a) = 0.5, {\lambda_1}^F(a) = 0.7, {\lambda_1}^T(b) = 0.6, {\lambda_1}^I(b) = 0.4, {\lambda_1}^F(b) = 0.3,$

${\lambda_1}^T = 0.5, {\lambda_1}^I(c) = 0.6, {\lambda_1}^F(c) = 0.4$

$\mu^T(a) = 0.6, \mu^I(a) = 0.5, \mu^F(a) = 0.4, \mu^T(b) = 0.6, \mu^I(b) = 0.5, \mu^F(b) = 0.4,$

$\mu^T(c) = 0.6, \mu^I(c) = 0.5, \mu^F(c) = 0.4$

${\mu_1}^T(a) = 0.6, {\mu_1}^I(a) = 0.4, {\mu_1}^F(a) = 0.3, {\mu_1}^T(b) = 0.7, {\mu_1}^I(b) = 0.5, {\mu_1}^F(b) = 0.2,$

${\mu_1}^T(c) = 0.8, {\mu_1}^I(c) = 0.4, {\mu_1}^F(c) = 0.4$

Clearly (X, τ) and (Y, σ) are neutrosophic topological spaces.

Define g : (X, τ) → (Y, σ), f_1 : X → X and f_2 : Y→ Y as g(a) = b, g(b) = c, g(c) =a, f_1(a) = c, f_1(b) =c, f_1(c) = c and f_2(a) = b, f_2(b) =c, f_2(c) = a.

Let $\alpha^T : X \rightarrow \left]^-0, 1^+\right[$

$\alpha^I : X \rightarrow \left]^-0, 1^+\right[$

$\alpha^F : X \rightarrow \left]^-0, 1^+\right[$ be any neutrosophic set such that

$\alpha^T(a) = 0, \alpha^I(a) = 1, \alpha^F(a) = 1, \alpha^T(b) = 0, \alpha^I(b) = 1, \alpha^F(b) = 1, \alpha^T(c) = 0.2, \alpha^I(c) = 0.8, \alpha^F(c) = 0.9$

For the neutrosophic orbit open set μ under the mapping f_2 in (Y, σ), g(α) $\leq \mu$.

Now the neutrosophic orbit open set λ under the mapping f_1 in (X, τ) with $\alpha \leq \lambda$ such that g(λ) $\leq \mu$. Then g is neutrosophic orbit* continuous.

Now the neutrosophic open sets μ_1 in (Y, σ), but $g^{-1}(\mu_1)$ is not neutrosophic orbit open under the mapping f_1 in (X, τ). Thus g is not neutrosophic orbit continuous.

4. Conclusions

In this paper, we study the collection of neutrosophic orbit open sets under the mapping f : X → X. The characterization of neutrosophic orbit continuous functions are studied. Some interrelations are discussed with suitable examples. This paper paves way in future to introduce and study the family of all neutrosophic orbit open sets constructs a neutrosophic topological space.

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Balanced Neutrosophic Graphs

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Abstract: In this paper, we introduce the concept of balanced neutrosophic graphs based on density functions and investigate some of their properties. The necessary conditions for a neutrosophic graph to be a balanced neutrosophic graph are identified if graph G is a self-complementary, regular, complete, and strong neutrosophic graph. Some properties of complement neutrosophic graphs are presented here.

Keywords: Density of a neutrosophic graphs, Balanced neutrosophic graphs.

1. Introduction

Euler was the first to establish the concept of graph theory in 1736. In mathematical history, Euler's approach to the well-known Konigsberg bridge problem is considered as the first theorem of graph theory. This is now widely accepted as a branch of combinatorial mathematics. In many domains, such as geometry, combinatorics, elliptic curves, topography, decision theory, optimization, and data science, the theory of graphs provides a strong tool for determining combinatorial challenges.
The density of a graph G (D(G)) is associated with the network's connectivity patterns. Because of the rapid growth in network size, graph problems become ambiguous, which we address using the fuzzy logic method. The density D(H) ≤ D(G) for all subgraphs H of G in balanced graphs. Balanced graphs [10] first appeared in the work of random graphs, and the term Balanced neutrosophic graph is represented here based on the density functions given in [5]. A complete graph has the highest density, while a null graph has the lowest density. Several papers on balanced graph extension [25][32][14] have been published, and it has numerous applications in computer networks, image analysis, robotic systems, artificial intelligence , and decision making. Lotfi A Zadeh [29][30][31] developed a fuzzy set theory in 1965, and the idea of a fuzzy set is welcomed because it addresses uncertainty and vagueness that crisp set cannot, and it provides a meaningful and powerful recognition of quantification of ambiguity. Rosenfeld [24] developed the theory of fuzzy graphs in 1975 after studying fuzzy relations on fuzzy sets. Atanassov's [6][7] intuitionistic fuzzy graphs (IFGs) provide a way to incorporate uncertainty with an additional degree. A bipolar fuzzy graph is a fuzzy graph extension with a membership degree range of [-1, 1]. Akram [1][2] introduced the concept of bipolar fuzzy graphs and defined various operations on them. Talal Al Hawary [4] investigated some fuzzy graph operations and defined balanced fuzzy graphs. Balanced fuzzy graphs are increasingly

being used to represent complex systems in which the amount of data and information varies with different levels of precision.

A neutrosophic graph can comply with the uncertainty of any real-world problem's inconsistent and indeterminate information, whereas fuzzy graphs may lack sufficient satisfactory results. Florentin Smarandache et al [12][26-28] defined neutrosophic graphs and single valued neutrosophic graphs (SVNS) as a new dimension of graph theory as a generalisation of the fuzzy graph and the intuitionistic fuzzy graph. Said Broumi et al [8][9] developed the concept of SVNG and investigated its components. Motivated by the concept of a balanced graph and its extensions [3] [13] [15-20] [22][23] [27], we focused on introducing balanced and strictly balanced, in single valued neutrosophic graphs. The important properties of a balanced neutrosophic graph are discussed in this paper. Section 2 discusses the fundamental definitions and theorems required. Section 3 discusses the necessary conditions for a neutrosophic graph to be a balanced neutrosophic graph if graph G is a self-complementary, regular, complete, and strong neutrosophic graph. We also discussed some of the properties of complementary and a self-complementary balanced neutrosophic graphs. The paper is concluded in Section 4.

2. Preliminaries

Definition 2.1 [12] A single valued neutrosophic graph (SVN-graph) with underlying set V is defined to be a pair $G = (A, B)$ where

1. The functions $T_A: V \to [0,1]$, $I_A: V \to [0,1]$, and $F_A: V \to [0,1]$, denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element $v_i \in V$, respectively, and $0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$ for all $v_i \in V$.

2. The functions $T_B: \mathrm{E} \subseteq \mathrm{V \, x \, V} \to [0,1]$, $I_B: \mathrm{E} \subseteq \mathrm{V \, x \, V} \to [0,1]$, and $F_B: \mathrm{E} \subseteq \mathrm{V \, x \, V} \to [0,1]$ are defined by $T_B(v_i, v_j) \leq T_A(v_i) \wedge T_A(v_j)$, $I_B(v_i, v_j) \geq I_A(v_i) \vee I_A(v_j)$ and $F_B(v_i, v_j) \geq F_A(v_i) \vee \mathrm{F}(v_j)$ denotes the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in E$ respectively, where $0 \leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$ for all $(v_i, v_j) \in E$ $(\mathrm{i}, \mathrm{j} = 1, 2, \ldots, \mathrm{n})$. We call A the single valued neutrosophic vertex set of V, B the single valued neutrosophic edge set of E, respectively.

Definition 2.2 [8] A partial SVN-subgraph of SVN-graph $G = (A, B)$ is a SVN-graph $H = (V', E')$ such that $V' \subseteq \mathrm{V}$, where $T'_A(v_i) \leq T_A(v_i)$, $I'_A(v_i) \geq I_A(v_i)$, and $F'_A(v_i) \geq F_A(v_i)$ for all $v_i \in V$ and $E' \subseteq \mathrm{E}$, where $T'_B(v_i, v_j) \leq T_B(v_i, v_j)$, $I'_B(v_i, v_j) \geq I_B(v_i, v_j)$, $F'_B(v_i, v_j) \geq F_B(v_i, v_j)$ for all $(v_i, v_j) \in E$.

Definition 2.3 [11] Let $G = (A, B)$ be an SVNG. G is said to be a strong SVNG if

$T_{\mathrm{B}}(u, v) = T_A(u) \wedge T_A(v)$,

$I_{\mathrm{B}}(u, v) = I_A(u) \vee I_A(v)$ and

$F_{\mathrm{B}}(u, v) = F_A(u) \vee F_A(v)$ for every $(u, v) \in E$.

Definition 2.4 [11] Let $G = (A, B)$ be an SVNG. G is said to be a complete SVNG if

$T_{\mathrm{B}}(u, v) = T_A(u) \wedge T_A(v)$,

$I_{\mathrm{B}}(u, v) = I_A(u) \vee I_A(v)$ and

$F_B(u,v) = F_A(u) \vee F_A(v)$ for every $u, v \in V$.

Definition 2.5 [11] Let $G = (A, B)$ be an SVNG. $\bar{G} = (\bar{A}, \bar{B})$ is the complement of an SVNG if $\bar{A} = A$ *and* $\bar{B}$ is computed as below.

$\overline{T_B(u,v)} = T_A(u) \wedge T_A(v) - T_B(u,v)$,

$\overline{I_B(u,v)} = I_A(u) \vee I_A(v) - I_B(u,v)$

and $\overline{F_B(u,v)} = F_A(u) \vee F_A(v) - F_B(u,v)$ for every $(u,v) \in E$.

Here, $\overline{T_B(u,v)}, \overline{I_B(u,v)}$ and $\overline{F_B(u,v)}$ denote the true, intermediate, and false membership degree for edge (u,v) of $\bar{G}$.

Definition 2.6 [11] Let $G = (A, B)$ be an SVNG. G is a regular neutrosophic graph if it satisfies the following conditions.

$$\sum_{u \neq v} T_B(u,v) = \text{constant}, \quad \sum_{u \neq v} I_B(u,v) = \text{constant}, \quad \text{and} \quad \sum_{u \neq v} F_B(u,v) = \text{constant}.$$

Definition 2.7 [11] Let $G = (A, B)$ be an SVNG. G is a regular strong neutrosophic graph if it satisfies the following conditions.

$T_B(u,v) = T_A(u) \wedge T_A(v)$ and $\sum_{u \neq v} T_B(u,v) = \text{constant}$,

$I_B(u,v) = I_A(u) \vee I_A(v)$ and $\sum_{u \neq v} I_B(u,v) = \text{constant}$,

$F_B(u,v) = F_A(u) \vee F_A(v)$ and $\sum_{u \neq v} F_B(u,v) = \text{constant}$.

Definition 2.8 [4] The density of the complete fuzzy graph $G = (V, E)$ is

$D(G) = \frac{2 \sum_{u,v \epsilon V} (\mu(u,v))}{\sum_{(u,v) \epsilon V} (\sigma(u) \wedge \sigma(v))}$, for all $u, v \epsilon V$.

Definition 2.9[4] A fuzzy graph $G = (V, E)$is balanced if $D(H) \leq D(G)$, for all sub graphs H of G.

Definition 2.10 [21] A fuzzy graph $G = (V, E)$is a self-complementary if $\mu(u,v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v))$ for all $u, v \epsilon V$.

Table 1: Some basic notations

Notation	Meaning
$G = (V, E)$	Fuzzy graph
$G = (A, B)$	Single Valued Neutrosophic Graph (SVNG)
V	Vertex Set
E	Edge set
$T_A(v), I_A(v), F_A(v)$	True membership value, indeterminacy membership value, falsity membership value of the vertex v of $G = (A, B)$
$T_B(u,v), I_B(u,v), F_B(u,v)$	True membership value, indeterminacy membership value, falsity membership value of the edge (u,v) of $G = (A, B)$
$\bar{G} = (\bar{A}, \bar{B})$	Complement of an SVNG
$\overline{T_B(u,v)}, \overline{I_B(u,v)}$, $\overline{F_B(u,v)}$	True membership value, indeterminacy

	membership value, falsity membership value of the edge (u, v) of $\bar{G} = (\bar{A}, \bar{B})$
$D_T(G), D_I(G), D_F(G)$	Density of true membership value, indeterminacy membership value, falsity membership value of $G = (A, B)$
$D(G) = \big(D_T(G), D_I(G), D_F(G)\big)$	Density of a SVNG $G = (A, B)$

3. Balanced Neutrosophic Graphs

Definition 3.1

The density of a single valued neutrosophic graph $G = (A, B)$ of $G^* = (V, E)$, is $D(G) = \big(D_T(G), D_I(G), D_F(G)\big)$, where

$D_T(G)$ is defined by $D_T(G) = \frac{2\sum_{u,v\epsilon V} T_{\mathrm{B}}(u,v)}{\sum_{(u,v)\epsilon V} T_A(u) \wedge T_A(v)}$, for $u, v\epsilon V$,

$D_I(G)$ is defined by $D_I(G) = \frac{2\sum_{u,v\epsilon V} I_{\mathrm{B}}(u,v)}{\sum_{(u,v)\epsilon V} I_A(u) \vee I_A(v)}$, for $u, v\epsilon V$ and

$D_F(G)$ is defined by $D_I(G) = \frac{2\sum_{u,v\epsilon V} F_{\mathrm{B}}(u,v)}{\sum_{(u,v)\epsilon V} F_A(u) \vee F_A(v)}$, for $u, v\epsilon V$.

Definition 3.2

A single valued neutrosophic graph $G = (A, B)$ is balanced if $D(H) \leq D(G)$, that is, $D_T(H) \leq D_T(G)$, $D_I(H) \leq D_I(G)$, $D_F(H) \leq D_F(G)$ for all sub graphs H of G.

Example 1. Consider a neutrosophic graph, $G = (V, E)$ **,such that** $V = \{(v_1, v_2, v_3, v_4)\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3)\}$.

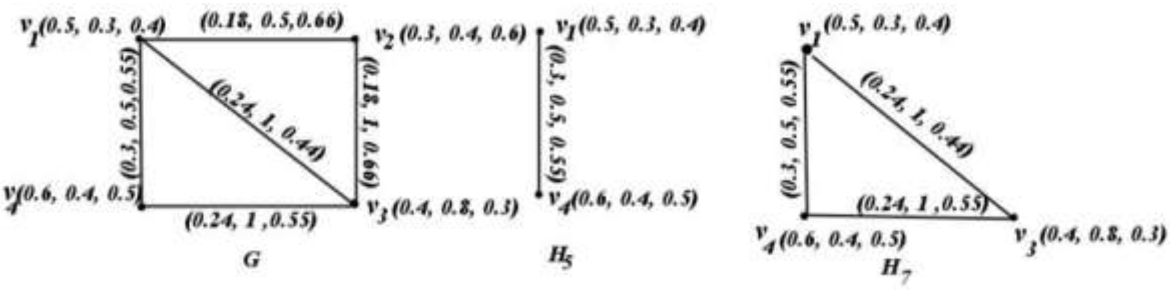

Fig.1 Balanced Neutrosophic Graph

T −density

$D_T(G) = 2\left(\frac{0.18+0.18+0.24+0.3+0.24}{0.3+0.3+0.4+0.5+0.4}\right) = 1.2$

I −density

$D_I(G) = 2\left(\frac{0.5+1+1+0.5+1}{0.4+0.8+0.8+0.4+0.8}\right) = 2.5$

F −density

$D_F(G) = 2\left(\frac{0.66+0.66+0.55+0.55+0.44}{0.6+0.6+0.5+0.5+0.4}\right) = 2.2$

$D(G) = \left(D_T(G), D_I(G), D_F(G)\right) = (1.2, 2.5, 2.2)$.

Let $H_1 = \{(v_1, v_2)\}$, $H_2 = \{(v_2, v_3)\}$, $H_3 = \{(v_3, v_4)\}$, $H_4 = \{(v_2, v_4)\}$, $H_5 = \{(v_1, v_4)\}$, $H_6 = \{(v_1, v_3)\}$, $H_7 = \{(v_1, v_3, v_4)\}$, $H_8 = \{(v_1, v_2, v_3)\}$, $H_9 = \{(v_1, v_2, v_4)\}$, $H_{10} = \{(v_2, v_3, v_4)\}$, $H_{11} = \{(v_1, v_2, v_3, v_4)\}$ be non-empty subgraphs of G. Density $\left(D_T(H), D_I(H), D_F(H)\right)$ is $D(H_1) = (1.2, 2.5, 2.2)$, $D(H_2) = (1.2, 2.5, 2.2)$, $D(H_3) = (1.2, 2.5, 2.2)$, $D(H_4) = (0, 0, 0)$, $D(H_5) = (1.2, 2.5, 2.2)$, $D(H_6) = (1.2, 2.5, 2.2)$, $D(H_7) = (1.2, 2.5, 2.2)$, $D(H_8) = (1.2, 2.5, 2.2)$, $D(H_9) = (1.2, 2.5, 2.2)$, $D(H_{10}) = (1.2, 2.5, 2.2)$, $D(H_{11}) = (1.2, 2.5, 2.2)$. So $D(H) \leq D(G)$ for all subgraphs H of G. Hence G is balanced neutrosophic graph.

Definition 3.3

A single valued neutrosophic graph $G = (A, B)$ is strictly balanced if for $u, v \in V$, $D(H) = D(G)$ for all sub graphs H of G.

Example 2. Consider a neutrosophic graph, $G = (V, E)$,such that $V = \{(v_1, v_2, v_3, v_4)\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3), (v_2, v_4)\}$.

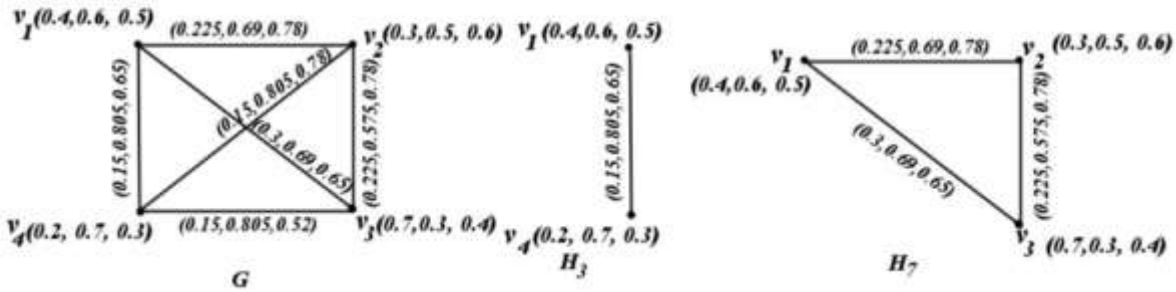

Fig. 2 Strictly Balanced Neutrosophic Graph

T −density

$D_T(G) = 2\left(\frac{0.225+0.225+0.15+0.15+0.15+0.3}{0.3+0.3+0.2+0.2+0.4+0.2}\right) = 1.5$

I −density

$D_I(G) = 2\left(\frac{0.69+0.575+0.805+0.805+0.69+0.805}{0.6+0.5+0.7+0.7+0.7+0.6}\right) = 2.3$

F −density

$D_F(G) = 2\left(\frac{0.78+0.78+0.78+0.65+0.78+0.65}{0.6+0.6+0.4+0.5+0.5+0.6}\right) = 2.6$

$D(G) = \left(D_T(G), D_I(G), D_F(G)\right) = (1.5, 2.3, 2.6)$.

Let $H_1 = \{(v_1, v_2)\}$, $H_2 = \{(v_2, v_3)\}$, $H_3 = \{(v_1, v_4)\}$, $H_4 = \{(v_2, v_4)\}$, $H_5 = \{(v_2, v_4)\}$, $H_6 = \{(v_1, v_3)\}$, $H_7 = \{(v_1, v_2, v_3)\}$,$H_8 = \{(v_1, v_3, v_4)\}$, $H_9 = \{(v_1, v_2, v_4)\}$, $H_{10} = \{(v_2, v_3, v_4)\}$, $H_{11} = \{(v_1, v_2, v_3, v_4)\}$ be non-empty subgraphs of G. Density $\left(D_T(H), D_I(H), D_F(H)\right)$ is $D(H_1) = (1.5, 2.3, 2.6)$, $D(H_2) = (1.5, 2.3, 2.6)$, $D(H_3) = (1.5, 2.3, 2.6)$, $D(H_4) = (1.5, 2.3, 2.6)$, $D(H_5) = (11.5, 2.3, 2.6)$, $D(H_6) = (1.5, 2.3, 2.6)$, $D(H_7) = (1.5, 2.3, 2.6)$, $D(H_8) = (1.5, 2.3, 2.6)$, $D(H_9) = (1.5, 2.3, 2.6)$, $D(H_{10}) =$

$(1.5, 2.3, 2.6)$, $D(H_{11}) = (1.5, 2.3, 2.6)$. So $D(H) = D(G)$ for all subgraphs H of G. Hence G is strictly balanced neutrosophic graph.

Theorem 3.4 Every complete single valued neutrosophic graph is balanced.

Proof:

Let $G = (A, B)$ be a complete single valued neutrosophic graph, then by the definition of complete neutrosophic graph, we have $T_B(u,v) = T_A(u) \wedge T_A(v)$, $I_B(u,v) = I_A(u) \vee I_A(v)$ and $F_B(u,v) = F_A(u) \vee F_A(v)$ for every $u, v \in V$.

$\therefore \sum_{u,v \in V} T_B(u,v) = \sum_{(u,v) \in V} T_A(u) \wedge T_A(v)$

$\sum_{u,v \in V} I_B(u,v) = \sum_{(u,v) \in V} I_A(u) \vee I_A(v)$ and

$\sum_{u,v \in V} F_B(u,v) = \sum_{(u,v) \in V} F_A(u) \vee F_A(v)$.

Now $D(G) = \left(\frac{2\sum_{u,v \in V} T_B(u,v)}{\sum_{(u,v) \in V} T_A(u) \wedge T_A(v)}, \frac{2\sum_{u,v \in V} I_B(u,v)}{\sum_{(u,v) \in V} I_A(u) \vee I_A(v)}, \frac{2\sum_{u,v \in V} F_B(u,v)}{\sum_{(u,v) \in V} F_A(u) \vee F_A(v)} \right)$

$$D(G) = \left(\frac{2\sum_{(u,v) \in V} T_A(u) \wedge T_A(v)}{\sum_{(u,v) \in V} T_A(u) \wedge T_A(v)}, \frac{2\sum_{(u,v) \in V} I_A(u) \vee I_A(v)}{\sum_{(u,v) \in V} I_A(u) \vee I_A(v)}, \frac{2\sum_{(u,v) \in V} F_A(u) \vee F_A(v)}{\sum_{(u,v) \in V} F_A(u) \vee F_A(v)} \right)$$

$$D(G) = (2, 2, 2).$$

Let H be a non-empty subgraph of G then, $D(H) = (2, 2, 2)$ for every $H \subseteq G$.

Thus, G is balanced.

Note 3.5. The converse of the preceding theorem do not have to be true. Each balanced neutrosophic graph does not have to be complete.

Example 3. Consider a neutrosophic graph, $G = (V, E)$,such that $V = \{(v_1, v_2, v_3, v_4)\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$.

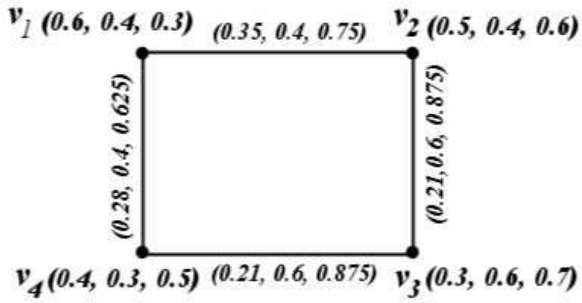

Fig. 3 Balanced but not complete neutrosophic graph

$D(G) = (D_T(G), D_I(G), D_F(G)) = (1.4, 2, 2.5)$.

Let $H_1 = \{(v_1, v_2)\}$, $H_2 = \{(v_2, v_3)\}$, $H_3 = \{(v_1, v_4)\}$, $H_4 = \{(v_2, v_4)\}$, $H_5 = \{(v_2, v_4)\}$, $H_6 = \{(v_1, v_3)\}$, $H_7 = \{(v_1, v_2, v_3)\}$, $H_8 = \{(v_1, v_3, v_4)\}$, $H_9 = \{(v_1, v_2, v_4)\}$, $H_{10} = \{(v_2, v_3, v_4)\}$, $H_{11} = \{(v_1, v_2, v_3, v_4)\}$ be non-empty subgraphs of G. Density $(D_T(H), D_I(H), D_F(H))$ is $D(H_1) = (1.4, 2, 2.5)$, $D(H_2) = (1.4, 2, 2.5)$, $D(H_3) = (1.4, 2, 2.5)$, $D(H_4) = (1.4, 2, 2.5)$, $D(H_5) = (1.4, 2, 2.5)$, $D(H_6) = (1.4, 2, 2.5)$, $D(H_7) = (1.4, 2, 2.5)$, $D(H_8) = (1.4, 2, 2.5)$, $D(H_9) = (1.4, 2, 2.5)$, $D(H_{10}) = (1.4, 2, 2.5)$, $D(H_{11}) = (1.5, 2.3, 2.6)$. So $D(H) \leq D(G)$ for all subgraphs H of G. Hence G is balanced neutrosophic graph.

From the above graph easy to see that:

$T_B(u,v) \neq T_A(u) \wedge T_A(v)$, $I_B(u,v) = I_A(u) \vee I_A(v)$ and $F_B(u,v) \neq F_A(u) \vee F_A(v)$ for every $u, v \in V$. Hence G is balanced not complete.

Corollary 3.6 Every strong single valued neutrosophic graph is balanced.

Theorem 3.7

Let G = (A, B) be a self-complementary neutrosophic graph. Then D(G) = (1,1,1).

Proof:

Let $G = (A, B)$ be a self-complementary neutrosophic graph, then

$$\sum_{u,v\epsilon V} T_{\mathrm{B}}(u,v) = \frac{1}{2}\sum_{(u,v)\epsilon V} T_A(u) \wedge T_A(v)$$

$$\sum_{u,v\epsilon V} I_{\mathrm{B}}(u,v) = \frac{1}{2}\sum_{(u,v)\epsilon V} I_A(u) \vee I_A(v) \text{ and}$$

$$\sum_{u,v\epsilon V} F_{\mathrm{B}}(u,v) = \frac{1}{2}\sum_{(u,v)\epsilon V} F_A(u) \vee F_A(v).$$

Now $D(G) = \left(\frac{2\sum_{u,v\epsilon V} T_{\mathrm{B}}(u,v)}{\sum_{(u,v)\epsilon V} T_A(u) \wedge T_A(v)}, \frac{2\sum_{u,v\epsilon V} I_{\mathrm{B}}(u,v)}{\sum_{(u,v)\epsilon V} I_A(u) \vee I_A(v)}, \frac{2\sum_{u,v\epsilon V} F_{\mathrm{B}}(u,v)}{\sum_{(u,v)\epsilon V} F_A(u) \vee F_A(v)}\right)$

$$D(G) = \left(\frac{2\sum_{(u,v)\epsilon V} T_A(u) \wedge T_A(v)}{2\sum_{(u,v)\epsilon V} T_A(u) \wedge T_A(v)}, \frac{2\sum_{(u,v)\epsilon V} I_A(u) \vee I_A(v)}{2\sum_{(u,v)\epsilon V} I_A(u) \vee I_A(v)}, \frac{2\sum_{(u,v)\epsilon V} F_A(u) \vee F_A(v)}{2\sum_{(u,v)\epsilon V} F_A(u) \vee F_A(v)}\right)$$

Hence $D(G) = (1, 1, 1)$.

Theorem 3.8

Let G = (A, B) be a strictly balanced neutrosophic graph and $\bar{G} = (\bar{A}, \bar{B})$ be its complement then D(G) + D($\bar{G}$) = (2, 2, 2).

Proof:

Let G = (A, B) be a strictly balanced neutrosophic graph and $\bar{G} = (\bar{A}, \bar{B})$ be its complement.

Let H be a subgraph of G which is non-empty. D(G) = D(H) for all H ⊆ G and u, v ∈ V since G is strictly balanced.

In $\bar{G}$, $\overline{T_B(u,v)} = T_A(u) \wedge T_A(v) - T_{\mathrm{B}}(u,v)$, (1)

$\overline{I_B(u,v)} = I_A(u) \vee I_A(v) - I_{\mathrm{B}}(u,v)$ (2)

and $\overline{F_B(u,v)} = F_A(u) \vee F_A(v) - F_{\mathrm{B}}(u,v)$ for every $(u,v) \in E$. (3)

Dividing (1) by $T_A(u) \wedge T_A(v)$

$$\frac{\overline{T_B(u,v)}}{T_A(u) \wedge T_A(v)} = 1 - \frac{T_{\mathrm{B}}(u,v)}{T_A(u) \wedge T_A(v)}, \quad \text{for every } u,v \in V$$

Similarly dividing (2) by $I_A(u) \vee I_A(v)$

$$\frac{\overline{I_B(u,v)}}{I_A(u) \vee I_A(v)} = 1 - \frac{I_{\mathrm{B}}(u,v)}{I_A(u) \vee I_A(v)}, \quad \text{for every } u,v \in V$$

and dividing (3) by $F_A(u) \vee F_A(v)$

$$\frac{\overline{F_B(u,v)}}{F_A(u) \vee F_A(v)} = 1 - \frac{F_{\mathrm{B}}(u,v)}{F_A(u) \vee F_A(v)}, \quad \text{for every } u,v \in V$$

then

$$\sum_{u,v\in V} \frac{\overline{T_B(u,v)}}{T_A(u) \wedge T_A(v)} = 1 - \sum_{u,v\in V} \frac{T_{\mathrm{B}}(u,v)}{T_A(u) \wedge T_A(v)}, \quad \text{for every } u,v \in V$$

$\sum_{u,v\in V} \frac{\overline{I_B(u,v)}}{I_A(u) \vee I_A(v)} = 1 - \sum_{u,v\in V} \frac{I_B(u,v)}{I_A(u) \vee I_A(v)}$, for every $u, v \in V$

$\sum_{u,v\in V} \frac{\overline{F_B(u,v)}}{F_A(u) \vee F_A(v)} = 1 - \sum_{u,v\in V} \frac{F_B(u,v)}{F_A(u) \vee F_A(v)}$, for every $u, v \in V$

Multiply the above equations by 2 on both sides

$2\sum_{u,v\in V} \frac{\overline{T_B(u,v)}}{T_A(u) \wedge T_A(v)} = 2 - 2\sum_{u,v\in V} \frac{T_B(u,v)}{T_A(u) \wedge T_A(v)}$, for every $u, v \in V$

$2\sum_{u,v\in V} \frac{\overline{I_B(u,v)}}{I_A(u) \vee I_A(v)} = 2 - 2\sum_{u,v\in V} \frac{I_B(u,v)}{I_A(u) \vee I_A(v)}$, for every $u, v \in V$

$2\sum_{u,v\in V} \frac{\overline{F_B(u,v)}}{F_A(u) \vee F_A(v)} = 2 - 2\sum_{u,v\in V} \frac{F_B(u,v)}{F_A(u) \vee F_A(v)}$, for every $u, v \in V$

$D_T(\bar{G})$ = 2 - $D_T(G)$, $D_I(\bar{G})$ = 2 - $D_I(G)$ and $D_F(\bar{G})$ = 2 - $D_F(G)$

Now, D(G) + D($\bar{G}$) = $(D_T(G), D_I(G), D_F(G)) + (D_T(\bar{G}), D_I(\bar{G}), D_F(\bar{G}))$

D(G) + D($\bar{G}$) =$\left((D_T(G) + D_T(\bar{G})), (D_I(G) + D_I(\bar{G})), (D_F(G) + D_F(\bar{G}))\right)$

Hence D(G) + D($\bar{G}$) =(2, 2, 2).

Theorem 3.9

The complement of a single valued neutrosophic graph that is strictly balanced is also strictly balanced.

Proof:

Let G = (A, B) be a strictly balanced neutrosophic graph and $\bar{G} = (\bar{A}, \bar{B})$ be its complement.

Let H be a subgraph of G which is non-empty. D(G) = D(H) for all H ⊆ G and u, v ∈ V since G is strictly balanced.

As G is strictly balanced by Theorem 3.7, D(G) + D($\bar{G}$) =(2, 2, 2)

Since D(H) + D($\bar{H}$) =(2, 2, 2) for every $H \subseteq G$.

Which implies D($\bar{H}$) = D($\bar{G}$)

Hence $\bar{G}$ is strictly balanced.

Theorem 3.10

The complement of strongly regular SVNG is balanced.

Proof:

Let G = (A, B) be a strongly regular neutrosophic graph and $\bar{G} = (\bar{A}, \bar{B})$ be its complement.

Since G is strongly, we have $T_B(u,v) = T_A(u) \wedge T_A(v)$, $I_B(u,v) = I_A(u) \vee I_A(v)$ and

$F_B(u,v) = F_A(u) \vee F_A(v)$ for every $(u,v) \in E$. (1)

In $\bar{G}$, $\overline{T_B(u,v)} = T_A(u) \wedge T_A(v) - T_B(u,v)$,

$\overline{I_B(u,v)} = I_A(u) \vee I_A(v) - I_B(u,v)$

and $\overline{F_B(u,v)} = F_A(u) \vee F_A(v) - F_B(u,v)$ for every $(u,v) \in E$.

Since G is strongly regular, we have $\overline{T_B(u,v)} = 0$, $\overline{I_B(u,v)} = 0$ and $\overline{F_B(u,v)} = 0$ by (1) for every $(u,v) \in E$ and

$\overline{T_B(u,v)} = T_A(u) \wedge T_A(v)$,

$\overline{I_B(u,v)} = I_A(u) \vee I_A(v)$

and $\overline{F_B(u,v)} = F_A(u) \vee F_A(v)$ for every $(u,v) \in \bar{E}$.

$\Longrightarrow \bar{G}$ is a strong neutrosophic graph. Then by Corollary 3.6, $\bar{G}$ is balanced.

Theorem 3.11

Let G = (A, B) be a SVNG and $\bar{G} = (\bar{A}, \bar{B})$ be its complement then $\bar{\bar{G}} = \mathrm{G}$.

Proof:

Let G = (A, B) be a SVNG $\bar{G} = (\bar{A}, \bar{B})$ be its complement.

In $\bar{G}$, $\overline{T_B(u,v)} = T_A(u) \wedge T_A(v) - T_{\mathrm{B}}(u,v)$, (1)

$\overline{I_B(u,v)} = I_A(u) \vee I_A(v) - I_{\mathrm{B}}(u,v)$ (2)

and $\overline{F_B(u,v)} = F_A(u) \vee F_A(v) - F_{\mathrm{B}}(u,v)$ for every $(u,v) \in E$. (3)

Taking complement for (1), we get $\overline{\overline{T_{\mathrm{B}}(u,v)}} = T_A(u) \wedge T_A(v) - \overline{T_B(u,v)}$

Substitute $T_A(u) \wedge T_A(v) = T_{\mathrm{B}}(u,v) + \overline{T_B(u,v)}$ from (1) weget, $\overline{\overline{T_{\mathrm{B}}(u,v)}} = T_{\mathrm{B}}(u,v)$

Similarly, $\overline{\overline{I_{\mathrm{B}}(u,v)}} = I_{\mathrm{B}}(u,v)$ and $\overline{\overline{F_{\mathrm{B}}(u,v)}} = F_{\mathrm{B}}(u,v)$

Hence $\bar{\bar{G}} = \mathrm{G}$.

4. Conclusion

Neutrosophic graph theory is now commonly used in numerous sciences and technology, most notably in cognitive science, genetic algorithms, optimization techniques, cluster analysis, medical diagnosis, and decision theory. Florentin Smarandache created a neutrosophic graph based on neutrosophic sets. When compared to other traditional and fuzzy models, neutrosophic models provide the system with greater precision, adaptability, and compatibility. We introduced the concept of balanced neutrosophic graphs in this paper and we plan to expand our work on the application of balancing social network connectivity using density functions in the neutrosophic environment.

Compliance with Ethical Standards

Conflict of Interest

The authors declare that they do not have any financial or associative interest indicating a conflict of interest in about submitted work.

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Within the Protection of COVID-19 Spreading: A Face Mask Detection Model Based on the Neutrosophic RGB with Deep Transfer Learning.

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Abstract: COVID-19's fast spread in 2020 compelled the World Health Organization (WHO) to declare COVID-19 a worldwide pandemic. According to the WHO, one of the preventative countermeasures against this type of virus is to use face masks in public places. This paper proposes a face mask detection model by extracting features based on the neutrosophic RGB with deep transfer learning. The suggested model is divided into three steps, the first step is the conversion to the neutrosophic RGB domain. This work is considered one of the first trails of applying neutrosophic RGB conversion to image domain, as it was commonly used in the conversion of grayscale images. The second step is the feature extraction using Alexnet, which has been small number of layers. The detection model is created in the third step using two traditional machine learning algorithms: decision trees classifier and Support Vector Machine (SVM). The Simulated Masked Face dataset (SMF) and the Real-World Mask Face dataset (RMF) are merged to a single dataset with two categories (a face with a mask, and a face without a mask). According to the experimental results, the SVM classifier with the True (T) neutrosophic domain achieved the highest testing accuracy with 98.37%.

Keywords: Neutrosophic RGB; COVID-19; Classical Machine Learning; Deep Learning; Face Mask Detection

1. Introduction

As COVID spread swiftly throughout the globe in 2020, the World Health Organization was forced to proclaim a worldwide pandemic. In more than 180 nations, more than seven million cases have been diagnosed with COVID-19 with a death rate of 3 percent, according to [1]. Extensive initiatives are underway around the globe to create innovative therapies and vaccinations for the disease. The new coronavirus known as severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) is a member of the pathogen family that caused respiratory illnesses during the 2002–2003 pandemic (SARS-CoV-1) [2].

COVID-19 is distinguished by a pre-symptomatic phase of transmission, as freshly infected persons may harm others inadvertently. The infection travels by direct touch and across polluted and overcrowded environments. The face mask is an effective way to prevent the COVID-19 spread of airborne particles [3]. According to [4], [5], having to wear face masks in crowded areas and public will help to reduce disease transmission all the time. In many societies, governments face tremendous obstacles and hazards in protecting people from coronavirus. When it comes to the dissemination and transmission of COVID-19, policymakers confront a slew of concerns and dangers [6]. In several countries, people are required by law to wear face masks in public [7]. These recommendations and rules were created as part of an attempt to combat the rapidly growing number of fatalities in several nations. However, managing a large group of people is getting increasingly difficult. One possible and realistic solution is to have a screening process that includes the identification of someone who does not wear a face mask. Despite a few screening processes, the search for a better approach is still an open research question, particularly in this COVID era.

Artificial intelligence contributes to the fight against the pandemic of COVID-19 in many ways. These days, several AI-powered initiatives focused on data analysis, 'machine learning' or 'big data' are being utilized in a broad variety of fields to anticipate, clarify and control the different health disaster scenarios [8]. AI technology and resources play a crucial role in every aspect of the COVID-19 crisis response, helping to prevent or slow down the spread of the virus through surveillance and contact tracking [9]. Due to significant developments in the domain of machine learning algorithms, face mask recognition technology looks to be effectively handled [10]. This type of technology is more relevant nowadays than it was previously because it's used for static picture recognition and on-monitoring as well as real-time inspection and supervision [11].

Numerous researches have been conducted on various issues relating to COVID-19 and have been solved by the computer science field, for example, tracking COVID-19 geographical infections using real-time tweets [12], investigating the role of developing technology in the fight against the COVID -19 pandemic [13], determining the COVID-19's influence on the electrical industry [14], Using machine learning and deep learning models to classify potential coronavirus treatments on a single human cell [15] and more. Numerous researches focus on the classification and categorization of COVID-19 CT and X-ray images [16]–[19].

Smarandache [20] proposed the principle of Neutrosophic logic in 1995 and then expanded it in 1999 [21]. Neutrosophic logic has been utilized in various disciplines of computer science since that time, including pattern recognition [22], image processing and segmentation [23], and more. This leads to the resolution of many scientific and practical real-life problems in a variety of areas, such as economics [24], [25], agriculture, and space satellite [26]. Neutrosophic [27] is the foundation of a wide family of current mathematical theories that describe both classical and fuzzy analogues. The term neutro-sophy refers to the feeling of neutral thinking, and it is this justification that distinguishes fuzzy and intuitive fuzzy logic from set theory. A neutrosophic set [28] can be a generic method for analyzing data set uncertainty and, in particular, images in the field of artificial intelligence and deep learning. Various works used Neutrsophic theory and set with medical image analysis as presented in [29].

Because people in some countries are compelled by law to wear face masks in public, masked face recognition is a must for dealing with apps such as object detection. To battle and ultimately win the war against the COVID-19 pandemic, policymakers require advice and surveillance of individuals in public areas, particularly crowds, to guarantee that the legislation requiring the use of face masks is implemented. This might be expanded by combining surveillance technology with Artificial Intelligence models.

The remainder of the paper is organized as follows. Section 2 is a synopsis of prior relevant works. Section 3 describes the data set's features. Section 4 describes the suggested model in detail. Section 5 summarizes and analyses the experimental data, and Section 6 offers the conclusions and future work options.

2. Related Works

In [30], the authors developed a novel method for identifying the face of the human characterized by the use of a mask or not. While wearing the face mask, they were able to distinguish three different sorts of conditions. Correct facemask wearing, improper facemask wearing, and no facemask wearing. The suggested technique achieved a face detection process accuracy of 98.70%. In [31], Convolutional neural networks were utilized to suggest a unique approach known as face emotion recognition (FERC). A two-part convolutional neural network served as the foundation for the FERC. The FERC was able to properly depict the emotion with 96% accuracy. Nizam et al. [32] proposed a GAN-based structure capable of automatically eliminating masks enclosing the facial region and reconstructing the image by filling in the missing area The introduced method produced a full, natural, and realistic picture of the face.. Khan et al [9] developed a framework for automatically separating a face image into face parts and subsequently classifying the gender. The scientists used hand-labeled facial pictures to train a segmentation design based on Conditional Random Fields (CRFs). The CRF-based model was utilised to segment a facial picture into six separate classes: mouth, hair, eyes, nose, skin, and back. The proposed framework was almost 93 percent accurate. In [33], the authors used the YOLOv3 with Darknet-53 algorithm for facial detection. The introduced approach was trained on two public datasets including more than 600k images and testing was on the Face Detection Data Set and Benchmark (FDDB) dataset [34]. The introduced approach had reached an accuracy of 93.9%. Canping et al [35] suggested an unique deep neural network (DNN) training framework to speed the training process of the triplet loss-based DNN while improving face recognition performance The suggested model obtained 97.3 percent accuracy on the LFW benchmark, according to experimental findings [36]. Most of the related work focuses on facial construction and recognition of faces, there are few research focuses if the human wear a mask or not on his/her face. The goal of this study is to identify persons who do not wear medical face masks in order to reduce COVID-19 transmission and spread.

3. Datasets Characteristics

To train and test the proposed approach two publicly available masked face dataset are being utilized. The first dataset is Real-World Mask Face (RMF) dataset [37]. The RMF dataset contains 95000 faces, both masked and unmasked. Figure 1 shows samples of peoples wearing masks and without masks. Only 5000 photos were chosen at random for this study's trials in order to balance the number of images for each class, as well as the second dataset.

Figure 1. RMF samples pictures

The Simulated Masked Face (SMF) [38] dataset is the second masked face dataset. The SMF dataset contains 1570 pictures, 785 of which are masked and unmasked faces. Figure 2 depicts several instances of SMF images. The combined dataset contains 3285 pictures for each class, for a total of 6570 images. Each class has 2500 RMF pictures and 785 SMF images.

Figure 2. SMF samples pictures

4. The Proposed Model Structure

The suggested model structure is divided into three parts. The first part involves converting the original RGB domain to neutrosophic domain. The second part involves extracting features from dataset pictures using the Alexnet (A. Krizhevsky et al., 2017) . The third part is the classification process, which employs traditional machine techniques such as decision trees and support vector machines. Figure 3 depicts the suggested model structure in a graphical form. The next three subsections go into the specifics of these parts.

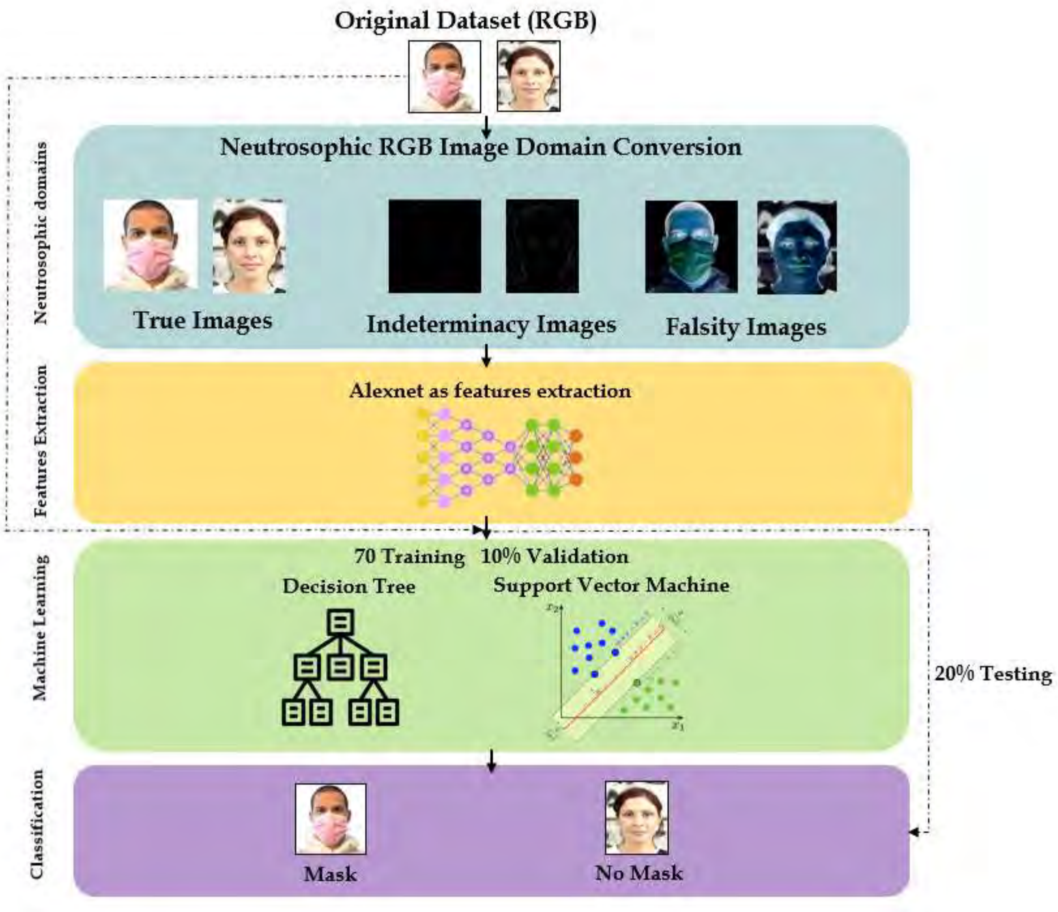

Figure 3. The proposed model structure

4.1. Neutrosophic RGB Conversion

The neutrosophic logic (NL) was created and implemented by Florentin Smarandache [39],[40]. In the NL, three neutrosophic subsets, true (T) value, indeterminacy (I) value and falsity (F) value are defined for any event. These neutrosophic values (T, I, F) are commonly used to transform a grayscale image into the neutrosophic image. The introduced research created a new neutrosophic definition on the masked face images, where T exemplify the masked face zone, I exemplify the masked face boundary, and F exemplify the background of image. The image converts to the neutrosophic image (NI) as illustrated in equations 1-4 [25] [41]:

$$\text{NI}\ (a,b) = \{T_{a,b}, I_{a,b}, F_{a,b}\} \tag{1}$$

$$T_{a,b} = \frac{v(a,b) - v_{min}}{v_{max} - v_{min}} \tag{2}$$

$$F_{a,b} = 1 - T_{a,b} \tag{3}$$

$$I_{a,b} = 1 - \frac{U(a,b) - U_{min}}{U_{max} - U_{min}} \tag{4}$$

Let $v(a,b)$ is the local mean value of related pixels. v_{max} and v_{min} are the maximum and minimum absolute difference pixels of the histogram. $U(a,b)$ is the homogeneity value of $T(a,b)$. While U_{max} and U_{min} are the maximum and minimum peaks respectively, measured from $U(a,b)$.

As mentioned above, the neutrosophic logics are commonly used with the grayscale domain. In this work, the authors introduced the neutrosophic RGB conversion. The main idea is to split the RGB domain into three domains (Red, Green, and Blue). After that, apply the equations of neutrosophic conversion in every domain separately. Then combine the resulted images again into the RGB domain. Figure 4 presents the flowchart of the neutrosophic RGB conversion.

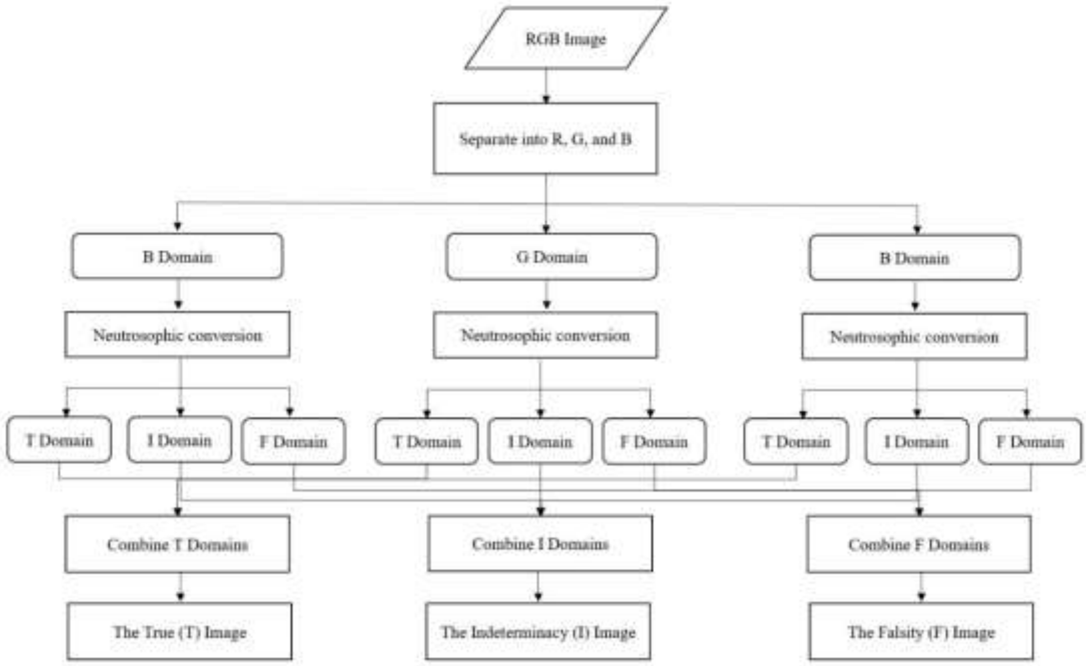

Figure 4. Neutrosophic RGB conversion flow chart

Figure 5 illustrates samples of ($T_{a,b}$, $I_{a,b}$, $F_{a,b}$) images after performing neutrosophic image transformation in the different domains (T, I, F). Where $T_{a,b}$ domain is masked face object, $I_{a,b}$ domain is the edges and $F_{a,b}$ domain is the background.

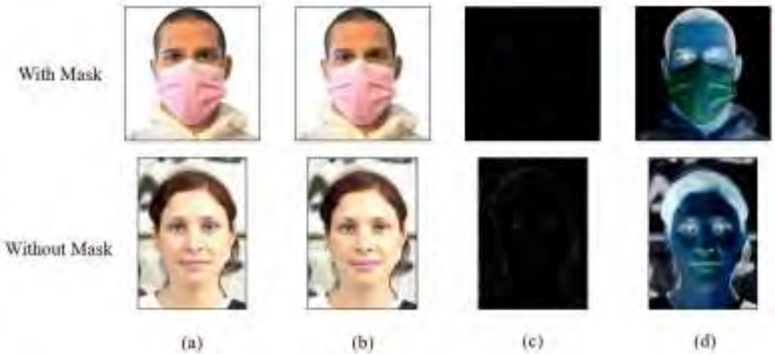

Figure 5. Different neutrosophic RGB pictures domains were (a) original RGB images, (b) True domain, (c) Indeterminacy, and (d) Falsity domain pictures for two classes in the dataset.

4.2. AlexNet as Features Extraction

AlexNet is a deep transfer learning technique based on the convolution and pooling [42]. AlexNet is an 8-layer deep network that begins with a convolution layer and ends with a fully linked layer, as illustrated in Figure 6. To enhance our model performance in classification, the final layer in AlexNet was changed and replaced with two machine learning classifiers, SVM and DT, as shown in Figure 7. This study's primary contribution is the development of SVM and DT that do not overfit the training process.

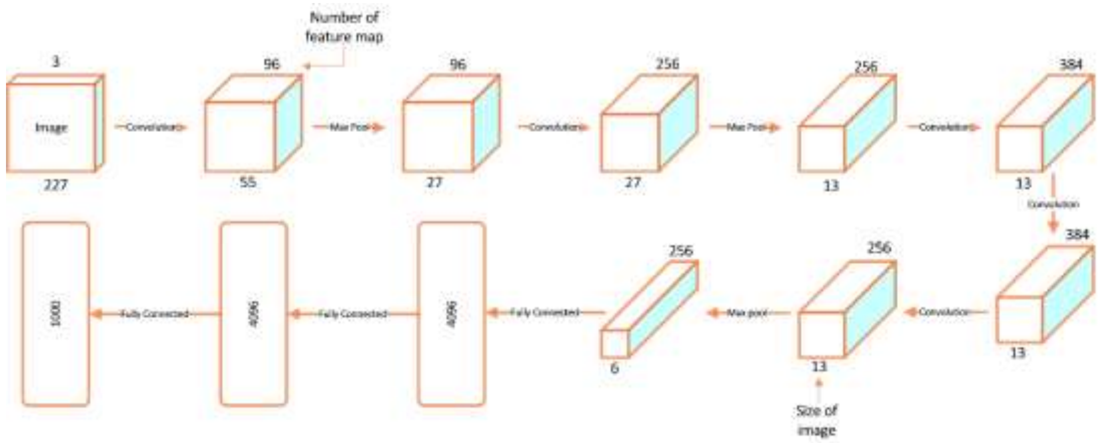

Figure 6. AlexNet Architecture

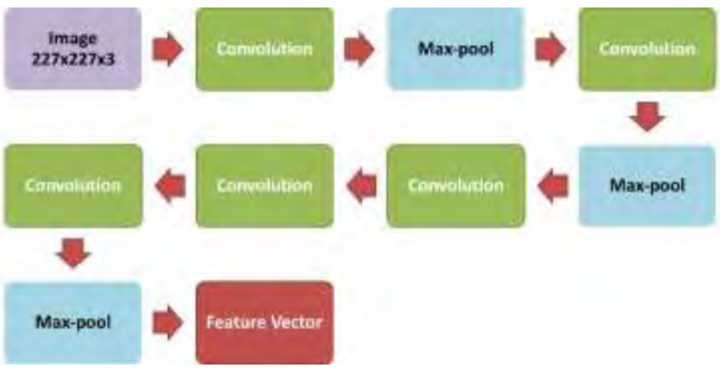

Figure 7. Proposed AlexNet as features extractor

4.3. Machine Learning Classifiers

SVM is a discriminative classifier based on hinge function H_v as illustrated in Equation 5. The output y is calculated based on w and d of linear classification as illustrated in equation 6. Where g is a class between 0 to 1. To minimize the SVM function, we implemented a loss function as shown in Equation 7 [43] , [44]. SVM maximize the distance between no mask and mask face class points as shown in Figure 8. DT is a graph of classification technique in the form of a tree model. Entropy and information gain are the main formula to calculate DT as illustrated in equation 8,9. Where v is related data, and u is a no masked face and masked face, and $p(u_i)$ is the degree of u class. Information Gain (IG) is calculated as shown in equation 9. Where d is a subset of related data [45] , [46].

$$H_v = \max(0, 1 - g_v y) \quad (5)$$

$$y = (w.x - d) \quad (6)$$

$$f = \frac{1}{u}\sum_{j=1}^{u} \max(0, H_j) \quad (7)$$

$$En(v) = \sum_{i=1}^{c} -p(u_i) . \log(p(u_i)) \quad (8)$$

$$IG = En(v) - \sum_{d \in v} p(v) En(v) \quad (9)$$

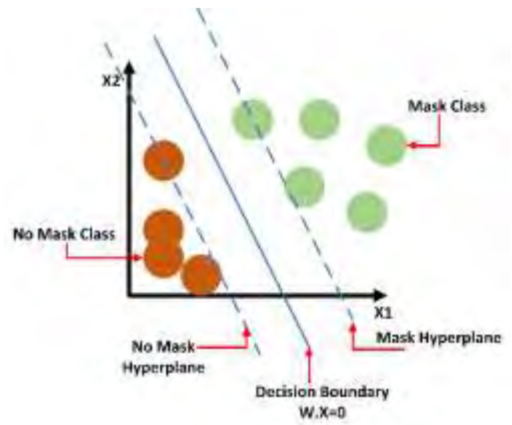

Figure 8. Illustrate how SVM work to classify masked face

5. Results and Discussions

All the experiments were conducted on a computer server outfitted with an Intel Xeon CPU (2 GHz) and 96 GB of RAM. The MATLAB software program was chosen for this study to create and implement the numerous experimental trails. During the experiments, the following specifications are chosen:

- Two classifiers (decision trees, and Support Vector Machine).
- Four domains of dataset images:
    - The original dataset domain (RGB).
    - The domain of the True (T) neutrosophic.
    - The domain of the Indeterminacy (I) neutrosophic.
    - The domain of the Falsity (F) neutrosophic.

- The dataset is divided into three components (70 percent of the data for the training process, 10 percent for the validation process, and 20 percent for the testing process).

Comprehensive research must examine how various classifiers perform on every dataset in order to investigate the ability of classifiers to generalize different datasets. The most common performance measures in machine learning evaluation models are, Accuracy, Precision, Recall, and F1 Score [47], and they are presented from Equation (10) to Equation (13).

$$\text{Accuracy} = \frac{\text{TPos+TNeg}}{(\text{TPos+FPos})+(TNeg+FNeg)} \quad (10)$$

$$\text{Precision} = \frac{\text{TPos}}{(\text{TPos+FPos})} \quad (11)$$

$$\text{Recall} = \frac{\text{TPos}}{(\text{TPos+FNeg})} \quad (12)$$

$$\text{F1 Score} = 2 * \frac{\text{Precision}*\text{Recall}}{(\text{Precision+Recall})} \quad (13)$$

Where TPos denotes the total number of True Positive samples, TNeg denotes the total number of True Negative samples, FPos denotes the total number of False Positive samples, and FNeg denotes the total number of False Negative samples from a confusion matrix. The findings of the experiments will be reported in three subsections. The first subsection will provide the findings acquired using a decision tree classifier, while the second subsection will give the results obtained using the SVM classifier. The third component will present a comparison outcome with similar research.

5.1. Experimental Results for DT Classifier

The first metric to be measured is the validation accuracy along with the other performance metrices. Validation accuracy is vital as it reflects the accuracy of the classifier during and after the training. The validation accuracy is calculated over 10% of the dataset [48]. Figure 9 depicts the validation accuracy of a decision tree classifier together with performance metrics for four domains of dataset images.

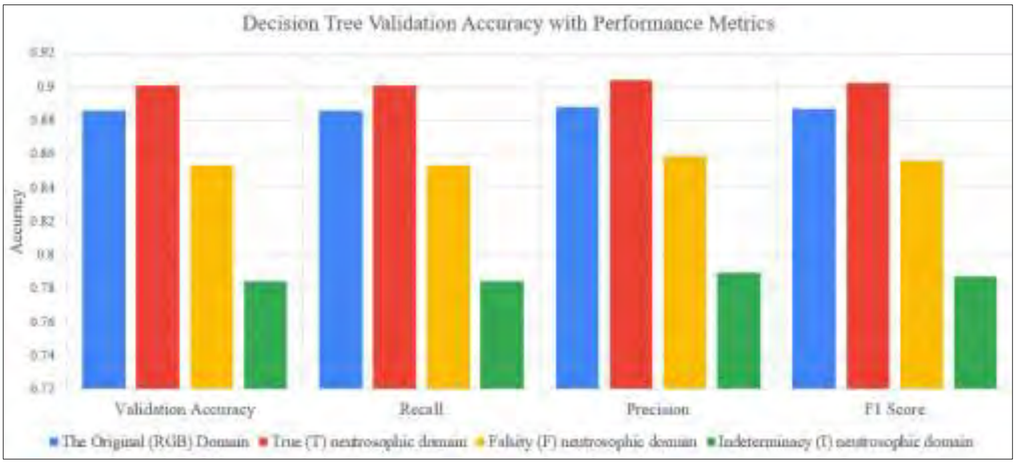

Figure 9. Validation accuracy and performance metrics for a DT classifier for four dataset image domains

Figure 9 illustrates that The True (T) neutrosophic domain achieved the highest possible validation accuracy with 90% while the original dataset validation accuracy is 88%. The improvement of validation accuracy is due to that the True (T) neutrosophic domain reflects the median actual pixel value depending on its neighbors' pixels. The performance metrics also support the obtained result for the achieved validation accuracy for True (T) neutrosophic domain with 0.9009, 0.9009, and 0.9039 for recall, precision, and F1 score accordingly. In Indeterminacy (I) neutrosophic domain achieved the least possible validation accuracy with performance metrics as according to the nature of the dataset, the borders of images which is the output result for the Indeterminacy (I) neutrosophic domain are not enough to improve the accuracy to differentiate between the masked and unmasked face images. Also, in the Falsity (F) neutrosophic domain, the validation accuracy is decreased than the validation accuracy for the original dataset. As in the Falsity (F) neutrosophic domain, some features are vanished due to the conversion process which reflected in the validation accuracy and other performance metrics. Validation accuracy does not reflect an accurate accuracy for the model as it is only present 10% of the dataset. So, the testing accuracy which is calculated over 20% will be more accurate and insightful for the proposed model. Figure 10 depicts the decision tree classifier's testing accuracy along with performance metrics for four domains of dataset images.

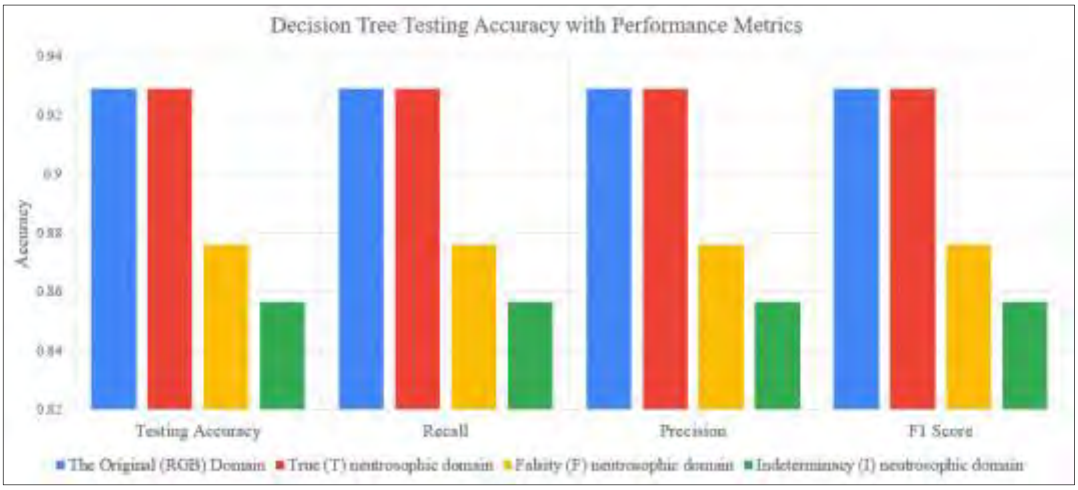

Figure 10. For four domains of dataset pictures, the DT classifier's accuracy was tested using performance measures.

Figure 10 illustrates that the testing accuracy with performance metrics for the original and the True (T) neutrosophic domain was the highest. The testing accuracy for both domains was 92.92%. With the same achieved performance metrics accuracy for both domains. True (T) neutrosophic domain doesn't improve the testing accuracy while the Indeterminacy (I) neutrosophic domain, and the Falsity (F) neutrosophic domain decrease the testing accuracy for the original dataset from 0.92 to 0.87 by using the Falsity (F) neutrosophic domain, and from 0.92 to 0.85 by using the Indeterminacy (I) neutrosophic domain. The reason is some of the important features in the images were disappeared due to the conversion process and the boundaries of the image for objects in the images are not enough to differentiate between the masked and the unmasked class.

To conclude this subsection concerning the decision tree classifier accuracy, the decision tree classifier was able to classify between the masked and unmasked face images using the original domain or the True (T) neutrosophic domain with a testing accuracy of 0.92 % along with performance metrics with the same value of accuracy of 0.92%.

5.2. Experimental Results for SVM Classifier

The validation accuracy with associated performance measures is the first statistic to be measured. Figure 11 depicts the validation accuracy of the SVM classifier together with performance characteristics for the four dataset picture domains.

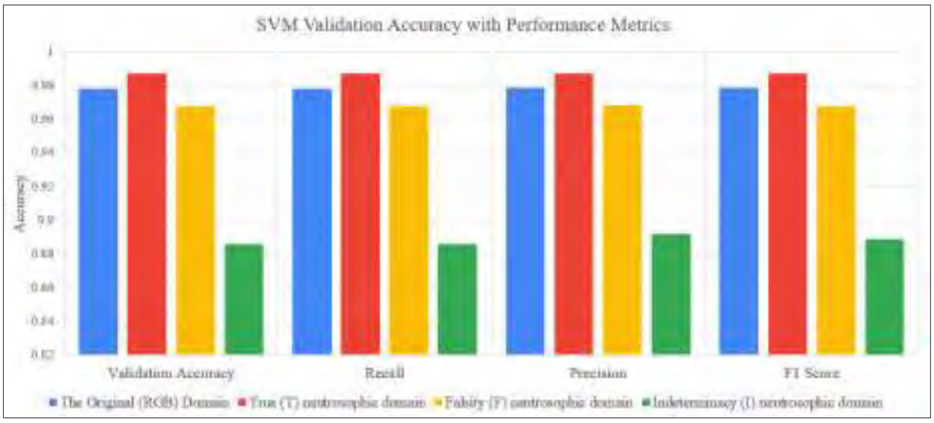

Figure 11. Validation accuracy of the SVM classifier using performance indicators for four dataset image domains.

Figure 11 shows that for each of the four domains, the SVM classifier outperforms the decision tree classifier in terms of validation accuracy. The SVM scores 0.9784 in validation accuracy in the original RGB domain, whereas the decision tree classifier reaches 0.8858. The same behavior is repeated in all the other neutrosophic domains. The highest accuracy possible achieved by the True (T) neutrosophic domain with 0.9871 validation accuracy. The performance metrics strengthen the validation accuracy for the True (T) neutrosophic domain with the same value of validation accuracy 0.9871 for recall, precision, and F1 score. Figure 12 depicts the SVM classifier's testing accuracy as well as performance metrics for four dataset image domains.

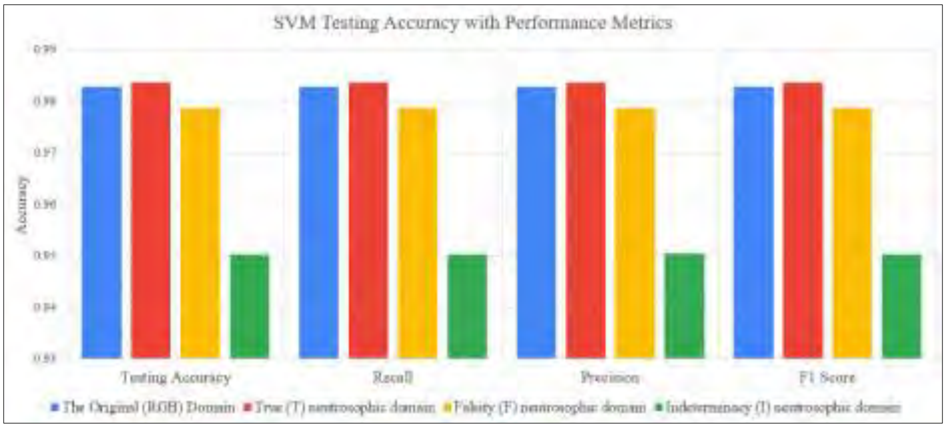

Figure 12. For four domains of image pictures, the SVM classifier's accuracy was tested using performance measures.

The testing accuracy is an accurate measure for the model accuracy is it present a large sample of data (20%). Figure 12 illustrates the SVM classifier achieves higher testing accuracy than the decision tree classifier in the four domains of images. In the True (T) neutrosophic domain, the SVM classifier achieves 0.9837 in the testing accuracy while the decision tree classifier achieved 0.9292. The improvement in testing accuracy by using the SVM classifier is notable and strengthen by the calculated performance metrics over the decision tree classifier.

It is clearly shown in Figure 12 that the True (T) neutrosophic domain achieves the highest accuracy possible with 0.9837 while the nearest accuracy achieved by the original RGB with 0.9829.

The difference is not very large, but it is considered an improvement for the testing accuracy of the proposed model. This improvement in the True (T) neutrosophic domain is due to that the True (T) neutrosophic domain correctly represent the features of the image which help in classifying between masked and unmasked face images correctly.

Figure 13 depicts the time spent by the various classifiers throughout the training process. It is well understood that the spent time is proportional to the dataset size and machine capabilities, yet it provides an indication of the classifier's performance.

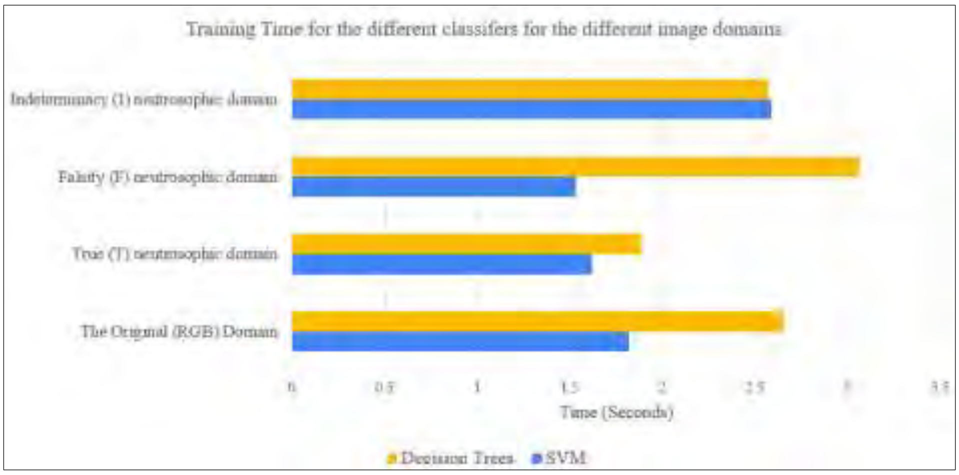

Figure 13. Training time spent by various classifiers for various image domains.

Figure 13 shows that the SVM classifier required less time to train in three of the four domains. The original domain RGB domain, the True (T) neutrosophic domain, and the Falsity (F) neutrosophic domain are the domains in which the SVM classifier obtained less time in training. To summaries this part, it is apparent that the SVM classifier outperforms the decision tree classifier in terms of validation, testing accuracy, performance metrics, and training time. With testing accuracy and performance metrics equal to 0.9837, the SVM classifier obtained the highest achievable accuracy in the True (T) neutrosophic domain.

The confusion matrix is also an excellent indicator of the performance of the model as it views more insights about the testing accuracy for every class in the dataset. Figure 14 presents the confusion matrix for the SVM classifier for the original RGB domain, and the True (T) neutrosophic domain.

Figure 14. Confusion matrix for (a) the original RGB domain, and (b) the True (T) neutrosophic domain

Figure 14 shows that the accuracy for the face "Mask" class is 98.4 percent for both the original RGB domain and the True (T) neutrosophic domain. The improvement is in the face “No Mask” class, where the testing accuracy is 98.5 for the True (T) neutrosophic domain and 98.2 percent for the original RGB domain.

5.3. Comparative Results

The study described in [37] employed the RMF dataset, and their model obtained testing accuracy ranging from 50% to 95%. The testing accuracy in the current work is 98.37% when utilizing the SVM classifier and the True (T) neutrosophic domain. According to the authors of the dataset [38], there is no documented accuracy for the simulated masked dataset SMF, there is no reported accuracy according to the author of the dataset. In this paper, we use the SVM classifier and the True (T) neutrosophic to achieve 98.37 percent testing accuracy. Table 1 compares related studies and prospective efforts that use the same datasets.

Table 1. A table comparing similar works and prospective efforts that use the same datasets

	Short description	Accuracy
[37]	key features extractions in visible parts of the masked face, such as face contour, ocular and periocular details, forehead with nearest neighbor algorithm.	50% to 95%
Proposed model	SVM classifier with the True (T) neutrosophic	98.37%

6. Conclusion and Future Works

A worldwide health catastrophe is triggered by the COVID-19 coronavirus pandemic. Governments all around the globe are battling to halt the spread of this sort of virus. Protection against COVID-19 infection, according to the World Health Organization (WHO), is a required countermeasure. Wearing a face mask in public places is one of the required countermeasures. A face mask classification model based on neutrosophic RGB with Convolutional Neural Network (CNN) for feature extraction and conventional machine learning was presented in this study. The suggested model was divided into three stages, the first of which was the conversion to the neutrosophic RGB domain. This study was regarded one of the earliest trails of using neutrosophic RGB conversion, since it was frequently utilized in grayscale picture conversion. The second state was the features extraction using Alexnet. It will be used as a feature extractor throughout the proposed model. The third phase was the detection model using classical machine learning. Two classical machine learning algorithms were investigated, and they were the decision tress classifier and Support Vector Machine (SVM). A dataset consisted of two different datasets, and they were the Real-World Mask Face dataset (RMF) and the Simulated Masked Face dataset (SMF). The combined dataset contained two classes (with a mask, and without a mask). The SVM classifier using the True (T) neutrosophic domain had the highest testing accuracy with 98.37 percent, according to the experimental findings. The acquired findings were validated by performance measures like as Precision, Recall, and F1 Score. At the end of the study, a comparison result was obtained, and the suggested model outperformed the findings of the related works in terms of testing accuracy. Deeper deep learning models for feature extraction, such as Resnet50 or Inception-ResNet-v2, may be included as one of the potential future efforts. In addition, other traditional machine learning techniques, such as Ensemble classifier, may be used to improve testing accuracy.

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An effective container inventory model under bipolar neutrosophic environment

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Abstract

The fuzzy set and its application play a major role to most of the uncertainty situations of inventory management problem. At present, as an enlargement of fuzzy set, the notion of neutrosophic set is initiated to implement in inventory models for uncertain parameters. In this paper, the Trapezoidal Bipolar Neutrosophic Number (TrBNN) is enforced to the container inventory model. Because of the imbalanced flow of containers, the container management organization faces the major issue of scarcity of containers. One-way Free Use (OFU) of container and renting of containers are employed to restore the shortfall units. An algorithm is designed to make a decision on various conditions to compute the expected total cost. Also, this paper scrutinizes a condition that some fraction of deficit containers is one-way free used and the remaining are leased. Unpredictable parameters such as the fraction of received containers after used, the fraction of amendable containers from received units, and the fraction of one-way free used containers are presumed as TrBNN. In the view of reduce the total cost, a neutrosophic container inventory model is framed to obtain the optimal duration of inspection process and the optimal duration of leasing process. To flourish this study more effective, the proposed container inventory model is compared with the model by presuming Triangular Bipolar Neutrosophic Number (TBNN).

Keywords: Trapezoidal bipolar neutrosophic number, Triangular bipolar neutrosophic number, Container inventory, One-way free use, Lease.

1. Introduction

The major aspect of any business trading is to attract the customers from the competitors throughout the globe. So that, the import and export trading has been developing day by day in most of the countries. The import and export traders spend lots of cost for cargoes transportation and thus for minimizing their cost and transporting cargoes safely, they chosen the Reusable Containers (RCs). In order to transporting goods from one destination to another, the non-vessel operating common carrier, or the shipping companies provides the RCs to the consigners. This study examines some of the issues and the various cost of maintaining the reusable container faced by the Container Management Organization (CMO).

The objective of any CMO is to satisfy the consigner's requirements. Because of imbalanced business trading, shortage of container will occur. This research prescribes the CMO to implement the following strategies in order to avoid from scarcity of containers. One of the main strategies is one-way free use; the OFU containers only make one trip. To return the containers at OFU vendor's depot, they allow OFU of containers from their surplus area to slack area instead of repositioning under free of cost. OFU containers help business to minimize the costs. Since, the OFU vendor offers only limited number of units, all the deficit containers are unable to replace under OFU option. Thus, the proportion of shortfall units are one-way free used and the remaining are restored from another CMO as rent. There is no cost spent for OFU when compared to leasing or purchasing a container. That is, OFU of container does not incurred costs like repositioning cost, carrying charge as well as repairing charge but it incurred only the screening charge.

Since the proportion of received containers after used, the proportion of amendable containers from received RCs, and the fraction of OFU RCs are uncertain, the present model framed under bipolar neutrosophic arena to reach the most approximate solutions. A container inventory model with price sensitive demand is developed and the costs under various strategies are framed. Neutrosophication of the various proportions of the proposed model leads the CMO to attain most approximate ratio. In this study, these proportions are considered as single type linear TrBNN and then removal area technique of de-bipolarization is applied. The proposed study is compared with TrBNN and TBNN. The research works related to these topics are discussed as follows.

Buchanan and Abad [9] framed the single and N-periods study on inventory model for routine system of reusable transport containers. A notion of repositioning empty containers on container inventory model along with leasing option under (*s,S*) policy is analyzed by Yun et al. [38]. Further, in 2014, Kim and Glock [22] scrutinized the RFID and its usage to the management of reusable containers by presuming the proportion of returned units as stochastic. Glock and Kim [16] studied an inventory model of combined finished goods and RTIs under some safety measures. Hariga et al. [19] designed the reusable container inventory model with finished goods of single vendor single retailer along with the renting option of RTIs for delay returns.

Cobb [13] studied the container inventory control problem to attain the optimal inspection length, the optimal mending period and the optimal purchasing. This study presumed that the investigation procedure as well as the mending procedure done subsequently and analyzed about the early returned containers which are stored as safety stock. The studies [15, 20, 23, 24] examine the container inventory model under various strategies. Further, the works [17] and [25] analyzed the container inventory model with the notion of repositioning of empty container. Maity et al. [26] analyzed the EOQ model under cloudy fuzzy logic. Rajeswari et al. [30] examined the work of Cobb [13] and presented a container management model with customer charge sensitive demand by utilizing the ECR as well as the renting option instead of buying new containers under fuzzy arena. Recently, [31 and 32] studied the notion of prepayment strategy in fuzzy EOQ model under various situations.

The uncertainty situation leads the researchers of various fields to use the approach of the fuzzy set and its applications. First of all, the fuzzy set and its approach were initiated by Zadeh [39]. Heilpern [21] formulated the expected value along with the expected value interval of fuzzy number. Then Atanassov [2] elongated the fuzzy set as intuitionistic fuzzy set and some researchers classified the intuitionistic fuzzy set under various types and applied to various situations. Shaw and Row [33] established the trapezoidal intuitionistic fuzzy set along with the arithmetic operations then applied to reach the accurate result.

As an enlargement of the fuzzy set and intuitionistic fuzzy set, the neutrosophic set is established by Smarandache [34]. For easy understanding, Wang et al. [36] designed the neutrosophic set as the singled valued neutrosophic set. For effective results, [3, 4, 8, 18, 29, 37] utilizes various measures on neutrosophic numbers to decision making models under various environments. A new de-netrosophication method of pentagonal neutrosophic number using removal area method is established by Chakraborty et al. [10] and utilized this method to a minimal spanning tree. Further, [27, 28] developed the inventory models with various strategies under neutrosophic arena.

By the view of negative and positive part of the human decision-making techniques, Bosc and Pivert [5] established the bipolar fuzzy set. Subsequently, Abdullah et al. [1] designed the bipolar fuzzy soft set and utilized to decision theory. Followed by these works, the concept bipolar neutrosophic set is initiated by Deli et al. [14] and applied its notion to decision making problem. In 2016, Broumi et al. [6] proposed the approach of bipolar single valued neutrosophic under graph theory and [7] utilizes the application of bipolar neutrosophic set and solved the shortest path problem. For effective outcomes, [11, 12, 35] framed the various measures on different bipolar neutrosophic numbers to decision making models under various situations.

The objective of this research is to help the CMO in order to supply the RC to their consigner without scarcity under minimum cost. The novelty of this study is to utilizing the OFU and leasing option to avoid the revenue loss from deficit containers instead of repositioning the empty container or purchasing new container. To attain the outcomes pore effective, this model is proposed under bipolar neutrosophic environment. The main contribution of this paper is the OFU option, which helps the CMO reduce the total cost. The advantage of utilizing the one-way free use strategy is the CMO compensates for the shortfall containers without spending much cost.

The elementary definitions on fuzzy set, Intuitionistic fuzzy set, Neutrosophic set and bipolar neutrosophic set are provided in section 2. In section 3, the container inventory costs under various strategies say serviceable container cost, OFU cost, ECR cost and least cost are computed, and an algorithm is given to make decision on choosing these strategies. Then the container inventory model under bipolar neutrosophic environment is framed by presuming the proportions of return rate, repair rate, and the OFU rate as TrBNN and also the de-bipolrization of TrBNN and TBNN are provided. The numerical computation and sensitivity analysis is performed in section 4. In the last section, the outcomes of the study are deliberated.

2. Preliminaries

Some of the preliminaries related to this research are as follows.

***Fuzzy set* [39]:** A set of 2-tuples $\tilde{\xi} = (z', \mu_{\tilde{\xi}}(z')): z' \in \mathcal{U}$ is said to be a a fuzzy set $\tilde{\xi}$ in $\mathcal{U}$ (Universe of discourse) where $\mu_{\tilde{\xi}}(z')$ represents the membership degree of z' such that $\mu_{\tilde{\xi}}(z') \in [0,1]$.

***Fuzzy number* [39]:** Let $\mathbb{R}$ be a real line then $\tilde{\xi} \subset \mathbb{R}$ is said to be a fuzzy number whose membership degree $\mu_{\tilde{\xi}}$ satisfies the given conditions

i. $\mu_{\tilde{\xi}}(z')$ is piecewise continuous in its domain.

ii. $\tilde{\xi}$ is normal, i.e., $\exists\, z_0' \in \tilde{\xi}$ such that $\mu_{\tilde{\xi}}(z_0') = 1$.

iii. $\tilde{\xi}$ is convex, i.e., $\mu_{\tilde{\xi}}(\epsilon z_1' + (1-\epsilon) z_2') \geq min\left(\mu_{\tilde{\xi}}(z_1'), \mu_{\tilde{\xi}}(z_2')\right) \forall\, z_1', z_2'$ in $\mathcal{U}$.

***Trapezoidal Fuzzy number*:[3]** Let $\tilde{\xi} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ be a trapezoidal fuzzy number then it has a membership degree

$$\mu_{\tilde{\xi}}(z') = \begin{cases} \frac{z' - \sigma_1}{\sigma_2 - \sigma_1}, & \sigma_1 \leq z' \leq \sigma_2 \\ 1, & \sigma_2 \leq z' \leq \sigma_3 \\ \frac{\sigma_4 - z'}{\sigma_4 - \sigma_3}, & \sigma_3 \leq z' \leq \sigma_4 \\ 0, & otherwise \end{cases}.$$

where, $\mu_{\tilde{\xi}}(z')$ satisfies the following conditions:

(i) $\mu_{\tilde{\xi}}(z')$ is a continuous mapping from $\mathbb{R}$ to [0,1],
(ii) $\mu_{\tilde{\xi}}(z') = 0$ for every $z' \in (-\infty, \sigma_1]$,
(iii) $\mu_{\tilde{\xi}}(z')$ is strictly increasing and continuous on $[\sigma_1, \sigma_2]$,
(iv) $\mu_{\tilde{\xi}}(z') = 0$ for every $z' \in [\sigma_2, \sigma_3]$,
(v) $\mu_{\tilde{\xi}}(z')$ is strictly decreasing and continuous on $[\sigma_3, \sigma_4]$,
(vi) $\mu_{\tilde{\xi}}(z') = 0$ for every $z' \in (\sigma_4, \infty,]$.

Intuitionistic Fuzzy Set **[2]:** An intuitionistic fuzzy set $\tilde{\xi}$ in $\mathcal{U}$ (finite universe of discourse) is given as $\tilde{\xi} = \{\langle z', \varphi_{\tilde{\xi}}(z'), \psi_{\tilde{\xi}}(z')\rangle \mid z' \in \mathcal{U}\}$, where $\varphi_{\tilde{\xi}}: \mathcal{U} \to [0,1]$ and $\psi_{\tilde{\xi}}: \mathcal{U} \to [0,1]$ satisfying the condition $0 \le \varphi_{\tilde{\xi}}(z') + \psi_{\tilde{\xi}}(z') \le 1$ and $\varphi_{\tilde{\xi}}(z')$ denotes the membership degree of $z' \in \mathcal{U}$ in $\tilde{\xi}$ and $\psi_{\tilde{\xi}}(z')$ represents the non-membership function of $z' \in \mathcal{U}$ in $\tilde{\xi}$. Along with this, the hesitance degree of $z' \in \mathcal{U}$ in $\tilde{\xi}$ is denoted as $\varepsilon_{\tilde{\xi}}(z')$ and is given as $\varepsilon_{\tilde{\xi}}(z') = 1 - \varphi_{\tilde{\xi}}(z') - \psi_{\tilde{\xi}}(z')$. For convenience, the intuitionistic fuzzy number is considered as $\tilde{\xi} = \left(\varphi_{\tilde{\xi}}(z'), \psi_{\tilde{\xi}}(z')\right)$.

Trapezoidal Intuitionistic Fuzzy Number **[33]:** A fuzzy number $\tilde{\mathcal{F}} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4), (\sigma_1^1, \sigma_2^1, \sigma_3^1, \sigma_4^1)$ is a trapezoidal intuitionistic fuzzy number where $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_1^1, \sigma_2^1, \sigma_3^1, \sigma_4^1$ are real and its membership degree $\mu_{\tilde{\mathcal{F}}}(g)$ and non-membership degree $\vartheta_{\tilde{\mathcal{F}}}(g)$ are given as below:

$$\mu_{\tilde{\mathcal{F}}}(g) = \begin{cases} \left(\frac{g-\sigma_1}{\sigma_2-\sigma_1}\right), & \sigma_1 \le g \le \sigma_2 \\ 1, & \sigma_2 \le g \le \sigma_3 \\ \left(\frac{\sigma_4-g}{\sigma_4-\sigma_3}\right), & \sigma_3 \le g \le \sigma_4 \\ 0, & otherwise \end{cases}$$

$$\text{and } \vartheta_{\tilde{\mathcal{F}}}(g) = \begin{cases} \left(\frac{\sigma_2^1-g}{\sigma_2^1-\sigma_1^1}\right), & \sigma_1^1 \le g \le \sigma_2^1 \\ 0, & \sigma_2^1 \le g \le \sigma_3^1 \\ \left(\frac{g-\sigma_3^1}{\sigma_4^1-\sigma_3^1}\right), & \sigma_3^1 \le g \le \sigma_4^1 \\ 1, & otherwise \end{cases}.$$

Neutrosophic Set **[34]:** A neutrosophic set $\tilde{\mathcal{A}}_N$ in $\mathcal{U}$ (Universe of discourse) is catogorized as functions of a truth membership $T_{\tilde{\mathcal{A}}_N}(g)$, an indeterminacy membership $I_{\tilde{\mathcal{A}}_N}(g)$ and a falsity membership $F_{\tilde{\mathcal{A}}_N}(g)$. The functions $T_{\tilde{\mathcal{A}}_N}$, $I_{\tilde{\mathcal{A}}_N}$ and $F_{\tilde{\mathcal{A}}_N}$ are real standard or non-standard subsets of $]^-0, 1^+[$ i.e., $T_{\tilde{\mathcal{A}}_N}: \mathcal{U} \to]^-0, 1^+[$; $I_{\tilde{\mathcal{A}}_N}: \mathcal{U} \to]^-0, 1^+[$; $F_{\tilde{\mathcal{A}}_N}: \mathcal{U} \to]^-0, 1^+[$. $T_{\tilde{\mathcal{A}}_N}(g)$, $I_{\tilde{\mathcal{A}}_N}(g)$, and $F_{\tilde{\mathcal{A}}_N}(g)$ satisfy the relation $^-0 \le \sup T_{\tilde{\mathcal{A}}_N}(g) \le \sup I_{\tilde{\mathcal{A}}_N}(g) \le \sup F_{\tilde{\mathcal{A}}_N}(g) \le 3^+$, where $g \in \mathcal{U}$

Fuzzy Neutrosophic Number **[3]:** Let $\tilde{\mathcal{A}}_N$ be a fuzzy neutrosophic number in the set of real numbers $\mathbb{R}$ then its validity membership degree, indeterminacy membership degree and negation membership degree are respectively written as below:

$$T_{\tilde{\mathcal{A}}_N}(g) = \left\langle \begin{matrix} T^L_{\tilde{\mathcal{A}}_N}(g) & \sigma_{11} \le g \le \sigma_{12} \\ 1 & \sigma_{12} \le g \le \sigma_{13} \\ T^U_{\tilde{\mathcal{A}}_N}(g) & \sigma_{13} \le g \le \sigma_{14} \\ 0 & otherwise \end{matrix} \right\rangle$$

$$I_{\tilde{\mathcal{A}}_N}(y) = \begin{cases} I^L_{\tilde{\mathcal{A}}_N}(g) & \sigma_{21} \le g \le \sigma_{22} \\ 0 & \sigma_{22} \le g \le \sigma_{23} \\ I^U_{\tilde{\mathcal{A}}_N}(g) & \sigma_{23} \le g \le \sigma_{24} \\ 1 & otherwise \end{cases}$$

$$F_{\tilde{\mathcal{A}}_N}(y) = \begin{cases} F^L_{\tilde{\mathcal{A}}_N}(g) & \sigma_{31} \le g \le \sigma_{32} \\ 0 & \sigma_{32} \le g \le \sigma_{33} \\ F^U_{\tilde{\mathcal{A}}_N}(g) & \sigma_{33} \le g \le \sigma_{34} \\ 1 & otherwise \end{cases}$$

where $0 \le \sup T_{\tilde{\mathcal{A}}_N}(g) \le \sup I_{\tilde{\mathcal{A}}_N}(g) \le \sup F_{\tilde{\mathcal{A}}_N}(g) \le 3$, $\forall\, g \in \mathcal{U}$ and for all $i = 1,2,3; j = 1,2,3,4$; $\sigma_{ij} \in$ R, such that $\sigma_{11} \le \sigma_{12} \le \sigma_{13} \le \sigma_{14}$; $\sigma_{21} \le \sigma_{22} \le \sigma_{23} \le \sigma_{24}$ and $\sigma_{31} \le \sigma_{32} \le \sigma_{33} \le \sigma_{34}$. Here $T^L_{\tilde{\mathcal{A}}_N}(g), I^U_{\tilde{\mathcal{A}}_N}(g), F^U_{\tilde{\mathcal{A}}_N}(g) \in [0,1]$ are continuous monotonic increasing functions and $T^U_{\tilde{\mathcal{A}}_N}(g), I^L_{\tilde{\mathcal{A}}_N}(g), F^L_{\tilde{\mathcal{A}}_N}(g) \in [0,1]$ are continuous monotonic decreasing functions.

***Single-valued Neutrosophic Set* [36]:** A single-valued neutrosophic set $\tilde{\mathcal{A}}_N$ in $\mathcal{U}$ (Universe of discourse) is catogorized as functions of a truth membership $T_{\tilde{\mathcal{A}}_N}(g)$, an indeterminacy membership $I_{\tilde{\mathcal{A}}_N}(g)$ and a falsity membership $F_{\tilde{\mathcal{A}}_N}(g)$and is given by

$\tilde{\mathcal{A}} = \{g, \langle T_{\tilde{\mathcal{A}}_N}(g),\, I_{\tilde{\mathcal{A}}_N}(g), F_{\tilde{\mathcal{A}}_N}(g)\rangle \mid g \in \mathcal{U}\}$.

Here $T_{\tilde{\mathcal{A}}_N}(g)$, $I_{\tilde{\mathcal{A}}_N}(g), F_{\tilde{\mathcal{A}}_N}(g) \in [0,1]$ and the relation $0 \le \sup T_{\tilde{\mathcal{A}}_N}(g) \le \sup I_{\tilde{\mathcal{A}}_N}(g) \le \sup F_{\tilde{\mathcal{A}}_N}(g) \le 3$ holds for all $g \in \mathcal{U}$.

***Trapezoidal neutrosophic number* [3]:**

A trapezoidal fuzzy neutrosophic number (TrFNN) $\tilde{\mathcal{A}}_N$ in $\mathcal{U}$ (Universe of discourse) is defined as follows:

$$\tilde{\mathcal{A}}_N = \langle(\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}), (\sigma_{21}, \sigma_{22}, \sigma_{23}, \sigma_{24}), (\sigma_{31}, \sigma_{32}, \sigma_{33}, \sigma_{34})\rangle$$

where $\sigma_{11} \le \sigma_{12} \le \sigma_{13} \le \sigma_{14}$; $\sigma_{21} \le \sigma_{22} \le \sigma_{23} \le \sigma_{24}$ and $\sigma_{31} \le \sigma_{32} \le \sigma_{33} \le \sigma_{34}$. Its truth member function is given as

$$T_{\tilde{\mathcal{A}}_N}(g) = \begin{cases} \frac{g-\sigma_{11}}{\sigma_{12}-\sigma_{11}} & \sigma_{11} \le g \le \sigma_{12} \\ 1 & \sigma_{12} \le g \le \sigma_{13} \\ \frac{\sigma_{14}-g}{\sigma_{14}-\sigma_{13}} & \sigma_{13} \le g \le \sigma_{14} \\ 0 & otherwise \end{cases}$$

Its indeterminacy membership function is written as

$$I_{\tilde{\mathcal{A}}_N}(g) = \begin{cases} \frac{\sigma_{22}-g}{\sigma_{22}-\sigma_{21}} & \sigma_{21} \le g \le \sigma_{22} \\ 0 & \sigma_{22} \le g \le \sigma_{23} \\ \frac{g-\sigma_{24}}{\sigma_{24}-\sigma_{23}} & \sigma_{23} \le g \le \sigma_{24} \\ 1 & otherwise \end{cases}$$

and its falsity membership function is written as

$$F_{\tilde{\mathcal{A}}_N}(g) = \begin{cases} \frac{\sigma_{32}-g}{\sigma_{32}-\sigma_{31}} & \sigma_{31} \le g \le \sigma_{32} \\ 0 & \sigma_{32} \le g \le \sigma_{33} \\ \frac{g-\sigma_{34}}{\sigma_{34}-\sigma_{33}} & \sigma_{33} \le g \le \sigma_{34} \\ 1 & otherwise \end{cases}$$

***Bipolar Neutrosophic Set* [14]:** A set $\tilde{\mathcal{A}}_{Bi}$ in $\mathcal{U}$ (finite universe of discourse) is said to be the bipolar neutrosophic set if

$\tilde{\mathcal{A}}_{Bi} = \{\langle g, [T_{\tilde{\mathcal{A}}_{Bi}}{}^{+}(g), I_{\tilde{\mathcal{A}}_{Bi}}{}^{+}(g), F_{\tilde{\mathcal{A}}_{Bi}}{}^{+}(g), T_{\tilde{\mathcal{A}}_{Bi}}{}^{-}(g), I_{\tilde{\mathcal{A}}_{Bi}}{}^{-}(g), F_{\tilde{\mathcal{A}}_{Bi}}{}^{-}(g)]\rangle \mid g \in \mathcal{U}\}$, where $T_{\tilde{\mathcal{A}}_{Bi}}{}^{+}(g): \mathcal{U} \to [0,1]$, $T_{\tilde{\mathcal{A}}_{Bi}}{}^{-}(g): \mathcal{U} \to [-1,0]$ represents the truth membership function, $I_{\tilde{\mathcal{A}}_{Bi}}{}^{+}(g): \mathcal{U} \to [0,1]$, $I_{\tilde{\mathcal{A}}_{Bi}}{}^{-}(g): \mathcal{U} \to [-1,0]$ represents the indeterminacy membership function and $F_{\tilde{\mathcal{A}}_{Bi}}{}^{+}(g): \mathcal{U} \to [0,1]$, $F_{\tilde{\mathcal{A}}_{Bi}}{}^{-}(g): \mathcal{U} \to [-1,0]$ represents the falsity membership function.

Triangular bipolar neutrosophic number **[3]:**

A triangular bipolar neutrosophic number (TBNN) $\tilde{\mathcal{A}}_{Bi}$ in $\mathcal{U}$ (Universe of discourse) is defined as follows:

$$\tilde{\mathcal{A}}_{Bi} = \langle(\sigma_{11}, \sigma_{12}, \sigma_{13}), (\sigma_{21}, \sigma_{22}, \sigma_{23}), (\sigma_{31}, \sigma_{32}, \sigma_{33})\rangle$$

where $\sigma_{11} \le \sigma_{12} \le \sigma_{13}$; $\sigma_{21} \le \sigma_{22} \le \sigma_{23}$ and $\sigma_{31} \le \sigma_{32} \le \sigma_{33}$. Its validity membership degree, indeterminacy membership degree and the negation membership degree of TBNN are written as

$$T_{\tilde{\mathcal{A}}_{Bi}}{}^{+}(g) = \begin{cases} \frac{g-\sigma_{11}}{\sigma_{12}-\sigma_{11}} & \sigma_{11} \le g < \rho_2 \\ 1 & g = \sigma_{12} \\ \frac{\sigma_{13}-g}{\sigma_{13}-\sigma_{12}} & \sigma_{12} < g \le \rho_4 \\ 0 & otherwise \end{cases}, \quad T_{\tilde{\mathcal{A}}_{Bi}}{}^{-}(g) = \begin{cases} \frac{\sigma_{12}-g}{\sigma_{12}-\rho_1} & \sigma_{11} \le g < \rho_2 \\ -1 & g = \sigma_{12} \\ \frac{g-\sigma_{13}}{\sigma_{13}-\sigma_{12}} & \sigma_{12} < g \le \sigma_{13} \\ 0 & otherwise \end{cases};$$

$$I_{\tilde{\mathcal{A}}_{Bi}}{}^{+}(g) = \begin{cases} \frac{\sigma_{22}-g}{\sigma_{22}-\sigma_{21}} & \sigma_{21} \le g < \sigma_{22} \\ 0 & g = \sigma_{22} \\ \frac{g-\sigma_{23}}{\sigma_{23}-\sigma_{22}} & \sigma_{22} < g \le \sigma_{23} \\ 1 & otherwise \end{cases}, \quad I_{\tilde{\mathcal{A}}_{Bi}}{}^{-}(g) = \begin{cases} \frac{g-\sigma_{22}}{\sigma_{22}-\sigma_{21}} & \sigma_{21} \le g < \sigma_{22} \\ 0 & g = \sigma_{22} \\ \frac{\sigma_{23}-g}{\sigma_{23}-\sigma_{22}} & \sigma_{22} < g \le \sigma_{23} \\ -1 & otherwise \end{cases} \text{ and}$$

$$F_{\tilde{\mathcal{A}}_{Bi}}{}^{+}(g) = \begin{cases} \frac{\sigma_{32}-g}{\sigma_{32}-\sigma_{31}} & \sigma_{31} \le g < \sigma_{32} \\ 0 & g = \sigma_{32} \\ \frac{g-\sigma_{33}}{\sigma_{33}-\sigma_{32}} & \sigma_{32} < g \le \sigma_{33} \\ 1 & otherwise \end{cases}, \quad F_{\tilde{\mathcal{A}}_{Bi}}{}^{-}(g) = \begin{cases} \frac{g-\vartheta_{2}}{\vartheta_{2}-\sigma_{31}} & \sigma_{31} \le g < \sigma_{32} \\ 0 & g = \sigma_{32} \\ \frac{\sigma_{33}-g}{\sigma_{33}-\vartheta_{3}} & \sigma_{32} < g \le \sigma_{33} \\ -1 & otherwise \end{cases}$$

where $-3 \le T_{\tilde{\mathcal{A}}_{Bi}}(g) + I_{\tilde{\mathcal{A}}_{Bi}}(g) + F_{\tilde{\mathcal{A}}_{Bi}}(g) \le 3+$.

Trapezoidal bipolar neutrosophic number **[12]:** A trapezoidal bipolar neutrosophic number (TrBNN) $\tilde{\boldsymbol{\mathcal{A}}}_{\boldsymbol{Bi}}$ in $\mathcal{U}$ (finite universe of discourse) characterized by three independent membership degrees as follows:

$$\tilde{\mathcal{A}}_{Bi} = \langle(\rho_1, \rho_2, \rho_3, \rho_4), (\sigma_1, \sigma_2, \sigma_3, \sigma_4), (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)\rangle$$

Where $\rho_1 \le \rho_2 \le \rho_3 \le \rho_4$; $\sigma_1 \le \sigma_2 \le \sigma_3 \le \sigma_4$ and $\vartheta_1 \le \vartheta_2 \le \vartheta_3 \le \vartheta_4$.

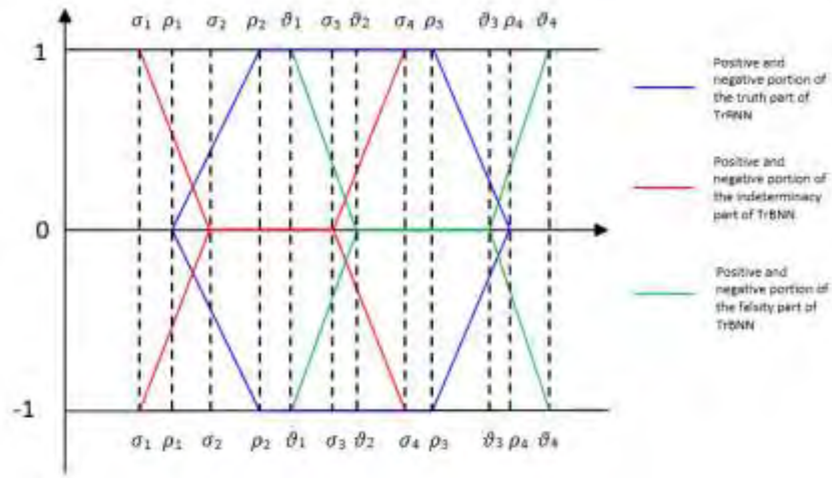

Fig. 1 Linear Trapezoidal Bipolar Neutrosophic Number

Let the positive and the negative portion of truth membership function are denoted by ${T_{\tilde{\mathcal{A}}_{Bi}}}^{+}$ and ${T_{\tilde{\mathcal{A}}_{Bi}}}^{-}$ respectively, the positive and the negative portion of indeterminacy membership function are represented by ${I_{\tilde{\mathcal{A}}_{Bi}}}^{+}$ and ${I_{\tilde{\mathcal{A}}_{Bi}}}^{-}$ respectively and the positive and the negative portion of negation membership function are denoted by ${F_{\tilde{\mathcal{A}}_{Bi}}}^{+}$ and ${F_{\tilde{\mathcal{A}}_{Bi}}}^{-}$ respectively then the validity membership degree, indeterminacy membership degree and the negation membership degree of TrBNN are written as

$${T_{\tilde{\mathcal{A}}_{Bi}}}^{+}(g) = \begin{cases} \frac{g-\rho_1}{\rho_2-\rho_1} & \rho_1 \le g < \rho_2 \\ 1 & \rho_2 \le g \le \rho_3 \\ \frac{\rho_4-g}{\rho_4-\rho_3} & \rho_3 < g \le \rho_4 \\ 0 & otherwise \end{cases}, \quad {T_{\tilde{\mathcal{A}}_{Bi}}}^{-}(g) = \begin{cases} \frac{\rho_2-g}{\rho_2-\rho_1} & \rho_1 \le g < \rho_2 \\ -1 & \rho_2 \le g \le \rho_3 \\ \frac{g-\rho_4}{\rho_4-\rho_3} & \rho_3 < g \le \rho_4 \\ 0 & otherwise \end{cases};$$

$${I_{\tilde{\mathcal{A}}_{Bi}}}^{+}(g) = \begin{cases} \frac{\sigma_2-g}{\sigma_2-\sigma_1} & \sigma_1 \le g < \sigma_2 \\ 0 & \sigma_2 \le g \le \sigma_3 \\ \frac{g-\sigma_4}{\sigma_4-\sigma_3} & \sigma_3 < g \le \sigma_4 \\ 1 & otherwise \end{cases}, \quad {I_{\tilde{\mathcal{A}}_{Bi}}}^{-}(g) = \begin{cases} \frac{g-\sigma_2}{\sigma_2-\sigma_1} & \sigma_1 \le g < \sigma_2 \\ 0 & \sigma_2 \le g \le \sigma_3 \\ \frac{\sigma_4-g}{\sigma_4-\sigma_3} & \sigma_3 < g \le \sigma_4 \\ -1 & otherwise \end{cases} \text{ and}$$

$${F_{\tilde{\mathcal{A}}_{Bi}}}^{+}(g) = \begin{cases} \frac{\vartheta_2-g}{\vartheta_2-\vartheta_1} & \vartheta_1 \le g < \vartheta_2 \\ 0 & \vartheta_2 \le g \le \vartheta_3 \\ \frac{g-\vartheta_4}{\vartheta_4-\vartheta_3} & \vartheta_3 < g \le \vartheta_4 \\ 1 & otherwise \end{cases}, \quad {F_{\tilde{\mathcal{A}}_{Bi}}}^{-}(g) = \begin{cases} \frac{g-\vartheta_2}{\vartheta_2-\vartheta_1} & \vartheta_1 \le g < \vartheta_2 \\ 0 & \vartheta_2 \le g \le \vartheta_3 \\ \frac{\vartheta_4-g}{\vartheta_4-\vartheta_3} & \vartheta_3 < g \le \vartheta_4 \\ -1 & otherwise \end{cases}$$

The de-bipolarization of single typed linear TrBNN is adopted from Chakraborty et al. [12] which is written as

$$S\left(\widetilde{De}_{\mathcal{A}_{Bi},0}\right) = \frac{\rho_1+\rho_2+\rho_3+\rho_4+\sigma_1+\sigma_2+\sigma_3+\sigma_4+\vartheta_1+\vartheta_2+\vartheta_3+\vartheta_4}{6}. \tag{1}$$

The de-bipolarization of single typed linear TBNN is adopted from Chakraborty et al. [11] which is written as

$$S\left(\widetilde{De}_{\mathcal{A}_{Bi},0}\right) = \frac{\sigma_{11}+2\sigma_{12}+\sigma_{13}+\sigma_{21}+2\sigma_{22}+\sigma_{23}+\sigma_{31}+2\sigma_{32}+\sigma_{33}}{6}. \tag{2}$$

3. Problem development:

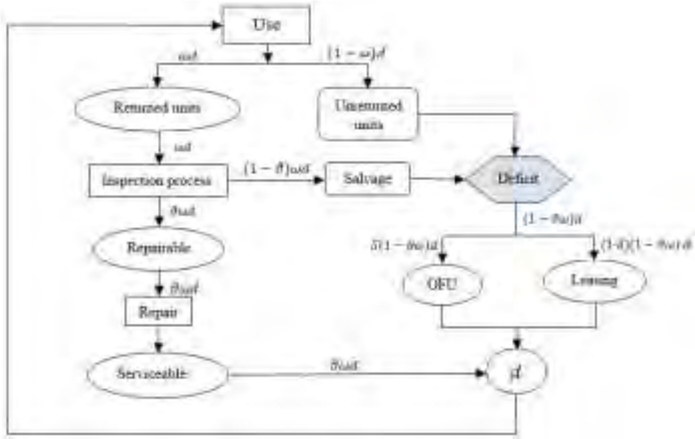

Fig.2 Diagrammatic representation of container flow per cycle

For the purpose of import and export trading, the container management organization affords the empty container to its consigner. Every CMO managing and supplies the large number of empty containers for trading. Fig. 2 clearly shows the container flow under various strategies on a closed loop supply chain. For the convenience, in this study, it is presumed that the CMO supplied the demand of d per supply chain, and it is assumed as price sensitive demand, that is $d = A - B\wp$, where $\wp$ represents the rent acquires from customer per container, $A > 0$ and $B > 0$ are constant and price dependent coefficients respectively. Because of the imbalance flow of containers, this study scrutinizes the shortage territory. According to [16], instead of d units, ω fraction of used containers received and the balance $(1-\omega)d$ units being unreturned in that supply chain. After receiving the used containers, they are subjected to the process of inspecting during the time τ_I at constant rate φ, as well as the process of repairing begins simultaneously at the rate R. While screening ωd units, it is observed that ϑfraction of received units is restorable but the remaining $(1-\vartheta)\omega d$ units are not reusable which are kept as salvaged units. Here, the unreturned and the salvaged units are considered as deficit containers. The duration to mend $\vartheta\omega d$ is considered as τ_R. The number of reusable containers inspected during τ_I is $\varphi\tau_I$ and $\tau_R = \frac{\vartheta\varphi T_i}{R}$ is the repairing process duration. After the repairing process, the $\vartheta\omega d$ RCs are assigned as ready to service units. To restore the deficit containers, the OFU of container is considered and OFU vendor allows only a limited unit for one-way free use. So that, the proportion of shortfall RCs say $\delta(1-\omega\vartheta)d$ is presumed as OFU and the remaining $(1-\delta)(1-\omega\vartheta)d$ RCs are leased from another CMO and stored with containers that are ready to service.

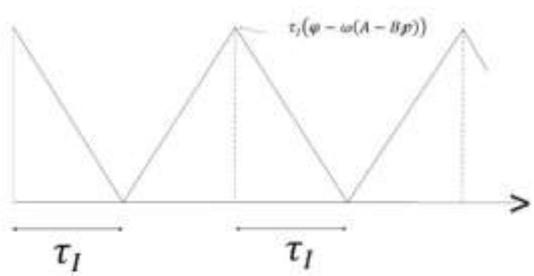

Fig. 3 (a)

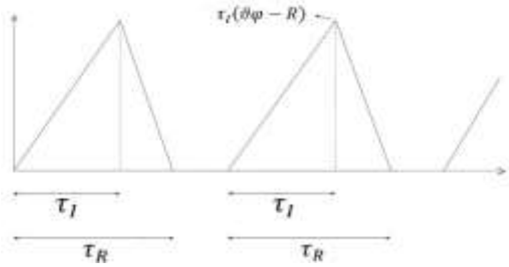

Fig. 3 (b)

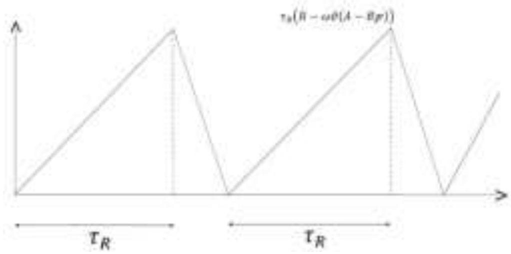

Fig. 3 (c)

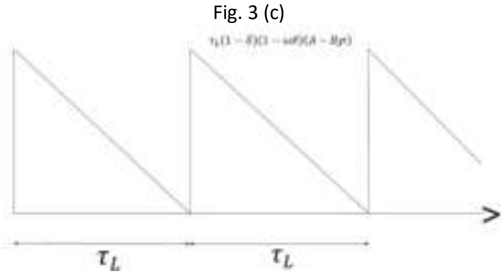

Fig. 3 (d)

Fig. 3 Inventory level of the Containers under various strategies

3.1. *Maximum inventory level*

The amount of received RCs, the number of amendable ones, the number of RCs that are ready to service and the amount of rented RCs and then the maximum units of all the above strategies are conferred as follows.

In this study, it is hypothesized that ωd reusable containers are received after the use and the left over $(1-\omega)d$ containers are not received in that cycle. Fig. 3(a) clearly indicates that the duration inspecting the received container is τ_I and the number of units examined per length is $\varphi\tau_I$. Thus, for each inspection procedure, the proportion of a year is $\frac{\varphi\tau_I}{\omega D}$, where $D = \mathcal{m}d$, is the annual demand and $\mathcal{m}$ denotes the total working days per annum. The idle time between the inspection procedure per unit time and the RCs stocked over that time is $\mathcal{m}\frac{\varphi\tau_I}{\omega D} - \tau_I$. Also, by observing Figs. 3(a)-3(c), it is noted that the examining and the mending works proceeds simultaneously. Once a container is screened, it has been sent for mending process and then kept as ready to service RCs.

$$\text{The maximum unit of received RCs} = \left(\mathcal{m}\frac{\varphi\tau_I}{\omega D} - \tau_I\right)\omega d$$
$$= \tau_I\big(\varphi - \omega(A - B\mathcal{p})\big) \quad (3)$$

Here, the inventory level reduces at a rate of $\varphi - \omega(A - B\mathcal{p})$ per day and the containers received at the fraction of $\omega(A - B\mathcal{p})$ per day when the inspection period is idle.

It is clear that the number of containers sent for repairing process is $\vartheta\varphi\tau_I = R\tau_R$. During the time τ_I, the rate of $\vartheta\varphi - R$ containers received for repairing process.

Thus, the maximum unit of amendable RCs $= \tau_I\ (\vartheta\varphi - R)$. (4)

The total days that mending process done per year is $\frac{\vartheta\varphi D}{R}$ and percentage of mending process per year is $\frac{\vartheta\varphi D}{mR}$. Clearly, the stock level reduces at the rate of R for one unit time during the period $\tau_I\ (\vartheta\varphi - R)/R$.

On observing Fig. 3(b) and Fig. 3(c), the RCs are stored as ready to service units after the completion of the mending process. Thus, the serviceable containers compile by $R - \vartheta\varphi(A - B\wp)$ for one unit time which reduces by the rate of $\vartheta\varphi(A - B\wp)$ when the mending time is idle. Hence,

The maximum stock of serviceable RCs $=\tau_R\big(R - \vartheta\varphi(A - B\wp)\big)$. (5)

To rectify the issue of unbalancing container flow, the CMO preferred some effective strategies such as OFU and leasing option. Since, the OFU vendor offers only limited number of containers for OFU, the fraction $\delta(1 - \omega\vartheta)d$ is considered as OFU and the remaining units are restored using leasing option. Therefore, the maximum number of containers for OFU are $\delta(1 - \omega\vartheta)(A - B\wp)$.

It is clearly noted that from Fig. 3(d), the time τ_L represents the successive leasing cycles. During this time period, the expected maximum number of rented RCs per cycle is given as follows.

The maximum stock of rented RCs $= \tau_L(1-\delta)(1-\omega\vartheta)(A - B\wp)$. (6)

Here, the inventory reduces at the rate of $(1-\delta)(1-\omega\vartheta)(A - B\wp)$ per leasing cycle.

3.2. *Cost model*

3.2.1. *Serviceable container cost*

The serviceable container cost is the charges for screening and repairing process of returned containers which includes the fixed charge and the variable charge of screened and repairable units and the storage charge of received, repaired and serviceable RCs. The variable charges of all the categories are computed with respect to number of RCs processed. The integrated fixed costs and the variable costs obtained from the stock of received RCs and amendable RCs is given as,

$$FC_{IR} + VC_{IR} = \frac{m\omega(A - B\wp)(W_i + W_r)}{\varphi\tau_I} + m\omega(A - B\wp)(w_i + \vartheta w_r). \quad (7)$$

The storage cost is the charge received by the empty yard for the stock of received RCs, amendable RCs and the ready to service RCs which is derived as,

$$HC_U = \frac{\tau_I}{2}\Big[\big(\varphi - \omega(A - B\wp)\big)\mathcal{H}_u\Big]. \quad (8)$$

$$HC_R = \frac{\tau_I}{2}\Big[\frac{\omega\vartheta(A-B\wp)(\varphi\vartheta - R)}{R}\mathcal{H}_r\Big]. \quad (9)$$

$$HC_S = \frac{\tau_I}{2}\Big[\frac{\varphi\vartheta\big(R - \vartheta\omega(A - B\wp)\big)}{R}\mathcal{H}_s\Big]. \quad (10)$$

The serviceable container cost for returned container is derived as

$$SCC = FC_{IR} + VC_{IR} + HC_U + HC_R + HC_S \quad (11)$$

$$= \frac{m\omega(A - B\wp)(W_i + W_r)}{\varphi\tau_I} + m\omega(A - B\wp)(w_i + \vartheta w_r) + \frac{\tau_I}{2}\Bigg[\big(\varphi - \omega(A - B\wp)\big)\mathcal{H}_u + \frac{\omega\vartheta(A - B\wp)(\varphi\vartheta - R)}{R}\mathcal{H}_r + \frac{\varphi\vartheta\big(R - \vartheta\omega(A - B\wp)\big)}{R}\mathcal{H}_s\Bigg]. \quad (12)$$

where W_i and w_i denotes the fixed and variable inspection charge per container; W_r and w_r represents the fixed and variable mending charge per container; $\mathcal{H}_u$, $\mathcal{H}_r$ and $\mathcal{H}_s$ are the unit storage charge for returned, repaired and serviceable containers respectively.

3.2.2. *One-way Free Use cost*

For OFU containers, the customer should pick the serviceable containers from the OFU vendor's empty depot. So that, there is no cost spent for repairing as well as carrying the OFU containers but for survey process, the inspection charge will arise. In OFU, the variable inspection charge per container is considered as $(1-\theta)$ proportion of unit variable screening charge of returned container w_i. Thus, the one-way free use cost is given as

$$OFUC = m\delta(1-\omega\vartheta)(A-B\wp)(1-\theta)w_i. \quad (13)$$

3.2.3. *Lease cost*

The fixed ordering charge and the rent for a leased container are W_ℓ and w_ℓ respectively and $\mathcal{H}_\ell$ denotes the unit storage charge per leased container. Thus, the combined fixed costs and the variable costs of leased containers is given as,

$$FC_L + VC_L = \frac{mW_\ell}{\tau_L} + m(1-\delta)(1-\omega\vartheta)(A-B\wp)w_\ell. \quad (14)$$

The carrying charge of leased containers is derived as

$$HC_L = \frac{\tau_L}{2}[(1-\delta)(1-\omega\vartheta)(A-B\wp)\mathcal{H}_\ell]. \quad (15)$$

Therefore, the lease cost is given as

$$LC = FC_L + VC_L + HC_L. \quad (16)$$

$$= \frac{mW_\ell}{\tau_L} + m(1-\delta)(1-\omega\vartheta)(A-B\wp)w_\ell + \frac{\tau_L}{2}[(1-\delta)(1-\omega\vartheta)(A-B\wp)\mathcal{H}_\ell]. \quad (17)$$

An algorithm is designed to make decision on various strategies for computing the total cost.

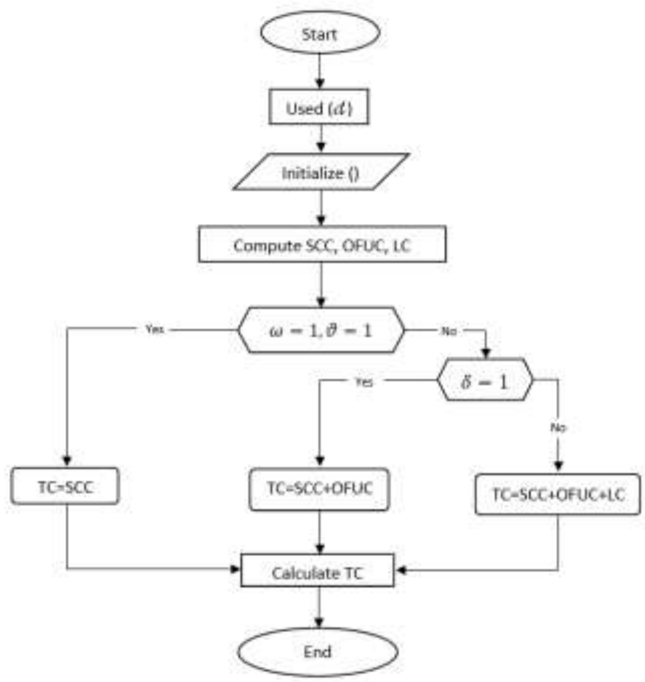

Fig. 4 Flow chart for decision making

Step 1: Initialize all parameters.

Step 2: Calculate SCC, $OFUC$ and LC.
Step 3: If $\omega = 1$ and $\vartheta = 1$, then the total cost $TC = SCC$, using Equation (12).
Step 4: If $\omega \neq 1$, $\vartheta \neq 1$ and $\delta = 1$, then the total cost $TC = SCC + OFUC$, using Equations (12) and (13).
Step 5: If $\omega \neq 1$, $\vartheta \neq 1$ and $\delta \neq 1$ then the total cost $TC = SCC + OFUC + LC$, using Equations (12), (13) and (17).
In this paper, it is considered that $\omega \neq 1$, $\vartheta \neq 1$ and $\delta \neq 1$.
Thus, the total cost, $TC = SCC + OFUC + LC.$ (18)
Therefore, the total cost is obtained as

$$TC(\tau_I,\tau_L) = \frac{m\omega(A-B\wp)(W_i+W_r)}{\varphi\tau_I} + \frac{mW_\ell}{\tau_L} + m(A-B\wp)[\omega(w_i+\vartheta(w_r-\delta(1-\theta)w_i-(1-\delta)w_\ell))+\delta(1-\theta)w_i + (1-\delta)w_\ell] + \frac{\tau_I}{2}\left[(\varphi-\omega(A-B\wp))\mathcal{H}_u + \frac{\omega\vartheta(A-B\wp)(\varphi\vartheta-R)}{R}\mathcal{H}_r + \frac{\varphi\vartheta(R-\omega\vartheta(A-B\wp))}{R}\mathcal{H}_s\right] + \frac{\tau_L}{2}[(1-\delta])(1-\omega\vartheta)(A - B\wp)\mathcal{H}_\ell]. \quad (19)$$

3.3 *Bipolar Neutrosophic container inventory model*

The container inventory model under bipolar neutrosophic arena is framed by presuming the proportion of received RCs, the fraction of repairable from received RCs and the fraction of OFU RCs as TrBNN. That is,

$$\widetilde{\omega} = \langle(\omega_{11},\omega_{12},\omega_{13},\omega_{14}),(\omega_{21},\omega_{22},\omega_{23},\omega_{24}),(\omega_{31},\omega_{32},\omega_{33},\omega_{34})\rangle;$$
$$\widetilde{\vartheta} = \langle(\vartheta_{11},\vartheta_{12},\vartheta_{13},\vartheta_{14}),(\vartheta_{21},\vartheta_{22},\vartheta_{23},\vartheta_{24}),(\vartheta_{31},\vartheta_{32},\vartheta_{33},\vartheta_{34})\rangle;$$
$$\widetilde{\delta} = \langle(\delta_{11},\delta_{12},\delta_{13},\delta_{14}),(\delta_{21},\delta_{22},\delta_{23},\delta_{24}),(\delta_{31},\delta_{32},\delta_{33},\delta_{34})\rangle .$$

Therefore, the total cost in bipolar neutrosophic sense, $TC_{Bi}(\widetilde{\tau_I,\tau_E},\tau_L)$ is obtained as,

$$TC_{\widetilde{Bi(\tau_I},\tau_L)} = \frac{m\widetilde{\omega}(A-B\wp)(W_i+W_r)}{\varphi\tau_I} + \frac{mW_\ell}{\tau_L} + m(A-B\wp)[\widetilde{\omega}\left(w_i+\widetilde{\vartheta}(w_r-\widetilde{\delta}(1-\theta)w_i-(1-\widetilde{\delta})w_\ell)\right)+\widetilde{\delta}(1-\theta)w_i + (1-\widetilde{\delta})w_\ell] + \frac{\tau_I}{2}\left[(\varphi-\widetilde{\omega}(A-B\wp))\mathcal{H}_u + \frac{\widetilde{\omega}\widetilde{\vartheta}(A-B\wp)(\varphi\vartheta-R)}{R}\mathcal{H}_r + \frac{\varphi\widetilde{\vartheta}\left(R-\widetilde{\omega}\widetilde{\vartheta}(A-B\wp)\right)}{R}\mathcal{H}_s\right] + \frac{\tau_L}{2}[(1-\widetilde{\delta}])(1-\widetilde{\omega}\widetilde{\vartheta})(A - B\wp)\mathcal{H}_\ell]. \quad (20)$$

From equation (1), the de-bipolarization of TrBNNs $\widetilde{\omega}$, $\widetilde{\vartheta}$ and $\widetilde{\delta}$ are obtained as follows:

$$\widetilde{\omega_{BneuD}} = \frac{\omega_{11}+\omega_{12}+\omega_{13}+\omega_{14}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{24}+\omega_{31}+\omega_{32}+\omega_{33}+\omega_{34}}{6}, \quad (21)$$

$$\widetilde{\vartheta_{BneuD}} = \frac{\vartheta_{11}+\vartheta_{12}+\vartheta_{13}+\vartheta_{14}+\vartheta_{21}+\vartheta_{22}+\vartheta_{23}+\vartheta_{24}+\vartheta_{31}+\vartheta_{32}+\vartheta_{33}+\vartheta_{34}}{6}, \quad (22)$$

$$\widetilde{\delta_{BneuD}} = \frac{\delta_{11}+\delta_{12}+\delta_{13}+\delta_{14}+\delta_{21}+\delta_{22}+\delta_{23}+\delta_{24}+\delta_{31}+\delta_{32}+\delta_{33}+\delta_{34}}{6} \quad (23)$$

Hence, by de-bipolarization it is obtained as

$$\begin{aligned} \widetilde{TC_{BneuD}(\tau_I,\tau_L)} = {} & \frac{m\widetilde{\omega_{BneuD}}(A-B\wp)(W_i+W_r)}{\varphi\tau_I} + \frac{mW_\ell}{\tau_L} \\ & + m(A-B\wp)\widetilde{\omega_{BneuD}}\left(w_i + \widetilde{\vartheta_{BneuD}}\left(w_r - \widetilde{\delta_{BneuD}}(1-\theta)w_i - \left(1-\widetilde{\delta_{BneuD}}\right)w_\ell\right)\right) \\ & + \widetilde{\delta_{BneuD}}(1-\theta)w_i + \left(1-\widetilde{\delta_{BneuD}}\right)w_\ell] \\ & + \frac{\tau_I}{2}\left[\left(\varphi - \widetilde{\omega_{BneuD}}(A-B\wp)\right)\mathcal{H}_u + \frac{\widetilde{\omega_{BneuD}}\widetilde{\vartheta_{BneuD}}(A-B\wp)\left(\varphi\widetilde{\vartheta_{BneuD}}-R\right)}{R}\mathcal{H}_r \right. \\ & \left. + \frac{\varphi\widetilde{\vartheta_{BneuD}}\left(R-\widetilde{\omega_{BneuD}}\widetilde{\vartheta_{BneuD}}(A-B\wp)\right)}{R}\mathcal{H}_s\right] \\ & + \frac{\tau_L}{2}\left[\left(1-\widetilde{\delta_{BneuD}}\right)\left(1-\widetilde{\omega_{BneuD}}\widetilde{\vartheta_{BneuD}}\right)(A \right. \\ & \left. - B\wp)\mathcal{H}_\ell\right]. \end{aligned} \quad (24)$$

To flourish this paper more effective, the proposed model is also computed by considering the unpredictable parameters ω, ϑ and δ as TBNN. That is,

$$\widetilde{\omega} = \langle(\omega'_{11}, \omega'_{12}, \omega'_{13}), (\omega'_{21}, \omega'_{22}, \omega'_{23}), (\omega'_{31}, \omega'_{32}, \omega'_{33})\rangle;$$
$$\widetilde{\vartheta} = \langle\left(\vartheta'_{11}, \vartheta'_{12}, \vartheta'_{13}\right), \left(\vartheta'_{21}, \vartheta'_{22}, \vartheta'_{23}\right), \left(\vartheta'_{31}, \vartheta'_{32}, \vartheta'_{33}\right)\rangle;$$
$$\widetilde{\delta} = \langle\left(\delta'_{11}, \delta'_{12}, \delta'_{13}\right), \left(\delta'_{21}, \delta'_{22}, \delta'_{23}\right), \left(\delta'_{31}, \delta'_{32}, \delta'_{33}\right)\rangle; .$$

Thus, from equation (2), the de-bipolarization of TBNNs $\widetilde{\omega}$, $\widetilde{\vartheta}$ and $\widetilde{\delta}$ are obtained as follows:

$$\widetilde{\omega_{BneuD}} = \frac{\omega'_{11}+2\omega'_{12}+\omega'_{13}+\omega'_{21}+2\omega'_{22}+\omega'_{23}+\omega'_{31}+2\omega'_{32}+\omega'_{33}}{6}, \quad (25)$$

$$\widetilde{\vartheta_{BneuD}} = \frac{\vartheta'_{11}+2\vartheta'_{12}+\vartheta'_{13}+\vartheta'_{21}+2\vartheta'_{22}+\vartheta'_{23}+\vartheta'_{31}+2\vartheta'_{32}+\vartheta'_{33}}{6}, \quad (26)$$

$$\widetilde{\delta_{BneuD}} = \frac{\delta'_{11}+2\delta'_{12}+\delta'_{13}+\delta'_{21}+2\delta'_{22}+\delta'_{23}+\delta'_{31}+2\delta'_{32}+\delta'_{33}}{6}. \quad (27)$$

The above set of equations is substituted in the equation (24) to obtain the result under TBNN environment.

Preposition:

(a) $\widetilde{TC_{BneuD}(\tau_I,\tau_L)}$ is strictly convex.

(b) The optimal inspection duration and the optimal leasing duration are

$$\tau_I{}^* = \sqrt{\frac{2Rm\widetilde{\omega_{BneuD}}(A-B\wp)(W_i+W_r)}{\varphi R\left(\varphi-\widetilde{\omega_{BneuD}}(A-B\wp)\right)\mathcal{H}_u + \varphi\widetilde{\omega_{BneuD}}\widetilde{\vartheta_{BneuD}}(A-B\wp)\left(\varphi\widetilde{\vartheta_{BneuD}}-R\right)\mathcal{H}_r + \varphi^2\widetilde{\vartheta_{BneuD}}\left(R-\widetilde{\omega_{BneuD}}\widetilde{\vartheta_{BneuD}}(A-B\wp)\right)\mathcal{H}_s}} \quad (28)$$

and $$\tau_L{}^* = \sqrt{\frac{2mW_\ell}{\left(1-\widetilde{\delta_{BneuD}}\right)\left(1-\widetilde{\omega_{BneuD}}\widetilde{\vartheta_{BneuD}}\right)(A-B\wp)\mathcal{H}_\ell}}. \quad (29)$$

Proof:

Differentiating eqn. (24) partially with respect to τ_I and τ_L, it is obtained as

$$\frac{\partial TC_{\widetilde{BneuD}}(\tau_I,\tau_L)}{\partial \tau_I} = \frac{-m\omega_{\widetilde{BneuD}}(A-B\wp)(W_i+W_r)}{\varphi {\tau_I}^2} + \frac{1}{2}\Bigg[\left(\varphi-\omega_{\widetilde{BneuD}}(A-B\wp)\right)\mathcal{H}_u + \frac{\omega_{\widetilde{BneuD}}\vartheta_{\widetilde{BneuD}}(A-B\wp)\left(\varphi\vartheta_{\widetilde{BneuD}}-R\right)}{R}\mathcal{H}_r + \frac{\varphi\vartheta_{\widetilde{BneuD}}\left(R-\omega_{\widetilde{BneuD}}\vartheta_{\widetilde{BneuD}}(A-B\wp)\right)}{R}\mathcal{H}_s\Bigg],$$

$$\frac{\partial TC_{\widetilde{BneuD}}(\tau_I,\tau_L)}{\partial \tau_L} = \frac{-mW_\ell}{{\tau_L}^2} + \frac{(1-\delta_{\widetilde{BneuD}})(1-\omega_{\widetilde{BneuD}}\vartheta_{\widetilde{BneuD}})(A-B\wp)\mathcal{H}_\ell}{2}.$$

Again, differentiating partially with respect to τ_I, τ_E and τ_L, it follows as

$$\frac{\partial^2 TC_{\widetilde{BneuD}}(\tau_I,\tau_L)}{\partial {\tau_I}^2} = \frac{2m\omega_{\widetilde{BneuD}}(A-B\wp)(W_i+W_r)}{\varphi {\tau_I}^3},$$

and $\frac{\partial^2 TC_{\widetilde{BneuD}}(\tau_I,\tau_L)}{\partial {\tau_L}^2} = \frac{2mW_\ell}{{\tau_L}^3}$

Also,

$$\frac{\partial^2 TC_{\widetilde{BneuD}}(\tau_I,\tau_L)}{\partial \tau_I \partial \tau_L} = \frac{\partial^2 TC_{\widetilde{BneuD}}(\tau_I,\tau_L)}{\partial \tau_L \partial \tau_I} = 0.$$

Therefore, the Hessian matrix for $TC_{\widetilde{BneuD}}(\tau_I,\tau_L)$ is

$$H = \frac{\partial^2 E[TC_{B\iota}(\widetilde{\tau_I,\tau_E,\tau_L})]}{\partial {\tau_I}^2}\frac{\partial^2 E[TC_{B\iota}(\widetilde{\tau_I,\tau_E,\tau_L})]}{\partial {\tau_L}^2} - \left[\frac{\partial^2 TC_{\widetilde{BneuD}}(\tau_I,\tau_L)}{\partial \tau_I \partial \tau_L}\right]^2 = \frac{2m\omega_{\widetilde{BneuD}}(A-B\wp)(W_i+W_r)}{\varphi {\tau_I}^3} . \frac{2mW_\ell}{{\tau_L}^3} > 0$$

Hence, the expected total cost $TC_{\widetilde{BneuD}}(\tau_I,\tau_L)$ is strictly convex.

By setting, $\frac{\partial E[TC_{B\iota}(\widetilde{\tau_I,\tau_L})]}{\partial \tau_I} = 0$ and $\frac{\partial E[TC_{B\iota}(\widetilde{\tau_I,\tau_L})]}{\partial \tau_L} = 0$, the optimal inspection duration and the optimal renting period of RCs are attained.

Hence,

$${\tau_I}^* = \sqrt{\frac{2Rm\omega_{\widetilde{BneuD}}(A-B\wp)(W_i+W_r)}{\varphi R\left(\varphi-\omega_{\widetilde{BneuD}}(A-B\wp)\right)\mathcal{H}_u + \varphi\omega_{\widetilde{BneuD}}\vartheta_{\widetilde{BneuD}}(A-B\wp)(\varphi\vartheta_{\widetilde{BneuD}}-R)\mathcal{H}_r + \varphi^2\vartheta_{\widetilde{BneuD}}\left(R-\omega_{\widetilde{BneuD}}\vartheta_{\widetilde{BneuD}}(A-B\wp)\right)\mathcal{H}_s}}$$

and $${\tau_L}^* = \sqrt{\frac{2mW_\ell}{(1-\delta_{\widetilde{BneuD}})(1-\omega_{\widetilde{BneuD}}\vartheta_{\widetilde{BneuD}})(A-B\wp)\mathcal{H}_\ell}}.$$

This completes the proof.

4. Numerical Analysis

Some parameters below are utilized from [13]:

$A = 6000, B = 20$, $\wp = \$50$/unit, $\varphi = 8000$, $R = 6000$, $W_i = \$200$, $W_r = \$400$, $W_\ell = \$100$, $w_i = \$2$/unit, $w_r = \$4$/unit, $w_\ell = \$10$, $w_h = \$5$/unit, $w_t = \$8$/unit, $\mathcal{H}_u = \$2$/unit, $\mathcal{H}_r = \$3$/unit, $\mathcal{H}_s = \$5$/unit, $\mathcal{H}_\ell = \$5$/unit, $\theta = 0.75$, $m = 240$ days.

The proportion of the received RCs, the proportion of repairable RCs and the fraction of OFU RCs as TrBNN which are given as

$\tilde{\omega} = \langle(0.5,0.75,1,1.5), (0.05,0.1,0.15,0.2), (0.15,0.2,0.4,0.55)\rangle$;

$\widetilde{\vartheta} = \langle(0.7,0.9,1,1.25),(0.05,0.075,0.1,0.125),(0.15,0.25,0.5,0.65)\rangle$;
$\widetilde{\delta} = \langle(0.08,0.12,0.25,0.75),(0.04,0.1,0.24,0.34),(0.05,0.13,0.5,0.55)\rangle$.
To compute the numerical studies of the proposed model using TBNN, the following parameters are used:
$\widetilde{\omega} = \langle(0.5,1,1.5),(0.075,0.1,0.15),(0.15,0.25,0.5)\rangle$;
$\widetilde{\vartheta} = \langle(0.75,1,1.5),(0.05,0.075,0.1),(0.15,0.25,0.5)\rangle$;
$\widetilde{\delta} = \langle(0.1,0.2,0.3),(0.2,0.25,0.4),(0.15,0.3,0.5)\rangle$.
According to the hypothesis of this study, it is considered that the values $\omega \neq 1, \vartheta \neq 1$ and $\delta \neq 1$. From equations (21) - (24), (28) and (29), the expected total cost under TrBNN is $\widetilde{TC_{Bi}(\tau_I,\tau_L)} = \$7{,}218{,}500$, the optimal inspection duration and the optimal leasing duration are $\tau_I{}^* = 2.8527$ days and $\tau_L{}^* = 5.9666$ days. From equations (24) - (29), the expected total cost under TBNN is $\widetilde{TC_{Bi}(\tau_I,\tau_L)} = \$7{,}233{,}000$, the optimal inspection duration and the optimal leasing duration are $\tau_I{}^* = 2.8713$ days and $\tau_L{}^* = 5.8704$ days.

4.1 Comparison study

Table 1: Comparison study of the three decision making strategy in the proposed study

Option	Condition	$\widetilde{TC_{Bi}(\tau_I,\tau_L)}$ (USD) [Using TrBNN]	$\widetilde{TC_{Bi}(\tau_I,\tau_L)}$ (USD) [Using TBNN]
I	$\omega = 1, \vartheta = 1$	7,249,400	7,249,400
II	$\omega \neq 1, \vartheta \neq 1, \delta = 1$	6,595,600	6,589,700
III	$\omega \neq 1, \vartheta \neq 1, \delta \neq 1$	7,218,500	7,233,000

The option I indicates that all the used containers are returned as well as there is no salvaged units found while inspection. In option II, it is considered that the fraction of containers are unreturned and some of the returned units are salvaged. All the shortfall RCs are one-way free used. In option III, a proportion of shortfall RCs that are unable to OFU are leased from the local dealer. From table 1, it is clear that the option II that is, all the deficit containers are replaced by OFU option is the best result when compared to other two options under both TrBNN as well as TBNN, which is shown in Fig 5. So that, the OFU is comparitively better option while leasing the containers. It is also observed that the model using TrBNN is provided the best outcomes when compared to the model under the TBNN environment.

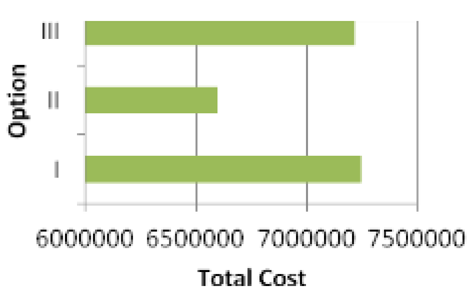

Fig. 5(a)

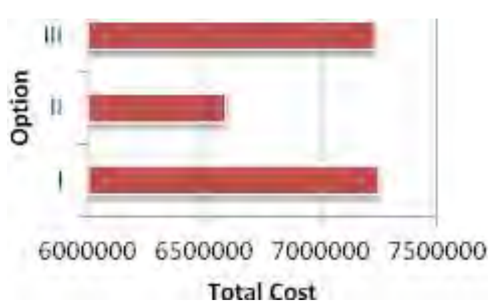

Fig. 5(b)

Fig.5 The total cost of various strategies

The below table shows the comparison of the outcome of the proposed study with the results of relative research works. As per the research works [13] and [30], the proposed research considers to sold the deficit RCs at scrap price.

Table 2: Comparison study of the proposed research with related studies

References	Implemented options to restore the deficit RC	Demand	Scrap price ($)	Total cost ($)	Result obtained in the proposed study		
					Demand	Scrap price ($)	Total cost ($) [Using TrBNN]
[13]	Purchase new Returnable Transport Items (RTIs)	8000	40	31,258,000	8000	40	8,570,100
[30]	Repositioning and leasing of RCs	5000	100	3,819,800	5000	100	2,593,500

Table 3: Descriptive comparison analysis of the proposed research with related studies

References	Implemented options to restore the deficit RC	Remarks when comparing with the proposed study
[13]	Purchase new RTIs	By setting $A = 10000, B = 40$, $\wp = \$50$/unit, $\varphi = 12000$ and $R = 10000$ and by considering the salvage RCs are sold at \$40 in the proposed study, the value of the parameters in this study is similar to [13]. Thus, the total cost is \$8,570,100 and the optimal screening length is 2.4698 days which is the better result when compared to [13]. In [13], the total cost is \$31,258,000 and the optimal screening length is 2.5 days.
[30]	Repositioning and leasing of RCs	By considering the salvage RCs are sold at \$100 in the proposed study, the total cost is \$2,593,500 but the total cost in [30] is \$3,819,800. Thus, the OFU option with ECR and the leasing of RCs helps the CMO to reduce the total cost when compare to [30].

The studies [15] and [22] consider purchasing new RTIs to restore the deficit RTIs. According to the present study, instead of buying new RTIs, the repositioning of RTIs and the leasing of RTIs, as well as the OFU option, lead the model to minimize the total cost. Also, the study [19] presumes the renting option to restore the deficit RCs, along with renting option, the implementation of the OFU option analyzed in the proposed study will lead the outcomes of [19] to better results.

4.2 Sensitivity analysis

The sensitivity analysis of the trapezoidal bipolar neutrosophic container inventory model is examined as follows.

Table 4: Effect of TrBNNs $\tilde{\omega}$, $\tilde{\vartheta}$ and $\tilde{\delta}$ on the optimal solutions

Parameter	Value of the parameter	τ_I^*	τ_L^*	$\widetilde{TC_{B\iota}(\tau_I,\tau_L)}$
$\widetilde{\omega}$	⟨(0.46,0.71,0.06,1.46), (0.01,0.06,0.11,0.16), (0.11,0.16,0.36,0.51)⟩	2.5501	4.6099	7,124,800
	⟨(0.48,0.73,0.08,1.48), (0.03,0.08,0.13,0.18), (0.13,0.18,0.38,0.53)⟩	2.6955	5.1589	7,171,800
	⟨(0.5,0.75,1,1.5), (0.05,0.1,0.15,0.2), (0.15,0.2,0.4,0.55)⟩	2.8527	5.9666	7,218,500
	⟨(0.52,0.77,1.02,1.52), (0.07,0.12,0.17,0.22), (0.17,0.22,0.42,0.57)⟩	3.0240	7.3311	7,264,900
$\tilde{\vartheta}$	⟨(0.66,0.86,0.96,1.21), (0.01,0.035,0.06,0.085), (0.11,0.21,0.46,0.61)⟩	2.8723	4.6425	7,310,300
	⟨(0.68,0.88,0.98,1.23), (0.03,0.055,0.08,0.105), (0.13,0.23,0.48,0.63)⟩	2.8611	5.1817	7,264,500
	⟨(0.7,0.9,1,1.25), (0.05,0.075,0.1,0.125), (0.15,0.25,0.5,0.65)⟩	2.8527	5.9666	7,218,500
	⟨(0.72,0.92,1.02,1.27), (0.07,0.095,0.12,0.145), (0.17,0.27,0.52,0.67)⟩	2.8472	7.2970	7,172,200
$\tilde{\delta}$	⟨(0.04,0.08,0.21,0.71), (0,0.06,0.2,0.3), (0.01,0.09,0.46,0.51)⟩	2.8527	5.5198	7,322,700
	⟨(0.06,0.1,0.23,0.73), (0.02,0.08,0.22,0.32), (0.03,0.11,0.48,0.53)⟩	2.8527	5.7302	7,270,600
	⟨(0.08,0.12,0.25,0.75), (0.04,0.1,0.24,0.34), (0.05,0.13,0.5,0.55)⟩	2.8527	5.9666	7,218,500
	⟨(0.1,0.14,0.27,0.77), (0.06,0.12,0.26,0.36), (0.07,0.15,0.52,0.57)⟩	2.8527	6.2349	7,166,400

From the table 4, it is observed that when the bipolar neutrosophic proportion of the returned containers $\widetilde{\omega}$ raises, the optimal screening period τ_I^* and the optimal leasing duration τ_L^* are increase but the total cost reduces. Also, when the bipolar neutrosophic proportion of the repaired units $\tilde{\vartheta}$ raises, the optimal renting period is increases but the optimal screening period and the total cost reduce. When the bipolar neutrosophic variable $\widetilde{\delta}$ raises, the optimal time length τ_L^* increase but the total cost reduces but the optimal screening cycle length, τ_I^*, remains unchange.

Table 5: Effect of customers' rent price on the optimal solutions

$\wp$	τ_I^*	τ_L^*	$\widetilde{TC_{B\iota}(\tau_I,\tau_L)}$ \$
40	3.0106	5.8507	7,503,600
45	2.9300	5.9078	7,361,100
50	2.8527	5.9666	7,218,500
55	2.7784	6.0272	7,076,000
60	2.7069	6.0896	6,933,400

In the notion of customers' rent price for a container, without loss of generality it is obtained that the increase of customer rent price per RC leads to increase the optimal leasing period whereas the optimal screening period and the total cost decreasing, which is clearly shown in Table 5.

Table 6: Effect of container storage costs $\mathcal{H}_u$ and $\mathcal{H}_r$ on the optimal solutions

$\mathcal{H}_u$	$\mathcal{H}_r$	τ_I^*	$\widetilde{TC_{B\iota}(\tau_I,\tau_L)}$ \$
1	3	3.1218	7211600
2	3	2.8527	7218500
3	3	2.6430	7224600
4	3	2.4737	7230100
5	3	2.3332	7235100
2	1	3.0416	7214900

2	2	2.9426	7216700
2	3	2.8527	7218500
2	4	2.7706	7220300
2	5	2.6951	7222200

The container storage cost is the major cost of any CMO for maintaining the empty containers. In this study, there is no storage charge for OFU containers because the customer used to pick the serviceable containers from the OFU vendor's empty depot but different storage cost is acquired for the used containers that are received after the usage, repairable containers, the serviceable containers and the leased units. Also, in this analysis, the storage cost for serviceable containers, ($\mathcal{H}_s$) and the storage charge for leased containers, ($\mathcal{H}_\ell$) are fixed as $5/unit. Here, the sensitive analysis is performed when the storage cost for received containers, ($\mathcal{H}_u$) and the storage charge for repairable from received units, ($\mathcal{H}_r$) are varies from $1 to $5 per container on optimal solutions. On observing from Table 6, when $\mathcal{H}_u$varies from $1 to $5 per container and the storage cost for repairable from received container is consider as in numerical analysis, the optimal inspection length reduces but the total cost increases. The same result attains when $\mathcal{H}_r$ varies from $1 to $5 per container and the storage cost for received containers after the usage is consider as in numerical analysis.

5. Conclusion

A container inventory model under uncertain situation is performed in this research. This uncertainty condition leads the present model to utilize the notion of bipolar neutrosophic arena. The serviceable container cost, OFU cost and the lease cost are obtained under price sensitive demand by computing the expected maximum inventory of received, restorable, ready to service units, OFU units and leased units. The framed algorithm helps to make decision on various strategies for computing the total cost. The total cost is framed by presuming that the fraction of deficit units is one-way free used, and the fraction of remaining containers are leased from local dealer. The bipolar neutrosophic container inventory model is developed by which the fraction of received RCs, the fraction of repairable from received RCs and the fraction of one-way free used RCs are considered as TrBNNs. The container inventory model is performed by presuming the received rate, the repairable rate, and the OFU rate as Triangular Bipolar Neutrosophic Number (TBNN), and then the outcomes under both TrBNN as well as TBNN are compared.

The optimal time lengths of inspection process and leasing procedure in order to minimize the total cost are attained. The numerical computation on the comparison of four decision making strategies shows that, the OFU of all the deficit containers minimizes the CMO's total cost. The bipolar neutrosophic received proportion, the bipolar neutrosophic repaired proportion and the bipolar neutrosophic OFU proportion and its impact on optimal solutions are shown in sensitivity analysis. Finally, the comparison result of this study with the work of [13] on optimal screening length is given. The comparison analysis of present model and [30] is also given and then suggests the studies [15] and [22] to utilizing the OFU option in order to minimize the container management costs.

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Multi-objective Mathematical Model for Asset Portfolio Selection using Neutrosophic Goal Programming Technique

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Abstract: In this paper we have considered a multi-objective asset portfolio selection optimization model with the objectives maximization of the expected return of the portfolio and simultaneously minimizing the overall risk of the asset portfolio. Our model is an improved and enlarged version in a particular direction. In our model we had incorporated transaction cost in the first objective. We had considered absolute deviation as risk measure. Our portfolio optimization model had been solved by generalized neutrosophic goal programming method.

For applicability of this technique and demonstration of the methodology we have illustrated it numerically by data taken from National Stock Exchange (NSE). And finally the result obtained using generalized neutrosophic goal programming approach is compared with that of the result obtained different method of aggregation for objective functions.

Keywords: Portfolio; Generalized Neutrosophic Goal Programming, Arithmetic Aggregation, Geometric Aggregation.

1. Introduction

Portfolio management is one of the most important aspects of economic management. Essentially, portfolio management is the process of building a portfolio with the goal of satisfying an investor's risk and return expectations. The primary goal of portfolio management is to select a proper combination of assets in order to provide the best predicted return while maintaining a suitable level of risk.

An investor's goal in portfolio optimization is to maximise portfolio return while maintaining a reasonable level of risk at the same time. Because risk will repay the return, investors will need to manage the risk-return trade-off for their investments. As a result, a single optimization portfolio is ineffective. As a result, when determining the best portfolio, one must consider the investor's risk-reward preferences.

The Mean-Variance (MV) model, established by Markowitz[1] in 1952, is considered the first model in the field of portfolio management. Markowitz trade-off between expected return and portfolio risk in the basic mean-variance model of the portfolio framework, where mean is represented by the average mean of the past performances, i.e. the mean of asset's return and the dispersion of the return as risk, respectively.

Over the last few years, the pioneer model proposed by Markowitz, mathematical programming approaches have grown to be vital tools to guide financial decision-making systems and have been widely deployed in real-world scenarios. There are numbers of well-known mathematical tools that are used to find the best solution in portfolio optimization. Forecasting, simulation, statistical models, and mathematical programming models are some examples. Among these approaches, mathematical programming is a good option for a decision maker looking for the best solution.

According to the existing literatures, a mathematical model for portfolio addressing transaction cost generally seeks to generate a changed portfolio from cash, i.e., preferring to pass from a present portfolio to a new one. The majority of the models add at least one more binary variable to the portfolio, as well as new constraints will be added. As a result, the majority of these transaction pricing components will add complexity to the problems. Let us now have a look at the available literature of the transaction cost. Angelelli et al. [2] used a mixed integer linear programming model that included transaction cost and cardinality constraints with CVaR and MAD model. In the generalised MV Markowitz model, Chen and Cai [3] added transaction cost. According to the assumptions, transaction costs are a V-shaped function that is known at the beginning of the period and paid at the conclusion. In the transaction cost model, Baule [4] took transaction cost into account as a non-convex function. In the mixed quadratic portfolio optimization model technique, Adcock and Meade [5] included a weighting factor to account for variable transaction costs. There are also a few additional journals, as well as the concept of transaction price in portfolio optimization.

Integer programming technique [6], goal programming technique [7], lexiographic goal programming technique [8], and other precise method based techniques were used to solve portfolio optimization models. Simulated annealing [9], genetic algorithm [10], particle swarm optimization [11], and ant colony optimization [12] are some of the meta-heuristics-based techniques used. However, in practice, if you want to make good portfolio decisions, you'll need to use a few vaguely defined financial characteristics like the return is greater than 20%, the risk is less than 10%, and so on. It's difficult to put together satisfying portfolios using crisp or interval numbers when the language is so hazy. In such a situation, the decision maker must enlist the help of fuzzy set theory in order to build portfolio selection models. Fuzzy set theory not only manages uncertainty and ambiguity, but it also helps decision makers make flexible choices by considering the choices of investors.

Financial risks are the component of the uncertainty that pertains to asset returns as a result of unforeseeable and unpredictable events. Risks cannot be quantified in portfolio selection or asset assessment for a variety of reasons, including a lack or plenty of information, subjective estimation and perception, insufficient knowledge, the complexity of the researched systems, and so on. In these instances, language judgements rather than numerical values are a more realistic approach.But there is a lot of uncertainty and ambiguity related with these linguistic expressions, such as, "high", "low", "moderate". So traditional two valued logic of probability is not enough to handle the dual presence of uncertainty and ambiguity. In this scenario, fuzzy set theory proposed by Professor L.A. Zadeh becomes a natural choice since it can define the linguistic information in a more logical and meaningful fashion. It is also quite impossible for decision maker to determine or estimate the movement in financial markets. So the decision maker faces the dilemma of guessing the market direction in order to meet the return target for asset under management. Under these circumstances, an uncertainty may be included in their estimation. Because of some uncertainty and ambiguity present in the Asset Liability Management and portfolio optimization, concept of fuzzy set theory is used in this area. Watada [13] had used fuzzy computational intelligence in portfolio selection problem. Yager [14] contributed in taking decisions on uncertain issue like portfolio selection using fuzzy mathematics. In [15] the authors described the selection of fuzzy portfolio using the concept like expected value of fuzzy numbers and ranking .

Bellmann and Zadeh [16] proposed the concept of fuzzy decision theory, which was based on Zadeh's 1965 [17] presentation of fuzzy sets. Several writers had also used the fuzzy framework to select the most efficient portfolio using the mean-variance model.

This is also a tough procedure due to elements like insufficient information that is frequently offered in real-life decision-making scenarios. Our major goal in this decision-making process is to identify a value from the chosen set that has the maximum degree of membership in the decision set and that agrees with the goals only under certain constraints. However, there may be many times when some of the selected values from the set are incompatible with the aim, i.e., those values are strongly opposed to the purpose due to limitations that cannot be accepted. Such values may be found in this case from the selected set with the lowest degree of non-membership in the choice set. In such instances, intuitionistic fuzzy can help the decision maker deal with partial data, but it is unable to deal with indeterminate and inconsistent data, which are also common in the systems. Atanassov [18],[19] developed the concept of intuitionistic fuzzy sets. Truth membership, falsity membership, and indeterminacy membership are all independent in the neutrosophic set presented by Smarandache [20], and indeterminacy can be quantified directly. As a result, it is evident that the value in the decision set from the chosen set with the highest degree of truth membership, falsity membership, and indeterminacy membership should be considered. As a result, we have chosen a neutrosophic environment to deal with asset liability management decisions for commercial banks. Different authors have used the concept of neutrosophic optimization in a variety of fields. This approach was used to the reliability problem by Sahidul Islam and Tanmay Kundu [21], to the multi-objective welded beam optimization by M. Sarkar and T.K. Roy [22], to the riser design problem by Pintu Das and T.K. Roy [23], and to optimization problems in a variety of other domains. S.Islam and Partha Ray [24] created a multi-objective portfolio selection model with entropy using the Neutrosophic optimization technique for portfolio selection.

With the above observation in mind, we will attempt to propose a multi-objective portfolio optimization model in this paper. In a specific direction, our model is a better and larger version. One of the objectives of our approach was to include transaction costs. We used absolute deviation as a risk indicator. The generalised neutrosophic goal programming method which is just a generalisation of Neutrosophic Goal programming method proposed by M.Abdel-Baset, I.M.Heza, and F.Smarandache [28] was used to solve our portfolio optimization model. The portfolio optimization model was validated in this research using data from the National Stock Exchange (NSE).

2. Mathematical Model:

In this section we will discuss about proposed optimization model for selection of portfolio. The notations used for this model are listed below:

n : the number of assets which are available for investment.

x_i :the proportion of the total fund invested in i-th asset, for $i = 1,2, \dots \dots \dots .., n$.

x_i^0 : the proportion of the total funds had been invested in i-th asset, for $i = 1,2, \dots \dots \dots .., n$.

R_i : the rate of return of i-th asset which is basically a random variable for $i = 1,2, \dots \dots \dots \dots \dots .., n$.

r_i : the expected rate of return on the i-th asset, for $i = 1,2, \dots \dots \dots .., n$. $r_i = E[R_i]$

r_{n+1} : the rate of return for the risk free asset.

λ_i : the rate of transaction cost on i-th asset , for $i = 1,2, \dots \dots \dots \dots \dots .., n$.

L_i : The lower limit of the fund that can be invested on the i-th asset for $i = 1,2, \dots \dots \dots \dots \dots .., n$.

U_i : The upper limit of the fund that can be invested on the i-th asset for $i = 1,2, \dots \dots \dots \dots \dots .., n$.

In this model we had considered absolute deviation as risk measure. Before introducing the mathematical model let us give some introduction to this measure of risk.

2.1 Absolute deviation

The main aim of every investor in portfolio selection is to get portfolio return $r(x_1, x_2, \dots \dots \dots , x_n)$ as high as possible. Also an investor would also prefer to have minimum variation or dispersion in the portfolio return. Variance is the most common measure to quantify risk of portfolio, which measures the variation from the expected return. Despite its shortcomings, researchers continue to choose variance as a prominent risk metric. The biggest disadvantage of utilizing variance as a risk indicator is that it penalizes extreme upside and downside deviations from the expected return. As a result, the variance will be a less appropriate measure of portfolio risk in the case of an asymmetric probability distribution of asset return. This is due to the fact that, in exchange for a larger predicted return, the obtained portfolio may provide a risk. As a result, a downside risk metric may be preferable to variance. Only negative deviations from a reference return level are included in this risk assessment. Another downside risk metric, known as semi variance, was established by Markowitz.

Both the above mentioned risk measure have some advantages and simultaneously have some limitations. In order to improve both the theoretical and computational performance of the mean-variance model or mean-Semi variance model Konno and Yamazaki [27] had considered an alternative risk measure namely absolute deviation to quantify risk and introduced a linear programming portfolio selection model. So far the formulation of the risk function was based on the notion of L_2 metric, we had discussed these earlier. The risk function namely absolute deviation is defined based on the notion of L_1 metric on $\mathbb{R}^n$. Normally this risk measure is applicable to the problems having a-symmetric distributions of the rate of return. L_1 risk function draw much attention of the researcher since a portfolio selection model with L_1 risk function can easily be converted into a scalar parametric linear programming problem. Another benefit of using absolute deviation in a portfolio optimization model is computational ease and simplicity even for large number of assets also.

The expected absolute for the difference between the random variables and its mean is known as absolute deviation of a random variable. This measure of portfolio risk is denoted by $m(x_1, x_2, \dots \dots \dots, x_n)$ and is expressed as:

$$m(x_1, x_2, \dots \dots \dots, x_n) = E[|\sum_{i=1}^{n} R_i x_i - E[\sum_{i=1}^{n} R_i x_i]|].$$

Since we shall approximate expected value of the random variable by the average derived from the past data, so we shall use $r_i = E[R_i] = \frac{\sum_{t=1}^{T} r_{it}}{T}$, the absolute deviation is approximated as

$$m(x_1, x_2, \dots \dots \dots, x_n) = E[|\sum_{i=1}^{n} R_i x_i - E[\sum_{i=1}^{n} R_i x_i]|] = \frac{1}{T}\sum_{t=1}^{T}|\sum_{i=1}^{n}(r_{it} - r_i)x_i|.$$

2.2 The proposed Mathematical model:

(P 1.1)

$$Maximize\ Z_1 = \sum_{i=1}^{n+1}\left(r_i x_i - \lambda_i \left|x_i - x_i^0\right|\right)$$

$$Minimize\ Z_2 = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(r_{it} - r_i)x_i\right|$$

$subject\ to:$

$$\sum_{i=1}^{n} x_i = 1,$$

$$x_i \geq 0\,,$$

$$L_i \leq x_i \leq U_i$$

$$i = 1,2, \dots \dots \dots \dots \dots \dots \dots \dots., n$$

Because of the existence of the absolute value function the above mathematical model is non-linear and non-smooth. For elimination the absolute value function the above mathematical model had been transformed into the following form

(P 1.2)

$$Maximize\ \ Er = \sum_{i=1}^{n+1}(r_i x_i - \lambda_i q_i)$$

$$Minimize\ \ Ad = \frac{1}{T}\sum_{t=1}^{T} p_t$$

$subject\ to:$

$$\sum_{i=1}^{n} x_i = 1,$$

$$q_i \geq (x_i - x_i^0)$$
$$q_i \geq -(x_i - x_i^0)$$
$$p_t \geq \sum_{i=1}^{n} (r_{it} - r_i) x_i$$

$$p_t \geq -\sum_{i=1}^{n} (r_{it} - r_i) x_i$$

$$x_i \geq 0\,,$$
$$p_t \geq 0$$
$$q_i \geq 0$$
$$L_i \leq x_i \leq U_i$$
$$i = 1,2, \dots \dots \dots \dots \dots \dots \dots, n$$

2.3 Descriptions of the Objectives and the Constraints

The first objective is maximization of expected return of the portfolio, which is difference between the rate of expected return of the portfolio and the transaction cost of the portfolio. In the first objective $\sum_{i=1}^{n+1}\left(r_i x_i - \lambda_i |x_i - x_i^0|\right)$, $\sum_{i=1}^{n+1} r_i x_i$ is the rate of expected return , and $\sum_{i=1}^{n+1} \lambda_i |x_i - x_i^0|$ is the transaction cost of the portfolio. And the second objective is minimization of absolute deviation. $\sum_{i=1}^{n} x_i = 1$ is the capital budget constraint. $L_i \leq x_i \leq U_i, i = 1,2, \dots \dots, n$ is the maximal and minimal fraction of the total capital to be invested in each asset.

3. Mathematical Analysis

In this section we will discuss about some preliminary concepts of the neutrosophic set and then the Neutrosophic goal programming technique which will be used in this paper to deal with the portfolio selection model.

3.1 Some definitions

Fuzzy Sets

Let $\tilde{B}$ is a fuzzy set and X be considered as universe of discourse. Then fuzzy set $\tilde{B}$-can be defined as follow-$\tilde{B} = \{< x, \mu_{\tilde{B}}(x) >: x \in X\}$; where $\mu_{\tilde{B}}(x)$ is a mapping from X to [0, 1], which is the membership function of the corresponding fuzzy set$\tilde{B}$.

Intuitionistic Fuzzy Sets

An intuitionistic fuzzy sets (IFS) $\tilde{B}^i$ in the universe of discourse X is defined by $\tilde{B}^i = \{\langle x, \mu_{\tilde{B}^i}(x), \nu_{\tilde{B}^i}(x)\rangle | x \in X\}$

Where, $\mu_{\tilde{B}^i}(x): X \to [0,1]$ is the degree of membership of $x \in X$ and $\nu_{\tilde{B}^i}(x): X \to [0,1]$ is the degree of non-membership of $x \in X$. Also for every-$x \in X$, $0 \leq \mu_{\tilde{B}^i}(x) + \nu_{\tilde{B}^i}(x) \leq 1$.

Now for each element-$x \in X$, the value of $\Pi_{\tilde{B}^i}(x) = 1 - \mu_{\tilde{B}^i}(x) - \nu_{\tilde{B}^i}(x)$ is said to be the degree of uncertainty of the element $x \in X$ to the IFS $\tilde{B}^i$.

Neutrosophic Sets

Let X be the universe of discourse and x be a generic element of this set. A neutrosophic set (NS) denoted by $\tilde{B}^N$ in X is characterized by a truth membership function $\mu_B(x)$, a falsity membership function $\nu_B(x)$and an indeterminacy membership function $\sigma_B(x)$ and having the form

$\tilde{B}^N = \{\langle x, \mu_B(x), \nu_B(x), \sigma_B(x)\rangle | x \in X\}$

Where,

$\mu_B(x): X \to]0^-, 1^+[$

$\nu_B(x): X \to]0^-, 1^+[$

$\sigma_B(x): X \to]0^-, 1^+[$

i.e. $\mu_B(x), \nu_B(x), \sigma_B(x)$ are real standard or non standard subsets of $]0^-, 1^+[$.

Also $0^- \leq \text{Sup}\, \mu_B(x) + Sup\, \nu_B(x) + Sup\, \sigma_B(x) \leq 3^+$.

The NS takes the value from the real standard or non-standard subsets of $]0^-, 1^+[$ from the philosophical point of view, but in application of real life in engineering and scientific problems it is difficult to use NS with value from the subsets of$]0^-, 1^+[$.

3.2 Neutrosophic Goal Programming

Let us consider a goal programming problem as

To find $X = (x_1, x_2, \dots \dots, x_{n-1}, x_n)^T$

to achieve :

$f_i = t_i, i = 1,2, \dots \dots, k$

Under the conditions, $x \in X$

where X is a feasible set of all the constraints,t_i are scalars representing level of achievement for the objective functions, which the decision maker want to attain in the feasible set.

More generally a non-linear goal programming problem can be expressed as

(P 1.3)

To find $X = (x_1, x_2, \dots \dots, x_{n-1}, x_n)^T$

In order to $Minimize\ f_i$,having the target value t_i , acceptance tolerance a_i, rejection tolerance c_i, and indeterminacy tolerance d_i

$g_j(x) \leq b_j,\ j = 1,2, \dots \dots \dots, m$

$x_i \geq 0, i = 1,2, \dots \dots, n$

The truth-membership functions, falsity-membership functions and indeterminacy-membership-functions as given by Mohamed Abdel-Baset et all [28] are respectively

$$T_i(f_i) = \begin{cases} 1 & if\ f_i \leq t_i \\ \left(\frac{t_i + a_i - f_i}{a_i}\right) & if\ t_i \leq f_i \leq t_i + a_i \\ 0 & if\ \ f_i \geq t_i + a_i \end{cases}$$

$$F_i(f_i) = \begin{cases} 0 & if\ f_i \leq t_i \\ \left(\frac{f_i - t_i}{c_i}\right) & if\ t_i \leq f_i \leq t_i + c_i \\ 1 & if\ \ f_i \geq t_i + c_i \end{cases}$$

$$I_i(f_i) = \begin{cases} 0 & if\ f_i \leq t_i \\ \left(\frac{f_i - t_i}{d_i}\right) & if\ t_i \leq f_i \leq t_i + a_i \\ \left(\frac{t_i + a_i - f_i}{a_i - d_i}\right) & if\ t_i + d_i \leq f_i \leq t_i + a_i \\ 0 & if\ \ f_i \geq t_i + a_i \end{cases}$$

Now the formulation to minimize the degree of rejection and maximize the degree of acceptance as well as the degree of the indeterminacy of objectives and constraints for a given nonlinear goal programming is as follow:

(P 1.4)

$Maximize\ T_{f_i}(f_i)\ , i = 1,2, \dots \dots , k$

$Maximize\ I_{f_i}(f_i)\ , i = 1,2, \dots \dots , k$

$Minimize\ F_{f_i}(f_i)\ , i = 1,2, \dots \dots , k$

Subject to

$0 \le T_{f_i}(f_i) + I_{f_i}(f_i) + F_{f_i}(f_i) \le 3, i = 1,2, \dots \dots , k$

$T_{f_i}(f_i) \ge 0, I_{f_i}(f_i) \ge 0, F_{f_i}(f_i) \ge 0\ , i = 1,2, \dots \dots , k$

$T_{f_i}(f_i) \ge I_{f_i}(f_i), i = 1,2, \dots \dots , k$

$T_{f_i}(f_i) \ge F_{f_i}(f_i), i = 1,2, \dots \dots , k$

$g_j(x) \le b_j,\ j = 1,2, \dots \dots \dots , m$

$x_i \ge 0, i = 1,2, \dots \dots , n$

Here the truth-membership function, falsity-membership function and indeterminacy-membership function of the corresponding neutrosophic decision set are respectively $T_{f_i}(f_i), F_{f_i}(f_i)$ and $I_{f_i}(f_i)$.

Now using the truth-membership function, falsity-membership function and indeterminacy-membership function in generating the corresponding crisp programming model of P(1.4) which is non-linear goal programming problem be expressed as follow

(P 1.5)

$Maximize\ A$

$Maximize\ C$

$Minimize\ B$

$T_{f_i}(f_i) \ge A,\ i = 1,2, \dots \dots , k$

$I_{f_i}(f_i) \ge C,\ i = 1,2, \dots \dots , k$

$F_{f_i}(f_i) \le B, i = 1,2, \dots \dots , k$

$f_i \le t_i, i = 1,2, \dots \dots , k$

$0 \le A + B + C \le 3;$

$A \ge 0, C \ge 0, B \le 1;$

$g_j(x) \le b_j,\ j = 1,2, \dots \dots \dots , m$

$x_i \ge 0, i = 1,2, \dots \dots , n$

3.3 Generalized Neutrosophic Goal Programming

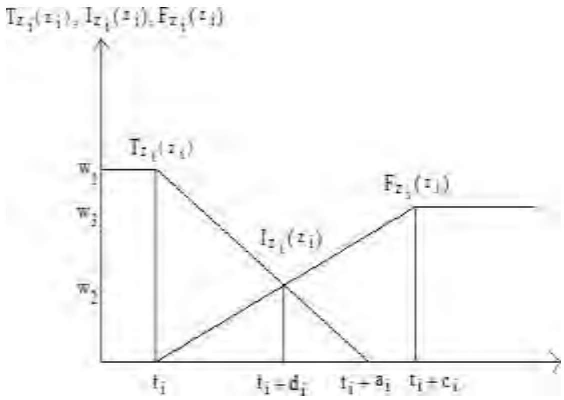
In the case of generalized neutrosophic goal programming, the truth-membership functions, falsity-membership functions and the indeterminacy-membership-functions as defined by Mridula Sarkar et all [29] are defined respectively as

$$T_i^{w_1}(f_i) = \begin{cases} w_1 & if\ f_i \le t_i \\ w_1\ \left(\frac{t_i + a_i - f_i}{a_i}\right) & if\ t_i \le f_i \le t_i + a_i \\ 0 & if\ \ f_i \ge t_i + a_i \end{cases}$$

$$F_i^{w_2}(f_i) = \begin{cases} 0 & if\ f_i \le t_i \\ w_2\ \left(\frac{f_i - t_i}{c_i}\right) & if\ t_i \le f_i \le t_i + c_i \\ w_2 & if\ \ f_i \ge t_i + c_i \end{cases}$$

$$I_i^{w_3}(f_i) = \begin{cases} 0 & if\ f_i \le t_i \\ w_3\left(\frac{f_i - t_i}{d_i}\right) & if\ t_i \le f_i \le t_i + a_i \\ w_3\left(\frac{t_i + a_i - f_i}{a_i - d_i}\right) & if\ t_i + d_i \le f_i \le t_i + a_i \\ 0 & if\ f_i \ge t_i + a_i \end{cases}$$

where w_1, w_2, w_3 are degree of gradations of the truth-membership functions, falsity-membership functions and the indeterminacy-membership-functions respectively. Also the target value is t_i, acceptance tolerance is a_i, rejection tolerance c_i, and indeterminacy tolerance is d_i



The general formulation of Neutrosophic goal programming is as follow:

(P 1.6)

$Maximize\ T_{f_i}(f_i)\ , i = 1,2, \dots \dots, k$

$Maximize\ I_{f_i}(f_i)\ , i = 1,2, \dots \dots, k$

$Minimize\ F_{f_i}(f_i)\ , i = 1,2, \dots \dots, k$

Subject to

$0 \le T_{f_i}(f_i) + I_{f_i}(f_i) + F_{f_i}(f_i) \le w_1 + w_2 + w_3, i = 1,2, \dots \dots, k$

$T_{f_i}(f_i) \ge 0, I_{f_i}(f_i) \ge 0, F_{f_i}(f_i) \ge 0\ , i = 1,2, \dots \dots, k$

$T_{f_i}(f_i) \ge I_{f_i}(f_i), i = 1,2, \dots \dots, k$

$T_{f_i}(f_i) \ge F_{f_i}(f_i), i = 1,2, \dots \dots, k$

$0 \le w_1 + w_2 + w_3 \le 3$

$w_1, w_2, w_3 \in [0,1]$

$g_j(x) \le b_j,\ j = 1,2, \dots \dots \dots, m$

$x_i \ge 0, i = 1,2, \dots \dots, n$

The above problem is equivalent to

(P 1.7)

$Maximize\ A$

$Minimize\ B$

$Maximize\ C$

$T_{f_i}(f_i) \ge A,\ i = 1,2, \dots \dots, k$

$I_{f_i}(f_i) \ge C,\ i = 1,2, \dots \dots, k$

$F_{f_i}(f_i) \leq B, i = 1,2, \dots \dots, k$

$f_i \leq t_i, i = 1,2, \dots \dots, k$

$0 \leq A + B + C \leq w_1 + w_2 + w_3$;

$A \in [0, w_1],\ B \in [0, w_2]\ , C \in [0, w_3]$;

$0 \leq w_1 + w_2 + w_3 \leq 3$

$w_1, w_2, w_3 \in [0,1]$

$g_j(x) \leq b_j,\ j = 1,2, \dots \dots \dots, m$

$x_i \geq 0, i = 1,2, \dots \dots, n$

Again using the corresponding membership function, finally this problem is equivalent to

(P 1.8)

$Maximize\ A$

$Minimize\ B$

$Maximize\ C$

$f_i \leq t_i + a_i\left(1 - \frac{A}{w_1}\right),\ i = 1,2, \dots \dots, k$

$f_i \leq t_i + \frac{c_i}{w_2}B,\ i = 1,2, \dots \dots, k$

$f_i \geq t_i + \frac{d_i}{w_3}C,\ i = 1,2, \dots \dots, k$

$f_i \leq t_i + a_i - \frac{1}{w_3}(a_i - d_i)C, i = 1,2, \dots \dots, k$

$f_i \leq t_i, i = 1,2, \dots \dots, k$

$0 \leq A + B + C \leq w_1 + w_2 + w_3$;

$A \in [0, w_1],\ B \in [0, w_2]\ , C \in [0, w_3]$;

$0 \leq w_1 + w_2 + w_3 \leq 3$

$w_1, w_2, w_3 \in [0,1]$

$g_j(x) \leq b_j,\ j = 1,2, \dots \dots \dots, m$

$x_i \geq 0, i = 1,2, \dots \dots, n$

Now using generalized truth, falsity and indeterminacy membership function and under the consideration of arithmetic aggregation operator the generalized neutrosophic goal programming can be formulated as

(P 1.9)

$Minimize\ \left\{\frac{(1-A)+B+(1-C)}{3}\right\}$

Under the same set of constraints as of (P 1.8)

Also using geometric aggregation operator same generalized neutrosophic goal programming can be formulated as :

(P 1.10)

$Minimize\ \sqrt[3]{(1-A)B(1-C)}$

Under the same set of constraints as of (P 1.8)

Finally to get the solution of multi-objective non-linear programming problem by generalized neutrosophic goal programming approach, we can take help of some appropriate mathematical programming to solve the non linear programming problem (P 1.8 or P 1.9 or P 1.10).

4. **Solution of Multi-Objective Portfolio Optimization Model by Generalized Neutrosophic Goal Programming**

Multi-objective neutrosophic portfolio optimization model can be expressed as

$Maximize\ Er(X)$, with target value E_0 , acceptance tolerance a_E, indeterminacy tolerance d_E , and rejection tolerance c_E.

$Minimize\ Ad(X)$, with target value A_0 , acceptance tolerance a_A, indeterminacy tolerance d_A , and rejection tolerance c_A.

$subject\ to:$

$\sum_{i=1}^{n} x_i = 1,$

$x_i \geq 0\ ,$

$L_i \leq x_i \leq U_i$

$i = 1,2, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ., n$

Where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$ are the decision variables.

In case of generalized neutrosophic goal programming the truth-membership functions, falsity-membership functions and indeterminacy-membership-functions for the objective functions are defined respectively as

$$T_{Er(X)}^{w_1}(Er(X)) = \begin{cases} w_1 & if\ Er(X) \geq E_0 \\ w_1 \left(\frac{Er(X)-E_0+a_E}{a_E}\right) & if\ E_0 - a_E \leq Er(X) \leq E_0 \\ 0 & if\ Er(X) \leq E_0 - a_E \end{cases}$$

$$F_{Er(X)}^{w_2}(Er(X)) = \begin{cases} 0 & if\ Er(X) \geq E_0 \\ w_2 \left(\frac{E_0-Er(X)}{c_E}\right) & if\ E_0 - c_E \leq Er(X) \leq E_0 \\ w_2 & if\ Er(X) \leq E_0 - c_E \end{cases}$$

$$I_{Er(X)}^{w_3}(Er(X)) = \begin{cases} 0 & if\ Er(X) \leq E_0 - a_E \\ w_3 \left(\frac{a_E+Er(X)-E_0}{a_E-d_E}\right) & if\ E_0 - a_E \leq Er(X) \leq E_0 - d_E \\ w_3 \left(\frac{-Er(X)+E_0}{d_E}\right) & if\ E_0 - d_E \leq Er(X) \leq E_0 \\ 0 & if\ Er(X) \geq E_0 \end{cases}$$

Where $d_E = \frac{w_1}{\frac{w_1}{a_E}+\frac{w_2}{c_E}}$

And

$$T_{Ad(X)}^{w_1}(Ad(X)) = \begin{cases} w_1 & if\ Ad(X) \leq A_0 \\ w_1 \left(\frac{A_0+a_A-Ad(X)}{a_A}\right) & if\ A_0 \leq Ad(X) \leq A_0 + a_A \\ 0 & if\ Ad(X) \geq A_0 + a_A \end{cases}$$

$$F_{Ad(X)}^{w_2}(Ad(X)) = \begin{cases} 0 & if\ Ad(X) \le A_0 \\ w_2\left(\frac{Ad(X)-A_0}{c_A}\right) & if\ A_0 \le Ad(X) \le A_0 + c_A \\ w_2 & if\ Ad(X) \ge A_0 + c_A \end{cases}$$

$$I_{Ad(X)}^{w_3}(Ad(X)) = \begin{cases} 0 & if\ Ad(X) \le A_0 \\ w_3\left(\frac{Ad(X)-A_0}{d_A}\right) & if\ A_0 \le Ad(X) \le A_0 + a_A \\ w_3\left(\frac{A_0+a_A-Ad(X)}{a_A-d_A}\right) & if\ A_0 + d_A \le Ad(X) \le A_0 + a_A \\ 0 & if\ Ad(X) \ge A_0 + a_A \end{cases}$$

Where $d_A = \frac{w_1}{\frac{w_1}{a_A}+\frac{w_2}{c_A}}$

Now using generalized neutrosophic goal programming technique and incorporating truth, falsity and indeterminacy membership functions the problem (P 1.2) can be formulated as the following (P 1.11)

(P 1.11)

$Maximize\ A$

$Minimize\ B$

$Maximize\ C$

$Er(X) \ge E_0 + a_E\left(\frac{A}{w_1} - 1\right),$

$Er(X) \ge E_0 - \frac{c_E}{w_2}B,$

$Er(X) \le E_0 - \frac{d_E}{w_3}C,$

$Er(X) \ge E_0 - a_E + \frac{C}{w_3}(a_E - d_E),$

$Er(X) \ge E_0,$

$\mathrm{Ad}(X) \le A_0 + a_A\left(1 - \frac{A}{w_1}\right),$

$\mathrm{Ad}(X) \le A_0 + \frac{c_A}{w_2}B,$

$\mathrm{Ad}(X) \ge A_0 + \frac{d_A}{w_3}C,$

$Ad(X) \le A_0 + a_A - \frac{C}{w_3}(a_A - d_A),$

$Ad(X) \le A_0,$

$0 \le A + B + C \le w_1 + w_2 + w_3\ ;$

$A \in [0, w_1],\ B \in [0, w_2]\ , C \in [0, w_3];$

$0 \le w_1 + w_2 + w_3 \le 3$

$w_1, w_2, w_3 \in [0,1]$

$g_j(x) \le b_j,\ j = 1,2, \dots \dots \dots, m$

$x_i \ge 0, i = 1,2, \dots \dots, n$

$\sum_{i=1}^{n} x_i = 1,$

$q_i \geq (x_i - x_i^0)$

$q_i \geq -(x_i - x_i^0)$

$p_t \geq \sum_{i=1}^{n}(r_{it} - r_i)x_i$

$p_t \geq -\sum_{i=1}^{n}(r_{it} - r_i)x_i$

$x_i \geq 0$,

$p_t \geq 0$

$q_i \geq 0$

$L_i \leq x_i \leq U_i$

$i = 1,2, \dots \dots \dots \dots \dots \dots \dots ., n$

5. Numerical Illustration

Our portfolio optimization model had been solved by generalized neutrosophic goal programming method. In this paper the portfolio optimization model had been validated by data taken from National Stock Exchange (NSE). For demonstration a data set of 10 randomly selected assets had been considered from NSE for an entire financial year i.e. 12 months, here each rows are data of any companies like ABL, ALL, etc for the entire financial year and columns are data for 1st month, 2nd month, etc of the financial year. The data is given below

Table1: Return of assets of some companies taken from National stock exchange

Company	1	2	3	4	5	6	7	8	9	10	11	12
ABL	0.072	0.32032	0.2971	0.236	-0.05161	0.50633	-0.02516	0.90484	0.03214	0.45968	0.227	-0.87871
ALL	-0.14433	0.19032	0.75032	0.03433	-0.33581	0.247	0.49968	0.27032	-0.32786	0.31968	0.11933	-0.50903
BHL	0.08667	1.05613	0.05516	0.27567	-0.21839	0.49233	1.11516	0.57613	0.17143	0.92258	0.22367	-0.67903
CGL	-0.18567	0.76774	0.16194	0.48633	-0.2071	0.47833	0.2571	0.59484	-0.02321	0.55387	0.07333	-0.11871
HHM	0.18233	0.33	0.13677	0.46533	-0.12774	0.56067	0.10839	0	0.14321	0.00968	-0.15767	-0.27258
HCC	-0.157	0.61226	1.23548	0.56067	-0.71065	0.97333	0.32839	0.61581	0.03286	0.49935	-0.03733	-0.59452
KMB	0.18567	0.27806	0.55097	0.02733	-0.46613	0.73333	0.20581	0.17065	-0.05286	0.6671	0.373	-0.08355
MML	0.37533	0.65903	0.1929	0.16533	-0.15226	0.80867	0.39097	0.29	0.1975	0.21839	0.031	-0.06548
SIL	-0.10467	0.200552	0.31161	0.43333	-0.3171	1.104	0.37194	0.73097	0.03321	0.75903	0.09467	-0.44903
UNL	0.26367	0.41581	0.24484	0.12967	-0.0829	0.54	0.93258	0.61871	0.2275	0.68968	0.65433	0.65258

Using this data set the problem reduces to $Maximize\ Er(X)$ with target value 0.28745, truth tolerance 0.1295, and indeterminacy tolerance $\frac{w_1}{7.72\, w_1 + 20\, w_2}$ and rejection tolerance 0.05.

and $Minimize\ \ Ad(X)$with target value 0.0877, truth tolerance 0.08, and indeterminacy tolerance $\frac{w_1}{12.5\, w_1 + 6.67\, w_2}$ and rejection tolerance 0.15.

Solving the portfolio optimization model by the above mentioned methods using LINGO the solutions so obtained is given below in tabular form.

Table2: Optimal solutions using different methods

	$Z_1(x)$	$Z_2(x)$
Generalized neutrosophic goal programming $w_1 = 0.3, w_2 = 0.5, w_3 = 0.7$	0.3159	0.0784
Generalized neutrosophic optimization based on arithmetic aggregation operator $w_1 = 0.3, w_2 = 0.5, w_3 = 0.7$	0.3255	0.0781
Generalized neutrosophic optimization based on geometric aggregation operator $w_1 = 0.3, w_2 = 0.5, w_3 = 0.7$	0.3491	0.0698

For different value of w_1, w_2, w_3 using different method of aggregation for objective functions the solutions so obtained are almost same. Although the best solutions have been obtained using geometric aggregation method for objective functions for different value of w_1, w_2, w_3.

It is clear from the above table that in neutrosophic goal programming method based upon distinct aggregation operator, all the objective functions attained their respective goal and also the restrictions of truth, falsity and indeterminacy membership functions. The sum of truth, falsity and indeterminacy membership function of each of the objective is less than sum of degree of gradiation $w_1 + w_2 + w_3$, which in turn satisfies the condition of neutrosophic set.

6. Conclusion

It was explored in this study that, when the neutrosophic goal programming considered as a method for determining the best portfolio the the best result obtained utilizing different aggregation methods for the mathematical model of this study was obtained by employing geometric aggregation method. The degree of truth membership function is defined using the neutrosophic optimization technique; however, it is not simply a complement of degree of falsehood; rather, these two degrees of membership are independent of degree of indeterminacy. Because we used the neutrosophic goal programming technique to optimize portfolios, it may also be applied to solve other optimization problems of several fields.

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On Nβ*-Closed sets in Neutrosophic Topological spaces

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Abstract: The aim of this paper is to introduce the concept of β*-closed sets in terms of neutrosophic topological spaces. We also study some of the properties of neutrosophic β*-closed sets. Further we introduce Nβ*-continuity and Nβ*-contra continuity in neutrosophic topological spaces.

Keywords: neutrosophic topology, Nβ*-closed set, Nβ*-Continuity and Nβ*-Contra Continuity.

1. Introduction

Zadeh [19] introduced and studied the fuzzy set theory. An intuitionistic fuzzy set was introduced by Atanassov [9]. Coker [10] developed intuitionistic fuzzy topology. Neutrality, the degree of indeterminacy, as an independent concept, was introduced by Smarandache [3,4] in 1998. He also defined the neutrosophic set on three components (t, f, i) = (truth, falsehood, indeterminacy). The Neutrosophic crisp set concept was converted into neutrosophic topological spaces by Salama etal. in [3]. This opened up a wide range of investigation in terms of neutosophic topology and its application in decision-making algorithms. Renu Thomas etal.[17] introduced and studied semi pre-open(or β-open) sets in neutrosophic topological spaces. R. Dhavaseelan and S. Jafari[11] introduced generalized neutrosophic closed sets. In this article, the neutrosophic β*-closed sets are introduced in neutrosophic topological space. Moreover, we introduce and investigate neutrosophic β*-continuous and neutrosophic contra β*-continuous mappings.

2. Preliminaries

Definition 2.1. [6] Let X be a non-empty fixed set. A neutrosophic set (NS) A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ where $\mu_A(x), \sigma_A(x), \nu_A(x)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element $x \in X$ to the set A.

A Neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ can be identified as an ordered triple $\langle \mu_A(x), \sigma_A(x), \nu_A(x)\rangle$ in $]{-0}, 1{+}[$ on X.

Definition 2.2. [6] Let $A =\langle \mu_A(x), \sigma_A(x), \nu_A(x)\rangle$ be a NS on X, then the complement C(A) may be defined as

1. $C(A) = \{\langle x, 1 - \mu_A(x), 1 - \nu_A(x)\rangle : x \in X\}$

2. C(A) = {⟨x, $\nu_A(x)$, σ_A (x), $\mu_A(x)$⟩: x ∈ X}

3. C(A) = {⟨x, $\nu_A(x)$, 1 − $\sigma_A(x)$, $\mu_A(x)$⟩: x ∈ X}

Note that for any two neutrosophic sets A and B,

4. C(A ∪ B) = C(A) ∩ C(B)

5. C(A ∩ B) = C(A) ∪ C(B).

Definition 2.3. [6] For any two neutrosophic sets A = {⟨x, $\mu_A(x)$, $\sigma_A(x)$, $\nu_A(x)$⟩: x ∈ X} and B = {⟨x, $\mu_B(x)$, $\sigma_B(x)$, $\nu_B(x)$⟩: x ∈ X} we may have

1. A ⊆ B ⇔ $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x) \forall$ x ∈ X

2. A ⊆ B ⇔ $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x) \forall$ x ∈ X

3. A ∩ B = ⟨x, $\mu_A(x) \wedge \mu_B(x)$, $\sigma_A(x) \wedge \sigma_B(x)$ and $\nu_A(x) \vee \nu_B(x)$⟩

4. A ∩ B = ⟨x, $\mu_A(x) \wedge \mu_B(x)$, $\sigma_A(x) \vee \sigma_B(x)$ and $\nu_A(x) \vee \nu_B(x)$⟩

5. A ∪ B = ⟨x, $\mu_A(x) \vee \mu_B(x)$, $\sigma_A(x) \vee \sigma_B(x)$ and $\nu_A(x) \wedge \nu_B(x)$⟩

6. A ∪ B = ⟨x, $\mu_A(x) \vee \mu_B(x)$, $\sigma_A(x) \wedge \sigma_B(x)$ and $\nu_A(x) \wedge \nu_B(x)$⟩

Definition 2.4. [6] A neutrosophic topology (NT) on a non-empty set X is a family τ of neutrosophic subsets in X satisfies the following axioms:
(NT1) 0_N, 1_N ∈ τ
(NT2) $G_1 \cap G_2$ ∈ τ for any G_1 , G_2 ∈ τ
(NT3) ∪ G_i ∈ τ ∀{G_i : i ∈ J} ⊆ τ

Definition 2.5. [5] Let A be an Neutrosophic Set in NTS X. Then Nint(A) = ∪ {G : G is an NOS in X and G ⊆ A} is called a neutrosophic interior of A Ncl(A) = ∩ {K : K is an NCS in X and A ⊆ K} is called a neutrosophic closure of A.

Definition 2.6. A NS A of a NTS X is said to be
(1) a neutrosophic semi-open set (NSOS)[15] if A ⊆ NCl(NInt(A)) and a neutrosophic semi-closed set (NSCS) if NInt(NCl(A)) ⊆ A.
(2) a neutrosophic α-open set (NαOS)[8] if A ⊆ NInt(NCl(NInt(A))) and a neutrosophic α-closed set (NαCS) if NCl(NInt(NCl(A))) ⊆ A.
(3) a neutrosophic semi-pre open set or β-open(NβOS) [17] if A ⊆ NCl(NInt(NCl(A))) and a neutrosophic semi-pre closed set or β-closed(NβCS) if NInt(NCl(NInt(A))) ⊆ A.

Definition 2.7. [17] Consider a NS A in a NTS (X, τ). Then the neutrosophic β interior and the neutrosophic β closure are defined as
Nβint(A) = ∪{G: G is a Nβ-open set in X and G ⊆ A}
Nβcl(A) = ∩ {K: K is a Nβ-closed set in X and A ⊆ K }

Definition 2.8. [16] A subset A of a neutrosophic topological space (X, τ) is called a neutrosophic generalized closed (Ng-closed) set if Ncl(A) ⊆ U whenever A ⊆ U and U is neutrosophic open set in (X, τ).

Definition 2.9. [18] A subset A of a neutrosophic topological space (X,τ) is called a neutrosophic ω– closed (Nω-closed) set if Ncl(A) ⊆ U whenever A ⊆ U and U is neutrosophic semi - open set in (X, τ).

3. Neutrosophic β*-closed set

In this section, the new concept of neutrosophic β*-closed sets in neutrosophic topological spaces was defined and studied.

Definition 3.1: A subset A of a neutrosophic topological space (X, τ) is called a neutrosophic β*- closed (Nβ*-closed) set if Nβcl(A) ⊆ U whenever A ⊆ U and U is neutrosophic g-open set in (X, τ).

Example 3.2. Let X = {a,b,c} with τ_N = {0_N,G,1_N} where G = ⟨(a,0.5,0.6,0.4),(b,0.4,0.5,0.2), (c,0.7,0.6,0.9)⟩. Here A = ⟨(a,0.2,0.2,0.1),(b,0,1,0.2),(c,0.9,0.4,0.7)⟩ , B = ⟨(a,0.1,0.7,0.3),(b,0.2,0,0),(c,0.8,0.3,0.9)⟩, C = ⟨(a,0.4,0.5,0.2),(b,0.2,0.1,0.4),(c,0.6,0.5,1)⟩, D = ⟨(a,0.3,0.2,0.8),(b,0.1,0.3,0.6),(c,0.7,0.2,0.8)⟩ are some examples of Nβ*-closed sets.

Theorem 3.3. Each Neutrosophic Closed Set is an Nβ*-closed set in X.
Proof. Let A ⊆ U where U is a neutrosophic g-open set in X. Since A is a neutrosophic closed set Ncl(A) = A. We have Nβcl(A) ⊆ Ncl(A) = A ⊆ U. Hence Nβcl(A) ⊆ U. Therefore A is a Nβ*-closed set in X.

The converse of the above theorem need not be true as shown in the following example.

Example 3.4. Let X = {a,b,c} with τ_N = {0_N,G,1_N} where G = ⟨(a,0.5,0.6,0.4),(b,0.4,0.5,0.2), (c,0.7,0.6,0.9)⟩. Here A = ⟨(a,0.4,0.5,0.5),(b,0.3,0.2,0.3),(c,0.5,0.4,1)⟩ is a Nβ*-closed set, however A is not a Neutrosophic Closed Set.

Theorem 3.5. Each Nβ − closed set is an Nβ*-closed set in X.
Proof. Let A ⊆ U where U is a neutrosophic g-open set in X. Let A be an Nβ−closed set in X. Hence Nβcl(A) = A ⊆ U. Hence Nβcl(A) ⊆ U. Therefore A is a Nβ*-closed set in X.

The converse of the above theorem need not be true as shown in the following example.
Example 3.6. Let X = {a,b,c} with τ_N = {0_N,A,B,1_N} where A = ⟨(a,0.5,0.5,0.4),(b,0.7,0.5,0.5), (c,0.4,0.5,0.5)⟩ and B = ⟨(a,0.3,0.4,0.4),(b,0.4,0.5,0.5),(c,0.3,0.4,0.6)⟩. Here C = ⟨(a,0.7,0.6,0.3),(b,0.9,0.7,0.2), (c,0.5,0.7,0.3)⟩ is a Nβ*-closed set, but C is not an Nβ − Closed Set.

Theorem 3.7. Each Nsemi-closed set is an Nβ*-closed set in X.
Proof. Let $A \subseteq U$ where U is a neutrosophic g-open set in X. Since A is an Nsemi-closed set in X, we have Nβcl(A) ⊆ Nscl(A) = A ⊆ U. Hence Nβcl(A) ⊆ U. Therefore A is an Nβ*-closed set in X.

The converse of the above theorem need not be true as shown in the following example.
Example 3.8. Let X = {a,b} with τ_N = {0_N,A,B,1_N} where A = ⟨(a,0.4,0.3,0.5),(b,0.1,0.2,0.5)⟩ and B = ⟨(a,0.4,0.4,0.5),(b,0.4,0.3,0.4)⟩. Here C = ⟨(a,0.4,0.6,0.5),(b,0.3,0.6,0.9)⟩ is a Nβ*-closed set, but C is not an N semi − Closed Set.

Theorem 3.9. Each Neutrosophic generalized-closed set is an Nβ*-closed set in X.

Proof. Let $A \subseteq U$ be a neutrosophic generalized closed set, where U is a neutrosophic open set in X. Since every neutrosophic open set in X is a neutrosophic g-open set, we have Ncl(A) ⊆ U. Also we have Nβcl(A) ⊆ Ncl(A) ⊆ U. Hence Nβcl(A) ⊆ U. Therefore A is an Nβ*-closed set in X.

The converse of the above theorem need not be true as shown in the following example.

Example 3.10. Let X = {a,b} with $\tau_N = \{0_N, A, B, 1_N\}$ where A = ⟨(a,0.4,0.3,0.5),(b,0.1,0.2,0.5)⟩ and B = ⟨(a,0.4,0.4,0.5),(b,0.4,0.3,0.4)⟩. Here C = ⟨(a,0.3,0.3,0.6),(b,0.3,0.2,0.5)⟩ is a Nβ*-closed set, but C is not a Neutrosophic g– Closed Set.

Theorem 3.11. Each Neutrosophic ω-closed set is an Nβ*-closed set in X.

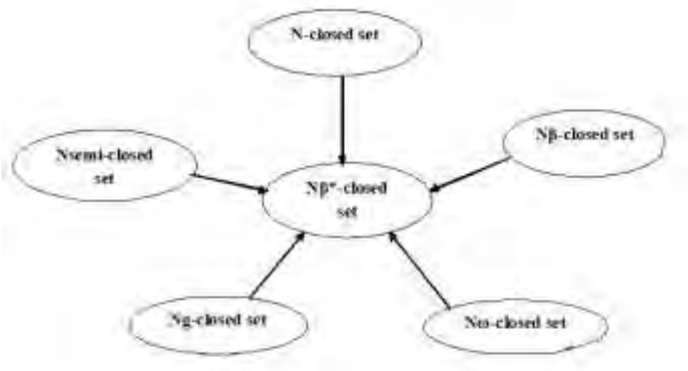
Proof. Let $A \subseteq U$ be a neutrosophic ω- closed set, where U is a neutrosophic semi-open set in X. Since every neutrosophic semi-open set in X is a neutrosophic g-open set, we have Ncl(A) ⊆ U. Also we have Nβcl(A) ⊆ Ncl(A) ⊆ U. Hence Nβcl(A) ⊆ U. Therefore A is an Nβ*-closed set in X.

The converse of the above theorem need not be true as shown in the following example.

Example 3.12. Let X = {a,b,c} with $\tau_N = \{0_N, A, B, 1_N\}$ where

A = ⟨(a,0.5,0.5,0.4),(b,0.7,0.5,0.5), (c,0.4,0.5,0.5)⟩ and B = ⟨(a,0.3,0.4,0.4),(b,0.4,0.5,0.5),(c,0.3,0.4,0.6)⟩. Here C = ⟨(a,0.2,0.3,0.5),(b,0.3,0.2,0.6), (c,0.1,0.2,0.9)⟩ is an Nβ*-closed set, but C is not an Nω– Closed Set.

Remark 3.13. The following diagram shows the relationships of Nβ*-closed set with other know existing sets. A ⟶ B represents A implies B but not conversely.



Theorem 3.14. If A and B are Nβ*-closed sets in (X, τ_N), then A ∪ B is an Nβ*-closed set in (X,τ_N).

Proof: Let A and B are Nβ*-closed sets in (X,τ_N). Then Nβcl(A) ⊆ U whenever A ⊆ U and U is neutrosophic g-open set in (X, τ_N) and Nβcl(B) ⊆ U whenever B ⊆ U and U is neutrosophic g-open set in (X, τ_N). Since A ⊆ U and B ⊆ U, which implies A ∪ B ⊆ U and U is neutrosophic g-open set, then Nβcl(A) ⊆ U and Nβcl(B) ⊆ U implies Nβcl(A) ∪ Nβcl(B) ⊆ U, hence Nβcl(A ∪ B) ⊆ U. Thus A ∪ B is an Nβ*-closed set in X.

Theorem 3.15. A neutrosophic set A is Nβ*-closed set then Nβcl(A) – A does not contain any nonempty Nβ-closed sets.

Proof: Suppose that A is an Nβ*-closed set. Let F be an Nβ-closed set such that F ⊆ Nβcl(A) – A which implies F ⊆ Nβcl(A)∩A^c. Then A ⊆ F^c. Since A is Nβ*-closed set, we have Nβcl(A) ⊆ F^c . Consequently F ⊆ $(N\beta cl(A))^c$. We have F ⊆ Nβcl(A). Thus F ⊆ Nβcl(A)∩$(N\beta cl(A))^c$ = ϕ. Hence F is empty.

Theorem 3.16. If A is an Nβ*-closed set in (X,τ_N) and A ⊆ B ⊆ Nβcl(A), then B is Nβ*-closed.

Proof: Let B ⊆ U where U is a Neutrosophic g-open set in (X,τ_N). Then A ⊆ B implies A ⊆ U. Since A is an Nβ*-closed set, we have Nβcl(A)⊆U. Also A ⊆ Nβcl(B) implies Nβcl(B) ⊆ Nβcl(A). Thus Nβcl(B)⊆ U and so B is an Nβ*-closed set in (X,τ_N).

Theorem 3.17. If A is Neutrosophic g-open and Nβ*-closed, then A is Nβ-closed set.
Proof: Since A is Neutrosophic g-open and Nβ*-closed, then Nβcl(A) ⊆ A. Therefore Nβcl(A) =A. Hence A is Nβ-closed.

4. On Nβ*-Continuity and Nβ*-Contra Continuity

Definition 4.1. Let f be a mapping from a neutrosophic topological space (X, τ) to a neutrosophic topological space (Y, σ). Then f is said to be a neutrosophic β*-continuous (Nβ*-continuous) mapping if f^{-1} (A) is a Nβ*-closed set in X, for each neutrosophic-closed set A in Y.

Theorem 4.2. Consider a mapping $f\colon (X, \tau) \to (Y, \sigma)$. Then the following statements are equal.
(1) f is Nβ*-continuous
(2) The inverse image of each neutrosophic-closed set A in Y is Nβ*-closed set in X.
Proof. The result is obvious from the Definition 4.1.

Theorem 4.3. Consider an Nβ*-continuous mapping $f : (X, \tau) \to (Y, \sigma)$ then the following assertions hold:
(1) for all neutrosophic sets A in X, $f(N\beta^*\text{-} Ncl(A)) \subseteq Ncl(f(A))$
(2) for all neutrosophic sets B in Y, $N\beta^*Ncl\ (f^{-1}(B)) \subseteq f^{-1}\ (Ncl(B))$.
Proof. (1) Let A be a neutrosophic set in X, then Ncl(f (A)) be a neutrosophic closed set in Y and f be Nβ*-continuous, then it follows that f^{-1}(Ncl(f(A))) is Nβ*-closed in X. In view that $A \subseteq f^{-1}(Ncl(f(A)))$ and $N\beta^*cl(A) \subseteq f^{-1}(Ncl(f(A)))$. Hence, $f(N\beta^*\text{-} Ncl(A)) \subseteq Ncl(f(A))$.
(2) We get $f(N\beta^*\text{-}Ncl(f^{-1}\ (B))) \subseteq Nclf(f^{\ -1}(B))) \subseteq Ncl(B)$. Hence, $N\beta^*\text{-}Ncl(f^{\ -1}\ (B)) \subseteq f^{\ -1}\ (Ncl(B))$ by way of changing A with B in (1).

Definition 4.4. Let f be a mapping from a neutrosophic topological space (X, τ) to a neutrosophic topological space (Y, σ). Then f is known as neutrosophic β*-contra continuous (Nβ*-contra continuous) mapping if f^{-1} (B) is a Nβ*-closed set in X for each neutrosophic-open set B in Y.

Theorem 4.5. Consider a mapping $f : (X, \tau) \to (Y, \sigma)$. Then the following assertions are equivalent:
(1) f is a Nβ*- contra continuous mapping
(2) $f^{\ -1}$(B) is an Nβ*-closed set in X, for each neutrosophic open set B in Y.
Proof. (1) ⟹ (2) Assume that f is Nβ*-contra continuous mapping and B is a NOS in Y. Then B^c is an NCS in Y. It follows that, $f^{\ -1}\ (B^c\ )$ is an Nβ*-open set in X. For this reason, $f^{\ -1}$(B) is an Nβ*closed set in X.
(2) ⟹ (1) The converse is similar.

Theorem 4.6. Consider a bijective mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ from a NTS(X, τ) into an NTS(Y, σ). If $Ncl(f(A)) \subseteq f(N\beta^{*}int(A))$, for each NS B in X, then the mapping f is Nβ*-contra continuous.

Proof. Consider a NCS B in Y. Then $Ncl(B) = B$ and f is onto, by way of assumption, $f(N\beta^{*}int(f^{-1}(B)))$ $\subseteq Ncl(f(f^{-1}(B))) = Ncl(B) = B$. Consequently, $f^{-1}(f(N\beta^{*}int(f^{-1}(B)))) \subseteq f^{-1}(B)$. Additionally due to the fact that f is an into mapping, we have $N\beta^{*}int(f^{-1}(B)) = f^{-1}(f(N\beta^{*}int(f^{-1}(B)))) \subseteq f^{-1}(B)$. Consequently, $N\beta^{*}int(f^{-1}(B)) = f^{-1}(B)$, so $f^{-1}(B)$ is an Nβ*-open set in X. Hence, f is an Nβ*-contra continuous mapping.

5. Conclusion and Future work

In this paper we have introduced Nβ*-closed set, Nβ*- continuous function, Nβ*- contra continuous function and discussed some of its properties and derived some contradicting examples. This idea can be developed and extended in the area of homeomorphisms, compactness and connectedness and so on.

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HSSM- MADM Strategy under SVPNS environment

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Abstract
In the present paper, we propose the Hyperbolic Sine Similarity Measure (HSSM) for pentapartitioned neutrosophic sets which is based on hyperbolic sine function. We also establish some properties of the similarity measures by providing some suitable examples. Further we develop an MADM (Multi-Attribute-Decision-Making) model for single valued pentapartitioned neutrosophic set (SVPNS) environment based on the similarity measure which we call HSSM-MADM strategy. We also validate our proposed model by solving a numerical example.
Keywords: *MADM*; *Neutrosophic Set*; *Pentapartitioned Neutrosophic Set*; *Similarity Measure*.

1. Introduction:

Smarandache grounded the idea of Neutrosophic Set (NS) [1] as an extension of Fuzzy Set (FS) [2], and Intuitionistic Fuzzy Set (IFS) [3] to deal with incomplete and indeterminate information. In NS theory, truth-membership, indeterminacy-membership, and falsity-membership values are independent of each other. The concept of Single Valued NS (SVNS) was presented by Wang et al. [4], which is the subclass of an NS. By using SVNS, we can represent incomplete, imprecise, and indeterminate information that helps in decision making in the real- world problems. NS and the various extensions of NSs were studied and used for model/algorithm in different areas of research such as medical diagnosis ([5-7], social problems [8], conflict resolution [9], decision making [10-27], etc. Detail theoretical development and applications of NS and its extensions can be found in the studies [28-37].

Chatterjee et al. [38] defined the Quadripartitioned SVNS (QSVNS) by introducing contradiction and ignorance membership functions in place of indeterminacy membership function. Mallick and Pramanik [39] defined Pentapartitioned Neutrosophic Set (PNS) by introducing unknown membership function in QSVNS to handle uncertainty and indeterminacy comprehensively.

Similarity measures [40-68] were defined in various NS environments and were utilized for decision, medical diagnosis, etc. Mondal and Pramanik [69] proposed Hyperbolic Sine Similarity Measure (HSSM) and proved their basic properties in SVNS environment. Receiving motivation from the work of Mondal and Pramanik [70], we extend the HSSM for Single Valued PNSs (SVPNSs) and prove their basic properties. Based on HSSM, we propose an HSSM based MADM strategy which we call the HSSM-MADM model under SVPNS environment. Also, we validate our model by solving an illustrative example of an MADM problem.

The remaining part of this paper is divided into several sections:

In section 2, we recall PNS, and some relevant properties of PNSs. In section 3, we introduce the notion of SVPNS and HSSM between them. In section 4, we develop the SVPNS- MADM strategy. In section 5, we validate the proposed strategy by solving an illustrative MADM problem. In section 6, we conclude the paper by stating the future scope of research.

2. Some Relevant Definitions:

Definition 2.1. [4] An SVNS K over a non-empty set L is defined as follows:

$K = \{(u, T_K(u), I_K(u), F_K(u)): u \in L\}$, where T_K, I_K, F_K are truth, indeterminacy, and falsity membership mappings from L to$]^-0,1^+[$, and $^-0 \leq T_K(u) + I_K(u) + F_K(u) \leq 3^+$.

Example 2.1. Let $L= \{q, w, e\}$ be a universe of discourse. Then $\{(q, 0.9, 0.6, 0.4), (w, 0.4, 0.6, 0.7), (e, 0.2, 0.7, 0.7)\}$ is an SVNS over L.

Definition 2.2. [4] Suppose that L be a universe of discourse. Then P, a pentapartitioned neutrosophic set (P-NS) over L is denoted as follows:

$P= \{(u, T_P(q), C_P(u), G_P(u), U_P(u), F_P(u)): u \in L\}$, where $T_P, C_P, G_P, U_P, F_P : L \rightarrow]0,1[$ are the truth, contradiction, ignorance, unknown, falsity membership functions and so $0 \leq T_P(q)+C_P(q)+G_P(q)+ U_P(q)+F_P(q) \leq 5$.

Example 2.2. Let $L = \{q, w\}$ be a universe of discourse. Then $\{(q, 0.9, 0.6, 0.4, 0.3, 0.5), (w, 0.4, 0.6, 0.7, 0.8, 0.2)\}$ is a PNS over L.

Definition 2.3.[4] Assume that $X= \{(q, T_X(q), C_X(q), G_X(q), U_X(q), F_X(q)): q \in W\}$ and $Y = \{(q, T_Y(q), C_Y(q), G_Y(q), U_Y(q), F_Y(q)): q \in W\}$ be two PNSs over W. Then $X \subseteq Y \Leftrightarrow T_X(q) \leq T_Y(q)$, $C_X(q) \leq C_Y(q)$, $G_X(q) \geq G_Y(q)$, $U_X(q) \geq U_Y(q)$, $F_X(q) \geq F_Y(q)$, for all $q \in W$.

Example 2.3. Let $L= \{q, w\}$ be a universe of discourse. Consider two PNSs $X=\{(q, 0.5, 0.6, 0.5, 0.7, 0.3), (w, 0.8, 0.8, 0.3, 0.3, 0.3)\}$ and $Y=\{(q, 0.9, 0.9, 0.3, 0.3, 0.3), (w, 1.0, 0.8, 0.2, 0.1, 0.3)\}$ over L. Then $X \subseteq Y$.

Definition 2.4.[4] Suppose that $X= \{(u, T_X(u), C_X(u), G_X(u), U_X(u), F_X(u)): u \in L\}$ and $Y= \{(u, T_Y(u), C_Y(u), G_Y(u), U_Y(u), F_Y(u)): u \in L\}$ be two PNSs over L. Then $X \cup Y= \{(u, \max \{T_X(u), T_Y(u)\}, \max \{C_X(u), C_Y(u)\}, \min \{G_X(u), G_X(u)\}, \min \{U_X(u), U_X(u)\}, \min \{F_X(u), F_X(u)\}): u \in L\}$.

Example 2.4. Suppose that $L= \{q, w\}$. Consider two PNSs $X= \{(q, 0.7, 0.5, 0.5, 0.7 ,0.7), (w, 0.5, 0.6, 0.7, 0.7, 0.6)\}$ and $Y= \{(q,1.0, 0.6, 0.8, 0.7, 0.7), (w, 0.6, 0.7, 0.8, 0.4, 0.6)\}$ over L. Then $X \cup Y=$
$\{(q,1.0, 0.6, 0.5, 0.7, 0.7), (w, 0.6, 0.7, 0.7, 0.4, 0.6)\}$.

Definition 2.5.[4] Suppose that $X= \{(u, T_X(u), C_X(u), G_X(u), U_X(u), F_X(u)): u \in W\}$ and $Y= \{(u, T_Y(u), C_Y(u), G_Y(u), U_Y(u), F_Y(u)): u \in L\}$ are two PNSs over L. Then $X^c= \{(u, F_X(u), U_X(u), 1\text{-}G_X(u), C_X(u), T_X(u)): u \in L\}$.

Example 2.5. Suppose that L= {q, w} be a universe of discourse and X= {(q, 0.5, 0.7, 0.7, 0.6, 1.0), (w, 1.0,0.5, 0.5, 0.5, 1.0)} be a PNS over L. Then X^c= {(q, 1.0, 0.6, 0.3, 0.7, 0.5), (w,1.0, 0.5, 0.5, 0.5, 1.0)}.

Definition 2.6.[4] Suppose that X= {(u, $T_X(u)$, $C_X(u)$, $G_X(u)$, $U_X(u)$, $F_X(u)$): $u\in L$} and Y= {(u, $T_Y(u)$, $C_Y(u)$, $G_Y(u)$, $U_Y(u)$, $F_Y(u)$): $u\in L$} be two PNSs over L. Then $X \cap Y$ = {(u, min {$T_X(u)$, $T_Y(u)$}, min {$C_X(u)$, $C_Y(u)$}, max {$G_X(u)$, $G_X(u)$}, max {$U_X(u)$, $U_X(u)$}, max {$F_X(u)$, $F_X(u)$}): $u\in L$}.

Example 2.6. Suppose that X and Y be two PNSs over a non-empty set L, as shown in Example 2.4. Then $X \cap Y$= {(q, 0.7, 0.5, 0.8, 0.7, 0.7), (w, 0.5, 0.6, 0.8, 0.7, 0.6)}.

Definition 2.7. [4] The null PNS (0_{PN}) and the absolute PNS (1_{PN}) over L are defined by

(i) 0_{PN} = {(u, 0, 0, 1, 1, 1): $u\in L$};

(ii) 1_{PN} = {(u,1, 1, 0, 0, 0): $u\in L$}.

3. Single Valued Pentapartitioned Neutrosophic Set (SVPNS):

Definition 3.1. [39] Assume that L be a universe of discourse. An *SVPNS Y* over L is characterized by a truth-membership function T_Y, a contradiction-membership function C_Y, an ignorance-membership function G_Y, an unknown-membership function U_Y, a falsity-membership function F_Y. For each element $u\in L$, $T_Y(u)$, $C_Y(u)$, $G_Y(u)$, $U_Y(u)$, $F_Y(u)\in[0,1]$.

The *SVPNS Y* is denoted as follows:

Y= {(u, $T_Y(u)$, $C_Y(u)$, $G_Y(u)$, $U_Y(u)$, $F_Y(u)$): $u\in L$}.

Definition 3.2. [39] Suppose that B= {(u, $T_B(u)$, $C_B(u)$, $G_B(u)$, $U_B(u)$, $F_B(u)$): $u\in L$} and A= {(u, $T_A(u)$, $C_A(u)$, $G_A(u)$, $U_A(u)$, $F_A(u)$): $u\in L$} be any two *SVPNSs* over L. Then

(i) B=A $\Leftrightarrow$ $T_B(u) = T_A(u)$, $C_B(u) = C_A(u)$, $G_B(u) = G_A(u)$, $U_B(u) = U_A(u)$, $F_B(u) = F_A(u)$, for each $u\in L$;

(ii) $B\subseteq Y \Leftrightarrow$ $T_B(u) \le T_A(u)$, $C_B(u) \le C_A(u)$, $G_B(u) \ge G_A(u)$, $U_B(u) \ge U_A(u)$, $F_B(u) \ge F_A(u)$, for each $u\in L$.

Definition 3.3. Suppose that M= {(u, $T_M(u)$, $C_M(u)$, $G_M(u)$, $U_M(u)$, $F_M(u)$): $u\in L$} and W = {(u, $T_W(u)$, $C_W(u)$, $G_W(u)$, $U_W(u)$, $F_W(u)$): $u\in L$} are any two *SVPNSs* over L. Then the hyperbolic sine similarity measure between M and W is defined by:

HSSM(M, W)=

$$1- \frac{1}{n}\sum_{i=1}^{n}\left(\frac{sinh(|T_M(u_i)-T_W(u_i)| + |C_M(u_i)-C_W(u_i)| + |G_M(u_i)-G_W(u_i)| + |U_M(u_i)-U_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{75}\right) \quad (1)$$

Definition 3.4. Suppose that M = {(u, $T_M(u)$, $C_M(u)$, $G_M(u)$, $U_M(u)$, $F_M(u)$): $u\in L$} and W = {(u, $T_W(u)$, $C_W(u)$, $G_W(u)$, $U_W(u)$, $F_W(u)$): $u\in L$} be any two SVPNSs over L. Then the weighted hyperbolic sine similarity measure between M and W is defined by:

WHSSM(M, W)=

$$1- \frac{1}{n}\sum_{i=1}^{n} w_i\left(\frac{sinh(|T_M(u_i)-T_W(u_i)| + |C_M(u_i)-C_W(u_i)| + |G_M(u_i)-G_W(u_i)| + |U_M(u_i)-U_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{75}\right) \quad (2)$$

where $0 \le w_i \le 1$ and $\sum_{i=}^{n} w_i$=1.

Theorem 3.1. Assume that HSSM(M, W) is the hyperbolic sine similarity measure between two SVPNSs M and W. Then 0 ≤ HSSM(M, W) ≤ 1.

Proof. Suppose that M={(u, $T_M(u)$, $C_M(u)$, $G_M(u)$, $U_M(u)$, $F_M(u)$): $u\in L$} and W={(u, $T_W(u)$, $C_W(u)$, $G_W(u)$, $U_W(u)$, $F_W(u)$): $u\in L$} aew any two SVPNSs over L.

Now $0\le T_M(u_i)$, $C_M(u_i)$, $G_M(u_i)$, $U_M(u_i)$, $F_M(u_i)$, $T_W(u_i)$, $C_W(u_i)$, $G_W(u_i)$, $U_W(u_i)$, $F_W(u_i)\le 1$.

$\Rightarrow 0 \leq |\, T_M(u_i) - T_W(u_i)\, | + |\, C_M(u_i) - C_W(u_i)\, | + |\, G_M(u_i) - G_W(u_i)\, | + |\, U_M(u_i) - U_W(u_i)\, | + |\, F_M(u_i) - F_W(u_i)\, | \leq 5.$

$\Rightarrow 0 \leq \left(\frac{sinh(|T_M(u_i)-T_W(u_i)| + |C_M(u_i)-C_W(u_i)| + |G_M(u_i)-G_W(u_i)| + |U_M(u_i)-U_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{75}\right) \leq 1.$

$\Rightarrow 0 \leq 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{sinh(|T_M(u_i)-T_W(u_i)| + |C_M(u_i)-C_W(u_i)| + |G_M(u_i)-G_W(u_i)| + |U_M(u_i)-U_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{75}\right) \leq 1.$

$\Rightarrow 0 \leq$ HSSM$(M, W) \leq 1$.

Theorem 3.2. Assume that HSSM(M, W) is the hyperbolic sine similarity measure between two SVPNSs M and W. Then HSSM$(M, W) = 1$ if $M = W$.

Proof. Suppose that $M = \{(u, T_M(u), C_M(u), G_M(u), U_M(u), F_M(u)): u \in L\}$ and $W = \{(u, T_W(u), C_W(u), G_W(u), U_W(u), F_W(u)): u \in L\}$ are any two SVPNSs over L such that $M=W$.

So $T_M(u_i) = T_W(u_i)$, $C_M(u_i) = C_W(u_i)$, $G_M(u_i) = G_W(u_i)$, $U_M(u_i) = U_W(u_i)$, $F_M(u_i) = F_W(u_i)$ for each $u_i \in L$.

$\Rightarrow |T_M(u_i) - T_W(u_i)| = 0, |C_M(u_i) - C_W(u_i)| = 0, |G_M(u_i) - G_W(u_i)| = 0, |U_M(u_i) - U_W(u_i)| = 0, |F_M(u_i) - F_W(u_i)| = 0$ for each $u_i \in L$.

$\Rightarrow sinh\begin{pmatrix}|T_M(u_i) - T_W(u_i)| + |C_M(u_i) - C_W(u_i)| + |G_M(u_i) - G_W(u_i)| \\ +|U_M(u_i) - U_W(u_i)| + |F_M(u_i) - F_W(u_i)|\end{pmatrix} = 0.$

$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\left(\frac{sinh(|T_M(u_i)-T_W(u_i)| + |C_M(u_i)-C_W(u_i)| + |G_M(u_i)-G_W(u_i)| + |U_M(u_i)-U_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{75}\right) = 0$

$\Rightarrow 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{sinh(|T_M(u_i)-T_W(u_i)| + |C_M(u_i)-C_W(u_i)| + |G_M(u_i)-G_W(u_i)| + |U_M(u_i)-U_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{75}\right) = 1$

$\Rightarrow$ HSSM$(M, W) = 1$.

Theorem 3.3. Assume that HSSM(M, W) is the hyperbolic sine similarity measure between two SVPNSs M and W. Then HSSM(M, W) = HSSM(W, M).

Proof. Suppose that $M = \{(u, T_M(u), C_M(u), G_M(u), U_M(u), F_M(u)): u \in L\}$ and $W = \{(u, T_W(u), C_W(u), G_W(u), U_W(u), F_W(u)): u \in L\}$ any two SVPNSs over L.

Now HSSM(M, W)=

$1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{sinh(|T_M(u_i)-T_W(u_i)| + |C_M(u_i)-C_W(u_i)| + |G_M(u_i)-G_W(u_i)| + |U_M(u_i)-U_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{75}\right)$

$= 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{sinh(|T_W(u_i)-T_M(u_i)| + |C_W(u_i)-C_M(u_i)| + |G_W(u_i)-G_M(u_i)| + |U_W(u_i)-U_M(u_i)| + |F_W(u_i)-F_M(u_i)|)}{75}\right)$

= HSSM(W, M).

Therefore HSSM(M, W)= HSSM(M, W).

Theorem 3.4. Assume that SSM(M, W) is the hyperbolic sine similarity measure between the SVPNSs M and W. If Q is an SVPNS over L such that $M \subseteq W \subseteq Q$, then HSSM$(M, W) \geq$ HSSM(M, Q), HSSM$(W, Q) \geq$ HSSM(M, Q).

Proof. Suppose that $M = \{(u, T_M(u), C_M(u), G_M(u), U_M(u), F_M(u)): u \in L\}$ and $W = \{(u, T_W(u), C_W(u), G_W(u), U_W(u), F_W(u)): u \in L\}$ are any two SVPNSs over L. Let Q be an SVPNS over L such that $M \subseteq W \subseteq Q$. Since $M \subseteq W \subseteq Q$, so $|T_M(u_i) - T_W(u_i)| \leq |T_M(u_i) - T_Q(u_i)|$, $|C_M(u_i) - C_W(u_i)| \leq |C_M(u_i) - C_Q(u_i)|$, $|G_M(u_i) - G_W(u_i)| \leq |G_M(u_i) - G_Q(u_i)|$, $|U_M(u_i) - U_W(u_i)| \leq |U_M(u_i) - U_Q(u_i)|$, $|F_M(u_i) - F_W(u_i)| \leq |F_M(u_i) - F_Q(u_i)|$.

Now HSSM(M, W)=

$1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{sinh(|T_M(u_i)-T_W(u_i)| + |C_M(u_i)-C_W(u_i)| + |G_M(u_i)-G_W(u_i)| + |U_M(u_i)-U_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{75}\right)$

$\geq 1- \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\sinh(|T_M(u_i)-T_Q(u_i)| + |C_M(u_i)-C_Q(u_i)| + |G_M(u_i)-G_Q(u_i)| + |U_M(u_i)-U_Q(u_i)| + |F_M(u_i)-F_Q(u_i)|)}{75}\right)$

= HSSM(M, Q).

Therefore, HSSM$(M, W) \geq$ HSSM(M, Q).

Again, from $M \subseteq W \subseteq Q$, we can say that $|T_W(u_i)-T_Q(u_i)| \leq |T_M(u_i)-T_Q(u_i)|$, $|C_W(u_i)-C_Q(u_i)| \leq |C_M(u_i)-C_Q(u_i)|$, $|G_W(u_i)-G_Q(u_i)| \leq |G_M(u_i)-G_Q(u_i)|$, $|U_W(u_i)-U_Q(u_i)| \leq |U_M(u_i)-U_Q(u_i)|$, $|F_M(u_i)-F_W(u_i)| \leq |F_M(u_i)-F_Q(u_i)|$.

Now, HSSM(W, Q)=

$1- \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\sinh(|T_W(u_i)-T_Q(u_i)| + |C_W(u_i)-C_Q(u_i)| + |G_W(u_i)-G_Q(u_i)| + |U_W(u_i)-U_Q(u_i)| + |F_W(u_i)-F_Q(u_i)|)}{75}\right)$

$\geq 1- \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\sinh(|T_M(u_i)-T_Q(u_i)| + |C_M(u_i)-C_Q(u_i)| + |G_M(u_i)-G_Q(u_i)| + |U_M(u_i)-U_Q(u_i)| + |F_M(u_i)-F_Q(u_i)|)}{75}\right)$

= HSSM(M, Q).

Therefore, HSSM$(M, W) \geq$ HSSM(M, Q).

4. SVPNS- MADM Strategy

Suppose that $Q = \{Q_1, Q_2, ..., Q_n\}$ is a finite set of possible alternatives from which a decision maker needs to choose the best alternative. Let $P = \{P_1, P_2, ..., P_m\}$ be the finite collection of attributes for every alternative. A decision maker provides their evaluation information of each alternative Q_i (i = 1, 2,..., n) against the attribute P_j (j = 1, 2, ..., m) in terms of single valued pentapartitioned numbers. The whole evaluation information of all alternatives can be expressed by a decision matrix. The steps of proposed HSSM-MADM strategy (see figure 1) are described as follows:

Step-1: Construct the decision matrix

The whole evaluation information of each alternative Q_i (i = 1, 2,..., n) based on the attributes P_j (j = 1, 2, ..., m) is expressed in terms of SVPNS E_{Q_i}= $\{(P_j, T_{ij}(Q_i, P_j), C_{ij}(Q_i, P_j), G_{ij}(Q_i, P_j), U_{ij}(Q_i, P_j), F_{ij}(Q_i, P_j)): P_j \in P\}$, where $(T_{ij}(Q_i, P_j), C_{ij}(Q_i, P_j), G_{ij}(Q_i, P_j), U_{ij}(Q_i, P_j), F_{ij}(Q_i, P_j))$ denotes the evaluation assessment of Q_i (i = 1, 2,..., n) against P_j (j = 1, 2, ..., m).

Then the Decision Matrix (DM$[Q|P]$) can be expressed as:

DM$[Q|P]$ =

	P_1	P_2	…… ….	…..…	P_m
Q_1	<$T_{11}(Q_1, P_1), C_{11}(Q_1, P_1), G_{11}(Q_1, P_1), U_{11}(Q_1, P_1), F_{11}(Q_1, P_1)$>	<$T_{12}(Q_1, P_2), C_{12}(Q_1, P_2), G_{12}(Q_1, P_2), U_{12}(Q_1, P_2), F_{12}(Q_1, P_2)$>	……. ..	…….	<$T_{1m}(Q_1, P_m), C_{1m}(Q_1, P_m), G_{1m}(Q_1, P_m), U_{1m}(Q_1, P_m), F_{1m}(Q_1, P_m)$>
Q_2	<$T_{21}(Q_2, P_1), C_{21}(Q_2, P_1), G_{21}(Q_2, P_1), U_{21}(Q_2, P_1), F_{21}(Q_2, P_1)$>	<$T_{22}(Q_2, P_2), C_{22}(Q_2, P_2), G_{22}(Q_2, P_2), U_{22}(Q_2, P_2), F_{22}(Q_2, P_2)$>	… … …..	…..…	<$T_{2m}(Q_2, P_m), C_{2m}(Q_2, P_m), G_{2m}(Q_2, P_m), U_{2m}(Q_2, P_m), F_{2m}(Q_2, P_m)$>
. .	. .	. .	. .	. .	. .

.	.	.	.	.	
Q_n	$<T_{n1}(Q_n, P_1), C_{n1}(Q_n, P_1), G_{n1}(Q_n, P_1), U_{n1}(Q_n, P_1), F_{n1}(Q_n, P_1)>$	$<T_{n2}(Q_n, P_2), C_{n2}(Q_n, P_2), G_{n2}(Q_n, P_2), U_{n2}(Q_n, P_2), F_{n2}(Q_n, P_2)>$	… … …..	…… ….	$<T_{nm}(Q_n, P_m), C_{nm}(Q_n, P_m), G_{nm}(Q_n, P_m), U_{nm}(Q_n, P_m), F_{nm}(Q_n, P_m)>$

Step-2: Determine the weights of the attributes

In an MADM strategy, the weights of the attributes play an important role in taking decision. When the weights of the attributes are totally unknown to the decision makers, then the attribute weights can be determined by using the compromise function defined in equation (3).

Compromise Function: The compromise function of Q is defined by:

$$\Omega_j=\sum_{i=1}^{n} (3+T_{ij}(Q_i, P_j)+C_{ij}(Q_i, P_j)-G_{ij}(Q_i, P_j)-U_{ij}(Q_i, P_j)-F_{ij}(Q_i, P_j))/5 \quad (3)$$

Then the desired weight of the jth attribute is defined by $w_j= \frac{\Omega_j}{\sum_{j=1}^{m}\Omega_j}$ (4)

Here $\sum_{j=1}^{m} w_j=1$.

Step-3: Determination of ideal solution

In every MADM process, the attributes chosen by the decision maker can be split into two different types. One is "benefit type" attribute and the other is "cost type" attribute. In our proposed SVPNS-MADM model, an ideal alternative can be identified by the decision maker using the following operators:

(i) For the cost type attributes (P_j), we use the maximum operator to determine the best value (P_j^*) of each attribute among all the alternatives. The best value (P_j^*) is defined by:

$$P_j^* = (\max\ T_{11}(Q_1, P_1), \max\ C_{11}(Q_1, P_1), \min\ G_{11}(Q_1, P_1), \min\ U_{11}(Q_1, P_1), \min\ F_{11}(Q_1, P_1)) \quad (5)$$

where j=1, 2, ….., m.

(ii) For the benefit type attributes (P_j), we use the minimum operator to determine the best value (P_j^*) of each attribute among all the alternatives. The best value (P_j^*) is defined by:

$$P_j^* = (\min\ T_{11}(Q_1, P_1), \min\ C_{11}(Q_1, P_1), \max\ G_{11}(Q_1, P_1), \max\ U_{11}(Q_1, P_1), \max\ F_{11}(Q_1, P_1)) \quad (6)$$

where j=1, 2, ….., m.

Then we define an ideal solution as follows:

$Q^*= \{P_1^*, P_2^*, ……., P_m^*\}$, which is also an SVPNS.

Step-4: Determination of hyperbolic sine similarity value.

After the formation of ideal solution in step-3, by using eq (1), we calculate the HSSM values for every alternative between the ideal solutions and the corresponding SVPNS from decision matrix DM[$Q|P$].

Step-5: Ranking order of the alternatives.

The rank of the alternatives Q_1, Q_2, ……., Q_n is determined based on the ascending order of hyper sine similarity values. The alternative with lowest hyper sine similarity value is the best alternative among the set of possible alternatives.

Step-6: End.

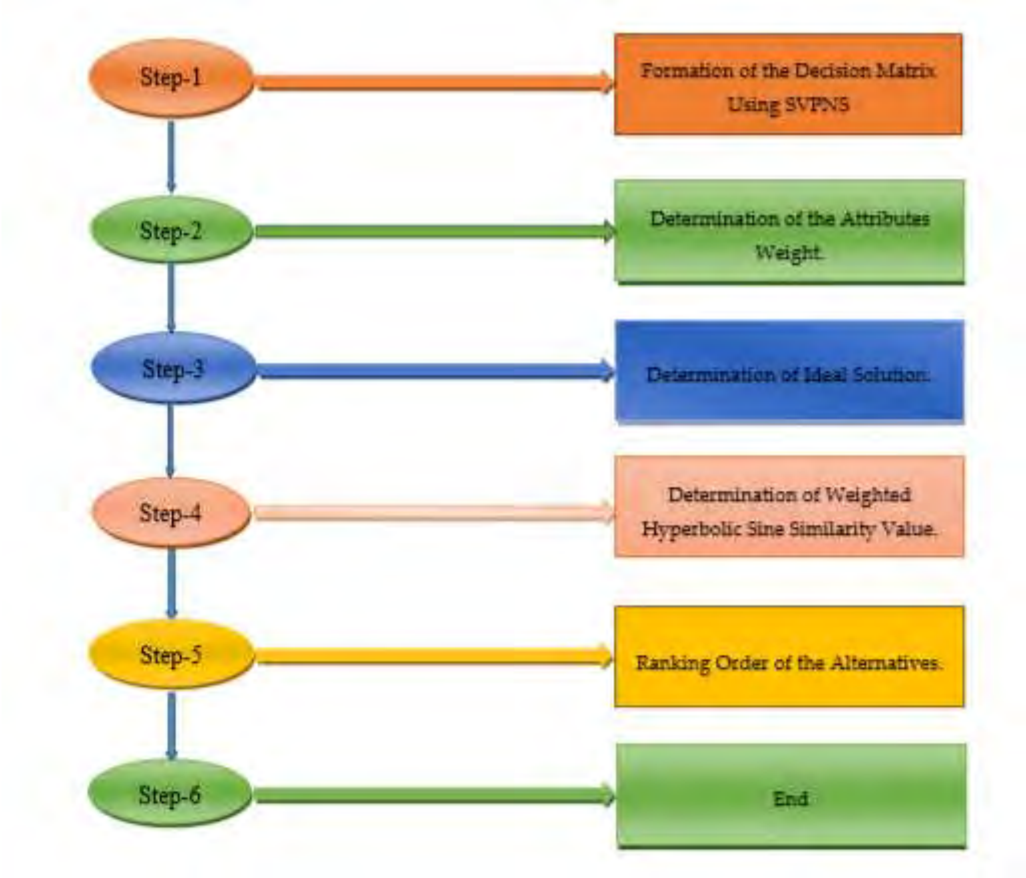

Figure 1: Flow chart of the SVPNS- MADM strategy

4. Validation of the Proposed Model:

In this section, we validate our proposed model / strategy by giving a numerical example.

4.1. Numerical example:

In this section, we demonstrate a numerical example as a real- life application of our proposed strategy. In our daily life time management is very important for everyone. Suppose a passenger needs to travel from the city-X to the city-Y by road. The passenger wants to book a car (alternative) by an online App to reach his/her destination. The selection of car by the passenger can be done based on some attributes, namely, Charges(P_1), Payment mode (P_2), AC / Non-AC(P_3), Rating(P_4). So, the selection of affordable car (for travelling) by an online App can be considered as a MADM approach.

Then the MADM strategy is presented by using the following steps.

Step-1: Construct the decision matrix under single valued pentapartitioned neutrosophic environment.

The decision matrix is shown in table 1.

Table-1: Decision matrix

	P_1	P_2	P_3	P_4
Q_1	(0.7,0.3,0.1,0.3,0.4)	(0.8,0.4,0.2,0.3,0.8)	(0.8,0.2,0.5,0.7,0.3)	(0.8,0.4,0.2,0.3,0.6)
Q_2	(0.7,0.4,0.3,0.6,0.2)	(0.7,0.4,0.4,0.7,0.5)	(0.6,0.2,0.4,0.5,0.7)	(0.9,0.3,0.9,0.2,0.3)
Q_3	(0.5,0.4,0.6,0.3,0.4)	(0.6,0.4,0.4,0.7,0.9)	(0.5,0.3,0.4,0.5,0.6)	(0.7,0.5,0.7,0.3,0.8)

Step-2: Determine the weights of attributes.

By using the eq. (3) and (4), we have the weight vector as follows:

(w_1, w_2, w_3, w_4) = (0.279, 0.234, 0.222, 0.263).

Step-3: Determine the ideal solution.

In this problem, the attribute P_1 is cost type attribute and P_2, P_3, P_4 are the benefit type attributes. The ideal solution is given in the table 2:

Table-2: The ideal solution

	${P_1}^*$	${P_2}^*$	${P_3}^*$	${P_4}^*$
Q^*	(0.7,0.4,0.1,0.3,0.2)	(0.6,0.4,0.4,0.4,0.7,0.9)	(0.5,0.2,0.5,0.7,0.7)	(0.7,0.3,0.9,0.7,0.8)

Step-4: Determine the weighted hyperbolic sine similarity values.

By using eq. (2), we calculate the similarity measure values for each alternative. The weighted hyperbolic sine similarity values are:

WHSSM(Q_1, Q^*) = 0.996488;

WHSSM(Q_2, Q^*) = 0.997482;

WHSSM(Q_3, Q^*) = 0.997881.

Step-5: Ranking the alternatives.

From the above step, we see that WHSSM(Q_1, Q^*) < WHSSM(Q_2, Q^*) < WHSSM(Q_3, Q^*). Therefore, Q_1 is the best suitable alternative (car) for the passenger to book for travelling.

5. Conclusions:

In the study, we propose a hyperbolic sine similarity measure and weighted hyperbolic sine similarity measures for single valued pentapartitioned neutrosophic set and prove some of their basic properties. We develop a novel HSSM-MADM strategy based on the proposed weighted hyperbolic sine similarity measure to solve MADM problems. We also validate the proposed strategy by solving an illustrative MADM problem to demonstrate the effectiveness of the proposed SVPNS-MADM strategy.

The proposed SVPNS-MADM strategy can also be used to deal with other decision-making problems such as teacher selection [71], weaver selection [72], brick selection [73], logistic center location selection [74], personnel selection [75], etc.

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Some Mappings in N-Neutrosophic Supra Topological Spaces

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Abstract: The main aim of this paper defines some N-neutrosophic supra topological continuous mappings and some N-neutrosophic supra topological open mappings by weak neutrosophic supra topological open sets and their different properties are discussed. The relation between these N-neutrosophic supra continuous mappings are established with suitable examples.

Keywords: N-neutrosophic continuous mapping, S^*-N-neutrosophic k-open mapping, S^*-N-neutrosophic k-continuous mapping, N-supra neutrosophic k-open mapping, N-supra neutrosophic k-continuous mapping.

1. Introduction

Zadeh [26] introduced the concept of fuzzy set theory by studying each element its membership values. The fuzzy topological space is a topological space defined on fuzzy sets, initiated by Chang [7]. Fuzzy supra topological spaces and their supra continuous mappings were defined by Abd El-Monsef and Ramadan [2]. Jayaparthasarathy [11] derived some contradicting examples of the statements of Abd El-Monsef and Ramadan [2] in fuzzy supra topological spaces. In 1986, Atanassov [4] introduced an intuitionistic fuzzy set as a generalization of the fuzzy set. Dogan Coker [9] extended the concept of fuzzy topological spaces into intuitionistic fuzzy topological spaces. The concept of intuitionistic fuzzy supra topological space was initiated by Turnal [19]. Florentin Smarandache [24] was the first one to develop the neutrosophic set theory, which is the generalization of Atanassov's intuitionistic fuzzy set theory. Recently many researchers [1, 6, 10, 25] developed the applications of neutrosophic sets in various fields such as artificial intelligence, biology, control systems, data analysis, economics, medical diagnosis, probability, etc. Salama et al. [22] defined the neutrosophic crisp sets and neutrosophic topological space.

In 1963, Levine [16] introduced semi-open sets and semi-continuous functions in classical topological spaces. Njastad [20] derived a classical topology using the α-open sets. Mashhour et al. [17] investigated the properties of pre-open sets. Andrijevic [3] established the behavior of β-open sets in classical topology. Mashhour et al. [18] introduced the concept of supra topological spaces by removing one topological condition and they further defined the supra semi-open set and supra semi-continuous function. Devi et al. [8] introduced the properties of α-open sets and α-continuous functions in supra topological spaces. Supra topological pre-open sets and their continuous functions are defined by Sayed [23]. Saeid Jafari et al. [21] investigated the properties of supra β-open sets and their continuity. In 2016, Lellis Thivagar et al. [13, 14, 27] originated the N-topological space with its own open sets. Apart from this, Lellis Thivagar et al. [15] introduced N-neutrosophic topological spaces with several properties.

Motivation of the work: The neutrosophic supra topological space is a new space developed by Jayaparthasarathy et al. [11]. In this area, some neutrosophic supra topological open sets, and their continuous mappings are defined. Arockia Dasan et al. [5], and Jayaparthasarathy et al. [12] further extended these neutrosophic supra topological spaces to N-neutrosophic supra topological spaces. In N-neutrosophic supra topological spaces, some weak open sets with some operators are only defined so far. Hence the motivation of this paper extends to define different properties of continuous and open mappings by using N-neutrosophic supra topological open sets as well as its weak open sets.

Organization of the paper: Section 2 of this paper presents some basic preliminaries of neutrosophic fuzzy sets and N-neutrosophic supra topological spaces. Section 3 introduces continuous mappings and open mappings using N-neutrosophic supra topological open sets. In section 4, we define some weak forms of continuous and weak open mappings in N-neutrosophic supra topological spaces, and the last section states summary and some of the future work in the conclusion and future work of this paper.

2 Preliminary

In this section, we discuss the basic definitions and properties of N-neutrosophic supra topological spaces which are useful in the sequel.

Definition 2.1 [24] Let X be a non-empty set. A neutrosophic set A having the form $A = \{(x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X\}$, where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x) \in]^-0, 1^+[$ represent the degree of membership (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) respectively of each $x \in X$ to the set A such that $^-0 \leq \mu_A(x) + \sigma_A(x) + \gamma_A(x) \leq 3^+$ for all $x \in X$. For X, $N(X)$ denotes the collection of all neutrosophic sets of X.

Definition 2.2 [24] The following statements are true for neutrosophic sets A and B on X:

1. $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$ if and only if $A \subseteq B$
2. $A \subseteq B$ and $B \subseteq A$ if and only if $A = B$.
3. $A \cap B = \{(x, min\{\mu_A(x), \mu_B(x)\}, min\{\sigma_A(x), \sigma_B(x)\}, max\{\gamma_A(x), \gamma_B(x)\}) : x \in X\}$.
4. $A \cup B = \{(x, max\{\mu_A(x), \mu_B(x)\}, max\{\sigma_A(x), \sigma_B(x)\}, min\{\gamma_A(x), \gamma_B(x)\}) : x \in X\}$.

More generally, the intersection and the union of a collection of neutrosophic sets $\{A_i\}_{i\in\Lambda}$, are defined by $\bigcap_{i\in\Lambda} A_i = \{(x, inf_{i\in\Lambda}\{\mu_{A_i}(x)\}, inf_{i\in\Lambda}\{\sigma_{A_i}(x)\}, sup_{i\in\Lambda}\{\gamma_{A_i}(x)\}) : x \in X\}$ and $\bigcup_{i\in\Lambda} A_i = \{(x, sup_{i\in\Lambda}\{\mu_{A_i}(x)\}, sup_{i\in\Lambda}\sigma_{A_i}\{(x)\}, inf_{i\in\Lambda}\{\gamma_{A_i}(x)\}) : x \in X\}$.

Definition 2.3 [11] Let A, B be two neutrosophic sets of X, then the difference of A and B is a neutrosophic set on X, defined as $A \setminus B = \{(x, |\mu_A(x) - \mu_B(x)|, |\sigma_A(x) - \sigma_B(x)|, 1 - |\gamma_A(x) - \gamma_B(x)|) : x \in X\}$. Clearly $X^c = X \setminus X = (x, 0,0,1) = \emptyset$ and $\emptyset^c = X \setminus \emptyset = (x, 1,1,0) = X$.

Notation 2.4 [11] Let X be a non-empty set. We consider the neutrosophic empty set as $\emptyset = \{(x, 0, 0, 1) : x \in X\}$ and the neutrosophic whole set as $X = \{(x, 1, 1, 0) : x \in X\}$.

Definition 2.5 [12] Let X be a non-empty set, $\tau_{n_1}{}^*, \tau_{n_2}{}^*, \dots, \tau_{n_N}{}^*$ be N arbitrary neutrosophic supra topologies defined on X. Then the collection $N\tau_n{}^* = \{S \in N(X) : S = \bigcup_{i=1}^{N} A_i,\ A_i \in \tau_{n_i}{}^*\}$ is said to be a N-neutrosophic supra topology if it satisfies the following axioms:

1. $X, \emptyset \in N\tau_n{}^*$
2. $\bigcup_{i=1}^{\infty} S_i \in N\tau_n{}^*$ for all $S_i \in N\tau_n{}^*$

Then the N-neutrosophic supra topological space is the non-empty set X together with the collection $N\tau_n{}^*$, denoted by $(X, N\tau_n{}^*)$ and its elements are known as $N\tau_n{}^*$-open sets on X. A neutrosophic subset A of X is said to be $N\tau_n{}^*$-closed on X if $X \setminus A$ is $N\tau_n{}^*$-open on X. The set of all $N\tau_n{}^*$-open sets on X and the set of all $N\tau_n{}^*$-closed sets on X are respectively denoted by $N\tau_n{}^*O(X)$ and $N\tau_n{}^*C(X)$.

Definition 2.6 [12] Let $(X, N\tau_n{}^*)$ be a N-neutrosophic supra topological space and A be a neutrosophic set of X. Then

1. The $N\tau_n{}^*$-interior of A is defined by $int_{N\tau_n{}^*}(A) = \cup\{G : G \subseteq A$ and G is $N\tau_n{}^*$-open$\}$.
2. The $N\tau_n{}^*$-closure of A is defined by $cl_{N\tau_n{}^*}(A) = \cap\{F : A \subseteq F$ and F is $N\tau_n{}^*$-closed$\}$.

Definition 2.7 [5] A neutrosophic set A of a N-neutrosophic supra topological space $(X, N\tau_n^*)$ is called

1. N-neutrosophic supra α-open set if $A \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(A)))$.
2. N-neutrosophic supra semi-open set if $A \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(A))$.
3. N -neutrosophic supra pre-open set if $A \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(A))$.
4. N -neutrosophic supra β-open set if $A \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(cl_{N\tau_n^*}(A)))$.
5. N -neutrosophic supra regular-open if $A = int_{N\tau_n^*}(cl_{N\tau_n^*}(A))$.

The set of all N-neutrosophic supra α-open (resp. N-neutrosophic supra semi-open, N-neutrosophic supra pre-open, N-neutrosophic supra β-open, N-neutrosophic supra regular-open) sets of $(X, N\tau_n^*)$ is denoted by $N\tau_n^*\alpha O(X)$ (resp. $N\tau_n^*SO(X), N\tau_n^*PO(X)$, $N\tau_n^*\beta O(X)$) and $N\tau_n^*RO(X)$. The complement of set of all N-neutrosophic supra α-open (resp. N-neutrosophic supra semi-open, N-neutrosophic supra pre-open and N-neutrosophic supra β-open) sets of $(X, N\tau_n^*)$ is called N-neutrosophic supra α-closed (resp. N-neutrosophic supra semi-closed, N-neutrosophic supra pre-closed, N-neutrosophic supra β-closed and N-neutrosophic supra regular closed) sets, denoted by $N\tau_n^*\alpha C(X)$ (resp. $N\tau_n^*SC(X), N\tau_n^*PC(X)$,$N\tau_n^*\beta C(X)$) and $N\tau_n^*RC(X)$. Hereafter N-neutrosophic supra k-open set (shortly $N\tau_n^*k$-open set) is can be any one of the following: $N\tau_n^*$-open set, $N\tau_n^*\alpha$-open set, $N\tau_n^*$semi-open set, $N\tau_n^*$pre-open set, $N\tau_n^*\beta$-open set and $N\tau_n^*r$- open set.

Definition 2.8 [5] Let $(X, N\tau_n^*)$ be a N-Neutrosophic supra topological space and A be a subset of X.

1. The $kN\tau_n^*$-interior of A, is defined by

$$kint_{N\tau_n^*}(A) = \cup \{G : G \subseteq A \text{ and } G \in N\tau_n^*kO(X)\}.$$

2. The $kN\tau_n^*$-closure of A, is defined by

$$kcl_{N\tau_n^*}(A) = \cap \{F : A \subseteq F \text{ and } F \in N\tau_n^*kC(X)\}.$$

Definition 2.9 [15] Let X be a non-empty set, then $\tau_{n_1}, \tau_{n_2}, \dots, \tau_{n_N}$ be N-arbitrary neutrosophic-topologies defined on X, then the collection $N_n\tau = \{S \subseteq X : S = (\cup_{i=1}^N A_i) \cup (\cap_{i=1}^N B_i), A_i, B_i \in \tau_{n_i}\}$ is called N-neutrosophic topology if the following axioms are satisfied.

1. $\emptyset, X \in N_n\tau$.
2. $\cup_{i=1}^{\infty} S_i \in N_n\tau$ for all $S_i \in N_n\tau$
3. $\cap_{i=1}^{n} S_i \in N_n\tau$ for all $S_i \in N_n\tau$.

Then $(X, N_n\tau)$ is called N_n-topological space on X. The element of $N_n\tau$ are known as N_n-open sets on X and its complement is called N_n-closed set on X.

Definition 2.10 [15] Let $(X, N_n\tau)$ and $(Y, N_n\sigma)$ be N-neutrosophic topological spaces. A mapping $f: X \to Y$ is said to be N-neutrosophic continuous on X if the inverse image of every $N_n\sigma$ -open set in Y is $N_n\tau$ -open in X.

3. Some Mappings in N-neutrosophic Supra Topological Spaces

In this section, we introduce continuous mappings in N-neutrosophic supra topological spaces and discuss their different properties.

Definition 3.1 Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces, $N\tau_n{}^*$ and $N\sigma_n{}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. A mapping $f: X \to Y$ is said to be N-supra neutrosophic continuous on X if the inverse image of every $N\sigma_n{}^*$-open set in Y is $N\tau_n{}^*$-open in X. If $N = 1$, then f is a supra neutrosophic continuous on X [11].

Definition 3.2 Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. $N\tau_n{}^*$and $N\sigma_n{}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. A mapping $f: X \to Y$ is said to be S^*-N-neutrosophic continuous if the inverse image of every N-neutrosophic open set in $(Y, N\sigma_n)$ is N-neutrosophic supra open in $(X, N\tau_n{}^*)$.If $N = 1$, then f is a S^*-neutrosophic continuous on X [11].

Lemma 3.3. i. Every N-neutrosophic continuous mapping is S^*-N-neutrosophic continuous, but the converse need not be true.

ii. Every N-supra neutrosophic continuous mapping is S^*-N-neutrosophic continuous, but the converse need not be true.

iii. N-supra neutrosophic continuous and N-neutrosophic continuous mappings are independent each other.

Proof. The proof follows from the definition the converse and the independency are shown in the following example.

Example 3.4.(i) For $N = 3$, let $X = \{a, b\}$ and $Y = \{x, y\}$ with the neutrosophic topologies are

$\tau_{n_1}O(X) = \{\emptyset, X, ((0.3, 0.4), (0.2, 0.5), (0.1, 0.6))\}$, $\tau_{n_2}O(X) = \{\emptyset, X, ((0.7, 0.2), (0.6, 0.1), (0.8, 0))\}$,

$\tau_{n_3}O(X) = \{\emptyset, X\}$ and $\sigma_{n_1}O(Y) = \{\emptyset, Y\}$, $\sigma_{n_2}O(Y) = \{\emptyset, Y, ((0.3, 0.4), (0.2, 0.5), (0.1, 0.6))\}$, $\sigma_{n_3}O(X)$ $= \{\emptyset, Y, ((0, 0.2), (0, 0), (1, 0.6))\}$. Then $3\tau_n O(X) = \{\emptyset, X, ((0.3, 0.4), (0.2, 0.5), (0.1, 0.6))$, $((0.7, 0.2), (0.6, 0.1), (0.8, 0))$, $((0.7, 0.4), (0.6, 0.5), (0.1, 0)), ((0.3, 0.2), (0.2, 0.1), (0.8, 0.6))\}$ and $3\sigma_n O(Y) = \{\emptyset, Y, ((0.3, 0.4), (0.2, 0.5), (0.1, 0.6)), ((0, 0.2), (0, 0), (1, 0.6))\}$. Let $3\tau_n{}^* O(X) = \{\emptyset, X,$ $((0.3, 0.4), (0.2, 0.5), (0.1, 0.6)), ((0.7, 0.2), (0.6, 0.1), (0.8, 0)), ((0.7, 0.4), (0.6, 0.5), (0.1, 0)),$ $((0.3, 0.2), (0.2, 0.1), (0.8, 0.6)), ((0, 0.2), (0, 0), (1, 0.6)), ((0.7, 0.6), (0.8, 0.5), (0.9, 0.4)),$ $((1, 0.8), (1, 1), (0, 0.4)), ((0.7, 0.6), (0.8, 0.5), (0.1, 0.4)), ((0.7, 0.6), (0.8, 0.5), (0.8, 0)),$ $((0.7, 0.6),$ $(0.8, 0.5), (0.1, 0)), ((0.7, 0.6), (0.8, 0.5), (0.8, 0.4)), ((1, 0.8), (1, 1), (0, 0))\}$ and$3\sigma_n{}^* O(Y) = 3\sigma_n O(Y)$ be the associated 2-neutrosophic supra topological space. Define $f: X \to Y$ by $f(a) = x$, $f(b) = y$. Clearly, f is S^*-3-neutrosophic continuous and 3-supra neutrosophic continuous mapping on X but it is not 3-neutrosophic continuous map.

(ii) For $N=1$, let $X=\{a,b\}$ and $Y=\{x,y\}$ with the neutrosophic topologies are $\tau_n O(X)=\{\emptyset, X, ((0.7,0.6),(0.8,0.5),(0.9,0.4)), ((0.3,0.8),(0.4,0.9),(0.2,1)), ((0.3,0.6),(0.4,0.5),(0.9,1)), ((0.7,0.8),(0.8,0.9),(0.2,0.4)), ((0,0.6),(0,0.5),(1,1))\}$ and $\sigma_n O(Y)=\{\emptyset, Y\}$. Let ${\tau_n}^* O(X)=\tau_n O(X)$ and ${\sigma_n}^* O(Y)=\{\emptyset, Y, ((0.7,0.6),(0.8,0.5),(0.9,0.4)), ((0,0.6),(0,0.5),(1,1)), ((0.3,0.2),(0.2,0.2),(1,0.4)), ((0.3,0.6),(0.2,0.5),(1,0.4))\}$ be the associated neutrosophic supra topological space. Define $f: X \to Y$ by $f(a)=x$, $f(b)=y$. Clearly, f is S^*-neutrosophic continuous and neutrosophic continuous mapping on X but it is not supra neutrosophic continuous.

Theorem 3.5. Let $(X, N{\tau_n}^*)$ and $(Y, N{\sigma_n}^*)$ be N-neutrosophic supra topological space. Then the following are equivalent:

i. A mapping $f:(X, N{\tau_n}^*) \to (Y, N{\sigma_n}^*)$ is N-supra neutrosophic continuous.
ii. The inverse image of every N-neutrosophic supra closed set in $(Y, N{\sigma_n}^*)$ is a N-neutrosophic supra closed set in $(X, N{\tau_n}^*)$.
iii. $cl_{N{\tau_n}^*}(f^{-1}(A)) \subseteq f^{-1}(cl_{N\sigma_n^*}(A)) \subseteq f^{-1}(cl_{N\sigma_n}(A))$ for every neutrosophic set A in Y.
iv. $f(cl_{N{\tau_n}^*}(B)) \subseteq cl_{N\sigma_n^*}(f(B) \subseteq cl_{N\sigma_n}(f(B))$ for every neutrosophic set B of X.
v. $f^{-1}(int_{N\sigma_n}(A)) \subseteq f^{-1}(int_{N{\sigma_n}^*}(A)) \subseteq int_{N{\tau_n}^*}(f^{-1}(A))$ for every neutrosophic subset A in Y.

Proof. $i \Rightarrow ii$: Assume that $f: X \to Y$ is N-supra neutrosophic continuous on X and let A be a $N{\sigma_n}^*$-closed set in Y. Then $Y-A$ is a $N{\sigma_n}^*$-open set in Y. Since f is N-supra neutrosophic continuous on X, then $f^{-1}(Y-A)$ is $N{\tau_n}^*$-open set in X. Then $X-f^{-1}(A)$ is $N{\tau_n}^*$-open set in X. Then $f^{-1}(A)$ is $N{\tau_n}^*$-closed set in X.

$ii \Rightarrow i$: Let A be $N{\sigma_n}^*$-open set in Y, then $Y-A$ is $N{\sigma_n}^*$-closed set in Y and by assumption, $f^{-1}(Y-A)=X-f^{-1}(A)$ is $N{\tau_n}^*$-closed in X. Thus $f^{-1}(A)$ is $N{\tau_n}^*$-open in X.

$ii \Rightarrow iii$: Since for each neutrosophic set A in Y, $cl_{N{\sigma_n}^*}(A)$ is a $N{\sigma_n}^*$-closed set in Y. Then $f^{-1}(cl_{N{\sigma_n}^*}(A))$ is $N{\tau_n}^*$ -closed in X. Thus $f^{-1}\left(cl_{N{\sigma_n}^*}(A)\right)=cl_{N{\tau_n}^*}(f^{-1}\left(cl_{N{\sigma_n}^*}(A)\right)) \supseteq cl_{N{\tau_n}^*}(f^{-1}(A))$ and implies $cl_{N{\tau_n}^*}(f^{-1}(A)) \subseteq f^{-1}(cl_{N\sigma_n^*}(A)) \subseteq f^{-1}(cl_{N\sigma_n}(A))$, since $cl_{N\sigma_n^*}(A) \subseteq cl_{N\sigma_n}(A)$.

$iii \Rightarrow iv$: Let B be the neutrosophic set in X, then

$f^{-1}(cl_{N\sigma_n}(f(B))) \supseteq f^{-1}(cl_{N{\sigma_n}^*}(f(B))) \supseteq cl_{N{\tau_n}^*}(f^{-1}(f(B))) \supseteq cl_{N{\tau_n}^*}(B)$ and so $(cl_{N{\tau_n}^*}(B)) \subseteq cl_{N\sigma_n^*}$

$(f(B) \subseteq cl_{N\sigma_n}(f(B))$.

$iv \Rightarrow ii$: Let A be $N{\sigma_n}^*$ -closed set in Y and $B=f^{-1}(A)$, then

$f\left(cl_{N{\tau_n}^*}(B)\right) \subseteq cl_{N{\sigma_n}^*}(f(B)) \subseteq cl_{N{\sigma_n}^*}(A)=A$ and $cl_{N{\tau_n}^*}(B) \subseteq f^{-1}\left(f\left(cl_{N{\tau_n}^*}(B)\right)\right) \subseteq f^{-1}(A)=B$.

Therefore $B=f^{-1}(A)$ is $N{\tau_n}^*$-closed in X.

$i \Rightarrow v$: Let A be a $N{\sigma_n}^*$ -open set in Y, then $f^{-1}(int_{N{\sigma_n}^*}(A))$ is $N{\tau_n}^*$ -open in X and $f^{-1}\left(int_{N{\sigma_n}^*}(A)\right)=int_{N{\tau_n}^*}(f^{-1}(int_{N{\sigma_n}^*}(A)) \subseteq int_{N{\tau_n}^*}(f^{-1}(A))$. Thus $f^{-1}(int_{N\sigma_n}(A)) \subseteq f^{-1}(int_{N{\sigma_n}^*}(A)) \subseteq int_{N{\tau_n}^*}(f^{-1}(A))$, since $int_{N\sigma_n^*}(A) \supseteq int_{N\sigma_n}(A)$.

$v \Rightarrow i$: Let A be $N\sigma_n{}^*$-open set in Y, then $f^{-1}(A) = f^{-1}(int_{N\sigma_n{}^*}(A)) \subseteq int_{N\tau_n{}^*}(f^{-1}(A))$ and so $f^{-1}(A)$ is $N\tau_n{}^*$-open in X.

Theorem 3.6. Let $(X, N\tau_n{}^*)$ and $(Y, N\sigma_n{}^*)$ be N-neutrosophic supra topological spaces. Then the following are equivalent:

i. A mapping $f : (X, N\tau_n{}^*) \rightarrow (Y, N\sigma_n{}^*)$ is S^*-N- neutrosophic continuous.
ii. The inverse image of every N-neutrosophic closed set in $(Y, N\sigma_n)$ is N-neutrosophic supra closed set in $(X, N\tau_n{}^*)$.
iii. $cl_{N\tau_n{}^*}(f^{-1}(A)) \subseteq f^{-1}(cl_{N\sigma_n}(A))$ for every neutrosophic set A in Y.
iv. $f(cl_{N\tau_n{}^*}(B)) \subseteq cl_{N\sigma_n}(f(B))$ for every neutrosophic set $B$of X.
v. $f^{-1}(int_{N\sigma_n}(A)) \subseteq int_{N\tau_n{}^*}(f^{-1}(A))$ for every neutrosophic subset A in Y.

Proof. The proof is similarly follows from the theorem 3.5.

Theorem 3.7. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are N-supra neutrosophic continuous mappings, then $g \circ f : X \rightarrow Z$ is N-supra neutrosophic continuous.

Proof. Let V be neutrosophic supra open set in Z then $g^{-1}(V)$ is neutrosophic supra open in Y and $f^{-1}(g^{-1}(V))$ is neutrosophic supra open in X, by hypothesis. Therefore $(g \circ f)^{-1}(V)$ is neutrosophic supra open in X and so $g \circ f$ is N-supra neutrosophic continuous.

Remark 3.8. The composition of two S^*-N-neutrosophic continuous mappings need not be S^*-N-neutrosophic continuous.

Example 3.9. For $N = 2$, let $X = \{a, b\}$, $Y = \{u, v\}$ and $Z = \{x, y\}$ with the neutrosophic topologies are $\tau_{n_1} = \{\emptyset, X\}$, $\tau_{n_2} = \{\emptyset, X, ((0.5, 0.5), (0.5, 0.5), (0.5, 0.5))\}$, $\sigma_{n_1} = \{\emptyset, Y\}$, $\sigma_{n_2} = \{\emptyset, Y, ((0.6, 0.6), (0.6, 0.6), (0.6, 0.6))\}$, $\eta_{n_1} = \{\emptyset, Z, ((0.3, 0.3), (0.3, 0.3), (0.3, 0.3))\}$ and $\eta_{n_2} = \{\emptyset, Z\}$ with the 2 -neutrosophic topologies are $2\tau_n O(X) = \{\emptyset, X, ((0.5, 0.5), (0.5, 0.5), (0.5, 0.5))\}$, $2\sigma_n O(Y) = \{\emptyset, Y, ((0.6, 0.6), (0.6, 0.6), (0.6, 0.6))\}$ and $2\eta_n O(Z) = \{\emptyset, Z, ((0.3, 0.3), (0.3, 0.3), (0.3, 0.3))\}$. Let $2\tau_n{}^* O(X) = \{\emptyset, X, ((0.5, 0.5), (0.5, 0.5), (0.5, 0.5)), ((0.6, 0.6), (0.6, 0.6), (0.6, 0.6)), ((0.4, 0.4), (0.4, 0.4), (0.4, 0.4))\}$ and $2\sigma_n{}^* O(Y) = \{\emptyset, Y, ((0.6, 0.6), (0.6, 0.6), (0.6, 0.6)), ((0.3, 0.3), (0.3, 0.3), (0.3, 0.3)), ((0.7, 0.7), (0.7, 0.7), (0.7, 0.7))\}$ be the associated 2-neutrosophic supra topologies with respect to $2\tau_n$ and $2\sigma_n$. Then the mapping $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are defined respectively by $f(a) = u, f(b) = v, g(u) = x, g(v) = y$ are S^*-2-neutrosophic continuous. But $g \circ f$ is not S^*-2-neutrosophic continuous.

Theorem 3.10. If $f : X \rightarrow Y$ is S^*-N-neutrosophic continuous and $g : Y \rightarrow Z$ is N-neutrosophic continuous, then $g \circ f : X \rightarrow Z$ is S^*-N-neutrosophic continuous.

Proof. Let V be N-neutrosophic open set in Z, then by hypothesis, $g^{-1}(V)$ is N-neutrosophic open in Y and $f^{-1}(g^{-1}(V))$ is N-neutrosophic supra open in X implies $(g \circ f)^{-1}(V)$ is N-neutrosophic supra open in X. Therefore $g \circ f$ is S^*-N-neutrosophic continuous.

Theorem 3.11. If $f : X \rightarrow Y$ is N -supra neutrosophic continuous and $g : Y \rightarrow Z$ is S^*-N-neutrosophic continuous, then $g \circ f : X \rightarrow Z$ is S^*-N-neutrosophic continuous.

Proof. Let V be N-neutrosophic open set in Z then $g^{-1}(V)$ is N-neutrosophic supra open in Y and $f^{-1}(g^{-1}(V))$ is N-neutrosophic supra open in X implies $(g \circ f)^{-1}(V)$ is N-neutrosophic supra open in X. Therefore $g \circ f$ is S^*-N-neutrosophic continuous.

Definition 3.12. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N -neutrosophic topological spaces. $N\tau_n{}^*$ and $N\sigma_n{}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. A mapping $f: X \to Y$ is said to be N-supra neutrosophic open if the image of every $N\tau_n{}^*$-open set in X is a $N\sigma_n{}^*$-open set in Y.

Definition 3.13. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. $N\tau_n{}^*$ and $N\sigma_n{}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. A mapping $f: X \to Y$ is said to be S^*-N-neutrosophic open if the image of every neutrosophic $N\tau_n$-open set in X is $N\sigma_n{}^*$-open set in Y.

Lemma 3.14. Every N-supra neutrosophic open mapping is S^*-N-neutrosophic open, but the converse need not be true.

Proof. The proof follows from the definitions, the converse part is shown in the following example.

Example 3.15. Consider the example 3.4 (ii), define $g: Y \to X$ by $g(x) = a$, $g(y) = b$. Clearly, g is S^*-N neutrosophic open map but it is not supra neutrosophic open map.

Theorem 3.16. Let $f: X \to Y$ be a N-supra neutrosophic open mapping. Then for each neutrosophic subset A of X,

i. $f(int_{N\tau_n{}^*}(A)) \subseteq int_{N\sigma_n{}^*}(f(A)$.

ii. $f(cl_{N\tau_n{}^*}(A)) \supseteq cl_{N\sigma_n{}^*}(f(A))$.

Proof.

i. Since $int_{N\tau_n{}^*}(A) \subseteq A$, then $f\left(int_{N\tau_n{}^*}(A)\right) \subseteq f(A)$ and $int_{N\sigma_n{}^*}\left(f\left(int_{N\tau_n{}^*}(A)\right)\right) \subseteq int_{N\sigma_n{}^*}(f(A))$. Since $int_{N\tau_n{}^*}(A)$ is $N\tau_n{}^*$-open in X, then $f(int_{N\tau_n{}^*}(A))$ is $N\tau_n{}^*$-open in Y. Therefore $int_{N\sigma_n{}^*}(f(int_{N\tau_n{}^*}(A))) = f(int_{N\tau_n{}^*}(A))$. Hence $f(int_{N\tau_n{}^*}(A)) \subseteq int_{N\sigma_n{}^*}(f(A))$.

ii. Since $cl_{N\tau_n{}^*}(A) \supseteq A$, then $f\left(cl_{N\tau_n{}^*}(A)\right) \supseteq f(A)$ and $cl_{N\sigma_n{}^*}\left(f\left(cl_{N\tau_n{}^*}(A)\right)\right) \supseteq cl_{N\sigma_n{}^*}(f(A))$. Since $cl_{N\tau_n{}^*}(A)$ is $N\tau_n{}^*$-closed set in X, then $X - cl_{N\tau_n{}^*}(A)$ is $N\tau_n{}^*$-open set in X and $f(X - cl_{N\tau_n{}^*}(A))$ is $N\sigma_n{}^*$-open in Y. That is $Y - f(cl_{N\tau_n{}^*}(A))$ is $N\sigma_n{}^*$-open in Y implies that $f(cl_{N\tau_n{}^*}(A))$ is $N\sigma_n{}^*$-closed in Y and so $cl_{N\sigma_n{}^*}(f(cl_{N\tau_n{}^*}(A))) = f(cl_{N\tau_n{}^*}(A))$. Therefore, $f(cl_{N\tau_n{}^*}(A)) \supseteq cl_{N\sigma_n{}^*}(f(A))$.

Theorem 3.17. Let $f: X \to Y$ be a S^*-N-neutrosophic open mapping. Then for each neutrosophic subset A of X,

i. $f(int_{N\tau_n}(A)) \subseteq int_{N\sigma_n{}^*}(f(A)$.

ii. $f(cl_{N\tau_n}(A)) \supseteq cl_{N\sigma_n{}^*}(f(A))$.

Proof. The proof follows directly from theorem 3.16.

4 Some Weak Mappings in $N\tau_n{}^*$-Topological Space

In this section, we introduce some weak forms of continuous functions in N-neutrosophic supra topological spaces and investigate the relationship between them. Throughout the section, N-neutrosophic supra k-open set (shortly $N\tau_n{}^*k$-open set) is can be any one of the following: $N\tau_n{}^*$-open set, $N\tau_n{}^*\alpha$-open set, $N\tau_n{}^*$semi-open set, $N\tau_n{}^*$pre-open set, $N\tau_n{}^*\beta$-open set, and $N\tau_n{}^*r$- open set.

Definition 4.1. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. $N\tau_n{}^*$ and $N\sigma_n{}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. A mapping $f: X \to Y$ is said to be N-supra neutrosophic k-continuous ((shortly $N\tau_n{}^*k$-continuous) is can be any one of the following: ${N\tau_n}^*\alpha$-continuous, ${N\tau_n}^*$ semi continuous, ${N\tau_n}^*$ pre continuous, ${N\tau_n}^*\beta$-continuous and ${N\tau_n}^*r$- continuous) on X if the inverse image of every ${N\sigma_n}^*$-open set in Y is a ${N\tau_n}^*k$-open in X.

Definition 4.2. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. ${N\tau_n}^*$ and ${N\sigma_n}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. A mapping $f : X \to Y$ is said to be S^*-N-neutrosophic k-continuous (is can be any one of the following: S^*-N-neutrosophic α-continuous, S^*-N-neutrosophic semi continuous, S^*-N-neutrosophic pre continuous, S^*-N-neutrosophic β-continuous and S^*-N-neutrosophic r-continuous) if the inverse image of every N-neutrosophic open set in $(Y, N\sigma_n)$ is N-neutrosophic supra k-open set in $(X, {N\tau_n}^*)$.

Lemma 4.3. Every N-supra neutrosophic k-continuous mapping is S^*-N-neutrosophic k-continuous, but the converse need not be true.

Proof. The proof follows from the definition; the converse part is shown in the following example.

Example 4.4. Consider the example 3.4(i), f is S^*-3-neutrosophic k-continuous and 3-supra neutrosophic k-continuous mapping on X.

Theorem 4.5. Let $(X, {N\tau_n}^*)$ and $(Y, {N\sigma_n}^*)$ be N-neutrosophic supra topological space. Then the following are equivalent:

i. A mapping $f : (X, {N\tau_n}^*) \to (Y, {N\sigma_n}^*)$ is N-supra neutrosophic k-continuous.
ii. The inverse image of every N-neutrosophic supra closed set in $(Y, {N\sigma_n}^*)$ is a N-neutrosophic supra k-closed set in $(X, {N\tau_n}^*)$.
iii. $kcl_{{N\tau_n}^*}(f^{-1}(A)) \subseteq f^{-1}(kcl_{{N\sigma_n}^*}(A))$ for every neutrosophic set A of Y.
iv. $f(kcl_{{N\tau_n}^*}(B)) \subseteq kcl_{{N\sigma_n}^*}(f(B))$ for every neutrosophic set B of X.
v. $f^{-1}(kint_{{N\sigma_n}^*}(A)) \subseteq kint_{{N\tau_n}^*}(f^{-1}(A))$ for every neutrosophic subset A of Y.

Proof. The proof can be similarly derive as that of theorem 3.5

Theorem 4.6. Let $(X, {N\tau_n}^*)$ and $(Y, {N\sigma_n}^*)$ be N-neutrosophic supra topological space. Then the following are equivalent:

i. A mapping $f : (X, {N\tau_n}^*) \to (Y, {N\sigma_n}^*)$ is S^*-N- neutrosophic k-continuous.
ii. The inverse image of every N-neutrosophic closed set in $(Y, N\sigma_n)$ is a N-neutrosophic supra k-closed set in $(X, {N\tau_n}^*)$.
iii. $kcl_{{N\tau_n}^*}(f^{-1}(A)) \subseteq f^{-1}(kcl_{N\sigma_n}(A))$ for every neutrosophic set A of Y.
iv. $f(kcl_{{N\tau_n}^*}(B)) \subseteq kcl_{N\sigma_n}(f(B))$ for every neutrosophic set B of X.
v. $f^{-1}(kint_{{N\sigma_n}^*}(A)) \subseteq kint_{{N\tau_n}^*}(f^{-1}(A))$ for every neutrosophic subset $A$of Y.

Proof. The proof is straightforward from theorem 3.5.

Theorem 4.7. The following statements are true for the mapping $f : (X, {N\tau_n}^*) \to (Y, {N\sigma_n}^*)$:

i. Every N-supra neutrosophic r-continuous is N-supra neutrosophic continuous.
ii. Every N-supra neutrosophic continuous is N-supra neutrosophic α-continuous.
iii. Every N-supra neutrosophic α-continuous is N-supra neutrosophic semi-continuous.
iv. Every N-supra neutrosophic α-continuous is N-supra neutrosophic pre-continuous.
v. Every N-supra neutrosophic semi-continuous is N-supra neutrosophic β-continuous.
vi. Every N-supra neutrosophic pre-continuous is N-supra neutrosophic β-continuous.

Proof. The proof follows directly from the fact that theorem 4.2 of [12] and theorem 14 of [5].

The converse of the above theorem need not be true as shown in the following example.

Example 4.8. (i) For $N = 4$. Let $X = \{a, b\}$ and $Y = \{x, y\}$ with the neutrosophic topologies are $\tau_{n_1}O(X) = \{\emptyset, X, ((0.6, 0.1), (0.7, 0.2), (0.8, 0))\}$, $\tau_{n_2}O(X) = \{\emptyset, X, ((0.6, 0.3), (0.5, 0.4), (0.3, 0.5))\}$, $\tau_{n_3}O(X) = \{\emptyset, X, ((0.6, 0.3), (0.7, 0.4), (0.3, 0))\}$, $\tau_{n_4}O(X) = \{\emptyset, X\}$ and $\sigma_{n_1}O(Y) = \{\emptyset, Y, ((0.6, 0.3), (0.7, 0.4), (0.3, 0))\}$, $\sigma_{n_2}O(Y) = \{\emptyset, Y, ((0.3, 0.1), (0.7, 0.2), (0.8, 0))\}$, $\sigma_3 O(Y) = \{\emptyset, Y\}$, and $\sigma_{n_4}O(Y) = \{\emptyset, Y, ((0.6, 0.3), (0.5, 0.4), (0.3, 0.5))\}$. Then $4\tau_n O(X) = \{\emptyset, X, ((0.6, 0.1), (0.7, 0.2), (0.8, 0)), ((0.6, 0.3), (0.5, 0.4), (0.3, 0.5)), ((0.6, 0.3), (0.7, 0.4), (0.3, 0)), ((0.6, 0.1), (0.5, 0.2), (0.8, 0.5)), ((0.6, 0.3), (0.7, 0.4), (0.3, 0))\}$ and $4\sigma_n O(Y) = \{\emptyset, Y, ((0.6, 0.1), (0.7, 0.2), (0.8, 0)), ((0.6, 0.3), (0.5, 0.4), (0.3, 0.5)), ((0.6, 0.3), (0.7, 0.4), (0.3, 0)), ((0.6, 0.1), (0.5, 0.2), (0.8, 0.5))\}$. Let $4\tau_n{}^*O(X) = 4\tau_n O(X)$, $4\sigma_n{}^*O(Y) = 4\sigma_n O(Y)$ be the associated 4-neutrosophic supra topological space. Define $f : X \to Y$ by $f(a) = x$, $f(b) = y$. Clearly, f is 4-supra neutrosophic continuous mapping on X but it is not 4-supra neutrosophic r-continuous mapping on X.

(ii) For $N = 2$, let $X = \{a, b\}$ and $Y = \{x, y\}$ with the neutrosophic topologies are $\tau_{n_1} = \{\emptyset, X, ((0.3, 0.4), (0.3, 0.4), (0.4, 0.5))\}$, $\tau_{n_2} = \{\emptyset, X, ((0.4, 0.4), (0.4, 0.4), (0.4, 0.4))\}$ and $\sigma_{n_1} = \{\emptyset, Y\}$, $\sigma_{n_2} = \{\emptyset, Y, ((0.4, 0.6), (0.4, 0.6), (0.3, 0.4))\}$. Then $2\tau_n O(X) = \{\emptyset, X, ((0.3, 0.4), (0.3, 0.4), (0.4, 0.5)), ((0.4, 0.4), (0.4, 0.4), (0.4, 0.4))\}$ and $2\sigma_n O(Y) = \{\emptyset, Y, ((0.4, 0.6), (0.4, 0.6), (0.3, 0.4)\}$. Let $2\tau_n{}^*O(X) = \{\emptyset, X, ((0.3, 0.4), (0.3, 0.4), (0.4, 0.5)), ((0.4, 0.2), (0.4, 0.2), (0.5, 0.4)), ((0.4, 0.4), (0.4, 0.4), (0.4, 0.4))\}$ and $2\sigma_n{}^*O(Y) = \{\emptyset, Y, ((0.4, 0.6), (0.4, 0.6), (0.3, 0.4)), ((0.4, 0.4), (0.4, 0.4), (0.4, 0.4))\}$ be the associated 2-neutrosophic supra topological space. Define $f : X \to Y$ by $f(a) = x$ and $f(b) = y$. Therefore, f is 2-supra neutrosophic α-continuous, 2-supra neutrosophic semi-continuous, 2-supra neutrosophic pre-continuous and 2-supra neutrosophic β-continuous on X but not 2-supra neutrosophic continuous.

(iii) For $N = 2$, let $X = \{a, b\}$ and $Y = \{x, y\}$. Consider $\tau_{n_1}O(X) = \{\emptyset, X, ((0.3, 0.5), (0.3, 0.5), (0.4, 0.5))\}$, $\tau_{n_2}O(X) = \{\emptyset, X, ((0.4, 0.3), (0.4, 0.3), (0.5, 0.6))\}$ and $\sigma_{n_1}O(Y) = \{\emptyset, Y\}$ and $\sigma_{n_2}O(Y) = \{\emptyset, Y, ((0.4, 0.4), (0.4, 0.4), (0.5, 0.4))\}$. Then $2\tau_n O(X) = \{\emptyset, X, ((0.3, 0.5), (0.3, 0.5), (0.4, 0.5)), ((0.4, 0.3), (0.4, 0.3), (0.5, 0.6)), ((0.4, 0.5), (0.4, 0.5), (0.4, 0.5)), ((0.3, 0.3), (0.3, 0.3), (0.5, 0.5))\}$ and $2\sigma_n O(Y) = \{\emptyset, Y, ((0.4, 0.4), ((0.4, 0.4), (0.5, 0.4))\}$. Let $2\tau_n{}^*O(X) = 2\tau_n O(X)$ and $2\sigma_n{}^*O(Y) = 2\sigma_n O(Y)$ be the associated 2-neutrosophic supra topological space. Define $f : X \to Y$ by $f(a) = x$ and $f(b) = y$. Then f is $2\tau_n{}^*$-supra neutrosophic semi-continuous and 2-supra neutrosophic β-continuous but it is not 2-supra neutrosophic α-continuous and not 2-supra neutrosophic pre-continuous.

(iv) For $N = 2$, let $X = \{a, b\}$ and $Y = \{x, y\}$. Consider $\tau_{n_1}O(X) = \{\emptyset, X, ((0.3, 0.5), (0.3, 0.5), (0.4, 0.5))\}$, $\tau_{n_2}O(X) = \{\emptyset, X, ((0.4, 0.5), (0.4, 0.5), (0.4, 0.2))\}$. $\sigma_{n_1}O(Y) = \{\emptyset, Y, ((0.4, 0.5), (0.4, 0.5), (0.5, 0.4))\}$, $\sigma_{n_2}O(Y) = \{\emptyset, Y\}$. Then $2\tau_n O(X) = \{\emptyset, X, ((0.3, 0.5), (0.3, 0.5), (0.4, 0.5)), ((0.4, 0.5), (0.4, 0.5), (0.4, 0.2))\}$, and $2\sigma_n O(Y) = \{\emptyset, Y, ((0.4, 0.5), (0.4, 0.5), (0.5, 0.4))\}$. Let $2\tau_n{}^*O(X) = \{\emptyset, X, ((0.3, 0.5), (0.3, 0.5), (0.4, 0.5)), ((0.4, 0.3), (0.4, 0.3), (0.5, 0.2)), ((0.4, 0.5), (0.4, 0.5), (0.4, 0.2))\}$ and $2\sigma_n{}^*O(Y) = \{\emptyset, Y, ((0.4, 0.5), (0.4, 0.5), (0.5, 0.4)), ((0.4, 0.5), (0.4, 0.5), (0.4, 0.2))\}$ be the associated 2-neutrosophic supra topological space. Define $f : X \to Y$ by $f(a) = x$ and $f(b) = y$. Then f is 2-supra neutrosophic pre-continuous and 2-supra neutrosophic β-continuous but it is not 2-supra neutrosophic α-continuous and not 2-supra neutrosophic semi-continuous.

Theorem 4.9. The following statements are true for the mapping $f : (X, N\tau_n{}^*) \to (Y, N\sigma_n{}^*)$:

i. Every S^*-N-neutrosophic r-continuous is S^*-N-neutrosophic continuous.

ii. Every S^*-N-neutrosophic continuous is S^*-N-neutrosophic α-continuous.
iii. Every S^*-N-neutrosophic α-continuous is S^*-N-neutrosophic semi-continuous.
iv. Every S^*-N-neutrosophic α-continuous is S^*-N-neutrosophic pre-continuous.
v. Every S^*-N-neutrosophic semi-continuous is S^*-N-neutrosophic β-continuous.
vi. Every S^*-N-neutrosophic pre-continuous is S^*-N-neutrosophic β-continuous.

Proof. The proof follows directly from the fact that theorem 4.2 of [12] and theorem 14 of [5].
The converse of the above theorem need not be true as shown in the following example.

Example 4.10. Consider the example 4.8(i) f is S^*-4-neutrosophic continuous mapping on X but it is not S^*4-neutrosophic r-continuous mapping on X.

Consider the example 4.8.(ii) , f is S^*-2-neutrosophic α-continuous, S^*-2-neutrosophic semi-continuous, S^*-2-neutrosophic pre-continuous and S^*-2-neutrosophic β-continuous on X but not S^*-2-neutrosophic continuous.

Consider the example 4.8.(iii), f is S^*-2-neutrosophic semi-continuous and S^*-2-neutrosophic β-continuous but it is not S^*-2-neutrosophic α-continuous and not S^*-2-neutrosophic pre-continuous.

Consider the example 4.8.(iv), Then f is S^*-2-neutrosophic pre-continuous and S^*-2-neutrosophic β-continuous but it is not S^*-2-neutrosophic α-continuous and not S^*-2-neutrosophic semi-continuous.

Theorem 4.11.A function $f : X \to Y$ is N-supra neutrosophic α-continuous on X if and only if N-supra neutrosophic semi-continuous and N-supra neutrosophic pre-continuous.

Proof. The proof can be derive from the fact of theorem 4.6 of [12].

Theorem 4.12.A function $f : X \to Y$ is S^*-N-neutrosophic α-continuous on X if and only if S^*-N-neutrosophic semi-continuous and S^*-N-neutrosophic pre-continuous.

Proof. The proof of the theorem is directly following from theorem 4.6 of [12].

Theorem 4.13. If $f : X \to Y$ and $g : Y \to Z$ are N-supra neutrosophic k-continuous mappings, then $g \circ f : X \to Z$ is N-supra neutrosophic k-continuous.

Proof. Let V be a N-neutrosophic supra k-open set in Z, then $g^{-1}(V)$ is neutrosophic supra k-open in Y and $f^{-1}(g^{-1}(V))$ is neutrosophic supra k-open in X implies $(g \circ f)^{-1}(V)$ is neutrosophic supra k-open in X. Therefore $g \circ f$ is N-supra neutrosophic k-continuous.

Remark 4.14. The composition of two S^*-N-neutrosophic k-continuous mappings need not be S^*-N-neutrosophic k-continuous.

Example 4.15. For $N = 2$, let $X = \{a, b\}$, $Y = \{u, v\}$ and $Z = \{x, y\}$ with the neutrosophic topologies are $\tau_{n_1} = \{\emptyset, X, ((0.3, 0.7), (0.4, 0.8), (0.5, 0.9))\}$, $\tau_{n_2} = \{\emptyset, X, ((0.3, 0.8), (0.5, 0.8), (0.5, 0.8))\}$, $\sigma_{n_1} = \{\emptyset, Y\}$, $\sigma_{n_2} = \{\emptyset, Y, ((0.3, 0.7), (0.4, 0.8), (0.5, 0.9))\}$, $\eta_{n_1} = \{\emptyset, Z, ((0.7, 0.3), (0.6, 0.2), (0.5, 0.1))\}$

and $\eta_{n_2} = \{\emptyset, Z\}$ with the 2-neutrosophic topologies are

$2\tau_n O(X) = \{\emptyset, X, ((0.3, 0.7), (0.4, 0.8), (0.5, 0.9)), ((0.3, 0.8), (0.5, 0.8), (0.5, 0.8))\}$, $2\sigma_n O(Y) =$

$\{\emptyset, Y, ((0.3, 0.7), (0.4, 0.8), (0.5, 0.9))\}$ and $2\eta_n O(Z) = \{\emptyset, Z, ((0.7, 0.3), (0.6, 0.2), (0.5, 0.1))\}$. Let

$2\tau_n{}^* O(X) = \{\emptyset, X, ((0.5, 0.8), (0.5, 0.8), (0.5, 0.8)), ((0.3, 0.7), (0.4, 0.8), (0.5, 0.9)), ((0.3, 0.8), (0.5, 0.8),$

$(0.5, 0.8))\}$ and $2\sigma_n{}^*O(Y) = \{\emptyset, Y, ((0.3, 0.7), (0.4, 0.8), (0.5, 0.9)), ((0.7, 0.3), (0.6, 0.2), (0.5, 0.1)), ((0.7, 0.7), (0.6, 0.8), (0.5, 0.1))\}$ be the associated 2 -neutrosophic supra topologies with respect to $2\tau_n$ and $2\sigma_n$. Then the mapping $f: X \to Y$ and $g: Y \to Z$ are defined respectively by $f(a) = u, f(b) = v, g(u) = x, g(v) = y$ are S^* - 2 -neutrosophic α -continuous S^* - 2 -neutrosophic semi-continuous S^* - 2 -neutrosophic pre-continuous, S^*-2-neutrosophic β-continuous. But $g \circ f$ is not S^*-2-neutrosophic k-continuous. Consider the example 3.9. f and g is S^*-2-neutrosophic r-continuous. But $g \circ f$ is not S^*-2-neutrosophic r-continuous.

Theorem 4.16. If $f: X \to Y$ be S^* - N -neutrosophic k -continuous and $g: Y \to Z$ is N-neutrosophic continuous, then $g \circ f: X \to Z$ is S^*-N-neutrosophic k-continuous.

Proof. Let V be a N-neutrosophic open set in Z, then $g^{-1}(V)$ is N-neutrosophic open in Y and $f^{-1}(g^{-1}(V))$ is N-neutrosophic supra k-open in X implies $(g \circ f)^{-1}(V)$ is N-neutrosophic supra k-open in X. Therefore $g \circ f$ is S^*-N-neutrosophic k-continuous.

Theorem 4.17. If $f: X \to Y$ is N -supra neutrosophic k -continuous and $g: Y \to Z$ is S^* - N -neutrosophic k -continuous (or N -neutrosophic continuous), then $g \circ f: X \to Z$ is S^*-N-neutrosophic k-continuous.

Proof. Let V be a N-neutrosophic open set in Z. Since g is S^*-N-neutrosophic k-continuous, then $g^{-1}(V)$ is N -neutrosophic supra k -open in Y. Since f is N -supra neutrosophic k-continuous, then $f^{-1}(g^{-1}(V))$ is N-neutrosophic supra k-open in X implies $(g \circ f)^{-1}(V)$ is N-neutrosophic supra k-open in X. Therefore $g \circ f$ is S^*-N-neutrosophic k-continuous.

Definition 4.18. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. $N\tau_n{}^*$ and $N\sigma_n{}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. A mapping $f: X \to Y$ is said to be N-supra neutrosophic k-open on X if the image of every $N\tau_n{}^*$-open set in X is a $N\sigma_n{}^*k$-open in Y.

Definition 4.19. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. $N\tau_n{}^*$ and $N\sigma_n{}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. A mapping $f: X \to Y$ is said to be N-supra neutrosophic k-closed on X if the image of every $N\tau_n{}^*$-closed set in X is a $N\sigma_n{}^*k$-closed in Y.

Definition 4.20. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N -neutrosophic topological spaces. $N\tau_n{}^*$ and $N\sigma_n{}^*$ be associated N -neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. A mapping $f: X \to Y$ is said to be S^*-N-neutrosophic k-open mapping on X if the image of every N-neutrosophic open set in $(X, N\tau_n)$ is N-neutrosophic supra k-open in $(Y, N\sigma_n{}^*)$.

Definition 4.21. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. $N\tau_n{}^*$ and $N\sigma_n{}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. A mapping

$f: X \to Y$ is said to be S^*-N-neutrosophic k-closed mapping on X if the image of every N-neutrosophic closed set in $(X, N\tau_n)$ is N-neutrosophic supra k-closed in $(Y, N\sigma_n{}^*)$.

Lemma 4.22. Every N-supra neutrosophic k-open mapping is S^*-N-neutrosophic k-open but the converse need not be true.

Proof. The proof is trivially true from the definition, the converse part is shown in the following example.

Example 4.23 Consider the example 3.15.(ii), g is S^*-2-neutrosophic k-open map on X but it is not 2-supra neutrosophic k-open map

Theorem 4.24. Let $f: X \to Y$ be a N- supra neutrosophic k-open mapping. Then for each neutrosophic subset A of X,

i. $f(kint_{N\tau_n^*}(A)) \subseteq kint_{N\sigma_n^*}(f(A))$.

ii. $f(kcl_{N\tau_n^*}(A)) \supseteq kcl_{N\sigma_n^*}(f(A))$.

Proof. The proof is similarly follows from theorem 3.16.

Theorem 4.25. Let $f: X \to Y$ be a S^*-N-neutrosophic open mapping. Then for each neutrosophic subset A of X,

i. $f(kint_{N\tau_n}(A)) \subseteq kint_{N\sigma_n^*}(f(A))$.

ii. $f\left(kcl_{N\tau_n}(A)\right) \supseteq kcl_{N\sigma_n^*}(f(A))$.

Proof. This proof is straightforward from theorem 3.17.

Theorem 4.26. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. $N\tau_n{}^*$ and $N\sigma_n{}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. Let A be then neutrosophic subset of X. Then

i. If $f: X \to Y$ is N-supra neutrosophic α-closed if and only if $cl_{N\sigma_n^*}(int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(A)))) \subseteq f(cl_{N\tau_n^*}(A))$.

ii. If $f: X \to Y$ is N-supra neutrosophic semi-closed if and only if $int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(A)) \subseteq f(cl_{N\tau_n^*}(A))$.

iii. If $f: X \to Y$ is N-supra neutrosophic pre-closed if and only if $cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A))) \subseteq f(cl_{N\tau_n^*}(A))$.

iv. If $f: X \to Y$ is N-supra neutrosophic β-closed if and only if $int_{N\sigma_n^*}(cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A)))) \subseteq f(cl_{N\tau_n^*}(A))$.

Proof. i. Assume that f be N-supra neutrosophic α-closed and $A \subseteq X$ and $cl_{N\tau_n^*}(A)$ is $N\tau_n{}^*$-closed in X. Then $f(cl_{N\tau_n^*}(A))$ is $N\sigma_n{}^*\alpha$-closed in Y and so, $f\left(cl_{N\tau_n^*}(A)\right) \supseteq cl_{N\sigma_n^*}\left(int_{N\sigma_n^*}\left(cl_{N\sigma_n^*}\left(f\left(cl_{N\tau_n^*}(A)\right)\right)\right)\right)$. Now $f(A) \subseteq f(cl_{N\tau_n^*}(A))$ then $cl_{N\sigma_n^*}(f(A)) \subseteq cl_{N\sigma_n^*}(f(cl_{N\tau_n^*}(A))$ and $cl_{N\sigma_n^*}(int_{N\sigma_n^*}\left(cl_{N\sigma_n^*}(f(A))\right) \subseteq cl_{N\sigma_n^*}\left(int_{N\sigma_n^*}\left(cl_{N\sigma_n^*}\left(f\left(cl_{N\tau_n^*}(A)\right)\right)\right)\right)$. Hence $cl_{N\sigma_n^*}(int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(A))) \subseteq f(cl_{N\tau_n^*}(A))$. Conversely, suppose the condition holds. Let F be any $N\tau_n{}^*$-closed set in X. Then $cl_{N\tau_n^*}(F) = F$. By the condition, $cl_{N\sigma_n^*}(int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(F)))) \subseteq f(cl_{N\tau_n^*}(F)) = f(F)$ which gives $f(F)$ is $N\sigma_n{}^*\alpha$-closed in Y and so f is N-supra neutrosophic α-closed.

ii. Assume that f be N-supra neutrosophic semi-closed and $A \subseteq X$ and $cl_{N\tau_n^*}(A)$ is $N\tau_n{}^*$-closed in X. Then $f(cl_{N\tau_n^*}(A))$ is $N\sigma_n{}^*$-semi closed in Y and $f\left(cl_{N\tau_n^*}(A)\right) \supseteq int_{N\sigma_n^*}\left(cl_{N\sigma_n^*}\left(f\left(cl_{N\tau_n^*}(A)\right)\right)\right)$. Now $f(A) \subseteq f\left(cl_{N\tau_n^*}(A)\right)$ then $cl_{N\sigma_n^*}(f(A)) \subseteq cl_{N\sigma_n^*}(f(cl_{N\tau_n^*}(A))$ and $int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(A))) \subseteq int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(cl_{N\tau_n^*}(A))))$. Then

$int_{N\sigma_n^*}\left(cl_{N\sigma_n^*}(f(A))\right) \subseteq f\left(cl_{N\tau_n^*}(A)\right)$. Conversely, suppose the condition holds. Let F be any $N\tau_n^*$ -closed set in X. Then $cl_{N\tau_n^*}(F) = F$. By the condition, $int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(F)) \subseteq f(cl_{N\tau_n^*}(A)) = f(F)$ which gives $f(F)$ is $N\sigma_n^*$-semi-closed in Y. So f is N-supra neutrosophic semi-closed.

iii. Assume that f be N -supra neutrosophic pre-closed and $A \subseteq X$. Then $cl_{N\tau_n^*}(A)$ is $N\tau_n^*$ -closed in X. Hence $f(cl_{N\tau_n^*}(A))$ is $N\sigma_n^*$ -pre closed in Y and so, $f(cl_{N\tau_n^*}(A)) \supseteq cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(cl_{N\tau_n^*}(A))))$. Now $f(A) \subseteq f(cl_{N\tau_n^*}(A))$. That is $int_{N\sigma_n^*}(f(A)) \subseteq int_{N\sigma_n^*}(f(cl_{N\tau_n^*}(A)))$. We have $cl_{N\sigma_n^*}\left(int_{N\sigma_n^*}(f(A))\right) \subseteq cl_{N\sigma_n^*}\left(int_{N\sigma_n^*}\left(f\left(cl_{N\tau_n^*}(A)\right)\right)\right)$. Hence $cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A))) \subseteq f(cl_{N\tau_n^*}(A))$. Conversely, suppose the condition holds. Let F be any $N\tau_n^*$ -closed set in X. Then $cl_{N\tau_n^*}(F) = F$. By the condition, $cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(F)) \subseteq f(cl_{N\tau_n^*}(A)) = f(F)$ which gives $f(F)$ is $N\sigma_n^*$-pre-closed in Y. So f is N-supra neutrosophic supra pre-closed.

iv. Assume that f be N-supra neutrosophic β-closed and $A \subseteq X$. Then $cl_{N\tau_n^*}(A)$ is $N\tau_n^*$-closed in X. Hence $f(cl_{N\tau_n^*}(A))$ is $N\sigma_n^*\beta$ -closed in Y and so, $f\left(cl_{N\tau_n^*}(A)\right) \supseteq int_{N\sigma_n^*}\left(cl_{N\sigma_n^*}\left(int_{N\sigma_n^*}\left(f\left(cl_{N\tau_n^*}(A)\right)\right)\right)\right)$. Now $f(A) \subseteq f(cl_{N\tau_n^*}(A))$. That is $int_{N\sigma_n^*}(f(A)) \subseteq int_{N\sigma_n^*}(f(cl_{N\tau_n^*}(A)))$. We have $int_{N\sigma_n^*}(cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A)))) \subseteq int_{N\sigma_n^*}(cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(cl_{N\tau_n^*}(A)))))$. Hence $int_{N\sigma_n^*}(cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A)))) \subseteq f(cl_{N\tau_n^*}(A))$. Conversely, suppose the condition holds. Let F be any $N\tau_n^*$ -closed set in X. Then $cl_{N\tau_n^*}(F) = F$. By the condition, $int_{N\sigma_n^*}(cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(F)))) \subseteq f(cl_{N\tau_n^*}(A)) = f(F)$ which gives $f(F)$ is $N\sigma_n^*$-pre closed in Y. So f is N-supra neutrosophic β-closed mapping.

Theorem 4.27. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. $N\tau_n^*$ and $N\sigma_n^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. Let A be then neutrosophic subset of X. Then

i. If $f : X \to Y$ is S^* - N -neutrosophic α -closed if and only $cl_{N\sigma_n^*}(int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(A)))) \subseteq f(cl_{N\tau_n}(A))$.

ii. If $f : X \to Y$ is S^* - N -neutrosophic semi-closed if and only if $int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(A)) \subseteq f(cl_{N\tau_n}(A))$.

iii. If $f : X \to Y$ is S^* - N -neutrosophic pre-closed if and only if $cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A))) \subseteq f(cl_{N\tau_n}(A))$.

iv. If $f : X \to Y$ is S^* - N -neutrosophic β -closed if and only if $int_{N\sigma_n^*}(cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A)))) \subseteq f(cl_{N\tau_n}(A))$.

Proof. This proof is similarly follows from theorem 4.26

Theorem 4.28. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. $N\tau_n^*$ and $N\sigma_n^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. Let A be then neutrosophic subset of X. Then

i. If $f : X \to Y$ is N -supra neutrosophic α -open if and only if $f(int_{N\tau_n^*}(A)) \subseteq int_{N\sigma_n^*}(cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A))))$.

ii. If $f : X \to Y$ is N -supra neutrosophic semi-open if and only if $f(int_{N\tau_n^*}(A)) \subseteq cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A))$.

iii. If $f : X \to Y$ is N - supra neutrosophic pre-open if and only if $f\left(int_{N\tau_n^*}(A)\right) \subseteq int_{N\sigma_n^*}\left(cl_{N\sigma_n^*}(f(A))\right)$.

iv. If $f : X \to Y$ is N -supra neutrosophic β -open $f(int_{N\tau_n^*}(A)) \subseteq cl_{N\sigma_n^*}(int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(A))))$.

Proof. This proof follows from theorem 4.26.

Theorem 4.29. Let $(X, N\tau_n)$ and $(Y, N\sigma_n)$ be N-neutrosophic topological spaces. $N\tau_n{}^*$ and $N\sigma_n{}^*$ be associated N-neutrosophic supra topologies with respect to $N\tau_n$ and $N\sigma_n$. Let A be then neutrosophic subset of X. Then

i. $f: X \to Y$ is S^* - N -neutrosophic α -open if and only if $f(int_{N\tau_n}(A)) \subseteq int_{N\sigma_n^*}(cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A))))$.

ii. $f: X \to Y$ is S^* - N -neutrosophic semi-open if and only if $f(int_{N\tau_n}(A)) \subseteq cl_{N\sigma_n^*}(int_{N\sigma_n^*}(f(A))$.

iii. A mapping $f: X \to Y$ is S^* - N -neutrosophic pre-open if and only if $f(int_{N\tau_n}(A)) \subseteq int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(A)))$.

iv. A mapping $f: X \to Y$ is S^* - N -neutrosophic β -open if and only if $f(int_{N\tau_n}(A)) \subseteq cl_{N\sigma_n^*}(int_{N\sigma_n^*}(cl_{N\sigma_n^*}(f(A))))$.

Proof. This proof is straightforward from theorem 4.26.

5 Conclusion and Future Work

Neutrosophic supra topological space is one of the new research areas to deal with the uncertainty concept and it is a generalized form of fuzzy supra topological spaces as well as intuitionistic fuzzy supra topological spaces. This paper theoretically introduced N-neutrosophic supra topological mappings with suitable examples. The properties and relationship between N-neutrosophic supra topological mappings are derived. We can construct the real-life application of these N-neutrosophic supra topological sets and mappings in the future and implement these concepts to other applicable research areas of topology such as Rough topology, Fuzzy topology, intuitionistic topology, Digital topology, and so on.

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Hyperbolic Cosine Similarity Measure Based MADM-Strategy under the SVNS Environment

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Abstract
In this article, we propose a MADM-strategy based on hyperbolic cosine similarity measures under the single valued neutrosophic set environment. Further, we also give some properties of the similarity measures by giving some suitable examples. We also solve a numerical example to validate our proposed MADM-model.

Keywords: MADM-Strategy; Neutrosophic Set; Similarity Measure; Distillation Unit.

1. Introduction

In the year 1998, Smarandache [18] grounded the concept of neutrosophic set (in short NS) as a generalization of the notion of fuzzy set [27] and intuitionistic fuzzy set (in short IFS) [1] theory to deal with incomplete and indeterminate information. In every NS, truth membership, indeterminacy membership, and falsity membership values of each element are independent of each other. Indeterminacy-membership plays a vital role in many real world multi attribute decision making (in short MADM) problems. In the year 2010, Wang et al. [21] presented the concept of single valued neutrosophic set (in short SVNS), which is the subclass of an NS. By using SVNS, we can represent incomplete, imprecise, and indeterminate information that helps in decision making in the real world. The notion of SVNS and the various extensions of SVNS have been used in the formation of MADM-model / MADM-algorithm in different branch (branches) of real world such as medical diagnosis, educational problem, social problems, decision making problems, conflict resolution, etc. In the year (In) 2014, Biswas et al. [2] proposed the entropy based grey relational analysis (in short GRA) method and developed a MADM-strategy under SVNS-environment. Afterwards, Dey et al. [4] proposed a MADM model for the select ion of weaver based on extended GRA method under the interval NS environment. Later on, Dey et al. [5] also proposed a MADM-strategy under the interval NS environment based on extended projection method. In the year 2016, Mondal et al. [11] studied the role of SVNS in data mining. In the year 2016, Pramanik et al. [13] proposed a MADM-strategy to choose the logistic center location. Later on, Mondal et al. [10] defined a similarity measure under the SVNS environment namely single valued neutrosophic hyperbolic sine similarity measure, and proposed a MADM-strategy based on it. In the year 2015, Pramanik and Mondal [15, 16] proposed two MADM-strategies under the rough neutrosophic set environment. Afterwards, several MADM-strategies has been developed by Ye [22-25], Ye and Zhang [26], etc. using different similarity measure under the SVNS environment.

In this study, we propose a MADM-strategy based using (on) the single valued weighted hyperbolic cosine similarity measure under the SVNS-environment. Further, we validate the proposed model by solving an illustrative numerical example entitled "Selection of the Most Suitable Distillation Unit under SVNS-Environment".

There is no study in the literature relating to MADM-strategy using single valued neutrosophic weighted hyperbolic cosine similarity measure under the SVNS-environment. To fill the research gap, we propose this MADM-strategy under SVNS-environment based on single valued neutrosophic weighted hyperbolic cosine similarity measure.

The rest of the paper has been split into the following sections:

In section-2, we recall SVNS and its different properties. In section-3, we introduce a new similarity measure namely single valued neutrosophic weighted hyperbolic cosine similarity measure of similarities between two single valued neutrosophic numbers. In section-4, we propose a MADM-strategy based on single valued neutrosophic weighted hyperbolic cosine similarity measure under the SVNS-environment. In section-5, we give a numerical example to show the applicability and effectiveness of the proposed MADM-strategy. In section-6, we conclude the work done in this paper by stating some future scope of research.

2. Preliminaries and Definitions

In this section, we give some basic definitions and results those are relevant for developing the main results of this article.

Definition 2.1. [18] A single valued neutrosophic set K over a fixed set L is defined by $K=\{(u, T_K(u), I_K(u), F_K(u)): u \in L\}$, where T_K, I_K, F_K are truth, indeterminacy and falsity membership mappings from L to [0, 1], and so $0 \leq T_K(u) + I_K(u) + F_K(u) \leq 3$.

The null SVNS (0_N) and the absolute SVNS (1_N) over a fixed set L are defined as follows:

(*i*) $0_N = \{(u, 0, 1, 1) : u \in L\}$,

(*ii*) $1_N = \{(u, 1, 0, 0) : u \in L\}$.

Example 2.1. Assume that $L=\{a, b\}$ be a fixed set. Then, $K=\{(a,0.3,0.2,0.6), (b,0.9,0.5,0.8)\}$ is a SVNS over L.

Definition 2.2. [18] Suppose that X={(u, $T_X(u)$, $I_X(u)$, $F_X(u)$): $u \in L$} and Y={(u, $T_Y(u)$, $I_Y(u)$, $F_Y(u)$): $u \in L$} be two SVNSs over L. Then, $X \subseteq Y$ if and only if $T_X(u) \leq T_Y(u)$, $I_X(u) \geq I_Y(u)$, $F_X(u) \geq F_Y(u)$, for all $u \in L$.

Example 2.2. Assume that L={a, b} be a fixed set. Let K={(a,0.3,0.5,0.6), (b,0.2,0.5,0.8)} and S={(a,0.4,0.3,0.6), (b,0.4,0.5,0.6)} be two SVNSs over L. Then, $K \subseteq S$.

Definition 2.3. [18] Assume that X = {(u, $T_X(u)$, $I_X(u)$, $F_X(u)$): $u \in L$} and Y = {(u, $T_Y(u)$, $I_Y(u)$, $F_Y(u)$): $u \in L$} be two SVN-Sets over L. Then, $X \cup Y$={(u, max {$T_X(u)$, $T_Y(u)$}, min {$I_X(u)$, $I_X(u)$}, min {$F_X(u)$, $F_X(u)$}): $u \in L$}.

Example 2.3. Suppose that K={(a,0.3,0.7,0.2), (b,0.9,0.4,0.8)} and S={(a,0.4,0.3,0.6), (b,0.4,0.5, 0.6)} be two SVNSs over a fixed set L={a, b}. Then, $K \cup S$={(a,0.4,0.3,0.2), (b,0.9,0.4,0.6)}.

Definition 2.4. [18] Suppose that X = {(u, $T_X(u)$, $I_X(u)$, $F_X(u)$): $u \in L$} and Y = {(u, $T_Y(u)$, $I_Y(u)$, $F_Y(u)$): $u \in L$} be two SVN-Sets over L. Then, $X \cap Y$ = {(u, min {$T_X(u)$, $T_Y(u)$}, max {$I_X(u)$, $I_X(u)$}, max {$F_X(u)$, $F_X(u)$}): $u \in L$}.

Example 2.4. Suppose that K and S be two SVNSs over a fixed set L={a, b} as shown in Example 2.3. Then, $K \cap S$={(a,0.3,0.7,0.6), (b,0.4,0.5,0.8)}.

Definition 2.5. [18] Suppose that X = {(u, $T_X(u)$, $I_X(u)$, $F_X(u)$): $u \in W$} and Y = {(u, $T_Y(u)$, $I_Y(u)$, $F_Y(u)$): $u \in L$} be two SVN-Sets over L. Then, X^c= {(u, 1-$T_X(u)$, 1-$I_X(u)$, 1-$F_X(u)$): $u \in L$}.

Example 2.5. Assume that K={(a,0.3,0.2,0.6), (b,0.9,0.5,0.8)} be a SVNS over L={a, b} as shown in Example 2.1. Then, K^c={(a,0.7,0.8,0.4), (b,0.1,0.5,0.2)}.

3. Single Valued Neutrosophic Hyperbolic Cosine Similarity Measure

In this section, we introduce a new similarity measure namely single valued neutrosophic weighted hyperbolic cosine similarity measure under the SVNS-environment. Then, we formulate some basic results based on it.

Definition 3.1. Suppose that M = {(u_i, $T_M(u_i)$, $I_M(u_i)$, $F_M(u_i)$): $u_i \in L$, i=1, 2, …, n} and W = {(u_i, $T_W(u_i)$, $I_W(u_i)$, $F_W(u_i)$): $u_i \in L$, i=1, 2, …, n } be two SVNS over a non-empty set L. Then, the single valued neutrosophic hyperbolic cosine similarity measure (in short SVNHCSM) of the similarity between the SVNSs M and W is defined by:

$$\text{SVNHCSM}(M, W) = 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\cosh(|T_M(u_i)-T_W(u_i)| + |I_M(u_i)-I_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{11}\right) \quad (1)$$

Example 3.1. Let M = {(a,0.5,0.3,0.5), (b,0.3,0.5,0.4)} and W = {(a,0.6,0.4,0.3), (b,0.7,0.5, 0.4)} be two SVNSs over a fixed set L={a, b}. Then, SVNHCSM (M, W) = 0.9017206935.

Definition 3.2. Suppose that M = {(u_i, $T_M(u_i)$, $I_M(u_i)$, $F_M(u_i)$): $u_i \in L$, i=1, 2, …, n} and W = {(u_i, $T_W(u_i)$, $I_W(u_i)$, $F_W(u_i)$): $u_i \in L$, i=1, 2, …, n} be two SVNSs over a fixed set L. Then, the single valued neutrosophic weighted hyperbolic cosine similarity measure (in short SVNWHCSM) of the similarity between the SVNSs M and W is defined by:

$$\text{SVNWHCSM}(M, W) = 1 - \frac{1}{n}\sum_{i=1}^{n} w_i\left(\frac{\cosh(|T_M(u_i)-T_W(u_i)| + |I_M(u_i)-I_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{11}\right), \quad (2)$$

where $0 \leq w_i \leq 1$ and $\sum_{i=}^{n} w_i$=1.

Example 3.2. Let us consider two SVNSs M and W as shown in Example 3.1. Assume that w_1= 0.5 and w_2= 0.4 be the corresponding weights of M and W. Then, SVNWHCSM (M, W) = 0.9557743121.

Theorem 3.1. Let SVNHCSM (M, W) be the single valued neutrosophic hyperbolic cosine similarity measure between the SVNSs M and W. Then, $0 \leq$ SVNHCSM (M, W) ≤ 1.

Proof. Suppose that $M = \{(u_i, T_M(u_i), I_M(u_i), F_M(u_i)): u\in L, i=1, 2, \dots, n\}$ and $W = \{(u_i, T_W(u_i), I_W(u_i), F_W(u_i)): u_i\in L, i=1, 2, \dots, n\}$ be two SVN-Sets over a fixed set L.

Now, $0 \le T_M(u_i), I_M(u_i), F_M(u_i), T_W(u_i), I_W(u_i), F_W(u_i) \le 1$, for each $i=1, 2, \dots, n$

$\Rightarrow 0 \le |T_M(u_i) - T_W(u_i)| + |I_M(u_i) - I_W(u_i)| + |F_M(u_i) - F_W(u_i)| \le 3$, for each $i=1, 2, \dots, n$

$\Rightarrow 0 \le \frac{cosh(|T_M(u_i)-T_W(u_i)| + |I_M(u_i)-I_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{11} \le 1$, for each $i=1, 2, \dots, n$

$\Rightarrow 0 \le 1 - \frac{1}{n}\sum_{i=1}^{n}\frac{cosh(|T_M(u_i)-T_W(u_i)| + |I_M(u_i)-I_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{11} \le 1$

$\Rightarrow 0 \le$ SVNHCSM $(M, W) \le 1$.

Theorem 3.2. Assume that SVNHCSM (M, W) be the single valued neutrosophic hyperbolic cosine similarity measure of the similarities between two SVPNSs M and W. If $M = W$, then SVNHCSM $(M, W) = 1$.

Proof. Suppose that $M = \{(u_i, T_M(u_i), I_M(u_i), F_M(u_i)): u_i\in L, i=1, 2, \dots, n\}$ and $W = \{(u_i, T_W(u_i), I_W(u_i), F_W(u_i)): u_i \in L, i=1, 2, \dots, n\}$ be two SVN-Sets over a fixed set L such that $M = W$.

So, $T_M(u_i) = T_W(u_i)$, $I_M(u_i) = I_W(u_i)$, $F_M(u_i) = F_W(u_i)$, for each $u_i\in L$ $(i=1, 2, \dots, n)$

$\Rightarrow |T_M(u_i) - T_W(u_i)| = 0$, $|I_M(u_i) - I_W(u_i)| = 0$, $|F_M(u_i) - F_W(u_i)| = 0$, for each $u_i\in L$ $(i=1, 2, \dots, n)$

$\Rightarrow cosh(|T_M(u_i) - T_W(u_i)| + |I_M(u_i) - I_W(u_i)| + |F_M(u_i) - F_W(u_i)|) = 0$, for each $u_i\in L$ $(i=1, 2, \dots, n)$

$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\frac{cosh(|T_M(u_i)-T_W(u_i)| + |I_M(u_i)-I_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{11} = 0$

$\Rightarrow 1 - \frac{1}{n}\sum_{i=1}^{n}\frac{cosh(|T_M(u_i)-T_W(u_i)| + |I_M(u_i)-I_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{11} = 1$

$\Rightarrow$ SVNHCSM $(M, W) = 1$.

Theorem 3.3. Assume that SVNHCSM (M, W) be the single valued neutrosophic hyperbolic cosine similarity measure of the similarities between two SVN-Sets M and W. Then, SVNHCSM (M, W) = SVNHCSM (W, M).

Proof. Suppose that $M = \{(u_i, T_M(u_i), I_M(u_i), F_M(u_i)): u\in L, i=1, 2, \dots, n\}$ and $W = \{(u_i, T_W(u_i), I_W(u_i), F_W(u_i)): u_i\in L, i=1, 2, \dots, n\}$ be two SVN-Sets over L.

Now, SVNHCSM (M, W)

$= 1- \frac{1}{n}\sum_{i=1}^{n}\left(\frac{cosh(|T_M(u_i)-T_W(u_i)| + |I_M(u_i)-I_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{11}\right)$

$= 1- \frac{1}{n}\sum_{i=1}^{n}\left(\frac{cosh(|T_W(u_i)-T_M(u_i)| + |I_W(u_i)-I_M(u_i)| + |F_W(u_i)-F_M(u_i)|)}{11}\right)$

= SVNHCSM (W, M).

Therefore, SVNHCSM (M, W) = SVNHCSM (M, W).

Theorem 3.4. Suppose that SVNHCSM (M, W) be the single valued neutrosophic hyperbolic cosine similarity measure of the similarity between the SVN-Sets M and W. If Q be a SVN-Set over L such that $M\subseteq W\subseteq Q$, then SVNHCSM $(M, W) \ge$ SVNHCSM (M, Q) and SVNHCSM $(W, Q) \ge$ SVNHCSM (M, Q).

Proof. Suppose that $M = \{(u_i, T_M(u_i), I_M(u_i), F_M(u_i)): u_i\in L, i=1, 2, \dots, n\}$ and $W = \{(u_i, T_W(u_i), I_W(u_i), F_W(u_i)): u_i\in L, i=1, 2, \dots, n\}$ be two SVN-Sets over L. Let Q be a SVN-Set over L such that $M\subseteq W\subseteq Q$. Since $M\subseteq W\subseteq Q$, so $|T_M(u_i)-T_W(u_i)| \le |T_M(u_i)-T_Q(u_i)|$, $|I_M(u_i)-I_W(u_i)| \le |I_M(u_i)-I_Q(u_i)|$, $|F_M(u_i)-F_W(u_i)| \le |F_M(u_i)-F_Q(u_i)|$, $\forall u_i\in L, i=1, 2, \dots, n$.

Now, we have

SVNHCSM (M, W)

$= 1- \frac{1}{n}\sum_{i=1}^{n}\left(\frac{cosh(|T_M(u_i)-T_W(u_i)| + |I_M(u_i)-I_W(u_i)| + |F_M(u_i)-F_W(u_i)|)}{11}\right)$

$\geq 1- \frac{1}{n}\sum_{i=1}^{n}\left(\frac{cosh(|T_M(u_i)-T_Q(u_i)| + |I_M(u_i)-I_Q(u_i)| + |F_M(u_i)-F_Q(u_i)|)}{11}\right)$

= SVNHCSM (M, Q).

Therefore, SVNHCSM (M, W) ≥ SVNHCSM (M, Q).

Again, from $M \subseteq W \subseteq Q$ it can be say that $|T_W(u_i)-T_Q(u_i)| \leq |T_M(u_i)-T_Q(u_i)|$, $|I_W(u_i)-I_Q(u_i)| \leq |I_M(u_i)-I_Q(u_i)|$, $|F_M(u_i)-F_W(u_i)| \leq |F_M(u_i)-F_Q(u_i)|$, $\forall u_i \in L$, i=1, 2, …, n.

Now, we have

SVNHCSM (W, Q)

$= 1- \frac{1}{n}\sum_{i=1}^{n}\left(\frac{cosh(|T_W(u_i)-T_Q(u_i)| + |I_W(u_i)-I_Q(u_i)| + |F_W(u_i)-F_Q(u_i)|)}{11}\right)$

$\geq 1- \frac{1}{n}\sum_{i=1}^{n}\left(\frac{cosh(|T_M(u_i)-T_Q(u_i)| + |I_M(u_i)-I_Q(u_i)| + |F_M(u_i)-F_Q(u_i)|)}{11}\right)$

= SVNHCSM (M, Q).

Therefore, SVNHCSM (M, W) ≥ SVNHCSM (M, Q).

4. SVNWHCSM Based MADM Strategy

Let $Q = \{Q_1, Q_2, ..., Q_n\}$ be the fixed set of possible alternatives and $P = \{P_1, P_2, ..., P_m\}$ be the collection of attributes for a multi attribute decision making (in short MADM) problem. Then, a decision maker can provide their evaluation information of each alternative Q_i (i = 1, 2, ..., n) against the attributes P_j (j = 1, 2, ..., m) in terms of SVNS. Then, the whole evaluation information of all alternatives can be expressed by a decision matrix.

The following are the steps of the proposed MADM-technique:

Step-1: Construct the Decision Matrix Using the SVNS

The whole evaluation information of each alternative Q_i (i = 1, 2,..., n) based on the attributes P_j (j = 1, 2, ..., m) is expressed in terms of SVN-Set $E_{Q_i} = \{(P_j, T_{ij}(Q_i, P_j), I_{ij}(Q_i, P_j), F_{ij}(Q_i, P_j)): P_j \in P\}$, where $(T_{ij}(Q_i, P_j), I_{ij}(Q_i, P_j), F_{ij}(Q_i, P_j))$ denotes the evaluation assessment of Q_i (i = 1, 2,..., n) against P_j (j = 1, 2, ..., m).

Then, we can build the decision matrix (DM[$Q|P$]) as follows:

	P_1	P_2	….	P_m
Q_1	$[T_{11}(Q_1, P_1), I_{11}(Q_1, P_1), F_{11}(Q_1, P_1)]$	$[T_{12}(Q_1, P_2), I_{12}(Q_1, P_2), F_{12}(Q_1, P_2)]$	….	$[T_{1m}(Q_1, P_m), I_{1m}(Q_1, P_m), F_{1m}(Q_1, P_m)]$
Q_2	$[T_{21}(Q_2, P_1), I_{21}(Q_2, P_1), F_{21}(Q_2, P_1)]$	$[T_{22}(Q_2, P_2), I_{22}(Q_2, P_2), F_{22}(Q_2, P_2)]$	….	$[T_{2m}(Q_2, P_m), I_{2m}(Q_2, P_m), F_{2m}(Q_2, P_m)]$
….	……………………..	………………………..	….	……………………..

Q_n	$[T_{n1}(Q_n, P_1), I_{n1}(Q_n, P_1), F_{n1}(Q_n, P_1)]$	$[T_{n2}(Q_n, P_2), I_{n2}(Q_n, P_2), F_{n2}(Q_n, P_2)]$	….	$[T_{nm}(Q_n, P_m), I_{nm}(Q_n, P_m), F_{nm}(Q_n, P_m)]$

Step-2: Determination of the Attributes Weight

In an MADM-strategy, the weights of the attributes play an important role in taking decision. When the weights of the attributes are totally unknown to the decision makers, then the attribute weights can be determined by using the compromise function defined in equation (3).

Compromise Function: The compromise function of Q is defined by:

$$\Omega_j=\sum_{i=1}^{n} (2+T_{ij}(Q_i, P_j)-I_{ij}(Q_i, P_j)-F_{ij}(Q_i, P_j))/3 \quad (3)$$

Then the desired weight of the j^{th} attribute is defined by $w_j= \frac{\Omega_j}{\sum_{j=1}^{m}\Omega_j}$ (4)

Here, $\sum_{j=1}^{m} w_j=1$.

Step-3: Determination of ideal solution

In any similarity measure based MADM-strategy, the selection of ideal solution is the key factor to find the most suitable alternative. In our proposed MADM-strategy, we take the absolute SVNS 1_N as an ideal solution to find the suitable alternative.

Step-4: Determination of single valued neutrosophic weighted hyperbolic cosine similarity value

After the formation of ideal solution in step-3, by using eq. (1), we calculate the SVNWHCSM values for every alternative between the ideal solution and the corresponding SVNS from decision matrix DM[$Q|P$] that formed in step-1.

Step-5: Ranking Order of the Alternatives

In this step, we arrange the all the single valued neutrosophic weighted hyperbolic cosine similarity value in ascending order. The alternative with the lowest single valued neutrosophic weighted hyperbolic cosine similarity value with the ideal solution is the most suitable alternative for selection.

Step-6: End.

5. Validation of the Proposed MADM-strategy

In this section, we demonstrate a numerical example to show the real life applicability of the proposed MADM-strategy.

Example 5.1. "Selection of the Most Suitable Distillation Unit under SVNS-Environment"

Distillation units are one of the essential laboratory equipment in modern day science. A solvent distillation unit or distilled machine comes in various designs, capacities and lab grade solvent purity level. The distillation process removes minerals and microbiological contaminants and can reduce levels of chemical contaminants through boiling the target solvent. The distillation apparatus structurally consists of flask with heating elements embedded in glass and a fused spiral coil tapered round glass, joints at the top double walled condenser with ground glass joints.

Successful distillation depends on several factors, including the difference in boiling points of the materials in the mixture, and therefore the difference in their vapor pressures, the type of apparatus used, and the care exercised by the experimentalist. In heating, the lowest boiling distills first (most volatile), having a maximum boiling point distills last, and others subsequently or not at all. Distillation is a simple apparatus with entirely satisfactory for the purification of a solvent containing nonvolatile material and is reasonably adequate for separating liquids of wide-ranged boiling points. Industrially, distillation is the basis for the separation of crude oil into the various, more useful hydrocarbon fractions. Chemically, distillation is the principal method for purifying liquids (e.g. samples, or solvents for performing reactions).

Structure:

A distilling flask, a source of heat or a hot bath, condenser, receiving flask to collect the condensed vapors or distillate are the basic structural units of an ideal distillation apparatus. For laboratory use, the apparatus is commonly made of glass and connected with corks, rubber bungs, or ground-glass joints, wherein in industrial applications, larger equipment of metal or ceramic is used. The underlying mechanism of distillation is the differences in volatility between individual components. With sufficient heat applied, a gas phase (vapor) is formed from the liquid solution. The liquid product is subsequently condensed from the gas phase by the removal of the heat.

Process:

There are many types of distillation units used in modern laboratories and industries based on their application. Some are simple distillation, fractional distillation, steam distillation, and vacuum distillation.

(i) **Simple distillation:** In simple distillation heating of the liquid mixture at the boiling point and immediately condensing the resulting vapors. This method is only effective for mixtures wherein the boiling points of the liquids are considerably different (~ 25^oC).

(ii) **Fractional distillation:** Simple distillation is not efficient for separating liquids whose boiling points lie close to one another. Fractional distillation is often used to separate mixtures of liquids that have similar boiling points. It involves several vaporization-condensation steps (which take place in a fractioning column). This process is also known as rectification.

(iii) **Steam distillation:** Steam distillation is often used to separate components from a mixture of heat-sensitive components. The process is processed by passing steam through the mixture (which is slightly heated) to vaporize it. It establishes a high heat transfer rate without the need for a source of high temperatures. The resultant vapor is condensed to afford the required distillate liquid. The process of steam distillation is used to obtain essential oil constituents and herbal distillates from several aromatic flowers/herbs.

(iv) **Vacuum distillation:** Vacuum distillation is ideal for separating mixtures of liquids with very high boiling points. To boil these compounds, heating to high temperatures is an inefficient method. Therefore, the pressure of the surroundings is lowered instead. The lowering of the pressure enables the component to boil at lower temperatures. Once the vapor pressure of the component is equal to the surrounding pressure, it is converted into vapor. These vapors are then condensed and collected as the distillate. The vacuum distillation method is also used to obtain high-purity samples of compounds that decompose at high temperatures.

Suppose that, a bio-science department of an institution needs a distillation unit for their laboratory research. In market there are several types of distillation units. But it is difficult to choose the most suitable distillation unit among the possible distillation units those are available in the market. For this the decision maker (institution) can choose the some attributes on which basis customers /users/ institutions are being interested to buy distillation unit for their laboratory purpose or other needs.

- **Capacity or productivity (E_1):** Production of required solvent as per hour is one of the most important criteria that buyers looking for. On average a laboratory distillation unit can produce 2.0-2.5 liters of distillate per hour where an industrial unit can produce much more than a laboratory distillation unit.
- **The material used in the vessel (E_2):** The distillation flask should preferably be round-bottomed rather than a flat-bottomed one for smoothness of boiling. The material used in the vessel should be very heat resistant and light-weighted. There are two major glass materials used maximum glass distillation units *i.e.* borosilicate glass and quartz glass material.
- **Automation Grade of machine (E_3):** Machine-operating systems are the most advanced technology for all of us where it works automatically and without human involvement. So the criteria should be either a semi-automatic or automatic process.
- **Usage/Application of machine (E_4):** The application of any machine defines the existence of that machine. This is one of the price-dependent criteria among all.
- **Temperature Control Range (E_5):** Temperature measurement is a common control parameter in distillation cooling and heating processes. Depending upon the application and process fluid, temperature control may be used for cooling distillate to condense high volatility products into liquid phase, or heating of process fluid to vaporize the high volatility components for easier separation. The lower limit of the range is the temperature indicated by the thermometer when the first drop of condensate leaves the tip of the condenser, and the upper limit is the temperature at which the last drop evaporates from the lowest point in the distillation flask.
- **Price (E_6):** Generally, there are two types of cost named fixed cost and variable cost, which are used along with numbers of units for determining the selling price of the product. Cost of materials plays a very significant role in their selection. The application and material of glass are charged with the cost of distillation units issued to them.

Hence, the selection of a suitable distillation unit for biological laboratory can be considered as a multi-attribute-decision-making problem.

Assume that, the decision maker select four alternatives after the initial screening. Let $\H{U}$ = $\{P_1, P_2, P_3, P_4\}$ be the universal set of available distillation units from which the decision maker will buy a suitable distillation unit. Let $E = \{E_1, E_2, E_3, E_4, E_5, E_6\}$ be the set of attributes based on which the decision maker will select the most suitable distillation units. Then, the tabular representation of the information of distillation units P_1, P_2, P_3, P_4 against the attributes $E_1, E_2, E_3, E_4, E_5, E_6$ are given in Table-1.

Table-1

	E_1	E_2	E_3	E_4	E_5	E_6
P_1	(0.8,0.1,0.2)	(0.9,0.2,0.3)	(0.7,0.1,0.1)	(0.9,0.0,0.1)	(1.0,0.3,0.1)	(0.9,0.2,0.1)
P_2	(0.2,0.6,0.5)	(0.1,0.4,0.3)	(0.0,0.2,0.3)	(0.1,0.2,0.0)	(0.5,0.1,0.1)	(0.0,0.1,0.2)
P_3	(0.8,0.4,0.2)	(0.6,0.5,0.5)	(0.5,0.2,0.1)	(0.4,0.5,0.4)	(0.6,0.4,0.2)	(0.2,0.3,0.1)
P_4	(0.8,0.2,0.2)	(0.7,0.3,0.2)	(0.9,0.0,0.1)	(1.0,0.2,0.1)	(0.9,0.2,0.1)	(0.8,0.1,0.1)

Now, by using the eq. (3) and eq. (4), we have w_1= 0.1607378, w_2 = 0.1528327, w_3 = 0.1712780, w_4 = 0.1699605, w_5 = 0.1778656, and w_6 = 0.1673254.

The ideal solution is $P^*= 1_N$ = {(E_1, 1, 0, 0), (E_2, 1, 0, 0), (E_3, 1, 0, 0), (E_4, 1, 0, 0), (E_5, 1, 0, 0), (E_6, 1, 0, 0)}. The single valued neutrosophic weighted hyperbolic cosine similarity measure of similarities between the possible alternatives (distillation units) and the ideal solution (ideal distillation unit) are:
SVNWHCSM (P_1, P^*) = 0.9833013,
SVNWHCSM (P_2, P^*) = 0.9669271,
SVNWHCSM (P_3, P^*) = 0.9734845,
and SVNWHCSM (P_4, P^*) = 0.9830226.

Here, SVNWHCSM (P_2, P^*) < SVNWHCSM (P_3, P^*) < SVNWHCSM (P_4, P^*) < SVNWHCSM (P_1, P^*). Therefore, the alternative P_2 is the most suitable alternative among the set of possible alternative. Hence, the institution can buy the distillation unit P_2 for their laboratory related work.

6. Conclusions

In the study, we have proposed a new similarity measure namely single valued neutrosophic weighted hyperbolic cosine similarity measures of similarities between two SVNSs and proved some of their basic properties. Further, we have developed a novel MADM-strategy based on the proposed single valued neutrosophic weighted hyperbolic cosine similarity measure under the SVNS environment. Then, we validate our proposed MADM-strategy by solving an illustrative MADM-problem namely "Selection of the Most Suitable Distillation Unit for Biological Laboratory under SVNS-environment" to demonstrate the applicability and effectiveness of our proposed MADM-strategy.

The proposed MADM-strategy also can be used to deal with other real life problems in real world such as decision making [3-4, 6-8, 13], data mining [11], medical diagnosis [15-16].

The data used in this paper has not taken from any source. We have considered these numbers for the verification of our algorithm. However, this algorithm can apply for any real source data.

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Neutrosophic Stable Random Variables

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Abstract: In this paper, the concept of a neutrosophic stable random variable is introduced. Two definitions of a neutrosophic random variable are presented. We introduced both the neutrosophic probability distribution function and the neutrosophic probability density function, and the convolution with the neutrosophic concept. In addition, we proved some properties of a neutrosophic stable random variable, and three examples are discussed.

Keywords: Random Variables; Stable Distributions; Gaussian Distribution; Cauchy Distribution; Lévy Distribution.

1. Introduction

The term stability in probability theory refers to a property of some probability distributions, which is that the random variable indicative of a sum of independent and identically distributed random variables has the same probability distribution for each of these variables. This property is true for a finite or infinite sum of random variables. Variables that achieve this specificity are called stable random variables. Stability in this concept is called classical stability, and stable distributions represent a large part of the family of all probability distributions. Regarding the tail of the distribution, all stable distributions are heavy-tailed except for the normal distribution, which is light-tailed.

In 1925, Paul Lévy [1] presented stable distributions as a generalization of the normal distribution in several ways. The theory of stable distributions was developed in the messages exchanged between Lévy (1937) [2] and Khintchine (1938) [3], and work on these results was expanded by Gnedenko and Kolmogorov (1949) [4] and then Feller (1970) [5]. Paul Lévy defined a stable distribution by defining its characteristic function and used a Lévy- Khintchine representation for the infinitely divisible distributions. The second definition is the definition related to the stability property of independent and identically distributed random variables, and the third is the generalized central limit theorem, in which the stable distributions appear as the end of a set of independent and identically distributed random variables without imposing the condition contained in the central limit theorem [4], which revolves around the limitation of variance. A recent and condensed overview of the theory of stable distributions can be found in [6–12].

Fuzzy logic can be generalized to Neutrosophic logic by adding the component of indeterminacy.

In probability theory, F. Smarandache defined the neutrosophic probability measure and the probability function. Some researchers introduced many other concepts through the neutrosophic concept such as queuing theory, time series prediction, and modeling in many cases such as linear models, moving averages, and logarithmic models, more information can be founded at [13–23].

In this paper, depending on the geometric isometry (AH-Isometry) [20] (Under publication in Neutrosophic Sets and Systems), the concept of a stable neutrosophic random variable is introduced and provides two definitions of a neutrosophic stable random variable. We also presented some basic properties and present several well-known examples.

2. Preliminaries

2.1. α -Stable distributions

Definition 2.1.1. A random variable X (which is non-degenerate) is said to have a stable distribution if for any positive numbers A and B, there is a positive number C and a real number D such that

$$AX_1 + BX_2 \overset{d}{=} CX + D,$$

where X_1, X_2 are independent copies of X, and where $\overset{d}{=}$ denotes equality in distribution. That X is called strictly stable if the relation $AX_1 + BX_2 \overset{d}{=} CX + D$ hold with $D = 0$.

Dedinition 2.1.2. (*equivalent to definition 2.1*). A random variable X (which is non-degenerate) is said to have a stable distribution if for any $n \geq 0$, there is a positive number C_n and a real number D_n such that

$$X_1 + X_2 + ... + X_n \overset{d}{=} C_n X + D_n,$$

where $X_1, X_2, ..., X_n$ are independent copies of X.

And X is called strictly stable if $X_1 + X_2 + ... + X_n \overset{d}{=} C_n X + D_n$ hold with $D_n = 0$.

Theorem 2.1.3. If $X_1 + X_2 + ... + X_n \overset{d}{=} C_n X$, C_n has the form $C_n = n^{1/\alpha}$. See [5,9] for a proof.

Theorem 2.1.4. If G is strictly stable with characteristic parameter α, then

$$A^{1/\alpha} X_1 + B^{1/\alpha} X_2 \overset{d}{=} \left(A + B\right)^{1/\alpha} X,$$

holds for all $A > 0,\ B > 0$. See [5] for a proof.

2.2. Neutrosophic Functions on $R(I)$

Depending on information in [20], here are some interesting facts :

Definition 2.2.1.

Let $R(I) = \{a + bI;\ a, b \in R\}$ where $I^2 = I$ be the neutrosophic field of reals. The one-dimensional isometry (AH-Isometry) is defined as follows: [19]

$$T : R(I) \to R \times R$$
$$T(a + bI) = (a, a + b).$$

Some properties of an algebraic isomorphism T :

1. T is bijective.
2. T is invertible by

$$T^{-1}: R \times R \to R(I)$$

$$T^{-1}(a,b) = a + (b-a)I$$

3.

$$T\left[\left(a+bI\right)+\left(c+dI\right)\right] = T\left(a+bI\right)+T\left(c+dI\right)$$

And

$$T\left[\left(a+bI\right).\left(c+dI\right)\right] = T\left(a+bI\right).T\left(c+dI\right).$$

And more can be found in [20].

3. Neutrosophic Stable Random Variables

Definition 3.1. Aneutrosophic random variable $X_N = X + YI$ is said to have a neutrosophic stable distribution if for any positive numbers $A_N = A_1 + A_2 I$ and $B_N = B_1 + B_2 I$, there is a positive number $C_N = C_1 + C_2 I$ and a number $D_N = D_1 + D_2 I$ such that

$$A_N X_N^{(1)} + B_N X_N^{(2)} \overset{d}{=} C_N X_N + D_N, \tag{1}$$

where $X_N^{(1)} = X_1 + Y_1 I$ and $X_N^{(2)} = X_2 + Y_2 I$ are independent copies of X_N, and where $"\overset{d}{=}"$ denotes equality in distribution.

Remark 3.1. The right hand side of (1) takes the form

$$C_N X_N + D_N = C_1 X + I[L\zeta - C_1 X] + D_N,$$

where $C_1 + C_2 = L,\ X + Y = \zeta$.

Proof By taking T for the left hand side of (1) we obtain

$$T\left[A_N X_N^{(1)} + B_N X_N^{(2)}\right] \overset{d}{=} (A_1, A_1 + A_2)(X_1, X_1 + Y_1) + (B_1, B_1 + B_2)(X_2, X_2 + Y_2),$$

$$\overset{d}{=} (A_1 X_1, (A_1 + A_2)(X_1 + Y_1)) + (B_1 X_2, (B_1 + B_2)(X_2 + Y_2)),$$

$$\overset{d}{=} (A_1 X_1 + B_1 X_2, (A_1 + A_2)(X_1 + Y_1) + (B_1 + B_2)(X_2 + Y_2)).$$

By taking T^{-1} for both sides we obtain

$$A_N X_N^{(1)} + B_N X_N^{(2)} \overset{d}{=} A_1 X_1 + B_1 X_2 + I[(A_1 + A_2)(X_1 + Y_1) + (B_1 + B_2)(X_2 + Y_2) - A_1 X_1 + B_1 X_2].$$

Since $A_1 X_1 + B_1 X_2 \overset{d}{=} C_1 X + D_1$, $(A_1 + A_2)(X_1 + Y_1) + (B_1 + B_2)(X_2 + Y_2) \overset{d}{=} (C_1 + C_2)(X + Y) + (D_1 + D_2)$ then

$$A_N X_N^{(1)} + B_N X_N^{(2)} \overset{d}{=} C_1 X + D_1 + I[(C_1 + C_2)(X + Y) + (D_1 + D_2) - (C_1 X + D_1)],$$

and $$A_N X_N^{(1)} + B_N X_N^{(2)} \overset{d}{=} C_1 X + I[(C_1 + C_2)(X+Y) - C_1 X] + D_1 + D_2 I \, .$$

Finally

$$C_N X_N + D_N = C_1 X + I[L\zeta - C_1 X] + D_N \, . \tag{2}$$

Definition 3.2. A neutrosophic stable random variable is called neutrosophic strictly stable if (1) holds with $D_N = 0_N = 0 + 0I$.

Definition 3.3. Aneutrosophic random variable $X_N = X + YI$ is referred to as neutrosophic stable if there exist constants $0 < A_N^{(n)} = A_1^{(n)} + A_2^{(n)} I$ and $B_N^{(n)} = B_1^{(n)} + B_2^{(n)} I$ such that

$$\sum_{i=1}^{n} X_N^{(n)} \overset{d}{=} B_N^{(n)} + A_N^{(n)} X_N \, , \tag{3}$$

where $X_N^{(1)}, X_N^{(2)}, \ldots$ are independent neutrosophic random variables each having the same distribution as X_N .

Again, if $B_N^{(n)} = 0_N$, then X_N in (3) is called neutrosophic strictly stable, i.e.

$$\sum_{i=1}^{n} X_N^{(n)} \overset{d}{=} A_N^{(n)} X_N \, , \tag{4}$$

Theorem 3.1. In relation (4), the constant $A_N^{(n)}$ has the form

$$A_N^{(n)} = n^{1_N/\alpha_N}, \quad n^{1_N/\alpha_N} = n^{1/\alpha} + I\left(n^{1/\alpha} - n^{1/\alpha}\right) = n^{1/\alpha} + 0I, \quad \alpha_N = \alpha + \alpha I.$$

Proof Rewriting (4) as the sequence of sums

$$X_N^{(1)} + X_N^{(2)} \overset{d}{=} A_N^{(2)} X_N$$
$$X_N^{(1)} + X_N^{(2)} + X_N^{(3)} \overset{d}{=} A_N^{(3)} X_N$$
$$X_N^{(1)} + X_N^{(2)} + X_N^{(3)} + X_N^{(4)} \overset{d}{=} A_N^{(4)} X_N$$
$$\ldots \tag{5}$$

We cosider only those sums which contain 2^k terms, $k = 1, 2, \ldots$:

$$X_N^{(1)} + X_N^{(2)} \overset{d}{=} A_N^{(2)} X_N$$
$$X_N^{(1)} + X_N^{(2)} + X_N^{(3)} + X_N^{(4)} \overset{d}{=} A_N^{(4)} X_N$$
$$X_N^{(1)} + X_N^{(2)} + X_N^{(3)} + X_N^{(4)} + X_N^{(5)} + X_N^{(6)} + X_N^{(7)} + X_N^{(8)} \overset{d}{=} A_N^{(8)} X_N$$
$$\ldots$$
$$X_N^{(1)} + X_N^{(2)} + \ldots + X_N^{(2^k - 1)} + X_N^{(2^k)} \overset{d}{=} A_N^{(2^k)} X_N$$
$$\ldots$$

Making use the first formula, we transform the second one as follows:

$$S_N^{(4)} = \left(X_N^{(1)} + X_N^{(2)}\right) + \left(X_N^{(3)} + X_N^{(4)}\right) \overset{d}{=} A_N^{(2)}\left(X_N^{(1)} + X_N^{(2)}\right) \overset{d}{=} \left(A_N^{(2)}\right)^2 X_N .$$

Here we keep in mind that $X_N^{(1)} + X_N^{(2)} \overset{d}{=} X_N^{(3)} + X_N^{(4)}$. Applying this reasoning to the third formula, we obtain

$$\begin{aligned} S_N^{(8)} &= \left(X_N^{(1)} + X_N^{(2)}\right) + \left(X_N^{(3)} + X_N^{(4)}\right) + \left(X_N^{(5)} + X_N^{(6)}\right) + \left(X_N^{(7)} + X_N^{(8)}\right) \\ &\overset{d}{=} A_N^{(2)}\left(X_N^{(1)} + X_N^{(2)}\right) + A_N^{(2)}\left(X_N^{(5)} + X_N^{(6)}\right) \\ &\overset{d}{=} \left(A_N^{(2)}\right)^2 X_N^{(1)} + \left(A_N^{(2)}\right)^2 X_N^{(5)} \\ &\overset{d}{=} \left(A_N^{(2)}\right)^2 \left(X_N^{(1)} + X_N^{(5)}\right) = \left(A_N^{(2)}\right)^3 X_N . \end{aligned}$$

For the sum of 2^k terms, we similarly obtain

$$S_N^{(2^k)} \overset{d}{=} A_N^{(2^k)} X_N \overset{d}{=} A_N^{(k)} X_N .$$

Comparing this with (4), with $n = 2^k$, we obtain:

$$A_N^{(n)} = \left(A_N^{(2)}\right)^k = \left(A_N^{(2)}\right)^{(\log n)/\log 2};$$

hence

$$\log A_N^{(n)} = [(\log n) / \log 2] \log A_N^{(2)} = \log n^{(\log A_N^{(2)})/\log 2} .$$

Thus, for the sequence of sums we obtain

$$A_N^{(n)} = n^{1/(\alpha_N^{(2)})}, \quad \alpha_N^{(2)} = \log 2 / \log A_N^{(2)}, \quad n=2^k, \quad k = 1, 2, \dots . \tag{6}$$

Choosing now from (5) those sums which contain 3^k terms, and repeating the above reasoning, we arrive at

$$A_N^{(n)} = n^{1/(\alpha_N^{(3)})}, \quad \alpha_N^{(3)} = \log 3 / \log A_N^{(3)}, \quad n = 3^k, \quad k = 1, 2, \dots \tag{7}$$

In the general case,

$$A_N^{(n)} = n^{1/(\alpha_N^{(m)})}, \quad \alpha_N^{(m)} = \log m / \log A_N^{(m)}, \quad n=m^k, \quad k = 1, 2, \dots \tag{8}$$

We set $m = 4$. By virtue of (8),

$$\alpha_N^{(4)} = \log 4 / \log A_N^{(4)},$$

whereas (6) with $k = 2$ yields

$$\log A_N^{(4)} = \left(1_N / \alpha_N^{(2)}\right) \log 4.$$

Comparing the two last formulae, we conclude that

$$\alpha_N^{(2)} = \alpha_N^{(4)}.$$

By induction, we come to the conlusion that all $\alpha_N^{(m)}$ are equal to each other:

$$\alpha_N^{(m)} = \alpha_N.$$

The following expression hence holds for the scale factors $A_N^{(n)}$:

$$A_N^{(n)} = n^{1_N/\alpha_N}, \qquad n=1,2,3,... \tag{9}$$

whereas (4) takes the form

$$S_N^{(n)} = \sum_{i=1}^{n} X_N^{(n)} \overset{d}{=} n^{1_N/\alpha_N} X_N, \tag{10}$$

$$T\left(n^{1_N/\alpha_N}\right) = T\left(n^{\frac{1+I}{(\alpha+\alpha I)}}\right) = T\left(n+0I\right)^{T\left(\frac{1+I}{\alpha+\alpha I}\right)} = (n,n)^{\frac{(1,2)}{(\alpha,2\alpha)}}$$

$$=(n,n)^{(1/\alpha,2/(2\alpha))} = \left(n^{1/\alpha}, n^{1/\alpha}\right).$$

By taking T^{-1} for both sides of the last relation, then

$$n^{1_N/\alpha_N} = n^{1/\alpha} + I\left(n^{1/\alpha} - n^{1/\alpha}\right). \qquad \square$$

Remark 3.2. The right hand side of the relation (4) takes the form

$$n^{1_N/\alpha_N} X_N = n^{1/\alpha} X + I\left(n^{1/\alpha}(X+Y) - n^{1/\alpha} X\right).$$

In fact

$$T\left(n^{1_N/\alpha_N} X_N\right) = T\left(n^{(\alpha+\alpha I)^{-1}(1_N)}\right) T\left(X+YI\right) = T\left(n+0I\right)^{T(\alpha+\alpha I)^{T(-1)} T(1_N)} T\left(X+YI\right)$$

$$= (n,n)^{(\alpha,2\alpha)^{(-1,-1)}(1,2)} \left(X, X+Y\right) = (n,n)^{(1/\alpha,1/2\alpha))(1,2)} \left(X, X+Y\right)$$

$$= \left(n^{1/\alpha}, n^{2/2\alpha}\right)\left(X, X+Y\right) = \left(n^{1/\alpha} X, n^{1/\alpha}(X+Y)\right).$$

Note that $1_N = 1 + I$, and in the neutrosophic field: $\frac{1_N}{\alpha_N} = 1_N . \alpha_N^{-1}$.

By taking T^{-1} for both sides of the last relation, the proof will be completed.

Let us prove the relation $\frac{1_N}{\alpha_N} = 1_N . \alpha_N^{-1}$ in the general case where $\alpha_N = \alpha_1 + \alpha_2 I$:

$$\frac{1_N}{\alpha_N}=\frac{1+I}{\alpha_1+\alpha_2 I}\Rightarrow T\left(\frac{1_N}{\alpha_N}\right)=T\left(\frac{1+I}{\alpha_1+\alpha_2 I}\right)=\frac{(1,2)}{(\alpha_1,\alpha_1+\alpha_2)}=\left(\frac{1}{\alpha_1},\frac{2}{\alpha_1+\alpha_2}\right),$$

$$1_N.\alpha_N^{-1}=(1+I)(\alpha_1+\alpha_2 I)^{-1}\Rightarrow T\left[(1+I)(\alpha_1+\alpha_2 I)^{-1}\right]=T(1+I)T(\alpha_1+\alpha_2 I)^{T(-1)}.$$

$$=(1,2)(\alpha_1,\alpha_1+\alpha_2)^{(-1,-1)}=(1,2)\left(\frac{1}{\alpha_1},\frac{1}{\alpha_1+\alpha_2}\right)=\left(\frac{1}{\alpha_1},\frac{2}{\alpha_1+\alpha_2}\right).$$

Theorem 3.2. If G_N is a neutrosophic strictly stable distribution with characteristic parameter $\alpha_N=\alpha+\alpha I$ then

$$(A)^{1_N/\alpha_N}X_N^{(1)}+(B)^{1_N/\alpha_N}X_N^{(2)}\overset{d}{=}(A+B)^{1/\alpha}X_N,$$

for $A,\ B>0$.

Proof By recognizing the relation (10), for any positive numbers $A,\ B$, let $X_N^{(i)}$, $i=\overbrace{1,2,\ldots,A}^{A\text{ number}},\overbrace{A+1,\ldots,n}^{\text{B number}}$. be neutrosophic strictly stable random variables.

Then, we have $S_N^{(A)}=\sum_{i=1}^{A}X_N^{(i)}$, $S_N^{(B)}=\sum_{i=A+1}^{n}X_N^{(i)}$, and $S_N^{(A+B)}=\sum_{i=1}^{A+B}X_N^{(i)}$, hence

$$S_N^{(A)}=\sum_{i=1}^{A}X_N^{(i)}\overset{d}{=}A^{1/\alpha}X_N^{(1)},\ S_N^{(B)}=\sum_{i=A+1}^{n}X_N^{(i)}\overset{d}{=}B^{1/\alpha}X_N^{(2)},\text{ and }S_N^{(A+B)}=\sum_{i=A+1}^{A+B}X_N^{(i)}\overset{d}{=}(A+B)^{1/\alpha}X_N.$$

Since $S_N^{(A)}+S_N^{(B)}=S_N^{(A+B)}$, then $(A)^{1_N/\alpha_N}X_N^{(1)}+(B)^{1_N/\alpha_N}X_N^{(2)}\overset{d}{=}(A+B)^{1/\alpha}X_N$. ⊔

The neutrosophic convolution

Let X_N be a neutrosophic random variable, its neutrosophic density function is $f_{X_N}(x_N)$. We stand for the neutrosophic probability distribution function by $F_{X_N}(x_N)$ and we define it as

$$F_{X_N}(x_N)=\mathrm{P}(X_N\le x_N)=\int_{-\infty_N}^{x_N}f_N(t_N)\,dt_N.$$

What the right hand side form is?

Suppose that $X_N=X+YI$, and the probability density functions of $X,\ Y$ are $f,\ g$ respectively. By taking T for both sides, we obtain

$$T\left(F_{X_N}(x_N)\right)=T\left(\int_{-\infty_N}^{x_N}f_N(t_N)\,dt_N\right). \tag{11}$$

$$T\left(\int_{-\infty_N}^{x_N}f_N(t_N)\,dt_N\right)\equiv T\left(\int_{-\infty_N}^{x_N}\right)T\left(f_N(t_N)\right)T\left(d(t_N)\right). \tag{12}$$

By taking $T\left(\int_{-\infty_N}^{x_N}\right) \equiv T\left(\int_{-\infty-\infty I}^{x+yI}\right) = \left(\left(\int_{-\infty}^{x}\right),\left(\int_{-\infty}^{x+y}\right)\right)$, $T\left(f_N(t_N)\right) \equiv \left(f_X(t_1),(f*g)_{X+Y}(t_1+t_2)\right)$,

and $T\left(d(t_N)\right) = T\left(d\left(t_1+t_2 I\right)\right) = T\left(dt_1+Idt_2\right) = \left(dt_1,dt_1+dt_2\right) = \left(dt_1,d(t_1+t_2)\right)$.

Hence, the right hand side of (12) becomes

$$T\left(\int_{-\infty_N}^{x_N}\right)T\left(f_N(t_N)\right)T\left(d(t_N)\right) = \left(\left(\int_{-\infty}^{x}\right),\left(\int_{-\infty}^{x+y}\right)\right)\left(f_X(t_1),(f*g)_{X+Y}(t_1+t_2)\right)\left(dt_1,d(t_1+t_2)\right)$$

$$= \left(\left(\int_{-\infty}^{x} f_X(t_1)dt_1\right),\left(\int_{-\infty}^{x+y}(f*g)_{X+Y}(t_1+t_2)d(t_1+t_2)\right)\right).$$

Now, the relation (11) becomes

$$T\left(F_{X_N}(x_N)\right) = \left(\left(\int_{-\infty}^{x} f_X(t_1)dt_1\right),\left(\int_{-\infty}^{x+y}(f*g)_{X+Y}(t_1+t_2)d(t_1+t_2)\right)\right)$$

By taking T^{-1} for both sides, we obtain

$$F_{X_N}(x_N) = \int_{-\infty}^{x} f_X(t_1)dt_1 + I\left(\int_{-\infty}^{x+y}(f_* g)_{X+Y}(t_1+t_2)d(t_1+t_2) - \int_{-\infty}^{x} f_X(t_1)dt_1\right). \tag{13}$$

Definition 3.4. Suppose that X_N, Y_N are two independent neutrosophic random variables. $F_{X_N}(x_N)$, $G_{Y_N}(y_N)$ and $f_N = f_{X_N}(x_N)$, $g_N = g_{Y_N}(y_N)$ are their neutrosophic probability distribution functions and neutrosophic probability density functions respectively. The neutrosophic convolution of $F_N = F_{X_N}(x_N)$ and $G_N = G_{Y_N}(y_N)$ can be defined as

$$F_N *_N G_N = \int_{-\infty_N}^{x_N}\left(f_N *_N g_N\right) d\left(t_N\right), \tag{14}$$

where

$$f_N *_N g_N = \int_{-\infty_N}^{\infty_N} f_N\left(t_N - y_N\right) g_N\left(y_N\right) dy_N. \tag{15}$$

Theorem 3.3. According to the above hypotheses, the relations (14) and (15) hold, and (15) takes the form

$$f_{N *_N} g_N = (f_{X_1} * g_{Y_1})(t_1) + I\left\lfloor (f_{X_1+X_2} * g_{Y_1+Y_2})(t_1+t_2) - (f_{X_1} * g_{Y_1})(t_1)\right\rfloor, \tag{16}$$

where $(f_{X_1+X_2} * g_{Y_1+Y_2})(t_1+t_2)$ is the convolution of the variables $X = X_1 + X_2$ and $Y = Y_1 + Y_2$.

Proof Because of the independence of X_N, Y_N:

$$F_N *_N G_N = \int_{-\infty_N}^{\infty_N}\int_{-\infty_N}^{\infty_N} f_N(x_N) g_N(y_N)\, d(x_N) d(y_N)$$

$$= \int_{-\infty_N}^{\infty_N} \left[\int_{-\infty_N}^{t_N - y_N} f_N(x_N)\, d(x_N) \right] g_N(y_N) d(y_N)$$

$$= \int_{-\infty_N}^{\infty_N} \left[\int_{-\infty_N}^{x_N} f_N(t_N - y_N)\, d(t_N) \right] g_N(y_N) d(y_N)$$

$$= \int_{-\infty_N}^{x_N} \left[\int_{-\infty_N}^{\infty_N} f_N(t_N - y_N)\, g_N(y_N) d(y_N) \right] d(t_N).$$

Prove the relation (16) is similar to prove (13). □

Based on the previous facts, the convolution can be generalized for n.

4. Applications

There are three fundamental and well-known examples of stable laws, let $q(x)$ is the probability density function of stable random variable X :

4.1. Gaussian Distribution

In (16), two classical convolutions are well-known for the gaussian distribution. Because of the independence and identically in distribution for stable random variables, $(f_{X_1+X_2} * g_{Y_1+Y_2})(t_1+t_2)$ becomes the convolution of four gaussian random variables with one dimensional. The same applies to the rest of the examples.

We have

$$q\left(x;a,\sigma^2\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-a)^2}{2\sigma^2} \right\}, \quad -\infty < x < \infty, \quad \sigma > 0.$$

Since (See [9])

$$q(x;a_1,\sigma_1) * q(x;a_2,\sigma_2) = q(x;a_1+a_2,\sqrt{\sigma_1^2+\sigma_2^2}),$$

$$q(x;a_1,\sigma_1) * q(x;a_2,\sigma_2) * q(x;a_3,\sigma_3) * q(x;a_4,\sigma_4) = q(x;a_1+a_2+a_3+a_4,\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}),$$

then

$$q_{N *_N} q_N = q(x;a_1+a_2,\sqrt{\sigma_1^2+\sigma_2^2}) + I\left\lfloor q(x;a_1+a_2+a_3+a_4,\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}) - q(x;a_1+a_2,\sqrt{\sigma_1^2+\sigma_2^2}) \right\rfloor.$$

4.2. Cauchy Distribution

Without losing generality, it is known that the convolution of a Cauchy probability density function with a scale parameter equal to one is

$$q(x) * q(x) = \frac{1}{2} q_{X_1+X_2}\left(\frac{x}{2}\right).$$

And

$$q(x)*q(x)*q(x)*q(x)=\frac{1}{4}q_{X_1+X_2+X_3+X_4}\left(\frac{x}{4}\right).$$

Then

$$q_{N*_N}q_N=\frac{1}{2}q_{X_1+X_2}\left(\frac{x}{2}\right)+I\left[\frac{1}{4}q_{X_1+X_2+X_3+X_4}\left(\frac{x}{4}\right)-\frac{1}{2}q_{X_1+X_2}\left(\frac{x}{2}\right)\right].$$

4.3. Lévy Distribution

We have for Lévy Distribution that

$$q(x)*q(x)=(1/4)q(x/4).$$

And

$$q(x)*q(x)=(1/4)q(x/4).$$
$$q(x)*q(x)*q(x)*q(x)=(1/16)q(x/16).$$

Then

$$q_{N*_N}q_N=(1/4)q(x/4)+I\left[(1/16)q(x/16)-(1/4)q(x/4)\right].$$

5. Conclusions

In this paper, we suggested some basic definitions of the neutrosophic stable random variable and generalize some of the main properties of the classical stable distributions to the neutrosophic field. We also defined both the neutrosophic probability distribution function and the neutrosophic probability density function, then we defined the convolution with the neutrosophic concept. Finally, we supported the article with three examples of stable distributions with the neutrosophical concept, which are famous distributions in classical stability. Later, we will extend the work in the field of neutrosophic stability and work to generalize and prove more profound facts.

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Neutrosophic Hypersoft Expert Set: Theory and Applications

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Abstract. Soft set-like models deal with single argument approximate functions while hypersoft set, an extension of the soft set, deals with multi-argument approximate functions. The soft set cannot handle situations when attributes are required to be further divided into disjoint attribute-valued sets. To overcome this situation, a hypersoft set has been developed. In different fields like decision making and medical diagnosis, many researchers developed models based on the soft set for the solution of many problems. But these models deal with only one expert who creates many problems for the users, primarily in designing questionnaires. To remove this discrepancy, we present a neutrosophic hypersoft expert set. This model not only solves the problem of dealing with one expert but also solves the problem of different parametric-valued sets parallel to different characteristics. In this study, we first introduce the concept of neutrosophic hypersoft expert sets, which is a amalgam of both structures i.e., neutrosophic set and hypersoft expert sets. Certain essential basic characteristics (i.e., subset, equal set, agree, disagree set, null set, whole relative set, and whole absolute set), aggregation operations (i.e., complement, restricted union, extended intersection, AND and OR), and results (i.e., idempotent, absorption, domination, identity, commutative, associative and distributive law) are discussed with examples. Some hybrid structures of the neutrosophic hypersoft expert set are developed with illustrated examples. In the end, a decision-making application is presented for the validity of the proposed theory.

Keywords: Soft Set; Soft Expert Set; Neutrosophic set; Hypersoft Set; Neutrosophic Hypersoft Expert Set.

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1. **Introduction**

In some real-life issues in professional and information systems where we have a situation to deal with the truth-membership along with the falsity-membership for a correct description of an object in an uncertain and an ambiguous environment. Smarandache [1–3] characterized neutrosophic set as a generalization of classical sets, fuzzy set, intuitionistic fuzzy set. Membership functions are use to define fuzzy sets [4], while membership and non-membership functions both are used for intuitionistic fuzzy sets [5] and are used for solving problems having the data of imprecise, indeterminacy and inconsistent. Neutrosophic set (NS) has wide applications in different fields like decision making, medical diagnosis, data bases, control theory and topology etc. Wang et al [6] introduced single valued neutrosophic set (SVNS) and presented its set operations and different properties. The use NS and its hybridized structures in various fields has been continuing quickly [7]- [32].

Molodtsov [34] constructed soft set by taking the advantage of parameterization tool. Rahman et al. [35, 36] conceptualized m-convexity (m-concavity) and (m, n)-convexity ((m, n)-concavity) on soft sets with some properties. Maji et al. [37] made development by introducing fuzzy soft set to solve parametrization problems with uncertainty. Many researchers [38]- [44] advanced this theory and used in many fields. Rahman et al. [45] conceptualized (m-n)-convexity(concavity)on fuzzy soft set with applications in first and second senses. Alkhazaleh et al. [46, 47] made extensions in soft set by introducing soft and fuzzy soft expert sets. They used these structures for applications in decision-making problems(DMPs). Ihsan et al. [48,49] conceptualized convexity on soft expert set and fuzzy soft expert set with certain properties. Broumi et al. [50] conceptualized intuitionistic fuzzy soft expert set and made its use in DMPs. Mehmet et al. [33] defined neutrosophic soft expert sets and applied it in DMPs.

In 2018, Smarandache [51] extended soft set to hypersoft set and used in daily life problems. In 2020, Saeed et al. [52] advanced this theory and explained its structures. In 2020, Rahman et al. [53], [54] worked on hypersoft set and introduced its some new structures like complex fuzzy hypersoft set. They also gave the concept of convexity (concavity) on it and proved its some basic properties. Ihsan et al. [58, 59] introduced the structures of hypersoft expert set and fuzzy hypersoft expert set with applications in DMPs. Kamaci and Saqlain [60] worked on n-ary fuzzy hypersoft expert set and applied in real life problem. Kamaci [61] gave hybrid structures of hypersoft set and rough set and applied in DMPs. He [62] introduced the structure of simplified neutrosophic multiplicative refined sets and their correlation coefficients with Application in medical pattern recognition. The neutrosophic soft set like structures have been investigated and applied in different fields like game theory and DMPs in [63–65].

Having motivation from [33]- [50], new notions of neutrosophic hypersoft expert set are developed and some hybrids of neutrosophic hypersoft expert set are established.

The remaining portion of the paper is constructed as: Section 2 describes the basic definitions of soft set, fuzzy set, intuitionistic fuzzy set, neutrosophic set, fuzzy soft expert set, hypersoft set and relevant definitions used in the proposed work. Section 3, presents notions of fuzzy hypersoft expert set, neutrosophic hypersoft expert set with properties. Section 4, describes the set theoretic operations of NHSES. Section 5, presents the basic properties and laws of

NHSES. Section 6 shows the hybrids of NHSES. Section 7, presents an application in decision making and section 8 contains the conclusions of the paper.

2. Preliminaries

In this portion, some elementary definitions are presented from the literature.
Suppose W be a set of experts and $\mathbf{O}$ be a set of opinions, $\mathbf{T} = \mathbf{F} \times W \times \mathbf{O}$. Taking $S \subseteq \mathbf{T}$ and Δ as a set of universe with $P(\Delta)$ is the power set of universe, while parameters set is $\mathbf{F}$.

Definition 2.1. [40] A set "F_z" is called a *fuzzy set* written as $F_z = \{(\hat{r}, B(\hat{r}))|\hat{r} \in \Delta\}$ with $B : \Delta \to \mathbb{I}$ and $B(\hat{r})$ represents the membership value of $\hat{r} \in F_z$.

Definition 2.2. [41] A set " $\mathcal{J}$" is called an *intuitionistic fuzzy set* written as $\mathcal{J} = \{(\check{a}, < Z_{\mathcal{J}}(\check{a}), X_{\mathcal{J}}(\check{a}) >), |(\check{a}) \in \Delta\}$ with $Z_{\mathcal{J}} : \mathbb{I} \to \Delta$, $X_{\mathcal{J}} : \mathbb{I} \to \Delta$ and $Z_{\mathcal{J}}(\check{a})$, $X_{\mathcal{J}}(\check{a})$ represent the truth, falsity membership functions of $\check{a} \in \Delta$ satisfying the inequality $0 \leq Z_{\mathcal{J}}(\check{a}) + X_{\mathcal{J}}(\check{a}) \leq 1$.

Definition 2.3. [39] A neutrosophic set $\mathbf{N}$ in Δ is defined by
$N = \{< y, (T_N(y), I_N(y), F_N(y)) >: y \in \mathbf{F}, T_N, I_N, F_N \in]^-0, 1^+[\}$
where T_N, I_N, F_N are truth, indeterminacy, and falsity membership functions and are real standard or nonstandard subsets of $]^-0, 1^+[$. Their sum does not have any restriction, that is, $0^- \leq T_N(y), I_N(y), F_N(y) \leq 3^+$. Here $]^-0, 1^+[$ is named the nonstandard subset, which is the extension of real standard subsets $[0, 1]$ where the nonstandard number $1^+ = 1 + \epsilon$, 1 is named the standard part, and ϵ is named the nonstandard part. $^-0 = 0 - \epsilon$, 0 is the standard part and ϵ is named the nonstandard part, where ϵ is closed to positive real number zero.

Definition 2.4. [34] A pair $(\Psi_M, \mathbf{F})$ is named as *soft set* and Ψ_M is characterized by a mapping

$$\Psi_M : \mathbf{F} \to P(\Delta)$$

where $P(\Delta)$ is the power set of universe of discourse.

Definition 2.5. [37] Let $C \subseteq \mathbf{F}$. A fuzzy soft set is a pair (R, C) and R is characterized as

$$R : C \to I^{\Delta}$$

where I^{Δ} represents the collection of all fuzzy subsets of Δ.

Definition 2.6. [46] A soft expert set is a pair (Φ_H, S) with Φ_H is characterized by a mapping

$$\Phi_H : S \to P(\Delta)$$

where $S \subseteq \mathbf{F} \times W \times \mathbf{O}$.

Definition 2.7. [47] A fuzzy soft expert set is a pair (Ψ_F, S) with Ψ_F is characterized by a mapping

$$\Psi_F : S \rightarrow I^{\Delta}$$

where $S \subseteq \mathbf{F} \times W \times \mathbf{O}$ and I^{Δ} represents the collection of all fuzzy subsets of Δ.

Definition 2.8. [6] Let $\mathcal{K}_{\eth}$,$\mathcal{L}_{\eth}$ and $\mathcal{M}_{\eth}$ represent truth, indeterminacy and falsity membership functions, then $\eth$ represents a single valued neutrosophic set such that $0 \leq \mathcal{K}_{\eth}(\beta) + \mathcal{L}_{\eth}(\beta) + \mathcal{M}_{\eth}(\beta) \leq 3$. While $\mathcal{K}_{\eth}$, $\mathcal{L}_{\eth}$, $\mathcal{M}_{\eth} \in [0, 1]$ for all β in Δ.

Definition 2.9. [51]
Let $\ltimes_1, \ltimes_2, \ltimes_3, ..., \ltimes_\epsilon$, with $\epsilon \geq 1$, be ϵ different characters having parrallel characteristics values are the sets $\bar{\barwedge}_1, \bar{\barwedge}_2, \bar{\barwedge}_3, ..., \bar{\barwedge}_\epsilon$, with $\bar{\barwedge}_p \cap \bar{\barwedge}_q = \emptyset$, for $p \neq q$, and $p, q \in \{1, 2, 3, ..., \epsilon\}$. Then hypersoft expert set is a pair (Υ, L) with Υ is characterized by a mapping

$$\Upsilon : L \rightarrow P(\Delta)$$

where $L = \bar{\barwedge}_1 \times \bar{\barwedge}_2 \times \bar{\barwedge}_3 \times ..... \times \bar{\barwedge}_\epsilon$.

3. Neutrosophic Hypersoft Expert Set (NHSES)

In this portion, neutrosophic hypersoft expert set has been developed with the help of existing concept of neutrosophic soft expert set and some basic properties are presented.

Definition 3.1. [59] Fuzzy Hypersoft Expert Set (FHSES)
A pair $(\pounds, \mathcal{F})$ represents a FHSES with $\pounds$ is characterized by a mapping

$$\xi : \mathcal{F} \rightarrow I^{\Delta}$$

- I^{Δ} is being used a collection of all fuzzy subsets of Δ
- $\mathcal{F} \subseteq \mathcal{H} = \text{Æ} \times \text{Œ} \times \text{Å}$
- $\text{Æ} = \text{Æ}_1 \times \text{Æ}_2 \times \text{Æ}_3 \times .... \times \text{Æ}_p$ where Æ_i are different characteristics-valued sets parallel to different characteristics $\text{æ}_i, i = 1, 2, 3, ..., p$
- Œ represents an expert set
- Å represents a conclusion set.

Definition 3.2. Neutrosophic Hypersoft Expert Set (NHSES)
A *neutrosophic hypersoft expert set* represents a pair $(\hbar, \mathbb{G})$ if

$$\hbar : \mathbb{G} \rightarrow NF^{\Delta}$$

with NF^{Δ} is being used as collection of all neutrosophic subsets of Δ and $\text{Æ} \subseteq \mathcal{H} = \text{Æ} \times \text{Œ} \times \text{Å}$.

Example 3.3. Assume that a worldwide organization expects to continue the assessment of specific experts about its sure items. Let $\Delta = \{w_1, w_2, w_3, w_4\}$ be a set of products and $Æ_1 = \{p_{11}, p_{12}\}$, $Æ_2 = \{p_{21}, p_{22}\}$, $Æ_3 = \{p_{31}, p_{32}\}$, be different characteristics sets for different characteristics p_1= simple to use, p_2= nature, p_3= modest. Now $Æ = Æ_1 \times Æ_2 \times Æ_3$

$$Æ = \left\{ \begin{array}{l} \upsilon_1 = (p_{11}, p_{21}, p_{31}), \upsilon_2 = (p_{11}, p_{21}, p_{32}), \upsilon_3 = (p_{11}, p_{22}, p_{31}), \upsilon_4 = (p_{11}, p_{22}, p_{32}), \\ \upsilon_5 = (p_{12}, p_{21}, p_{31}), \upsilon_6 = (p_{12}, p_{21}, p_{32}), \upsilon_7 = (p_{12}, p_{22}, p_{31}), \upsilon_8 = (p_{12}, p_{22}, p_{32}) \end{array} \right\}$$

Now $\mathcal{H} = Æ \times Œ \times Å$.

$$\mathcal{H} = \left\{ \begin{array}{l} (\upsilon_1, c, 0), (\upsilon_1, c, 1), (\upsilon_1, d, 0), (\upsilon_1, d, 1), (\upsilon_1, e, 0), (\upsilon_1, e, 1), (\upsilon_2, c, 0), (\upsilon_2, c, 1), \\ (\upsilon_2, d, 0), (\upsilon_2, d, 1), (\upsilon_2, e, 0), (\upsilon_2, e, 1), (\upsilon_3, c, 0), (\upsilon_3, c, 1), (\upsilon_3, d, 0), (\upsilon_3, d, 1), \\ (\upsilon_3, e, 0), (\upsilon_3, e, 1), (\upsilon_4, c, 0), (\upsilon_4, c, 1), (\upsilon_4, d, 0), (\upsilon_4, d, 1), (\upsilon_4, e, 0), (\upsilon_4, e, 1), \\ (\upsilon_5, c, 0), (\upsilon_5, c, 1), (\upsilon_5, d, 0), (\upsilon_5, d, 1), (\upsilon_5, e, 0), (\upsilon_5, e, 1), (\upsilon_6, c, 0), (\upsilon_6, c, 1), \\ (\upsilon_6, d, 0), (\upsilon_6, d, 1), (\upsilon_6, e, 0), (\upsilon_6, e, 1), (\upsilon_7, c, 0), (\upsilon_7, c, 1), (\upsilon_7, d, 0), (\upsilon_7, d, 1), \\ (\upsilon_7, e, 0), (\upsilon_7, e, 1), (\upsilon_8, c, 0), (\upsilon_8, c, 1), (\upsilon_8, d, 0), (\upsilon_8, d, 1), (\upsilon_8, e, 0), (\upsilon_8, e, 1) \end{array} \right\}.$$

let

$$\mathbb{G} = \left\{ \begin{array}{l} (\upsilon_1, c, 0), (\upsilon_1, c, 1), (\upsilon_1, d, 0), (\upsilon_1, d, 1), (\upsilon_1, e, 0), (\upsilon_1, e, 1), \\ (\upsilon_2, c, 0), (\upsilon_2, c, 1), (\upsilon_2, d, 0), (\upsilon_2, d, 1), (\upsilon_2, e, 0), (\upsilon_2, e, 1), \\ (\upsilon_3, c, 0), (\upsilon_3, c, 1), (\upsilon_3, d, 0), (\upsilon_3, d, 1), (\upsilon_3, e, 0), (\upsilon_3, e, 1), \end{array} \right\}$$

be a subset of $\mathcal{H}$ and $Œ = \{c, d, e, \}$ be a set of specialists.

Following check relates the varieties of three specialists:

$$\hbar_1 = \hbar(\upsilon_1, c, 1) = \left\{ \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.7,0.2,0.5>}, \frac{w_3}{<0.5,0.4,0.6>}, \frac{w_4}{<0.1,0.3,0.6>} \right\},$$

$$\hbar_2 = \hbar(\upsilon_1, d, 1) = \left\{ \frac{w_1}{<0.4,0.2,0.3>}, \frac{w_2}{<0.8,0.1,0.5>}, \frac{w_3}{<0.4,0.5,0.6>}, \frac{w_4}{<0.2,0.5,0.3>} \right\},$$

$$\hbar_3 = \hbar(\upsilon_1, e, 1) = \left\{ \frac{w_1}{<0.7,0.2,0.3>}, \frac{w_2}{<0.5,0.3,0.6>}, \frac{w_3}{<0.6,0.3,0.7>}, \frac{w_4}{<0.3,0.5,0.6>} \right\},$$

$$\hbar_4 = \hbar(\upsilon_2, c, 1) = \left\{ \frac{w_1}{<0.9,0.1,0.3>}, \frac{w_2}{<0.4,0.5,0.4>}, \frac{w_3}{<0.7,0.2,0.6>}, \frac{w_4}{<0.3,0.4,0.8>} \right\},$$

$$\hbar_5 = \hbar(\upsilon_2, d, 1) = \left\{ \frac{w_1}{<0.4,0.5,0.6>}, \frac{w_2}{<0.8,0.1,0.7>}, \frac{w_3}{<0.3,0.6,0.5>}, \frac{w_4}{<0.2,0.6,0.7>} \right\},$$

$$\hbar_6 = \hbar(\upsilon_2, e, 1) = \left\{ \frac{w_1}{<0.5,0.4,0.7>}, \frac{w_2}{<0.3,0.6,0.4>}, \frac{w_3}{<0.6,0.2,0.5>}, \frac{w_4}{<0.8,0.1,0.6>} \right\},$$

$$\hbar_7 = \hbar(\upsilon_3, c, 1) = \left\{ \frac{w_1}{<0.2,0.7,0.5>}, \frac{w_2}{<0.9,0.1,0.4>}, \frac{w_3}{<0.4,0.5,0.7>}, \frac{w_4}{<0.5,0.4,0.8>} \right\},$$

$$\hbar_8 = \hbar(\upsilon_3, d, 1) = \left\{ \frac{w_1}{<0.4,0.3,0.2>}, \frac{w_2}{<0.6,0.3,0.1>}, \frac{w_3}{<0.7,0.2,0.3>}, \frac{w_4}{<0.9,0.1,0.4>} \right\},$$

$$\hbar_9 = \hbar(\upsilon_3, e, 1) = \left\{ \frac{w_1}{<0.7,0.2,0.6>}, \frac{w_2}{<0.3,0.5,0.7>}, \frac{w_3}{<0.5,0.4,0.5>}, \frac{w_4}{<0.2,0.7,0.8>} \right\},$$

$$\hbar_{10} = \hbar(\upsilon_1, c, 0) = \left\{ \frac{w_1}{<0.3,0.2,0.1>}, \frac{w_2}{<0.2,0.4,0.5>}, \frac{w_3}{<0.4,0.5,0.8>}, \frac{w_4}{<0.1,0.8,0.3>} \right\},$$

$$\hbar_{11} = \hbar(\upsilon_1, d, 0) = \left\{ \frac{w_1}{<0.1,0.8,0.4>}, \frac{w_2}{<0.9,0.1,0.2>}, \frac{w_3}{<0.6,0.3,0.4>}, \frac{w_4}{<0.2,0.7,0.5>} \right\},$$

$$\hbar_{12} = \hbar(\upsilon_1, e, 0) = \left\{ \frac{w_1}{<0.2,0.7,0.5>}, \frac{w_2}{<0.1,0.8,0.6>}, \frac{w_3}{<0.3,0.5,0.7>}, \frac{w_4}{<0.5,0.4,0.6>} \right\},$$

$$\hbar_{13} = \hbar(\upsilon_2, c, 0) = \left\{ \frac{w_1}{<0.8,0.1,0.6>}, \frac{w_2}{<0.3,0.6,0.7>}, \frac{w_3}{<0.5,0.4,0.8>}, \frac{w_4}{<0.7,0.2,0.9>} \right\},$$

$$\hbar_{14} = \hbar(\upsilon_2, d, 0) = \left\{ \frac{w_1}{<0.7,0.2,0.5>}, \frac{w_2}{<0.2,0.6,0.4>}, \frac{w_3}{<0.9,0.1,0.6>}, \frac{w_4}{<0.4,0.5,0.7>} \right\},$$

$$\hbar_{15} = \hbar(\upsilon_2, e, 0) = \left\{ \frac{w_1}{<0.6,0.2,0.5>}, \frac{w_2}{<0.7,0.2,0.4>}, \frac{w_3}{<0.3,0.5,0.4>}, \frac{w_4}{<0.2,0.7,0.6>} \right\},$$

$$\hbar_{16} = \hbar(\upsilon_3, c, 0) = \left\{ \frac{w_1}{<0.1,0.7,0.5>}, \frac{w_2}{<0.4,0.5,0.7>}, \frac{w_3}{<0.7,0.2,0.9>}, \frac{w_4}{<0.8,0.2,0.4>} \right\},$$

$$\hbar_{17} = \hbar(\upsilon_3, d, 0) = \left\{ \frac{w_1}{<0.2,0.7,0.4>}, \frac{w_2}{<0.9,0.1,0.6>}, \frac{w_3}{<0.8,0.2,0.4>}, \frac{w_4}{<0.3,0.5,0.7>} \right\},$$

$$\hbar_{18} = \hbar(\upsilon_3, e, 0) = \left\{ \frac{w_1}{<0.5,0.4,0.2>}, \frac{w_2}{<0.3,0.6,0.1>}, \frac{w_3}{<0.6,0.3,0.2>}, \frac{w_4}{<0.1,0.8,0.3>} \right\}.$$

The NHSES can be described as $(\hbar, \mathbb{G}) =$

$$\left\{ \begin{array}{l}
\left((\upsilon_1, c, 1), \left\{ \begin{array}{l} \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.7,0.2,0.5>}, \\ \frac{w_3}{<0.5,0.4,0.6>}, \frac{w_4}{<0.1,0.3,0.6>} \end{array} \right\} \right), \left((\upsilon_1, d, 1), \left\{ \begin{array}{l} \frac{w_1}{<0.4,0.2,0.3>}, \frac{w_2}{<0.8,0.1,0.5>}, \\ \frac{w_3}{<0.4,0.5,0.6>}, \frac{w_4}{<0.2,0.5,0.3>} \end{array} \right\} \right), \\
\left((\upsilon_1, e, 1), \left\{ \begin{array}{l} \frac{w_1}{<0.7,0.2,0.3>}, \frac{w_2}{<0.5,0.3,0.6>}, \\ \frac{w_3}{<0.6,0.3,0.7>}, \frac{w_4}{<0.3,0.5,0.6>} \end{array} \right\} \right), \left((\upsilon_2, c, 1), \left\{ \begin{array}{l} \frac{w_1}{<0.9,0.1,0.3>}, \frac{w_2}{<0.4,0.5,0.4>}, \\ \frac{w_3}{<0.7,0.2,0.6>}, \frac{w_4}{<0.3,0.4,0.8>} \end{array} \right\} \right), \\
\left((\upsilon_2, d, 1), \left\{ \begin{array}{l} \frac{w_1}{<0.4,0.5,0.6>}, \frac{w_2}{<0.8,0.1,0.7>}, \\ \frac{w_3}{<0.3,0.6,0.5>}, \frac{w_4}{<0.2,0.6,0.7>} \end{array} \right\} \right), \left((\upsilon_2, e, 1), \left\{ \begin{array}{l} \frac{w_1}{<0.5,0.4,0.7>}, \frac{w_2}{<0.3,0.6,0.4>}, \\ \frac{w_3}{<0.6,0.2,0.5>}, \frac{w_4}{<0.8,0.1,0.6>} \end{array} \right\} \right), \\
\left((\upsilon_3, c, 1), \left\{ \begin{array}{l} \frac{w_1}{<0.2,0.7,0.5>}, \frac{w_2}{<0.9,0.1,0.4>}, \\ \frac{w_3}{<0.4,0.5,0.7>}, \frac{w_4}{<0.5,0.4,0.8>} \end{array} \right\} \right), \left((\upsilon_3, d, 1), \left\{ \begin{array}{l} \frac{w_1}{<0.4,0.3,0.2>}, \frac{w_2}{<0.6,0.3,0.1>}, \\ \frac{w_3}{<0.7,0.2,0.3>}, \frac{w_4}{<0.9,0.1,0.4>} \end{array} \right\} \right), \\
\left((\upsilon_3, e, 1), \left\{ \begin{array}{l} \frac{w_1}{<0.7,0.2,0.6>}, \frac{w_2}{<0.3,0.5,0.7>}, \\ \frac{w_3}{<0.5,0.4,0.5>}, \frac{w_4}{<0.2,0.7,0.8>} \end{array} \right\} \right), \left((\upsilon_1, c, 0), \left\{ \begin{array}{l} \frac{w_1}{<0.3,0.2,0.1>}, \frac{w_2}{<0.2,0.4,0.5>}, \\ \frac{w_3}{<0.4,0.5,0.8>}, \frac{w_4}{<0.1,0.8,0.3>} \end{array} \right\} \right), \\
\left((\upsilon_1, d, 0), \left\{ \begin{array}{l} \frac{w_1}{<0.1,0.8,0.4>}, \frac{w_2}{<0.9,0.1,0.2>}, \\ \frac{w_3}{<0.6,0.3,0.4>}, \frac{w_4}{<0.2,0.7,0.5>} \end{array} \right\} \right), \left((\upsilon_1, e, 0), \left\{ \begin{array}{l} \frac{w_1}{<0.2,0.7,0.5>}, \frac{w_2}{<0.1,0.8,0.6>}, \\ \frac{w_3}{<0.3,0.5,0.7>}, \frac{w_4}{<0.5,0.4,0.6>} \end{array} \right\} \right), \\
\left((\upsilon_2, c, 0), \left\{ \begin{array}{l} \frac{w_1}{<0.8,0.1,0.6>}, \frac{w_2}{<0.3,0.6,0.7>}, \\ \frac{w_3}{<0.5,0.4,0.8>}, \frac{w_4}{<0.7,0.2,0.9>} \end{array} \right\} \right), \left((\upsilon_2, d, 0), \left\{ \begin{array}{l} \frac{w_1}{<0.7,0.2,0.5>}, \frac{w_2}{<0.2,0.6,0.4>}, \\ \frac{w_3}{<0.9,0.1,0.6>}, \frac{w_4}{<0.4,0.5,0.7>} \end{array} \right\} \right), \\
\left((\upsilon_2, e, 0), \left\{ \begin{array}{l} \frac{w_1}{<0.6,0.2,0.5>}, \frac{w_2}{<0.7,0.2,0.4>}, \\ \frac{w_3}{<0.3,0.5,0.4>}, \frac{w_4}{<0.2,0.7,0.6>} \end{array} \right\} \right), \left((\upsilon_3, c, 0), \left\{ \begin{array}{l} \frac{w_1}{<0.1,0.7,0.5>}, \frac{w_2}{<0.4,0.5,0.7>}, \\ \frac{w_3}{<0.7,0.2,0.9>}, \frac{w_4}{<0.8,0.2,0.4>} \end{array} \right\} \right), \\
\left((\upsilon_3, d, 0), \left\{ \begin{array}{l} \frac{w_1}{<0.2,0.7,0.4>}, \frac{w_2}{<0.9,0.1,0.6>}, \\ \frac{w_3}{<0.8,0.2,0.4>}, \frac{w_4}{<0.3,0.5,0.7>} \end{array} \right\} \right), \left((\upsilon_3, e, 0), \left\{ \begin{array}{l} \frac{w_1}{<0.5,0.4,0.2>}, \frac{w_2}{<0.3,0.6,0.1>}, \\ \frac{w_3}{<0.6,0.3,0.2>}, \frac{w_4}{<0.1,0.8,0.3>} \end{array} \right\} \right),
\end{array} \right\}.$$

Definition 3.4. Neutrosophic Hypersoft Expert Subset

A NHSES $(\hbar_1, \mathbb{G})$ is said to be NHSE subset of $(\hbar_2, \mathbb{P})$, if

(i) $\mathbb{G} \subseteq \mathbb{P}$, (ii) $\forall\ \gamma \in \mathbb{G}, \hbar_1(\gamma) \subseteq \hbar_2(\gamma)$ and denoted by $(\hbar_1, \mathbb{G}) \subseteq (\hbar_2, \mathbb{P})$.

Example 3.5. Considering Example 3.3, with two NHSESs

$$\mathbb{G}_1 = \Big\{ (\upsilon_1, c, 1), (\upsilon_3, c, 0), (\upsilon_1, d, 1), (\upsilon_3, d, 1), (\upsilon_3, d, 0), (\upsilon_1, e, 0), (\upsilon_3, e, 1) \Big\}$$

$$\mathbb{G}_2 = \Big\{ (\upsilon_1, c, 1), (\upsilon_3, c, 0), (\upsilon_3, c, 1), (\upsilon_1, d, 1), (\upsilon_3, d, 1), (\upsilon_1, d, 0), (\upsilon_3, d, 0), (\upsilon_1, e, 0), (\upsilon_3, e, 1), (\upsilon_1, e, 1) \Big\}.$$

It is clear that $\mathbb{G}_1 \subset \mathbb{G}_2$. Suppose $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ be defined as following

$$(\hbar_1,\mathbb{G}_1)=\left\{\begin{array}{l}
\left((\upsilon_1,c,1),\left\{\frac{w_1}{<0.1,0.6,0.7>},\frac{w_2}{<0.6,0.5,0.8>},\frac{w_3}{<0.4,0.6,0.9>},\frac{w_4}{<0.1,0.8,0.6>}\right\}\right),\\
\left((\upsilon_1,d,1),\left\{\frac{w_1}{<0.3,0.4,0.5>},\frac{w_2}{<0.6,0.4,0.6>},\frac{w_3}{<0.2,0.5,0.7>},\frac{w_4}{<0.1,0.5,0.6>}\right\}\right),\\
\left((\upsilon_3,d,1),\left\{\frac{w_1}{<0.2,0.6,0.4>},\frac{w_2}{<0.5,0.4,0.7>},\frac{w_3}{<0.6,0.5,0.8>},\frac{w_4}{<0.8,0.6,0.4>}\right\}\right),\\
\left((\upsilon_3,e,1),\left\{\frac{w_1}{<0.6,0.4,0.3>},\frac{w_2}{<0.2,0.7,0.6>},\frac{w_3}{<0.4,0.5,0.3>},\frac{w_4}{<0.1,0.7,0.4>}\right\}\right),\\
\left((\upsilon_1,e,0),\left\{\frac{w_1}{<0.1,0.6,0.3>},\frac{w_2}{<0.1,0.7,0.4>},\frac{w_3}{<0.2,0.7,0.6>},\frac{w_4}{<0.1,0.6,0.7>}\right\}\right),\\
\left((\upsilon_3,c,0),\left\{\frac{w_1}{<0.1,0.8,0.6>},\frac{w_2}{<0.3,0.6,0.5>},\frac{w_3}{<0.6,0.3,0.4>},\frac{w_4}{<0.7,0.2,0.6>}\right\}\right),\\
\left((\upsilon_3,d,0),\left\{\frac{w_1}{<0.1,0.7,0.4>},\frac{w_2}{<0.6,0.3,0.6>},\frac{w_3}{<0.7,0.2,0.5>},\frac{w_4}{<0.2,0.7,0.4>}\right\}\right)
\end{array}\right\}$$

$$(\hbar_2,\mathbb{G}_2)=\left\{\begin{array}{l}
\left((\upsilon_1,c,1),\left\{\frac{w_1}{<0.2,0.3,0.6>},\frac{w_2}{<0.7,0.4,0.7>},\frac{w_3}{<0.5,0.4,0.8>},\frac{w_4}{<0.2,0.4,0.5>}\right\}\right),\\
\left((\upsilon_1,d,1),\left\{\frac{w_1}{<0.4,0.3,0.4>},\frac{w_2}{<0.8,0.3,0.5>},\frac{w_3}{<0.4,0.3,0.6>},\frac{w_4}{<0.2,0.6,0.5>}\right\}\right),\\
\left((\upsilon_3,c,1),\left\{\frac{w_1}{<0.1,0.3,0.4>},\frac{w_2}{<0.9,0.1,0.3>},\frac{w_3}{<0.4,0.5,0.4>},\frac{w_4}{<0.5,0.3,0.4>}\right\}\right),\\
\left((\upsilon_3,d,1),\left\{\frac{w_1}{<0.4,0.2,0.3>},\frac{w_2}{<0.6,0.3,0.6>},\frac{w_3}{<0.7,0.4,0.5>},\frac{w_4}{<0.9,0.5,0.2}\right\}\right),\\
\left((\upsilon_1,e,1),\left\{\frac{w_1}{<0.7,0.2,0.4>},\frac{w_2}{<0.5,0.2,0.6>},\frac{w_3}{<0.6,0.2,0.7>},\frac{w_4}{<0.3,0.5,0.8}\right\}\right),\\
\left((\upsilon_3,e,1),\left\{\frac{w_1}{<0.7,0.3,0.1>},\frac{w_2}{<0.3,0.5,0.4>},\frac{w_3}{<0.5,0.4,0.2>},\frac{w_4}{<0.2,0.6,0.3>}\right\}\right),\\
\left((\upsilon_1,e,0),\left\{\frac{w_1}{<0.2,0.5,0.1>},\frac{w_2}{<0.2,0.6,0.3>},\frac{w_3}{<0.3,0.5,0.4>},\frac{w_4}{<0.5,0.3,0.5>}\right\}\right),\\
\left((\upsilon_1,d,0),\left\{\frac{w_1}{<0.1,0.6,0.4>},\frac{w_2}{<0.9,0.1,0.6>},\frac{w_3}{<0.6,0.3,0.8>},\frac{w_4}{<0.2,0.6,0.8>}\right\}\right),\\
\left((\upsilon_3,c,0),\left\{\frac{w_1}{<0.2,0.7,0.4>},\frac{w_2}{<0.4,0.5,0.3>},\frac{w_3}{<0.7,0.2,0.1>},\frac{w_4}{<0.8,0.1,0.5>}\right\}\right),\\
\left((\upsilon_3,d,0),\left\{\frac{w_1}{<0.2,0.5,0.1>},\frac{w_2}{<0.7,0.2,0.3>},\frac{w_3}{<0.8,0.2,0.4>},\frac{w_4}{<0.3,0.5,0.2>}\right\}\right)
\end{array}\right\}$$

which shows that $(\hbar_1, \mathbb{G}_1) \subseteq (\hbar_2, \mathbb{G}_2)$.

Definition 3.6. Two NHSESs $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ over Δ are said to be equal if $(\hbar_1, \mathbb{G}_1)$ is a NHSE subset of $(\hbar_2, \mathbb{G}_2)$ and $(\hbar_2, \mathbb{G}_2)$ is a neutrosophic hypersoft expert subset of $(\hbar_1, \mathbb{G}_1)$.

Definition 3.7. The complement of a NHSES is characterized by as
$(\hbar, \mathbb{G})^c = \tilde{c}(\hbar(\varsigma))\ \forall\ \varsigma \in \Delta$ while $\tilde{c}$ is a neutrosophic complement.

Example 3.8. Finding complement of NHSES find in 3.3, we have

$$(\hbar, \mathbb{G})^{c} = \left\{ \begin{array}{l}
\left((\upsilon_1, c, 1), \left\{\frac{w_1}{<0.4,0.5,0.2>}, \frac{w_2}{<0.5,0.8,0.7>}, \frac{w_3}{<0.6,0.6,0.5>}, \frac{w_4}{<0.6,0.7,0.1>}\right\}\right), \\
\left((\upsilon_1, d, 1), \left\{\frac{w_1}{<0.3,0.8,0.6>}, \frac{w_2}{<0.5,0.9,0.8>}, \frac{w_3}{<0.6,0.5,0.4>}, \frac{w_4}{<0.3,0.5,0.2>}\right\}\right), \\
\left((\upsilon_1, e, 1), \left\{\frac{w_1}{<0.3,0.8,0.7>}, \frac{w_2}{<0.6,0.7,0.5>}, \frac{w_3}{<0.7,0.7,0.6>}, \frac{w_4}{<0.6,0.5,0.3>}\right\}\right), \\
\left((\upsilon_2, c, 1), \left\{\frac{w_1}{<0.3,0.9,0.9>}, \frac{w_2}{<0.4,0.5,0.4>}, \frac{w_3}{<0.6,0.8,0.7>}, \frac{w_4}{<0.8,0.6,0.3>}\right\}\right), \\
\left((\upsilon_2, d, 1), \left\{\frac{w_1}{<0.6,0.5,0.4>}, \frac{w_2}{<0.7,0.9,0.8>}, \frac{w_3}{<0.5,0.4,0.3>}, \frac{w_4}{<0.7,0.4,0.2>}\right\}\right), \\
\left((\upsilon_2, e, 1), \left\{\frac{w_1}{<0.7,0.6,0.5>}, \frac{w_2}{<0.4,0.4,0.3>}, \frac{w_3}{<0.5,0.8,0.6>}, \frac{w_4}{<0.6,0.9,0.8>}\right\}\right), \\
\left((\upsilon_3, c, 1), \left\{\frac{w_1}{<0.5,0.3,0.2>}, \frac{w_2}{<0.4,0.9,0.9>}, \frac{w_3}{<0.7,0.5,0.4>}, \frac{w_4}{<0.8,0.6,0.5>}\right\}\right), \\
\left((\upsilon_3, d, 1), \left\{\frac{w_1}{<0.2,0.7,0.4>}, \frac{w_2}{<0.1,0.7,0.6>}, \frac{w_3}{<0.3,0.8,0.7>}, \frac{w_4}{<0.4,0.9,0.9>}\right\}\right), \\
\left((\upsilon_3, e, 1), \left\{\frac{w_1}{<0.6,0.8,0.7>}, \frac{w_2}{<0.7,0.5,0.3>}, \frac{w_3}{<0.5,0.6,0.5>}, \frac{w_4}{<0.8,0.3,0.2>}\right\}\right), \\
\left((\upsilon_1, c, 0), \left\{\frac{w_1}{<0.1,0.8,0.3>}, \frac{w_2}{<0.5,0.6,0.2>}, \frac{w_3}{<0.8,0.5,0.4>}, \frac{w_4}{<0.3,0.2,0.1>}\right\}\right), \\
\left((\upsilon_1, d, 0), \left\{\frac{w_1}{<0.4,0.2,0.1>}, \frac{w_2}{<0.2,0.9,0.9>}, \frac{w_3}{<0.4,0.7,0.6>}, \frac{w_4}{<0.5,0.3,0.2>}\right\}\right), \\
\left((\upsilon_1, e, 0), \left\{\frac{w_1}{<0.5,0.3,0.2>}, \frac{w_2}{<0.6,0.2,0.1>}, \frac{w_3}{<0.7,0.5,0.3>}, \frac{w_4}{<0.6,0.6,0.5>}\right\}\right), \\
\left((\upsilon_2, c, 0), \left\{\frac{w_1}{<0.6,0.9,0.8>}, \frac{w_2}{<0.7,0.4,0.3>}, \frac{w_3}{<0.8,0.6,0.5>}, \frac{w_4}{<0.9,0.8,0.7>}\right\}\right), \\
\left((\upsilon_2, d, 0), \left\{\frac{w_1}{<0.5,0.8,0.7>}, \frac{w_2}{<0.4,0.4,0.2>}, \frac{w_3}{<0.6,0.9,0.9>}, \frac{w_4}{<0.7,0.5,0.4>}\right\}\right), \\
\left((\upsilon_2, e, 0), \left\{\frac{w_1}{<0.5,0.8,0.6>}, \frac{w_2}{<0.4,0.8,0.7>}, \frac{w_3}{<0.4,0.5,0.3>}, \frac{w_4}{<0.6,0.3,0.2>}\right\}\right), \\
\left((\upsilon_3, c, 0), \left\{\frac{w_1}{<0.5,0.3,0.1>}, \frac{w_2}{<0.7,0.5,0.3>}, \frac{w_3}{<0.9,0.8,0.8>}, \frac{w_4}{<0.4,0.8,0.8>}\right\}\right), \\
\left((\upsilon_3, d, 0), \left\{\frac{w_1}{<0.4,0.3,0.2>}, \frac{w_2}{<0.6,0.9,0.9>}, \frac{w_3}{<0.4,0.8,0.8>}, \frac{w_4}{<0.7,0.5,0.3>}\right\}\right), \\
\left((\upsilon_3, e, 0), \left\{\frac{w_1}{<0.2,0.6,0.5>}, \frac{w_2}{<0.1,0.4,0.3>}, \frac{w_3}{<0.2,0.7,0.6>}, \frac{w_4}{<0.3,0.2,0.1>}\right\}\right)
\end{array} \right\}.$$

Definition 3.9. An agree-NHSES is described by as $(\hbar, \text{Æ})_{ag} = \{\hbar_{ag}(\varsigma) : \varsigma \in \text{Æ} \times \text{Œ} \times \{1\}\}$.

Example 3.10. Finding agree-NHSES calculated in 3.3, we get

$$(\hbar, \mathbb{G}) = \left\{ \begin{array}{l}
\left((\upsilon_1, c, 1), \left\{\frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.7,0.2,0.5>}, \frac{w_3}{<0.5,0.4,0.6>}, \frac{w_4}{<0.1,0.3,0.6>}\right\}\right), \\
\left((\upsilon_1, d, 1), \left\{\frac{w_1}{<0.4,0.2,0.3>}, \frac{w_2}{<0.8,0.1,0.5>}, \frac{w_3}{<0.4,0.5,0.6>}, \frac{w_4}{<0.2,0.5,0.3>}\right\}\right), \\
\left((\upsilon_1, e, 1), \left\{\frac{w_1}{<0.7,0.2,0.3>}, \frac{w_2}{<0.5,0.3,0.6>}, \frac{w_3}{<0.6,0.3,0.7>}, \frac{w_4}{<0.3,0.5,0.6>}\right\}\right), \\
\left((\upsilon_2, c, 1), \left\{\frac{w_1}{<0.9,0.1,0.3>}, \frac{w_2}{<0.4,0.5,0.4>}, \frac{w_3}{<0.7,0.2,0.6>}, \frac{w_4}{<0.3,0.4,0.8>}\right\}\right), \\
\left((\upsilon_2, d, 1), \left\{\frac{w_1}{<0.4,0.5,0.6>}, \frac{w_2}{<0.8,0.1,0.7>}, \frac{w_3}{<0.3,0.6,0.5>}, \frac{w_4}{<0.2,0.6,0.7>}\right\}\right), \\
\left((\upsilon_2, e, 1), \left\{\frac{w_1}{<0.5,0.4,0.7>}, \frac{w_2}{<0.3,0.6,0.4>}, \frac{w_3}{<0.6,0.2,0.5>}, \frac{w_4}{<0.8,0.1,0.6>}\right\}\right), \\
\left((\upsilon_3, c, 1), \left\{\frac{w_1}{<0.2,0.7,0.5>}, \frac{w_2}{<0.9,0.1,0.4>}, \frac{w_3}{<0.4,0.5,0.7>}, \frac{w_4}{<0.5,0.4,0.8>}\right\}\right), \\
\left((\upsilon_3, d, 1), \left\{\frac{w_1}{<0.4,0.3,0.2>}, \frac{w_2}{<0.6,0.3,0.1>}, \frac{w_3}{<0.7,0.2,0.3>}, \frac{w_4}{<0.9,0.1,0.4>}\right\}\right), \\
\left((\upsilon_3, e, 1), \left\{\frac{w_1}{<0.7,0.2,0.6>}, \frac{w_2}{<0.3,0.5,0.7>}, \frac{w_3}{<0.5,0.4,0.5>}, \frac{w_4}{<0.2,0.7,0.8>}\right\}\right)
\end{array} \right\}.$$

Definition 3.11. A disagree-NHSES is described by as
$(\hbar, \text{Æ})_{dag} = \{\hbar_{dag}(\varsigma) : \varsigma \in \text{Æ} \times \text{Œ} \times \{0\}\}$.

Example 3.12. Getting disagree-NHSES calculated in 3.3,

$$(\hbar, \mathbb{G}) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 0), \left\{\frac{w_1}{<0.3,0.2,0.1>}, \frac{w_2}{<0.2,0.4,0.5>}, \frac{w_3}{<0.4,0.5,0.8>}, \frac{w_4}{<0.1,0.8,0.3>}\right\}\right), \\ \left((\upsilon_1, d, 0), \left\{\frac{w_1}{<0.1,0.8,0.4>}, \frac{w_2}{<0.9,0.1,0.2>}, \frac{w_3}{<0.6,0.3,0.4>}, \frac{w_4}{<0.2,0.7,0.5>}\right\}\right), \\ \left((\upsilon_1, e, 0), \left\{\frac{w_1}{<0.2,0.7,0.5>}, \frac{w_2}{<0.1,0.8,0.6>}, \frac{w_3}{<0.3,0.5,0.7>}, \frac{w_4}{<0.5,0.4,0.6>}\right\}\right), \\ \left((\upsilon_2, c, 0), \left\{\frac{w_1}{<0.8,0.1,0.6>}, \frac{w_2}{<0.3,0.6,0.7>}, \frac{w_3}{<0.5,0.4,0.8>}, \frac{w_4}{<0.7,0.2,0.9>}\right\}\right), \\ \left((\upsilon_2, d, 0), \left\{\frac{w_1}{<0.7,0.2,0.5>}, \frac{w_2}{<0.2,0.6,0.4>}, \frac{w_3}{<0.9,0.1,0.6>}, \frac{w_4}{<0.4,0.5,0.7>}\right\}\right), \\ \left((\upsilon_2, e, 0), \left\{\frac{w_1}{<0.6,0.2,0.5>}, \frac{w_2}{<0.7,0.2,0.4>}, \frac{w_3}{<0.3,0.5,0.4>}, \frac{w_4}{<0.2,0.7,0.6>}\right\}\right), \\ \left((\upsilon_3, c, 0), \left\{\frac{w_1}{<0.1,0.7,0.5>}, \frac{w_2}{<0.4,0.5,0.7>}, \frac{w_3}{<0.7,0.2,0.9>}, \frac{w_4}{<0.8,0.2,0.4>}\right\}\right), \\ \left((\upsilon_3, d, 0), \left\{\frac{w_1}{<0.2,0.7,0.4>}, \frac{w_2}{<0.9,0.1,0.6>}, \frac{w_3}{<0.8,0.2,0.4>}, \frac{w_4}{<0.3,0.5,0.7>}\right\}\right), \\ \left((\upsilon_3, e, 0), \left\{\frac{w_1}{<0.5,0.4,0.2>}, \frac{w_2}{<0.3,0.6,0.1>}, \frac{w_3}{<0.6,0.3,0.2>}, \frac{w_4}{<0.1,0.8,0.3>}\right\}\right), \end{array} \right\}.$$

Definition 3.13. A NHSES $(\hbar_1, \mathbb{G}_1)$ is called a relative null NHSES w.r.t $\mathbb{G}_1 \subset \mathbb{G}$, denoted by $(\hbar_1, \mathbb{G}_1)$, if $\hbar_1(g) = \emptyset$, $\forall$ g $\in \mathbb{G}_1$.

Example 3.14. Considering Example 3.3, we
$(\hbar_1, \mathbb{G}_1) = \{((w_1, c, 1), \emptyset), ((w_2, d, 1), \emptyset), ((w_3, e, 1), \emptyset)\}$.

Definition 3.15. A NHSES $(\hbar_2, \mathbb{G}_2)$ is called a relative whole NHSES w.r.t $\mathbb{G}_2 \subset \mathbb{G}$, denoted by $(\hbar_2, \mathbb{G}_2)_\Delta$, if $\hbar_1(g) = \Delta$, $\forall$ g $\in \mathbb{G}_2$.

Example 3.16. Considering Example 3.3, we have
$(\hbar_2, \mathbb{G}_2)_\Delta = \{((w_1, c, 1), \Delta), ((w_2, d, 1), \Delta), ((w_3, e, 1), \Delta)\}$ where $\mathbb{G}_2 \subseteq \mathbb{G}$.

Definition 3.17. A NHSES $(\hbar, \mathbb{G})$ is called absolute whole NHSES denoted by $(\hbar, \mathbb{G})_\Delta$, if $\hbar(g) = \Delta$, $\forall$ g $\in \mathbb{G}$.

Example 3.18. Considering Example 3.3, we have

$$(\Psi, \mathbb{S})_\Delta = \left\{ \begin{array}{l} ((w_1, c, 1), \Delta), ((w_1, d, 1), \Delta), ((w_1, e, 1), \Delta), ((w_3, c, 1), \Delta), \\ ((w_3, d, 1), \Delta), ((w_3, e, 1), \Delta), ((w_5, c, 1), \Delta), ((w_5, d, 1), \Delta), \\ ((w_5, e, 1), \Delta), ((w_1, c, 0), \Delta), ((w_1, d, 0), \Delta), ((w_1, e, 0), \Delta), \\ ((w_3, c, 0), \Delta), ((w_3, d, 0), \Delta), ((w_3, e, 0), \Delta), ((w_5, c, 0), \Delta), \\ ((w_5, d, 0), \Delta), ((w_5, e, 0), \Delta) \end{array} \right\}.$$

Proposition 3.19. *Suppose $(\hbar_1, \mathbb{G}_1)_\Delta$, $(\hbar_2, \mathbb{G}_2)_\Delta$, $(\hbar_3, \mathbb{G}_3)_\Delta$, be three NHSES-sets over Δ, then*

- $(\hbar_1, \mathbb{G}_1) \subset (\hbar_2, \mathbb{G}_2)_\Delta$,
- $(\hbar_1, \mathbb{G}_1)_\hbar \subset (\hbar_1, \mathbb{G}_1)$,
- $(\hbar_1, \mathbb{G}_1) \subset (\hbar_1, \mathbb{G}_1)$,
- *If* $(\hbar_1, \mathbb{G}_1) \subset (\hbar_2, \mathbb{G}_2)$, *and* $(\hbar_2, \mathbb{G}_2) \subset (\hbar_3, \mathbb{G}_3)$, *then* $(\hbar_1, \mathbb{G}_1) \subset (\hbar_3, \mathbb{S}_3)$.

- *If* $(\hbar_1, \mathbb{G}_1) = (\hbar_2, \mathbb{G}_2)$, *and* $(\hbar_2, \mathbb{G}_2) = (\hbar_3, \mathbb{G}_3)$, *then* $(\hbar_1, \mathbb{G}_1) = (\hbar_3, \mathbb{G}_3)$.

Proposition 3.20. *If* $(\hbar, \mathbb{G})$ *is a NHSES over* Δ, *then*

(1) $((\hbar, \mathbb{G})^c)^c = (\hbar, \mathbb{G})$
(2) $(\hbar, \mathbb{G})^c_{ag} = (\hbar, \mathbb{G})_{dag}$
(3) $(\hbar, \mathbb{G})^c_{dag} = (\hbar, \mathbb{G})_{ag}$.

4. Set Theoretic Operations of NHSES

In this portion, some set theoretic operations are presented with detailed examples.

Definition 4.1. The union of $(\hbar_1, \mathbb{G})$ and $(\hbar_2, \mathbb{R})$ over Δ is $(\hbar_3, \mathbb{L})$ with $\mathbb{L} = \mathbb{G} \cup \mathbb{R}$, defined as

$$\hbar_3(\varsigma) = \begin{cases} \hbar_1(\varsigma) & ; \varsigma \in \mathbb{G} - \mathbb{R} \\ \hbar_2(\varsigma) & ; \varsigma \in \mathbb{R} - \mathbb{G} \\ \cup(\hbar_1(\varsigma), \hbar_2(\varsigma)) & ; \varsigma \in \mathbb{G} \cap \mathbb{R}. \end{cases}$$

where $\cup(\hbar_1(\varsigma), \hbar_2(\varsigma)) = \{< u, \max\{\upsilon_1(\varsigma), \upsilon_2(\varsigma)\}, 1/2\{\nu_1(\varsigma) + \nu_2(\varsigma)\}, \min\{\omega_1(\varsigma), \omega_2(\varsigma)\} >: u \in \Delta\}$.

Example 4.2. Considering Example 3.3, we see

$$\mathbb{G}_1 = \left\{ (\upsilon_1, c, 1), (\upsilon_3, c, 0), (\upsilon_1, d, 1), (\upsilon_3, d, 1), (\upsilon_3, d, 0), (\upsilon_1, e, 0), (\upsilon_3, e, 1) \right\}$$

$$\mathbb{G}_2 = \left\{ (\upsilon_1, c, 1), (\upsilon_3, c, 0), (\upsilon_3, c, 1), (\upsilon_1, d, 1), (\upsilon_3, d, 1), (\upsilon_1, e, 1), (\upsilon_3, d, 0), (\upsilon_1, e, 0), (\upsilon_3, e, 1), (\upsilon_1, d, 0) \right\}.$$

Suppose $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ over Δ are two NHSESs such that

$$(\hbar_1, \mathbb{G}_1) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.1,0.6,0.4>}, \frac{w_2}{<0.6,0.3,0.2>}, \frac{w_3}{<0.4,0.5,0.1>}, \frac{w_4}{<0.1,0.8,0.5>} \right\}\right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.3,0.4,0.5>}, \frac{w_2}{<0.6,0.2,0.3>}, \frac{w_3}{<0.2,0.5,0.6>}, \frac{w_4}{<0.1,0.5,0.3>} \right\}\right), \\ \left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.2,0.6,0.7>}, \frac{w_2}{<0.5,0.2,0.3>}, \frac{w_3}{<0.6,0.3,0.5>}, \frac{w_4}{<0.8,0.1,0.9>} \right\}\right), \\ \left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.6,0.2,0.4>}, \frac{w_2}{<0.2,0.7,0.6>}, \frac{w_3}{<0.4,0.3,0.5>}, \frac{w_4}{<0.1,0.5,0.4>} \right\}\right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.1,0.3,0.5>}, \frac{w_2}{<0.1,0.7,0.6>}, \frac{w_3}{<0.2,0.7,0.4>}, \frac{w_4}{<0.4,0.6,0.8>} \right\}\right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.1,0.6,0.9>}, \frac{w_2}{<0.3,0.6,0.7>}, \frac{w_3}{<0.6,0.1,0.2>}, \frac{w_4}{<0.7,0.2,0.3>} \right\}\right), \\ \left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.1,0.7,0.3>}, \frac{w_2}{<0.8,0.1,0.2>}, \frac{w_3}{<0.7,0.2,0.4>}, \frac{w_4}{<0.2,0.7,0.6>} \right\}\right) \end{array} \right\}$$

$$(\hbar_2, \mathbb{G}_2) = \left\{ \begin{array}{l}
\left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.2,0.3,0.4>}, \frac{w_2}{<0.7,0.4,0.5>}, \frac{w_3}{<0.5,0.4,0.6>}, \frac{w_4}{<0.2,0.4,0.7>} \right\} \right), \\
\left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.4,0.3,0.8>}, \frac{w_2}{<0.8,0.3,0.5>}, \frac{w_3}{<0.4,0.3,0.5>}, \frac{w_4}{<0.2,0.6,0.7>} \right\} \right), \\
\left((\upsilon_3, c, 1), \left\{ \frac{w_1}{<0.1,0.3,0.6>}, \frac{w_2}{<0.9,0.1,0.7>}, \frac{w_3}{<0.4,0.5,0.8>}, \frac{w_4}{<0.5,0.3,0.5>} \right\} \right), \\
\left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.4,0.2,0.3>}, \frac{w_2}{<0.6,0.3,0.5>}, \frac{w_3}{<0.7,0.4,0.5>}, \frac{w_4}{<0.9,0.5,0.7} \right\} \right), \\
\left((\upsilon_1, e, 1), \left\{ \frac{w_1}{<0.7,0.2,0.3>}, \frac{w_2}{<0.5,0.2,0.4>}, \frac{w_3}{<0.6,0.2,0.4>}, \frac{w_4}{<0.3,0.5,0.6} \right\} \right), \\
\left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.7,0.3,0.7>}, \frac{w_2}{<0.3,0.5,0.6>}, \frac{w_3}{<0.5,0.4,0.3>}, \frac{w_4}{<0.2,0.6,0.4>} \right\} \right), \\
\left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.2,0.6,0.3>}, \frac{w_3}{<0.3,0.5,0.6>}, \frac{w_4}{<0.5,0.3,0.7>} \right\} \right), \\
\left((\upsilon_1, d, 0), \left\{ \frac{w_1}{<0.1,0.6,0.3>}, \frac{w_2}{<0.9,0.1,0.2>}, \frac{w_3}{<0.6,0.3,0.4>}, \frac{w_4}{<0.2,0.6,0.3>} \right\} \right), \\
\left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.2,0.7,0.5>}, \frac{w_2}{<0.4,0.5,0.6>}, \frac{w_3}{<0.7,0.2,0.3>}, \frac{w_4}{<0.8,0.1,0.4>} \right\} \right), \\
\left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.7,0.2,0.3>}, \frac{w_3}{<0.8,0.2,0.6>}, \frac{w_4}{<0.3,0.5,0.7>} \right\} \right)
\end{array} \right\}.$$

Then $(\hbar_1, \mathbb{G}_1) \cup (\hbar_2, \mathbb{G}_2) = (\hbar_3, \mathbb{G}_3)$

$$(\hbar_3, \mathbb{G}_3) = \left\{ \begin{array}{l}
\left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.2,0.45,0.4>}, \frac{w_2}{<0.7,0.35,0.2>}, \frac{w_3}{<0.5,0.45,0.1>}, \frac{w_4}{<0.2,0.6,0.5>} \right\} \right), \\
\left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.4,0.35,0.5>}, \frac{w_2}{<0.8,0.25,0.3>}, \frac{w_3}{<0.4,0.4,0.5>}, \frac{w_4}{<0.2,0.55,0.3>} \right\} \right), \\
\left((\upsilon_3, c, 1), \left\{ \frac{w_1}{<0.1,0.30,0.6>}, \frac{w_2}{<0.9,0.10,0.7>}, \frac{w_3}{<0.4,0.5,0.8>}, \frac{w_4}{<0.5,0.3,0.5>} \right\} \right), \\
\left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.4,0.40,0.3>}, \frac{w_2}{<0.6,0.25,0.3>}, \frac{w_3}{<0.7,0.35,0.5>}, \frac{w_4}{<0.9,0.30,0.7} \right\} \right), \\
\left((\upsilon_1, e, 1), \left\{ \frac{w_1}{<0.7,0.20,0.3>}, \frac{w_2}{<0.5,0.20,0.4>}, \frac{w_3}{<0.6,0.20,0.4>}, \frac{w_4}{<0.3,0.50,0.6} \right\} \right), \\
\left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.7,0.20,0.4>}, \frac{w_2}{<0.3,0.50,0.6>}, \frac{w_3}{<0.5,0.3,0.3>}, \frac{w_4}{<0.2,0.5,0.4>} \right\} \right), \\
\left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.2,0.40,0.4>}, \frac{w_2}{<0.2,0.65,0.3>}, \frac{w_3}{<0.3,0.6,0.4>}, \frac{w_4}{<0.5,0.45,0.7>} \right\} \right), \\
\left((\upsilon_1, d, 0), \left\{ \frac{w_1}{<0.1,0.60,0.3>}, \frac{w_2}{<0.9,0.10,0.2>}, \frac{w_3}{<0.6,0.30,0.4>}, \frac{w_4}{<0.2,0.60,0.3>} \right\} \right), \\
\left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.2,0.65,0.5>}, \frac{w_2}{<0.4,0.55,0.6>}, \frac{w_3}{<0.7,0.15,0.2>}, \frac{w_4}{<0.8,0.15,0.3>} \right\} \right), \\
\left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.2,0.60,0.3>}, \frac{w_2}{<0.8,0.15,0.2>}, \frac{w_3}{<0.8,0.20,0.4>}, \frac{w_4}{<0.3,0.60,0.6>} \right\} \right)
\end{array} \right\}.$$

Definition 4.3. Restricted Union of two NHSESs $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ over Δ is $(\hbar_3, \mathbb{L})$ with $\mathbb{L} = \mathbb{G}_1 \cap \mathbb{G}_2$, defined as $\hbar_3(\varsigma) = \hbar_1(\varsigma) \cup_{\mathbb{R}} \hbar_2(\varsigma)$ for $\varsigma \in \mathbb{G}_1 \cap \mathbb{G}_2$.

Example 4.4. Considering Example 3.3, we see
$\mathbb{G}_1 = \left\{ (\upsilon_1, c, 1), (\upsilon_3, c, 0), (\upsilon_1, d, 1), (\upsilon_3, d, 1), (\upsilon_3, d, 0), (\upsilon_1, e, 0), (\upsilon_3, e, 1) \right\}$, $\mathbb{G}_2 = \left\{ (\upsilon_1, c, 1), (\upsilon_3, c, 0), (\upsilon_3, c, 1), (\upsilon_1, d, 1), (\upsilon_3, d, 1), (\upsilon_1, d, 0), (\upsilon_3, d, 0), (\upsilon_1, e, 0), (\upsilon_3, e, 1), (\upsilon_1, e, 1) \right\}$.

Suppose $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ over Δ are two NHSESs such that

$$(\hbar_1, \mathbb{G}_1) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.1,0.6,0.4>}, \frac{w_2}{<0.6,0.3,0.2>}, \frac{w_3}{<0.4,0.5,0.1>}, \frac{w_4}{<0.1,0.8,0.5>} \right\} \right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.3,0.4,0.5>}, \frac{w_2}{<0.6,0.2,0.3>}, \frac{w_3}{<0.2,0.5,0.6>}, \frac{w_4}{<0.1,0.5,0.3>} \right\} \right), \\ \left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.2,0.6,0.7>}, \frac{w_2}{<0.5,0.2,0.3>}, \frac{w_3}{<0.6,0.3,0.5>}, \frac{w_4}{<0.8,0.1,0.9>} \right\} \right), \\ \left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.6,0.2,0.4>}, \frac{w_2}{<0.2,0.7,0.6>}, \frac{w_3}{<0.4,0.3,0.5>}, \frac{w_4}{<0.1,0.5,0.4>} \right\} \right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.1,0.3,0.5>}, \frac{w_2}{<0.1,0.7,0.6>}, \frac{w_3}{<0.2,0.7,0.4>}, \frac{w_4}{<0.4,0.6,0.8>} \right\} \right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.1,0.6,0.9>}, \frac{w_2}{<0.3,0.6,0.7>}, \frac{w_3}{<0.6,0.1,0.2>}, \frac{w_4}{<0.7,0.2,0.3>} \right\} \right), \\ \left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.1,0.7,0.3>}, \frac{w_2}{<0.8,0.1,0.2>}, \frac{w_3}{<0.7,0.2,0.4>}, \frac{w_4}{<0.2,0.7,0.6>} \right\} \right) \end{array} \right\}$$

$$(\hbar_2, \mathbb{G}_2) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.2,0.3,0.4>}, \frac{w_2}{<0.7,0.4,0.5>}, \frac{w_3}{<0.5,0.4,0.6>}, \frac{w_4}{<0.2,0.4,0.7>} \right\} \right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.4,0.3,0.8>}, \frac{w_2}{<0.8,0.3,0.5>}, \frac{w_3}{<0.4,0.3,0.5>}, \frac{w_4}{<0.2,0.6,0.7>} \right\} \right), \\ \left((\upsilon_3, c, 1), \left\{ \frac{w_1}{<0.1,0.3,0.6>}, \frac{w_2}{<0.9,0.1,0.7>}, \frac{w_3}{<0.4,0.5,0.8>}, \frac{w_4}{<0.5,0.3,0.5>} \right\} \right), \\ \left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.4,0.2,0.3>}, \frac{w_2}{<0.6,0.3,0.5>}, \frac{w_3}{<0.7,0.4,0.5>}, \frac{w_4}{<0.9,0.5,0.7} \right\} \right), \\ \left((\upsilon_1, e, 1), \left\{ \frac{w_1}{<0.7,0.2,0.3>}, \frac{w_2}{<0.5,0.2,0.4>}, \frac{w_3}{<0.6,0.2,0.4>}, \frac{w_4}{<0.3,0.5,0.6} \right\} \right), \\ \left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.7,0.3,0.7>}, \frac{w_2}{<0.3,0.5,0.6>}, \frac{w_3}{<0.5,0.4,0.3>}, \frac{w_4}{<0.2,0.6,0.4>} \right\} \right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.2,0.6,0.3>}, \frac{w_3}{<0.3,0.5,0.6>}, \frac{w_4}{<0.5,0.3,0.7>} \right\} \right), \\ \left((\upsilon_1, d, 0), \left\{ \frac{w_1}{<0.1,0.6,0.3>}, \frac{w_2}{<0.9,0.1,0.2>}, \frac{w_3}{<0.6,0.3,0.4>}, \frac{w_4}{<0.2,0.6,0.3>} \right\} \right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.2,0.7,0.5>}, \frac{w_2}{<0.4,0.5,0.6>}, \frac{w_3}{<0.7,0.2,0.3>}, \frac{w_4}{<0.8,0.1,0.4>} \right\} \right), \\ \left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.7,0.2,0.3>}, \frac{w_3}{<0.8,0.2,0.6>}, \frac{w_4}{<0.3,0.5,0.7>} \right\} \right) \end{array} \right\}.$$

Then $(\hbar_1, \mathbb{G}_1) \cup_{\mathbb{R}} (\hbar_2, \mathbb{G}_2) = (\hbar_3, \mathbb{L})$

$$(\hbar_3, \mathbb{L}) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.2,0.45,0.4>}, \frac{w_2}{<0.7,0.35,0.2>}, \frac{w_3}{<0.5,0.45,0.1>}, \frac{w_4}{<0.2,0.6,0.5>} \right\} \right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.4,0.35,0.5>}, \frac{w_2}{<0.8,0.25,0.3>}, \frac{w_3}{<0.4,0.4,0.5>}, \frac{w_4}{<0.2,0.55,0.3>} \right\} \right), \\ \left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.4,0.40,0.3>}, \frac{w_2}{<0.6,0.25,0.3>}, \frac{w_3}{<0.7,0.35,0.5>}, \frac{w_4}{<0.9,0.30,0.7} \right\} \right), \\ \left((\upsilon_1, e, 1), \left\{ \frac{w_1}{<0.7,0.20,0.3>}, \frac{w_2}{<0.5,0.20,0.4>}, \frac{w_3}{<0.6,0.20,0.4>}, \frac{w_4}{<0.3,0.50,0.6} \right\} \right), \\ \left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.7,0.20,0.4>}, \frac{w_2}{<0.3,0.50,0.6>}, \frac{w_3}{<0.5,0.3,0.3>}, \frac{w_4}{<0.2,0.5,0.4>} \right\} \right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.2,0.60,0.4>}, \frac{w_2}{<0.2,0.10,0.3>}, \frac{w_3}{<0.3,0.30,0.4>}, \frac{w_4}{<0.5,0.60,0.7>} \right\} \right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.2,0.65,0.5>}, \frac{w_2}{<0.4,0.55,0.6>}, \frac{w_3}{<0.7,0.15,0.2>}, \frac{w_4}{<0.8,0.15,0.3>} \right\} \right), \\ \left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.2,0.60,0.3>}, \frac{w_2}{<0.8,0.15,0.2>}, \frac{w_3}{<0.8,0.20,0.4>}, \frac{w_4}{<0.3,0.60,0.6>} \right\} \right) \end{array} \right\}.$$

Proposition 4.5. *If $(\hbar_1, \mathbb{G}_1)$,$(\hbar_2, \mathbb{G}_2)$ and $(\hbar_3, \mathbb{G}_3)$ are three NHSESs, then*

(1) $(\hbar_1, \mathbb{G}_1) \cup (\hbar_2, \mathbb{G}_2) = (\hbar_2, \mathbb{G}_2) \cup (\hbar_1, \mathbb{G}_1)$

(2) $((\hbar_1, \mathbb{G}_1) \cup (\hbar_2, \mathbb{G}_2)) \cup (\hbar_3, \mathbb{G}_3) = (\hbar_1, \mathbb{G}_1) \cup ((\hbar_2, \mathbb{G}_2) \cup (\hbar_3, N_3))$

(3) $(\hbar, \mathbb{G}) \cup \Phi = (\hbar, \mathbb{G})$.

Definition 4.6. The intersection of two NHSESs $(\hbar_1, \mathbb{G})$ and $(\hbar_2, \mathbb{R})$ over Δ is $(\hbar_3, \mathbb{L})$ with $\mathbb{L} = \mathbb{G} \cap \mathbb{R}$, defined as

$$\hbar_3(\varsigma) = \begin{cases} \hbar_1(\varsigma) & ; \varsigma \in \mathbb{G} - \mathbb{R} \\ \hbar_2(\varsigma) & ; \varsigma \in \mathbb{R} - \mathbb{G} \\ \cap(\hbar_1(\varsigma), \hbar_2(\varsigma)) & ; \varsigma \in \mathbb{G} \cap \mathbb{R} \end{cases}$$

where $\cap(\hbar_1(\varsigma), \hbar_2(\varsigma)) = \{< u, \min\{\upsilon_1(\varsigma), \upsilon_2(\varsigma)\}, 1/2\{\nu_1(\varsigma) + \nu_2(\varsigma)\}, \max\{\omega_1(\varsigma), \omega_2(\varsigma)\} >: u \in \Delta\}$.

Example 4.7. Reconsidering Example 3.3, we have

$$\mathbb{G}_1 = \Big\{ (\upsilon_1, c, 1), (\upsilon_3, c, 0), (\upsilon_1, d, 1), (\upsilon_3, d, 1), (\upsilon_3, d, 0), (\upsilon_1, e, 0), (\upsilon_3, e, 1) \Big\}$$

$$\mathbb{G}_2 = \Big\{ (\upsilon_1, c, 1), (\upsilon_3, c, 0), (\upsilon_3, c, 1), (\upsilon_1, d, 1), (\upsilon_3, d, 1), (\upsilon_1, d, 0), (\upsilon_3, d, 0), (\upsilon_1, e, 0), (\upsilon_3, e, 1), , (\upsilon_1, e, 1) \Big\}$$

Suppose $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ are two NHSESs over Δ such that

$$(\hbar_1, \mathbb{G}_1) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.1,0.6,0.4>}, \frac{w_2}{<0.6,0.3,0.2>}, \frac{w_3}{<0.4,0.5,0.1>}, \frac{w_4}{<0.1,0.8,0.5>} \right\}\right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.3,0.4,0.5>}, \frac{w_2}{<0.6,0.2,0.3>}, \frac{w_3}{<0.2,0.5,0.6>}, \frac{w_4}{<0.1,0.5,0.3>} \right\}\right), \\ \left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.2,0.6,0.7>}, \frac{w_2}{<0.5,0.2,0.3>}, \frac{w_3}{<0.6,0.3,0.5>}, \frac{w_4}{<0.8,0.1,0.9>} \right\}\right), \\ \left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.6,0.2,0.4>}, \frac{w_2}{<0.2,0.7,0.6>}, \frac{w_3}{<0.4,0.3,0.5>}, \frac{w_4}{<0.1,0.5,0.4>} \right\}\right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.1,0.3,0.5>}, \frac{w_2}{<0.1,0.7,0.6>}, \frac{w_3}{<0.2,0.7,0.4>}, \frac{w_4}{<0.4,0.6,0.8>} \right\}\right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.1,0.6,0.9>}, \frac{w_2}{<0.3,0.6,0.7>}, \frac{w_3}{<0.6,0.1,0.2>}, \frac{w_4}{<0.7,0.2,0.3>} \right\}\right), \\ \left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.1,0.7,0.3>}, \frac{w_2}{<0.8,0.1,0.2>}, \frac{w_3}{<0.7,0.2,0.4>}, \frac{w_4}{<0.2,0.7,0.6>} \right\}\right) \end{array} \right\}$$

$$(\hbar_2, \mathbb{G}_2) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.2,0.3,0.4>}, \frac{w_2}{<0.7,0.4,0.5>}, \frac{w_3}{<0.5,0.4,0.6>}, \frac{w_4}{<0.2,0.4,0.7>} \right\}\right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.4,0.3,0.8>}, \frac{w_2}{<0.8,0.3,0.5>}, \frac{w_3}{<0.4,0.3,0.5>}, \frac{w_4}{<0.2,0.6,0.7>} \right\}\right), \\ \left((\upsilon_3, c, 1), \left\{ \frac{w_1}{<0.1,0.3,0.6>}, \frac{w_2}{<0.9,0.1,0.7>}, \frac{w_3}{<0.4,0.5,0.8>}, \frac{w_4}{<0.5,0.3,0.5>} \right\}\right), \\ \left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.4,0.2,0.3>}, \frac{w_2}{<0.6,0.3,0.5>}, \frac{w_3}{<0.7,0.4,0.5>}, \frac{w_4}{<0.9,0.5,0.7} \right\}\right), \\ \left((\upsilon_1, e, 1), \left\{ \frac{w_1}{<0.7,0.2,0.3>}, \frac{w_2}{<0.5,0.2,0.4>}, \frac{w_3}{<0.6,0.2,0.4>}, \frac{w_4}{<0.3,0.5,0.6} \right\}\right), \\ \left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.7,0.3,0.7>}, \frac{w_2}{<0.3,0.5,0.6>}, \frac{w_3}{<0.5,0.4,0.3>}, \frac{w_4}{<0.2,0.6,0.4>} \right\}\right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.2,0.6,0.3>}, \frac{w_3}{<0.3,0.5,0.6>}, \frac{w_4}{<0.5,0.3,0.7>} \right\}\right), \\ \left((\upsilon_1, d, 0), \left\{ \frac{w_1}{<0.1,0.6,0.3>}, \frac{w_2}{<0.9,0.1,0.2>}, \frac{w_3}{<0.6,0.3,0.4>}, \frac{w_4}{<0.2,0.6,0.3>} \right\}\right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.2,0.7,0.5>}, \frac{w_2}{<0.4,0.5,0.6>}, \frac{w_3}{<0.7,0.2,0.3>}, \frac{w_4}{<0.8,0.1,0.4>} \right\}\right), \\ \left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.7,0.2,0.3>}, \frac{w_3}{<0.8,0.2,0.6>}, \frac{w_4}{<0.3,0.5,0.7>} \right\}\right) \end{array} \right\}$$

Then $(\hbar_1, \mathbb{G}_1) \cap (\hbar_2, \mathbb{G}_2) = (\hbar_3, \mathbb{G}_3)$

$$(\hbar_3, \mathbb{G}_3) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.1,0.45,0.4>}, \frac{w_2}{<0.6,0.35,0.5>}, \frac{w_3}{<0.4,0.45,0.6>}, \frac{w_4}{<0.1,0.6,0.7>} \right\} \right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.3,0.35,0.8>}, \frac{w_2}{<0.6,0.25,0.5>}, \frac{w_3}{<0.2,0.4,0.6>}, \frac{w_4}{<0.1,0.55,0.7>} \right\} \right), \\ \left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.2,0.4,0.7>}, \frac{w_2}{<0.5,0.25,0.5>}, \frac{w_3}{<0.6,0.35,0.5>}, \frac{w_4}{<0.8,0.30,0.7>} \right\} \right), \\ \left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.6,0.25,0.7>}, \frac{w_2}{<0.2,0.60,0.6>}, \frac{w_3}{<0.4,0.35,0.5>}, \frac{w_4}{<0.1,0.55,0.4>} \right\} \right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.1,0.40,0.5>}, \frac{w_2}{<0.1,0.65,0.6>}, \frac{w_3}{<0.2,0.60,0.6>}, \frac{w_4}{<0.4,0.45,0.8>} \right\} \right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.1,0.65,0.9>}, \frac{w_2}{<0.3,0.55,0.7>}, \frac{w_3}{<0.6,0.15,0.3>}, \frac{w_4}{<0.7,0.15,0.4>} \right\} \right), \\ \left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.1,0.60,0.4>}, \frac{w_2}{<0.8,0.15,0.3>}, \frac{w_3}{<0.7,0.20,0.6>}, \frac{w_4}{<0.2,0.60,0.7>} \right\} \right), \end{array} \right\}.$$

Definition 4.8. Extended intersection of two NHSESs $(\hbar_1, \mathbb{S})$ and $(\hbar_2, \mathbb{R})$ over Δ is $(\hbar_3, \mathbb{L})$ with $\mathbb{L} = \mathbb{S} \cup \mathbb{R}$, defined as

$$\hbar_3(\varsigma) = \begin{cases} \hbar_1(\varsigma) & ; \varsigma \in \mathbb{S} - \mathbb{R} \\ \hbar_2(\varsigma) & ; \varsigma \in \mathbb{R} - \mathbb{S} \\ \hbar_1(\varsigma) \cap \hbar_2(\varsigma) & ; \varsigma \in \mathbb{S} \cap \mathbb{R}. \end{cases}$$

Example 4.9. Considering Example 3.3, we have

$$\mathbb{G}_1 = \Big\{ (\upsilon_1, c, 1), (\upsilon_3, c, 0), (\upsilon_1, d, 1), (\upsilon_3, d, 1), (\upsilon_3, d, 0), (\upsilon_1, e, 0), (\upsilon_3, e, 1) \Big\}$$

$$\mathbb{G}_2 = \Big\{ (\upsilon_1, c, 1), (\upsilon_3, c, 0), (\upsilon_3, c, 1), (\upsilon_1, d, 1), (\upsilon_3, d, 1), (\upsilon_1, d, 0), (\upsilon_3, d, 0), (\upsilon_1, e, 0), (\upsilon_3, e, 1), (\upsilon_1, e, 1) \Big\}.$$

Suppose $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ are two NHSESs over Δ such that

$$(\hbar_1, \mathbb{G}_1) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.1,0.6,0.4>}, \frac{w_2}{<0.6,0.3,0.2>}, \frac{w_3}{<0.4,0.5,0.1>}, \frac{w_4}{<0.1,0.8,0.5>} \right\} \right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.3,0.4,0.5>}, \frac{w_2}{<0.6,0.2,0.3>}, \frac{w_3}{<0.2,0.5,0.6>}, \frac{w_4}{<0.1,0.5,0.3>} \right\} \right), \\ \left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.2,0.6,0.7>}, \frac{w_2}{<0.5,0.2,0.3>}, \frac{w_3}{<0.6,0.3,0.5>}, \frac{w_4}{<0.8,0.1,0.9>} \right\} \right), \\ \left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.6,0.2,0.4>}, \frac{w_2}{<0.2,0.7,0.6>}, \frac{w_3}{<0.4,0.3,0.5>}, \frac{w_4}{<0.1,0.5,0.4>} \right\} \right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.1,0.3,0.5>}, \frac{w_2}{<0.1,0.7,0.6>}, \frac{w_3}{<0.2,0.7,0.4>}, \frac{w_4}{<0.4,0.6,0.8>} \right\} \right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.1,0.6,0.9>}, \frac{w_2}{<0.3,0.6,0.7>}, \frac{w_3}{<0.6,0.1,0.2>}, \frac{w_4}{<0.7,0.2,0.3>} \right\} \right), \\ \left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.1,0.7,0.3>}, \frac{w_2}{<0.8,0.1,0.2>}, \frac{w_3}{<0.7,0.2,0.4>}, \frac{w_4}{<0.2,0.7,0.6>} \right\} \right) \end{array} \right.$$

$$(\hbar_2, \mathbb{G}_2) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.2,0.3,0.4>}, \frac{w_2}{<0.7,0.4,0.5>}, \frac{w_3}{<0.5,0.4,0.6>}, \frac{w_4}{<0.2,0.4,0.7>} \right\} \right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.4,0.3,0.8>}, \frac{w_2}{<0.8,0.3,0.5>}, \frac{w_3}{<0.4,0.3,0.5>}, \frac{w_4}{<0.2,0.6,0.7>} \right\} \right), \\ \left((\upsilon_3, c, 1), \left\{ \frac{w_1}{<0.1,0.3,0.6>}, \frac{w_2}{<0.9,0.1,0.7>}, \frac{w_3}{<0.4,0.5,0.8>}, \frac{w_4}{<0.5,0.3,0.5>} \right\} \right), \\ \left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.4,0.2,0.3>}, \frac{w_2}{<0.6,0.3,0.5>}, \frac{w_3}{<0.7,0.4,0.5>}, \frac{w_4}{<0.9,0.5,0.7} \right\} \right), \\ \left((\upsilon_1, e, 1), \left\{ \frac{w_1}{<0.7,0.2,0.3>}, \frac{w_2}{<0.5,0.2,0.4>}, \frac{w_3}{<0.6,0.2,0.4>}, \frac{w_4}{<0.3,0.5,0.6} \right\} \right), \\ \left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.7,0.3,0.7>}, \frac{w_2}{<0.3,0.5,0.6>}, \frac{w_3}{<0.5,0.4,0.3>}, \frac{w_4}{<0.2,0.6,0.4>} \right\} \right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.2,0.6,0.3>}, \frac{w_3}{<0.3,0.5,0.6>}, \frac{w_4}{<0.5,0.3,0.7>} \right\} \right), \\ \left((\upsilon_1, d, 0), \left\{ \frac{w_1}{<0.1,0.6,0.3>}, \frac{w_2}{<0.9,0.1,0.2>}, \frac{w_3}{<0.6,0.3,0.4>}, \frac{w_4}{<0.2,0.6,0.3>} \right\} \right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.2,0.7,0.5>}, \frac{w_2}{<0.4,0.5,0.6>}, \frac{w_3}{<0.7,0.2,0.3>}, \frac{w_4}{<0.8,0.1,0.4>} \right\} \right), \\ \left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.7,0.2,0.3>}, \frac{w_3}{<0.8,0.2,0.6>}, \frac{w_4}{<0.3,0.5,0.7>} \right\} \right) \end{array} \right\}.$$

Then $(\hbar_1, \mathbb{G}_1) \cap_E (\hbar_2, \mathbb{G}_2) = (\hbar_3, \mathbb{L})$

$$(\hbar_3, \mathbb{L}) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.1,0.6,0.4>}, \frac{w_2}{<0.6,0.4,0.5>}, \frac{w_3}{<0.4,0.5,0.6>}, \frac{w_4}{<0.1,0.8,0.7>} \right\} \right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.3,0.4,0.8>}, \frac{w_2}{<0.6,0.3,0.5>}, \frac{w_3}{<0.2,0.5,0.6>}, \frac{w_4}{<0.1,0.6,0.7>} \right\} \right), \\ \left((\upsilon_3, d, 1), \left\{ \frac{w_1}{<0.2,0.6,0.7>}, \frac{w_2}{<0.5,0.4,0.5>}, \frac{w_3}{<0.6,0.4,0.5>}, \frac{w_4}{<0.8,0.1,0.7>} \right\} \right), \\ \left((\upsilon_3, e, 1), \left\{ \frac{w_1}{<0.6,0.3,0.7>}, \frac{w_2}{<0.2,0.7,0.6>}, \frac{w_3}{<0.4,0.4,0.5>}, \frac{w_4}{<0.1,0.6,0.4>} \right\} \right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.1,0.5,0.5>}, \frac{w_2}{<0.1,0.6,0.6>}, \frac{w_3}{<0.2,0.7,0.6>}, \frac{w_4}{<0.4,0.6,0.8>} \right\} \right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.1,0.7,0.9>}, \frac{w_2}{<0.3,0.6,0.7>}, \frac{w_3}{<0.6,0.2,0.3>}, \frac{w_4}{<0.7,0.2,0.4>} \right\} \right), \\ \left((\upsilon_3, d, 0), \left\{ \frac{w_1}{<0.1,0.7,0.4>}, \frac{w_2}{<0.8,0.2,0.3>}, \frac{w_3}{<0.7,0.2,0.6>}, \frac{w_4}{<0.2,0.7,0.7>} \right\} \right), \\ \left((\upsilon_1, d, 0), \left\{ \frac{w_1}{<0.1,0.6,0.3>}, \frac{w_2}{<0.9,0.1,0.2>}, \frac{w_3}{<0.6,0.3,0.4>}, \frac{w_4}{<0.2,0.6,0.3>} \right\} \right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<0.2,0.5,0.4>}, \frac{w_2}{<0.2,0.6,0.3>}, \frac{w_3}{<0.3,0.5,0.6>}, \frac{w_4}{<0.5,0.3,0.7>} \right\} \right), \\ \left((\upsilon_3, c, 1), \left\{ \frac{w_1}{<0.1,0.3,0.6>}, \frac{w_2}{<0.9,0.1,0.7>}, \frac{w_3}{<0.4,0.5,0.8>}, \frac{w_4}{<0.5,0.3,0.5>} \right\} \right), \end{array} \right\}.$$

Proposition 4.10. *If* $(\hbar_1, \mathbb{G}_1)$,$(\hbar_2, \mathbb{G}_2)$ *and* $(\hbar_3, \mathbb{G}_3)$ *are three NHSESs then*

(1) $(\hbar_1, \mathbb{G}_1) \cap (\hbar_2, \mathbb{G}_2) = (\hbar_2, \mathbb{G}_2) \cap (\hbar_1, \mathbb{G}_1)$

(2) $((\hbar_1, \mathbb{G}_1) \cap (\hbar_2, \mathbb{G}_2)) \cap (\hbar_3, \mathbb{G}_3) = (\hbar_1, \mathbb{G}_1) \cap ((\hbar_2, \mathbb{G}_2) \cap (\hbar_3, \mathbb{G}_3))$

(3) $(\hbar, \mathbb{G}) \cap \phi = \phi$.

Proposition 4.11. *If* $(\hbar_1, \mathbb{G}_1)$,$(\hbar_2, \mathbb{G}_2)$ *and* $(\hbar_3, \mathbb{G}_3)$ *are three NHSESs, then*

(1) $(\hbar_1, \mathbb{G}_1) \cup ((\hbar_2, \mathbb{G}_2) \cap (\hbar_3, \mathbb{G}_3)) =$
$((\hbar_1, \mathbb{G}_1) \cup ((\hbar_2, \mathbb{G}_2)) \cap ((\hbar_1, \mathbb{G}_1) \cup (\hbar_3, \mathbb{G}_3))$

(2) $(\hbar_1, \mathbb{G}_1) \cap ((\hbar_2, \mathbb{G}_2) \cup (\hbar_3, \mathbb{G}_3)) = ((\hbar_1, \mathbb{G}_1) \cap ((\hbar_2, \mathbb{G}_2)) \cup ((\hbar_1, \mathbb{G}_1) \cap (\hbar_3, \mathbb{G}_3))$.

Definition 4.12. If $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ are two NHSESs over Δ then $(\hbar_1, \mathbb{G}_1)$ AND $(\hbar_2, \mathbb{G}_2)$ denoted by $(\hbar_1, \mathbb{G}_1) \wedge (\hbar_2, \mathbb{G}_2)$ is defined by
$(\hbar_1, \mathbb{G}_1) \wedge (\hbar_2, \mathbb{G}_2) = (\hbar_3, \mathbb{G}_1 \times \mathbb{G}_2)$, while $\hbar_3(\varsigma, \gamma) = \hbar_1(\varsigma) \cap \hbar_2(\gamma), \forall (\varsigma, \gamma) \in \mathbb{G}_1 \times \mathbb{G}_2$.

Example 4.13. Considering Example 3.3, we have
$\mathbb{G}_1 = \Big\{ (\upsilon_1, c, 1), (\upsilon_1, d, 1), (\upsilon_3, c, 0) \Big\}$, $\mathbb{G}_2 = \Big\{ (\upsilon_1, c, 0), (\upsilon_3, c, 1) \Big\}$.
Suppose $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ over Δ are two NHSESs such that

$$(\hbar_1, \mathbb{G}_1) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.1,0.6,0.4>}, \frac{w_2}{<0.6,0.4,0.5>}, \frac{w_3}{<0.4,0.5,0.6>}, \frac{w_4}{<0.1,0.8,0.7>} \right\} \right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.3,0.4,0.8>}, \frac{w_2}{<0.6,0.3,0.5>}, \frac{w_3}{<0.2,0.5,0.6>}, \frac{w_4}{<0.1,0.6,0.7>} \right\} \right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.1,0.6,0.9>}, \frac{w_2}{<0.3,0.6,0.7>}, \frac{w_3}{<0.6,0.1,0.2>}, \frac{w_4}{<0.7,0.2,0.3>} \right\} \right), \end{array} \right\}.$$

$$(\hbar_2, \mathbb{G}_2) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 0), \left\{ \frac{w_1}{<0.2,0.1,0.3>}, \frac{w_2}{<0.7,0.2,0.4>}, \frac{w_3}{<0.5,0.2,0.5>}, \frac{w_4}{<0.2,0.3,0.6>} \right\} \right), \\ \left((\upsilon_3, c, 1), \left\{ \frac{w_1}{<0.1,0.5,0.6>}, \frac{w_2}{<0.4,0.2,0.5>}, \frac{w_3}{<0.7,0.1,0.2>}, \frac{w_4}{<0.8,0.1,0.4>} \right\} \right), \end{array} \right\}.$$

Then $(\hbar_1, \mathbb{G}_1) \wedge (\hbar_2, \mathbb{G}_2) = (\hbar_3, \mathbb{G}_1 \times \mathbb{G}_2)$,

$$(\hbar_3, \mathbb{G}_1 \times \mathbb{G}_2) = \left\{ \begin{array}{l} \left(((\upsilon_1, c, 1), (\upsilon_1, c, 0)), \left\{ \frac{w_1}{<0.1,0.35,0.4>}, \frac{w_2}{<0.6,0.30,0.5>}, \frac{w_3}{<0.4,0.35,0.6>}, \frac{w_4}{<0.1,0.55,0.7>} \right\} \right), \\ \left(((\upsilon_1, d, 1), (\upsilon_1, c, 0)), \left\{ \frac{w_1}{<0.2,0.25,0.8>}, \frac{w_2}{<0.6,0.25,0.5>}, \frac{w_3}{<0.2,0.35,0.6>}, \frac{w_4}{<0.1,0.45,0.7>} \right\} \right), \\ \left(((\upsilon_1, d, 1), (\upsilon_3, c, 1)), \left\{ \frac{w_1}{<0.1,0.45,0.8>}, \frac{w_2}{<0.4,0.25,0.5>}, \frac{w_3}{<0.2,0.30,0.6>}, \frac{w_4}{<0.1,0.35,0.7>} \right\} \right), \\ \left(((\upsilon_1, c, 1), (\upsilon_3, c, 1)), \left\{ \frac{w_1}{<0.1,0.55,0.6>}, \frac{w_2}{<0.4,0.30,0.5>}, \frac{w_3}{<0.4,0.30,0.6>}, \frac{w_4}{<0.1,0.45,0.7>} \right\} \right), \\ \left(((\upsilon_3, c, 0), (\upsilon_1, c, 0)), \left\{ \frac{w_1}{<0.1,0.35,0.9>}, \frac{w_2}{<0.3,0.40,0.7>}, \frac{w_3}{<0.5,0.15,0.5>}, \frac{w_4}{<0.2,0.25,0.6>} \right\} \right), \\ \left(((\upsilon_3, c, 0), (\upsilon_3, c, 1)), \left\{ \frac{w_1}{<0.1,0.55,0.9>}, \frac{w_2}{<0.3,0.40,0.7>}, \frac{w_3}{<0.6,0.10,0.2>}, \frac{w_4}{<0.7,0.15,0.4>} \right\} \right), \end{array} \right\}.$$

Definition 4.14. If $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ are two NHSESs over Δ, then $(\hbar_1, \mathbb{G}_1)$ OR $(\hbar_2, \mathbb{G}_2)$ denoted by $(\hbar_1, \mathbb{G}_1) \vee (\hbar_2, \mathbb{G}_2)$ is defined by $(\hbar_1, \mathbb{G}_1) \vee (\hbar_2, \mathbb{G}_2) = (\hbar_3, \mathbb{G}_1 \times \mathbb{G}_2)$, while $\hbar_3(\delta, \gamma) = \hbar_1(\delta) \cup \hbar_2(\gamma), \forall (\delta, \gamma) \in \mathbb{G}_1 \times \mathbb{G}_2$.

Example 4.15. Considering Example 3.3, we see
$\mathbb{G}_1 = \Big\{ (\upsilon_1, c, 1), (\upsilon_1, d, 1), (\upsilon_3, c, 0) \Big\}$, $\mathbb{G}_2 = \Big\{ (\upsilon_1, c, 0), (\upsilon_3, c, 1) \Big\}$.
Suppose $(\hbar_1, \mathbb{G}_1)$ and $(\hbar_2, \mathbb{G}_2)$ over Δ are two NHSESs such that

$$(\hbar_1, \mathbb{G}_1) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<0.1,0.6,0.4>}, \frac{w_2}{<0.6,0.4,0.5>}, \frac{w_3}{<0.4,0.5,0.6>}, \frac{w_4}{<0.1,0.8,0.7>} \right\} \right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<0.3,0.4,0.8>}, \frac{w_2}{<0.6,0.3,0.5>}, \frac{w_3}{<0.2,0.5,0.6>}, \frac{w_4}{<0.1,0.6,0.7>} \right\} \right), \\ \left((\upsilon_3, c, 0), \left\{ \frac{w_1}{<0.1,0.6,0.9>}, \frac{w_2}{<0.3,0.6,0.7>}, \frac{w_3}{<0.6,0.1,0.2>}, \frac{w_4}{<0.7,0.2,0.3>} \right\} \right) \end{array} \right\}$$

$$(\hbar_2, \mathbb{G}_2) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 0), \left\{ \frac{w_1}{<0.2,0.1,0.3>}, \frac{w_2}{<0.7,0.2,0.4>}, \frac{w_3}{<0.5,0.2,0.5>}, \frac{w_4}{<0.2,0.3,0.6>} \right\} \right), \\ \left((\upsilon_3, c, 1), \left\{ \frac{w_1}{<0.1,0.5,0.6>}, \frac{w_2}{<0.4,0.2,0.5>}, \frac{w_3}{<0.7,0.1,0.2>}, \frac{w_4}{<0.8,0.1,0.4>} \right\} \right), \end{array} \right\}.$$

Then $(\hbar_3, \mathbb{G}_3) \vee (\hbar_2, \mathbb{G}_2) = (\hbar_3, \mathbb{G}_1 \times \mathbb{G}_2)$,

$$(\hbar_3, \mathbb{G}_1 \times \mathbb{G}_2) = \left\{ \begin{array}{l} \left(((\upsilon_1, c, 1), (\upsilon_1, c, 0)), \left\{ \frac{w_1}{<0.2,0.35,0.3>}, \frac{w_2}{0<0.7,0.30,0.4>}, \frac{w_3}{<0.5,0.35,0.5>}, \frac{w_4}{<0.2,0.55,0.6>} \right\}\right), \\ \left(((\upsilon_1, d, 1), (\upsilon_1, c, 0)), \left\{ \frac{w_1}{<0.3,0.25,0.3>}, \frac{w_2}{<0.7,0.25,0.4>}, \frac{w_3}{<0.5,0.35,0.5>}, \frac{w_4}{<0.2,0.45,0.6>} \right\}\right), \\ \left(((\upsilon_1, d, 1), (\upsilon_3, c, 1)), \left\{ \frac{w_1}{<0.3,0.45,0.6>}, \frac{w_2}{<0.6,0.25,0.5>}, \frac{w_3}{<0.7,0.30,0.2>}, \frac{w_4}{<0.8,0.35,0.4>} \right\}\right), \\ \left(((\upsilon_1, c, 1), (\upsilon_3, c, 1)), \left\{ \frac{w_1}{<0.1,0.55,0.4>}, \frac{w_2}{<0.6,0.30,0.5>}, \frac{w_3}{<0.7,0.30,0.2>}, \frac{w_4}{<0.8,0.45,0.4>} \right\}\right), \\ \left(((\upsilon_3, c, 0), (\upsilon_1, c, 0)), \left\{ \frac{w_1}{<0.2,0.35,0.3>}, \frac{w_2}{<0.7,0.40,0.4>}, \frac{w_3}{<0.6,0.15,0.2>}, \frac{w_4}{<0.7,0.25,0.3>} \right\}\right), \\ \left(((\upsilon_3, c, 0), (\upsilon_3, c, 1)), \left\{ \frac{w_1}{<0.1,0.55,0.6>}, \frac{w_2}{<0.4,0.40,0.5>}, \frac{w_3}{<0.7,0.10,0.2>}, \frac{w_4}{<0.8,0.15,0.4>} \right\}\right), \end{array} \right\}.$$

Proposition 4.16. *If* $(\hbar_1, \mathbb{G}_1)$,$(\hbar_2, \mathbb{G}_2)$ *and* $(\hbar_3, \mathbb{G}_3)$ *are three NHSESs over* Δ*, then*

(1) $((\hbar_1, \mathbb{G}_1) \wedge (\hbar_2, \mathbb{G}_2))^c = ((\hbar_1, \mathbb{G}_1))^c \vee ((\hbar_2, \mathbb{G}_2))^c$

(2) $((\hbar_1, \mathbb{G}_1) \vee (\hbar_2, \mathbb{G}_2))^c = ((\hbar_1, \mathbb{G}_1))^c \wedge ((\hbar_2, \mathbb{G}_2))^c$.

Proposition 4.17. *If* $(\hbar_1, \mathbb{G}_1)$,$(\hbar_2, \mathbb{G}_2)$ *and* $(\hbar_3, \mathbb{G}_3)$ *are three NHSESs over* Δ*, then*

(1) $((\hbar_1, \mathbb{G}_1) \wedge (\hbar_2, \mathbb{G}_2)) \wedge (\hbar_3, \mathbb{G}_3) = (\hbar_1, \mathbb{G}_1) \wedge ((\hbar_2, \mathbb{G}_2) \wedge (\hbar_3, \mathbb{G}_3))$

(2) $((\hbar_1, \mathbb{G}_1) \vee (\hbar_2, \mathbb{G}_2)) \vee (\hbar_3, \mathbb{G}_3) = (\hbar_1, \mathbb{G}_1) \vee ((\hbar_2, \mathbb{G}_2) \vee (\hbar_3, \mathbb{G}_3))$

(3) $(\hbar_1, \mathbb{G}_1) \vee ((\hbar_2, \mathbb{G}_2) \wedge (\hbar_3, \mathbb{G}_3) = ((\hbar_1, \mathbb{G}_1) \vee ((\hbar_2, \mathbb{G}_2)) \wedge ((\hbar_1, \mathbb{G}_1) \vee (\hbar_3, \mathbb{G}_3))$

(4) $(\hbar_1, \mathbb{G}_1) \wedge ((\hbar_2, \mathbb{G}_2) \vee (\hbar_3, \mathbb{G}_3)) = ((\hbar_1, \mathbb{G}_1) \wedge ((\hbar_2, \mathbb{G}_2)) \vee ((\hbar_1, \mathbb{G}_1) \wedge (\hbar_3, \mathbb{G}_3))$.

5. Basic Properties and Laws of Neutrosophic Hypersoft Expert Set Operations

In this important part of the paper, certain important characteristics and laws are explained for NHSES.

Here $(\hbar, \mathbb{G}), (\hbar, \mathbb{G}_1), (\hbar, \mathbb{G}_2), (\hbar, \mathbb{G}_3)$ and $(\hbar_1, \mathbb{G})$ are NHSESs over Δ

- Idempotent Laws

  (a) $(\hbar, \mathbb{G}) \cup (\hbar, \mathbb{G}) = (\hbar, \mathbb{G}) = (\hbar, \mathbb{G}) \cup_R (\hbar, \mathbb{G})$

  (b) $(\hbar, \mathbb{G}) \cap (\hbar, \mathbb{G}) = (\hbar, \mathbb{G}) = (\hbar, \mathbb{G}) \cap_\varepsilon (\hbar, \mathbb{G})$

- Identity Laws

  (a) $(\hbar, \mathbb{G}) \cup (\hbar, \mathbb{G})_\Phi = (\hbar, \mathbb{G}) = (\hbar, \mathbb{G}) \cup_R (\hbar, \mathbb{G})_\Phi$

  (b) $(\hbar, \mathbb{G}) \cap (\hbar, \mathbb{G})_\Delta = (\hbar, \mathbb{G}) = (\hbar, \mathbb{G}) \cap_\varepsilon (\hbar, \mathbb{G})_\Delta$.

- Domination Laws

  (a) $(\hbar, \mathbb{G}) \cup (\hbar, \mathbb{G})_\Delta = (\hbar, \mathbb{G})_\Delta = (\hbar, \mathbb{G}) \cup_R (\hbar, \mathbb{G})_\Delta$

  (b) $(\hbar, \mathbb{G}) \cap (\hbar, \mathbb{G})_\Phi = (\hbar, \mathbb{G})_\Phi = (\hbar, \mathbb{G}) \cap_\varepsilon (\hbar, \mathbb{G})_\Phi$.

- Characteristic of Exclusion

  $(\hbar, \mathbb{G}) \cup (\hbar, \mathbb{G})^c = (\hbar, \mathbb{G})_\Delta = (\hbar, \mathbb{G}) \cup_R (\hbar, \mathbb{G})^c$.

- Characteristic of Contradiction

  $(\hbar, \mathbb{G}) \cap (\hbar, \mathbb{G})^c = (\hbar, \mathbb{G})_\Phi = (\hbar, \mathbb{G}) \cap_\varepsilon (\hbar, \mathbb{G})^c$.

- Absorption Laws
  (a) $(\hbar, \mathbb{G}_1) \cup ((\hbar, \mathbb{G}_1) \cap (\hbar, \mathbb{G}_1)) = (\hbar, \mathbb{G}_1)$
  (b) $(\hbar, \mathbb{G}_1) \cap ((\hbar, \mathbb{G}_1) \cup (\hbar, \mathbb{G}_1)) = (\hbar, G_1)$
  (c) $(\hbar, \mathbb{G}_1) \cup_R ((\hbar, \mathbb{G}_1) \cap_\varepsilon (\hbar, \mathbb{G}_1)) = (\hbar, \mathbb{G}_1)$
  (d) $(\hbar, \mathbb{G}_1) \cap_\varepsilon ((\hbar, \mathbb{G}_1) \cup_R (\hbar, \mathbb{G}_1)) = (\hbar, \mathbb{G}_1)$.

- Absorption Laws
  (a) $((\hbar, \mathbb{G}_1) \cup (\hbar, \mathbb{G}_2)) = ((\hbar, \mathbb{G}_1) \cup (\hbar, \mathbb{G}_2))$
  (b) $((\hbar, \mathbb{G}_1) \cup_R (\hbar, \mathbb{G}_2)) = ((\hbar, \mathbb{G}_1) \cup_R (\hbar, \mathbb{G}_2))$
  (c) $((\hbar, \mathbb{G}_1) \cap (\hbar, \mathbb{G}_2)) = ((\hbar, \mathbb{G}_1) \cap (\hbar, \mathbb{G}_2))$
  (d) $((\hbar, \mathbb{G}_1) \cap_\varepsilon (\hbar, \mathbb{G}_2)) = ((\hbar, \mathbb{G}_1) \cap_\varepsilon (\hbar, \mathbb{G}_2))$.

- Associative Laws
  (a) $(\hbar, \mathbb{G}_1) \cup ((\hbar, \mathbb{G}_2) \cup (\hbar_1, G_3)) = ((\hbar, \mathbb{G}_1) \cup (\hbar, \mathbb{G}_2)) \cup (\hbar_1, \mathbb{G}_3)$
  (b) $(\hbar, \mathbb{G}_1) \cup_R ((\hbar, \mathbb{G}_2) \cup_R (\hbar_1, \mathbb{G}_3)) = ((\hbar, \mathbb{G}_1) \cup_R (\hbar, G_2)) \cup_R (\hbar_1, \mathbb{G}_3)$
  (c) $(\hbar, \mathbb{G}_1) \cap ((\hbar, \mathbb{G}_2) \cap (\hbar_1, G_3)) = ((\hbar, \mathbb{G}_1) \cap (\hbar, \mathbb{G}_2)) \cap (\hbar_1, \mathbb{G}_3)$
  (d) $(\hbar, \mathbb{G}_1) \cap_\varepsilon ((\hbar, \mathbb{G}_2) \cap_\varepsilon (\hbar_1, \mathbb{G}_3)) = ((\hbar, \mathbb{G}_1) \cap_\varepsilon (\hbar, \mathbb{G}_2)) \cap_\varepsilon (\hbar_1, \mathbb{G}_3)$
  (e) $(\hbar, \mathbb{G}_1) \bigvee ((\hbar, \mathbb{G}_2) \bigvee (\hbar_1, G_3)) = ((\hbar, \mathbb{G}_1) \bigvee (\hbar, \mathbb{G}_2)) \bigvee (\hbar_1, \mathbb{G}_3)$
  (e) $(\hbar, \mathbb{G}_1) \bigwedge ((\hbar, \mathbb{G}_2) \bigwedge (\hbar_1, \mathbb{G}_3)) = ((\hbar, \mathbb{G}_1) \bigwedge (\hbar, \mathbb{G}_2)) \bigwedge (\hbar_1, \mathbb{G}_3)$.

- De Morgan's Laws
  (a) $((\hbar, \mathbb{G}_1) \cup (\hbar, \mathbb{G}_2))^c = (\hbar, \mathbb{G}_1)^c \cap_\varepsilon (\hbar, \mathbb{G}_2)^c$
  (b) $((\hbar, \mathbb{G}_1) \cap_\varepsilon (\hbar, \mathbb{G}_2))^c = (\hbar, \mathbb{G}_1)^c \cup (\hbar, \mathbb{G}_2)^c$
  (c) $((\hbar, \mathbb{G}_1) \bigvee (\hbar, \mathbb{G}_2))^c = (\hbar, \mathbb{G}_1)^c \bigwedge (\hbar, \mathbb{G}_2)^c$
  (d) $((\hbar, \mathbb{G}_1) \bigwedge (\hbar, \mathbb{G}_2))^c = (\hbar, \mathbb{G}_1)^c \bigvee (\hbar, \mathbb{G}_2)^c$.

- Distributive Laws
  (a) $(\hbar, \mathbb{G}_1) \cup ((\hbar, \mathbb{G}_2) \cap (\hbar_1, \mathbb{G}_3)) = ((\hbar, \mathbb{G}_1) \cup (\hbar, \mathbb{G}_2)) \cap ((\hbar, \mathbb{G}_1) \cup (\hbar_1, \mathbb{G}_3))$
  (b) $(\hbar, \mathbb{G}_1) \cap ((\hbar, \mathbb{G}_2) \cup (\hbar_1, \mathbb{G}_3)) = ((\hbar, \mathbb{G}_1) \cap (\hbar, \mathbb{G}_2)) \cup ((\hbar, \mathbb{G}_1) \cap (\hbar_1, \mathbb{G}_3))$
  (c) $(\hbar, \mathbb{G}_1) \cup_R ((\hbar, \mathbb{G}_2) \cap_\varepsilon (\hbar_1, \mathbb{G}_3)) = ((\hbar, \mathbb{G}_1) \cup_R (\hbar, \mathbb{G}_2)) \cap_\varepsilon ((\hbar, \mathbb{G}_1) \cup_R (\hbar_1, \mathbb{G}_3))$
  (d) $(\hbar, \mathbb{G}_1) \cap_\varepsilon ((\hbar, \mathbb{G}_2) \cup_R (\hbar_1, \mathbb{G}_3)) = ((\hbar, \mathbb{G}_1) \cap_\varepsilon (\hbar, \mathbb{G}_2)) \cup_R ((\hbar, \mathbb{G}_1) \cap_\varepsilon (\hbar_1, \mathbb{G}_3))$
  (c) $(\hbar, \mathbb{G}_1) \cup_R ((\hbar, \mathbb{G}_2) \cap (\hbar_1, \mathbb{G}_3)) = ((\hbar, \mathbb{G}_1) \cup_R (\hbar, G_2)) \cap ((\hbar, \mathbb{G}_1) \cup_R (\hbar_1, \mathbb{G}_3))$
  (e) $(\hbar, \mathbb{G}_1) \cap ((\hbar, \mathbb{G}_2) \cup_R (\hbar_1, \mathbb{G}_3)) = ((\hbar, \mathbb{G}_1) \cap (\hbar, \mathbb{G}_2)) \cup_R ((\hbar, \mathbb{G}_1) \cap (\hbar_1, \mathbb{G}_3))$.

6. Hybrids of Neutrosophic Hypersoft Expert Set

In this study, some hybridized structures of NHSES are presented. Suppose $\mathbf{Y}$ denotes the set of expert and $\mathbf{O}$ be a set of opinions, $T = \mathbf{F} \times \mathbf{Y} \times \mathbf{O}$. Taking $A \subseteq T$ and Δ denotes the universe, while $\mathbf{F}$ used for parameters.

Definition 6.1. A bipolar neutrosophic hypersoft expert set is a pair $(\mathbb{B}, A)$ and is characterized by a mapping

$$\mathbb{B} : A \to P(\Delta)$$

where

$$(\mathbb{B}, A) = \{\langle x, \upsilon^{+}_{B(e)}(x), \nu^{+}_{B(e)}(x), \omega^{+}_{B(e)}(x), \upsilon^{-}_{B(e)}(x), \nu^{-}_{B(e)}(x), \omega^{-}_{B(e)}(x)\rangle : \forall e \in A, x \in \Delta\}$$

, where $\upsilon^{+}_{B(e)}, \nu^{+}_{B(e)}, \omega^{+}_{B(e)} : \Delta \to [0, 1]$, $\upsilon^{-}_{B(e)}, \nu^{-}_{B(e)}, \omega^{-}_{B(e)} : \Delta \to [0, 1]$.

Example 6.2. Considering Example 3.3 with $\Delta = \{w_1, w_2\}$, we have bipolar neutrosophic hypersoft expert set as

$$(\mathbb{B}, \mathrm{A}) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{\frac{w_1}{<0.2,0.5,0.4,-0.1,-0.2,-0.3>}, \frac{w_2}{<0.1,0.3,0.6,-0.2,-0.3,-0.2>}\right\}\right), \\ \left((\upsilon_1, d, 1), \left\{\frac{w_1}{<0.4,0.2,0.3,-0.1,-0.1,-0.2>}, \frac{w_2}{<0.2,0.5,0.3,-0.1,-0.2,-0.5>}\right\}\right), \\ \left((\upsilon_1, e, 1), \left\{\frac{w_1}{<0.7,0.2,0.3,-0.3,-0.1,-0.2>}, \frac{w_2}{<0.3,0.5,0.6,-0.2,-0.3,-0.4>}\right\}\right), \\ \left((\upsilon_2, c, 1), \left\{\frac{w_1}{<0.9,0.1,0.3,-0.3,-0.2,-0.1>}, \frac{w_2}{<0.3,0.4,0.8,-0.1,-0.7,-0.4>}\right\}\right), \\ \left((\upsilon_2, d, 1), \left\{\frac{w_1}{<0.4,0.5,0.6,-0.2,-0.3,-0.4>}, \frac{w_2}{<0.2,0.6,0.7,-0.1,-0.3,-0.4>}\right\}\right), \\ \left((\upsilon_2, e, 1), \left\{\frac{w_1}{<0.5,0.4,0.7,-0.1,-0.2,-0.3>}, \frac{w_2}{<0.8,0.1,0.6,-0.2,-0.2,-0.3>}\right\}\right), \\ \left((\upsilon_1, c, 0), \left\{\frac{w_1}{<0.3,0.2,0.1,-0.1,-0.2,-0.3>}, \frac{w_2}{<0.1,0.8,0.3,-0.1,-0.7,-0.2>}\right\}\right), \\ \left((\upsilon_1, d, 0), \left\{\frac{w_1}{<0.1,0.8,0.4,-0.1,-0.2,-0.3>}, \frac{w_2}{<0.2,0.7,0.5,-0.1,-0.3,-0.4>}\right\}\right), \\ \left((\upsilon_1, e, 0), \left\{\frac{w_1}{<0.2,0.7,0.5,-0.1,-0.3,-0.4>}, \frac{w_2}{<0.5,0.4,0.6,-0.1,-0.3,-0.4>}\right\}\right), \end{array} \right\}.$$

Definition 6.3. A complex neutrosophic hypersoft expert set $(\mathcal{C}, A)$ is characterized by a mapping

$$\mathcal{C} : A \to \mathcal{C}N^{\Delta}$$

where $\mathcal{C}N^{\Delta}$ denotes the collection of all complex neutrosophic subsets of Δ and $(\mathbb{C}, A) = \{\langle x, \upsilon_{C(e)}(x), \nu_{C(e)}(x), \omega_{C(e)}(x)\rangle : \forall e \in A, x \in \Delta\}$, where

$$\upsilon_{C(e)}(x) = aC(e)(x).e^{jC(e)(x)}, \nu_{C(e)}(x) = bC(e)(x).e^{jC(e)(x)}, \omega_{C(e)}(x) = cC(e)(x).e^{jC(e)(x)}$$

for all $u \in \Delta$ while $\upsilon_{C(e)}, \nu_{C(e)}, \omega_{C(e)}$ are complex-valued truth, indeterminacy and falsity membership functions and these values lie within the unit circle in the complex plane and both the amplitude terms $aC(e)(x), bC(e)(x), cC(e)(x)$ and the phase terms $\upsilon C(e)(x), \nu C(e)(x), \omega C(e)(x)$ are real valued such that $0 \leq aC(e)(x)+bC(e)(x)+cC(e)(x) \leq 3$ while $aC(e)(x), bC(e)(x), cC(e)(x) \in [0, 1]$.

Example 6.4. Considering Example 3.3, we have complex neutrosophic hypersoft expert as

$$(\mathbb{C}, \mathrm{A}) = \left\{ \begin{array}{l} \left((\upsilon_1, c, 1), \left\{ \frac{w_1}{\langle 0.1e^{j2\pi(0.3)}, 0.6e^{j2\pi(0.3)}, 0.7e^{j2\pi(0.3)}\rangle}, \frac{w_2}{\langle 0.9e^{j2\pi(0.3)}, 0.7e^{j2\pi(0.3)}, 0.8e^{j2\pi(0.3)}\rangle} \right\}\right), \\ \left((\upsilon_1, d, 1), \left\{ \frac{w_1}{\langle 0.2e^{j2\pi(0.3)}, 0.6e^{j2\pi(0.3)}, 0.3e^{j2\pi(0.3)}\rangle}, \frac{w_2}{\langle 0.1e^{j2\pi(0.3)}, 0.8e^{j2\pi(0.3)}, 0.2e^{j2\pi(0.3)}\rangle} \right\}\right), \\ \left((\upsilon_1, e, 1), \left\{ \frac{w_1}{\langle 0.6e^{j2\pi(0.3)}, 0.7e^{j2\pi(0.3)}, 0.8e^{j2\pi(0.3)}\rangle}, \frac{w_2}{\langle 0.6e^{j2\pi(0.3)}, 0.9e^{j2\pi(0.3)}, 0.2e^{j2\pi(0.3)}\rangle} \right\}\right), \\ \left((\upsilon_2, c, 1), \left\{ \frac{w_1}{\langle 0.4e^{j2\pi(0.3)}, 0.5e^{j2\pi(0.3)}, 0.4e^{j2\pi(0.3)}\rangle}, \frac{w_2}{\langle 0.1e^{j2\pi(0.3)}, 0.8e^{j2\pi(0.3)}, 0.5e^{j2\pi(0.3)}\rangle} \right\}\right), \\ \left((\upsilon_2, d, 1), \left\{ \frac{w_1}{\langle 0.4e^{j2\pi(0.3)}, 0.7e^{j2\pi(0.3)}, 0.4e^{j2\pi(0.3)}\rangle}, \frac{w_2}{\langle 0.6e^{j2\pi(0.3)}, 0.8e^{j2\pi(0.3)}, 0.1e^{j2\pi(0.3)}\rangle} \right\}\right), \\ \left((\upsilon_2, e, 1), \left\{ \frac{w_1}{\langle 0.4e^{j2\pi(0.3)}, 0.7e^{j2\pi(0.3)}, 0.2e^{j2\pi(0.3)}\rangle}, \frac{w_2}{\langle 0.4e^{j2\pi(0.3)}, 0.6e^{j2\pi(0.3)}, 0.6e^{j2\pi(0.3)}\rangle} \right\}\right), \\ \left((\upsilon_1, c, 0), \left\{ \frac{w_1}{\langle 0.3e^{j2\pi(0.3)}, 0.3e^{j2\pi(0.3)}, 0.2e^{j2\pi(0.3)}\rangle}, \frac{w_2}{\langle 0.1e^{j2\pi(0.3)}, 0.6e^{j2\pi(0.3)}, 0.1e^{j2\pi(0.3)}\rangle} \right\}\right), \\ \left((\upsilon_1, d, 0), \left\{ \frac{w_1}{\langle 0.7e^{j2\pi(0.3)}, 0.4e^{j2\pi(0.3)}, 0.2e^{j2\pi(0.3)}\rangle}, \frac{w_2}{\langle 0.1e^{j2\pi(0.3)}, 0.6e^{j2\pi(0.3)}, 0.2e^{j2\pi(0.3)}\rangle} \right\}\right), \\ \left((\upsilon_1, e, 0), \left\{ \frac{w_1}{\langle 0.6e^{j2\pi(0.3)}, 0.7e^{j2\pi(0.3)}, 0.4e^{j2\pi(0.3)}\rangle}, \frac{w_2}{\langle 0.5e^{j2\pi(0.3)}, 0.9e^{j2\pi(0.3)}, 0.4e^{j2\pi(0.3)}\rangle} \right\}\right), \\ \left((\upsilon_2, c, 0), \left\{ \frac{w_1}{\langle 0.6e^{j2\pi(0.3)}, 0.7e^{j2\pi(0.3)}, 0.5e^{j2\pi(0.3)}\rangle}, \frac{w_2}{\langle 0.7e^{j2\pi(0.3)}, 0.9e^{j2\pi(0.3)}, 0.3e^{j2\pi(0.3)}\rangle} \right\}\right) \end{array} \right\}.$$

Definition 6.5. A pair (F, H) is called a fuzzy parameterized complex neutrosophic hypersoft expert set(FP-CNHSES) over Δ, where F is a mapping given by

$$F : H \to CN^{\Delta}$$

where CN^{Δ} is the collection of all complex neutrosophic subsets of Δ.
It can also be written as $(F,\ H) = \left\{\left(t, \left\{\frac{w}{F(t)(x)} : x \in \Delta\right\}\right) : t \in H\right\}$
where $H \subseteq \mathcal{G} \times \mathcal{D} \times \mathbb{C} = \left\{\left(\frac{\alpha}{\Im(\alpha)} : \beta,\ \gamma \in \Delta\right) : \alpha \in \mathcal{G},\ \beta \in \mathcal{D}, \gamma \in \mathcal{C}\right\}$ with $\Im$ is a corresponding membership function of fuzzy set and

$$(F, H) = \langle x, \upsilon_{C(e)}(x), \nu_{C(e)}(x), \omega_{C(e)}(x)\rangle : \forall e \in H, x \in \Delta,$$

where

$$\upsilon_{C(e)}(x) = aC(e)(x).e^{jC(e)(x)}, \nu_{C(e)}(x) = bC(e)(x).e^{jC(e)(x)}, \omega_{C(e)}(x) = cC(e)(x).e^{jC(e)(x)}$$

for all $x \in \Delta$ while $\upsilon_{C(e)}, \nu_{C(e)}, \omega_{C(e)}$ are complex-valued truth, indeterminacy and falsity membership functions for or the FP-CNHSES and these values lie within the unit circle in the complex plane and both the amplitude terms $aC(e)(x), bC(e)(x), cC(e)(x)$ and the phase terms $\upsilon C(e)(x), \nu C(e)(x), \omega C(e)(x)$ are real valued such that $0 \leq aC(e)(x)+bC(e)(x)+cC(e)(x) \leq 3$.

Example 6.6. Considering Example 3.3 with $\Im = \left\{\frac{\upsilon_1}{0.2}, \frac{\upsilon_2}{0.3}, \frac{\upsilon_3}{0.5}\right\}$ as a fuzzy subset of FZ(E). We can define FP-CNHSES as

$$(\mathrm{F},\mathrm{H}) = \left\{ \begin{array}{l}
\left((\frac{\upsilon_1}{0.2}, c, 1), \left\{ \frac{w_1}{\langle 0.1e^{2\pi(0.2)},\, 0.6e^{2\pi(0.2)},\, 0.7e^{2\pi(0.2)} \rangle}, \frac{w_2}{\langle 0.9e^{2\pi(0.2)},\, 0.7e^{2\pi(0.2)},\, 0.8e^{2\pi(0.2)} \rangle} \right\} \right), \\
\left((\frac{\upsilon_2}{0.3}, d, 1), \left\{ \frac{w_1}{\langle 0.2e^{2\pi(0.2)},\, 0.6e^{2\pi(0.2)},\, 0.3e^{2\pi(0.2)} \rangle}, \frac{w_2}{\langle 0.1e^{2\pi(0.2)},\, 0.8e^{2\pi(0.2)},\, 0.2e^{2\pi(0.2)} \rangle} \right\} \right), \\
\left((\frac{\upsilon_3}{0.5}, e, 1), \left\{ \frac{w_1}{\langle 0.6e^{2\pi(0.2)},\, 0.7e^{2\pi(0.2)},\, 0.8e^{2\pi(0.2)} \rangle}, \frac{w_2}{\langle 0.6e^{2\pi(0.2)},\, 0.9e^{2\pi(0.2)},\, 0.2e^{2\pi(0.2)} \rangle} \right\} \right), \\
\left((\frac{\upsilon_1}{0.2}, c, 1), \left\{ \frac{w_1}{\langle 0.4e^{2\pi(0.2)},\, 0.5e^{2\pi(0.2)},\, 0.4e^{2\pi(0.2)} \rangle}, \frac{w_2}{\langle 0.1e^{2\pi(0.3)},\, 0.8e^{2\pi(0.2)},\, 0.5e^{2\pi(0.2)} \rangle} \right\} \right), \\
\left((\frac{\upsilon_2}{0.3}, d, 1), \left\{ \frac{w_1}{\langle 0.4e^{2\pi(0.2)},\, 0.7e^{2\pi(0.2)},\, 0.4e^{2\pi(0.2)} \rangle}, \frac{w_2}{\langle 0.6e^{2\pi(0.2)},\, 0.8e^{2\pi(0.2)},\, 0.1e^{2\pi(0.2)} \rangle} \right\} \right), \\
\left((\frac{\upsilon_3}{0.5}, e, 1), \left\{ \frac{w_1}{\langle 0.4e^{2\pi(0.2)},\, 0.7e^{2\pi(0.2)},\, 0.2e^{2\pi(0.2)} \rangle}, \frac{w_2}{\langle 0.4e^{2\pi(0.2)},\, 0.6e^{2\pi(0.2)},\, 0.6e^{2\pi(0.2)} \rangle} \right\} \right), \\
\left((\frac{\upsilon_1}{0.2}, c, 0), \left\{ \frac{w_1}{\langle 0.3e^{2\pi(0.3)},\, 0.3e^{2\pi(0.2)},\, 0.2e^{2\pi(0.2)} \rangle}, \frac{w_2}{\langle 0.1e^{2\pi(0.3)},\, 0.6e^{2\pi(0.2)},\, 0.1e^{2\pi(0.2)} \rangle} \right\} \right), \\
\left((\frac{\upsilon_2}{0.3}, d, 0), \left\{ \frac{w_1}{\langle 0.7e^{2\pi(0.2)},\, 0.4e^{2\pi(0.2)},\, 0.2e^{2\pi(0.2)} \rangle}, \frac{w_2}{\langle 0.1e^{2\pi(0.2)},\, 0.6e^{2\pi(0.2)},\, 0.2e^{2\pi(0.2)} \rangle} \right\} \right), \\
\left((\frac{\upsilon_3}{0.5}, e, 0), \left\{ \frac{w_1}{\langle 0.6e^{2\pi(0.2)},\, 0.7e^{2\pi(0.2)},\, 0.4e^{22\pi(0.2)} \rangle}, \frac{w_2}{\langle 0.5e^{2\pi(0.2)},\, 0.9e^{2\pi(0.2)},\, 0.4e^{2\pi(0.2)} \rangle} \right\} \right), \\
\left((\frac{\upsilon_1}{0.2}, c, 0), \left\{ \frac{w_1}{\langle 0.6e^{2\pi(0.2)},\, 0.7e^{2\pi(0.2)},\, 0.5e^{2\pi(0.2)} \rangle}, \frac{w_2}{\langle 0.7e^{2\pi(0.3)},\, 0.9e^{2\pi(0.2)},\, 0.3e^{2\pi(0.2)} \rangle} \right\} \right)
\end{array} \right\}.$$

Definition 6.7. A pair (V, A) is called a neutrosophic vague hypersoft expert set is a pair (V, A), with V representing a mapping $V : A \rightarrow NV^{\Delta}$, and NV^{Δ} is being used for the power neutrosophic vague set of Δ. Let mapping V is defined by as $V(t) = V(t)(x),\ \ \mathrm{x} \in \Delta$. For each $t_i \in A$, $V(t_i) = V(t_i)(x)$, where $V(t_i)$ represents the truth, indeterminacy and falsity membership functions of Δ in $V(t_i)$. Hence $V(t_i)$ can be written as
$V(t_i) = \ \left\{\frac{x_i}{V(t_i)x_i}\right\}$, for $i = 1, 2, 3, ...$
where $V(t_i)(x_i) = [\upsilon - \omega(t_i)(x_i), \upsilon + \omega(t_i)(x_i)], [I - \omega(t_i)(x_i), I + \omega(t_i)(x_i)], [\omega - \omega(t_i)(x_i), \omega + \omega(t_i)(x_i)]$ and $\upsilon + \omega(t_i)(x_i) = 1 - \omega(t_i)(x_i), \omega + \omega(t_i)(x_i) = 1 - \upsilon - \omega(t_i)(x_i)$ with $[\upsilon\omega(t_i)(x_i), \upsilon + \omega(t_i)(x_i)], [I - \omega(t_i)(x_i), I + \omega(t_i)(x_i)]$ representing the truth, indeterminacy and falsity-membership functions of each of the elements $x_i \in \Delta$, respectively.

Example 6.8. Considering Example 3.3 with $\Delta = \{w_1, w_2\}$, we have neutrosophic vague hypersoft expert set as

$$(\mathbb{B},\mathrm{A}) = \left\{ \begin{array}{l}
\left((\upsilon_1, c, 1), \left\{ \frac{w_1}{<[0.2,0.5],[0.4,0.1],[0.2,0.3]>}, \frac{w_2}{<[0.1,0.3],[0.6,0.2],[0.3,0.2]>} \right\} \right), \\
\left((\upsilon_1, d, 1), \left\{ \frac{w_1}{<[0.4,0.2],[0.3,0.1],[0.1,0.2]>}, \frac{w_2}{<[0.2,0.5],[0.3,0.1],[0.2,0.5]>} \right\} \right), \\
\left((\upsilon_1, e, 1), \left\{ \frac{w_1}{<[0.7,0.2],[0.3,0.8],[0.1,0.6]>}, \frac{w_2}{<[0.3,0.5],[0.6,0.2],[0.3,0.4]>} \right\} \right), \\
\left((\upsilon_2, c, 1), \left\{ \frac{w_1}{<[0.9,0.1],[0.3,0.3],[0.2,0.1]>}, \frac{w_2}{<[0.3,0.4],[0.8,0.1],[0.7,0.4]>} \right\} \right), \\
\left((\upsilon_2, d, 1), \left\{ \frac{w_1}{<[0.4,0.5],[0.6,0.2],[0.3,0.4]>}, \frac{w_2}{<[0.2,0.6],[0.7,0.1],[0.3,0.4]>} \right\} \right), \\
\left((\upsilon_2, e, 1), \left\{ \frac{w_1}{<[0.5,0.4],[0.7,0.1],[0.2,0.3]>}, \frac{w_2}{<[0.8,0.1],[0.6,0.2],[0.2,0.3]>} \right\} \right), \\
\left((\upsilon_1, c, 0), \left\{ \frac{w_1}{<[0.3,0.2],[0.1,0.1],[0.2,0.3]>}, \frac{w_2}{<[0.1,0.8],[0.3,0.1],[0.7,0.2]>} \right\} \right), \\
\left((\upsilon_1, d, 0), \left\{ \frac{w_1}{<[0.1,0.8],[0.4,0.1],[0.2,0.3]>}, \frac{w_2}{<[0.2,0.7],[0.5,0.1],[0.3,0.4]>} \right\} \right), \\
\left((\upsilon_1, e, 0), \left\{ \frac{w_1}{<[0.2,0.7],[0.5,0.1],[0.3,0.4]>}, \frac{w_2}{<[0.5,0.4],[0.6,0.1],[0.3,0.4]>} \right\} \right),
\end{array} \right\}.$$

7. An Application to Neutrosophic Hypersoft Expert Set

An application of NHSES theory related to the decision-making problem is presented while using an algorithmic technique..

Statement of the problem

Mr Jay needs to buy a mask from a business opportunity for his own wellbeing. He takes help from his a few companions (Henry, John and Watson) who have skill in mask buying.

Proposed Algorithm For Selection Of Mask

The accompanying calculation is embraced for this choice (purchase).

(1) Construct NHSES $(\hbar, \mathbb{G})$,
(2) Determine an Agree and Disagree-NHSES,
(3) Compute $d_i = \sum_i t_{ij}$ for Agree-NHSES,
(4) Determine $q_i = \sum_i t_{ij}$ for Disagree-NHSES,
(5) Determine $g_j = d_j - q_j$ for Agree and Disaree-NHSES,
(6) Compute n, for which $p_n = \max p_j$ for best solution of the product.

Step-1

Let eight categories of mask which are being used for the universe of discourse $\Omega = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\}$ and X $= \{\rho_1 = Henry, \rho_2 = John, \rho_3 = Watson\}$ be a set of experts. The prescribed attributes for the attribute-valued sets are :

$O_1 = Brand = \{o_1 = new, o_2 = old\}$

$O_2 = Price = \{o_3 = l00dollar, o_4 = 50dollar\}$

$O_3 = Colour = \{o_5 = black, o_6 = blue\}$

$O_4 = Quality = \{o_7 = good, o_8 = better\}$

$O_5 = Shape = \{o_9 = circular, o_{10} = square\}$

and then $O = O_1 \times O_2 \times O_3 \times O_4 \times O_5$

$$O = \left\{ \begin{array}{l} (o_1, o_3, o_5, o_7, o_9), (o_1, o_3, o_5, o_7, o_{10}), (o_1, o_3, o_5, o_8, o_9), (o_1, o_3, o_5, o_8, o_{10}), (o_1, o_3, o_6, o_7, o_9), \\ (o_1, o_3, o_6, o_7, o_{10}), (o_1, o_3, o_6, o_8, o_9), (o_1, o_3, o_6, o_8, o_{10}), (o_1, o_4, o_5, o_7, o_9), (o_1, o_4, o_5, o_7, o_{10}), \\ (o_1, o_4, o_5, o_8, o_9), (o_1, o_4, o_5, o_8, o_{10}), (o_1, o_4, o_6, o_7, o_9), (o_1, o_4, o_6, o_7, o_{10}), (o_1, o_4, o_6, o_8, o_9), \\ (o_1, o_4, o_6, o_8, o_{10}), (o_2, o_3, o_5, o_7, o_9), (o_2, o_3, o_5, o_7, o_{10}), (o_2, o_3, o_5, o_8, o_9), (o_2, o_3, o_5, o_8, o_{10}), \\ (o_2, o_3, o_6, o_7, o_9), (o_2, o_3, o_6, o_7, o_{10}), (o_2, o_3, o_6, o_8, o_9), (o_2, o_3, o_6, o_8, o_{10}), (o_2, o_4, o_5, o_7, o_9), \\ (o_2, o_4, o_5, o_7, o_{10}), (o_2, o_4, o_5, o_8, o_9), (o_2, o_4, o_5, o_8, o_{10}), (o_2, o_4, o_6, o_7, o_9), (o_2, o_4, o_6, o_7, o_{10}), \\ (o_2, o_4, o_6, o_8, o_9), (o_2, o_4, o_6, o_8, o_{10}) \end{array} \right\}$$

Now take $Q \subseteq O$ as

$Q = \{q_1 = (o_1, o_3, o_5, o_7, o_9), q_2 = (o_1, o_3, o_6, o_7, o_{10}), q_3 = (o_1, o_4, o_6, o_8, o_9), q_4 =$

$(o_2, o_3, o_6, o_8, o_9), q_5 = (o_2, o_4, o_6, o_7, o_{10})\}$

$$(\hbar, \mathbb{G}) = \left\{ \begin{array}{l} ((q_1,\rho_1,1) = \{o_2,o_3,o_4,o_5,o_6,o_8\}),((q_1,\rho_2,1) = \{o_1,o_2,o_3,o_7\}),((q_2,\rho_1,1) = \{o_5,o_8\}), \\ ((q_2,\rho_2,1) = \{o_1,o_2,o_3,o_4,o_5,o_6,o_8\}),((q_3,\rho_1,1) = \{o_4,o_7\}),((q_3,\rho_2,1) = \{o_1,o_2,o_4,o_5,o_8\}), \\ ((q_3,\rho_3,1) = \{o_1,o_5,o_7,o_8\}), \\ ((q_4,\rho_1,1) = \{o_1,o_7,o_8\}),((q_4,\rho_2,1) = \{o_1,o_4,o_8\}),((q_4,\rho_3,1) = \{o_1,o_6,o_7,o_8\}), \\ ((q_5,\rho_1,1) = \{o_3,o_7,o_8\}),((q_5,\rho_2,1) = \{o_1,o_2,o_3,o_4,o_5,o_8\}),((q_5,\rho_3,0) = \{o_1,o_3,o_6\}) \\ ((q_5,\rho_3,1) = \{o_2,o_3,o_5,o_7,o_8\}),((q_1,\rho_1,0) = \{o_3,o_5,o_6\}),((q_1,\rho_2,0) = \{o_2,o_3,o_6,o_7\}), \\ ((q_1,\rho_3,0) = \{o_3,o_4\}),((q_2,\rho_1,0) = \{o_1,o_2,o_4,o_5,o_6,o_7\}),((q_2,\rho_2,0) = \{o_2,o_7\}), \\ ((q_2,\rho_3,0) = \{o_2,o_3,o_4,o_5,o_6\}),((q_3,\rho_1,0) = \{o_1,o_2,o_6,o_8\}),((q_3,\rho_2,0) = \{o_3,o_4,o_6,o_7\}), \\ ((q_3,\rho_3,0) = \{o_2,o_3,o_4,o_5,o_7\}),((q_5,\rho_1,0) = \{o_4,o_6,o_7\}),((q_4,\rho_1,0) = \{o_2,o_3,o_3,o_4,o_5,o_7\}), \\ ((q_4,\rho_3,0) = \{o_2,o_3,o_4,o_5\}),((q_5,\rho_2,0) = \{o_2,o_3,o_6,o_7\}), \\ ((q_1,\rho_3,1) = \{o_1,o_3,o_4,o_6,o_7,o_8\}),((q_4,\rho_2,0) = \{o_2,o_3,o_6,o_7\}), \\ ((q_2,\rho_3,1) = \{o_1,o_2,o_4,o_7,o_8\}), \end{array} \right\}$$

is a NHSES.

Step-2

The Agree and Disagree-NHSES are represented by Table 1 and Table 2 respectively, also when $o_i \in F_1(\beta)$ then $o_{ij} = \checkmark = 1$ diversely $o_{ij} = \times = 0$, and if

$$o_i \in F_0(\beta)$$

then $o_{ij} = \checkmark = 1$ diversely $o_{ij} = \times = 0$ while o_{ij} are being used as members of Tables 1 and 2.

Step-(3-5)

presents The $d_i = \sum_i o_{ij}$ for Agree-NHSES, $q_i = \sum_i o_{ij}$ for Disagree-NHSES are presented in Table 3 and $g_j = d_j - q_j$ have been shown and to choose product $p_n = \max p_j$ for solution.

Step-6-Decision

Since g_8 is maximum in above Table 3, so category b_8 is preferred to be selected for purchase.

8. Conclusions

In this paper,

- The fundamentals of neutrosophic hypersoft expert set are established and some necessary properties like subset, equal set, agree and disagree set, relative whole and relative null set, absolute whole set are explained with detailed examples.
- Some theoretic operations like union, restricted union, intersection, extended intersection, complement, AND and OR are generalized.
- Some basic laws such as idempotent, absorption, domination, identity, associative and distributive are discussed with examples.

TABLE 1. Agree-NHSES

B	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
(p_1, ρ_1)	×	✓	✓	✓	✓	✓	×	×
(p_2, ρ_1)	×	✓	×	×	×	×	✓	×
(p_3, ρ_1)	×	×	×	✓	×	×	✓	×
(p_4, ρ_1)	✓	×	×	×	×	×	✓	✓
(p_5, ρ_1)	✓	✓	✓	✓	×	×	×	✓
(p_1, ρ_2)	✓	×	✓	×	×	×	✓	×
(p_2, ρ_2)	✓	✓	✓	×	✓	✓	×	✓
(p_3, ρ_2)	×	✓	✓	×	×	×	✓	×
(p_4, ρ_2)	✓	×	×	×	×	×	×	✓
(p_5, ρ_2)	×	×	×	×	✓	×	×	×
(p_1, ρ_3)	✓	×	✓	✓	×	✓	✓	×
(p_2, ρ_3)	✓	✓	×	✓	✓	×	✓	×
(p_3, ρ_3)	✓	×	✓	×	✓	×	✓	×
(p_4, ρ_3)	✓	✓	✓	✓	×	✓	✓	×
(p_5, ρ_3)	×	✓	✓	✓	✓	×	✓	×
$d_j = \sum_i n_{ij}$	$d_1 = 09$	$d_2 = 08$	$d_3 = 9$	$d_4 = 7$	$d_5 = 06$	$d_6 = 4$	$d_7 = 10$	$d_8 = 11$

TABLE 2. Disagree-NHSES

B	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
(p_1, ρ_1)	×	×	✓	×	×	✓	×	×
(p_2, ρ_1)	✓	✓	×	✓	×	✓	✓	×
(p_3, ρ_1)	✓	✓	×	×	×	✓	×	✓
(p_4, ρ_1)	×	✓	✓	✓	✓	✓	×	×
(p_5, ρ_1)	×	×	×	✓	×	✓	✓	×
(p_1, ρ_2)	×	✓	✓	×	×	✓	✓	×
(p_2, ρ_2)	×	✓	×	×	×	×	✓	×
(p_3, ρ_2)	✓	✓	×	×	×	✓	×	✓
(p_4, ρ_2)	×	✓	✓	×	×	✓	✓	×
(p_5, ρ_2)	×	✓	✓	×	×	✓	✓	×
(p_1, ρ_3)	×	×	✓	✓	×	×	×	×
(p_2, ρ_3)	×	×	✓	×	✓	✓	×	×
(p_3, ρ_3)	×	✓	✓	✓	✓	✓	×	×
(p_4, ρ_3)	×	✓	✓	✓	✓	×	×	×
(p_5, ρ_3)	×	✓	×	✓	×	✓	×	×
$p_i = \sum_i n_{ij}$	$p_1 = 3$	$p_2 = 11$	$p_3 = 9$	$p_4 = 7$	$p_5 = 4$	$p_6 = 12$	$p_7 = 6$	$p_8 = 2$

- Some hybridized structures of neutrosophic hypersoft expert set are established with illustrative examples.

TABLE 3. Optimal

$d_i = \sum_i n_{ij}$	$q_i = \sum_i n_{ij}$	$g_j = d_j - q_j$
$d_1 = 09$	$q_1 = 3$	$g_1 = 6$
$d_2 = 8$	$q_2 = 11$	$g_2 = -3$
$d_3 = 9$	$q_3 = 9$	$g_3 = 0$
$d_4 = 7$	$q_4 = 7$	$g_4 = 0$
$d_5 = 06$	$q_5 = 4$	$g_5 = 2$
$d_6 = 4$	$q_6 = 12$	$g_6 = -8$
$d_7 = 10$	$q_7 = 6$	$g_7 = 4$
$d_8 = 11$	$q_8 = 2$	$g_8 = 9$

- An algorithm is developed to explain the procedure of decision making problem.
- An application related to the mask purchasing is described with the help of proposed algorithm.
- Future task may include the extension of the existing work for other neutrosophic hypersoft expert-like hybrids i.e., generalized neutrosophic, generalized interval valued neutrosophic, neutrosophic vague , interval-valued neutrosophic, etc. This new work will give an outstanding extension to existing theories for dealing with truthness, indeterminacy and falsity membership functions.

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Neutrosophic $\mathcal{N}-$structures on Sheffer stroke BCH-algebras

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Abstract. The aim of the study is to introduce a neutrosophic $\mathcal{N}-$subalgebra and neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebras. We prove that the level-set of a neutrosophic $\mathcal{N}-$subalgebra (neutrosophic $\mathcal{N}-$ideal) of a Sheffer stroke BCH-algebra is its subalgebra (ideal) and vice versa. Then it is shown that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCH-algebra forms a complete distributive modular lattice. Also, we state that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse is generally not true. We examine relationships between neutrosophic $\mathcal{N}-$ideals of Sheffer stroke BCH-algebras by means of a surjective homomorphism between these algebras. Finally, certain subsets of a Sheffer stroke BCH-algebra are defined by means of $\mathcal{N}-$functions on this algebraic structure and some properties are investigated.

Keywords: Sheffer stroke BCH-algebra; subalgebra; neutrosophic $\mathcal{N}-$ subalgebra; neutrosophic $\mathcal{N}-$ideal.

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1. **Introduction**

Sheffer stroke (or Sheffer operation) introduced by H. M. Sheffer is one of the two operators that can be used by itself, without any other logical operators to build a logical formal system [22]. Since it provides new, basic and easily applicable axiom systems for many algebraic structures, this operation has many applications in algebraic structures such as orthoimplication algebras [1], ortholattices [3], Boolean algebras [15], strong Sheffer stroke non-associative MV-algebras [4] and their neutrosophic N-structures [19], Sheffer Stroke Hilbert algebras [16] and their neutrosophic N-structures [17]. Besides, the concepts of BCK-algebras and BCI-algebras were introduced by Y. Imai and K. Iséki ([10], [11]) and BCK-algebras are proper subclasses of BCI-algebras. Also, a new class of these algebras so-called BCH-algebras is introduced and studied by Hu and Li ([8], [9]) and the new class contains BCK-algebras and BCI-algebras. Some properties of these new algebraic structures have been investigated by Chaudhry ([5], [6]), Dudek and Thomys [7]. Recently, BCH-algebras with Sheffer stroke, subalgebras, minimal and medial elements and BCA-parts of these algebras are studied by Oner et al [18].

On the other side, Zadeh introduced the fuzzy set theory [26] which is a generalization of ordinary sets, has the truth (t) (membership) function and positive meaning of information. Hence, scientists have been interested in negative meaning of information, and so, Atanassov introduced the intuitionistic fuzzy set theory [2] which is a generalization of fuzzy sets, has truth (t) (membership) and the falsehood (f) (nonmembership) functions. Besides, Smarandache introduced the neutrosophic set theory which is a generalization of the intuitionistic fuzzy set theory and has the indeteminacy/neutrality (i) function with membership and nonmembership functions [23, 24]. Thus, neutrosophic sets are defined on three components (t, i, f) [27]. In recent times, neutrospohic sets are applied to the algebraic structures such as BCK/BCI-algebras and BE-algebras [12–14, 20, 25].

We give basic definitions and notions on Sheffer stroke BCH-algebras, neutrosophic $\mathcal{N}-$functions and neutrosophic $\mathcal{N}-$structures. Also, neutrosophic $\mathcal{N}-$subalgebra, a neutrosophic $\mathcal{N}-$ideal and a level set on neutrosophic $\mathcal{N}-$structures are introduced on Sheffer stroke BCH-algebras. Then we prove that the level set of a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke BCH-algebra is its subalgebra and the inverse always is true, and that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCH-algebra forms a complete distributive modular lattice. Moreover, it is shown that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse does not mostly hold. Finally, we define special subsets of a Sheffer stroke BCH-algebra by means of the $\mathcal{N}-$functions T_N, I_N and F_N and its any elements a_t, a_i, a_f and show that these subsets are ideals of this algebraic structure if a neutrosophic $\mathcal{N}-$structure on this algebraic structure is the neutrosophic $\mathcal{N}-$ideal.

2. **Preliminaries**

In this section, we give basic definitions and notions about Sheffer stroke BCH-algebras and neutrosophic $\mathcal{N}-$structures.

Definition 2.1. [3] Let $\mathcal{A} = \langle A, |\rangle$ be a groupoid. The operation $\circ$ on S is said to be a *Sheffer operation (or Sheffer stroke)* if it satisfies the following conditions for all $a, b, c \in A$:
$(S1)$ $a|b = b|a$,
$(S2)$ $(a|a)|(a|b) = a$,
$(S3)$ $a|((b|c)|(b|c)) = ((a|b)|(a|b))|c$,
$(S4)$ $(a|((a|a)|(b|b)))|(a|((a|a)|(b|b))) = a$.

Definition 2.2. [18] A Sheffer stroke BCH-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A the following axioms are satisfied:
$(sBCH.1)$ $(a|(a|a))|(a|(a|a)) = 0$,

$(sBCH.2)$ $(a|(b|b))|(a|(b|b)) = (b|(a|a))|(b|(a|a)) = 0$ imply $a = b$,
$(sBCH.3)$ $((a|(b|b))|(a|(b|b)))|(c|c) = ((a|(c|c))|(a|(c|c)))|(b|b)$,
for all $a, b, c \in A$.

Definition 2.3. [18] Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then a relation $\leqslant$ on A defined by

$$a \leqslant b \text{ if and only if } (a|(b|b))|(a|(b|b)) = 0,$$

is a partial order on A.

Lemma 2.4. *[18] Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then the following features hold for all $a, b, c \in A$:*

(1) $(a|(a|a))|(a|a) = a$,
(2) $a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b))) = 0|0$,
(3) $(0|0)|(a|a) = a$,
(4) $(a|(0|0))|(a|(0|0)) = a$,
(5) $a|((b|(c|c))|(b|(c|c))) = b|((a|(c|c))|(a|(c|c)))$,
(6) $((a|(a|(b|b)))|(a|(a|(b|b))))|(b|b) = 0|0$,
(7) $((a|(b|b))|(a|(b|b)))|(a|a) = 0|(b|b)$,
(8) $0|(a|(b|b)) = ((0|(a|a))|(0|(a|a)))|(0|(b|b))$,
(9) $a \leqslant b$ *implies* $0|(a|a) = 0|(b|b)$.

Definition 2.5. [18] Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then a nonempty subset S of A is called a subalgebra of A, if $(a|(b|b))|(a|(b|b)) \in S$, for all $a, b \in S$.

Definition 2.6. [18] A nonempty subset I of a Sheffer stroke BCH-algebra $(A, |, 0)$ is called an ideal of A if it satisfies
(I1) $0 \in I$,
(I2) $(a|(b|b))|(a|(b|b)) \in I$ and $a_2 \in I$ imply $a_1 \in I$,
for all $a, b \in A$.

Definition 2.7. [21] A modular lattice is any lattice which satisfies $a \leqslant b \longrightarrow a \vee (b \wedge c) = b \wedge (a \vee c)$.

Theorem 2.8. *[21] Every distributive lattice is a modular lattice.*

Definition 2.9. [12] $\mathcal{F}(A, [-1, 0])$ denotes the collection of functions from a set A to $[-1, 0]$ and a element of $\mathcal{F}(A, [-1, 0])$ is called a negative-valued function from A to $[-1, 0]$ (briefly, $\mathcal{N}-$function on A). An $\mathcal{N}-$structure refers to an ordered pair (A, f) of A and $\mathcal{N}-$function f on A.

Definition 2.10. [14] A neutrosophic $\mathcal{N}-$structure over a nonempty universe A is defined by

$$A_N := \frac{A}{(T_N, I_N, F_N)} = \{\frac{A}{(T_N(a), I_N(a), F_N(a))} : a \in A\}$$

where T_N, I_N and F_N are $\mathcal{N}-$function on A, called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively.

Every neutrosophic $\mathcal{N}-$structure A_N over A satisfies the condition

$$(\forall a \in A)(-3 \leqslant T_N(a) + I_N(a) + F_N(a) \leqslant 0).$$

3. Neutrosophic $\mathcal{N}-$structures

In this section, neutrosophic $\mathcal{N}-$subalgebras and neutrosophic $\mathcal{N}-$ideals of Sheffer stroke BCH-algebras. Unless indicated otherwise, A denotes a Sheffer stroke BCH-algebra.

Definition 3.1. A neutrosophic $\mathcal{N}-$subalgebra A_N of a Sheffer stroke BCH-algebra A is a neutrosophic $\mathcal{N}-$structure on A satisfying the condition

$$\begin{aligned}
&T_N((a|(b|b))|(a|(b|b))) \leqslant \max\{T_N(a), T_N(b)\},\\
&\min\{I_N(a), I_N(b)\} \leqslant I_N((a|(b|b))|(a|(b|b)))\\
&\qquad\qquad and\\
&\min\{F_N(a), F_N(b)\} \leqslant F_N((a|(b|b))|(a|(b|b))),
\end{aligned} \tag{1}$$

for all $a, b \in A$.

Example 3.2. Consider the Sheffer stroke BCH-algebra A where $A = \{0, x, y, 1\}$ and Sheffer stroke $|$ on A has Cayley table in Table 1 [18]:

TABLE 1. Cayley table of Sheffer stroke $|$ on A

$\circ$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{a}{(-0.63, -0.3, -0.08)} : \ a \in A - \{1\}\} \cup \{\frac{1}{(0, -0.98, -0.84)}\}$$

on A is a neutrosophic $\mathcal{N}-$subalgebra of A.

Definition 3.3. Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. For the sets

$$T_N^{\alpha} := \{a \in A : T_N(a) \leqslant \alpha\},$$

$$I_N^{\beta} := \{a \in A : \beta \leqslant I_N(a)\}$$

and

$$F_N^{\gamma} := \{a \in A : \gamma \leqslant F_N(a)\},$$

the set $A_N(\alpha, \beta, \gamma) := \{a \in A : T_N(a) \leqslant \alpha, \beta \leqslant I_N(a) \text{ and } \gamma \leqslant F_N(x)\}$ is called the $(\alpha, \beta, \gamma)-$level set of A_N. Moreover, $A_N(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$.

Theorem 3.4. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A and α, β, γ be any elements of $[-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. If A_N is a neutrosophic $\mathcal{N}-$subalgebra of A, then the nonempty level set $A_N(\alpha, \beta, \gamma)$ of A_N is a subalgebra of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of A and a, b be any elements of $A_N(\alpha, \beta, \gamma)$, for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. Then $T_N(a), T_N(b) \leqslant \alpha; \beta \leqslant I_N(a), I_N(b)$ and $\gamma \leqslant F_N(a), F_N(b)$. Since

$$T_N((a|(b|b))|(a|(b|b))) \leqslant \max\{T_N(a), T_N(b)\} \leqslant \alpha,$$

$$\beta \leqslant \min\{I_N(a), I_N(b)\} \leqslant I_N((a|(b|b))|(a|(b|b)))$$

and

$$\gamma \leqslant \min\{F_N(a), F_N(b)\} \leqslant F_N((a|(b|b))|(a|(b|b))),$$

for all $a, b \in A$, it follows that $(a|(b|b))|(a|(b|b)) \in T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$. Then

$$(a|(b|b))|(a|(b|b)) \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma} = A_N(\alpha, \beta, \gamma).$$

Thus, $A_N(\alpha, \beta, \gamma)$ is a subalgebra of A. □

Theorem 3.5. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A and $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} be subalgebras of A, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. Then A_N is a neutrosophic $\mathcal{N}-$subalgebra of A.*

Proof. Let $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} be subalgebras of A, for all $\alpha, \beta, \gamma \in [-1, 0]$ with

$$-3 \leqslant \alpha + \beta + \gamma \leqslant 0.$$

Assume that

$$\alpha_1 = \max\{T_N(a), T_N(b)\} < T_N((a|(b|b))|(a|(b|b))) = \alpha_2,$$

$$\beta_1 = I_N((a|(b|b))|(a|(b|b))) < \min\{I_N(a), I_N(b)\} = \beta_2$$

and

$$\gamma_1 = F_N((a|(b|b))|(a|(b|b))) < \min\{F_N(a), F_N(b)\} = \gamma_2.$$

If $\alpha = \frac{1}{2}(\alpha_1 + \alpha_2), \beta = \frac{1}{2}(\beta_1 + \beta_2), \gamma = \frac{1}{2}(\gamma_1 + \gamma_2) \in [-1, 0)$, then $\alpha_1 < \alpha < \alpha_2$, $\beta_1 < \beta < \beta_2$ and $\gamma_1 < \gamma < \gamma_2$. Hence, $a, b \in T_N^\alpha, I_N^\beta, F_N^\gamma$ but $(a|(b|b))|(a|(b|b)) \notin T_N^\alpha, I_N^\beta, F_N^\gamma$ which is a contradiction. Thus,

$$T_N((a|(b|b))|(a|(b|b))) \leqslant \max\{T_N(a), T_N(b)\},$$

$$\min\{I_N(a), I_N(b)\} \leqslant I_N((a|(b|b))|(a|(b|b)))$$

and

$$\min\{F_N(a), F_N(b)\} \leqslant F_N((a|(b|b))|(a|(b|b))),$$

for all $a, b \in A$. Thereby, A_N is a neutrosophic $\mathcal{N}-$subalgebra of A. □

Theorem 3.6. *Let $\{A_{N_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCH-algebra A. Then $\{A_{N_i} : i \in \mathbb{N}\}$ forms a complete distributive modular lattice.*

Proof. Let S be a nonempty subset of $\{A_{N_i} : i \in \mathbb{N}\}$. Since every A_{N_i} is a neutrosophic $\mathcal{N}-$subalgebra of A, for all $i \in \mathbb{N}$, it satisfies the condition (1), for all $a, b \in A$. Then $\bigcap S$ satisfies the condition (1), and so, $\bigcap S$ is a neutrosophic $\mathcal{N}-$subalgebra of A. Let B be a family of all neutrosophic $\mathcal{N}-$subalgebras of A containing $\bigcup\{A_{N_i} : i \in \mathbb{N}\}$. Thus, $\bigcap B$ is a neutrosophic $\mathcal{N}-$subalgebra of A. If $\bigwedge_{i\in\mathbb{N}} A_{N_i} = \bigcap_{i\in\mathbb{N}} A_{N_i}$ and $\bigvee_{i\in\mathbb{N}} A_{N_i} = \bigcap B$, then $(\{A_{N_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$ forms a complete lattice. Also, this lattice is distibutive by the definitions of $\bigvee$ and $\bigwedge$, and so, it is modular from Theorem 2.8. □

Lemma 3.7. *Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke BCH-algebra A. Then*

$$T_N(0) \leqslant T_N(a), I_N(a) \leqslant I_N(0) \text{ and } F_N(a) \leqslant F_N(0), \tag{2}$$

for all $a \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of A. Then it is obtained from (sBCH.1) that

$$T_N(0) = T_N((a|(a|a))|(a|(a|a))) \leqslant \max\{T_N(a), T_N(a)\} = T_N(a),$$

$$I_N(a) = \min\{I_N(a), I_N(a)\} \leqslant I_N((a|(a|a))|(a|(a|a))) = I_N(0)$$

and

$$F_N(a) = \min\{F_N(a), F_N(a)\} \leqslant F_N((a|(a|a))|(a|(a|a))) = F_N(0),$$

for all $a \in A$. □

The inverse of Lemma 3.7 is not true in general.

Example 3.8. Consider the Sheffer stroke BCH-algebra A in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{x}{(-0.05, -0.3, -0.29)}\} \cup \{\frac{a}{(-1, -0.03, -0.08)} : a \in A - \{x\}\}$$

on A satisfies the condition (2) but it is not a neutrosophic $\mathcal{N}-$subalgebra of A since $I_N((x|(0|0))|(x|(0|0))) = I_N(x) = -0.3 < -0.03 = \min\{I_N(x), I_N(0)\}$.

Lemma 3.9. *A neutrosophic $\mathcal{N}-$subalgebra A_N of a Sheffer stroke BCH-algebra A satisfies*

$$\begin{aligned} &T_N((a|(b|b))|(a|(b|b))) \leqslant T_N(b), \\ &I_N(b) \leqslant I_N((a|(b|b))|(a|(b|b))) \\ &\qquad \textit{and} \\ &F_N(b) \leqslant F_N((a|(b|b))|(a|(b|b))), \end{aligned} \tag{3}$$

for all $a, b \in A$ if and only if T_N, I_N and F_N are constant.

Proof. Let A_N be a a neutrosophic $\mathcal{N}-$subalgebra of A satisfying the condition (3). Since $T_N(a) = T_N((a|(0|0))|(a|(0|0))) \leqslant T_N(0)$, $I_N(0) \leqslant I_N((a|(0|0))|(a|(0|0))) = I_N(a)$ and $F_N(0) \leqslant F_N((a|(0|0))|(a|(0|0))) = F_N(a)$ from Lemma 2.4 (4), it is obtained from Lemma 3.7 that $T_N(a) = T_N(0)$, $I_N(a) = I_N(0)$ and $F_N(a) = F_N(0)$, for all $a \in A$. Therefore, T_N, I_N and F_N are constant. Conversely, it is clear since T_N, I_N and F_N are constant. □

Definition 3.10. A neutrosophic $\mathcal{N}-$structure A_N on a Sheffer stroke BCH-algebra A is called a neutrosophic $\mathcal{N}-$ideal of A if

$$\begin{aligned} &T_N(0) \leqslant T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\}, \\ &\min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a) \leqslant I_N(0) \\ &\qquad \textit{and} \\ &\min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a) \leqslant F_N(0), \end{aligned} \tag{4}$$

for all $a, b \in A$.

Example 3.11. Consider the Sheffer stroke BCH-algebra A in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{a}{(-0.71, -0.11, -0.07)} : x = 0, x\} \cup \{\frac{a}{(-0.48, -0.35, -1)} : a = y, 1\}$$

on A is a neutrosophic $\mathcal{N}-$ideal of A.

Lemma 3.12. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A if and only if*

(1) *$a \leqslant b$ implies $T_N(a) \leqslant T_N(b)$, $I_N(b) \leqslant I_N(a)$ and $F_N(b) \leqslant F_N(a)$,*
(2) *$T_N((a|a)|(b|b)) \leqslant \max\{T_N(a), T_N(b)\}$, $\min\{I_N(a), I_N(b)\} \leqslant I_N((a|a)|(b|b))$ and $\min\{F_N(a), F_N(b)\} \leqslant F_N((a|a)|(b|b))$,*

for all $a, b \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A.

(1) Suppose that $a \leqslant b$. Then $(a|(b|b))|(a|(b|b)) = 0$. Thus, we have from Lemma 3.7 that

$$T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} = \max\{T_N(0), T_N(b)\} = T_N(b),$$

$$I_N(b) = \min\{I_N(0), I_N(b)\} = \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$F_N(b) = \min\{F_N(0), F_N(b)\} = \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a),$$

for all $a, b \in A$.

(2) Since $(((((a|a)|(b|b))|(b|b))|(((a|a)|(b|b))|(b|b)))|(a|a))|(((((a|a)|(b|b))|(b|b))|(((a|a)|(b|b))|(b|b)))|(a|a)) = (((a|a)|(b|b))|(((a|a)|(b|b))|((a|a)|(b|b))))|(((a|a)|(b|b))|(((a|a)|(b|b))|((a|a)|(b|b)))) = 0$ from (S1), (S3) and (sBCH.1), we obtain from Definition 2.3 and (1) that

$$\begin{aligned} T_N((a|a)|(b|b)) &\leqslant \max\{T_N((((a|a)|(b|b))|(b|b))|(((a|a)|(b|b))|(b|b))), T_N(b)\} \\ &\leqslant \max\{T_N(a), T_N(b)\}, \end{aligned}$$

$$\begin{aligned} \min\{I_N(a), I_N(b)\} &\leqslant \min\{I_N((((a|a)|(b|b))|(b|b))|(((a|a)|(b|b))|(b|b))), I_N(b)\} \\ &\leqslant I_N((a|a)|(b|b)) \end{aligned}$$

and

$$\begin{aligned} \min\{F_N(a), F_N(b)\} &\leqslant \min\{F_N((((a|a)|(b|b))|(b|b))|(((a|a)|(b|b))|(b|b))), F_N(b)\} \\ &\leqslant F_N((a|a)|(b|b)), \end{aligned}$$

for all $a, b \in A$.

Conversely, let A_N be a neutrosophic $\mathcal{N}-$structure on A satisfying (1) and (2). Since $(0|(a|a))|(0|(a|a)) = (((0|0)|(0|0))|((0|0)|a))|(((0|0)|(0|0))|((0|0)|a)) = 0$ from (S1)-(S2) and Lemma 2.4 (4), we get that $0 \leqslant a$, for all $a \in A$. Then it follows from (1) that $T_N(0) \leqslant T_N(a)$, $I_N(a) \leqslant I_N(0)$ and $F_N(a) \leqslant F_N(0)$, for all $a \in A$. Since $(a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b))))|(a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b)))) = 0$ from Lemma 2.4 (2) and (S2), we have from Definition 2.3 that $a \leqslant (a|(b|b))|(b|b)$, for all $a, b \in A$. Hence, it is obtained (1), (2) and (S2) that

$$\begin{aligned} T_N(a) &\leqslant T_N((a|(b|b))|(b|b)) \\ &= T_N((((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b))))(b|b)) \\ &\leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\}, \end{aligned}$$

$$\begin{aligned} \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} &\leqslant I_N((((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b))))(b|b)) \\ &= I_N((a|(b|b))|(b|b)) \\ &\leqslant I_N(a) \end{aligned}$$

and

$$\begin{aligned}\min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} &\leqslant F_N((((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b))))|(b|b)) \\ &= F_N((a|(b|b))|(b|b)) \\ &\leqslant F_N(a),\end{aligned}$$

for all $a, b \in A$. Thus, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Lemma 3.13. *Let A_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra A. Then*

(1) $T_N(b) \leqslant T_N(a|(b|b))$, $I_N(a|(b|b)) \leqslant I_N(b)$ *and* $F_N(a|(b|b)) \leqslant F_N(b)$,

(2) $T_N((a|(b|b))|(a|(b|b))) \leqslant \max\{T_N(a), T_N(b)\}$, $\min\{I_N(a), I_N(b)\} \leqslant I_N((a|(b|b))|(a|(b|b)))$ *and* $\min\{F_N(a), F_N(b)\} \leqslant F_N((a|(b|b))|(a|(b|b)))$,

(3) $T_N(a) \leqslant T_N((a|(b|b))|(b|b))$, $I_N((a|(b|b))|(b|b)) \leqslant I_N(a)$ *and* $F_N((a|(b|b))|(b|b)) \leqslant F_N(a)$.

for all $a, b, c \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A. Then

(1) Since

$$\begin{aligned}&(b|((a|(b|b))|(a|(b|b))))|(b|((a|(b|b))|(a|(b|b)))) \\ &= (a|((b|(b|b))|(b|(b|b))))|(a|((b|(b|b))|(b|(b|b)))) \\ &= (a|0)|(a|0) \\ &= (((0|0)|(0|0))|((0|0)|(a|a)))|(((0|0)|(0|0))|((0|0)|(a|a))) \\ &= 0\end{aligned}$$

from (sBCH.1), (S1)-(S2), Lemma 2.4 (3) and (5), we obtain that $b \leqslant a|(b|b)$, for all $a, b \in A$. Thus, it follows from Lemma 3.12 (1) that

$$T_N(b) \leqslant T_N(a|(b|b)), I_N(a|(b|b)) \leqslant I_N(b) \text{ and } F_N(a|(b|b)) \leqslant F_N(b),$$

for all $a, b \in A$.

(2) Since

$$\begin{aligned}&(((a|(b|b))|(a|(b|b)))|(a|a))|(((a|(b|b))|(a|(b|b)))|(a|a)) \\ &= ((b|b)|((a|(a|a))|(a|(a|a))))|((b|b)|((a|(a|a))|(a|(a|a)))) \\ &= ((b|b)|0)|((b|b)|0) \\ &= (((0|0)|(0|0))|((0|0)|b))|(((0|0)|(0|0))|((0|0)|b)) \\ &= 0\end{aligned}$$

from (S1)-(S3), Lemma 2.4 (3), it is obtained that $(a|(b|b))|(a|(b|b)) \leqslant a$, for all $a, b \in A$. Hence, we have from Lemma 3.12 (1) that

$$T_N((a|(b|b))|(a|(b|b))) \leqslant T_N(a) \leqslant \max\{T_N(a), T_N(b)\},$$

$$\min\{I_N(a), I_N(b)\} \leqslant I_N(a) \leqslant I_N((a|(b|b))|(a|(b|b)))$$

and

$$\min\{F_N(a), F_N(b)\} \leqslant F_N(a) \leqslant F_N((a|(b|b))|(a|(b|b))),$$

for all $a, b \in A$.

(3) Since

$$\begin{aligned}
&(a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b))))|(a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b))))\\
&= ((a|(b|b))|((a|(b|b))|(a|(b|b))))|((a|(b|b))|((a|(b|b))|(a|(b|b))))\\
&= 0
\end{aligned}$$

from Lemma 2.4 (5) and (sBCH.1), it follows from Lemma 3.12 (1) that

$$T_N(a) \leqslant T_N((a|(b|b))|(b|b)),\ I_N((a|(b|b))|(b|b)) \leqslant I_N(a) \text{ and } F_N((a|(b|b))|(b|b)) \leqslant F_N(a),$$

for all $a, b \in A$. □

Theorem 3.14. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A if and only if*

$$\begin{aligned}
&((b|(c|c))|(b|(c|c)))|(a|a) = 0|0 \ \textit{ implies } \ T_N(b) \leqslant \max\{T_N(a), T_N(c)\},\\
&\min\{I_N(a), I_N(c)\} \leqslant I_N(b) \ \textit{ and } \ \min\{F_N(a), F_N(c)\} \leqslant F_N(b),
\end{aligned} \tag{5}$$

for all $a, b, c \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A and $((b|(c|c))|(b|(c|c)))|(a|a) = 0|0$. Since $(((b|(c|c))|(b|(c|c)))|(a|a))|(((b|(c|c))|(b|(c|c)))|(a|a)) = 0$ from (S2), it follows that $(b|(c|c))|(b|(c|c)) \leqslant a$. Then it is obtained from Lemma 3.12 (1) that

$$T_N(b) \leqslant \max\{T_N((b|(c|c))|(b|(c|c))), T_N(c)\} \leqslant \max\{T_N(a), T_N(c)\},$$

$$\min\{I_N(a), I_N(c)\} \leqslant \min\{I_N((b|(c|c))|(b|(c|c))), I_N(c)\} \leqslant I_N(b)$$

and

$$\min\{F_N(a), F_N(c)\} \leqslant \min\{F_N((b|(c|c))|(b|(c|c))), F_N(c)\} \leqslant F_N(b),$$

for all $a, b, c \in A$.

Conversely, let A_N be a neutrosophic $\mathcal{N}-$structure on A satisfying the condition (5). Since

$$\begin{aligned}
((0|(a|a))|(0|(a|a)))|(a|a) &= 0|(a|a)\\
&= ((0|0)|(0|0))|((0|0)|a)\\
&= 0|0
\end{aligned}$$

from (S2), (S3) and Lemma 2.4 (3), we have from the condition (5) that

$$T_N(0) \leqslant \max\{T_N(a), T_N(a)\} = T_N(a),$$

$$I_N(a) = \min\{I_N(a), I_N(a)\} \leqslant I_N(0)$$

and

$$F_N(a) = \min\{F_N(a), F_N(a)\} \leqslant F_N(0),$$

for all $a \in A$. Since

$$\begin{aligned}&((a|(b|b))|(a|(b|b)))|(((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b))))\\&=(a|(b|b))|((a|(b|b))|(a|(b|b)))\\&=0|0\end{aligned}$$

from (S1), (S2) and (sBCH.1), it follows from the condition (5) that

$$T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\},$$

$$\min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$\min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a),$$

for all $a, b \in A$. Therefore, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Theorem 3.15. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A and α, β, γ be any elements of $[-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. If A_N is a neutrosophic $\mathcal{N}-$ideal of A, then the nonempty (α, β, γ)-level set $A_N(\alpha, \beta, \gamma)$ of A_N is an ideal of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A and $A_N(\alpha, \beta, \gamma) \neq \emptyset$, for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. Since $T_N(0) \leqslant T_N(a) \leqslant \alpha$, $\beta \leqslant I_N(a) \leqslant I_N(0)$ and $\gamma \leqslant F_N(a) \leqslant F_N(0)$, for all $a \in A$, it is obtained that $0 \in T_N(\alpha, \beta, \gamma)$. Let $(a|(b|b))|(a|(b|b)), b \in A_N(\alpha, \beta, \gamma)$. Since

$$T_N((a|(b|b))|(a|(b|b))), T_N(b) \leqslant \alpha,$$

$$\beta \leqslant I_N((a|(b|b))|(a|(b|b))), I_N(b)$$

and

$$\gamma \leqslant F_N((a|(b|b))|(a|(b|b))), F_N(b),$$

it follows that

$$T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} \leqslant \alpha,$$

$$\beta \leqslant \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$\gamma \leqslant \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a),$$

for all $a, b \in A$, which imply that $a \in A_N(\alpha, \beta, \gamma)$. Thus, $A_N(\alpha, \beta, \gamma)$ is an ideal of A. □

Theorem 3.16. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A and $T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$ be ideals of A, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$structure on A and $T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$ be ideals of A, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. Assume that $T_N(a) < T_N(0)$, $I_N(0) < I_N(a)$ and $F_N(0) < F_N(a)$, for some $a \in A$. If $\alpha = \frac{1}{2}(T_N(0) + T_N(a))$, $\beta = \frac{1}{2}(I_N(0) + I_N(a))$ and $\gamma = \frac{1}{2}(F_N(0) + F_N(a))$ in $[-1, 0)$, then $T_N(a) < \alpha < T_N(0)$, $I_N(0) < \beta < I_N(a)$ and $F_N(x0) < \gamma < F_N(a)$. Thus, $0 \notin T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$ which is a contradiction with (I1). Hence, $T_N(0) \leqslant T_N(a)$, $I_N(a) \leqslant I_N(0)$ and $F_N(a) \leqslant F_N(0)$, for all $a \in A$. Suppose that

$$\alpha_1 = \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} < T_N(a) = \alpha_2,$$

$$\beta_1 = I_N(a) < \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} = \beta_2,$$

and

$$\gamma_1 = F_N(a) < \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} = \gamma_2.$$

If $\alpha^* = \frac{1}{2}(\alpha_1 + \alpha_2)$, $\beta^* = \frac{1}{2}(\beta_1 + \beta_2)$ and $\gamma^* = \frac{1}{2}(\gamma_1 + \gamma_2)$ in $[-1, 0)$, then $\alpha_1 < \alpha^* < \alpha_2$, $\beta_1 < \beta^* < \beta_2$ and $\gamma_1 < \gamma^* < \gamma_2$. Thus, $(a|(b|b))|(a|(b|b))$, $b \in T_N^{\alpha^*}, I_N^{\beta^*}, F_N^{\gamma^*}$ but $a \notin T_N^{\alpha^*}, I_N^{\beta^*}, F_N^{\gamma^*}$, which is a contradiction with (I2). Thus,

$$T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b)), T_N(b)\},$$

$$\min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$\min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant I_N(a),$$

for all $a, b \in A$. Hence, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Definition 3.17. Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BCH-algebras. Then a mapping $f : A \longrightarrow B$ is called a homomorphism if $f(a|_A b) = f(a)|_B f(b)$, for all $a, b \in A$ and $f(0_A) = 0_B$.

Theorem 3.18. *Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BCH-algebras, $f : A \longrightarrow B$ be a surjective homomorphism and $B_N = \frac{B}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}-$structure on B. Then B_N is a neutrosophic $\mathcal{N}-$ideal of B if and only if $B_N^f = \frac{A}{(T_N^f, I_N^f, F_N^f)}$ is a neutrosophic $\mathcal{N}-$ideal of A where the $\mathcal{N}-$functions $T_N^f, I_N^f, F_N^f : A \longrightarrow [-1, 0]$ on A are defined by $T_N^f(a) = T_N(f(a))$, $I_N^f(a) = I_N(f(a))$ and $F_N^f(a) = F_N(f(a))$, for all $a \in A$, respectively.*

Proof. Let $(A, |, 0)$ and $(B, |, 0)$ be Sheffer stroke BCH-algebras, $f : A \longrightarrow B$ be a surjective homomorphism and $B_N = \frac{B}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}-$ideal of B. Then $T_N^f(0_A) = T_N(f(0_A)) = T_N(0_B)) \leqslant T_N(x) = T_N(f(a)) = T_N^f(a)$, $I_N^f(a) = I_N(f(a)) = I_N(x) \leqslant I_N(0_B) =$

$I_N(f(0_A)) = I_N^f = (0_A)$ and $F_N^f(a) = F_N(f(a)) = F_N(x) \leqslant F_N(0_B) = F_N(f(0_A)) = F_N^f = (0_A)$, for all $a \in A$. Moreover,

$$\begin{aligned} T_N^f(a) &= T_N(f(a)) \\ &\leqslant \max\{T_N((f(a)|_B(f(b)|_B f(b)))|_B(f(a)|_B(f(b)|_B f(b)))), T_N(f(b))\} \\ &= \max\{T_N(f((a|_A(b|_A b))|_A(a|_A(b|_A b)))), T_N(f(b))\} \\ &= \max\{T_N^f((a|_A(b|_A b))|_A(a|_A(b|_A b))), T_N^f(b)\}, \end{aligned}$$

$$\begin{aligned} &\min\{I_N^f((a|_A(b|_A b))|_A(a|_A(b|_A b))), I_N^f(b)\} \\ &= \min\{I_N(f((a|_A(b|_A b))|_A(a|_A(b|_A b)))), I_N(f(b))\} \\ &= \min\{I_N((f(a)|_B(f(b)|_B f(b)))|_B(f(a)|_B(f(b)|_B f(b)))), I_N(f(b))\} \\ &\leqslant I_N(f(a)) \\ &= I_N^f(a) \end{aligned}$$

and

$$\begin{aligned} &\min\{F_N^f((a|_A(b|_A b))|_A(a|_A(b|_A b))), F_N^f(b)\} \\ &= \min\{F_N(f((a|_A(b|_A b))|_A(a|_A(b|_A b)))), F_N(f(b))\} \\ &= \min\{F_N((f(a)|_B(f(b)|_B f(b)))|_B(f(a)|_B(f(b)|_B f(b)))), F_N(f(b))\} \\ &\leqslant F_N(f(a)) \\ &= F_N^f(a), \end{aligned}$$

for all $a, b \in A$. Hence, $B_N^f = \dfrac{A}{(T_N^f, I_N^f, F_N^f)}$ is a neutrosophic $\mathcal{N}-$ideal of A.

Conversely, let B_N^f be a neutrosophic $\mathcal{N}-$ideal of A. Thus,

$$T_N(0_B) = T_N(f(0_A)) = T_N^f(0_A) \leqslant T_N^f(a) = T_N(f(a)) = T_N(x),$$

$$I_N(x) = I_N(f(a)) = I_N^f(a) \leqslant I_N^f(0_A) = I_N(f(0_A)) = I_N(0_B)$$

and

$$F_N(x) = F_N(f(a)) = F_N^f(a) \leqslant F_N^f(0_A) = F_N(f(0_A)) = F_N(0_B),$$

for all $x \in B$. Also,

$$\begin{aligned} T_N(x) &= T_N(f(a)) \\ &= T_N^f(a) \\ &\leqslant \max\{T_N^f((a|_A(b|_A b))|_A(a|_A(b|_A b))), T_N^f(b)\} \\ &= \max\{T_N(f((a|_A(b|_A b))|_A(a|_A(b|_A b)))), T_N(f(b))\} \\ &= \max\{T_N((f(a)|_B(f(b)|_B f(b)))|_B(f(a)|_B(f(b)|_B f(b)))), T_N(f(b))\} \\ &= \max\{T_N((x|_B(y|_B y))|_B(x|_B(y|_B y))), T_N(y)\}, \end{aligned}$$

$$\begin{aligned}
&\min\{I_N((x|_B(y|_By))|_B(x|_B(y|_By))), I_N(y)\}\\
&= \min\{I_N((f(a)|_B(f(b)|_Bf(b)))|_B(f(a)|_B(f(b)|_Bf(b)))), I_N(f(b))\}\\
&= \min\{I_N(f((a|_A(b|_Ab))|_A(a|_A(b|_Ab)))), I_N(f(b))\}\\
&= \min\{I_N^f((a|_A(b|_Ab))|_A(a|_A(b|_Ab))), I_N^f(b)\}\\
&\leqslant I_N^f(a)\\
&= I_N(f(a))\\
&= I_N(x)
\end{aligned}$$

and

$$\begin{aligned}
&\min\{F_N((x|_B(y|_By))|_B(x|_B(y|_By))), F_N(y)\}\\
&= \min\{F_N((f(a)|_B(f(b)|_Bf(b)))|_B(f(a)|_B(f(b)|_Bf(b)))), F_N(f(b))\}\\
&= \min\{F_N(f((a|_A(b|_Ab))|_A(a|_A(b|_Ab)))), F_N(f(b))\}\\
&= \min\{F_N^f((a|_A(b|_Ab))|_A(a|_A(b|_Ab))), F_N^f(b)\}\\
&\leqslant F_N^f(a)\\
&= F_N(f(a))\\
&= F_N(x)
\end{aligned}$$

for all $x, y \in B$. Therefore, $B_N = \dfrac{B}{(T_N, I_N, F_N)}$ is a neutrosophic $\mathcal{N}-$ideal of B. □

Theorem 3.19. *Every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra A is a neutrosophic $\mathcal{N}-$subalgebra of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A. Since

$$\begin{aligned}
&(((a|(b|b))|(a|(b|b)))|(a|a))|(((a|(b|b))|(a|(b|b)))|(a|a))\\
&= (0|(b|b))|(0|(b|b))\\
&= (((0|0)|(0|0))|((0|0)|b))|(((0|0)|(0|0))|((0|0)|b))\\
&= 0
\end{aligned}$$

from (S2), Lemma 2.4 (3) and (7), it follows that $(a|(b|b))|(a|(b|b)) \leqslant a$, for all $a, b \in A$. Then it is obtained from Lemma 3.12 (1) that

$$T_N((a|(b|b))|(a|(b|b))) \leqslant T_N(a) \leqslant \max\{T_N(a), T_N(b)\},$$

$$\min\{I_N(a), I_N(b)\} \leqslant I_N(a) \leqslant I_N((a|(b|b))|(a|(b|b)))$$

and

$$\min\{F_N(a), F_N(b)\} \leqslant F_N(a) \leqslant F_N((a|(b|b))|(a|(b|b))),$$

for all $a, b \in A$. Thereby, A_N is a neutrosophic $\mathcal{N}-$subalgebra of A. □

The inverse of Theorem 3.19 does notusually hold.

Example 3.20. Consider the Sheffer stroke BCH-algebra S in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{0}{(-0.82,-0.49,-0.17)}, \frac{1}{(-0.1,-0.91,-0.5)}\} \cup \{\frac{a}{(-0.61,-0.54,-0.3)} : a = x, y\}$$

on A is a neutrosophic $\mathcal{N}-$subalgebra of A but it is not a neutrosophic $\mathcal{N}-$ideal of A since $I_N(1) = -0.91 < -0.54 = \min\{I_N((1|(y|y))|(1|(y|y))), I_N(y)\}$.

Lemma 3.21. *Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke BCH-algebra A satisfying*

$$\begin{array}{c} T_N(a|(b|b)) \leqslant \max\{T_N((a|((b|(c|c))|(b|(c|c))))|(a|((b|(c|c))|(b|(c|c))))), T_N(a|(c|c))\} \\ \min\{I_N((a|((b|(c|c))|(b|(c|c))))|(a|((b|(c|c))|(b|(c|c))))), I_N(a|(c|c))\} \leqslant I_N(a|(b|b)) \\ \text{and} \\ \min\{F_N((a|((b|(c|c))|(b|(c|c))))|(a|((b|(c|c))|(b|(c|c))))), F_N(a|(c|c))\} \leqslant F_N(a|(b|b)), \end{array} \quad (6)$$

for all $a, b, c \in A$. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A.

Proof. Let S_N be a neutrosophic $\mathcal{N}-$subalgebra of A satisfying the condition (6). By Lemma 3.7, $T_N(0) \leqslant T_N(a), I_N(a) \leqslant I_N(0)$ *and* $F_N(a) \leqslant F_N(0)$, for all $a \in A$. By substituting $[a := 0|0]$, $[b := a]$ and $[c := b]$ in the condition (6), simultaneously, it follows from Lemma 2.4 (3) that

$$\begin{aligned} T_N(a) &= T_N((0|0)|(a|a)) \\ &\leqslant \max\{T_N(((0|0)|((a|(b|b))|(a|(b|b))))|((0|0)|((a|(b|b))|(a|(c|c))))), T_N((0|0)|(b|b))\} \\ &= \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\}, \end{aligned}$$

$$\begin{aligned} \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} &= \min\{I_N(((0|0)|((a|(b|b))|(a|(b|b))))|((0| \\ &\quad 0)|((a|(b|b))|(a|(b|b))))), I_N((0|0)|(b|b))\} \\ &\leqslant I_N((0|0)|(a|a)) \\ &= I_N(a) \end{aligned}$$

and

$$\begin{aligned} \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} &= \min\{F_N(((0|0)|((a|(b|b))|(a|(b|b))))|((0| \\ &\quad 0)|((a|(b|b))|(a|(b|b))))), F_N((0|0)|(b|b))\} \\ &\leqslant F_N((0|0)|(a|a)) \\ &= F_N(a), \end{aligned}$$

for all $a, b \in A$. Thus, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Lemma 3.22. *Let A_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra A. Then the subsets $A_{T_N} = \{a \in A : T_N(a) = T_N(0)\}$, $A_{I_N} = \{a \in A : I_N(a) = I_N(0)\}$ and $A_{F_N} = \{a \in A : F_N(a) = F_N(0)\}$ of A are ideals of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A. Then it is clear that $0 \in A_{T_N}, A_{I_N}, A_{F_N}$. Suppose that $(a|(b|b))|(a|(b|b)), b \in A_{T_N}, A_{I_N}, A_{F_N}$. Since

$$T_N(b) = T_N(0) = T_N((a|(b|b))|(a|(b|b))),$$

$$I_N(b) = I_N(0) = I_N((a|(b|b))|(a|(b|b)))$$

and

$$F_N(b) = F_N(0) = F_N((a|(b|b))|(a|(b|b))),$$

it follows that

$$T_N(a) = T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} = \max\{T_N(0), T_N(0)\} = T_N(0),$$

$$I_N(0) = \min\{I_N(0), I_N(0)\} = \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$F_N(0) = \min\{F_N(0), F_N(0)\} = \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a).$$

Thus, $T_N(a) = T_N(0)$, $I_N(a) = I_N(0)$ and $F_N(a) = F_N(0)$, and so, $a \in A_{T_N}, A_{I_N}, A_{F_N}$. Hence, A_{T_N}, A_{I_N} and A_{F_N} are ideals of A. □

Definition 3.23. Let A be a Sheffer stroke BCH-algebra. Define the subsets

$$A_N^{a_t} := \{a \in A : T_N(a) \leqslant T_N(a_t)\},$$

$$A_N^{a_i} := \{a \in A : I_N(a_i) \leqslant I_N(a)\}$$

and

$$A_N^{a_f} := \{a \in A : F_N(a_f) \leqslant F_N(a)\}$$

of S, for all $a_t, a_i, a_f \in A$. Also, it is obvious that $a_t \in A_N^{a_t}, a_i \in A_N^{a_i}$ and $a_f \in A_N^{a_f}$.

Example 3.24. Consider the Sheffer stroke BCH-algebra A in Example 3.2. Let

$$T_N(a) = \begin{cases} 0, & \text{if } a = 0, 1 \\ -0.46, & \text{if } a = x \\ -0.23, & a = y, \end{cases} \qquad I_N(a) = \begin{cases} -0.17, & \text{if } a = 1 \\ 0, & \text{otherwise}, \end{cases}$$

$$F_N(a) = \begin{cases} -1, & \text{if } a = 0 \\ -0.4, & \text{otherwise}, \end{cases} \qquad a_t = x, a_i = 0 \text{ and } a_f = 1.$$

Then

$$A_N^{a_t} = \{a \in A : T_N(a) \leqslant T_N(x)\} = \{x\},$$

$$A_N^{a_i} = \{a \in A : I_N(0) \leqslant I_N(a)\} = \{0, x, y\}$$

and

$$A_N^{a_f} = \{a \in A : F_N(1) \leqslant F_N(a)\} = \{x, y, 1\}.$$

Theorem 3.25. *Let a_t, a_i and a_f be any elements of a Sheffer stroke BCH-algebra A. If A_N is a neutrosophic $\mathcal{N}-$ideal of A, then $A_N^{a_t}, A_N^{a_i}$ and $A_N^{a_f}$ are ideals of A.*

Proof. Let a_t, a_i and a_f be any elements of A and A_N be a neutrosophic $\mathcal{N}-$ideal of A. since $T_N(0) \leqslant T_N(a_t)\}$, $I_N(a_i) \leqslant I_N(0)$ and $F_N(a_f) \leqslant F_N(0)$, for all $a_t, a_i, a_f \in A$, it is obtained that $0 \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Suppose that $(a|(b|b))|(a|(b|b)), b \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Since

$$T_N((a|(b|b))|(a|(b|b))), T_N(b) \leqslant T_N(a_t),$$

$$I_N(a_i) \leqslant I_N((a|(b|b))|(a|(b|b))), I_N(b)$$

and

$$F_N(a_f) \leqslant F_N((a|(b|b))|(a|(b|b))), F_N(b),$$

it follows that

$$T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} \leqslant T_N(a_t),$$

$$I_N(a_i) \leqslant \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$F_N(a_f) \leqslant \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a),$$

which imply that $a \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Hence, $A_N^{a_t}, A_N^{a_i}$ and $A_N^{a_f}$ are ideals of A. □

Example 3.26. Consider the Sheffer stroke BCH-algebra A in Example 3.2. For a neutrosophic $\mathcal{N}-$ideal

$$A_N = \{\frac{a}{(-1, -0.47, -0.81)} : \ a = 0, x\} \cup \{\frac{a}{(-0.34, -0.69, -0.95)} : \ a = y, 1\}$$

of A and $a_t = 0, a_i = x, a_f = y \in S$, the subsets

$$A_N^{a_t} = \{a \in A : T_N(a) \leqslant T_N(0)\} = \{0, x\},$$

$$A_N^{a_i} = \{a \in A : I_N(x) \leqslant I_N(a)\} = \{0, x\}$$

and

$$A_N^{a_f} = \{a \in A : F_N(y) \leqslant F_N(a)\} = A$$

of A are ideals of A.

Theorem 3.27. *Let a_t, a_i and a_f be any elements of a Sheffer stroke BCH-algebra A and A_N be a neutrosophic $\mathcal{N}-$structure on A.*

(1) *If $A_N^{a_t}, A_N^{a_i}$ and $A_N^{a_f}$ are ideals of A, then*

$$\max\{T_N((b|(c|c))|(b|(c|c))), T_N(c)\} \leqslant T_N(a) \Rightarrow T_N(b) \leqslant T_N(a),$$

$$I_N(a) \leqslant \min\{I_N((b|(c|c))|(b|(c|c))), I_N(c)\} \Rightarrow I_N(a) \leqslant I_N(b) \quad and \tag{7}$$

$$F_N(a) \leqslant \min\{F_N((b|(c|c))|(b|(c|c))), F_N(c)\} \Rightarrow F_N(a) \leqslant F_N(b),$$

for all $a, b, c \in A$.

(2) *If A_N satisfies the condition (7) and*

$$T_N(0) \leqslant T_N(a), \quad I_N(a) \leqslant I_N(0) \quad and \quad F_N(a) \leqslant F_N(0), \tag{8}$$

for all $a \in A$, then $A_N^{a_t}, A_N^{a_i}$ and $A_N^{a_f}$ are ideals of A, for all $a_t \in T_N^{-1}$, $a_i \in I_N^{-1}$ and $a_f \in F_N^{-1}$.

Proof. Let a_t, a_i and a_f be any elements of A and A_N be a neutrosophic $\mathcal{N}-$structure on A.

(1) Suppose that $A_N^{a_t}, A_N^{a_i}$ and $A_N^{a_f}$ are ideals of A and

$$\max\{T_N((b|(c|c))|(b|(c|c))), T_N(c)\} \leqslant T_N(a),$$

$$I_N(a) \leqslant \min\{I_N((b|(c|c))|(b|(c|c))), I_N(c)\}$$

and

$$F_N(a) \leqslant \min\{F_N((b|(c|c))|(b|(c|c))), F_N(c)\}.$$

Since $(b|(c|c))|(b|(c|c)), c \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$ where $a_t = a_i = a_f = a$, it is obtained that $b \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$ where $a_t = a_i = a_f = a$. Thus, $T_N(b) \leqslant T_N(a)$, $I_N(a) \leqslant I_N(b)$ and $F_N(a) \leqslant F_N(b)$, for all $a, b, c \in A$.

(2) Let A_N be a neutrosophic $\mathcal{N}-$structure on A satisfying the conditions (7) and (8), for any $a_t \in T_N^{-1}$, $a_i \in I_N^{-1}$ and $a_f \in F_N^{-1}$. Then it follows from the condition (8) that $0 \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Assume that $(a|(b|b))|(a|(b|b)), b \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Thus,

$$T_N((a|(b|b))|(a|(b|b))), T_N(b) \leqslant T_N(a_t),$$

$$I_N(a_i) \leqslant I_N((a|(b|b))|(a|(b|b))), I_N(b)$$

and

$$F_N(a_f) \leqslant F_N((a|(b|b))|(a|(b|b))), F_N(b).$$

Since

$$\max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} \leqslant T_N(a_t),$$

$$I_N(a_i) \leqslant \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\}$$

and

$$F_N(a_f) \leqslant \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\},$$

we have from the condition (7) that

$$T_N(a) \leqslant T_N(a_t), I_N(a_i) \leqslant I_N(a) \text{ and } F_N(a_f) \leqslant F_N(a),$$

which imply that $a \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Hence, $A_N^{a_t}, A_N^{a_i}$ and $A_N^{a_f}$ are ideals of A. □

Example 3.28. Consider the Sheffer stroke BCH-algebra A in Example 3.2. Let

$$T_N(a) = \begin{cases} -1, & \text{if } a = 0, y \\ -0.05, & \text{otherwise,} \end{cases} \qquad I_N(a) = \begin{cases} -0.79, & \text{if } a = x, 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$F_N(a) = \begin{cases} -0.02, & \text{if } a = 0 \\ -0.72, & \text{otherwise,} \end{cases} \qquad \text{and} \quad a_t = 0, a_i = 1\ a_f = y \in A.$$

Then the ideals

$$A_N^{a_t} = \{0, y\}, A_N^{a_i} = A \text{ and } A_N^{a_f} = A$$

of A satisfy the condition (7).

Let

$$A_N = \{\frac{a}{(-0.8, -0.32, 0)} : \ a = 0, y\} \cup \{\frac{a}{(-0.27, -0.45, -0.51)} : \ a = x, 1\}$$

be a neutrosophic $\mathcal{N}-$structure on A satisfying the conditions (7) and (8). Then the subsets $A_N^{a_t} = A, A_N^{a_i} = \{0, y\}$ and $A_N^{a_f} = \{0, y\}$ of A are ideals of A, where $a_t = x, a_i = y$ and $a_f = 0$.

4. Conclusion

In this study, we introduce a neutrosophic $\mathcal{N}-$subalgebra, a neutrosophic $\mathcal{N}-$ideal and a level-set of neutrosophic $\mathcal{N}-$structures on Sheffer stroke BCH-algebras. Then we show that the level-set of a neutrosophic $\mathcal{N}-$subalgebra (a neutrosophic $\mathcal{N}-$ideal) of a Sheffer stroke BCH-algebra is its subalgebra (an ideal) and vice versa. Also, we prove that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCH-algebra forms a complete distributive modular lattice. We analyze the situations which $\mathcal{N}-$functions are constant. Moreover, we present new statements equivalent to the definition of a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra and its properties. By defining a homomorphism on a Sheffer stroke BCH-algebra, we demonstrate relationships between neutrosophic $\mathcal{N}-$ideals of two Sheffer stroke BCH-algebras by means of a surjective homomorphism. We propound that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse is not true in general. Besides, the subsets A_{T_N}, A_{I_N} and A_{F_N} of a Sheffer stroke BCH-algebra are its ideals for the neutrosophic $\mathcal{N}-$ideal which is defined by means of the $\mathcal{N}-$functions T_N, I_N and F_N. After that we describe the subsets $A_N^{a_t}$, $A_N^{a_i}$ and $A_N^{a_f}$ of a Sheffer stroke BCH-algebra for its any elements a_t, a_i, a_f and state that these subsets are ideals of this algebraic structure if a neutrosophic $\mathcal{N}-$structure on this algebraic structure is the neutrosophic $\mathcal{N}-$ideal.

In future works, we wish to study on fuzzy and plithogenic structures on Sheffer stroke BCH-algebras.

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Introduction to Neutrosophic Restricted SuperHyperGraphs and Neutrosophic Restricted SuperHyperTrees

and several of their properties

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Abstract: In this article, we first provide a modified definition of SuperHyperGraphs (SHG) and we call it Restricted SuperHyperGraphs (R-SHG). We then generalize the R-SHG to the neutrosophic graphs and then define the corresponding trees. In the following, we examine the Helly property for subtrees of SuperHyperGraphs.

Keywords: SuperHyperGraphs; Restricted SuperHyperGraphs; Neutrosophic SuperHyperGraphs; Neutrosophic SuperHyperTrees; Helly property; chordal graph; subtree.

1. Introduction

Hypergraph theory is one of the most widely used theories in modeling large and complex problems. In recent years, many efforts have been made to find different properties of these graphs [1-5]. One of these features that is also very important is the property of Helly. To read more about this property, you can refer to [4, 5]. Here we first rewrite the definition of SuperHyperGraphs from [1], which has the advantage that we have reduced the empty set from the set of vertices because in practice the empty vertex is not much applicable, and we have also categorized the set of vertices and edges according to its type. Then the adjacency matrix. We define the incidence matrix and the Laplacian matrix.

Obviously, if a super hyper power graph contains a triangle, it will not have a highlight feature. We show here that some defined super hyper power graphs have subtrees that have Helly property. There are algorithms for detecting Helly property in subtrees that the reader can refer to [4] to view.

In graph theory, a chordal graph is a graph in which each cycle is four or more lengths and contains at least one chord. In other words, each induction cycle in these graphs has a maximum of three vertices. Chord graphs have unique features and applications. To study an example of the applications of chordal graphs, you can refer to [7].

Definition 1 [4]. Let A be a set. We say that A has Helly property if and only if, for every non-empty set S such that $S \subseteq A$ and for all sets x, y such that $x, y \in S$ holds x meets y holds $\cap S \neq \emptyset$.

Proposition 1 [4]. Let T be a tree and X be a finite set such that for every set x such that $x \in X$ there exists a subtree t of T such that x is equal the vertices of t. Then X has Helly property.

2. Neutrosophic Restricted SuperHyperGraphs

In this section, we provide a modified definition of Restricted SuperHyperGraphs (RSHG), and then generalize this definition to neutrosophic graphs.

Definition 2. SuperHyperGraph (SHG))[1]

A Super Hyper Graph (SHG) is an ordered pair $SHG = (X \subseteq P(V)\backslash\emptyset, E \subseteq P(V) \times P(V))$, where

i. $V = \{v_1, v_2, \dots, v_n\}$ is a finite set of $n \geq 0$ vertices, or an infinite set.

ii. $P(V)$ is the power set of V (all subset of V). therefore, an SHG-vertex may be a **single** (classical) **vertex** (V_{Si}), or a **super-vertex** (V_{Su}) (a subset of many vertices) that represents a group (organization), or even an **indeterminate-vertex** (V_I) (unclear, unknown vertex);

iii. $E = \{e_1, e_2, \dots, e_m\}$, for $m \geq 1$, is a family of subsets of $V \times V$, and each e_i is an SHG −edge, $e_i \in P(V) \times P(V)$. An SHG −edge may be a (classical) **edge**, or a **super-edge** (edge between super vertices) that represents connections between two groups (organizations), or **hyper-super-edge** that represents connections between three or more groups (organizations), or even an **indeterminate-edge** (unclear, unknown edge); $\emptyset$ represents the null-edge (edge that means there is no connection between the given vertices).

Definition 2-1(2-Restricted SuperHyperGraphs)

2-Restricted SuperHyperGraphs are a special case of SuperHyperGraphs, where we look at the system from the part to the whole. So, according to definition 2, we have

1. Single Edges (E_{Si}), as in classical graphs.
2. Hyper Edges (E_H), edges connecting three or more single- vertices.
3. Super Edges (E_{Su}), edges connecting only two SHG- vertices and at least one vertex is super Vertex.
4. Hyper Super Edges (E_{HS}), edges connecting three or more single- vertices (and at least one vertex is super vertex.
5. Indeterminate Edges (E_I), either we do not know their value, or we do not know what vertices they might connect.

Then, $G = (X, E)$ where $X = (V_{Si}, V_{Su}, V_I) \subseteq P(V)\backslash\emptyset$, and $E = (E_{Si}, E_H, E_{Su}, E_{HS}, E_I) \subseteq P(V) \times P(V)$.

Definition 3. (Neutrosophic Restricted SuperHyperGraphs) Let $G = (X, E)$ be a Restricted SuperHyperGraph. If all vertices and edges of G belong to the neutrosophic set, then the SHG is a Neutrosophic Restricted SuperHyperGraphs (NRSHG). If x is a neutrosophic super vertex containing vertices $\{v_1, v_2, \dots, v_k\}$, where $v_i \in V$ for $1 \leq i \leq k$, then

$$T_X(x) = \min\{T_X(v_i), 1 \leq i \leq k\},$$
$$I_X(x) = \min\{I_X(v_i), 1 \leq i \leq k\},$$
$$F_X(x) = \max\{F_X(v_i), 1 \leq i \leq k\}.$$

Definition 4. Let $G = (X, E)$ be a 2-Restricted SuperHyperGraph, with $X = (V_{Si}, V_{Su}, V_I) \subseteq P(V)\backslash\emptyset$, and $E = (E_{Si}, E_H, E_{Su}, E_{HS}, E_I) \subseteq P(V) \times P(V)$. Then, the ***adjacency matrix*** $A(G) = (a_{ij})$ of G is defined as a square matrix which columns and rows its, is shown by the vertices of G and for each $v_i, v_j \in X$,

$$a_{ij} = \begin{cases} 0 & \text{there should be no edge between vertices } v_i \text{ and } v_j; \\ 1 & \text{there is a single edge between vertices } v_i \text{ and } v_j; \\ \text{S} & \text{there is a super edge between vertices } v_i \text{ and } v_j; \\ \text{H} & \text{there is a hyper edge between vertices } v_i \text{ and } v_j; \\ \text{SH} & \text{there is a super hyper edge between vertices } v_i \text{ and } v_j. \end{cases}$$

Note that in the adjacency matrix A, a value of one can be placed instead of non-numeric values (S, H and SH) if necessary for calculations. So that, since A is a symmetric and values of A is positive, eigenvalues of A are real.

Definition 5. Let $G = (X, E)$ be a Restricted SuperHyperGraph, with $X = (V_{Si}, V_{Su}, V_I) \subseteq P(V)\backslash\emptyset$, and $E = (E_{Si}, E_H, E_{Su}, E_{HS}, E_I) \subseteq P(V) \times P(V)$. If $E = (e_1, e_2, \dots, e_m)$ then an ***incidence matrix*** $B(G) = (b_{ij})$ define as

$$b_{ij} = \begin{cases} 1 & if\ v_i \in e_j, \\ 0 & otherwise. \end{cases}$$

Definition 6. Let $G = (X, E)$ be a Restricted SuperHyperGraph, with $X = (V_{Si}, V_{Su}, V_I) \subseteq P(V)\backslash\emptyset$, and $E = (E_{Si}, E_H, E_{Su}, E_{HS}, E_I) \subseteq P(V) \times P(V)$. If $D = diag(D(v_1), D(v_2), \dots, D(v_n))$ where $D(v_i) = \sum_{v_j \in X} a_{v_i v_j}$, then, a ***laplacian matrix*** define as

$$L(G) = D - A(G).$$

Example 1. Consider $G = (X, E)$ as shown in figure 1 (This figure is selected from reference [1]). Where $X = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, IV_9, SV_{4,5}, SV_{1,2,3}\}$ and $E = \{SiE_{5,6}, IE_{7,8}, SE_{123,45}, HE_{459,3}, HSE_{123,7,8}\}$. We now obtain the SuperHyperGraph – related matrices in figure 1 using the above definitions.

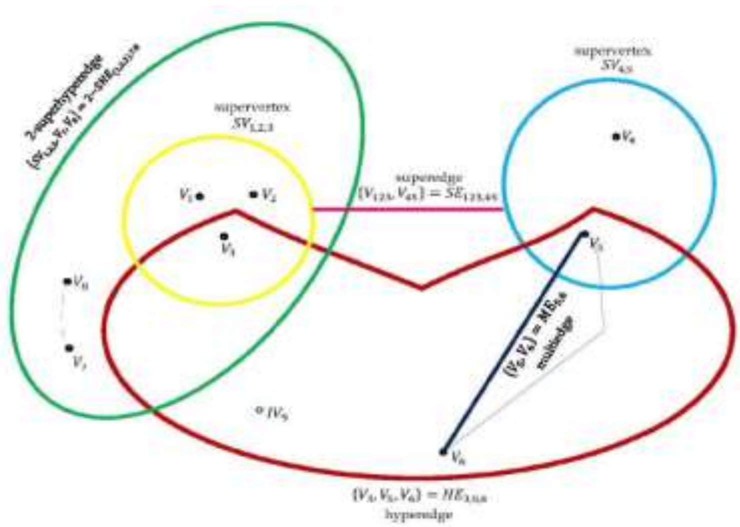

Figure 1. a Restricted SuperHyperGraph $G = (X, E)$

a. Adjacency matrix

$$\mathbb{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ Iv_9 \\ Sv_{4,5} \\ Sv_{1,2,3} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & Iv_9 & Sv_{4,5} & Sv_{1,2,3} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H & H & 0 & 0 & H & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 0 & 2,H & 0 & 0 & H & 0 & 0 \\ 0 & 0 & H & 0 & 2,H & 0 & 0 & 0 & H & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & SH \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & SH \\ 0 & 0 & H & 0 & H & H & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S \\ 0 & 0 & 0 & 0 & 0 & 0 & SH & SH & 0 & S & 0 \end{pmatrix} \end{array}$$

b. incidence matrix

$$B = \begin{array}{c} \\ E_{5,6} \\ SE_{123,45} \\ HE_{3,5,6,9} \\ SHE_{123,7,8} \\ IE_{7,8} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & Iv_9 & Sv_{4,5} & Sv_{1,2,3} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

c. Laplacian matrix

To calculate the Laplacian matrix, we first obtain the diameter matrix D, in which the vertices on the principal diameter, the degree of vertices, and the other vertices are 0. Then its Laplacian matrix is calculated as follows.

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 5 & -3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -3 & 5 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{pmatrix}$$

3. Neutrosophic SuperHyperTree

In this section, we first provide a definition of Neutrosophic SuperHyperTree. We then define the subtree for Neutrosophic SuperHyperGraphs. In the following, we will examine the Helly property in this type of power graphs.

Definition 7. Let $G = (X, E)$ be a Neutrosophic SuperHyperGraph. Then G is called a Neutrosophic SuperHyperTree (NSHG) if G be a connected Neutrosophic SuperHyperGraph without a neutrosophic cycle.

Definition 8. Let $H = (A, B)$ be a Neutrosophic SuperHyperGraph. Then H is called a subtree NSHG if there exists a tree T with the same vertex set V such that each hyperedge, superedge, or hypersuperedge $e \in E$ induces a subtree in T.

Note. Here we consider the underlying graph H^* to find the subtree of NSHG.

Example 2. Consider $G = (X, E)$ a Restricted SuperHyperGraph as shown in figure 2.

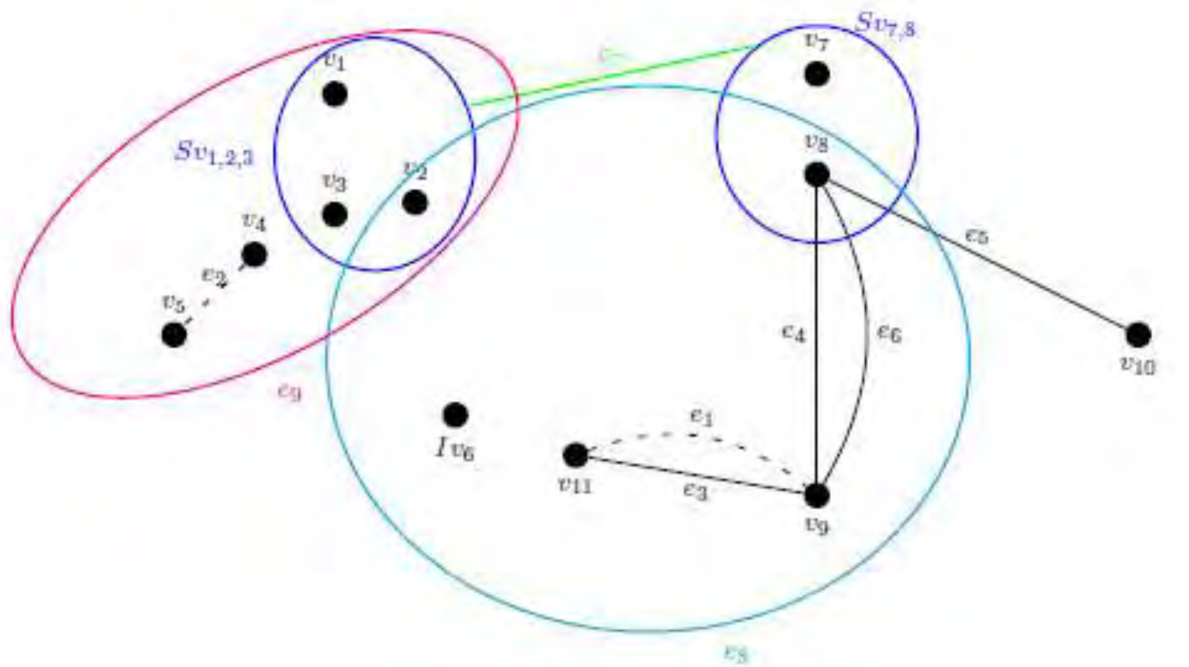

Figure 2. A Restricted SuperHyperGraph

As you can see, since G contains the cycle, so that G is not a Restricted SuperHyperTree. An RSH −subgraph induced by the subset $\{e_7, e_8, e_9, e_5\}$ of X, is a RSHT.

Example 3. Consider $G = (X, E)$ a Neutrosophic Super Hyper Power Graph as shown in figure 3. Note that in this example all vertices and edges belong to the neutrosophic sets. As you can see, G is a Restricted SuperHyperTree.

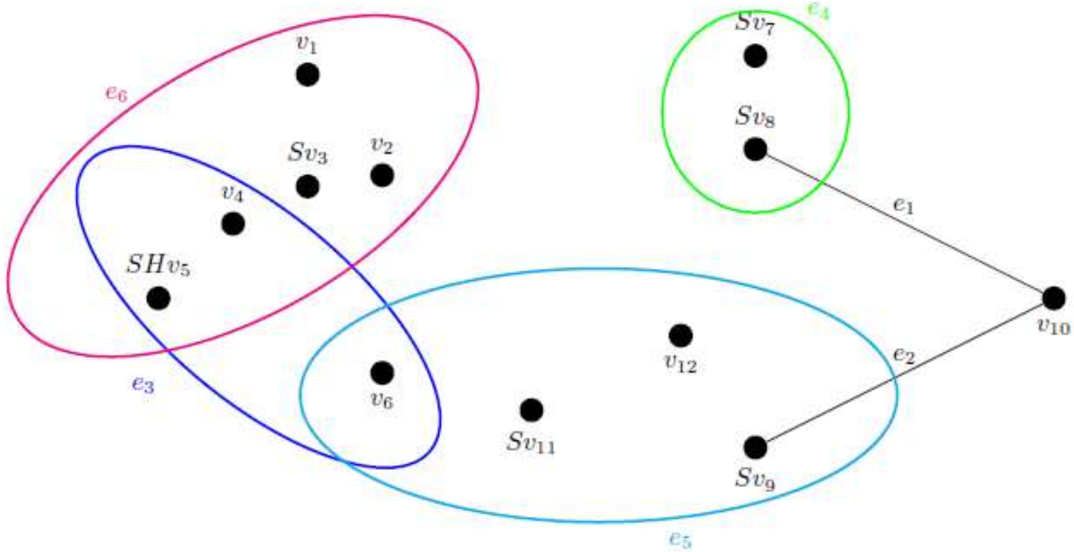

Figure 3. A Neutrosophic Restricted SuperHyperTree G

Now we find a subtree according to definition 7 for G.

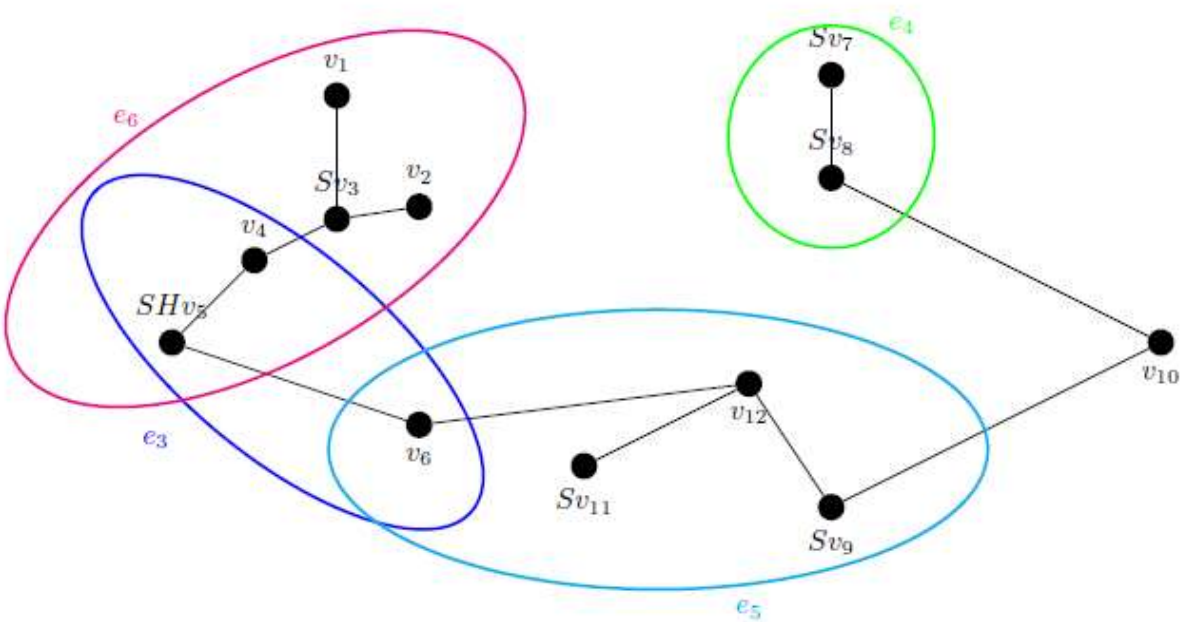

Figure 4. A subtree for NRSHG G

Now, let $T = (A, C)$ be a tree, that is, T is a connected neutrosophic graph without cycle. Then, we build a connected NRSHGraph H in the following way:

1. The set of vertices of H is the set of vertices of T;
2. The set of edges (hyperedges, superedges or superhyperedges) are a family E of subset V such that induced subgraph T_i is a subtree of T where T_i is produced by vertices located on edge $e_i \in E$. so that subgraph T_i is a tree.

Theorem 1. Let $T = (V, E')$ be a tree. Also, H is a subtree Restricted SuperHyperGraph according to T. Then H has the Helly property.

Proof. Since for each tree there exist exactly one path between the two vertices v_i, v_j. The path between two vertices v_i, v_j denoted $P[v_i, v_j]$.suppose that, v_i, v_j and v_k are three vertices of H. The paths $P[v_i, v_j]$, $P[v_j, v_k]$ and $P[v_k, v_i]$ have one common vertex. Now, using theorem 1, for each family of edges (hyperedges, superedges and superhyperedges) where the edge contains at least two of the vertices v_i, v_j and v_k have a non-empty intersection.
□

Theorem 2. Let $T = (V, E')$ be a tree. Also, H is a subtree Restricted SuperHyperGraph according to T. Then $L(H)$ is a chordal graph.

Proof. Consider $T = (V, E')$ is a tree. Suppose H is a subtree Restricted SuperHyperGraph according to T. If $|V| = 1$, then H include exactly one vertex and one hyperdege, so that, the linegraph of H has only one vertex hence H is a clique. It turns out that H is a chordal graph. Next, assume that the assertion is true for each tree with $|V| = n - 1, n > 1$.

Now we have to show that the problem assumption is valid for n vertices as well. For that, suppose $v \in V$ is a vertex leaf on H. remember that in a tree with at least two vertices there exist at least two leaves. If $T_1 = (V - \{v\}, E_1')$, where T_1 is the subgraph on $V - \{v\}$, and

$$H_1(V - \{v\}) = (V - \{v\}, E_1),\ |V| > 1.$$

The $T_1 = (V - \{v\}, E_1')$ is a tree moreover $H_1 = (V - \{v\}, E_1)$ is an induced subtree Restricted SuperHyperGraph associated with T_1. Hence $L(H_1)$ is chordal.

Now, if the number of edges should be the same, that is, $|E'| = |E_1'|$ then we have $L(H) \approx L(H_1)$ so that $L(H)$ is a chordal graph.

If $|E'| \neq |E_1'|$ then we have

$$\{v\} \in E'\ and\ |E'| > |E_1'|.$$

It is easy to show that a neighborhood from $\{v\}$ in $L(H)$ is a clique. Hence any cycle passing through {v} is chordal in $L(H)$ and so $L(H)$ is chordal.
□

Corollary 1. A Restricted SuperHyperGraph G is a subtr Restricted SuperHyperGraph if and only if G has the Helly property and its line graph is a chordal graph.

4. Conclusions

In this article, we have defined a SuperHyperTree and Neutrosophic SuperHyperTree, and examined the Helly property, which is one of the most important and practical properties in subtrees, for the super hyper tree introduced in this article. There are also algorithms for detecting Helly property that we have omitted here.

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Soft Neutrosophic Quasigroups

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Abstract. This paper introduced and studied for the first time the object of neutrosophic quasigroup $\mathcal{Q}_{(Q,\cdot)}$ over a quasigroup $(Q,\cdot)$. It was shown that the direct product of any two neutrosophic quasigroups is a neutrosophic quasigroup. Also, it was established that the holomorph of any neutrosophic quasigroup is a neutrosophic quasigroup. The soft set theory which Molodtsov innovated is a potent mathematical tool used for solving mathematical problems with uncertainties and things that are not clearly defined. We broaden soft set theory by introducing soft neutrosophic quasigroup $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}}$ over a neutrosophic quasigroup $\mathcal{Q}_{(Q,\cdot)}$. We introduced and established the order of a finite soft neutrosophic quasigroup with varied mathematical inequality expressions which exist between the order of a finite neutrosophic quasigroup and that of its soft neutrosophic quasigroup.

Keywords: Soft set, Neutrosophic set, Quasigroup, Neutrosophic quasigroup, Neutrosophic subquasigroup, Soft neutrosophic quasigroup, soft neutrosophic subquasigroup

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1. **Introduction**

Molodtsov [14] introduced a better and more potent generalisation of set theory in solving classic mathematical problems represented by problems involving data structure that are deficient. Before then there exists some mathematical tools like rough set, fuzzy set, vague set, probability theory, intuitionistic set and neutrosophic sets among others. However, they have some limitations due to absence of adequate parametrisation tools that exists when solving mathematical uncertainties. Soft set has characteristics that makes it different from other mathematical tools. For instance, it's effectiveness is by using parametrisation in solving problems that involves incomplete data where grade of membership that is imperative in fuzzy set and grade of estimation in rough set are not needed.

Smarandache [18] launched the object of neutrosophic set in an attempt to generalise set theory involving uncertainties. Neutrosophic set is a mathematical tool that probe the origin, nature and the range of neutralities that is used for the generalisation of classical set, fuzzy set, interval fuzzy set and rough set etc. Neutrosophic set is used to solve mathematical problems involving imprecision, indeterminate, and inconsistencies.

In the concept of neutrosophy, each suggestion or idea is approximated to have certain degree of subset of truthfulness $\mathcal{T}^\star$, indeterminacy $\mathcal{I}^\star$ and falsehood $\mathcal{F}^\star$. Neutrosophic set theory is used in solving problems involving informations that are imprecise, indeterminate, false and not properly defined which exist mainly in belief methodology. Since its introduction, many authors have published works in it, such as Vasantha and Smarandache [25] introduced certain algebraic neutrosophic structures and N-Algebraic neutrosophic structures, neutrosophic vector space and neutrosophic loops. Ali et al. [15] introduced soft neutrosophic loops and their generalizations, and Ali et al. [16] worked on soft neutrosophic groupiod and their generalisations. Maji [13] worked on neutrosophic soft sets.

Quasigroups are structures that generalises groups but they are not associative like groups. Quasigroups theory was introduced more than two centuries ago. Effectiveness of the application of the theory of quasigroups is based on its "generalized permutations" of some sort and the number of quasigroups of order n is larger than $n!$ - (Denes and Keedwell [17]). Namely, both left and right translations properties in quasigroups are permutations.

We refer readers to Aktas and Cagman [3], Molodtsov [14], Maji et. al. [13] and studies in [4, 5, 8, 12, 19, 20, 23, 24] for works on soft sets.

After the introduction of neutrosophic sets as generalization of intuitionistic fuzzy sets by Smarandache [18] in 2002, Vasantha and Smarandache [25] did a comprehensive introduction of algebraic neutrosophic structures and N-algebraic neutrosophic structures in 2006. Thereafter, Ali et al. [15, 16] studied neutrosophic groupoid, neutrosophic quasigroup and their soft sets deeply. Some recent developments in the study of 'soft neutrosophic' versions of various algebraic structures have been reported in [28, 29, 36] while some latest developments in the study of 'neutrosophic soft' versions of some algebraic structures have been reported in [30–32, 37].

The exploits done by different authors on algebraic characteristics of soft sets in general, and recent exploits on soft quasigroups in Oyem et al. [33–35] inspired us to institute the research on soft neutrosophic quasigroup structures. In this work, we introduced neutrosophic quasigroups and their soft sets.

2. Preliminaries

We start by reviewing some results concerning neutrosophic sets, quasigroups and soft sets. Various algebraic structures of neutrosophic sets have been introduced and studied in [6, 13, 15, 16, 18, 25].

Definition 2.1. *(Neutrosophic Set) If $\mathcal{X}$ is a universal set of discourse, then the set $\mathcal{A}$ on $\mathcal{X}$ is regarded as a neutrosophic set and denoted as;*

$$\mathcal{A} = \{< x, \mathcal{T}_{\mathcal{A}}(x), \mathcal{I}_{\mathcal{A}}(x)\}, \mathcal{F}_{\mathcal{A}}(x) >, x \in \mathcal{X}$$
$$\text{where } \mathcal{T}^{\star}, \mathcal{I}^{\star}, \mathcal{F}^{\star} : \mathcal{X} \longrightarrow]^{-}0, 1^{+}[\text{ and } ^{-}0 \leq \mathcal{T}_{\mathcal{A}}(x) + \mathcal{I}_{\mathcal{A}}(x) + \mathcal{F}_{\mathcal{A}}(x) \leq 3^{+}$$

Quasigroups and loops have been studied in [1, 2, 7, 9–11, 22, 26].

Definition 2.2. *(Groupoid, Quasigroup)*

If G is a non-empty set with a binary operation $(\cdot)$ on G, the pair $(G, \cdot)$ is called a groupoid or Magma if for all $x, y \in G$, $x \cdot y \in G$. If for any $m, n \in G$, the equations:

$$m \cdot x = n \qquad \text{and} \qquad y \cdot m = n$$

have unique solutions x and y in G respectively, then $(G, \cdot)$ is called a quasigroup.

Let $(G, \cdot)$ be a quasigroup. $(G, \cdot)$ is called a loop if $e \in G$ such that for any $x \in G$, $x \cdot e = e \cdot x = x$.

Assuming x is a member of a groupoid $(G, \cdot); x \in G$, such that the left and right translation maps of G represented as L_x and R_x are defined as

$$yL_x = x \cdot y \qquad \text{and} \qquad yR_x = y \cdot x.$$

If in the groupoid $(G, \cdot)$, the left and right translation maps are permutations, then the groupoid $(G, \cdot)$ becomes a quasigroup. Thus, their inverse mappings L_x^{-1} and R_x^{-1} exist. Therefore

$$x\backslash y = yL_x^{-1} \qquad \text{and} \qquad x/y = xR_y^{-1}$$

and note that

$$x\backslash y = z \Leftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Leftrightarrow z \cdot y = x.$$

Definition 2.3. *(Subquasigroup [22])*

Assuming $(Q, \cdot)$ is a non-empty quasigroup and $H \subset Q$. Then H will be regarded as a subquasigroup of $(Q, \cdot)$ if $(H, \cdot)$ is closed under the operation of $(Q, \cdot)$ and it is a quasigroup on its own right.

In quasigroups, the cancellation rule holds, that is, if for $x, y, z \in Q$, $x \cdot y = x \cdot z$ or $y \cdot x = z \cdot x$ then $y = z$. It means that in the Cayley table for a quasigroup, each element appears exactly once in each row and in each column, so the table forms a Latin square. The body of any finite quasigroup represented in Cayley table represents a Latin square.

The following results and definitions will be used for our main results.

Proposition 2.1. *([26])*

Take Q as a quasigroup of order n and P as a proper subquasigroup of Q or order p. Then, $2p \leq n$.

Proof. Let $x \in Q - P$, if $y \in P$, then $xy \in Q - P$. So $xP \subset Q - P$. But xP has the order p since $Q - P$ has order $n - p$; which implies that $p \leq n - p = 2p \leq n$.

Therefore the order of a subquasigroup is equal to or less than half of order of the quasigroup.

□

Proposition 2.2. *Take Q to be a quasigroup, and let P and H be subquasigroups of Q, so that $Q = P \cup H$. Then, either $P = Q$ or $H = Q$.*

Proof. Assuming $P \neq H$. If $p \in P - H$ and $h \in H$, then $ph \notin H$, and therefore $ph \in P$ and $h \in P$. So, $H \leq P$, therefore $P = Q$. □

Lemma 2.1. *([26])*

Take P to be a proper subquasigroup of $(Q, \cdot)$, so if

(1) *$a \in Q$ and $P \subset Q$, then $|P| = |a \cdot P| = |P \cdot a|$.*
(2) *$(P, \cdot)$ is a groupoid and $P \subset Q$, then $P \subset Q$.*
(3) *$a \in P$ and $P \subset Q$, $a \in P$ and $b \notin P$ means $ab \notin P$.*

Theorem 2.1. *([26])*

Take Q be a quasigroup with a proper subquasigroup P. Then,

$$2|P| \leq |Q|.$$

Definition 2.4. *(Lagrange Property [22])*

Take Q to be a finite quasigroup such that $P \subset Q$. Then the subquasigroup P of Q is said to be Lagrange-like if $|P|$ divides $|Q|$.

Definition 2.5. *(Weak Lagrange Property [22])*

Take Q to be a finite quasigroup and let $P \subseteq Q$. Then Q is said to satisfy the weak Lagrange property if every subquasigroup P of Q is Lagrange-like, that is $|P|$ divides $|Q|$ for all $P \subset Q$.

Definition 2.6. *(Strong Lagrange Property [22])*

Take Q to be a finite quasigroup and $P \subseteq Q$. Q is said to have a strong Lagrange property if P satisfies the weak Lagrange property for all the $P \subset Q$.

Remark 2.1. *The order of a subquasigroup is not necessarily a factor of the order of the quasigroup, that is the Lagrange properties does not in general hold for quasigroups. For example, let Q be a quasigroup of order 10 with only two subquasigroups H, K of orders $|H| = 2$ and $|K| = 5$ respectively such that $H \leq K \leq Q$. Since $2, 5$ divide 10, then Q has the weak Lagrange property. But Q does not have the strong Lagrange property since $|H|$ is not a divisor of $|K|$.*

We now introduce soft sets and operations defined on them. Throughout this subsection, Q^* denotes an initial universe, E is the set of defined parameters and $A \subseteq E$.

Definition 2.7. *(Soft Sets, Soft Subset, Equal Soft Sets [3, 5, 12, 14, 19, 23, 33])*

Assume Q^* is a universal set of discourse and E is a set of defined parameters such that $C \subset E$. The couple (G, C) is called a soft set over Q^*, whenever $G(c) \subset Q^* \ \forall \ c \in C$; and F is a function mappings C to all the non-empty subsets of Q^*, i.e $G : C \longrightarrow 2^{Q^*} \backslash \{\emptyset\}$. A soft set (G, C) over a set Q^* is described as a set of ordered pairs: $(G, C) = \{(c, G(c)) : c \in C \ \text{ and } G(a) \in 2^{Q^*}\}$. The set of all soft sets, over Q^* under a defined set of parameters C, is denoted by $SS(Q^*_C)$.

Suppose that (G^*, C) and (K^*, D) are two soft sets defined over Q^*, then (K^*, D) will be regarded as a soft subset of (G^*, C) if,

(1) *$D \subseteq C$; and*

(2) *$K^*(c) \subseteq G^*(c) \ \forall \ c \in D$.*

Definition 2.8. *(Restricted Intersection)*

Consider (G^*, C) and (K^*, D) to be two soft sets over Q^* so that $C \cap D \neq \emptyset$. We define their restricted intersection as a soft set $(G^*, C) \cap (K^*, D) = (Z^*, E)$ where (Z^*, E) is represented as $Z^*(e) = G^*(e) \cap K^*(e) \ \forall \ e \in E$ and $E = C \cap D$.

Definition 2.9. *(Extended Intersection)*

Consider (G^*, C) and (K^*, D) be two soft sets over Q^* so that $C \cap D$ is not empty. Their extended intersection is a soft set (Z^*, E), where $E = C \cup D \ \forall \ e \in E$, $Z^*(e)$ can be defined as;

$$Z^*(e) = \begin{cases} G^*(e) & \text{whenever} \quad e \in C - D \\ K^*(e) & \text{whenever} \quad e \in D - C \\ G^*(e) \cap K^*(e) & \text{whenever} \quad e \in C \cap D. \end{cases}$$

Definition 2.10. *(Union)*

The extended union of two soft sets (G^*, C) *and* (K^*, D) *over* Q^* *is defined as* $(G^*, C) \bigcup (K^*, D)$ *and it is called a soft set* (Z, E) *over* Q^*, *as* $E = C \cup D \; \forall \; e \in E$ *and*

$$Z^*(e) = \begin{cases} G^*(e) & \textit{whenever} \quad e \in C - D \\ K^*(e) & \textit{whenever} \quad e \in D - C \\ G^*(e) \cup K^*(e) & \textit{whenever} \quad e \in C \cap D. \end{cases}$$

3. MAIN RESULTS

3.1. *Neutrosophic Quasigroup*

Definition 3.1. *(Neutrosophic Quasigroup)*

Let $(Q, \cdot)$ *be a quasigroup. The neutrosophic quasigroup over a quasigroup* Q *is* $\mathcal{Q} = \left\langle Q \cup \mathcal{N}_\rceil \right\rangle$ *generated by* Q *and neutrosophic element* $\mathcal{N}_\rceil$ *coupled with a binary operation* $\odot$ *such that* $\mathcal{Q} = \left(\left\langle Q \cup \mathcal{N}_\rceil \right\rangle, \odot\right)$ *is a quasigroup.* $\mathcal{Q}$ *being a neutrosophic quasigroup over* $(Q, \cdot)$ *will sometimes represented by* $\mathcal{Q}_{(Q,\cdot)}$ *or* $\mathcal{Q}_Q$.

Remark 3.1. *If* $\mathcal{Q}_Q$ *is neutrosophic quasigroup over* Q, *then* $\mathcal{Q}$ *contains* Q *as a subquasigroup.*

Example 3.1. *Let* $(Q, \cdot)$ *be a quasigroup of order* 4 *where* $Q = \{1, 2, 3, 4\}$ *and let*

$$\mathcal{Q} = \left\langle Q \cup \mathcal{N}_\rceil \right\rangle = \{1, 2, 3, 4, 1\mathcal{N}_\rceil, 2\mathcal{N}_\rceil, 3\mathcal{N}_\rceil, 4\mathcal{N}_\rceil\}$$

be represented by the multiplication Table 1. Then $\mathcal{Q}_{(Q,\cdot)} = \mathcal{Q}_Q = \mathcal{Q} = \left(\left\langle Q \cup \mathcal{N}_\rceil \right\rangle, \odot\right)$ *is a the neutrosophic quasigroup over* Q.

TABLE 1. Neutrosophic quasigroup of order 8

$\odot$	1	2	3	4	$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$
1	*1*	*2*	*4*	*3*	$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$
2	*2*	*1*	*3*	*4*	$2\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$
3	*3*	*4*	*2*	*1*	$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$
4	*4*	*3*	*1*	*2*	$4\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$
$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	*2*	*1*	*4*	*3*
$2\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	*1*	*2*	*3*	*4*
$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	*3*	*4*	*1*	*2*
$4\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	*4*	*3*	*2*	*1*

Example 3.2. *Consider* $(G, +)$ *to be a quasigroup of order* 3 *where* $G = \{i, j, k\}$ *and let*

$$\mathcal{G} = \left\langle G \cup \mathcal{N}_\rceil \right\rangle = \{i, j, k, i\mathcal{N}_\rceil, j\mathcal{N}_\rceil, k\mathcal{N}_\rceil\}$$

be represented by the multiplication Table 2 . Then $\mathcal{G}_{(G,+)} = \mathcal{G}_G = \mathcal{G} = \left(\left\langle G \cup \mathcal{N}_\rceil \right\rangle, \oplus\right)$ *is a the neutrosophic quasigroup over* G.

TABLE 2. Neutrosophic quasigroup of order 6

$\oplus$	i	j	k	$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$
i	i	j	k	$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$
j	j	k	i	$j\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$	$i\mathcal{N}_\rceil$
k	k	i	j	$k\mathcal{N}_\rceil$	$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$
$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$	$i\mathcal{N}_\rceil$	i	j	k
$j\mathcal{N}_\rceil$	$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$	k	i	j
$k\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$	$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$	j	k	i

Definition 3.2. *(Neutrosophic Subquasigroup)*

Consider $\mathcal{Q}_Q = \left(\left\langle Q \cup \mathcal{N}_\rceil \right\rangle, \odot\right)$ *to be a neutrosophic quasigroup over* Q *and* $\emptyset \neq \mathcal{H} \subseteq \mathcal{Q}$*. Then,* $\mathcal{H}_H$ *will be regarded as a neutrosophic subquasigroup of* $\mathcal{Q}$ *if there exists* $H \leq Q$ *such that* $\mathcal{H}_H = \left(\left\langle H \cup \mathcal{N}_\rceil \right\rangle, \odot\right)$ *is a neutrosophic quasigroup over* H*. This will often be expressed as* $\mathcal{H}_H \leq_{\mathcal{N}_\rceil} \mathcal{Q}_Q$*..*

Remark 3.2. *In Definition 3.2, if* $\mathcal{H}_H = H$*, then* $\mathcal{H}_H$ *will be called a trivial neutrosophic subquasigroup of* $\mathcal{Q}$*. Also,* $\mathcal{H}_H = \mathcal{Q}_Q$ *will be regarded as a trivial neutrosophic subquasigroup of* $\mathcal{Q}$*.*

Example 3.3.

(1) *In Example 3.1,* $\mathcal{H}_H = \{1, 2, 1\mathcal{N}_\rceil, 2\mathcal{N}_\rceil\}$ *is a neutrosophic subquasigroup of* $\mathcal{Q}_Q$ *i.e.* $\mathcal{H}_H \leq_{\mathcal{N}_\rceil} \mathcal{Q}_Q$ *because* $(\mathcal{H}_H, \odot)$ *is a neutrosophic quasigroup over* H *going by Table 1. However,* $\mathcal{K} = \{1, 2, 3\mathcal{N}_\rceil, 4\mathcal{N}_\rceil\}$ *is not a neutrosophic subquasigroup of* $\mathcal{Q}_Q$ *even though* $K \leq \mathcal{Q}_Q$*. This is because by Table 1, there is no* $K \leq Q$*, such that* $\mathcal{K}_K = \left\langle K \cup \mathcal{N}_\rceil \right\rangle = \{1, 2, 3\mathcal{N}_\rceil, 4\mathcal{N}_\rceil\}$*. Neither* $\{1, 2\}$ *nor* $\{3, 4\}$ *can be* K*.*

(2) *In Example 3.2,* $\mathcal{Q}_Q$ *has no nontrivial neutrosophic subquasigroup judging by Table 2.*

Remark 3.3. *Based on Example 3.3(1), not every subquasigroup of a neutrosophic quasigroup* $\mathcal{Q}_Q$ *is a neutrosophic subquasigroup of* $\mathcal{Q}_Q$*. Ofcourse, every neutrosophic subquasigroup of a neutrosophic quasigroup* $\mathcal{Q}_Q$ *is a subquasigroup of* $\mathcal{Q}_Q$*.*

3.2. *Direct Product of Neutrosophic Quasigroups*

The direct product $Q \times H$ of two groups (quasigroups,loops) Q, H is a group (quasigroup, loop). For group (loop), it clearly contains at least one subgroup (subloop) isomorphic to Q, namely $Q \times \{e\}$. However, this is not the case for a direct product of two quasigroups. Bruck [7] gave examples of finite nontrivial quasigroups Q and H whose direct product has no proper subquasigroup. Foguel [21] considered when the direct product $Q \times H$ of two quasigroups

Q, H contains a subquasigroup isomorphic Q. We shall now consider the direct product of two neutrosophic quasigroups.

Theorem 3.1. *(Direct Product of Neutrosophic Quasigroups)*

Take $\mathcal{Q}_{(Q,\cdot)} = \left(\left\langle Q \cup \mathcal{N}_{\rceil} \right\rangle, \odot\right)$ *and* $\mathcal{H}_{(H,*)} = \left(\left\langle H \cup \mathcal{N}_{\lfloor} \right\rangle, \circledast\right)$ *to be any two neutrosophic quasigroups. Their direct product*

$$(\mathcal{Q} \times \mathcal{H})_{(Q\times H,(\cdot,*))} = (\mathcal{Q} \times \mathcal{H})_{Q\times H} = \mathcal{Q} \times \mathcal{H} = \left(\left\langle Q \times H \cup \left(\mathcal{N}_{\rceil}, \mathcal{N}_{\lfloor}\right)\right\rangle, (\cdot, *)\right)$$

is a neutrosophic quasigroup.

Proof.

$$\mathcal{Q}_Q \times \mathcal{H}_H = \left\langle Q \cup \mathcal{N}_{\rceil} \right\rangle \times \left\langle H \cup \mathcal{N}_{\lfloor} \right\rangle =$$

$$\left\{(q,h), (q\mathcal{N}_{\rceil}, h), (q, h\mathcal{N}_{\lfloor}), (q\mathcal{N}_{\rceil}, h\mathcal{N}_{\lfloor}) | q \in Q, h \in H\right\} = \left\langle Q \times H \cup \left(\mathcal{N}_{\rceil}, \mathcal{N}_{\lfloor}\right)\right\rangle.$$

So, $\mathcal{Q}_Q \times \mathcal{H}_H$ is a generated by $Q \times H$ and $\left(\mathcal{N}_{\rceil}, \mathcal{N}_{\lfloor}\right)$, and thus $\left(\left\langle Q \times H \cup \left(\mathcal{N}_{\rceil}, \mathcal{N}_{\lfloor}\right)\right\rangle, (\cdot, *)\right)$ is a quasigroup and has $Q \times H$ as a subquasigroup. □

Corollary 3.1. *Let* $\mathcal{Q}_{(Q,\cdot)} = \left(\left\langle Q \cup \mathcal{N}_{\rceil} \right\rangle, \odot\right)$ *and* $\mathcal{H}_{(H,*)} = \left(\left\langle H \cup \mathcal{N}_{\lfloor} \right\rangle, \circledast\right)$ *be any two neutrosophic quasigroups with neutrosophic subquasigroups* ${\mathcal{Q}'}_{(Q',\cdot)}$ *and* ${\mathcal{H}'}_{(H',*)}$ *respectively. Then,*

$$(\mathcal{Q}' \times \mathcal{H}')_{(Q'\times H',(\cdot,*))} \leq_{(\mathcal{N}_{\rceil}, \mathcal{N}_{\lfloor})} (\mathcal{Q} \times \mathcal{H})_{(Q\times H,(\cdot,*))}$$

Proof. Going by Theorem 3.1, $(\mathcal{Q} \times \mathcal{H})_{(Q\times H,(\cdot,*))}$. Since ${\mathcal{Q}'}_{(Q',\cdot)}$ and ${\mathcal{H}'}_{(H',*)}$ are neutrosophic quasigroups, then by Theorem 3.1. Thus,

$$(\mathcal{Q}' \times \mathcal{H}')_{(Q'\times H',(\cdot,*))} \leq_{(\mathcal{N}_{\rceil}, \mathcal{N}_{\lfloor})} (\mathcal{Q} \times \mathcal{H})_{(Q\times H,(\cdot,*))}$$

because $\mathcal{Q}' \times \mathcal{H}' \subseteq \mathcal{Q} \times \mathcal{H}$ and $Q' \times H' \leq Q \times H$. □

Example 3.4. *By considering the direct product of the neutrosophic quasigroups* $\mathcal{Q}_{(Q,\cdot)}$ *and* $\mathcal{G}_{(G,+)}$ *in Example 3.1 and Example 3.2 respectively, the multiplication table of the neutrosophic quasigroup* $(\mathcal{Q} \times \mathcal{G})_{(Q\times G,(\cdot,+))}$ *can be constructed by using the multiplication Table 1 and Table 2.*

3.3. *Holomorph of Neutrosophic Quasigroups*

We recall the definition of the holomorph of a quasigroup.

Definition 3.3. *If we take* $(Q, \cdot)$ *to be a quasigroup and let* $A(Q, \cdot) = A(Q) \leq AUM(Q, \cdot) = AUM(Q)$ *be the subgroup of the automorphism group of* $(Q, \cdot)$*. Let* $H(Q, \cdot) = H(Q) = A(Q) \times Q$ *and define* $\circ$ *on* $H(Q)$ *as follows* $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$ *for all* x, y *in* Q *and for all* $\alpha, \beta \in A(Q)$*. Then, the pair* $(H(Q), \circ)$ *(or* $H(Q)$*) is called the* $A(Q)$*-holomorph (or holomorph)* *of* Q. $(H(Q), \circ)$ *is a quasigroup. The* $A(Q)$*-holomorph* $H(Q)$ *of a quasigroup* Q *is a semi-direct product of* Q *and an automorphism group* $A(Q)$ *of it.*

Moreover, many authors such as [38, 40–42, 44, 52, 53] have considered the holomorphs of A-loops, Bruck loops, Bol loops, conjugacy closed loops, extra loops, inverse property loops, weak inverse property loops. Jaiyéọlá [11, 47, 48] also derived some results on the holomorph of certain varieties of loops. Adeniran et. al. [39] studied the holomorph of generalized Bol Loops. Results on the holomorphy of Osborn loops and the holomorphy of middle Bol loops can be seen on Jaiyéọlá and Popoola [49]; Isere et. al. [46] and Jaiyéọlá et al. [50]. Recently, Ogunrinade et al. [51] studied the holomorphy of self distributive quasigroup, Ilojide et al. [45] studied the holomorphy of Fenyves BCI-algebras; Oyebo et al. [54] considered the Holomorphy of (r, s, t)-inverse loops while Effiong et al. [43] considered the holomorphy of Basarab loops. Specifically, we cite the work of Bruck [40] on inverse property loops (IPL), where he established that the holomorph of IPL is an IPL. Also, Huthnance Jr. [44] established that the holomorph of WIP loops is a WIP loop.

Let us now introduce the holomorph of a neutrosophic quasigroup and investigate if it is a neutrosophic quasigroup.

Theorem 3.2. *Take* $(Q, \cdot)$ *to be a quasigroup with* $A(Q)$*-holomorph* $(H(Q, \cdot), \circ)$*. Let* $\mathcal{Q}_{(Q,\cdot)} = \left(\left\langle Q \cup \mathcal{N}_\rceil \right\rangle, \odot\right)$ *be a neutrosophic quasigroup over* $(Q, \cdot)$ *then,*

(1) $\left(\mathcal{H}_{(H(Q,\cdot),\circ)}, \circledcirc\right)$ *is a quasigroup and the* $A\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right)$*-holomorph of* $\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right)$*; and*
(2) $\left(\mathcal{H}_{(H(Q,\cdot),\circ)}, \circledcirc\right)$ *is a neutrosophic quasigroup over* $(H(Q, \cdot), \circ)$.

Proof.

(1) Note that since $A(Q, \cdot) \leq AUM(Q, \cdot)$, then $A\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right) \leq AUM\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right)$. Thus, since $(H(Q, \cdot), \odot)$ is a quasigroup, then with $\mathcal{H} = \left\langle Q \cup \mathcal{N}_\rceil \right\rangle \cup A\left(\mathcal{Q}_{(Q,\cdot)}\right) = \mathcal{Q}_{(Q,\cdot)} \cup A\left(\mathcal{Q}_{(Q,\cdot)}\right)$, $\left(\mathcal{H}_{(H(Q,\cdot),\circ)}, \circledcirc\right)$ is a quasigroup, and so, $A\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right)$- is holomorph of $\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right)$.
(2) Notice that $A(Q, \cdot) = \left\{\alpha \in A\left(\mathcal{Q}_{(Q,\cdot)}\right) | \alpha = \alpha|_{(Q,\cdot)}\right\} \leq A\left(\mathcal{Q}_{(Q,\cdot)}\right)$. Thus, $(H(Q, \cdot), \circ) \leq \left(\mathcal{H}_{(H(Q,\cdot),\circ)}, \circledcirc\right)$. Therefore, $\left(\mathcal{H}_{(H(Q,\cdot),\circ)}, \circledcirc\right)$ is a neutrosophic quasigroup over $(H(Q, \cdot), \circ)$. □

3.4. *Soft Neutrosophic Quasigroup*

Definition 3.4. *(Soft Neutrosophic Quasigroup)*

Assuming that $\mathcal{Q}_{(Q,\cdot)}$ *is a neutrosophic quasigroup, and suppose that* E *is a set of defined parameters and* $A \subset E$*. The couple* $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}}$ *is regarded as a soft neutrosophic quasigroup over* $\mathcal{Q}_{(Q,\cdot)}$ *if* $N_F(a)$ *is neutrosophic subquasigroup of* $\mathcal{Q}_Q$ $\forall\ a \in A$*, where* $N_F : A \longrightarrow 2^{\mathcal{Q}_Q}$.

We shall sometimes write $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}} = \{N_F(a)|a \in A\}$.

Example 3.5. *Table 1 defines a finite neutrosophic quasigroup as a Latin square table; where*

$$\mathcal{Q}_{(\mathcal{Q},\cdot)} = \langle Q \cup \mathcal{N}_\rceil \rangle = \{1, 2, 3, 4, 1\mathcal{N}_\rceil, 2\mathcal{N}_\rceil, 3\mathcal{N}_\rceil, 4\mathcal{N}_\rceil\}$$

Assume $A = \{\beta_1, \beta_2, \beta_3\}$ *to be a set of parameters and let*

$$N_F : A \to 2^{\mathcal{Q}} \uparrow N_F(\beta_1) = \{1, 2\},\ N_F(\beta_2) = \{1, 2, 3, 4\},\ N_F(\beta_3) = \{1, 2, 1\mathcal{N}_\rceil, 2\mathcal{N}_\rceil\}.$$

Then, $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}}$ *is regarded as soft neutrosophic quasigroup over neutrosophic quasigroup* $\mathcal{Q}_{(Q,\cdot)}$ *because* $N_F(\beta_i) \leq_{\mathcal{N}_\rceil} \mathcal{Q}_Q$ *for* $i = 1, 2, 3$.

Now assume $B = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ *be set of defined parameters and*

$$N_F : B \to 2^{\mathcal{Q}} \uparrow N_F(\beta_1) = \{1, 2\},\ N_F(\beta_2) = \{1, 2, 3, 4\},$$
$$N_F(\beta_3) = \{1, 2, 1\mathcal{N}_\rceil, 2\mathcal{N}_\rceil\},\ N_F(\beta_4) = \{1, 2, 3\mathcal{N}_\rceil, 4\mathcal{N}_\rceil\}.$$

Then, $(N_F, B)_{\mathcal{Q}_{(Q,\cdot)}}$ *is not a soft neutrosophic quasigroup over neutrosophic quasigroup* $\mathcal{Q}_{(Q,\cdot)}$ *because* $N_F(\beta_i) \leq_{\mathcal{N}_\rceil} \mathcal{Q}_Q$ *for* $i = 1, 2, 3$ *but* $N_F(\beta_4) \not\leq_{\mathcal{N}_\rceil} \mathcal{Q}_Q$.

Definition 3.5. *(Soft sub-neutrosophic quasigroup)*

If $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}}$ *and* $(N_G, B)_{\mathcal{Q}_{(Q,\cdot)}}$ *are two soft neutrosophic quasigroups over a common neutrosophic quasigroup* $\mathcal{Q}_{(Q,\cdot)}$. $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}}$ *is called soft neutrosophic subquasigroup of* $(N_G, B)_{\mathcal{Q}_{(Q,\cdot)}}$ *if*

(1) $A \subseteq B$, *and*

(2) $N_F(a) \leq_{\mathcal{N}_\rceil} N_G(a)$, *for all* $a \in A$.

This will be expressed as $(N_F, A) \leq_{\mathcal{N}_\rceil} (N_G, B)$.

3.5. *Order of Soft Neutrosophic Quasigroup*

Pflugfelder [22] and Wall [26] established that quasigroups might not neccessarily obey Lagrange's theorem. We extend some results of Wall [26] to soft neutrosophic quasigroup. The existence of identity element in the definition of the order of soft group in Aktas [3] was considered. Hence we introduced new definition for the order of a soft neutrosophic quasigroup that is independent of identity element and associative property. We introduce the order of a soft neutrosophic quasigroup (N_F, χ) over a finite neutrosophic quasigroup $\mathcal{Q}_Q$ and to check for divisibility properties between $|\mathcal{Q}_Q|$ and $|(N_F, \chi)|$, and prove that there are some algebraic connections existing between the orders of a neutrosophic quasigroup $\mathcal{Q}_Q$ and its soft neutrosophic quasigroup (N_F, χ).

Definition 3.6. *(The Order of Soft Neutrosophic Quasigroups)*

Consider $(N_F, \chi)_{\mathcal{Q}_{(Q,\cdot)}}$ *to be a soft neutrosophic quasigroup over a finite neutrosophic quasi-group* $\mathcal{Q}_{(Q,\cdot)}$. $(N_F, \chi)_{\mathcal{Q}_{(Q,\cdot)}}$ *will be called a finite soft neutrosophic quasigroup. The order of* *a finite soft neutrosophic quasigroup* $(N_F,\chi)_{\mathcal{Q}_{(Q,\cdot)}}$*, where* χ *is the set of parameters, will be defined as;*

$$|(N_F,\chi)|_{\mathcal{Q}_{(Q,\cdot)}} = |(N_F,\chi)|_{\mathcal{Q}_Q} = |(N_F,\chi)|_{\mathcal{Q}} = |(N_F,\chi)| = \sum_{a\in\chi} |N_F(a)|,\ N_F(a)\in(N_F,\chi),\ a\in\chi.$$

Definition 3.7. *Consider* $(N_F,\chi)_{\mathcal{Q}_Q}$ *to be soft neutrosophic quasigroup over a neutrosophic quasigroup* $\mathcal{Q}_Q$*. Then, we defined the arithmetic mean of a soft neutrosophic quasigroup and the geometric mean of soft neutrosophic quasigroup* (N_F,χ)*, where* $\chi\neq 0$ *as;*

$$\mathcal{AM}_{\mathcal{Q}}(N_F,\chi) = \frac{1}{|\chi|}\sum_{a\in\chi}|N_F(a)|; \qquad \mathcal{GM}_{\mathcal{Q}}(N_F,\chi) = \sqrt[|\chi|]{\prod_{a\in\chi}|N_F(a)|};$$

Remark 3.4. $(N_F,\chi)_{\mathcal{Q}}$ *is a soft neutrosophic quasigroup over a finite neutrosophic quasigroup* $\mathcal{Q}$ *as in Example 3.5. We note that* $|N_F(a)|\Big||(N_F,\chi)|$ *and* $|N_F(a)|\Big||\mathcal{Q}|$ *occurred for just one case of* $a\in\chi$*,* $|N_F(a)|\Big||(N_F,\chi)|$ *occurred one case of* $a\in\chi$ *and* $|N_F(a)|\Big||\mathcal{Q}|$ *occurred in all cases for* $a\in\chi$*.*

Lemma 3.1. *Consider* $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ *to be a finite neutrosophic quasigroup, then*

- i *If* $(N_F,\chi)_{\mathcal{Q}}$ *is a finite soft neutrosophic quasigroup over* $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$*. For any* $a\in\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$*,* $|N_F(a)| = |\alpha\odot N_F(a)| = |N_F(a)\odot\alpha|\forall\alpha\in\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$*.*
- ii *If* $(N_F,\chi)_{\mathcal{Q}}$ *is a soft set over* $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$*. Then* $(N_F,\chi)_{\mathcal{Q}}$ *is soft neutrosophic quasigroup iff* $(N_F,\chi)_{\mathcal{Q}}$ *is soft neutrosophic groupoid.*
- iii *Consider* $(N_F,\chi)_{\mathcal{Q}}$ *to be soft neutrosophic quasigroup. Hence,*
  - (a) *if for any* $\alpha\in\chi,\ \alpha\in N_F(\alpha)$ *and* $\beta\notin N_F(\alpha)$ *means* $\alpha\odot\beta\notin\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$*.*
  - (b) $N_F(\alpha)\odot\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\setminus N_F(\alpha)\subset\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\setminus N_F(\alpha)$ *for all* $\alpha\in\chi$*.*

Proof.

(1) Assume (N_F,χ) to be a soft neutrosophic quasigroup over a finite neutrosophic quasigroup $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$. From (i) of Lemma 2.1; $N_F(a)\subset\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ for all $a\in\chi$, for any $\alpha\in\mathcal{Q}_{(\mathcal{Q},\cdot)}$, $|N_F(a)| = |\alpha\odot N_F(a)| = |N_F(a)\odot\alpha|\ \forall\ \alpha\in\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$.

(2) If (N_F,χ) is soft set over $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$, and $(N_F,\chi)_{\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)}$ is soft neutrosophic quasigroup, and $(N_F,\chi)_{\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)}$ a soft neutrosophic groupoid. Hence, if $(N_F,\chi)_{\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)}$ is soft neutrosophic groupoid, then $N_F(a)$ is also neutrosophic subgroupoid of $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\ \forall a\in\chi$. From (ii) of Lemma 2.1, $N_F(a)$ is neutrosophic subquasigroup of $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\ \forall\ a\in\chi$. Therefore, $(N_F,\chi)_{\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)}$ is soft neutrosophic quasigroup.

(3) If $(N_F,\chi)_{\mathcal{Q}_{(Q,\cdot)}}$ is soft neutrosophic quasigroup, therefore
  - (a) $N_F(a)\leq_{\mathcal{N}_\rceil}\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\ \forall a\in\chi$, from (iii) of Lemma 2.1, for any $a\in\chi,\ \alpha\in N_F(a)$ and $\beta\notin N_F(a)$ imply $\alpha\odot\beta\notin\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$.
  - (b) This follows from (a) above.□

Theorem 3.3. *Consider* $(N_F,\chi)_{(\mathcal{Q}_{(Q,\cdot)},\odot)}$ *to be a finite soft neutrosophic quasigroup. Then the following holds;*

1. $|(N_F,\chi)| = |\chi|\mathcal{AM}(N_F,\chi)$; 2. $2|(N_F,\chi)| \leq |\chi|\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right|$; 3. $\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right| \geq 2\mathcal{AM}(N_F,\chi)$.

Proof. We derived $|(N_F,\chi)| = |\chi|\mathcal{AM}(N_F,\chi)$ from the combination of the definition of $|(N_F,\chi)|$ and $\mathcal{AM}(N_F,\chi)$. If $(N_F,\chi)_{(\mathcal{Q}_{(Q,\cdot)},\odot)}$ is a soft neutrosophic quasigroup, then $N_F(a) \leq_{\mathcal{N}_\urcorner} \left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ $\forall a \in \chi$. From Theorem 2.1, $2|N_F(a)| \subset \left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right|$ for all $a \in \chi$. So from $\chi = \{a_1, a_2, \cdots, a_n\}$,

$$2|N_F(a_1)|+2|N_F(a_2)|+\cdots+2|N_F(a_n)| \leq |\chi|\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right| \Rightarrow 2\sum_{a\in\chi}|N_F(a)| \leq |\chi|\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right| \Rightarrow$$

$$2|(N_F,\chi)| \leq |\chi|\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right|.$$

Also, $2\sum_{a\in\chi}|N_F(a)| \leq |\chi|\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right| \Rightarrow \left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right| \geq \dfrac{2}{|\chi|}\sum_{a\in\chi}|N_F(a)| \Rightarrow \left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right| \geq 2\mathcal{AM}(N_F,\chi)$. □

Remark 3.5.

(1) From Theorem 3.3, if equation $|(N_F,\chi)| = |\chi|\mathcal{AM}(F,\chi)$ is considered as a Lagrange's Formula for finite soft neutrosophic quasigroup. We let $|\chi|$ and $\mathcal{AM}(N_F,\chi)$ to take the character of the order of subgroup and its index in the group theory, which may not be an integer.

(2) In Theorem 3.3 both $|(N_F,\chi)| = |\chi|\mathcal{AM}(N_F,\chi)$;and $2|(N_F,\chi)| \subset |\chi|\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right|$ gives both an upper and lower bound for the order of a finite soft neutrosophic quasigroup.

Also in Theorem 3.3, the second part from is proved from 1 of Lemma 3.1, such that for any $a \in \left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$, $|N_F(a)| = |\alpha\odot N_F(a)| = |N_F(a)\odot\alpha| \forall\alpha \in \left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$. Hence, if $\alpha \in \left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ and $\alpha \notin N_F(a)$, clearly from 3 of Lemma 3.1,

$$|(N_F,\chi)| \leq \sum_{a\in\chi}\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\setminus N_F(a)\right| \Rightarrow |(N_F,\chi)| \leq \sum_{a\in\chi}\left(\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right| - |N_F(a)|\right) =$$

$$\sum_{a\in\chi}\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right| - \sum_{a\in A}|N_F(a)| \Rightarrow |(N_F,\chi)| \leq |\chi|\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right| - |(N_F,\chi)| \Rightarrow$$

$$2|(N_F,\chi)| \leq |\chi|\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right|.$$

From Example 3.5, if we consider (N_F,χ) as a soft neutrosophic quasigroup over a finite neutrosophic quasigroup $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$. Then it can be observed that if $|\chi| = 3$, $\left|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\right| = 8$, $|(N_F,\chi)| = 10$, then the equations in Theorem 3.3 will be satisfied.

Theorem 3.4. *Consider* $(N_F,\chi)_{(\mathcal{Q}_{(Q,\cdot)},\odot)}$ *to be a finite soft neutrosophic quasigroup. We have,*

$$(i)\ |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\times \sqrt[|\chi|]{\prod_{a\in\chi}|N_F(a)|},\quad (ii)\ |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\mathcal{AM}(N_F,\chi)\quad \textit{and}$$

$$(iii)\ |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq \mathcal{AM}(N_F,\chi)+\mathcal{GM}(N_F,\chi).$$

Proof. From Theorem 2.1, we have,

$$\prod_{a\in\chi}2|N_F(a)| \leq \prod_{i=1}^{|\chi|}|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \Rightarrow 2^{|\chi|}\times\prod_{a\in\chi}|N_F(a)| \leq \prod_{i=1}^{|\chi|}|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|$$

$$\Rightarrow 2^{|\chi|}\times\prod_{a\in\chi}|N_F(a)| \leq |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|^{|\chi|} \Rightarrow \left(\frac{|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|}{2}\right)^{|\chi|} \geq \prod_{a\in\chi}|F(a)|$$

$$\Rightarrow \frac{|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|}{2} \geq \sqrt[|\chi|]{\prod_{a\in\chi}|N_F(a)|} \Rightarrow |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\times\sqrt[|\chi|]{\prod_{a\in\chi}|N_F(a)|} \Rightarrow$$

$$|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\mathcal{GM}(N_F,\chi).$$

By Theorem 3.3, $|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\mathcal{AM}(N_F,\chi)$, therefore $2|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\mathcal{AM}(N_F,\chi)+2\mathcal{GM}(N_F,\chi) \Rightarrow |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq \mathcal{AM}(N_F,\chi)+\mathcal{GM}(N_F,\chi)$. □

Remark 3.6. *The (i) and (ii) of Theorem 3.4 defines the lower bounds of the soft neutrosophic quasigroup by taking into consideration the order of the soft neutrosophic quasigroup in relations to both the arithmetic and geometric means of the soft neutrosophic quasigroup.*

Example 3.6. *Based on Table 1 and Example 3.5,* (N_F,χ) *is a soft neutrosophic quasigroup over a finite neutrosophic quasigroup* $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ *. It will be noticed that;*

$$|\chi|=3,\ |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|=8,\ |(N_F,\chi)|=10,\ \mathcal{AM}(N_F,\chi)=\frac{10}{3},\ \mathcal{GM}(N_F,\chi)=\sqrt[3]{32}.$$

Therefore, all the inequalities in Theorem 3.4 are satisfied.

4. Conclusion

In conclusion, we introduced and studied the abstraction of neutrosophic quasigroup $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ over a quasigroup $(Q,\cdot)$. It was discovered that the direct product of any two neutrosophic quasigroups is neutrosophic quasigroup and that the holomorph of any neutrosophic quasigroup is a neutrosophic quasigroup. Furthermore, soft set theory was broadened by studying soft neutrosophic quasigroup $(N_F,\chi)_{(\mathcal{Q}_{(Q,\cdot)},\odot)}$ over a neutrosophic quasigroup $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$. From the study of order of finite soft neutrosophic quasigroup, we introduced and established the order of finite soft neutrosophic quasigroup with varied mathematical inequality expressions that exist among the order of finite neutrosophic quasigroup and the order of soft neutrosophic quasigroup over the same quasigroup. From the study of their arithmetic mean $\mathcal{AM}(N_F, \chi)$ and geometric mean $\mathcal{GM}(N_F, \chi)$ of finite soft neutrosophic quasigroup (N_F, χ), Lagrange's like Formula $|(N_F, \chi)| = |\chi|\mathcal{AM}(N_F, \chi)$ for finite soft neutrosophic quasigroup was established. In future work, Definition 2.8, Definition 2.9 and Definition 2.10 will be studied for soft neutrosophic quasigroups.

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Rough Neutrosophic Ideals in a Ring

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Abstract. Aim of this paper is to introduce the notion of rough neutrosophic sets in rings also we discuss the sum and product of rough neutrosophic ideal in a ring. Also we prove the upper and lower approximation of neutrosophic subring is also a neutrosophic subring and some examples are discussed.

Keywords: Rough set; Neutrosophic set; Rough neutrosophic set; Neutrosophic ideal; Rough neutrosophic ideal.)

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1. **Introduction**

Fuzzy set is introduced and described using membership functions by Zadeh in 1965 [14] in 1965. The notion of Rough sets was introduced by Pawlak [7] in his seminal paper of 1982. Crisp set and equivalence relation are the basic elements of Rough set theory. Rough set is based on result of approximating crisp sets known as the lower approximation and the upper approximation of a set introduced by Biswas and Nanda [2] in 1994. Approximation spaces are sets with multiple memberships but, fuzzy sets are with partial memberships. Many scholars Dubios et al [4], Gong et al [5], Leung et al [6], Sun et al [12] has developed many models upon different aspects. Rough sets and fuzzy sets, vague set and Intuitionistic fuzzy sets combine with various notions such as Generalized fuzzy rough sets, Intuitionistic fuzzy rough sets, Rough Intuitionistic fuzzy sets, and Rough vague sets were introduced.
Atanassov (1983) [1] introduced the notion of Intuitionistic fuzzy sets. They are the sets whose elements having degrees of membership and non-membership. Selvan, Senthil Kumar [8–10],

introduced the notion of rough intuitionistic fuzzy ideal(prime ideal) in rings in 2012. The generalizations of the theory of intuitionistic fuzzy sets is the theory of neutrosophic sets. The words neutrosophy and neutrosophic were introduced by Smarandache [11]. Neutrosophic concepts are very much useful in real life problem. For example, If a opinion is asked about a statement one may assign that the possibility that the statement true is 0.6 and the statement false is 0.8 and if it is not sure is 0.2. This idea is very much needful in various problems in real life situation. Neutrosophic sets are characterized by truth membership function , indeterminacy membership function and falsity membership function . Vildan cetkin and Halis Aygun [13] introduced an approach to single valued Neutrosophic ideals over a classical ring and Neutrosophic subring in 2018. Rough neutrosophic set is introduced by Broumi, Smarandache, and Dhar [3].

In this paper, we prove that any neutrosophic subring (ideal) of a ring is an upper and lower rough neutrosophic subring (ideal) of the ring.

2. Preliminaries

See [3], [7], [13] for basic concepts which are used in this work.

3. Operations on Rough Neutrosophic sets in a Ring

In this section we introduce the notion of RNI in a ring. Some basic properties of these ideals are proved and examples are given. Let C_R denote the congruence relation on R, throughout this section.

Theorem 3.1. *Let C_R and $C_R{}'$ be the two congruence relations on R. If P and Q are any two NS of R, then the following properties are,*

$(a)\underline{C_R}(P) \subseteq P \subseteq \overline{C_R}(P)$

$(b)\underline{C_R}(\underline{C_R}(P)) = \underline{C_R}(P)$

$(c)\overline{C_R}(\overline{C_R}(P)) = \overline{C_R}(P)$

$(d)\overline{C_R}(\underline{C_R}(P)) = \underline{C_R}(P)$

$(e)\underline{C_R}(\overline{C_R}(P)) = \overline{C_R}(P)$

$(f)(\overline{C_R}(P^c))^c = \underline{C_R}(P)$

$(g)(\underline{C_R}(P^c))^c = \overline{C_R}(P)$

$(h)\underline{C_R}(P \cap Q) = \underline{C_R}(P) \cap \underline{C_R}(Q)$

$(i)\overline{C_R}(P \cap Q) \subseteq \overline{C_R}(P) \cap \overline{C_R}(Q)$

$(j)\overline{C_R}(P \cup Q) = \overline{C_R}(P) \cup \overline{C_R}(Q)$

$(k)\underline{C_R}(P \cup Q) \supseteq \underline{C_R}(P) \cup \underline{C_R}(Q)$

$(l)P \subseteq Q \Rightarrow \overline{C_R}(P) \subseteq \overline{C_R}(Q)$

$(m)P \subseteq Q \Rightarrow \underline{C_R}(P) \subseteq \underline{C_R}(Q)$

$(n) C_R \subseteq C_R{}' \Rightarrow \underline{C_R}(P) \supseteq \underline{C_R}'(P)$
$(o) C_R \subseteq C_R{}' \Rightarrow \overline{C_R}(P) \subseteq \overline{C_R}'(P).$

Proof. Proof is obvious. □

Theorem 3.2. *If P and Q are any two NS of R, then $\overline{C_R}(P) + \overline{C_R}(Q) \subseteq \overline{C_R}(P + Q)$.*

Proof. Since P and Q be any two NS of R. Then

$$\overline{C_R}(P) + \overline{C_R}(Q) = \{[\overline{C_R}(P(n_t)) + \overline{C_R}(Q(n_t))], [\overline{C_R}(P(n_i)) + \overline{C_R}(Q(n_i))], [\overline{C_R}(P(n_f)) + \overline{C_R}(Q(n_f))]\}.$$

$$\overline{C_R}(P + Q) = \{[\overline{C_R}(P(n_t) + Q(n_t))], [\overline{C_R}(P(n_i) + Q(n_i))], [\overline{C_R}(P(n_f) + Q(n_f))]\}.$$

we've to prove, $\overline{C_R}(P) + \overline{C_R}(Q) \subseteq \overline{C_R}(P + Q)$.
For this we want to prove, $\forall\ \alpha \in R$

$(\overline{C_R}(P(n_t)) + \overline{C_R}(Q(n_t)))(\alpha) \leq \overline{C_R}(P(n_t) + Q(n_t))(\alpha)$
$(\overline{C_R}(P(n_i)) + \overline{C_R}(Q(n_i)))(\alpha) \geq \overline{C_R}(P(n_i) + Q(n_i))(\alpha)$
$(\overline{C_R}(P(n_f)) + \overline{C_R}(Q(n_f)))(\alpha) \geq \overline{C_R}(P(n_f) + Q(n_f))(\alpha)$

Consider,

$$\begin{aligned}
(\overline{C_R}(P(n_t)) + \overline{C_R}(Q(n_t)))(\alpha) &= \bigvee_{\alpha=\beta+\gamma} [\overline{C_R}(P(n_t))(\beta) \wedge \overline{C_R}(Q(n_t))(\gamma)] \\
&= \bigvee_{\alpha=\beta+\gamma} [(\bigvee_{x\in[\beta]_{C_R}} (P(n_t)(x)) \wedge (\bigvee_{y\in[\gamma]_{C_R}} (Q(n_t)(y))] \\
&= \bigvee_{\alpha=\beta+\gamma} [(\bigvee_{\substack{x\in[\beta]_{C_R} \\ y\in[\gamma]_{C_R}}} (P(n_t)(x) \wedge Q(n_t)(y))] \\
&\leq \bigvee_{\alpha=\beta+\gamma} [(\bigvee_{x+y\in[\beta+\gamma]_{C_R}} (P(n_t)(x) \wedge Q(n_t)(y))] \\
&= \bigvee_{x+y\in[\alpha]_{C_R}} (P(n_t)(x) \wedge Q(n_t)(y)) \\
&= \bigvee_{\substack{z\in[\alpha]_{C_R} \\ z=x+y}} (P(n_t)(x) \wedge Q(n_t)(y)) \\
&= \bigvee_{z\in[\alpha]_{C_R}} \bigvee_{z=x+y} (P(n_t)(x) \wedge Q(n_t)(y)) \\
&= \bigvee_{z\in[\alpha]_{C_R}} [P(n_t) + Q(n_t)](z) \\
&= \overline{C_R}(P(n_t) + Q(n_t))(z)
\end{aligned}$$

And

$$\begin{aligned}
(\overline{C_R}(P(n_i)) + \overline{C_R}(Q(n_i)))(\alpha) &= \bigwedge_{\alpha=\beta+\gamma} [\overline{C_R}(P(n_i))(\beta) \vee \overline{C_R}(Q(n_i))(\gamma)] \\
&= \bigwedge_{\alpha=\beta+\gamma} [(\bigwedge_{x\in[\beta]_{C_R}} (P(n_i)(x)) \vee (\bigwedge_{y\in[\gamma]_{C_R}} (Q(n_i)(y))] \\
&= \bigwedge_{\alpha=\beta+\gamma} [(\bigwedge_{\substack{x\in[\beta]_{C_R} \\ y\in[\gamma]_{C_R}}} (P(n_i)(x) \vee Q(n_i)(y))] \\
&\geq \bigwedge_{\alpha=\beta+\gamma} [(\bigwedge_{x+y\in[\beta+\gamma]_{C_R}} (P(n_i)(x) \vee Q(n_i)(y))] \\
&= \bigwedge_{x+y\in[\alpha]_{C_R}} (P(n_i)(x) \vee Q(n_i)(y)) \\
&= \bigwedge_{\substack{z\in[\alpha]_{C_R} \\ z=x+y}} (P(n_i)(x) \vee Q(n_i)(y)) \\
&= \bigwedge_{z\in[\alpha]_{C_R}} \bigwedge_{z=x+y} (P(n_i)(x) \vee Q(n_i)(y)) \\
&= \bigwedge_{z\in[\alpha]_{C_R}} [P(n_i) + Q(n_i)](z) \\
&= \overline{C_R}(P(n_i) + Q(n_i))(z)
\end{aligned}$$

Also,

$$\begin{aligned}
(\overline{C_R}(P(n_f)) + \overline{C_R}(Q(n_f)))(\alpha) &= \bigwedge_{\alpha=\beta+\gamma} [\overline{C_R}(P(n_f))(\beta) \vee \overline{C_R}(Q(n_f))(\gamma)] \\
&= \bigwedge_{\alpha=\beta+\gamma} [(\bigwedge_{x\in[\beta]_{C_R}} (P(n_f)(x)) \vee (\bigwedge_{y\in[\gamma]_{C_R}} (Q(n_f)(y))] \\
&= \bigwedge_{\alpha=\beta+\gamma} [(\bigwedge_{\substack{x\in[\beta]_{C_R} \\ y\in[\gamma]_{C_R}}} (P(n_f)(x) \vee Q(n_f)(y))] \\
&\geq \bigwedge_{\alpha=\beta+\gamma} [(\bigwedge_{x+y\in[\beta+\gamma]_{C_R}} (P(n_f)(x) \vee Q(n_f)(y))] \\
&= \bigwedge_{x+y\in[\alpha]_{C_R}} (P(n_f)(x) \vee Q(n_f)(y)) \\
&= \bigwedge_{\substack{z\in[\alpha]_{C_R} \\ z=x+y}} (P(n_f)(x) \vee Q(n_f)(y)) \\
&= \bigwedge_{z\in[\alpha]_{C_R}} \bigwedge_{z=x+y} (P(n_f)(x) \vee Q(n_f)(y)) \\
&= \bigwedge_{z\in[\alpha]_{C_R}} [P(n_f) + Q(n_f)](z) \\
&= \overline{C_R}(P(n_f) + Q(n_f))(z)
\end{aligned}$$

Hence Proved.

□

Theorem 3.3. *If P and Q are any two NS of R, then $\underline{C_R}(P) + \underline{C_R}(Q) \subseteq \underline{C_R}(P+Q)$.*

Proof. This proof is similar to Theorem 3.2. □

Theorem 3.4. *Let P and Q are any two NS of R, then $\overline{C_R}(P).\overline{C_R}(Q) \subseteq \overline{C_R}(P.Q)$.*

Proof. Since P and Q be any two NS of R. Then,
$\overline{C_R}(P).\overline{C_R}(Q) = \{[\overline{C_R}(P(n_t)).\overline{C_R}(Q(n_t))], [\overline{C_R}(P(n_i)).\overline{C_R}(Q(n_i))], [\overline{C_R}(P(n_f)).\overline{C_R}(Q(n_f))]\}$
$\overline{C_R}(P.Q) = \{[\overline{C_R}(P(n_t).Q(n_t))], [\overline{C_R}(P(n_i).Q(n_i))], [\overline{C_R}(P(n_f).Q(n_f))]\}$
To prove, $\overline{C_R}(P).\overline{C_R}(Q) \subseteq \overline{C_R}(P.Q)$.
It is enough to prove that, $\forall\ \alpha \in R$
$(\overline{C_R}(P(n_t)).\overline{C_R}(Q(n_t)))(\alpha) \leq \overline{C_R}(P(n_t).Q(n_t))(\alpha)$
$(\overline{C_R}(P(n_i)).\overline{C_R}(Q(n_i)))(\alpha) \geq \overline{C_R}(P(n_i).Q(n_i))(\alpha)$
$(\overline{C_R}(P(n_f)).\overline{C_R}(Q(n_f)))(\alpha) \geq \overline{C_R}(P(n_f).Q(n_f))(\alpha)$
Consider,

$$\begin{aligned}
(\overline{C_R}(P(n_t)).\overline{C_R}(Q(n_t)))(\alpha) &= \bigvee_{\alpha=\beta\gamma} [\overline{C_R}(P(n_t))(\beta) \wedge \overline{C_R}(Q(n_t))(\gamma)] \\
&= \bigvee_{\alpha=\beta\gamma} [(\bigvee_{x\in[\beta]_{C_R}} (P(n_t)(x)) \wedge (\bigvee_{y\in[\gamma]_{C_R}} (Q(n_t)(y))] \\
&= \bigvee_{\alpha=\beta\gamma} [(\bigvee_{\substack{x\in[\beta]_{C_R} \\ y\in[\gamma]_{C_R}}} (P(n_t)(x) \wedge Q(n_t)(y))] \\
&\leq \bigvee_{\alpha=\beta\gamma} [(\bigvee_{xy\in[\beta\gamma]_{C_R}} (P(n_t)(x) \wedge Q(n_t)(y))] \\
&= \bigvee_{xy\in[\alpha]_{C_R}} (P(n_t)(x) \wedge Q(n_t)(y)) \\
&= \bigvee_{\substack{z\in[\alpha]_{C_R} \\ z=xy}} (P(n_t)(x) \wedge Q(n_t)(y)) \\
&= \bigvee_{z\in[\alpha]_{C_R}} \bigvee_{z=xy} (P(n_t)(x) \wedge Q(n_t)(y)) \\
&= \bigvee_{z\in[\alpha]_{C_R}} [P(n_t).Q(n_t)](z) \\
&= \overline{C_R}(P(n_t).Q(n_t))(z)
\end{aligned}$$

$$\begin{aligned}
(\overline{C_R}(P(n_i)).\overline{C_R}(Q(n_i)))(\alpha) &= \bigwedge_{\alpha=\beta\gamma} [\overline{C_R}(P(n_i))(\beta) \vee \overline{C_R}(Q(n_i))(\gamma)] \\
&= \bigwedge_{\alpha=\beta\gamma} [(\bigwedge_{x\in[\beta]_{C_R}} (P(n_i)(x)) \vee (\bigwedge_{y\in[\gamma]_{C_R}} (Q(n_i)(y))] \\
&= \bigwedge_{\alpha=\beta\gamma} [(\bigwedge_{\substack{x\in[\beta]_{C_R} \\ y\in[\gamma]_{C_R}}} (P(n_i)(x) \vee Q(n_i)(y))] \\
&\geq \bigwedge_{\alpha=\beta\gamma} [(\bigwedge_{xy\in[\beta\gamma]_{C_R}} (P(n_i)(x) \vee Q(n_i)(y))] \\
&= \bigwedge_{xy\in[\alpha]_{C_R}} (P(n_i)(x) \vee Q(n_i)(y)) \\
&= \bigwedge_{\substack{z\in[\alpha]_{C_R} \\ z=xy}} (P(n_i)(x) \vee Q(n_i)(y)) \\
&= \bigwedge_{z\in[\alpha]_{C_R}} \bigwedge_{z=xy} (P(n_i)(x) \vee Q(n_i)(y)) \\
&= \bigwedge_{z\in[\alpha]_{C_R}} [P(n_i).Q(n_i)](z) \\
&= \overline{C_R}(P(n_i).Q(n_i))(z)
\end{aligned}$$

$$\begin{aligned}
(\overline{C_R}(P(n_f)) \cdot \overline{C_R}(Q(n_f)))(\alpha) &= \bigwedge_{\alpha=\beta+\gamma} [\overline{C_R}(P(n_f))(\beta) \vee \overline{C_R}(Q(n_f))(\gamma)] \\
&= \bigwedge_{\alpha=\beta\gamma} [(\bigwedge_{x\in[\beta]_{C_R}} (P(n_f)(x)) \vee (\bigwedge_{y\in[\gamma]_{C_R}} (Q(n_f)(y))] \\
&= \bigwedge_{\alpha=\beta\gamma} [(\bigwedge_{\substack{x\in[\beta]_{C_R} \\ y\in[\gamma]_{C_R}}} (P(n_f)(x) \vee Q(n_f)(y))] \\
&\geq \bigwedge_{\alpha=\beta\gamma} [(\bigwedge_{xy\in[\beta\gamma]_{C_R}} (P(n_f)(x) \vee Q(n_f)(y))] \\
&= \bigwedge_{xy\in[\alpha]_{C_R}} (P(n_f)(x) \vee Q(n_f)(y)) \\
&= \bigwedge_{\substack{z\in[\alpha]_{C_R} \\ z=xy}} (P(n_f)(x) \vee Q(n_f)(y)) \\
&= \bigwedge_{z\in[\alpha]_{C_R}} \bigwedge_{z=xy} (P(n_f)(x) \vee Q(n_f)(y)) \\
&= \bigwedge_{z\in[\alpha]_{C_R}} [P(n_f).Q(n_f)](z) \\
&= \overline{C_R}(P(n_f).Q(n_f))(z)
\end{aligned}$$

Hence proved. □

Theorem 3.5. *Let P and Q are any two NS of R, then*
$\underline{C_R}(P).\underline{C_R}(Q) \subseteq \underline{C_R}(P.Q)$.

Proof. This proof is similar to Theorem 3.4. □

4. **Rough Neutrosophic Subring (RNSR) of Ring**

Definition 4.1. A NSR is called an $RNSR$ if it is both upper $RNSR$ and lower $RNSR$ of R.

Definition 4.2. A NSR is said to be an lower (upper) $RNSR$ of R if its lower(upper) approximation is also a NSR of R.

Theorem 4.3. *If K be a NSR of R, then K is an upper $RNSR$ of R.*

Proof. Since K is a NSR of R. Now,$\forall\ a, b \in R$

$$\begin{aligned}
\overline{C_R}(K(n_t))(a-b) &= \bigvee_{c\in[a-b]_{C_R}} K(n_t)(c) \\
&= \bigvee_{x-y\in[a]_{C_R}-[b]_{C_R}} K(n_t)(x-y) \\
&\geq \bigvee_{\substack{x\in[a]_{C_R}\\ y\in[b]_{C_R}}} [K(n_t)(x)\wedge K(n_t)(y)] \\
&= [\bigvee_{x\in[a]_{C_R}} K(n_t)(x)] \wedge [\bigvee_{y\in[b]_{C_R}} K(n_t)(y)] \\
&= \overline{C_R}(K(n_t))(x) \wedge \overline{C_R}(K(n_t))(y)
\end{aligned}$$

$$\begin{aligned}
\overline{C_R}(K(n_i))(a-b) &= \bigwedge_{c\in[a-b]_{C_R}} K(n_i)(c) \\
&= \bigwedge_{x-y\in[a]_{C_R}-[b]_{C_R}} K(n_i)(x-y) \\
&\leq \bigwedge_{\substack{x\in[a]_{C_R}\\ y\in[b]_{C_R}}} [P(n_i)(x)\vee K(n_i)(y)] \\
&= [\bigwedge_{x\in[a]_{C_R}} K(n_i)(x)] \vee [\bigwedge_{y\in[b]_{C_R}} K(n_i)(y)] \\
&= \overline{C_R}(K(n_i))(x) \vee \overline{C_R}(K(n_i))(y)
\end{aligned}$$

$$\begin{aligned}
\overline{C_R}(K(n_f))(a-b) &= \bigwedge_{c\in[a-b]_{C_R}} K(n_f)(c) \\
&= \bigwedge_{x-y\in[a]_{C_R}-[b]_{C_R}} K(n_f)(x-y) \\
&\leq \bigwedge_{\substack{x\in[a]_{C_R}\\ y\in[b]_{C_R}}} [K(n_f)(x)\vee K(n_f)(y)] \\
&= [\bigwedge_{x\in[a]_{C_R}} K(n_f)(x)] \vee [\bigwedge_{y\in[b]_{C_R}} K(n_f)(y)] \\
&= \overline{C_R}(K(n_f))(x) \vee \overline{C_R}(K(n_f))(y)
\end{aligned}$$

$\forall a, b \in R$

$$\begin{aligned}
\overline{C_R}(K(n_t))(ab) &= \bigvee_{c\in[ab]_{C_R}} K(n_t)(c) \\
&= \bigvee_{xy\in[a]_{C_R}[b]_{C_R}} K(n_t)(xy) \\
&\geq \bigvee_{x\in[a]_{C_R}, y\in[b]_{C_R}} [K(n_t)(x) \wedge K(n_t)(y)] \\
&= [\bigvee_{x\in[a]_{C_R}} K(n_t)(x)] \wedge [\bigvee_{y\in[b]_{C_R}} K(n_t)(y)] \\
&= \overline{C_R}(K(n_t))(x) \wedge \overline{C_R}(K(n_t))(y)
\end{aligned}$$

$$\begin{aligned}
\overline{C_R}(K(n_i))(ab) &= \bigwedge_{c\in[ab]_{C_R}} K(n_i)(c) \\
&= \bigwedge_{xy\in[a]_{C_R}[b]_{C_R}} K(n_i)(xy) \\
&\leq \bigwedge_{x\in[a]_{C_R}, y\in[b]_{C_R}} [K(n_i)(x) \vee K(n_i)(y)] \\
&= [\bigwedge_{x\in[a]_{C_R}} K(n_i)(x)] \vee [\bigwedge_{y\in[b]_{C_R}} K(n_i)(y)] \\
&= \overline{C_R}(K(n_i))(x) \vee \overline{C_R}(K(n_i))(y)
\end{aligned}$$

$$\begin{aligned}
\overline{C_R}(K(n_f))(ab) &= \bigwedge_{c\in[ab]_{C_R}} K(n_f)(c) \\
&= \bigwedge_{xy\in[a]_{C_R}[b]_{C_R}} K(n_f)(xy) \\
&\leq \bigwedge_{x\in[a]_{C_R}, y\in[b]_{C_R}} [K(n_f)(x) \vee K(n_f)(y)] \\
&= [\bigwedge_{x\in[a]_{C_R}} K(n_f)(x)] \vee [\bigwedge_{y\in[b]_{C_R}} K(n_f)(y)] \\
&= \overline{C_R}(K(n_f))(x) \vee \overline{C_R}(K(n_f))(y)
\end{aligned}$$

Hence, $\overline{C_R}(K)$ is a NSR of R. Thus K is an upper $RNSR$ of R. □

Theorem 4.4. *If K be a NSR of R, then K is a lower $RNSR$ of R.*

Proof. This proof is similar to Theorem 4.3 □

Corollary 4.5. *Let K be the NSR of R. Then K is a rough $RNSR$ of R.*

Proof. By applying Theorem 4.3 and 4.4 we get the result. □

Definition 4.6. A NI is called an RNI if it is both upper RNI and lower RNI of R.

Definition 4.7. A NI is said to be an lower (upper) RNI of R if its lower(upper) approximation is also an NI of R.

Theorem 4.8. *If K be a NI of R, then K is an upper RNI of R.*

Proof. Since K is a NI of R. We've to prove that, $\forall\ a, b \in R$
$\overline{C_R}(K(n_t))(ab) \geq \overline{C_R}(K(n_t))(a) \vee \overline{C_R}(K(n_t))(b)$
$\overline{C_R}(K(n_i))(ab) \leq \overline{C_R}(K(n_i))(a) \wedge \overline{C_R}(K(n_i))(b)$
$\overline{C_R}(K(n_f))(ab) \leq \overline{C_R}(K(n_f))(a) \wedge \overline{C_R}(K(n_f))(b)$
Now,

$$\begin{aligned}
\overline{C_R}(K(n_i))(ab) &= \bigvee_{c\in[ab]_{C_R}} K(n_t)(c) \\
&\geq \bigvee_{c\in[a]_{C_R}[b]_{C_R}} K(n_t)(c) \\
&= \bigvee_{xy\in[a]_{C_R}[b]_{C_R}} K(n_t)(xy) \\
&\geq \bigvee_{x\in[a]_{C_R},y\in[b]_{C_R}} [K(n_t)(x) \vee K(n_t)(y)] \\
&= [\bigvee_{x\in[a]_{C_R}} K(n_t)(x)] \vee [\bigvee_{y\in[b]_{C_R}} K(n_t)(y)] \\
&= \overline{C_R}(K(n_t))(x) \vee \overline{C_R}(K(n_t))(y)
\end{aligned}$$

$$\begin{aligned}
\overline{C_R}(K(n_i))(ab) &= \bigwedge_{c\in[ab]_{C_R}} K(n_i)(c) \\
&\leq \bigwedge_{c\in[a]_{C_R}[b]_{C_R}} K(n_i)(c) \\
&= \bigwedge_{xy\in[a]_{C_R}[b]_{C_R}} K(n_i)(xy) \\
&\leq \bigwedge_{x\in[a]_{C_R},y\in[b]_{C_R}} [K(n_i)(x) \wedge K(n_i)(y)] \\
&= [\bigwedge_{x\in[a]_{C_R}} K(n_i)(x)] \wedge [\bigwedge_{y\in[b]_{C_R}} K(n_i)(y)] \\
&= \overline{C_R}(K(n_i))(x) \wedge \overline{C_R}(K(n_i))(y)
\end{aligned}$$

$$
\begin{aligned}
\overline{C_R}(K(n_f))(ab) &= \bigwedge_{c\in[ab]_{C_R}} K(n_f)(c) \\
&\leq \bigwedge_{c\in[a]_{C_R}[b]_{C_R}} K(n_f)(c) \\
&= \bigwedge_{xy\in[a]_{C_R}[b]_{C_R}} K(n_f)(xy) \\
&\leq \bigwedge_{x\in[a]_{C_R},y\in[b]_{C_R}} [K(n_f)(x) \wedge K(n_f)(y)] \\
&= [\bigwedge_{x\in[a]_{C_R}} K(n_f)(x)] \wedge [\bigwedge_{y\in[b]_{C_R}} K(n_f)(y)] \\
&= \overline{C_R}(K(n_f))(x) \wedge \overline{C_R}(K(n_f))(y)
\end{aligned}
$$

Hence, $\overline{C_R}(K)$ is a NI of R. Thus K is an upper RNI of R. □

Theorem 4.9. *If K be a NI of R, then K is a lower RNI of R.*

Proof. This proof is similar to Theorem 4.8. □

Corollary 4.10. *If K be the NI of R. then K is a RNI of R.*

Proof. By applying Theorem 4.8 and 4.9 we get the result. □

5. Conclusion

In this paper, we discussed the notion of rough neutrosophic set in a ring and their properties. Also, we proved that any neutrosophic ideal of a ring is an rough neutrosophic ideal of a ring. For further research one can extend this to other algebraic systems.

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Neutrosophic ℵ-filters in semigroups

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Abstract. Models of universe problems are brimming with complexities and uncertainties in almost every field of study, including engineering, mathematics, medical sciences, computer science, physics, management sciences, artificial intelligence, and operations research. To address these uncertainties, various theories have been developed, including probability, rough sets, fuzzy sets, soft ideals, and neutrosophic sets. Neutrosophic set theory is the focus of this paper. In this paper, we introduce the notions of neutrosophic ℵ-filters and neutrosophic ℵ-bi-filters in a semigroup and investigate several properties. Moreover, the relations of prime bi-ideal subset and prime neutrosophic ℵ-bi- ideal structure; neutrosophic ℵ-bi-filter and neutrosophic ℵ-bi-ideal structure; left (resp., right) filter and neutrosophic ℵ-left (resp., right) filter; neutrosophic ℵ-left(resp., right) filter and prime neutrosophic ℵ-left (resp., right) ideals in semigroups are discussed. Finally we prove that: let X be a semigroup and X_N be any neutrosophic structure. Then X_N is a neutrosophic ℵ-bi-filter of X if and only if X_{N^c} is a prime neutrosophic ℵ-bi-ideal of X.

Keywords: Semigroup; fuzzy sets; filter; bi-ideal; neutrosophic ℵ-bi-ideals.

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1. Introduction

In 1965, L.A. Zadeh [22] introduced the idea of Fuzzy sets which were represented using membership functions. Rather than a classic set, in the case of a fuzzy set A, x is an object that belong to this set with varying membership degrees in the range $[0, 1]$, where 0 and 1 denote, respectively, lack of membership and full membership. The investigation of algebraic structures has begun with the presentation of the idea of fuzzy subgroups in the spearheading paper of Rosenfeld [18]. Subsequently, many authors further studied fuzzy concept in semigroups(See [9–11, 19]). Of several higher-order fuzzy sets, the intuitionistic fuzzy set presented by Atanassov [3] has been seen as a profoundly useful idea in managing vagueness. Following the introduction of the intuitionistic fuzzy set concept, mathematicians published several papers extending classical and fuzzy mathematical concepts to the case of intuitionistic fuzzy mathematics.

In 1999, F. Smarandache [20] introduced the concept of neutrosophic set, which is the generalizations of fuzzy sets and intuitionistic fuzzy set. Neutrosophic set is a useful mathematical tool for dealing with incomplete, inconsistent and indeterminate information. The neutrosophic set theory is applied to algebraic structures, multiple attribute decision-making, and so on [1, 2, 6, 7, 12–17, 21].

For additional informations about neutrosophic set theory, we refer the readers to the below website *http://fs.unm.edu/neutrosophy.htm*.

In [12], M. Khan et al. introduced and investigated the concept of a neutrosophic $\aleph$-sub semigroup of a semigroup. The conditions for neutrosophic $\aleph$-structure to be neutrosophic $\aleph$-subsemigroup were given, and the characterization of neutrosophic $\aleph$-subsemigroup was discussed using neutrosophic $\aleph$-product. They also proved that the homomorphic preimage of a neutrosophic $\aleph$-subsemigroup is a neutrosophic $\aleph$-subsemigroup and that the onto homomorphic image of a neutrosophic $\aleph$-subsemigroup is a neutrosophic $\aleph$-subsemigroup. The notions of neutrosophic $\aleph$-ideals and neutrosophic $\aleph$- bi-ideals were defined to semigroups and obtained many useful results (See [5, 17]).

As a follow-up, in this paper we define the concept of neutrosophic $\aleph$-left (resp., bi-)filters in semigroup and describe the semigroup in terms of these notions. We also define prime neutrosophic $\aleph$-left ideals and prime neutrosophic $\aleph$-bi-ideal structures of semigroup and characterize the relations of neutrosophic $\aleph$-left filters and prime neutrosophic $\aleph$-left ideals in semigroups.

Throughout this paper, X denotes a semigroup and for $K, S \subseteq X$, we denote $KS := \{ks : k \in K, s \in S\}$.

Definition 1.1. [4] Let X be a semigroup and $\phi \neq K \subseteq X$. Then

(i) K is called a *subsemigroup* of X if $K^2 \subseteq K$.
(ii) K is called a *left (resp., right) ideal* of X if $XK \subseteq K$ (resp., $KX \subseteq K$).
(iii) If K is both a left and a right ideal of X, then it is called an *ideal* of X.
(iv) K is called a *bi-ideal subset* of X if $k \in K$ and $s \in X$ imply $ksk \in K$.

Definition 1.2. [10] Let X be a semigroup and K a subsemigroup of X. Then

(i) K is called *left (resp., right) filter* of X if $r, s \in X$, $rs \in K$ implies $s \in K$ (resp., $r \in K$).

(ii) K is called a *bi-filter* of X if $r, s \in X$, $rsr \in K$ implies $r \in K$.

Definition 1.3. [11] Let X be a semigroup and $\phi \neq K \subseteq X$. Then

(i) K is called a *prime subset* of X if $r, s \in X$, $rs \in K$ implies $r \in K$ or $s \in K$.

Equivalently, $S, T \subseteq X$, $ST \subseteq K$ implies $S \subseteq K$ or $T \subseteq K$.

(ii) K is called a *semiprime subset* of X if $r \in X$, $r^2 \in K$ implies $r \in K$.

Equivalently, $S \subseteq X$, $S^2 \subseteq K$ implies $S \subseteq K$.

2. Preliminary definitions and results of Neutrosophic ℵ- structure

In this section, we present the necessary fundamental concepts of neutrosophic ℵ-structures of X that we need in the sequel.

For a semigroup X, $\mathcal{F}(X, [-1, 0])$ is the collection of negative-valued functions from a set X to $[-1, 0]$. An element $g \in \mathcal{F}(X, [-1, 0])$ is called a ℵ-function on X and ℵ-structure means (X, g) of X.

Definition 2.1. [12] A *neutrosophic* ℵ- *structure* of X is defined to be the structure:

$$X_M := \frac{X}{(T_M, I_M, F_M)} = \left\{ \frac{l}{T_M(l), I_M(l), F_M(l)} \ : \ l \in X \right\}$$

where T_M is the negative truth membership function on X, I_M is the negative indeterminacy membership function on X and F_M is the negative falsity membership function on X.

Note that for any $k \in X, X_M$ fulfills the condition $-3 \leq T_M(k) + I_M(k) + F_M(k) \leq 0$.

Definition 2.2. For a subset K of X, consider the neutrosophic ℵ-structure

$$\chi_K(X_N) = \frac{X}{(\chi_K(T)_N, \chi_K(I)_N, \chi_K(F)_N)}$$

where

$$\chi_K(T)_N : X \to [-1, 0], \ x \to \begin{cases} -1 & if \ x \in K \\ 0 & if \ x \notin K, \end{cases}$$

$$\chi_K(I)_N : X \to [-1, 0], \ x \to \begin{cases} 0 & if \ x \in K \\ -1 & if \ x \notin K, \end{cases}$$

$$\chi_K(F)_N : X \to [-1, 0], \ x \to \begin{cases} -1 & if \ x \in K \\ 0 & if \ x \notin K, \end{cases}$$

which is called the *characteristic neutrosophic* ℵ*-structure* of K over X.

Definition 2.3. [12] Let X be a semigroup. Then for any $X_N := \frac{X}{(T_N, I_N, F_N)}$ and $X_M := \frac{X}{(T_M, I_M, F_M)}$.

(i) X_M is called a *neutrosophic* $\aleph$*-substructure* of X_N, denoted by $X_N \subseteq X_M$, if it satisfies the below condition for any $l \in X$,

$$T_N(l) \geq T_M(l), I_N(l) \leq I_M(l), F_N(l) \geq F_M(l).$$

If $X_N \subseteq X_M$ and $X_M \subseteq X_N$, then we say that $X_N = X_M$.

(ii) The union of X_N and X_M is a neutrosophic $\aleph$-structure over X is defined as

$$X_N \cup X_M = X_{N\cup M} = (X; T_{N\cup M}, I_{N\cup M}, F_{N\cup M}),$$

where

$$(T_N \cup T_M)(k) = T_{N\cup M}(k) = T_N(k) \wedge T_M(k),$$
$$(I_N \cup I_M)(k) = I_{N\cup M}(k) = I_N(k) \vee I_M(k),$$
$$(F_N \cup F_M)(k) = F_{N\cup M}(k) = F_N(k) \wedge F_M(k) \ for\ any\ k \in X.$$

(iii) The intersection of X_N and X_M is a neutrosophic $\aleph$-structure over X is defined as

$$X_N \cap X_M = X_{N\cap M} = (X; T_{N\cap M}, I_{N\cap M}, F_{N\cap M}),$$

where

$$(T_N \cap T_M)(k) = T_{N\cap M}(k) = T_N(k) \vee T_M(k),$$
$$(I_N \cap I_M)(k) = I_{N\cap M}(k) = I_N(k) \wedge I_M(k),$$
$$(F_N \cap F_M)(k) = F_{N\cap M}(k) = F_N(k) \vee F_M(k) \ for\ any\ k \in X.$$

Definition 2.4. [12] Let $X_N = \frac{X}{(T_N, I_N, F_N)}$. Then the *complement of* X_N, denoted by X_{N^c} over U, is defined to be a neutrosophic $\aleph$-structure

$$X_{N^c} := \frac{X}{(T_{N^c}, I_{N^c}, F_{N^c})},$$

over X, where $T_{N^c}(l) = -1 - T_N(l); I_{N^c}(l) = -1 - I_N(l)$ and $F_{N^c}(l) = -1 - F_N(l)\ \forall l \in X$.

Definition 2.5. [12] Let $X_N = \frac{X}{(T_N, I_N, F_N)}$ and $\mu, \lambda, \nu \in [-1, 0]$ with $-3 \leq \mu + \lambda + \nu \leq 0$. Consider the following sets:

$$\begin{aligned} T_N^{\mu} = & \ \{k \in X \mid T_N(k) \leq \mu\}, \\ I_N^{\lambda} = & \ \{k \in X \mid I_N(k) \geq \lambda\}, \\ I_N^{\nu} = & \ \{k \in X | F_N(k) \leq \nu\}. \end{aligned}$$

Then the set $X_N(\mu, \lambda, \nu) = \{k \in X | T_N(k) \leq \mu, I_N(k) \geq \lambda, F_N(k) \leq \nu\}$ is called a (μ, λ, ν)-*level set* of X_N. Note that $X_N(\mu, \lambda, \nu) = T_N^{\mu} \cap I_N^{\lambda} \cap F_N^{\nu}$.

Definition 2.6. [12] A neutrosophic $\aleph$-structure X_M of X is called a *neutrosophic* $\aleph$-*subsemigroup* if it satisfies:

$$(\forall\ k, s \in X) \begin{pmatrix} T_M(ks) \leq T_M(k) \vee T_M(s) \\ I_M(ks) \geq I_M(k) \wedge I_M(s) \\ F_M(ks) \leq F_M(k) \vee F_M(s) \end{pmatrix}.$$

Definition 2.7. [5] A neutrosophic $\aleph$-structure X_M of X is called a *neutrosophic* $\aleph$-*left (resp., right) ideal* if it satisfies the below condition: for any $k, s \in X$

$$\begin{pmatrix} T_M(ks) \leq T_M(s)(resp., T_M(ks) \leq T_M(k)) \\ I_M(ks) \geq I_M(s)(resp., I_M(ks) \geq I_M(k)) \\ F_M(ks) \leq F_M(s)(resp., F_M(ks) \leq F_M(k)) \end{pmatrix}.$$

If X_M is both a neutrosophic $\aleph$-left and a neutrosophic $\aleph$-right ideal of X, then it is called a *neutrosophic* $\aleph$-*ideal* of X.

Definition 2.8. A neutrosophic $\aleph$-subsemigroup X_M is called a *neutrosophic* $\aleph$-*left(resp., right) filter* of X if it satisfies the below condition: for any $k, s \in X$

$$\begin{pmatrix} T_M(ks) \geq T_M(s)(resp., T_M(ks) \geq T_M(k)) \\ I_M(ks) \leq I_M(s)(resp., I_M(ks) \leq I_M(k)) \\ F_M(ks) \geq F_M(s)(resp., F_M(ks) \geq F_M(k)) \end{pmatrix}.$$

Definition 2.9. A neutrosophic $\aleph$-subsemigroup X_M is called a *neutrosophic* $\aleph$-*filter* if it both a neutrosophic $\aleph$-left filter and a neutrosophic $\aleph$-right filter of X.

Equivalently, a neutrosophic $\aleph$-subsemigroup X_M over X is called a *neutrosophic* $\aleph$-*filter* of X if it satisfies:

$$(\forall k, s \in X) \begin{pmatrix} T_M(ks) = T_M(k) \vee T_M(s) \\ I_M(ks) = I_M(k) \wedge I_M(s) \\ F_M(ks) = F_M(k) \vee F_M(s) \end{pmatrix}.$$

The following example shows that there are some neutrosophic $\aleph$-subsemigroups in X, which are neither neutrosophic $\aleph$-left filters nor neutrosophic $\aleph$-right filters of X.

Example 2.10. Consider the semigroup X, the set of all positive integers, with respect to multiplication. Then $X_N = \left\{ \frac{k}{(-\frac{1}{k}, 0, -\frac{1}{k})} : k \in X \right\}$ is a neutrosophic $\aleph$-subsemigroup of X, but not a neutrosophic $\aleph$-left filter as well as not a neutrosophic $\aleph$-right filter of X. □

Example 2.11. Let $X = \{1, 2, 3, 4, 5\}$ be a finite semigroup with the below multiplication table:

.	1	2	3	4	5
1	1	1	1	1	1
2	1	2	3	1	1
3	1	1	1	2	3
4	1	4	5	1	1
5	1	1	1	4	5

Then $X_N = \left\{ \frac{1}{(-0.5,-0.7,-0.4)}, \frac{2}{(-0.4,-0.8,-0.3)}, \frac{3}{(-0.4,-0.8,-0.3)}, \frac{4}{(-0.4,-0.7,-0.3)}, \frac{5}{(-0.4,-0.7,-0.3)} \right\}$ is a neutrosophic $\aleph$-subsemigroup of X. Here $I_N(3.3) \nleq I_N(3)$. So X_N is neither a neutrosophic $\aleph$-left filter nor a neutrosophic $\aleph$-right filter of X. □

Example 2.12. Let $X = \{k, r, s\}$ be a semigroup with the below multiplication table:

.	k	r	s
k	k	k	k
r	r	r	r
s	s	s	s

Then $X_N = \left\{ \frac{k}{(-0.5,-0.5,-0.7)}, \frac{r}{(-0.4,-0.6,-0.6)}, \frac{s}{(-0.3,-0.7,-0.5)} \right\}$ is a neutrosophic ℵ-right filter, but not a neutrosophic ℵ-left filter of X as $T_N(kr) \ngeq T_N(r), I_N(kr) \nleq I_N(r)$ and $F_N(kr) \ngeq F_N(r)$. □

Definition 2.13. A neutrosophic structure X_N of X is a *neutrosophic ℵ-bi-ideal structure* if it satisfies:

$$(\forall k, s \in X) \begin{pmatrix} T_N(ksk) \leq T_N(k) \\ I_N(ksk) \geq I_N(k) \\ F_N(ksk) \leq F_N(k) \end{pmatrix}.$$

Definition 2.14. A neutrosophic ℵ-subsemigroup X_N of X is called a *neutrosophic ℵ-bi-filter* if it satisfies:

$$(\forall k, s \in X) \begin{pmatrix} T_N(ksk) \geq T_N(k) \\ I_N(ksk) \leq I_N(k) \\ F_N(ksk) \geq F_N(k) \end{pmatrix}.$$

Example 2.15. Let X be the set of all non-negative integers except one. Then X is a semigroup with usual multiplication.

Consider $X_M = \left\{ \begin{matrix} \frac{0}{(-0.1,-0.8,-0.1)}, \frac{2}{(-0.6,-0.5,-0.6)}, \frac{3}{(-0.7,-0.4,-0.8)}, \frac{6}{(-0.8,-0.3,-0.9)}, \\ \frac{otherwise}{(-0.2,-0.6,-0.3)} \end{matrix} \right\}$.

Then X_M is a neutrosophic ℵ-bi-filter of X, but not a filter as $T_N(2.3) = T_N(6) = -0.8 \ngeq T_N(3)$. □

Definition 2.16. Let $X_N = \frac{X}{(T_N, I_N, F_N)}$. Then X_N is called *prime neutrosophic ℵ-structure* of X if it satisfies:

$$(\forall k, s \in X) \begin{pmatrix} T_N(ks) \geq T_N(k) \wedge T_N(s) \\ I_N(ks) \leq I_N(k) \vee I_N(s) \\ F_N(ks) \geq F_N(k) \wedge F_N(s) \end{pmatrix}.$$

Definition 2.17. Let $X_N = \frac{X}{(T_N, I_N, F_N)}$. Then X_N is called *semiprime neutrosophic ℵ-structure* of X if it satisfies:

$$(\forall k \in X) \begin{pmatrix} T_N(k^2) \geq T_N(k) \\ I_N(k^2) \leq I_N(k) \\ F_N(k^2) \geq F_N(k) \end{pmatrix}.$$

Note 2.18. *Clearly every prime neutrosophic ℵ-structure of X is a semi prime neutrosophic ℵ-structure of X, but converse is not true.*

Example 2.19. Let $X = \{0, k, r, s\}$ be a semigroup with the following multiplication table:

.	0	k	r	s
0	0	0	0	0
k	0	0	s	r
r	0	s	0	k
s	0	r	k	0

Then $X_N = \left\{ \frac{0}{(-0.1,-0.9,-0.2)}, \frac{k}{(-0.4,-0.5,-0.6)}, \frac{r}{(-0.5,-0.6,-0.7)}, \frac{s}{(-0.6,-0.4,-0.8)} \right\}$ is a semi-prime neutrosophic ℵ- structure of X, but it is not a prime neutrosophic ℵ-structure of X since $T_N(kr) \ngeq T_N(k) \wedge T_N(r); I_N(kr) \nleq I_N(k) \vee I_N(r)$ and $F_N(kr) \ngeq F_N(k) \wedge F_N(r)$. □

3. **Neutrosophic ℵ-filters and Neutrosophic ℵ-bi-filters**

Lemma 3.1. *Let* $X_N = \frac{X}{(T_N, I_N, F_N)}$; $X_M = \frac{X}{(T_M, I_M, F_M)}$ *and* $X_O = \frac{X}{(T_O, I_O, F_O)}$. *Then*

(i) $X_N \subseteq X_M$ *if and only if* $X_{N^c} \supseteq X_{M^c}$.

(ii) $X_O \subseteq X_N \cup X_M$ *if and only if* $X_{O^c} \supseteq X_{N^c} \cap X_{M^c}$.

(iii) $X_O \subseteq X_N \cap X_M$ *if and only if* $X_{O^c} \supseteq X_{N^c} \cup X_{M^c}$.

Proof: (i) For any $a \in X$, we have

$$X_N \subseteq X_M \Leftrightarrow \begin{pmatrix} T_N(a) \geq T_M(a) \\ I_N(a) \leq I_M(a) \\ F_N(a) \geq F_M(a) \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -T_N(a) \leq -T_M(a) \\ -I_N(a) \geq -I_M(a) \\ -F_N(a) \leq -F_M(a) \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -1-T_N(a) \leq -1 - T_M(a) \\ -1-I_N(a) \geq -1-I_M(a) \\ -1 - F_N(a) \leq -1 - F_M(a) \end{pmatrix}$$

$$\Leftrightarrow X_{N^c} \supseteq X_{M^c}.$$

(ii) For any $a \in X$, we have

$$X_O \subseteq X_M \cup X_N$$

$$\Leftrightarrow \begin{pmatrix} T_O(a) \geq T_M(a) \wedge T_N(a) \\ I_O(a) \leq I_M(a) \vee I_N(a) \\ F_O(a) \geq F_M(a) \wedge F_N(a) \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -T_O(a) \leq -(T_M(a) \wedge T_N(a)) \\ -I_O(a) \geq -(I_M(a) \vee I_N(a)) \\ -F_O(a) \leq -(F_M(a) \wedge F_N(a)) \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -T_O(a) \leq -T_M(a) \vee -T_N(a)) \\ -I_O(a) \geq -I_M(a) \wedge -I_N(a)) \\ -F_O(a) \leq -F_M(a) \vee -F_N(a)) \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -1 - T_O(a) \leq (-1 - T_M(a)) \vee (-1 - T_N(a)) \\ -1 - I_O(a) \geq (-1 - I_M(a)) \wedge (-1 - I_N(a)) \\ -1 - F_O(a) \leq (-1 - F_M(a)) \vee (-1 - F_N(a)) \end{pmatrix}$$

$$\Leftrightarrow X_{O^c} \supseteq X_{M^c} \cap X_{N^c}.$$

(iii) For any $a \in X$, we have

$$X_O \subseteq X_M \cap X_N$$

$$\Leftrightarrow \begin{pmatrix} T_O(a) \geq T_M(a) \ \vee \ T_N(a) \\ I_O(a) \leq I_M(a) \ \wedge \ I_N(a) \\ F_O(a) \geq F_M(a) \ \vee \ F_N(a) \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -T_O(a) \leq -(T_M(a) \ \vee \ T_N(a)) \\ -I_O(a) \geq -(I_M(a) \ \wedge \ I_N(a)) \\ -F_O(a) \leq -(F_M(a) \ \vee \ F_N(a)) \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -T_O(a) \leq -T_M(a) \ \wedge -T_N(a)) \\ -I_O(a) \geq -I_M(a) \ \vee \ - \ I_N(a)) \\ -F_O(a) \leq -F_M(a) \ \wedge \ -F_N(a)) \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -1 - T_O(a) \leq (-1 - T_M(a)) \ \wedge (\ -1 - T_N(a)) \\ -1 - I_O(a) \geq (-1 - I_M(a)) \vee \ (-1 - \ I_N(a)) \\ -1 - F_O(a) \leq (-1 - F_M(a)) \ \wedge (-1 - F_N(a)) \end{pmatrix}$$

$$\Leftrightarrow X_{O^c} \supseteq X_{M^c} \cup X_{N^c}.$$

So $X_O \subseteq X_N \cup X_M$ if and only if $X_{O^c} \supseteq X_{N^c} \cap X_{M^c}$. □

Theorem 3.2. *For $\Phi \neq K \subseteq X$ and $X_N = \frac{X}{(T_N, I_N, F_N)}$, the below assertions are equivalent:*

(i) $\chi_K(X_N)$ of X is a neutrosophic $\aleph$-subsemigroup,

(ii) K of X is a subsemigroup.

Proof: Suppose $\chi_K(X_N)$ is a neutrosophic $\aleph$-subsemigroup of X. Let $k, s \in K$. Then

$$\begin{aligned} \chi_K(T)_N(ks) \leq & \quad \chi_K(T)_N(k) \vee \chi_K(T)_N(s) = -1, \\ \chi_K(I)_N(ks) \geq & \quad \chi_K(I)_N(k) \wedge \chi_K(I)_N(s) = 0, \\ \chi_K(F)_N(ks) \leq & \quad \chi_K(F)_N(k) \vee \chi_K(F)_N(s) = -1. \end{aligned}$$

Thus $ks \in K$ and hence K is a subsemigroup of X.

Conversely, suppose that K is a subsemigroup of X and let $k, s \in X$.

If $k, s \in K$, then $ks \in K$. Now

$$\begin{aligned} \chi_K(T)_N(ks) = & \quad -1 = \chi_K(T)_N(k) \vee \chi_K(T)_N(s), \\ \chi_K(I)_N(ks) = & \quad 0 = \chi_K(I)_N(k) \wedge \chi_K(I)_N(k), \\ \chi_K(F)_N(ks) = & \quad -1 = \chi_K(F)_N(k) \vee \chi_K(F)_N(k). \end{aligned}$$

If $k \notin K$ or $s \notin K$, then

$$\begin{aligned} \chi_K(T)_N(ks) \leq & \quad 0 = \chi_K(T)_N(k) \vee \chi_K(T)_N(s), \\ \chi_K(I)_N(ks) \geq & \quad -1 = \chi_K(I)_N(k) \wedge \chi_K(I)_N(s), \\ \chi_K(F)_N(ks) \leq & \quad 0 = \chi_K(F)_N(k) \vee \chi_K(F)_N(s). \end{aligned}$$

So $\chi_K(X_N)$ of X is a neutrosophic $\aleph$-subsemigroup. □

Theorem 3.3. *For $\Phi \neq K \subseteq X$ and $X_N = \frac{X}{(T_N, I_N, F_N)}$, the below assertions are equivalent:*

(i) $\chi_K(X_N)$ of X is a neutrosophic $\aleph$-bi-ideal structure,

(ii) K is a bi-ideal subset of X.

Proof: Suppose $\chi_K(X_N)$ is a neutrosophic $\aleph-$bi-ideal structure of X. Let $k \in K$ and $s \in X$. Then

$$\begin{aligned} \chi_K(T)_N(ksk) \leq & \quad \chi_K(T)_N(k) = -1, \\ \chi_K(I)_N(ksk) \geq & \quad \chi_K(I)_N(k) = 0, \\ \chi_K(F)_N(ksk) \leq & \quad \chi_K(F)_N(k) = -1. \end{aligned}$$

Thus $ksk \in K$ and hence K is a bi-ideal subset of X.

Conversely, suppose K is a bi-ideal subset of X. Let $k, s \in X$.

If $k \in K$, then $ksk \in K$. Now

$$\begin{aligned} \chi_K(T)_N(ksk) = & \quad -1 = \chi_K(T)_N(k), \\ \chi_K(I)_N(ksk) = & \quad 0 = \chi_K(I)_N(k), \\ \chi_K(F)_N(ksk) = & \quad -1 = \chi_K(F)_N(k). \end{aligned}$$

If $k \notin K$, then

$$\begin{aligned} \chi_K(T)_N(ksk) \leq & \quad 0 = \chi_K(T)_N(k), \\ \chi_K(I)_N(ksk) \geq & \quad -1 = \chi_K(I)_N(k) \\ \chi_K(F)_N(ksk) \leq & \quad 0 = \chi_K(F)_N(k). \end{aligned}$$

Therefore $\chi_K(X_N)$ of X is a neutrosophic $\aleph-$bi-ideal structure. □

Theorem 3.4. *For $\Phi \neq K \subseteq X$ and $X_N = \frac{X}{(T_N, I_N, F_N)}$, the below assertions are equivalent:*

(i) $\chi_K(X_N)$ *of* X *is a neutrosophic* $\aleph$*-bi-filter,*

(ii) K *of* X *is a bi-filter.*

Proof: Suppose $\chi_K(X_N)$ of X is a neutrosophic $\aleph$-bi-ideal. Then by Theorem 3.2, K is a subsemigroup of X. Let $k \in K$ and $s \in X$ with $ksk \in K$. Then

$$-1 = \chi_K(T)_N(ksk) \leq \chi_K(T)_N(k) = -1,$$
$$0 = \chi_K(I)_N(ksk) \geq \chi_K(I)_N(k) = 0,$$
$$-1 = \chi_K(F)_N(ksk\ ) \leq \chi_K(F)_N(k) = -1.$$

Thus $k \in K$ and hence K is a bi-filter of X,

Conversely, suppose K of X is a bi-filter. Then by Theorem 3.2, we have $\chi_K(X_N)$ of X is a neutrosophic $\aleph$-subsemigroup.

Let $s, k \in X$.

If k$\in K$, then $ksk \in K$. Now

$$\begin{aligned} \chi_K(T)_N(ksk) = &\quad -1 = \chi_K(T)_N(k), \\ \chi_K(I)_N(ksk) = &\quad 0 = \chi_K(I)_N(k), \\ \chi_K(F)_N(ksk) = &\quad -1 = \chi_K(F)_N(k). \end{aligned}$$

If $k \notin K$, then

$$\begin{aligned} \chi_K(T)_N(ksk) \leq &\quad 0 = \chi_K(T)_N(k), \\ \chi_K(I)_N(ksk) \geq &\quad -1 = \chi_K(I)_N(k), \\ \chi_K(F)_N(ksk) \leq &\quad 0 = \chi_K(F)_N(k). \end{aligned}$$

So $\chi_K(X_N)$ of X is a neutrosophic $\aleph$-bi-filter. □

Theorem 3.5. *For* $X_N = \frac{X}{(T_N, I_N, F_N)}$, *the below assertions are equivalent:*

(i) X_N *of* X *is a neutrosophic* $\aleph-$ *left (resp., right) ideal,*

(ii) The non-empty sets T_N^{α}, I_N^{β} *and* F_N^{γ} *are left (resp., right) ideals of* X $\forall \alpha, \beta, \gamma \in [-1, 0]$.

Proof: Suppose X_N is a neutrosophic $\aleph$-left ideal of X and $\alpha, \beta, \gamma \in [-1, 0]$.

Let $k \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}; s \in X$. Then

$$T_N(sk) \leq T_N(k) \leq \alpha,$$
$$I_N(sk) \geq I_N(k) \geq \beta,$$
$$F_N(sk) \leq F_N(k) \leq \gamma$$

which imply $sk \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$. So T_N^{α}, I_N^{β} and F_N^{γ} are left ideals of X.

Conversely, assume that T_N^{α}, I_N^{β} and F_N^{γ} are left ideals of X for any $\alpha, \beta, \gamma \in [-1, 0]$. Then by Theorem 3.2 of [5], X_N of X is a neutrosophic $\aleph$-left ideal. □

Theorem 3.6. *For* $\Phi \neq K \subseteq X$ *and* $X_N = \frac{X}{(T_N, I_N, F_N)}$, *the below statements are equivalent:*

(i) K *is a prime left (resp., right) ideal of* X,

(ii) $\chi_K(X_N)$ *is a prime neutrosophic* $\aleph$*-left (resp., right) ideal of* X.

Proof: Suppose that K is a prime left ideal of X. Then by Theorem 3.2 of [5], $\chi_K(X_N)$ of X is a neutrosophic $\aleph$-left ideal. Let $k, s \in X$.

If $ks \notin K$, then

$$\chi_K(T)_N(ks) = 0 \geq \chi_K(T)_N(k) \wedge \chi_K(T)_N(s),$$
$$\chi_K(I)_N(ks) = -1 \leq \chi_K(I)_N(k) \vee \chi_K(I)_N(s),$$
$$\chi_K(F)_N(ks) = 0 \geq \chi_K(F)_N(k) \wedge \chi_K(F)_N(s).$$

If $ks \in K$, then $k \in K$ or $s \in K$. So

$$\chi_K(T)_N(ks) = \; -1 = \chi_K(T)_N(k) \wedge \chi_K(T)_N(s),$$
$$\chi_K(I)_N(ks) = \; 0 = \chi_K(I)_N(k) \vee \chi_K(I)_N(s),$$
$$\chi_K(F)_N(ks) = \; -1 = \chi_K(F)_N(k) \wedge \chi_K(F)_N(s).$$

Hence $\chi_K(X_N)$ is a prime neutrosophic $\aleph-$ left ideal of X.

Conversely, suppose $\chi_K(X_N)$ of X is a prime neutrosophic $\aleph-$ left (resp., right) ideal. Then by Theorem 3.2 of [5], K of X is a left ideal.

Let $k, s \in S$ with $ks \in K$. Suppose that $k \notin K$ and s$\notin K$. Then

$$-1 = \chi_K(T)_N(ks) \geq \chi_K(T)_N(k) \wedge \chi_K(T)_N(s) = 0,$$
$$0 = \chi_K(I)_N(ks) \leq \chi_K(I)_N(k) \vee \chi_K(T)_N(s) = -1,$$
$$-1 = \chi_K(F)_N(ks) \geq \chi_K(F)_N(k) \wedge \chi_K(T)_N(s) = 0$$

which are not possible. Thus $k \in K$ or $s \in K$, and hence K of X is a prime left ideal. □

Theorem 3.7. *Let $X_N = \frac{X}{(T_N, I_N, F_N)}$. Then the below assertions are equivalent:*

(i) X_N of X is a prime neutrosophic $\aleph$- left (resp., right) ideal,

(ii) The non-empty sets T_N^α, I_N^β and F_N^γ are prime left (resp., right) ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$.

Proof: Suppose X_N of X is a prime neutrosophic $\aleph$-left ideal. Then by Theorem 3.5, T_N^α, I_N^β and F_N^γ are left ideals of X for $\alpha, \beta, \gamma \in [-1, 0]$.

Let $k, s \in X$ with $ks \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Then $\alpha \geq T_N(ks) \geq T_N(k) \wedge T_N(s)$ implies $\alpha \geq T_N(k)$ or $\alpha \geq T_N(s)$. So $k \in T_N^\alpha$ or $s \in T_N^\alpha$. Also $\beta \leq I_N(ks) \leq I_N(k) \vee I_N(s)$ gives $\beta \leq I_N(k)$ or $\beta \leq I_N(s)$. So $k \in I_N^\beta$ or $s \in I_N^\beta$. Also $\gamma \geq F_N(ks) \geq F_N(k) \wedge F_N(s)$ implies $\gamma \geq F_N(k)$ or $\gamma \geq F_N(s)$. So $k \in F_N^\gamma$ or $s \in F_N^\gamma$.

Therefore T_N^α, I_N^β and F_N^γ are prime left ideals of X.

Conversely, suppose T_N^α, I_N^β and F_N^γ are prime left ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$. Then by Theorem 3.5, X_N of X is a neutrosophic $\aleph$-left ideal.

Let $k, s \in X$. Then $T_N(ks) = \alpha_1$; $I_N(ks) = \beta_1$ and $F_N(ks) = \gamma_1$ for some $\alpha_1, \beta_1, \gamma_1 \in [-1, 0]$ which imply $s \in T_N^{\alpha_1} \cap I_N^{\beta_1} \cap F_N^{\gamma_1}$. Since $T_N^{\alpha_1}$ is prime, we have $k \in T_N^{\alpha_1}$ or $s \in T_N^{\alpha_1}$ which implies $T_N(k) \leq \alpha_1$ or $T_N(s) \leq \alpha_1$. Since $I_N^{\beta_1}$ is prime, we have $k \in I_N^{\beta_1}$ or $s \in I_N^{\beta_1}$ which implies

$I_N(k) \geq \beta_1$ or $I_N(s) \geq \beta_1$. Since $F_N^{\gamma_1}$ is prime, we have $k \in F_N^{\gamma_1}$ or $s \in F_N^{\gamma_1}$ which implies $F_N(k) \leq \gamma_1$ or $F_N(s) \leq \gamma_1$. Now

$$\begin{aligned} T_N(ks) = & \quad \alpha_1 \geq T_N(k) \wedge T_N(s), \\ I_N(ks) = & \quad \beta_1 \leq I_N(k) \vee I_N(s), \\ F_N(ks) = & \quad \gamma_1 \geq F_N(k) \wedge F_N(s). \end{aligned}$$

So X_N of X is a prime neutrosophic $\aleph$- left ideal. □

Theorem 3.8. *For $X_N = \frac{X}{(T_N, I_N, F_N)}$, the below assertions are equivalent:*

(i) X_N of X is a semiprime neutrosophic $\aleph$- left (resp., right) ideal,

(ii) The non-empty sets T_N^α, I_N^β and I_N^γ are semiprime left (resp., right) ideals of X for any $\alpha, \beta, \gamma \in [-1, 0]$.

Proof: Suppose X_N of X is a semiprime neutrosophic $\aleph$- left ideal. Then by Theorem 3.5, T_N^α, I_N^β and F_N^γ are left ideals of X for $\alpha, \beta, \gamma \in [-1, 0]$.

Let $r \in X$ with $r^2 \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Then $\alpha \geq T_N(r^2) \geq T_N(r)$ implies $\alpha \geq T_N(r)$. So $r \in T_N^\alpha$. Also $\beta \leq I_N(r^2) \leq I_N(r)$ implies $\beta \leq I_N(r)$. So $r \in I_N^\beta$. Also $\gamma \geq F_N(r^2) \geq F_N(r)$ implies $\gamma \geq F_N(r)$. So $r \in F_N^\gamma$. Hence T_N^α, I_N^β and F_N^γ are semiprime left ideals of X.

Conversely, suppose T_N^α, I_N^β and F_N^γ are semiprime left ideals of X $\forall \alpha, \beta, \gamma \in [-1,\ 0]$. Then by Theorem 3.5, X_N of X is a neutrosophic $\aleph$- left ideal. Let $r \in X$. Then $T_N(r^2) = \alpha_1$; $I_N(r^2) = \beta_1$ and $F_N(r^2) = \gamma_1$ for some $\alpha_1, \beta_1, \gamma_1 \in [-1, 0]$ which imply $r^2 \in T_N^{\alpha_1} \cap I_N^{\beta_1} \cap F_N^{\gamma_1}$. Since $T_N^{\alpha_1}$, $I_N^{\beta_1}$ and $F_N^{\gamma_1}$ are semiprime, we have $r \in T_N^{\alpha_1}$ gives $T_N(r) \leq \alpha_1$; $r \in I_N^{\beta_1}$ gives $I_N(r) \geq \beta_1$ and $r \in F_N^{\gamma_1}$ gives $F_N(r) \leq \gamma_1$.

Now

$$\begin{aligned} T_N(r^2) = & \quad \alpha_1 \geq T_N(r), \\ I_N(r^2) = & \quad \beta_1 \leq I_N(r), \\ F_N(r^2) = & \quad \gamma_1 \geq F_N(r). \end{aligned}$$

So X_N is semiprime neutrosophic $\aleph$-left ideal. □

Theorem 3.9. *Let $X_N = \frac{X}{(T_N, I_N, F_N)}$. Then the below assertions are equivalent:*

(i) X_N of X is a neutrosophic $\aleph$-bi-ideal structure,

(ii) The non-empty sets T_N^α, I_N^β and F_N^γ are bi-ideal subsets of X for all $\alpha, \beta, \gamma \in [-1, 0]$.

Proof: Suppose X_N of X is a neutrosophic $\aleph$-bi-ideal structure and $\alpha, \beta, \gamma \in [-1, 0]$.

Let $k \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma; s \in X$. Then

$$\begin{aligned} T_N(ksk) \leq & \quad T_N(k) \leq \alpha, \\ I_N(ksk) \geq & \quad I_N(k) \geq \beta, \\ F_N(ksk) \leq & \quad F_N(k) \leq \gamma \end{aligned}$$

which imply $ksk \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. So T_N^α, I_N^β and F_N^γ are bi-ideal subsets of X.

Conversely, suppose T_N^α, I_N^β and F_N^γ are bi-ideal subsets of X for all $\alpha, \beta, \gamma \in [-1, 0]$.

If there are $r, s \in X$ such that $T_N(rsr) > T_N(r)$, then $T_N(rsr) > t_\alpha \geq T_N(r)$ for some $t_\alpha \in [-1, 0)$ which implies $r \in T_N^{t_\alpha}(r)$ and $rsr \notin T_N^{t_\alpha}(r)$, a contradiction. So $T_N(rsr) \leq T_N(r)$.

If there are $r, s \in X$ such that $I_N(rsr) < I_N(r)$, then $I_N(rsr) < t_\beta \leq I_N(r)$ for some $t_\beta \in (-1, 0]$ which implies $r \in I_N^{t_\beta}(r)$ and $rsr \notin I_N^{t_\beta}(r)$, a contradiction. So $I_N(rsr) \geq I_N(r)$.

If there are $r, s \in X$ such that $F_N(rsr) > F_N(r)$, then $F_N(rsr) > t_\gamma \geq F_N(r)$ for some $t_\gamma \in [-1, 0)$ which implies $r \in F_N^{t_\gamma}(r)$ and $rsr \notin F_N^{t_\gamma}(r)$, a contradiction. So $F_N(rsr) \leq F_N(r)$.

Therefore X_N is a neutrosophic $\aleph$-bi-ideal structure. □

Theorem 3.10. *Let $X_N = \frac{X}{(T_N, I_N, F_N)}$. Then the below assertions are equivalent:*

(i) X_N of X is a prime neutrosophic $\aleph$-bi-ideal structure,

(ii) The non-empty sets T_N^α, I_N^β and F_N^γ are prime bi-ideal subsets of X for any $\alpha, \beta, \gamma \in [-1, 0]$.

Proof: Suppose X_N of X is a prime neutrosophic $\aleph$- bi-ideal structure and $\alpha, \beta, \gamma \in [-1, 0]$. Let $k, s \in X$ with $ks \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Then $\alpha \geq T_N(ks) \geq T_N(k) \wedge T_N(s)$ implies $\alpha \geq T_N(k)$ or $\alpha \geq T_N(s)$. So $k \in T_N^\alpha$ or $s \in T_N^\alpha$. Also $\beta \leq I_N(ks) \leq I_N(k) \vee I_N(s)$ implies $\beta \leq I_N(k)$ or $\beta \leq I_N(s)$. So $k \in I_N^\beta$ or $s \in I_N^\beta$. Also $\gamma \geq F_N(ks) \geq F_N(k) \wedge F_N(s)$ implies $\gamma \geq F_N(k)$ or $\gamma \geq F_N(s)$. So $k \in F_N^\gamma$ or $s \in F_N^\gamma$. Hence T_N^α, I_N^β and F_N^γ are prime left ideals of X.

Conversely, suppose T_N^α, I_N^β and F_N^γ are prime bi-ideal subsets of X $\forall \alpha, \beta, \gamma \in [-1, 0]$. Then by Theorem 3.9, X_N of X is a neutrosophic $\aleph$-bi-ideal. Let $k, s \in X$. Then $T_N(ks) = \alpha_1$;$I_N(ks) = \beta_1$ and $F_N(ks) = \gamma_1$ for some $\alpha_1, \beta_1, \gamma_1 \in [-1, 0]$ which imply $ks \in T_N^{\alpha_1} \cap I_N^{\beta_1} \cap F_N^{\gamma_1}$. Since $T_N^{\alpha_1}$ is prime bi-ideal, $k \in T_N^{\alpha_1}$ or $s \in T_N^{\alpha_1}$ which implies $T_N(k) \leq \alpha_1$ or $T_N(s) \leq \alpha_1$.

Since $I_N^{\beta_1}$ is prime bi-ideal, $k \in I_N^{\beta_1}$ or $s \in I_N^{\beta_1}$ which implies $I_N(k) \geq \beta_1$ or $I_N(s) \geq \beta_1$. Also $F_N^{\gamma_1}$ is prime bi-ideal, $k \in F_N^{\gamma_1}$ or $s \in F_N^{\gamma_1}$ which implies $F_N(k) \leq \gamma_1$ or $F_N(s) \leq \gamma_1$. Now

$$\begin{aligned} T_N(ks) &= \alpha_1 \geq T_N(k) \wedge T_N(s), \\ I_N(ks) &= \beta_1 \leq I_N(k) \vee I_N(s), \\ F_N(ks) &= \gamma_1 \geq F_N(k) \wedge F_N(s). \end{aligned}$$

Therefore X_N is a prime neutrosophic $\aleph$-bi-ideal structure of X. □

Theorem 3.11. *For $X_N = \frac{X}{(T_N, I_N, F_N)}$, the below assertions are equivalent:*

(i) X_N is a semiprime neutrosophic $\aleph$-bi- ideal structure of X,

(ii) The non-empty sets T_N^α, I_N^β and F_N^γ are semiprime bi-ideal subsets of X for all $\alpha, \beta, \gamma \in [-1, 0]$.

Proof: It is similar to the proof of Theorem 3.10. □

Theorem 3.12. *For $X_N = \frac{X}{(T_N, I_N, F_N)}$ and $\Phi \neq K \subseteq X$, the below statements are equivalent:*

(i) K is a prime bi-ideal subset of X,

(ii) $\chi_K(X_N)$ of X is a prime neutrosophic $\aleph$-bi-ideal structure.

Proof: It is similar to the proof of Theorem 3.6. □

Theorem 3.13. *For* $X_N = \frac{X}{(T_N, I_N, F_N)}$, *the below assertions are equivalent:*

(i) X_N *of* X *is a neutrosophic* $\aleph$*-bi-filter,*

(ii) X_{N^c} *of* X *is a neutrosophic* $\aleph$*-bi-ideal structure.*

Proof: It is trivial as for $k, s \in X$, we have

$$\begin{pmatrix} T_N(ksk) \geq T_N(k) \\ I_N(ksk) \leq I_N(k) \\ F_N(ksk) \geq F_N(k) \end{pmatrix} \Leftrightarrow \begin{pmatrix} T_{N^c}(ksk) \leq T_{N^c}(k) \\ I_{N^c}(ksk) \geq I_{N^c}(k) \\ F_{N^c}(ksk) \leq F_{N^c}(k) \end{pmatrix}.$$

□

Theorem 3.14. *For* $\Phi \neq K \subseteq X$ *and* $X_N = \frac{X}{(T_N, I_N, F_N)}$, *the below assertions are equivalent:*

(i) K *is a left (resp., right) filter of X,*

(ii) $\chi_K(X_N)$ *is a neutrosophic* $\aleph$*-left (resp., right) filter of X.*

Proof: Suppose K of X is a left filter. Then by Theorem 3.12, $\chi_K(X_N)$ of X is a neutrosophic $\aleph$-subsemigroup. Let k, $t \in X$.

If $kt \notin K$, then

$$\begin{aligned} \chi_K(T)_N(kt) = &\quad 0 \geq \chi_K(T)_N(t), \\ \chi_K(I)_N(kt) = &\quad -1 \leq \chi_K(I)_N(t), \\ \chi_K(F)_N(kt) = &\quad 0 \geq \chi_K(F)_N(t). \end{aligned}$$

If $kt \in K$, then $k \in K$. So

$$\begin{aligned} \chi_K(T)_N(kt) = &\quad -1 = \chi_K(T)_N(t), \\ \chi_K(I)_N(kt) = &\quad 0 = \chi_K(I)_N(t), \\ \chi_K(F)_N(kt) = &\quad -1 = \chi_K(F)_N(t). \end{aligned}$$

Hence $\chi_K(X_N)$ of X is a neutrosophic $\aleph$-left filter.

Conversely, suppose $\chi_K(X_N)$ of X is a neutrosophic $\aleph$-left (resp., right) filter. Then by Theorem 3.12, K is a subsemigroup of X.

Let $r, s \in S$ such that $rs \in K$. Suppose that $s \notin K$. Then

$$\begin{aligned} -1 &\quad = \chi_K(T)_N(rs) \geq \chi_K(T)_N(s) = 0, \\ 0 &\quad = \chi_K(I)_N(rs) \leq \chi_K(I)_N(s) = -1, \\ -1 &\quad = \chi_K(F)_N(xy) \geq \chi_K(F)_N(s) = 0, \end{aligned}$$

which are not possible.

Thus $s \in K$ and hence K of X is a left filter. □

Theorem 3.15. *Let* $X_N = \frac{X}{(T_N,\ I_N,\ F_N)}$. *Then the below statements are equivalent:*

(i) X_N *is a neutrosophic* $\aleph-$ *left (resp., right) filter of X,*

(ii) X_{N^c} *is a prime neutrosophic* $\aleph-$ *left (resp., right) ideal of X.*

Proof: Suppose X_N of X is a neutrosophic $\aleph-$ left filter. Then X_N of X is a neutrosophic $\aleph-$ subsemigroup. For k, $s \in X$, we have

$$\begin{pmatrix} T_N(ks) \geq T_N(s) \\ I_N(ks) \leq I_N(s) \\ F_N(ks) \geq F_N(s) \end{pmatrix} \Leftrightarrow \begin{pmatrix} T_{N^c}(ks) \leq T_{N^c}(s) \\ I_{N^c}(ks) \geq I_{N^c}(s) \\ F_{N^c}(ks) \leq F_{N^c}(s) \end{pmatrix} \quad (a)$$

So X_{N^c} of X is a neutrosophic $\aleph-$ left ideal.

Since X_N is neutrosophic $\aleph$- subsemigroup, we have

$$\begin{pmatrix} T_N(ks) \leq T_N(k) \vee T_N(s) \\ I_N(ks) \geq I_N(k) \wedge I_N(s) \\ F_N(ks) \leq F_N(k) \vee F_N(s) \end{pmatrix} \Leftrightarrow \begin{pmatrix} T_{N^c}(ks) \geq T_{N^c}(k) \wedge T_{N^c}(s) \\ I_{N^c}(ks) \leq I_{N^c}(k) \vee I_{N^c}(s) \\ F_{N^c}(ks) \geq F_{N^c}(k) \wedge F_{N^c}(s) \end{pmatrix}.$$

Therefore X_{N^c} is a prime neutrosophic $\aleph$-left ideal of X.

Conversely, suppose X_{N^c} of X is a prime neutrosophic $\aleph$- left ideal. Then X_{N^c} of X is a neutrosophic $\aleph$- left ideal. Then by (a), we have X_N of X is a neutrosophic $\aleph$-left filter. □

Theorem 3.16. *Let $X_N = \frac{X}{(T_N, I_N, F_N)}$. Then the below statements are equivalent:*

(i) X_N is a neutrosophic $\aleph$-bi-filter of X,

(ii) X_{N^c} is a prime neutrosophic $\aleph$-bi-ideal structure of X.

Proof: Suppose X_N is a neutrosophic $\aleph$-bi-filter of X. Then X_N is a neutrosophic $\aleph$-subsemigroup of X. For any $k, s \in X$, we have

$$\begin{pmatrix} T_N(ksk) \leq T_N(k) \\ I_N(ksk) \geq I_N(k) \\ F_N(ksk) \leq F_N(k) \end{pmatrix} \Leftrightarrow \begin{pmatrix} T_{N^c}(ksk) \geq T_{N^c}(k) \\ I_{N^c}(ksk) \leq I_{N^c}(k) \\ F_{N^c}(ksk) \geq F_{N^c}(k) \end{pmatrix} \quad (1)$$

So X_{N^c} is a neutrosophic $\aleph$-bi-ideal structure of X.

Since X_N is a neutrosophic $\aleph$-subsemigroup of X, we have

$$\begin{pmatrix} T_N(ks) \leq T_N(k) \vee T_N(s) \\ I_N(ks) \geq I_N(k) \wedge I_N(s) \\ F_N(ks) \leq F_N(k) \vee F_N(s) \end{pmatrix} \Leftrightarrow \begin{pmatrix} T_{N^c}(ks) \geq T_{N^c}(k) \wedge T_{N^c}(s) \\ I_{N^c}(ks) \leq I_{N^c}(k) \vee I_{N^c}(s) \\ F_{N^c}(ks) \geq F_{N^c}(k) \wedge F_{N^c}(s) \end{pmatrix}$$

Therefore X_{N^c} is a prime neutrosophic $\aleph$-bi-ideal structure of X.

Conversely, suppose X_{N^c} of X is a prime neutrosophic $\aleph$-bi-ideal structure. Then X_{N^c} of X is a neutrosophic $\aleph$-bi-ideal structure. Then by (1), we have X_N of X is a neutrosophic $\aleph$-bi-filter. □

4. Conclusion

In this paper, we have characterized the concept neutrosophic $\aleph$-bi-filter of X and described semigroup as far as neutrosophic $\aleph$-bi-ideal and neutrosophic $\aleph$-bi-filter of X. We likewise characterized the notions neutrosophic ℵ-left filters and prime neutrosophic ℵ-left ideals of X and portrayed semigroup in terms of these notions.

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A Novel Intelligent Multi-Attributes Decision-Making Approach Based on Generalized Neutrosophic Vague Hybrid Computing

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Abstract. Neutrosophic vague hypersoft set (nVHs-set) is a novel hybrid model that is projected to address the limitations of existing fuzzy vague set-like structures for degree of indeterminacy and multi argument approximate function. This function maps the cartesian product of disjoint attribute valued sets to power set of initial universe. This study aims to characterize nVHs-set to tackle uncertainties more efficiently. Some essential properties and set-theoretic cum aggregation operations of nVHs-set are characterized by employing axiomatic and analytical approaches respectively and explained with the help of suitable examples. An algorithm is proposed based on aggregations of nVHs-set for dealing real-world decision-making issues and problems. The proposed algorithm is validated by its implementation in real-world decision-making problem for the optimal selection of farmhouse. Moreover advantageous aspects of proposed model are assessed with the help of evaluating features through comparison analysis.

Keywords: Soft set; Vague set; Hypersoft set; Neutrosophic vague soft set; Neutrosophic vague hypersoft set; Decision making.

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1. Introduction

The concept of fuzzy set was generated by Zadeh [1] to address uncertainty and vagueness in daily life problems. Real life problems involving indecisive and ambiguous environment under fuzzy sets and fuzzy logic were addressed by different authors [2–5]. Gau et al. [6], Atanassov [7] and Pawlak [8] also worked on research problems under uncertain situations. Neutrosophic set theory [9] generalized the concept of classical set, fuzzy set and intuitionistic fuzzy set. Neutrosophic logic is a logic where every proposition has different values for truth, falsehood, and indeterminacy which means that there exists some neutral part which is neither true, nor false, rather it is vague. Soft set theory was conceptualized by Molodtsov [10] to handle vagueness and uncertainty in data. The idea of soft set and fuzzy set with their amplified impacts on set theory were undertaken by Maji et al. [11], Feng et al. [12] Zhang et al. [13], Salleh et al. [14] and Alkhazaleh et al. [15]. Xu et al. [16] developed the innovative concept of vague soft set where as Alhazaymeh et al. [17] generalized the concept. Alhazaymeh et al. [18] also discussed vague soft sets relations and functions. Intuitionistic fuzzy soft set and neutrosophic soft set were initiated by Maji et al. [19, 20]. Broumi et al. [21] and Deli [22] depicted the idea of intuitionistic neutrosophic soft set and interval-valued neutrosophic soft sets and applied the concept in decision making. Recent work on interval-valued vague soft sets by Alhazaymeh et al. [23–26] has created many slits for researchers [27–31]. Different vague soft set variants were discussed by Hassan et al. [32, 33]. Vague set and neutrosophic set were hybridized to form neutrosophic vague set [34] which became an efficient tool to discuss and solve problems with uncertain, incomplete and inconsistent data.

Al Quran et al. [35] developed neutrosophic vague soft set nVs-set as hybrid model of soft set and neutrosophic vague set which made it more effective and efficient for solving decision making problems. nVs-set deals with uncertain, incomplete and indeterminate type of data.

1.1. *Research gap and Motivation*

In many real life problems, it is essential to partition attributes into sets of sub attributive values. Soft set theory is incompatible and inadequate to deal with such type of problems. The concept of hypersoft set [36] introduces multi argument approximate function which fulfills the insufficiency of soft set. Fundamentals of hypersoft set have been elaborated in [37]. Many hypersoft set variants under uncertain environment have already been examined by Rahman et al. [38–44] and Saeed et al. [45–50]. Recently the researchers [51–59] made rich contributions towards the characterization of various hybrids of hypersoft set ad their application in decision making and other fields.

The question arises "Can we mingle the concept of neutrosophic vague soft set and hypersoft set"? In other words "How is multi argument approximate function applicable to neutrosophic vague soft sets?" and "How can this new hybrid structure of hypersoft set and neutrosophic vague soft set be more effective and useful than existing models"? The research paper aims to answer these questions.

1.2. *Main Contributions*

The major contributions of the study are given hereafter:

(1) The existing models [34–36] are made adequate with nVHs-set,
(2) the scenario where parameters are divided into sub-parameters, is dealt,

(3) multi-attribute decision making is discussed based on nVHs-set through algorithmic approach,
(4) real life decision making problem is solved using nVHs-set,
(5) proposed model is compared with existing relevant models,
(6) validity and generalization of proposed model is discussed.

1.3. *Paper Layout*

The research paper is divided into different sections as given below:
Some basic definitions are discussed in section 2. The concept of nVHs-set is originated in section 3 whereas a decision making problem is solved in section 4. Comparison analysis of proposed model with existing models is done in section 5. Merits of proposed model are discussed in section 6. Finally section 7 concludes the paper with future directions.

2. **Preliminaries**

In this section, some basic definitions from literature are recalled. In this paper $\mathcal{Z}$ will represent universe of discourse.

Definition 2.1. [6] Let z be a generic element of $\mathcal{Z}$. Let $\mathcal{V}$ in $\mathcal{Z}$ denote vague set which contains a truth membership function $\mathcal{T}_V$ whereas $\mathcal{T}_V(z) \in [0,1]$ is lower bound on grade of membership taken from the evidence for z and false membership function $\mathcal{F}_V$ whereas $\mathcal{F}_V(z) \in [0,1]$ is lower bound on grade of non-membership taken from the evidence against z with condition $\mathcal{T}_V(z) + \mathcal{F}_V(z) \leq 1$.

Definition 2.2. [9] A neutrosophic set $\mathcal{N}$ defined on universal set $\mathcal{Z}$ is given by

$$\mathcal{N} = \{< z; \mathcal{T}_{\mathcal{N}}(z); \mathcal{I}_{\mathcal{N}}(z); \mathcal{F}_{\mathcal{N}}(z) >; z \in \mathcal{Z}\},$$

such that $\mathcal{T}; \mathcal{I}; \mathcal{F} : \mathcal{Z} \to]^-0, 1^+[$ with $^-0 \leq \mathcal{T}_{\mathcal{N}}(z) + \mathcal{I}_{\mathcal{N}}(z) + \mathcal{F}_{\mathcal{N}}(z) \leq 3^+$.

Definition 2.3. [34] Let $\mathcal{Z}$ be universe of discourse. A neutrosophic vague set $\mathcal{N}_{\mathcal{V}}$ on $\mathcal{Z}$ denoted by nV-set can be given by

$$\mathcal{N}_{\mathcal{V}} = \{< z; \mathcal{T}_{\mathcal{N}_{\mathcal{V}}}(z); \mathcal{I}_{\mathcal{N}_{\mathcal{V}}}(z); \mathcal{F}_{\mathcal{N}_{\mathcal{V}}}(z) >; z \in \mathcal{Z}\},$$

were $\mathcal{T}_{\mathcal{N}_{\mathcal{V}}}(z) = [\mathcal{T}^-, \mathcal{T}^+]$, $\mathcal{I}_{\mathcal{N}_{\mathcal{V}}}(z) = [\mathcal{I}^-, \mathcal{I}^+]$ and $\mathcal{F}_{\mathcal{N}_{\mathcal{V}}}(z) = [\mathcal{F}^-, \mathcal{F}^+]$ are truth membership, indeterminacy and false membership respectively and satisfy following conditions $\mathcal{T}^+ = 1 - \mathcal{F}^-$, $\mathcal{F}^+ = 1 - \mathcal{T}^-$, $^-0 \leq \mathcal{T} + \mathcal{I} + \mathcal{F} \leq 2^+$.

Definition 2.4. [34] For two nV-sets $\mathcal{N}_{\mathcal{V}_1}$ and $\mathcal{N}_{\mathcal{V}_2}$, $\mathcal{N}_{\mathcal{V}_1}$ is called nV-subset of $\mathcal{N}_{\mathcal{V}_2}$ if following conditions hold for all $z_i \in \mathcal{Z}$ and $i = 1, 2, 3, ..., n$; $\mathcal{T}_{\mathcal{N}_{\mathcal{V}_1}}(z_i) \leq \mathcal{T}_{\mathcal{N}_{\mathcal{V}_2}}(z_i)$, $\mathcal{I}_{\mathcal{N}_{\mathcal{V}_1}}(z_i) \geq \mathcal{I}_{\mathcal{N}_{\mathcal{V}_2}}(z_i)$ and $\mathcal{F}_{\mathcal{N}_{\mathcal{V}_1}}(z_i) \geq \mathcal{F}_{\mathcal{N}_{\mathcal{V}_2}}(z_i)$.

Definition 2.5. [34] Two nV-sets $\mathcal{N}_{\mathcal{V}_1}$ and $\mathcal{N}_{\mathcal{V}_2}$ are nV-set equal if following conditions hold for all $z_i \in \mathcal{Z}$ and $i = 1, 2, 3, ..., n$; $\mathcal{T}_{\mathcal{N}_{\mathcal{V}_1}}(z_i) = \mathcal{T}_{\mathcal{N}_{\mathcal{V}_2}}(z_i)$, $\mathcal{I}_{\mathcal{N}_{\mathcal{V}_1}}(z_i) = \mathcal{I}_{\mathcal{N}_{\mathcal{V}_2}}(z_i)$ and $\mathcal{F}_{\mathcal{N}_{\mathcal{V}_1}}(z_i) = \mathcal{F}_{\mathcal{N}_{\mathcal{V}_2}}(z_i)$.

Definition 2.6. [34] $\mathcal{N}^c_{\mathcal{V}}$, the complement of nV-set $\mathcal{N}_{\mathcal{V}}$ on $\mathcal{Z}$ is given by

$$\mathcal{N}^c_{\mathcal{V}} = \left\{< z; \mathcal{T}^c_{\mathcal{N}_{\mathcal{V}}}(z); \mathcal{I}^c_{\mathcal{N}_{\mathcal{V}}}(z); \mathcal{F}^c_{\mathcal{N}_{\mathcal{V}}}(z) >; z \in \mathcal{Z}\right\},$$

where $\mathcal{T}^c_{\mathcal{N}_{\mathcal{V}}}(z) = [1-\mathcal{T}^+, 1-\mathcal{T}^-]$, $\mathcal{I}^c_{\mathcal{N}_{\mathcal{V}}}(z) = [1-\mathcal{I}^+, 1-\mathcal{I}^-]$, and $\mathcal{F}^c_{\mathcal{N}_{\mathcal{V}}}(z) = [1-\mathcal{F}^+, 1-\mathcal{F}^-]$.

Definition 2.7. [34] The intersection $\mathcal{N}_{\mathcal{V}}$ of two nV-sets $\mathcal{N}_{\mathcal{V}_1}$ and $\mathcal{N}_{\mathcal{V}_2}$ denoted by $\mathcal{N}_{\mathcal{V}} = \mathcal{N}_{\mathcal{V}_1} \cap \mathcal{N}_{\mathcal{V}_2}$ is nV-set with following conditions $\forall z_i \in \mathcal{Z}$ and $i = 1, 2, 3, ..., n$; $\mathcal{T}_{\mathcal{N}_{\mathcal{V}}}(z_i) = \left[\min(\mathcal{T}^-_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{T}^-_{\mathcal{N}_{\mathcal{V}_2}}), \min(\mathcal{T}^+_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{T}^+_{\mathcal{N}_{\mathcal{V}_2}})\right]$, $\mathcal{I}_{\mathcal{N}_{\mathcal{V}}}(z_i) = \left[\max(\mathcal{I}^-_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{I}^-_{\mathcal{N}_{\mathcal{V}_2}}), \max(\mathcal{I}^+_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{I}^+_{\mathcal{N}_{\mathcal{V}_2}})\right]$ and $\mathcal{F}_{\mathcal{N}_{\mathcal{V}}}(z_i) = \left[\max(\mathcal{F}^-_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{F}^-_{\mathcal{N}_{\mathcal{V}_2}}), \max(\mathcal{F}^+_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{F}^+_{\mathcal{N}_{\mathcal{V}_2}})\right]$.

Definition 2.8. [34] The union $\mathcal{N}_{\mathcal{V}}$ of two nV-sets $\mathcal{N}_{\mathcal{V}_1}$ and $\mathcal{N}_{\mathcal{V}_2}$ denoted by $\mathcal{N}_{\mathcal{V}} = \mathcal{N}_{\mathcal{V}_1} \cup \mathcal{N}_{\mathcal{V}_2}$ is nV-set with following conditions $\forall z_i \in \mathcal{Z}$ and $i = 1, 2, 3, ..., n$; $\mathcal{T}_{\mathcal{N}_{\mathcal{V}}}(z_i) = \left[\max(\mathcal{T}^-_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{T}^-_{\mathcal{N}_{\mathcal{V}_2}}), \max(\mathcal{T}^+_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{T}^+_{\mathcal{N}_{\mathcal{V}_2}})\right]$, $\mathcal{I}_{\mathcal{N}_{\mathcal{V}}}(z_i) = \left[\min(\mathcal{I}^-_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{I}^-_{\mathcal{N}_{\mathcal{V}_2}}), \min(\mathcal{I}^+_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{I}^+_{\mathcal{N}_{\mathcal{V}_2}})\right]$ and $\mathcal{F}_{\mathcal{N}_{\mathcal{V}}}(z_i) = \left[\min(\mathcal{F}^-_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{F}^-_{\mathcal{N}_{\mathcal{V}_2}}), \min(\mathcal{F}^+_{\mathcal{N}_{\mathcal{V}_1}}, \mathcal{F}^+_{\mathcal{N}_{\mathcal{V}_2}})\right]$.

Definition 2.9. [35] Let $\mathcal{E}$ be set of parameters for $\mathcal{Z}$ and $\Lambda \subset \mathcal{E}$. A neutrosophic vague soft set $(\mathcal{N}_{\mathcal{VS}}, \Lambda)$ on $\mathcal{Z}$ denoted by nVs-set can be given by $\mathcal{N}_{\mathcal{VS}} : \Lambda \to \mathcal{N}_{\mathcal{V}}(\mathcal{Z})$ where $\mathcal{N}_{\mathcal{V}}(\mathcal{Z})$ represents set of all nV-subsets of $\mathcal{Z}$.

Definition 2.10. [35] For two nVs-sets $(\mathcal{N}_{\mathcal{VS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VS}_2}\Delta)$, $(\mathcal{N}_{\mathcal{VS}_1}, \Lambda)$ is called nVs-subset of $(\mathcal{N}_{\mathcal{VS}_2}, \Delta)$ i.e. $(\mathcal{N}_{\mathcal{VS}_1}, \Lambda) \subseteq (\mathcal{N}_{\mathcal{VS}_2}\Delta)$ if following conditions hold: $\Lambda \subseteq \Delta$, $\mathcal{N}_{\mathcal{VS}_1}(\theta)$ is nV-subset of $\mathcal{N}_{\mathcal{VS}_2}(\theta)$ for all $\theta \in \Lambda$.

Definition 2.11. [35] Two nVs-sets $(\mathcal{N}_{\mathcal{VS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VS}_2}\Delta)$, are nVs-set equal if $(\mathcal{N}_{\mathcal{VS}_1}, \Lambda) \subseteq (\mathcal{N}_{\mathcal{VS}_2}, \Delta)$ and $(\mathcal{N}_{\mathcal{VS}_2}, \Delta) \subseteq (\mathcal{N}_{\mathcal{VS}_1}\Lambda)$.

Definition 2.12. [35] A nVs-set $(\mathcal{N}_{\mathcal{VS}}, \Lambda)$ is null nVs-set written as $\Phi_{\mathcal{N}_{\mathcal{VS}}}$ if following conditions holds for all values $\pi \in \mathcal{Z}$ and $\theta \in \Lambda$; $\mathcal{T}_{\mathcal{N}_{\mathcal{VS}(\theta)}}(\pi) = [0, 0]$, $\mathcal{I}_{\mathcal{N}_{\mathcal{VS}(\theta)}}(\pi) = [1, 1]$, and $\mathcal{F}_{\mathcal{N}_{\mathcal{VS}(\theta)}}(\pi) = [1, 1]$.

Definition 2.13. [35] A nVs-set $(\mathcal{N}_{\mathcal{VS}}, \Lambda)$ is absolute nVs-set written as $\Psi_{\mathcal{N}_{\mathcal{VS}}}$ if following conditions holds for all values $\pi \in \mathcal{Z}$ and $\theta \in \Lambda$; $\mathcal{T}_{\mathcal{N}_{\mathcal{VS}(\theta)}}(\pi) = [1, 1]$, $\mathcal{I}_{\mathcal{N}_{\mathcal{VS}(\theta)}}(\pi) = [0, 0]$, and $\mathcal{F}_{\mathcal{N}_{\mathcal{VS}(\theta)}}(\pi) = [0, 0]$.

Definition 2.14. [35] The compliment $(\mathcal{N}_{\mathcal{VS}}, \Lambda)^{\varsigma}$ of nVs-set $(\mathcal{N}_{\mathcal{VS}}, \Lambda)$ is given by $(\mathcal{N}_{\mathcal{VS}}, \Lambda)^{\varsigma} = (\mathcal{N}^{\varsigma}_{\mathcal{VS}}, \Lambda)$ where $\mathcal{N}^{\varsigma}_{\mathcal{VS}} : \Lambda \to \mathcal{N}_{\mathcal{V}}(\mathcal{Z})$ is defined as $\mathcal{N}^{\varsigma}_{\mathcal{VS}}(\pi) = \varsigma(\mathcal{N}_{\mathcal{VS}}(\pi)), \forall \pi \in \Lambda$ such that $\mathcal{N}_{\mathcal{V}}(\mathcal{Z})$ represents set of all nV-subsets of $\mathcal{Z}$ and ς is neutrosophic vague compliment.

Definition 2.15. [35] The intersection $(\mathcal{N}_{\mathcal{VS}}, \Upsilon)$ of two nVs-sets $(\mathcal{N}_{\mathcal{VS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VS}_2}, \Delta)$ denoted by $\mathcal{N}_{\mathcal{VS}} = \mathcal{N}_{\mathcal{VS}_1} \hat{\cap} \mathcal{N}_{\mathcal{VS}_2}$ where $\Upsilon = \Lambda \cup \Delta$ and $\forall \theta \in \Upsilon$, is given by

$$(\mathcal{N}_{\mathcal{VS}}, \Upsilon) = \left\{ \begin{array}{cc} \mathcal{N}_{\mathcal{VS}_1}(\theta) & ;\theta \in \Lambda - \Delta \\ \mathcal{N}_{\mathcal{VS}_2}(\theta) & ;\theta \in \Delta - \Lambda \\ \mathcal{N}_{\mathcal{VS}_1}(\theta) \hat{\cap} \mathcal{N}_{\mathcal{VS}_2}(\theta) & ;\theta \in \Lambda \cap \Delta \end{array} \right\},$$

where $\hat{\cap}$ is nV-set intersection.

Definition 2.16. [35] The union $(\mathcal{N}_{\mathcal{VS}}, \Upsilon)$ of two nVs-sets $(\mathcal{N}_{\mathcal{VS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VS}_2}, \Delta)$ denoted by $\mathcal{N}_{\mathcal{VS}} = \mathcal{N}_{\mathcal{VS}_1} \check{\cup} \mathcal{N}_{\mathcal{VS}_2}$ where $\Upsilon = \Lambda \cup \Delta$ and $\forall \theta \in \Upsilon$, is given by

$$(\mathcal{N}_{\mathcal{VS}}, \Upsilon) = \left\{ \begin{array}{cc} \mathcal{N}_{\mathcal{VS}_1}(\theta) & ;\theta \in \Lambda - \Delta \\ \mathcal{N}_{\mathcal{VS}_2}(\theta) & ;\theta \in \Delta - \Lambda \\ \mathcal{N}_{\mathcal{VS}_1}(\theta) \check{\cap} \mathcal{N}_{\mathcal{VS}_2}(\theta) & ;\theta \in \Lambda \cap \Delta \end{array} \right\},$$

where $\check{\cup}$ is nV-set union.

3. NEUTROSOPHIC VAGUE HYPERSOFT SET (nVHs-set)

Neutrosophic vague hypersoft set (nVHs-set) is introduced in this section. Some basic operations of (nVHs-set) are also discussed.

Definition 3.1. For a universal set $\mathcal{Z}$, let $\mathcal{E}$ be set of parameters and $\Lambda \subseteq \mathcal{E}$. The pair $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)$ is called neutrosophic vague hypersoft set (nVHs-set) over $\mathcal{Z}$ where $\mathcal{N}_{\mathcal{VHS}}$ is defined by $\mathcal{N}_{\mathcal{VHS}} : \Lambda \to NV(\mathcal{Z})$ such that $\Lambda = \Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$ with $\Lambda_i, i = 1, 2, 3, .., n$ are disjoint attribute-valued sets corresponding to distinct attributes $\varepsilon_i, i = 1, 2, 3, ..., n$ respectively and θ is a n-tuple element of Λ and $\mathcal{N}_{\mathcal{VHS}}(\theta)$ is an approximate element of nVHs-set over $\mathcal{Z}$.

Example 3.2. A company wants to supply antibacterial soap for cure of Covid-19 patient in a hospital. Let $\mathcal{Z} = \{z_1, z_2, ..., z_5\}$ be the universal set consisting of five kinds of antibacterial soap for cure of Covid-19 patient available in market. Let $\mathcal{E}$ be the set of parameters. Let Λ_i be the nonempty subset of $\mathcal{E}$ for each $i = 1, 2, 3$ represent multi attribute set corresponding to each element of $\mathcal{E}$ and $\Lambda = \Lambda_1 \times \Lambda_2 \times \Lambda_3$, where $\Lambda_1 = \{a_{11}\}, \Lambda_2 = \{b_{11}, b_{12}\}, \Lambda_3 = \{c_{11}\}$. Let $\Lambda = \{\theta_1, \theta_2, \theta_3\}$ i.e. we have three criteria for evaluation of material where θ_1 stands for ingredient of soap which triclosan, triclocarban and benzalkonium chloride, θ_2 stands for color of soap which is blue, green and white, and θ_3 stands for price which is low, medium and high. A mapping is defined as follows $\mathcal{N}_{\mathcal{VHS}} : \Lambda \to NV(\mathcal{Z})$. Consider

$$\mathcal{N}_{\mathcal{VHS}}(\theta_1) = \left\{ \begin{array}{l} z_1/ < [0.1, 0.3]; [0.2, 0.4]; [0.7, 0.9] >, z_2/ < [0.2, 0.5]; [0.3, 0.4]; [0.5, 0.8] >, \\ z_3/ < [0.2, 0.6]; [0.2, 0.4]; [0.4, 0.8] >, z_4/ < [0.1, 0.7]; [0.4, 0.5]; [0.3, 0.9] >, \\ z_5/ < [0.2, 0.4]; [0.4, 0.5]; [0.6, 0.8] > \end{array} \right\}$$

Table 1. nVHs-set $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)$

$\mathcal{Z}$	θ_1	θ_2	θ_3
z_1	$<[0.1,0.3];[0.2,0.4];[0.7,0.9]>$	$<[0.2,0.5];[0.3,0.5];[0.5,0.8]>$	$<[0.3,0.5];[0.1,0.3];[0.5,0.7]>$
z_2	$<[0.2,0.5];[0.3,0.4];[0.5,0.8]>$	$<[0.2,0.3];[0.2,0.4];[0.7,0.8]>$	$<[0.2,0.7];[0.3,0.4];[0.3,0.8]>$
z_3	$<[0.2,0.6];[0.2,0.4];[0.4,0.8]>$	$<[0.4,0.5];[0.2,0.6];[0.5,0.6]>$	$<[0.1,0.3];[0.5,0.8];[0.7,0.9]>$
z_4	$<[0.1,0.7];[0.4,0.5];[0.3,0.9]>$	$<[0.1,0.3];[0.1,0.5];[0.7,0.9]>$	$<[0.2,0.5];[0.3,0.7];[0.5,0.8]>$
z_5	$<[0.2,0.4];[0.4,0.5];[0.6,0.8]>$	$<[0.3,0.6];[0.4,0.7];[0.4,0.7]>$	$<[0.3,0.6];[0.2,0.3];[0.4,0.7]>$

$$\mathcal{N}_{\mathcal{VHS}}(\theta_2) = \left\{ \begin{array}{l} z_1/<[0.2,0.5];[0.3,0.5];[0.5,0.8]>, z_2/<[0.2,0.3];[0.2,0.4];[0.7,0.8]>, \\ z_3/<[0.4,0.5];[0.2,0.6];[0.5,0.6]>, z_4/<[0.1,0.3];[0.1,0.5];[0.7,0.9]>, \\ z_5/<[0.3,0.6];[0.4,0.7];[0.4,0.7]> \end{array} \right\}$$

$$\mathcal{N}_{\mathcal{VHS}}(\theta_3) = \left\{ \begin{array}{l} z_1/<[0.3,0.5];[0.1,0.3];[0.5,0.7]>, z_2/<[0.2,0.7];[0.3,0.4];[0.3,0.8]>, \\ z_3/<[0.1,0.3];[0.5,0.8];[0.7,0.9]>, z_4/<[0.2,0.5];[0.3,0.7];[0.5,0.8]>, \\ z_5/<[0.3,0.6];[0.2,0.3];[0.4,0.7]> \end{array} \right\}$$

It can also be written as

$$(\mathcal{N}_{\mathcal{VHS}}, \Lambda) = \left\{ \begin{array}{l} \left(\theta_1, \left\{ \begin{array}{l} z_1/<[0.1,0.3];[0.2,0.4];[0.7,0.9]>, z_2/<[0.2,0.5];[0.3,0.4];[0.5,0.8]>, \\ z_3/<[0.2,0.6];[0.2,0.4];[0.4,0.8]>, z_4/<[0.1,0.7];[0.4,0.5];[0.3,0.9]>, \\ z_5/<[0.2,0.4];[0.4,0.5];[0.6,0.8]> \end{array} \right\} \right), \\ \left(\theta_2, \left\{ \begin{array}{l} z_1/<[0.2,0.5];[0.3,0.5];[0.5,0.8]>, z_2/<[0.2,0.3];[0.2,0.4];[0.7,0.8]>, \\ z_3/<[0.4,0.5];[0.2,0.6];[0.5,0.6]>, z_4/<[0.1,0.3];[0.1,0.5];[0.7,0.9]>, \\ z_5/<[0.3,0.6];[0.4,0.7];[0.4,0.7]> \end{array} \right\} \right), \\ \left(\theta_3, \left\{ \begin{array}{l} z_1/<[0.3,0.5];[0.1,0.3];[0.5,0.7]>, z_2/<[0.2,0.7];[0.3,0.4];[0.3,0.8]>, \\ z_3/<[0.1,0.3];[0.5,0.8];[0.7,0.9]>, z_4/<[0.2,0.5];[0.3,0.7];[0.5,0.8]>, \\ z_5/<[0.3,0.6];[0.2,0.3];[0.4,0.7]> \end{array} \right\} \right) \end{array} \right\}$$

nVHs-set $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)$ can also be represented in the form of table 1

Definition 3.3. For two nVHs-sets $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2}\Delta)$, $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ is called nVHs-subset of $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ i.e. $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda) \subseteq (\mathcal{N}_{\mathcal{VHS}_2}\Delta)$ if following conditions hold; $\Lambda \subseteq \Delta$ and $\mathcal{N}_{\mathcal{VHS}_1}(\theta)$ is nVs-subset of $\mathcal{N}_{\mathcal{VHS}_2}(\theta)$ for all $\theta \in \Lambda$.

Example 3.4. Consider Example 3.2 where $\Lambda = \{\theta_2, \theta_3\}$ for $\theta_i \in \Lambda_1 \times \Lambda_2 \times \Lambda_3, i = 2, 3$ and $\Delta = \{\theta_1, \theta_2, \theta_3\}$ for $\theta_i \in \Delta_1 \times \Delta_2 \times \Delta_3, i = 1, 2, 3$. Suppose $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ are two nVHs-sets of defined as follow and demonstrated in table 2 and table 3:

$$(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda) =$$

TABLE 2. nVHs-set $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$

$\mathcal{Z}$	θ_2	θ_3
z_1	$< [0.2, 0.4]; [0.3, 0.6]; [0.6, 0.8] >$	$< [0.2, 0.4]; [0.2, 0.5]; [0.6, 0.8] >$
z_2	$< [0.1, 0.3]; [0.2, 0.5]; [0.7, 0.9] >$	$< [0.2, 0.6]; [0.3, 0.5]; [0.4, 0.8] >$
z_3	$< [0.3, 0.5]; [0.2, 0.7]; [0.5, 0.7] >$	$< [0.1, 0.2]; [0.6, 0.9]; [0.8, 0.9] >$
z_4	$< [0.1, 0.3]; [0.1, 0.7]; [0.7, 0.9] >$	$< [0.2, 0.4]; [0.4, 0.8]; [0.6, 0.8] >$
z_5	$< [0.2, 0.5]; [0.4, 0.8]; [0.5, 0.8] >$	$< [0.2, 0.5]; [0.4, 0.7]; [0.5, 0.8] >$

TABLE 3. nVHs-set $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$

$\mathcal{Z}$	θ_1	θ_2	θ_3
z_1	$< [0.1, 0.3]; [0.2, 0.4]; [0.7, 0.9] >$	$< [0.2, 0.5]; [0.3, 0.5]; [0.5, 0.8] >$	$< [0.3, 0.5]; [0.1, 0.3]; [0.5, 0.7] >$
z_2	$< [0.2, 0.5]; [0.3, 0.4]; [0.5, 0.8] >$	$< [0.2, 0.3]; [0.2, 0.4]; [0.7, 0.8] >$	$< [0.2, 0.7]; [0.3, 0.4]; [0.3, 0.8] >$
z_3	$< [0.2, 0.6]; [0.2, 0.4]; [0.4, 0.8] >$	$< [0.4, 0.5]; [0.2, 0.6]; [0.5, 0.6] >$	$< [0.1, 0.3]; [0.5, 0.8]; [0.7, 0.9] >$
z_4	$< [0.1, 0.7]; [0.4, 0.5]; [0.3, 0.9] >$	$< [0.1, 0.3]; [0.1, 0.5]; [0.7, 0.9] >$	$< [0.2, 0.5]; [0.3, 0.7]; [0.5, 0.8] >$
z_5	$< [0.2, 0.4]; [0.4, 0.5]; [0.6, 0.8] >$	$< [0.3, 0.6]; [0.4, 0.7]; [0.4, 0.7] >$	$< [0.3, 0.6]; [0.2, 0.3]; [0.4, 0.7] >$

$$\left\{ \begin{array}{l} \left(\theta_2, \left\{ \begin{array}{l} z_1/<[0.2,0.4];[0.3,0.6];[0.6,0.8]>, z_2/<[0.1,0.3];[0.2,0.5];[0.7,0.9]>, \\ z_3/<[0.3,0.5];[0.2,0.7];[0.5,0.7]>, z_4/<[0.1,0.3];[0.1,0.7];[0.7,0.9]>, \\ z_5/<[0.2,0.5];[0.4,0.8];[0.5,0.8]> \end{array} \right\}\right), \\ \left(\theta_3, \left\{ \begin{array}{l} z_1/<[0.2,0.4];[0.2,0.5];[0.6,0.8]>, z_2/<[0.2,0.6];[0.3,0.5];[0.4,0.8]>, \\ z_3/<[0.1,0.2];[0.6,0.9];[0.8,0.9]>, z_4/<[0.2,0.4];[0.4,0.8];[0.6,0.8]>, \\ z_5/<[0.2,0.5];[0.4,0.7];[0.5,0.8]> \end{array} \right\}\right) \end{array} \right\}$$

$$(\mathcal{N}_{\mathcal{VHS}_2}, \Delta) =$$

$$\left\{ \begin{array}{l} \left(\theta_1, \left\{ \begin{array}{l} z_1/<[0.1,0.3];[0.2,0.4];[0.7,0.9]>, z_2/<[0.2,0.5];[0.3,0.4];[0.5,0.8]>, \\ z_3/<[0.2,0.6];[0.2,0.4];[0.4,0.8]>, z_4/<[0.1,0.7];[0.4,0.5];[0.3,0.9]>, \\ z_5/<[0.2,0.4];[0.4,0.5];[0.6,0.8]> \end{array} \right\}\right), \\ \left(\theta_2, \left\{ \begin{array}{l} z_1/<[0.2,0.5];[0.3,0.5];[0.5,0.8]>, z_2/<[0.2,0.3];[0.2,0.4];[0.7,0.8]>, \\ z_3/<[0.4,0.5];[0.2,0.6];[0.5,0.6]>, z_4/<[0.1,0.3];[0.1,0.5];[0.7,0.9]>, \\ z_5/<[0.3,0.6];[0.4,0.7];[0.4,0.7]> \end{array} \right\}\right), \\ \left(\theta_3, \left\{ \begin{array}{l} z_1/<[0.3,0.5];[0.1,0.3];[0.5,0.7]>, z_2/<[0.2,0.7];[0.3,0.4];[0.3,0.8]>, \\ z_3/<[0.1,0.3];[0.5,0.8];[0.7,0.9]>, z_4/<[0.2,0.5];[0.3,0.7];[0.5,0.8]>, \\ z_5/<[0.3,0.6];[0.2,0.3];[0.4,0.7]> \end{array} \right\}\right) \end{array} \right\}$$

It can easily be seen that nVHs-set $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda) \subseteq$ nVHs-set $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ where as $\Lambda \subseteq \Delta$.

Definition 3.5. Two nVHs-sets $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2}\Delta)$, are nVHs-set equal if $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda) \subseteq (\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ and $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta) \subseteq (\mathcal{N}_{\mathcal{VHS}_1}\Lambda)$

TABLE 4. $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)^{\varsigma}$

$\mathcal{Z}$	θ_1	θ_2	θ_3
z_1	$<[0.7,0.9];[0.6,0.8];[0.1,0.3]>$	$<[0.5,0.8];[0.5,0.8];[0.2,0.5]>$	$<[0.5,0.7];[0.7,0.9];[0.3,0.5]>$
z_2	$<[0.5,0.8];[0.6,0.7];[0.2,0.5]>$	$<[0.7,0.8];[0.6,0.8];[0.2,0.3]>$	$<[0.3,0.8];[0.6,0.7];[0.2,0.7]>$
z_3	$<[0.4,0.8];[0.6,0.8];[0.2,0.6]>$	$<[0.5,0.6];[0.4,0.8];[0.4,0.5]>$	$<[0.7,0.9];[0.2,0.5];[0.1,0.3]>$
z_4	$<[0.3,0.9];[0.5,0.6];[0.1,0.7]>$	$<[0.7,0.9];[0.5,0.9];[0.1,0.3]>$	$<[0.5,0.8];[0.3,0.7];[0.2,0.5]>$
z_5	$<[0.6,0.8];[0.5,0.6];[0.2,0.4]>$	$<[0.4,0.7];[0.3,0.6];[0.3,0.6]>$	$<[0.4,0.7];[0.7,0.8];[0.3,0.6]>$

Definition 3.6. A nVHs-set $(\mathcal{N}_{\mathcal{VS}}, \Lambda)$ is null nVHs-set written as $\Phi_{\mathcal{N}_{\mathcal{VHS}}}$ if following conditions holds for all values $\pi \in \mathcal{Z}$ and $\theta \in \Lambda$; $\mathcal{T}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(\pi) = [0,0]$, $\mathcal{I}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(\pi) = [1,1]$, and $\mathcal{F}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(\pi) = [1,1]$.

Definition 3.7. A nVHs-set $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)$ is absolute nVHs-set written as $\Psi_{\mathcal{N}_{\mathcal{VHS}}}$ if following conditions holds for all values $\pi \in \mathcal{Z}$ and $\theta \in \Lambda$; $\mathcal{T}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(\pi) = [1,1]$, $\mathcal{I}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(\pi) = [0,0]$, and $\mathcal{F}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(\pi) = [0,0]$.

Definition 3.8. The compliment $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)^{\varsigma}$ of nVs-set $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)$ is given by $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)^{\varsigma} = (\mathcal{N}^{\varsigma}_{\mathcal{VHS}}, \Lambda)$ where $\mathcal{N}^{\varsigma}_{\mathcal{VHS}} : \Lambda \to \mathcal{N}_{\mathcal{V}}(\mathcal{Z})$ is defined as $\mathcal{N}^{\varsigma}_{\mathcal{VHS}}(\pi) = \varsigma(\mathcal{N}_{\mathcal{VHS}}(\pi)), \forall \pi \in \Lambda$ such that $\mathcal{N}_{\mathcal{V}}(\mathcal{Z})$ represents set of all nVHs-subsets of $\mathcal{Z}$ and ς is neutrosophic vague compliment.

Example 3.9. Consider $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)$ is nVHs-set defined as in Example 3.2 where $\Lambda = \{\theta_1, \theta_2, \theta_3\}$ for $\theta_i \in \Lambda_1 \times \Lambda_2 \times \Lambda_3, i = 1,2,3$. The compliment $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)^{\varsigma}$ of nVHs-set $(\mathcal{N}_{\mathcal{VHS}}, \Lambda)$ is demonstrated in table 4 and given by:

$$(\mathcal{N}_{\mathcal{VHS}}, \Lambda)^{\varsigma} = \left\{ \begin{array}{l} \left(\theta_1, \left\{ \begin{array}{l} z_1/<[0.7,0.9];[0.6,0.8];[0.1,0.3]>, z_2/<[0.5,0.8];[0.6,0.7];[0.2,0.5]>, \\ z_3/<[0.4,0.8];[0.6,0.8];[0.2,0.6]>, z_4/<[0.3,0.9];[0.5,0.6];[0.1,0.7]>, \\ z_5/<[0.6,0.8];[0.5,0.6];[0.2,0.4]> \end{array} \right\} \right), \\ \left(\theta_2, \left\{ \begin{array}{l} z_1/<[0.5,0.8];[0.5,0.8];[0.2,0.5]>, z_2/<[0.7,0.8];[0.6,0.8];[0.2,0.3]>, \\ z_3/<[0.5,0.6];[0.4,0.8];[0.4,0.5]>, z_4/<[0.7,0.9];[0.5,0.9];[0.1,0.3]>, \\ z_5/<[0.4,0.7];[0.3,0.6];[0.3,0.6]> \end{array} \right\} \right), \\ \left(\theta_3, \left\{ \begin{array}{l} z_1/<[0.5,0.7];[0.7,0.9];[0.3,0.5]>, z_2/<[0.3,0.8];[0.6,0.7];[0.2,0.7]>, \\ z_3/<[0.7,0.9];[0.2,0.5];[0.1,0.3]>, z_4/<[0.5,0.8];[0.3,0.7];[0.2,0.5]>, \\ z_5/<[0.4,0.7];[0.7,0.8];[0.3,0.6]> \end{array} \right\} \right) \end{array} \right\}.$$

Definition 3.10. The intersection $(\mathcal{N}_{\mathcal{VHS}}, \Upsilon)$ of two nVHs-sets $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ denoted by $\mathcal{N}_{\mathcal{VHS}} = \mathcal{N}_{\mathcal{VHS}_1} \hat{\cap} \mathcal{N}_{\mathcal{VHS}_2}$ where $\Upsilon = \Lambda \cup \Delta$ and $\forall \theta \in \Upsilon$, is given by

$$(\mathcal{N}_{\mathcal{VHS}}, \Upsilon) = \left\{ \begin{array}{cc} \mathcal{N}_{\mathcal{VHS}_1}(\theta) & , if \theta \in \Lambda - \Delta \\ \mathcal{N}_{\mathcal{VHS}_2}(\theta) & , if \theta \in \Delta - \Lambda \\ \mathcal{N}_{\mathcal{VHS}_1}(\theta) \hat{\cap} \mathcal{N}_{\mathcal{VHS}_2}(\theta) & , if \theta \in \Lambda \cap \Delta, \end{array} \right\},$$

where $\hat{\cap}$ is nV-set intersection.

TABLE 5. nVHs-set $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$

$\mathcal{Z}$	θ_1	θ_2
z_1	$<[0.1,0.3];[0.2,0.4];[0.7,0.9]>$	$<[0.3,0.6];[0.2,0.5];[0.4,0.7]>$
z_2	$<[0.2,0.5];[0.3,0.4];[0.5,0.8]>$	$<[0.1,0.3];[0.2,0.5];[0.7,0.9]>$
z_3	$<[0.2,0.6];[0.2,0.4];[0.4,0.8]>$	$<[0.3,0.6];[0.2,0.4];[0.4,0.7]>$
z_4	$<[0.1,0.7];[0.4,0.5];[0.3,0.9]>$	$<[0.2,0.3];[0.3,0.5];[0.7,0.8]>$
z_5	$<[0.2,0.4];[0.4,0.5];[0.6,0.8]>$	$<[0.3,0.4];[0.5,0.7];[0.6,0.7]>$

TABLE 6. nVHs-set $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$

$\mathcal{Z}$	θ_2	θ_3
z_1	$<[0.2,0.5];[0.3,0.5];[0.5,0.8]>$	$<[0.3,0.5];[0.1,0.3];[0.5,0.7]>$
z_2	$<[0.2,0.3];[0.2,0.4];[0.7,0.8]>$	$<[0.2,0.7];[0.3,0.4];[0.3,0.8]>$
z_3	$<[0.4,0.5];[0.2,0.6];[0.5,0.6]>$	$<[0.1,0.3];[0.5,0.8];[0.7,0.9]>$
z_4	$<[0.1,0.3];[0.1,0.5];[0.7,0.9]>$	$<[0.2,0.5];[0.3,0.7];[0.5,0.8]>$
z_5	$<[0.3,0.6];[0.4,0.7];[0.4,0.7]>$	$<[0.3,0.6];[0.2,0.3];[0.4,0.7]>$

Example 3.11. Let $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ are two nVHs-sets where $\Lambda = \{\theta_1, \theta_2\}$ for $\theta_i \in \Lambda_1 \times \Lambda_2 \times \Lambda_3, i = 1, 2$ and $\Delta = \{\theta_2, \theta_3\}$ for $\theta_i \in \Delta_1 \times \Delta_2 \times \Delta_3, i = 2, 3$, as discussed in Example 3.2 and demonstrated in table 5 and table 6 and given as following:

$$(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda) = \left\{ \begin{array}{l} \left(\theta_1, \left\{ \begin{array}{l} z_1/<[0.1,0.3];[0.2,0.4];[0.7,0.9]>, z_2/<[0.2,0.5];[0.3,0.4];[0.5,0.8]>, \\ z_3/<[0.2,0.6];[0.2,0.4];[0.4,0.8]>, z_4/<[0.1,0.7];[0.4,0.5];[0.3,0.9]>, \\ z_5/<[0.2,0.4];[0.4,0.5];[0.6,0.8]> \end{array} \right\} \right), \\ \left(\theta_2, \left\{ \begin{array}{l} z_1/<[0.3,0.6];[0.2,0.5];[0.4,0.7]>, z_2/<[0.1,0.3];[0.2,0.5];[0.7,0.9]>, \\ z_3/<[0.3,0.6];[0.2,0.4];[0.4,0.7]>, z_4/<[0.2,0.3];[0.3,0.5];[0.7,0.8]>, \\ z_5/<[0.3,0.4];[0.5,0.7];[0.6,0.7]> \end{array} \right\} \right), \end{array} \right\}.$$

$$(\mathcal{N}_{\mathcal{VHS}_2}, \Delta) = \left\{ \begin{array}{l} \left(\theta_2, \left\{ \begin{array}{l} z_1/<[0.2,0.5];[0.3,0.5];[0.5,0.8]>, z_2/<[0.2,0.3];[0.2,0.4];[0.7,0.8]>, \\ z_3/<[0.4,0.5];[0.2,0.6];[0.5,0.6]>, z_4/<[0.1,0.3];[0.1,0.5];[0.7,0.9]>, \\ z_5/<[0.3,0.6];[0.4,0.7];[0.4,0.7]> \end{array} \right\} \right), \\ \left(\theta_3, \left\{ \begin{array}{l} z_1/<[0.3,0.5];[0.1,0.3];[0.5,0.7]>, z_2/<[0.2,0.7];[0.3,0.4];[0.3,0.8]>, \\ z_3/<[0.1,0.3];[0.5,0.8];[0.7,0.9]>, z_4/<[0.2,0.5];[0.3,0.7];[0.5,0.8]>, \\ z_5/<[0.3,0.6];[0.2,0.3];[0.4,0.7]> \end{array} \right\} \right) \end{array} \right\}.$$

The intersection $(\mathcal{N}_{\mathcal{VHS}}, \Upsilon)$ of two nVHs-sets $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ represented by $\mathcal{N}_{\mathcal{VHS}} = \mathcal{N}_{\mathcal{VHS}_1} \hat{\cap} \mathcal{N}_{\mathcal{VHS}_2}$ where $\Upsilon = \Lambda \cup \Delta$ and $\forall \theta \in \Upsilon$, is demonstrated in table 7 and given by $(\mathcal{N}_{\mathcal{VHS}}, \Upsilon) = (\mathcal{N}_{\mathcal{VHS}_1}, \Lambda) \hat{\cap} (\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$

$$(\mathcal{N}_{\mathcal{VHS}}, \Upsilon) = \left\{ \begin{array}{l} \left(\theta_1, \left\{ \begin{array}{l} z_1/<[0.1,0.3];[0.2,0.4];[0.7,0.9]>, z_2/<[0.2,0.5];[0.3,0.4];[0.5,0.8]>, \\ z_3/<[0.2,0.6];[0.2,0.4];[0.4,0.8]>, z_4/<[0.1,0.7];[0.4,0.5];[0.3,0.9]>, \\ z_5/<[0.2,0.4];[0.4,0.5];[0.6,0.8]> \end{array} \right\} \right), \\ \left(\theta_2, \left\{ \begin{array}{l} z_1/<[0.2,0.5];[0.3,0.5];[0.5,0.8]>, z_2/<[0.1,0.3];[0.2,0.5];[0.7,0.9]>, \\ z_3/<[0.3,0.5];[0.2,0.6];[0.5,0.7]>, z_4/<[0.1,0.3];[0.3,0.5];[0.7,0.9]>, \\ z_5/<[0.3,0.4];[0.5,0.7];[0.6,0.7]> \end{array} \right\} \right), \\ \left(\theta_3, \left\{ \begin{array}{l} z_1/<[0.3,0.5];[0.1,0.3];[0.5,0.7]>, z_2/<[0.2,0.7];[0.3,0.4];[0.3,0.8]>, \\ z_3/<[0.1,0.3];[0.5,0.8];[0.7,0.9]>, z_4/<[0.2,0.5];[0.3,0.7];[0.5,0.8]>, \\ z_5/<[0.3,0.6];[0.2,0.3];[0.4,0.7]> \end{array} \right\} \right) \end{array} \right\}$$

TABLE 7. $(\mathcal{N}_{\mathcal{VHS}}, \Upsilon) = (\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$

$\mathcal{Z}$	θ_1	θ_2	θ_3
z_1	$<[0.1, 0.3]; [0.2, 0.4]; [0.7, 0.9]>$	$<[0.2, 0.5]; [0.3, 0.5]; [0.5, 0.8]>$	$<[0.3, 0.5]; [0.1, 0.3]; [0.5, 0.7]>$
z_2	$<[0.2, 0.5]; [0.3, 0.4]; [0.5, 0.8]>$	$<[0.1, 0.3]; [0.2, 0.5]; [0.7, 0.9]>$	$<[0.2, 0.7]; [0.3, 0.4]; [0.3, 0.8]>$
z_3	$<[0.2, 0.6]; [0.2, 0.4]; [0.4, 0.8]>$	$<[0.3, 0.5]; [0.2, 0.6]; [0.5, 0.7]>$	$<[0.1, 0.3]; [0.5, 0.8]; [0.7, 0.9]>$
z_4	$<[0.1, 0.7]; [0.4, 0.5]; [0.3, 0.9]>$	$<[0.1, 0.3]; [0.3, 0.5]; [0.7, 0.9]>$	$<[0.2, 0.5]; [0.3, 0.7]; [0.5, 0.8]>$
z_5	$<[0.2, 0.4]; [0.4, 0.5]; [0.6, 0.8]>$	$<[0.3, 0.4]; [0.5, 0.7]; [0.6, 0.7]>$	$<[0.3, 0.6]; [0.2, 0.3]; [0.4, 0.7]>$

TABLE 8. nVHs-set $(\mathcal{N}_{\mathcal{VHS}}, \Upsilon)$

$\mathcal{Z}$	θ_1	θ_2	θ_3
z_1	$<[0.1, 0.3]; [0.2, 0.4]; [0.7, 0.9]>$	$<[0.2, 0.5]; [0.3, 0.5]; [0.5, 0.8]>$	$<[0.3, 0.5]; [0.1, 0.3]; [0.5, 0.7]>$
z_2	$<[0.2, 0.5]; [0.3, 0.4]; [0.5, 0.8]>$	$<[0.1, 0.3]; [0.2, 0.5]; [0.7, 0.9]>$	$<[0.2, 0.7]; [0.3, 0.4]; [0.3, 0.8]>$
z_3	$<[0.2, 0.6]; [0.2, 0.4]; [0.4, 0.8]>$	$<[0.3, 0.5]; [0.2, 0.6]; [0.5, 0.7]>$	$<[0.1, 0.3]; [0.5, 0.8]; [0.7, 0.9]>$
z_4	$<[0.1, 0.7]; [0.4, 0.5]; [0.3, 0.9]>$	$<[0.1, 0.3]; [0.3, 0.5]; [0.7, 0.9]>$	$<[0.2, 0.5]; [0.3, 0.7]; [0.5, 0.8]>$
z_5	$<[0.2, 0.4]; [0.4, 0.5]; [0.6, 0.8]>$	$<[0.3, 0.4]; [0.5, 0.7]; [0.6, 0.7]>$	$<[0.3, 0.6]; [0.2, 0.3]; [0.4, 0.7]>$

Definition 3.12. The union $(\mathcal{N}_{\mathcal{VHS}}, \Upsilon)$ of two nVHs-sets $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ denoted by $\mathcal{N}_{\mathcal{VHS}} = \mathcal{N}_{\mathcal{VHS}_1}\breve{\cup}\mathcal{N}_{\mathcal{VHS}_2}$ where $\Upsilon = \Lambda \cup \Delta$ and $\forall \theta \in \Upsilon$, is given by

$$(\mathcal{N}_{\mathcal{VHS}}, \Upsilon) = \left\{ \begin{array}{cc} \mathcal{N}_{\mathcal{VHS}_1}(\theta) & , if \theta \in \Lambda - \Delta \\ \mathcal{N}_{\mathcal{VHS}_2}(\theta) & , if \theta \in \Delta - \Lambda \\ \mathcal{N}_{\mathcal{VHS}_1}(\theta)\breve{\cap}\mathcal{N}_{\mathcal{VHS}_2}(\theta) & , if \theta \in \Lambda \cap \Delta, \end{array} \right\},$$

where $\breve{\cup}$ is nV-set union.

Example 3.13. Let $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ are two nVHs-sets as discussed in example 3.2 and defined in example 3.11
The union $(\mathcal{N}_{\mathcal{VHS}}, \Upsilon)$ of $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ represented by $\mathcal{N}_{\mathcal{VHS}} = \mathcal{N}_{\mathcal{VHS}_1}\breve{\cup}\mathcal{N}_{\mathcal{VHS}_2}$ where $\Upsilon = \Lambda \cup \Delta$ and $\forall \theta \in \Upsilon$, is demonstrated in table 8 and given by $(\mathcal{N}_{\mathcal{VHS}}, \Upsilon) = (\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_2}, \Delta) =$

$$(\mathcal{N}_{\mathcal{VHS}}, \Upsilon) = \left\{ \begin{array}{c} \left(\theta_1, \left\{ \begin{array}{c} z_1/<[0.1, 0.3]; [0.2, 0.4]; [0.7, 0.9]>, z_2/<[0.2, 0.5]; [0.3, 0.4]; [0.5, 0.8]>, \\ z_3/<[0.2, 0.6]; [0.2, 0.4]; [0.4, 0.8]>, z_4/<[0.1, 0.7]; [0.4, 0.5]; [0.3, 0.9]>, \\ z_5/<[0.2, 0.4]; [0.4, 0.5]; [0.6, 0.8]> \end{array} \right\}\right), \\ \left(\theta_2, \left\{ \begin{array}{c} z_1/<[0.3, 0.6]; [0.2, 0.5]; [0.4, 0.7]>, z_2/<[0.2, 0.3]; [0.2, 0.4]; [0.7, 0.8]>, \\ z_3/<[0.4, 0.6]; [0.2, 0.4]; [0.4, 0.6]>, z_4/<[0.2, 0.3]; [0.1, 0.5]; [0.7, 0.8]>, \\ z_5/<[0.3, 0.6]; [0.4, 0.7]; [0.4, 0.7]> \end{array} \right\}\right), \\ \left(\theta_3, \left\{ \begin{array}{c} z_1/<[0.3, 0.5]; [0.1, 0.3]; [0.5, 0.7]>, z_2/<[0.2, 0.7]; [0.3, 0.4]; [0.3, 0.8]>, \\ z_3/<[0.1, 0.3]; [0.5, 0.8]; [0.7, 0.9]>, z_4/<[0.2, 0.5]; [0.3, 0.7]; [0.5, 0.8]>, \\ z_5/<[0.3, 0.6]; [0.2, 0.3]; [0.4, 0.7]> \end{array} \right\}\right) \end{array} \right\}.$$

Proposition 3.14. *Let $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2}, \Delta)$ are two nVHs-sets over $\mathcal{Z}$, the following laws hold.*

(1) $(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_1}, \Lambda) = (\mathcal{N}_{\mathcal{VHS}_1}, \Lambda)$

(2) $(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)=(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)$
(3) $(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_2},\Delta)=(\mathcal{N}_{\mathcal{VHS}_2},\Delta)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)$
(4) $(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_2},\Delta)=(\mathcal{N}_{\mathcal{VHS}_2},\Delta)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)$

Proposition 3.15. *Let $(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)$, $(\mathcal{N}_{\mathcal{VHS}_2},\Delta)$ and $(\mathcal{N}_{\mathcal{VHS}_3},\Upsilon)$ are three nVHs-sets over $\mathcal{Z}$, the following laws hold.*

(1) $(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\hat{\cap}((\mathcal{N}_{\mathcal{VHS}_2},\Delta)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_1},\Upsilon))=((\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_2},\Delta))\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_1},\Upsilon)$,
(2) $(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\breve{\cup}((\mathcal{N}_{\mathcal{VHS}_2},\Delta)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_1},\Upsilon))=((\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_2},\Delta))\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_1},\Upsilon)$,
(3) $(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\hat{\cap}((\mathcal{N}_{\mathcal{VHS}_2},\Delta)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_3},\Upsilon))=$
$((\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_2},\Delta))\breve{\cup}((\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_3},\Upsilon))$,
(4) $(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\breve{\cup}((\mathcal{N}_{\mathcal{VHS}_2},\Delta)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_3},\Upsilon))=$
$((\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_2},\Delta))\hat{\cap}((\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_3},\Upsilon))$.

Proposition 3.16. *Let $(\mathcal{N}_{\mathcal{VHS}},\Lambda)$ be nVHs-set over $\mathcal{Z}$, the following laws hold.* 1) $(\mathcal{N}_{\mathcal{VHS}},\Lambda)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}},\Lambda)^{\varsigma}=(\Phi_{\mathcal{N}_{\mathcal{VHS}}},\Lambda)$ *where* $(\Phi_{\mathcal{N}_{\mathcal{VHS}}},\Lambda)$ *is null nVHs-set* 2) $(\mathcal{N}_{\mathcal{VHS}},\Lambda)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}},\Lambda)^{\varsigma}=(\Psi_{\mathcal{N}_{\mathcal{VHS}}},\Lambda)$ *where* $(\Psi_{\mathcal{N}_{\mathcal{VHS}}},\Lambda)$ *is called absolute nVHs-set*

Proposition 3.17. *Let $(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)$ and $(\mathcal{N}_{\mathcal{VHS}_2},\Delta)$ are two nVHs-sets over $\mathcal{Z}$, the following laws hold.*

$$1)\ ((\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_2},\Delta))^{\varsigma}=(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)^{\varsigma}\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_2},\Delta)^{\varsigma}$$

$$2)\ ((\mathcal{N}_{\mathcal{VHS}_1},\Lambda)\breve{\cup}(\mathcal{N}_{\mathcal{VHS}_2},\Delta))^{\varsigma}=(\mathcal{N}_{\mathcal{VHS}_1},\Lambda)^{\varsigma}\hat{\cap}(\mathcal{N}_{\mathcal{VHS}_2},\Delta)^{\varsigma}$$

4. Decision Making Technique based on nVHs-set

A decision making nVHs-set based problem is discussed and an approach is made to address and solve this problem but first of all concept of level soft set is discussed.

Definition 4.1. Let $\bar{L}=\{(\bar{\alpha},\bar{\beta},\bar{\gamma})\}$ where $\bar{\alpha},\bar{\beta},\bar{\gamma}\in\bar{I}$ and $\bar{I}$ is set of all close subintervals of $[0,1]$, $\bar{\alpha}=[\bar{\alpha_1},\bar{\alpha_2}]$, $\bar{\beta}=[\bar{\beta_1},\bar{\beta_2}]$ and $\bar{\gamma}=[\bar{\gamma_1},\bar{\gamma_2}]$, $0\le\bar{\alpha_2}+\bar{\beta_2}+\bar{\gamma_2}\le 2$. The relation $\gtrapprox_{\bar{L}}$ on set $\bar{L}$ is called partial ordering on $\bar{L}$ if it satisfies following conditions:
$\forall(\bar{\alpha},\bar{\beta},\bar{\gamma}),(\bar{\mu},\bar{\nu},\bar{\omega})\in\bar{L}$, $(\bar{\alpha},\bar{\beta},\bar{\gamma})\gtrapprox_{\bar{L}}(\bar{\mu},\bar{\nu},\bar{\omega})\in\bar{L}\Leftrightarrow\bar{\alpha}\ge\bar{\mu},\bar{\beta}\le\bar{\nu},\bar{\gamma}\le\bar{\omega}$, which means $[\bar{\alpha_1},\bar{\alpha_2}]\ge[\bar{\mu_1},\bar{\mu_2}]$ i.e. $\bar{\alpha_1}\ge\bar{\mu_1}$ and $\bar{\alpha_2}\ge\bar{\mu_2}$. Similarly $\bar{\beta_1}\le\bar{\nu_1},\bar{\beta_2}\le\bar{\nu_2}$ and $\bar{\gamma_1}\le\bar{\omega_1},\bar{\gamma_2}\le\bar{\omega_2}$

Definition 4.2. For a universal set $\mathcal{Z}$, let $\mathcal{E}$ be set of parameters and $\Lambda\subseteq\mathcal{E}$. $\digamma=(\mathcal{N}_{\mathcal{VHS}},\Lambda)$ be nVHs-set over $\mathcal{Z}$. $(\bar{\alpha},\bar{\beta},\bar{\gamma})-$level hypersoft set of $\digamma$ is crisp hypersoft set $\bar{L}(\digamma;\bar{\alpha},\bar{\beta},\bar{\gamma})=(\mathcal{N}_{\mathcal{VHS}_{(\bar{\alpha},\bar{\beta},\bar{\gamma})}},\Lambda)$, for $\bar{\alpha},\bar{\beta},\bar{\gamma}\in\bar{L}$ and is given by

$$\mathcal{N}_{\mathcal{VHS}_{(\bar{\alpha},\bar{\beta},\bar{\gamma})}}(\theta)=\left\{\mathcal{N}_{\mathcal{VHS}(\theta)}(z)\gtrapprox_{\bar{L}}(\bar{\alpha},\bar{\beta},\bar{\gamma})\forall z\in\mathcal{Z}\right\}$$

$$=\left\{\mathcal{T}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z)\ge\bar{\alpha},\mathcal{I}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z)\le\bar{\beta},\mathcal{F}_{\mathcal{N}_{\mathcal{VHS}_{(\theta)}}}(z)\le\bar{\gamma},\forall z\in\mathcal{Z},\theta\in\Lambda\right\}$$

The above definition was restated by replacing threshold parameter constant value triplets by function as thresholds on truth membership, indeterminacy and false membership values.

Definition 4.3. For a universal set $\mathcal{Z}$, let $\mathcal{E}$ be set of parameters and $\Lambda \subseteq \mathcal{E}$. $\mathcal{F} = (\mathcal{N}_{\mathcal{VHS}}, \Lambda)$ be nVHs-set over $\mathcal{Z}$. Let $\chi : \Lambda \to I \times I \times I$ be nVHs-set where $I = [0,1]$. On the basis of χ, the level hypersoft set of $\mathcal{F}$ is a crisp hypersoft set $\bar{L}(\mathcal{F}, \chi) = (\mathcal{N}_{\mathcal{VHS}}, \Lambda)$ given by

$$\mathcal{N}_{\mathcal{VHS}_\chi}(\theta) = \bar{L}(\mathcal{N}_{\mathcal{VHS}}(\theta); \chi(\theta))$$
$$= \left\{\mathcal{N}_{\mathcal{VHS}(\theta)}(z) \gtrapprox_{\bar{L}} \chi(\theta) \forall z \in \mathcal{Z}\right\}$$
$$= \left\{\mathcal{T}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z) \geq \mathcal{T}_\chi(\theta), \mathcal{I}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z) \leq \mathcal{I}_\chi(\theta), \mathcal{F}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z) \leq \mathcal{F}_\chi(\theta), \forall z \in \mathcal{Z}, \theta \in \Lambda\right\}$$

Consider the following example

Example 4.4. For a universal set $\mathcal{Z}$, let $\mathcal{E}$ be set of parameters and $\Lambda \subseteq \mathcal{E}$. $\mathcal{F} = (\mathcal{N}_{\mathcal{VHS}}, \Lambda)$ be nVHs-set over $\mathcal{Z}$. Let $avg_{\mathcal{F}} : \Lambda \to I \times I \times I$ be nV-set where $I = [0,1]$ and is given by

$$\mathcal{T}^L_{avg_{\mathcal{F}}}(\theta) = \frac{1}{|\mathcal{Z}|}\sum_{z\in\mathcal{Z}} \mathcal{T}^L_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z), \mathcal{T}^R_{avg_{\mathcal{F}}}(\theta) = \frac{1}{|\mathcal{Z}|}\sum_{z\in\mathcal{Z}} \mathcal{T}^R_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z)$$
$$\mathcal{I}^L_{avg_{\mathcal{F}}}(\theta) = \frac{1}{|\mathcal{Z}|}\sum_{z\in\mathcal{Z}} \mathcal{I}^L_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z), \mathcal{I}^R_{avg_{\mathcal{F}}}(\theta) = \frac{1}{|\mathcal{Z}|}\sum_{z\in\mathcal{Z}} \mathcal{I}^R_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z)$$
$$\mathcal{F}^L_{avg_{\mathcal{F}}}(\theta) = \frac{1}{|\mathcal{Z}|}\sum_{z\in\mathcal{Z}} \mathcal{F}^L_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z), \mathcal{F}^R_{avg_{\mathcal{F}}}(\theta) = \frac{1}{|\mathcal{Z}|}\sum_{z\in\mathcal{Z}} \mathcal{F}^R_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z)$$

Here nV-set $avg_{\mathcal{F}}$ is known as $avg-$threshold of nVHs-set $\mathcal{F}$. $\bar{L}(\mathcal{F}; avg\mathcal{F}) = (\mathcal{N}_{\mathcal{VHS}_{avg\mathcal{F}}}, \Lambda)$ is called $avg-$level hypersoft set of $\mathcal{F}$ and can be given as

$$\mathcal{N}_{\mathcal{VHS}_{avg\mathcal{F}}}(\theta) = \bar{L}(\mathcal{N}_{\mathcal{VHS}}(\theta); avg_{\mathcal{F}}(\theta)) = \{\mathcal{N}_{\mathcal{VHS}\theta}(z) \gtrapprox_{\bar{L}} avg_{\mathcal{F}}(\theta)\}$$
$$= \left\{\mathcal{T}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z) \geq \mathcal{T}_{avg_{\mathcal{F}}}(\theta), \mathcal{I}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z) \leq \mathcal{I}_{avg_{\mathcal{F}}}(\theta), \mathcal{F}_{\mathcal{N}_{\mathcal{VHS}(\theta)}}(z) \leq \mathcal{F}_{avg_{\mathcal{F}}}(\theta), z \in \mathcal{Z}, \theta \in \Lambda\right\}$$

Example 4.5. An individual wants to buy a farmhouse from a real estate agent. He can construct a nVHs-set $\mathcal{F} = (\mathfrak{F}, \Lambda)$ according to his preference which describes characteristics of farmhouse. Let $\mathcal{Z} = \{z_1, z_2, ..., z_5\}$ be the universal set consisting of five farmhouses under consider-
ation. Let $\mathcal{E} = \{covered\,area = \theta_1, beautiful = \theta_2, cheap = \theta_3, location = \theta_4, altitude = \theta_5\}$ be the set of parameters. Let Λ_i be the nonempty subset of $\mathcal{E}$ for each $i = 1, 2, 3$ represent multi attribute set corresponding to each element of $\mathcal{E}$ and $\Lambda = \Lambda_1 \times \Lambda_2 \times \Lambda_3$, where $\Lambda_1 = \{a_{11}, a_{12}\}, \Lambda_2 = \{b_{11}\}, \Lambda_3 = \{c_{11}\}$. Let $\Lambda = \{\theta_1, \theta_2, \theta_3\}$ i.e. we have three criteria for evaluation where θ_1 stands for price which is low, high, very high, θ_2 stands for covered area which is less than 1 sq. mile, between 1 sq. mile to 5 sq. mile, more than 5 sq. miles and θ_3 stands for location which is sea shore, hilly area, desert.

TABLE 9. $(\mathfrak{F}, \Lambda)$

$\mathcal{Z}$	θ_1	θ_2	θ_3
z_1	$<[0.2,0.3];[0.2,0.4];[0.8,0.9]>$	$<[0.2,0.5];[0.2,0.5];[0.5,0.8]>$	$<[0.2,0.5];[0.1,0.3];[0.5,0.8]>$
z_2	$<[0.3,0.5];[0.3,0.4];[0.5,0.7]>$	$<[0.1,0.3];[0.2,0.4];[0.7,0.9]>$	$<[0.2,0.4];[0.3,0.4];[0.6,0.8]>$
z_3	$<[0.2,0.6];[0.2,0.4];[0.4,0.8]>$	$<[0.4,0.5];[0.2,0.6];[0.5,0.6]>$	$<[0.1,0.3];[0.4,0.8];[0.7,0.9]>$
z_4	$<[0.1,0.7];[0.4,0.6];[0.3,0.9]>$	$<[0.2,0.4];[0.1,0.5];[0.6,0.8]>$	$<[0.2,0.5];[0.3,0.7];[0.5,0.8]>$
z_5	$<[0.1,0.4];[0.4,0.5];[0.6,0.9]>$	$<[0.3,0.6];[0.3,0.7];[0.4,0.7]>$	$<[0.2,0.6];[0.2,0.3];[0.4,0.8]>$

Consider

$$\mathfrak{F}(\theta_1) = \left\{ \begin{array}{l} z_1/<[0.2,0.3];[0.2,0.4];[0.8,0.9]>, z_2/<[0.3,0.5];[0.3,0.4];[0.5,0.7]>, \\ z_3/<[0.2,0.6];[0.2,0.4];[0.4,0.8]>, z_4/<[0.1,0.7];[0.4,0.6];[0.3,0.9]>, \\ z_5/<[0.1,0.4];[0.4,0.5];[0.6,0.9]> \end{array} \right\}$$

$$\mathfrak{F}(\theta_2) = \left\{ \begin{array}{l} z_1/<[0.2,0.5];[0.2,0.5];[0.5,0.8]>, z_2/<[0.1,0.3];[0.2,0.4];[0.7,0.9]>, \\ z_3/<[0.4,0.5];[0.2,0.6];[0.5,0.6]>, z_4/<[0.2,0.4];[0.1,0.5];[0.6,0.8]>, \\ z_5/<[0.3,0.6];[0.3,0.7];[0.4,0.7]> \end{array} \right\}$$

$$\mathfrak{F}(\theta_3) = \left\{ \begin{array}{l} z_1/<[0.2,0.5];[0.1,0.3];[0.5,0.8]>, z_2/<[0.2,0.4];[0.3,0.4];[0.6,0.8]>, \\ z_3/<[0.1,0.3];[0.4,0.8];[0.7,0.9]>, z_4/<[0.2,0.5];[0.3,0.7];[0.5,0.8]>, \\ z_5/<[0.2,0.6];[0.2,0.3];[0.4,0.8]> \end{array} \right\}$$

It can also be written as

$$(\mathfrak{F}, \Lambda) = \left\{ \begin{array}{l} \left(\theta_1, \left\{ \begin{array}{l} z_1/<[0.2,0.3];[0.2,0.4];[0.8,0.9]>, z_2/<[0.3,0.5];[0.3,0.4];[0.5,0.7]>, \\ z_3/<[0.2,0.6];[0.2,0.4];[0.4,0.8]>, z_4/<[0.1,0.7];[0.4,0.6];[0.3,0.9]>, \\ z_5/<[0.1,0.4];[0.4,0.5];[0.6,0.9]> \end{array} \right\}\right), \\ \left(\theta_2, \left\{ \begin{array}{l} z_1/<[0.2,0.5];[0.2,0.5];[0.5,0.8]>, z_2/<[0.1,0.3];[0.2,0.4];[0.7,0.9]>, \\ z_3/<[0.4,0.5];[0.2,0.6];[0.5,0.6]>, z_4/<[0.2,0.4];[0.1,0.5];[0.6,0.8]>, \\ z_5/<[0.3,0.6];[0.3,0.7];[0.4,0.7]> \end{array} \right\}\right), \\ \left(\theta_3, \left\{ \begin{array}{l} z_1/<[0.2,0.5];[0.1,0.3];[0.5,0.8]>, z_2/<[0.2,0.4];[0.3,0.4];[0.6,0.8]>, \\ z_3/<[0.1,0.3];[0.4,0.8];[0.7,0.9]>, z_4/<[0.2,0.5];[0.3,0.7];[0.5,0.8]>, \\ z_5/<[0.2,0.6];[0.2,0.3];[0.4,0.8]> \end{array} \right\}\right) \end{array} \right\}.$$

The nVHs-set $(\mathfrak{F}, \Lambda)$ can also be represented in the form of table 9 $avg-$threshold of $\digamma = (\mathfrak{F}, \Lambda)$ can easily be calculated as:

$$avg_(\mathfrak{F}, \Lambda) = \left\{ \begin{array}{l} <[0.18,0.50];[0.30,0.46];[0.50,0.72]\backslash\, \theta_1>, \\ <[0.24,0.46];[0.20,0.67];[0.54,0.76]\backslash\, \theta_2>, \\ <[0.18,0.46];[0.26,0.50];[0.54,0.72]\backslash\, \theta_3> \end{array} \right\}$$

$\bar{L}(\digamma, avg)$, the $avg-$level hypersoft set of $\digamma = (\mathfrak{F}, \Lambda)$ can be evaluated as:

$$\mathfrak{F}_{avg_{\digamma}}(\theta_1) = \bar{L}(\mathfrak{F}(\theta_1); avg_{\digamma}(\theta_1)) = \{z_2, z_3\}$$

$$\mathfrak{F}_{avg_{\digamma}}(\theta_2) = \bar{L}(\mathfrak{F}(\theta_2); avg_{\digamma}(\theta_2)) = \{z_3\}$$

$$\mathfrak{F}_{avg_{\digamma}}(\theta_3) = \bar{L}(\mathfrak{F}(\theta_3); avg_{\digamma}(\theta_3)) = \{z_1, z_5\}$$

4.1. *Level hypersoft set based approach*

An algorithm based on nVHs-set is developed for decision making

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Algorithm I

▷ **Start**

▷ **Input Stage**:

———1. Consider $\mathcal{Z}$ as universe of discourse

———2. Consider Λ as subset of set of parameters

———3. Classify parameters into disjoint parametric valued sets $\Lambda_1, \Lambda_2, \Lambda_3, ..., \Lambda_n$

———4. $\Lambda = \Lambda_1 \times \Lambda_2 \times \Lambda_3 \times ... \times \Lambda_n$

▷ **Construction Stage**:

———5. Construct nVHs-set $\digamma = (\mathfrak{F}, \Lambda)$

———6. Choose threshold value triple $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in \bar{L}$

OR

———6. Construct threshold nV-set $\chi : \Lambda \rightarrow I \times I \times I$ where $I = [0, 1]$

OR

———6. Choose $avg-$level decision rule.

▷ **Computation Stage**:

———7. Compute $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})-$level hypersoft set $\bar{L}(\digamma, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$

OR

———7. Compute level hypersoft set $\bar{L}(\digamma, \chi)$

OR

———7. Compute $avg-$level hypersoft set $\bar{L}(\digamma, avg)$

▷ **Output Stage**:

———8. Present $\bar{L}(\digamma, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ or $\bar{L}(\digamma, \chi)$ or $\bar{L}(\digamma, avg)$ in tabular form.

———9. Compute choice value c_p of z_p for any $z_p \in \mathcal{Z}$

———10. Select z_m if $c_m = \max_{z_p \in \mathcal{Z}}(c_p)$.

———11. Choose any value z_m if m has more than one values

▷ **End**

Example 4.6. Let $\digamma = (\mathfrak{F}, \Lambda)$ is nVHs-set as discussed in example 4.5 and demonstrated in table 9. By $avg-$level rule, $\bar{L}((\mathfrak{F}, \Lambda); avg)$ is obtained which is demonstrated in table 10. The elements of table 10 are represented by $z_{pq} = 1$ if $z_p \in \mathfrak{F}_{avg_{\digamma}}(\theta_q)$, otherwise $z_{pq} = 0$.

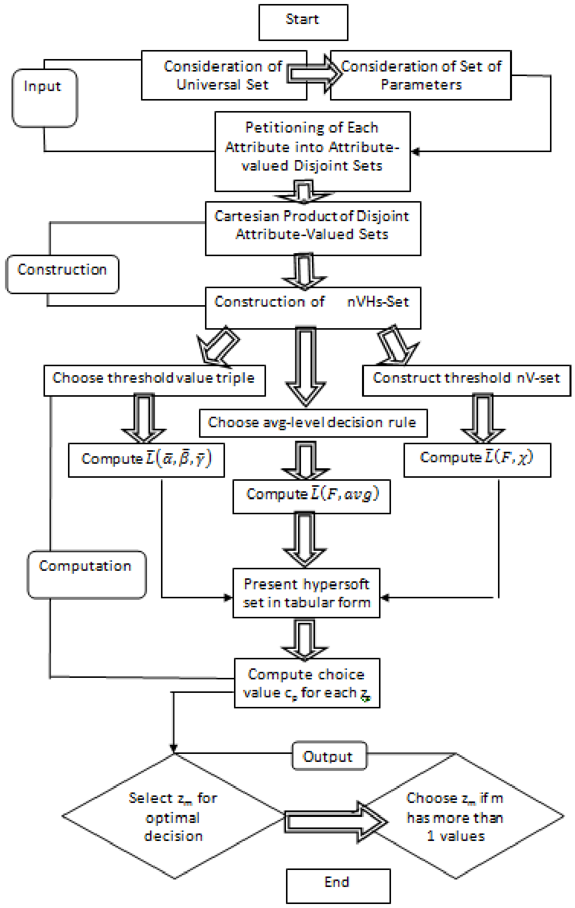

FIGURE 1. Algorithm I : Optimal Selection of material for manufacturing of Surgical Masks

TABLE 10. $\bar{L}((\mathfrak{F},\Lambda);avg)$ with choice vales

$\mathcal{Z}$	θ_1	θ_2	θ_3	choice value
z_1	0	0	1	1
z_2	1	0	0	1
z_3	1	1	0	2
z_4	0	0	0	0
z_5	0	0	1	1

Choice value can be obtained by $c_p = \sum_{q=1}^{5} z_{pq}$ i.e. $c_1 = 1$, $c_2 = 1$, $c_3 = 2$, $c_4 = 0$ and $c_5 = 1$
Farmhouse z_3 is selected as $c_3 = \max_{z_p \in \mathcal{Z}}(c_p)$

5. Comparison Analysis

Different decision making approaches have already been discussed in literature [12, 13, 17, 22, 23] that were based on hybridized structures of fuzzy set, intuitionistic fuzzy soft set and

Table 11. The advantage of the proposed study

Sr. No.	Author	Structure	Multi-argument approximate function
1	Smarandache [9]	Neutrosophic set	Insufficient
2	Molodtsov [10]	Soft set	Insufficient
3	Xu et al. [16]	Vague soft set	Insufficient
4	Alhazaymeh et al. [17]	Generalized vague soft set	Insufficient
5	Maji et al. [20]	Neutrosophic soft set	Insufficient
6	Alhazaymeh et al. [23]	Interval valued vague soft set	Insufficient
7	Alkhazaleh [34]	Neutrosophic vague set	Insufficient
8	Al Quran et al. [35]	Neutrosophic vague soft set	Insufficient
9	Smarandache [36]	Hypersoft set	Insufficient
10	Proposed Structure	Neutrosophic vague hypersoft set	Sufficient

neutrosophic set. Decision making is greatly affected due to many factors where attributes are not further classified into their disjoint attributive valued sets. The above mentioned existing decision making models are insufficient either for vague soft sets or for multi-argument approximate function but in proposed model, the inadequacies of these models have been addressed. The consideration of neutrosophic vague hypersoft set will make the decision making process more reliable and trust-worthy.

6. Discussion and Merits

In this section some merits of proposed structure are discussed:

The introduced approach took the significance of the idea of nVHs-set to deal with current decision making issues. The presented idea enables the researchers to deal with real-world scenario where problems involving indeterminacy and vagueness needs more attention. The core idea in this association has tremendous potential in the genuine depiction inside the space of computational incursions. As the proposed structure emphasizes on in-depth study of attributes (i.e. further partitioning of attributes) rather than focussing on attributes merely therefore it makes the decision-making process better, flexible and more reliable. It covers the characteristics and properties of the existing relevant structures i.e. fuzzy set, soft set, fuzzy soft set, intuitionistic fuzzy set, neutrosophic set, vague set, vague soft set, hypersoft set, neutrosophic vague soft set etc., so one can call it the generalized form of all these structures. The advantage of the proposed study can easily be judged from the table 11.

7. Conclusion

The summary of the proposed study is highlighted as:

(1) The relevant literature on fuzzy set, neutrosophic set, soft set, vague set and hypersoft set has been reviewed to support the main results.

(2) Some axiomatic and algebraic properties, set-theoretic operations and laws of nVHs-set have been investigated and explained with the help of illustrative examples.

(3) An algorithm based on set-theoretic operational concept of nVHs-set has been proposed to asses the role of proposed model in real-world decision-making scenario.

(4) A real-world decision-making application has been discussed by implementing the steps of proposed algorithm which opted the best farmhouse from real estate dealer.

(5) The advantageous aspects of the proposed model have been judged by comparing it with most relevant existing models.

(6) Many other real-world decision-making problems can be resolved with the help of the proposed algorithm.

Conflict of interest

The authors declare that they have no conflict of interest.

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An Enumeration Technique for Transshipment Problem in Neutrosophic Environment

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Abstract. Neutrosophic sets, which are the generalization of fuzzy, and intuitionistic fuzzy sets, have been introduced to express uncertain, incomplete, and indeterminacy knowledge regarding a real-world problem. This paper is intended for the first time to introduce a transshipment problem mathematically in a neutrosophic environment. The neutrosophic transshipment problem is a special type of neutrosophic transportation problem in which available commodities regularly travel from one origin to other origins/destinations before arriving at their final destination. This article provides a technique for solving transshipment problems in a neutrosophic environment. A fully neutrosophic transshipment problem is considered in this article and the parameters (transshipment cost, supply and demand) are expressed in trapezoidal neutrosophic numbers. The possibility mean ranking function is used in the proposed technique. The proposed technique gives a direct optimal solution. The proposed technique is simple to implement and can be used to find the neutrosophic optimal solution to real-world transshipment problems. A numerical example is provided to demonstrate the efficacy of the proposed technique in the neutrosophic environment.

Keywords: Decision-Making Problem, Transshipment Problem, Neutrosophic Transshipment Problem, Single-Valued Trapezoidal Neutrosophic Number

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When a particular product needs to be transported from source to sink in a network, transportation is one of the most important engineering challenges. A common transportation problem arises when a certain bulk of commodity needs to shipped from their origins to their destinations through multiple intermediate points (transshipment points). This classic form of transporters problem is called a transshipment problem. This problem was first proposed by

Orden [1]. The concept of the transshipment problem can also be applied to determining the shortest path between two nodes in a network. As an application of transshipment problem, King and Logan [2] established a mathematical method for simultaneously identifying threads in a network linking to product processor market points, while Rhody [3] suggested a method based on a reduced matrix. Judges et al. [4] proposed a general linear model extension of the transshipment problem to a multi-plant, multi-product, and multiregional problem. The alternative formulations of transshipment problems under the transport model was discussed by Hurt and Tramel [5] , which would allow for answers to common difficulties that King and Logan articulated without the requirement for artificial variables to be subtracted. "The time-minimizing transshipment problem" was investigated by Garg and Prakash [6]. Subsequently, Herer and Tzur [7] examined the dynamic transshipment problem. Ozdemir et al. [8] then looked on the problem of multilocation transshipment with capacitated manufacturing and lost sales.

Transshipment problem formulation requires the understanding of parameters such as demand, supply, associated cost, time, stock space, budget, etc. Traditional methods can be used to solve the transshipment problem when the decision parameters are known. However, in real-world scenarios, numerous types of uncertainty arise mathematically when designing transshipment due to factors such as a lack of precise information, information that cannot be obtained, rapid changes in the fuel rate or traffic jam, or whether conditions. Therefore, the transshipment problem with impricise information cannot be solved by traditional mathematical techniques. Zadeh [9] introduced the idea of fuzzy sets to deal with uncertainties. In order to handle unsure information, Zadeh effectively applied the theory of fuzzy set (FS) in various fields. The applications of this theory are rapidly growing in the field of optimization after the foremost work by Bellman and Zadeh [10]. Zimmermann [11] demonstrated that the solutions generated by fuzzy linear programming are always optimal and efficient. Fuzzy transshipment problem is the name given to the transshipment problem that is explored in fuzzy theory, which has been discussed by many researchers ([12]- [16]). Only the membership degrees are insufficient to indicate the element's marginal attainment in the fuzzy decision set, as was shown later on in the research. The intuitionistic fuzzy set (IFS), which incorporates both a membership and a non-membership function, was developed by Atanassov [17]. It is recommended that the sum of an element's membership and non-membership degrees does not exceed 1 in an intuitionistic fuzzy set. The transportation problem discussed in IFS is known as intuitionistic fuzzy transportation problem. Paramanik and Roy [18–20] discussed transportation and goal programming in IFS. Later, the transshipment problem in IFS has been discussed by many researchers ([21], [22]).

As a result of the presence of neutral ideas in the decision-making process, the extension of FS and IFS were required. Smarandache [23] introduced the neutrosophic set (NS) as a way to deal with the degree of indeterminacy and neutrality. Truth (degree of belongingness), indeterminacy (degree of belongingness up to a certain extant), and falsity (degree of non-belongingness) are three different membership functions for the element into a feasible solution set that the NS evaluates. But NS is difficult to implement without explicit detail in real-life problems. A single-valued neutrosophic set (SVNS) has been proposed for NS extension by Wang et al. [24]. By combining trapezoidal fuzzy numbers with a single-valued neutrosophic set, Ye [25] introduced single-valued trapezoidal neutrosophic (SVTrN) numbers. Many researchers such as Ahmad et al. [26], Garai et al. [27], Ahmad [28], Touqeer et al. [29], have recently used the concept of NS in decision-making problems. The effects of ignoring the values of propositions between the truth and falsity degrees are indeterminacy/neutral thoughts. As a result, when dealing with transshipment problems, it is important to consider the degree of indeterminacy.

Despite the fact that many researchers ([30]- [33]) applied the concept of neutrosophic theories to transportation problems. Neuosophic logic has not been applied to existing supply chain theories of transshipment models, to the best of our knowledge. This article aims to provide a simple yet effective method for solving neutrosophic transshipment problems in a day-to-day situation. There are a number of advantages to using the technique:

- All parameters are represented as trapezoidal neutrosophic numbers in a fully neutrosophic transshipment problem.
- The proposed technique is based on the possiblility mean ranking function.
- The technique proposed produces an optimal solution directly.
- The proposed method is simple to comprehend and can be used to solve real-life transshipment issues.

The following is how this article is organised. The neutrosophic set and neutrosophic numbers are introduced in Section 2. In the Section 3 formulates the airthmetic operations on single valued neutrosophic numbers, while the Section 4 presents the possilibity mean and ranking function on SVTrN-numbers. The mathematical structure of the transshipment problem in a neutrosophic environment was formulated in Section 5. The proposed technique's steps were addressed in Section 6. In Section 7, an example is given to show the effectiveness of the proposed solution strategy. The paper comes to a close with the conclusion.

1. **Mathematical Preliminaries**

This section provides an overview of key conceptions and definitions related to neutrosophic sets.

Definition 1. *[23] Let M be a universe and y in M. The neutrosophic set N over M is defined by $N = \langle y, T_N(y), I_N(y), F_N(y) : y \in M\rangle$, where the functions $T_N, I_N, F_N : P \to]^-0, 1^+[$ represent the truth-membership, indeterminate-membership, falsity-membership respectively such that $^-0 \le T_N(y) + I_N(y) + F_N(y) \le 3^+$*

Definition 2. *[23] Let M be a universe and y in M. Then a single valued neutrosophic set N is characterized by truth-membership T_N, indeterminacy-membership function I_N, falsity-membership function F_N, where $T_N, I_N, F_N : M \to [0,1]$ are functions such that $0 \le T_N(y) + I_N(y) + F_N(y) \le 3$.*

Definition 3. *[27] A single valued trapezoidal neutrosophic number is defined by $\tilde{m} = \langle (m_1, m_2, m_3, m_4); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}}\rangle$, where $t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} \in [0,1]$ and m_1, m_2, m_3, m_4 in $\mathbb{R}$ with condition that $m_1 \le m_2 \le m_3 \le m_4$. The truth-membership, indeterminacy-membership, and falsity-membership functions of $\tilde{m}$ are given as follows:*

$$\mu_{\tilde{m}}(y) = \begin{cases} t_{\tilde{m}}(\frac{y-m_1}{m_2-m_1}); & m_1 \le y \le m_2 \\ t_{\tilde{m}}; & m_2 \le y \le m_3 \\ t_{\tilde{m}}(\frac{m_4-y}{m_4-m_3}); & m_3 \le y \le m_4 \\ 0; & otherwise, \end{cases}$$

$$\nu_{\tilde{m}}(y) = \begin{cases} \frac{m_2-y+i_{\tilde{m}}(y-m_1)}{m_2-m_1}; & m_1 \le y \le m_2 \\ i_{\tilde{m}}; & m_2 \le y \le m_3 \\ \frac{y-m_3+i_{\tilde{m}}(m_4-y)}{m_4-m_3}; & m_3 \le y \le m_4 \\ 1; & otherwise \end{cases}$$

$$\lambda_{\tilde{m}} = \begin{cases} \frac{m_2-y+f_{\tilde{m}}(y-m_1)}{m_2-m_1}; & m_1 \le y \le m_2 \\ f_{\tilde{m}}; & m_2 \le y \le m_3 \\ \frac{y-m_3+f_{\tilde{m}}(m_4-y)}{m_4-m_3}; & m_3 \le y \le m_4 \\ 1; & otherwise, \end{cases}$$

where $t_{\tilde{m}}$, $i_{\tilde{m}}$ and $f_{\tilde{m}}$ are represents the maximum truth-membership degree, minimum-indeterminacy membership degree, minimum falsity-membership degree respectively. The geometrical representation of SVTrNF-number is shown by Fig. 1.

Definition 4. *[34] Let $\tilde{m} = \langle (m_1, m_2, m_3, m_4); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}}\rangle$ and $\tilde{n} = \langle (n_1, n_2, n_3, n_4); t_{\tilde{n}}, i_{\tilde{n}}, f_{\tilde{n}}\rangle$ be two single valued trapezoidal neutrosophic numbers and $k \neq 0$ be any number and $\wedge =$ min, $\vee =$ max, then the operations on them are defined as follows :*

(1) $\tilde{m} \oplus \tilde{n} = \langle (m_1 + n_1, m_2 + n_2, m_3 + n_3, m_4 + n_4); t_{\tilde{m}} \wedge t_{\tilde{n}}, i_{\tilde{m}} \vee i_{\tilde{n}}, f_{\tilde{m}} \vee f_{\tilde{n}}\rangle$,

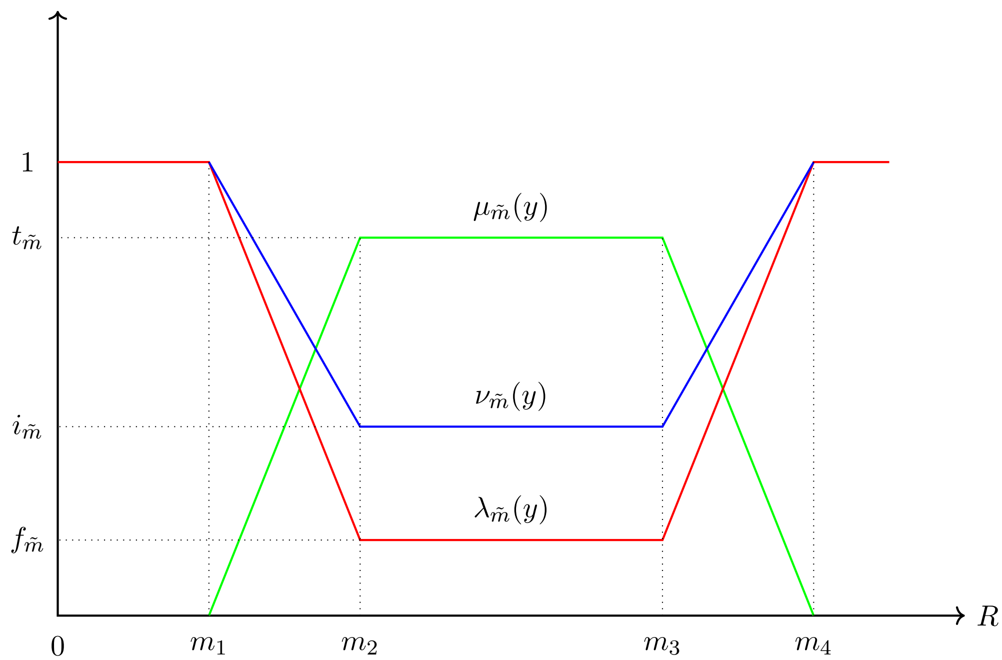

Figure 1. SVTrN-number

(2) $\tilde{m} \ominus \tilde{n} = \langle (m_1 - n_4, m_2 - n_3, m_3 - n_2, m_4 - n_1); t_{\tilde{m}} \wedge t_{\tilde{n}}, i_{\tilde{m}} \vee i_{\tilde{n}}, f_{\tilde{m}} \vee f_{\tilde{n}} \rangle$,

(3)

$$\tilde{m} \otimes \tilde{n} = \begin{cases} \langle (\frac{m_1}{n_4}, \frac{m_2}{n_3}, \frac{m_3}{n_2}, \frac{m_4}{n_1}); t_{\tilde{m}} \wedge t_{\tilde{n}}, i_{\tilde{m}} \vee i_{\tilde{n}}, f_{\tilde{m}} \vee f_{\tilde{n}} \rangle & \textit{if } m_4 > 0, n_4 > 0 \\ \langle (\frac{m_4}{n_4}, \frac{m_3}{n_3}, \frac{m_2}{n_2}, \frac{m_1}{n_1}); t_{\tilde{m}} \wedge t_{\tilde{n}}, i_{\tilde{m}} \vee i_{\tilde{n}}, f_{\tilde{m}} \vee f_{\tilde{n}} \rangle & \textit{if } m_4 < 0, n_4 > 0 \\ \langle (\frac{m_4}{n_1}, \frac{m_3}{n_2}, \frac{m_2}{n_3}, \frac{m_1}{n_4}); t_{\tilde{m}} \wedge t_{\tilde{n}}, i_{\tilde{m}} \vee i_{\tilde{n}}, f_{\tilde{m}} \vee \rangle & \textit{if } m_4 < 0, n_4 < 0 \end{cases}$$

(4)

$$c\tilde{m} = \begin{cases} \langle (cm_1, cm_2, cm_3, cm_4); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} \rangle & \textit{if } c > 0 \\ \langle (cm_4, cm_3, cm_2, cm_1); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} \rangle & \textit{if } c < 0 \end{cases}$$

(5) $\tilde{m}^{-1} = \langle (\frac{1}{m_4}, \frac{1}{m_3}, \frac{1}{m_2}, \frac{1}{m_1}); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} \rangle$, *where* $\tilde{m} \neq 0$.

2. The Possibility Mean and The Ranking Function for SVTrN-numbers

Sometimes, decision information supplied by a decision maker in difficult decision-making situations is vague or inaccurate due to time restrictions, a lack of facts, or the restricted attention and information processing capacity of the decision maker. As a result, incorporating the possibility mean into the neutrosophic decision-making process in transshipment is critical for scientific study and real-world application. Therefore, in this section the possibilty mean and the ranking function based on the possiblity mean are defined.

2.1. *The Possibility Mean Functions for SVTrN-Number*

Let $\tilde{m} = \langle (m_1, m_2, m_3, m_4); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} \rangle$ be any SVTrN-number. Then the possibilty mean functions are defined as follows: [27]

1. α-cut set of the SVTrN-number $\tilde{m}$ for truth-membership function is obtained as

$$\tilde{m}_\alpha = [M_L, M_R] = [m_1 + \tfrac{\alpha(m_2-m_1)}{t_{\tilde{m}}}, m_3 - \tfrac{\alpha(m_3-m_2)}{t_{\tilde{m}}}]$$

where $\alpha \in [0, t_{\tilde{m}}]$. The possibility mean of truth-membership function for SVTrN-number $\tilde{m}$ is given by

$$P_\mu(\tilde{m}) = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6} t_{\tilde{m}}^2$$

2. β-cut set of the SVTrN-number $\tilde{m}$ for indeterminacy membership function is obtained as

$$\tilde{m}_\beta = [M_L, M_R] = [m_1 + \tfrac{(1-\beta)(m_2-m_1)}{1-i_{\tilde{m}}}, m_3 - \tfrac{(1-\beta)(m_3-m_1)}{1-i_{\tilde{m}}}]$$

where $\beta \in [\nu_{\tilde{m}}, 1]$. The possibility mean of indeterminacy-membership function for SVTrN-number $\tilde{m}$ is given by

$$P_\nu(\tilde{m}) = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6} (1 - i_{\tilde{m}})^2$$

3. γ-cut set of the SVTrN-number $\tilde{m}$ for falsity-membership function is obtained as

$$\tilde{m}_\gamma = [M_L, M_R] = [m_1 + \tfrac{(1-\gamma)(m_2-m_1)}{1-i_{\tilde{m}}}, m_3 - \tfrac{(1-\gamma)(m_3-m_1)}{1-i_{\tilde{m}}}]$$

where $\gamma \in [f_{\tilde{m}}, 1]$. The possibility mean of falsity-membership function for SVTrN-number $\tilde{m}$ is given by

$$P_\lambda(\tilde{m}) = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6} (1 - f_{\tilde{m}})^2$$

2.2. *The Ranking Function Based on The Possibility Mean Function*

The ranking fuction based on possiblity mean values for a SVTrN-number $\tilde{m} = \langle (m_1, m_2, m_3, m_4);\ t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} \rangle$ is given by

$$P_\theta(\tilde{m}) = \theta P_\mu(\tilde{m}) + (1-\theta) P_\nu(\tilde{m}) + (1-\theta) P_\lambda(\tilde{m})$$

Theorem 1. *Let $\tilde{m} = \langle (m_1, m_2, m_3, m_4); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} \rangle$ and $\tilde{n} = \langle (n_1, n_2, n_3, n_4); t_{\tilde{n}}, i_{\tilde{n}}, f_{\tilde{n}} \rangle$ be two SVTrN-numbers and $\theta \in [0, 1]$. For the possibility mean values of the SVTrN-numbers $\tilde{m}$ and $\tilde{n}$, the following illustrantions hold true.*

(1) *If $P_\theta(\tilde{m}) > P_\theta(\tilde{n})$, than $\tilde{m} \succ \tilde{n}$.*
(2) *If $P_\theta(\tilde{m}) < P_\theta(\tilde{n})$, than $\tilde{m} \prec \tilde{n}$.*
(3) *If $P_\theta(\tilde{m}) = P_\theta(\tilde{n})$, than $\tilde{m} \approx \tilde{n}$.*

Proof. It is evident from the definition of ranking function. □

3. Mathematical Formulation of SVNTrP

We have mathematically formulated a transshipment problem in a neutrosophic environment in this section. The parameters of the problem under consideration are single-valued trapezoidal neutrosophic numbers. i.e., the decision maker is unsure about the cost of transshipment, supply and demand. The primary goal of the transshipment problem is to transport any item/product from one origin or destination to another origin or destination while minimising total transshipment costs. In a neutrosophic environment, the mathematical structure of the transshipment problem is as follows:

$$
\begin{aligned}
& min\tilde{z}^N = \sum_{i=1}^{m+n}\sum_{j=1}^{m+n} \tilde{C}_{ij}^N \otimes \tilde{X}_{ij}^N \\
& \text{Subject to } \sum_{j=1}^{m+n} \tilde{X}_{ij}^N - \sum_{j=1}^{m+n} \tilde{X}_{ji}^N = \tilde{a}_i^N,\ i = 1,2,...,m. \\
& \qquad\qquad \sum_{i=1}^{m+n} \tilde{X}_{ij}^N - \sum_{i=1}^{m+n} \tilde{X}_{ji}^N = \tilde{b}_j^N,\ j = m+1, m+2, ..., m+n. \\
& \qquad\qquad \tilde{X}_{ij}^N \geq 0,\ i,j = 1,2,...,m+n;\ i \neq j.
\end{aligned}
$$

The problem is said to be balanced if $\sum_{i=1}^{m} \tilde{a}_i^N \approx \sum_{j=1}^{n} \tilde{b}_j^N$, otherwise it is know as unbalance problem. Where,

- m and n denote total number of supply sources and total number of demand points, respectively.
- a_i^N denotes available commodity at ith source.
- b_j^N denotes demand of the commodity at jth destination.
- $\tilde{C}_{ij}^N = (c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4}\ ; w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}})$ denotes the neutrosophic transshipment cost of a unit commodity from ith source to jth destination.
- The number of units of the commodity to be carried from the ith source to the jth destination is denoted by X_{ij}.

4. Methodology

In this section, a noval transshipment problem technique is presented, that uses the possibility mean ranking function to obtain the optimal solution. The technique is explained in detail below in a step-by-step manner.

Step 1 Construct a neutrosophic transshipment problem in Table form in which either all parameters are taken as SVTrNF-numbers.

Step 2 Put zeros where demand and supply are unknown.

Step 3 Assign zero values to each digonal cell and delete the rows/columns whose demand has been met.

Step 4 The transshipment cost is then converted to a crisp number using the possibility mean based ranking function that is discussed in Section 4.

Step 5 From the matrix obtained after Step 4, choose minimum element from each row then subtract it from each element of corresponding row.

Step 6 From the matrix obtained after Step 5, choose minimum element from each column then subtract it from each element of corresponding column.

Step 7 In this manner, each row and column will have at least one zero value. Then, for each cell having a zero value, use the following formula to determine the zero average value O_{ij}.

$$O_{ij} = \text{the average of the ith row's and jth column's minimum values.}$$

Step 8 Select the maximum zero average value and assign it to the appropriate cell with the minimum demand/supply, then delete the row/column whose supply/demand has reached its limit.

Step 9 Pick an allocation that assigns the highest feasible demand in the same rank case.

Step 10 Follow steps 7 to 9 until the total demands are not fulfilled.

Step 11 Add the product of the assigned demand/supply and the cost value for each cell to get the total transshipment cost. The neutrosophic optimal solution is provided by this total transshipment cost.

5. Numerical Example

We provide an example of our proposed solution methodology in this section. A neutrosophic transshipment problem with two origins (A,B) and two destinations (C,D) has been considered. Table 1 shows the availability at the origins, the requirements at the destinations, and the transshipment costs.

Table 1. SVTrN transshipment problem

Destination → Sources ↓	A	B	C	D	Supply
A	(0, 0, 0, 0)	(5, 7, 9, 11 ; 0.4, 0.8, 0.5)	(4, 6, 9, 11 ; 0.9, 0.3, 0.7)	(1, 3, 8, 10; 0.3, 0.9, 0.4)	(14, 20, 21, 27 ; 0.2, 0.7, 0.9)
B	(7, 10, 12, 15 ; 0.1, 0.6, 0.8)	(0, 0, 0, 0)	(2, 5, 9, 12 ; 0.6, 0.3, 0.1)	(6, 9, 12, 15 ; 0.5, 0.2, 0.6)	(13, 18, 23, 28 ; 0.5, 0.3, 0.6)
C	(3, 7, 9, 13 ; 0.4, 0.7, 0.3)	(5, 8, 12, 15 ; 0.2, 0.4, 0.7)	(0, 0, 0, 0)	(7, 12, 14, 19 ; 0.8, 0.3, 0.2)	–
D	(2, 6, 9, 13 ; 0.9, 0.7, 0.8)	(6, 7, 8, 9 ; 0.3, 0.8, 0.6)	(1, 5, 7, 11 ; 0.2, 0.9, 0.7)	(0, 0, 0, 0)	–
Demand	–	–	(12, 18, 20, 26 ; 0.4, 0.3, 0.5)	(15, 20, 24, 29 ; 0.7, 0.8, 0.4)	

For each column or row, write zero value for unknown demand/supply.

Table 2. Balance tansshipment problem

Destination → Sources ↓	A	B	C	D	Supply
A	$(0,0,0,0)$	$(5,7,9,11\ ;\ 0.4,0.8,0.5)$	$(4,6,9,11\ ;\ 0.9,0.3,0.7)$	$(1,3,8,10;0.3,0.9,0.4)$	$(14,20,21,27\ ;\ 0.2,0.7,0.9)$
B	$(7,10,12,15\ ;\ 0.1,0.6,0.8)$	$(0,0,0,0)$	$(2,5,9,12\ ;\ 0.6,0.3,0.1)$	$(6,9,12,15\ ;\ 0.5,0.2,0.6)$	$(13,18,23,28\ ;\ 0.5,0.3,0.6)$
C	$(3,7,9,13\ ;\ 0.4,0.7,0.3)$	$(5,8,12,15\ ;\ 0.2,0.4,0.7)$	$(0,0,0,0)$	$(7,12,14,19\ ;\ 0.8,0.3,0.2)$	$(0,0,0,0)$
D	$(2,6,9,13\ ;\ 0.9,0.7,0.8)$	$(6,7,8,9\ ;\ 0.3,0.8,0.6)$	$(1,5,7,11\ ;\ 0.2,0.9,0.7)$	$(0,0,0,0)$	$(0,0,0,0)$
Demand	$(0,0,0,0)$	$(0,0,0,0)$	$(12,18,20,26\ ;\ 0.4,0.3,0.5)$	$(15,20,24,29\ ;\ 0.7,0.8,0.4)$	

The lowest cost value in each row is zero. We discover that zero is the smallest unit cost in each row, so we place zero in the diagonal cell of the transshipment matrix. Table 2 illustrates this.

Table 3. Reduced tansshipment problem

Destination → Sources ↓	A	B	C	D	Supply
A	$(0,0,0,0)$ $(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$	$(5,7,9,11\ ;\ 0.4,0.8,0.5)$	$(4,6,9,11\ ;\ 0.9,0.3,0.7)$	$(1,3,8,10;0.3,0.9,0.4)$	$(14,20,21,27\ ;\ 0.2,0.7,0.9)$
B	$(7,10,12,15\ ;\ 0.1,0.6,0.8)$	$(0,0,0,0)$ $(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$	$(2,5,9,12\ ;\ 0.6,0.3,0.1)$	$(6,9,12,15\ ;\ 0.5,0.2,0.6)$	$(13,18,23,28\ ;\ 0.5,0.3,0.6)$
C	$(3,7,9,13\ ;\ 0.4,0.7,0.3)$	$(5,8,12,15\ ;\ 0.2,0.4,0.7)$	$(0,0,0,0)$ $(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$	$(7,12,14,19\ ;\ 0.8,0.3,0.2)$	$(0,0,0,0)$
D	$(2,6,9,13\ ;\ 0.9,0.7,0.8)$	$(6,7,8,9\ ;\ 0.3,0.8,0.6)$	$(1,5,7,11\ ;\ 0.2,0.9,0.7)$	$(0,0,0,0)$ $(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$	$(0,0,0,0)$
Demand	$(0,0,0,0)$	$(0,0,0,0)$	$(12,18,20,26\ ;\ 0.4,0.3,0.5)$	$(15,20,24,29\ ;\ 0.7,0.8,0.4)$	

Delete the rows or columns whose demands have been met.

Table 4. New tansshipment problem

Destination → Sources ↓	C	D	Supply
A	$(4,6,9,11\ ;\ 0.9,0.3,0.7)$	$(1,3,8,10;0.3,0.9,0.4)$	$(14,20,21,27\ ;\ 0.2,0.7,0.9)$
B	$(2,5,9,12\ ;\ 0.6,0.3,0.1)$	$(6,9,12,15\ ;\ 0.5,0.2,0.6)$	$(13,18,23,28\ ;\ 0.5,0.3,0.6)$
Demand	$(12,18,20,26\ ;\ 0.4,0.3,0.5)$	$(15,20,24,29\ ;\ 0.7,0.8,0.4)$	

Now, using the possibility mean-based ranking function discussed in the section 3, apply it to the cost of transshipment.

Table 5. De-neutrosophic transshipment problem

Destination → Sources ↓	C	D	Supply
A	5.21	1.26	$(14, 20, 21, 27\ ;\ 0.2, 0.7, 0.9)$
B	5.81	5051	$(13, 18, 23, 28\ ;\ 0.5, 0.3, 0.6)$
Demand	$(12, 18, 20, 26\ ;\ 0.4, 0.3, 0.5)$	$(15, 20, 24, 29\ ;\ 0.7, 0.8, 0.4)$	

We obtained the final optimal table (Table 6) by completing the remaining steps of the proposed technique .

Table 6. Final Optimal Table

Destination → Sources ↓	A	B	C	D
A	$(0, 0, 0, 0)$ $\mathbf{(0, 0, 0, 0)}$	$(5, 7, 9, 11\ ;\ 0.4, 0.8, 0.5)$	$(4, 6, 9, 11\ ;\ 0.9, 0.3, 0.7)$	$(1, 3, 8, 10; 0.3, 0.9, 0.4)$ $\mathbf{(14, 20, 21, 27\ ;\ 0.2, 0.7, 0.9)}$
B	$(7, 10, 12, 15\ ;\ 0.1, 0.6, 0.8)$	$(0, 0, 0, 0)$ $\mathbf{(0, 0, 0, 0)}$	$(2, 5, 9, 12\ ;\ 0.6, 0.3, 0.1)$ $\mathbf{(12, 18, 20, 26\ ;\ 0.4, 0.2, 0.6)}$	$(6, 9, 12, 15\ ;\ 0.5, 0.2, 0.6)$ $\mathbf{(-12, -1, 4, 15\ ;\ 0.2, 0.8, 0.9)}$
C	$(3, 7, 9, 13\ ;\ 0.4, 0.7, 0.3)$	$(5, 8, 12, 15\ ;\ 0.2, 0.4, 0.7)$	$(0, 0, 0, 0)$ $\mathbf{(0, 0, 0, 0)}$	$(7, 12, 14, 19\ ;\ 0.8, 0.3, 0.2)$
D	$(2, 6, 9, 13\ ;\ 0.9, 0.7, 0.8)$	$(6, 7, 8, 9\ ;\ 0.3, 0.8, 0.6)$	$(1, 5, 7, 11\ ;\ 0.2, 0.9, 0.7)$	$(0, 0, 0, 0)$ $\mathbf{(0, 0, 0, 0)}$

The optimal solution of trapezoidal neutrosophic transshipment problem, given in Table 1, is $(1, 3, 8, 10; 0.3, 0.9, 0.4) \otimes (14, 20, 21, 27; 0.2, 0.7, 0.9) \oplus (2, 5, 9, 12; 0.6, 0.3, 0.1) \otimes (12, 18, 20, 26; 0.4, 0.2, 0.6) \oplus (6, 9, 12, 15; 0.5, 0.2, 0.6) \otimes (-12, -1, 4, 15; 0.2, 0.8, 0.9) = (-34, 141, 396, 807; 0.2, 0.9, 0.9)$.

6. **Conclusion**

Neutrosophic sets, a generalisation of intuitionistic fuzzy sets, can represent both indeterminacy and uncertainty. Though many decision-making problems have been studied with various forms of input data, this study looked at solutions to the transshipment problem in a neutrosophic environment. The proposed method has proven to be effective in solving transshipment problems involving single-valued trapezoidal neutrosophic numbers. The proposed technique has been based on the possibility mean ranking function. The technique is simple to implement in real-world transshipment problems. Further, the technique produces an optimal solution directly. While the proposed technique analyses the solutions to neutrosophic transshipment problem in concrete form, the prediction of qualitative and complex data solutions does have certain limitations. Genetic algorithm approaches can overcome computational complexity in the management of higher dimensional problems. The research can be further expanded to address multiobjective transshipment problems in neutrosophic environment.

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An Innovative Neutrosophic Combinatorial Approach Towards the Fusion and Edge Detection of MR Brain Medical Images

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Abstract. This research proposes the idea of implementing an innovative mechanism to detect the edges in distinct MR brain medical images based on the aspect of Neutrosophic sets (NSs). NS-based entropy is one of the emerging tools to procure a neutrosophic image from the crisp image. Followed by the aforementioned procedure, fusion has been done for the neutrosophic image in order to acquire fused neutrosophic images (FNI) then the FNI's are again regenerated to form a fused crisp image. Later, the Bell-Shaped (BS) function and the Sobel operator works on the FNI to obtain a combination of three subsets. After determining the neutrosophic subsets, various entropies such as Norm, Threshold, Sure, and Shannon act on it to provide their threshold values, and the computed subsets along with thresholds are incorporated to produce a new binarized image. Subsequently, morphological operations were implemented to construct the image edges. The resultant images with different entropies are compared by using the performance measurement factors. Based on the measurement factors, the proposed Norm entropy image edge detection innovations have proven to be an efficient tool with reference to other entropies. In addition, the Norm entropy-based proposed method was compared with some of the other existing edge detection methods inclusive of Sobel, Chan, Tian, and Wu. FOM and PSNR factors have been applied to estimate the results of edge detection achieved through five distinct methods. The findings confirmed that the implementation of the proposed object edge detection mechanism is much stronger compared to other existing methods.

Keywords: Neutrosophic set; Image Fusion; Segmentation; Entropy; Edge detection; Homogeneity

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1. Introduction

Over the past few decades, two key applications such as image fusion and segmentation have become much clearer and they are significant in the field of image processing and computer vision, see references [1, 2] for more details. Nowadays, a substantial number of analytically validated image fusion and segmentation, that have been applied to fuse and segment the image objects, and also have been defined to eliminate the noise and its related uncertainty factors. In general, the scheme of fusion is characterized into three parts such as feature, pixel, and decision level. One of the fusion methods is, every pixel of the source images is summed and takes the average then gets the fused image, this algorithm gives the lower performance. So, various types of multi-resolution transform appeared to deal with this problem and they are employed in image fusion. Many image fusion schemes are available, but some schemes are frequently handled, they are spatial fusion, transform fusion, contourlet transform, gradient pyramid, Laplacian pyramid, MSVD, wavelet transform, curvelet transform, etc. The image is fused by the notion of SVD called singular value decomposition that execution is more similar to wavelets. The MSVD based image fusion has been viewed as a faster performance than approximated SVD, see the reference [3] for further details.

Edge detection is a kind of crucial step in the human visual system and image analysis, which is the wild development research area in image processing. The edge detection process substantially alleviates the amount of information because it divides the meaningful data and shields the foremost geometric features. To illustrate the object edges, the object data is performed by either analog / digital computation. The edge is noted as a significant local change in the intensity of the image. In general, the edge is correlated with pixel discontinuity and it occurs between distinct gray level regions of the image. It should be noted that many Gradient and Laplacian operators such as Roberts [4], Canny [5], Prewitt [6], Sobel [7], and others, have been presented for classical edge detection. And these operators based edge detection methods are listed as follows: (a) Cellular Automata [8] (b) particle swarm optimization (PSO) [9] (c) Wavelet [10], (d) Anisotropic Gaussian Kernels (AGK) [11], (e) Ant colony optimization (ACO) [12]. Besides, the swarm intelligence-based ACO algorithm has been offered by Dorigo et al. and it was motivated by the universal collective behavior of ants in the present environment [13], an important method derived from the development of ACO modifications. Zhang et al. suggested topology-preserving 3D image segmentation based On hyperelastic regularization [14]. The authors Liu and Li [15] recommended a fabric defect detection technique based on the aspect of low-rank decomposition with structural constraints. Further information on edge detection method through active contour without edge has also been available in the literature [16]. In addition, new edge detection approaches have been published in peer-reviewed journals [17]. The main limitations of the edge detection algorithms include illumination sensitivity, noise sensitivity, and unadjusted parameters, according to [18, 19]. Hence, there are distinct edge detection methods that have so far been framed to reduce the limitations of the aforementioned edge detection approaches while maximizing their enforcement [20]. Methods of edge detection based on the fuzzy notion, which have some difficulties in the formation of fuzzy rules. The use of NS-based edge detection helps to alleviate these issues. Just a few works are done in NS based framework and it exposes the originality of the current research. Therefore, the goal of this research is to contribute to future studies that use NS for segmentation and edge detection.

In this continuation, the Neutrosophic Set (NS) was discovered by the premise of Neutrosophy theory and it was first introduced by Florentin Smarandache in the year 2003 [21]. With the help of the NS approach, the origin, scope, and nature of neutralities are discussed. NS is a new philosophical branch [22] and it is a recent method that successfully addresses the problem of vagueness and indeterminacy in circumstances [23] such as biomedical, stock exchange, weather, law, and so on. Hence, the NS domain act as a decision support system in order to overcome the limitations of vagueness. Nonetheless, the neutrosophication functions and their application by MATLAB were recommended by the authors, Bakro et al. [24]. Meanwhile, the same authors offered the neutrosophic method to digital images in their paper [25]. However, NS has been applied successfully in a wide range of domains, including filtering, image processing, segmentation, edge detection, and so on. For remarkable efficiencies in the analysis of neutrosophic information, neutrosophic-based edge detection of medical images is a specialized area of research.

In the literature, only a few works can be obtained from the premise of NS-based segmentation technique [26], which is a popular segmentation methodology for obtaining indeterminate situations of the images, where the indeterminacy of the image is approximated by disturbances like noise, low quality of an image, and so on. This method leads to the problem of uncertainties and inconsistent information. According to [27], the authors used Chan-Vese approaches and NS to segment the images. Dhar and Kundu [28] utilized the two concepts like NS and digital shearlet transform (DST) to segment (text region) the images. Antera and Hassenian (2018) [29] designed an NS segmentation framework for CT liver tumors by using the fast fuzzy C-means algorithm (FFCM) and particle swarm optimization (PSO). We were able to obtain a better CT image with less noise by using NS-based pre-processing. Guo et al. (2017) [30] demonstrated the new method to identify the myocardial contrast of the myocardial echocardiography in the left ventricle and the method is computed by combining two strategies such as active contour method and neutrosophic similarity score. Besides that, employing chest X-ray images, Yasser et al [31]. developed a hybrid automated intelligent COVID-19 classification based on Neutrosophic logic and machine learning approaches. Singh, after that, proffered a multiple thresholding technique depending on type-2 neutrosophic entropy-fusion for the classification of brain tumor tissue structures [32]. Following that, Dhar [33] described a technique for accurately sectioning multi-class images using weak continuity constraints and a neutrosophic set.

In addition, Mehrdad et al. [34] formed a new framework and it is applied in the liver dome in which the liver is automatically detected by utilizing the random walker method. The authors of [35] devised a multi-atlas extraction method for fast automated (with/without abnormality) liver image segmentation from computational tomography angiography (CTA) and this method contains the local decision fusion. Platero et al. [36] constructed a unique segmentation process, it is to extract the liver image from a CT scan, where the process is formed by the combination of an affine probabilistic atlas, low-level operations, and a multi atlas-based segmentation. Li et al. [37] presented the automatic 3D (liver) object extraction procedure, which is to divide the Object and Background of the image via the concept of convolution neural network and the graph cut. In [38], the author proposed the thyroid nodule segmentation process and it is fundamentally based on the perspective like level-sets and spatial information with clustering. Further, Salah et al. [39] suggested a new method based on the convolutional neural network for human skin detection.

On the other hand, MR image analysis has become a remarkable research field due to the rapid advancement in computer vision and image analysis techniques. However, due to the presence of objects that are inconsistent with their edges and textures, it is necessary to develop a method of recognition and vision for MR images but it is complicated. In order to address this issue, many researchers are now focusing their greater efforts on the separation approach. The varying distribution of gray level pixels can be applied to distinguish various areas (i.e., gray matter, white matter, and cerebrospinal fluid) of the MR image separation. The MR image separation has gone through various incarnations since its advances. In addition, extraction of gray matter into the spine, Datta et al. [40] a threshold-based technique (TBT) is furnished, that is primarily based on the morphological geodetic active contour (MGAC) technique. Taheri et al. [41] advised TBT for extracting the tumor, which has been based on the technique in a level set. The authors [42] recommended an automated and adaptive technique for extracting the MR image of the liver vessels. For the past few decades, MR image analysis is not examined in terms of NS theory.

Inspired by the before talks and compared to the present research accomplishments, the great contributions of this study are suggested by being given at the upcoming points.

(1) As a first attempt, the NS-based edge detection system is framed in this research article for images with not only noises but also unstable situations. The concept of NS, NS-based entropy, and the fusion rule were combined in the design of NS to create a new fusion mechanism.

(2) A new powerful edge detection methodology for finding edges in the fused images is elaborated in a supervised manner, and it ensures the rejection of disturbances inclusive of noise and unstable information. The proposed NS-based edge detection method can be easily implemented, and it has the amount of truth, false and indeterminant degrees that is simply obtained through a combination of Bell Shaped (BS) function, Sobel operator, and Neutrosophic Theory.

(3) The resulting subsets are changed into the form of binary using the threshold values and these values of threshold are calculated by applying different entropies such as Norm, Threshold, Sure, and Shannon. Applying the morphological operations on the generated images, then acquire the edge detected images.

(4) For the experimental purposes, distinct performance measurement factors such as FOM and PSNR are employed. Ultimately, the statistical values, including comparative research, are provided to reveal the efficacy of the Norm entropy-based image edge detection scheme introduced in this work.

(5) Additionally, the aforementioned method has been compared to some of the methods found in the literature via Sobel [7], Tian [13], Chan [16], and Wu [18]. As a result, these process activities demonstrated the reliability of the industry and the systematic superiority of the proposed method.

The scheme of the research paper is arranged below. Section 2 gathers the theoretical background of NS, which consists of some subsections such as Basics of neutrosophy, Preprocessing, Neutrosophic image fusion, and Neutrosophic edge detection. Section 3 presents the performance measurement factors. Section 4 describes the statistical findings of the present work in terms of NS-based edge detection of different images. Lastly, numerous closing quotes are briefly showed in the final section.

2. Theoretical Background

2.1. ***Basics of Neutrosophy***

Smarandache [21] firstly initiated the new idea of neutrosophy motivated by the aspect of philosophical precedence, which simultaneously separates each concept from a certain degree of truth, falsity, and indeterminacy. Since, neutrosophy has laid solid foundations for new mathematical theories such as NS, neutrosophic probability, neutrosophic logic, and neutrosophic statistics. In this, each statement is evaluated as belonging to the amount of true subset T, false subset F, and indeterminant subset I, respectively. The NS holds some sets like the dialetheist set, paraconsistent set, fuzzy set, tautological set, and intuitionistic fuzzy set. In a multidisciplinary way, NS has solved many problems. Further, neutrosophic logic delivers the structure of neutrosophic connectives such as conjunction, negation, and disjunction. Furthermore, the neutrosophic images are mathematically formulated and it is being shown in the upcoming part of the research article.

Moreover, MR brain changes can be noticed in terms of gray matter, white matter, and cerebrospinal fluid. The before-mentioned changes are extremely imprecise and cannot be exactly specified at the idea of probability computation. Since NS theory can be recognized as a suitable approach. This explains some of the ambiguities and these ambiguities correlated with the MR images can be represented with three distinct membership degrees. In this, MR brain image is split into three subsets respectively, T, F, and I in the NS domain. The subset T specifies the image object, the subset F describes the image background and the subset I refers to the image edge. Figure. 1 exhibits the flow diagram of the NS-based image edge detection process. In Figure. 1, the test image can be initially transferred to grey scale form, then the gray image is processed (removing disturbances) by employing a median filter. As a result, the filtered image is changed into the NS domain (partitioned into T, F, and I subsets). Later, NS-based entropy is implemented on these subsets to create a new image. Moreover, the gained image is decomposed into the blocks and applies the fusion rule to form a neutrosophic fused image. Then the obtained image is processed by edge detection. In the flow chart 1, the object having the edges is gained at the final stage.

2.2. ***Preprocessing***

The initial image (original) changes to a grayscale domain by using direct Matlab code and then it is drained by applying a median filter (MF). MF is a noise removal approach that helps to eliminate the disturbances in the image and gives exclusive results in the image segmentation process. This offers a superior noiseless edge detected image during the segmentation process. Additionally, it protects the features of the edges in the image. More number of filters were available in the literature but this study preferred MF, because of the simplest one and it yields good results in our edge detection mechanism.

2.3. ***Neutrosophic image fusion***

To employ NS design for image processing, the image must be converted to a neutrosophic field. Particularly, we expand an NS-based image processing system that selects three degrees via., one for the T and the others F and I. They are degrees of truth, false and indeterminate subsets respectively. In consequence, a test image ($\mathbb{TI}$) whose length is specified by the symbol 'l'and the width is symbolized by 'w'. Each image element $\mathbb{TI}(l, w)$ in the image is reset by a neutrosophic field design [21], and that can be visualized by the following format: $\mathbb{TI}_{NS} = \{T(l, w), F(l, w), I(l, w)\}$ or $\mathbb{TI}_{NS} = \mathbb{TI}(t, f, i)$, where the image element t indicates the true

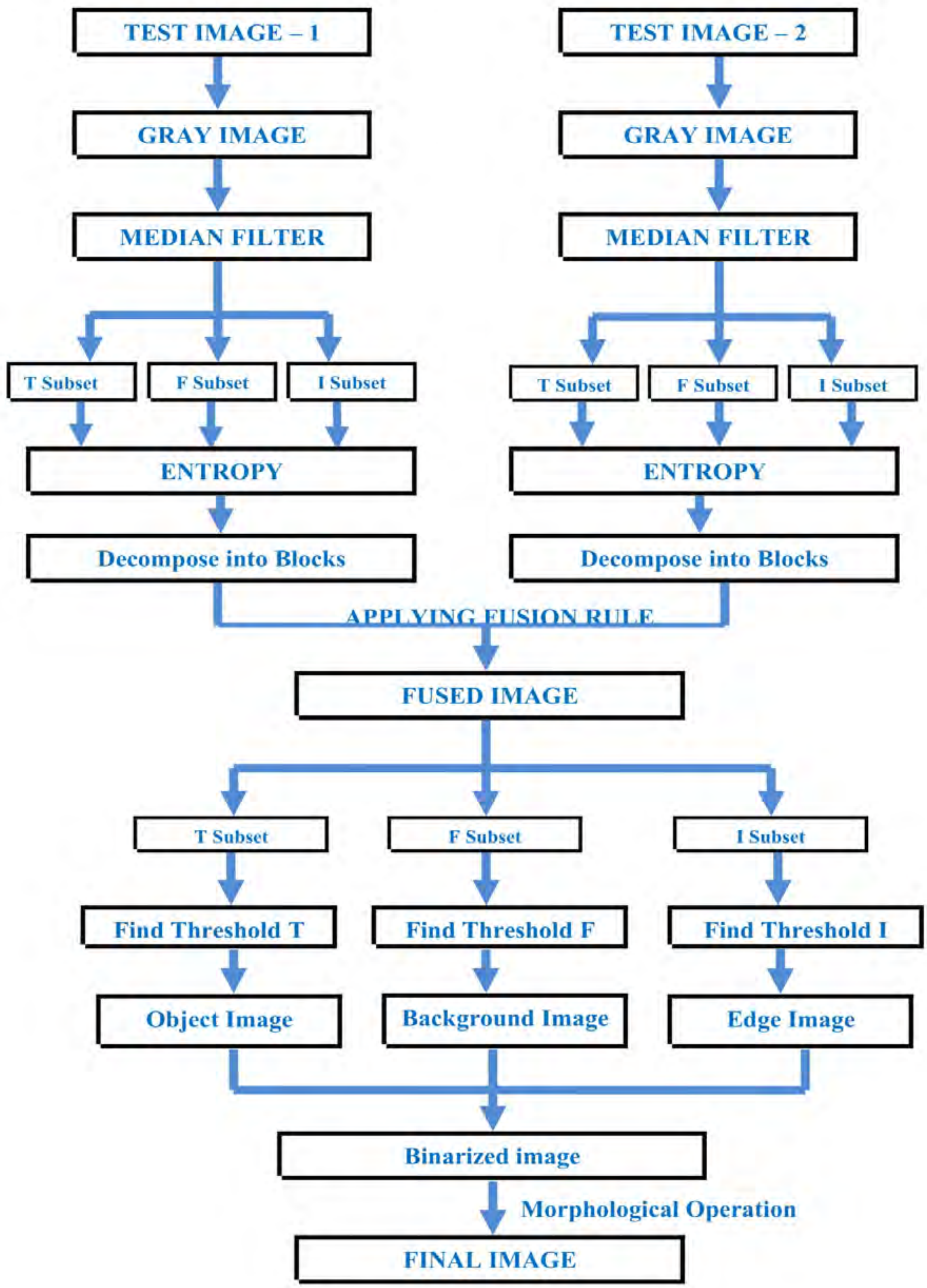


FIGURE 1. **Flowchart of the proposed mechanism**

subset, f mentions the false subset and i refers the indeterminate subset. Here, t differs in the white pixel set T, f varies in the black pixel set F and i fluctuate in the noise pixel set I. These are described as follows:

$$T(l,w) = \frac{\mathbb{G}(l,w) - \mathbb{G}_{min}}{\mathbb{G}_{max} - \mathbb{G}_{min}} \tag{1}$$

$$F(l,w) = 1 - T(l,w) \tag{2}$$

$$I(l,w) = \sqrt{T(l,w)^2 + F(l,w)^2} \tag{3}$$

In equ (1), the notations $\mathbb{G}_{max}$, $\mathbb{G}_{min}$, respectively depicts the maximum and minimum values of the image ($\mathbb{TI}$) also the symbol $\mathbb{G}(l,w)$ indicates the $(l,w)^{th}$ gray level of the test image. Then, each image element (l,w) belongs to the image ($\mathbb{TI}$), which can then be defined as NS using the aforementioned equation. For this depiction, the maximum and minimum values from the gray level $\mathbb{G}$ in the image ($\mathbb{TI}$) are applied, and they can be derived from $\mathbb{G}_{min}$ and $\mathbb{G}_{max}$. The main benefit of those received formulas is that they can control truth, false, and indeterminate subsets between the ranges 0 and 1. Combining the previously obtained neutrosophic components (T, F, and I) can provide complete information about inherited uncertainty at the problem space. Now, the entropy can be applied to compute an inherited ambiguity in indefinite circumstances. If such ambiguities are reported using NS, their estimation is also achievable from entropy. The neutrosophic entropy information (NEI) function [43] is used for this specific purpose, which can determine the values of each Neutrosophic Information ($\mathbb{NI}$) entropy and it is illustrated as follows.

The function $\mathbb{ENT}(\mathbb{NI}) : \mathbb{ENT}(\mathbb{NI}) \to [0,1]$ is known as NEI of an $\mathbb{NI}$, it can be signified by the following design

$$\mathbb{ENT}(\mathbb{NI}) = 1 - \frac{1}{3} \sum_{l,w \in \mathbb{G}} (T(l,w) + F(l,w) + I(l,w)) \times E_1 E_2 E_3 \tag{4}$$

Where, $E_1 = |T(l,w) - T^c(l,w)|$, $E_2 = |F(l,w) - F^c(l,w)|$, $E_3 = |I(l,w) - I^c(l,w)|$, $T^c = F(l,w)$, $F^c = T(l,w)$ and $I^c = 1 - I(l,w)$.

2.3.1. ***NS-based image fusion algorithm***

The image fusion algorithm is drawn by the notion of NS [21], which is augmented by the upcoming steps.

(1) The representation of two test images given by $\mathbb{TI}_1$ and $\mathbb{TI}_2$. These test images have $\mathbb{L}$ levels of grayness and $\mathbb{G}(l,w)$ is the intensity of the image element at the particular position (l,w), where the (l,w) varies from 0 to 255. The test images can be written

in the following matrix design.

$$\mathbb{TI}_1 = \begin{bmatrix} \mathbb{G}(1,1) & \mathbb{G}(1,2) & \cdots & \mathbb{G}(1,w) \\ \mathbb{G}(2,1) & \mathbb{G}(2,2) & \cdots & \mathbb{G}(2,w) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{G}(l,1) & \mathbb{G}(l,2) & \cdots & \mathbb{G}(l,w) \end{bmatrix} \tag{5}$$

Here, 'l' and 'w' signifies the length and width of the test image $\mathbb{TI}_1$ and $\forall \mathbb{G}(l,w) \in \mathbb{TI}_1$. This formation is same for test image $\mathbb{TI}_2$.

(2) At first, each image element ($\forall \mathbb{G}(l,w)$) of the test images characterized individually in $\mathbb{NI}$ format (this contains true, false and indeterminate degrees) that can be mathematically encoded as $\mathbb{NI}(l,w)$, which is defined by the following matrix form.

$$\mathbb{TI}(\mathbb{NI}) = \begin{bmatrix} \langle T(1,1), F(1,1), I(1,1)\rangle & \langle T(1,2), F(1,2), I(1,2)\rangle & \cdots & \langle T(1,w), F(1,w), I(1,w)\rangle \\ \langle T(2,1), F(2,1), I(2,1)\rangle & \langle T(2,2), F(2,2), I(2,2)\rangle & \cdots & \langle T(2,w), F(2,w), I(2,w)\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle T(l,1), F(l,1), I(l,1)\rangle & \langle T(l,2), F(l,2), I(l,2)\rangle & \cdots & \langle T(l,w), F(l,w), I(l,w)\rangle \end{bmatrix}$$

(OR) (6)

$$\mathbb{TI}(\mathbb{NI}) = \begin{bmatrix} \mathbb{NI}(1,1) & \mathbb{NI}(1,2) & \cdots & \mathbb{NI}(1,w) \\ \mathbb{NI}(2,1) & \mathbb{NI}(2,2) & \cdots & \mathbb{NI}(2,w) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{NI}(l,1) & \mathbb{NI}(l,2) & \cdots & \mathbb{NI}(l,w) \end{bmatrix}$$

(3) Thereafter, each $\mathbb{NI}(l,w)^{th}$ values of test images $\mathbb{TI}_1$ and $\mathbb{TI}_2$ are assessed by the fundamental concept of NEI function in equ (4) and it is generally notified by the symbol is $\mathbb{ENT}(\mathbb{NI}(l,w))$, then their matrix representation is given in the upcoming equation

$$\mathbb{ENT}(\mathbb{NI}) = \begin{bmatrix} \mathbb{ENT}(\mathbb{NI}(1,1)) & \mathbb{ENT}(\mathbb{NI}(1,2)) & \cdots & \mathbb{ENT}(\mathbb{NI}(1,w)) \\ \mathbb{ENT}(\mathbb{NI}(2,1)) & \mathbb{ENT}(\mathbb{NI}(2,2)) & \cdots & \mathbb{ENT}(\mathbb{NI}(2,w)) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{ENT}(\mathbb{NI}(l,1)) & \mathbb{ENT}(\mathbb{NI}(l,2)) & \cdots & \mathbb{ENT}(\mathbb{NI}(l,w)) \end{bmatrix} \tag{7}$$

(4) The entropy values in the above matrix $\mathbb{ENT}(\mathbb{NI}(l,w))$ for the test images $\mathbb{TI}_1$ and $\mathbb{TI}_2$ are decomposed into $\mathfrak{p} \times \mathfrak{q}$ blocks. In general, the $\mathfrak{m}^{th}$ blocks of decomposed images are specified by the variables $\mathbb{TI}_{NS1\mathfrak{m}}$ and $\mathbb{TI}_{NS2\mathfrak{m}}$ respectively.

(5) Evaluate the sum of amount of whiteness and blackness of the two associated blocks.

(6) Each block is combined with each other with the help of the following fusion rule

$$\mathbb{TI}_{NS\mathfrak{m}}(l,w) = \begin{cases} \min\{\mathbb{TI}_{NS1\mathfrak{m}}(l,w), \mathbb{TI}_{NS2\mathfrak{m}}(l,w)\}, & \text{if count(blackness)¿count(whiteness)} \\ \max\{\mathbb{TI}_{NS1\mathfrak{m}}(l,w), \mathbb{TI}_{NS2\mathfrak{m}}(l,w)\}, & \text{if count(blackness)¡count(whiteness)} \\ \frac{\mathbb{TI}_{NS1\mathfrak{m}}(l,w)+\mathbb{TI}_{NS2\mathfrak{m}}(l,w)}{2}, & \text{otherwise} \end{cases} \tag{8}$$

Here, max & min in equ (8) indicate the maximum and minimum actions on the NS domain images respectively.

(7) The gained blocks are re-modeled to an image, then attain a fused neutrosophic image without ambiguities.

(8) In the previous step, the neutrosophic image was discovered, and then the image was regenerated to the crisp format ($\mathbb{I}_{Fused}$), by applying the reverse function of Equ (1).

2.4. *Neutrosophic Edge detection*

2.4.1. *The procedure for finding T and F subsets*

By introducing the design of bell-shaped function (BS-function), which acts as a suitable and a virtual soft computer tool for addressing the brightness level of the gray image, hence studies of BS-function are growing rapidly in many realistic areas [27]. Here, the BS-function is employed to fused image ($\mathbb{I}_{Fused}$), as a result of T subset is found out. According to the design of BS-function, the mathematical formulation of neutrosophic T subset is explored below.

$$T(l,w) = \pi(\mathbb{I}_{Fused}(l,w), b_1, b_2, b_3, b_4) \begin{cases} 0 \text{ if } 0 \leq \mathbb{I}_{Fused}(l,w) \leq b_1 \\ \frac{(\mathbb{I}_{Fused}(l,w)-b_1)^2}{(b_4-b_1)(b_4-b_1)} \text{ if } b_1 \leq \mathbb{I}_{Fused}(l,w) \leq b_2 \\ 1 - \frac{(\mathbb{I}_{Fused}(l,w)-b_2)}{(b_4-b_3)(b_4-b_3)} \text{ if } b_2 \leq \mathbb{I}_{Fused}(l,w) \leq b_3 \\ \frac{(\mathbb{I}_{Fused}-b_4)^2}{(b_4-b_3)(b_4-b_3)} \text{ if } b_3 \leq \mathbb{I}_{Fused}(l,w) \leq b_4 \\ 0 \text{ if } \mathbb{I}_{Fused}(l,w) > b_4 \end{cases} \quad (9)$$

$$F(l,w) = 1 - T(l,w) \quad (10)$$

Where $\mathbb{I}_{Fused}(l,w)$ denotes the grayness of $(l,w)^{th}$ pixel of the fused image $\mathbb{I}_{Fused}$. The input parameters b_1, b_2, b_3 and b_4 of the BS-function computes the shape of the function. The action of the BS-function is shown in Algorithm 1.

Obtaining b_1, b_2, b_3 and b_4 Parameters: The parameters b_1, b_2, b_3 and b_4 are to be computed by employing a histogram based method as explored below:

(1) Compute the histogram of the ($\mathbb{I}_{Fused}$) fused image.

(2) Obtain local maxima of the histogram
$\mathbb{I}_{Fused}\max(g_1), \mathbb{I}_{Fused}\max(g_2), \mathbb{I}_{Fused}\max(g_3), ..., \mathbb{I}_{Fused}\max(g_n)$.

(3) Obtain the mean of local maxima
$\overline{\mathbb{I}_{Fused}\max} = \left(\sum_{i=1}^{\mathbb{N}} \mathbb{I}_{Fused}\max(g_i)\right) / \mathbb{N}$

(4) Evaluate (peak values ¿$\overline{\mathbb{I}_{Fused}\max}$) then
$b_2 \leftarrow$ First peak value
$b_3 \leftarrow$ Last peak value

Algorithm 1 BS-Function Algorithm

if $0 \leq \mathbb{I}_{Fused}(l,w) \leq b_1$ **then**

$\quad T(l,w) \leftarrow 0$

else if $b_1 \leq \mathbb{I}_{Fused}(l,w) \leq b_2$ **then**

$\quad T(l,w) \leftarrow \frac{(\mathbb{I}_{Fused}(l,w)-b_1)^2}{(b_4-b_1)(b_4-b_1)}$

else if $b_2 \leq \mathbb{I}_{Fused}(l,w) \leq b_3$ **then**

$\quad T(l,w) \leftarrow 1 - \frac{(\mathbb{I}_{Fused}(l,w)-b_2)}{(b_4-b_3)(b_4-b_3)}$

else if $b_3 \leq \mathbb{I}_{Fused}(l,w) \leq b_4$ **then**

$\quad T(l,w) \leftarrow \frac{(\mathbb{I}_{Fused}(l,w)-b_4)^2}{(b_4-b_3)(b_4-b_3)}$

else if $\mathbb{I}_{Fused}(l,w) > b_4$ **then**

$\quad T(l,w) \leftarrow 0$

end if

end if

(5) Find out the standard deviation (std) of the fused image

$$std = \left(\frac{1}{\mathbb{N}}\sum_{i=1}^{\mathbb{N}}(\mathbb{I}_{Fused}(i) - \overline{\mathbb{I}_f})^2\right)^{1/2}, \text{ where } \overline{\mathbb{I}_f} = \frac{1}{\mathbb{N}}\sum_{i=1}^{\mathbb{N}}\mathbb{I}_{Fused}(i)$$

(6) Obtaining the values of b_1 and b_4 are given below

$b_1 \leftarrow b_2 - std$

$b_4 \leftarrow b_3 + std.$

2.4.2. ***The procedure for finding I subset***

The neutrosophic subset I was determined by utilizing Zhang et al. [11] method. One of the significant research domains is Homogeneity ($homo$) because it plays a vital role in edge detection schemes, and it is correlated with local information. In some situations like background transitions, color transitions, and edge regions, this domain ($homo$) value is increased. Algorithm 2 refers to the process for determining the I subset. The $homo$ consists of two main parts such as standard deviation ($std.div$) and the discontinuity (dis) of the gray level pixels. The $std.div$ is one of the most effective methods for representing the contrast level of the pixels, whereas the changes in the gray pixels are symbolized in dis. The $std.div$ and dis are enumerated at beginning of algorithm 2. The edge values are illustrated by utilizing the *dis* of the pixel (l, w). The Sobel operator is a frequently used system to provide *dis* of a pixel in the image due to its appropriateness.

Algorithm 2 The process of obtaining the I subset

1: Determine the each coordinate pixel $(l, w)^{th}$ *std.div* of the $\mathbb{I}_{Fused}$ image is given by

$$std.div(l,w) = \sqrt{\frac{\sum_{p=l-(D-1)/2}^{l+(D-1)/2} \sum_{q=w-(D-1)/2}^{w+(D-1)/2} \left(\mathbb{I}_{Fused}(l,w) - \frac{\sum_{p=l-(D-1)/2}^{l+(D-1)/2} \sum_{q=w-(D-1)/2}^{w+(D-1)/2} \mathbb{I}_{Fused}(l,w)}{D^2} \right)^2}{D^2}} \tag{11}$$

2: Illustrate every pixel coordinate $(l, w)^{th}$ *dis* of the $\mathbb{I}_{Fused}$ image by applying the Sobel operator, which is given below

$$dis(l,w) = \sqrt{G_x^2 + G_y^2} \tag{12}$$

where, G_y and G_x indicates the vertical and horizontal derivatives.

3: Find out the $homo(l, w)$ of each pixel coordinate (l, w) of the $\mathbb{I}_{Fused}$ and it is represented as

$$homo(l,w) = 1 - \frac{std.div(l,w)}{max(std.div)} \times \frac{dis(l,w)}{max(dis)} \tag{13}$$

4: Compute the indeterminant subset $I(l, w)$ of the $\mathbb{I}_{Fused}$ image and it is defined by

$$I(l,w) = 1 - homo(l,w) \tag{14}$$

2.4.3. ***Finding binarized T, F and I Subsets***

The computed subsets T, F, and I (previous step) are in the grayscale domain, hence the subsets are transformed into a binarized form (Black and White image) by employing the threshold values. The thresholds are an important and main kind of tool in segmentation schemes and they perform automatically and spontaneously in image processing. The threshold values are directly determined by the following entropies.

Implemented entropy types:

In our segmentation process, some specific entropies are considered to estimate the threshold value within the subsets T, F, and I. Then, the edge detection assessment is tested in terms of selected entropies. The implemented entropies are namely Norm, Threshold, Sure, and Shannon. The mathematical design of these entropies is displayed in the upcoming points.

(1) Norm Entropy= $\frac{1}{L \times W} \sum_{l=1}^{L} \sum_{w=1}^{W} |T(l,w)|^p$, where $1 \leq p < 2$

(2) Threshold Entropy=$\frac{1}{L\times W}\sum_{l=1}^{L}\sum_{w=1}^{W}\begin{cases}Thres(l,w)=1 \text{ if } |T(l,w)|>\epsilon\\ Thres(l,w)=0 \text{ if } |T(l,w)|\leq\epsilon\end{cases}$, where ϵ is a positive threshold value.

(3) Sure Entropy=$\frac{1}{L\times W}\sum_{l=1}^{L}\sum_{w=1}^{W}\begin{cases}Sure(l,w)=min(T^2(l,w),\epsilon^2), \text{ if } |T(l,w)|\leq\epsilon\\ Sure(l,w)=0, \quad otherwise\end{cases}$, where ϵ is a positive threshold value.

(4) Shannon Entropy=$\frac{-1}{L\times W}\sum_{l=1}^{L}\sum_{w=1}^{W}T^2(l,w).log_2(T^2(l,w))$

The same points are repeated in the remaining subsets F and I to get all threshold values of the subsets T, F, and I. The variables $Object$, $Edge$ and $Background$ are received in this step. The process of these variables is depicted in Algorithm 3. In this segmentation analysis, the respective parameters T_t, F_f, and I_i are implemented for $Object$, $Edge$ and $Background$ variables to be acquired. The threshold values of the parameters T_t, F_f and I_i are determined with the help of neutrosophic subsets (T, F, I) and before mentioned four entropies. Subsequently, the $\mathbb{I}_{Fused}$ image is divided into three sub-images namely Object, Background and Edge by using the calculated thresholds. The achieved sub-images are combined with each other to get a new $\mathbb{I}_{Binary}$ image, which returns the 0 (Black), 1 (White) values from the image.

2.4.4. ***Edge detection process***

Edge detection is accomplished by using the images $\mathbb{I}_{Fused}$ and $\mathbb{I}_{Binary}$. The scheme of edge detection is expressed in Algorithm 4. Morphological operations were achieved on the image $\mathbb{I}_{Binary}$ then the edge of the image is acquired. Further, the obtained edges are reassigned to the variable $\mathbb{I}_{Edge}$. Ultimately, to earn a new edge detected image $\mathbb{I}_{ED}$, the $\mathbb{I}_{Edge}$ is exaggerated in the fused image, and its edges are detected.

3. **Performance measurement factors**

In this research, performance measurement factors such as Figure of Merit (FOM), Peak Signal to Noise Ration (PSNR) were taken into account, which was applied to find the success of the edge detection method. The FOM [44] is computed using the given form

$$FOM=\frac{1}{\max(N_{ED},N_{EA})}\sum_{i=1}^{N_{ED}}\frac{1}{1+dm^2(i)} \tag{15}$$

Here, the symbols N_{ED} and N_{EA} denote the number of detected edge pixels by the edge detection method and actual edge pixels. The notations m(i) and d respectively denote the closest distance to the actual edge and scaling constant. The FOM value is directly proportional to the quality of the discovered edges. Although the FOM result fluctuates between 0 and 1, the efficiency of edge detection is much better if the above value meets 1.

Algorithm 3 Edge detection process

If $(T(l,w) \geq T_t$ **AND** $I(l,w)$¡I_i**)**
  $Object(l,w) \leftarrow$ true
else
  $Object(l,w) \leftarrow$ false
end
If $(T(l,w)$ ¡ T_t **OR** $F(l,w)$¡F_f**)** **AND** $I(l,w) \geq I_i$**)**
  $Edge(l,w) \leftarrow$ true
else
  $Edge(l,w) \leftarrow$ false
end
If ($F(l,w) \geq F_f$ **AND** $I(l,w)$¡I_i**)**
  $Background(l,w) \leftarrow$ true
else
  $Background(l,w) \leftarrow$ false
end
If $(Object(l,w)$ **OR** $Edge(l,w)$ **OR** $\overline{Background(l,w)})$=**true**
  $\mathbb{I}_{Binary}(l,w) \leftarrow$ true
else
  $\mathbb{I}_{Binary}(l,w) \leftarrow$ false
end

Algorithm 4 Edge detection algorithm

$\mathbb{I}_{Dilation} \leftarrow$ Image dilation activation $\mathbb{I}_{Binary}$
$\mathbb{I}_{Fill} \leftarrow$ Fills the internal gaps on the $(\mathbb{I}_{Dilation})$ image
$\mathbb{I}_{Clear} \leftarrow$ Extract the connected objects on the $(\mathbb{I}_{Fill})$ image
$\mathbb{I}_{Skeleton} \leftarrow$ Identify the skeleton image $(\mathbb{I}_{Clear})$
$\mathbb{I}_{Edge} \leftarrow$ Determines the complement of the $(\mathbb{I}_{Skeleton})$ image
$\mathbb{I}_{ED} \leftarrow$ Fused image $(\mathbb{I}_{Fused})$
$[Length\ Width] \leftarrow$ Computes the $(\mathbb{I}_{Edge})$ image size
for l=1:$Length$
for w=1:$Width$
if $\mathbb{I}_{Edge}(l,w) \leftarrow 0$
  $\mathbb{I}_{ED}(l,w,1) \leftarrow 255$
  $\mathbb{I}_{ED}(l,w,2) \leftarrow 0$
  $\mathbb{I}_{ED}(l,w,3) \leftarrow 0$
end
end
end

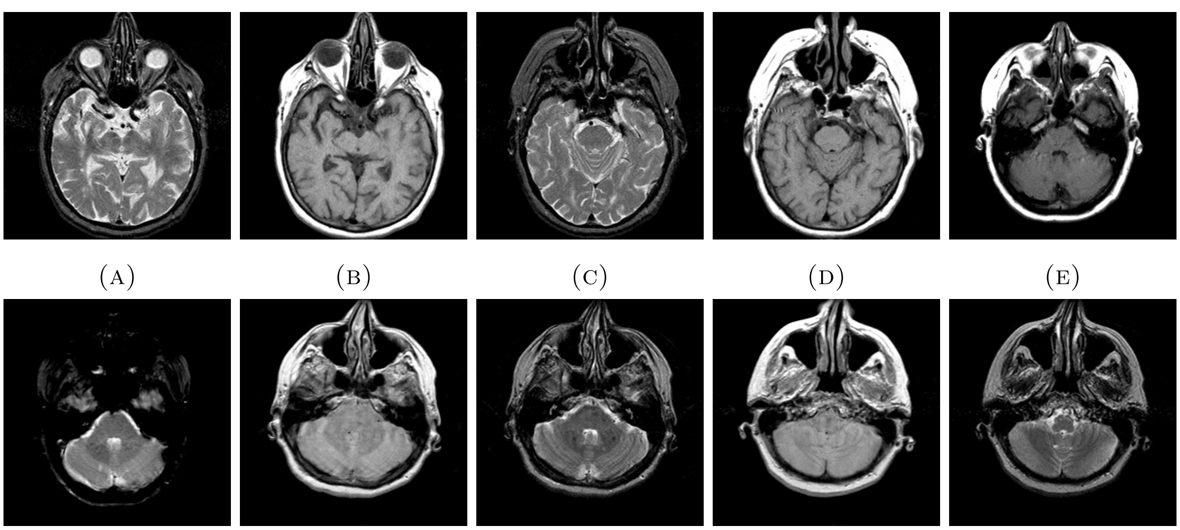
On the other hand, PSNR [45], which was recommended by the author Pratt, is a remarkable factor in evaluating the effectiveness of edge detection. Equ. (16) is the formula used to do the PSNR analysis. PSNR is determined using the formula below.

$$PSNR = 10.log_{10}\left[\frac{255^2}{\frac{1}{L\times W}\sum_{l=1}^{L}\sum_{w=1}^{W}(N_{EA}(l,w)-N_{ED}(l,w))^2}\right] \tag{16}$$

where L, W specifies the dimensions of the image. $N_{EA}(l,w)$ and $N_{ED}(l,w)$ indicates the $(l,w)^{th}$ image element of the actual edges and the detected image edges obtained by using proposed technique. The largest value of PSNR specifies the high similarity between actual and detected image edges. If the PSNR reaches its minimum, then the similarity will be reduced.

4. Experimental results

In this section, different test images of MR brain were considered for our experiment and are aligned in the proper manner as displayed in Figure. 2. The MR brain medical images were gathered from the link address http://www.med.harvard.edu/AANLIB/home.html. The medical images of MR brain (Figure. 2 (A) and 2 (B)) are provided for the purpose of fusion by the proposed fusion technique, and then the fused image result is given in the Figure. 3 (A). The another MR brain images (Figure 2 (C) and 2 (D)) are utilized for the proposed fusion technique and their fused result is presented in the Figure. 3 (B). Similarly, the fusion results of remaining MR brain medical images (Figure 2 (E) - (F), Figure 2 (G) - (H) and Figure 2 (I) - (J)) from the proposed NS-technique were shown in Figure. 3 (C) - 3 (E).



(A) (B) (C) (D) (E)
(F) (G) (H) (I) (J)

FIGURE 2. **Test images of medical MR brain**

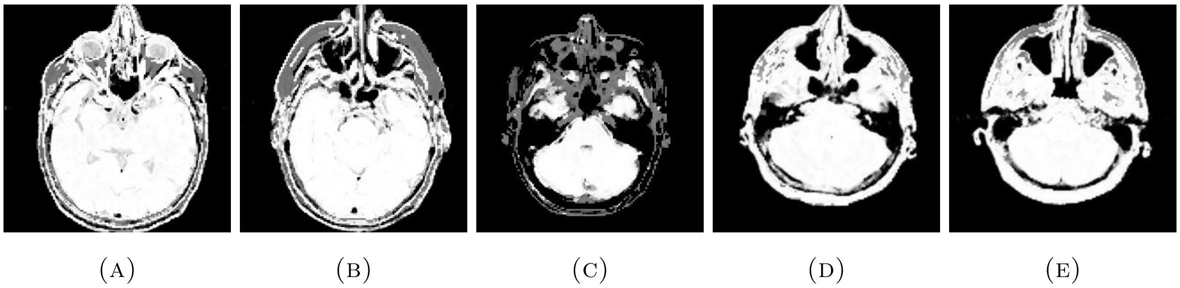


FIGURE 3. **Fusion results of medical MR brain images**

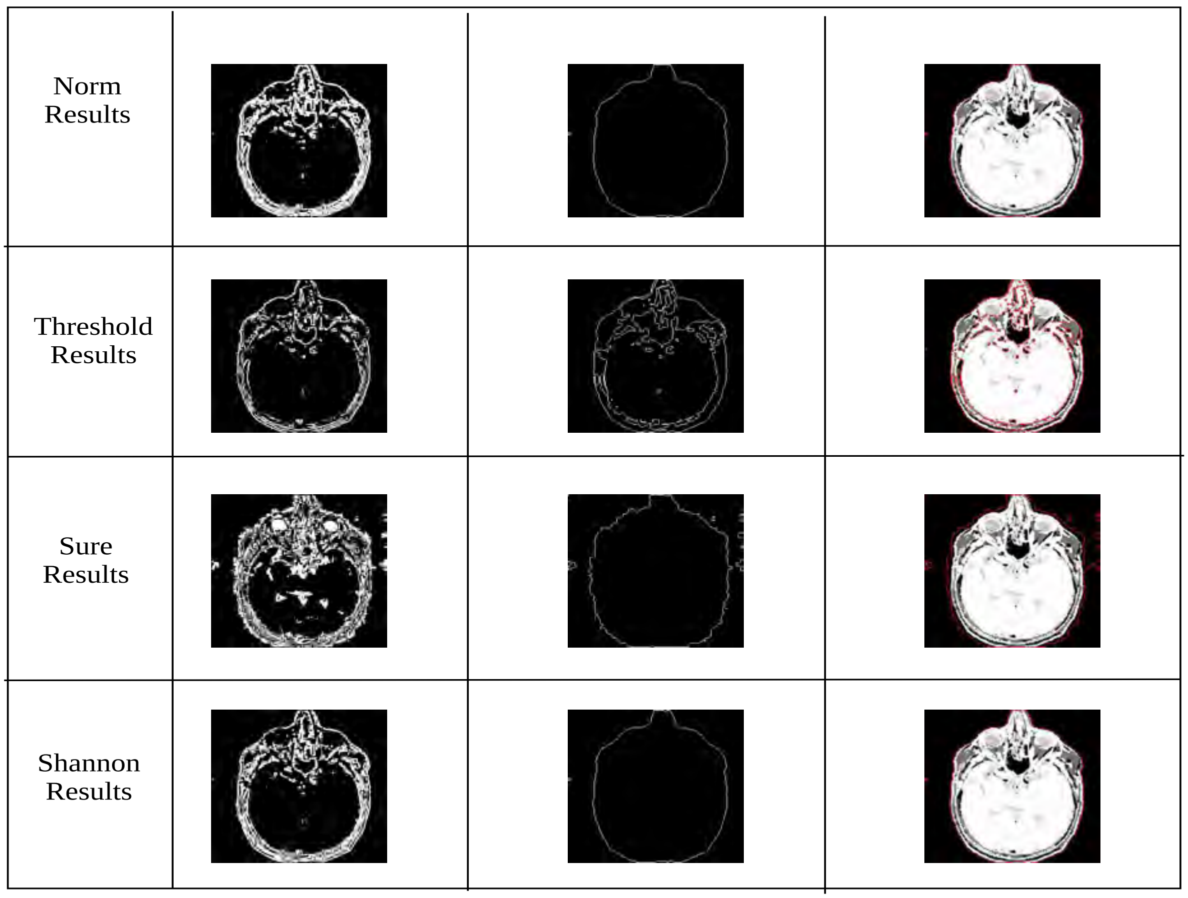


FIGURE 4. **Edge detection of fused image 3(A) using different entropies**

4.1. *Entropy performance test*

To examine the achievement of edge detection for the Norm entropy used in the proposed method, which was compared with some other entropies in NS-based edge detection methods including Threshold, Sure, and Shannon. During this manner, five fused images that are tough to find on the edge are applied. In the initial method analysis, the first fused image of MR

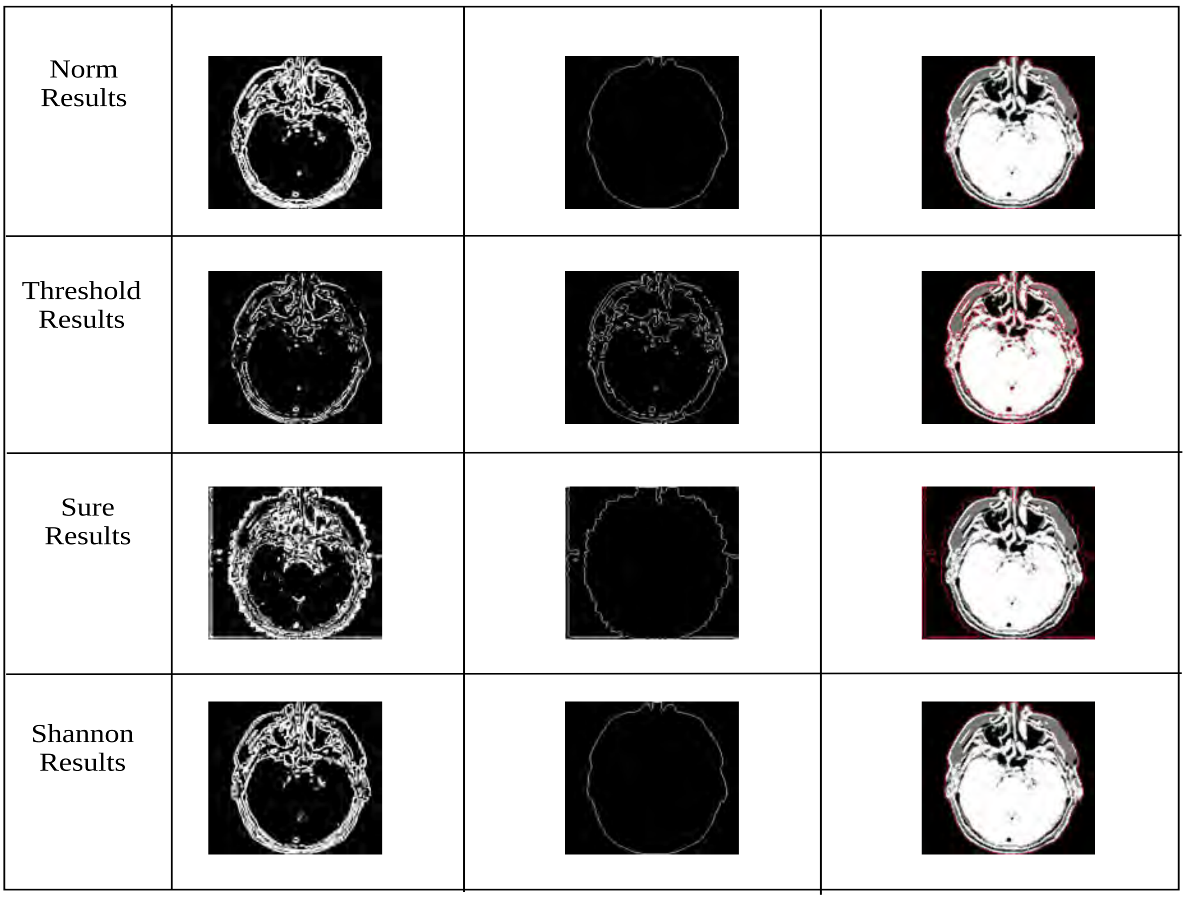


Figure 5. **Edge detection of fused image 3(B) using different entropies**

Table 1. **The edge detection results for fused images by using FOM and PSNR analysis.**

Fused	**Norm entropy**		**Threshold entropy**		**Sure entropy**		**Shannon entropy**	
Images	FOM	PSNR	FOM	PSNR	FOM	PSNR	FOM	PSNR
Image-1	**0.93**	**33.01**	0.86	29.79	0.83	28.94	0.90	31.16
Image-2	**0.96**	**34.20**	0.92	32.30	0.90	31.48	0.95	33.40
Image-3	**0.91**	**31.65**	0.85	29.48	0.83	28.70	0.83	30.36
Image-4	**0.95**	**33.88**	0.91	31.06	0.88	30.52	0.93	32.50
Image-5	**0.88**	**30.85**	0.82	28.95	0.80	27.61	0.85	29.99

brain (Figure. 3 (A)) is utilized, which is then converted by a neutrosophic field in terms of three subsets by applying the BS-function and Sobel operator. Then, the subset thresholds of fused MR brain image are calculated by implementing the above four entropies. Further, the subsets are to be generated in the binary form with the help of obtained thresholds for each

TABLE 2. **Statistical effects about edge detection results.**

	Norm entropy		**Threshold entropy**		**Sure entropy**		**Shannon entropy**	
	FOM	PSNR	FOM	PSNR	FOM	PSNR	FOM	PSNR
Minimum	**0.88**	**30.85**	0.82	28.95	0.80	27.61	0.85	29.99
Average	**0.93**	**32.72**	0.87	30.32	0.85	29.45	0.89	31.48
Maximum	**0.96**	**34.20**	0.92	32.30	0.90	31.48	0.95	33.40

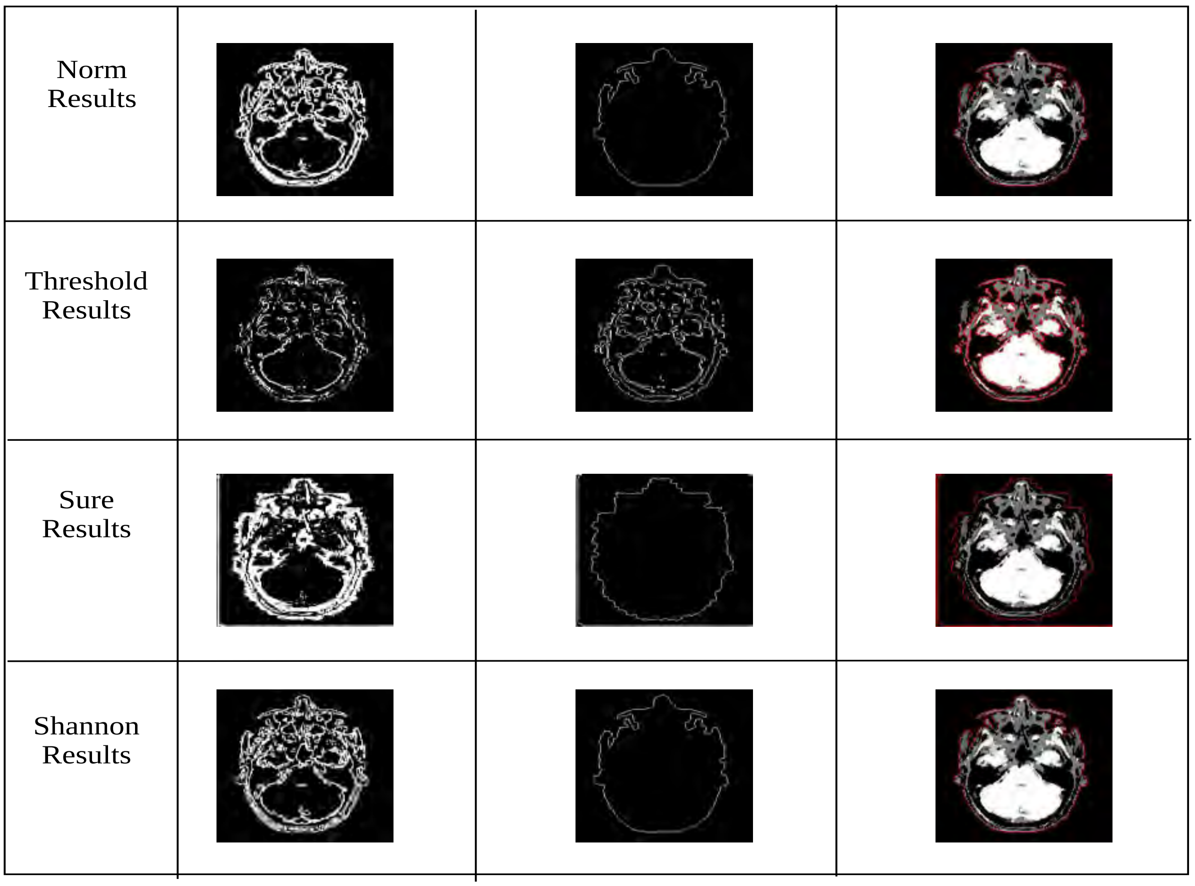


FIGURE 6. **Edge detection of fused image 3(C) using different entropies**

entropy. By using algorithm 3, the binarised subsets of each entropies were combined with each other in order to obtain $\mathbb{I}_{Binary}$ images, which are presented in the first column of Figure. 4. Furthermore, the morphological operations are executed in the obtained $\mathbb{I}_{Binary}$ images, and the detected results are also shown in the last column of the Figure. 4. In addition, the performance measurement factors such as FOM, and PSNR were analyzed for the edge detected images at different entropies to confirm edge detection performance. The statistical values of the performance measurement factors with different entropies of the proposed mechanism are

TABLE 3. **The edge detection results for fused images by using FOM and PSNR analysis.**

Fused	**Sobel Method**		**Tian Method**		**Chan Method**		**Wu Method**		**Proposed Method**	
Images	FOM	PSNR	FOM	PSNR	FOM	PSNR	FOM	PSNR	FOM	PSNR
Image-1	0.89	31.41	0.90	32.29	0.92	32.45	0.93	33.50	**0.95**	**33.88**
Image-2	0.90	31.47	0.91	31.94	0.92	32.01	0.94	32.95	**0.96**	**33.58**
Image-3	0.89	30.89	0.91	31.87	0.94	33.12	0.95	33.80	**0.96**	**34.20**
Image-4	0.81	25.35	0.83	28.70	0.85	28.24	0.86	31.71	**0.89**	**32.66**
Image-5	0.86	30.11	0.87	30.43	0.88	30.88	0.91	31.36	**0.93**	**32.92**

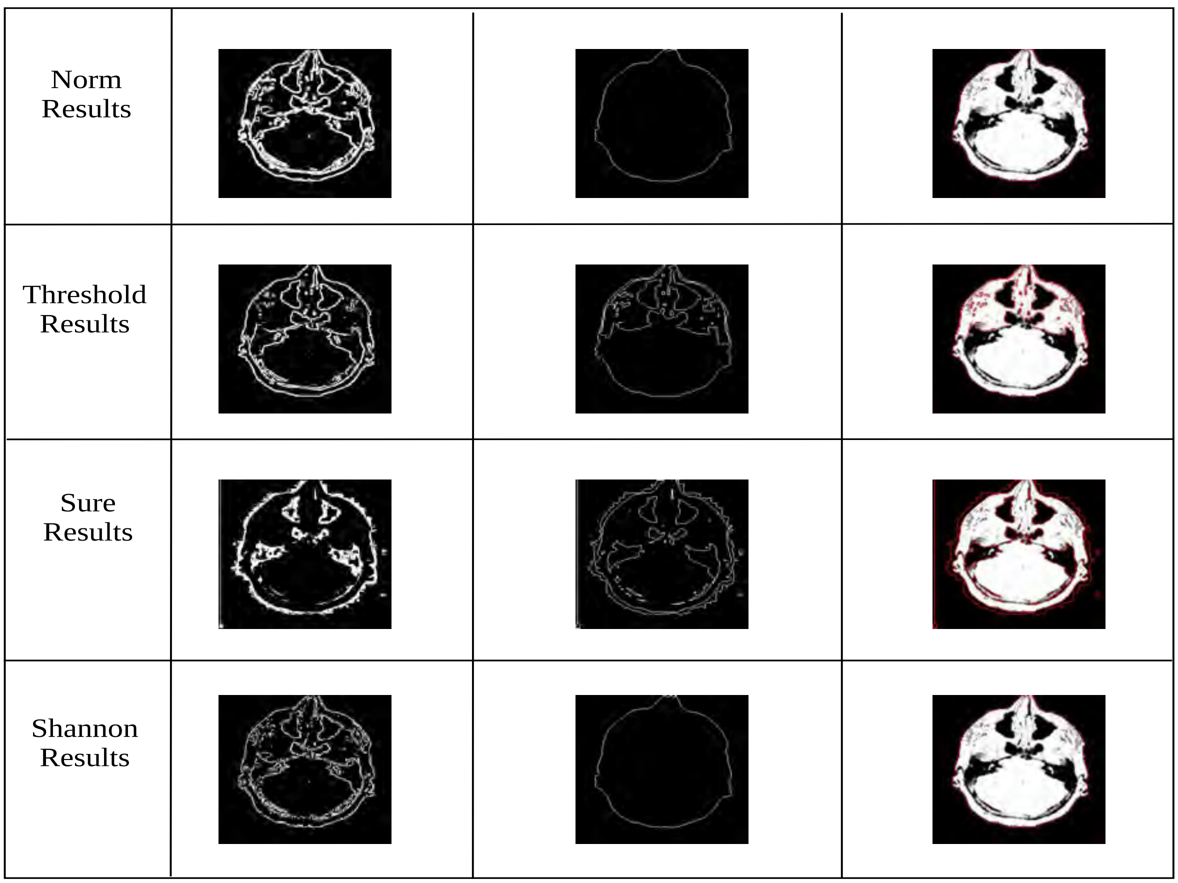


FIGURE 7. **Edge detection of fused image 3(D) using different entropies**

displayed in Tables 1 - 2. From the Tables. 1 and 2 , the high-performance entropy of the proposed edge detection mechanism is highlighted in bold characters and the tabulated values are plotted in Figures. 10 (a)- 10 (b). At last, Table. 1 and Figure. 10 shows that the proposed Norm entropy in the edge detection mechanism gives the significant performance with reference to other entropies.

TABLE 4. **Statistical effects about edge detection results.**

	Sobel Method		Tian Method		Chan Method		Wu Method		Proposed Method	
	FOM	PSNR	FOM	PSNR	FOM	PSNR	FOM	PSNR	FOM	PSNR
Minimum	0.81	25.35	0.83	28.70	0.85	29.24	0.86	31.36	**0.89**	**32.66**
Average	0.87	29.85	0.88	31.05	0.90	31.34	0.92	32.67	**0.94**	**33.45**
Maximum	0.90	31.47	0.91	31.94	0.94	33.12	0.95	33.80	**0.96**	**34.20**

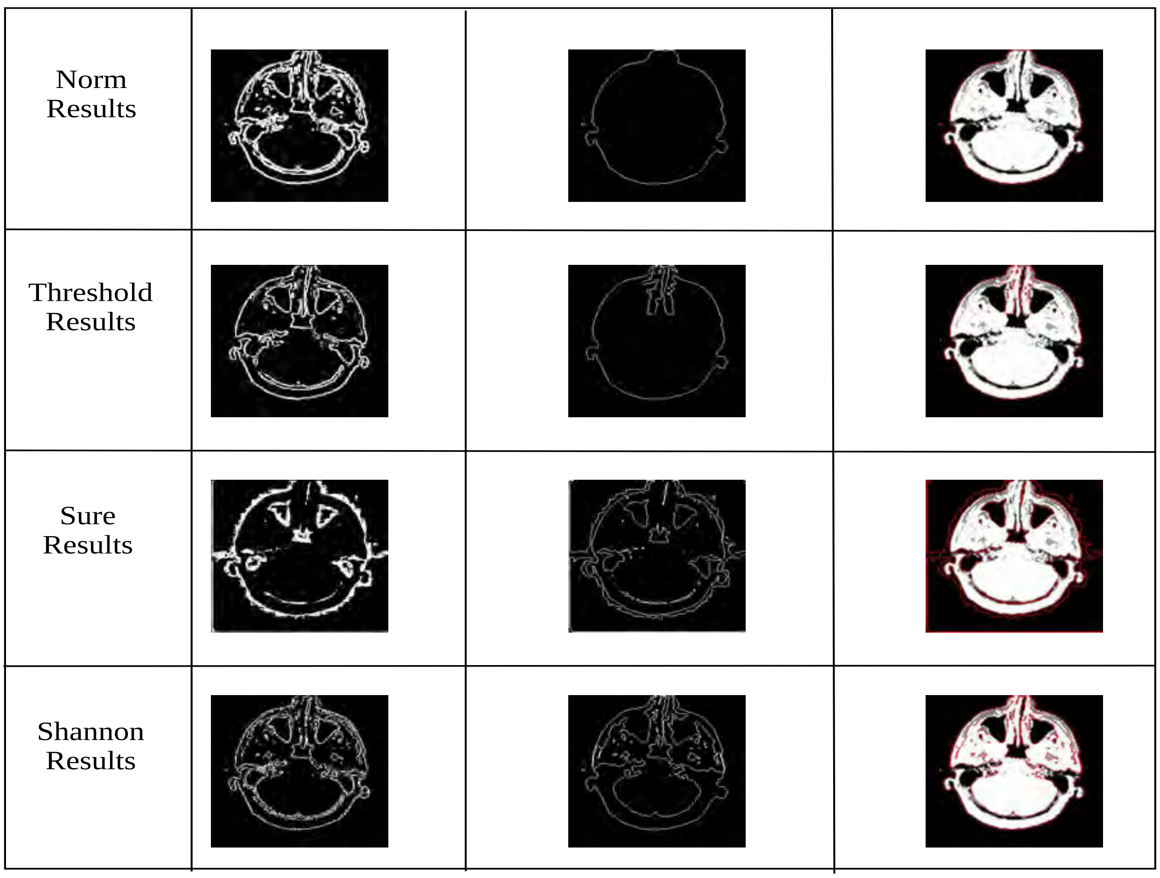


FIGURE 8. **Edge detection of fused image 3(E) using different entropies**

Subsequently, the second fused image of the MR brain (Figure. 3 (B)) is considered for the edge detection process. Firstly, the taken image is transferred to the domain of NS and it includes the terms of three subsets. Subsets can be calculated using the BS function and the Sobel operator. Each subset threshold value is determined by the concept of the different

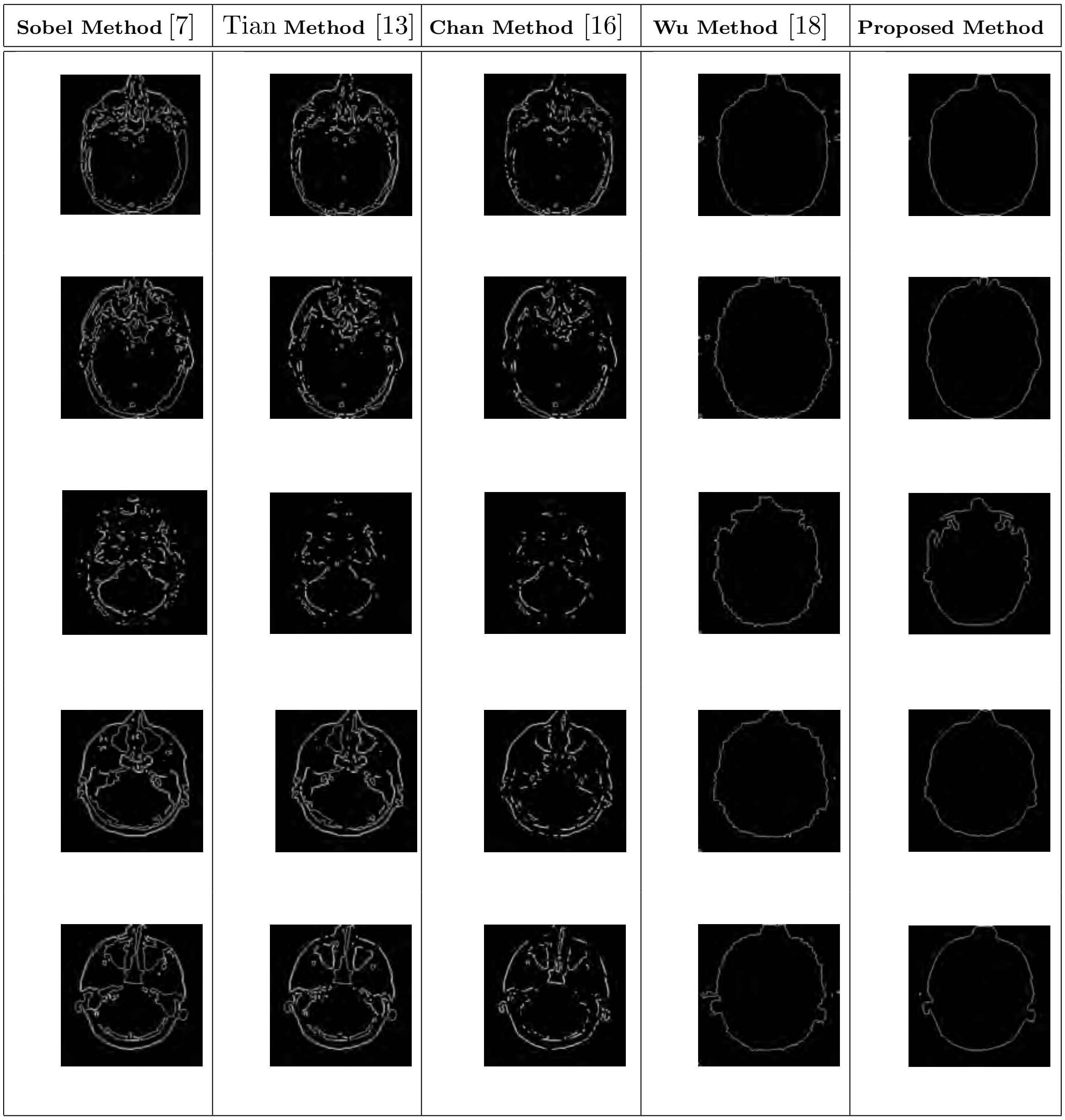
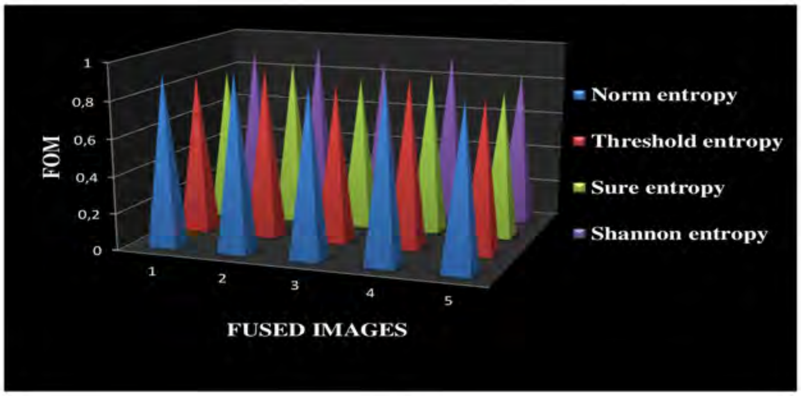
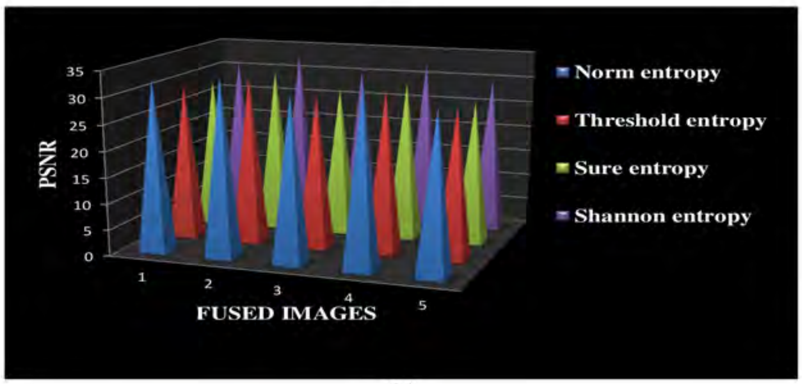


FIGURE 9. **Edge detection results for different methods**

entropies. Using each entropy threshold value, three subsets are generated by the form of binary images and the resulting images are combined by using algorithm 3 and it produces the new image called as $\mathbb{I}_{Binary}$. Each entropy of the $\mathbb{I}_{Binary}$ images is exhibited in the first column of the Figure. 5. Moreover, the computed $\mathbb{I}_{Binary}$ images are processed by the morphological operations, and the gained images are shown in the last column of the Figure. 5. Images detected in distinct entropy are intended to confirm the effectiveness of the proposed edge detection by the measurement factors and the statistical values of the measurement factors that are presented in Tables. 1 - 2. The table values (Table 1) of FOM and PSNR are plotted, which are given in Figure. 10 (a) - (b). Finally, the Figures 10 (a) and 10 (b) provide the result as a Norm entropy with the proposed approach offers higher accuracy than other existing entropy because it offers higher FOM and PSNR values.



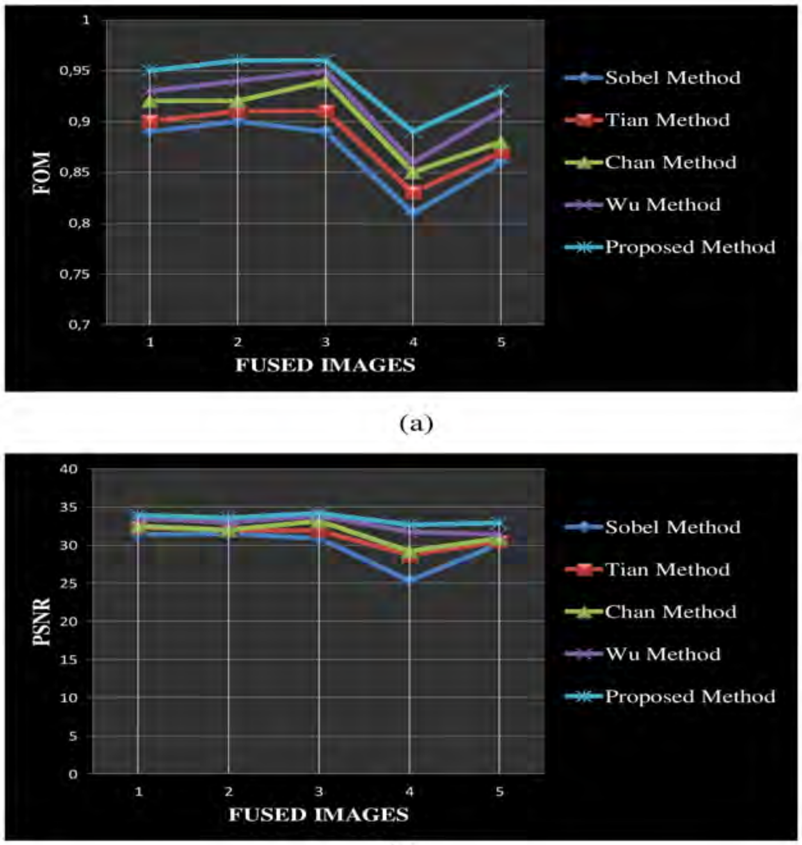
(a)



(b)

FIGURE 10. **Geometric representation of FOM and PSNR analysis for various entropies.**

In this continuation, 4 distinct entropies and remaining medical MR brain images are considered for the analysis. The complete information of the previously mentioned test process is employed in the considered images, which are presented in Figure. 2 (E) - 2 (J). At this end, 5 edge detected images were found utilizing the proposed mechanism and these images are given in Figures. 6 - 8. In consequence, the statistical values of the measurement factors in the edge detected images are gained under the proposed edge detection mechanism with 5 MR brain fused images and 4 individual entropies, which are displayed in the aforementioned Tables 1 and 2. In Tables 1 - 2, the Norm entropy produces maximum values in the measurement factors FOM and PSNR values whereas Sure entropy has the minimal FOM and PSNR values.



(b)

FIGURE 11. **Geometric representation of FOM and PSNR analysis for different methods.**

Then, the tabulated results (Table 1) of the remaining resultant images are also made available in the Figures. 10 (a) - (b). Thus with the acquired results, it is evidenced that the norm entropy provides constructive performance. While Sure entropy reveals contrast results which have been evidenced in Figure. 10 and Tables. 1 - 2. Hence our study implicates the efficacy of norm entropy as a sustainable tool in the edge detection process of MR brain images.

4.2. *Comparative applications of the proposed and different object edge detection methods*

Norm entropy with the proposed method was compared to other existing methods which includes Sobel [7], Tian [13], Chan [16] and Wu [18] to assess the proposed edge detection performance. In the aforementioned methods, the edges are seen to be thicker. As a result, the final image was subjected to morphological operations. To examine the edge detection efficiency, five fused images utilized in entropy analysis were applied, while the five edge detection methods mentioned earlier were employed to these same images. Further, the fused image edges are depicted in Figure 9. The proposed Norm entropy, as shown in Figure. 9, was discovered to be the most effective edge detection method. The performance measurement factors such as FOM and PSNR were executed to compare the real performance of detected edges of the mentioned fused images. Table 3 also includes FOM and PSNR factor data, which are plotted in Figures. 11 (a) - (b), respectively. The proposed method's FOM and PSNR factor values are superior to those acquired by the other four edge detection methods. Moreover, statistical analysis of the FOM and PSNR outcomes earned by the five edge detection methods was performed, and these outcomes are given in Table 4. Also, in Table 4 it can be observed that the minimum, arithmetic average, and maximum values related to the FOM and PSNR effects achieved in the suggested method are larger than the other method. Those effects intimate that this proposed method offers edge detection through a greater level compared via distinct methods.

For the presented Table 2, the average values of the Norm entropy of FOM and PSNR are 0.93 and 32.72, respectively. For these values are larger compared to the different other entropies that can be recommended that the Norm entropy under NS improves the edge detection efficacy. Similarly, in Table 4 given, the average values based on the proposed method of FOM and PSNR are 0.94 and 33.45 respectively. These results are extremely high when compared to the other methods. The most important finding from the experiment is that the suggested method significantly outperforms other methods in detecting the object's edges because these effects are significantly larger when compared to other methods.

5. Conclusion

In this report, a new edge detection mechanism based on the category of the NS scheme has been nominated. This was achieved by the frame of image fusion with edge detection. This mechanism is capable to handle the uncertainties and indeterminant situations of the images. First, the proposed image fusion has been found with the help of NS structure and fusion rules. As a result, images are reproduced by the NS framework using the BS function and the Sobel operator. The resultant images contain three subsets and their values of the threshold are found using distinct entropies. Afterward, the computed thresholds and the subsets are integrated and it produces a new binarised image. Morphological operations are accomplished in the binarised image then the edges of the image are acquired. The same process acts on each entropy and it presents several detected edges. The efficacy of this mechanism is well established through the measurement factors. The given edge detection scheme is also appropriately implemented when a norm entropy appears. Further, the proposed method based on norm entropy was compared with other methods including Sobel, Chan, Tian, and Wu. According to the statistical results of performance measurement factors such as FOM and PSNR, the mentioned factor values of the proposed method are greater compared to the other existing four methods, which illustrate the superiority of the proposed mechanism.

In future research, the neutrosophic set and its extensions will be further applied in medical image processing such as image denoising, segmentation, etc.

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Conflicts of Interest

The authors declare no conflict of interest.

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Eigenspace of a Circulant Fuzzy Neutrosophic Soft Matrix

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Abstract. The eigen problem of a Circulant Fuzzy Neutrosophic Soft Matrix (CFNSM) in max-min Fuzzy Neutrosophic Soft Algebra (FNSA) is analyzed. By describing all possible types of Fuzzy Neutrosophic Soft Eigenvectors (FNSEvs), eigen space structures of CFNSM is characterized.

Keywords: Fuzzy Neutrosophic Soft Set (FNSS), Fuzzy Neutrosophic Soft Matrices(FNSMs), Circulant Fuzzy Neutrosophic Soft Matrix (CFNSM), Fuzzy Neutrosophic Soft Eigenvectors (FNSEvs).

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1. **Introduction**

In real world, we face so many uncertainties in all walks of life. However most of the existing mathematical tools for formal modeling, reasoning and computing are crisp and precise in character. There are theories viz,theory of probability, evidence, fuzzy set [31], intuitionistic fuzzy set [3], neutrosophic set [26], vague set, interval mathematics, rough set for dealing with uncertainties. These theories have their own difficulties as pointed out by Molodtsov [21]. In 1999, Molodtsov [21] initiated a novel concept of soft set theory, which is completely a new approach for modeling vagueness and uncertainties. Soft set theory has a rich potential for application in solving practical problems in economics, social science, medical science etc.. Later on Maji et al. [22] have proposed the theory of fuzzy soft set. Maji et al. [18,19] extended soft sets to intuitionistic fuzzy soft sets and neutrosophic soft sets.

Eigenvectors of a max-min matrix characterize stable state of the corresponding discrete-events system. Investigation of the max-min eigenvectors of a given matrix is therefore of a great practical importance. The eigenproblem in max-min algebra has been studied by many authors. Interesting results were found in describing the structure of the eigenspace, and algorithms for computing the maximal eigenvector of a given matrix were sugested, see e.g. [5, 6, 23, 24, 31, 32]. The structure of the eigenspace as a union of intervals of increasing eigenvectors is described in [7].

Fuzzy matrices defined first time by Thomason in 1977 [25] and he discussed about the convergence of the powers of a fuzzy matrix. The theory of fuzzy matrices were developed by Kim and Roush [16] as an extension of Boolean matrices. Manoj Bora *et al.* [20] have applied intuitionistic fuzzy soft matrices in the medical diagnosis problem. Arockiarani and Sumathi [1, 2] introduced Fuzzy Neutrosophic Soft Matrix (FNSM) and used them in decision making problems. Broumi *et al.* [4] proposed the concept of generalized interval neutrosophic soft set and studied their operations. Also, they presented an application of it in decision making problem. First time Kavitha *et al.* [10–13, 15] introduced the concept of unique solvability of max-min operation through FNSM equation $Ax = b$ and explained strong regularity of FNSMs over fuzzy neutrosophic soft algebra and computing the greatest X-eigenvector of fuzzy neutrosophic soft matrix. They also introduced the power of FNSM and Periodicity of Interval Fuzzy Neutrosophic Soft Matrices. Murugadas *et al.* proposed the ideas of the Monotone interval fuzzy neutrosophic soft eigenproblem and Solveability of System of Netrosophic Soft Linear Equations in [17]. In [30], Uma et. al, introduced the concept of FNSMs of Type-1 and Type-2.

By max-min FNSA we understand a triplet $(\mathcal{N}, \oplus, \otimes)$, where $\mathcal{N}$ is a linearly ordered FNSS, and $\oplus = \max$, $\otimes = \min$ are binary operations on $\mathcal{N}$. The notation $\mathcal{N}_{(n,n)}, \mathcal{N}_{(n)}$ denotes the set of all Fuzzy Neutrosophic Soft Square Matrices(FNSSMs) (all FNSVs) of given dimension n over $\mathcal{N}$. Operations $\oplus, \otimes$ are extended to FNSMs and FNSVs in formal way.

The eigenproblem for a given FNSM $A \in \mathcal{N}_{(n,n)}$ in max-min FNSA consists of finding a FNSV $\langle x^T, x^I, x^F\rangle \in \mathcal{N}_{(n)}$ (FNSEv) such that the equation
$A \otimes \langle x^T, x^I, x^F\rangle = \langle x^T, x^I, x^F\rangle$ holds true. By the eigenspace of a given FNSM we mean the set of all its FNSEvs.

In this paper the eigenspace structure for a special case of so-called CFNSMs is studied. The paper presents a detailed description of all possible types of FNSEvs of any given CFNSM.

2. Preliminaries

In this section, some basic notions related to this topics are recalled.

Definition 2.1. [26] A neutrosophic set A on the universe of discourse X is defined as $A = \{\langle x,\ T_A(x),\ I_A(x),\ F_A(x)\rangle, x \in X\}$, where $T, I, F : X \to\ ]^-0, 1^+[$ and
$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$. (1)]

From philosophical point of view the NS set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application especially in Scientific and Engineering problems it is difficult to use NS with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the NS which takes the value from the subset of $[0, 1]$. Therefore we can rewrite equation (1) as $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. In short an element $\widetilde{a}$ in the NS A, can be written as $\widetilde{a} = \langle a^T, a^I, a^F \rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy, a^F denotes degree of falsity such that $0 \leq a^T + a^I + a^F \leq 3$.

Definition 2.2. [1] A NS A on the universe of discourse X is defined as $A = \{x, \langle T_A(x), I_A(x), F_A(x) \rangle, x \in X\}$, where $T, I, F : X \rightarrow [0,1]$ and $0 \leq T_A(x)+I_A(x)+F_A(x) \leq 3$.

Definition 2.3. [21] Let U be the initial universe set and E be a set of parameter. Consider a non-empty set $A, A \subset E$. Let $P(U)$ denotes the set of all NSs of U. The collection (F, A) is termed to be the NSS over U, where F is a mapping given by $F : A \rightarrow P(U)$. Here after we simply consider A as NSS over U instead of (F, A).

Definition 2.4. [2] Let $U = \{c_1, c_2, ..., c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1, e_2, ..., e_m\}$. Let $A \subset E$. A pair (F, A) be a NSS over U. Then the subset of $U \times E$ is defined by $R_A = \{(u, e);\ e \in A,\ u \in F_A(e)\}$
which is called a relation form of $(F_A,\ E)$. The membership function, indeterminacy membership function and non membership function are written by
$T_{R_A} : U \times E \rightarrow [0,1]$, $I_{R_A} : U \times E \rightarrow [0,1]$ and $F_{R_A} : U \times E \rightarrow [0,1]$ where $T_{R_A}(u,e) \in [0,1]$, $I_{R_A}(u,e) \in [0,1]$ and $F_{R_A}(u,e) \in [0,1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$.
If $[(T_{ij},\ I_{ij},\ F_{ij})] = [T_{ij}(u_i, e_j),\ I_{ij}(u_i, e_j)\ , F_{ij}(u_i, e_j)]$ we define a matrix

$$[\langle T_{ij}\ , I_{ij},\ F_{ij} \rangle]_{m \times n} = \begin{bmatrix} \langle T_{11},\ I_{11},\ F_{11} \rangle & \cdots & \langle T_{1n},\ I_{1n},\ F_{1n} \rangle \\ \langle T_{21},\ I_{21},\ F_{21} \rangle & \cdots & \langle T_{2n},\ I_{2n},\ F_{2n} \rangle \\ \vdots & \vdots & \vdots \\ \langle T_{m1},\ I_{m1},\ F_{m1} \rangle & \cdots & \langle T_{mn},\ I_{mn},\ F_{mn} \rangle \end{bmatrix}.$$

Which is called an $m \times n$ $FNSM$ of the NSS (F_A, E) over U.

Definition 2.5. [30] Let $A = (\langle a^T_{ij},\ a^I_{ij},\ a^F_{ij} \rangle),\ B = \langle (b^T_{ij},\ b^I_{ij},\ b^F_{ij} \rangle) \in \mathcal{N}_{(m,n)}$, NSM of order $m \times n$) and $\mathcal{N}_{(n)}$-denotes a square NSM of order n. The component wise addition and component wise multiplication is defined as
$A \oplus B = (sup\{a^T_{ij},\ b^T_{ij}\},\ sup\{a^I_{ij},\ b^I_{ij}\},\ inf\{a^F_{ij},\ b^F_{ij}\})$
$A \otimes B = (inf\{a^T_{ij},\ b^T_{ij}\},\ inf\{a^I_{ij},\ b^I_{ij}\},\ sup\{a^F_{ij},\ b^F_{ij}\})$

Definition 2.6. Let $A \in \mathcal{N}_{(m,n)},\ B \in \mathcal{N}_{(n,p}$, the composition of A and B is defined as

$$A \circ B = \left(\sum_{k=1}^{n} (a_{ik}^{T} \wedge\ b_{kj}^{T}),\ \sum_{k=1}^{n} (a_{ik}^{I} \wedge\ b_{kj}^{I}),\ \prod_{k=1}^{n} (a_{ik}^{F} \vee\ b_{kj}^{F}) \right)$$

equivalently we can write the same as

$$= \left(\bigvee_{k=1}^{n} (a_{ik}^{T} \wedge b_{kj}^{T}),\ \bigvee_{k=1}^{n} (a_{ik}^{I} \wedge b_{kj}^{I}),\ \bigwedge_{k=1}^{n} (a_{ik}^{F} \vee b_{kj}^{F}) \right).$$

The product $A \circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. Then A and B are said to be conformable for multiplication. We shall use AB instead of $A \circ B$.

Where $\sum (a_{ik}^{T} \wedge\ b_{kj}^{T})$ means max-min operation and

$\prod_{k=1}^{n} (a_{ik}^{F} \vee\ b_{kj}^{F})$ means min-max operation.

3. **Eigenvectors of CFNSM**

The characterization of the eigenspace structure for a CFNSM is discussed in this section.

Circulancy of FNSM is analogous to circulancy of classical matrix. Formally, FNSM $A \in \mathcal{N}_{(n,n)}$ is circulant if

$$\langle a_{ij}^{T}, a_{ij}^{I}, a_{ij}^{F} \rangle = \langle a_{i'j'}^{T}, a_{i'j'}^{I}, a_{i'j'}^{F} \rangle \text{ whenever } i - i' \equiv j - j' \ (mod\ n).$$

Hence, CFNSM A is totally determined by its inputs $\langle a_0^T, a_0^I, a_0^F \rangle, \langle a_1^T, a_1^I, a_1^F \rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F \rangle$ in the first row. $\langle a_0^T, a_0^I, a_0^F \rangle$ is the common in all diagonal , and similarly each $\langle a_i^T, a_i^I, a_i^F \rangle$ is common in a line parallel to the FNSM diagonal,

$A(\langle a_0^T, a_0^I, a_0^F \rangle, \langle a_1^T, a_1^I, a_1^F \rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F \rangle) =$

$$\begin{bmatrix} \langle a_0^T\ a_0^I\ a_0^F \rangle & \langle a_1^T\ a_1^I\ a_1^F \rangle & \langle a_2^T\ a_2^I\ a_2^F \rangle & \cdots & \langle a_{n-1}^T\ a_{n-1}^I\ a_{n-1}^F \rangle \\ \langle a_{n-1}^T\ a_{n-1}^I\ a_{n-1}^F \rangle & \langle a_0^T\ a_0^I\ a_0^F \rangle & \langle a_1^T\ a_1^I\ a_1^F \rangle & \cdots & \langle a_{n-2}^T\ a_{n-2}^I\ a_{n-2}^F \rangle \\ \langle a_{n-2}^T\ a_{n-2}^I\ a_{n-2}^F \rangle & \langle a_{n-1}^T\ a_{n-1}^I\ a_{n-1}^F \rangle & \langle a_0^T\ a_0^I\ a_0^F \rangle & \cdots & \langle a_{n-3}^T\ a_{n-3}^I\ a_{n-3}^F \rangle \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \langle a_1^T\ a_1^I\ a_1^F \rangle & \langle a_2^T\ a_2^I\ a_2^F \rangle & \langle a_3^T\ a_3^I\ a_3^F \rangle & \cdots & \langle a_0^T\ a_0^I\ a_0^F \rangle \end{bmatrix}.$$

Set $N = \{1, 2, ..., n\}$ and $N_0 = \{0, 1, ..., n-1\}$. Further for a given CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F \rangle, \langle a_1^T, a_1^I, a_1^F \rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F \rangle)$, a strictly non increasing sequence $M(A) = (s_1, s_2, ...)$ of the length $l(A)$ by repetition

$$s_r = \begin{cases} \max\{\langle a_i^T, a_i^I, a_i^F \rangle;\ i \in N_0\}\ for\ r = 1 \\ \max\{\langle a_i^T, a_i^I, a_i^F \rangle < s_{r-1};\ i \in N_0\}\ for\ r > 1 \end{cases}$$

Here $s_r = \langle s_r^T, s_r^I, s_r^F \rangle$. Henceforth $s_1 > s_2 > ...$ and $l(A)$ the length of the sequence $M(A)$ is the first l satisfying $\{\langle a_i^T, a_i^I, a_i^F \rangle;\ i \in N_0\} = \{s_r; 1 \leq r \leq l\}$. Use the notation $L(A) = \{1, 2, ..., l(A)\}$. Denote P_r as the set of all positions of the value s_r in the first row of the FNSM A, for any $r \in L(A)$ i.e.

$$P_r = \{i \in N_0;\ \langle a_i^T, a_i^I, a_i^F \rangle = s_r\}$$

and we set the highest common factors(HCF) d_r, e_r as follows

$$d_r = HCF(P_r \cup \{n\}),\ e_r = HCF(d_1, d_2, ..., d_r) = HCF(e_{r-1}, d_r).$$

Remark 3.1. The indices of FNSM values $\langle a_i^T, a_i^I, a_i^F \rangle$, and their placements, are numbers in $N_0 = \{0, 1, ..., n-1\}$, while the row and columns of the FNSM are indexed between 1 and n. Thus, for all $k \in N$, the kth row of A will be like this

$$A_k = (..., \langle a_{kk}^T, a_{kk}^I, a_{kk}^F \rangle, \langle a_{kk+1}^T, a_{kk+1}^I, a_{kk+1}^F \rangle, \langle a_{kk+2}^T, a_{kk+2}^I, a_{kk+2}^F \rangle, ...)$$

and for any position $p \in P_r$, we have $\langle a_{kk+p}^T, a_{kk+p}^I, a_{kk+p}^F \rangle = s_r$ (here the column index is computed modulo $k + p\ n$).

The next two lemmas are vital in this work.

Lemma 3.2. Let CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F \rangle, \langle a_1^T, a_1^I, a_1^F \rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F \rangle)$ be given, let $(\langle x^T, x^I, x^F \rangle) = (\langle x_1^T, x_1^I, x_1^F \rangle, \langle x_2^T, x_2^I, x_2^F \rangle, ..., \langle x_n^T, x_n^I, x_n^F \rangle)$ be FNSEv of A, let $k \in N, r \in L(A)$ and $p \in P_r(A)$. If $\langle x_k^T, x_k^I, x_k^F \rangle < s_r$, then $\langle x_k^T, x_k^I, x_k^F \rangle = \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle$.

Proof. Assume that $\langle x_k^T, x_k^I, x_k^F \rangle < \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle$. Then by Remark 3.1

$$\langle x_k^T, x_k^I, x_k^F \rangle < s_r \otimes \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle = \langle a_{kk+p}^T, a_{kk+p}^I, a_{kk+p}^F \rangle \otimes \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle \leq A_k \otimes (\langle x^T, x^I, x^F \rangle),$$

i.e $(\langle x^T, x^I, x^F \rangle)$ cannot be eigenvector of A, a contradiction. Then $\langle x_k^T, x_k^I, x_k^F \rangle \geq \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle$. Repeating like this we get, due to the cyclicity of A,

$$\langle x_k^T, x_k^I, x_k^F \rangle \geq \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle \geq \langle x_{k+2p}^T, x_{k+2p}^I, x_{k+2p}^F \rangle \geq ... \geq \langle x_k^T, x_k^I, x_k^F \rangle$$

hence, $\langle x_k^T, x_k^I, x_k^F \rangle = \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle$ must be hold true.

Lemma 3.3. Let CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F \rangle, \langle a_1^T, a_1^I, a_1^F \rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F \rangle)$ be given. Let $(\langle x^T, x^I, x^F \rangle)$ be FNSEv of A, let $k, l \in N, r \in L(A)$. If $\langle x_k^T, x_k^I, x_k^F \rangle < s_r$, then the following result hold

(i) if $k \equiv l\ mod\ d_r$ then $\langle x_k^T, x_k^I, x_k^F \rangle = \langle x_l^T, x_l^I, x_l^F \rangle$,

(ii)if $k \equiv l\ mod\ e_r$ then $\langle x_k^T, x_k^I, x_k^F \rangle = \langle x_l^T, x_l^I, x_l^F \rangle$.

Proof. (i) Clearly d_r can be expressed as a linear combination of values in $P_r \cup \{n\}$ with non-negative coefficients from number theory. By repeated use of Lemma 3.2 (i) is obtained.
(ii) follows directly from the definition of e_r and (i).

Theorem 3.4. Let CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F\rangle, \langle a_1^T, a_1^I, a_1^F\rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F\rangle)$ be given, let $\langle x^T, x^I, x^F\rangle$ be FNSEv of A. Then $\langle x_k^T, x_k^I, x_k^F\rangle < s_1$, holds true for every $k \in N$.

Proof: By contradiction, that if$\langle x_k^T, x_k^I, x_k^F\rangle > s_1$ for some $k \in N$. Then, the inequality $\langle x_k^T, x_k^I, x_k^F\rangle > \langle a_i^T, a_i^I, a_i^F\rangle$ holds for every $i \in N_0$, by definition of s_1, which gives $\langle x_k^T, x_k^I, x_k^F\rangle > \langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle$ for every $j \in N$. Hence

$$\langle x_k^T, x_k^I, x_k^F\rangle > \bigoplus_{j\in N}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) = A_k \otimes (\langle x^T, x^I, x^F\rangle),$$

i.e. $\langle x_k^T, x_k^I, x_k^F\rangle \neq A_k \otimes (\langle x^T, x^I, x^F\rangle)$ and, thus, $(\langle x^T, x^I, x^F\rangle)$ is not a eigenvector of A.

Theorem 3.5. Let CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F\rangle, \langle a_1^T, a_1^I, a_1^F\rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F\rangle)$ be given, such that the diagonal input $\langle a_0^T, a_0^I, a_0^F\rangle$ is greater than all other inputs of the FNSM. If a FNSV $(\langle x^T, x^I, x^F\rangle) \in \mathcal{N}_{(n)}$ has inputs fulfilling the inequalities $s_2 \leq \langle x_k^T, x_k^I, x_k^F\rangle \leq s_1$ for every $k \in N$, then $(\langle x^T, x^I, x^F\rangle)$ is FNSEv of A.

Proof: By definition of P_r, the hypothesis of the theorem gives $P_1 = \{0\}$ and thus

$$A_k \otimes (\langle x^T, x^I, x^F\rangle) = \bigoplus_{j\in N}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) =$$
$$(\langle a_{kk}^T, a_{kk}^I, a_{kk}^F\rangle \otimes \langle x_k^T, x_k^I, x_k^F\rangle) \oplus \bigoplus_{j\in N\backslash\{k\}}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle).$$

Further, we have $\langle a_{kk}^T, a_{kk}^I, a_{kk}^F\rangle \otimes \langle x_k^T, x_k^I, x_k^F\rangle = s_1 \otimes \langle x_k^T, x_k^I, x_k^F\rangle = \langle x_k^T, x_k^I, x_k^F\rangle$,

$\bigoplus_{j\in N\backslash\{k\}}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle \leq \bigoplus_{j\in N\backslash\{k\}}(s_2 \otimes \langle x_j^T, x_j^I, x_j^F\rangle) = s_2$,

hence $\langle x_k^T, x_k^I, x_k^F\rangle = \langle a_{kk}^T, a_{kk}^I, a_{kk}^F\rangle \otimes \langle x_k^T, x_k^I, x_k^F\rangle \leq A_k \otimes \langle x^T, x^I, x^F\rangle \leq \langle x_k^T, x_k^I, x_k^F\rangle \oplus s_2 = \langle x_k^T, x_k^I, x_k^F\rangle$.

for every $k \in N$, i.e. $A \otimes (\langle x^T, x^I, x^F\rangle) = (\langle x^T, x^I, x^F\rangle)$.

Remark 3.6. Theorem 3.5 is a special case of the sufficient part of Theorem 3.8. The assertions of Lemma 3.3 are fulfilled, as in Theorem 3.5 we have $P_1 = \{0\}$ and $d_1 = e_1 = n$, hence, the equivalence relation modulo n is the identity relation on N_0.

Remark 3.7. If the maximal input of the CFNSM is not unique, or if it is placed on other position than the diagonal one, then $0 < e_1 < n$ and the equivalence modulo e_1 differs from the identity relation on N_0. Hence, the inputs of any FNSEv cannot be arbitrary value in the interval $\langle s_2, s_1\rangle$ but according to Lemma 3.3, some repetitions must occur, see Example 4.2.

Theorem 3.8. Let CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F\rangle, \langle a_1^T, a_1^I, a_1^F\rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F\rangle)$ be given. A FNSV $(\langle x^T, x^I, x^F\rangle) \in \mathcal{N}_{(n)}$ is FNSEv of A if and only if there is a partition $\mathcal{T}$, on N, such that for every class $t \in \mathcal{T}$ there exist $(\langle x^T(t), x^I(t), x^F(t)\rangle) \in \mathcal{N}$ and $r(t) \in L(A)$, satisfying the following conditions

(i) $\langle x_k^T, x_k^I, x_k^F\rangle = \langle x^T(t), x^I(t), x^F(t)\rangle \leq s_1$ for every $k \in t$,

(ii)$r(t) = \max\{r \in S(A); x(t) < s_r\}$,

(iii) t is an equivalence class in $\mathcal{N}$ modulo $e_r(t)$.

Proof:($\Rightarrow$) The conditions (i)-(iii) follow from Lemma 3.3 and Theorem 3.4.

($\Leftarrow$) Let (i)-(iii) be satisfied. If $(\langle x^T(t), x^I(t), x^F(t)\rangle) = s_1$, then according to (ii), $r(t)$ is the maximum of the $\emptyset$, which is the least element in $S(A)$, i.e. $r(t) = 1$ in this case.

For arbitrary, but fixed $k \in N$, there is $t \in \mathcal{T}$ with $k \in t$ and $P_1 \neq \emptyset$ by definition, hence there is $p \in P_1$, and $\langle a_p^T, a_p^I, a_p^F\rangle = s_1$. Therefore, $k \equiv k + p \ mod \ e_{r(t)}$ and by conditions (i),(iii), we have

$\langle x_k^T, x_k^I, x_k^F\rangle = \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F\rangle = s_1 \otimes \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F\rangle = \langle a_{kk+p}^T, a_{kk+p}^I, a_{kk+p}^F\rangle \otimes \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F\rangle \leq \bigoplus_{j\in N}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) = A_k \otimes \langle x^T, x^I, x^F\rangle$.

To prove the otherside, consider any $j \in N$. If $j \in t$, then $\langle x_j^T, x_j^I, x_j^F\rangle = \langle x_k^T, x_k^I, x_k^F\rangle$, by (i). Thus,

$\bigoplus_{j\in t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) = \bigoplus_{j\in t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_k^T, x_k^I, x_k^F\rangle) \leq \langle x_k^T, x_k^I, x_k^F\rangle$.

If $j \notin t$, then $j, k \not\cong mode_{r(t)}$. Therefore, $p = j - k$ is not a multiple of the HCF $e_{r(T)}$, and so, the difference p cannot be expressed as a linear combination with integer coefficients, of the values in $P_1 \cup P_2 \cup ... \cup P_{r(t)} \cup \{n\}$, from definition of $e_{r(t)}$. As a result we have $\langle a_p^T, a_p^I, a_p^F\rangle = s_q$ for some $q > r(t)$, which implies $s_q \leq \langle x^T(t), x^I(t), x^F(t)\rangle$, by assumption (ii). Therefore $\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle = \langle a_{kk+p}^T, a_{kk+p}^I, a_{kk+p}^F\rangle = s_q \leq \langle x_k^T, x_k^I, x_k^F\rangle$. Thus we have

$\bigotimes_{j\in N\backslash t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) \leq \bigotimes_{j\in N\backslash t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \leq \langle x_k^T, x_k^I, x_k^F\rangle$.

Summarizing we get

$\langle x_k^T, x_k^I, x_k^F\rangle \leq A_k \otimes (\langle x^T, x^I, x^F\rangle) = \bigotimes_{j\in t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) \oplus \bigotimes_{j\in N\backslash t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle$.

As $k \in N$ is arbitrary, we have

$A \otimes (\langle x^T, x^I, x^F\rangle) = (\langle x^T, x^I, x^F\rangle)$.

4. **Examples of FNSEvs**

Examples of FNSEvs of CFNSM are illustrated here.

Example 4.1. Let $n = 6$ and let

$A = A(\langle 1, 1, 0\rangle, \langle 0.1, 0.1, 0.9\rangle, \langle 0.3, 0.2, 0.7\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.3, 0.2, 0.7\rangle, \langle 0, 0, 1\rangle)$ be a CFNSM generated by inputs on positions $(0, 1, ..., 5)$ in the first row. Then $M(A) = (s_1, s_2, ..., s_5) = (\langle 1, 1, 0\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.3, 0.2, 0.7\rangle, \langle 0.1, 0.1, 0.9\rangle, \langle 0, 0, 1\rangle)$. The maximal input $s_1 = \langle 1, 1, 0\rangle$ is on the diagonal, i.e. on position 0 and nowhere else, the second largest input has value $s_2 = \langle 0.7, 0.6, 0.3\rangle$. Hence, in view of Theorem 3.5, any FNSV with arbitrary inputs from interval $[\langle 0.7, 0.6, 0.3\rangle, \langle 1, 1, 0\rangle]$, e.g.

$(\langle x^T, x^I, x^F\rangle) = (\langle 0.9, 0.8, 0.1\rangle, \langle 0.8, 0.7, 0.2\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.8, 0.7, 0.2\rangle, \langle 0.8, 0.7, 0.2\rangle,$
$\langle 0.7, 0.6, 0.3\rangle)^t$ is an FNSEv of A.

$A =$

$$\begin{bmatrix} \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle \\ \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle \end{bmatrix} \otimes$$

$$\begin{bmatrix} \langle 0.9,0.8,0.1\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.7,0.6,0.3\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.7,0.6,0.3\rangle \end{bmatrix} = \begin{bmatrix} \langle 0.9,0.8,0.1\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.7,0.6,0.3\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.7,0.6,0.3\rangle \end{bmatrix}.$$

Example 4.2. In this example we show further FNSEvs of the FNSM $A = A(\langle 1,1,0\rangle, \langle 0.1,0.1,0.9\rangle, \langle 0.3,0.2,0.7\rangle, \langle 0.7,0.6,0.3\rangle, \langle 0.3,0.2,0.7\rangle, \langle 0,0,1\rangle)$ from the previous example. If an FNSEv should contain inputs not belonging to the interval $\langle s_2, s_1\rangle = \langle\langle 0.7,0.6,0.3\rangle, \langle 1,1,0\rangle\rangle$, then in view of Theorem 3.4, such inputs con not be large then $s_1 = \langle 1,1,0\rangle$. Hence such inputs must be less than the value $s_2 = \langle 0.7,0.6,0.3\rangle$ and some repetitions must occur, by Lemma 3.3.

The position sets for particular inputs are $P_1 = \{0\}$ for $s_1 = \langle 1,1,0\rangle, P_2 = \{3\}$ for $s_2 = \langle 0.7,0.6,0.3\rangle, P_3 = \{2,4\}$ for $s_3 = \langle 0.3,0.2,0.7\rangle, P_4 = \{1\}$ for $s_4 = \langle 0.1,0.1,0.9\rangle, P_5 = \{1\}$ for $s_5 = \langle 0,0,1\rangle$. By definition of the HCF d_r, e_r we get

$$d_1 = HCF(P_1 \cup \{n\}) = HCF(0,6) = 6 \ e_1 = 6$$
$$d_2 = HCF(P_2 \cup \{n\}) = HCF(3,6) = 3 \ e_2 = HCF(d_1,d_2) = HCF(6,3) = 3$$
$$d_3 = HCF(P_3 \cup \{n\}) = HCF(2,4,6) = 2 \ e_3 = HCF(e_2,d_3) = HCF(3,2) = 1$$
$$d_4 = HCF(P_4 \cup \{n\}) = HCF(1,6) = 1 \ e_4 = HCF(e_3,d_4) = HCF(1,1) = 1$$

Further $e_5 = 1$. By Lemma 3.3, any input $\langle x_k^T, x_k^I, x_k^F\rangle < s_r$ must be repeated in $\langle x^T, x^I, x^F\rangle$ after e_r positions. In particular, inputs less than value $s_2 = \langle 0.7,0.6,0.3\rangle$ must be repeated after 3rd positions, inputs less than $s_3 = \langle 0.3,0.2,0.7\rangle$ must be repeated on every second position. However, inputs which are not less than $s_2 = \langle 0.7,0.6,0.3\rangle$ can be arbitrary. The above conditions are satisfied e.g. by FNSV $(\langle x^T, x^I, x^F\rangle) = (\langle 0.4,0.3,0.6\rangle, \langle 0.5,0.4,0.6\rangle, \langle 0.6,0.5,0.4\rangle, \langle 0.4,0.3,0.6\rangle, \langle 0.5,0.4,0.6\rangle, \langle 0.6,0.5,0.4\rangle)^t$ which is therefore an FNSEv of A, in the view of Theorem 3.8

$$A = \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle \\ \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle \end{bmatrix} \otimes$$

$$\begin{bmatrix} \langle 0.4,0.3,0.6\rangle \\ \langle 0.5,0.4,0.6\rangle \\ \langle 0.6,0.5,0.4\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.5,0.4,0.6\rangle \\ \langle 0.6,0.5,0.4\rangle \end{bmatrix} = \begin{bmatrix} \langle 0.4,0.3,0.6\rangle \\ \langle 0.5,0.4,0.6\rangle \\ \langle 0.6,0.5,0.4\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.5,0.4,0.6\rangle \\ \langle 0.6,0.5,0.4\rangle \end{bmatrix}.$$

We may note that if an FNSEv $(\langle x^T, x^I, x^F\rangle)$ of A should contain an input $\langle x_k^T, x_k^I, x_k^F\rangle < s_3 = \langle 0.3, 0.2, 0.7\rangle$, then such an input would be repeated after every $e_2 = 1$ position, in other words the FNSEv would have only that single input, i.e. it would be a constant FNSV.

Example 4.3. labelE3 This example illustrates Remark 3.7 by analyzing FNSEvs of the FNSM $B = B(\langle 1,1,0\rangle, \langle 0.1,0.1,0.9\rangle, \langle 1,1,0\rangle, \langle 0.7,0.6,0.3\rangle, \langle 0.3,0.2,0.7\rangle, \langle 0,0,1\rangle)$ which differs from FNSM A in a single input, namely $\langle b_3^T, b_3^I, b_3^F\rangle = \langle 1,1,0\rangle$. Thus, the maximal input of the FNSM B is placed on the diagonal position 0 and also on a non-diagonal position 3. We have $P_1 = \{0, 3\}$ for $s_1 = \langle 1,1,0\rangle$ and $e_1 = d_1 = HCF(0,2,6) = 2$. Theorem 3.5 can not be applied, and the input values belonging to the interval $\langle s_2, s_1\rangle = \langle\langle 0.7,0.6,0.3\rangle, \langle 1,1,0\rangle\rangle$ must be repeated after $e_1 = 2$ positions. In fact, the same is true for all input values in the interval $\langle s_3, s_1\rangle$, because it can be easily computed that $e_1 = e_2 = 2$.

$$B = \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 1,1,0\rangle & \langle 0.3.0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 1,1,0\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 1,1,0\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 1,1,0\rangle \\ \langle 1,1,0\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle \\ \langle 0.1,0.1,0.9\rangle & \langle 1,1,0\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle \end{bmatrix} \otimes$$

$$\begin{bmatrix} \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \end{bmatrix} = \begin{bmatrix} \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \end{bmatrix}.$$

5. Conclusion

We study the eigenspace of a circulant max-min matrix, and propose the characterization of eigenspace structure for circulant fuzzy neutrosophic soft matrix. Further examples are given for all possible types of fuzzy neutrosophic soft eigenvectors.

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Types of Semi Continuous functions in Linguistic Neutrosophic Topological Spaces

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Abstract. A number of types of semi-continuous linguistic neutrosophic functions are introduced in this paper. Furthermore, these types of functions are demonstrated with appropriate examples. Theorems and properties are discussed in great detail.

Keywords: Linguistic neutrosophic semi continuous function; Linguistic neutrosophic quasi semi continuous function; Linguistic neutrosophic perfectly semi continuous function; Linguistic neutrosophic totally continuous function; Linguistic neutrosophic strongly semi continuous function; Linguistic neutrosophic slightly semi continuous function; Linguistic neutrosophic semi totally continuous function;

1. **Introduction**

In many system oriented implementations in which typical and standard logic was not suitable due to contradictory circumstances or unpredictability, the Fuzzy logic was utilized primarily, which was discovered by Zadeh(1965) [7]. This idea concerned with the membership or truth value of every elements of the fuzzy set. Along with truth value, the false value or non-membership was adjoined in intuitionistic fuzzy sets which was found by Atanassov [1]. Furthermore, in a new class of sets called neutrosophic sets which was given by Smarandache(1999) [5], possesses an additional membership called indeterminate membership. Neutrosophic sets have a wide range of applications over many real life fields.

Linguistic sets were invented by Fang [3], which has a variety of applications in day to day life. Gayathri and Helen(2021) [4] have found a topological space merging linguistic neutrosophic sets and topological spaces, termed as linguistic neutrosophic topology. The concept of linguistic neutrosophic semi continuous function is recasted into many forms of continuity by using linguistic neutrosophic semi open sets. Interrelations are analyzed among these types of linguistic neutrosophic continuity and counter examples are established to vindicate that the reverse implication of the result is not holds true.

2. Preambles

Definition 2.1. [7] Let S be a space of points (objects), with a generic element in x denoted by S. A neutrosophic set A in S is characterized by a truth-membership function T_A, an indeterminacy membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is
$T_A : S \rightarrow]0^-, 1^+[, I_A : S \rightarrow]0^-, 1^+[, F_A : S \rightarrow]0^-, 1^+[$
There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2.2. [7] Let S be a space of points (objects), with a generic element in x denoted by S. A single valued neutrosophic set (SVNS) A in S is characterized by truth-membership function T_A, indeterminacy-membership function I_A and falsity-membership function F_A. For each point S in S, $T_A(x), I_A(x), F_A(x) \in [0, 1]$.
When S is continuous, a SVNS A can be written as $A = \int \langle T(x), I(x), F(x)\rangle / x \in S$.
When S is discrete, a SVNS A can be written as $A = \sum \langle T(x_i), I(x_i), F(x_i)\rangle / x_i \in S$.

Definition 2.3. [3] Let $S = \{s_\theta | \theta = 0, 1, 2, ..., \tau\}$ be a finite and totally ordered discrete term set, where τ is the even value and s_θ represents a possible value for a linguistic variable.

Su [8] extended the discrete linguistic term set S into a continuous term set $S = \{s_\theta | \theta \in [0, q]\}$, where, if $s_\theta \in S$, then w call s_θ the original term, otherwise it is called as a virtual term.

Definition 2.4. [3] Let $Q = \{s_0, s_1, s_2, ..., s_t\}$ be a linguistic term set (LTS) with odd cardinality $t+1$ and $\overline{Q} = \{s_h / s_0 \leq s_h \leq s_t, h \in [0, t]\}$. Then, a linguistic single valued neutrosophic set A is defined by,
$A = \{\langle x, s_\theta(x), s_\psi(x), s_\sigma(x)\rangle | x \in S\}$, where $s_\theta(x), s_\psi(x), s_\sigma(x) \in \overline{Q}$ represent the linguistic truth, linguistic indeterminacy and linguistic falsity degrees of S to A, respectively, with condition $0 \leq \theta + \psi + \sigma \leq 3t$. This triplet $(s_\theta, s_\psi, s_\sigma)$ is called a linguistic single valued neutrosophic number.

Definition 2.5. [3] Let $\alpha = (s_\theta, s_\psi, s_\sigma), \alpha_1 = (s_{\theta 1}, s_{\psi_1}, s_{\sigma 1}), \alpha_2 = (s_{\theta 2}, s_{\psi_2}, s_{\sigma 2})$ be three LSVNNs, then

(1) $\alpha^c = (s_\sigma, s_\psi, s_\theta)$;
(2) $\alpha_1 \cup \alpha_2 = (max(\theta_1, \theta_2), max(\psi_1, \psi_2), min(\sigma_1, \sigma_2))$;
(3) $\alpha_1 \cap \alpha_2 = (min(\theta_1, \theta_2), min(\psi_1, \psi_2), max(\sigma_1, \sigma_2))$;

(4) $\alpha_1 = \alpha_2$ iff $\theta_1 = \theta_2, \psi_1 = \psi_2, \sigma_1 = \sigma_2$;

Definition 2.6. [4] For a linguistic neutrosophic topology τ, the collection of linguistic neutrosophic sets should obey,

(1) $0_{LN}, 1_{LN} \in \tau$

(2) $K_1 \bigcap K_2 \in \tau$ for any $K_1, K_2 \in \tau$

(3) $\bigcup K_i \in \tau, \forall \{K_i : i \in J\} \subseteq \tau$

We call, the pair (S_{LN}, τ_{LN}), a linguistic neutrosophic topological space.

Definition 2.7. [4] Let (S_{LN}, τ_{LN}) be a linguistic neutrosophic topological space (LNTS). Then,

- $(S_{LN}, \tau_{LN})^c$ is the dual linguistic neutrosophic topology, whose elements are ${K^C}_{LN}$ for $K_{LN} \in (S_{LN}, \tau_{LN})$.
- Any open set in τ_{LN} is known as linguistic neutrosophic open set(LNOS).
- Any closed set in τ_{LN} is known as linguistic neutrosophic closed set(LNCS) if and only if it's complement is linguistic neutrosophic open set.

3. **Types of Linguistic Neutrosophic Semi continuous Functions**

Definition 3.1. A mapping from $f : S_{LN} \to T_{LN}$ is a linguistic neutrosophic quasi semi continuous if the inverse image $f^{-1}(K_{LN})$ of every linguistic neutrosophic semi open set K_{LN} of T_{LN} is a linguistic neutrosophic open set in S_{LN}.

Theorem 3.2. *A mapping from $f : S_{LN} \to T_{LN}$ is a linguistic neutrosophic quasi semi continuous if and only if the inverse image $f^{-1}(K_{LN})$ of every linguistic neutrosophic semi closed set K_{LN} of T_{LN} is a linguistic neutrosophic closed set in S_{LN}.*

Proof: Necessity Part: Let $f : S_{LN} \to T_{LN}$ be linguistic neutrosophic quasi semi continuous and V_{LN} be any linguistic neutrosophic semi closed set in T_{LN}. Then $T_{LN} \backslash V_{LN}$ is linguistic neutrosophic semi open set in T_{LN}. As f is linguistic neutrosophic quasi semi continuous, $f^{-1}(T_{LN} \backslash V_{LN}) = S_{LN} \backslash f^{-1}(V_{LN})$ is linguistic neutrosophic semi open set in S_{LN}. Hence, the set $f^{-1}(V_{LN})$ is linguistic neutrosophic semi closed and thus the function f is linguistic neutrosophic quasi semi continuous.
Sufficiency Part: Let the set $f^{-1}(V_{LN})$ be linguistic neutrosophic semi closed in S_{LN} for each linguistic neutrosophic closed set in T_{LN}. Let V_{LN} be any linguistic neutrosophic open set in T_{LN}, then $T_{LN} \backslash V_{LN}$ is linguistic neutrosophic closed set in T_{LN}.
By assumption, the set $f^{-1}(T_{LN} \backslash V_{LN}) = S_{LN} \backslash f^{-1}(V_{LN})$ is linguistic neutrosophic semi closed in S_{LN}, which implies $f^{-1}(V_{LN})$ is linguistic neutrosophic semi open in S_{LN}. So, the mapping f is linguistic neutrosophic quasi semi continuous.

Remark 3.3. The above theorem is established by the following example.

Example 3.4. Let the universe of discourse be $U = \{p, q, r, s, t\}$ and let $S_{LN} = \{q, r\} = T_{LN}$. The set of all linguistic term set be L= {never familiar(l_0), almost never familiar(l_1), slightly familiar(l_2), some what familiar(l_3),occasionally familiar(l_4),moderately familiar(l_5), almost every time familiar(l_6), frequently familiar(l_7), extremely familiar(l_8)}. Let $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ be the linguistic neutrosophic identity mapping, where $\tau_{LN} = \{0_{LN}, 1_{LN}, K_{LN}\}$ and $\eta_{LN} = \{0_{LN}, 1_{LN}, H_{LN}\}$. The linguistic neutrosophic sets K_{LN} and H_{LN} are given by $K_{LN} = \{\langle, q, (l_3, l_6, l_3)\rangle, \langle r, (l_4, l_3, l_2)\rangle\}$ and $H_{LN} = \{\langle q, (l_3, l_6, l_3)\rangle, \langle r, (l_2, l_3, l_4)\rangle\}$ respectively. Here the inverse image $f^{-1}(H_{LN})$ is linguistic neutrosophic closed in S_{LN}.

Theorem 3.5. *If the mapping $f : S_{LN} \rightarrow T_{LN}$ is linguistic neutrosophic strongly semi continuous, then it is linguistic neutrosophic quasi semi continuous.*

Proof: Let V_{LN} be a linguistic neutrosophic semi open set in T_{LN}. Since f is linguistic neutrosophic strongly semi continuous, $f^{-1}(V_{LN})$ is linguistic neutrosophic semi cl-open in S_{LN}. Thus, f is linguistic neutrosophic quasi semi continuous.

Remark 3.6. The converse part of the above theorem need not be true in general, which is given by a counter example.

Example 3.7. Let the universe of discourse be as in example (3.4). And let $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ be a linguistic neutrosophic mapping defined by $f(a) = c, f(c) = a$, where $\tau_{LN} = \{0_{LN}, 1_{LN}, K_{LN}, H_{LN}\}$ and $\eta_{LN} = \{0_{LN}, 1_{LN}, M_{LN}\}$. The linguistic neutrosophic sets K_{LN}, H_{LN} and M_{LN} are given by $K_{LN} = \{\langle q, (l_4, l_5, l_2)\rangle, \langle r, (l_3, l_2, l_4)\rangle\}, H_{LN} = \{\langle q, (l_4, l_6, l_4)\rangle, \langle r, (l_4, l_3, l_8)\rangle\}$ and $M_{LN} = \{\langle q, (l_4, l_6, l_4)\rangle, \langle r, (l_8, l_3, l_4)\rangle\}$ respectively. Now, the mapping f is linguistic neutrosophic quasi continuous but not linguistic neutrosophic strongly semi continuous.

Definition 3.8. A mapping from $f : S_{LN} \rightarrow T_{LN}$ is said to be a linguistic neutrosophic perfectly semi continuous mapping if the inverse image $f(E_{LN})$ of every linguistic neutrosophic semi open set E_{LN} of T_{LN} is linguistic neutrosophic cl-open set in S_{LN}.

Example 3.9. Let the universe of discourse be as in example (3.4). And let $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ be a linguistic neutrosophic mapping defined by $f(a) = c, f(c) = a$, where $\tau_{LN} = \{0_{LN}, 1_{LN}, E_{LN}, F_{LN}\}$ and $\eta_{LN} = \{0_{LN}, 1_{LN}, G_{LN}\}$. The linguistic neutrosophic sets E_{LN}, F_{LN} and G_{LN} are given by $E_{LN} = \{\langle q, (l_3, l_6, l_3)\rangle, \langle r, (l_2, l_5, l_2)\rangle\}, F_{LN} = \{\langle q, (l_3, l_5, l_3)\rangle, \langle r, (l_4, l_5, l_8)\rangle\}$ and $G_{LN} = \{\langle q, (l_3, l_6, l_3)\rangle, \langle r, (l_2, l_5, l_2)\rangle\}$ respectively. Now, the mapping f is linguistic neutrosophic perfectly semi continuous.

Theorem 3.10. *A mapping $f : S_{LN} \to T_{LN}$ is linguistic neutrosophic perfectly semi continuous if and only if the inverse image $f^{-1}(E_{LN})$ of every linguistic neutrosophic semi closed set E_{LN} of T_{LN} is linguistic neutrosophic cl-open set in S_{LN}.*

Proof:

Necessity Part: Let $f : S_{LN} \to T_{LN}$ be linguistic neutrosophic perfectly semi continuous and V_{LN} be any linguistic neutrosophic semi closed set in T_{LN}. Then $T_{LN}\backslash V_{LN}$ is linguistic neutrosophic semi open set in T_{LN}. As f is linguistic neutrosophic perfectly semi continuous, $f^{-1}(T_{LN}\backslash V_{LN}) = S_{LN}\backslash f^{-1}(V_{LN})$ is linguistic neutrosophic cl-open set in S_{LN}. Hence, the set $f^{-1}(V_{LN})$ is linguistic neutrosophic cl-open and thus the function f is linguistic neutrosophic perfectly semi continuous.

Sufficiency Part: Let the set $f^{-1}(V_{LN})$ be linguistic neutrosophic cl-open in S_{LN} for each linguistic neutrosophic semi closed set in T_{LN}. Let V_{LN} be any linguistic neutrosophic semi open set in T_{LN}, then $T_{LN}\backslash V_{LN}$ is linguistic neutrosophic semi closed set in T_{LN}. By assumption, the set $f^{-1}(T_{LN}\backslash V_{LN}) = S_{LN}\backslash f^{-1}(V_{LN})$ is linguistic neutrosophic cl-open in S_{LN}, which implies $f^{-1}(V_{LN})$ is linguistic neutrosophic cl-open in S_{LN}. So, the mapping f is linguistic neutrosophic perfectly semi continuous.

Theorem 3.11. *If the mapping $f : S_{LN} \to T_{LN}$ is linguistic neutrosophic perfectly semi continuous, then it is linguistic neutrosophic quasi semi continuous.*

Proof: Let V_{LN} be a linguistic neutrosophic semi open set in T_{LN}. Since f is linguistic neutrosophic perfectly semi continuous, $f^{-1}(V_{LN})$ is linguistic neutrosophic cl-open in S_{LN}. Thus, f is linguistic neutrosophic quasi semi continuous.

Remark 3.12. The converse part of the above theorem need not be true in general, which is given by a counter example.

Example 3.13. Let the linguistic term set be as in example (3.5) and $f : S_{LN} \to T_{LN}$ be any linguistic neutrosophic mapping. Now, the mapping f is linguistic neutrosophic quasi continuous but not linguistic neutrosophic perfectly semi continuous.

Theorem 3.14. *If the mapping $f : S_{LN} \to T_{LN}$ is linguistic neutrosophic strongly continuous, then it is linguistic neutrosophic perfectly semi continuous.*

Proof: Let V_{LN} be a linguistic neutrosophic semi open set in T_{LN}. Since f is linguistic neutrosophic strongly semi continuous, $f^{-1}(V_{LN})$ is linguistic neutrosophic cl-open in S_{LN}. Thus, f is linguistic neutrosophic perfectly semi continuous.

Remark 3.15. The reverse part of the above theorem need not be true in general, which is given by a counter example.

Example 3.16. Let the universe of discourse be $U = \{p, q, r, s, t\}$ and let $S_{LN} = \{p, q, r\} = T_{LN}$ and the linguistic term set be as in example (3.5). Let $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ be a linguistic neutrosophic identity mapping, where $\tau_{LN} = \{0_{LN}, 1_{LN}, E_{LN}\}$ and $\eta_{LN} = \{0_{LN}, 1_{LN}, A_{LN}, B_{LN}\}$. The linguistic neutrosophic sets E_{LN}, A_{LN} and B_{LN} are given by $E_{LN} = \{\langle p, (l_3, l_5, l_3)\rangle, \langle q, (l_2, l_1, l_2)\rangle, \langle r, (l_8, l_5, l_8)\rangle\}, A_{LN} = \{\langle p, (l_3, l_5, l_3)\rangle, \langle q, (l_2, l_1, l_2)\rangle, \langle r, (l_8, l_5, l_8)\rangle\}$ and $B_{LN} = \{\langle p, (l_7, l_5, l_2)\rangle, \langle q, (l_6, l_6, l_3)\rangle, \langle r, (l_3, l_2, l_3)\rangle\}$ respectively. The inverse image of H_{LN} in T_{LN}, is E_{LN} in S_{LN}, which is a linguistic neutrosophic cl-open set. Then the mapping f is linguistic neutrosophic perfectly semi continuous but not linguistic neutrosophic strongly continuous.

Theorem 3.17. *Let (S_{LN}, τ_{LN}) be a discrete linguistic neutrosophic topological space and (T_{LN}, η_{LN}) be any linguistic neutrosophic topological space such that $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ is a mapping. Then the following are equivalent.*

(1) *f is linguistic neutrosophic perfectly semi continuous*
(2) *f is linguistic neutrosophic quasi semi continuous*

Proof: (1) $\Rightarrow$ (2): Let U_{LN} be linguistic neutrosophic semi open set in (T_{LN}, η_{LN}) and the function f be linguistic neutrosophic perfectly semi continuous, (i.e) the inverse image $f^{-1}(U_{LN})$ of any linguistic neutrosophic semi open set in (T_{LN}, η_{LN}), is linguistic neutrosophic cl-open in (S_{LN}, τ_{LN}). This implies the function f is linguistic neutrosophic quasi semi continuous.
(2) $\Rightarrow$ (1): Let U_{LN} be linguistic neutrosophic semi open set in (T_{LN}, η_{LN}), then the set $f^{-1}(U_{LN})$ is linguistic neutrosophic open in (S_{LN}, τ_{LN}), since the function f is linguistic neutrosophic quasi semi continuous. Thus, $f^{-1}(U_{LN})$ is linguistic neutrosophic closed as (S_{LN}, τ_{LN}) is a discrete linguistic neutrosophic topological space, (i.e) $f^{-1}(U_{LN})$ is linguistic neutrosophic cl-open which implies the mapping f is linguistic neutrosophic perfectly semi continuous.

Theorem 3.18. *Let $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ and $g : (T_{LN}, \eta_{LN}) \rightarrow (P_{LN}, \mu_{LN})$ be any two mappings. Then their composition $g \circ f$ is*

(1) *linguistic neutrosophic semi continuous if g is linguistic neutrosophic strongly continuous and f is linguistic neutrosophic semi continuous.*
(2) *linguistic neutrosophic perfectly semi continuous if g is linguistic neutrosophic perfectly semi continuous and f is linguistic neutrosophic continuous.*

Proof: (1): Let $g : (T_{LN}, \eta_{LN}) \rightarrow (P_{LN}, \mu_{LN})$ be linguistic neutrosophic strongly continuous and f is linguistic neutrosophic semi continuous. Let U_{LN} be any linguistic neutrosophic closed set in (P_{LN}, μ_{LN}). Then, $g^{-1}(U_{LN})$ is linguistic neutrosophic cl-open set in (T_{LN}, η_{LN}) as g

is linguistic neutrosophic strongly semi continuous. Now, $f^{-1}(g^{-1}(U_{LN})) = (g \circ f)^{-1}(U_{LN})$ is linguistic neutrosophic semi closed set in (S_{LN}, τ_{LN}), since f is linguistic neutrosophic semi continuous. Thus, $g \circ f : (S_{LN}, \tau_{LN}) \to (P_{LN}, \mu_{LN})$ is linguistic neutrosophic semi continuous.
(2): Let f be linguistic neutrosophic continuous and g be linguistic neutrosophic perfectly semi continuous. Let U_{LN} be any linguistic neutrosophic semi closed set in (P_{LN}, μ_{LN}). Then, $g^{-1}(U_{LN})$ is linguistic neutrosophic cl-open set in (T_{LN}, η_{LN}) as g is linguistic neutrosophic perfectly semi continuous. Since f is linguistic neutrosophic continuous, $f^{-1}(g^{-1}(U_{LN})) = (g \circ f)^{-1}(U_{LN})$ is linguistic neutrosophic semi closed set in (S_{LN}, τ_{LN}), which implies $g \circ f : (S_{LN}, \tau_{LN}) \to (P_{LN}, \mu_{LN})$ is linguistic neutrosophic perfectly semi continuous.

Definition 3.19. A function $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is called as linguistic neutrosophic totally semi continuous if the inverse image of every linguistic neutrosophic open subset of (T_{LN}, η_{LN}) is a linguistic neutrosophic semi cl-open subset of (S_{LN}, τ_{LN}).

Remark 3.20. It is clear that every linguistic neutrosophic totally continuous function is linguistic neutrosophic totally semi continuous but the reverse implication is not true which can be seen from the counter example.

Example 3.21. Let the universe of discourse be $U = \{x, y, z, w\}$. The set of all linguistic terms be L= { very strongly disagree(l_0), strongly disagree(l_1), disagree(l_2), mostly disagree(l_3), slightly disagree(l_4), neither disagree nor agree(l_5), slightly agree(l_6), mostly agree(l_7), agree(l_8), strongly agree(l_9), very strongly agree(l_{10})}.
And $S_{LN} = \{y, z\} = T_{LN}, \tau_{LN} = \{0_{LN}, 1_{LN}, E_{LN}, F_{LN}\}$ and $\eta_{LN} = \{0_{LN}, 1_{LN}, K_{LN}\}$, defines linguistic neutrosophic topology where, $E_{LN} = \{\langle y, (l_3, l_4, l_2)\rangle, \langle z, (l_3, l_5, l_3)\rangle\}, F_{LN} = \{\langle y, (l_2, l_4, l_3)\rangle, \langle z, (l_3, l_5, l_3)\rangle), K_{LN} = (\langle y, l_4, l_2, l_3\rangle, \langle z, l_5, l_3, l_3\rangle)$. The mapping A function $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN}$ is defined by $f(a) = c, f(b) = a, f(c) = b$. Then f is a linguistic neutrosophic totally semi continuous function but not linguistic neutrosophic totally continuous.

Definition 3.22. A function $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is called as linguistic neutrosophic strongly semi continuous if the inverse image of every linguistic neutrosophic subset of (T_{LN}, η_{LN}) is a linguistic neutrosophic semi cl-open subset of (S_{LN}, τ_{LN}).

Remark 3.23. Obviously, LN strong semi continuity $\Rightarrow$ LN totally semi continuity $\Rightarrow$ LN semi continuity.
Given below is an example of a linguistic neutrosophic function which is linguistic neutrosophic totally semi continuous but not linguistic neutrosophic strongly semi continuous.

Example 3.24. In example (3.5), the mapping f is linguistic neutrosophic semi continuous. Clearly, the inverse image $f^{-1}(H_{LN})$ is not linguistic neutrosophic closed and hence it is not linguistic neutrosophic semi cl-open. Thus, f is not linguistic neutrosophic totally semi continuous and linguistic neutrosophic strongly semi continuous.

Theorem 3.25. *Every linguistic neutrosophic totally semi continuous function into T_1 space is linguistic neutrosophic strongly semi continuous.*

Proof:
In a T_1 space all linguistic neutrosophic singleton sets are closed. Hence, $f^{-1}(A_{LN})$ is linguistic neutrosophic semi cl-open in S_{LN} for every linguistic neutrosophic subset A_{LN} of T_{LN}.

Definition 3.26. A function $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is called as linguistic neutrosophic slightly semi continuous if for every $s \in S_{LN}$ and for each cl-open subset V_{LN} of T_{LN} containing $f(s)$, there exists a linguistic neutrosophic semi open subset U_{LN} of S_{LN} such that $s \in U_{LN}$ and $f(U_{LN}) \subseteq V_{LN}$.

Theorem 3.27. *Every linguistic neutrosophic slightly semi continuous function into a linguistic neutrosophic discrete space is linguistic neutrosophic strongly semi continuous.*

Proof:
Let $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ be a linguistic neutrosophic slightly semi continuous function from a linguistic neutrosophic space S_{LN} into a linguistic neutrosophic discrete space T_{LN}. Let A_{LN} be any linguistic neutrosophic subset of T_{LN}, then A_{LN} is a linguistic neutrosophic cl-open subset of T_{LN}. Hence $f^{-1}(A_{LN})$ is linguistic neutrosophic cl-open set of S_{LN}. Thus, f is linguistic neutrosophic strongly semi continuous.

Theorem 3.28. *If $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is linguistic neutrosophic slightly semi continuous function and $g : (T_{LN}, \eta_{LN}) \to (R_{LN}, \mu_{LN})$ is linguistic neutrosophic totally continuous, then $g \circ f$ is linguistic neutrosophic totally semi continuous.*

Proof:
Let A_{LN} be linguistic neutrosophic open subset of (R_{LN}, μ_{LN}). Then $g^{-1}(A_{LN})$ is a linguistic neutrosophic semi cl-open subset of (T_{LN}, η_{LN}). As f is linguistic neutrosophic slightly semi continuous, we have, $f^{-1}(g^{-1}(A_{LN})) = (g \circ f)^{-1}(A_{LN})$ is linguistic neutrosophic semi cl-open subset of (S_{LN}, τ_{LN}). Hence $g \circ f$ is linguistic neutrosophic totally semi continuous.

Definition 3.29. A function $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is called as linguistic neutrosophic totally continuous if the inverse image of every linguistic neutrosophic open subset of (T_{LN}, η_{LN}) is a linguistic neutrosophic cl-open subset of (S_{LN}, τ_{LN}).

Definition 3.30. A function $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is called as linguistic neutrosophic semi totally continuous if the inverse image of every linguistic neutrosophic semi open subset of (T_{LN}, η_{LN}) is a linguistic neutrosophic cl-open subset of (S_{LN}, τ_{LN}).

Example 3.31. Let the universe of discourse be $U = \{a, b, c, d, e\}$ and let $S_{LN} = \{b\} = T_{LN}$. The set of all linguistic term set be L= {no healing(l_0), deterioting(l_1), chronic(l_2), some what chronic(l_3), extremely chronic(l_4), very ill(l_5), ill(l_6), no healing(l_7), healing(l_8), slowly healing(l_9), fastly healing(l_{10})}. Let $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ be the linguistic neutrosophic mapping defined by $f(b) = c, f(c) = b$. And $\tau_{LN} = \{0_{LN}, 1_{LN}, \langle b, (l_3, l_1, l_3)\rangle\}$ and $\eta_{LN} = \{0_{LN}, 1_{LN}, \langle b, (l_3, l_2, l_1)\rangle, \langle b, (l_1, l_1, l_2)\rangle\}$ be linguistic neutrosophic topologies. The set $E_{LN} = \langle b, (l_3, l_2, l_1)\rangle$ is linguistic neutrosophic semi open subset of (T_{LN}, η_{LN}). The inverse image $f^{-1}(E_{LN})$ in (S_{LN}, τ_{LN}) is both linguistic neutrosophic semi closed and linguistic neutrosophic semi open. Thus, the map f is linguistic neutrosophic semi totally continuous.

Theorem 3.32. *A function $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is linguistic neutrosophic semi totally continuous if and only if the inverse image of each linguistic neutrosophic semi closed subset of (T_{LN}, η_{LN}) is a linguistic neutrosophic cl-open subset of (S_{LN}, τ_{LN}).*

Proof:
Let K_{LN} be any linguistic neutrosophic semi closed subset in (T_{LN}, η_{LN}), then $T_{LN} \backslash K_{LN}$ is linguistic neutrosophic semi open subset of T_{LN}. Then $f^{-1}(T_{LN} \backslash K_{LN})$ is linguistic neutrosophic cl-open in S_{LN}, (i.e) $S_{LN} \backslash f^{-1}(K_{LN})$ is linguistic neutrosophic cl-open in S_{LN}. Therefore, $f^{-1}(K_{LN})$ is linguistic neutrosophic cl-open in S_{LN}.
Conversely, if H_{LN} is linguistic neutrosophic semi open subset of (T_{LN}, η_{LN}), then $T_{LN} \backslash H_{LN}$ is linguistic neutrosophic semi closed subset of T_{LN}. Then, $f^{-1}(T_{LN} \backslash H_{LN}) = S_{LN} \backslash f^{-1}(H_{LN})$ is linguistic neutrosophic cl-open in S_{LN} and hence $f^{-1}(H_{LN})$ is linguistic neutrosophic cl-open in S_{LN}. Therefore, the inverse image of every linguistic neutrosophic semi open subset of T_{LN} is a linguistic neutrosophic cl-open subset of S_{LN}. Thus, f is linguistic neutrosophic semi totally continuous.

Theorem 3.33. *Every linguistic neutrosophic semi totally continuous mapping is a linguistic neutrosophic totally continuous mapping.*

Proof:
Let H_{LN} be any linguistic neutrosophic open subset of T_{LN}, where $f : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is linguistic neutrosophic semi totally continuous mapping. As each linguistic neutrosophic open set is linguistic neutrosophic semi open, H_{LN} is linguistic neutrosophic semi open in T_{LN} and $f^{-1}(H_{LN})$ is linguistic neutrosophic cl-open subset of S_{LN}. Thus, the inverse image of every open subset of T_{LN} is linguistic neutrosophic cl-open in S_{LN} which implies, f is totally continuous. The converse part is not holds true which is given by a counter example.

Example 3.34. Let the universe of discourse be $U = \{x, y, z\}$ and $S_{LN} = \{y, z\} = T_{LN}$. The set of all linguistic term set be L= {no healing(l_0), deterioting(l_1), chronic(l_2), some what chronic(l_3), extremely chronic(l_4), very ill(l_5), ill(l_6), no healing(l_7), healing(l_8), slowly healing(l_9), fastly healing(l_{10})}.

Let $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ be a linguistic neutrosophic mapping,defined by $f(b) = c, f(c) = b$ where $\tau_{LN} = \{0_{LN}, 1_{LN}, K_{LN}\}$ and $\eta_{LN} = \{0_{LN}, 1_{LN}, H_{LN}\}$. The linguistic neutrosophic sets K_{LN} and H_{LN} are given by $K_{LN} = \{\langle y, (l_4, l_5, l_4)\rangle, \langle z, (l_9, l_9, l_9)\rangle\}$ and $H_{LN} = \{\langle y, (l_5, l_4, l_4)\rangle, \langle z, (l_9, l_9, l_9)\rangle\}$ respectively. The inverse image $f^{-1}(H_{LN})$ is linguistic neutrosophic cl-open in S_{LN}. Thus, the map f is linguistic neutrosophic totally continuous.

Let $D_{LN} = \langle y, (l_2, l_4, l_5)\rangle, \langle z, (l_6, l_5, l_9)\rangle$ be any linguistic neutrosophic set in T_{LN}. Then D_{LN} is linguistic neutrosophic semi open but the inverse image is not linguistic neutrosophic cl-open subset of S_{LN}. Thus, the map f is not linguistic neutrosophic semi totally continuous.

Theorem 3.35. *Every linguistic neutrosophic semi totally continuous mapping is a linguistic neutrosophic totally semi continuous mapping.*

Proof:

Let H_{LN} be any linguistic neutrosophic open subset of T_{LN}, where $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ is linguistic neutrosophic semi totally continuous mapping. As each linguistic neutrosophic open set is linguistic neutrosophic semi open, H_{LN} is linguistic neutrosophic semi open in T_{LN} and $f^{-1}(H_{LN})$ is linguistic neutrosophic semi cl-open subset of S_{LN}, as f is linguistic neutrosophic semi totally continuous mapping. Thus, the inverse image of every open subset of T_{LN} is linguistic neutrosophic semi cl-open in S_{LN} which implies, f is totally semi continuous. The converse part is not holds true which is given by a counter example.

Example 3.36. In example (3.5), let $S_{LN} = \{q, t\} = T_{LN}$. Let $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ be the linguistic neutrosophic mapping defined by $f(b) = c, f(c) = b$. And $\tau_{LN} = \{0_{LN}, 1_{LN}, K_{LN}\}$ and $\eta_{LN} = \{0_{LN}, 1_{LN}, H_{LN}\}$ be linguistic neutrosophic topologies.The linguistic neutrosophic sets K_{LN} and H_{LN} are given by $K_{LN} = \{\langle q, (l_3, l_4, l_2)\rangle, \langle t, (l_5, l_5, l_2)\rangle\}$ and $H_{LN} = \{\langle q, (l_3, l_3, l_2)\rangle, \langle ts, (l_4, l_4, l_5)\rangle\}$ respectively. Linguistic neutrosophic semi open sets in S_{LN} are $\{0_{LN}, 1_{LN}, \langle q, (l_3, l_2, l_3)\rangle, \langle t, (l_4, l_5, l_4)\rangle\}$ and in T_{LN} are $\{0_{LN}, 1_{LN}, \langle q, (l_4, l_5, l_0)\rangle, \langle t, (l_4, l_6, l_2)\rangle\}$.

Then the inverse image of the linguistic neutrosophic open set in (T_{LN}, η_{LN}) is linguistic neutrosophic semi cl-open in (S_{LN}, τ_{LN}) whereas the inverse image of the linguistic neutrosophic semi open set in T_{LN} is not linguistic neutrosophic semi cl-open. Thus, the map f is linguistic neutrosophic totally semi continuous but not linguistic neutrosophic semi totally continuous.

Theorem 3.37. *Every linguistic neutrosophic strongly continuous mapping is a linguistic neutrosophic semi totally continuous mapping.*

Proof:

Let $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ is linguistic neutrosophic strongly continuous mapping and

M_{LN} be any linguistic neutrosophic semi open subset of T_{LN}. Now, $f^{-1}(M_{LN})$ is linguistic neutrosophic semi cl-open subset of S_{LN}, by definition. Thus, the inverse image of every semi open subset of T_{LN} is linguistic neutrosophic cl-open in S_{LN} which implies,f is semi totally continuous. The converse part is not holds true which is given by a counter example.

Example 3.38. In example (3.5), let $S_{LN} = \{q, t\} = T_{LN}$. Let $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ be the linguistic neutrosophic identity mapping. And $\tau_{LN} = \{0_{LN}, 1_{LN}, K_{LN}\}$ and $\eta_{LN} = \{0_{LN}, 1_{LN}, H_{LN}\}$ be linguistic neutrosophic topologies.The linguistic neutrosophic sets K_{LN} and H_{LN} are given by $K_{LN} = \{\langle q, (l_3, l_2, l_3)\rangle, \langle t, (l_4, l_5, l_4)\rangle\}$ and $H_{LN} = \{\langle q, (l_3, l_4, l_2)\rangle, \langle t, (l_5, l_5, l_3)\rangle\}$ respectively. Linguistic neutrosophic semi open sets in T_{LN} are $\{0_{LN}, 1_{LN}, \langle q, (l_3, l_2, l_3)\rangle, \langle t, (l_4, l_5, l_4)\rangle\}$.
Now, the inverse image of the linguistic neutrosophic semi open set in T_{LN} is linguistic neutrosophic open in S_{LN}. Let $\{\langle q, (l_3, l_6, l_0)\rangle, \langle t, (l_3, l_3, l_1)\rangle\}$ be any linguistic neutrosophic set whose inverse image is neither linguistic neutrosophic open nor linguistic neutrosophic closed in S_{LN}. Therefore, the map f is linguistic neutrosophic semi totally continuous but not linguistic neutrosophic strongly continuous mapping.

Theorem 3.39. *Let $f : (S_{LN}, \tau_{LN}) \rightarrow (T_{LN}, \eta_{LN})$ be a mapping, from a linguistic neutrosophic topological space (S_{LN}, τ_{LN}) into a linguistic neutrosophic topological space (T_{LN}, η_{LN}). Then the following statements are equivalent.*

(1) *f is linguistic neutrosophic semi totally continuous mapping.*
(2) *for every $s \in S_{LN}$ and for each linguistic neutrosophic semi open set M_{LN} in (T_{LN}, η_{LN}) with $f(s) \in T_{LN}$, there exists a linguistic neutrosophic cl-open set K_{LN} in S_{LN} such that $s \in K_{LN}$ and $f(K_{LN}) \subset M_{LN}$.*

Proof:
(1) $\Rightarrow$ (2): If f is linguistic neutrosophic semi totally continuous and M_{LN} be any linguistic neutrosophic semi open set in (T_{LN}, η_{LN}) containing $f(s)$ so that $s \in f^{-1}(M_{LN})$. Since f is linguistic neutrosophic semi totally continuous, $f^{-1}(M_{LN})$ is linguistic neutrosophic cl-open in S_{LN} and $s \in M_{LN}$. Also, let $K_{LN} = f^{-1}(M_{LN})$, then $f(K_{LN}) = f(f^{-1}(M_{LN})) \subset M_{LN}$ which implies $f(K_{LN}) \subset M_{LN}$.
(2) $\Rightarrow$ (1): Let M_{LN} be linguistic neutrosophic semi open set in T_{LN} and $s \in f^{-1}(M_{LN})$ be any arbitrary linguistic neutrosophic point, then $f(s) \in M_{LN}$. Thus, from the assumption, there exists a linguistic neutrosophic cl-open set $f(G_{LN}) \in S_{LN}$ containing s such that $f(G_{LN}) \subset M_{LN}$, which implies $s \in G_{LN} \subset f^{-1}(M_{LN})$.
Now, $f^{-1}(M_{LN})$ is linguistic neutrosophic cl-open neighborhood of s. As s is arbitrary, $f^{-1}(M_{LN})$ is linguistic neutrosophic cl-open neighborhood of each of its points. Thus,

$f^{-1}(M_{LN})$ is linguistic neutrosophic cl-open set in S_{LN} and hence f is linguistic neutrosophic semi totally continuous.

Remark 3.40. The implications of all linguistic neutrosophic continuous functions are given below.

$$\begin{bmatrix} & A & B & C & D & E & F & G \\ A & - & 1 & 0 & 1 & 0 & 0 & 1 \\ B & 0 & - & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 1 & - & 1 & 0 & 0 & 0 \\ D & 0 & 0 & 0 & - & 0 & 0 & 0 \\ E & 1(indiscrete) & 0 & 0 & 0 & - & 0 & 0 \\ F & 0 & 0 & 0 & 0 & 0 & - & 0 \\ G & 1(T1) & 0 & 0 & 0 & 0 & 0 & - \end{bmatrix}$$

where, A - LN strongly semi continuous, B - LN quasi semi continuous, C - LN perfectly semi continuous, D - LN contra semi continuous, E - LN slightly semi continuous, F - LN semi totally continuous, G - LN totally semi continuous, respectively.

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On Characterizations of $(\varpi, \varepsilon, \varsigma)$-Single Valued Neutrosophic Hyperrings and Hyperideals

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Abstract. This study included concepts for $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic hyperring ($(\varpi, \varepsilon, \varsigma)$-$SVNHR$) and $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic hyperideal ($(\varpi, \varepsilon, \varsigma)$-$SVNHI$). $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic hyperrings ($(\varpi, \varepsilon, \varsigma)$-$SVNHRs$) and $(\varpi, \varepsilon, \varsigma)$-single valued hyperideals ($(\varpi, \varepsilon, \varsigma)$-$SVNHIs$) are examined and validated in terms of algebraic features and of structural characteristics.

Keywords: Hyperring, Hyperideal, Single valued neutrosophic set, Single valued neutrosophic hyperring, Single valued neutrosophic hyperideal, $(\varpi, \varepsilon, \varsigma)$-Single valued neutrosophic hyperring and $(\varpi, \varepsilon, \varsigma)$-Single valued neutrosophic hyperideal.

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1. Introduction

The theory of hyperstructure came into existence in 1934 when Marty [1] defined hypergroups as being generalized. To propose an overlying homomorphism, Corsini [2] developed the concept of hypering and general forms of hypering. The H_v-ring, the H_v-subring, and the H_v-ideals of the H_v-ring, all these are modifying the thoughts introduced by Corsini [2], have been invented by Vougiouklis [3, 4]. Generally, [5] offers a variety of rates in $[0, 1]$ stated by a single real number. In order to relieve ambiguities, a fuzzy set model was created by the Turksen [6], which was utilized to assess membership of the fuzzy set framework. An enhancement of fuzzy sets is intuitionistic sets, suggested in 1986 by Atanassov [7]. This approach was analogous to the interval-valued fuzzy sets described in [8]. Intuitionistic fuzzy sets can execute flawed data and not inexhaustible information, frequently in real-life [8]. Rosenfeld [9] launched the fuzzy algebra work, extending it to several fuzzy models such as intuitionistic fuzzy sets, fuzzy soft sets, and imprecise soft sets. Some artworks related to soft, fuzzy rings and ideal vague soft groups, vague soft rings, and soft ideals are also found in [10–13]. In 1998, to attain these goals, Smarandache suggested the neutrosophic paradigm in [14]. In [15–20], numerous new neutrosophic theoretical fads were launched.

Wang et al. [8] pioneered the theory of a single-valued neutrosophic set ($SVNS$), whereas Smaranache plithogenic was presented into [21] as a refinement of neutrosophic structure. Hyperstructure theory is often used in numerous mathematical ideas. Algebraic hyperstructures have a wide range of applications, including fuzzy sets, design and data, artificial intelligence, lattices, automation, and combinatorics, and etc. As a result of fuzzy algebra research, fuzzy hyperalgebraic theory was produced. Liu [22] created the idea of fuzzy ideals of a ring. A lot of hyperstructure work has been done over the last two decades, such as fuzzy hyperalgebras [23], fuzzy hyperrings [24], fuzzy topological F-polygroups [25], Bipolar-valued fuzzy soft hyper BCK ideals [26], fuzzy hypergroup degree [27], fuzzy hypergraphs [28], hyper-spectral image analysis [29], fundamental relations on fuzzy hypermodules [30], and so on.

There are works available of hyperstructures related to hyperrings in these manuscripts: fuzzy hyperings [31], Γ-hyperrings [32], soft hyperrings [33], topological hyperrings [34], and topological structures of lower and upper rough subsets in a hyperring [35], etc. In [36], Davvaz initiated the generalization of fuzzy hyperideal. Bharathi and Vimala subsequently established the notions of fuzzy l-ideal in [37], and the fuzzy l-ideal was then expanded in [38]. In [39–41], Selvachandran et al. introduced the hypergroup and hyperring theory for imprecise soft sets, and some other important works on fuzzy sets are studied in [42–45].

In this paper, we focus on the theories of $(\varpi, \varepsilon, \varsigma)$-$SVNHR$s and $(\varpi, \varepsilon, \varsigma)$-$SVNHIs$ in order to contribute to the advancement of the neutrosophic theory of hyperalgebraic.

2. Preliminaries

Let Ξ be a set of points where $\hat{n}$ refers to a generic element of Ξ.

Definition 2.1. [8] A $SVNS$ Υ neutrosophic set that is characterized by a truth membership function $\tau_\Upsilon(\hat{n})$, an indeterminacy-membership function $\iota_\Upsilon(\hat{n})$, and a falsity-membership function $F_\Upsilon(\hat{n})$, where $\tau_\Upsilon(\hat{n}), \iota_\Upsilon(\hat{n}), F_\Upsilon(\hat{n}) \in [0, 1]$. This set Υ can thus be written as:

$$\Upsilon = \{\langle \hat{n}, \tau_\Upsilon(\hat{n}), \iota_\Upsilon(\hat{n}), F_\Upsilon(\hat{n})\rangle : \hat{n} \in \Xi)\}.$$

The sum of $\tau_\Upsilon(\hat{n})$, $\iota_\Upsilon(\hat{n})$ and $F_\Upsilon(\hat{n})$ must fulfill the clause $0 \leq \tau_\Upsilon(\hat{n}) + \iota_\Upsilon(\hat{n}) + F_\Upsilon(\hat{n}) \leq 3$. For a $SVNS$ Υ in Ξ, the triplet $(\tau_\Upsilon(\hat{n}), \iota_\Upsilon(\hat{n}), F_\Upsilon(\hat{n}))$ is referred to as a single valued neutrosophic number (SVNN). Let $\hat{n} = (\tau_{\hat{n}}, \iota_{\hat{n}}, F_{\hat{n}})$ stand for a SVNN.

Definition 2.2. [8] Assume Υ and Γ are two $SVNSs$ in a universe Ξ.

(1) Υ is contained in Γ, if $\tau_\Upsilon(\hat{n}) \leq \tau_\Gamma(\hat{n}), \iota_\Upsilon(\hat{n}) \leq \iota_\Gamma(\hat{n})$, and $F_\Upsilon(\hat{n}) \geq F_\Gamma(\hat{n})$, $\forall\ \hat{n} \in \Xi$. This relationship is denoted as $\Upsilon \subseteq \Gamma$.

(2) Υ=Γ if $\Upsilon \subseteq \Gamma$ and $\Gamma \subseteq \Upsilon$.

(3) $\Upsilon^c = (\hat{n}, (F_\Upsilon(\hat{n}), 1 - \iota_\Upsilon(\hat{n}), \tau_\Upsilon(\hat{n})), \forall\ \hat{n} \in \Xi$.

(4) $\Upsilon \cup \Gamma = (\hat{n}, (\bigvee(\tau_\Upsilon, \tau_\Gamma), \bigvee(\iota_\Upsilon, \iota_\Gamma), \bigwedge(F_\Upsilon, F_\Gamma))), \forall\ \hat{n} \in \Xi$.

(5) $\Upsilon \cap \Gamma = (\hat{n}, (\bigwedge(\tau_\Upsilon, \tau_\Gamma), \bigwedge(\iota_\Upsilon, \iota_\Gamma), \bigvee(F_\Upsilon, F_\Gamma))), \forall\ \hat{n} \in \Xi$.

Definition 2.3. [1] A hypergroup $\langle H, \circ \rangle$ is a set H with an associative hyperoperation $(\circ) : H * H \longrightarrow P(H)$ which satisfies $\hat{n} \circ H = H \circ \hat{n} = H$, $\forall\ \hat{n} \in H$ (reproduction axiom).

Definition 2.4. [36] If the following properties satisfy, a hyperstructure $\langle H, \circ \rangle$ is termed a H_v-group:

(1) $\hat{n} \circ (\hat{o} \circ \hat{p}) \cap (\hat{n} \circ \hat{o}) \circ \hat{p} \neq \phi$, $\forall\ \hat{n}, \hat{o}, \hat{p} \in H$, ($H_v$-semigroup).

(2) $\hat{n} \circ H = H \circ \hat{n} = H$, $\forall\ \hat{n} \in H$.

Definition 2.5. [1] A subset W of H is termed as subhypergroup if $\langle W, \circ \rangle$ is a hypergroup.

Definition 2.6. [2] A H_v-ring is a multi-valued system $(R, +, \circ)$ that satisfies the following axioms:

(1) $(R, +)$ must a H_v-group,

(2) $(R, \circ)$ must a H_v-semigroup,

(3) The hyperoperation "$\circ$" is weak distributive over the hyperoperation "$+$", that is for each $\hat{n}, \hat{o}, \hat{p} \in R$ the clauses $\hat{n} \circ (\hat{o} + \hat{p}) \cap ((\hat{n} \circ \hat{o}) + (\hat{n} \circ \hat{p})) \neq \phi$ and $(\hat{n} + \hat{o}) \circ \hat{p} \cap ((\hat{n} \circ \hat{p}) + (\hat{o} \circ \hat{p})) \neq \phi$ must satisfy.

Definition 2.7. [2] A nonempty subset $R^{'}$ of R is a subhyperring of $(R, +, \circ)$ if $(R^{'}, +)$ is a subhypergroup of $(R, +)$ and $\forall\ \hat{n}, \hat{o}, \hat{p} \in R^{'}, \hat{n} \circ \hat{o} \in P^*(R^{'})$, where $P^*(R^{'})$ denotes the set of all non-empty subsets of $R^{'}$.

Definition 2.8. [2] Suppose H_v-ring be R. a nonempty subset I of R is called a left (resp. right) H_v-ideal if the following axioms hold:

(1) $(I, +)$ be a H_v-subgroup of $(R, +)$,

(2) $R \circ I \subseteq I$(resp. $I \circ R \subseteq I$).

If I is both a left and right H_v-ideal of R, then I is called H_v-ideal of R.

3. $(\varpi, \varepsilon, \varsigma)$-Single Valued Neutrosophic Hyperrings

We represent hyperring $(R, +, \circ)$ by R throughout this section.

Definition 3.1. If Υ be a single valued neutrosophic subset of Ξ then $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic Υ subset of Ξ is categorize as,

$$\Upsilon^{(\varpi,\varepsilon,\varsigma)} = \Big\{\langle \hat{n}, \tau_\Upsilon^\varpi(\hat{n}), \iota_\Upsilon^\varepsilon(\hat{n}), F_\Upsilon^\varsigma(\hat{n})\rangle | \tau_\Upsilon^\varpi(\hat{n}) = \bigwedge\{\tau_\Upsilon(\hat{n}), \varpi\}, \iota_\Upsilon^\varepsilon(\hat{n}) = \bigwedge\{\iota_\Upsilon(\hat{n}), \varepsilon\}, F_\Upsilon^\varsigma(\hat{n}) = \bigvee\{F_\Upsilon(\hat{n}), \varsigma\}, \hat{n} \in \Xi\Big\},$$

and $0 \leq \tau^{\varpi}_{\Upsilon}(\hat{n})+\iota^{\varepsilon}_{\Upsilon}(\hat{n})+F^{\varsigma}_{\Upsilon}(\hat{n}) \leq 3$, where $\varpi, \varepsilon, \varsigma \in [0,1]$ also $\tau, \iota, F : \Upsilon \to [0,1]$, such that τ^{ϖ}_{Υ}, $\iota^{\varepsilon}_{\Upsilon}$, F^{ς}_{Υ} represents the functions of truth, indeterminacy, and falsity-membership, respectively.

Definition 3.2. Let Υ be a $(\varpi, \varepsilon, \varsigma)$-$SVNS$ over R. Then Υ is called a $(\varpi, \varepsilon, \varsigma)$-$SVNHR$ over R, if,

(1) $\forall\ \hat{k}, \hat{l} \in R$,
$\bigwedge\{\tau^{\varpi}_{\Upsilon}(\hat{k}), \tau^{\varpi}_{\Upsilon}(\hat{l})\} \leq \inf\{\tau^{\varpi}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k}+\hat{l}\}$,
$\bigvee\{\iota^{\varepsilon}_{\Upsilon}(\hat{k}), \iota^{\varepsilon}_{\Upsilon}(\hat{l})\} \geq \sup\{\iota^{\varepsilon}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k}+\hat{l}\}$, and
$\bigvee\{F^{\varsigma}_{\Upsilon}(\hat{k}), F^{\varsigma}_{\Upsilon}(\hat{l})\} \geq \sup\{F^{\varsigma}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k}+\hat{l}\}$.

(2) $\forall\ \hat{n}, \hat{k} \in R$, $\exists\ \hat{l} \in R$ such that $\hat{k} \in \hat{n}+\hat{l}$ and
$\bigwedge\{\tau^{\varpi}_{\Upsilon}(\hat{n}), \tau^{\varpi}_{\Upsilon}(\hat{k})\} \leq \tau^{\varpi}_{\Upsilon}(\hat{l})$,
$\bigvee\{\iota^{\varepsilon}_{\Upsilon}(\hat{n}), \iota^{\varepsilon}_{\Upsilon}(\hat{k})\} \geq \iota^{\varepsilon}_{\Upsilon}(\hat{l})$, and
$\bigvee\{F^{\varsigma}_{\Upsilon}(\hat{n}), F^{\varsigma}_{\Upsilon}(\hat{k})\} \geq F^{\varsigma}_{\Upsilon}(\hat{l})$

(3) $\forall\ \hat{n}, \hat{k} \in R$, $\exists\ \hat{m} \in R$ such that $\hat{k} \in \hat{m}+\hat{n}$ and
$\bigwedge\{\tau^{\varpi}_{\Upsilon}(\hat{n}), \tau^{\varpi}_{\Upsilon}(\hat{k})\} \leq \tau^{\varpi}_{\Upsilon}(\hat{m})$,
$\bigvee\{\iota^{\varepsilon}_{\Upsilon}(\hat{n}, \iota^{\varepsilon}_{\Upsilon}(\hat{k})\} \geq \iota^{\varepsilon}_{\Upsilon}(\hat{m})$, and
$\bigvee\{F^{\varsigma}_{\Upsilon}(\hat{n}, F^{\varsigma}_{\Upsilon}(\hat{k})\} \geq F^{\varsigma}_{\Upsilon}(\hat{m})$.

(4) $\forall\ \hat{k}, \hat{l} \in R$,
$\bigwedge\{\tau^{\varpi}_{\Upsilon}(\hat{k}), \tau^{\varpi}_{\Upsilon}(\hat{l}\} \leq \inf\{\tau^{\varpi}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$,
$\bigvee\{\iota^{\varepsilon}_{\Upsilon}(\hat{k}), \iota^{\varepsilon}_{\Upsilon}(\hat{l})\} \geq \sup\{\iota^{\varepsilon}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$, and
$\bigvee\{F^{\varsigma}_{\Upsilon}(\hat{k}), F^{\varsigma}_{\Upsilon}(\hat{l})\} \geq \sup\{F^{\varsigma}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$.

Example 3.3. The family of t-level sets of $(\varpi, \varepsilon, \varsigma)$-$SVNSs$ over R is a subhyperring of R is resulting below:

$$\Upsilon_t^{(\varpi,\varepsilon,\varsigma)} = \{\hat{k} \in R : \tau^{\varpi}_{\Upsilon}(\hat{k}) \geq t,\ \iota^{\varepsilon}_{\Upsilon}(\hat{k}) \leq t,\ F^{\varsigma}_{\Upsilon}(\hat{k}) \leq t\}, \forall\ t \in [0,1].$$

Then Υ over R is a $(\varpi, \varepsilon, \varsigma)$-$SVNHR$.

Theorem 3.4. *Υ is a $(\varpi, \varepsilon, \varsigma)$-$SVNS$ over R. Then Υ is a $(\varpi, \varepsilon, \varsigma)$-$SVNHR$ over R if and only if Υ is $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic semi hypergroup over $(R, \circ)$ and also a $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic hypergroup over $(R, +)$.*

Proof. The definition 3.2 readily indicates this proof. □

Proposition 3.5. *If Υ and Γ be two $(\varpi, \varepsilon, \varsigma)$-single-valued neutrosophic subset of ring R then $(\Upsilon \cap \Gamma)^{(\varpi,\varepsilon,\varsigma)} = \Upsilon^{(\varpi,\varepsilon,\varsigma)} \cap \Gamma^{(\varpi,\varepsilon,\varsigma)}$.*

Proof. Assume that Υ and Γ are two $(\varpi, \varepsilon, \varsigma)$-single-valued neutrosophic subset of ring R.

$$(\Upsilon \cap \Gamma)^{(\varpi,\varepsilon,\varsigma)}(\hat{n}) = \Big\{ \min\{\min\{\tau_\Upsilon(\hat{n}), \tau_\Gamma(\hat{n})\}, \varpi\}, \min\{\min\{\iota_\Upsilon(\hat{n}),\ \iota_\Gamma(\hat{n})\}, \varepsilon\}, \max\{\max\{F_\Upsilon(\hat{n}), F_\Gamma(\hat{n})\}, \varsigma\} \Big\}$$

$$= \Big\{ \min\{\min\{\tau_\Upsilon(\hat{n}), \varpi\}, \min\{\tau_\Gamma(\hat{n}), \varpi\}\}, \min\{\min\{\iota_\Upsilon(\hat{n}), \varepsilon\}, \max\{\iota_\Gamma(\hat{n}), \varepsilon\}\}, \max\{\max\{F_\Upsilon(\hat{n}), \varsigma\}, \max\{F_\Gamma(\hat{n}), \varsigma\}\} \Big\}$$

$$= \Big\{ \min(\{\tau^\varpi_\Upsilon(\hat{n})\}, \{\tau^\varpi_\Gamma(\hat{n})\}), \min(\{\iota^\varepsilon_\Upsilon(\hat{n})\}, \{\iota^\varepsilon_\Gamma(\hat{n})\}), \max(\{F^\varsigma_\Upsilon(\hat{n})\}, \{F^\varsigma_\Gamma(\hat{n})\}) \Big\} = \Upsilon^{(\varpi,\varepsilon,\varsigma)}(\hat{n}) \cap \Gamma^{(\varpi,\varepsilon,\varsigma)}(\hat{n}),\ \forall\, \hat{n} \in R.$$

□

Theorem 3.6. *Let Υ and Γ be $(\varpi, \varepsilon, \varsigma)$-$SVNHRs$ over R. Then $\Upsilon \cap \Gamma$ is a $(\varpi, \varepsilon, \varsigma)$-$SVNHR$ over R if it is non-null.*

Proof. Let Υ and Γ are $(\varpi, \varepsilon, \varsigma)$-$SVNHR$s over R. By using Definition 3.2, and Proposition 3.5

$$(\Upsilon \cap \Gamma)^{(\varpi,\varepsilon,\varsigma)} = \Upsilon^{(\varpi,\varepsilon,\varsigma)} \cap \Gamma^{(\varpi,\varepsilon,\varsigma)} = \{\langle \hat{k}, (\tau^\varpi_\Upsilon \bigwedge \tau^\varpi_\Gamma)(\hat{k}), (\iota^\varepsilon_\Upsilon \bigwedge \iota^\varepsilon_\Gamma)(\hat{k}), (F^\varsigma_\Upsilon \bigvee F^\varsigma_\Gamma)(\hat{k})\rangle : \hat{k} \in R\},$$

where

$$\begin{aligned}
(\tau^\varpi_\Upsilon \bigwedge \tau^\varpi_\Gamma)(\hat{k}) &= \bigwedge(\tau^\varpi_\Upsilon(\hat{k}), \tau^\varpi_\Gamma(\hat{k})),\\
(\iota^\varepsilon_\Upsilon \bigwedge \iota^\varepsilon_\Gamma)(\hat{k}) &= \bigwedge(\iota^\varepsilon_\Upsilon(\hat{k}), \iota^\varepsilon_\Gamma(\hat{k})),\\
(F^\varsigma_\Upsilon \bigvee F^\varsigma_\Gamma)(\hat{k}) &= \bigvee(F^\varsigma_\Upsilon(\hat{k}), F^\varsigma_\Gamma(\hat{k})).
\end{aligned}$$

Assuming $\forall\, \hat{k}, \hat{l} \in R$, we are only proven to include all four clauses for membership terms τ^ϖ_Υ, τ^ϖ_Γ and indeterminacy terms $\iota^\varepsilon_\Upsilon$, ι^ε_Γ. Indications for falsity functions of F^ς_Υ, F^ς_Γ correspondingly derived.

$$\begin{aligned}
(1)\ \bigwedge\{(\tau^\varpi_\Upsilon \bigwedge \tau^\varpi_\Gamma)(\hat{k}), (\tau^\varpi_\Upsilon \bigwedge \tau^\varpi_\Gamma)(\hat{l})\} &= \bigwedge\{\bigwedge(\tau^\varpi_\Upsilon(\hat{k}), \tau^\varpi_\Gamma(\hat{k})), \bigwedge(\tau^\varpi_\Upsilon(\hat{l}), \tau^\varpi_\Gamma(\hat{l}))\}\\
&\le \bigwedge\{\bigwedge(\tau^\varpi_\Upsilon(\hat{k}), \tau^\varpi_\Upsilon(\hat{l})), \bigwedge(\tau^\varpi_\Gamma(\hat{k}), \tau^\varpi_\Gamma(\hat{l}))\}\\
&\le \bigwedge\{\inf\{\tau^\varpi_\Upsilon(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}, \inf\{\tau^\varpi_\Gamma(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}\}\\
&\le \inf\{\bigwedge(\tau^\varpi_\Upsilon(\hat{m}), \tau^\varpi_\Gamma(\hat{m})) : \hat{m} \in \hat{k} + \hat{l}\}\\
&= \inf\{(\tau^\varpi_\Upsilon \bigwedge \tau^\varpi_\Gamma)(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}.\\
\Rightarrow \bigwedge\{(\tau^\varpi_\Upsilon \bigwedge \tau^\varpi_\Gamma)(\hat{k}), (\tau^\varpi_\Upsilon \bigwedge \tau^\varpi_\Gamma)(\hat{l})\} &\le \inf\{(\tau^\varpi_\Upsilon \bigwedge \tau^\varpi_\Gamma)(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}.
\end{aligned}$$

Also

$$\begin{aligned}
\bigvee\{(\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{k}),(\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})\}(\hat{l})\} &= \bigvee\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{k}),\iota^{\varepsilon}_{\Gamma}(\hat{k})),\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{l}),\iota^{\varepsilon}_{\Gamma}(\hat{l}))\} \\
&\geq \bigwedge\{\bigvee(\iota^{\varepsilon}_{\Upsilon}(\hat{k}),\iota^{\varepsilon}_{\Upsilon}(\hat{l})),\bigvee(\iota^{\varepsilon}_{\Gamma}(\hat{k}),\iota^{\varepsilon}_{\Gamma}(\hat{l}))\} \\
&\geq \bigwedge\{\sup\{\iota^{\varepsilon}_{\Upsilon}(\hat{m}):\hat{m}\in\hat{k}+\hat{l}\},\sup\{\iota^{\varepsilon}_{\Gamma}(\hat{m}):\hat{m}\in\hat{k}+\hat{l}\}\} \\
&\geq \sup\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{m}),\iota^{\varepsilon}_{\Gamma}(\hat{m})):\hat{m}\in\hat{k}+\hat{l}\} \\
&= \sup\{(\iota^{\varepsilon}_{\Upsilon}(\hat{m})\bigwedge\iota^{\varepsilon}_{\Gamma}(\hat{m})):\hat{m}\in\hat{k}+\hat{l}\}. \\
\Rightarrow \bigvee\{(\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{k}),(\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})\}(\hat{l})\} &\geq \sup\{(\iota^{\varepsilon}_{\Upsilon}(\hat{m})\bigwedge\iota^{\varepsilon}_{\Gamma}(\hat{m})):\hat{m}\in\hat{k}+\hat{l}\}.
\end{aligned}$$

Similarly,

$$\bigvee\{(F^{\varsigma}_{\Upsilon} \bigvee F^{\varsigma}_{\Gamma})(\hat{k}),(F^{\varsigma}_{\Upsilon} \bigvee F^{\varsigma}_{\Gamma})\}(\hat{l})\} \geq \sup\{(F^{\varsigma}_{\Upsilon}(\hat{m})\bigvee F^{\varsigma}_{\Gamma}(\hat{m})):\hat{m}\in\hat{k}+\hat{l}\}.$$

(2) $\exists\ \forall\ \hat{n},\hat{k}\in R$ such that $\hat{k}\in\hat{n}+\hat{l}$ then it argues that:

$$\begin{aligned}
\bigwedge\{(\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{n}),(\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{k})\} &= \bigwedge\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{n}),\tau^{\varpi}_{\Gamma}(\hat{n})),\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{k}),\tau^{\varpi}_{\Gamma}(\hat{k}))\} \\
&\leq \bigwedge\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{n}),\tau^{\varpi}_{\Upsilon}(\hat{k})),\bigwedge(\tau^{\varpi}_{\Gamma}(\hat{n}),\tau^{\varpi}_{\Gamma}(\hat{k}))\} \\
&\leq \{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{l}),\tau^{\varpi}_{\Gamma}(\hat{l}))\} \\
&= (\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{l}). \\
\Rightarrow \bigwedge\{(\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{n}),(\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{k})\} &\leq \{(\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{l}):\hat{k}\in\hat{n}+\hat{l}\}.
\end{aligned}$$

Also, $\exists\ \forall\ \hat{n},\hat{k}\in R$ such that $\hat{k}\in\hat{n}+\hat{l}$ then it argues that:

$$\begin{aligned}
\bigvee\{(\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{n}),(\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{k})\} &= \bigvee\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{n}),\iota^{\varepsilon}_{\Gamma}(\hat{n})),\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{k}),\iota^{\varepsilon}_{\Gamma}(\hat{k}))\} \\
&\geq \bigvee\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{n}),\iota^{\varepsilon}_{\Upsilon}(\hat{k})),\bigwedge(\iota^{\varepsilon}_{\Gamma}(\hat{n}),\iota^{\varepsilon}_{\Gamma}(\hat{k}))\} \\
&\geq \{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{l}),\iota^{\varepsilon}_{\Gamma}(\hat{l}))\} \\
&= \{(\iota^{\varepsilon}_{\Upsilon}(\hat{l})\bigwedge\iota^{\varepsilon}_{\Gamma}(\hat{l}))\}. \\
\Rightarrow \bigvee\{(\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{n}),(\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{k})\} &\geq \{(\iota^{\varepsilon}_{\Upsilon}(\hat{l})\bigwedge\iota^{\varepsilon}_{\Gamma}(\hat{l})):\hat{k}\in\hat{n}+\hat{l}\}
\end{aligned}$$

Similarly,

$$\bigvee\{(F^{\varsigma}_{\Upsilon} \bigvee F^{\varsigma}_{\Gamma})(\hat{n}),(F^{\varsigma}_{\Upsilon} \bigvee F^{\varsigma}_{\Gamma})(\hat{k})\} \geq \{(F^{\varsigma}_{\Upsilon}(\hat{l})\bigvee F^{\varsigma}_{\Gamma}(\hat{l})):\hat{k}\in\hat{n}+\hat{l}\}$$

(3) $\forall\ \hat{n}, \hat{k} \in R\ \exists\ \hat{m} \in R$ where $\hat{k} \in \hat{m} + \hat{n}$ can be readily proved that

$$\begin{aligned} \bigwedge\{(\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{n}), (\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{k})\} &= \bigwedge\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{n}), \tau^{\varpi}_{\Gamma}(\hat{n})), \bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{k}), \tau^{\varpi}_{\Gamma}(\hat{k}))\} \\ &\leq \bigwedge\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{n}), \tau^{\varpi}_{\Upsilon}(\hat{k})), \bigwedge(\tau^{\varpi}_{\Gamma}(\hat{n}), \tau^{\varpi}_{\Gamma}(\hat{k}))\} \\ &\leq \{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{m}), \tau^{\varpi}_{\Gamma}(\hat{m}))\} \\ &= (\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{m}). \\ \Rightarrow \bigwedge\{(\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{n}), (\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{k})\} &\leq \{(\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{m}) : \hat{k} \in \hat{m} + \hat{n}\}. \end{aligned}$$

Also, $\forall\ \hat{n}, \hat{k} \in R\ \exists\ \hat{m} \in R$ where $\hat{k} \in \hat{m} + \hat{n}$ then it argues that:

$$\begin{aligned} \bigvee\{(\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{n}), (\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{k})\} &= \bigvee\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{n}), \iota^{\varepsilon}_{\Gamma}(\hat{n})), \bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{k}), \iota^{\varepsilon}_{\Gamma}(\hat{k}))\} \\ &\geq \bigvee\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{n}), \iota^{\varepsilon}_{\Upsilon}(\hat{k})), \bigwedge(\iota^{\varepsilon}_{\Gamma}(\hat{n}), \iota^{\varepsilon}_{\Gamma}(\hat{k}))\} \\ &\geq \{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{m}), \iota^{\varepsilon}_{\Gamma}(\hat{m}))\} \\ &= \{(\iota^{\varepsilon}_{\Upsilon}(\hat{m}) \bigwedge \iota^{\varepsilon}_{\Gamma}(\hat{m}))\}. \\ \Rightarrow \bigvee\{(\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{n}), (\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{k})\} &\geq \{(\iota^{\varepsilon}_{\Upsilon}(\hat{m}) \bigwedge \iota^{\varepsilon}_{\Gamma}(\hat{m})) : \hat{k} \in \hat{m} + \hat{n}\} \end{aligned}$$

Similarly,

$$\bigvee\{(F^{\varsigma}_{\Upsilon} \bigvee F^{\varsigma}_{\Gamma})(\hat{n}), (F^{\varsigma}_{\Upsilon} \bigvee F^{\varsigma}_{\Gamma})(\hat{k})\} \geq \{(F^{\varsigma}_{\Upsilon}(\hat{m}) \bigvee F^{\varsigma}_{\Gamma}(\hat{m})) : \hat{k} \in \hat{m} + \hat{n}\}$$

(4) $\forall\ \hat{k},\ \hat{l} \in R$,

$$\begin{aligned} \bigwedge\{(\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{k}), (\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{l})\} &\leq \inf\{(\tau^{\varpi}_{\Upsilon} \bigwedge \tau^{\varpi}_{\Gamma})(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}, \\ \bigvee\{(\iota^{\varepsilon}_{\Upsilon} \bigwedge \tau^{\varepsilon}_{\Gamma})(\hat{k}), (\iota^{\varepsilon}_{\Upsilon} \bigwedge \iota^{\varepsilon}_{\Gamma})(\hat{l})\} &\geq \sup\{(\iota^{\varepsilon}_{\Upsilon} \bigwedge \tau^{\varepsilon}_{\Gamma})\ (\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}, \\ \bigvee\{(F^{\varsigma}_{\Upsilon} \bigwedge F^{\varsigma}_{\Gamma})(\hat{k}), (F^{\varsigma}_{\Upsilon} \bigwedge F^{\varsigma}_{\Gamma})(\hat{l})\} &\geq \sup\{(F^{\varsigma}_{\Upsilon} \bigwedge F^{\varsigma}_{\Gamma})(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}. \end{aligned}$$

Hence, $\Upsilon \cap \Gamma$ is $(\varpi, \varepsilon, \varsigma)$-$SVNHR$ over R. □

Theorem 3.7. *Let Υ be a $(\varpi, \varepsilon, \varsigma)$-$SVNHR$ over R. Then for every $t \in [0, 1]$, $\Upsilon_t^{(\varpi,\varepsilon,\varsigma)} \neq \phi$ is a subhyperring over R.*

Proof. Let Υ be a $(\varpi, \varepsilon, \varsigma)$-$SVNHR$ over R. $\forall\ t \in [0, 1]$, let $\hat{k}, \hat{l} \in \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$.
Then $\tau^{\varpi}_{\Upsilon}(\hat{k}), \tau^{\varpi}_{\Upsilon}(\hat{l}) \geq t,\ \iota^{\varepsilon}_{\Upsilon}(\hat{k}), \iota^{\varepsilon}_{\Upsilon}(\hat{l}) \leq t$ and $F^{\varsigma}_{\Upsilon}(\hat{k}), F^{\varsigma}_{\Upsilon}(\hat{l}) \leq t$.
Since Υ is a $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic subhypergroup of $(R, +)$, we have the following

$$\inf\{\tau^{\varpi}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\} \geq \bigwedge\{\tau^{\varpi}_{\Upsilon}(\hat{k}), \tau^{\varpi}_{\Upsilon}(\hat{l})\} \geq \bigwedge\{t, t\} = t,$$

$$\sup\{\iota^{\varepsilon}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\} \leq \bigvee\{\iota^{\varepsilon}_{\Upsilon}(\hat{k}), \iota^{\varepsilon}_{\Upsilon}(\hat{l})\} \leq \bigvee\{t, t\} = t,$$

and

$$\sup\{F^{\varsigma}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\} \leq \bigvee\{F^{\varsigma}_{\Upsilon}(\hat{k}), F^{\varsigma}_{\Upsilon}(\hat{l})\} \leq \bigvee\{t, t\} = t.$$

This implies that $\hat{m} \in \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$ and then for every $\hat{m} \in \hat{k} + \hat{l}$, we obtain $\hat{k} + \hat{l} \subseteq \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$.
As such, for every $\hat{m} \in \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$, we obtain $\hat{m} + \Upsilon_t^{(\varpi,\varepsilon,\varsigma)} \subseteq \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$.
Now let $\hat{k},\ \hat{m} \in \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$. Then $\tau_\Upsilon^\varpi(\hat{k}),\ \tau_\Upsilon^\varpi(\hat{m}) \geq t,\ \iota_\Upsilon^\varepsilon(\hat{k}),\ \iota_\Upsilon^\varepsilon(\hat{m}) \leq t$ and $F_\Upsilon^\varsigma(\hat{k}),\ F_\Upsilon^\varsigma(\hat{m}) \leq t$. Υ is a $(\varpi,\varepsilon,\varsigma)$-single valued neutrosophic subhypergroup of $(R,+)$, $\exists\ \hat{l} \in R$ such that $\hat{k} \in \hat{m} + \hat{l}$ and $\tau_\Upsilon^\varpi(\hat{l}) \geq \bigwedge(\tau_\Upsilon^\varpi(\hat{k}),\ \tau_\Upsilon^\varpi(\hat{m})) \geq t,\ \iota_\Upsilon^\varepsilon(\hat{l}) \leq \bigvee(\iota_\Upsilon^\varepsilon(\hat{k}),\ \iota_\Upsilon^\varepsilon(\hat{m})) \leq t,\ F_\Upsilon^\varsigma(\hat{l}) \leq \bigvee(F_\Upsilon^\varsigma(\hat{k}),\ F_\Upsilon^\varsigma(\hat{m})) \leq t$, and this implies that $\hat{l} \in \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$. Therefore, we obtain $\Upsilon_t^{(\varpi,\varepsilon,\varsigma)} \subseteq \hat{m} + \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$.
As such, we obtain $\hat{m} + \Upsilon_t^{(\varpi,\varepsilon,\varsigma)} = \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$. As a result, $\Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$ is a subhypergroup of $(R,+)$.
Let $\hat{k},\hat{l} \in \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$, then $\tau_\Upsilon^\varpi(\hat{k}),\ \tau_\Upsilon^\varpi(\hat{l}) \geq t,\ \iota_\Upsilon^\varepsilon(\hat{k}),\ \iota_\Upsilon^\varepsilon(\hat{l}) \leq t$ and $F_\Upsilon^\varsigma(\hat{k}),\ F_\Upsilon^\varsigma(\hat{l}) \leq t$. Since Υ is a $(\varpi,\varepsilon,\varsigma)$-single valued neutrosophic sub-semihypergroup of $(R,\circ)$, then $\forall\ \hat{k},\hat{l} \in R$, we have the following:

$$\inf\{\tau_\Upsilon^\varpi(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\} \geq \bigwedge\{\tau_\Upsilon^\varpi(\hat{k}),\ \tau_\Upsilon^\varpi(\hat{l})\} = t,$$

$$\sup\{\iota_\Upsilon^\varepsilon(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\} \leq \bigvee(\iota_\Upsilon^\varepsilon(\hat{k}),\ \iota_\Upsilon^\varepsilon(\hat{l})) = t,$$

and

$$\sup\{F_\Upsilon^\varsigma(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l} \leq \bigvee(F_\Upsilon^\varsigma(\hat{k}),\ F_\Upsilon^\varsigma(\hat{l}) = t.$$

This implies that $\hat{m} \in \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$ and consequently $\hat{k} \circ \hat{l} \in \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$.
Therefore, for every $\hat{k},\hat{l} \in \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$ we obtain $\hat{k} \circ \hat{l} \in P^*(R)$. Hence $\Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$ is a subhyperring over R. □

Theorem 3.8. *Let Υ be a $(\varpi,\varepsilon,\varsigma)$-single valued neutrosophic set over R. Then the following statements are equivalent:*

(1) *Υ is a $(\varpi,\varepsilon,\varsigma)$-$SVNHR$ over R.*
(2) *$\forall\ t \in [0,1]$, a non-empty $\Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$ is a subhyperring over R.*

Proof. (1)⇒(2) $\forall\ t \in [0,1]$, by Theorem 3.7, $\Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$ is subhyperring over R.
(2)⇒(1) Assume that $\Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$ is a subhyperring over R. Let $\hat{k},\hat{l} \in \Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$ and therefore $\hat{k} + \hat{l} \subseteq \Upsilon_{t_0}^{(\varpi,\varepsilon,\varsigma)}$. Then for every $\hat{m} \in \hat{k} + \hat{l}$ we have $\tau_\Upsilon^\varpi(\hat{m}) \geq t_0,\ \iota_\Upsilon^\varepsilon(\hat{m}) \leq t_0$ and $F_\Upsilon^\varsigma(\hat{m}) \leq t_0$, which implies that:

$$\bigwedge(\tau_\Upsilon^\varpi(\hat{k}),\tau_\Upsilon^\varpi(\hat{l})) \leq \inf\{\tau_\Upsilon^\varpi(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\},$$

$$\bigvee(\iota_\Upsilon^\varepsilon(\hat{k}),\iota_\Upsilon^\varepsilon(\hat{l})) \geq \sup\{\iota_\Upsilon^\varepsilon(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\},$$

and

$$\bigvee(F_\Upsilon^\varsigma(\hat{k}),F_\Upsilon^\varsigma(\hat{l})) \geq \sup\{F_\Upsilon^\varsigma(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}.$$

Thus, clause (1) of Definition 3.2 has been fulfilled.
Next, let $\hat{n},\hat{k} \in \Upsilon_{t_1}^{(\varpi,\varepsilon,\varsigma)}$ for every $t_1 \in [0,1]$ which means that $\exists\ \hat{l} \in \Upsilon_{t_1}^{(\varpi,\varepsilon,\varsigma)}$ such that $\hat{k} \in \hat{n} \circ \hat{l}$.

Since $\hat{l} \in \Upsilon_{t_1}^{(\varpi,\varepsilon,\varsigma)}$, we have $\tau_\Upsilon^\varpi(\hat{l}) \geq t_1,\ \iota_\Upsilon^\varepsilon(\hat{l}) \leq t_1$ and $F_\Upsilon^\varsigma(\hat{l}) \leq t_1$, and thus we have

$$\tau_\Upsilon^\varpi(\hat{l}) \geq t_1 = \bigwedge(\tau_\Upsilon^\varpi(\hat{k}), \tau_\Upsilon^\varpi(\hat{m})),$$

$$\iota_\Upsilon^\varepsilon(\hat{l}) \leq t_1 = \bigvee(\iota_\Upsilon^\varepsilon(\hat{k}), \iota_\Upsilon^\varepsilon(\hat{m})),$$

and

$$F_\Upsilon^\varsigma(\hat{l}) \leq t_1 = \bigvee(F_\Upsilon^\varsigma(\hat{k}), F_\Upsilon^\varsigma(\hat{m})).$$

Thus, clause (2) of Definition 3.2 has been fulfilled.

Assurance of (3) of Definition 3.2 can be satisfied in a similar way.

As a result, Υ is a $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic subhypergroup of $(R, +)$.

Now since $\Upsilon_t^{(\varpi,\varepsilon,\varsigma)}$ is a sub-semihypergroup of the semihypergroup $(R, \circ)$, we got the following. Let $\hat{k}, \hat{l} \in \Upsilon_{t_2}^{(\varpi,\varepsilon,\varsigma)}$ and therefore we have $\hat{k} \circ \hat{l} \in \Upsilon_{t_2}^{(\varpi,\varepsilon,\varsigma)}$. Thus for every $\hat{m} \in \hat{k} \circ \hat{l}$, we obtain $\tau_\Upsilon^\varpi(\hat{m}) \geq t_2,\ \iota_\Upsilon^\varepsilon(\hat{m}) \leq t_2$ and $F_\Upsilon^\varsigma(\hat{m}) \leq t_2$, and therefore it follows that:

$$\bigwedge(\tau_\Upsilon^\varpi(\hat{k}),\ \tau_\Upsilon^\varpi(\hat{l})) \leq \inf\{\tau_\Upsilon^\varpi(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\},$$

$$\bigvee(\iota_\Upsilon^\varepsilon(\hat{k}),\ \iota_\Upsilon^\varepsilon(\hat{l})) \geq \sup\{\iota_\Upsilon^\varepsilon(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\},$$

and

$$\bigvee(F_\Upsilon^\varsigma(\hat{k}),\ F_\Upsilon^\varsigma(\hat{l})) \geq \sup\{F_\Upsilon^\varsigma(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}.$$

which reveals that clause (4) of Definition 3.2 is verified.

Hence Υ is a $(\varpi, \varepsilon, \varsigma)$-$SVNHR$ over R. □

4. $(\varpi, \varepsilon, \varsigma)$-Single Valued Neutrosophic Hyperideals

Definition 4.1. Let Υ be a $(\varpi, \varepsilon, \varsigma)$-$SVNS$ over R. Then Υ is $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic left (resp. right) hyperideal over R, if,

(1) $\forall\ \hat{k}, \hat{l} \in R$,
$\bigwedge\{\tau_\Upsilon^\varpi(\hat{k}),\ \tau_\Upsilon^\varpi(\hat{l})\} \leq \inf\{\tau_\Upsilon^\varpi(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}$,
$\bigvee\{\iota_\Upsilon^\varepsilon(\hat{k}),\ \iota_\Upsilon^\varepsilon(\hat{l})\} \geq \sup\{\iota_\Upsilon^\varepsilon(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}$, and
$\bigvee\{F_\Upsilon^\varsigma(\hat{k}),\ F_\Upsilon^\varsigma(\hat{l})\} \geq \sup\{F_\Upsilon^\varsigma(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}$.

(2) $\forall\ \hat{n}, \hat{k} \in R$, $\exists\ \hat{l} \in R$ such that $\hat{k} \in \hat{n} + \hat{l}$, and
$\bigwedge\{\tau_\Upsilon^\varpi(\hat{n}),\ \tau_\Upsilon^\varpi(\hat{k})\} \leq \tau_\Upsilon^\varpi(\hat{l})$,
$\bigvee\{\iota_\Upsilon^\varepsilon(\hat{n}),\ \iota_\Upsilon^\varepsilon(\hat{k})\} \geq \iota_\Upsilon^\varepsilon(\hat{l})$, and
$\bigvee\{F_\Upsilon^\varsigma(\hat{n}),\ F_\Upsilon^\varsigma(\hat{k})\} \geq F_\Upsilon^\varsigma(\hat{l})$.

(3) $\forall\ \hat{n}, \hat{k} \in R$, $\exists\ \hat{m} \in R$ such that $\hat{k} \in \hat{m} + \hat{n}$, and
$\bigwedge\{\tau_\Upsilon^\varpi(\hat{n}),\ \tau_\Upsilon^\varpi(\hat{k})\} \leq \tau_\Upsilon^\varpi(\hat{m})$,
$\bigvee\{\iota_\Upsilon^\varepsilon(\hat{n}),\ \iota_\Upsilon^\varepsilon(\hat{k})\} \geq \iota_\Upsilon^\varepsilon(\hat{m})$, and
$\bigvee\{F_\Upsilon^\varsigma(\hat{n}),\ F_\Upsilon^\varsigma(\hat{k})\} \geq F_\Upsilon^\varsigma(\hat{m})$.

(4) $\forall\ \hat{k}, \hat{l} \in R$,
$\tau^{\varpi}_{\Upsilon}(\hat{l}) \leq \inf\{\tau^{\varpi}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$ (resp. $\tau^{\varpi}_{\Upsilon}(\hat{k}) \leq \inf\{\tau^{\varpi}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$),
$\iota^{\varepsilon}_{\Upsilon}(\hat{l}) \geq \sup\{\iota^{\varepsilon}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$ (resp. $\iota^{\varepsilon}_{\Upsilon}(\hat{k}) \geq \sup\{\iota^{\varepsilon}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$), and
$F^{\varsigma}_{\Upsilon}(\hat{l}) \geq \sup\{F^{\varsigma}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$ (resp. $F^{\varsigma}_{\Upsilon}(\hat{k}) \geq \sup\{F^{\varsigma}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$).

If Υ is a $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic left (resp. right) hyperideal of R then Υ is a $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic subhypergroup of $(R, +)$ by clauses (1), (2) and (3).

Definition 4.2. Let Υ be a $(\varpi, \varepsilon, \varsigma)$-$SVNS$ over R. Then Υ is a $(\varpi, \varepsilon, \varsigma)$-$SVNHI$ over R, if aforementioned clauses are met:

(1) $\forall\ \hat{k}, \hat{l} \in R$,
$\bigwedge\{\tau^{\varpi}_{\Upsilon}(\hat{k}),\ \tau^{\varpi}_{\Upsilon}(\hat{l})\} \leq \inf\{\tau^{\varpi}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}$,
$\bigvee\{\iota^{\varepsilon}_{\Upsilon}(\hat{k}),\ \iota^{\varepsilon}_{\Upsilon}(\hat{l})\} \geq \sup\{\iota^{\varepsilon}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}$, and
$\bigvee\{F^{\varsigma}_{\Upsilon}(\hat{k}),\ F^{\varsigma}_{\Upsilon}(\hat{l})\} \geq \sup\{F^{\varsigma}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} + \hat{l}\}$.

(2) $\forall\ \hat{n}, \hat{k} \in R$, $\exists\ \hat{l} \in R$ such that $\hat{k} \in \hat{n} + \hat{l}$, and
$\bigwedge\{\tau^{\varpi}_{\Upsilon}(\hat{n}),\ \tau^{\varpi}_{\Upsilon}(\hat{k})\} \leq \tau^{\varpi}_{\Upsilon}(\hat{l})$,
$\bigvee\{\iota^{\varepsilon}_{\Upsilon}(\hat{n}),\ \iota^{\varepsilon}_{\Upsilon}(\hat{k})\} \geq \iota^{\varepsilon}_{\Upsilon}(\hat{l})$, and
$\bigvee\{F^{\varsigma}_{\Upsilon}(\hat{n}),\ F^{\varsigma}_{\Upsilon}(\hat{k})\} \geq F^{\varsigma}_{\Upsilon}(\hat{l})$.

(3) $\forall\ \hat{n}, \hat{k} \in R$, $\exists\ \hat{m} \in R$ such that $\hat{k} \in \hat{m} + \hat{n}$, and
$\bigwedge\{\tau^{\varpi}_{\Upsilon}(\hat{n}),\ \tau^{\varpi}_{\Upsilon}(\hat{k})\} \leq \tau^{\varpi}_{\Upsilon}(\hat{m})$,
$\bigvee\{\iota^{\varepsilon}_{\Upsilon}(\hat{n}),\ \iota^{\varepsilon}_{\Upsilon}(\hat{k})\} \geq \iota^{\varepsilon}_{\Upsilon}(\hat{m})$, and
$\bigvee\{F^{\varsigma}_{\Upsilon}(\hat{n}),\ F^{\varsigma}_{\Upsilon}(\hat{k})\} \geq F^{\varsigma}_{\Upsilon}(\hat{m})$.

(4) $\forall\ \hat{k}, \hat{l} \in R$,
$\bigvee\{\tau^{\varpi}_{\Upsilon}(\hat{k}), \tau^{\varpi}_{\Upsilon}(\hat{l})\} \leq \inf\{\tau^{\varpi}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$,
$\bigvee\{\iota^{\varepsilon}_{\Upsilon}(\hat{k}), \iota^{\varepsilon}_{\Upsilon}(\hat{l})\} \geq \sup\{\iota^{\varepsilon}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$, and
$\bigvee\{F^{\varsigma}_{\Upsilon}(\hat{k}), F^{\varsigma}_{\Upsilon}(\hat{l})\} \geq \sup\{F^{\varsigma}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}$.

Υ is a $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic subhypergroup of $(R, +)$ by clauses (1), (2) and (3). Clause (4) indicate that Υ is both $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic left hyperideal and $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic right hyperideal.
$\Rightarrow \Upsilon$ is a $(\varpi, \varepsilon, \varsigma)$-$SVNHI$ of R.

Theorem 4.3. *Let Υ be a non-null $(\varpi, \varepsilon, \varsigma)$-$SVNS$ over R. Υ is a $(\varpi, \varepsilon, \varsigma)$-$SVNHI$ over R if and only if Υ is a $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic hypergroup over $(R, +)$ and also Υ is both a $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic left hyperideal and a $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic right hyperideal of R.*

Proof. With the help of Definitions 4.1 and 4.2, we get the required proof. □

Theorem 4.4. *Let Υ and Γ be two $(\varpi,\varepsilon,\varsigma)$-$SVNHIs$ over R. Then $\Upsilon\cap\Gamma$ is a $(\varpi,\varepsilon,\varsigma)$-$SVNHI$ over R if it is non-null.*

Proof. Let Υ and Γ are $(\varpi,\varepsilon,\varsigma)$-$SVNHIs$ over R. By using Definition 3.2, and Proposition 3.5

$$(\Upsilon\cap\Gamma)^{(\varpi,\varepsilon,\varsigma)}=\Upsilon^{(\varpi,\varepsilon,\varsigma)}\cap\Gamma^{(\varpi,\varepsilon,\varsigma)}=\{\langle\hat{k},(\tau^{\varpi}_{\Upsilon}\bigwedge\tau^{\varpi}_{\Gamma})(\hat{k}),(\iota^{\varepsilon}_{\Upsilon}\bigwedge\iota^{\varepsilon}_{\Gamma})(\hat{k}),(F^{\varsigma}_{\Upsilon}\bigvee F^{\varsigma}_{\Gamma})(\hat{k})\rangle:\hat{k}\in R\},$$

where

$$\begin{aligned}(\tau^{\varpi}_{\Upsilon}\bigwedge\tau^{\varpi}_{\Gamma})(\hat{k})&=\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{k}),\tau^{\varpi}_{\Gamma}(\hat{k})),\\(\iota^{\varepsilon}_{\Upsilon}\bigwedge\iota^{\varepsilon}_{\Gamma})(\hat{k})&=\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{k}),\iota^{\varepsilon}_{\Gamma}(\hat{k})),\\(F^{\varsigma}_{\Upsilon}\bigvee F^{\varsigma}_{\Gamma})(\hat{k})&=\bigvee(F^{\varsigma}_{\Upsilon}(\hat{k}),F^{\varsigma}_{\Gamma}(\hat{k})).\end{aligned}$$

Assuming $\forall\ \hat{k},\hat{l}\in R$, we are only proven to include all four clauses for membership terms τ^{ϖ}_{Υ}, τ^{ϖ}_{Γ} and indeterminacy terms $\iota^{\varepsilon}_{\Upsilon}$, $\iota^{\varepsilon}_{\Gamma}$.

$$\begin{aligned}(1)\ \bigwedge\{(\tau^{\varpi}_{\Upsilon}\bigwedge\tau^{\varpi}_{\Gamma})(\hat{k}),(\tau^{\varpi}_{\Upsilon}\bigwedge\tau^{\varpi}_{\Gamma})(\hat{l})\}&=\bigwedge\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{k}),\tau^{\varpi}_{\Gamma}(\hat{k})),\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{l}),\tau^{\varpi}_{\Gamma}(\hat{l}))\}\\&\le\bigwedge\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{k}),\tau^{\varpi}_{\Upsilon}(\hat{l})),\bigwedge(\tau^{\varpi}_{\Gamma}(\hat{k}),\tau^{\varpi}_{\Gamma}(\hat{l}))\}\\&\le\bigwedge\{\inf\{\tau^{\varpi}_{\Upsilon}(\hat{m}):\hat{m}\in\hat{k}+\hat{l}\},\inf\{\tau^{\varpi}_{\Gamma}(\hat{m}):\hat{m}\in\hat{k}+\hat{l}\}\}\\&\le\inf\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{m}),\tau^{\varpi}_{\Gamma}(\hat{m})):\hat{m}\in\hat{k}+\hat{l}\}\\&=\inf\{(\tau^{\varpi}_{\Upsilon}\bigwedge\tau^{\varpi}_{\Gamma})(\hat{m}):\hat{m}\in\hat{k}+\hat{l}\}.\\\Rightarrow\ \bigwedge\{(\tau^{\varpi}_{\Upsilon}\bigwedge\tau^{\varpi}_{\Gamma})(\hat{k}),(\tau^{\varpi}_{\Upsilon}\bigwedge\tau^{\varpi}_{\Gamma})(\hat{l})\}&\le\inf\{(\tau^{\varpi}_{\Upsilon}\bigwedge\tau^{\varpi}_{\Gamma})(\hat{m}):\hat{m}\in\hat{k}+\hat{l}\}.\end{aligned}$$

Also

$$\begin{aligned}\bigvee\{(\iota^{\varepsilon}_{\Upsilon}\bigwedge\iota^{\varepsilon}_{\Gamma})(\hat{k}),(\iota^{\varepsilon}_{\Upsilon}\bigwedge\iota^{\varepsilon}_{\Gamma})\}(\hat{l})\}&=\bigvee\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{k}),\iota^{\varepsilon}_{\Gamma}(\hat{k})),\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{l}),\iota^{\varepsilon}_{\Gamma}(\hat{l}))\}\\&\ge\bigwedge\{\bigvee(\iota^{\varepsilon}_{\Upsilon}(\hat{k}),\iota^{\varepsilon}_{\Upsilon}(\hat{l})),\bigvee(\iota^{\varepsilon}_{\Gamma}(\hat{k}),\iota^{\varepsilon}_{\Gamma}(\hat{l}))\}\\&\ge\bigwedge\{\sup\{\iota^{\varepsilon}_{\Upsilon}(\hat{m}):\hat{m}\in\hat{k}+\hat{l}\},\sup\{\iota^{\varepsilon}_{\Gamma}(\hat{m}):\hat{m}\in\hat{k}+\hat{l}\}\}\\&\ge\sup\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{m}),\iota^{\varepsilon}_{\Gamma}(\hat{m})):\hat{m}\in\hat{k}+\hat{l}\}\\&=\sup\{(\iota^{\varepsilon}_{\Upsilon}(\hat{m})\bigwedge\iota^{\varepsilon}_{\Gamma}(\hat{m})):\hat{m}\in\hat{k}+\hat{l}\}.\\\Rightarrow\ \bigvee\{(\iota^{\varepsilon}_{\Upsilon}\bigwedge\iota^{\varepsilon}_{\Gamma})(\hat{k}),(\iota^{\varepsilon}_{\Upsilon}\bigwedge\iota^{\varepsilon}_{\Gamma})\}(\hat{l})\}&\ge\sup\{(\iota^{\varepsilon}_{\Upsilon}(\hat{m})\bigwedge\iota^{\varepsilon}_{\Gamma}(\hat{m})):\hat{m}\in\hat{k}+\hat{l}\}.\end{aligned}$$

Similarly,

$$\bigvee\{(F^{\varsigma}_{\Upsilon}\bigvee F^{\varsigma}_{\Gamma})(\hat{k}),(F^{\varsigma}_{\Upsilon}\bigvee F^{\varsigma}_{\Gamma})\}(\hat{l})\}\ \ge\ \sup\{(F^{\varsigma}_{\Upsilon}(\hat{m})\bigvee F^{\varsigma}_{\Gamma}(\hat{m})):\hat{m}\in\hat{k}+\hat{l}\}.$$

(2) $\exists\ \forall\ \hat{n},\hat{k}\in R$ such that $\hat{k}\in\hat{n}+\hat{l}$ then it argues that:

$$\begin{aligned}
\bigwedge\{(\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{n}),(\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{k})\} &= \bigwedge\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{n}),\tau^{\varpi}_{\Gamma}(\hat{n})),\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{k}),\tau^{\varpi}_{\Gamma}(\hat{k}))\}\\
&\leq \bigwedge\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{n}),\tau^{\varpi}_{\Upsilon}(\hat{k})),\bigwedge(\tau^{\varpi}_{\Gamma}(\hat{n}),\tau^{\varpi}_{\Gamma}(\hat{k}))\}\\
&\leq \{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{l}),\tau^{\varpi}_{\Gamma}(\hat{l}))\}\\
&= (\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{l}).\\
\Rightarrow \bigwedge\{(\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{n}),(\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{k})\} &\leq \{(\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{l}):\hat{k}\in\hat{n}+\hat{l}\}.
\end{aligned}$$

Also, $\exists\ \forall\ \hat{n},\hat{k}\in R$ such that $\hat{k}\in\hat{n}+\hat{l}$ then it argues that:

$$\begin{aligned}
\bigvee\{(\iota^{\varepsilon}_{\Upsilon}\ \bigwedge\ \iota^{\varepsilon}_{\Gamma})(\hat{n}),(\iota^{\varepsilon}_{\Upsilon}\ \bigwedge\ \iota^{\varepsilon}_{\Gamma})(\hat{k})\} &= \bigvee\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{n}),\iota^{\varepsilon}_{\Gamma}(\hat{n})),\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{k}),\iota^{\varepsilon}_{\Gamma}(\hat{k}))\}\\
&\geq \bigvee\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{n}),\iota^{\varepsilon}_{\Upsilon}(\hat{k})),\bigwedge(\iota^{\varepsilon}_{\Gamma}(\hat{n}),\iota^{\varepsilon}_{\Gamma}(\hat{k}))\}\\
&\geq \{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{l}),\iota^{\varepsilon}_{\Gamma}(\hat{l}))\}\\
&= \{(\iota^{\varepsilon}_{\Upsilon}(\hat{l})\bigwedge\iota^{\varepsilon}_{\Gamma}(\hat{l}))\}.\\
\Rightarrow \bigvee\{(\iota^{\varepsilon}_{\Upsilon}\ \bigwedge\ \iota^{\varepsilon}_{\Gamma})(\hat{n}),(\iota^{\varepsilon}_{\Upsilon}\ \bigwedge\ \iota^{\varepsilon}_{\Gamma})(\hat{k})\} &\geq \{(\iota^{\varepsilon}_{\Upsilon}(\hat{l})\bigwedge\iota^{\varepsilon}_{\Gamma}(\hat{l})):\hat{k}\in\hat{n}+\hat{l}\}
\end{aligned}$$

Similarly,

$$\bigvee\{(F^{\varsigma}_{\Upsilon}\ \bigvee\ F^{\varsigma}_{\Gamma})(\hat{n}),(F^{\varsigma}_{\Upsilon}\ \bigvee\ F^{\varsigma}_{\Gamma})(\hat{k})\} \geq \{(F^{\varsigma}_{\Upsilon}(\hat{l})\bigvee F^{\varsigma}_{\Gamma}(\hat{l})):\hat{k}\in\hat{n}+\hat{l}\}$$

(3) $\forall\ \hat{n},\hat{k}\in R\ \exists\ \hat{m}\in R$ where $\hat{k}\in\hat{m}+\hat{n}$ can be readily proved that

$$\begin{aligned}
\bigwedge\{(\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{n}),(\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{k})\} &= \bigwedge\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{n}),\tau^{\varpi}_{\Gamma}(\hat{n})),\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{k}),\tau^{\varpi}_{\Gamma}(\hat{k}))\}\\
&\leq \bigwedge\{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{n}),\tau^{\varpi}_{\Upsilon}(\hat{k})),\bigwedge(\tau^{\varpi}_{\Gamma}(\hat{n}),\tau^{\varpi}_{\Gamma}(\hat{k}))\}\\
&\leq \{\bigwedge(\tau^{\varpi}_{\Upsilon}(\hat{m}),\tau^{\varpi}_{\Gamma}(\hat{m}))\}\\
&= (\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{m}).\\
\Rightarrow \bigwedge\{(\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{n}),(\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{k})\} &\leq \{(\tau^{\varpi}_{\Upsilon}\ \bigwedge\ \tau^{\varpi}_{\Gamma})(\hat{m}):\hat{k}\in\hat{m}+\hat{n}\}.
\end{aligned}$$

Also, $\forall\ \hat{n},\hat{k}\in R\ \exists\ \hat{m}\in R$ where $\hat{k}\in\hat{m}+\hat{n}$ then it argues that:

$$\begin{aligned}
\bigvee\{(\iota^{\varepsilon}_{\Upsilon}\ \bigwedge\ \iota^{\varepsilon}_{\Gamma})(\hat{n}),(\iota^{\varepsilon}_{\Upsilon}\ \bigwedge\ \iota^{\varepsilon}_{\Gamma})(\hat{k})\} &= \bigvee\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{n}),\iota^{\varepsilon}_{\Gamma}(\hat{n})),\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{k}),\iota^{\varepsilon}_{\Gamma}(\hat{k}))\}\\
&\geq \bigvee\{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{n}),\iota^{\varepsilon}_{\Upsilon}(\hat{k})),\bigwedge(\iota^{\varepsilon}_{\Gamma}(\hat{n}),\iota^{\varepsilon}_{\Gamma}(\hat{k}))\}\\
&\geq \{\bigwedge(\iota^{\varepsilon}_{\Upsilon}(\hat{m}),\iota^{\varepsilon}_{\Gamma}(\hat{m}))\}\\
&= \{(\iota^{\varepsilon}_{\Upsilon}(\hat{m})\bigwedge\iota^{\varepsilon}_{\Gamma}(\hat{m}))\}.\\
\Rightarrow \bigvee\{(\iota^{\varepsilon}_{\Upsilon}\ \bigwedge\ \iota^{\varepsilon}_{\Gamma})(\hat{n}),(\iota^{\varepsilon}_{\Upsilon}\ \bigwedge\ \iota^{\varepsilon}_{\Gamma})(\hat{k})\} &\geq \{(\iota^{\varepsilon}_{\Upsilon}(\hat{m})\bigwedge\iota^{\varepsilon}_{\Gamma}(\hat{m})):\hat{k}\in\hat{m}+\hat{n}\}
\end{aligned}$$

Similarly,

$$\bigvee\{(F^{\varsigma}_{\Upsilon}\ \bigvee\ F^{\varsigma}_{\Gamma})(\hat{n}),(F^{\varsigma}_{\Upsilon}\ \bigvee\ F^{\varsigma}_{\Gamma})(\hat{k})\} \geq \{(F^{\varsigma}_{\Upsilon}(\hat{m})\bigvee F^{\varsigma}_{\Gamma}(\hat{m})):\hat{k}\in\hat{m}+\hat{n}\}$$

(4) $\forall\ \hat{k},\ \hat{l} \in R$,

$$\begin{aligned} \bigvee\{\tau^{\varpi}_{\Upsilon}(\hat{k}), \tau^{\varpi}_{\Upsilon}(\hat{l})\} &\leq \inf\{\tau^{\varpi}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}, \\ \bigvee\{\iota^{\varepsilon}_{\Upsilon}(\hat{k}), \iota^{\varepsilon}_{\Upsilon}(\hat{l})\} &\geq \sup\{\iota^{\varepsilon}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\},\ and \\ \bigvee\{F^{\varsigma}_{\Upsilon}(\hat{k}), F^{\varsigma}_{\Upsilon}(\hat{l})\} &\geq \sup\{F^{\varsigma}_{\Upsilon}(\hat{m}) : \hat{m} \in \hat{k} \circ \hat{l}\}. \end{aligned}$$

Hence, it is verified that $\Upsilon \cap \Gamma$ is a $(\varpi, \varepsilon, \varsigma)$-$SVNHI$ over R. □

5. Conclusions

This research has introduced the novel concepts of the $(\varpi, \varepsilon, \varsigma)$-single valued neutrosophic theory of hyperrings and hyperideals through the introduction of a few hyperalgebraic structures and the analysis of some basic properties, outcomes, and structural characteristics of these concepts. We plan to meld more hyperalgebraic theory with real-world applications in the future for plithogenic sets for $(\varpi, \varepsilon, \varsigma)$-single-valued neutrosophic sets and $(\varpi, \varepsilon, \varsigma)$-interval-valued neutrosophic sets.

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Introduction to the IndetermSoft Set and IndetermHyperSoft Set

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Abstract: In this paper one introduces for the first time the *IndetermSoft Set*, as extension of the classical (determinate) Soft Set, that deals with indeterminate data, and similarly the HyperSoft Set extended to *IndetermHyperSoft Set*, where 'Indeterm' stands for 'Indeterminate' (uncertain, conflicting, not unique outcome). They are built on an *IndetermSoft Algebra* that is an algebra dealing with *IndetermSoft Operators* resulted from our real world. Afterwards, the corresponding Fuzzy / Intuitionistic Fuzzy / Neutrosophic / and other fuzzy-extension IndetermSoft Set & IndetermHyperSoft Set are presented together with their applications.

Keywords: Soft Set; HyperSoft Set; IndetermSoft Set; IndetermHyperSoft Set; IndetermSoft Operators; IndetermSoft Algebra.

1. Introduction

The classical Soft Set is based on a determinate function (whose values are certain, and unique), but in our world there are many sources that, because of lack of information or ignorance, provide indeterminate (uncertain, and not unique – but hesitant or alternative) information.

They can be modelled by operators having some degree of indeterminacy due to the imprecision of our world.

The paper recalls the definitions of the classical Soft Set and HyperSoft Set, then shows the distinction between determinate and indeterminate soft functions.

The neutrosophic triplets <*Function, NeutroFunction, AntiFunction*> and <*Operator, NeutroOperator, AntiOperator*> are brought into discussion, as parts of the <*Algebra, NeutroAlgebra, AntiAlgebra*> (Smarandache, 2019).

Similarly, distinctions between determinate and indeterminate operators are taken into consideration.

Afterwards, an IndetermSoft Algebra is built, using a determinate soft operator (*joinAND*), and three indeterminate soft operators *(disjoinOR, exclussiveOR, NOT)*, whose properties are further on studied.

IndetermSoft Algebra and IndetermHyperSoft Algebra are subclasses of the IndetermAlgebra.

The IndetermAlgebra is introduced as an algebra whose space or operators have some degree of indeterminacy ($I > 0$), and it is a subclass of the NeutroAlgebra.

It was proved that the IndetermSoft Algebra and IndetermHyperSoft Algebra are non-Boolean Algebras, since many Boolean Laws fail.

2. Definition of Classical Soft Set

Let U be a universe of discourse, H a non-empty subset of U, with $P(H)$ the powerset of H, and a an attribute, with its set of attribute values denoted by A. Then the pair (F, H), where $F: A \to P(H)$, is called a Classical Soft Set over H.

Molodtsov [1] has defined in 1999 the Soft Set, and Maji [2] the Neutrosophic Soft Set in 2013.

3. Definition of the Determinate (Classical) Soft Function

The above function $F: A \to P(H)$, where for each $x \in A, f(x) \in P(H)$, and f(x) is certain and unique, is called a Determinate (Classical) Function.

4. Definition of the IndetermSoft Function

One introduces it for the first time. Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the powerset of H. Let a be an attribute, and A be a set of this attribute values.

A function $F: A \to P(H)$ is called an *IndetermSoft Function* if:

i. the set A has some indeterminacy;
ii. or $P(H)$ has some indeterminacy;
iii. or there exist at least an attribute value $v \in A$, such that $F(v)$ = indeterminate (unclear, uncertain, or not unique);
iv. or any two or all three of the above situations.

The IndetermSoft Function has some degree of indeterminacy, and as such it is a particular case of the NeutroFunction [6, 7], defined in 2014 – 2015, that one recalls below.

5. <Function, NeutroFunction, AntiFunction>

We have formed the above neutrosophic triplet [10, 11].

i. **(Classical) Function**, which is a function well-defined (inner-defined) for all elements in its domain of definition, or (T, I, F) = (1,0,0).
ii. **NeutroFunction** (or Neutrosophic Function), which is a function partially well-defined (degree of truth T), partially indeterminate (degree of indeterminacy I), and partially outer-defined (degree of falsehood F) on its domain of definition, where $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$.
iii. **AntiFunction**, which is a function outer-defined for all the elements in its domain of definition, or (T, I, F) = (0, 0, 1).

6. Applications of the Soft Set

A detective must find the criminal(s) out of a crowd of suspects. He uses the testimonies of several witnesses.

Let the crowd of suspects be the set $S = \{s_1, s_2, s_3, s_4, s_5\} \cup \{\phi\}$, where $\{\phi\}$ is the empty (null) element, and the attribute *c* = *criminal*,

which has two attribute-values $C = \{yes, no\}$.

i. Let the function $F_1: C \to P(S)$, where $P(S)$ is the powerset of S, represent the information provided by the witness W_1.

For example,

$F_1(yes) = s_3$, which means that, according to the witness W_1, the suspect s_3 is the criminal,

and $F_1(no) = s_4$, which similarly means, according to the witness W_1, that the suspect s_4 is not the criminal.

These are determined (exact) information, provided by witness W_1, therefore this is a classical Soft Set.

ii. Further on, let the function $F_2 : C \to P(S)$, where $P(S)$ is the powerset of *S*, represent the information provided by the witness W_2.

For example,

$F_2(yes) = \{\phi\}$, the null-element, which means that according to the witness W_2, none of the suspects in the set *S* is the criminal. This is also a determinate information as in classical Soft Set.

7. Indeterminate Operator as Extension of the Soft Set

iii. Again, let the function $F_3 : C \to P(S)$, where $P(S)$ is the powerset of *S*, represent the information provided by the witness W_3.

This witness is not able to provide a certain and unique information, but some indeterminate (uncertain, not unique but alternative) information.

For example:

$F_3(yes) = NOT(s_2)$

and $F_3(no) = s_3$ *OR* s_4

The third source (W_3) provides indeterminate (unclear, not unique) information, since $NOT(s_2)$ means that s_2 is not the criminal, then consequently: either one, or two, or more suspects from the remaining set of suspects $\{s_1, s_3, s_4, s_5\}$ may be the criminal(s), or $\{\phi\}$ (none of the remaining suspects is the criminal), whence one has:

$C_4^1 + C_4^2 + C_4^3 + C_4^4 + 1 = 2^4 = 16$ possibilities (alternatives, or outcomes), resulted from a single input, to chose from, where C_n^m means combinations of *n* elements taken into groups of *m* elements, for integers $0 \le m \le n$.

Indeterminate information again, since:

s_3 *OR* s_4 means: either {s_3 *yes*, and s_4 *no*}, or {s_3 *no*, and s_4 *yes*}, or {s_3 *yes*, and s_4 *yes*}, therefore 3 possible (alternatives) outcomes to chose from.

Thus, $F_3 : C \to P(S)$ is an Indeterminate Soft Function (or renamed/contracted as IndetermSoft Function).

8. Indeterminate Attribute-Value Extension of the Soft Set

Let's extend the previous Applications of the Soft Set with the crowd of suspects being the set $S = \{s_1, s_2, s_3, s_4, s_5\} \cup \{\phi\}$, where $\{\phi\}$ is the empty (null) element, and the attribute *c* = *criminal*, but the attribute *c* has this time three attribute-values *K* = *{yes, no, maybe}*, as in the new branch of philosophy, called neutrosophy, where between the opposites <*A*> = *yes*, and <*antiA*> = *no*, there is the indeterminacy (or neutral) <*neutA*> = *maybe*.

And this is provided by witness W_4 and defined as:

$F_4 : K \to P(S)$

For example: $F_4(maybe) = s_5$, which means that the criminal is maybe s_5.

There also is some indeterminacy herein as well because the attribute-value "maybe" means unsure, uncertain.

One can transform this one into a Fuzzy (or Intuitionistic Fuzzy, or Neutrosophic, or other Fuzzy-Extension) Soft Sets in the following ways:

$F_4(maybe) = s_5$ is approximately equivalent to $F_4(yes) = s_5$*(some appurtenance degree)*

or

$F_4(maybe) = s_5$ is approximately equivalent to $F_4(no) = s_5$*(some non-appurtenance degree)*

Let's consider the bellow example.

Fuzzy Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(yes) = s_5(0.6)*, or the chance that s_5 be a criminal is *60%;*

Intuitionistic Fuzzy Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(yes) = s_5(0.6, 0.3)*, or the chance that *s_5* be a criminal is *60%*, and chance that *s_5* not be a criminal is *30%*;

Neutrosophic Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(yes) = s_5(0.6, 0.2, 0.3)*, or the chance that s_5 is a criminal is *60%*, indeterminate-chance of criminal-noncriminal is *20%*, and chance that *s_5* not be a criminal is *30%*.

And similarly for other *Fuzzy-Extension Soft Set*.

Or, equivalently, employing the attribute-value "no", one may consider:

Fuzzy Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(no) = s_5(0.4)*, or the chance that s_5 is not a criminal is *40%;*

Intuitionistic Fuzzy Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(no) = s_5(0.3, 0.6)*, or the chance that s_5 is not a criminal is *30%*, and chance that *s_5* is a criminal is *60%*;

Neutrosophic Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(no) = s_5(0.3, 0.2, 0.6)*, or the chance that s_5 is not a criminal is *30%*, indeterminate-chance of criminal-noncriminal is *20%*, and chance that *s_5* is a criminal is *60%*.

And similarly for other *Fuzzy-Extension Soft Set*.

9. HyperSoft Set

Smarandache has extended in 2018 the Soft Set to the HyperSoft Set [3, 4] by transforming the function *F* from a uni-attribute function into a multi-attribute function.

9.1. Definition of HyperSoft Set

Let $\mathcal{U}$ be a universe of discourse, H a non-empty set included in U, and $P(H)$ the powerset of H. Let $a_1, a_2, \dots a_n$, where $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, and $i, j \in \{1, 2, \dots, n\}$. Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $A_1 \times A_2 \times \dots \times A_n$ represents a Cartesian product, with

$F: A_1 \times A_2 \times \dots \times A_n \rightarrow P(H)$

is called a HyperSoft Set.

For example,

let

$$(e_1, e_2, ..., e_n) \in A_1 \times A_2 \times ... \times A_n$$

then

$$F(e_1, e_2, ..., e_n) = G \in P(H).$$

9.2. Classification of HyperSoft Sets

With respect to the types of sets, such as: classical, fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and all other fuzzy-extension sets, one respectively gets: the Crisp HyperSoft Set, Fuzzy HyperSoft Set, Intuitionistic Fuzzy HyperSoft Set, Neutrosophic HyperSoft Set, Plithogenic HyperSoft Set, and all other fuzzy-extension HyperSoft Sets [3, 5-9].

The HyperSoft degrees of T = truth, I = indeterminacy, F = falsehood, H = hesitancy, N = neutral etc. assigned to these Crisp HyperSoft Set, Fuzzy HyperSoft Set, Intuitionistic Fuzzy HyperSoft Set, Neutrosophic HyperSoft Set, Plithogenic HyperSoft Set, and all other fuzzy-extension HyperSoft Sets verify the same conditions of inclusion and inequalities as in their corresponding fuzzy and fuzzy-extension sets.

9.3. Applications of HyperSoft Set and its corresponding Fuzzy / Intuitionistic Fuzzy / Neutrosophic HyperSoft Set

Let $H = \{h_1, h_2, h_3, h_4\}$ be a set of four houses, and two attributes:
s = *size*, whose attribute values are S = *{small, medium, big}*,
and l = *location*, whose attribute values are L = *{central, peripherical}.*
Then $F: S \times L \rightarrow P(H)$ is a HyperSoft Set.

i. For example, *F(small, peripherical)* = $\{h_2, h_3\}$, which means that the houses that are *small and peripherical* are h_2 and h_3.
ii. A Fuzzy HyperSoft Set may assign some <u>fuzzy degrees</u>, for example:

F(small, peripherical) = $\{h_2(0.7), h_3(0.2)\}$, which means that with respect to the attributes' values *small and peripherical* <u>all together</u>, h_2 meets the requirements of being both small and peripherical in a fuzzy degree of *70%*, while h_3 in a fuzzy degree of *20%*.

iii. Further on, a Intuitionistic Fuzzy HyperSoft Set may assign some <u>intuitionistic fuzzy degrees</u>, for example:

F(small, peripherical) = $\{h_2(0.7, 0.1), h_3(0.2, 0.6)\}$, which means that with respect to the attributes' values *small and peripherical* <u>all together</u>, h_2 meets the requirements of being both small and peripherical in a intuitionistic fuzzy degree of *70%*, and does not meet it in a intuitionistic fuzzy degree of *10%*; and similarly for h_3.

iv. Further on, a Neutrosophic HyperSoft Set may assign some <u>neutrosophic degrees</u>, for example:

F(small, peripherical) = $\{h_2(0.7, 0.5, 0.1), h_3(0.2, 0.3, 0.6)\}$, which means that with respect to the attributes' values *small and peripherical* <u>all together</u>, h_2 meets the requirements of being both small and peripherical in a neutrosophic degree of *70%*, the indeterminate-requirement in a neutrosophic degree of *50%*, and does not meet the requirement in a neutrosophic degree of *10%*. And similarly, for h_3.

v. In the same fashion for other fuzzy-extension HyperSoft Sets.

10. Operator, NeutroOperator, AntiOperator

Let U be a universe of discourse and H a non-empty subset of U.
Let $n \geq 1$ be an integer, and ω be an operator defined as:
$\omega: H^n \rightarrow H$
Let's take a random *n-tuple* $(x_1, x_2, ..., x_n) \in H^n$.
There are three possible cases:

i. $\omega(x_1, x_2, ..., x_n) \in H$ and $\omega(x_1, x_2, ..., x_n)$ is a determinate (clear, certain, unique) output; this is called degree of well-defined (inner-defined), or degree of Truth (*T*).
ii. $\omega(x_1, x_2, ..., x_n)$ is an indeterminate (unclear, uncertain, undefined, not unique) output; this is called degree of Indeterminacy (*I*).
iii. $\omega(x_1, x_2, ..., x_n) \in U - H$; this is called degree of outer-defined (since the output is outside of *H*), or degree of Falsehood (*F*).

Consequently, one has a Neutrosophic Triplet of the form

<Operator, NeutroOperator, AntiOperator>

defined as follows [12, 13, 14]:

10.1. (Classical) Operator

For any *n-tuple* $(x_1, x_2, ..., x_n) \in H^n$, one has $\omega(x_1, x_2, ..., x_n) \in H$ and $\omega(x_1, x_2, ..., x_n)$ is a determinate (clear, certain, unique) output. Therefore *(T, I, F) = (1, 0, 0).*

10.2. NeutroOperator

There are some *n-tuples* $(x_1, x_2, ..., x_n) \in H^n$ such that $\omega(x_1, x_2, ..., x_n) \in H$ and $\omega(x_1, x_2, ..., x_n)$ are determinate (clear, certain, unique) outputs (degree of truth T);

other *n-tuples* $(y_1, y_2, ..., y_n) \in H^n$ such that $\omega(y_1, y_2, ..., y_n) \in H$ and $\omega(y_1, y_2, ..., y_n)$ are indeterminate (unclear, uncertain, not unique) output (degree of indeterminacy I);

and other *n-tuples* $(z_1, z_2, ..., z_n) \in H^n$ such that $\omega(z_1, z_2, ..., z_n) \in U - H$ (degree of falsehood F);

where $(T, I, F) \neq \{(1,0,0), (0,0,1)\}$ that represent the first (Classical Operator), and respectively the third case (AntiOperator).

10.3. AntiOperator

For any *n-tuple* $(x_1, x_2, ..., x_n) \in H^n$, one has $\omega(x_1, x_2, ..., x_n) \in U - H$. Therefore *(T, I, F)* = (0, 0, 1).

11. Particular Cases of Operators

11.1. Determinate Operator

A *Determinate Operator* is an operator whose degree of indeterminacy *I = 0*, while the degree of truth *T = 1* and degree of falsehood *F = 0*.

Therefore, only the Classical Operator is a Determinate Operator.

11.2. IndetermOperator

As a subclass of the above NeutroOperator, there is the *IndetermOperator* (*Indeterminate Operator*), which is an operator that has some degree of indeterminacy (*I > 0).*

12. Applications of the IndetermOperators to the Soft Sets

Let *H* be a set of finite number of houses (or, in general, objects, items, etc.):

$H = \{h_1, h_2, ..., h_n\} \cup \{\emptyset\}, 1 \leq n < \infty,$

where h_1 = house1, h_2 = house2, etc.

and ∅ is the empty (or null) element (no house).

13. Determinate and Indeterminate Soft Operators

Let us define four soft operators on H.

13.1. joinAND

joinAND, or put together, denoted by ⩕, defined as:

x ⩕ y = x and y, or put together x and y; herein the conjunction "and" has the common sense from the natural language.

$x \Cap y = \{x, y\}$ is a set of two objects.

For example:

$h_1 \Cap h_2$ = *house1* $\Cap$ *house2 = house1 and house2*

= put together *house1* and *house2 = {house1, house2} =* $\{h_1, h_2\}$.

joinAND is a Determinate Soft Operator since one gets one clear (certain) output.

13.2. disjoinOR

disjoinOR, or separate in parts, denoted by $\Cup$, defined as:

x disjoinOR $y = x \Cup y = \{x\}$, or $\{y\}$, or both $\{x, y\}$

= x, or y, or both x and y;

herein, similarly, the disjunction "or" (and the conjunction "and" as well) have the common sense from the natural language.

But there is some indeterminacy (uncertainty) to choose among three alternatives.

For example:

$h_1 \Cup h_2$ = *house1* $\Cup$ *house2 = house1*, or *house2*, or both houses together *{house1 and house2}.*

disjoinOR is an IndetermSoft Operator, since it does not have a clear unique output, but three possible alternative outputs to choose from.

13.3. exclusiveOR

exclusiveOR, meaning either one, or the other**;** it is an IndetermSoft Operator (to choose among two alternatives).

$h_1 \Cup_E h_2$ = either h_1, or h_2, and no both $\{h_1, h_2\}$.

13.4. NOT

NOT, or no, or sub-negation/sub-complement, denoted by $\neg$, where

$\mathrm{NOT}(h) = \neg h = no\ h$, in other words all elements from H, except h, either single elements, or two elements, …, or $n-1$ elements from $H - \{h\}$, or the empty element $\emptyset$.

The "not" negation has the common sense from the natural language; when we say "not John" that means "someone else" or "many others".

13.4.1. Theorem 1

Let the cardinal of the set H-$\{h\}$ be $|H - \{h\}| = m \geq 1$.

Then $NOT(h) = \{x, x \in P(H - \{h\})\}$ and the cardinal $|\mathrm{NOT(h)}| = 2^{n-1}$.

Proof:

Because $NOT(h)$ means all elements from H, except h,
either by single elements, or by two elements, …, or by $n-1$ elements from $H - \{h\}$, or the empty element $\emptyset$, then one obtains:

$C_{n-1}^1 + C_{n-1}^2 + \cdots + C_{n-1}^{n-1} + 1 = (2^{n-1} - 1) + 1 = 2^{n-1}$ possibilities (alternatives to h).

The *NOT* operator has as output a multitude of sub-negations (or sub-complements).

NOT is also an IndetermSoft Operator.

13.4.2. Example

Let $H = \{x_1, x_2, x_3, x_4\}$

Then,

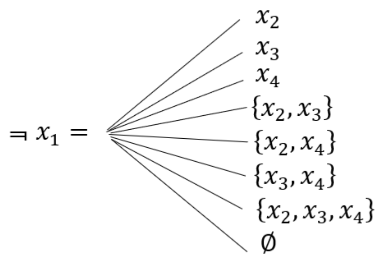
$NOT(x_1) = \neg x_1$ = either x_2, or x_3, or x_4,

or $\{x_2, x_3\}$, or $\{x_2, x_4\}$, or $\{x_3, x_4\}$,

or $\{x_2, x_3, x_4\}$,
or $\emptyset$;

therefore $C_3^1 + C_3^2 + C_3^3 + 1 = 3 + 3 + 1 + 1 = 8 = 2^3$ possibilities/alternatives.

Graphic representations:



Or another representation (equivalent to the above) is below:

$$\neg x_1 = \begin{cases} x_2 \\ x_3 \\ x_4 \\ \{x_2, x_3\} \\ \{x_2, x_4\} \\ \{x_3, x_4\} \\ \{x_2, x_3, x_4\} \\ \phi \end{cases}$$

The *NOT* operator is equivalent to $(2^{n-1} - 1)$ *OR* disjunctions (from the natural language).

14. Similarities between IndetermSoft Operators and Classical Operators

(i) joinAND is similar to the classical logic AND operator ($\wedge$) in the following way.
Let *A, B, C* be propositions, where $C = A \wedge B$.
Then the proposition *C* is true, if both: *A* = true, and *B* = true.
(ii) disjoinOR is also similar to the classical logic OR operator ($\vee$) in the following way.
Let *A, B, D* be propositions, where $D = A \vee B$.
Then the proposition *D* is true if:
either *A* = true,
or *B* = true,
or both *A* = true and *B* = true
(therefore, one has three possibilities).

(iii) exclusiveOR is also similar to the classical logic exclusive OR operator ($\vee_E$) in the following way.
Let *A, B, D* be propositions, where D = A $\vee_E$ B
Then the proposition *D* is true if:
either *A* = true,
or *B* = true,
and not both *A* and *B* are true simultaneously
(therefore, one has two possibilities).

(iv) NOT resembles the classical set, or complement operator ($\neg$), in the following way.

Let *A, B, C, D* be four sets, whose intersections two by two are empty, from the universe of discourse $\mathcal{U} = A \cup B \cup C \cup D$.

Then $\neg A = NotA = \mathcal{U} \setminus A$ = the complement of A with respect to $\mathcal{U}$.

While $\neg A$ has only one exact output ($\mathcal{U} \setminus A$) in the classical set theory, the NOT operator ⫬ A has 8 possible outcomes: either the empty set ($\emptyset$), or *B*, or *C*, or *D*, or $\{B, C\}$, or $\{B, D\}$, or $\{C, D\}$, or or $\{B, C, D\}$.

15. Properties of Operators

Let $x, y, z \in$ H(⩓,⩔, ⩔E, ⫬).

15.1. Well-Defined Operators

Let consider the set H closed under these four operators: H(⩓,⩔, ⩔E, ⫬).

Therefore, for any $x, y \in H$ one has:

x ⩓ y ∈ H(⩓,⩔, ⩔E, ⫬), because $\{x, y\} \in H$(⩓,⩔, ⩔E, ⫬),

and x ⩔ y ∈ H(⩓,⩔, ⩔E, ⫬), because each of $\{x\}$, $\{y\}$, $\{x, y\} \in H$(⩓,⩔, ⩔E, ⫬),

also x ⩔E y ∈ H(⩓,⩔, ⩔E, ⫬), because each of $\{x\}$, $\{y\} \in H$(⩓,⩔, ⩔E, ⫬),

Then the NOT operator is also well-defined because it is equivalent to a multiple of disjoinOR operators.

Thus:

⩓ : $H^2 \to H$(⩓,⩔, ⩔E, ⫬)

⩔ : $H^2 \to H$(⩓,⩔, ⩔E, ⫬)

⩔E: $H^2 \to H$(⩓,⩔, ⩔E, ⫬)

⫬ : $H \to H$(⩓,⩔, ⩔E, ⫬)

15.2. Commutativity

x ⩓ y = y ⩓ x, and x ⩔ y = y ⩔ x, and x ⩔E y = y ⩔E x

Proof

x ⩓ y = $\{x, y\} = \{y, x\}$ = y ⩓ x

x ⩔ y = ($\{x\}$, or $\{y\}$, or $\{x, y\}$) = ($\{y\}$ or $\{x\}$, or $\{y, x\}$) = y ⩔ x

x ⩔E y = (either $\{x\}$, or $\{y\}$, but not both x and y) =

= (either $\{y\}$, or $\{x\}$, but not both y and x) = y ⩔E x.

15.3. Associativity

x ⩓ (y ⩓ z) = (x ⩓ y) ⩓ z,

and x ⩔ (y ⩔ z) = (x ⩔ y) ⩔ z, and x ⩔E (y ⩔E z = (x ⩔E y) ⩔E z

Proof

x ⩓ (y ⩓ z) = $\{x, y$ ⩓ $z\} = \{x, \{y, z\}\}$

$= \{x, y, z\} = \{\{x, y\}, z\}$

= (x ⩓ y) ⩓ z.

x ⩔ (y ⩔ z) = (x ⩔ y) ⩔ z

$$x \text{ or } (y \text{ or } z) = x \text{ or } \begin{cases} y \\ z \\ y \text{ or } z \end{cases} = x \text{ or } \begin{cases} y \\ z \\ y \text{ or } z \begin{cases} y \\ z \\ \{y, z\} \end{cases} \end{cases}$$

$$= \begin{cases} x \text{ or } y = \begin{cases} x \\ y \\ \{x,y\} \end{cases} \\ x \text{ or } z = \begin{cases} x \\ z \\ \{x,z\} \end{cases} \\ x \text{ or } y \\ x \text{ or } z \\ x \text{ or } \{y,z\} = \begin{cases} x \\ \{y,z\} \\ \{x,y,z\} \end{cases} \end{cases}$$

$$= x, y, z, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}.$$

$$(x \text{ or } y) \text{ or } z = \begin{cases} x \\ y \\ \{x,y\} \end{cases} \text{ or } z = \begin{cases} x \text{ or } z \begin{cases} x \\ z \\ \{x,z\} \end{cases} \\ y \text{ or } z \begin{cases} y \\ z \\ \{y,z\} \end{cases} \\ \{x,z\} \text{ or } z \begin{cases} \{x,y\} \\ z \\ \{x,y,z\} \end{cases} \end{cases}$$

$$= x, y, z, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}.$$

Therefore, *(x or y) or z = x or (y or z) = x, y, z, {x, y}, {y, z}, {z, x}, {x, y, z}* with $2^3 - 1 = 8 - 1 = 7$ possibilities.

$x \veebar_E (y \veebar_E z) =$

either x, or (y ⩒E z), and no both x and (y ⩒E z) = either x, or (y, or z, and no both y and z), and no both x and (either y or z) = either x, or y, or z, and no both {y, z}, and (no x and no (either y or z)) = either x, or y, or z, and no {y, z}, no {x, y}, no {x, z} = (x ⩒E y) ⩒E z

15.4. Distributivity of joinAND over disjoinOR and exclusiveOR

$x \wedge\!\!\wedge (y \vee\!\!\vee z) = (x \wedge\!\!\wedge y) \vee\!\!\vee (x \wedge\!\!\wedge z)$

P roof

$x \wedge\!\!\wedge (y \vee\!\!\vee z) = x$ and $(y$ or $z) = x$ and $(y,$ or $z,$ or $\{y,z\})$

$= x$ and y, or x and z, or x and $\{y,z\}$

$= \{x,y\}$, or $\{x,z\}$, or $\{x,y,z\}$

$= \{z,y\}$, $\{x,z\}$, $\{x,y,z\}$.

$(x \wedge\!\!\wedge y) \vee\!\!\vee (x \wedge\!\!\wedge z) = \{x,y\}$

or $\{x,z\} = \{x,y\}, \{x,z\}, \{x,y,x,z\} = \{x,y\}, \{x,z\}, \{x,y,z\}$.

x ⩓ (y ⩒E z) = x and (either y, or z, and no both {y, z}) = either x and y, or x and z, and x and no both {y, z} = either {x, y}, or {x, z}, and {x, no { y, z} } = = either {x, y}, or {x, z}, and no {x, y, z} = (x ⩓ y) ⩒E (x ⩓ z)

15.5. No distributivity of disjoinOR and exclusiveOR over joinAND

$x \vee\!\!\vee (y \wedge\!\!\wedge z) \neq (x \vee\!\!\vee y) \wedge\!\!\wedge (x \vee\!\!\vee z)$

$x \vee\!\!\vee (y \wedge\!\!\wedge z) = x$ or $(y$ and $z) = x$ *or* $\{y,z\} = x, \{y,z\}, \{x,y,z\}$

But

$(x \vee\!\!\vee y) \wedge\!\!\wedge (x \vee\!\!\vee z) = (x, y, \{x,y\})$ and $(x, z, \{x,z\})$

$= \{x,x\}, \{x,z\}, \{x,z\}, \{y,z\}, \{y,x\}, \{x,y,z\}, \{x,y\}, \{x,y,z\}, \{x,y,z\}$

$= x, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}$.

Whence in general $x, \{y,z\}, \{x,y,z\} \neq x, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}$.

While in classical Boolean Algebra the distribution of or over and is valid:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

x $\veebar_E$ *(y* $\wedge$ *z) = either x, or {y, z}, and no {x, y, z}* $\neq$
$\neq$ *(x* $\veebar_E$ *y)* $\wedge$ *(x* $\veebar_E$ *y) = (either x, or y, and no {x, y}) and (either x, or z, and no {x, z})*

15.6. Idempotence

$x \wedge x = \{x, x\} = x$
$x \vee x =$ either x, or x, or $\{x, x\}$
= x, or x, or x
= x.
$x \veebar_E x$ = either x, or x, and no $\{x, x\}$ = impossible.

15.6.1. Theorem 2

Let $x_1, x_2, \dots, x_n \in (H, \wedge, \vee, =)$, for $n \geq 2$. Then:
(i) $x_1 \wedge x_2 \wedge \dots \wedge x_n = \{x_1, x_2, \dots, x_n\}$,
and
(ii) $x_1 \vee x_2 \vee \dots \vee x_n = x_1, x_2, \dots, x_n$,
$\{x_1, x_2\}$, $\{x_1, x_3\}$, …, $\{x_{n-1}, x_n\}$,
$\{x_1, x_2, x_3\}$, …
… … … … … …
$\{x_1, x_2, \dots, x_{n-1}\}$, …
$\{x_1, x_2, \dots, x_{n-1}, x_n\}$.
There are: $C_n^1 + C_n^2 + \dots + C_n^{n-1} + C_n^n = 2^n - 1$ possibilities/alternatives.
The bigger is *n*, the bigger the indeterminacy.
(iii) $x_1 \veebar_E x_2 \veebar_E \dots \veebar_E x_n = x_1, x_2, \dots, x_n =$
= *either* x_1, *or* x_2, …, *or* x_n,
and no two or more variables be true simultaneously.
There are: $C_n^1 = n$ possibilities.
The bigger is *n*, the bigger the indeterminacy due to many alternatives.

Proof

(i) The joinAND equality is obvious.

(ii) The disjoinOR outputs from the fact that for the disjunction of n proposition to be true, it is enough to have at least one which is true. As such, we may have only one proposition true, or only two propositions true, and so on, only $n - 1$ propositions true, up to all n propositions true.

(iii) It is obvious.

15.7. The classical Boolean Absorption Law1

$x \wedge (x \vee y) = x$ does not work in this structure, since $x \wedge (x \vee y) \neq x$.

Proof

$$x \wedge (x \vee y) = x \text{ and } (x \text{ or } y)$$
$$= x \text{ and } \begin{cases} x \\ y \\ \{x, y\} \end{cases}$$
$$= \{x, x\} \text{ or } \{x, y\} \text{ or } \{x, x, y\}$$
$$= x \text{ or } \{x, y\} \text{ or } \{x, y\}$$
$$= x \text{ or } \{x, y\}$$

$$= \begin{cases} x \\ \{x,y\} \\ \{x,x,y\} \end{cases} = \begin{cases} x \\ \{x,y\} \\ \{x,y\} \end{cases} = \begin{cases} x \\ \{x,y\} \end{cases} \neq x.$$

But this one work:

$x \curlywedge$ (x ⩔E y) = *x and (either x, or y, and no {x, y}) =*

= (x and x), or (x and y), and (x and no{x, y}) = x.

15.8. The classical Boolean Absorption Law2

$x \vee (x \wedge y) = x$ does not work in this structure, since $x \curlyvee (x \curlywedge y) \neq x$.

Proof

$x \curlywedge (x \curlyvee y) = x$ and (x or y)

x or (x and y) = x or $\{x, y\}$

$$= \begin{cases} x \\ \{x,y\} \\ \{x,x,y\} \end{cases} = \begin{cases} x \\ \{x,y\} \\ \{x,y\} \end{cases}$$

$$= \begin{cases} x \\ \{x,y\} \end{cases} \neq x.$$

But this one work:

x ⩔E *(x* ⩕ y *) =* (either x), or {x, y}, and (no {x, y}) = x.

15.9. Annihilators and Identities for IndetermSoft Algebra

While 0 is an annihilator for conjunction $\wedge$ in the classical Boolean Algebra, $x \wedge 0 = 0$, in IndetermSoft Algebra $\emptyset$ is an identity for ⩕, while for the others it does not work.

Proof

$x \curlywedge \emptyset = x$ and $\emptyset$

= x and nothing

= x put together with nothing

= x.

15.10. $\emptyset$ is neither an identity, nor an annihilator for disjoinOR nor for exclusiveOR

While 0 is an identity for the $\vee$ in the classical Boolean Algebra, $x \vee 0 = x$ in IndetermAlgebra $\emptyset$ is neither an identity, nor an annihilator.

Proof

$x \curlyvee \emptyset = x$, or $\emptyset$ (nothing), or $\{x, \emptyset\}$

= x, or $\emptyset$, or x

= x, or $\emptyset$.

x ⩔E $\emptyset$ = either x, or $\emptyset$, and no {x, $\emptyset$}.

15.11. The negation of $\emptyset$ has multiple solutions

While in the classical Boolean Algebra the negation of 0 is 1 (one solution only), $\neg 0 = 1$, in IndetermAlgebra the negation of $\emptyset$ has multiple solutions.

Proof

⫬ $\emptyset = NOT(\emptyset)$,

= not nothing

= one or more elements from the set that the operator ⫬ is defined upon.

Example

Let $H = \{x_1, x_2, x_3\} \cup \emptyset$.

Then, ⫬ $\emptyset = x_1$, or x_2, or x_3, or $\{x_1, x_2\}$, or $\{x_1, x_3\}$, or $\{x_2, x_3\}$, or $\{x_1, x_2, x_3\}$, therefore 7 alternative solutions.

15.12. The Double Negation is invalid on IndetermSoft Algebra

While in the classical Boolean Algebra the Double Negation Law is valid: $\neg(\neg x) = x$, in IndetermAlgebra it is not true:

In general, $\eqcirc (\eqcirc x) \neq x$.

Proof

A counter-example:

Let $H = \{x_1, x_2, x_3\} \cup \emptyset$.

$\eqcirc x_1$ = what is not x_1 or does not contain x_1

$= x_2, x_3, \{x_2, x_3\}, \emptyset$.

Thus one has 4 different values of the negation of x_1.

Let us choose $\eqcirc x_1 = x_2$; then $\eqcirc (\eqcirc x_1) = x_2 = (x_1, x_3, \{x_1, x_3\}, \emptyset) \neq x_1$.

Similarly for taking other values of $\eqcirc x_1$.

Let $H = \{x_1, x_2, \ldots, x_n\} \cup \emptyset$, $n \geq 2$. Let $x \in H$.

Minimum and Maximum elements with respect to the relation of inclusion are:

$\emptyset$ = the empty (null) element

and respectively

$$x_1 \Cap x_2 \Cap \ldots \Cap x_n \overset{notation}{=} \{x_1, x_2, \ldots, x_n\} = H,$$

but in the Boolean Algebra they are 0 and 1 respectively.

15.13. The whole set H is an annihilator for joinAND

While in the classical Boolean Algebra the identity for $\wedge$ is 1, since $x \wedge 1 = x$, in the IndetermSoft Algebra for $\Cap$ there is an annihilator H, since $x \Cap H = H$, since $x \Cap H = \{x_1, x_2, \ldots, x_n, x\} = H$, because $x \in H$ so x is one of $x_1, x_2, \ldots, x_n$.

16. The maximum *(H)* is neither annihilator nor identity

While in the classical Boolean Algebra the annihilator for $\vee$ is 1, because $x \vee 1 = 1$, in the IndetermSoft Algebra for $\Cup$ the maximum H is neither annihilator nor identity,

$x \Cup H = x$ or $H = x, H, \{x, H\} = x, H, H = x, H$.

$x \Cup_E H$ = either x, or H, and (no x and no H).

17. Complementation1

In the classical Boolean Algebra, Complementation1 is: $x \wedge \neg x = 0$.

In the IndetermSoft Algebra, $x \Cap (\eqcirc x) \neq \emptyset$, and $x \Cap (\eqcirc x) \neq H$.

Counter-Example

$M = \{x_1, x_2, x_3\} \cup \emptyset$

$\eqcirc x_1 = x_2, x_3, \{x_2, x_3\}, \emptyset$

$x_1 \Cap (\eqcirc x_1) = x_1 \Cap (x_2, x_3, \{x_2, x_3\}, \emptyset) =$

$= (x_1$ and $x_2)$ or $(x_1$ or $x_3)$

or $(x_1$ and $\{x_2, x_3\})$

or $(x_1$ and $\emptyset) =$

$= (x_1, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_2, x_3\}) \neq \emptyset \neq M$.

18. Complementation2

In the classical Boolean Algebra, <u>Complementation2</u> is: $x \vee \neg x = 1$.

In the IndetermSoft Algebra, $x \Cup \dashv x \neq H$, and $x \Cup \dashv x \neq \emptyset$.

Counter-Example

The above $H = \{x_1, x_2, x_3\} \cup \emptyset$

and $\dashv x_1 = x_2, x_3, \{x_2, x_3\}, \emptyset$, then

$$x_1 \Cup \dashv x_1 = x_1 \Cup (x_2, x_3, \{x_2, x_3\}, \emptyset) = \begin{cases} x_1 \\ x_2, x_3, \{x_2, x_3\}, \emptyset \\ x_1, x_2, x_3, \{x_2, x_3\}, \emptyset \end{cases}$$

$= x_1$, or $(x_2, x_3, \{x_2, x_3\}, \emptyset)$, or $(x_1, x_2, x_3, \{x_2, x_3\}, \emptyset)$

which is different from H and from $\emptyset$.

And:

$$x_1 \Cup_E \dashv x_1 = x_1 \Cup_E (x_2, x_3, \{x_2, x_3\}, \emptyset) = \begin{Bmatrix} x_1 \\ x_2, x_3, \{x_2, x_3\}, \phi \end{Bmatrix}$$ and no $(x_1, x_2, x_3, \{x_2, x_3\}, \phi)$,

which is different from H and from $\emptyset$.

19. De Morgan Law1 in the IndetermSoft Algebra

De Morgan Law1 from Classical Boolean Algebra is:

$\neg(x \vee y) = (\neg x) \wedge (\neg y)$

is also true in the IndetermSoft Algebra:

$\dashv (x \Cup y) = (\dashv x) \Cap (\dashv y)$

Proof

$\dashv (x \Cup y) = \dashv (x, \text{or } y, \text{or } \{x \text{ and } y\})$

$= \dashv x, \text{and } \dashv y, \text{and } \dashv \{x_1 \text{ and y}\}$

$= \dashv x_1, \text{and } \dashv y, \text{and } (\dashv x, \text{or } \dashv \text{y})$

$= \dashv x, \text{and } \dashv y$

$= (\dashv x) \Cap (\dashv y)$.

Example

$M = \{x_1, x_2, x_3\} \cup \emptyset$

$\dashv (x_1 \Cup x_2) \quad = \dashv (x_1, \text{or } x_2, \text{or } \{x_1 \text{ and } x_2\})$

$= \dashv x_1, \text{and } \dashv x_2, \text{and } (\dashv x_1 \text{ or } \dashv x_2)$

$= \dashv x_1, \text{and } \dashv x_2$

$= (\dashv x_1) \Cap (\dashv x_2)$.

$\dashv x_1 = (x_2, x_3, \{x_2, x_3\}, \emptyset)$

$\dashv x_2 = (x_1, x_3, \{x_1, x_3\}, \emptyset)$

$(\dashv x_1) \Cap (\dashv x_2) = (x_2, x_3, \{x_2, x_3\}, \emptyset) \Cap (x_1, x_3, \{x_1, x_3\}, \emptyset)$

$= x_1, x_2, x_3, \{x_1, x_3\}, \{x_2, x_3\}, \emptyset$.

20. De Morgan Law2 in the IndetermSoft Algebra

De Morgan Law2 in the classical Boolean Algebra is

$\neg(x \wedge y) = (\neg x) \vee (\neg y)$

is also true in the new structure called IndetermSoft Algebra:

$\dashv (x \Cap y) = (\dashv x) \Cup (\dashv y)$

Proof

$\dashv (x \Cap y) = \dashv (\{x \text{ and } y\}) = \dashv x, \text{or } \dashv y, or\ \{\dashv x, and\ \dashv y\} = (\dashv x) \Cup (\dashv y)$

Example

$\dashv (x_1 \Cap x_2) \quad = \dashv (\{x_1, x_2\})$

$= (\dashv x_1, or \dashv x_2, \text{or } (\dashv x_1 \text{ and } \dashv x_2))$

$= (x_2, x_3, \{x_2, x_3\}, \emptyset) =$

or $(x_2, x_3, \{x_2, x_3\}, \emptyset)$
or $(x_1, x_3, \{x_1, x_3\}, \emptyset)$
or $(x_1, x_2, x_3, \{x_1, x_3\}, \{x_2, x_3\}, \emptyset) =$
$= (x_1, x_2, x_3, \{x_1, x_3\}, \{x_2, x_3\}, \emptyset)$

$(\neg x_1) \veebar (\neg x_2) = \neg x_1$, or $\neg x_2$, or $(\neg x_1 \wedge\!\!\!\wedge \neg x_2) =$
$(x_2, x_3, \{x_2, x_3\}, \emptyset)$
or $(x_1, x_3, \{x_1, x_3\}, \emptyset)$
or $(x_1, x_2, x_3, \{x_1, x_3\}, \{x_2, x_3\}, \emptyset)$
$= (x_1, x_2, x_3, \{x_1, x_3\}, \{x_2, x_3\}, \emptyset)$
$= \neg (x_1 \wedge\!\!\!\wedge x_2)$

*

This IndetermSoft Algebra is not a Boolean Algebra because many of Boolean Laws do not work, such as:

- Identity for ⩓
- Identity for ⩔
- Identity for ⩔$_E$
- Annihilator for ⩓
- Annihilator for ⩔
- Annihilator for ⩔$_E$
- Absorption1 [x ⩓ (x ⩔ y) = x]
- Absorption2 [x ⩔ (x ⩓ y) = x]
- Double Negation
- Complementation1 [x ⩓ ⫤ $x = \emptyset$]
- Complementation2 { [x ⩔ ⫤ $x = H$] and [x ⩔$_E$ ⫤ $x = H$] }

21. Practical Applications of Soft Set and IndetermSoft Set

Let $H = \{h_1, h_2, h_3, h_4\}$ a set of four houses, and the attribute *a = color*, whose values are *A = {white, green, blue, red}*.

21.1. Soft Set

The function
$F: A \rightarrow \mathcal{P}(H)$
where $\mathcal{P}(H)$ is the powerset of H,
is called a classical Soft Set.

For example,
F (white) = h_3, i.e. the house h_3 is painted white;
F (green) = $\{h_1, h_2\}$, i.e. both houses h_1 and h_2 are painted green;
F (blue) = h_4, i.e. the house h_4 is painted blue;
F (red) = $\emptyset$, i.e. no house is painted red.
Therefore, the information about the houses' colors is well-known, certain.

21.2. IndetermSoft Set

But there are many cases in our real life when the information about the attributes' values of the objects (or items – in general) is unclear, uncertain.

That is why we need to extend the classical (Determinate) Soft Set to an Indeterminate Soft Set.

The determinate (exact) soft function
$F: A \rightarrow \mathcal{P}(H)$

is extended to an indeterminate soft function

$F: A \to H(⩓, ⩔, ⩔_E, ⫬)$,

where $(⩓, ⩔, ⩔_E, ⫬)$ is a set closed under ⩓, ⩔, $⩔_E$, and ⫬, and *f(x)* is not always determinate.

For example,

F (white) = h_3 ⩔ h_4,

means the houses h_3 or h_4 are white, but we are not sure which one,

whence one has three possibilities/outcomes/alternatives:

either h_3 is white (and h_4 is not),

or h_4 is white (and h_3 is not),

or both h_3 and h_4 are white.

This is an indeterminate information.

We may also simply write:

$$F\ (white) = \begin{cases} h_3 \\ h_4 \\ \{h_3, h_4\} \end{cases}$$

or *F(white)* = h_3, h_4, $\{h_3, h_4\}$,

where $\{h_3, h_4\}$ means $\{h_3$ and $h_4\}$,

that we read as: either h_3, or h_4, or $\{h_3$ and $h_4\}$.

Another example:

F(blue) = ⫬ h_2, or the house h_2 is not blue,

therefore other houses amongst $\{h_1, h_3, h_4\}$ may be blue,

or no house (∅) may be blue.

This is another indeterminate information.

The negation of h_2 (denoted as $\text{NOT}(h_2) =$ ⫬ h_2) is not equal to the classical complement of $C(h_2)$ of the element h_2 with respect to the set H, since

$C(h_2) = H \setminus \{h_2\} = \{h_1, h_3, h_4\}$,

but ⫬ h_2 may be any subset of $H \setminus \{h_2\}$, or any sub-complement of $C(h_2)$,

again many (in this example 8) possible outcomes to choose from:

⫬ h_2 = h_1, h_3, h_4, $\{h_1, h_3\}$, $\{h_1, h_4\}$, $\{h_3, h_4\}$, $\{h_1, h_3, h_4\}$, ∅ =

= either h_1, or h_3, or h_4,

or $\{h_1$ and $h_3\}$, $\{h_1$ and $h_4\}$, $\{h_3$ and $h_4\}$,

or $\{h_1$ and h_3 and $h_4\}$,

or ∅ (null element, i.e. no other house is blue).

The negation (⫬ h_2) produces a higher degree of indeterminacy than the previous unions: (h_3 ⩔ h_4) and respectively (h_3 $⩔_E$ h_4).

The intersection (⩓) is a determinate (certain) operator.

For example,

F (green) = h_1 ⩓ h_2, which is equal to $\{h_1, h_2\}$, i.e. h_1 and h_2 put together, $\{h_1$ and $h_2\}$.

A combination of these operators may occur, so the indeterminate (uncertain) soft function becomes more complex.

Example again.

F (green) = h_1 ⩓ (⫬ h_4), where of course ⫬ $h_4 \neq h_1$, which means that:

the house h_1 is green,

and other houses amongst $\{h_2, h_3\}$ may be blue,

or ∅ (no other house is blue).

h_1 ⩓ (⫬ h_4) = h_1 and ($\text{NOT}h_4$)

= h_1 and $(h_1, h_2, h_3, \{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}, \emptyset)$

[one cuts h_1 since ⫬ h_4 suppose to be different from h_1]

$= h_1$ and $(h_2, h_3, \{h_2, h_3\}, \emptyset)$
$= (h_1$ and $h_2)$ or $(h_1$ and $h_3)$
or $(h_1$ and $\{h_2, h_3\})$
or $\emptyset$
$= (h_1$ and $h_2)$ or $(h_1$ and $h_3)$ or $(h_1$ and h_2 and $h_3)$ or $\emptyset$

$$\overset{notation}{=} \{h_1, h_2\}, \{h_1, h_3\}, \{h_1, h_2, h_3\}, \emptyset.$$

Thus, 4 possibilities.

22. Definitions of <Algebra, NeutroAlgebra, AntiAlgebra>

Let $\mathcal{U}$ be a universe of discourse, and H a non-empty set included in $\mathcal{U}$. Also, H is endowed with some operations and axioms.

22.1. Algebra

An algebraic structure whose all operations are well-defined, and all axioms are totally true, is called a classical Algebraic Structure (or **Algebra**). Whence *(T, I, F) = (1, 0, 0).*

22.2. NeutroAlgebra

If at least one operation or one axiom has some degree of truth (*T*), some degree of indeterminacy (*I*), and some degree of falsehood (*F*), where *(T, I, F)* $\notin \{(1,0,0), (0,0,1)\}$, and no other operation or axiom is totally false (*F = 1*), then this is called a NeutroAlgebra.

22.3. AntiAlgebra

An algebraic structure that has at least one operation that is totally outer-defined (F = 1) or at least one axiom that is totally false (F = 0), is called AntiAlgebra.

23. Definition of IndetermAlgebra

We introduce now for the first time the concept of IntermAlgebra (= Indeterminate Algebra), as a subclass of NeutroAlgebra.

IndetermAlgebra results from real applications, as it will be seen further.

Let $\mathcal{U}$ be a universe of discourse, and H a non-empty set included in $\mathcal{U}$.

If at least one operation or one axiom has some degree of indeterminacy ($I > 0$), the degree of falsehood $F = 0$, and all other operations and axioms are totally true, then H is an IndetermAlgebra.

24. Definition of IndetermSoft Algebra

The set $H(⩕, ⩔, ⩔_E, ⫬)$ closed under the following operators:

joinAND (denoted by ⩕), which is a determinate operator;

disjoinOR (denoted by ⩔), which is an indeterminate operator;

exclusiveOR (denoted by $⩔_E$), which is an indeterminate operator,

and sub-negation/sub-complement *NOT* (denoted by ⫬), which is an indeterminate operator;
is called an IndetermSoft Algebra.

The IndetermSoft Algebra extends the classical Soft Set Algebra.

The IndetermSoft Algebra is a particular case of the IndetermAlgebra, and of the NeutroAlgebra.

The operator *joinAND*

$⩕: H^2 \longrightarrow H(⩕, ⩔, ⩔_E, ⫬)$

is determinate (in the classical sense):

$\forall x, y \in H, x \neq y, x \wedge y = x$ joinAND $y = \{x, y\} \in H(\wedge, \vee, \vee_E, \neg)$

therefore, the aggregation of x and y by using the operator ⩓ gives a clear and unique output, i.e. the classical set of two elements: $\{x, y\}$.

But the operator *disjoinOR*

⩔: $H^2 \longrightarrow$ H(⩓, ⩔, ⩔E, ⫬)

is indeterminate because:

$$\forall x, y \in H, x \neq y, x \vee y = x \text{ disjoinOR } y = \begin{cases} \text{either } x \\ \text{or } y \\ \text{or both } \{x \text{ and } y\} \end{cases} = \begin{cases} x \\ y \\ \{x, y\} \end{cases}$$

Thus, the aggregation of x and y by using the operator ⩔ gives an unclear output, with three possible alternative solutions (either x, or y, or $\{x$ and $y\}$).

The exclusiveOR operator is also indeterminate:

$\forall x, y \in H, x \neq y$, x ⩔E y $= x$ exclusiveOR $y =$ either x, or y, and no {x, y},

therefore two possible solutions:

⩔E: $H^2 \longrightarrow$ H(⩓,⩔, ⩔E, ⫬).

Similarly, the operator sub-negation / sub-complement NOT

⫬: $H \longrightarrow H$(⩓, ⩔, ⩔E, ⫬)

is indeterminate because of many elements $x \in H$,

NOT(x) = ⫬ x = a part of the complement of x with respect to H

= a subset of $H \setminus \{x\}$.

But there are many subsets of $H \setminus \{x\}$, therefore there is an unclear (uncertain, ambiguous) output, with multiple possible alternative solutions.

25. Definition of IndetermSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and H(⩓, ⩔, ⩔E, ⫬) the IndetermSoft Algebra generated by closing the set H under the operators ⩓, ⩔, ⩔E, and ⫬.

Let a be an attribute, with its set of attribute values denoted by A. Then the pair

(F, A), where F: $A \rightarrow H$(⩓,⩔, ⩔E, ⫬), is called an IndetermSoft Set over H.

26. Fuzzy / Intuitionistic Fuzzy / Neutrosophic / and other fuzzy-extension / IndetermSoft Set

One may associate fuzzy / intuitionistic fuzzy / neutrosophic etc. degrees and extend the IndetermSoft Set to some **Fuzzy / Intuitionistic Fuzzy / Neutrosophic / and other fuzzy-extension / IndetermSoft Set**.

26.1. Applications of (Fuzzy/ Intuitionistic Fuzzy / Neutrosophic / and other fuzzy-extension) IndetermSoft Set

Let $H = \{h_1, h_2, h_3, h_4\}$ be a set of four houses, and the IndetermSoft Algebra generated by closing the set H under the previous soft operators, H(⩓, ⩔, ⩔E, ⫬).

Let the attribute c = *color*, and its attribute values be the set C = *{white, green, blue}*.

The IndetermSoft Function $F : A \longrightarrow H$(⩓, ⩔, ⩔E, ⫬) forms an IndetermSoft Set.

Let an element $h \in H$, and one denotes by:

$d^{\circ}(h)$ = any type of degree (either fuzzy, or intuitionistic fuzzy, or neutrosophic, or any other fuzzy-extension) of the element h.

We extend the soft operators ⩓, ⩔, ⩔E, ⫬ by assigning some degree $d^0(.) \in [0,1]^p$, where:

$p = 1$ for classical and fuzzy degree, $p = 2$ for intuitionistic fuzzy degree, $p = 3$ for neutrosophic degree, and so on $p = n$ for n-valued refined neutrosophic degree, to the elements involved in the operators, where $\wedge, \vee, \neg$ represent the conjunction, disjunction, and negation respectively of these degrees in their corresponding fuzzy-extension sets or logics.

For examples:

i. From *F(white)* = h_1 ⩓ h_2 as in IndetermSoft Set, one extends to:

F(white) = $h_1(d_1^\circ)$ ⩓ $h_2(d_2^\circ)$, which means the degree (chance) that h_1 be white is d_1° and the degree (chance) that h_2 be white is d_2°, whence:

F(white) = $h_1(d_1^\circ)$ ⩓ $h_2(d_2^\circ)$ = $\{h_1, h_2\}(d_1^\circ \wedge d_2^\circ)$

As such, the degree of both houses $\{h_1, h_2\} = \{h_1$ *and* $h_2\}$ be white is $d_1^\circ \wedge d_2^\circ$.

ii. Similarly, *F(white)* = $h_1(d_1^\circ)$ ⩔ $h_2(d_2^\circ)$ = $\{h_1$ *or* $h_2\}$ $(d_1^\circ \vee d_2^\circ)$,

or the degree of at least one house $\{h_1$ *or* $h_2\}$ be white is $(d_1^\circ \vee d_2^\circ)$.

iii. *F(white)* = $h_1(d_1^\circ)$ ⩔$_E$ $h_2(d_2^\circ)$ =

= { h_1 *and* (*no* h_2)}, *or* { (*no* h_1) *and* h_2 }, *and* { (*no* h_1) *and* (*no* h_2) }

= (*either* h_1 *is white, or* h_2 *is white, and* [*no both* $\{h_1, h_2\}$] *are white simultaneously*) has the degree of $(d_1^\circ \vee d_2^\circ) - (d_1^\circ \wedge d_2^\circ)$.

iv. *F(white)* = (⫬h_1)(d_1°), which means that the degree (chance) for h_1 not to be white is d_1°.

(⫬ h_1 = *NOT*(h_1) = *either* h_2, *or* h_3, *or* h_4,

or $\{h_2, h_3\}$, $\{h_2, h_4\}$, $\{h_3, h_4\}$,

or $\{h_2, h_3, h_4\}$,

or ϕ (*no house*).

There are 8 alternatives, thus NOT(h_1) *is one of them.*

Let's assume that NOT(h_1) = $\{h_3, h_4\}$. *Then the degree of both houses* $\{h_3, h_4\}$ *be white is* $\neg d_1^\circ$.

27. Definition of IndetermHyperSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and H(⩓, ⩔, ⩔$_E$, ⫬) the IndetermSoft Algebra generated by closing the set H under the operators ⩓, ⩔, ⩔$_E$, and ⫬.

Let $a_1, a_2, \ldots a_n$, where $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$. Then the pair $(F, A_1 \times A_2 \times \ldots \times A_n)$, where $A_1 \times A_2 \times \ldots \times A_n$ represents a Cartesian product, with

$F: A_1 \times A_2 \times \ldots \times A_n \to H$(⩓, ⩔, ⩔$_E$, ⫬), is called an IndetermHyperSoft Set.

Similarly, one may associate fuzzy / intuitionistic fuzzy / neutrosophic etc. degrees and extend the IndetermHyperSoft Set to some *Fuzzy / Intuitionistic Fuzzy / Neutrosophic etc. IndetermHyperSoft Set.*

28. Applications of the IndetermHyperSoft Set

Let's again $H = \{h_1, h_2, h_3, h_4\}$ be a set of four houses, and the attribute *c = color*, whose values are *C = {white, green, blue, red}*, and another attribute *p = price*, whose values are *P = {cheap, expensive}*.

The function

$F: C \times P \to \mathcal{P}(H)$

where $\mathcal{P}(H)$ is the powerset of H, is a HyperSoft Set.

$F: C \times P \to H$(⩓, ⩔, ⩔$_E$, ⫬), is called an IndetermHyperSoft Set.

Examples:

F(white, cheap) = h_2 ⩔ h_4

F(green, expensive) = h_1 ⩔$_E$ h_2

F(red, expensive) = ⫬ h_3

For a Neutrosophic IndetermHyperSoft Set one has neutrosophic degrees, for example:

F(white, cheap) = $h_2(0.4, 0.2, 0.3) \Vvee h_4 (0.5, 0.1, 0.4)$

In the same way as above (Section 26.1), one extends the HyperSoft operators ⩓, ⩔, $⩔_E$, ⫬ by assigning some degree $d^0(.) \in [0,1]^p$, where: $p = 1$ for classical and fuzzy degree, $p = 2$ for intuitionistic fuzzy degree, $p = 3$ for neutrosophic degree, and so on $p = n$ for n-valued refined neutrosophic degree, to the elements involved in the operators, where $\wedge, \vee, \neg$ represent the conjunction, disjunction, and negation respectively of these degrees in their corresponding fuzzy-extension sets or logics.

29. Definition of Neutrosophic Triplet Commutative Group

Let $\mathcal{U}$ be a universe of discourse, and $(H,*)$ a non-empty set included in $\mathcal{U}$, where * is a binary operation (law) on H.

(i) The operation * on H is well-defined, associative, and commutative.

(ii) For each element $x \in H$ there exist an element $y \in H$, called the neutral of x, such that y is different from the unit element (if any), with $x * y = y * x = x$, and there exist an element $z \in H$, called the inverse of x, such that $x * z = z * x = y$, then $\langle x, y, z \rangle$ is called a neutrosophic triplet.

Then $(H, *)$ is Neutrosophic Triplet Commutative Group.

In general, a Neutrosophic Triplet Algebra is different from a Classical Algebra.

29.1. Theorem 3

The joinAND Algebra $(H, ⩕)$ and the disjoinOR Algebra $(H, ⩖)$ are Neutrosophic Triplet Commutative Groups.

Proof

We have previously proved that the operators ⩕ and ⩖ are each of them: well-defined, associative, and commutative.

We also proved that the two operators are idempotent:

$\forall x \in H$, $x ⩕ x = x$ and $x ⩖ x = x$.

Therefore, for $(H,⩕)$ and respectively $(H,⩖)$ one has neutrosophic triplets of the form: $\langle x, x, x \rangle$.

30. Enriching the IndetermSoft Set and IndetermHyperSoft Set

The readers are invited to extend this research, since more determinate and indeterminate soft operators may be added to the IndetermSoft Algebra or IndetermHyperSoft Algebra, resulted from, or needed to, various real applications - as such one gets stronger soft and hypersoft structures.

A few suggestions:

F(white) = at least k houses;

or *F(white) = at most k houses;*

or *F(green, small) = between* k_1 *and* k_2 *houses;*

where k, k_1 and k_2 are positive integers, with $k_1 \le k_2$.

Etc.

31. Conclusions

The indeterminate soft operators, presented in this paper, have resulted from our real-world applications. An algebra closed under such operators was called an indeterminate soft algebra.

IndetermSoft Set and IndetermHyperSoft Set, and their corresponding Fuzzy / Intuitionistic Fuzzy / Neutrosophic forms, constructed on this indeterminate algebra, are introduced for the first time as extensions of the classical Soft Set and HyperSoft Set.

Many applications and examples are showed up.

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MADM Technique Using Tangent Trigonometric SvNN Aggregation Operators for the Teaching Quality Assessment of Teachers

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Abstract: In current Chinese higher education, the teaching quality assessment (TQA) of teachers in colleges/universities is an essential way to promotes the improvement of teacher teaching quality in the teaching process. In the TQA process of teachers, the evaluation information of experts/decision makers implies incompleteness, uncertainty and inconsistency corresponding to experts' cognition and judgment on evaluation indicators. Neutrosophic multiple attribute decision making (MADM) is one of key research topics in indeterminate and inconsistent decision-making problems. This paper presents a novel MADM technique using tangent trigonometric aggregation operators for single-valued neutrosophic numbers (SvNNs) to assess the teaching quality of teachers. First, we propose novel operational laws of tangent trigonometric SvNNs based on tangent trigonometric function. In view of the tangent trigonometric SvNN operational laws, we present tangent trigonometric SvNN weighted averaging and geometric operators to aggregate tangent trigonometric SvNNs. Then, we establish the MADM technique using the proposed two aggregation operators to perform MADM problems, and provide an actual example about the TQA of teachers and the comparison of existing related MADM techniques in the SvNN environment to reveal the efficiency and suitability of the proposed technique.

Keywords: single-valued neutrosophic number; tangent trigonometric operation; tangent trigonometric aggregation operator; decision making; teaching quality assessment

1. Introduction

The teaching quality of teachers reveals significance in the training and competition of modern talents in current Chinese higher education. In this case, the teaching quality assessment (TQA) mechanism in colleges and universities plays an important role in the teaching process and promotes the improvement of teachers' teaching quality. Since the TQA of teachers contains many evaluation indicators/attributes, such as teaching level and skill, teaching means and method, teaching attitude, TQA is a multiple attribute decision-making (MADM) issue. Then, in the TQA process of teachers, the evaluation information of experts implies incompleteness, uncertainty, and inconsistency corresponding to experts' cognition and judgment on evaluation indicators.

In the environment of incompleteness, uncertainty and inconsistency, a neutrosophic set (NS) was presented by Smarandache [1] in view of the true, false and indeterminate membership functions subject to a non-standard interval $]^-0, 1^+[$. Then, some researchers introduced an interval NS (INS) [2], a single-valued NS (SvNS) [3], and a simplified NS [4] based on a real-standard interval [0, 1] to suit the application of engineering and science fields. Due to the merit of simplified NSs, including INSs and SvNSs, the simplified NSs have been used for many MADM issues [5-8]. Recently, neutrosophic MADM models have been applied to the TQA of teachers [9-11].

Neutrosophic decision making is a current research hotspot. It is very vital to establish reasonable information representations and operations in decision-making models. It is worth noting that the neutrosophic aggregation operation plays an important role in neutrosophic MADM issues. Some researchers [12-14] proposed the operational laws of logarithmic single-valued neutrosophic numbers (SvNNs) and sine trigonometric SvNNs (ST-SvNNs) and their weighted aggregation operators for MADM problems in the SvNS setting. Then, the merit of the sine trigonometric function is its periodicity and symmetry about the origin, which meets the preference of decision-makers for multiple time phase parameters. Another periodic function, except the sine trigonometric function, is the tangent trigonometric function. In terms of the superiority of the tangent trigonometric function, this paper needs to build up some new operational laws of tangent trigonometric SvNNs (TT-SvNNs) and studies their aggregation operators, then establishes the MADM technique to perform the assessment mechanism of teaching quality in Shaoxing University in China under the environment of SvNSs.

The remainder of this article is arranged as follows. Section 2 introduces some preliminaries related to SvNNs. In Section 3, we give the definition of TT-SvNN and some novel operational laws of TT-SvNNs. In Section 4, we propose the TT-SvNN weighted averaging (TT-SvNNWA) and TT-SvNN weighted geometric (TT-SvNNWG) operators, along with the related proof of their properties. Section 5 establishes the MADM technique in terms of the TT-SvNNWA and TT-SvNNWG operators to perform MADM problems with SvNN information. Section 6 applies the established MADM technique to an actual example about the TQA problem of teachers in Shaoxing University in China and conducts the comparison of existing related MADM techniques to show the efficiency and suitability of the established MADM technique in the environment of SvNSs. The article ends with conclusions and future research in Section 7.

2. Some Preliminaries of SvNNs

2.1 Operations and sorting rules of SvNNs

The SvNS S_N in a universe set Y is denoted as $S_N = \{\langle y, Nt(y), Nu(y), Nf(y)\rangle | y \in Y\}$ [3], where $Nt(y), Nu(y), Nf(y) \in [0, 1]$ are the true, indeterminate, and false membership functions subject to $0 \le Nt(y) + Nu(y) + Nf(y) \le 3$ for $y \in Y$. Then, $\langle y, Nt(y), Nu(y), Nf(y)\rangle$ in S_N is denoted as $Ns = \langle Nt, Nu, Nf\rangle$ for simplicity, which is named SvNN.

Set two SvNNs as $Ns_1 = \langle Nt_1, Nu_1, Nf_1\rangle$ and $Ns_2 = \langle Nt_2, Nu_2, Nf_2\rangle$ with $h > 0$. Then, their operations are defined below [4, 7]:

(1) $Ns_1 \supseteq Ns_2 \Leftrightarrow Nt_1 \ge Nt_2$, $Nu_1 \le Nu_2$, and $Nf_1 \le Nf_2$;

(2) $Ns_1 = Ns_2 \Leftrightarrow Ns_1 \supseteq Ns_2$ and $Ns_2 \supseteq Ns_1$;

(3) $Ns_1 \bigcup Ns_2 = \langle Nt_1 \vee Nt_2, Nu_1 \wedge Nu_2, Nf_1 \wedge Nf_2 \rangle$;

(4) $Nt_1 \bigcap Nt_2 = \langle Nt_1 \wedge Nt_2, Nu_1 \vee Nu_2, Nf_1 \vee Nf_2 \rangle$;

(5) $(Ns_1)^c = \langle Nf_1, 1 - Nu_1, Nt_1 \rangle$ (Complement of Ns_1);

(6) $Ns_1 \oplus Ns_2 = \langle Nt_1 + Nt_2 - Nt_1 Nt_2, Nu_1 Nu_2, Nf_1 Nf_2 \rangle$;

(7) $Ns_1 \otimes Ns_2 = \langle Nt_1 Nt_2, Nu_1 + Nu_2 - Nu_1 Nu_2, Nf_1 + Nf_2 - Nf_1 Nf_2 \rangle$;

(8) $h \cdot Ns_1 = \left\langle 1-(1-Nt_1)^h, Nu_1^h, Nf_1^h \right\rangle$;

(9) $Ns_1^h = \left\langle Nt_1^h, 1-(1-Nu_1)^h, 1-(1-Nf_1)^h \right\rangle$.

Set Ns_g = <Nt_g, Nu_g, Nf_g> (g = 1, 2, …, n) as a group of SvNNs with their weight vector $\boldsymbol{H} = (h_1, h_2, \ldots, h_n)$ subject to $0 \le h_g \le 1$ and $\sum_{g=1}^{n} h_g = 1$. Then, the weighted averaging and geometric operators of SvNNs are denoted as SvNNWA and SvNNWG and defined by the following equations [7]:

$$SvNNWA(Ns_1, Ns_2, \ldots, Ns_n) = \sum_{g=1}^{n} h_g Ns_g = \left\langle 1-\prod_{g=1}^{n}\left(1-Nt_g\right)^{h_g}, \prod_{g=1}^{n}\left(Nu_g\right)^{h_g}, \prod_{g=1}^{n}\left(Nf_g\right)^{h_g} \right\rangle, \quad (1)$$

$$SvNNWG(Ns_1, Ns_2, \ldots, Ns_n) = \prod_{g=1}^{n}\left(Ns_g\right)^{h_g} = \left\langle \prod_{g=1}^{n}\left(Nt_g\right)^{h_g}, 1-\prod_{g=1}^{n}\left(1-Nu_g\right)^{h_g}, 1-\prod_{g=1}^{n}\left(1-Nf_g\right)^{h_g} \right\rangle. \quad (2)$$

To compare two SvNNs Ns_g = <Nt_g, Nu_g, Nf_g> (g = 1, 2), the score and accuracy functions of SvNNs are defined as follows [7]:

$$F(Ns_g) = (2 + Nt_g - Nu_g - Nf_g)/3 \text{ for } F(Ns_g) \in [0,1], \quad (3)$$

$$G(Ns_g) = Nt_g - Nf_g \text{ for } G(Ns_g) \in [-1,1]. \quad (4)$$

In terms of the score and accuracy functions, a sorting method of two SvNNs is defined by the following rules:

(1) If $F(Ns_1) > F(Ns_2)$, then $Ns_1 > Ns_2$;

(2) If $F(Ns_1) = F(Ns_2)$ and $G(Ns_1) > G(Ns_2)$, then $Ns_1 > Ns_2$;

(3) If $F(Ns_1) = F(Ns_2)$ and $G(Ns_1) = G(Ns_2)$, then $Ns_1 \cong Ns_2$.

2.2 Operational Laws of ST-SvNNs

Set SvNN as Ns = <Nt, Nu, Nf>. Then, ST-SvNN is defined as $\sin(Ns) = \left\langle \sin(\frac{\pi}{2}Nt), 1-\sin(\frac{\pi}{2}-Nu), 1-\sin(\frac{\pi}{2}-Nf) \right\rangle$ [14], where the true, indeterminate, and false membership degrees are $\sin(\frac{\pi}{2}Nt): Y \to [0,1]$, $1-\sin(\frac{\pi}{2}-Nu): Y \to [0,1]$, and $1-\sin(\frac{\pi}{2}-Nf): Y \to [0,1]$, respectively.

Set two ST-SvNNs as $\sin(Ns_1) = \left\langle \sin(\frac{\pi}{2}Nt_1), 1-\sin(\frac{\pi}{2}-Nu_1), 1-\sin(\frac{\pi}{2}-Nf_1) \right\rangle$ and $\sin(Ns_2) = \left\langle \sin(\frac{\pi}{2}Nt_2), 1-\sin(\frac{\pi}{2}-Nu_2), 1-\sin(\frac{\pi}{2}-Nf_2) \right\rangle$ with $h > 0$. Then, their operational laws are defined below [14]:

(1) $$\sin(Ns_1) \oplus \sin(Ns_2) = \left\langle \begin{matrix} 1-(1-\sin(\frac{\pi}{2}Nt_1))(1-\sin(\frac{\pi}{2}Nt_2)), \\ (1-\sin(\frac{\pi}{2}-Nu_1))(1-\sin(\frac{\pi}{2}-Nu_2)), \\ (1-\sin(\frac{\pi}{2}-Nf_1))(1-\sin(\frac{\pi}{2}-Nf_2)) \end{matrix} \right\rangle,$$

(2) $$\sin(Ns_1) \otimes \sin(Ns_2) = \left\langle \begin{matrix} \sin(\frac{\pi}{2}Nt_1)\sin(\frac{\pi}{2}Nt_2), \\ 1-\sin(\frac{\pi}{2}-Nu_1)\sin(\frac{\pi}{2}-Nu_2), \\ 1-\sin(\frac{\pi}{2}-Nf_1)\sin(\frac{\pi}{2}-Nf_2) \end{matrix} \right\rangle,$$

(3) $h\cdot\sin(Ns_1)=\left\langle 1-(1-\sin(\frac{\pi}{2}Nt_1))^h,(1-\sin(\frac{\pi}{2}-Nu_1))^h,(1-\sin(\frac{\pi}{2}-Nf_1))^h\right\rangle$,

(4) $(\sin(Ns_1))^h=\left\langle (\sin(\frac{\pi}{2}Nt_1))^h,1-(\sin(\frac{\pi}{2}-Nu_1))^h,1-(\sin(\frac{\pi}{2}-N_{f1}))^h\right\rangle$.

Set Ns_g = <Nt_g, Nu_g, Nf_g> (g = 1, 2, …, n) as a group of SvNNs with their weight vector $\boldsymbol{H}$ = (h_1, h_2, …, h_n) subject to $0 \le h_g \le 1$ and $\sum_{g=1}^{n} h_g = 1$. Then, the ST-SvNN weighted averaging and geometric operators are denoted as ST-SvNNWA and ST-SvNNWG and defined by the following equations [14]:

$$ST-SvNNWA(Ns_1,Ns_2,...,Ns_n)=\sum_{g=1}^{n}h_g\sin(Ns_g)$$
$$=\left(1-\prod_{g=1}^{n}\left(1-\sin\left(\frac{\pi}{2}Nt_g\right)\right)^{h_g},\prod_{g=1}^{n}\left(1-\sin\left(\frac{\pi}{2}-Nu_g\right)\right)^{h_g},\prod_{g=1}^{n}\left(1-\sin\left(\frac{\pi}{2}-Nf_g\right)\right)^{h_g}\right), \quad (5)$$

$$ST-SvNNWG(Ns_1,Ns_2,...,Ns_n)=\prod_{g=1}^{n}\left(\sin(Ns_g)\right)^{h_g}$$
$$=\left(\prod_{g=1}^{n}\left(\sin(\frac{\pi}{2}Nt_g)\right)^{h_g},1-\prod_{g=1}^{n}\left(\sin(\frac{\pi}{2}-Nu_g)\right)^{h_g},1-\prod_{g=1}^{n}\left(\sin(\frac{\pi}{2}-Nf_g)\right)^{h_g}\right). \quad (6)$$

3. Operational Laws of TT-SvNNs

This section defines TT-SvNN and some operational laws of TT-SvNNs.

First, we give the definition of TT-SvNN.

Definition 1. Set SvNN as Ns = <Nt, Nu, Nf>. Then, TT-SvNN is defined below:

$$\tan(Ns)=\left\langle \tan(\frac{\pi}{4}Nt),1-\tan(\frac{\pi}{4}(1-Nu),1-\tan(\frac{\pi}{4}(1-Nf)\right\rangle,$$

where the true, false, and indeterminate membership degrees are given, respectively, by

$\tan(\frac{\pi}{4}Nt):Y\to[0,1],\quad 0\le\tan(\frac{\pi}{4}Nt)\le 1$,

$1-\tan(\frac{\pi}{4}(1-Nf)):Y\to[0,1],\quad 0\le 1-\tan(\frac{\pi}{4}(1-Nf))\le 1$,

$1-\tan(\frac{\pi}{4}(1-Nu)):Y\to[0,1],\quad 0\le 1-\tan(\frac{\pi}{4}(1-Nu))\le 1$.

Definition 2. Set SvNN as Ns = <Nt, Nu, Nf>. If $\tan(Ns)=\left\langle \tan(\frac{\pi}{4}Nt),1-\tan(\frac{\pi}{4}(1-Nu)),1-\tan(\frac{\pi}{4}(1-Nf))\right\rangle$, then tan($Ns$) is named the tangent trigonometric operator and its value is named TT-SvNN.

Definition 3. Set two TT-SvNNs as $\tan(Ns_1)=\left\langle \tan(\frac{\pi}{4}Nt_1),1-\tan(\frac{\pi}{4}(1-Nu_1)),1-\tan(\frac{\pi}{4}(1-Nf_1))\right\rangle$ and $\tan(Ns_2)=\left\langle \tan(\frac{\pi}{4}Nt_2),1-\tan(\frac{\pi}{4}(1-Nu_2)),1-\tan(\frac{\pi}{4}(1-Nf_2))\right\rangle$. Then, their operational laws are defined below:

(1) $$\tan(Ns_1)\oplus\tan(Ns_2)=\left\langle \begin{array}{l} 1-(1-\tan(\frac{\pi}{4}Nt_1))(1-\tan(\frac{\pi}{4}Nt_2)), \\ (1-\tan(\frac{\pi}{4}(1-Nu_1)))(1-\tan(\frac{\pi}{4}(1-Nu_2))), \\ (1-\tan(\frac{\pi}{4}(1-Nf_1)))(1-\tan(\frac{\pi}{4}(1-Nf_2))) \end{array} \right\rangle,$$

(2) $\tan(Ns_1)\otimes\tan(Ns_2)=\left\langle\begin{array}{l}\tan(\frac{\pi}{4}Nt_1)\tan(\frac{\pi}{4}Nt_2),\\ 1-\tan(\frac{\pi}{4}(1-Nu_1))\tan(\frac{\pi}{4}(1-Nu_2)),\\ 1-\tan(\frac{\pi}{4}(1-Nf_1))\tan(\frac{\pi}{4}(1-Nf_2))\end{array}\right\rangle$,

(3) $h\cdot\tan(Ns_1)=\left\langle 1-(1-\tan(\frac{\pi}{4}Nt_1))^h,(1-\tan(\frac{\pi}{4}(1-Nu_1)))^h,(1-\tan(\frac{\pi}{4}(1-Nf_1)))^h\right\rangle$,

(4) $(\tan(Ns_1))^h=\left\langle(\tan(\frac{\pi}{4}Nt_1))^h,1-(\tan(\frac{\pi}{4}(1-Nu_1)))^h,1-(\tan(\frac{\pi}{4}(1-Nf_1)))^h\right\rangle$.

4. TT-SvNN Aggregation Operators

This section proposes two weighted aggregation operators of TT-SvNNWA and TT-SvNNWG in terms of the proposed operational laws of TT-SvNNs and indicates their properties.

3.1 TT-SvNNWA Operator

Definition 4. Set Ns_g = <Nt_g, Nu_g, Nf_g> (g = 1, 2, …, n) as a group of SvNNs with their weight vector $\boldsymbol{H}$ = (h_1, h_2, …, h_n) subject to $0 \le h_g \le 1$ and $\sum_{g=1}^{n}h_g=1$. Then, the TT-SvNN weighted averaging operator is denoted by TT-SvNNWA and defined below:

$$TT-SvNNWA(Ns_1,Ns_2,...,Ns_n)=h_g\tan(Ns_1)\oplus h_g\tan(Ns_2)\oplus...\oplus h_n\tan(Ns_n)=\sum_{g=1}^{n}h_g\tan(Ns_g). \quad (7)$$

Theorem 1. Set Ns_g = <Nt_g, Nu_g, Nf_g> (g = 1, 2, …, n) as a group of SvNNs with their weight vector $\boldsymbol{H}$ = (h_1, h_2, …, h_n) subject to $0 \le h_g \le 1$ and $\sum_{g=1}^{n}h_g=1$. Then, the value of the TT-SvNNWA operator is obtained by the following equation:

$$TT-SvNNWA(Ns_1,Ns_2,...,Ns_n)=\sum_{g=1}^{n}h_g\tan(Ns_g)=\left(1-\prod_{g=1}^{n}\left(1-\tan\left(\tfrac{\pi}{4}Nt_g\right)\right)^{h_g},\prod_{g=1}^{n}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nu_g)\right)\right)^{h_g},\prod_{g=1}^{n}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nf_g)\right)\right)^{h_g}\right). \quad (8)$$

Proof: We can verify Theorem 1 in view of mathematical induction and Definition 3.

For n = 2, we obtain the operational result:

$$TT-SvNNWA(Ns_1,Ns_2)=\sum_{g=1}^{2}h_g\tan(Ns_g)$$
$$=h_1\tan(Ns_1)\oplus h_2\tan(Ns_2)=\begin{pmatrix}1-(1-\tan(\frac{\pi}{4}Nt_1))^{h_1},\\(1-\tan(\frac{\pi}{4}(1-Nu_1)))^{h_1},\\(1-\tan(\frac{\pi}{4}(1-Nf_1)))^{h_1}\end{pmatrix}\oplus\begin{pmatrix}1-(1-\tan(\frac{\pi}{4}Nt_2))^{h_2},\\(1-\tan(\frac{\pi}{4}(1-Nu_2)))^{h_2},\\(1-\tan(\frac{\pi}{4}(1-Nf_2)))^{h_2}\end{pmatrix}$$
$$=\left(1-\prod_{g=1}^{2}(1-\tan(\tfrac{\pi}{4}Ns_g))^{h_g},\prod_{g=1}^{2}(1-\tan(\tfrac{\pi}{4}(1-Nu_g)))^{h_g},\prod_{g=1}^{2}(1-\tan(\tfrac{\pi}{4}(1-Nf_g)))^{h_g}\right).$$

Assume that Eq. (8) holds for n = p as follows:

$$TT-SvNNWA(Ns_1, Ns_2, ..., Ns_p) = \sum_{g=1}^{p} h_g \tan(Ns_g)$$
$$= \left(1-\prod_{g=1}^{p}\left(1-\tan\left(\tfrac{\pi}{4}Nt_g\right)\right)^{h_g}, \prod_{g=1}^{p}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nu_g)\right)\right)^{h_g}, \prod_{g=1}^{p}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nf_g)\right)\right)^{h_g} \right).$$

For $n = p + 1$, we have the following result:

$$TT-SvNNWA(Ns_1, Ns_2, ..., Ns_p, Ns_{p+1}) = \sum_{g=1}^{p} h_g \tan(Ns_g) \oplus h_{g+1} \tan(Ns_{p+1})$$
$$= \left(1-\prod_{g=1}^{p}\left(1-\tan\left(\tfrac{\pi}{4}Nt_g\right)\right)^{h_g}, \prod_{g=1}^{p}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nu_g)\right)\right)^{h_g}, \prod_{g=1}^{p}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nf_g)\right)\right)^{h_g} \right)$$
$$\oplus\left(1-(1-\tan(\tfrac{\pi}{4}Ns_{p+1}))^{h_{p+1}}, (1-\tan(\tfrac{\pi}{4}(1-Nu_{p+1})))^{h_{p+1}}, (1-\tan(\tfrac{\pi}{4}(1-Nf_{p+1})))^{h_{p+1}}\right)$$
$$= \left(1-\prod_{g=1}^{p+1}\left(1-\tan\left(\tfrac{\pi}{4}Nt_g\right)\right)^{h_g}, \prod_{g=1}^{p+1}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nu_g)\right)\right)^{h_g}, \prod_{g=1}^{p+1}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nf_g)\right)\right)^{h_g} \right).$$

Thus, Eq. (8) can hold for $n = p+1$.

Therefore, Eq. (8) also holds for any n. This proof is finished. □

Example 1. Set three SvNNs as Ns_1 = <0.76, 0.23, 0.2>, Ns_2 = <0.83, 0.3, 0.22>, and Ns_3 = <0.67, 0.12, 0.15> with the weight vector $\boldsymbol{H}$ = <0.35, 0.2, 0.45>. Using Eq. (8), we give the following calculational process:

$$TT-SvNNWA(Ns_1, Ns_2, Ns_3) = \sum_{g=1}^{3} h_g \tan(Ns_g)$$
$$= \left\langle \begin{array}{l} 1-\left(1-\tan\left(\tfrac{\pi}{4}\times 0.76\right)\right)^{0.35}\times\left(1-\tan\left(\tfrac{\pi}{4}\times 0.83\right)\right)^{0.2}\times\left(1-\tan\left(\tfrac{\pi}{4}\times 0.67\right)\right)^{0.45}, \\ \left(1-\tan\left(\tfrac{\pi}{4}(1-0.23)\right)\right)^{0.35}\times\left(1-\tan\left(\tfrac{\pi}{4}(1-0.3)\right)\right)^{0.2}\times\left(1-\tan\left(\tfrac{\pi}{4}(1-0.12)\right)\right)^{0.45}, \\ \left(1-\tan\left(\tfrac{\pi}{4}(1-0.2)\right)\right)^{0.35}\times\left(1-\tan\left(\tfrac{\pi}{4}(1-0.22)\right)\right)^{0.2}\times\left(1-\tan\left(\tfrac{\pi}{4}(1-0.15)\right)\right)^{0.45} \end{array} \right\rangle$$
$$=< 0.6596, 0.2488, 0.2478 >.$$

Theorem 2. The proposed TT-SvNNWA operator contains some properties based on the tangent trigonometric function as follows:

(i) Idempotency: If Ns_g = <Nt_g, Nu_g, Nf_g> = <Nt, Nu, Nf> = Ns (g = 1, 2, …, n), there is $TT-SvNNWA(Ns_1, Ns_2, ..., Ns_n) = \tan(Ns)$.

(ii) Boundedness: Set $Ns^- = \left\langle \min_g(Nt_g), \max_g(Nu_g), \max_g(Nf_g) \right\rangle$ and $Ns^+ = \left\langle \max_g(Nt_g), \min_g(Nu_g), \min_g(Nf_g) \right\rangle$ as the minimum SvNN and the maximum SvNN, respectively. Then, there is $\tan(Ns^-) \le TT-SvNNWA(Ns_1, Ns_2, ..., Ns_n) \le \tan(Ns^+)$.

(iii) Monotonicity: Set $Ns_g = \left\langle Nt_g, Nu_g, Nf_g \right\rangle$ and $Ns^* = \left\langle Nt_g^*, Nu_g^*, Nf_g^* \right\rangle$ (g = 1, 2, …, n) as two groups of SvNNs. Then $TT-SvNNWA(Ns_1, Ns_2, \cdots, Ns_n) \le TT-SvNNWA(Ns_1^*, Ns_2^*, \cdots, Ns_n^*)$ exists when $Ns_g \le Ns_g^*$.

Proof:

(i) For $Ns_g = Ns$, using Eq. (8), we obtain

$$TT-SvNNWA(Ns_1, Ns_2, ..., Ns_n) = \sum_{g=1}^{n} h_g \tan(Ns_g)$$
$$= \left(1-\prod_{g=1}^{n}\left(1-\tan\left(\tfrac{\pi}{4}Nt_g\right)\right)^{h_g}, \prod_{g=1}^{n}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nu_g)\right)\right)^{h_g}, \prod_{g=1}^{n}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nf_g)\right)\right)^{h_g}\right)$$
$$= \left(1-\prod_{g=1}^{n}\left(1-\tan\left(\tfrac{\pi}{4}Nt\right)\right)^{h_g}, \prod_{g=1}^{n}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nu)\right)\right)^{h_g}, \prod_{g=1}^{n}\left(1-\tan\left(\tfrac{\pi}{4}(1-Nf)\right)\right)^{h_g}\right)$$
$$= \left(1-\left(1-\tan\left(\tfrac{\pi}{4}Nt\right)\right)^{\sum_{g=1}^{n}h_g}, \left(1-\tan\left(\tfrac{\pi}{4}(1-Nu)\right)\right)^{\sum_{g=1}^{n}h_g}, \left(1-\tan\left(\tfrac{\pi}{4}(1-Nf)\right)\right)^{\sum_{g=1}^{n}h_g}\right)$$
$$= \left(\tan\left(\tfrac{\pi}{4}Nt\right), 1-\tan\left(\tfrac{\pi}{4}(1-Nu)\right), 1-\tan\left(\tfrac{\pi}{4}(1-Nf)\right)\right) = \tan(Ns).$$

(ii) When $Ns^- \le Ns_g \le Ns^+$, $\tan(Ns^-) \le \tan(Ns_g) \le \tan(Ns^+)$ exists since tan (x) for $0 \le x \le \pi/4$ is an increasing function. Then, there is also $\sum_{g=1}^{n} h_g \tan(Ns^-) \le \sum_{g=1}^{n} h_g \tan(Ns_g) \le \sum_{g=1}^{n} h_g \tan(Ns^+)$. Therefore, based on the property (i), there is $\tan(Ns^-) \le TT-SvNNWA(Ns_1, Ns_2, ..., Ns_n) \le \tan(Ns^+)$.

(iii) When $Ns_g \le Ns_g^*$, there is $\tan(Ns_g) \le \tan(Ns_g^*)$ since tan (x) for $0 \le x \le \pi/4$ is an increasing function. $\sum_{g=1}^{n} h_g \tan(Ns_g) \le \sum_{g=1}^{n} h_g \tan(Ns_g^*)$ can hold in view of the property (ii). Thus, $TT-SvNNWA(Ns_1, Ns_2, \cdots, Ns_n) \le TT-SvNNWA(Ns_1^*, Ns_2^*, \cdots, Ns_n^*)$ exists.

3.2 TT-SvNNWG Operator

Definition 5. Set Ns_g = <Nt_g, Nu_g, Nf_g> (g = 1, 2, …, n) as a group of SvNNs with their weight vector $\boldsymbol{H}$ = (h_1, h_2, …, h_n) subject to $0 \le h_g \le 1$ and $\sum_{g=1}^{n} h_g = 1$. Then, the TT-SvNN weighted geometric operator is denoted by TT-SvNNWG and defined below:

$$TT-SvNNWG(Ns_1, Ns_2, ..., Ns_n) = \left(\tan(Ns_1)\right)^{h_1} \otimes \left(\tan(Ns_2)\right)^{h_2} \otimes ... \otimes \left(\tan(Ns_n)\right)^{h_n}$$
$$= \prod_{g=1}^{n}\left(\tan(Ns_g)\right)^{h_g}. \quad (9)$$

Theorem 3. Set Ns_g = <Nt_g, Nu_g, Nf_g> (g = 1, 2, …, n) as a group of SvNNs with their weight vector $\boldsymbol{H}$ = (h_1, h_2, …, h_n) subject to $0 \le h_g \le 1$ and $\sum_{g=1}^{n} h_g = 1$. Then, the value of the TT-SvNNWG operator is obtained by the following equation:

$$TT-SvNNWG(Ns_1, Ns_2, ..., Ns_n) = \prod_{g=1}^{n}\left(\tan(Ns_g)\right)^{h_g}$$
$$= \left(\prod_{g=1}^{n}\left(\tan(\tfrac{\pi}{4}Nt_g)\right)^{h_g}, 1-\prod_{g=1}^{n}\left(\tan(\tfrac{\pi}{4}(1-Nu_g))\right)^{h_g}, 1-\prod_{g=1}^{n}\left(\tan(\tfrac{\pi}{4}(1-Nf_g))\right)^{h_g}\right). \quad (10)$$

In view of the similar proof process of Theorem 1, we can easily verify Theorem 3, which is omitted.

Example 2. Set three SvNNs as Ns_1 = <0.8, 0.2, 0.1>, Ns_2 = <0.7, 0.2, 0.2>, and Ns_3 = <0.9, 0.1, 0.1> with the weight vector $\boldsymbol{H}$ = (0.35, 0.25, 0.4). Using Eq. (10), we give the following calculational process:

$$TT-SvNNWG(Ns_1,Ns_2,Ns_3)=\prod_{g=1}^{3}\left(\tan(Ns_g)\right)^{h_g}$$
$$=\left\langle \begin{array}{l} \left(\tan\left(\frac{\pi}{4}\times 0.8\right)\right)^{0.35}\times\left(\tan\left(\frac{\pi}{4}\times 0.7\right)\right)^{0.25}\times\left(\tan\left(\frac{\pi}{4}\times 0.9\right)\right)^{0.4}, \\ 1-\left(\tan\left(\frac{\pi}{4}(1-0.2)\right)\right)^{0.35}\times\left(\tan\left(\frac{\pi}{4}(1-0.2)\right)\right)^{0.25}\times\left(\tan\left(\frac{\pi}{4}(1-0.1)\right)\right)^{0.4}, \\ 1-\left(\tan\left(\frac{\pi}{4}(1-0.1)\right)\right)^{0.35}\times\left(\tan\left(\frac{\pi}{4}(1-0.2)\right)\right)^{0.25}\times\left(\tan\left(\frac{\pi}{4}(1-0.1)\right)\right)^{0.4} \end{array}\right\rangle$$
$$=<0.7428,\ 0.2249,\ 0.1798>.$$

Theorem 4. The proposed TT-SvNNWG operator contains some properties based on the tangent trigonometric function as follows:

(i) Idempotency: If Ns_g = $<Nt_g, Nu_g, Nf_g>$ = $<Nt, Nu, Nf>$ = Ns (g = 1, 2, …, n), there is
$$TT-SvNNWG(Ns_1,Ns_2,...,Ns_n)=\tan(Ns).$$

(ii) Boundedness: Set $Ns^-=\left\langle \min_g(Nt_g),\max_g(Nu_g),\max_g(Nf_g)\right\rangle$ and $Ns^+=\left\langle \max_g(Nt_g),\min_g(Nu_g),\min_g(Nf_g)\right\rangle$ as the minimum SvNN and the maximum SvNN, respectively. Then, there is $\tan(Ns^-)\le TT-SvNNWG(Ns_1,Ns_2,...,Ns_n)\le\tan(Ns^+)$.

(iii) Monotonicity: Set $Ns_g=\left\langle Nt_g,Nu_g,Nf_g\right\rangle$ and $Ns^*=\left\langle Nt_g^*,Nu_g^*,Nf_g^*\right\rangle$ (g = 1, 2, …, n) as two groups of SvNNs. Then $TT-SvNNWG(Ns_1,Ns_2,\cdots,Ns_n)\le TT-SvNNWG(Ns_1^*,Ns_2^*,\cdots,Ns_n^*)$ exists when $Ns_g\le Ns_g^*$.

Obviously, the proof process of Theorem 4 is similar to that of Theorem 2, which is omitted.

5. MADM Technique

This section establishes a MADM technique using the proposed TT-SvNNWA and TT-SvNNWG operators in the SvNS setting.

MADM problems usually contain a set of p alternatives K = {K_1, K_2, …, K_p} and a set of n attributes L = {L_1, L_2, …, L_n} and then indicate decision matrix $M = (Ns_{kg})_{p\times n}$, where Ns_{kg} (k = 1, 2, …, p; g = 1, 2, …, n) are SvNNs corresponding to satisfactory assessments of an alternative K_k over attributes L_g given by decision makers. The weight vector of the attributes is presented by $\boldsymbol{H}$ = (h_1, h_2, …, h_n) subject to $0 \le h_g \le 1$ and $\sum_{g=1}^{n}h_g=1$. Regarding MADM problems with SvNN information, the MADM algorithm is composed of the following steps.

Step 1: In view of the satisfactory levels of each teacher with respect to the teaching quality indicators, the experts give the decision matrix of SvNNs $M = (Ns_{kg})_{p\times n}$.

Step 2: The aggregated values Ns_k for K_k (k = 1, 2, …, p) are calculated by the following TT-SvNNWA or TT-SvNNWG operator:

$$Ns_k=TT-SvNNWA(Ns_{k1},Ns_{k2},...,Ns_{kn})=\sum_{g=1}^{n}h_g\tan(Ns_{kg})$$
$$=\left(1-\prod_{g=1}^{n}\left(1-\tan\left(\frac{\pi}{4}Nt_{kg}\right)\right)^{h_g},\prod_{g=1}^{n}\left(1-\tan\left(\frac{\pi}{4}(1-Nu_{kg})\right)\right)^{h_g},\prod_{g=1}^{n}\left(1-\tan\left(\frac{\pi}{4}(1-Nf_{kg})\right)\right)^{h_g}\right). \quad (11)$$

$$Ns_k = TT - SvNNWG(Ns_{k1}, Ns_{k2}, ..., Ns_{kn}) = \prod_{g=1}^{n}\left(\tan(Ns_{kg})\right)^{h_g}$$

or

$$= \left(\prod_{g=1}^{n}\left(\tan(\tfrac{\pi}{4} Nt_{kg})\right)^{h_g}, 1 - \prod_{g=1}^{n}\left(\tan(\tfrac{\pi}{4}(1 - Nu_{kg}))\right)^{h_g}, 1 - \prod_{g=1}^{n}\left(\tan(\tfrac{\pi}{4}(1 - Nf_{kg}))\right)^{h_g} \right). \quad (12)$$

Step 3: The score values of $F(Ns_k)$ (the accuracy values of $G(Ns_k)$) are calculated by Eq. (3) (Eq. (4)).

Step 4: Alternatives are sorted in descending order in terms of the score values (accuracy values) and the best one and the worst one are determined.

Step 5: End.

6. Actual Example about the TQA of Teachers

6.1. TQA Example of Teachers

In current Chinese higher education, the teaching quality of teachers is becoming more and more important in the training and competition of modern talents. In this case, the assessment mechanism of teaching quality in colleges and universities reveals its importance and necessary in the teaching process. Since the TQA problem of teachers contain many evaluation indicators/attributes, liking teaching level and skill, teaching means and methods, teaching attitude, etc. Therefore, TQA is a MADM issue, where evaluation data of the indicators/attributes contain incomplete, uncertain and inconsistent information in the evaluation process. This section applies the established MADM technique to an actual example about the TQA of teachers to show the efficiency and suitability of the established MADM technique in the environment of SvNSs.

Shaoxing University in China needs to establish the TQA system of teachers as an effective teaching management strategy. To evaluate the teaching quality of teachers, the teaching management department preliminarily chooses five teachers as the evaluated objects, which are denoted as a set of the five alternatives $K = \{K_1, K_2, K_3, K_4, K_5\}$. In the TQA process, they must evaluate the satisfactory levels of each teacher with respect to the indicators/attributes of teaching quality, including the teaching level and skill (L_1), the teaching means and method (L_2), the teaching attitude (L_3), the teaching innovation (L_4), and the satisfaction of students (L_5), which are denoted as a set of the attributes $L = \{L_1, L_2, L_3, L_4, L_5\}$. The weight vector of the five attributes is presented by $\boldsymbol{H}$ = (0.23, 0.2, 0.2, 0.17, 0.2).

In this MADM problem with SvNNs, the established MADM technique is applied to this actual example. Then, the decision steps are given below.

Step 1: In view of the satisfactory levels of each teacher with respect of the teaching quality indicators, the experts give the following decision matrix of SvNNs:

$$M = \begin{bmatrix} <0.6,0.2,0.3> & <0.7,0.3,0.2> & <0.7,0.3,0.3> & <0.8,0.1,0.2> & <0.6,0.1,0.4> \\ <0.7,0.1,0.2> & <0.6,0.1,0.1> & <0.9,0.4,0.3> & <0.9,0.2,0.2> & <0.8,0.2,0.4> \\ <0.8,0.1,0.2> & <0.9,0.3,0.3> & <0.6,0.1,0.3> & <0.8,0.4,0.3> & <0.8,0.3,0.3> \\ <0.9,0.1,0.2> & <0.7,0.4,0.2> & <0.9,0.4,0.5> & <0.7,0.1,0.5> & <0.6,0.4,0.4> \\ <0.7,0.4,0.5> & <0.8,0.3,0.3> & <0.9,0.3,0.3> & <0.6,0.3,0.1> & <0.9,0.1,0.2> \end{bmatrix}.$$

Step 2: Using Eq. (11) or Eq. (12), the aggregated values of Ns_k for K_k (k = 1, 2, …, p) are given below:

Ns_1 = <0.5960, 0.2491, 0.3569>, Ns_2 = <0.7361, 0.2346, 0.2906>, Ns_3 = <0.7289, 0.2649, 0.3574>, Ns_4 = <0.7332, 0.3020, 0.4074>, and Ns_5 = <0.7455, 0.3363, 0.3365>.

Or N_{S1} = <0.5827, 0.2794, 0.3710>, N_{S2} = <0.6909, 0.2745, 0.3244>, N_{S3} = <0.6990, 0.3150, 0.3627>, N_{S4} = <0.6812, 0.3735, 0.4503>, and N_{S5} = <0.7017, 0.3723, 0.3869>.

Step 3: Applying Eq. (3), the score values of $F(N_{Sk})$ are obtained as follows:

$F(N_{S1})$ = 0.6633, $F(N_{S2})$ = 0.7370, $F(N_{S3})$ = 0.7022, $F(N_{S4})$ = 0.6746, and $F(N_{S5})$ = 0.6909.

Or $F(N_{S1})$ = 0.6441, $F(N_{S2})$ = 0.6973, $F(N_{S3})$ = 0.6738 $F(N_{S4})$ = 0.6191, and $F(N_{S5})$ = 0.6475.

Step 4: The sorting order of the five teachers is $K_2 > K_3 > K_5 > K_4 > K_1$ or $K_2 > K_3 > K_5 > K_1 > K_4$, and the best one is K_2 and the worst one is K_1 or K_4 in the TQA process of the teachers.

It is obvious that the sorting orders of the five teachers obtained based on the proposed TT-SvNNWA and TT-SvNNWG operators in the SvNS setting reveal some differences, which show that different aggregation algorithms may affect the sorting order.

6.2. Related Comparison

To reveal the efficiency and suitability of the established MADM technique for the TQA problem of teachers, this part compares the established MADM technique with the related techniques in the environment of SvNSs.

Using Eqs. (1)–(6), the evaluation results of the five teachers are given by existing MADM techniques [7, 14]. Then, all the decision results based on the established MADM technique and the existing MADM techniques [7, 14] are shown in Table 1.

Table 1. Decision results of various MADM techniques

MADM technique	Score value	Sorting order	The best one	The worst one
Existing MADM technique using Eq. (1) [7]	0.7425, 0.8058, 0.7768, 0.7499, 0.7658	$K_2 > K_3 > K_5 > K_4 > K_1$	K_2	K_1
Existing MADM technique using Eq. (2) [7]	0.7251, 0.7708, 0.7516, 0.6989, 0.7264	$K_2 > K_3 > K_5 > K_1 > K_4$	K_2	K_4
Existing MADM technique using Eq. (5) [14]	0.9418, 0.9719, 0.9650, 0.9582, 0.9638	$K_2 > K_3 > K_5 > K_4 > K_1$	K_2	K_1
Existing MADM technique using Eq. (6) [14]	0.9324, 0.9534, 0.9516, 0.9309, 0.9429	$K_2 > K_3 > K_5 > K_1 > K_4$	K_2	K_4
Established MADM technique using Eq. (11)	0.6633, 0.7370, 0.7022, 0.6746, 0.6909	$K_2 > K_3 > K_5 > K_4 > K_1$	K_2	K_1
Established MADM technique using Eq. (12)	0.6441, 0.6973, 0.6738, 0.6191, 0.6475	$K_2 > K_3 > K_5 > K_1 > K_4$	K_2	K_4

In Table 1, the sorting orders given by the established MADM technique using Eqs. (11) and (12) and the existing MADM techniques using Eqs. (1) and (2) and Eqs. (5) and (6) are the same. In the meantime, the best one is K_2 and the worst one is K_1 or K_4 in the MADM problem. In the existing MADM techniques [7, 14] and the established MADM technique, different aggregation operators will affect the sorting order of the five teachers. In general, the weighted average aggregation operators mainly tend to group opinions, while the weighted geometric aggregation operators mainly tend to individual opinions. However, one of the aggregation operators is selected depending on decision makers' preference and some actual requirements in the TQA process.

7. Conclusions

Since the TQA of teachers shows its importance and necessity to improve the teaching quality in colleges/universities, it is critical to establish a suitable assessment mechanism in the teaching process. Then, the evaluation information for the teaching quality of teachers implies incompleteness, indeterminacy, and inconsistency due to the indeterminacy and inconsistency of human cognition and judgements to the evaluated objects. In this case, this research proposed an MADM technique for the TQA of teachers in the SvNN situation. To perform this task, we proposed the operational laws of TT-SvNNs and the TT-SvNNWA and TT-SvNNWG operators, and then established an MADM technique using the TT-SvNNWA and TT-SvNNWG operators in the SvNS setting. Consequently, the established MADM technique was applied in an actual MADM example about the TQA problem of teachers and compared with existing related MADM techniques. The comparative results revealed the efficiency and suitability of the established MADM technique in the SvNS setting.

However, the established MADM technique is another complement to existing MADM techniques. Since the tangent trigonometric function shows the main superiority of its periodicity and symmetry about the origin, fitting the preference of decision makers for multiple time phase parameters, this new technique will also be extended to new aggregation operations and applied to the areas of slope stability/risk assessment and medical diagnosis in the environment of simplified NSs (SvNSs and INSs).

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