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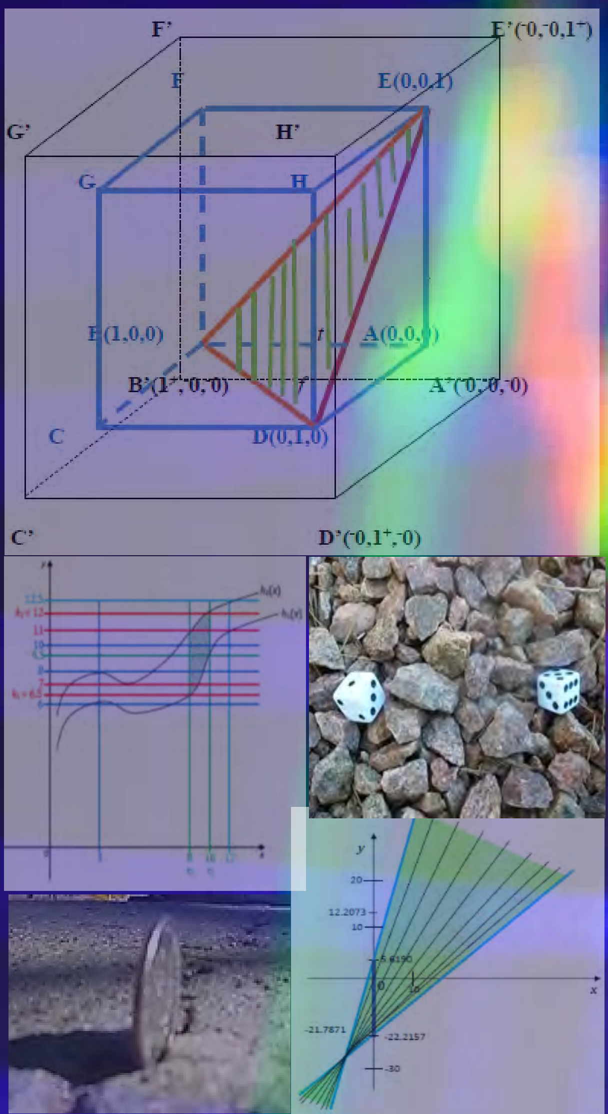

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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A Neutrosophic Evaluation Method of Engineering Certification Teaching Effect Based on Improved Entropy Optimization Model and Its Application in Student Clustering

Jingyuan Li [1], Fangwei Zhang [1], Jun Ye [2, 3, *], Yuanhong Liu [1], and Jianbiao Hu [4]

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Abstract: To realize the all-around assessment of teaching quality in the context of engineering education accreditation, this study proposes a single-valued neutrosophic information entropy and a novel assessment method of teaching effectiveness. In this proposed assessment model, an optimization model is structured based on the minimum information entropy value of single-valued neutrosophic sets (SvNSs). In this study, the primary innovation is that by using the structured model, the attribute weights are extracted from the given deterministic information. The main work of this study is summarized as follows. Firstly, aiming at the engineering certification problem, this study proposes an improved single-value neutrosophic information entropy formula to enhance the effectiveness of the engineering certification results. Secondly, this paper establishes an optimization model based on the minimum information entropy value and provides a method for assigning weight values. Thirdly, this study proposes an improved evaluation method for teaching effect, which provides a novel idea for engineering certification. Thereafter, a case study is presented to demonstrate the effectiveness and practicality of the proposed model.

Keywords: single-valued neutrosophic set; single-valued neutrosophic information entropy; engineering education certification; student clustering

1. Introduction

Engineering accreditation serves as a globally recognized system for ensuring the quality of engineering education, and is essential for achieving international recognition of both engineering education and qualifications. The central aspect of engineering certification is to ensure that engineering graduates meet industry-recognized standards [1]. In 2016, China formally joined the Washington Agreement, introduced advanced results-oriented education concepts, and promoted the certification of undergraduate engineering education [2]. However, the current evaluation of teaching quality is mainly the evaluation of course achievement degrees, which predominantly test the student's mastery of the course content. The evaluation method is single and subjective, and the evaluation of graduation requirement achievement is not comprehensive. In addition, assessment methods to be met for graduation requirements have some problems, such as difficulty in quantifying graduation requirements and unscientific assessment methods. Therefore, aiming at the fuzzy and uncertain information in the teaching evaluation process, this paper gives an improved information entropy function based on the theory of neutrosophic sets. Three evaluation indexes are used to comprehensively assess the teaching effectiveness, which provides a novel method for university engineering accreditation.

The accreditation of engineering education programs began in the 1930s as a form of professional certification, and has become the most influential certification system in the world. To evaluate whether students have met the graduation requirements, scholars typically use the 12 graduation requirements outlined in the Washington Accord as a benchmark. Khan et al. [3] took the civil engineering major of King Saudi University as an example. They mapped the students' learning achievements one by one and established a mapping table of learning achievements and curriculum achievements. Jiao et al. [4] established 12 graduation requirements according to the training objectives of surveying and mapping engineering professionals in their colleges and universities and decomposed each graduation requirement into lots of evaluation indicators. One approach used in their study was to calculate the achievement degree of graduation requirements as the minimum value among all the evaluation values of the indicators. Qu and Fan [5] constructed an evaluation index system based on the network analytic hierarchy process from four aspects: teaching links, teacher quality, teaching resources, and teaching preparation.

Smarandache proposed the concept of neutrosophic sets in 1998 [6]. Neutrosophic sets consist of truth-membership degrees, indeterminacy-membership degrees, and falsity-membership degrees, which can more clearly express uncertain and inconsistent information. After that, neutrosophic sets have received the extensive attention and research of scholars. Ye [7] developed the notion of simplified neutrosophic sets, and provided their set relationships and rules for operations. Then, single-valued neutrosophic sets (SvNSs) are also widely used in group decision-making problems involving multiple attributes. Ye et al. [8] put forward the distance formula of SvNSs and gave a novel approach for multi-attribute decision-making based on SvNSs. Aydogdu [9] gave the measurement formula of entropy and similarity. Many scholars also combined the neutrosophic set with a variety of traditional multiple criteria decision-making methods. Peng et al. [10] introduced new operations for simplified neutrosophic numbers and devised a comparison approach based on the existing research on intuitionistic fuzzy numbers, and then applied them to the medical field. Peng et al. [11] proposed a novel approach to tackle multi-criteria group decision-making problems where weight information is unknown, and evaluation values are expressed as probability multi-valued neutrosophic numbers. They successfully applied this method to solve the vendor selection problem. Ye [12] introduced a novel method for measuring the vector similarity between simplified neutrosophic sets and implemented it in the domain of investment and risk management.

In dealing with process on uncertain information, information measurement is a very important content, which has attracted the extensive attention of scholars. Tan et al. [13] introduced the notion of hesitant fuzzy index entropy, which enabled the construction of a multi-attribute decision-making model based on an entropy weight method. Wei et al. [14] introduced a new model for hesitant fuzzy entropy that is based on the mean and variance of hesitation fuzzy elements. Hu et al. [15] proposed an alternative hesitation fuzzy entropy model that is derived from the perspective of hesitation fuzzy similarity. Xu et al. [16] proposed several measurement formulae for fuzzy entropy and cross entropy of hesitation fuzzy sets. Liang et al. [17] constructed a model for multi-attribute decision-making that utilized the scoring function and minimum relative entropy principle. In the field of fuzzy decision-making, Vlachos and Sergiadis [18] presented a discrimination information method for intuitionistic fuzzy sets and introduced the concept of cross entropy. Liu et al. [19] proposed three novel formulae for probability hesitation fuzzy entropy and provided axiomatic definitions for these measures. They applied these measures to solve multi-attribute decision-making problems.

To introduce the work efficiently, the following sections of this study are organized as follows. Section 2 provides a review of concepts related to SvNSs and introduces an improved single-valued neutrosophic information entropy function. Section 3 presents an evaluation model based on single-valued neutrosophic information entropy and explains how the evaluation mechanism is optimized. Section 4 verifies the feasibility of the presented model through an example. Section 5 summarizes the full text. The specific research framework is shown in Figure 1.

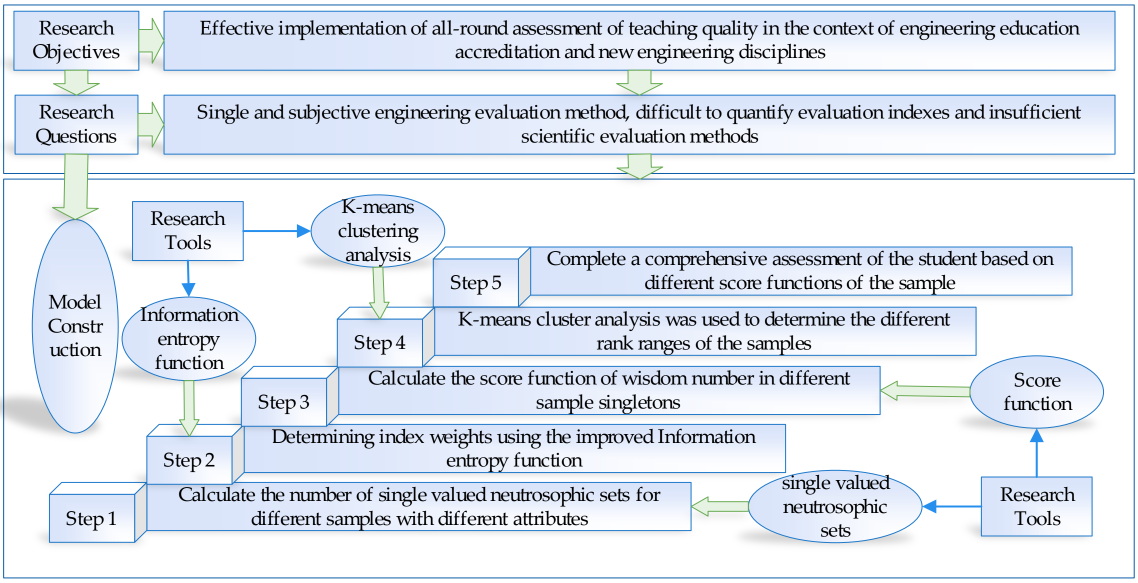

Figure 1. Research framework

2. Preliminary Knowledge

In this section, the preliminary concepts on SvNSs and single-valued neutrosophic information entropy are introduced.

2.1. Concepts on SvNSs

Definition 1. If X is a universe set and x is any one element in X, then the SvNS A in X is expressed as $A = \{< x, \alpha_1(x),\ \alpha_2(x),\ \alpha_3(x) > | x \in X\}$ [20], where $\alpha_1(x),\ \alpha_2(x),\ \alpha_3(x) \in [0,\ 1]$ are the truth-membership degree, the indeterminacy-membership degree, and the falsity-membership degree, respectively. Then, the element $< x, \alpha_1(x),\ \alpha_2(x),\ \alpha_3(x) >$ in A is simply denoted as the single-valued neutrosophic number (SvNN) $a =< \alpha_1, \alpha_2, \alpha_3 >$.

Definition 2. If $a_1 =< \alpha_1^1, \alpha_2^1, \alpha_3^1 >$ and $a_2 =< \alpha_1^2, \alpha_2^2, \alpha_3^2 >$ are two SvNNs, then there are the following algorithms [21]:

(1) $a_1 \cup a_2 = \left\langle \max\left(\alpha_1^1, \alpha_1^2\right), \min\left(\alpha_2^1, \alpha_2^2\right), \min\left(\alpha_3^1, \alpha_3^2\right) \right\rangle$;

(2) $a_1 \cap a_2 = \left\langle \min\left(\alpha_1^1, \alpha_1^2\right), \max\left(\alpha_2^1, \alpha_2^2\right), \max\left(\alpha_3^1, \alpha_3^2\right) \right\rangle$;

(3) $a_1 \oplus a_2 = \left\langle \alpha_1^1 + \alpha_1^2 - \alpha_1^1, \alpha_1^2, \alpha_2^1\alpha_2^2, \alpha_3^1\alpha_3^2 \right\rangle$;

(4) $a_1 \otimes a_2 = \left\langle \alpha_1^1\alpha_1^2, \alpha_2^1 + \alpha_2^2 - \alpha_2^1, \alpha_2^2, \alpha_3^1 + \alpha_3^2 - \alpha_3^1\alpha_3^2 \right\rangle$;

(5) $\lambda a_1 = \left\langle 1 - \left(1 - \alpha_1^1\right)^{\lambda}, (\alpha_2^1)^{\lambda}, (\alpha_3^1)^{\lambda} \right\rangle, \lambda > 0$;

(6) $a_1^{\lambda} ⊚ \left\langle (\alpha_1^1)^{\lambda}, 1-\left(1-\alpha_2^1\right)^{\lambda}, 1-\left(1-\alpha_3^1\right)^{\lambda} \right\rangle, \lambda > 0$;

(7) The complement of a_1 is $a_1^c = \left\langle \alpha_3^1, 1-\alpha_2^1, \alpha_1^1 \right\rangle$.

2.2 Single-Valued Neutrosophic Information Entropy

Definition 3. Inspired by the classical hesitant fuzzy entropy function [22] , this study proposes the information entropy function of the SvNN. Let $a =< \alpha_1, \alpha_2, \alpha_3 >$ be SvNN. According to the trigonometric functions, the information entropy function of the SvNN a is represented as

$$E_1(a) = \frac{1}{3}\sum_{t=1}^{3} \frac{\left(\sqrt{2}\sin\frac{\left|\alpha_t - (\alpha_t)^c\right| ⊚ 1}{4}\pi + \sqrt{2}\cos\frac{\left|\alpha_t - (\alpha_t)^c\right| ⊚ 1}{4}\pi - \sqrt{2}\right)}{2-\sqrt{2}} \quad . \tag{1}$$

Definition 4. Let $a =< \alpha_1, \alpha_2, \alpha_3 >$ be SvNN, then the single-valued neutrosophic information entropy function $E_1(a)$ satisfies the following properties:

(E1) When $\alpha_t \in [0, 0.5]$ for $t = 1,2,3,$ the function $E_1(a)$ is a monotonic increasing function. When $\alpha_t \in [0.5, 1]$ for $t = 1,2,3,$ the function $E_1(a)$ is a monotonic decreasing function.

(E2) When $a =< \alpha_1, \alpha_2, \alpha_3 > \; = \; < 0.5, 0.5, 0.5 >$, the function $E_1(a)$ has the maximum value.

(E3) $E_1(a) = 0$ iff a is a crisp set.

(E4) $E_1(a) = E_1(a^c)$.

Proof:

First, the following functions are constructed:

$$f(x) = \left(\sqrt{2}\sin\frac{x}{4}\pi + \sqrt{2}\cos\frac{x}{4}\pi - \sqrt{2}\right), x \in [1,2]. \tag{2}$$

Then

$$f'(x) = \frac{df(x)}{dx} = \frac{\pi}{2\sqrt{2}}\left(\cos\frac{x}{4}\pi - \sin\frac{x}{4}\pi\right), x \in [1,2]. \tag{3}$$

$$f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{-\pi^2}{8\sqrt{2}}\left(\sin\frac{x}{4}\pi + \cos\frac{x}{4}\pi\right), x \in [1,2]. \tag{4}$$

Here $x = |a_t - (a_t)^c| + 1$ for $t = 1, 2, 3$. When $x \in [1,2]$, $f'(x)$ is always less than or equal to 0, then $f(x)$ is a monotone decreasing function. Therefore, if $x = 1$, $|a_t - (a_t)^c| = 0,$ and $a_t = 0.5$, $f(x)$ takes the maximum value $f_{max}(x) = 2-\sqrt{2}$ and $E_1(a) = 1$. If and only if $x = 2$ and $|a_t - (a_t)^c| = 1$, it is clear that $a_t = 0$ or 1. Then $f(x)$ takes the minimum value $f_{min}(x) = 0$, i.e, $E_1(a) = 0$. When the $\alpha_t \in [0, 0.5]$ for $t = 1,2,3,$ the function $E_1(a)$ is a monotonic increasing function. When the $\alpha_t \in [0.5, 1]$ for $t = 1,2,3,$ the function $E(a)$ is a monotonic decreasing function. Therefore, (E1) is obtained.

When $x = 1$ and $a = a^c = \; < 0.5, 0.5, 0.5 >$, there is $|a_t - (a_t)^c| = 0$ $(t = 1,2,3)$. Then, the information entropy $E_1(a)$ has the maximum value. Therefore, (E2) is obtained.

When a is a crisp set, i.e., $|a_t - (a_t)^c| = 0$ $(t = 1,2,3)$ and $x = 2$. It implies that either $a = <1, 0, 0>$ and $a^c =< 0,1,1 >$ or $a = \; < 0,0,1 >$ and $a^c = \; < 1,1,0 >$, then the information entropy value $E_1(a) = 0$. Obviously, it also gets $|a_t - (a_t)^c| = |(a_t)^c - a_t|$ and $E_1(a) = E_1(a^c)$. Therefore, (E3) and (E4) are obtained.

3. Model Construction

To improve the evaluation mechanism of colleges and universities, this study proposes three evaluation indicators to achieve a comprehensive evaluation of students. In the process of evaluation, the graduation requirements of different majors are divided into several indicators, and curriculum objectives are set for the indicators of graduation requirements. Each curriculum objective corresponds to different graduation indicators. Teachers set exam questions for different graduation indicators and course objectives. For the sake of clarity, we use an evaluation sample as an illustration of the above correspondence. The corresponding relationship of the sample evaluation bases is shown in Table 1.

In this study, the comprehensive evaluation of college students is regarded as a multi-attribute evaluation problem, and all the students in the class are regarded as the evaluation set $X = \{x_1, x_2, \cdots, x_m\}$. Then, $C = \{c_1, c_2, \cdots, c_n\}$ is denoted as an attribute indicator set. Through the three main attribute indicators, we evaluate the comprehensive quality of students, including the overall fulfillment level of the graduation requirements c_1, the achievement degree of curriculum objectives c_2 and the overall achievement degree c_3. Therefore, the attribute indicator set is denoted as $C = \{c_1, c_2, c_3\}$, and then the weight vector of the attribute indicator set is specified as $\omega_j = \{\omega_1, \omega_2, \omega_3\}$, where $0 \leq \omega_j \leq 1$, and $\sum_{j=1}^{3} \omega_j = 1$. The school evaluates each student according to the SvNN, the SvNNs of different students form the SvNN decision matrix $D = (a_{ij})_{m\times n}$. Since the attribute indicators in this evaluation problem are all benefit-based attributes and have no cost-based attributes, it is unnecessary to normalize the matrix D. The evaluation process of the teaching effect is as follows.

Step 1: Calculate the SvNN of different samples. $O_i (i = 1, 2, \ldots, k)$ is denoted as the original score, G_i is denoted as the achievement degree of each assessment basis, and G_i is named the total achievement degree of the curriculum graduation requirement index. R_i is denoted as the assessment result, W_i is denoted as the index weight of the corresponding graduation requirements, A_i is named the average score of each major question, and l is denoted as the index point of the graduation requirements. M_i is denoted as the sub-item weight of the curriculum objectives. Thus, $R_i = W_i \cdot A_j$. Then, the truth-membership degrees of c_1 and c_2 are defined as

$$\alpha_1^{c_1} = \frac{1}{n}\sum_{i=1}^{k}\frac{R_i}{O_i}, \quad \sum_{i=1}^{k} O_i = 100, \tag{5}$$

$$\alpha_1^{c_2} = \frac{1}{l}\sum_{j=1}^{l}\sum_{i=j}^{j+1}\frac{O_i G_i}{O_i + O_{i+1}}. \tag{6}$$

The assessment results of this study are divided into final exam scores and usual performance. The weight of the final exam scores is 70%, while the weight of the usual performance is 30%. The average score of the usual performance is recorded as A_i. Then, the truth-membership degree of c_3 is recorded as

$$\alpha_1^{c_3} = \frac{1}{100}\left(\sum_{i=1}^{k} R_i \cdot 0.7 + \sum_{i=1}^{k} A_{si} \cdot 0.3\right). \tag{7}$$

The falsity-membership degrees of c_1, c_2 and c_3 are defined as

$$\alpha_3^{c_1} = 1 - \frac{1}{k}\sum_{i=1}^{k}\frac{W_i}{G_i}, \tag{8}$$

$$\alpha_3^{c_2} = 1 - \frac{1}{100}\sum_{i=1}^{k} R_i \cdot M_i, \tag{9}$$

$$\alpha_3^{c_3} = 1 - \frac{1}{100}\left(\sum_{i=1}^{k} R_i - \sum_{i=1}^{k} R_i W_i\right). \tag{10}$$

The indeterminacy-membership degrees of c_1, c_2 and c_3 are defined as

$$\alpha_2^{c_1} = \cos\left[\frac{\pi}{2}\cos\left(\frac{\pi(\alpha_1^{c_1} - \alpha_3^{c_1})}{2}\right)\right], \tag{11}$$

$$\alpha_2^{c_2} = \cos\left[\frac{\pi}{2}\cos\left(\frac{\pi(\alpha_1^{c_2} - \alpha_3^{c_2})}{2}\right)\right], \tag{12}$$

$$\alpha_2^{c_3} = \cos\left[\frac{\pi}{2}\cos\left(\frac{\pi(\alpha_1^{c_3} - \alpha_3^{c_3})}{2}\right)\right]. \tag{13}$$

Therefore, the corresponding SvNNs of different students are obtained.

Table 1. Corresponding relationship of sample evaluation bases

Course objective	Indicator of graduation requirements		Assessment basis			
	Index point of graduation requirements	**Weight**	**Basis**	**Score**	**Examination result : Usual result**	**Achievement degree of the assessment basis**
Course objective 1	1-4	100%	First question	10	40% : 60%	0.71
	8-3	40%	Third question	28		0.86
Course objective 2	4-3	50%	Second question	10		0.5
	5-1	40%	Third question	28		0.86
Course objective 3	4-1	50%	Second question	10		0.5
	11-1	20%	Third question	14		0.86

Step 2: Determine the index weight. To enhance the rationality of the evaluation process, each index weight is determined by calculating neutrosophic entropy. To maximize the reliance of the evaluation process on the available information, the objective function is established by determining the indicator weight. It uses the minimum entropy value of neutrosophic information to ensure that the evaluation process relies on the most determinate information possibility. The number of evaluation objects is m. The information entropy function $E(c_j), j = \{1, 2, 3\}$ of each attribute is defined as

$$E(c_j) = \frac{1}{m}\sum_{i=1}^{m} E_1(a_{ij}),\ j = 1, 2, 3, \tag{14}$$

where a_{ij} is defined as the SvNN of the j-th indicator on the i-th assessment object and $E_1(a_{ij})$ is the information entropy function of the SvNN a_{ij}.

To reduce the influence of uncertain information on the evaluation outcomes, we establish an optimization model with the goal function J of minimizing the entropy value. According to Eq. (1), the optimization model is defined as

$$\min J ≜ \sum_{j=1}^{3} E(c_j)\omega_j$$
$$s.t.\ \begin{cases} \sum_{j=1}^{3}\omega_j = 1, \omega_j \geq 0. \\ 0 \leq E(c_j) \leq 1 \end{cases} \tag{15}$$

The optimization model is solved by the Lagrange multiplier method, and then the weight of each attribute is obtained finally.

Step 3: Calculate the score function of the SvNN [23]. The score function of the SvNN $a_{ij} = < \alpha_1^{ij}, \alpha_2^{ij}, \alpha_3^{ij} >$ is denoted as $S(a_{ij})$ for $S(a_{ij}) \in [0,100]$ and defined as follows:

$$S(a_{ij}) = 50\sqrt{2}\cdot \mathrm{sgn}(\alpha_1^{ij} - \alpha_2^{ij} + \left(\frac{\alpha_1^{ij} - \alpha_2^{ij}}{\alpha_1^{ij} + \alpha_2^{ij}}\right)\alpha_3^{ij})\sqrt{\left|\alpha_1^{ij} - \alpha_2^{ij} + \left(\frac{\alpha_1^{ij} - \alpha_2^{ij}}{\alpha_1^{ij} + \alpha_2^{ij}}\right)\alpha_3^{ij}\right|}. \tag{16}$$

For convenience, set $\alpha_1^{ij} - \alpha_2^{ij} + (\alpha_1^{ij} - \alpha_2^{ij})\alpha_3^{ij}/(\alpha_1^{ij} + \alpha_2^{ij}) = x$. Then, the score function is defined as

$$S(\alpha_{ij}) = 50\sqrt{2}\cdot \mathrm{sgn}(x)\sqrt{|x|},$$

where $\mathrm{sgn}(x)$ is a symbolic function, which is defined as

$$\mathrm{sgn}(x) = \begin{cases} 1, x \neq 0 \\ 0, x = 0 \end{cases}.$$

According to Eq. (16), the comprehensive ability evaluation matrix R of students is indicated by

$$R = \begin{bmatrix} S(a_{11}) & S(a_{12}) & S(a_{13}) & \ldots & S(a_{1m}) \\ S(a_{21}) & S(a_{22}) & S(a_{23}) & \ldots & S(a_{2m}) \\ S(a_{31}) & S(a_{32}) & S(a_{33}) & \ldots & S(a_{3m}) \end{bmatrix}^T.$$

Then, the evaluation result vector D is defined as

$$D = R\cdot\omega_j ≜ (d_1, d_2, d_3, \cdots, d_m). \tag{17}$$

Step 4: Determine the value range of the evaluation grade. To determine the levels of different evaluation indicators, this study employs clustering analysis to sample data. In this situation, students are divided into five grades according to different indicators. The evaluation grades are set as $V = \{v_1, v_2, v_3, v_4, v_5\}$. The five grades correspond to five evaluations: excellent, good, fair, poor, and very poor. Then, the K-means clustering analysis method [24] is adopted in this study, and the specific calculation method is given as follows.

First, five sample data are randomly selected as the initial cluster center $k_i (i = 1, 2, 3, 4, 5)$ in the evaluation matrix. Each element in the evaluation matrix is recorded as t_m. The distance d_i from the element t_m in row m to the cluster center k_i is marked as

$$d_i(t_m, k_i) ≜ \sqrt{(t_{1m} - k_{i1})^2 + (t_{2m} - k_{i2})^2 + (t_{3m} - s_{i3})^2}. \tag{18}$$

Once the distance between each sample and each cluster center is calculated, allocate each sample to the nearest cluster center category. After all samples are allocated, recalculate the positions of the five cluster centers. The calculation formula of cluster centers is defined as

$$k_i = \frac{1}{N}\sum_{i=1}^{N} d_i \,, \tag{19}$$

where k_i is named the new cluster center, d_i is named the the distance from the sample in the cluster center to the cluster center, and the number of samples belonging to the cluster center is denoted as N. Once the new cluster center is obtained, the distance between each sample and the new cluster center is calculated again. The process of recalculating the new cluster center and the distance is repeated until the cluster centers no longer change significantly or reach a predetermined convergence criterion. The final position of the cluster center is the critical point of the evaluation grade.

Step 5: The Euclidean distance between each sample point and the five cluster centers is calculated by using Equation (16). The cluster center with the smallest distance from the sample point is the cluster of the category to which the sample belongs. Based on the aforementioned five steps, the neutrosophic assessment of engineering certification teaching effect is realized.

4. Case Analysis

This section presents an example to demonstrate the model described above. We select 10 students as the evaluation set $X = \{x_1, x_2, \cdots, x_{10}\}$. Three evaluation indicators are selected as the attribute indicator set $C = \{c_1, c_2, c_3\}$. Score registration form is shown in Table 2. The evaluation process is addressed as follows.

Step 1: Calculate SvNNs of different samples. According to Eqs. (5)-(7), we obtain the truth-membership values of $\alpha_1^{c_1}$, $\alpha_1^{c_2}$, and $\alpha_1^{c_3}$. According to Eqs. (8)-(10), we get the falsity-membership values of $\alpha_3^{c_1}$, $\alpha_3^{c_2}$, and $\alpha_3^{c_3}$. According to Eqs. (11)-(13), we obtain that the indeterminacy-membership value of $\alpha_2^{c_1}$, $\alpha_2^{c_2}$, and $\alpha_2^{c_3}$. The course achievement rating scale is shown in Table 3. Then, the neutrosophic set matrix A of 10 samples is

$$A = \begin{bmatrix} <0.773,0.057,0.601> & <0.778,0.164,0.483> & <0.847,0.081,0.642> \\ \circledcirc 0.776,0.063,0.596 \circledcirc & <0.775,0.160,0.485> & <0.857,0.089,0.642> \\ <0.807,0.096,0.583> & <0.808,0.172,0.507> & <0.887,0.129,0.627> \\ <0.816,0.107,0.580> & <0.819,0.177,0.513> & <0.900,0.152,0.623> \\ <0.867,0.186,0.553> & <0.875,0.209,0.541> & <0.947,0.229,0.597> \\ <0.879,0.207,0.547> & <0.882,0.212,0.546> & <0.959,0.250,0.593> \\ <0.880,0.204,0.550> & <0.884,0.217,0.544> & <0.961,0.254,0.030> \\ <0.883,0.210,0.546> & <0.884,0.212,0.548> & <0.973,0.270,0.592> \\ <0.890,0.223,0.545> & <0.893,0.220,0.551> & <0.992,0.301,0.588> \\ <0.880,0.217,0.540> & <0.882,0.212,0.546> & <0.990,0.291,0.593> \end{bmatrix}.$$

Step 2: The information entropy functions are calculated according to Eqs. (1) and (14). We get $E(c_1) = 0.688$, $E(c_2) = 0.701$, and $E(c_3) = 0.572$. When the calculation results are substituted into the Eq. (15), the index weight results are obtained as follows:

$$\omega = (0.2764, 0.2593, 0.4643)^T.$$

Step 3: According to Eq. (16), we obtain that the score function values of SvNNs, and then the evaluation matrix R is obtained as follows:

$$R \circledcirc \begin{bmatrix} 78.56 & 78.09 & 76.49 & 75.98 & 72.06 & 71.08 & 71.38 & 70.78 & 70.48 & 70.33 \\ 68.15 & 68.34 & 69.46 & 69.74 & 70.65 & 70.87 & 70.59 & 70.99 & 70.92 & 70.85 \\ 80.50 & 80.29 & 78.29 & 77.17 & 73.57 & 72.69 & 72.51 & 72.03 & 70.89 & 73.57 \end{bmatrix}^T.$$

According to Eq. (17), the evaluation result vector D is obtained as follows:

$$D = R \cdot \omega = (75.76, 76.58, 75.50, 74.91, 72.40, 71.77, 71.69, 71.41, 71.97)^T.$$

Step 4: Determine the value range of the evaluation grade. Students are divided into five grades according to different indicator values. The set of evaluation grades is represented as $V = \{v_1, v_2, v_3, v_4, v_5\}$. The five grades $v_1, v_2, \ldots, v_5$ correspond to five evaluations: excellent, good, general, poor, and very poor. In order to better present the clustering effect, 300 SvNNs are selected for clustering analysis. The clustering results are shown in Figures 2, 3, 4. By iterating through Eq. (18) and Eq. (19), sample clustering is achieved. The clustering result of SvNFNs is shown in Table 4, and then the final evaluation results are shown in Figure 5.

Step 5: Eq. (16) is used to calculate the score of cluster center points, and then the results are given as follows:

$$S_{v_1} = 67.91, S_{v_2} = 69.28, S_{v_3} = 69.51, S_{v_4} = 72.3, \text{ and } S_{v_5} = 72.69.$$

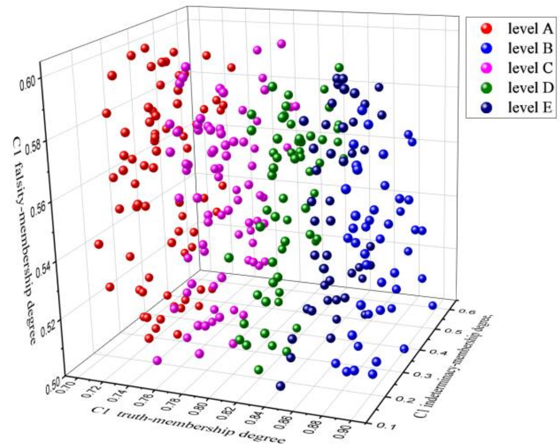

Figure 2. Clustering results of the attribute c_1

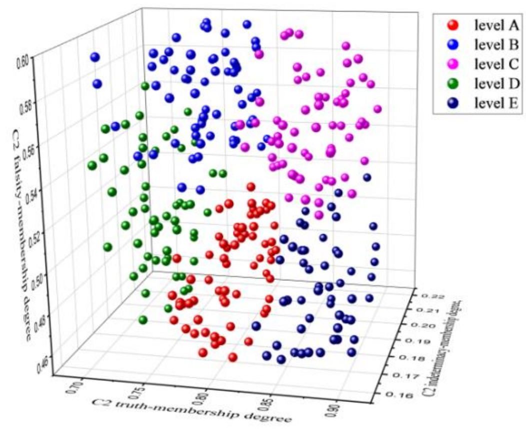

Figure 3. Clustering results of the attribute c_2

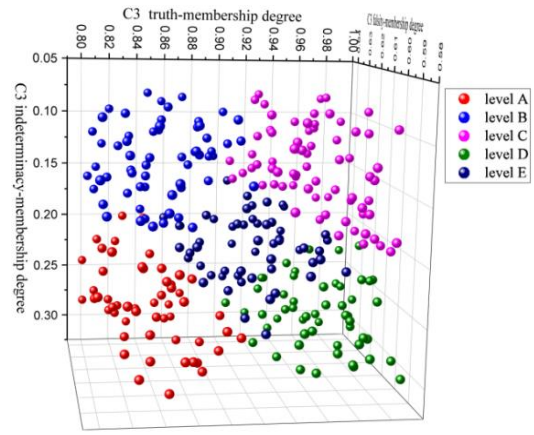

Figure 4. Clustering results of the attribute c_3

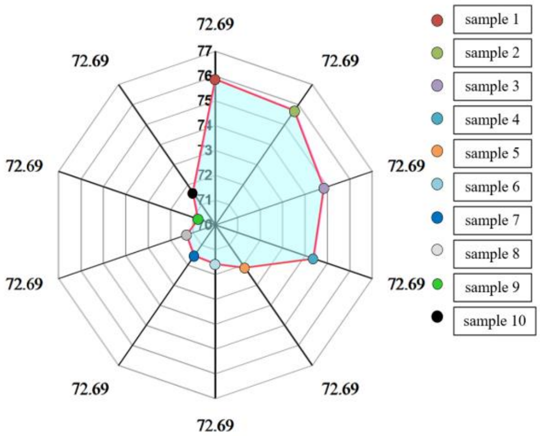

Figure 5. Comparison of the sample results

We calculate the distance between the score function of each sample point and the score function of each cluster center point based on Eq. (18). The shortest distance value is used as the category determination criterion so as to achieve clustering and evaluation of different samples. The maximum score function value of randomly selected sample points is 72.69. The larger the sample point value, the closer to the maximum value. The evaluation results indicate that the samples 1, 2, 3, 4 belong to the grade v_1, the samples 5, 6, 7, 8, 10 belong to the grade v_2, and the sample 9 belongs to the grade v_3. Finally, the comprehensive evaluation of students is completed.

Table 2. Score registration form

Sample		1	2	3	4	5	6	7	8	9	10
Course	Question1	5.672	5.698	6.703	6.749	6.823	6.697	7.031	7.092	5.642	5.643
objective 1	Question2	5.667	5.693	6.078	6.234	6.476	6.897	6.903	7.112	6.345	6.345
Course	Question 3	5.612	5.758	5.972	6.284	6.172	6.223	6.012	5.685	5.712	5.63
objective 2	Question 4	6.742	6.983	6.982	6.832	7.932	7.842	7.844	7.761	7.851	7.851
Course	Question 3	5.923	5.783	6.234	6.756	6.823	6.881	7.516	7.722	7.541	7.341
objective 3	Question 1	9.629	9.742	9.923	9.472	9.827	10.639	11.923	10.963	11.685	11.477
Course	Question 2	8.645	8.923	8.912	9.042	9.142	9.972	9.662	10.294	10.354	10.354
objective 4	Question 5	5.424	4.623	4.816	4.923	5.823	5.743	5.623	5.256	5.864	5.625
Course	Question 4	12.735	13.803	12.995	13.823	14.886	12.953	13.843	14.263	14.852	14.867
objective 5	Question 5	11.983	11.472	12.727	12.043	13.862	14.512	11.482	12.345	14.872	14.532
Peacetime performance		97.832	97.992	98.872	98.982	99.123	99.623	99.113	99.276	97.872	97.525

Table 3. Course achievement rating scale

Basis	Score (percentage system)	Score (original)	Assessment result (original score)	Achievement degree of each assessment basis	Sub-item weight of course objectives	Achievement of course objectives by item	Sub-item weight of index points required by course graduation
Question 1	5.6	8	6.375	0.8	50%	0.797	33%
Question 2	5.6	8	6.375	0.8	50%		33%
Question 3	5.6	8	5.906	0.74	50%	0.836	33%
Question 4	5.6	8	7.462	0.93	50%		50%
Question 3	5.6	8	6.852	0.86	40%	0.869	50%
Question 1	8.4	12	10.528	0.88	60%		40%
Question 2	8.4	12	9.53	0.79	66.70%	0.828	40%
Question 5	4.2	6	5.372	0.9	33.30%		20%
Question 4	10.5	15	13.902	0.93	50%	0.896	50%
Question 5	10.5	15	12.983	0.87	50%		50%

Table 4. Student clustering results

Attribute	Iteration	Membership degree	Level A	Level B	Level C	Level D	Level E
c_1	10	Truth-membership degree	0.734	0.799	0.852	0.755	0.868
		Indeterminacy-membership degree	0.193	0.188	0.196	0.189	0.189
		Falsity-membership degree	0.495	0.486	0.557	0.567	0.479
c_2	15	Truth-membership degree	0.791	0.768	0.855	0.725	0.866
		Indeterminacy-membership degree	0.188	0.19	0.197	0.191	0.189
		Falsity-membership degree	0.483	0.57	0.553	0.51	0.478
c_3	10	Truth-membership degree	0.843	0.941	0.937	0.95	0.846
		Indeterminacy-membership degree	0.134	0.113	0.186	0.266	0.248
		Falsity-membership degree	0.612	0.614	0.61	0.611	0.61

5. Conclusions

According to the introduced case, the primary contributions of this study are as follows.

Firstly, three indicators of students' performance were obtained as the achievement of graduation requirements, the achievement of course objectives, and the achievement of grades. By using the three indicators, a kind of comprehensive assessment method for teaching effectiveness was proposed in SvNN setting.

Secondly, we proposed an improved single-valued neutrosophic information entropy function and an optimization model for determining the weights of the three indicators. However, the weights of the three indicators depend entirely on the objective information given, without any subjective information.

Thirdly, we proposed an improved evaluation method for teaching effect, which realized the comprehensive evaluation and cluster of students. This approach offered a novel idea for engineering certification.

The proposed effect of this study is that students can adjust their learning programs according to their achievement of graduation requirements and promote their overall development. Based on the achievement of the indicators, teaching managers can assess the rationality of the training program and the pedagogical effectiveness of the course so as to further improve the student training system. At present, there are some limitations to this study. The selection of the initial cluster center can affect the clustering results during the clustering process. Therefore, in future research, the clustering method for students needs to be further improved.

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Analysis of Teaching-Learning Efficiency Using Attribute Based Double Bounded Rough Neutrosophic Set Driven Random Forests

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Abstract
Face-on-Face interaction constitutes an integral part of the classroom atmosphere as it provides teachers with an opportunity to understand their students intimately. Hence, this study deals with attribute based double bounded rough neutrosophic set driven random forests using Gini Impurity based split to arrive at a decision regarding the teaching-learning efficiency. A mathematical model is constructed using double bounded rough neutrosophic set which is utilised to evaluate the expression of the students with the help of a real-time data by capturing the images of the students against different subjects. The decisions made are then used to fit a random forest model to establish inferences regarding the teaching-learning efficiency for different subjects. The constructed model is then validated using newly added test objects.

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Keywords: Image processing, Neutrosophic image processing, Image segmentation, DICOM image**:** ouble Bounded Rough Sets, Rough Sets, Neutrosophic Sets, Fuzzy Sets, Face Expressions, Student expression detection, Facial key points, Random forests, Decision Trees, Gini Index Impurity s.

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1. Introduction

"Actions speak louder than words"

The notion that facial expressions make-up the roads leading to several significant observations, both psychological and physical trace back to the pioneer of facial expression detection, Paul Ekman [1]. Ekman, a contemporary psychologist is known well to have established the concept of facial expressions as a universal phenomenon. In 1973, he along with Tomkins [2], Gellhorn [3], Izard [4] popularised the facial feedback theory [5] put forward by James [6], almost a century back before the theory revolutionised psychology.

The idea was further shaped and bettered by Buck [7] to pave the stepping stones towards providing much serious considerations towards facial expression detection models. The facial feedback theory suggests that an individual's experience of emotions is often influenced by feedback from their facial movements. With increasing developments in the study of the facial feedback hypothesis, researchers suggest that observations play a pivotal role in contributing towards target affective judgements and expression manipulation [8]. Hence, it can be said that the excitement behind face recognition and expression analysis has successfully initiated a crossover between facial feedback theory in psychology and expression recognition [11,12,13,14].

Drawing on these conclusions, the authors have utilised the same concepts to construct a model that'd help derive decisions from the observations a student makes while listening to a subject and making judgements to manipulate their expressions and evaluate the purity of each decision made. The proposed model would be an extension of the attribute based double-bounded rough neutrosophic set [9] system for decision making wherein a random forest [10] is utilised to establish the purity-count of every attribute against the subject for which the image was taken.

1. Double Bounded Rough Neutrosophic Set Driven Random Forest in Deriving Inferences

In this section, a mathematical model is developed by combining the Double Bounded Rough Neutrosophic Set Theory [9] and random forest techniques to arrive at the conclusion and draw inferences from the decisions made for the objects present in the approximation space.

1.1. The Double Bounded Rough Neutrosophic Set For an N-information System

Throughout this section, there is consideration of a covering-based N-information system given by $I = (U, A, F, N)$ where U defines the universe and is a non-empty finite set of objects. A signifies the non-empty finite set of fuzzy attributes defined by the mapping

$$\mu_a: U \to [0,1]\ \forall a \in A$$

while F is a function defined by the mapping $F: A \to \rho(U)\ \forall\ a \in A$ such that $F(a) \subseteq U$ would contain those elements of U that possess the attribute a with the assumption that $U\ F(a) = U, a \in A$.

Equation 2.1.1: For every $a \in A$ and $x \in X$, the evaluation of F is carried out by the equation given by

$$\alpha_{\mathrm{i}} = \frac{(\sum_{i=1}^{n}(w(a_j) * \mu_p(x_i))}{\sum_{j=1}^{k} w(a_j)}$$

$$F(a) = \{x \in X | \alpha_{\mathrm{i}} \leq \delta\}$$

Here, n denotes the number of elements in U, k denotes the number of attributes in the universe, $w(a_{\mathrm{j}})$ signifies the weights assigned to the attributes that lies in the range [0,1], $\mu(pi)$ denotes the mean of the feature points that describe the dataset, and δ signifies the threshold to group the attributes by making the choice of an appropriate interval spacing.

Algorithm 2.1.1: Consider a set X defined by $X \subseteq U$. For any $x \in X$, the neighbourhood function $N(x)$ could be defined by $U\ N(x)$ and is computed using the shortest distance by computing the mean algorithm [9] given by

1. Let s denote the object chosen and S be the set of all objects in U
2. This object is identified by the value it possesses against every parameter Pi that can be represented by t.
3. Taking the absolute difference between the t for every s against the mean of every parameter Pi will be represented by b.
4. The absolute difference between the t' for every $s' \in S$ against the mean of every parameter Pi will be represented by c.
5. Whenever the absolute difference between b and c is greater than a threshold, the count variable for every object would be incremented for every iteration.
6. The objects in s' whose count is greater than a chosen threshold given by δ' would be declared as the neighbours of the chosen object s

Both the functions F and N would be constructed from the scenario or systems under consideration with expert knowledge and interference. Sometimes, the neighbourhood function could also be devised using any relation that has been observed while studying the data in hand amongst the elements of U.

For any set P' given by $P' \subseteq A$ and fuzzy membership function μ_a, a double bounded rough set that has three distinct elements in the set namely the lower approximation, the right upper approximation and the left upper approximation can be constructed. Here, the lower approximation set would be composed of the elements in P' that have an I-relationship defined with both x and y while the right and the left upper approximations would deal with the elements in P that have an I-relationship with either x or y [19].

The subset X of U defines the double bounded rough set $DRS(a\sim X)$ as the collection of the lower approximation $DR\_(a\sim X)$, right upper approximation $DR^{-}(a\sim X)$ and left upper approximation $^{-}DR\ (a\sim X)$ of X with respect to the attribute a. Here, the lower, left upper, and right upper approximations are defined by

$$DR\_(a\sim X) = N(F(a) \cap N(x))$$

$$^{-}DR(a \sim X) = N(X) \cup (N(F(a)) \cap N(X))$$

$$DR^{-}(a \sim X) = N(F(a)) \cup \left(N(F(a)) \cap N(X)\right)$$

Conclusively, the double bounded rough set $DRS(a \sim X)$ is given by

$$DRS(a \sim X) = (DR\_(a \sim X),\ ^{-}DR\ (a \sim X), DR^{-}(a \sim X))$$

The double bounded rough set provides the definite, possible, plausible, unascertainable elements of X that possess the attribute a. It must be noted that for every $a \in A, DRS(a \sim X)$ can be achieved.

For the set P', assuming $\forall\ p\ \in P'$, define a fuzzy set μ_P given by the mapping $\mu_P: U \rightarrow [0,1]$ that provides the degree of membership of a parameter on the elements of U. Developing on the knowledge of indeterminacy, one can define a neutrosophic set for every double bounded rough set by

$$DR = \{DRS(a \sim X) |\ X \subseteq U, a \in A\}$$

$$DR\_ = \{DR\_(a \sim X| \subseteq U, a\ \in A\}$$

$$DR^{-} = \{DR^{-}(a \sim X| \subseteq U, a\ \in A\}$$

$$^{-}DR = \{\ ^{-}DR(a \sim X| \subseteq U, a\ \in A\}$$

With the definitions of the DBRS established, a fuzzy set $\mu_{\infty}: DR\_ \rightarrow [0,1]$ can be defined for the lower approximation, right upper approximation, and left upper approximation as

$$\mu\_(DR(a \sim X)) ⊚ \max\{\min(\mu_{\mathrm{p}}(x))\}\ \ x \in DR\_(a \sim X)$$

$$\mu^{-}(DR(a \sim X)) ⊚ \max\{\min(\mu_{\mathrm{p}}(x))\}\ \ x \in DR^{-}(a \sim X)$$

$$^{-}\mu(DR(a \sim X)) ⊚ \max\{\min(\mu_{\mathrm{p}}(x))\}\ \ x \in\ ^{-}DR(a \sim X)$$

Definition 2.1.1: To construct a double bounded rough neutrosophic set, one needs to handle the three fundamental functions that define any neutrosophic set namely the truth membership, the indeterminacy and the non-membership which are almost always independently related.

A neutrosophic set takes the form

$$N = \{(x, \alpha N(x), \beta N(x), \gamma N(x) | x \in X\}$$

where $\alpha N, \beta N\ and\ \gamma N$ represent the three membership functions identified by the mapping $X \rightarrow]0-, 1+[$ which possess a sum that falls in the range defined by $\rightarrow]0- \leq \alpha N(x) + \beta N(x) + \gamma N(x) \leq 3+[$.

A neutrosophic fuzzy set defined in the N information space over the relation $\bar{\mu}: DR \rightarrow [0,1]\ X\ [0,1]\ X\ [0,1]$ can be given by

$$\bar{\mu}(DRS(a{\sim}X)) = (\mu\_(DR(a{\sim}X)), \mu^{-}(DR(a{\sim}X)), {}^{-}\mu(DR(a{\sim}X)))$$

Hence, the double bounded rough neutrosophic set $\overline{\mu}(DRS(a{\sim}X))$ in an N-information system identified by $I = (U, A, F, N)$ can be utilised to evaluate any attribute $a \in A$ defined for any dataset with no requirement of a training dataset to fit the model for validation and testing and can be given by

Equation 2.1.2:

$$V(a) = 2(max\left(\left(\frac{Ta + Ia}{2}\right), \left(\frac{1 + Ia - Fa}{2}\right)\right) - min\left(\left(\frac{Ta + Ia}{2}\right), \left(\frac{1 + Ia - Fa}{2}\right)\right))$$

where $T_a\ represents\ \ \mu\_(DR(a{\sim}X))$

$I_a\ representes\ \mu^{-}\big(DR(a{\sim}X)\big)$

$F_a\ represents\ {}^{-}\mu(DR(a{\sim}X)))$

Here, $V(a)$ is computed for every $a \in A$ and the values of all the attributes for a single object constitute a list given by V. The final decision corresponding to the object would be the maximum of all the elements present in V given by:

$$V = [V(a1), V(a2), \dots , V(ak)]$$

$$decision(x) = \max(V)\ \forall x \in X$$

Here, V is a fuzzy set obtained using the attribute based double bounded rough neutrosophic set method defined by the function $\mu_{v} : V \rightarrow [0,1]$.

In a similar fashion, the decision is evaluated for every object in the universe U and the collection of decisions are defined by the set D given by

$$D = \{\max(v1)\,, \max(v2)\,, \dots, \max(vn)\}$$

which is then appended to a relational universe U' that has the same number of objects as in U and shares a link with the set U that was utilised to compute the attribute values. The relational universe U' would then be fit into a random forest for evaluating newly added objects and draw inferences

1.2.Attribute Driven Random Forest in an N-Information System

In this section, a random forest is constructed for the N-information system defined by $I' = (U\ , A')$. Here, A' denotes the set of attributes that would describe the decisions made using the double bounded rough neutrosophic set model to generate inferences regarding the decisions made and evaluate a new object being added to the information system.

Moving forward, the set of unique decisions from the set of decisions D that was constructed using the double bounded neutrosophic set model would constitute the set S that defines the state any element $x \in X$ is in at any given time t where $X \subseteq U$.

$$S = \{S_1, S_2, \dots, S_m\}$$

where m denotes the number of unique decisions in D.

The objective of this section is to predict the state of each object and evaluate any new object added in the given N-information system using Random Forest Technique ◎18◎. The construction of the individual decision trees that would be bagged and aggregated would be performed using Gini Impurity Indexing.

Equation 2.2.1: For any $a' \in A'$ and $s \in S$, the Gini Impurity is given by the expression

$$G(a''_i) = \sum_{j=1}^{n}((n_j/n) * (1 - P(a''_i|s)^2); \; i = 1,2, \dots, k$$

where n denotes the number of elements in U.

Here, $P(a_i|s)$ denotes the probability of the number of objects in U possessing the attribute a'_i with respect to the decision d. The root nodes and subsequent nodes with the best split would then be given by:

$$G_{\text{bestSplit}} = \min\big(G(f1), G(f2), \dots, G(fl)\big)$$

With respect to the information system, the process to identify the best node from the set of attributes in U by estimating the best split using Gini impurity indexing takes the form:

```
def gini_impurity (val):
    n ◎ val.sum()
    p_sum ◎ 0
    for key in val.keys():
        p_sum ◎ p_sum ◎ (val◎key◎◎n ) ◎ (val◎key◎◎n )
        gini ◎ 1 - p_sum
return gini
```

Finally, the leaf node present at the maximum depth of the tree would contain the number of states an attribute a' is in.

In a similar fashion, several decision trees may be constructed which are then bagged i.e., bootstrapped and aggregated to give rise to a random forest. This attribute based neutrosophic rough set driven random forest model can now be utilised to validate the decision of any newly added object in the relational universe U'.

2. Application of The ADBRS Driven Random Forest Model to Draw Inferences Regarding the Teaching-Learning Efficiency Using Student Expressions

Objective: To implement the attribute based double bounded rough neutrosophic set driven random forest model to analyse the facial expressions of students while in class with reference to different subjects and infer the efficiency of the teaching-learning process by evaluating the various expressions displayed by the object against the different subjects for which the images were taken and validate the decision for a newly added object.

Data: The paper utilises real-time data for analysis by capturing the photographs of students while in class against four subject periods. There were 100 images in the dataset chosen with explicit age variation present between the different images chosen so as to increase the diversity in establishing the efficiency of the teaching learning process.

***Facial Key Points*:** An online face detector and key point marker titled makesense.ai [20] was employed to evaluate the feature points for determining the attributes. Since the dataset is relatively smaller when compared against a dataset with 1000 entries, but large enough to train, validate, test, and make satisfactory predictions, manual marking of the 15 essential facial key points was employed.

The fifteen feature points were marked on the pivotal points of the face. These fifteen features were divided into their respective x and y coordinates to get 30 parameter values and would constitute the parameter set P.

fig1: A sample to indicate how the facial points were chosen

The fifteen facial feature points thus constructed are as follows:

Table1: Facial Feature Points

Point	Co-ordinates	Parameters
0	(left_eye_center_x, left_eye_center_y)	(P1,P2)
1	(right_eye_center_x, right_eye_center_y)	(P3,P4)

2	(right_eye_inner_corner_x, right_eye_inner_corner_y)	(P5,P6)
3	(right_eye_outer_corner_x,right_eye_outer_corner_y	(P7,P8)
4	(left_eye_inner_corner_x, left_eye_inner_corner_y)	(P9,P10)
5	(left_eye_outer_corner_x, left_eye_outer_corner_y)	(P11,P12)
6	(right_eyebrow_inner_x, right_eyebrow_inner_y)	(P13,P14)
7	(right_eyebrow_outer_x,right_eyebrow_outer_y)	(P15,P16)
8	(left_eyebrow_inner_x, left_eyebrow_inner_y)	(P17,P18)
9	(left_eyebrow_outer_x, left_eyebrow_outer_y)	(P19,P20)
10	(nose_tip_x,nose_tip_y)	(P21,P22)
11	(mouth_left_corner_x, mouth_left_corner_y)	(P23,P24)
12	(mouth_right_corner_x, mouth_right_corner_y)	(P25,P26)
13	(mouth_center_top_lip_x, mouth_center_top_lip_y)	(P27,P28)
14	(mouth_center_bottomlip_x, mouth_center_bottomlip_y)	(P29,P30)

The diversity of the dataset with respect to the ages of the individuals chosen could be presented through the following chart

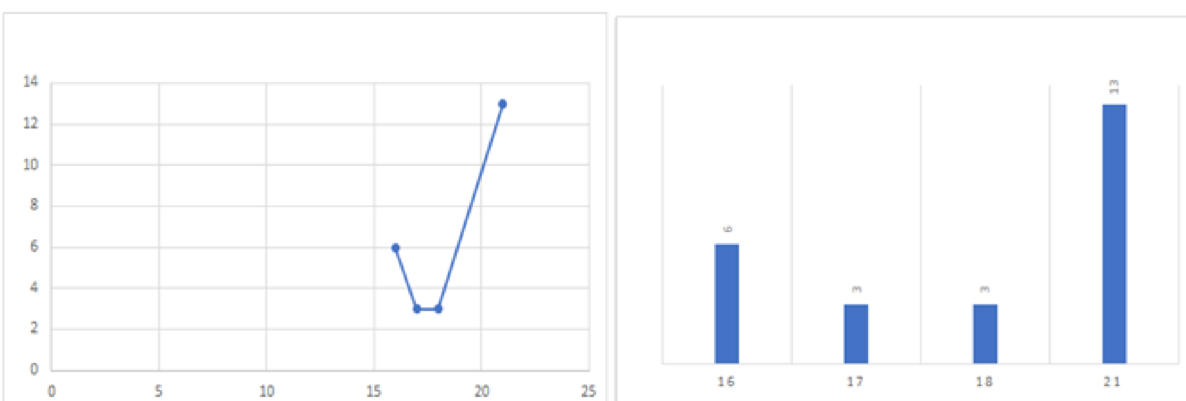


fig2: Age of the students vs. count: Linear and Bar Graph Representation

Throughout this section, there is consideration of a covering based N-information system I defined by $I = (U, A, F, N)$ where U describes the universe and is a collection of the 100 images in the dataset given by

$$U = \{0,1, \dots, 100\}$$

A is the collection of the attributes given by

$$A = \{C, B, E, S, U\}$$

where: C denotes concentrating

B denotes bored

E denotes excited

S denotes sleepy

U denotes uncertainty

The fuzzy set for A, A_{fuzzy} is given by:

$\{(\text{concentrating}, \mu_{\text{concentrating}}), (\text{bored}, \mu_{\text{bored}}), (\text{excited}, \mu_{\text{excited}}), (\text{sleepy}, \mu_{\text{sleepy}}), (\text{uncertainty}, \mu_{\text{uncertainty}})\}$

$$A_{\text{fuzzy}} = \{(C, 0.45), (B, 0.22), (E, 0.36), (S, 0.54), (U, 0.53)\}$$

Where:

$\mu_{\text{concentrating}}$ ⊚ w(concentrating) ⊚ 0.45

μ_{bored} ⊚ w(bored) ⊚ 0.22

μ_{excited} ⊚ w(excited) ⊚ 0.36

μ_{sleepy} ⊚ w(sleepy) ⊚ 0.54

$\mu_{\text{uncertainty}}$ ⊚ w(uncertainty) ⊚ 0.53

F is a function defined by the mapping $F: A \rightarrow \rho(U)\ \forall\ a \in A$ such that $F(a) \subseteq U$ would contain those elements of U that possess the attribute a with the assumption that $U\ F(a) = U, a \in A$. To estimate $F(a)$, the mean of each of the objects were evaluated following which the individual means were multiplied against the individual weights of the attributes.

The sum of the product of the object-mean and attribute weights were divided against the sum of the attribute weights. The result was compared against the threshold which is the median of the dataset chosen. To give an idea regarding the process followed in **equation 2.1.1**, an example has been provided with respect to the first object in the dataset.

Table2: Computation of F using αi for the first attribute

$\mu_p(x_1)$	**$\mu_p(x_1)$ * w(a_1)**	**$\mu_p(x_1)$ * w(a_2)**	**$\mu_p(x_1)$ * w(a_3)**	**$\mu_p(x_1)$ * w(a_4)**	**$\mu_p(x_1)$ * w(a_1)**	$\sum_{j=1}^{k}(w((a_j)) * \mu_p(x_1)))$	$\sum_{j=1}^{k} w(a_j)$	**αi**
158.82	71.475	34.94333	57.18	85.77	84.18167	333.5397	2.1	155.23

Similarly, the object-mean and attribute weight product was computed and its ratio against the sum of the attribute weights was compared against the threshold. The threshold for αi has been chosen to be the median of the dataset by dividing it into appropriate partitions according to the attribute weights.

With these definitions, the categorisation of the 100 images in the dataset has been done into the five attributes chosen namely C(concentrating), B(bored), S(sleepy), E(excited), and U(uncertainty).

Table3: F(a) vs A

a	**F(a)**
C	{0, 2, 3, 6, 13, 14, 15, 16, 17, 18, 20, 27, 39, 42, 43, 45, 46, 47, 48, 50, 54, 56, 57, 58, 59, 60, 62, 64, 67, 68, 70, 73, 75, 77, 80, 82, 89, 92, 96, 98, 99}
B	{7, 9}
E	{10, 11, 12, 23, 25, 38, 65, 72, 74, 78, 84, 87, 88}
S	{4, 5, 8, 26, 29, 30, 31, 32, 33, 35, 49, 51, 52, 53, 61, 63, 69, 76, 79, 81, 83, 85, 86, 90, 91, 93, 94, 95, 97}
U	{1, 19, 21, 22, 24, 28, 34, 36, 37, 40, 41, 44, 55, 66, 71}

N denotes the neighbourhood function and is more than often defined by the mapping $N: U \rightarrow \rho(U)$ which associates every $x \in U$ to a subset $N(x)$ and would contain the neighbours of the object x.

The neighbours of x are then evaluated using the **algorithm 2.1.1**. The value of the threshold to declare an xm as a neighbour of xn has been calculated through thorough experimentation and has been decided upon as 25. The neighbours of xn are defined by $N(x)$ which is a subset of the universal set U.

Depending on the system under study, the value of the threshold can suitably vary. To give an idea about the computation followed, there is consideration of the first ten images in the dataset.

Table4: Neighbours of xn

Object (xi)	**N(xi)**
x0	{0, 1, 6, 13, 14, 15, 16, 17, 18, 20, 27, 32, 42, 43, 48, 52, 54, 56, 57, 59, 61, 62, 63, 64, 67, 68, 69, 70, 73, 75, 93, 94, 98, 99}
x1	{0, 1, 2, 3, 5, 6, 14, 15, 16, 17, 18, 20, 21, 22, 24, 27, 28, 32, 35, 42, 43, 48, 49, 50, 51, 52, 56, 57, 59, 60, 61, 62, 64, 66, 68, 69, 70, 71, 77, 93, 94, 95, 97, 98, 99}
x2	{1, 2, 6, 14, 15, 16, 17, 18, 27, 42, 43, 48, 54, 56, 57, 58, 59, 60, 63, 64, 67, 68, 70, 75, 77, 85, 86, 90, 91, 95
x3	{1, 3, 10, 11, 12, 14, 16, 18, 21, 22, 24, 27, 28, 32, 35, 48, 49, 51, 52, 53, 56, 57, 61, 62, 64, 69, 70, 72, 93, 94, 97, 99}
x4	{4, 53, 96}
x5	{1, 5, 10, 11, 12, 19, 21, 22, 24, 28, 32, 33, 35, 36, 40, 44, 46, 49, 50, 51, 53, 60, 66, 71, 72, 77, 80, 82, 93, 94, 97, 99}
x6	{0, 1, 2, 6, 13, 14, 15, 16, 17, 18, 20, 27, 42, 43, 48, 52, 54, 56, 57, 58, 59, 60, 61, 62, 63, 64, 67, 68, 69, 70, 73, 75, 77, 85, 86, 90, 91, 93, 95, 98}
x7	{7}
x8	{8, 15, 17, 21, 22, 24, 28, 31, 45, 48, 56, 57, 60, 63, 77, 99}
x9	{9, 38, 65}
x10	{3, 5, 10, 11, 12, 19, 21, 22, 24, 28, 32, 33, 35, 49, 51, 53, 71, 72, 93, 94, 97}

Next, there is construction of the double bounded neutrosophic rough set to estimate the decisions for every $x \in U$. As the computation of 100 images may crowd the region here, an image is chosen to give an illustration of the procedure.

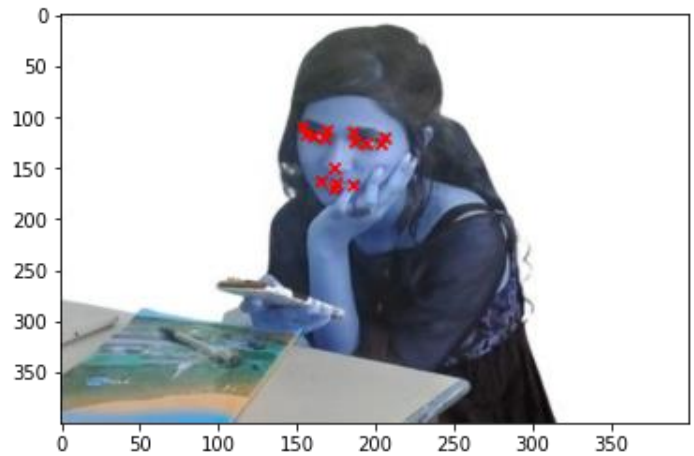


figure3: Image Considered For Evaluation

The index of the image chosen is 20. The neighbours of the chosen image, $N(20)$ are given by

$$\{0,1,6,13,14,15,16,17,18,20,27,32,35,42,43,48,51,52,$$

$$54,56,57,59,61,62,64,67,68,69,70,73,75,93,94,98,99\}$$

After construction of the double bounded rough set using the equations given in section 3.1, the neutrosophic values of the image chosen are computed using the **equation 2.1.2**.

$$V(concentrating) = 0.525$$

$$V(bored) = 0.54$$

$$V(sleepy) = 0.525$$

$$V(excited) = 0.459$$

$$V(uncertainty) = 0.495$$

Hence, the expression detected in the face is given by

$$decision(20) = \max(V(concetrating), V(bored), V(sleepy), V(excited), V(uncertainty))$$

Therefore, the expression detected is **bored.**

In a similar fashion, the expressions are detected for all the images in the dataset and the decisions made are appended to the dataset containing the scores for the subject IDs and image IDs for which the images were captured. This dataset, also known as the relational dataset, was taken from Kaggle [21].

By utilising the subject IDs, essential parameters like gender and age of the student, and the decisions made using attribute based DBRS, the following data frame was constructed.

Table5: Constructed data frame for the first ten entries

StudentId	SubjectId	Gender	Age	Marks	SubjectExpression
0	A	Female	16	72	excited
1	A	Male	16	69	excited
2	B	Female	16	72	excited
3	C	Female	16	74	bored
4	D	Female	16	97	attentive
5	A	Female	21	90	bored
6	A	Female	21	47	excited
7	B	Female	21	95	attentive
8	A	Female	21	76	uncertain
9	A	Male	16	71	attentive

The Gini Index was computed for the attributes $a' \in A'$ given by the list [‘SubjectId’, ‘Gender’, ‘Age’, ‘Marks’] against the individual student expression that was evaluated using the formula $V(a)$ and by computing the formula $decision(x) = \min(V)\ \forall x \in X$ using the **equation 2.2.1**.

The Gini values to estimate the root node that were obtained are as follows:

$$Gini\ for\ SubjectId = 0.759$$

$$Gini\ for\ Gender = 0.745$$

$$Gini\ for\ Age = 0.709$$

$$Gini\ for\ Marks = 0.341$$

Even though the Gini for the feature ‘Marks’ is minimum, the choice was made to construct a random forest using individual decision trees with the root as the feature ‘Age’ which has the next minimum in the list. The reason for dropping the feature ‘Marks’ and proceeding to construct a decision tree with the other feature lay in the lack of distinction the corresponding feature provided. Constructing a decision tree with a feature that has minimal distinction might not be the suitable solution in a system where we seek to use as many minimum decision trees as possible to draw the inferences.

So, the root of the decision tree chosen would be the feature **‘Age’** with an impurity index of **0.709** and a maximum entropy of **0.291**. The evaluation of the subsequent nodes was performed in reference to the root node against the decision made. In the next depth, division was carried out with the Gini for **Gender** as **0.542** and for **SubjectID** as **0.531**. In this fashion, several decision trees were aggregated and boosted i.e., bagged to give rise to a random forest. The tree in the forest with the most likely interference was chosen for three cases: The overall dataset, the 16-18 age group, and the 20-21 age group.

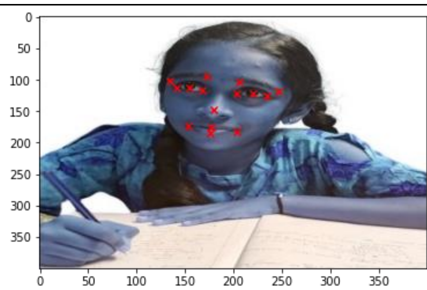
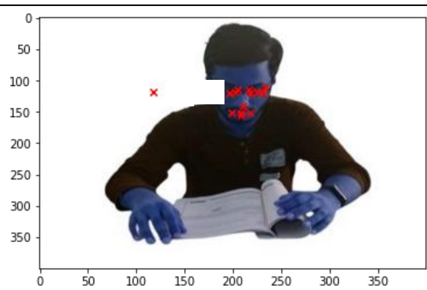
The results and inferences drawn from the construction of the ADBRS driven random forest are presented in detail in Section 4 with an illustration of applying a new object to validate its decision utilising the model constructed.

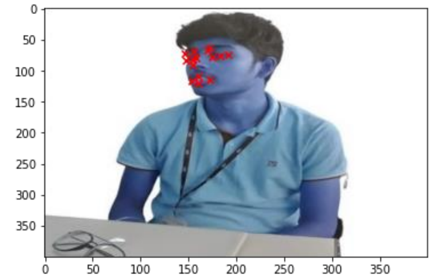
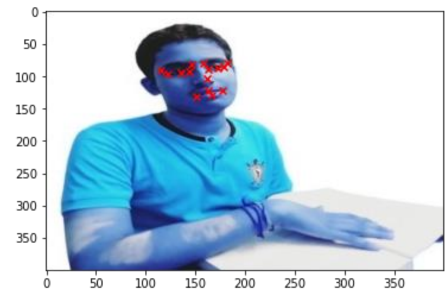
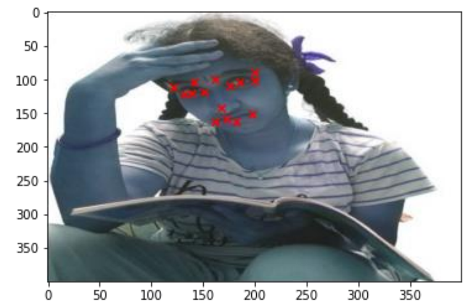
The greatest advantage of the model proposed comes down to its ability to make decisions without any prior knowledge of the decisions with respect to the object involved and to validate a newly added observation using the already existing model without the need to handle the photographs of the newly added object against the existing DBRNS.

3. Results

3.1. Data was taken from the images and the expressions of the students against different subject durations for which the photos were taken were computed

Table6-1: Images and The Corresponding decisions made

Image	Values evaluated	Maximum Value	Expression Detected
	$V(A) = 0.525$ $V(B) = 0.525$ $V(E) = 0.540$ $V(S) = 0.459$ $V(C) = 0.495$	$V(E)$	The person is ***excited*** to be in class
	$V(A) = 0.525$ $V(B) = 0.525$ $V(E) = 0.495$ $V(S) = 0.540$ $V(C) = 0.459$	$V(S)$	The person is feeling ***sleepy*** in class

	$V(A) = 0.679$ $V(B) = 0.555$ $V(E) = 0.545$ $V(S) = 0.540$ $V(C) = 0.525$	$V(A)$	The person is ***attentive*** in class
	$V(A) = 0.525$ $V(B) = 0.615$ $V(E) = 0.495$ $V(S) = 0.585$ $V(C) = 0.525$	$V(B)$	The person is ***bored*** in the class
	$V(A) = 0.540$ $V(B) = 0.492$ $V(E) = 0.540$ $V(S) = 0.525$ $V(C) = 0.615$	$V(C)$	The person is ***uncertain*** of the things being taught in the class

Due to the relatively larger size of the dataset, few images have been utilised in the table illustrated above to give an idea about the process executed

3.2. Construction of the individual decision trees to be bagged to draw inferences regarding the teaching-learning process efficiency.

Gini Index Impurity based split was employed to construct the individual decision trees that constitute the random forest. The visualization of the individual decision tree has been done by considering the following three cases.

Case 1: Individual decision tree for the dataset as a whole

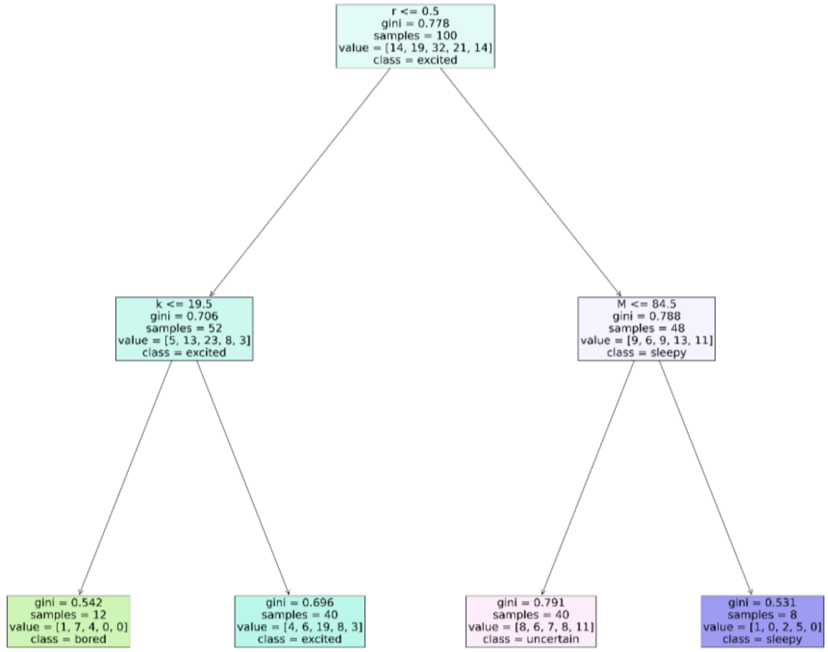


fig4-1: Individual Decision Tree From The Random Forest: Without Age Distinction

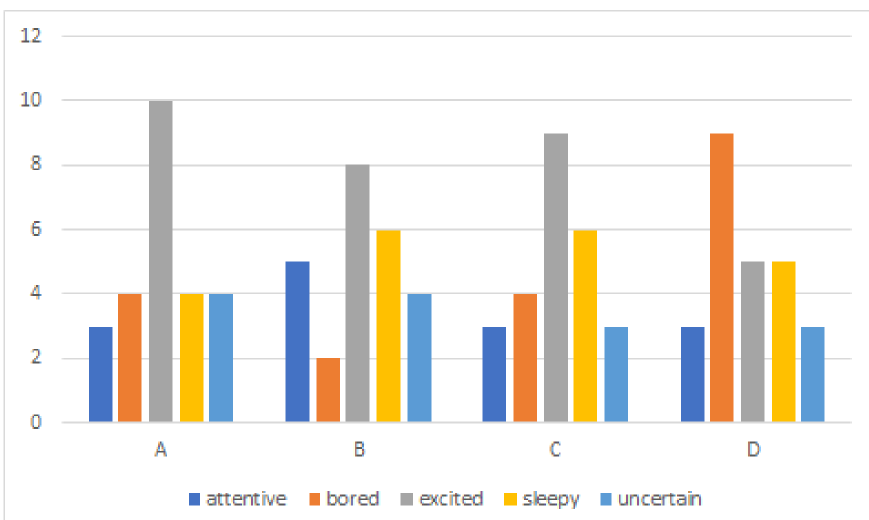


fig4-2: Distribution of the various expressions observed across the dataset against the subject for which the expressions were computed.

There is construction of individual decision tree for the dataset as a whole without age distinction and the inferences can be visualized using the following tree from the random forest which has been evaluated to be the most likely estimation.

Inference: Across the age groups, subject A has the highest excitement levels. Subject B and C have comparable excitement levels and almost equal number of students who fell asleep in class hours. Hence, it can be concurred that special measures from the teacher's side can improve the reception for both the subjects mentioned. Subject D indicates high boredom levels and it can be concluded that either the subject is too dry or the learning process isn't efficient.

Case 2: Individual Decision Tree for the age group 16-18

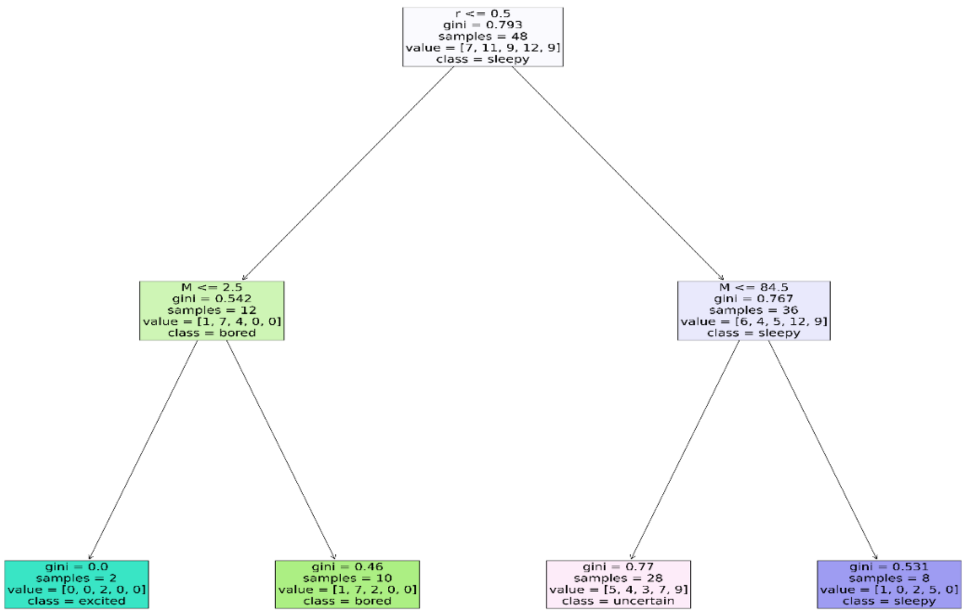


fig5-3: Individual Decision Tree from the Random Forest for the age group 16-18

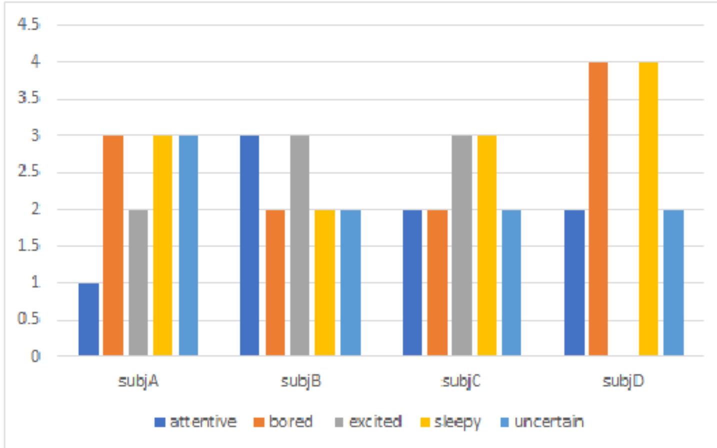


fig5-2: Distribution of the various expressions observed for the age group [16-18] against the subject for which the expressions were computed.

Inference: For High Schoolers, Subjects A and D seem to have the highest count of people who were either bored or uncertain. Hence, it can be concurred that either the subject is too dry or the teaching-learning flow was inefficient. Subjects B and C, on the other, have higher counts of people who are attentive or bored and excited or sleepy respectively. This could mean that if the teacher looks into the matter, they can try and engage more people to like the subject.

Case 3: Individual Decision Tree for the age group 20-21

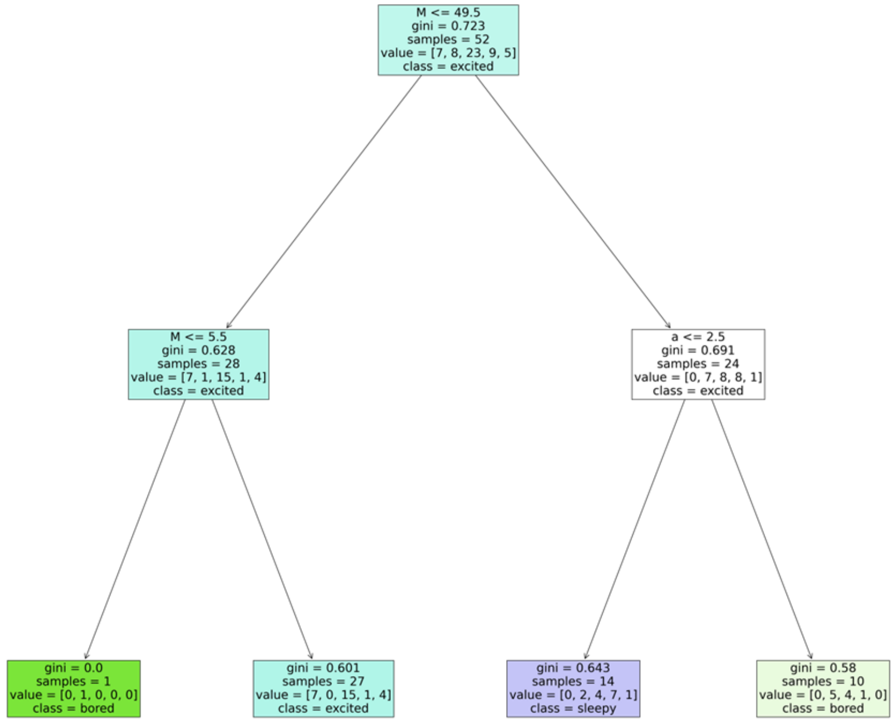


fig6-1: Individual Decision Tree from the Random Forest for the age group 20-21

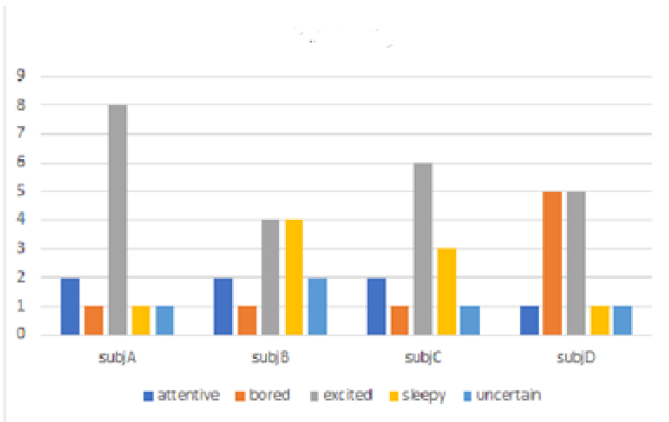


fig6-2: Distribution of the various expressions observed for the age group [20-21] against the subject for which the expressions were computed.

Inferences: For University Sophomores, Subjects A and C have a minimum distribution with respect to the uncertainty, boredom, and sleepy faces of the student. This means that the teaching-learning process is efficient. Subjects B and D, on the other, have equal distributions of boredom, excitement, and sleepiness with respect to the student. This could mean that if the teacher looks into the matter, they can improve the engageability and interactivity of the classroom atmosphere.

Case 4: Evaluation of a new object using the ADBNS-driven-random forest model

Let there be addition of a new object a given by ⟨2,1,17,76⟩. The list a is encoded, meaning the first index indicates that the subjectID is B (as there are four subject IDs, A, B, C and D, encoding them would give values of 0,1,2,3,4), the second indicates that the person is male (as there is presence of two sexes, male and female, encoding would give values of 1 and 0), the third indicates that the new object's age is 17 and the fourth index indicates that the person has scored 76 marks in subject B

Decision Evaluated: The person is ***uncertain*** about the subject studied.

Inference: The person has scored higher marks, but hasn't had an excellent grasp of the subject due to his expressions of uncertainty during that period. There is a need to establish better learning efficiency for the student mentioned.

4. Applications

The model constructed is extremely advantageous in the sense that it doesn't require prior information with respect to the decisions made for the dataset. With existing features, ADBRNS can help determine the decision and the decisions made can be utilised to both evaluate the impurity and draw inferences as well as fit into a classifier for training any similar models for future use.

Combined with frequency estimation algorithms like Apriori, FP Growth, and ECLAT, the model can help teachers and schools make early estimates on students who suffer from attention deficiency, belong to the spectrum, or display signs of ADHD and provide them with the adequate help they need.

The constructed model may also be used to approximately narrow down convicts who are guilty of committing a crime by simply utilising their mugshots, thereby saving the task of having to investigate a bigger crowd.

5. Results

The facial expressions of various students were detected using attribute based double-bounded rough neutrosophic set method. The decisions made were used against the scores of every student to evaluate the decision impurity and draw inferences regarding the teaching-learning efficiency.

6. Acknowledgement

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Generalized OWA Operator of Orthopair Neutrosophic Numbers and Their Application in Multiple Attribute Decision-Making Problems

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Abstract: A neutrosophic number (NN) is a useful mathematical tool in indeterminacy theory. As the mixed form of an intuitionistic fuzzy set and NN, an orthopair neutrosophic number (ONN) can express the true indeterminate degree and the false indeterminate degree. In view of generalized ordered weighted operators, this article presents two generalized ordered weighted operators of ONNs, including an orthopair neutrosophic number generalized ordered weighted average (ONNGOWA) operator and an orthopair neutrosophic number generalized ordered weighted geometric (ONNGOWG) operator, and their characteristics. A multi-attribute decision-making (MADM) model is established by the weighted operation of the ONNGOWA and ONNGOWG operators. Finally, an example on the selection problem of electric vehicle design schemes is given to reflect the effectivity of the proposed MADM model in the scenario of ONNs.

Keywords: orthopair neutrosophic number; generalized ordered weighted operator; multi-attribute decision-making

1. Introduction

In practical applications, it is difficult for decision makers to provide accurate evaluation values for complex decision-making problems in uncertain and incomplete circumstances. In this case, Zadeh presented the concept of fuzzy sets (FSs) [1]. On the basis of an extension of FS, Atanassov added a new parameter named a non-membership degree and defined an intuitionistic fuzzy set (IFS) [2]. Then, some scholars [3, 4] developed some intuitionistic fuzzy decision-making methods. Since various aggregation operators reveal important mathematical tools in multi-attribute decision-making (MADM) process, various aggregation operators of intuitionistic fuzzy numbers (IFNs) were proposed by many scholars. For example, Xu and Cai [5] and Xu and Yager [6] proposed intuitionistic fuzzy weighted aggregation operators, and then some researchers introduced the generalized aggregation operator of IFNs [7], the generalized geometric aggregation operator of IFNs [8], the induced generalized aggregation operators of IFNs [9], the power average operators of trapezoidal IFNs [10], and the Heronian aggregation operators of IFNs [11]. However, IFS/IFN cannot reasonably represent uncertain problems with uncertain membership and non-membership degrees. To express uncertain information, Smarandache proposed the concept of a neutrosophic number (NN) [12-14]. It is denoted by $N = g + hI$ for $I \in [I^-, I^+]$, where g is the determinate part and hI is the indeterminate part. Since NNs are very suitable for dealing with real problems with indeterminacy $I \in [I^-, I^+]$, they were currently used in production planning problems [15], fault diagnosis [16], medicine assessment [17], prediction of traffic volume [18]. Recently, Ye et al. [19] defined the concept of an orthopair neutrosophic number (ONN) as a mixed form of IFN and NN, which can represent the hybrid information of true and false indeterminate degrees, and then proposed the score and accuracy functions of ONN and the ONN weighted arithmetic and geometric averaging (ONNWAA and ONNWGA) operators for MADM.

With the complexity of the social and economic environment, it is difficult for a single decision-maker to consider all aspects of a MADM problem and to give a reasonable decision result. Accordingly, multiple decision makers are needed to provide decision information together and to construct a group decision-making result. Then, the aggregation algorithm of group decision information is very critical in group decision-making problems. Since the generalized ordered weighted averaging (GOWA) aggregation operators [20] consider not only the importance of parameters but also the importance of parameter positions, they reveal better aggregation algorithms in information aggregations. However, the GOWA operators have not been investigated for aggregating ONN information. On the basis of an extension of the GOWA operators, this article proposes the GOWA and generalized ordered weighted geometric (GOWG) operators of ONNs and a MADM model using the weighted operation of the GOWA and GOWG operators of ONNs.

The rest of the article consists of the following parts. The second part describes the related notions of ONNs, including the definition of ONN, the related operations of ONNs, as well as the score and accurate functions of ONNs and their sorting rules. The third part proposes an ONN generalized ordered weighted averaging (ONNGOWA) operator and an ONN generalized ordered weighted geometric (ONNGOWG) operator and indicates the characteristics of idempotency, boundedness, and monotonicity. The fourth part establishes a MADM model through the weighted operation of the ONNGOWA and ONNGOWG operators and addresses its decision steps. The fifth part applies the established MADM model to the choice problem of manufacturing schemes. The sixth part compares the established MADM model with the MADM model proposed in the previous literature [19]. The seventh part summarizes the conclusions and future research.

2. Preliminaries of ONNs

This section introduces the relevant notions of ONNs presented by Ye et al. [19].

Definition 1 [19]. Each ONN n_j (j = 1, 2, ..., m) is given by

$$n_j = \langle A_j(I), B_j(I)\rangle = \langle e_j + f_jI, g_j + h_jI\rangle, \tag{1}$$

where $e_j + f_jI \subseteq [0, 1]$ and $g_j + h_jI \subseteq [0, 1]$ for $I \in [I^-, I^+]$ are the true indeterminate degree and the false indeterminate degree, such that the condition $0 \le \sup A_j(I) + \sup B_j(I) \le 1$.

Definition 2 [19]. Let $n_1 = \langle A_1(I), B_1(I)\rangle = \langle e_1 + f_1I, g_1 + h_1I\rangle$ and $n_2 = \langle A_2(I), B_2(I)\rangle = \langle e_2 + f_2I, g_2 + h_2I\rangle$ for $I \in [I^-, I^+]$ be two ONNs. Then the operation rules of ONNs are presented as follows:

(1) $n_1 \supseteq n_2 \Leftrightarrow A_1(I) \supseteq A_2(I)$ and $B_1(I) \subseteq B_2(I)$;

(2) $n_1 = n_2 \Leftrightarrow n_1 \subseteq n_2$ and $n_1 \supseteq n_2$;

(3) $(n_1)^c = \langle B_1(I), A_1(I)\rangle$ (Complement of n_1);

(4) $n_1 \oplus n_2 = \left\langle \begin{array}{l} [\inf A_1(I) + \inf A_2(I) - \inf A_1(I)\inf A_2(I), \\ \sup A_1(I) + \sup A_2(I) - \sup A_1(I)\sup A_2(I)], \\ [\inf B_1(I)\inf B_2(I), \sup B_1(I)\sup B_2(I)] \end{array} \right\rangle$;

(5) $n_1 \otimes n_2 = \left\langle \begin{array}{l} [\inf A_1(I)\inf A_2(I), \sup A_1(I)\sup A_2(I)], \\ [\inf B_1(I) + \inf B_2(I) - \inf B_1(I)\inf B_2(I), \\ \sup B_1(I) + \sup B_2(I) - \sup B_1(I)\sup B_2(I)] \end{array} \right\rangle$;

(6) $\alpha n_1 = <[(1 - (1 - \inf A_1(I))^{\alpha}, 1 - (1 - \sup A_1(I))^{\alpha}], [(\inf B_1(I))^{\alpha}, (\sup B_1(I))^{\alpha}]>$ for $\alpha > 0$;

(7) $(n_1)^{\alpha} = <[(\inf A_1(I))^{\alpha}, (\sup A_1(I))^{\alpha}], [(1 - (1 - \inf B_1(I))^{\alpha}, 1 - (1 - \sup B_1(I))^{\alpha}]>$ for $\alpha > 0$.

To rank ONNs $n_j = <A_j(I), B_j(I)> = <e_j + f_jI, g_j + h_jI>$ ($j = 1, 2$) with $I \in [I^-, I^+]$, the accuracy function of ONN is given as [19]

$$T(n_j) = \{\inf A_1(I) + \inf B_1(I) + \sup A_1(I) + \sup B_1(I)\}/2$$
$$= \{[2e_j + f_j(I^- + I^+)] + [2g_j + h_j(I^- + I^+)]\}/2, \text{ for } T(n_j) \in [0, 1]. \quad (2)$$

The score function of ONN is given as [19]

$$S(n_j) = \{\inf A_1(I) - \inf B_1(I) + \sup A_1(I) - \sup B_1(I)\}/2$$
$$= \{[2e_j + f_j(I^- + I^+)] - [2g_j + h_j(I^- + I^+)]\}/2, \text{ for } S(n_j) \in [-1, 1]. \quad (3)$$

The ranking rules are described as follows [19]:

(1) If $S(n_1) > S(n_2)$, then $n_1 > n_2$;

(2) If $S(n_1) = S(n_2)$ and $T(n_1) > T(n_2)$, then $n_1 > n_2$;

(3) If $S(n_1) = S(n_2)$ and $T(n_1) = T(n_2)$, then $n_1 = n_2$.

3. Two Generalized Ordered Weighted Aggregation Operators of ONNs

This section proposes the ONNGOWA and ONNGOWG operators through the operation rules in Definition 2.

3.1. ONNGOWA Operator

The ONNGOWA operator for a group of ONNs can be derived from the operation rules in Definition 2.

Definition 3. Set $n_j = <A_j(I), B_j(I)> = <e_j + f_jI, g_j + h_jI>$ ($j = 1, 2, ..., m$) as a group of ONNs. Thus, the ONNGOWA operator is defined below:

$$\text{ONNGOWA}(n_1, n_2, ..., n_m) = \left(\sum_{j=1}^{m} v_j n_j^{\delta}\right)^{\frac{1}{\delta}}, \quad (4)$$

where v_j ($j = 1, 2, ..., m$) is the weight of n_j for $0 \le v_j \le 1$ and $\sum_{j=1}^{m} v_j = 1$.

Theorem 1. Set $n_j = <A_j(I), B_j(I)> = <e_j + f_jI, g_j + h_jI>$ ($j = 1, 2, ..., m$) as a group of ONNs. Thus, the value of the ONNGOWA operator is still ONN, which is obtained by the following formula:

$$\text{ONNGOWA}(n_1, n_2, ..., n_m) = \left(\sum_{j=1}^{m} v_j n_j^{\delta}\right)^{\frac{1}{\delta}}$$

$$= \left\langle \begin{array}{l} \left[\left(1 - \prod_{j=1}^{m}\left(1 - \left(e_j + f_j I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1 - \prod_{j=1}^{m}\left(1 - \left(e_j + f_j I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right], \\ \left[1 - \left(1 - \prod_{j=1}^{m}\left(1 - \left(1 - g_j - h_j I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1 - \left(1 - \prod_{j=1}^{m}\left(1 - \left(1 - g_j - h_j I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right] \end{array} \right\rangle, \quad (5)$$

where v_j ($j = 1, 2, ..., m$) is the weight of n_j for $0 \le v_j \le 1$ and $\sum_{j=1}^{m} v_j = 1$.

Proof:

According to the relevant operation rules in Definition 2, Eq. (5) can be verified below.

$$v_j n_j^{\delta} = \left\langle \begin{array}{l} \left[\left(1-\left(1-\left(e_j+f_j I^-\right)^{\delta}\right)^{v_j}\right), \left(1-\left(1-\left(e_j+f_j I^+\right)^{\delta}\right)^{v_j}\right)\right], \\ \left[1-\left(1-\left(1-\left(1-g_j-h_j I^-\right)^{\delta}\right)^{v_j}\right), 1-\left(1-\left(1-\left(1-g_j-h_j I^+\right)^{\delta}\right)^{v_j}\right)\right] \end{array} \right\rangle. \quad (6)$$

Then, we get the following equation:

$$\sum_{j=1}^{m} v_j n_j^{\delta} = \left\langle \begin{array}{l} \left[\left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_j I^-\right)^{\delta}\right)^{v_j}\right), \left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_j I^+\right)^{\delta}\right)^{v_j}\right)\right], \\ \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_j I^-\right)^{\delta}\right)^{v_j}\right), 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_j I^+\right)^{\delta}\right)^{v_j}\right)\right] \end{array} \right\rangle. \quad (7)$$

We can further get the result:

$$\left(\sum_{j=1}^{m} v_j n_j^{\delta}\right)^{\frac{1}{\delta}} =$$

$$\left\langle \begin{array}{l} \left[\left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_j I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_j I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right], \\ \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_j I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_j I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right] \end{array} \right\rangle. \quad (8)$$

So, the proof of the ONNGOWA operator is completed.

Theorem 2. The ONNGOWA operator expressed by Eq. (5) has the following properties:

(a) Idempotency: Set $n_j = <A_j(I), B_j(I)> = <e_j + f_jI, g_j + h_jI>$ ($j = 1, 2, ..., m$) as a group of ONNs. If $n_j = n$ ($j = 1, 2, \ldots, m$), then ONNGOWA($n_1, n_2, ..., n_m$) = n.

(b) Boundedness: Set $n_j = <A_j(I), B_j(I)> = <e_j + f_jI, g_j + h_jI>$ ($j = 1, 2, ..., m$) as a group of ONNs, and then let the maximum and minimum ONNs be the following values:

$$n_{\max} = \left\langle \left[\max_j\left(e_j + f_j I^-\right), \max_j\left(e_j + f_j I^+\right)\right], \left[\min_j\left(g_j + h_j I^-\right), \min_j\left(g_j + h_j I^+\right)\right] \right\rangle,$$

$$n_{\min} = \left\langle \left[\min_j\left(e_j + f_j I^-\right), \min_j\left(e_j + f_j I^+\right)\right], \left[\max_j\left(g_j + h_j I^-\right), \max_j\left(g_j + h_j I^+\right)\right] \right\rangle. \quad (9)$$

Thus, the inequality $n_{\min} \le$ ONNGOWA($n_1, n_2, ..., n_m$) $\le n_{\max}$ exists.

(c) Monotonicity: Let $n_j = <A_j(I), B_j(I)> = <e_j + f_jI, g_j + h_jI>$ and $n_j^* = <A_j^*(I), B_j^*(I)>$ ($j = 1, 2, ..., m$) be two groups of ONNs. If $n_j \le n_j^*$, then there is the inequality ONNGOWA($n_1, n_2, ..., n_m$) $\le$ ONNGOWA($n_1^*, n_2^*, ..., n_m^*$).

Proof:

(a) When $n_j = n$ ($j = 1, 2, ..., m$), the result of Eq. (5) is obtained below:

$$\text{ONNGOWA}(n_1, n_2, ..., n_m) = \left(\sum_{j=1}^{m} v_j n_j^{\delta}\right)^{\frac{1}{\delta}}$$

$$= \left\langle \begin{array}{l} \left[\left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_jI^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_jI^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}} \right], \\ \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_jI^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_jI^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\left(1-\left(1-\left(e+fI^-\right)^{\delta}\right)^{\sum_{j=1}^{m}v_j}\right)^{\frac{1}{\delta}}, \left(1-\left(1-\left(e+fI^+\right)^{\delta}\right)^{\sum_{j=1}^{m}v_j}\right)^{\frac{1}{\delta}} \right], \\ \left[1-\left(1-\left(1-\left(1-g-hI^-\right)^{\delta}\right)^{\sum_{j=1}^{m}v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\left(1-\left(1-g-hI^+\right)^{\delta}\right)^{\sum_{j=1}^{m}v_j}\right)^{\frac{1}{\delta}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\left(1-\left(1-\left(e+fI^-\right)^{\delta}\right)\right)^{\frac{1}{\delta}}, \left(1-\left(1-\left(e+fI^+\right)^{\delta}\right)\right)^{\frac{1}{\delta}} \right], \\ \left[1-\left(1-\left(1-\left(1-g-hI^-\right)^{\delta}\right)\right)^{\frac{1}{\delta}}, 1-\left(1-\left(1-\left(1-g-hI^+\right)^{\delta}\right)\right)^{\frac{1}{\delta}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\left(1-1+\left(e+fI^-\right)^{\delta}\right)^{\frac{1}{\delta}}, \left(1-1+\left(e+fI^+\right)^{\delta}\right)^{\frac{1}{\delta}} \right], \\ \left[1-\left(1-1+\left(1-g-hI^-\right)^{\delta}\right)^{\frac{1}{\delta}}, 1-\left(1-1+\left(1-g-hI^+\right)^{\delta}\right)^{\frac{1}{\delta}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\left(\left(e+fI^-\right)^{\delta}\right)^{\frac{1}{\delta}}, \left(\left(e+fI^+\right)^{\delta}\right)^{\frac{1}{\delta}} \right], \\ \left[1-\left(\left(1-g-hI^-\right)^{\delta}\right)^{\frac{1}{\delta}}, 1-\left(\left(1-g-hI^+\right)^{\delta}\right)^{\frac{1}{\delta}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\left(e+fI^-\right), \left(e+fI^+\right)\right], \\ \left[\left(g+hI^-\right), \left(g+hI^+\right)\right] \end{array} \right\rangle = n. \quad (10)$$

(b) Since n_{max} and n_{min} are the maximum and minimum ONNs, there is $n_{min} \le n_j \le n_{max}$. Hence, the inequality $\sum_{j=1}^{m} v_j n_{min} \le \sum_{j=1}^{m} v_j n_j \le \sum_{j=1}^{m} v_j n_{max}$ is established. According to the property (a), there exists $n_{min} \le \sum_{j=1}^{m} v_j n_j \le n_{max}$, i.e., $n_{min} \le$ ONNGOWA(n_1, n_2, ..., n_m) $\le n_{max}$.

(c) If $n_j \le n_j^*$, then the inequality $\sum_{j=1}^{m} v_j n_j \le \sum_{j=1}^{m} v_j n_j^*$ is established, i.e., the inequality ONNGOWA(n_1, n_2, ..., n_m) $\le$ ONNGOWA(n_1^*, n_2^*, ..., n_m^*) holds.

Thus, we complete the proof of Theorem 2.

3.2. ONNGOWG Operator

The ONNGOWG operator for a group of ONNs can be derived from the operation rules in Definition 2.

Definition 4. Set n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> (j = 1, 2, ..., m) as a group of ONNs. Thus, the ONNGOWG operator is defined below:

$$\text{ONNGOWG}(n_1, n_2, ..., n_m) = \left(\prod_{j=1}^{m} n_j^{\delta v_j}\right)^{\frac{1}{\delta}}, \quad (11)$$

where v_j (j = 1, 2, ..., m) is the weight of n_j for $0 \le v_j \le 1$ and $\sum_{j=1}^{m} v_j = 1$.

Theorem 3. Set n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> (j = 1, 2, ..., m) as a group of ONNs. Thus, the value of the ONNGOWG operator is still ONN, which is obtained by the following formula:

$$\text{ONNGOWG}(n_1, n_2, ..., n_m) = \left(\prod_{j=1}^{m} n_j^{\delta v_j}\right)^{\frac{1}{\delta}}$$

$$= \left\langle \begin{array}{l} \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-e_j-f_jI^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-e_j-f_jI^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right], \\ \left[\left(1-\prod_{j=1}^{m}\left(1-\left(g_j+h_jI^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{m}\left(1-\left(g_j+h_jI^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right] \end{array} \right\rangle, \quad (12)$$

where v_j (j = 1, 2, …, m) is the weight of n_j for $0 \le v_j \le 1$ and $\sum_{j=1}^{m} v_j = 1$.

The verification process of Eq. (12) is similar to that of Theorem 1, so it is omitted.

Theorem 4. The ONNGOWG operator of Eq. (12) has the following properties:

(a) Idempotency: Set n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> (j = 1, 2, ..., m) as a group of ONNs. If $n_j = n$ (j = 1, 2, ..., m), then ONNGOWG(n_1, n_2, ..., n_m) = n.

(b) Boundedness: Set n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> (j = 1, 2, ..., m) as a group of ONNs, and then let the maximum and minimum ONNs:

$$n_{\max} = \left\langle \left(\max_j\left(e_j + f_jI^-\right), \max_j\left(e_j + f_jI^+\right)\right), \left(\min_j\left(g_j + h_jI^-\right), \min_j\left(g_j + h_jI^+\right)\right) \right\rangle,$$

$$n_{\min} = \left\langle \left(\min_j\left(e_j + f_jI^-\right), \min_j\left(e_j + f_jI^+\right)\right), \left(\max_j\left(g_j + h_jI^-\right), \max_j\left(g_j + h_jI^+\right)\right) \right\rangle. \quad (13)$$

Thus, $n_{min} \le$ ONNGOWG(n_1, n_2, ..., n_m) $\le n_{max}$.

(c) Monotonicity: Let n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> and n_j^* = <$A_j^*(I)$, $B_j^*(I)$> (j = 1, 2, ..., m) be two groups of ONNs. If $n_j \le n_j^*$, then ONNGOWG(n_1, n_2, ..., n_m) ≤ ONNGOWG(n_1^*, n_2^*, ..., n_m^*).

4. MADM Model Based on the ONNGOWA and ONNGOWG Operators

In this section, a MADM model are established based on the weighted operation of the ONNGOWA and ONNGOWG operators to perform MADM problems with ONNs.

For a MADM problem, D = {D_1, D_2, ..., D_q} represents a set of q alternatives and then F = {f_1, f_2, ..., f_m} represents a set of m attributes. The importance of each attribute f_j (j = 1, 2, ..., m) is determined by the weight v_j. Experts/decision makers evaluate the satisfactory levels of each alternative D_i (i = 1, 2, ..., q) relative to the attributes f_j (j = 1, 2, ..., m) through true and falsity indeterminate degrees, which are expressed as the ONNs $n_{ij} = \langle A_{ij}(I), B_{ij}(I)\rangle = \langle e_{ij} + f_{ij}I, g_{ij} + h_{ij}I\rangle$ for $A_{ij}(I), B_{ij}(I) \in [0, 1]$, $I \in [I^-, I^+]$, and $0 \le \sup A_{ij}(I) + \sup B_{ij}(I) \le 1$. Thus, the decision matrix of ONNs can be expressed as $N = (n_{ij})_{q\times m}$. Therefore, the MADM model according to the weighted operation of the ONNGOWA and ONNGOWG operators is established through the following steps:

Step 1: Based on Eqs. (5) and (12), the aggregated ONNs n_{1i} and n_{2i} are obtained by the following equations:

$$n_{1i} = \text{ONNGOWA}(n_{i1}, n_{i2}, ..., n_{im}) = \left(\sum_{j=1}^{m} v_j n_{ij}^{\delta}\right)^{\frac{1}{\delta}}$$

$$= \left\langle \begin{array}{l} \left[\left(1-\prod_{j=1}^{m}\left(1-\left(e_{ij}+f_{ij}I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{m}\left(1-\left(e_{ij}+f_{ij}I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right], \\ \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_{ij}-h_{ij}I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_{ij}-h_{ij}I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right] \end{array} \right\rangle, \quad (14)$$

$$n_{2i} = \text{ONNGOWG}(n_{i1}, n_{i2}, ..., n_{im}) = \left(\prod_{j=1}^{m} n_{ij}^{\delta v_j}\right)^{\frac{1}{\delta}}$$

$$= \left\langle \begin{array}{l} \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-e_{ij}-f_{ij}I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-e_{ij}-f_{ij}I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right], \\ \left[\left(1-\prod_{j=1}^{m}\left(1-\left(g_{ij}+h_{ij}I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{m}\left(1-\left(g_{ij}+h_{ij}I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right] \end{array} \right\rangle.$$

(15)

Step 2: The weighted operation of the ONNGOWA and ONNGOWG operators with the weights ψ_1 and $\psi_2 = 1 - \psi_1$ for $\psi_1 \in [0, 1]$ is obtained by the following equation:

$$H_i = \psi_1 n_{1i} \oplus \psi_2 n_{2i} = \psi_1 n_{1i} \oplus (1-\psi_1) n_{2i}$$

$$= \left\langle \begin{array}{l} \left[1-\left(1-\inf A_{1i}(I)\right)^{\psi_1}, 1-\left(1-\sup A_{1i}(I)\right)^{\psi_1}\right], \\ \left[\inf(B_{1i}(I))^{\psi_1}, \sup(B_{1i}(I))^{\psi_1}\right] \end{array} \right\rangle \oplus \left\langle \begin{array}{l} \left[1-\left(1-\inf A_{2i}(I)\right)^{1-\psi_1}, 1-\left(1-\sup A_{2i}(I)\right)^{1-\psi_1}\right], \\ \left[\inf(B_{2i}(I))^{1-\psi_1}, \sup(B_{2i}(I))^{1-\psi_1}\right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[1-\left(1-\inf A_{1i}(I)\right)^{\psi_1}\left(1-\inf A_{2i}(I)\right)^{1-\psi_1}, 1-\left(1-\sup A_{1i}(I)\right)^{\psi_1}\left(1-\sup A_{2i}(I)\right)^{1-\psi_1}\right], \\ \left[\inf(B_{1i}(I))^{\psi_1}\cdot\inf(B_{2i}(I))^{1-\psi_1}, \sup(B_{1i}(I))^{\psi_1}\cdot\sup(B_{2i}(I))^{1-\psi_1}\right] \end{array} \right\rangle. \quad (16)$$

Step 3: The values of $S(H_i)$ and $T(H_i)$ (i = 1, 2, ..., q) are obtained by Eqs. (2) and (3).

Step 4: The alternatives are sorted according to the sorting rules and the best one is chosen.

Step 5: End.

5. Illustrative Example

In this section, the MADM model based on the weighted operation of the ONNGOWA and ONNGOWG operators is applied to the selection of electric vehicle design schemes.

A manufacturing company needs to choose the best design scheme of electric vehicles, the technique department preliminarily provides four design schemes of electric vehicles as a set of alternatives $D = \{D_1, D_2, D_3, D_4\}$. Each alternative is satisfactorily assessed by the three attributes: charging rate (f_1), driving range (f_2), and manufacturing cost (f_3). The weight vector of the three attributes is specified by v = (0.36, 0.3, 0.34). Therefore, experts/decision makers evaluate the four alternatives that satisfy these attributes by ONNs $n_{ij} = \langle A_{ij}(I), B_{ij}(I)\rangle = \langle e_{ij} + f_{ij}I, g_{ij} + h_{ij}I\rangle$ (i = 1, 2, 3, 4 and j= 1, 2, 3) for $A_{ij}(I), B_{ij}(I) \in [0, 1]$, $I \in [I^-, I^+]$, and $0 \le \sup A_{ij}(I) + \sup B_{ij}(I) \le 1$. Thus, the ONN decision matrix is listed in Table 1.

Table 1. The decision matrix of ONNs.

	f_1	f_2	f_3
D_1	<0.5 + 0.2I, 0.1 + 0.1I>	<0.6 + 0.1I, 0.1 + 0.1I>	<0.5 + 0.1I, 0.1 + 0.2I>
D_2	<0.6 + 0.1I, 0.1 + 0.1I>	<0.6 + 0.2I, 0.1 + 0.1I>	<0.6 + 0.1I, 0.1 + 0.1I>
D_3	<0.6 + 0.1I, 0.1 + 0.1I>	<0.6 + 0.1I, 0.1 + 0.2I>	<0.5 + 0.2I, 0.1 + 0.2I>
D_4	<0.5 + 0.2I, 0.1 + 0.1I>	<0.6 + 0.2I, 0.1 + 0.1I>	<0.7 + 0.1I, 0.1 + 0.1I>

Regarding the MADM problem in an ONN environment, the MADM steps are given below.

Step 1: Using Eqs. (14) and (15) with δ = 0.5 and $I \in [I^-, I^+]$ = [0, 0.3], the aggregated ONNs n_{1i} and n_{2i} (i = 1, 2, 3, 4) are obtained below:

$$\begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \\ n_{14} \end{bmatrix} = \begin{bmatrix} <0.5318 + 0.5724I, 0.1000 + 0.1395I> \\ <0.6000 + 0.6392I, 0.1000 + 0.1300I> \\ <0.5679 + 0.6073I, 0.1000 + 0.1485I> \\ <0.6053 + 0.6540I, 0.1000 + 0.1300I> \end{bmatrix},$$

$$\begin{bmatrix} n_{21} \\ n_{22} \\ n_{23} \\ n_{24} \end{bmatrix} = \begin{bmatrix} <0.5288 + 0.5700I, 0.1000 + 0.1401I> \\ <0.6000 + 0.6389I, 0.1000 + 0.1300I> \\ <0.5647 + 0.6057I, 0.1000 + 0.1491I> \\ <0.5948 + 0.6461I, 0.1000 + 0.1300I> \end{bmatrix}.$$

Step 2: By Eq. (16) for ψ_1 = 0.5 and $I \in [I^-, I^+]$ = [0, 0.3], the values of H_i are given below:
H_1 = <0.5303 + 0.7017I, 0.1000 + 0.1419I>, H_2 = <0.6000 + 0.7917I, 0.1000 + 0.1390I>,
H_3 = <0.5663 + 0.7483I, 0.1000 + 0.1446I>, and H_4 = <0.6001 + 0.7952I, 0.1000 + 0.1390I>.

Step 3: Using Eq. (3), the values of $S(H_i)$ for the alternatives D_i (i = 1, 2, 3, 4) are given as follows: $S(H_1)$ = 0.495, $S(H_2)$ = 0.5763, $S(H_3)$ = 0.535, and $S(H_4)$ = 0.5781.

Step 4: Since $S(H_4) > S(H_2) > S(H_3) > S(H_1)$, the sorting order of the four alternatives is $D_4 > D_2 > D_3 > D_1$, then the best one is D_4.

In order to reflect the influence of δ *and* ψ_1 on the decision results of the proposed MADM model, the corresponding ranking results are shown in Table 2.

In view of the ranking results shown in Table 2, different parameter values of δ and different weight values of ψ_1 can influence the ranking order of the four alternatives, which reveals the flexibility of the decision results.

Table 2. Values of $S(H_i)$ and ranking orders corresponding to δ = 0.3, 0.7, 1 *and* ψ_1 = 0, 0.1, 0.3, 0.5, 0.7, 1.

δ	ψ_1	$[I^-, I^+]$	$S(H_1)$, $S(H_2)$, $S(H_3)$, $S(H_4)$	Ranking order	The best one
δ = 0.3	ψ_1 = 0.0	[0, 0.3]	0.4935, 0.5763, 0.5335, 0.5733	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.1	[0, 0.3]	0.4939, 0.5763, 0.5338, 0.5744	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.3	[0, 0.3]	0.4944, 0.5763, 0.5344, 0.5764	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 0.5	[0, 0.3]	0.4950, 0.5763, 0.5350, 0.5783	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 0.7	[0, 0.3]	0.4956, 0.5763, 0.5356, 0.5803	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 1.0	[0, 0.3]	0.4965, 0.5764, 0.5365, 0.5832	$D_4 > D_2 > D_3 > D_1$	D_4
δ = 0.7	ψ_1 = 0.0	[0, 0.3]	0.4929, 0.5763, 0.5328, 0.5711	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.1	[0, 0.3]	0.4933, 0.5763, 0.5333, 0.5725	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.3	[0, 0.3]	0.4941, 0.5763, 0.5340, 0.5752	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.5	[0, 0.3]	0.4949, 0.5763, 0.5349, 0.5779	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 0.7	[0, 0.3]	0.4958, 0.5763, 0.5357, 0.5806	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 1.0	[0, 0.3]	0.4969, 0.5764, 0.5369, 0.5846	$D_4 > D_2 > D_3 > D_1$	D_4
δ = 1	ψ_1 = 0.0	[0, 0.3]	0.4924, 0.5763, 0.5323, 0.5691	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.1	[0, 0.3]	0.4929, 0.5763, 0.5328, 0.5709	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.3	[0, 0.3]	0.4939, 0.5763, 0.5338, 0.5743	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.5	[0, 0.3]	0.4949, 0.5763, 0.5348, 0.5775	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 0.7	[0, 0.3]	0.4959, 0.5764, 0.5358, 0.5808	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 1.0	[0, 0.3]	0.4974, 0.5764, 0.5374, 0.5856	$D_4 > D_2 > D_3 > D_1$	D_4

6. Comparative Analysis

To prove the effectiveness of the proposed model, the proposed MADM model based on the weighted operation of the ONNGOWA and ONNGOWG operators is compared with the MADM model proposed in [19]. The decision results of the existing MADM model for the above example are summarized in Table 3.

Table 3. The best one and ranking order corresponding to the existing MADM model [19].

Aggregation operator	Aggregated value	Score value	Ranking order	The best one
ONNWGA operator for I = [0, 0.3] [19]	0.5281, 0.6989, 0.1000, 0.1448 0.6342, 0.8363, 0.1000, 0.1390 0.5639, 0.7480, 0.1000, 0.1453 0.6323, 0.8344, 0.1000, 0.1390	$S(H_i)$ = (0.4924, 0.5763, 0.5323, 0.5691)	$D_2 > D_4 > D_3 > D_1$	D_2
ONNWAA operator for I = [0, 0.3] [19]	0.5324, 0.6274, 0.1000, 0.1911 0.6000, 0.6928, 0.1000, 0.1700 0.5685, 0.6601, 0.1000, 0.2120 0.6373, 0.7506, 0.1000, 0.1700	$S(p_j, I)$ = (0.4974, 0.5764, 0.5374, 0.5857)	$D_4 > D_2 > D_3 > D_1$	D_4

Regarding the decision results in Tables 2 and 3, the ranking results of the design schemes and the best one based on the proposed MADM model with δ = 1, ψ_1 = 0, 1, and I = [0, 0.3] are the same as those based on the existing MADM model [19] because the ONNWAA and ONNWGA operators [19] are the special cases of the ONNGOWA and ONNGOWG operators with $\delta = 1$ and $\psi_1 = 0, 1$. However, the proposed MADM model contains the advantage of flexible decision making, while the existing MADM model [19] lacks flexibility in the decision process. Therefore, the proposed MADM model reveals the obvious superiority over the existing MADM model [19] in an ONN circumstance.

7. Conclusions

In this paper, we presented the ONNGOWA and ONNGOWG operators based on the concepts of ONNs and the GOWA operators to reach more flexible aggregation operations than the existing ONNWAA and ONNWGA operators [19]. Then, the proposed MADM model based on the weighted operation of the ONNGOWA and ONNGOWG operators was established to solve flexible MAGM problems in an uncertain circumstance. However, the application of the proposed MADM model in an illustrative example demonstrated its effectivity, and then the comparative results reflected that the proposed MADM model revealed the advantage of flexible decision making in an ONN circumstance.

However, there are still many aggregation operators of ONNs for MADM to need further research and to apply them in practical areas, including supplier selection, fault diagnosis, medical diagnosis, etc.

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Neutrosophic speech recognition Algorithm for speech under stress by Machine learning

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Abstract

It is well known that the unpredictable speech production brought on by stress from the task at hand has a significant negative impact on the performance of speech processing algorithms. Speech therapy benefits from being able to detect stress in speech. Speech processing performance suffers noticeably when perceptually produced stress causes variations in speech production. Using the acoustic speech signal to objectively characterize speaker stress is one method for assessing production variances brought on by stress. Real-world complexity and ambiguity make it difficult for decision-makers to express their conclusions with clarity in their speech. In particular, the Neutrosophic speech algorithm is used to encode the language variables because they cannot be computed directly. Neutrosophic sets are used to manage indeterminacy in a practical situation. Existing algorithms are used except for stress on Neutrosophic speech recognition. The creation of algorithms that calculate, categorize, or differentiate between different stress circumstances. Understanding stress and developing strategies to combat its effects on speech recognition and human-computer interaction system are the goals of this recognition.

Keywords: speech recognition, categorization of stress in speech, linguistic technology, Neutrosophic, Machine learning

1.Introduction

In order to produce speech, a series of intricately synchronised articulator movements, respiratory system airflow, and timing of the vocal system physiology are all required. While the posture of the articulator changes to create speech, not all utterances made by a speaker will be identical in every way. This is due to the fact that the subject is frequently experiencing some sort of emotional stress, which will affect the utterance and cause an error in the articulator motions. Listeners can handle or interpret these subtle variations in human communications much better than the automatic human-machine interface. The features of stress, and its effects on human speech production, perception, and automatic speed systems, are still not fully understood. Speech is therefore a complex signal that contains information about the speaker. The speaker's intent, language history, features of their accent and dialect, and additional paralinguistic information. Stress can cause a change in speech output that can large and will consequently affect how well speech processing apps function [1],[2]. Numerous research has examined how stress affects speech production variability[3],[4], and [5]. Moreover, a stress-based expansion of multi-style training Additionally, token generation has improved anxious speech recognition [6]. Then, five stress-sensitive targeted feature sets are chosen.

stress situations such as the cockpit of an Apache helicopter, anger, clarity, the Lombard effect, loudness, etc. features that are frequently employed Include the cepstral characteristic for speaker identification [7]. When doing cepstral analysis, speaker recognition software often ignores the excitation source data that appears as a high-time component of the cestrum[8]. The Mel-Frequency Cepstral Coefficient, a phonetic characteristic, was retrieved from the voice signals, and the stress was identified using a neural network that was programmed into the system using Python[9]. serve as a resource for decision-makers in many real-world scenarios and application domains, particularly from a technical standpoint, for both academic and business experts[10]. In the research described in this paper, stress during applicant screening interviews is identified via voice analysis. The mean energy, mean intensity, and Mel-Frequency Cepstral Coefficients are employed as classification features in machine learning to identify stress in speech[11]. This study uses an EEG signal to suggest a stress classification system. 35 individuals' EEG signals were analyzed after being collected using a commercially available 4-electrode Muse EEG headgear with four EEG sensors [12]. In this study, it is expected that risk factors would, both cross-sectionally and longitudinally, predict mental health issues after controlling for sociodemographic traits and intent to become pregnant[13]. This investigation uses brain signals to look at how stress levels are affected by English and Urdu language music tracks[14]. This project looks into methods for sensing stress that is used to identify hardware[15]. The high-level features are combined into one unified representation using a proposed model-level fusion technique, which classifies the stress states into baseline, stress, and amusement[16]. The heart rate was measured and classified into three categories of positive, negative, and neutral emotions using the Geneva affective picture database. The support vector machine is a machine learning technique that has been built to predict the mental stress situation from the measured heart rate[17]. The development of a model for measuring stress levels makes use of several sensors, including those that measure body temperature, blood pressure (BP), heart rate, and CO2 concentration[18]. Studies show that combining IoT and AI with deep learning (DL) technology makes it possible to take preventative measures. Recognise stress well before its effects on human health become apparent[19]. In order to assess teaching effectiveness, enhance education, and limit risks from human errors that could occur as a result of workers' stressful circumstances, stress detection is crucial in both education and industry [20]. has good classification performance in this study and is able to gauge the stress levels of kids with accuracy. The growth of students' mental health has a strong foundation thanks to the precise measurement of stress, which also has important practical ramifications[21]. This research explores the concept of the intervention effect of physical activity on college students using an integrated evaluation-based algorithm. College students are used as an example of stress groups. The findings indicate that regular physical activity can significantly reduce college students' stress levels[22]. This study employs Neutrosophic logicto provide a valid ranking of hospital construction assets based on their changeable criticality and to lessen the subjectivity pertaining to expert-driven judgements[23]. This document compiles all research on machine learning mapping.

methods from the sharp number space to the neutrosophic environment. We also talk about contributions and combining single-valued neutrosophic numbers with machine learning methods

Modeling faulty information using (SVNs)[24]. In this paper, a brand-new paradigm for incorporating neutrosophy into deep learning models is given. To further comprehend the feelings, we quantified them using three membership functions as opposed to simply predicting a single class as the outcome. The two components of our suggested model are feature extraction and feature categorization[25]. The proposed framework would be an appropriate progression in the future by eliminating ineffective qualities through feature selection [26]. Stress is a psychological condition that results from an alleged threat or work demand and is accompanied by a variety of feelings. Finding linguistic cues of stress could be one of the verbal signs of stress. verbal indicators of stress are perceived by the listener, markers range in visibility from very visible to invisible. Consciously and unconsciously, these signals are watched continuously [27]. Speech recognition is the ability of a system to recognise the words and phrases of the speech and convert them to readable or written format. Speech recognition is typically carried out through processes including call routing, speech-to-text conversion, voice dialling, voice audibility, and language modelling. Although there are numerous techniques and algorithms for voice recognition, none of them is handling all factors including word length, speaker independence, a wide vocabulary, comprehension of speech, time complexity, noisy surroundings, and conversational speech. Neutrosophic can be integrated to analyse the acoustic signal of an unknown speaker and the decision-making process when indeterminacy occurs, respectively, to solve these issues.

2. Preliminaries

A neutrosophic set $\tilde{\mathcal{A}}_N$ in $\mathcal{U}$ (Universe of discourse) is catogorized as functions of a truth membership $T_{\tilde{\mathcal{A}}_N}(g)$, an indeterminacy membership $I_{\tilde{\mathcal{A}}_N}(g)$ and a falsity membership $F_{\tilde{\mathcal{A}}_N}(g)$and is given by

$\tilde{\mathcal{A}} = \{g, \langle T_{\tilde{\mathcal{A}}_N}(g),\ I_{\tilde{\mathcal{A}}_N}(g), F_{\tilde{\mathcal{A}}_N}(g)\rangle \mid g \in \mathcal{U}\}$.

Here $T_{\tilde{\mathcal{A}}_N}(g),\ I_{\tilde{\mathcal{A}}_N}(g), F_{\tilde{\mathcal{A}}_N}(g) \in [0,1]$ and the relation $0 \leq \sup T_{\tilde{\mathcal{A}}_N}(g) \leq \sup I_{\tilde{\mathcal{A}}_N}(g) \leq \sup F_{\tilde{\mathcal{A}}_N}(g) \leq 3$ holds for all $g \in \mathcal{U}$.

Definition 2.1[27,28 and 29]

Let X be the universal set, then Neutrosophic set is defined as $S = \{(T_S(x), I_S(x), F_S(x)), x \in X\}$ where $T_S(x), I_S(x), F_S(x) \in [0,1]$ and $0 \leq T_S(x) + I_S(x) + F_S(x) \leq 3$.

3.Database

The assessments carried out in this study are based on information previously gathered for speech analysis in noise and stress analysis and algorithm formulation. Because the task at hand entails mapping audio single value Neutrosophic sets(SVNS) to text SVNS for comparison, a dataset that included audio translation was necessary. LibriSpeech dataset was chosen as a result. The following two folders were utilised for the project demonstration: Dev-clean (337 MB) and Train-clean-100 (6.3 GB).

4.Methodology

4.1 Audio

converting.flac audio files to.wav

The dataset could be downloaded in FLAC format. These files had to be converted into.wav format in order to be processed further and have features extracted.

4.2 Features Extraction and Preprocessing

The python feature extraction script was then run on the audio files, extracting 193 features for each audio file. As a result, the npy files X dev.npy (2703 x 193) and X train.npy were created (28539 x 193). Then, sklearn was used to normalise these files.

4.3 Text

Using VADER, analyse the sentiment of translated text.

For each input sentence, the sentiment analysis programme VADER delivers a score for the truth,indeterminacy and falsity. Each audio file's text translation was examined using VADER, and SVNS were produced.

5. Speech recognition in to text conversion

5.1 Algorithm:1
Step 1: Import library
Step 2:Import speech recognition
Step 3:Initialize recognizer class
Step 4; Reading Microphone source
Step 5: Convert audio to text
Step 6: Adjust for ambient noise.
Step 7: Recognize the error
Step 8:Type the text.

5.2 Programme for Speech to Text

```
r ◎ sr.Recognizer()
   print("Talk")
```

```
r.adjust_for_ambient_noise(source, duration◎0.2)
audio_text ◎ r.listen(source)
print("Time over, thanks")
  print("Text: "◎r.recognize_google(audio_text))
    print("Sorry, I did not get that")
```

Once the programe is over, then run the programe. The out put is
Talk
Speak through microphone then it will showing. In this experiment speech word is "very good"
Time over, thanks
The output in the screen is
Text: very good

6. Neutrosophic speech stress analysis

6.1 Algorithm:2

Step 1: Import SentimentIntensityAnalyzer class
Step 2: Function to print sentiments
Step 3: Score for sentiment speech

Step 4: Which contains Truth, Falsity, Indeterminacy, and compound scores.

Step 5: Decide sentiment as Truth, Falsity and Indeterminacy se.
Step 6: Print the value of the compound score
Step 7: Print overall the stress statement is truth ,falsity or indeterminacy.

6.2 Programme for text to stress analysis by Neutrosophic speech algorithm

```
def sentiment_scores(sentence):

sid_obj ◎ SentimentIntensityAnalyzer()
C ◎ sid_obj.polarity_scores(sentence)
print
("Overall sentiment dictionary is : ", C)
Print
("sentence was rated as ", C◎'falsity'◎◎100, "◎  Negative")
Print
("sentence was rated as ", C◎'indeterminacy'◎◎100, "◎  Neutral")
Print
("sentence was rated as ", C◎'Truth'◎◎100, "◎  Positive")
Print
("Sentence Overall Rated As", end ◎ " ")
```

The following sentence "Very Good.","Not bad","Bad", "happy birth day." "god bless you.","beautiful."

In this algorithm 2 , include the output of the algorithm 1 statements. Once run the programme.

6.3 The output of the programme

1st statement is Very Good the output of the programme is
◎'Falsity': 0.0, 'Indeterminacy': 0.238, 'Truth': 0.762◎
0.0 ◎ Falsity, 23.799999999999997 ◎ Indeterminacy, 76.2 ◎ Truth and the speech is not under strees
2nd Statement : Not bad
◎'Falsity': 0.0, 'Indeterminacy': 0.26, 'Truth': 0.74◎
0.0 ◎ Falsity, 26.0 ◎ Indeterminacy, 74.0 ◎ Truth and the speech is not under stress.
3rd Statement :Bad
◎'Falsity': 1.0, 'indeterminacy': 0.0, 'Truth': 0.0◎
100.0 ◎ Falsity, 0.0 ◎ Indeterminacy, 0.0 Truth and the speech is under stress.

4th statement :Happy Birthday
◎'Falsity': 0.0, 'Indeterminacy': 0.351, 'Truth': 0.649◎
0.0 ◎ Falsity, 35.099999999999994 ◎ Indeterminacy, 64.9 ◎ Truth and the speech is not under stress.

5th Statement : god bless you
◎'Falsity': 0.0, 'Indeterminacy': 0.169, 'Truth': 0.831◎
0.0 ◎ Falsity,16.900000000000002 Indeterminacy, 83.1 Truth and the speech is not under stress.
6th Statement : beautiful
{'Falsity': 0.0, 'Indeterminacy': 0.0, 'truth': 1.0}
0.0 Falsity, 0.0 ◎ Indeterminacy, 100.0 ◎ Truth the speech is not under stress.

7th Statement :Please help me
◎'Falsity': 0.0, 'Indeterminacy': 0.167, 'Truth': 0.833◎
0.0 ◎ Falsity,16.7 ◎ Indeterminacy, 83.3 ◎ Truth and the speech is not under stress.

8th Statement :hate
◎'Falsity': 1.0, 'Indeterminacy': 0.0, 'Truth': 0.0◎
100.0 ◎ Falsity,0.0 ◎ Indeterminacy, 0.0 ◎ Truth and the speech is under stress.

9th Statement :Great
◎'Falsity': 0.0, 'Indeterminacy': 0.0, 'Truth': 1.0◎
0.0 ◎ Falsity, 0.0 ◎ Indeterminacy,100.0 ◎ Truth and the speech is not under stress.
Fig:1 Stress Analysis using Neutrosophic speech recognition

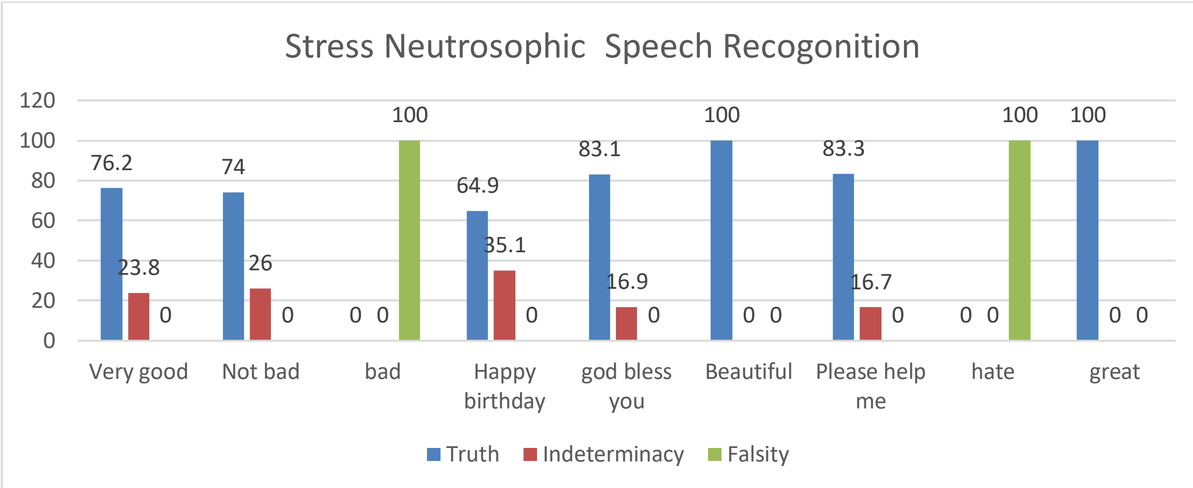

Fig:1 reveal that the percentage of the speech shows the truth, indeterminacy and falsity value .That means the probability of the stress in the speech. The probability value is give the statement is the speech is under stress or not.

Fig:2 Overall rated for Speech

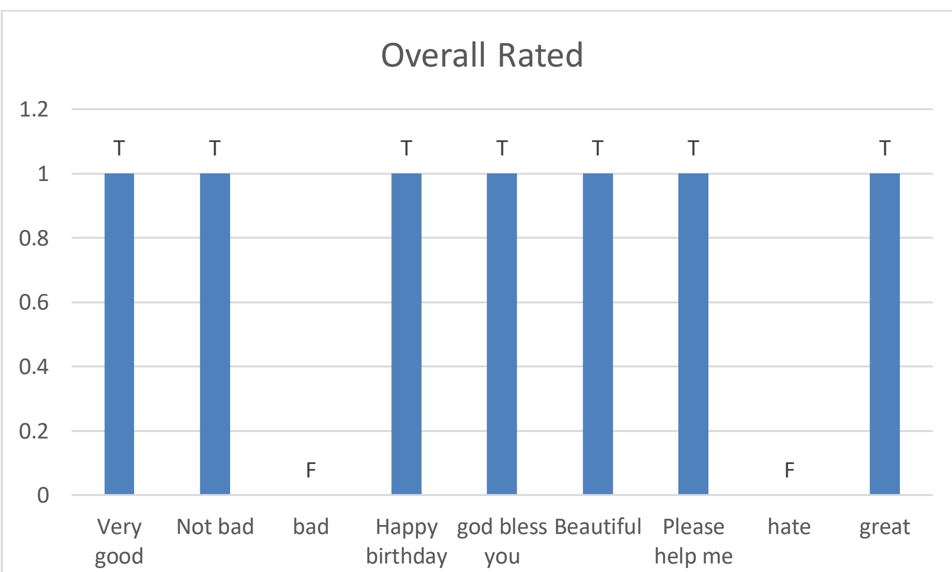

Fig :2 shows that the speech is under stress or not under stress. From the analysis of the speech verygood, notbad, happy birthday, god bless you, beautiful and great is positive speech text. Bad and hate is negative speech .

Conclusion

The requirement to accurately analyse, model, encode, identify, and categorise speech under stress will become increasingly important as speech and language technology develops. The condition of the speaker can be useful information for human-machine and dialogue systems that use voice interaction. This information can be utilised to create speaker and speech recognition technologies, leading to the development of systems that function better in actual multi-tasking environments.The difficulty, though, lies in finding a framework that can effectively analyse and model such speech technologies. The issue of better stress classification utilising targeted speech features has been taken into consideration in this work. categorization of stress The estimation of a probability vector that represents the level of speaker stress is proposed using neurosophic algorithms. Machine learning has demonstrated context-sensitive stress classification. The output stress probability vector can also be used to quantify combinations of speaker stress, such as speech that is both fast and loud. It is claimed that a stress mixture model could be helpful for tasks like sorting emergency phone messages or enhancing the efficiency of traditional speech processing systems. In conclusion, it has been demonstrated that stress classification utilising focused features in Neutrosophic speech recognition algorithm is effective for estimating the level of speaker stress and for providing helpful information for enhancing the performance of a voice recognition algorithm.

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Water Quality Evaluation Using Generalized Correlation Coefficient for M-Polar Neutrosophic Hypersoft Sets

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Abstract: Decision-making is a complex issue, especially for attributes being more than one and further bifurcated. Correlation analysis plays an important role in decision-making problems. For neutrosophic hypersoft sets (NHSSs), they have bifurcated sub-attributes so that we cannot compare the attributive values. Thus, Correlation Coefficient (CC) should be a good tool for decision-making in NHSSs. Moreover, in decision-making problems, most of the times opinion of more than one expert is involved. For dealing this, m-polar values can be better used. The basic purpose of this paper is to propose the concept of CC and Weighted CC (WCC) for m-polar NHSSs with some aggregation operators, theorems, and propositions. Algorithms, based on CC and WCC are also been proposed to solve decision-making problems. Two case studies have been solved by applying the proposed algorithms. The results obtained are compared with existing approaches. The experiment and comparison results reveal the validity and superiority of the proposed methods. They are more accurate and precise. In the future, the proposed methods can be applied to case studies, in which attributes are more than one and further bifurcated along with more than one decision-maker. They can be extended for several existing approaches, like TOPSIS, VIKOR, AHP, and many others.

Keywords: Aggregation operators, Correlation Coefficient (CC); Multi-Criteria Decision-Making (MCDM); Neutrosophic Hypersoft Sets (NHSSs); Weighted Correlation Coefficients (WCC).
2020 MSC classification: 00A69, 03B52, 90B50, 03E72

1. Introduction

Fuzzy Set (FS) theory with the concept of membership was proposed by Zadeh [1]. Nowadays, this theory is at its boom that a gadget used for the ease in our life or even the luxury we feel can be based on the FS theory. The FS had been extended to the new types of set structures. For more accuracy, falsity value is considered, and so the FS was extended to an Intuitionistic FS (IFS) by Atanassov [2] that has membership and non-membership values. A generalization of IFS was given by Yager and Abbasov [3] as Pythagorean FS (PFS). These FS, IFS and PFS had various applications, such as [4-6].

For an extension of FS, IFS and PFS for dealing a scientific gadget with truth membership, falsity membership, and indeterminacy membership, Smarandache [7] proposed a new concept of Neutrosophic Set (NS). The NS added an indeterminacy membership and then extended IFS to truth membership, indeterminacy membership, and falsity membership with a triple (T, I, F) component of memberships. This concept is important because indeterminacy exists extraordinarily in application systems. The NS with (T, I, F) memberships was used by a Decision-Maker (DM) and applied to Multi-Criteria Decision-Making (MCDM) problems. More extensions of NS can refer Awang et al. [8].

On the other hand, Molodtsov [9] first proposed Soft Set (SS) in 1999 in which the SS is a mapping from attributes to the power set of a universal set. The SS can be used for handling issues of indefinite circumstances with a parameterized family of the power set of the universal set. Afterwards, Ali et al. [10] and Cagman and Enginoglu [11] offered new operations and applications of soft sets in a decision making. By combining NS with soft set, Maji [12] proposed Neutrosophic Soft Set (NSS). By extending SS so that it is usable in the cases when attributes are more bi-furcated, Smarandache [13] came up with a new set structure known as HyperSoft Set (HSS). Basically, HSS is a mapping from the product of attributes which are further bi-furcated to the power set of universal set. To deal with truthiness, indeterminacy, and falsity, NHSS was considered in Saqlain et al. [14] where they also applied NHSS to TOPSIS using accuracy function. Saqlain et al. [15] gave similarity measures for NHSSs and Jafar *et al.* [16] proposed trigonometric similarity measures for NHSSs with application to renewable energy source selection.

On the other hand, the importance of bipolarity cannot be ignored in various real-life problems. Bipolarity can give positive and negative information for an object. Zhang [17] first considered a bipolar FS (BFS) for handling fuzziness with bipolarity. The BFS assigns each alternative to a positive membership degree and a negative membership degree between 0 and 1. Alghamdi et al. [18] applied BFS in multi-criteria decision-making and Zhang [19] applied BFS to quantum intelligence machinery. Furthermore, Akram et al. [20] considered m-polar FS and used it in decision making where the m-polar FS is an extension of BFS.

The joint connection between two variables may be used to analyze the interdependence of two or more variables. The correlation analysis can be used as a connection measure which is important in statistics and engineering. The correlation coefficient (CC) between random variables is generally used in correlation analysis. The CC for IFSs was first presented by Gerstenkorn and J. Mafiko [21], and then Bustince and Burillo [22] presented CC for the interval-valued IFSs. Ye [23] proposed CC for the single-valued NS (SVNS) along with an algorithm to solve decision-making problems. Samad et al. [24] considered the CC for NHSSs and applied it to the selection of an effective hand sanitizer to reduce covid-19 effects. Saqlain [25] proposed interval-valued, m-polar and m-polar interval-valued neutrosophic hypersoft set, Irfan et al. [26] later developed the similarity measures of m-polar NHSSs (m-p-NHSSs). However, there is no any CC method for m-p-NHSSs. In this paper, we propose the generalized CC for m-p-NHSSs. Thus, we fill the research gap of the CC methods for m-p-NHSSs. We then use the proposed CC to create the algorithms to solve multi-criteria decision-making (MCDM) problems under the m-p-NHSSs environment. In future, this can be used to create a high machine IQ and hybrid intelligent system by combining the m-polar hypersoft set with other soft computing techniques like bipolar fuzzy, Pythagorean set, and other hybrid structures. These techniques can be used in image processing, expert systems, and cognitive maps.

The remainder of the paper is organized as follows. In Section 2, some basic definitions are reviewed to understand the rest of the article i.e. SSs, NSs, NSSs, HSSs, NHSSs, and m-p-NSSs. In Section 3, we establish the generalized CC for m-p-NSSs, and then some examples and their desirable properties will be also considered in detail. We next develop an algorithm based on the generalized CC for m-p-NHSSs to solve decision-making problems in Section 4. In Section 5, by using these algorithms, we will solve the decision-making problem (case studies) to the m-p-NHSSs environment. In Section 6, results, discussion, and comparison will be discussed. Finally, the conclusion along with future directions will be presented in the last section.

2. Preliminary Section

In this section, we review essential concepts: Soft Sets (SSs), Neutrosophic Sets (NSs), Neutrosophic Soft Sets (NSSs), Hypersoft set (HSS), Neutrosophic Hypersoft Set (NHSSs), and m-polar NHSSs (m-p-NHSSs).

Definition 2.1 [9]. Assume $\mathbb{E}$ is a set of parameters, and $\mathbb{U}$ is a universe set. Suppose the power set of $\mathbb{U}$ is denoted by P($\mathbb{U}$), and $\mathbb{A}\subseteq \mathbb{E}$. A Soft Set (SS) over $\mathbb{U}$ is a pair (ζ, $\mathbb{A}$) where $\zeta: \mathbb{A} \rightarrow \mathbb{P}(\mathbb{U})$ is the mapping of the given set $\mathbb{A}$. To put it another way, the SS (ζ, $\mathbb{A}$) over $\mathbb{U}$ is said to be parameterized subset of $\mathbb{U}$. For $\mathbb{A}$ and $\zeta(\boldsymbol{e})$, the SS of e-approximate or e-elements might be considered (ζ, $\mathbb{A}$), and so (ζ, $\mathbb{A}$) can be given as;

$$(\zeta, \mathbb{A}) = \{\zeta(\boldsymbol{e}) \in \mathbb{P}(\mathbb{U}):\ \boldsymbol{e} \in \mathbb{E}, \zeta(\boldsymbol{e}) = \emptyset \text{ if } \boldsymbol{e} \neq \mathbb{A}\}$$

Definition 2.2 [12]. Assume that $\mathbb{U}$ is a universe set and a collection of attributes that apply to $\mathbb{U}$ is the set of attributes. Suppose that P($\mathbb{U}$) represents the collection of Neutrosophic values of $\mathbb{U}$. A pair (ζ, $\mathbb{A}$) is said to be a Neutrosophic SS (NSS) over $\mathbb{U}$ where ζ is a mapping with $\zeta: \mathbb{A} \rightarrow \mathbb{P}(\mathbb{U})$.

Definition 2.3 [13]. Suppose that the universe set and its power set are given as $\mathbb{U}$ and $\mathbb{P}(\mathbb{U})$, respectively. Let $\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{K}_1}, \boldsymbol{\mathcal{K}_2}, \ldots, \boldsymbol{\mathcal{K}_n}$ for $n \geq 1$ where $\boldsymbol{\mathcal{K}_i}$ specifies the collection of attributes and sub-attributes that are included in them with $\boldsymbol{\mathcal{K}_i} \cap \boldsymbol{\mathcal{K}_j} = \emptyset$, $i \neq j$ for $i, j \in \{1,2,3 \ldots n\}$. Let $\boldsymbol{\mathcal{K}_1} \times \boldsymbol{\mathcal{K}_2} \times \ldots \times \boldsymbol{\mathcal{K}_n} = \mathbb{A}$. Then, a pair $(\zeta, \boldsymbol{\mathcal{K}_1} \times \boldsymbol{\mathcal{K}_2} \times \ldots \times \boldsymbol{\mathcal{K}_n})$ is called a hypersoft set (HSS) over $\mathbb{U}$ defined as $\zeta: \boldsymbol{\mathcal{K}_1} \times \boldsymbol{\mathcal{K}_2} \times \ldots \times \boldsymbol{\mathcal{K}_n} = \mathbb{A} \rightarrow \mathbb{P}(\mathbb{U})$. It is also described as $(\zeta, \mathbb{A}) = \{(a, \zeta_{\mathbb{A}}(a)):\ a \in \mathbb{A}, \zeta_{\mathbb{A}}(a) \in \mathbb{P}(\mathbb{U})\}$.

Definition 2.4 [14]. Suppose $\mathbb{U}$ and P($\mathbb{U}$) are a universal set and power set, respectively. Assumed the well define attributes are $\mathbb{L}^1, \mathbb{L}^2, \ldots, \mathbb{L}^m$ with corresponding attributive values $\mathbb{I}^1, \mathbb{I}^2, \ldots, \mathbb{I}^m$ $for\ m \geq 1$ such that $\mathbb{L}^j \cap \mathbb{L}^k = \emptyset\ for\ j \neq k\ and\ j, k \in \{1,2, \ldots, m\}$ and the relation is $\mathbb{L}^1 \times \mathbb{L}^2 \times \ldots \times \mathbb{L}^m = \delta$. The pair of (ζ, δ) is known as a Neutrosophic HSS (NHSS) over $\mathbb{U}$ with $\zeta: \mathbb{L}^1 \times \mathbb{L}^2 \times \ldots \times \mathbb{L}^m \rightarrow$ P($\mathbb{U}$) and $\zeta(\mathbb{L}^1 \times \mathbb{L}^2 \times \ldots \times \mathbb{L}^m) = \{< x, T(\zeta(\delta)), I(\zeta(\delta)), F(\zeta(\delta)) >, x \in \mathbb{U}\}$, where T is the truthiness, I is the indeterminacy, and F is the falsity membership value with T, I, F: $\mathbb{U} \rightarrow [0,1]$ and also $0 \leq T(\zeta(\delta)) + I(\zeta(\delta)) + F(\zeta(\delta)) \leq 3$.

Definition 2.5 [15]. Let $\mathbb{U}$ be a universe set and let P($\mathbb{U}$) be the power set of $\mathbb{U}$. Let $\mathbb{E}$ be a set of attributes and consider $\mathbb{A} \subseteq \mathbb{E}$. The pair $(\zeta, \mathbb{A})$ is called multi-valued NHSS (MVNHSS) over $\mathbb{U}$ where ζ is a mapping with $\zeta: \mathbb{A} \to \mathbb{P}(\mathbb{U})$ and $(\zeta, \mathbb{A}) = \left\{\frac{\langle \mathbb{T}^x(\zeta(\mathbb{A})), \mathbb{I}^y(\zeta(\mathbb{A})), F^z(\zeta(\mathbb{A}))\rangle}{\mathcal{U}}, \mathcal{U} \in \mathrm{U}\right\}$, where $\mathbb{T}^x(\zeta(\mathbb{A})) \subseteq [0,1]$, $\mathbb{I}^y(\zeta(\mathbb{A})) \subseteq [0,1]$ and $F^z(\zeta(\mathbb{A})) \subseteq [0,1]$ are the multi-valued numbers and they are given as

$\mathbb{T}^x(\zeta(\mathbb{A})) = \mathbb{T}^1(\zeta(\mathbb{A})), \mathbb{T}^2(\zeta(\mathbb{A})), \ldots, \mathbb{T}^x(\zeta(\mathbb{A}))$

$\mathbb{I}^y(\zeta(\mathbb{A})) = \mathbb{I}^1(\zeta(\mathbb{A})), \mathbb{I}^2(\zeta(\mathbb{A})), \ldots, \mathbb{I}^y(\zeta(\mathbb{A}))$

$F^z((\mathbb{A})) = F^1(\zeta(\mathbb{A})), F^2(\zeta(\mathbb{A})), \ldots, F^z(\zeta(\mathbb{A}))$

$\mathbb{T}(\zeta(\mathbb{A})), \mathbb{I}(\zeta(\mathbb{A}))$, and $\mathrm{F}(\zeta(\mathbb{A}))$ represent the truthiness, indeterminacy and falsity of $\mathcal{U}$ to $\mathbb{A}$, respectively.

Definition 2.6 [25]. Let $\mathbb{U} = \{u_1, u_2, \ldots, u_n\}$ be a universe set and $\mathbb{P}(\mathbb{U})$ be the power set of $\mathbb{U}$. Let $\mathbb{L}_1, \mathbb{L}_2, \ldots, \mathbb{L}_b$ for b≥1 be b well-defined attributes whose corresponding attribute values are $\mathbb{L}_1^1, \mathbb{L}_2^2, \ldots, \mathbb{L}_b^n$, respectively, and their relation is $\mathbb{L}_1^a \times \mathbb{L}_2^b \times \ldots \times \mathbb{L}_b^z$ where $a,b,c,\ldots,z = 1,2,\ldots,n$. Then, the pair $(\zeta, \mathbb{L}_1^a \times \mathbb{L}_2^b \times \ldots \times \mathbb{L}_b^z)$ is called to be a m-polar PHSS (m-p-NHSS) over $\mathbb{U}$ where ζ is a mapping with $\zeta: \mathbb{L}_1^a \times \mathbb{L}_2^b \times \ldots \times \mathbb{L}_b^z \to \mathbb{P}(\mathbb{U})$; $\zeta(\mathbb{L}_1^a \times \mathbb{L}_2^b \times \ldots \times \mathbb{L}_b^z) = \{< u, \mathbb{T}_l^i(u), \mathbb{I}_l^j(u), \mathrm{F}_l^k(u) >: u \in \mathbb{U}; \ell \in \mathbb{L}_1^a \times \mathbb{L}_2^b \times \ldots \times \mathbb{L}_b^z$ where $\mathrm{i,j,k} = 1,2,\ldots,\mathrm{n}\}$ and $0 \leq \sum_{i=1}^{p} \mathbb{T}_l^i(u) \leq 1$, $0 \leq \sum_{j=1}^{q} \mathbb{I}_l^j(u) \leq 1$, $0 \leq \sum_{k=1}^{r} \zeta_l^k(u) \leq 1$, where $\mathbb{T}_l^i(u) \subseteq [0,1], \mathbb{I}_l^j(u) \subseteq [0,1]$, and $\zeta_l^k(u) \subseteq [0,1]$ are the numbers with $0 \leq \sum_{i=1}^{p} \mathbb{T}_l^i(u) + \sum_{j=1}^{q} \mathbb{I}_l^j(u) + \sum_{k=1}^{r} \mathrm{F}_l^k(u) \leq 3$.
For convenience, we assume that

$\mathbb{T}_l^i(u) = \mathbb{T}_{l1}^1(u), \mathbb{T}_{l2}^2(u), \mathbb{T}_{l3}^3(u), \ldots, \mathbb{T}_{lp}^p(u)$

$\mathbb{I}_l^j(u) = \mathbb{I}_{l1}^1(\mathrm{u}), \mathbb{I}_{l2}^2(\mathrm{u}), \mathbb{I}_{l3}^3(\mathrm{u}), \ldots, \mathbb{I}_{lq}^q(\mathrm{u})$

$F_l^k(u) = \mathrm{F}_{l1}^1(u), \mathrm{F}_{l2}^2(u), \mathrm{F}_{l3}^3(u), \ldots, \mathrm{F}_{lr}^r(u)$

3. Calculations

In this section, we propose informational energies, generalized CC and aggregation operators for m-polar NHSSs (m-p-NHSSs).

Definition 3.1. Informational energies for m-p-NHSSs

Let $(\wp, \ddot{A})$ *and* $((Q, \ddot{B}))$ be two m-p-NHSSs with

$$(\wp, \ddot{A}) = \left\{\left(v_i, \tau_{\wp(\breve{d}_k)}(v_i)^i, \mathfrak{J}_{\wp(\breve{d}_k)}(v_i)^j, \mathfrak{G}_{\wp(\breve{d}_k)}(v_i)^k\right) \middle| v_i \in u\right\}$$

$$(Q, \ddot{B}) = \left\{\left(v_i, \tau_{Q(\breve{d}_k)}(v_i)^i, \mathfrak{J}_{Q(\breve{d}_k)}(v_i)^j, \mathfrak{G}_{Q(\breve{d}_k)}(v_i)^k\right) \middle| v_i \in u\right\}.$$

Then, their informational energies are defined as

$$\varsigma_{m-p-NHSS}(\wp, \ddot{A}) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sum_{i=1}^{p}\left(\tau_{\wp(\breve{d}_k)i}^{i}(v_i)\right)^2 + \sum_{j=1}^{q}\left(\mathfrak{J}_{\wp(\breve{d}_k)j}^{j}(v_i)\right)^2 + \sum_{k=1}^{r}\left(\mathfrak{G}_{\wp(\breve{d}_k)k}^{k}(v_i)\right)^2\right) \quad (3.1)$$

$$\varsigma_{m-p-NHSS}(Q,\ddot{B}) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sum_{i=1}^{p}\left(\tau^{i}_{Q(\check{d}_k)i}\ (v_i)\right)^2 + \sum_{j=1}^{q}\left(\mathfrak{I}^{j}_{Q(\check{d}_k)j}\ (v_i)\right)^2 + \sum_{k=1}^{r}\left(\mathfrak{G}^{k}_{Q(\check{d}_k)k}\ (v_i)\right)^2\right) \quad (3.2)$$

Definition 3.2. Covariance for two m-p-NHSSs

Let $(\wp,\ddot{A})$ ***and*** $((Q,\ddot{B}))$ be two m-p-NHSSs with

$$(\wp,\ddot{A}) = \left\{\left(v_i, \tau_{\wp(\check{d}_k)}(v_i)^i, \mathfrak{I}_{\wp(\check{d}_k)}(v_i)^j, \mathfrak{G}_{\wp(\check{d}_k)}(v_i)^k\right)\middle| v_i \in u\right\}$$

$$(Q,\ddot{B}) = \left\{\left(v_i, \tau_{Q(\check{d}_k)}(v_i)^i, \mathfrak{I}_{Q(\check{d}_k)}(v_i)^j, \mathfrak{G}_{Q(\check{d}_k)}(v_i)^k\right)\middle| v_i \in u\right\}.$$

Then, the covariance between $(\wp,\ddot{A})$ ***and*** $((Q,\ddot{B}))$ is defined as

$$C_{m-p-NHSS}\left((\wp,\ddot{A}),(Q,\ddot{B})\right) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sum_{i=1}^{p}\left(\tau^{i}_{\wp(\check{d}_k)i}\ (v_i) * \mathfrak{I}^{j}_{Q(\check{d}_k)j}\ (v_i)\right) + \sum_{j=1}^{q}\left(\mathfrak{I}^{j}_{\wp(\check{d}_k)j}\ (v_i) * \mathfrak{I}^{j}_{Q(\check{d}_k)j}\ (v_i)\right) + \sum_{k=1}^{r}\left(\mathfrak{G}^{k}_{\wp(\check{d}_k)k}\ (v_i) * \mathfrak{G}^{k}_{Q(\check{d}_k)k}\ (v_i)\right)\right) \quad (3.3)$$

Definition 3.3. Correlation coefficient for two m-p-NHSSs

Let $(\wp,\ddot{A})$ **and** $((Q,\ddot{B}))$ be two m-p-NHSSs with

$$(\wp,\ddot{A}) = \left\{\left(v_i, \tau_{\wp(d_k)}(v_i)^i, \mathfrak{I}_{\wp(d_k)}(v_i)^j, \mathfrak{G}_{\wp(d_k)}(v_i)^k\right)\middle| v_i \in u\right\}$$

$(Q,\ddot{B}) = \left\{\left(v_i, \tau_{Q(d_k)}(v_i)^i, \mathfrak{I}_{Q(d_k)}(v_i)^j, \mathfrak{G}_{Q(d_k)}(v_i)^k\right)\middle| v_i \in u\right\}$. Then, CC between them is defined as

$$\delta_{m-p-NHSS}((\wp,A^{\cdots}),(Q,B^{\cdots})) = \frac{c_{m-p-NHSS}\ ((\wp,A^{\cdots}),(Q,B^{\cdots}))}{\sqrt{\varsigma_{m-p-NHSS}((\wp,\ddot{A})}*\sqrt{\varsigma_{m-p-NHSS}((Q,\ddot{B})}} \quad (3.4)$$

Example 3.4. Let $U = \{s^1, s^2, s^3, s^4, s^5\}$ be the set of nominated schools and consider the set of attributes with $E = \{\textit{teaching standard}, \textit{organization}, \textit{ongoing evaluation}, \textit{goals}\}$ Let $A \subseteq E$ with $A = \{A_1, A_2, A_3, A_4\}$ such that $A_1 = \textbf{Teaching standard}$, $A_2 = \textbf{Organization}$, $A_3 = \textit{ongoing evaluation}, A_4 =$ **goals**. These attributes are further bifurcated as $A_1^a \rightarrow A_1 \rightarrow$ teaching standard → (High, mediocre, low); $A_2^b \rightarrow A_2 \rightarrow$ organization → (good, average, poor); $A_3^c \rightarrow A_3 \rightarrow$ ongoing evaluation → (yes, no); $A_4^d \rightarrow A_4 \rightarrow$ Goals → (effective, committed, up to date). Define a mapping with **F**(high, average, yes, effective)= $\{s^1, s^5\}$. Then, $(\wp,\ddot{A}) =$

$$\left\{\begin{pmatrix} s^1 < A_1^a\{(0.4,0.3,0.2),(0.3,0.2,0.4),(0.1,0.3,0.5)\}, A_2^b\{(0.5,0.4,0.2),(0.4,0.3,0.5),(0.1,0.4,0.5)\}, \\ A_3^c\{(0.6,0.2,0.1),(0.3,0.4,0.2),(0.1,0.3,0.4)\}, A_4^d\{(0.2,0.3,0.1),(0.4,0.1,0.2),(0.1,0.2,0.4)\} > \\ s^5 < A_1^a\{(0.3,0.2,0.4),(0.4,0.2,0.3),(0.1,0.3,0.2)\}, A_2^b\{(0.4,0.3,0.2),(0.4,0.1,0.5),(0.1,0.3,0.5)\}, \\ A_3^c\{(0.6,0.3,0.1),(0.3,0.1,0.2),(0.1,0.2,0.4)\}, A_4^d\{(0.3,0.2,0.1),(0.4,0.3,0.2),(0.1,0.3,0.4)\} > \end{pmatrix}\right\}$$

Also, $B \subseteq E$, $B = \{B_1, B_2, B_3, B_4\}$. Further, bi-furcated attributes of "B" are $B_1^a \rightarrow B_1 \rightarrow$ teaching standard → (High, mediocre, low); $B_2^b \rightarrow B_2 \rightarrow$ organization → (good, average, poor); $B_3^c \rightarrow B_3 \rightarrow$ ongoing evaluation → (yes, no); $B_4^d \rightarrow B_4 \rightarrow$ Goals → (effective, committed, up-to-date). Consider another mapping **G** (high, good, yes, up-to-date)= $\{s^2, s^3\}$. Then, we have

$(Q,\ddot{B}) =$

$$\left\{\begin{pmatrix} s^2 < B_1^a\{(0.5,0.3,0.2),(0.4,0.2,0.1),(0.1,0.3,0.4)\}, B_2^b\{(0.5,0.3,0.2),(0.4,0.3,0.2),(0.1,0.3,0.5)\}, \\ B_3^c\{(0.6,0.1,0.2),(0.3,0.4,0.1),(0.1,0.3,0.2)\}, B_4^d\{(0.4,0.3,0.1),(0.4,0.3,0.1),(0.1,0.2,0.5)\} > \\ s^3 < B_1^a\{(0.4,0.2,0.3),(0.4,0.2,0.3),(0.1,0.3,0.4)\}, B_2^b\{(0.4,0.1,0.2),(0.4,0.1,0.3),(0.1,0.3,0.4)\}, \\ B_3^c\{(0.6,0.2,0.1),(0.3,0.4,0.2),(0.1,0.2,0.5)\}, B_4^d\{(0.3,0.2,0.4),(0.4,0.1,0.2),(0.1,0.2,0.4)\} > \end{pmatrix}\right\}$$

Thus, we have $\delta_{m-p-NHSS}(\wp,\ddot{A}) = 7.26$; $\delta_{m-p-NHSS}(Q,\ddot{B}) = 6.78$, and $\delta_{m-p-NHSS}\left((\wp,\ddot{A})(Q,\ddot{B})\right) = \frac{6.54}{\sqrt{7.26}*\sqrt{6.78}} = 0.93 \in [0,1]$.It shows that $(\wp,\ddot{A})$ and $(Q,\ddot{B})$ have a good positive relation.

Proposition 3.5. Let $(\wp,\ddot{A}) = \left\{\left(v_i, \tau_{\wp(\ddot{d}_k)}(v_i)^i, \mathfrak{J}_{\wp(\ddot{d}_k)}(v_i)^j, \mathfrak{G}_{\wp(\ddot{d}_k)}(v_i)^k\right) \middle| v_i \in u\right\}$ and $(Q,\ddot{B}) = \left\{\left(v_i, \tau_{Q(\ddot{d}_k)}(v_i)^i, \mathfrak{J}_{Q(\ddot{d}_k)}(v_i)^j, \mathfrak{G}_{Q(\ddot{d}_k)}(v_i)^k\right) \middle| v_i \in u\right\}$ be two m-p-NHSSs and let $\mathcal{C}_{m-p-NHSS}((\wp,\ddot{A}),(Q,\ddot{B}))$ be a CC between them. It satisfies the following properties:

1. $\mathcal{C}_{m-p-NHSS}(\wp,\ddot{A}),(\wp,\ddot{A}) = \varsigma_{m-p-NHSS}(\wp,\ddot{A})$.

2. $\mathcal{C}_{m-p-NHSS}(Q,\ddot{B}),(Q,\ddot{B}) = \varsigma_{m-p-NHSS}(Q,\ddot{B}))$.

Theorem 3.6. Let $(\wp,\ddot{A}) = \{(v_i, T_{\wp(d_k)}(v_i)^i, I_{\wp(d_k)}(v_i)^j, C_{\wp(d_k)}(v_i)^k) \mid v_i \in U\}$ and

$(Q,\ddot{B}) = \{(v_i, T_{Q(d_k)}(v_i)^i, I_{Q(d_k)}(v_i)^j, C_{Q(d_k)}(v_i)^k) \mid v_i \in U\}$ be two m-p-NHSSs, then CC between them satisfies the following properties:

$0 \leq \delta_{m-p-NHSS}\left((\wp,\ddot{A}),(Q,\ddot{B})\right) \leq 1$

$\delta_{m-p-NHSS}\left((\wp,\ddot{A}),(Q,\ddot{B})\right) = \delta_{m-p-NHSS}\left((\wp,\ddot{A}),(Q,\ddot{B})\right)$

iff $\left((\wp,\ddot{A}) = (Q,\ddot{B})\right)$.

If $T_{\wp(d_k)}(v_i)^i = T_{Q(d_k)}(v_i)^i$, $I_{\wp(d_k)}(v_i)^j = I_{Q(d_k)}(v_i)^j$, and

$C_{\wp(d_k)}(v_i)^k = C_{Q(d_k)}(v_i)^k$, then $\delta_{m-p-NHSS}((\wp,\ddot{A}), (Q,\ddot{B})) = 1$.

Whenever experts regulate distinctive weights for every alternative, the choice might be dissimilar. So, it is precisely to plot the weights for experts preceding assembling a decision. Assume the weights of experts can be expressed as $\Omega = \{\Omega_1, \Omega_2, \Omega_3, \ldots, \Omega_m\}^T$, where $\Omega_k > 0$, $\sum_{k=1}^{m} \Omega_k = 1$. Similarly, assume that the weights for sub-attributes are as follows $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n\}^T$, where $\gamma_i > 0$, $\sum_{i=1}^{n} \gamma_i = 1$.

Definition 3.7. Weighted CC for two m-p-NHSSs

Let $(\wp,\ddot{A}) = \{(v_i, T_{\wp(d_k)}(v_i)^i, I_{\wp(d_k)}(v_i)^j, C_{\wp(d_k)}(v_i)^k) \mid v_i \in U\}$ and $(Q,\ddot{B}) = \{(v_i, T_{Q(d_k)}(v_i)^i, I_{Q(d_k)}(v_i)^j, C_{Q(d_k)}(v_i)^k) \mid v_i \in U\}$ be two m-p-NHSSs, then, a weighted CC (WCC) among them is expressed as $\delta_{m-p-wNHSS}((\wp,\ddot{A}),(Q,\ddot{B}))$ and defined as follows:

$$\delta_{m-p-wNHSS}((\wp,\ddot{A}),(Q,\ddot{B})) = \frac{\mathcal{C}_{m-p-wNHSS}((\wp,\ddot{A}),(Q,\ddot{B}))}{\sqrt{\varsigma_{m-p-wNHSS}(\wp,\ddot{A})} * \sqrt{\varsigma_{m-p-wNHSS}(Q,\ddot{B})}} \quad (3.5)$$

i.e.

$$\delta_{m-p-wNHSS}\left((\wp,\ddot{A}),(Q,\ddot{B})\right)=$$

$$\frac{\sum_{k=1}^{m}\Omega_k\left(\left(\left(\begin{array}{c}\sum_{i=1}^{p}\left(\sum_{i=1}^{n}\sqrt{\gamma_i}\mathcal{T}^{i}_{\wp(\breve{d}_k)i}(v_i)*\sum_{i=1}^{n}\sqrt{\gamma_i}\sum_{i=1}^{n}\gamma_i\mathcal{T}^{i}_{Q(\breve{d}_k)i}(v_i)\right)+\\ \sum_{j=1}^{q}\left(\sum_{i=1}^{n}\sqrt{\gamma_i}\mathcal{I}^{j}_{\wp(\breve{d}_k)j}(v_i)*\sum_{i=1}^{n}\sqrt{\gamma_i}\ \mathcal{I}^{j}_{Q(\breve{d}_k)j}(v_i)\right)+\sum_{k=1}^{r}\left(\sum_{i=1}^{n}\sqrt{\gamma_i}\ \mathfrak{C}^{k}_{\wp(\breve{d}_k)k}(v_i)*\sum_{i=1}^{n}\sqrt{\gamma_i}\ \mathfrak{C}^{k}_{Q(\breve{d}_k)k}(v_i)\right)\end{array}\right)\right)\right)}{\sqrt{\sum_{k=1}^{m}\Omega_k\left(\left(\sum_{i=1}^{p}\left(\sum_{i=1}^{n}\gamma_i\mathcal{T}^{i}_{\wp(\breve{d}_k)i}(v_i)\right)^2+\sum_{j=1}^{q}\left(\sum_{i=1}^{n}\gamma_i\mathcal{I}^{j}_{\wp(\breve{d}_k)j}(v_i)\right)^2+\sum_{k=1}^{r}\left(\sum_{i=1}^{n}\gamma_i\mathfrak{C}^{k}_{\wp(\breve{d}_k)k}(v_i)\right)^2\right)\right)}\sqrt{\sum_{k=1}^{m}\Omega_k\left(\left(\sum_{i=1}^{p}\left(\sum_{i=1}^{n}\gamma_i\mathcal{T}^{i}_{Q(\breve{d}_k)i}(v_i)\right)^2+\sum_{j=1}^{q}\left(\sum_{i=1}^{n}\gamma_i\mathcal{I}^{j}_{Q(\breve{d}_k)j}(v_i)\right)^2+\sum_{k=1}^{r}\left(\sum_{i=1}^{n}\gamma_i\mathfrak{C}^{k}_{\wp(\breve{d}_k)k}(v_i)\right)^2\right)\right)}}$$

Theorem 3.8. Let $(\wp,\ddot{A})=\left\{\left(v_i,\tau_{\wp(\breve{d}_k)}(v_i)^i,\mathfrak{I}_{\wp(\breve{d}_k)}(v_i)^j,\mathfrak{G}_{\wp(\breve{d}_k)}(v_i)^k\right)\middle|v_i\in u\right\}$ and

$(Q,\ddot{B})=\left\{\left(v_i,\tau_{Q(\breve{d}_k)}(v_i)^i,\mathfrak{I}_{Q(\breve{d}_k)}(v_i)^j,\mathfrak{G}_{Q(\breve{d}_k)}(v_i)^k\right)\middle|v_i\in u\right\}$ be two m-p-NHSSs, then WCC between them satisfies the following properties:

$0\ \leq\ \delta_{m-p-wNHSS}((\wp,\ddot{A})(Q,\ddot{B}))\ \leq 1$

$$\delta_{m-p-wNHSS}\left((\wp,\ddot{A})(Q,\ddot{B})\right)=\ \delta_{m-p-wNHSS}((Q,\ddot{B}),(\wp,\ddot{A}))\ \ iff\ \ (\wp,\ddot{A})=(Q,\ddot{B})$$

$T_{\wp(d_k)}(v_i)^i=T_{Q(d_k)}(v_i)$, $I_{\wp(d_k)}(v_i)^j=\ I_{Q(d_k)}(v_i)^j$ and $C_{\wp(d_k)}(v_i)^k=\ C_{Q(d_k)}(v_i)^k$
then $\delta_{m-p-wNHSS}\left((\wp,\ddot{A}),(Q,\ddot{B})\right)=1$

Proposition 3.9. Let $(\wp,\ddot{A})=\left\{\left(v_i,\tau_{\wp(\breve{d}_k)}(v_i)^i,\mathfrak{I}_{\wp(\breve{d}_k)}(v_i)^j,\mathfrak{G}_{\wp(\breve{d}_k)}(v_i)^k\right)\middle|v_i\in u\right\}$

Consider $J_{d_k}=\ \langle {T_{F(d_{ij})}}^i,{I_{F(d_{ij})}}^j,{C_{F(d_{ij})}}^k\rangle$, $J_{d_{11}}=\ \langle {T_{F(d_{11})}}^i,{I_{F(d_{11})}}^j,{C_{F(d_{11})}}^k\rangle$ and $J_{d_{12}}=$ $\langle {T_{F(d_{11})}}^i,{I_{F(d_{12})}}^j,{C_{F(d_{12})}}^k\rangle$ as three m-p-NHSSs and α be a positive real number, by algebraic norms, then

$${J_{d_{11}}}^i\oplus {J_{\breve{d}_{12}}}^i=\langle {T_{F(d_{11})}}^i+{T_{F(d_{12})}}^i-{T_{F(d_{11})}}^i{T_{F(d_{12})}}^i,{J_{F(d_{11})}}^j{J_{F(d_{12})}}^j,\quad {C_{F(d_{11})}}^k{C_{F(d_{12})}}^k\rangle$$

${J_{d_{11}}}^i\otimes {J_{\breve{d}_{12}}}^i=\langle {T_{F(d_{11})}}^i{T_{F(d_{12})}}^i,{J_{F(d_{11})}}^j+{J_{F(d_{12})}}^j-{J_{F(d_{11})}}^j{J_{F(d_{12})}}^j,\ {C_{F(d_{11})}}^k+{C_{F(d_{12})}}^k-$
${C_{F(d_{11})}}^k{C_{F(d_{12})}}^k\rangle$

$\alpha J_{d_k}=\langle 1-\left(1-{T_{d_k}}^i\right)^{\alpha},{J_{d_k}}^{j^{\alpha}},{C_{d_k}}^{k^{\alpha}}\rangle$

$$J\ \ {}_{d_k}{}^{i^{\alpha}}=\langle,1-\left(1-{J_{d_k}}^j\right)^{\alpha},1-\left(1-{C_{d_k}}^k\right)^{\alpha}\rangle$$

Definition 3.10. Aggregate operator for m-p-NHSSs

Let $(\wp,\ddot{A})=\left\{\left(v_i,\tau_{\wp(\breve{d}_k)}(v_i)^i,\mathfrak{I}_{\wp(\breve{d}_k)}(v_i)^j,\mathfrak{G}_{\wp(\breve{d}_k)}(v_i)^k\right)\middle|v_i\in u\right\}$ and $J_{d_k}=$ $\langle {T_{F(d_{ij})}}^i,{I_{F(d_{ij})}}^j,{C_{F(d_{ij})}}^k\rangle$ be an m-p-NHSS. Ω_i and γ_j are weight vector for expert's and sub-attributes of the considered attributes correspondingly along with specified circumstances $\Omega_i>0$, $\sum_{i=1}^{n}\Omega_i=1$, $\gamma_j>0$, $\sum_{j=1}^{m}\gamma_j=1$. Then m-p-NHSS aggregate operator is defined as $\mathbf{m-PNHSWA}$:

$\Delta^n \rightarrow \Delta$ where $(\mathfrak{J}_{\breve{d}_{11}}, \mathfrak{J}_{\breve{d}_{12}}, \dots, \mathfrak{J}_{\breve{d}_{nm}}) = \oplus^{m}_{j=1} \boldsymbol{\gamma_j} \left(\oplus^{n}_{i=1} \Omega_i \mathfrak{J}_{\breve{d}_{ij}}\right) = \left\langle \mathbf{1} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\mathbf{1} - \mathcal{T}_{\breve{d}_{ij}}{}^{i}\right)^{\Omega_i}\right)^{\gamma_j}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\mathcal{J}_{\breve{d}_{ij}}{}^{j}\right)^{\Omega_i}\right)^{\gamma_j}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\mathfrak{C}_{\breve{d}_{ij}}{}^{k}\right)^{\Omega_i}\right)^{\gamma_j} \right\rangle$ (3.6)

4. Proposed Algorithms

In this section, we develop the algorithm based on Correlation Coefficient (CC) and Weighted Correlation Coefficient (WCC) under m-PNHSs and utilize the proposed approach for decision making in real life problems.

Algorithm 4.1.

The proposed algorithm 4.1, can be used solve MCDM problems based on CC of m-PNHSs and shown in Figure 1.

Step 1: Select Hypersoft sets $(\wp, \ddot{A})$ and $(\boldsymbol{Q}, \ddot{\boldsymbol{B}})$

Step 2: Construction of m-PNHSs by assigning m-PNHSN to each sub-attribute and solve them to get SVNHSs.

Step 3: Find the informational energies of the selected m-PNHSs using the formula;

$$\varsigma_{m-PNHSs}(\wp, \ddot{\boldsymbol{A}}) = \sum_{k=1}^{m} \sum_{i=1}^{n} \left(\sum_{i=1}^{p} \left(\mathcal{T}^{i}_{\wp(\breve{d}_k)i}(v_i) \right)^2 + \sum_{j=1}^{q} \left(\mathcal{I}^{j}_{\wp(\breve{d}_k)j}(v_i) \right)^2 + \sum_{k=1}^{r} \left(\mathfrak{C}^{k}_{\wp(\breve{d}_k)k}(v_i) \right)^2 \right)$$

Step 4: Calculate the correlation between the selected m-PNHS sets $(\wp, \ddot{A})$ and $(\boldsymbol{Q}, \ddot{\boldsymbol{B}})$ by using the formula;

$$\mathcal{C}_{m-PNHSs}\ (\wp, \ddot{\boldsymbol{A}}), (\boldsymbol{Q}, \ddot{\boldsymbol{B}}) = \sum_{k=1}^{m} \sum_{i=1}^{n} \left(\sum_{i=1}^{p} \left(\mathcal{T}^{i}_{\wp(\breve{d}_k)i}(v_i) * \mathcal{T}^{i}_{Q(\breve{d}_k)i}(v_i) \right) + \sum_{j=1}^{q} \left(\mathcal{I}^{j}_{\wp(\breve{d}_k)j}(v_i) * \mathcal{I}^{j}_{Q(\breve{d}_k)j}(v_i) \right) + \sum_{k=1}^{r} \left(\mathfrak{C}^{k}_{\wp(\breve{d}_k)k}(v_i) * \mathfrak{C}^{k}_{Q(\breve{d}_k)k}(v_i) \right) \right)$$

Step 5: Calculate correlation coefficients of the selected m-PNHS sets $(\wp, \ddot{\boldsymbol{A}})$ and $(\wp, \ddot{\boldsymbol{B}})$ by using the formula;

$$\boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\wp, \ddot{\boldsymbol{A}}), (\boldsymbol{Q}, \ddot{\boldsymbol{B}}) = \frac{\mathcal{C}_{\mathbf{m-PNHSs}}((\wp, \ddot{\boldsymbol{A}}), (\boldsymbol{Q}, \ddot{\boldsymbol{B}}))}{\sqrt{\varsigma_{\mathbf{m-PNHSs}}(\wp, \ddot{A})} * \sqrt{\varsigma_{\mathbf{m-PNHSs}}(\mathbf{Q}, \ddot{\boldsymbol{B}})}}$$

Step 6: Arrange the alternatives in descending order of the CC values.

Step 7: Rank the alternatives from largest to smallest CC values.

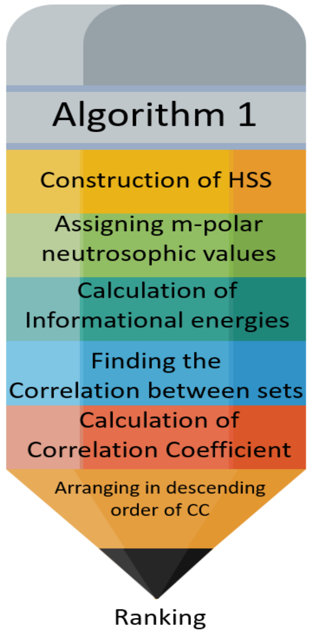

Figure 1. Algorithm based on Correlation Coefficient form-PNHSs

Algorithm 4.2.

The proposed algorithm 4.2, can be used solve MCDM problems based on WCC of m-PNHSs and shown in Figure 2.

Step 1: Construction of Hypersoft set and sub-attribute parameters.

Step 2: Assigning m-PNHSNs to the selected sets.

Step 3: Find the weighted informational energies for m-PNHSs using the formula;

$$\varsigma_{m-PWNHSs}(\wp,\ddot{A}) = \sum_{k=1}^{m} \Omega_k \left(\left(\sum_{i=1}^{p} \left(\sum_{i=1}^{n} \gamma_i \mathcal{T}^{i}_{\wp(\breve{a}_k)_i}(v_i) \right)^2 + \sum_{j=1}^{q} \left(\sum_{i=1}^{n} \gamma_i \mathcal{J}^{j}_{\wp(\breve{a}_k)_j}(v_i) \right)^2 + \sum_{k=1}^{r} \left(\sum_{i=1}^{n} \gamma_i \mathfrak{C}^{k}_{\wp(\breve{a}_k)k}(v_i) \right)^2 \right) \right)$$

Step 4. Calculate the Weighted Correlation between two m-PNHSs by using the formula;

$$\mathcal{C}_{m-PWNHSs}((\wp,\ddot{A}),(\boldsymbol{Q},\ddot{\boldsymbol{B}})) =$$

$$\sum_{k=1}^{m} \Omega_k \left(\left(\left(\sum_{i=1}^{p} \left(\sum_{i=1}^{n} \sqrt{\gamma_i}\, \mathcal{T}^{i}_{\wp(\check{a}_k)i}(v_i) * \sum_{i=1}^{n} \sqrt{\gamma_i} \sum_{i=1}^{n} \gamma_i\, \mathcal{T}^{i}_{\mathcal{Q}(\check{a}_k)i}(v_i) \right) + \sum_{j=1}^{q} \left(\sum_{i=1}^{n} \sqrt{\gamma_i}\, \mathcal{I}^{j}_{\wp(\check{a}_k)j}(v_i) * \sum_{i=1}^{n} \sqrt{\gamma_i}\ \ \mathcal{I}^{j}_{\mathcal{Q}(\check{a}_k)j}(v_i) \right) + \sum_{k=1}^{r} \left(\sum_{i=1}^{n} \sqrt{\gamma_i}\ \ \mathfrak{C}^{k}_{\wp(\check{a}_k)k}(v_i) * \sum_{i=1}^{n} \sqrt{\gamma_i}\ \ \mathfrak{C}^{k}_{\mathcal{Q}(\check{a}_k)k}(v_i) \right) \right) \right) \right)$$

Step 5: Calculate the WCC between two m-PNHSs by using the formula;

$$\boldsymbol{\delta}_{m-PWNHSs}((\wp,\ddot{A}),(\boldsymbol{Q},\ddot{\boldsymbol{B}})) = \frac{\mathcal{C}_{m-PWNHSs}((\wp,\ddot{A}),(\mathbf{Q},\ddot{\boldsymbol{B}}))}{\sqrt{\varsigma_{m-PWNHSs}(\boldsymbol{Q},\ddot{A})} * \sqrt{\varsigma_{m-PWNHSs}(\boldsymbol{Q},\ddot{\boldsymbol{B}})}}$$

Step 6: Arrange the alternative in descending order of WCC and rank the alternative from highest to the lowest.

5. Experiment

Lahore Garrison University (LGU) wanted to hire a teacher in Mathematics department, let $\mathbb{P} = \{\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^4\}$ be a set of candidates (alternatives) who has been shortlisted for the interview. The Interview committee consists of four decision-makers (DM), $\boldsymbol{\mathcal{D}} = \{\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{\sigma_3}, \boldsymbol{\sigma_4}\}$. The committee will decide the criteria (attributes) to fill up the said post which are $\boldsymbol{\mathcal{L}} = \{\boldsymbol{\ell_1} = \boldsymbol{Experience}, \boldsymbol{\ell_2} = \boldsymbol{Dealing\ \ skills}, \boldsymbol{\ell_3} = \boldsymbol{Qualification}\}$ be the set of attributes and their corresponding sub-attributes are given by;

$$\boldsymbol{\ell_1} = \boldsymbol{Experience} = \{\boldsymbol{a_{11}} = \boldsymbol{less\ \ than\ \ 20}, \boldsymbol{a_{12}} = \boldsymbol{more\ \ than\ \ 20}\ \}$$

Figure 2. Algorithm based on Weighted Correlation Coefficient for m-PNHSs

$$\boldsymbol{\ell_2 = Dealing\ \ skills} = \begin{Bmatrix} \boldsymbol{a_{21} = Good\ \ Communication\ \ skills,} \\ \boldsymbol{a_{22} = Good\ \ Teaching\ \ skills,} \\ \boldsymbol{a_{23} = Certified\ \ skills} \end{Bmatrix},$$

$$\boldsymbol{\ell_3 = Qualifaication} = \{\boldsymbol{a_{31} = M.Phil., a_{32} = PhD. a_{33} = Post\ \ Doctorate}\}$$

Solved example using Algorithm 4.1

Assume case study formulated above. The DM will assign values in term of m-PNHS numbers, based on their knowledge and expertise to each candidate.

Step 1: Define a mapping, and select Hypersoft set.

$$\boldsymbol{F}:\ \boldsymbol{\ell_1} \times\ \ \boldsymbol{\ell_2} \times \boldsymbol{\ell_3} = \boldsymbol{\mathcal{L}'} \rightarrow \boldsymbol{P}(\beth) = \mathbb{P}^1, \mathbb{P}^2$$

Step 2: Assigning values to the selected Hypersoft set in term of m-polar Neutrosophic number by considering m=3 as presented in Table 1-10.

Table 1. Neutrosophic m-polar values for alternative ℵ

ℵ	δ_1	δ_2	δ_3	δ_4

$\breve{a}_1$	(<0.2,0.1,0.2>,<0.1,0.2, 0.1>, <0.3,0.2,0.1>)	(<0.2,0.2,0.2>,<0.1,0.0, 0.2>, <0.2,0.3,0.1>)	(<0.2,0.1,0.1>,<0.2,0.2, 0.2>, <0.2,0.3,0.1>)	(<0.1,0.2,0.0>,<0.3,0.2, 0.2>, <0.1,0.1,0.1>)
$\breve{a}_2$	(<0.1,0.0,0.2>,<0.4,0.2, 0.1 ,<0.1,0.2,0.0>)	(<0.3,0.4,0.0>,<0.2,0.0, 0.1>,<0.2,0.1,0.3>)	(<0.2,0.1,0.1>,<0.1,0.3, 0.1>,<0.1,0.2,0.2>)	(<0.1,0.2,0.1>,<0.2,0.1, 0.2>,<0.0,0.3,0.4>)
$\breve{a}_3$	(<0.3,0.1,0.2>,<0.1,0.0, 0.2>,<0.2,0.1,0.1>)	(<0.2,0.1,0.1>,<0.1,0.0, 0.3>,<0.2,0.1,0.2>)	(<0.1,0.2,0.3>,<0.1,0.2, 0.0>,<0.1,0.3,0.0>)	(<0.1,0.2,0.1>,<0.2,0.1, 0.2>,<0.2,0.1,0.3>)
$\breve{a}_4$	(<0.1,0.2,0.1>,<0.2,0.1, 0.2>,<0.3,0.1,0.2>)	(<0.2,0.4,0.2>,<0.1,0.0, 0.1>,<0.2,0.1,0.2>)	(<0.2,0.1,0.2>,<0.1,0.1, 0.1>,<0.2,0.2,0.2>)	(<0.2,0.3,0.2>,<0.0,0.2, 0.0>,<0.2,0.2,0.2>)
$\breve{a}_5$	(<0.1,0.0,0.2>,<0.3,0.0, 0.1>,<0.3,0.1,0.1>)	(<0.2,0.2,0.2>,<0.2,0.1, 0.2>,<0.2,0.1,0.1>)	(<0.2,0.2,0.0>,<0.3,0.2, 0.0>,<0.3,0.2,0.2>)	(<0.2,0.2,0.2>,<0.1,0.1, 0.1>,<0.1,0.2,0.1>)
$\breve{a}_6$	(<0.2,0.2,0.2>,<0.1,0.1, 0.1>,<0.2,0.1,0.1>)	(<0.2,0.1,0.2>,<0.1,0.3, 0.1>,<0.1,0.2,0.0>)	(<0.2,0.2,0.1>,<0.4,0.0, 0.1>,<0.2,0.0,0.1>)	(<0.1,0.0,0.1>,<0.3,0.1, 0.4>,<0.2,0.1,0.1>)
$\breve{a}_7$	(<0.3,0.1,0.3>,<0.2,0.0, 0.1>,<0.2,0.1,0.1>)	(<0.1,0.0,0.1>,<0.4,0.1, 0.3>,<0.2,0.1,0.1>)	(<0.2,0.0,0.2>,<0.3,0.3, 0.2>,<0.0,0.2,0.0>)	(<0.2,0.0,0.2>,<0.2,0.1, 0.1>,<0.2,0.1,0.2>)
$\breve{a}_8$	(<0.2,0.1,0.2>,<0.2,0.4, 0.2>,<0.1,0.0,0.1>)	(<0.1,0.3,0.3>,<0.0,0.2, 0.0>,<0.2,0.1,0.3>)	(<0.0,0.1,0.1>,<0.3,0.3, 0.2>,<0.1,0.2,0.1>)	(<0.1,0.2,0.3>,<0.2,0.0, 0.1>,<0.2,0.2,0.2>)

Table 2. Neutrosophic m-polar values for alternative $\mathbb{P}^1$

$\mathbb{P}^1$	δ_1	δ_2	δ_3	δ_4
$\breve{a}_1$	(<0.2,0.1,0.1>,<0.2,0.1, 0.2>,<0.2,0.1,0.1>)	(<0.1,0.2,0.1>,<0.2,0.1, 0.2>,<0.3,0.1,0.3>)	(<0.1,0.2,0.3>,<0.0,0.1, 0.2>,<0.3,0.1,0.1>)	(<0.2,0.2,0.1>,<0.1,0.1, 0.2>,<0.4,0.1,0.2>)
$\breve{a}_2$	(<0.1,0.2,0.1>,<0.2,0.1, 0.2>,<0.3,0.1,0.1>)	(<0.2,0.2,0.3>,<0.0,0.1, 0.1>,<0.1,0.1,0.0>)	(<0.3,0.2,0.2>,<0.0,0.1, 0.1>,<0.1,0.1,0.2>)	(<0.1,0.2,0.1>,<0.1,0.1, 0.2>,<0.2,0.1,0.2>)
$\breve{a}_3$	(<0.2,0.2,0.2>,<0.0,0.1, 0.2>,<0.1,0.1,0.0>)	(<0.1,0.2,0.2>,<0.2,0.1, 0.2>,<0.1,0.1,0.0>)	(<0.3,0.2,0.3>,<0.0,0.1, 0.1>,<0.1,0.1,0.2>)	(<0.1,0.2,0.1>,<0.2,0.4, 0.2>,<0.0,0.1,0.1>)
$\breve{a}_4$	(<0.1,0.2,0.1>,<0.2,0.1, 0.3>,<0.0,0.1,0.1>)	(<0.2,0.2,0.3>,<0.1,0.1, 0.0>,<0.1,0.1,0.2>)	(<0.1,0.2,0.0>,<0.2,0.1, 0.2>,<0.2,0.1,0.2>)	(<0.4,0.2,0.1>,<0.0,0.1, 0.1>,<0.1,0.1,0.0>)

$\breve{a}_5$	(<0.1,0.1,0.1>,<0.2,0.3,0.2>,<0.0,0.1,0.2>)	(<0.3,0.2,0.3>,<0.1,0.1,0.0>,<0.2,0.1,0.2>)	(<0.1,0.2,0.1>,<0.1,0.1,0.2>,<0.1,0.1,0.0>)	(<0.1,0.2,0.3>,<0.1,0.1,0.1>,<0.3,0.0,0.2>)
$\breve{a}_6$	(<0.2,0.2,0.1>,<0.2,0.1,0.2>,<0.1,0.0,0.2>)	(<0.1,0.2,0.1>,<0.2,0.4,0.2>,<0.0,0.1,0.1>)	(<0.1,0.2,0.2>,<0.2,0.1,0.1>,<0.2,0.1,0.2>)	(<0.1,0.2,0.1>,<0.2,0.1,0.2>,<0.3,0.1,0.2>)
$\breve{a}_7$	(<0.4,0.2,0.1>,<0.0,0.1,0.1>,<0.2,0.1,0.0>)	(<0.1,0.2,0.1>,<0.2,0.1,0.2>,<0.3,0.1,0.2>)	(<0.1,0.2,0.2>,<0.1,0.1,0.2>,<0.3,0.1,0.2>)	(<0.1,0.2,0.3>,<0.2,0.1,0.0>,<0.3,0.1,0.1>)
$\breve{a}_8$	(<0.3,0.2,0.1>,<0.0,0.1,0.2>,<0.2,0.1,0.2>)	(<0.1,0.2,0.1>,<0.2,0.1,0.2>,<0.2,0.1,0.1>)	(<0.1,0.2,0.3>,<0.0,0.1,0.2>,<0.3,0.0,0.2>)	(<0.1,0.2,0.0>,<0.2,0.1,0.2>,<0.2,0.1,0.2>)

Table 3. Neutrosophic m-polar values for alternative $\mathbb{P}^2$

$\mathbb{P}^2$	δ_1	δ_2	δ_3	δ_4
$\breve{a}_1$	(<0.2,0.1,0.1>,<0.2,0.2,0.2>,<0.5,0.1,0.1>)	<0.2,0.2,0.1>,<0.3,0.1,0.1>, <0.0,0.1,0.1>)	<0.0,0.1,0.3>,<0.4,0.1,0.1>,<0.2,0.1,0.1>)	<0.1,0.2,0.1>,<0.2,0.1,0.2>,<0.3,0.3,0.2>)
$\breve{a}_2$	(<0.1,0.2,0.1>,<0.2,0.2,0.2>,<0.3,0.1,0.1>)	<0.2,0.2,0.3>,<0.1,0.1,0.1>, <0.2,0.1,0.1>)	<0.1,0.2,0.1>,<0.2,0.1,0.1>,<0.3,0.1,0.1>)	<0.2,0.2,0.1>,<0.3,0.0,0.2>,<0.3,0.3,0.1>)
$\breve{a}_3$	(<0.2,0.0,0.0>,<0.3,0.4,0.1>,<0.1,0.1,0.3>)	<0.0,0.2,0.1>,<0.5,0.1,0.1>, <0.3,0.2,0.1>)	<0.1,0.2,0.2>,<0.2,0.2,0.1>,<0.0,0.1,0.1>)	<0.1,0.1,0.0>,<0.3,0.1,0.1>,<0.3,0.2,0.1>)
$\breve{a}_4$	(<0.1,0.2,0.2>,<0.2,0.0,0.3>,<0.0,0.1,0.1>)	<0.1,0.2,0.1>,<0.3,0.1,0.1>, <0.3,0.2,0.1>)	<0.1,0.0,0.1>,<0.2,0.1,0.2>,<0.2,0.3,0.1>)	<0.2,0.2,0.0>,<0.2,0.1,0.1>,<0.0,0.1,0.1>)
$\breve{a}_5$	(<0.1,0.1,0.1>,<0.2,0.3,0.1>,<0.2,0.1,0.2>)	<0.1,0.2,0.1>,<0.2,0.1,0.1>, <0.0,0.1,0.1>)	<0.2,0.2,0.1>,<0.3,0.1,0.1>,<0.3,0.3,0.1>)	<0.2,0.0,0.3>,<0.3,0.1,0.1>,<0.0,0.1,0.1>)
$\breve{a}_6$	(<0.1,0.0,0.1>,<0.2,0.4,0.2>,<0.1,0.2,0.2>)	<0.0,0.1,0.1>,<0.3,0.1,0.1>, <0.3,0.2,0.1>)	<0.2,0.1,0.2>,<0.2,0.2,0.1>,<0.3,0.2,0.2>)	<0.1,0.0,0.1>,<0.3,0.1,0.1>,<0.3,0.2,0.1>)
$\breve{a}_7$	(<0.4,0.2,0.2>,<0.0,0.1,0.1>,<0.2,0.1,0.0>)	<0.2,0.2,0.0>,<0.3,0.1,0.1>, <0.3,0.2,0.1>)	<0.1,0.1,0.0>,<0.3,0.1,0.1>,<0.3,0.2,0.1>)	<0.2,0.2,0.1>,<0.2,0.2,0.1>,<0.0,0.1,0.1>)
$\breve{a}_8$	(<0.0,0.2,0.1>,<0.4,0.1,0.2>,<0.2,0.1,0.2>)	<0.2,0.1,0.1>,<0.1,0.3,0.1>, <0.3,0.1,0.2>)	<0.2,0.2,0.3>,<0.0,0.1,0.1>,<0.2,0.1,0.1>)	<0.0,0.1,0.1>,<0.3,0.1,0.1>,<0.3,0.2,0.1>)

Table 4. Neutrosophic m-polar values for alternative $\mathbb{P}^3$

$\mathbb{P}^3$	δ_1	δ_2	δ_3	δ_4

$\breve{a}_1$	<0.2,0.2,0.3>,<0.0,0.1,0.1>,<0.3,0.1,0.1>)	<0.3,0.2,0.2>,<0.2,0.2,0.0>,<0.1,0.1,0.1>)	<0.2,0.0,0.0>,<0.2,0.0,0.2>,<0.3,0.1,0.1>)	<0.3,0.4,0.1>,<0.2,0.0,0.2>,<0.0,0.1,0.1>)
$\breve{a}_2$	<0.2,0.4,0.2>,<0.1,0.1,0.0>,<0.1,0.1,0.1>)	<0.2,0.2,0.3>,<0.2,0.1,0.2>,<0.1,0.1,0.1>)	<0.2,0.2,0.3>,<0.1,0.1,0.1>,<0.4,0.2,0.1>)	<0.5,0.1,0.1>,<0.1,0.0,0.1>,<0.3,0.1,0.1>)
$\breve{a}_3$	<0.2,0.3,0.2>,<0.2,0.1,0.2>,<0.1,0.1,0.0>)	<0.2,0.2,0.2>,<0.0,0.2,0.2>,<0.0,0.1,0.1>)	<0.2,0.2,0.2>,<0.2,0.0,0.1>,<0.0,0.1,0.1>)	<0.1,0.0,0.1>,<0.2,0.1,0.2>,<0.4,0.1,0.1>)
$\breve{a}_4$	<0.3,0.3,0.2>,<0.1,0.0,0.1>,<0.3,0.1,0.1>)	<0.1,0.4,0.3>,<0.2,0.2,0.1>,<0.1,0.1,0.1>)	<0.2,0.2,0.1>,<0.2,0.1,0.2>,<0.4,0.2,0.1>)	<0.2,0.3,0.1>,<0.1,0.1,0.2>,<0.0,0.1,0.1>)
$\breve{a}_5$	<0.2,0.2,0.3>,<0.0,0.1,0.1>,<015,0.0,0.1>)	<0.2,0.0,0.0>,<0.1,0.2,0.2>,<0.4,0.1,0.1>)	<0.5,0.2,0.1>,<0.0,0.2,0.0>,<0.0,0.1,0.1>)	<0.1,0.1,0.1>,<0.1,0.1,0.2>,<0.2,0.2,0.1>)
$\breve{a}_6$	<0.4,0.2,0.2>,<0.1,0.0,0.1>,<0.2,0.1,0.1>)	<0.1,0.,20.1>,<0.3,0.1,0.1>,<0.2,0.2,0.2>)	<0.2,0.4,0.2>,<0.0,0.0,0.2>,<0.2,0.1,0.1>)	<0.1,0.1,0.1>,<0.3,0.2,0.2>,<0.3,0.2,0.0>)
$\breve{a}_7$	<0.1,0.1,0.1>,<0.2,0.3,0.2>,<0.3,0.1,0.1>)	<0.2,0.1,0.3>,<0.1,0.1,0.1>,<0.1,0.0,0.1>)	<0.1,0.1,0.0>,<0.1,0.3,0.1>,<0.4,0.1,0.1>)	<0.1,0.6,0.1>,<0.0,0.0,0.2>,<0.2,0.2,0.1>)
$\breve{a}_8$	<0.2,0.4,0.2>,<0.0,0.1,0.1>,<0.1,0.1,0.1>)	<0.2,0.2,0.2>,<0.1,0.2,0.0>,<0.0,0.1,0.1>)	<0.2,0.5,0.1>,<0.0,0.1,0.1>,<0.1,0.1,0.1>)	<0.2,0.3,0.1>,<0.1,0.1,0.1>,<0.0,0.1,0.1>)

Table 5. Neutrosophic m-polar values for alternative $\mathbb{P}^4$

$\mathbb{P}^4$	δ_1	δ_2	δ_3	δ_4
$\breve{a}_1$	<0.1,0.1,0.3>,<0.2,0.1,0.1>,<0.0,0.1,0.1>)	<0.1,0.2,0.2>,<0.5,0.1,0.1>,<0.2,0.1,0.1>)	<0.2,0.2,0.0>,<0.2,0.1,0.1>,<0.3,0.1,0.1>)	<0.1,0.0,0.1>,<0.3,0.1,0.1>,<0.3,0.2,0.1>)
$\breve{a}_2$	<0.2,0.2,0.2>,<0.1,0.0,0.1>,<0.3,0.1,0.3>)	<0.2,0.2,0.3>,<0.3,0.1,0.1>,<0.3,0.1,0.1>)	<0.2,0.2,0.2>,<0.1,0.1,0.1>,<0.3,0.1,0.1>)	<0.2,0.2,0.3>,<0.2,0.1,0.1>,<0.1,0.1,0.1>)
$\breve{a}_3$	<0.2,0.2,0.3>,<0.2,0.1,0.1>,<0.1,0.1,0.1>)	<0.2,0.2,0.4>,<0.0,0.1,0.1>,<0.1,0.1,0.1>)	<0.2,0.0,0.3>,<0.3,0.1,0.1>,<0.0,0.1,0.1>)	<0.2,0.2,0.1>,<0.2,0.2,0.1>,<0.1,0.1,0.1>)
$\breve{a}_4$	<0.3,0.2,0.3>,<0.2,0.1,0.1>,<0.3,0.1,0.1>)	<0.2,0.2,0.0>,<0.1,0.1,0.1>,<0.3,0.1,0.1>)	<0.2,0.2,0.1>,<0.4,0.1,0.1>,<0.3,0.3,0.1>)	<0.2,0.2,0.2>,<0.3,0.1,0.1>,<0.2,0.1,0.1>)
$\breve{a}_5$	<0.2,0.2,0.3>,<0.2,0.1,0.1>,<0.2,0.1,0.1>)	<0.1,0.1,0.3>,<0.1,0.1,0.1>,<0.3,0.1,0.1>)	<0.2,0.2,0.3>,<0.0,0.1,0.1>,<0.3,0.1,0.1>)	<0.2,0.2,0.4>,<0.2,0.1,0.1>,<0.1,0.1,0.1>)

$\breve{a}_6$	<0.2,0.2,0.2>,<0.3,0.1,0.1>,<0.2,0.1,0.1>)	<0.1,0.2,0.1>,<0.1,0.1,0.1>,<0.4,0.1,0.1>)	<0.2,0.2,0.2>,<0.3,0.1,0.1>,<0.0,0.1,0.1>)	<0.2,0.2,0.2>,<0.1,0.1,0.1>,<0.0,0.1,0.1>)
$\breve{a}_7$	<0.1,0.2,0.0>,<0.5,0.1,0.1>,<0.2,0.1,0.1>)	<0.2,0.4,0.2>,<0.4,0.1,0.1>,<0.1,0.1,0.1>)	<0.2,0.2,0.3>,<0.0,0.1,0.1>,<0.3,0.1,0.1>)	<0.2,0.2,0.3>,<0.3,0.1,0.1>,<0.2,0.1,0.1>)
$\breve{a}_8$	<0.2,0.2,0.3>,<0.2,0.1,0.1>,<0.3,0.1,0.1>)	<0.2,0.1,0.3>,<0.1,0.1,0.1>,<0.0,0.1,0.1>)	<0.2,0.2,0.1>,<0.1,0.1,0.1>,<0.3,0.2,0.1>)	<0.2,0.2,0.0>,<0.2,0.1,0.1>,<0.3,0.1,0.1>)

Table 6. Neutrosophic values for alternative $\aleph$

$\aleph$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	**(0.5, 0.4, 0.**	**(0.3, 0.7, 0.**	**(0.6, 0.3, 0.**	**(0.4, 0.5, 0.**	**(0.3, 0.4, 0.**	**(0.6, 0.3, 0.**	**(0.7, 0.3, 0.**	**(0.5, 0.8, 0**
δ_2	**(0.6, 0.3, 0.**	**(0.7, 0.3, 0.**	**(0.4, 0.4, 0.**	**(0.8, 0.2, 0.**	**(0.6, 0.5, 0.**	**(0.5, 0.5, 0.**	**(0.2, 0.8, 0.**	**(0.7, 0.2, 0**
δ_3	**(0.4, 0.6, 0**	**(0.4, 0.4, 0.**	**(0.6, 0.3, 0.**	**(0.5, 0.3, 0.**	**(0.4, 0.5, 0.**	**(0.5, 0.5, 0.**	**(0.4, 0.8, 0.**	**(0.2, 0.8, 0**
δ_4	**(0.3, 0.7, 0.**	**(0.4, 0.5, 0.**	**(0.4, 0.5, 0.**	**(0.7, 0.2, 0.**	**(0.6, 0.3, 0.**	**(0.2, 0.8, 0.**	**(0.4, 0.4, 0.**	**(0.6, 0.3, 0**

Table 7. Neutrosophic values for alternative $\mathbb{P}^1$

$\mathbb{P}^1$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	**(0.4, 0.5, 0.**	**(0.4, 0.5, 0.**	**(0.6, 0.3, 0.**	**(0.4, 0.6, 0.**	**(0.3, 0.7, 0.**	**(0.5, 0.5, 0.**	**(0.7, 0.2, 0.**	**(0.6, 0.3, 0.**
δ_2	**(0.4, 0.5, 0.**	**(0.7, 0.2, 0.**	**(0.5, 0.5, 0.**	**(0.7, 0.2, 0.**	**(0.8, 0.2, 0.**	**(0.4, 0.8, 0.**	**(0.4, 0.5, 0.**	**(0.4, 0.5, 0.**
δ_3	**(0.6, 0.3, 0**	**(0.7, 0.2, 0.**	**(0.8, 0.2, 0.**	**(0.3, 0.5, 0.**	**(0.4, 0.4, 0.**	**(0.5, 0.4, 0.**	**(0.5, 0.4, 0.**	**(0.7, 0.2, 0.**
δ_4	**(0.5, 0.4, 0.**	**(0.4, 0.4, 0.**	**(0.4, 0.8, 0.**	**(0.7, 0.2, 0.**	**(0.6, 0.3, 0.**	**(0.4, 0.5, 0.**	**(0.6, 0.3, 0.**	**(0.3, 0.5, 0.**

Table 8. Neutrosophic values for alternative $\mathbb{P}^2$

$\mathbb{P}^2$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	**(0.4, 0.6, 0.**	**(0.3, 0.6, 0**	**(0.2, 0.8, 0**	**(0.5, 0.5, 0**	**(0.3, 0.6, 0**	**(0.2, 0.8, 0**	**(0.8, 0.2, 0**	**(0.3, 0.7, 0**
δ_2	**(0.5, 0.5, 0.**	**(0.7, 0.2, 0**	**(0.3, 0.7, 0**	**(0.4, 0.5, 0**	**(0.4, 0.4, 0**	**(0.2, 0.5, 0**	**(0.4, 0.5, 0**	**(0.4, 0.5, 0**
δ_3	**(0.4, 0.6,**	**(0.4, 0.4, 0**	**(0.5, 0.5, 0**	**(0.2, 0.5, 0**	**(0.5, 0.5, 0**	**(0.5, 0.5, 0**	**(0.2, 0.5, 0**	**(0.7, 0.2, 0**

δ_4	(0.4, 0.5, 0.	(0.5, 0.5, 0	(0.2, 0.5, 0	(0.4, 0.4, 0	(0.5, 0.5, 0	(0.2, 0.5, 0	(0.5, 0.5, 0	(0.2, 0.5, 0

Table 9. Neutrosophic values for alternative $\mathbb{P}^3$

$\mathbb{P}^3$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	(0.7, 0.2, 0.	(0.8, 0.2, 0	(0.7, 0.5, 0	(0.8, 0.2, 0	(0.7, 0.2, 0	(0.8, 0.2, 0	(0.3, 0.7, 0	(0.8, 0.2, 0
δ_2	(0.7, 0.4, 0.	(0.7, 0.5, 0	(0.6, 0.4, 0	(0.8, 0.5, 0	(0.2, 0.5, 0	(0.4, 0.5, 0	(0.6, 0.3, 0	(0.6, 0.3, 0
δ_3	(0.4, 0.4, (	(0.7, 0.3, 0	(0.6, 0.3, 0	(0.5, 0.5, 0	(0.8, 0.2, 0	(0.8, 0.2, 0	(0.2, 0.5, 0	(0.8, 0.2, 0
δ_4	(0.8, 0.4, 0.	(0.7, 0.2, 0	(0.2, 0.5, 0	(0.6, 0.4, 0	(0.4, 0.4, 0	(0.3, 0.7, 0	(0.8, 0.2, 0	(0.6, 0.3, 0

Table 10. Neutrosophic values for alternative $\mathbb{P}^4$

$\mathbb{P}^4$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	(0.5, 0.4, 0	(0.6, 0.2, 0	(0.7, 0.4, 0	(0.8, 0.4, 0	(0.7, 0.4, 0	(0.6, 0.5, 0	(0.3, 0.7, 0	(0.7, 0.4, 0
δ_2	(0.5, 0.7, 0	(0.7, 0.5, 0	(0.8, 0.2, 0	(0.4, 0.3, 0	(0.5, 0.3, 0	(0.4, 0.3, 0	(0.8, 0.6, 0	(0.6, 0.3, 0
δ_3	(0.4, 0.4, 0	(0.6, 0.3, 0	(0.5, 0.5, 0	(0.5, 0.6, 0	(0.7, 0.2, 0	(0.6, 0.5, 0	(0.7, 0.2, 0	(0.5, 0.3, 0
δ_4	(0.2, 0.5, 0	(0.7, 0.4, 0	(0.5, 0.5, 0	(0.6, 0.5, 0	(0.8, 0.4, 0	(0.6, 0.3, 0	(0.7, 0.5, 0	(0.4, 0.4, 0

Step 3: Find informational energies of $\aleph$ ***and*** $\mathbb{P}^2$ using the formula;

$$\varsigma_{m-PNHSs}(\wp,\ddot{A}) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sum_{i=1}^{p}\left(\mathcal{T}^{i}_{\wp(\breve{a}_k)i}(v_i)\right)^2 + \sum_{j=1}^{q}\left(\mathcal{I}^{j}_{\wp(\breve{a}_k)j}(v_i)\right)^2 + \sum_{k=1}^{r}\left(\mathfrak{C}^{k}_{\wp(\breve{a}_k)k}(v_i)\right)^2\right)$$

and we get,

$\varsigma_{m-PNHSs}(\aleph) = 23.7$

$\varsigma_{m-PNHSs}(\mathbb{P}^1) = 21.8$

Step 4: Now we'll calculate correlation by using the formula;

$$\mathcal{C}_{m-PNHSs}((\wp,\ddot{A}),(\mathcal{Q},\ddot{B})) =$$

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\sum_{i=1}^{p}\left(\mathcal{T}^{i}_{\wp(\breve{a}_k)i}(v_i) * \mathcal{T}^{i}_{\mathcal{Q}(\breve{a}_k)i}(v_i)\right) + \sum_{j=1}^{q}\left(\mathcal{I}^{j}_{\wp(\breve{a}_k)j}(v_i) * \mathcal{I}^{j}_{\mathcal{Q}(\breve{a}_k)j}(v_i)\right) + \sum_{k=1}^{r}\left(\mathfrak{C}^{k}_{\wp(\breve{a}_k)k}(v_i) * \mathfrak{C}^{k}_{\mathcal{Q}(\breve{a}_k)k}(v_i)\right)\right)$$

$= \mathcal{C}_{m-PNHSs}((\aleph, \mathbb{P}^1) = 19.95$

Step 5: Now we'll find CC by using the formula;

$$\boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^1) = \frac{\mathcal{C}_{m-PNHSs}((\aleph,\mathbb{P}^1)}{\sqrt{\varsigma_{m-PNHSs}(\aleph)} * \sqrt{\varsigma_{m-PNHSs}(\mathbb{P}^1)}} = \frac{\mathbf{19.95}}{\sqrt{\mathbf{23.7}} * \sqrt{\mathbf{21.8}}} = \mathbf{0.877}$$

Repeating the step 3, and step 4 to calculate CC of the given candidates;

$\boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^1) = \mathbf{0.877}$,

$\boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^2) = \mathbf{0.885}$,

$\boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^3) = \mathbf{0.774}$,

$\boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^4) = \mathbf{0.880}$,

Step 6: Arrange the CC values in descending order,

$\boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^2) = \mathbf{0.885} > \boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^4) = \mathbf{0.880} > \boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^1) = \mathbf{0.877} >$
$\boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^3) = \mathbf{0.774}$

Step 7: Rank the alternatives from largest to smallest CC and informational energy values, from above results, $\boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^2) = \mathbf{0.885}$ has highest CC. Therefore, the position of Mathematics teacher at LGU can be filled by hiring $\mathbb{P}^2$ alternative.

Real Application for Water quantity evaluation (Problem formulation)

Water that is safe to drink is a basic requirement for good health. Water supply organizations place a high priority on quantity-related issues while paying little attention to drinking water quality concerns. We must supply safe drinking water (not necessarily excellent tasting) as well as appetizing food (pleasing to drink). The following four criteria are used to characterize the quality of drinking water: Physical, chemical, microbiological, and radiological. Due to water quality and quantity difficulties in Pakistan, access to clean drinking water is one of the country's public health concerns. A large percentage of the country's drinking water (almost 70%) originates from underground aquifers. Toxic metals such as arsenic, iron, and mercury have been found in some places due to bacterial contamination. Fluorides are a serious danger to the country's water quality. Microbial pollution of drinking water has been identified as a major source of sickness and mortality among Pakistanis, particularly youngsters, who are particularly sensitive. Water contamination is estimated to be the cause of 30% of all diseases and 40% of all fatalities in the country. Unfortunately, the drinking water quality issue in the country receives little attention, and most people consume water without understanding if it is safe or hazardous for them. The drinking water standards in Pakistan were evaluated by the ministry of health, the Government of Pakistan, and the World Health Organization (WHO). Pakistan adheres to WHO drinking water quality norms and standards (Pak-EPA-2008, And the Gazette of Pakistan 2010) and the data is listed in Table 11.

Table 11. Comparison of National and International water quality standards

Parameters	Pakistan standards (mg/L)	WHO standards (mg/L)

Color	$\leq \mathbf{15}$ TCU	$\leq \mathbf{15}$ TCU
Odor	**resinous / fragrant**	**resinous / fragrant**
Turbidity	5 NTU	5 NTU
pH	6.5-8.5	6.5-8.5
Chloride	250	250
Fluoride	$\leq \mathbf{1.5}$	1.5
Lead	$\leq \mathbf{0.05}$	0.01
Manganese	$\leq \mathbf{0.5}$	0.5
Zinc	5.0	3
Arsenic	0.05	0.01
Magnesium	$\leq \mathbf{100}$	30
Calcium	$\leq \mathbf{100}$	60-120
Sulfate	$<$ 250	$\leq \mathbf{250}$
Sodium	100	$\leq \mathbf{200}$
Iron	0.3	0.3

Consider U = $\{\boldsymbol{S_1}, \boldsymbol{S_2}, \boldsymbol{S_3}\}$ are there samples of water, we've to check which sample of water is safe for drinking purposes according to world health organization standards and we have taken a WHO standard parameter and represented with $\boldsymbol{\omega}$ ideal water (safe to drink). Consider the parameters describe above in Table 11. P = $\{\wp_1 =$ Color, $\wp_2 =$ Turbidity, $\wp_3 =$ pH, $\wp_4 =$ odour, $\wp_5 =$ Chloride, $\wp_6 = \boldsymbol{Fluroide}$, $\wp_7 =$ Magnesium, $\wp_8 =$ Calcium, $\wp_9 =$ Sulphate, $\wp_{10} =$ Sodium, $\wp_{11} =$Iron, $\wp_{12} =$ Arsenic, $\wp_{13} =$Manganese , $\wp_{14} =$ Lead, $\wp_{15} =$Zinc$\}$. These attributes are further divided as:

$\wp_1 =$ Color $\rightarrow \{\boldsymbol{a_{11} \leq 15}$ TCU, $\boldsymbol{a_{11} \geq 15\ TCU}\}$

$\wp_2 =$ Turbidity $\rightarrow \{\boldsymbol{a_{21} \leq 5}$ **NTU** $, \boldsymbol{a_{22} > 5NTU}\}$

$\wp_3 =$ pH $\rightarrow \{\ \boldsymbol{6.5 \leq a_{31} \leq 8.5}\ , \boldsymbol{a_{32} = other}\}$

$\wp_4 =$ Odour $\rightarrow \{a_{41} = \textbf{resinous}\ , a_{42} = \textbf{fragrant}\}$

$\wp_5 =$ Chloride$\rightarrow \{a_{51} = \textbf{250mg/L},\ a_{52} > \textbf{250 mg/ L}\}$

$\wp_6 =$Fluoride$\rightarrow \left\{a_{61} \leq 1.5\frac{mg}{L}, a_{62} > 1.5\ mg/L\right\}$

$\wp_7 =$ Magnesium $\rightarrow \left\{a_{71} \leq \frac{100mg}{L},\ a_{72} > 100\ mg/L\right\}$

$\wp_8 =$ Calcium $\rightarrow \left\{a_{81} \leq \frac{100mg}{L}, a_{82} > 100\ mg/L\right\}$

$\wp_9 =$ Sulfate $\rightarrow \left\{a_{91} \leq \frac{250mg}{L}, a_{92} > 250\ mg/L\right\}$

$\wp_{10} =$ Sodium$\rightarrow \left\{a_{10} = \frac{100mg}{L}, a_{10,2} > 100mg/L\right\}$

$\wp_{11} =$ Iron $\rightarrow \left\{a_{11,1} = \frac{0.3mg}{L}, a_{11,2} > 0.3mg/L\right\}$

$\wp_{12} =$ Arsenic $\rightarrow \left\{a_{12,1} < \frac{0.05mg}{L}, a_{12,2} > 0.05mg/L\right\}$

$\wp_{13} =$ Manganese $\rightarrow \left\{a_{13,1} \leq \frac{100mg}{L}, a_{13,2} > 100mg/L\right\}$

$\wp_{14} =$ Lead $\rightarrow \left\{a_{14,1} < \frac{0.05mg}{L}, a_{14,2} > 0.05mg/L\right\}$

$\wp_{15} =$ Zinc $\left\{a_{15,1} < \frac{5mg}{L}, a_{15,2} > 5mg/L\right\}$

The ideal water $\omega = F\Big(\textbf{Color} \leq \textbf{15TCU}, \textbf{Turbidity} = \textbf{5 NTU}, \textbf{PH} = 6.5 - 8.5, \textbf{Odour} = \textbf{Acceptable}, \textbf{Chloride} = \textbf{250mg/L}, \textbf{Fluoride} \leq 1.5\frac{mg}{L}, \textbf{Magnesium} \leq \frac{100mg}{L}, \textbf{Calcium} \leq \frac{100mg}{L}, \textbf{Sulfate} \leq \frac{250mg}{L}, \textbf{\textit{Sodium}} = \frac{100mg}{L}, \textbf{Iron} = \frac{0.3mg}{L}, \textbf{Arsenic} < \frac{0.05mg}{L}, \textbf{Manganese} \leq \frac{100mg}{L}, \textbf{Lead} < \frac{0.05mg}{L}, \textbf{Zinc} < 5mg/L\Big)$ (a)

And $\mathcal{DM} = \{\mathcal{DM}_1, \mathcal{DM}_2\}$ and $\Omega = \{\Omega_1 = 0.6, \Omega_1 = 0.4\}^T$ be the set of decision makers and their weights respectively.

Solved Example using Algorithm 4.2

Step 1: Define a mapping, and select Hypersoft set;

$$F: \wp_1 \times \wp_2 \times \wp_3 \times \ldots\ldots \times \wp_{15} = \gamma' \rightarrow P(\aleph) = S^1, S^2, S^3$$

compute the Weighted Correlation Coefficient (WCC) between $\delta_{m-PWNHSS}(\omega, S_1)$, $\delta_{m-PWNHSs}(\omega, S_2)$, $\delta_{m-PWNHSs}(\omega, S_3)$, to check that whether taken sample of water has positive correlation with the safe water ω, or not, if yes then it means the sample of water which is taken is safe to use for drinking purposes if the value of correlation coefficient is less than 0.50 then it means that water requires treatment before use, check the Truthiness, Indeterminacy and falsity values of sample regarding each attribute, those attributes which has unbalance ratio according to National standard for safe water (ω). Now, 1st we'll find $\delta_{m-PNHSs}(\omega, S_1)$ i.e. (Correlation coefficient between ω(safe water) and S_1(Sample 1). Let $\{\omega\}$, and $\{S_1\}$ be the two sets having sub-attributes, Step 2: We construct m-PNHSs in the form of m-PNHSNs by assigning Neutrosophic values to the selected alternatives of Hypersoft set i.e. $\boldsymbol{S^1}, \boldsymbol{S^2}, \boldsymbol{S^3}$. Also, DM will assign m=3 neutrosophic values to the ideal water i.e. $\boldsymbol{\omega}$ and shown in Table 12-15. Their simplified Neutrosophic form is shown in Table 16-19.

Table 12. Neutrosophic m-polar values for alternative $\boldsymbol{\omega}$

ω	$\mathcal{DM}_1$	$\mathcal{DM}_2$
$Color \leq$ 15TCU	(<0.2,0.4,0.3>,<0.02,0.01,0.02>,<0.03,0.01,0.01>)	(<0.20,0.40,0.25>,<0.10,0.5,0.0>,<0.0,0.1,0.0>)
$Turbidity = 5\ NTU$	(<0.2,0.4,0.2>,<0.1,0.0,0.0>,<0.0,0.0,0.1>)	(<0.2,0.4,0.1>,<0.1,0.1,0.0>,<0.0,0.0,0.1>)
$pH = 6.5 - 8.5$	(<0.3,0.3,0.3>,<0.02,0.01,0.02>,<0.02,0.02,0.01)	(<0.20,0.40,0.25>,<0.1,0.,0.0>,<0.03,0.02,0.0>)
Odor = $resinous$	(<0.25,0.25,0.35>,<0.1,0.0,0.0>,<0.03,0.01,0.01>)	(<0.2,0.4,0.2>,<0.1,0.0,0.0>,<<0.0,0.1,0.0>)
$Chloride = 250\ mg/\ L$	(<0.2,0.4,0.2>,<0.1,0.0,0.0>,<<0.03,0.01,0.01>)	(<0.20,0.40,0.25>,<0.1,0.0,0.0>,<0.01,0.01,0.03>)
Fluoride $\leq$ 1.5mg/L	(<0.2,0.4,0.2>,<0.1,0.0,0.0>,<<0.0,0.1,0.0>)	(<0.2,0.4,0.1>,<0.1,0.0,0.0>,<0.1,0.1,0.0>)
$Magnesium \leq 100mg/L$	(<0.2,0.4,0.2>,<0.03,0.01,0.01>,<0.0,0.1,0.0>)	(<0.2,0.4,0.2>,<0.1,0.0,0.0>,<0.0,0.1,0.0>)
Calcium $\leq$ 100 mg/L	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.1,0.0>)	(<0.2,0.4,0.2>,<0.1,0.0,0.0>,<0.0,0.1,0.0>)
$Sulfate \leq 250mg/L$	(<0.2,0.4,0.2>,<0.03,0.01,0.01>,<0.0,0.1,0.0>)	(<0.2,0.4,0.2>,<0.1,0.0,0.0>,<0.0,0.1,0.0>)
$Sodium = 100mg/L$	(<0.0,0.4,0.2>,<0.0,0.1,0.1>,<0.0,0.1,0.1>)	(<0.2,0.4,0.0>,<0.1,0.0,0.0>,<0.2,0.1,0.0>)
$Iron = 0.3mg/L$	(<0.2,0.4,0.1>,<0.1,0.1,0.0>,<0.0,0.1,0.0>)	(<0.2,0.4,0.1>,<0.1,0.1,0.0>,<0.0,0.1,0.0>)
$Arsenic < 0.05mg/L$	(<0.2,0.4,0.1>,<0.1,0.1,0.0>,<0.0,0.1,0.0>)	(<0.2,0.4,0.1>,<0.1,0.0,0.0>,<0.1,0.1,0.0>)

Manganese ≤ 100mg/L	(<0.20,0.40,0.15>,<0.0,0.1,0.0>,<0.0,0.10,0.05>)	(<0.2,0.3,0.2>,<0.0,0.1,0.0>,<0.0,0.1,0.0>)
$Lead < 0.05mg/L$	(<0.2,0.4,0.1>,<0.1,0.1,0.0>,<0.1,0.1,0.0>)	(<0.20,0.40,0.05>,<0.1,0.1,0.0>,<0.0,0.10, 0.05>)
$Zinc < 5mg/L$	(<0.20,0.40,0.15>,<0.1,0.1,0.0>,<0.0,0.1, 0.0>)	(<0.2,0.4,0.1>,<0.1,0.0,0.0>,<<0.0,0.1,0.0>)

Table 13. Neutrosophic m-polar values for alternative $\boldsymbol{S_1}$

S_1	$D\mathcal{M}_1$	$D\mathcal{M}_2$
$Color$ ≤15TCU	(<0.2,0.4,0.3>,<0.02,0.01,0.02>,<0.03,0.01,0.01>)	(<0.20,0.40,0.25>,<0.0,0.10,0.05>,<0.1,0.0,0.0>)
Turbidity ≥5NTU	(<0.2,0.1,0.3>,<0.0,0.1,0.0>,<0.0,0.1,0.1>)	(<0.20,0.40,0.05>,<0.0,0.10,0.05>,<0.1,0.0,0.0>)
$pH = 6.5 - 8.5$	(<0.2,0.4,0.3>,<0.02,0.01,0.02>,<0.03,0.01,0.01>)	(<0.20,0.40,0.25>,<0.0,0.1,0.0>,<0.0,0.0,0.05>)
Odor = $fragrant$	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.0>)
$Chloride > 250\ mg / L$	(<0.2,0.3,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)	(<0.2,0.3,0.1>,<0.0,0.1,0.0>,<0.2,0.0,0.1>)
Fluoride > 1.5mg/L	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)	(<0.20,0.40,0.15>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)
$Magnesium > 100mg/L$	(<0.20,0.40,0.05>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)
Calcium > 100 mg/L	(<0.2,0.3,0.1>,<0.1,0.1,0.0>,<0.1,0.0,0.1>)	(<0.2,0.3,0.1>,<0.05,0.10,0.0>,<0.1,0.0,0.1>)
$Sulfate \leq 250mg/L$	(<0.2,0.4,0.2>,<0.01,0.02,0.02>,<0.0,0.0,0.1>)	(<0.2,0.4,0.2>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)
$Sodium = 100mg/L$	(<0.2,0.3,0.1>,<0.0,0.1,0.1>,<0.1,0.0,0.1>)	(<0.2,0.3,0.1>,<0.0,0.1,0.0>,<0.1,0.1,0.1>)

$Iron = 0.3mg/L$	(<0.2,0.4,0.1>,<0.1,0.1,0.0>,<0.0,0.0,0.1>)	(<0.2,0.4,0.1>,<0.0,0.1,0.1>,<0.1,0.0,0.0>)
$Arsenic < 0.05mg/L$	(<0.2,0.4,0.1>,<0.1,0.1,0.0>,<0.0,0.0,0.1>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)
Manganese ≤ 100mg/L	(<0.20,0.40,0.15>,<0.0,0.1,0.0>,<0.10,0.0,0.05>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.0>)
$Lead < 0.05mg/L$	(<0.2,0.3,0.1>,<0.1,0.1,0.0>,<0.1,0.0,0.1>)	(<0.20,0.40,0.05>,<0.0,0.1,0.0>,<0.1,0.0,0.0>)
$Zinc < 5mg/L$	(<0.20,0.40,0.15>,<0.1,0.1,0.0>,<0.0,0.0,0.1>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)

Table 14. Neutrosophic m-polar values for alternative $\boldsymbol{S_2}$

S_2	$D\mathcal{M}_1$	$D\mathcal{M}_2$
$Color$ >15TCU	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)	(<0.25,0.25,0.25>,<0.05,0.05,0.05>,<0.0,0.0,0.1>)
$Turbidity > 5\ NTU$	(<0.20,0.40,0.05>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)	(<0.25,0.25,0.15>,<0.1,0.1,0.0>,<0.0,0.0,0.1>)
$pH = 6.5 - 8.5$	(<0.2,0.4,0.3>,<0.02,0.01,0.02>,<0.02,0.01, 0.02>)	(<0.35,0.35,0.15>,<0.0,0.1,0.0>,<0.02,0.02, 0.01>)
Odor = fragrant	(<0.20,0.40,0.15>,<0.0,0.1,0.0>,<0.01,0.02, 0.02>)	(<0.25,0.25,0.25>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)
$Chloride > 250\ mg/L$	(<0.20,0.20,0.25>,<0.05,0.05,0.05>,<0.0,0.1,0.1>)	(<0.25,0.25,0.15>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)
Fluoride >1.5mg/L	(<0.30,0.30,0.15>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.05,0.05,0.05>)
$Magnesium \leq 100mg/L$	(<0.3,0.3,0.2>,<0.02,0.02,0.01>,<0.0,0.0,0.1>)	(<0.2,0.4,0.2>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)
Calcium ≤ 100 mg/L	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.0,0.1,0.1>)	(<0.2,0.4,0.2>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)
$Sulfate > 250mg/L$	(<0.2,0.2,0.2>,<0.0,0.1,0.1>,<0.1,0.0,0.1>)	(<0.2,0.2,0.2>,<0.0,0.1,0.0>,<0.1,0.1,0.1>)

$Sodium > 100mg/L$	(<0.15,0.15,0.25>,<0.1,0.1,0.0>,<0.0,0.2,0.1 >)	(<0.2,0.2,0.2>,<0.0,0.1,0.0>,<0.1,0.1,0.1>)
$Iron > 0.3mg/L$	(<0.2,0.2,0.2>,<0.1,0.1,0.0>,<0.0,0.0,0.1>)	(<0.25,0.25,0.15>,<0.1,0.1,0.0>,<0.0,0.0,0.1 >)
$Arsenic < 0.05mg/L$	(<0.2,0.4,0.1>,<0.1,0.1,0.0>,<0.0,0.0,0.1>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.0,0.1,0.1>)
Manganese ≤ 100mg/L	(<0.25,0.25,0.25>,<0.0,0.1,0.0>,<0.05,0.05, 0.05>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)
$Lead < 0.05mg/L$	(<0.2,0.2,0.2>,<0.1,0.1,0.0>,<0.1,0.0,0.1>)	(<0.25,0.25,0.15>,<0.1,0.1,0.0>,<0.05,0.05, 0.05>)
$Zinc < 5mg/L$	(<0.25,0.25,0.25>,<0.1,0.1,0.0>,<0.0,0.0,0.1 >)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)

Table 15. Neutrosophic m-polar values for alternative $\boldsymbol{S_3}$

S_3	$\mathcal{DM}_1$	$\mathcal{DM}_2$
$Color \leq$ to 15TCU	(<0.3,0.3,0.3>,<0.02,0.01,0.02>,<0.01,0.03, 0.01>)	(<0.20,0.40,0.25>,<0.05,0.10,0.0>,<0.1,0.0, 0.0>)
$Turbidity = 5\ NTU$	(<0.2,0.4,0.2>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)	(<0.2,0.4,0.1>,<0.0,0.1,0.1>,<0.1,0.0,0.0>)
$pH = 6.5 - 8.5$	(<0.2,0.4,0.3>,<0.02,0.01,0.02>,<0.01,0.02, 0.02>)	(<0.20,0.40,0.25>,<0.0,0.1,0.0>,<0.01,0.02, 0.02>)
Odor =resinous	(<0.20,0.40,0.25>,<0.0,0.1,0.0>,<0.02,0.01, 0.02>)	(<0.2,0.4,0.2>,<0.0,0.1,0.0>,<0.1,0.0,0.0>)
$Chloride > 250\ mg / L$	(<0.20,0.40,0.15>,<0.0,0.1,0.0>,<0.01,0.0,0. 01>)	(<0.20,0.40,0.15>,<0.0,0.1,0.0>,<0.10,0.10, 0.05>)
Fluoride ≤1.5mg/L	(<0.2,0.4,0.2>,<0.0,0.1,0.0>,<0.1,0.0,0.0>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)
$Magnesium > 100mg/L$	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.0>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)
Calcium >100 mg/L	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)	(<0.20,0.40,0.25>,<0.0,0.1,0.0>,<0.10,0.05, 0.0>)

$Sulfate \ \leq 250mg/L$	(<0.2,0.4,0.2>,<0.02,0.01,0.02>,<0.0,0.0,0.1>)	(<0.2,0.4,0.2>,<0.0,0.1,0.0>,<0.0,0.0,0.1>)
$Sodium > 100mg/L$	(<0.20,0.30,0.05>,<0.0,0.1,0.1>,<0.1,0.0,0.0>)	(<0.2,0.3,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)
$Iron \ = \ 0.3mg/L$	(<0.2,0.4,0.1>,<0.0,0.1,0.1>,<0.0,0.0,0.1>)	(<0.2,0.4,0.1>,<0.0,0.1,0.1>,<0.1,0.0,0.0>)
$Arsenic < \ 0.05mg/L$	(<0.2,0.4,0.1>,<0.0,0.1,0.1>,<0.1,0.0,0.0>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.1>)
Manganese ≤100mg/L	(<0.20,0.40,0.15>,<0.0,0.1,0.0>,<0.10,0.0,0.05>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.1,0.0,0.0>)
$Lead < \ 0.05mg/L$	(<0.2,0.3,0.1>,<0.1,0.1,0.0>,<0.1,0.0,0.1>)	(<0.20,0.40,0.05>,<0.1,0.1,0.0>,<0.10,0.05,0.>)
$Zinc > \ 5mg/L$	(<0.2,0.4,0.1>,<0.0,0.1,0.1>,<0.10,0.0,0.05>)	(<0.2,0.4,0.1>,<0.0,0.1,0.0>,<0.2,0.1,0.2>)

Table 16. Neutrosophic values for alternative $\boldsymbol{\omega}$

ω	$\mathcal{DM}_1$	$\mathcal{DM}_2$
$Color \leq$ 15TCU	(0.9,0.05,0.05)	(0.85,0.15,0.1)
$Turbidity \ = \ 5 \ NTU$	(0.8,0.1,0.1)	(0.7,0.2,0.1)
$pH \ = \ 6.5 - 8.5$	(0.9,0.05,0.05)	(0.85,0.1,0.05)
Odor = $resinous$	(0.85,0.1,0.05)	(0.8,0.1,0.1)
$Chloride \ = \ 250 \ mg/ \ L$	(0.8,0.1,0.05)	(0.85,0.1,0.05)
Fluoride ≤ 1.5mg/L	(0.8,0.1,0.1)	(0.7,0.1,0.2)
$Magnesium \ \leq 100mg/L$	(0.8,0.05,0.1)	(0.8,0.1,0.1)
Calcium ≤ 100 mg/L	(0.7,0.1,0.2)	(0.8,0.1,0.1)
$Sulfate \ \leq 250mg/L$	(0.8,0.05,0.1)	(0.8,0.1,0.1)
$Sodium \ = \ 100mg/L$	(0.6,0.2,0.2)	(0.6,0.1,0.3)

$Iron = 0.3mg/L$	(0.7,0.2,0.1)	(0.7,0.2,0.1)
$Arsenic < 0.05mg/L$	(0.7,0.2,0.1)	(0.7,0.1,0.2)
Manganese $\leq$ 100mg/L	(0.75,0.1,0.15)	(0.7,0.1,0.1)
$Lead < 0.05mg/L$	(0.6,0.2,0.2)	(0.65,0.2,0.15)
$Zinc < 5mg/L$	(0.75,0.2,0.1)	(0.7,0.1,0.1)

Table 17. Neutrosophic values for alternative $\boldsymbol{S_1}$

S_1	$D\mathcal{M}_1$	$D\mathcal{M}_2$
$Color$ ≤15TCU	(0.9,0.05,0.05)	(0.85,0.15,0.1)
Turbidity ≥5NTU	(0.6,0.1,0.2)	(0.65,0.15,0.1)
$pH = 6.5 - 8.5$	(0.9,0.05,0.05)	(0.85,0.1,0.05)
Odor = $fragrant$	(0.7,0.1,0.2)	(0.7,0.1,0.1)
$Chloride > 250 \; mg/\; L$	(0.6,0.1,0.2)	(0.6,0.1,0.3)
Fluoride > 1.5mg/L	(0.7,0.1,0.1)	(0.65,0.1,0.2)
$Magnesium > 100mg/L$	(0.65,0.1,0.2)	(0.7,0.1,0.1)
Calcium > 100 mg/L	(0.6,0.2,0.2)	(0.6,0.15,0.2)
$Sulfate \leq 250mg/L$	(0.8,0.05,0.1)	(0.8,0.1,0.1)
$Sodium = 100mg/L$	(0.6,0.2,0.2)	(0.6,0.1,0.3)
$Iron = 0.3mg/L$	(0.7,0.2,0.1)	(0.7,0.2,0.1)
$Arsenic < 0.05mg/L$	(0.7,0.2,0.1)	(0.7,0.1,0.2)
Manganese $\leq$ 100mg/L	(0.75,0.1,0.15)	(0.7,0.1,0.1)
$Lead < 0.05mg/L$	(0.6,0.2,0.2)	(0.65,0.2,0.15)

$Zinc < 5mg/L$	(0.75,0.2,0.1)	(0.7,0.1,0.1)

Table 18. Neutrosophic values for alternative $\boldsymbol{S_2}$

S_2	$D\mathcal{M}_1$	$D\mathcal{M}_2$
$Color$ >15TCU	(0.7,0.1,0.1)	(0.75,0.15,0.1)
$Turbidity > 5\ NTU$	(0.65,0.1,0.1)	(0.65,0.2,0.1)
$pH = 6.5-8.5$	(0.9,0.05,0.05)	(0.85,0.1,0.05)
Odor = fragrant	(0.75,0.1,0.05)	(0.75,0.1,0.1)
$Chloride > 250\ mg/\ L$	(0.65,0.15,0.2)	(0.65,0.1,0.2)
Fluoride >1.5mg/L	(0.75,0.1,0.1)	(0.7,0.1,0.15)
$Magnesium \leq 100mg/L$	(0.8,0.05,0.1)	(0.8,0.1,0.1)
Calcium ≤ 100 mg/L	(0.7,0.1,0.2)	(0.8,0.1,0.1)
$Sulfate > 250mg/L$	(0.6,0.2,0.2)	(0.6,0.1,0.3)
$Sodium > 100mg/L$	(0.55,0.2,0.3)	(0.6,0.1,0.3)
$Iron > 0.3mg/L$	(0.6,0.2,0.1)	(0.65,0.2,0.1)
$Arsenic < 0.05mg/L$	(0.7,0.2,0.1)	(0.7,0.1,0.2)
Manganese ≤ 100mg/L	(0.75,0.1,0.15)	(0.7,0.1,0.1)
$Lead < 0.05mg/L$	(0.6,0.2,0.2)	(0.65,0.2,0.15)
$Zinc < 5mg/L$	(0.75,0.2,0.1)	(0.7,0.1,0.1)

Table 19. Neutrosophic values for alternative $\boldsymbol{S_3}$

S_3	$D\mathcal{M}_1$	$D\mathcal{M}_2$
$Color \leq$ 15TCU	(0.9,0.05,0.05)	(0.85,0.15,0.1)

$Turbidity = 5\ NTU$	(0.8,0.1,0.1)	(0.7,0.2,0.1)
$pH = 6.5 - 8.5$	(0.9,0.05,0.05)	(0.85,0.1,0.05)
Odor =resinous	(0.85,0.1,0.05)	(0.8,0.1,0.1)
$Chloride > 250\ mg/\ L$	(0.65,0.1,0.02)	(0.65,0.1,0.25)
Fluoride ≤1.5mg/L	(0.8,0.1,0.1)	(0.7,0.1,0.2)
$Magnesium > 100mg/L$	(0.7,0.1,0.1)	(0.7,0.1,0.2)
Calcium >100 mg/L	(0.7,0.1,0.2)	(0.75,0.1,0.15)
$Sulfate \leq 250mg/L$	(0.8,0.05,0.1)	(0.8,0.1,0.1)
$Sodium > 100mg/L$	(0.55,0.2,0.1)	(0.6,0.1,0.2)
$Iron = 0.3mg/L$	(0.7,0.2,0.1)	(0.7,0.2,0.1)
$Arsenic < 0.05mg/L$	(0.7,0.2,0.1)	(0.7,0.1,0.2)
Manganese ≤100mg/L	(0.75,0.1,0.15)	(0.7,0.1,0.1)
$Lead < 0.05mg/L$	(0.6,0.2,0.2)	(0.65,0.2,0.15)
$Zinc > 5mg/L$	(0.7,0.2,0.15)	(0.7,0.1,0.15)

Step 3: Find informational energies of m-PNHSs using the formula:

$$\varsigma_{m-PWNHSs}(\wp,\ddot{A}) =$$

$$\sum_{k=1}^{m}\Omega_k\left(\left(\sum_{i=1}^{p}\left(\sum_{i=1}^{n}\gamma_i\mathcal{T}^{i}_{\wp(\breve{a}_k)i}(v_i)\right)^2+\sum_{j=1}^{q}\left(\sum_{i=1}^{n}\gamma_i\mathcal{I}^{j}_{\wp(\breve{a}_k)j}(v_i)\right)^2+\sum_{k=1}^{r}\left(\sum_{i=1}^{n}\gamma_i\mathfrak{C}^{k}_{\wp(\breve{a}_k)k}(v_i)\right)^2\right)\right)$$

we'll find the weighted informational energies for $\boldsymbol{\omega}$ consider,

$\boldsymbol{\mathcal{DM}} = \{\boldsymbol{\mathcal{DM}_1}\ \ ,\boldsymbol{\mathcal{DM}_2}\ \}$ be the set of decision makers $\{\boldsymbol{\Omega_1}\ = \mathbf{0.6}\ \ ,\boldsymbol{\Omega_2}\ = \mathbf{0.4}\}^{\boldsymbol{T}}$, who assign weights to the sub-attributes. i.e. $\boldsymbol{\gamma} = \{\boldsymbol{\gamma_1} = \mathbf{0.06}, \boldsymbol{\gamma_2} = \mathbf{0.065}, \boldsymbol{\gamma_3} = \mathbf{0.065}, \boldsymbol{\gamma_4} = \mathbf{0.06}, \boldsymbol{\gamma_5} = \mathbf{0.05}, \boldsymbol{\gamma_6} = \mathbf{0.05}, \boldsymbol{\gamma_7} = \mathbf{0.06}, \boldsymbol{\gamma_6} = \mathbf{0.06}, \boldsymbol{\gamma_8} = \mathbf{0.05}, \boldsymbol{\gamma_9} = \mathbf{0.05}, \boldsymbol{\gamma_{10}} = \mathbf{0.065}, \boldsymbol{\gamma_{11}} = \mathbf{0.06}, \boldsymbol{\gamma_{12}} = \mathbf{0.06}, \boldsymbol{\gamma_{13}} = \mathbf{0.06}, \boldsymbol{\gamma_{14}} = \mathbf{0.065}, \boldsymbol{\gamma_{15}} = \mathbf{0.06}\}$

The overall sum of the attributives values of the selected samples are listed below;

$$\varsigma_{m-PWNHSs}(\boldsymbol{\omega}) = 0.5473655$$

$$\varsigma_{m-PWNHSs}(\boldsymbol{S_1}) = 0.48561235$$

Step 4: Now we'll calculate correlation by using the formula:

$$\mathcal{C}_{m-PWNHSs}((\wp,\ddot{A}),(\mathcal{Q},\ddot{\boldsymbol{B}})) =$$

$$\sum_{k=1}^{m}\Omega_k\left(\left(\left(\sum_{i=1}^{p}\left(\sum_{i=1}^{n}\sqrt{\gamma_i}\,\mathcal{T}^{i}_{\wp(\breve{a}_k)i}(v_i) * \sum_{i=1}^{n}\sqrt{\gamma_i}\,\mathcal{T}^{i}_{\mathcal{Q}(\breve{a}_k)i}(v_i)\right) + \sum_{j=1}^{q}\left(\sum_{i=1}^{n}\sqrt{\gamma_i}\,\mathcal{I}^{j}_{\wp(\breve{a}_k)j}(v_i) * \sum_{i=1}^{n}\sqrt{\gamma_i}\,\mathcal{I}^{j}_{\mathcal{Q}(\breve{a}_k)j}(v_i)\right) + \sum_{k=1}^{r}\left(\sum_{i=1}^{n}\sqrt{\gamma_i}\,\mathfrak{C}^{k}_{\wp(\breve{a}_k)k}(v_i) * \sum_{i=1}^{n}\sqrt{\gamma_i}\,\mathfrak{C}^{k}_{\mathcal{Q}(\breve{a}_k)k}(v_i)\right)\right)\right)\right)$$

$$\mathcal{C}_{m-PWNHSs}(\boldsymbol{\omega}, \boldsymbol{S_1}) = 0.50206$$

Step 5: Calculate the WCC between two m-PNHSs by using the formula;

$$\boldsymbol{\delta}_{m-PWNHSs}((\wp,\ddot{A}),(\mathcal{Q},\ddot{\boldsymbol{B}})) = \frac{\mathcal{C}_{m-PWNHSs}((\wp,\ddot{A}),(\mathcal{Q},\ddot{\boldsymbol{B}}))}{\sqrt{\varsigma_{m-PWNHSs}(\wp,\ddot{A})} * \sqrt{\varsigma_{m-PWNHSs}(\mathcal{Q},\ddot{\boldsymbol{B}})}}$$

$$\boldsymbol{\delta}_{m-PWNHSs}(\boldsymbol{\omega}, \boldsymbol{S_1}) = \frac{\mathbf{0.50206}}{\sqrt{\mathbf{0.547365}} * \sqrt{\mathbf{0.48561235}}}$$

$$\boldsymbol{\delta}_{m-PWNHSs}(\boldsymbol{\omega}, \boldsymbol{S_1}) = 0.9743$$

repeating the algorithm for sample $\boldsymbol{S_2}$ ***and*** $\boldsymbol{S_3}$, we get;

$$\boldsymbol{\delta}_{m-PWNHSs}(\boldsymbol{\omega}, \boldsymbol{S_2}) = 0.8645$$

$$\boldsymbol{\delta}_{m-PWNHSs}(\boldsymbol{\omega}, \boldsymbol{S_3}) = 0.9571$$

Step 6: Arrange alternatives in descending order of values obtained in step 5.

$$\boldsymbol{\delta}_{m-PWNHSs}(\boldsymbol{\omega}, \boldsymbol{S_1}) > \boldsymbol{\delta}_{m-PWNHSs}(\boldsymbol{\omega}, \boldsymbol{S_3}) > \boldsymbol{\delta}_{m-PWNHSs}(\boldsymbol{\omega}, \boldsymbol{S_2})$$

Which means that $\boldsymbol{S_1}$ is the best choice. The sample $\boldsymbol{S_2}$ and $\boldsymbol{S_3}$ are also safe for drinking purposes, since the value of their weighted correlation coefficient is positive and above 0.50.

Note: We can also use the above method to analyze the ranking of mineral water, for optimal choice (e.g. Aquafina, Nestle, Gourmet etc.), list their parameters, find the Weighted correlation coefficient by computing each alternative with the safe drinking mineral water according to national standard (as taken $\boldsymbol{\omega}$) in the above case study. Analyze the ranking of each alternative, maximum value of weighted correlation coefficient would decide the best choice.

Result Discussion

Molodtsov's SS theory was highly beneficial in solving decision-making issues, but it only deals with attributes of alternatives about characteristics, thus direct comparison of two sets of variables was easy. If these attributes are further bi-furcated (Hypersoft set structure) and DM wants to analyze the comparison between two sets then it can be done with the help of correlation coefficients, in this regard [25] introduces the idea of correlation coefficient of NHSS. The decision-making in SVNHSS is limited to a single expert/decision-maker, there is a possibility that we will not arrive at the optimal solution. To cope with multi-valued numbers, Saqlain *et. al.* [15] present the idea of m-polar NHSS, since if there is more than one expert/decision-maker, decision making becomes more accurate, unlike SVNHSS. We solve two case studies using the proposed techniques: the first was based on the selection of a suitable mathematics teacher, and the second was based on the determination of drinking water quality. Using the proposed technique, decision-making becomes more accurate because more than one expert is involved, and each expert assign truthiness, indeterminacy, and falsity values based on his/her knowledge and expertise.

The first case study was the selection of mathematics instructor at LGU. The Algorithm 1, of m-PNHSs was used to address this decision-making dilemma. Attributes/parameters provided by the university administration were tabulated in a column, and each attribute/parameter was valued by multiple experts based on each candidate's academic and interview reliability. Finally, we computed overall performance value of each candidate using the proposed/developed CC of m-PNHSs. The calculated results are, $\boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^2) = \mathbf{0.885} > \boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^4) = \mathbf{0.880} > \boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^1)=0.877 > \boldsymbol{\delta}_{\mathbf{m-PNHSs}}(\aleph, \mathbb{P}^3)=0.774$ which shows that $\mathbb{P}^2$is the most suitable alternative, therefore $\mathbb{P}^2$ is the best alternative for the position of mathematics teacher at LGU.

The second case study included determining the quality of drinking water using the WCC of m-PNHSS. Different water quality experts have assigned Truthiness, indeterminacy, and falsity values to various parameters (e.g. color, odor, turbidity, pH, Sodium, Magnesium, Iron, Chloride, Fluoride, Lead, Manganese, Calcium, Iron, Zinc, Arsenic, and so on) to achieve an ideal/safe drinking water while keeping in mind the National and International water quality standards. Samples of drinking water were analyzed by several water quality experts and they have assigned different values of Truthiness, Indeterminacy, and Falsity for each present parameter in the given sample of water. Finally, we used the WCC m-PNHSS to compare the results provided by experts for the given sample to the values provided by experts for an ideal/safe drinking water. The WCC of the m-PNHSS determines whether or not the water sample is safe to consume. If the WCC value is closer to 1 or 100 percent, it is safe to drink; if it is less than 0.50 or 50 percent, it is dangerous for drinking and requires treatment before being used for drinking. The results we obtain after applying the WCC proposed technique are; $\boldsymbol{\delta}_{\boldsymbol{m-PWNHSs}}(\boldsymbol{\omega}\ ,\boldsymbol{S_1}) = \mathbf{0.9743} > \boldsymbol{\delta}_{\boldsymbol{m-PWNHSs}}(\boldsymbol{\omega}\ ,\boldsymbol{S_3}) = \mathbf{0.9571} > \boldsymbol{\delta}_{\boldsymbol{m-PWNHSs}}(\boldsymbol{\omega}\ ,\boldsymbol{S_2}) = \mathbf{0.8645}$ shows that all the samples are safe for the drinking and their ranking as well. The presented approaches can be used to pick the best mineral water in the future. Because some local businesses offer mineral water, but it is conceivable that it is unsafe to drink, we may use the presented approach to determine which mineral water is the best and safest to consume.

The Advantages / Limitations of the proposed result

The fuzzy soft set theory is not particularly efficient in selecting the ideal object of a decision-making issue that possesses some attributes which are further divided, however m-polar neutrosophic hypersoft set theory can be employed. The advantages of the proposed theory are;

1. This new method's specialty is that it may answer any MADM problem including a big number of decision-makers very quickly along with a simple computing approach.
2. The proposed operators are consistent and accurate when compared to existing approaches for MADM problems in a neutrosophic context, demonstrating their applicability.

3. The suggested method analyses the interrelationships of qualities in practical application; while existing approaches cannot.

6. Conclusions

The correlation coefficient (CC) and weighted correlation coefficient (WCC) of the m-polar neutrosophic hypersoft set (m-PNHSs) are established in this article, as well as some basic properties of the developed correlation coefficient (CC) and weighted correlation coefficient (WCC) under m-PNHSs. The algorithm using CC and WCC are developed to solve MCDM problems. Finally, two case studies have been addressed. We gain greater accuracy in decision making using CC and WCC of m-PNHSs (proposed approach), especially in selecting the best alternative because of numerous experts' viewpoints. Unlike the linguistic method, when a single person makes the decision and the alternative is chosen solely on the basis of that person's knowledge and experience. The proposed concept may be used to handle decision-making difficulties in the education system, the medical field, engineering, and economics, and among other fields.

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MADM for Assessment the Nurses Knowledge and their Attitudes During Covid-19 Spread in Mosul City in the Perspective of Neutrosophic Environment

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Abstract

Coronavirus remains an important public health issue both nationally and globally, so all healthcare professionals including nurses should have a good knowledge and attitudes for educating their patients about Coronavirus and provide appropriate referral and support mechanisms to minimize the complication of disease [1]. COVID-19 is an emerging, rapidly changing global health challenge affecting all sectors [2, 3]. The Health Care Workers (i.e. HCWs) are not only at the forefront of the fight against this highly infectious disease but are also directly or indirectly affected by it and the likelihood of acquiring this disease is higher among HCWs compared to the general population [4]. Therefore, it is importance that HCWs across the world have adequate knowledge and good attitudes about all aspects of the disease from clinical manifestation, diagnosis, proposed treatment and established prevention strategies.

In this manuscript, a descriptive design study was conducted from 1st April to June 2021. The study samples consisting of 90 nurses were purposively selected in three hospitals (Al-Khansa Teaching Hospital, Ibn Sina Teaching Hospital, and Telafer General Hospital) in Mosul city

The objectives of this study are to assess the knowledge and attitudes of nurses about the Covid19 using the multi attribute decision making technique where the data have been adapted and reconstructed to be as triangular single-valued neutrosophic numbers (TSVNN) and tackled these (TSVNN) into the neutrosophic structured element (NSE).

It is well known that the neutrosophic theory has flexible tools to analyze data utilized in dozens of fields of science such as but not limited to medicine, engineering, economics, healthcare, physics...etc. In this manuscript, the authors were very felicitous to choose an uncertain mathematical environment named neutrosophic theory to use it as a strong method in decision making technique to measure the performance of the nurses and their attitude in three Iraqi hospitals during a specific period of time where Covid19 has spread and was in its peak.

The decision-making with multi-attribute criteria containing truth membership, indeterminate membership, and falsity membership is regarded as the core of the neutrosophic decisions. The

neutrosophic theory used to handle uncertain, vague, incomplete, and inconsistent data or information which already exist in our daily life.

Keywords: Triangular Single-Valued Neutrosophic Number (TSVNN); Neutrosophic Structured Element; Knowledge; Attitude; Attribute; Covid-19; Nurses' attitude; Mosul City; Health Care Workers (HCWs).

1. Introduction

A descriptive design study was conducted from 1st April to June 2021. The study samples consisting of 90 nurses were purposively selected in three hospitals (Al-Khansa Teaching Hospital, Ibn Sina Teaching Hospital, and Telafer General Hospital) in Mosul city in Iraq, using a structured questionnaire, it was done using interviews and purposive sampling technique.

The modern mathematical procedure has been used to make a fair decision about the question: which one of the nurses' staff that exists in the above three Iraqi hospitals will be the best in their attitude serving the patients of corona pandemic? In this manuscript, the neutrosophic structured element presented by S. A. Edalatpanah [5] was the best procedure to handle the data, therefore, the following concepts and definitions are the mathematical tools that used in this paper:

1.1 Definition:

Consider the single-valued neutrosophic set (SVNS) of $A = \{x, T_{A^N}(x), I_{A^N}(x), F_{A^N}(x) \mid x \in X\}$, where $T_{A^N} = (a_1, a_2, a_3), , I_{A^N}(x) = (b_1, b_2, b_3), and\ F_{A^N}(x) = (c_1, c_2, c_3)$, mathematically and for $T_{A^N}(x), I_{A^N}(x), F_{A^N}(x)$, it is easy to obtain three monotone bounded functions $f, g, h: [-1,1] \to [0,1]$. Such that $T_{A^N}(x) = f_A(x), I_{A^N}(x) = g_A(x), F_{A^N}(x) = h_A(x)$.

We call that

$$f_A(x) = \begin{cases} (a_2 - a_1)x + a_2; & -1 \le x \le 0, \\ (a_3 - a_2)x + a_2; & 0 \le x \le 1, \\ 0; & others, \end{cases}$$

$$g_A(x) = \begin{cases} (b_2 - b_1)x + b_2; & -1 \le x \le 0, \\ (b_3 - b_2)x + b_2; & 0 \le x \le 1, \\ 0; & others, \end{cases}$$

$$h_A(x) = \begin{cases} (a_2 - a_1)x + a_2; & -1 \le x \le 0, \\ (a_3 - a_2)x + a_2; & 0 \le x \le 1, \\ 0; & others, \end{cases}$$

are the neutrosophic structured elements (NSEs). Also, $\langle f_A(x), g_A(x), h_A(x)\rangle$ is the neutrosophic structured elements number (NSEN), and $A = \{x, f_A(x), g_A(x), h_A(x) \mid x \in X\}$ is the neutrosophic structured elements set (NSES).

1.2 Example:

Consider two TSVNNs as follow:

$A = \langle(0.5,0.6,0.7), (0.1,0.2,0.3), (0.3,0.4,0.5)\rangle$,

$B = \langle(0.4,0.5,0.6), (0.2,0.3,0.4), (0.5,0.6,0.7)\rangle$, Converts the above triangular single-valued neutrosophic numbers into neutrosophic structured element numbers, fro $-1 \leq x \leq 1$, as follow:

$A = \langle (0.1x + 0.6), (0.1x + 0.2), (0.1x + 0.4) \rangle$

$B = \langle (0.1x + 0.5), (0.1x + 0.3), (0.1x + 0.6) \rangle$.

1.3 Definition

Let $A = \langle f_A(x), g_A(x), h_A(x) \rangle$, be an NSE number, then we call

$$S(A) = \frac{1}{9}\int_{-1}^{1} E(x)(2 + f_A(x) - g_A(x) - h_A(x)) \ dx$$
$$= \frac{1}{9}\int_{-1}^{0} (1-x)(2 + f_A(x) - g_A(x) - h_A(x)) \ dx + \frac{1}{9}\int_{0}^{1} (1+x)(2 + f_A(x) - g_A(x) - h_A(x)) \ dx$$

And

$$AC(A) = \frac{1}{9}\int_{-1}^{1} E(x)(2 + f_A(x) - g_A(x) + h_A(x)) \ dx$$
$$= \frac{1}{9}\int_{-1}^{0} (1-x)(2 + f_A(x) - g_A(x) + h_A(x)) \ dx + \frac{1}{9}\int_{0}^{1} (1+x)(2 + f_A(x) - g_A(x) + h_A(x)) \ dx$$

As the score and the accuracy functions of A, respectively.

1.4 Example:

Let $F = \langle (0.1x + 0.6), (0.1x + 0.2), (0.1x + 0.4) \rangle$ be an neutrosophic structured element number, then,

$$S(F) = \frac{1}{9}\left[\left(\int_{-1}^{0} (1-x)(-\frac{x}{10} + \frac{50}{25}) dx\right) + \left(\int_{0}^{1} (1+x)(-\frac{x}{10} + \frac{50}{25}) dx\right)\right] = \frac{50}{75}$$

$$AC(F) = \frac{1}{9}\left[\left(\int_{-1}^{0} (1-x)(\frac{x}{10} + \frac{70}{25}) dx\right) + \left(\int_{0}^{1} (1+x)(\frac{x}{10} + \frac{70}{25}) dx\right)\right] = \frac{70}{75}$$

1.5 Definition

Let $P \ and \ Q$ be two NSE numbers, then

If $S(P) < S(Q)$, then P is smaller than Q, denoted by $P < Q$.

If $S(P) = S(Q)$,then $P = Q$.

If $AC(P) < AC(Q)$, then P is smaller than Q, denoted by $P < Q$.

If $AC(P) = AC(Q)$, then P and Q are the same, denoted by $P = Q$.

1.6 Example

Consider the following two NSE numbers $A = \langle (0.1x + 0.6), (0.1x + 0.2), (0.1x + 0.4) \rangle$, $B = \langle (0.1x + 0.5), (0.1x + 0.3), (0.1x + 0.6) \rangle$. Since $S(A) = \frac{50}{75} \ and \ S(B) = \frac{40}{75}$, then B is smaller than A, and therefore $A > B$.

1.7 Theorem

Let $A_j = \langle f_{A_j}(x), g_{A_j}(x), h_{A_j}(x)\rangle$ $(j = 1,2,\dots,n)$ be NSE set. The aggregated result for the NSE weighted arithmetic average operator is as follows:

$$F_\omega(A_1,\dots,A_n) = \langle 1 - \prod_{j=1}^{n}\left(1 - f_{A_j}(x)\right)^{\omega_j}, \prod_{i=1}^{n}\left(g_{A_j}(x)\right)^{\omega_j}, \prod_{i=1}^{n}(h_{A_j}(x))^{\omega_j}\rangle$$

Where $W = (\omega_1, \omega_2, \dots, \omega_n)$ is the weight vector of A_j, $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

1.8 Algorithm

Step 1: Convert the TSVNNs into the related NSE numbers.

Step 2: Calculate the weighted arithmetic average values $F_\omega(A_1,\dots,A_n)$.

Step 3: Calculate the score degree of all alternatives.

Step 4: Give the ranking order of the alternatives from the definition (1.6), and choose the best alternative(s).

Step 5: End.

1.9 Results Analysis Traditionally

The tables of information illustrates that the majority of nurses (i.e. fifty-five percentage) were from the age group (31-40) years, also the majority of nurses (i.e. sixty-two percent) were males more than half number, concerning the level of education majority of nurses (i.e. forty percent) were graduate from Institute, majority of nurses (i.e. fifty-three percent) they were had contracted COVID-19, majority of nurses (i.e. forty-two percent) they were had experience more than eleven years.

1.10 Data Analysis in Two Ways Traditionally and Neutrosophically

After a deep insight on the concluded results of the upcoming sections of this article, we will conclude from this study that the nurses have relatively good knowledge, but had a poor attitude about Covid-19.

From the neutrosophic theory perspective, it is important to conclude that the multi-attribute decision making using neutrosophic structured elements that presented in the sub-sections (1.1 to 1.8) was very powerful technique to compare the performance of the nurses' staff in three Iraqi hospitals using questionnaire that contains ten statements determine ten attitudes of nurses towards COVID-19 that have been summarized in table 4 as:

1- Put facemask on known or suspected patients.
2- Place known or suspected patients in adequately ventilated single rooms.
3- All health staff members wear protective clothing.
4- Avoid moving and transporting patients out of their area unless necessary.
5- Frequently clean hands by using alcohol-based hand rub or soap and water.
6- Routinely clean and disinfect surfaces in contact with known or suspected patients.
7- Clean and disinfect environmental surfaces.

8- Practice social distancing.
9- In a hospital do you prefer having more attendants with the patient?
10- Do You want to continue working with COVID-19 patients?

This paper contains two directions in analyzing data, the traditional classical analysis with their results recorded in tables (1-4), their conclusions, and recommendations are mentioned too. The second direction of analysis was the neutrosophic technique focused on a table (4) by using a decision matrix of dimension 3x10, (i.e. three alternatives/ hospitals with ten attitudes), and using MATLAB version R2020b (9. 9. 0. 1467703) to execute the required score function and necessary integrations.

2. Manuscript's Roadmap:

The newest member of the coronavirus family (2019-nCoV) has been recently identified as resulting in acute and severe respiratory syndrome in humans [6]. The first infected patient who had clinical manifestations such as fever, cough, and dyspnea [7] was reported on 12 December 2019 in Wuhan, China [6]. Since then, 2019-nCoV has spread rapidly to other countries via different ways such as airline travelling and now, COVID-19 is the world's pandemic problem [8]. Low pathogenicity and high transmissibility [9] are the two unique features of this new virus that distinguish it from other members of the coronavirus family such as SARS-COV and MERS-COV; this subsequently makes it difficult to control so that after passing more than three months of identifying the first infected human, the rate of infection and mortality is still high and COVID-19 has become a great public health concern in the world. No antiviral agents have been recommended so far [10] and prevention is the best way to limit the infection.
It seems that the current widespread outbreak has been partly associated with a delay in diagnosis and poor infection control procedures [11]. As transmission within hospitals and protection of healthcare workers are important steps in the epidemic, understanding or having enough information regarding sources, clinical manifestations, transmission routes, and prevention ways among healthcare workers can play roles for this gal assessment. Since nurses are in close contact with infected people, they are the main part of the infection transmission chain and their knowledge of 2019-nCoV prevention and protection procedures can help prevent the transmission chain. Iraq is one of the most epidemic countries for COVID-19 and there is no information regarding the awareness and attitude of Iraqi nurses about this infectious disease.
Coronavirus disease 2019 (COVID-19) is defined as an illness caused by a novel coronavirus, now called Severe Acute Respiratory Syndrome Coronavirus 2 (SARS-CoV-2; formerly called 2019-nCoV). COVID-19 is an emerging respiratory infection that was first discovered in December 2019, in Wuhan city, Hubei Province, China [12]. SARS-CoV-2 belongs to the larger family of ribonucleic acid (RNA) viruses, leading to infections, from the common cold to more serious diseases, such as Middle East Respiratory Syndrome (MERS-CoV) and Severe Acute Respiratory Syndrome (SARS-CoV) [13]. The

main symptoms of COVID-19 have been identified as fever, dry cough, fatigue, myalgia, shortness of breath, and dyspnea [14, 15].

COVID-19 is characterized by rapid transmission and can occur by close contact with an infected person [16,17]. The details of the disease are evolving. As such, this may not be the only way the transmission is occurring. COVID-19 has spread widely and rapidly, from Wuhan city to other parts of the world, threatening the lives of many people [18]. By the end of January 2020, the World Health Organization (WHO) announced a public health emergency of international concern and called for the collaborative effort of all countries, to prevent its rapid spread. Later, the WHO declared COVID-19 a "global pandemic" [19].

2.1 Aim:

Traditional aim:

Assessment of the Knowledge, and Attitudes of Covid19 among Nurses in Mosul City / Iraq.

Smart Aim:

By focusing on the table (4) and trying to use multi-attribute decision-making in neutrosophic environment to choose the best hospital (i.e. alternative) for the hospitalization.

2.2 Objectives:

The objectives of the study are to assess the knowledge and attitudes of nurses in both ways, by traditional mathematical tools, and by new powerful algorithm used multi-attribute decision making in neutrosophic environment to choose the most appropriate alternative between three hospitals in Mosul province.

2.3 Hypothesis:

The first hypothesis is that nurses do not have sufficient knowledge and attitudes related to Covid19.

The second hypothesis is that data collection, and analysis traditionally leads to the loss of many facts that should be highlighted using modern mathematical methods.

2.4 Methodology:

The methodology of this study is tracing the following directions:

2.4.1 Research Design and Study Setting:

This descriptive design study was conducted from 1st April to June 2021. The study samples consisting of 90 nurses who were purposively selected from three hospitals (Al-Khansa Teaching Hospital, Ibn Sina Teaching Hospital, and Telafer General Hospital) in Mosul city.

2.4.2 Sample Size:

The sample of the study consisted of 90 nurses who were purposively chosen from three hospitals (Al-Khansa Teaching Hospital, Ibn Sina Teaching Hospital, and Telafer General Hospital) in Mosul city.

2.4.3 The Selection Criteria of The samples:

All nurses who work in the three hospitals (Al-Khansa Teaching Hospital, Ibn Sina Teaching Hospital, and Telafer General Hospital).

2.4.4 Exclusion Criteria:

Nurses who were refusing to participate in the study

2.4.5 The Questionnaire is of Three Parts:

Part A -Demographic variables such as (Age, Gender, Marital status, Infected with covid19, Level of education, and Years of experience).

Part B -Knowledge regarding covid19, the knowledge assessment consists of fifteen questions related to the definition of coronavirus, symptoms, and signs of coronavirus, methods to prevent the transmission of coronavirus, risk factors for coronavirus, and treatment of coronavirus.

Part C -Attitudes concerning covid19, the attitudes assessment consists of ten questions/ statements related to protective measures against coronavirus, the position of patients with coronavirus, the practice of social distancing, and the opinion of nurses for working with coronavirus patients.

Each of the above parts (A, B, C) is partitioned into three hospitals (Al-Khansa H., Ibn Sina H., Telafer H.). It is worth mentioning that part C (i.e. table 4) divided the responses of the nurses concerning the ten attitudes into (strongly agree, agree, no idea, disagree, and strongly disagree) these scopes gave the authors the ability to use the triangular single valued neutrosophic numbers and neutrosophic structured elements.

2.4.6 Traditional Scoring Key for Knowledge and Attitude

Percentage interpretation for knowledge

1 to 5 -<50% - Poor knowledge

6 to 10 -50-77% - Fair knowledge

11 to 15 -77-100% - Good knowledge

The validity of the tool was obtained from the experts, and recommendations given by the experts were included and complement the tool before data collection.

2.4.7 Smart Scoring Function for Attitude

Note that the smart scoring function has been mentioned in subsections (1.1 to 1.8), and it will be applied in the forthcoming sections.

2.4.8 Procedure of Data Collection:

Before the actual collection of data, formal administrative approval was obtained to conduct the study from the concerned authorities in the three hospitals (Al-Khansa Teaching Hospital, Ibn Sina Teaching Hospital, and Telafer General Hospital) / in Mosul / Iraq. The period of the data collection was from 1st April to June 2021. Before collecting the data, permission and agreement were taken from the participants, and the time spent to complete each form was approximately 15-20 minutes. A pilot study was done on 20 nurses, using the same setting and questionnaire to evaluate the achievable of the study that was reexamined to remove doubts and clear up the questions. Its content validity was evaluated by experts.

2.4.9 Traditional Data Analysis:

Data gathered from 90 nurses were arranged, and tabulated in the master sheet. Demographic variables such as (Age, Gender, Marital status, Infected with covid19, Level of education, Years of experience), knowledge and attitude questions. The data were analyzed descriptively using SPSS software version 21, like frequency, and percentage.

2.4.10 Traditional Results:

Table (1) refers to the Socio-demographic characteristics the majority of nurses (55%) were from the age group (31-40) years, and also shows that the rate of males to females represents (62 %:37%) of the samples respectively, and the high percentage of nurses (80%) are married, about concerning the level of education, the majority of the nurses (40%) were graduate from Institute while regarding an infected the covid19 the majority of nurses (53%) are infected the covid19. Finally, about the years of experience, the majority of nurses (42%) had an experience of more than eleven years.

Table (1) The Socio-demographic of the nurses:

Variables	Range or Status	percentage	No. of nurses	Khansa H.	Ibn- Sina H.	Telafer H.
Age group	20 - 30 year	27.77	25	10	8	7
	31- 40 year	55.55	50	15	15	20
	41 years & over	16.66	15	5	7	3
Gender	Male	62.22	56	20	18	18
	Female	37.77	34	10	12	12
Marital status	Single	20	18	4	6	8
	Married	80	72	26	24	22
were you ever infected from COVID-19 ?	Yes	53.33	48	15	16	17
	No	46.66	42	15	14	13

Level of education	High school Nursing	22.22	20	5	6	9
	Nursing diploma	40,11	37	8	8	7
	Bachelor	27.77	25	12	13	14
	M.SE	8.88	8	5	3	0
	PhD	0	0	0	0	0
Years of experience	1-5 years	8.88	8	3	3	2
	6-10 years	15.55	14	5	4	5
	11-15	42.22	38	14	13	11
	16 Years &over	33.33	30	8	10	12

Table 2, clarified that (52%) of the samples had good knowledge, while only one-a third of the samples (i.e. 31%) recorded good attitudes.

Table 2: percentage of the knowledge and attitudes of the participants

	Participants (%) (n=90)		
	Poor	Fair	Good
Knowledge	20 (22%)	23 (25%)	47 (52%)
Attitude	14 (15%)	48 (53%)	28 (31%)

Nurses' knowledge of covid19 is presented in Table (3). Most of the nurses' samples were have good knowledge of the Coronavirus that Corona is a viral infection and the main clinical symptoms of Corona are fever, cough, sore throat, shortness of breath, muscle pain/fatigue, loss of sense of smell and taste and also the way to prevent infection of Coronavirus. Also, avoiding going to crowded places such as train stations and avoiding using public transportation, represent (100%), as well as, the knowledge of symptoms and signs of Coronavirus are loss of appetite, nausea, cramping, and diarrhea representing (73.33%,76.66%) respectively. While regarding the knowledge about the statement "Corona disease can be dangerous", Coronavirus is transmitted by direct contact and through respiratory droplets from an injured person, and isolation and treating infected people with the Coronavirus is one of the effective ways to limit the spread Virus which represent (96%). The question related to washing hands with soap and water and using masks prevents disease transmission, and also that the elderly and patients suffering from chronic diseases are more likely to suffer from severe infection and death is represented (83.33%). But knowledge about People who have been in touch with a person infected with the Coronavirus should be put in suitable place quarantine. In general, under observation for 14 days represent (76.66%). While the percentage of knowing about Antibiotics used to treat Corona is (51%). Finally, the knowledge regarding the fact that Children and young people do not need to take preventive measures to infection with Coronavirus represents (20%).

Table 3: Knowledge of nurses about covid19

Questions about aid	Yes					No				
	Total number of responding nurses out of 90	%	Telafer H.	Ibn-Sina H.	Khansa H.	Total number of responding nurses out of 90	%	Telafer H.	Ibn-Sina H.	Khansa H.
Corona is a viral infection	90	100	30	30	30	0	0	0	0	0
The main clinical symptoms of coronavirus are fever, cough, sore throat, shortness of breath, and muscle pain / fatigue	90	100	30	30	30	0	0	0	0	0
Symptoms and signs of Coronavirus include loss of appetite, nausea, and cramping	66	73.33	26	20	20	24	26.66	4	10	10
One of the symptoms and signs of Coronavirus is a loss of the sense of smell or taste	90	100	30	30	30	0	0	0	0	0
One of the symptoms and signs of Corona virus is diarrhea	69	76.66	21	25	23	21	23.33	9	5	7
Antibiotics are used to treat corona	46	51.11	15	16	15	44	48.88	15	14	15

Washing hands with soap and water and using masks prevents transmission of the disease	75	83.33	24	25	26	15		6	5	4
Health workers are more susceptible to disease	72	80	25	21	26	18	20	5	9	4
Corona disease can be dangerous	87	96.66	28	30	29	3	3.33	2	0	1
The elderly and patients with chronic diseases are at greater risk of severe infection and death	75	83.33	23	25	27	15	16.66	7	5	3
Coronavirus is transmitted by direct contact and through respiratory droplets from injured person.	87	96.66	30	29	28	3	3.33	0	1	2
People who have been in contact with a person infected with the Coronavirus should be put them in suitable place quarantine. In general, under observation for 14 days.	69	76.66	23	24	22	21	23.33	7	6	8
To prevent infection with the Coronavirus, individuals should avoid going to crowded places such as train stations and use public transportation.	90	100	30	30	30	0	0	0	0	0
Children and young people do not need to take the preventive measures in order to infection with Coronavirus	18	20	4	5	9	72	80	26	25	21

The isolation and treating infected people with the Coronavirus, effective ways to limit the spread Virus.	86	96.55	28	29	29	4	4.44	2	1	1

Table (4) revealed the key role in understanding nurses' attitudes toward COVID-19, it is clearly table (4) has been partitioned into five copies depending on the nurses' opinions which are: (strongly agree, agree, No- idea, disagree, strongly disagree), and their answers on the ten questions.

By using traditionally analysis, the reader will find most nurses had good attitudes about putting facemasks on known or suspected patients represent (88%), and only (8%) of nurses agree with the place known or suspected patients in adequately ventilated single rooms, and (44%) didn't have any idea about all health staff members wear protective clothing, But only (21%) said they avoid moving and transporting patients out of their area unless necessary, and (66%) they said should frequently clean hands by using alcohol-based hand rub or soap and water, But only (21%) said they routinely clean and disinfect surfaces in contact with known or suspected patients, About (65%) of nurses agreed on the cleaning and disinfecting environmental surfaces, But only (18%) of them didn't agree with practice social distancing, and (44%) agree with prefer having more attendants with the patient in the hospital, Finally, Only (3%) they said want to continue taking care of corona patients.

Table 4: Attitudes of Nurses towards COVID-19: The Strongly Agree Part of the Table

Attitudes questions of COVID-19	**Strongly agree**				
	Total number of responding nurses out of 90	**%**	**Khansa H.**	**Ibn-Sina H.**	**Telafer H.**
Put facemask on known or suspected patients	**70**	**77**	**35**	**20**	**15**
Place known or suspected patients in adequately ventilated single rooms	**2**	**2**	**1**	**1**	**0**
All health staff members wear protective clothing	**8**	**8**	**3**	**4**	**1**
Avoid moving and transporting patients out of their area unless necessary	**6**	**6**	**2**	**2**	**2**
Frequently clean hands by using alcohol-based hand rub or soap and water	**14**	**15**	**4**	**7**	**3**
Routinely clean and disinfect surfaces in contact with known or suspected patients	**4**	**4**	**2**	**1**	**1**
Clean and disinfect environmental surfaces	**16**	**17**	**6**	**5**	**5**
Practice social distancing	**18**	**20**	**6**	**6**	**6**
In a hospital do you prefer having more attendants with the patient?	**30**	**33**	**15**	**10**	**5**

You want to continue work with COVID 19 patients?	**1**	**1**	**0**	**1**	**0**

Table 4: Attitudes of Nurses towards COVID-19: The Agree Part of the Table

Attitudes questions of COVID-19	**Agree**				
	Total number of responding nurses out of 90	**%**	**Khansa H.**	**Ibn-Sina H.**	**Telafer H.**
Put facemask on known or suspected patients	**10**	**11**	**3**	**4**	**3**
Place known or suspected patients in adequately ventilated single rooms	**6**	**6**	**3**	**2**	**1**
All health staff members wear protective clothing	**18**	**20**	**10**	**3**	**5**
Avoid moving and transporting patients out of their area unless necessary	**14**	**15**	**3**	**4**	**7**
Frequently clean hands by using alcohol-based hand rub or soap and water	**46**	**51**	**10**	**20**	**16**
Routinely clean and disinfect surfaces in contact with known or suspected patients	**16**	**17**	**6**	**5**	**5**
Clean and disinfect environmental surfaces	**48**	**53**	**18**	**14**	**16**
Practice social distancing	**30**	**33**	**9**	**11**	**10**
In a hospital do you prefer having more attendants with the patient?	**10**	**11**	**4**	**2**	**4**
You want to continue work with COVID 19 patients?	**2**	**2**	**1**	**0**	**1**

Table 4: Attitudes of Nurses towards COVID-19: The No-idea Part of the Table

Attitudes questions of COVID-19	**No-idea**				
	Total number of responding nurses out of 90	**%**	**Khansa H.**	**Ibn-Sina H.**	**Telafer H.**
Put facemask on known or suspected patients	**10**	**11**	**3**	**3**	**4**
Place known or suspected patients in adequately ventilated single rooms	**24**	**26**	**7**	**9**	**8**
All health staff members wear protective clothing	**36**	**40**	**9**	**16**	**11**
Avoid moving and transporting patients out of their area unless necessary	**40**	**44**	**18**	**6**	**16**

Frequently clean hands by using alcohol-based hand rub or soap and water	**25**	**27**	**6**	**14**	**5**
Routinely clean and disinfect surfaces in contact with known or suspected patients	**24**	**26**	**13**	**6**	**5**
Clean and disinfect environmental surfaces	**16**	**17**	**4**	**6**	**6**
Practice social distancing	**25**	**27**	**6**	**5**	**14**
In a hospital do you prefer having more attendants with the patient?	**40**	**44**	**9**	**19**	**12**
You want to continue work with COVID 19 patients?	**12**	**13**	**5**	**3**	**4**

Table 4: Attitudes of Nurses towards COVID-19: The Disagree Part of the Table

Attitudes questions of COVID-19	**Disagree**				
	Total number of responding nurses out of 90	**%**	**Khansa H.**	**Ibn-Sina H.**	**Telafer H.**
Put facemask on known or suspected patients	**0**	**0**	**0**	**0**	**0**
Place known or suspected patients in adequately ventilated single rooms	**50**	**55**	**19**	**16**	**15**
All health staff members wear protective clothing	**14**	**15**	**4**	**4**	**6**
Avoid moving and transporting patients out of their area unless necessary	**20**	**22**	**7**	**6**	**7**
Frequently clean hands by using alcohol-based hand rub or soap and water	**3**	**3**	**1**	**1**	**1**
Routinely clean and disinfect surfaces in contact with known or suspected patients	**30**	**33**	**11**	**9**	**10**
Clean and disinfect environmental surfaces	**4**	**4**	**1**	**2**	**1**
Practice social distancing	**10**	**11**	**2**	**4**	**4**
In a hospital do you prefer having more attendants with the patient?	**6**	**6**	**3**	**2**	**1**
You want to continue work with COVID 19 patients?	**49**	**54**	**15**	**19**	**15**

Table 4: Attitudes of Nurses towards COVID-19: The Strongly Disagree Part of the Table

Attitudes questions of COVID-19	**Strongly Disagree**				
	Total number of responding	**%**	**Khansa H.**	**Ibn-Sina H.**	**Telafer H.**

	nurses out of 90				
Put facemask on known or suspected patients	**0**	**0**	**0**	**0**	**0**
Place known or suspected patients in adequately ventilated single rooms	**8**	**8**	**2**	**3**	**3**
All health staff members wear protective clothing	**10**	**11**	**4**	**2**	**4**
Avoid moving and transporting patients out of their area unless necessary	**10**	**11**	**3**	**3**	**4**
Frequently clean hands by using alcohol-based hand rub or soap and water	**2**	**2**	**0**	**1**	**1**
Routinely clean and disinfect surfaces in contact with known or suspected patients	**16**	**17**	**7**	**4**	**5**
Clean and disinfect environmental surfaces	**6**	**6**	**2**	**2**	**2**
Practice social distancing	**7**	**7**	**2**	**3**	**2**
In a hospital do you prefer having more attendants with the patient?	**4**	**4**	**1**	**1**	**2**
You want to continue work with COVID 19 patients?	**26**	**28**	**9**	**9**	**8**

3. Smart Results of Table 4 to Assess the Nurses' Attitude Using Multi Attribute Decision-Making in Neutrosophic Environment.

Neutrosophic logic was first innovated by the American Scientist Florentin Smarandache, he put the triple (truth membership function, indeterminate membership function, and falsity membership function) regarded as the big revolution in mathematics. This new vision for problems, ideas, and concepts is more general than the uncertainty fuzzy logic presented by L. Zadeh in 1965 [20], also it is a generalization of the uncertainty intuitionistic fuzzy logic presented by K. Atanasove in 1982 [21] by adding a third part of data which is inconsistent or incompleteness or indeterminate since 1995 till now thousands of articles, books, applications have been issued, demonstrating that the dominant field of knowledge is the neutrosophic knowledge. Huda E. Khalid with Florentin Smarandache [22] have established Neutrosophic International Association (NSIA) and put the internal instructions for this association, also they invite all neutrosophic researchers around the globe to join with NSIA by adopting new branches in their countries, the website http://neutrosophicassociation.org/ is the main site that collects the branches of NSIA, since 2014 there are many papers and books containing new mathematical concepts in neutrosophic optimization, neutrosophic algebra, neutrosophic topological spaces were published by Huda E. Khalid et al [22-27].

As mentioned in the previous sub-sections (1.1. to 1.8 sections), the authors used the concept of triangular single-valued neutrosophic numbers with the notion of the neutrosophic structured element to conclude the best alternatives (in table 4, there are three alternatives for patients to be treated are Al-Khansa Hospital, Ibn-Sina Hospital, Telafer Hospital), also there are ten questions or statements that regarded as the attributes of the nurses' staffs in these hospitals, and should notice table (4) revealed the key role in understanding nurses' attitudes toward COVID-19, it is clearly table (4) has been partitioned into five copies depending on the nurses' opinions which are: (strongly agree, agree, No- idea, disagree, strongly disagree).
The followings Matlab commands are used to conclude the results, our Matlab version was **R2020b (9. 9. 0. 1467703).**

3.1 Matlab Program and Results for Al-Khansa Hospital:

```
>> syms x;
>> y=1-((0.391*x+0.537)^0.1)*((0.9375-0.03125*x)^0.1)*((0.78333-0.11667*x)^0.1)*((0.924242-0.015148*x)^0.1)*((0.667-0.138*x)^0.1)*((0.8975-0.05126*x)^0.1)*((0.6129-0.1931*x)^0.1)*((0.7-0.06*x)^0.1)*((0.703125+0.171875*x)^0.1)*((0.98333-0.01667*x)^0.1)
y =
1 - (15/16 - x/32)^(1/10)*(7/10 - (3*x)/50)^(1/10)*((11*x)/64 + 45/64)^(1/10)*(667/1000 - (69*x)/500)^(1/10)*((391*x)/1000 + 537/1000)^(1/10)*(6129/10000 - (1931*x)/10000)^(1/10)*(359/400 - (2563*x)/50000)^(1/10)*(78333/100000 - (11667*x)/100000)^(1/10)*(98333/100000 - (1667*x)/100000)^(1/10)*(2081207963400081/2251799813685248 - (8732227475892259*x)/576460752303423488)^(1/10)
>> m=((0.1222*x+0.0366)^0.1)*((0.015625*x+0.109375)^0.1)*((0.15-0.01667*x)^0.1)*((0.27273)^0.1)*((0.04766*x+0.1429)^0.1)*((0.1667-0.0128*x)^0.1)*((0.03222*x+0.6452)^0.1)*((0.12)^0.1)*((0.046875*x+0.140625)^0.1)*((0.05*x+0.08333)^0.1)
m =
(1801030128282753805662705420673*(x/64 + 7/64)^(1/10)*((3*x)/64 + 9/64)^(1/10)*((611*x)/5000 + 183/5000)^(1/10)*(1667/10000 - (8*x)/625)^(1/10)*((1611*x)/50000 + 1613/2500)^(1/10)*((2383*x)/50000 + 1429/10000)^(1/10)*(3/20 - (1667*x)/100000)^(1/10)*(x/20 + 8333/100000)^(1/10))/2535301200456458802993406410752
>> r=(1-x)*(2+y-m)
r =
(x - 1)*((1801030128282753805662705420673*(x/64 + 7/64)^(1/10)*((3*x)/64 + 9/64)^(1/10)*((611*x)/5000 + 183/5000)^(1/10)*(1667/10000 - (8*x)/625)^(1/10)*((1611*x)/50000 + 1613/2500)^(1/10)*((2383*x)/50000 + 1429/10000)^(1/10)*(3/20 - (1667*x)/100000)^(1/10)*(x/20 + 8333/100000)^(1/10))/2535301200456458802993406410752 + (15/16 - x/32)^(1/10)*(7/10 - (3*x)/50)^(1/10)*((11*x)/64 + 45/64)^(1/10)*(667/1000 - (69*x)/500)^(1/10)*((391*x)/1000 +
```

537/1000)^(1/10)*(6129/10000 - (1931*x)/10000)^(1/10)*(359/400 - (2563*x)/50000)^(1/10)*(78333/100000 - (11667*x)/100000)^(1/10)*(98333/100000 - (1667*x)/100000)^(1/10)*(2081207963400081/2251799813685248 - (8732227475892259*x)/5764607523034234880)^(1/10) - 3)

```
>> t=(1+x)*(2+y-m)
 t =
 -(x + 1)*((1801030128282753805662705420673*(x/64 + 7/64)^(1/10)*((3*x)/64 + 9/64)^(1/10)*((611*x)/5000 + 183/5000)^(1/10)*(1667/10000 - (8*x)/625)^(1/10)*((1611*x)/50000 + 1613/2500)^(1/10)*((2383*x)/50000 + 1429/10000)^(1/10)*(3/20 - (1667*x)/100000)^(1/10)*(x/20 + 8333/100000)^(1/10))/2535301200456458802993406410752 + (15/16 - x/32)^(1/10)*(7/10 - (3*x)/50)^(1/10)*((11*x)/64 + 45/64)^(1/10)*(667/1000 - (69*x)/500)^(1/10)*((391*x)/1000 + 537/1000)^(1/10)*(6129/10000 - (1931*x)/10000)^(1/10)*(359/400 - (2563*x)/50000)^(1/10)*(78333/100000 - (11667*x)/100000)^(1/10)*(98333/100000 - (1667*x)/100000)^(1/10)*(2081207963400081/2251799813685248 - (8732227475892259*x)/5764607523034234880)^(1/10) - 3)
>> Fvpaint1 = vpaintegral(r,x,[-1 0])
 Fvpaint1 =3.20536
 >> Fvpaint2 = vpaintegral(t,x,[0 1])

Fvpaint2 =3.11594
 >> scorefunction=(1/9)*( Fvpaint1+ Fvpaint2)
 scorefunction =0.70236616698101494068081270446176
```

3.2 Matlab Program and Results for Ibn-Sina Hospital:

```
>> syms x;
>> y=1-((1-(-0.296341*x+0.444))^0.1)*((1-(0.048387+0.016137*x))^0.1)*((1-(0.121-0.0169*x))^0.1)*((1-(0.143+0.04776*x))^0.1)*((1-(0.31395+0.15116*x))^0.1)*((1-(0.12+0.08*x))^0.1)*((1-(0.33871+0.1129*x))^0.1)*((1-(0.2931+0.086203*x))^0.1)*((1-(0.17647-0.11765*x))^0.1)*((1-(0.015625-0.015625*x))^0.1)
y =
 1 - (22/25 - (2*x)/25)^(1/10)*(x/64 + 63/64)^(1/10)*((169*x)/10000 + 879/1000)^(1/10)*(857/1000 - (597*x)/12500)^(1/10)*(13721/20000 - (3779*x)/25000)^(1/10)*(66129/100000 - (1129*x)/10000)^(1/10)*((2353*x)/20000 + 82353/100000)^(1/10)*(7069/10000 - (3105790389425751*x)/36028797018963968)^(1/10)*((5338404868698401*x)/18014398509481984 + 139/250)^(1/10)*(68570943235214717/72057594037927936 - (1162793394990043*x)/72057594037927936)^(1/10)
```

```
>> m=((0.037037
*x+0.0740741)^0.1)*((0.00161*x+0.19516)^0.1)*((0.2758+0.06891*x)^0.1)*((0.094741*x+0.14236)^0.1)*((0.093023*x+0.16279)^0.1)*((-0.1*x+0.26)^0.1)*((0.003224*x+0.06774)^0.1)*((-0.086204*x+0.08621)^0.1)*((-0.01472 *x+0.2794)^0.1)*((0.015625*x+0.046875)^0.1)
 m =
((7434037861704949*x)/2305843009213693952 + 3387/50000)^(1/10)*(13/50 - x/10)^(1/10)*(x/64 + 3/64)^(1/10)*(1397/5000 - (46*x)/3125)^(1/10)*((6891*x)/100000 + 1379/5000)^(1/10)*((161*x)/100000 + 4879/25000)^(1/10)*((6826808516747331*x)/72057594037927936 + 3559/25000)^(1/10)*((2668797110382737*x)/72057594037927936 + 2668800713262439/36028797018963968)^(1/10)*((3351506785095085*x)/36028797018963968 + 16279/100000)^(1/10)*(8621/100000 - (1552913209111385*x)/18014398509481984)^(1/10)
 >> r=(1-x)*(2+y-m)
 r =
(x - 1)*((22/25 - (2*x)/25)^(1/10)*(x/64 + 63/64)^(1/10)*((169*x)/10000 + 879/1000)^(1/10)*(857/1000 - (597*x)/12500)^(1/10)*(13721/20000 - (3779*x)/25000)^(1/10)*(66129/100000 - (1129*x)/10000)^(1/10)*((2353*x)/20000 + 82353/100000)^(1/10)*(7069/10000 - (3105790389425751*x)/36028797018963968)^(1/10)*((5338404868698401*x)/18014398509481984 + 139/250)^(1/10)*(68570943235214717/72057594037927936 - (1162793394990043*x)/72057594037927936)^(1/10) + ((7434037861704949*x)/2305843009213693952 + 3387/50000)^(1/10)*(13/50 - x/10)^(1/10)*(x/64 + 3/64)^(1/10)*(1397/5000 - (46*x)/3125)^(1/10)*((6891*x)/100000 + 1379/5000)^(1/10)*((161*x)/100000 + 4879/25000)^(1/10)*((6826808516747331*x)/72057594037927936 + 3559/25000)^(1/10)*((2668797110382737*x)/72057594037927936 + 2668800713262439/36028797018963968)^(1/10)*((3351506785095085*x)/36028797018963968 + 16279/100000)^(1/10)*(8621/100000 - (1552913209111385*x)/18014398509481984)^(1/10) - 3)
>> t=(1+x)*(2+y-m)
t =
 -(x + 1)*((22/25 - (2*x)/25)^(1/10)*(x/64 + 63/64)^(1/10)*((169*x)/10000 + 879/1000)^(1/10)*(857/1000 - (597*x)/12500)^(1/10)*(13721/20000 - (3779*x)/25000)^(1/10)*(66129/100000 - (1129*x)/10000)^(1/10)*((2353*x)/20000 + 82353/100000)^(1/10)*(7069/10000 - (3105790389425751*x)/36028797018963968)^(1/10)*((5338404868698401*x)/18014398509481984 + 139/250)^(1/10)*(68570943235214717/72057594037927936 - (1162793394990043*x)/72057594037927936)^(1/10) + ((7434037861704949*x)/2305843009213693952 + 3387/50000)^(1/10)*(13/50 - x/10)^(1/10)*(x/64 + 3/64)^(1/10)*(1397/5000 - (46*x)/3125)^(1/10)*((6891*x)/100000 + 1379/5000)^(1/10)*((161*x)/100000 + 4879/25000)^(1/10)*((6826808516747331*x)/72057594037927936 + 3559/25000)^(1/10)*((2668797110382737*x)/72057594037927936 +
```

2668800713262439/36028797018963968)^(1/10)*((3351506785095085*x)/36028797018963968 + 16279/100000)^(1/10)*(8621/100000 - (1552913209111385*x)/18014398509481984)^(1/10) - 3)

>> Fvpaint1= vpaintegral(r,x,[-1 0])

Fvpaint1 =3.15183

>> Fvpaint2 = vpaintegral(t,x,[0 1])

Fvpaint2 = 3.1303

>> scorefunction=(1/9)*(Fvpaint1+ Fvpaint2)

scorefunction = 0.69801434253081771214022310800829

3.3 Matlab Program and Results for Telafer Hospital:

>> syms x;

>> y=1-((1-(-0.27272*x+0.4091))^0.1)*((1-(0.185185+0.185185*x))^0.1)*((1-(0.111+0.0736*x))^0.1)*((1-(0.125+0.06944*x))^0.1)*((1-(0.3654+0.25*x))^0.1)*((1-(0.1154+0.082938*x))^0.1)*((1-(0.35+0.1833*x))^0.1)*((1-(0.222+0.0553*x))^0.1)*((1-(-0.02083*x+0.1875))^0.1)*((1-(0.01786*x+0.01786))^0.1)

y =

1 - (14678402121503563/18014398509481984 - (3335996387978421*x)/18014398509481984)^(1/10)*(889/1000 - (46*x)/625)^(1/10)*(7/8 - (217*x)/3125)^(1/10)*(3173/5000 - x/4)^(1/10)*(389/500 - (553*x)/10000)^(1/10)*(13/20 - (1833*x)/10000)^(1/10)*((3409*x)/12500 + 5909/10000)^(1/10)*((2083*x)/100000 + 13/16)^(1/10)*(49107/50000 - (893*x)/50000)^(1/10)*(4423/5000 - (5976312734317667*x)/72057594037927936)^(1/10)

>> m=((-0.09091091*x+0.09091091)^0.1)*((-0.047285*x+0.1379)^0.1)*((0.018515*x+0.2037)^0.1)*((0.222)^0.1)*((0.01925*x+0.09615)^0.1)*((-0.05765*x+0.0962)^0.1)*((0.06667*x+0.1)^0.1)*((0.08344*x+0.19444)^0.1)*((-0.125*x+0.25)^0.1)*((0.07143*x+0.07143)^0.1)

m =

(3874316284374853*(6550821446398603/72057594037927936 - (6550821446398603*x)/72057594037927936)^(1/10)*(1/4 - x/8)^(1/10)*((77*x)/4000 + 1923/20000)^(1/10)*(481/5000 - (1153*x)/20000)^(1/10)*((1043*x)/12500 + 4861/25000)^(1/10)*((6667*x)/100000 + 1/10)^(1/10)*((7143*x)/100000 + 7143/100000)^(1/10)*(1379/10000 - (6814486668166845*x)/144115188075855872)^(1/10)*((5336585414448943*x)/288230376151711744 + 2037/10000)^(1/10))/4503599627370496

>> r=(1-x)*(2+y-m)

r =

(x - 1)*((3874316284374853*(6550821446398603/72057594037927936 - (6550821446398603*x)/72057594037927936)^(1/10)*(1/4 - x/8)^(1/10)*((77*x)/4000 +

1923/20000)^(1/10)*(481/5000 - (1153*x)/20000)^(1/10)*((1043*x)/12500 +
4861/25000)^(1/10)*((6667*x)/100000 + 1/10)^(1/10)*((7143*x)/100000 +
7143/100000)^(1/10)*(1379/10000 -
(6814486668166845*x)/144115188075855872)^(1/10)*((5336585414448943*x)/288230376151711744 +
2037/10000)^(1/10))/4503599627370496 + (14678402121503563/18014398509481984 -
(3335996387978421*x)/18014398509481984)^(1/10)*(889/1000 - (46*x)/625)^(1/10)*(7/8 -
(217*x)/3125)^(1/10)*(3173/5000 - x/4)^(1/10)*(389/500 - (553*x)/10000)^(1/10)*(13/20 -
(1833*x)/10000)^(1/10)*((3409*x)/12500 + 5909/10000)^(1/10)*((2083*x)/100000 +
13/16)^(1/10)*(49107/50000 - (893*x)/50000)^(1/10)*(4423/5000 -
(5976312734317667*x)/72057594037927936)^(1/10) - 3)

>> t=(1+x)*(2+y-m)

t =

-(x + 1)*((3874316284374853*(6550821446398603/72057594037927936 -
(6550821446398603*x)/72057594037927936)^(1/10)*(1/4 - x/8)^(1/10)*((77*x)/4000 +
1923/20000)^(1/10)*(481/5000 - (1153*x)/20000)^(1/10)*((1043*x)/12500 +
4861/25000)^(1/10)*((6667*x)/100000 + 1/10)^(1/10)*((7143*x)/100000 +
7143/100000)^(1/10)*(1379/10000 -
(6814486668166845*x)/144115188075855872)^(1/10)*((5336585414448943*x)/288230376151711744 +
2037/10000)^(1/10))/4503599627370496 + (14678402121503563/18014398509481984 -
(3335996387978421*x)/18014398509481984)^(1/10)*(889/1000 - (46*x)/625)^(1/10)*(7/8 -
(217*x)/3125)^(1/10)*(3173/5000 - x/4)^(1/10)*(389/500 - (553*x)/10000)^(1/10)*(13/20 -
(1833*x)/10000)^(1/10)*((3409*x)/12500 + 5909/10000)^(1/10)*((2083*x)/100000 +
13/16)^(1/10)*(49107/50000 - (893*x)/50000)^(1/10)*(4423/5000 -
(5976312734317667*x)/72057594037927936)^(1/10) - 3)

>> Fvpaint1= vpaintegral(r,x,[-1 0])

Fvpaint1 = 3.10822

>> Fvpaint2 = vpaintegral(t,x,[0 1])

Fvpaint2 =3.20678

>> scorefunction=(1/9)*(Fvpaint1+ Fvpaint2)

scorefunction =

0.70166648434761336943715153640571

From the definition (1.3), the all score degrees of the above alternatives are:

0.70236616698101494068081270446176 is the score degree of Al-Khansa hospital.

0.69801434253081771214022310800829 is the score degree of Ibn Sina hospital.

0.70166648434761336943715153640571 is the score degree of Telafer hospital.

Consequently, the ranking order of the above three alternatives are

Al-Khansa hospital > Telafer hospital > Ibn Sina hospital.

So the best hospital was Al-Khansa hospital, while Telafer hospital ranked as second hospital, the final hospital was Ibn Sina hospital.

4. **Discussion:**

In the current study, the participants were the nurses who were directly involved with COVID-19 patients. We assessed their knowledge, and attitude to protect them and prevent the further spread of the infection. It is reported that nurses are more prone to infection due to close contact with the patients [28]. In this study, the data of 90 participants were analyzed.

The main finding of the present study showed in table (1) the majority of nurses (55%) were from the age group (31-40) years, this is in agreement with Gaudencia C. et al 2020 were showed the majority of nurses (21%) were from the age group (≥40) years, also shows that the total number of males were 62 % and females were 37% . Usually in other studies, a higher female-to-male ratio has been observed see ref. [29] but contrary to other studies in our study we found a higher number of males. This is because of the reason that in Iraq country males are usually the bread earner and they have no other option, on the other hand, many female nurses have quit their jobs and resigned due to the family burden and the wrong view of some families towards the nursing profession [30], also from our study the table (1) showed a high percentage of nurses (80%) are married, About concerning the level of education, the majority of the nurses (40%) were graduate from Institute, this is disagreement with Gaudencia C, et al 2020 where it showed the majority of nurses (55%) were they had a bachelors.

While regarding an infected the covid19 the result showed the majority of nurses (53%) are infected the covid19. Finally, in terms of experience years, the majority of nurses (42%) had an experience of more than eleven years this is in agreement with [31] Yaling Peng, et al 2020 most of the nurses who participated had more than 8 years of experience.

Since the outbreak in epicenter Wuhan in December 2019, COVID-19 has rapidly become a threat to global public health and led to substantial socioeconomic damages in the whole world. Vigorous measurements have been enforcedly implemented including the lockdown of Wuhan and community quarantine by Chinese central and local governments since the outbreak to mitigate the disease effectively. In addition, public health education has been recognized as an effective measure to prevent and control public health emergencies for public preparedness against such situations. It will lead the public to acquire appropriate knowledge, mitigate panic and seek a positive attitude, and comply with aligned and desired practices. All these KAP elements have been considered crucial to ensure effective prevention and control of the pandemic [32].

Also, the main finding of these studies showed that the majority of the nurses had good knowledge (52%), while the majority of the nurses had fair attitudes (53%). these results agree with other findings that suggest people tend to express negative emotions, such as anxiety and panic, during a pandemic that could affect their attitude ◎33◎ Nevertheless, our results show that the participants' high knowledge of COVID-19 translates into good and safe practices, during the COVID-19 pandemic, which suggests that the practices of Iraq residents are very cautious. Almost 100◎ of respondents refrained from attending social events, 100◎ avoided crowded places, and 96 ◎ said the coronavirus is transmitted by direct contact and through respiratory droplets from an injured person these are positive things about preventing the spread of the disease. and 83◎ of Respondents said using personal protective measures such as Washing hands with soap and water and using masks prevents transmission of the disease, as a result of Iraqi health authorities providing education and outreach materials, to increase public understating of the disease and influence behavioral change.

Nurses' knowledge of covid19 is presented in Table (3). Most of the nurses' samples were good knowledge of the Coronavirus that Corona is a viral infection and the main clinical symptoms of Corona are fever, cough, sore throat, shortness of breath, muscle pain◎fatigue, loss of sense of smell and taste and also the way to prevent infection of Coronavirus, is avoid going to crowded places such as train stations and avoid using public transportation, these criteria represented (100◎), also the knowledge of symptoms and signs of Coronavirus are loss of appetite, nausea, and cramping, and diarrhea which represents (73.33◎ ,76.66◎) respectively. This finding is consistent with other studies that have shown satisfactory levels of knowledge, among the Iraqi population, for epidemics, such as MERS ◎34, 35◎ In our study, the high rate of correct answers to knowledge-related questions among participants was not surprising. This may be due to the characteristics of the sample, as 40◎ had a diploma of Nursing degree. It may also be due to the distribution of the questionnaire, amid the COVID-19 outbreak. In that particular period, people may have gained awareness and knowledge about the disease and its transmission, via television, news and social media, to protect themselves and their families. The positive association found between knowledge, educational background and age, supports our claim. Also, this study agrees with ◎36◎ and with Mohammed K. Al-Hanawi et al 2020 ◎25◎ who indicated that most of the participants in the study (98◎) were aware of the clinical symptoms, and 96◎ knew that there is no clinically approved treatment for COVID-19 as of the date of this manuscript. Viral infections have been documented to be highly contagious among people nearby ◎37◎ However, approximately half of the respondents were unaware that SARS-CoV-2 could spread from person to person nearby. also in the same study, they found (44◎) of the population had little knowledge of when and whom to wear masks to prevent infection. According to the WHO and the CDC, faces mask should only be worn by those who are sick or caring for people suspected of having COVID-19 ◎37,38◎ These findings highlight the need to continue to encourage and emphasize maintaining social distancing, as a means of preventing the spread of the virus.

This manuscript indicates that ignorance about the Corona disease can be dangerous, Coronavirus is transmitted by direct contact and through respiratory droplets from the injured person, and the isolation and treating infected people with the Coronavirus, effective ways to limit the spread Virus reaches to represent (96%). The question related to washing hands with soap and water and using masks prevents disease transmission, and also that the elderly and patients suffering from chronic diseases are more likely to suffer from severe infection and death is represented (83.33%). But knowledge about People who have been in contact with a person infected with the Coronavirus should be put in suitable place quarantine. In general, under observation for 14 days represent (76.66%). While the knowledge about Antibiotics is used to treat corona (51%). Finally, the knowledge regarding the fact that Children and young people do not need to take preventive measures to infection with Coronavirus represents (20%). It is important to note that there has been plenty of efforts at all levels by the government, including public awareness campaigns. The Iraqi Ministry of Health (MOH) has conducted an intensive awareness campaign, communicated via its website, television and various social media. The MOH has produced a guide to COVID-19, to provide residents with facts and precautionary messages in more than 10 languages. The MOH also works with the public and the media, especially via social media platforms. These early actions on engaging the public in prevention and control measures, as well as efforts to combat rumours and misinformation, have been greatly expanded. This unique experience has helped the Arab and international governments in taking prompt response and precautionary measures against COVID-19 to control its spread [39].

Table (4) revealed the key to understanding nurses' attitudes toward COVID-19, most nurses had good attitudes about the put facemask on known or suspected patients represent (88%), This study agrees with Mohammed K. Al-Hanawi et al 2020 [35], where they indicated that most of the participants showed a positive and optimistic attitude toward COVID-19. Approximately 94% concur that the virus can be successfully controlled, and 97% are convinced that the government will control the pandemic. Positive attitudes and high confidence in the control of COVID-19 can be explained by the government's unprecedented actions and prompt response in taking stringent control and precautionary measures against COVID-19, to safeguard citizens and ensure their well-being. These measures include the lockdown, and the suspension of all domestic and international flights, prayer at mosques, schools and universities, and the national curfew imposed on citizens. This finding is consistent with a recent study conducted in China, where the majority of participants were convinced that the disease is curable and that their country will combat the disease [40].

Also, in Table (4) there were two kinds of analysis, the classical analysis shows that only (8%) of nurses agree to the place known or suspected patients in adequately ventilated single rooms, and (44%) didn't have any idea about all health staff members wear protective clothing, But only (21%) said the avoid moving and transporting patients out of their area unless necessary, and (66%) they said should frequently clean hands by using alcohol-based hand rub or soap and water, But only (21%) said the routinely clean and disinfect surfaces in contact with known or suspected patients these results consistent with Blendon RJ. et al 2004, that suggests people tend to express negative emotions, such as anxiety and panic, during a pandemic could affect their attitude [41], the new modern analysis that specified to measure the performance of nurses' staffs in three Iraqi hospitals, in this technique the authors used triangular single-valued neutrosophic numbers and neutrosophic structured elements in multi-attribute decision making to decide which of the hospital is the best hospital in its nurses' staff, it is clearly table (4) has been partitioned into five copies depending on the nurses' opinion which are: (strongly agree, agree, No- idea, disagree, strongly disagree), and their answers on the ten questions.

When asking more questions concerning attitudes, about (65%) of nurses agreed on cleaning and disinfecting environmental surfaces, But only (18%) of them didn't agree with the practice of social distancing, and (44%) agree with prefer having more attendants with the patient in the hospital, Finally, Only (3%) they said want to continue working with corona patients. Patients with poor knowledge were more likely to have poor practice. This finding is consistent with a study in China. [42] This might be due to the reason that knowledge is the main modifier of positive attitudes toward COVID-19 preventive practices and these activities are practised after having awareness and knowledge of the activities to be performed. Knowledge of COVID-19 decreases the risk of infection by improving patient practices [42].

5. Conclusion and Recommendation:

The majority of nurses fifty-five percentage were from the age group (31-40) years, also the majority of nurses (sixty-two percent) were males more than half number, concerning the level of education majority of nurses (forty percentage) were graduates from Institute, the majority of nurses fifty-three percent they were had contracted COVID-19, majority of nurses forty-two percent they were had experience of more than eleven years, we can be concluded from this study that the nurses have relatively good knowledge, but had a poor attitude about the COVID-19.

According to the traditional results, the study recommended holding seminars, lectures and educational conferences in hospitals about the Coronavirus to improve the nurses' knowledge, especially in hospitals where there are cases of Coronavirus, because the nurses will be in direct contact with patients suffering from the pandemic. also raising awareness about the spread of disease by including Covid19 education in schools and colleges curricula is highly needed to prevent the transmission of the disease. While the results gained from section (3) that used intelligent neutrosophic technique illustrate that the attributes of nurses' staffs in Al-Khansa hospital were preferable for patients to be as healthcare staff, the second best hospital was Telafer hospital, the third-ranked order went for Ibn-Sina hospital.

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Expression and Analysis of Scale Effect and Anisotropy of Joint Roughness Coefficient Values Using Confidence Neutrosophic Number Cubic Values

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Abstract: The JRC data collected from a rock mass joint surface difficultly obtain enough large-scale JRC sample data, but small-scale JRC sample data, which usually contain indeterminate and incomplete information due to the limitation of the measurement environment, measurement technology, and other factors. In this case, the existing representation and analysis methods of the JRC sample data almost all lack the measures of confidence levels in the sample data analysis. In this paper, we propose the concept and expression method of confidence neutrosophic number cubic values (CNNCVs), and then establish CNNCVs of joint roughness coefficient (JRC) (JRC-CNNCVs) from the limited/small-scale JRC sample data subject to the normal distribution and confidence level of the JRC sample data to analyze the scale effect and anisotropy of JRC values. In the analysis process, the JRC-CNNCVs are first conversed from the JRC sample data (multi-valued sets) in view of their distribution characteristics and confidence level. Next, JRC-CNNCVs are applied to analyze the scale effect and anisotropy of the JRC values by an actual case, and then the effectiveness and rationality of the proposed expression and analysis method using JRC-CNNCVs are proved by the actual case in a JRC multi-valued environment. From a perspective of probabilistic estimation, the established expression and analysis method makes the JRC expression and analysis more reasonable and reliable under the condition of small-scale sample data.

Keywords: confidence neutrosophic number; confidence neutrosophic number cubic value; joint roughness coefficient; scale effect; anisotropy

1. Introduction

Joint roughness coefficient (JRC) was first proposed by Barton [1] and estimated through experience. Then, JRC is a key index that affects the shear strength of the rock joint. To make the JRC value more reasonable and accurate, researchers have proposed many calculation and expression methods of the JRC value, such as statistical parameter methods [2, 3], straight edge methods [4–7], fractal dimension methods [8–11], etc. However, the indeterminate and incomplete information contained in the JRC values is not considered in the above studies. Due to the irregularity of the rock mass joint surface, the JRC values at different positions on the same joint are different, which also means that the JRC values imply some uncertainty. Numerous studies have shown that the JRC values reflect their scale effect [12–15] and decrease with increasing sample scale. Another obvious characteristic of the JRC values is anisotropy [16–19], that is, the JRC values in different measurement directions of the same rock mass joint is different. Both of these characteristics reflect incomplete and indeterminate information contained in the JRC values. Furthermore, some studies [20, 21] have shown that sampling bias is also an important factor causing the indeterminacy of JRC values.

As an important branch of neutrosophic theory, a neutrosophic number (NN) was first proposed by Smarandache [22–24] from the perspective of the symbol. Then, Ye [25–27] gave the calculation rules of NNs from the perspective of the numerical value and generalized their application in practical problems. Subsequently, related theories of NN have been applied to the decision making/evaluation of investment projects, manufacturing schemes, software testing, goal programming, air quality, etc. [28–33]. NN can generally be expressed as $E(I) = v + \eta I$, where v is the determinate part and ηI is the indeterminate part, the indeterminacy $I \in [I^L, I^U]$, and $v, \eta \in R$ (all real numbers). According to an indeterminate range of $I \in [I^L, I^U]$, NN can represent all values in an interval. Therefore, it is very suitable for the expression of the JRC value because NN can express incomplete and indeterminate information flexibly and conveniently. Yong et al. [34] applied NN to the expression of the JRC value and utilized the NN function to analyze the anisotropy and scale effect of the JRC values. Although this research effectively considers the uncertainty in the JRC values, this method requires the use of a fitting function, where may loss some useful information in the fitting process. To avoid this defect, some scholars combined the theory of neutrosophic statistics with NN to express the JRC values by JRC-NNs [35–37]. Furthermore, Chen et al. [38] combined neutrosophic probability with NN and proposed neutrosophic interval probability (NIP) and neutrosophic interval statistical number (NISN) to express JRC values. Although this method makes the JRC-NN/interval value confident to a certain degree, this method still lacks some probabilistic estimation since the confidence level/interval of the neutrosophic probabilities (P_T, P_I, P_F) is not considered in NIP. It is difficult to ensure the probabilistic credibility of the JRC values within the NIP obtained from the limited JRC sample data. Then, Zhang and Ye [39] presented (fuzzy) confidence neutrosophic number cubic sets (CNNCSs) in a fuzzy multi-valued setting and used them for group decision-making problems with fuzzy multi-valued sets. Motivated by the notion of the fuzzy CNNCS, this paper introduces a confidence NN cubic value (CNNCV) in light of the probability distribution and confidence level of multi-valued sets. Then, considering the probability distribution and confidence level of the JRC values in the actual environment of small-scale JRC sample data, we convert the JRC multi-valued sets obtained from the rock mass in Changshan County (Zhejiang Province, China) into JRC-CNNCVs as the mixed representation form of the JRC confidence intervals and the JRC average values. The proposed expression method of JRC-CNNCVs can ensure that the JRC values fall within confidence neutrosophic numbers (CNNs) (confidence intervals with some confidence level of (1–α)%) from a probabilistic point of view and reveal the magnitude of the JRC mean. Finally, the scale effect and anisotropy of the JRC values are analyzed by JRC-CNNCVs to verify the validity and rationality of the proposed expression and analysis method in the actual environment of the limited/small-scale JRC sample data. Under the condition of small-scale JRC sample data, the expression and analysis method proposed in this study reflects the obvious advantage, as it is more suitable for engineering applications.

The rest of this paper is organized as follows. Section 2 gives the definition of CNNCV in view of the fuzzy CNNCS. Section 3 converts the actual measured JRC multi-valued sets into JRC-CNNCVs in terms of the normal distribution and confidence level of the JRC values, and then analyzes the scale effect and anisotropy of the JRC values by JRC-CNNCVs. Finally, conclusions and further research are given in Section 4.

2. CNNCVs

In this section, we give the definition of CNNCV in terms of the normal distribution of a multi-valued set and the confidence level of (1–α)% for a level α as an extension of the fuzzy CNNCS.

First, we introduce the notions of NN [22–24], NN probability [40], and CNN [39, 40]. The NN $E(I) = v + \eta I$ consists of two parts, including the determinate part ω and the indeterminate part ηI subject to the indeterminacy $I \in [I^L, I^U]$ and $v, \eta \in R$. Obviously, NN (changeable interval number for

$I \in [I^L, I^U]$) can conveniently express both the determinate information and the indeterminate information contained in the indeterminate situation by $E(I) = [v + \eta I^L, v + \eta I^U]$. Especially when considering $E(I)$ as the value of a random variable t in $[v + \eta I^L, v + \eta I^U]$ with the distribution function $p(t)$ (e.g., normal distribution function), the definition of NN probability is introduced as follows [40]:

$$P(t) = p(v+\eta I^L \le t \le v+\eta I^U) = \int_{v+\eta I^L}^{v+\eta I^U} p(t)dt . \quad (1)$$

The larger the NN probability for the variable t, the larger the range of indeterminacy I, that is, the larger the indeterminate interval.

Assuming that there is a multi-valued set $X = \{x_1, x_2, \ldots, x_n\}$ and x_i ($i = 1, 2, \ldots, n$) in X obeys the normal distribution, then the average value v and the standard deviation k of the data in X are given as follows:

$$v = \frac{1}{n}\sum_{i=1}^{n} x_i , \quad (2)$$

$$k = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - v)^2} . \quad (3)$$

Thus, the multi-valued set X with a confidence level of $(1-\alpha)\%$ can be converted into the CNNCV $E_X(I_\alpha)$ by the following equation:

$$E_X(I_\alpha) = \left\langle \left[E^L(I_\alpha), E^U(I_\alpha)\right], v\right\rangle = \left\langle \left[v+\eta I^L, v+\eta I^U\right], v\right\rangle = \left\langle \left[v - \frac{k}{\sqrt{n}} t_{\alpha/2}, v + \frac{k}{\sqrt{n}} t_{\alpha/2}\right], v\right\rangle , \quad (4)$$

where the indeterminate range of I_α is $[I^L, I^U] = [-t_{\alpha/2}, t_{\alpha/2}]$ and $t_{\alpha/2}$ is the critical value that is adopted from [39, 40] in view of confidence levels of $(1-\alpha)\%$ (commonly take $t_{\alpha/2}$ = 1.645, 1.96, 2.576 for the confidence levels of 90%, 95% and 99% [40]).

Example 1. There is a multi-valued set B = {6.32, 1.56, 2.39, 18.35, 10.32, 2.33, 5.77, 3.98, 8.82, 16.32, 9.35, 15.98, 5.58, 11.90, 10.06, 9.33, 5.52, 12.48, 4.46, 10.28} with the normal distribution. Then, the conversing process from the multi-valued set B to the CNNCV E_B is shown below.

First, the mean and standard deviation of the multi-valued set B can be calculated by Eqs. (2) and (3):

(i) $v_B = \frac{1}{n}\sum_{i=1}^{n} b_i = \frac{1}{20}\sum_{i=1}^{20} b_i = 8.56$;

(ii) $k_B = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(b_i - v_B)^2} = 4.82$.

Using Eq. (4) with the most common confidence level of 95% and the critical value $t_{\alpha/2}$ = 1.96 [40], the CNNCV E_B corresponding to B can be obtained below:

$$E_B = \left\langle \left[v_B + \eta_B I^L, v_B + \eta_B I^U\right], v_B\right\rangle = \left\langle \left[v_B - \frac{k_B}{\sqrt{n}} t_{\alpha/2}, v_b + \frac{k_B}{\sqrt{n}} t_{\alpha/2}\right], v_B\right\rangle$$

$$= \left\langle \left[8.56 - 1.96 \times 4.82 / \sqrt{20}, 8.56 + 1.96 \times 4.82 / \sqrt{20}\right], 8.56\right\rangle = \left\langle [6.45, 10.67], 8.56\right\rangle ,$$

where the indeterminate range of I is $[I^L, I^U] = [-t_{\alpha/2}, t_{\alpha/2}] = [-1.96, 1.96]$.

From this example, we get the CNNCV E_B = <[6.45, 10.67], 8.56>, which is composed of the CNN/confidence interval [6.45, 10.67] and the mean 8.56 of B at the confidence level of 95%. Then, we can see that converting the multi-valued set into CNNCV can ensure that 95% probability of the data in B will fall within the CNN/confidence interval [6.45, 10.67], and then 5% probability of the data in B will be outside the CNN/confidence interval [6.45, 10.67], while the mean 8.56 of B reveals the magnitude of the data. Therefore, this conversion approach reflects the advantages of rationality and credibility from the perspective of probabilistic estimation under the condition of small-scale sample data.

3. JRC-CNNCV expression and analysis approach for JRC values

The shear strength of the rock joint surface is recognized as a key parameter in the stability evaluation of engineering rock mass, then JRC is the most important factor affecting the shear strength of the rock mass joint. In practical engineering, the JRC values usually contain a lot of indeterminate and incomplete information, and the measurement of the JRC values is also often limited by the rock joint surface. Therefore, CNNCVs are very suitable for expressing the limited/small-scale JRC sample data.

In this section, we first express the JRC values collected from the rock mass in Changshan County (Zhejiang Province, China) [38] by CNNCVs to give JRC-CNNCVs (including the JRC confidence intervals and the average values). To do so, we introduce the measured data of 240 JRC sample data under 10 sample sizes in 24 measurement directions, and the number of sample data in each multi-valued set is 35. During the measurement process, the measurement directions are divided into 24 directions from 0° to 345° at 15° intervals, and the sample scales are divided into 10 sizes from 10 cm to 100 cm at 10 cm intervals [38]. Then using Eqs. (2) and (3), we calculated the mean v and the standard deviation k of each JRC multi-valued set, which are shown in Table 1.

Many existing studies [41–44] have noted that the distribution of JRC values approximates the normal distribution or left-biased normal distribution after statistical analysis of the JRC values of large-scale sample data. Therefore, in this study, we regard the distribution of the JRC values related to the limited sample data as the normal distribution. In view of the mean and standard deviation of the JRC values, the JRC values (multi-valued sets) are converted into the JRC-CNNCVs at the confidence level of 95% by Eqs. (2)-(4).

Taking the JRC values with the measurement direction of 0° and the sample size of 10 cm as an example, the JRC values are converted into JRC-CNNCV by the following calculation process.

First, it can be seen from Table 1 that the average value v of the JRC values corresponding to the 10 cm sample size in the 0° measurement direction is 10.5861 and the standard deviation k is 2.3026 subject to the 35sample data.

Then using Eq. (4) with the confidence level of 95% and $I_\alpha = [I^L, I^U] = [-1.96, 1.96]$, we can get the following JRC-CNNCV:

$$E_{JRC} = \left\langle [E^L(I_\alpha), E^U(I_\alpha)], v \right\rangle = \left\langle \left[v+\eta I^L, v+\eta I^U\right], v \right\rangle = \left\langle \left[v-\frac{k}{\sqrt{n}}t_{\alpha/2}, v+\frac{k}{\sqrt{n}}t_{\alpha/2}\right], v \right\rangle$$

$$= \left\langle \left[10.5861-1.96\times 2.3026/\sqrt{35}, 10.5861-1.96\times 2.3026/\sqrt{35}\right], 10.5861 \right\rangle = \left\langle [9.8233, 11.3490], 10.5861 \right\rangle.$$

By the similar calculation way, JRC-CNNCVs of E_{JRC} corresponding to the JRC values in other measurement directions and sample sizes are shown in Table 2.

Table 1. The mean v and the standard deviation k of JRC values obtained from 24 different directions under 10 different sample sizes

Direction (°)	Size (cm)	v	k	Direction (°)	Size (cm)	v	k
0	10	10.5861	2.3026	180	10	9.8462	2.1651
	20	9.6833	1.7374		20	9.9489	1.8742
	30	9.3136	1.5113		30	8.7877	1.7512
	40	9.0054	1.7304		40	8.6400	1.6939
	50	8.8621	1.6416		50	8.3278	1.6074
	60	8.8322	1.6281		60	8.1673	1.6464
	70	8.6922	1.6222		70	7.9951	1.5076
	80	8.6070	1.5109		80	7.9080	1.3551
	90	8.5757	1.3621		90	7.8390	1.2001
	100	8.4684	1.2872		100	7.8343	1.0682
15	10	10.7113	2.2212	195	10	9.7585	2.2466

	20	9.9985	1.7591		20	9.2766	1.7717
	30	9.3839	1.6341		30	8.7089	1.7003
	40	9.3013	1.2955		40	8.8393	1.4742
	50	9.2764	1.3036		50	8.5611	1.5763
	60	9.0033	1.3283		60	8.1420	1.5784
15	70	8.8430	1.1862	195	70	7.9523	1.2771
	80	8.5922	0.9463		80	7.6661	0.9830
	90	8.3672	0.7829		90	7.4662	0.8110
	100	8.1451	0.6422		100	7.3181	0.7462
	10	10.5447	2.3948		10	9.6262	2.0233
	20	9.9596	2.0498		20	8.9812	1.5484
	30	9.6129	1.6851		30	8.6833	1.6439
	40	9.1511	1.4519		40	8.3000	1.5812
	50	9.1326	1.4305		50	8.2374	1.5650
30	60	8.6311	1.0111	210	60	7.4231	1.2693
	70	8.7700	1.2245		70	7.7831	1.3201
	80	8.5761	1.0826		80	7.5425	1.1640
	90	8.3001	1.0984		90	7.2541	1.1126
	100	8.1092	1.0718		100	7.0531	0.9608
	10	9.8744	2.3957		10	8.9373	1.9976
	20	9.2311	1.7149		20	8.2956	1.4442
	30	9.0481	1.6650		30	8.1636	1.4963
	40	8.5387	1.1588		40	7.7412	1.2010
45	50	8.3741	1.4496	225	50	7.7188	1.4714
	60	8.6547	1.3639		60	7.4770	1.1934
	70	8.3362	1.2340		70	7.3487	1.2298
	80	8.0820	1.3067		80	7.1410	1.2905
	90	7.8533	1.2252		90	6.8703	1.2240
	100	7.5786	1.1344		100	6.6791	1.1621
	10	9.0755	2.5092		10	7.8881	1.8668
	20	8.4351	2.0025		20	7.3432	1.4171
	30	7.9250	1.8385		30	6.8544	1.1838
	40	7.8246	1.9041		40	6.7833	1.2208
60	50	7.2272	1.1859	240	50	6.3559	0.8483
	60	8.2981	1.8042		60	6.8582	1.1309
	70	7.3770	1.6112		70	6.3833	1.0642
	80	7.1431	1.4132		80	6.1620	1.0109
	90	6.8791	1.2334		90	5.9195	0.8986
	100	6.7181	0.9677		100	6.6900	0.7379
	10	7.9356	2.1883		10	7.2477	1.9553
	20	7.4933	1.7968		20	6.9045	1.4087
	30	6.8131	1.4339		30	6.3656	1.2917
	40	6.3361	1.0453		40	6.1451	1.0536
75	50	6.5859	1.1926	255	50	6.0632	0.9883
	60	6.5293	1.3320		60	6.1090	1.1380
	70	6.2540	1.1064		70	5.9224	0.9629
	80	6.0981	0.8921		80	5.7226	0.8309
	90	5.9603	0.7467		90	5.7850	0.8648
	100	5.8373	0.5905		100	5.4003	0.5677
90	10	7.0272	2.4874	270	10	6.8523	2.1377
	20	6.7210	1.8694		20	6.3523	1.6560

	30	6.3784	1.4929		30	6.0337	1.3998
	40	6.0293	1.1912		40	5.9224	1.3886
	50	6.1884	1.2206		50	5.8177	1.1995
	60	6.1190	1.2062		60	6.0111	1.3044
	70	5.9641	1.1177		70	5.8833	1.2923
	80	5.8982	0.9680		80	5.7481	1.2242
	90	5.8332	0.9337		90	5.8310	0.9231
	100	5.8276	0.8405		100	5.5914	0.9523
	10	7.8275	2.4935		10	7.0764	1.5386
	20	7.2524	1.7772		20	6.5345	1.1169
	30	6.7179	1.2794		30	6.1413	0.9798
	40	6.3534	1.0062		40	5.8853	1.0007
105	50	6.5030	1.2474	285	50	5.7866	0.9245
	60	6.4911	1.5241		60	6.1012	1.3789
	70	6.1771	1.3392		70	5.8737	1.2926
	80	5.9972	1.1066		80	5.6500	1.1515
	90	5.9050	1.0024		90	5.4772	1.0424
105	100	5.8414	0.8529	285	100	5.3685	0.9571
	10	9.1127	2.4071		10	8.5022	1.7660
	20	8.5513	1.9175		20	7.8511	1.3717
	30	8.2402	1.5978		30	7.5667	1.2339
	40	7.9977	1.4306		40	7.3211	1.0433
	50	7.3614	1.0404		50	6.9833	1.1301
120	60	7.8541	1.2019	300	60	7.1079	0.9066
	70	7.2572	1.0793		70	6.8333	0.9414
	80	7.0704	0.9557		80	6.6517	0.8883
	90	6.8619	0.8278		90	6.4512	0.8484
	100	6.6964	0.7785		100	6.3154	0.8254
	10	9.3165	2.0524		10	10.1736	2.5002
	20	8.5978	1.5624		20	9.4947	2.1335
	30	8.1356	1.3338		30	8.9945	1.7520
	40	7.8496	1.0122		40	8.6100	1.5135
	50	7.4142	1.0034		50	8.1522	1.4301
135	60	7.6961	1.3057	315	60	8.7262	1.5348
	70	7.3952	1.1764		70	8.3963	1.6146
	80	7.0922	1.1639		80	7.6686	1.3967
	90	6.9227	1.0501		90	7.4693	1.1613
	100	6.7641	0.9207		100	7.3590	1.1010
	10	10.5180	2.5185		10	9.8695	2.3056
	20	9.5954	1.9277		20	9.0412	1.6325
	30	8.9545	1.7049		30	8.3925	1.6217
	40	8.9364	1.4774		40	8.3692	1.3418
	50	8.4334	1.2041		50	7.9014	1.2522
150	60	8.8462	1.6082	330	60	8.0931	1.3041
	70	8.2161	1.3588		70	7.9430	1.1421
	80	8.0202	1.1037		80	7.6601	1.0313
	90	7.6638	1.0257		90	7.3525	1.0324
	100	7.4492	0.9130		100	7.1028	0.9392
	10	10.6543	2.2913		10	9.7433	2.0098
165	20	9.9955	1.6818	345	20	9.2146	1.6491
	30	9.5722	1.5881		30	8.8033	1.1898

40	8.9070	1.6206	40	8.5143	1.2073
50	8.6527	1.5085	50	7.8935	1.1648
60	8.6762	1.6154	60	7.8888	1.0518
70	8.4030	1.3793	70	7.7577	1.0386
80	8.1164	1.2253	80	7.4773	0.9410
90	7.9124	1.1049	90	7.1833	0.8261
100	7.7224	0.9357	100	7.0093	0.7396

Table 2. JRC-CNNCVs of E_{JRC} in 24 different directions under 10 different sample sizes

Direction (°)	Size (cm)	E_{JRC}	Direction (°)	Size (cm)	E_{JRC}
0	10	<[9.8233, 11.3490], 10.5861>	180	10	<[9.1289, 10.5635], 9.8462>
	20	<[9.1077, 10.2589], 9.6833>		20	<[9.3279, 10.5698], 9.9489>
	30	<[8.8129, 9.8143], 9.3136>		30	<[8.2076, 9.3679], 8.7877>
	40	<[8.3183, 9.4060], 9.0054>		40	<[8.0788, 9.2012], 8.6400>
	50	<[8.4322, 9.5787], 8.8621>		50	<[7.7952, 8.8603], 8.3278>
	60	<[8.2928, 9.3715], 8.8322>		60	<[7.6219, 8.7128], 8.1673>
	70	<[8.1547, 9.2296], 8.6922>		70	<[7.4956, 8.4945], 7.9951>
	80	<[8.1064, 9.1075], 8.6070>		80	<[7.4591, 8.3570], 7.9080>
	90	<[8.1244, 9.0270], 8.5757>		90	<[7.4414, 8.2366], 7.8390>
	100	<[8.0419, 8.8948], 8.4684>		100	<[7.4804, 8.1882], 7.8343>
15	10	<[9.9754, 11.4472], 10.7113>	195	10	<[9.0142, 10.5028], 9.7585>
	20	<[9.4157, 10.5813], 9.9985>		20	<[8.6896, 9.8636], 9.2766>
	30	<[8.8425, 9.9253], 9.3839>		30	<[8.1456, 9.2722], 8.7089>
	40	<[8.8721, 9.7305], 9.3013>		40	<[8.3509, 9.3277], 8.8393>
	50	<[8.8445, 9.7083], 9.2764>		50	<[8.0389, 9.0834], 8.5611>
	60	<[8.5633, 9.4434], 9.0033>		60	<[7.6191, 8.6649], 8.1420>
	70	<[8.4500, 9.2360], 8.8430>		70	<[7.5291, 8.3754], 7.9523>
	80	<[8.2787, 8.9057], 8.5922>		80	<[7.3405, 7.9918], 7.6661>
	90	<[8.1078, 8.6266], 8.3672>		90	<[7.1975, 7.7349], 7.4662>
	100	<[7.9324, 8.3579], 8.1451>		100	<[7.0709, 7.5653], 7.3181>
30	10	<[9.7513, 11.3380], 10.5447>	210	10	<[8.9559, 10.2965], 9.6262>
	20	<[9.2805, 10.6387], 9.9596>		20	<[8.4682, 9.4941], 8.9812>
	30	<[9.0547, 10.1712], 9.6129>		30	<[8.1387, 9.2280], 8.6833>
	40	<[8.6701, 9.6321], 9.1511>		40	<[7.7762, 8.8239], 8.3000>
	50	<[8.6587, 9.6066], 9.1326>		50	<[7.7189, 8.7559], 8.2374>
	60	<[8.2961, 8.9661], 8.6311>		60	<[7.0026, 7.8436], 7.4231>
	70	<[8.3644, 9.1757], 8.7700>		70	<[7.3457, 8.2204], 7.7831>
	80	<[8.2175, 8.9348], 8.5761>		80	<[7.1569, 7.9281], 7.5425>
	90	<[7.9362, 8.6640], 8.3001>		90	<[6.8855, 7.6227], 7.2541>
	100	<[7.7541, 8.4643], 8.1092>		100	<[6.7348, 7.3714], 7.0531>
45	10	<[9.0807, 10.6681], 9.8744>	225	10	<[8.2755, 9.5991], 8.9373>
	20	<[8.6630, 9.7993], 9.2311>		20	<[7.8172, 8.7741], 8.2956>
	30	<[8.4965, 9.5997], 9.0481>		30	<[7.6679, 8.6594], 8.1636>
	40	<[8.1548, 8.9227], 8.5387>		40	<[7.3433, 8.1391], 7.7412>
	50	<[7.8939, 8.8544], 8.3741>		50	<[7.2313, 8.2063], 7.7188>
	60	<[8.2028, 9.1065], 8.6547>		60	<[7.0817, 7.8724], 7.4770>
	70	<[7.9274, 8.7450], 8.3362>		70	<[6.9412, 7.7561], 7.3487>
	80	<[7.6491, 8.5149], 8.0820>		80	<[6.7135, 7.5686], 7.1410>
	90	<[7.4474, 8.2592], 7.8533>		90	<[6.4648, 7.2758], 6.8703>

	100	<[7.2028, 7.9544], 7.5786>		100	<[6.2941, 7.0641], 6.6791>
	10	<[8.2442, 9.9068], 9.0755>		10	<[7.2697, 8.5066], 7.8881>
	20	<[7.7717, 9.0986], 8.4351>		20	<[6.8737, 7.8127], 7.3432>
	30	<[7.3159, 8.5341], 7.9250>		30	<[6.4622, 7.2466], 6.8544>
	40	<[7.1938, 8.4554], 7.8246>		40	<[6.3789, 7.1878], 6.7833>
60	50	<[6.8343, 7.6200], 7.2272>	240	50	<[6.0748, 6.6369], 6.3559>
	60	<[7.7004, 8.8958], 8.2981>		60	<[6.4835, 7.2329], 6.8582>
	70	<[6.8432, 7.9108], 7.3770>		70	<[6.0308, 6.7359], 6.3833>
	80	<[6.6749, 7.6113], 7.1431>		80	<[5.8271, 6.4969], 6.1620>
	90	<[6.4704, 7.2877], 6.8791>		90	<[5.6218, 6.2172], 5.9195>
	100	<[6.3975, 7.0386], 6.7181>		100	<[6.4456, 6.9345], 6.6900>
	10	<[7.2106, 8.6605], 7.9356>		10	<[6.5999, 7.8954], 7.2477>
	20	<[6.8980, 8.0885], 7.4933>		20	<[6.4378, 7.3712], 6.9045>
	30	<[6.3381, 7.2882], 6.8131>		30	<[5.9376, 6.7935], 6.3656>
	40	<[5.9898, 6.6824], 6.3361>		40	<[5.7960, 6.4941], 6.1451>
75	50	<[6.1908, 6.9810], 6.5859>	255	50	<[5.7358, 6.3907], 6.0632>
	60	<[6.0880, 6.9706], 6.5293>		60	<[5.7320, 6.4861], 6.1090>
	70	<[5.8875, 6.6206], 6.2540>		70	<[5.6034, 6.2414], 5.9224>
	80	<[5.8025, 6.3937], 6.0981>		80	<[5.4474, 5.9979], 5.7226>
	90	<[5.7129,6.2077], 5.9603>		90	<[5.4985, 6.0715], 5.7850>
	100	<[5.6417, 6.0330], 5.8373>		100	<[5.2122, 5.5884], 5.4003>
	10	<[6.2031, 7.8512], 7.0272>		10	<[6.1440, 7.5605], 6.8523>
	20	<[6.1017, 7.3404], 6.7210>		20	<[5.8037, 6.9009], 6.3523>
	30	<[5.8838, 6.8730], 6.3784>		30	<[5.5699, 6.4974], 6.0337>
	40	<[5.6347, 6.4239], 6.0293>		40	<[5.4623, 6.3824], 5.9224>
90	50	<[5.7840, 6.5928], 6.1884>	270	50	<[5.4203, 6.2151], 5.8177>
	60	<[5.7194, 6.5186], 6.1190>		60	<[5.5790, 6.4433], 6.0111>
	70	<[5.5938, 6.3344], 5.9641>		70	<[5.4552, 6.3115], 5.8833>
	80	<[5.5775, 6.2189], 5.8982>		80	<[5.3425, 6.1537], 5.7481>
	90	<[5.5238, 6.1425], 5.8332>		90	<[5.5252, 6.1369], 5.8310>
	100	<[5.5491, 6.1060], 5.8276>		100	<[5.2759, 5.9069], 5.5914>
	10	<[7.0014, 8.6536], 7.8275>		10	<[6.5667, 7.5862], 7.0764>
	20	<[6.6636, 7.8411], 7.2524>		20	<[6.1644, 6.9045], 6.5345>
	30	<[6.2941, 7.1418], 6.7179>		30	<[5.8166, 6.4659], 6.1413>
	40	<[6.0200, 6.6867], 6.3534>		40	<[5.5538, 6.2169], 5.8853>
105	50	<[6.0897, 6.9163], 6.5030>	285	50	<[5.4803, 6.0929], 5.7866>
	60	<[5.9862, 6.9960], 6.4911>		60	<[5.6444, 6.5580], 6.1012>
	70	<[5.7334, 6.6208], 6.1771>		70	<[5.4454, 6.3019], 5.8737>
	80	<[5.6306, 6.3638], 5.9972>		80	<[5.2685, 6.0315], 5.6500>
	90	<[5.5729, 6.2371], 5.9050>		90	<[5.1318, 5.8225], 5.4772>
	100	<[5.5588, 6.1239], 5.8414>		100	<[5.0514, 5.6855], 5.3685>
	10	<[8.3152, 9.9102], 9.1127>		10	<[7.9171, 9.0873], 8.5022>
	20	<[7.9160, 9.1865], 8.5513>		20	<[7.3966, 8.3055], 7.8511>
120	30	<[7.7108, 8.7695], 8.2402>	300	30	<[7.1579, 7.9755], 7.5667>
	40	<[7.5238, 8.4717], 7.9977>		40	<[6.9754, 7.6667], 7.3211>
	50	<[7.0167, 7.7061], 7.3614>		50	<[6.6089, 7.3577], 6.9833>
	60	<[7.4559, 8.2523], 7.8541>		60	<[6.8075, 7.4082], 7.1079>
	70	<[6.8996, 7.6148], 7.2572>		70	<[6.5214, 7.1451], 6.8333>
120	80	<[6.7538, 7.3871], 7.0704>	300	80	<[6.3574, 6.9460], 6.6517>
	90	<[6.5877, 7.1362], 6.8619>		90	<[6.1701, 6.7323], 6.4512>
	100	<[6.4384, 6.9543], 6.6964>		100	<[6.0419, 6.5889], 6.3154>

135	10	<[8.6366, 9.9965], 9.3165>	315	10	<[9.3453, 11.0019], 10.1736>
	20	<[8.0802, 9.1154], 8.5978>		20	<[8.7879, 10.2015], 9.4947>
	30	<[7.6937, 8.5775], 8.1356>		30	<[8.4140, 9.5749], 8.9945>
	40	<[7.5142, 8.1849], 7.8496>		40	<[8.1086, 9.1114], 8.6100>
	50	<[7.0817, 7.7466], 7.4142>		50	<[7.6784, 8.6260], 8.1522>
	60	<[7.2635, 8.1286], 7.6961>		60	<[8.2178, 9.2347], 8.7262>
	70	<[7.0055, 7.7850], 7.3952>		70	<[7.8614, 8.9312], 8.3963>
	80	<[6.7065, 7.4778], 7.0922>		80	<[7.2059, 8.1313], 7.6686>
	90	<[6.5748, 7.2706], 6.9227>		90	<[7.0846, 7.8541], 7.4693>
	100	<[6.4591, 7.0692], 6.7641>		100	<[6.9943, 7.7238], 7.3590>
150	10	<[9.6836, 11.3523], 10.5180>	330	10	<[9.1057, 10.6334], 9.8695>
	20	<[8.9568, 10.2341], 9.5954>		20	<[8.5003, 9.5820], 9.0412>
	30	<[8.3896, 9.5193], 8.9545>		30	<[7.8552, 8.9297], 8.3925>
	40	<[8.4469, 9.4258], 8.9364>		40	<[7.9246, 8.8137], 8.3692>
	50	<[8.0344, 8.8323], 8.4334>		50	<[7.4865, 8.3162], 7.9014>
	60	<[8.3134, 9.3790], 8.8462>		60	<[7.6610, 8.5251], 8.0931>
	70	<[7.7660, 8.6663], 8.2161>		70	<[7.5647, 8.3214], 7.9430>
	80	<[7.6545, 8.3858], 8.0202>		80	<[7.3184, 8.0018], 7.6601>
	90	<[7.3240, 8.0036], 7.6638>		90	<[7.0105, 7.6946], 7.3525>
	100	<[7.1468, 7.7517], 7.4492>		100	<[6.7917, 7.4140], 7.1028>
165	10	<[9.8952, 11.4134], 10.6543>	345	10	<[9.0775, 10.4091], 9.7433>
	20	<[9.4383, 10.5527], 9.9955>		20	<[8.6683, 9.7610], 9.2146>
	30	<[9.0461, 10.0983], 9.5722>		30	<[8.4091, 9.1975], 8.8033>
	40	<[8.3701, 9.4439], 8.9070>		40	<[8.1143, 8.9143], 8.5143>
	50	<[8.1530, 9.1525], 8.6527>		50	<[7.5076, 8.2794], 7.8935>
	60	<[8.1410, 9.2114], 8.6762>		60	<[7.5403, 8.2372], 7.8888>
	70	<[7.9461, 8.8600], 8.4030>		70	<[7.4136, 8.1018], 7.7577>
	80	<[7.7104, 8.5223], 8.1164>		80	<[7.1656, 7.7891], 7.4773>
	90	<[7.5463, 8.2784], 7.9124>		90	<[6.9096, 7.4570], 7.1833>
	100	<[7.4124, 8.0324], 7.7224>		100	<[6.7643, 7.2544], 7.0093>

As shown in Table 2, JRC-CNNCV reflects the mixed information of the confidence interval and the mean of the JRC values at the confidence level of 95%, which is different from the traditional expression methods of JRC-NNs. Furthermore, JRC-CNNCV reveals that 95% probability of the JRC data will fall within CNNs corresponding the confidence level of 95% and the mean magnitude of the JRC data. In this case, the confidence level can effectively guarantee the rationality and credibility of E_{JRC} from a probabilistic point of view. From a perspective of probabilistic estimation, the JRC-CNNCVs of E_{JRC} in Table 2 can contain 95% probability of the actual JRC values, but cannot contain 5% probability of them based on the probability estimation of the JRC values corresponding to different measurement directions and sample sizes.

To analyze the scale effect and anisotropy of the JRC values by the expression method of JRC-CNNCVs, we give Figures 1-3 and their analysis in detail.

Figure 1 shows the E_{JRC} values at different sizes in the measurement directions of 0°, 90°, 180°, and 270° from Table 2 and the average values of the corresponding JRC values in Table 1. As shown in Figure 1, the upper and lower bounds of JRC-CNNs and the JRC average values in the same measurement direction show a decreasing trend with the increase of the sample size, which is in line with the scale effect of the JRC values. At the same time, we can find that the standard deviation of the JRC values corresponding to each measurement direction generally shows a decreasing trend with the increase of the sample size. In Figure 2, taking the measurement direction of 15° as an example, the confidence intervals in E_{JRC} shrink with the increase of the sample size in the same direction, and then the JRC-CNNs and the JRC average values decrease with the increase of the sample size. This case also means that the uncertainty about the JRC values is diminishing with the increase of the sample size. In addition, we select the confidence intervals and the average values in E_{JRC} in the measurement directions from 0° to 345° under the sample sizes of 10 cm, 40 cm, 70 cm, and 100 cm to draw polar plots in Figure 3. As shown in Figure 3, the interval values of $[E^L, E^U]$ and the average values of v in different measurement directions under the same size are different, which reflect the anisotropy of the JRC values. Meanwhile, with increasing sample size, the upper and lower bounds of CNNs in different measurement directions under the same size are also close to each other, and the interval ranges of CNNs and the average values in E_{JRC} are decreasing, which indicates the scale effect of the anisotropy of the JRC values. The above conclusions show that the JRC values expressed by JRC-CNNCVs can also reflect indeterminate and incomplete information contained in the anisotropy of the JRC values. Therefore, it is obvious that the expression and analysis method using JRC-CNNCVs proposed in this study can effectively reveal the scale effect and anisotropy of the JRC values, then the proposed method is obviously superior to the existing methods regarding their rationality and credibility in the application scenarios of small-scale sample data.

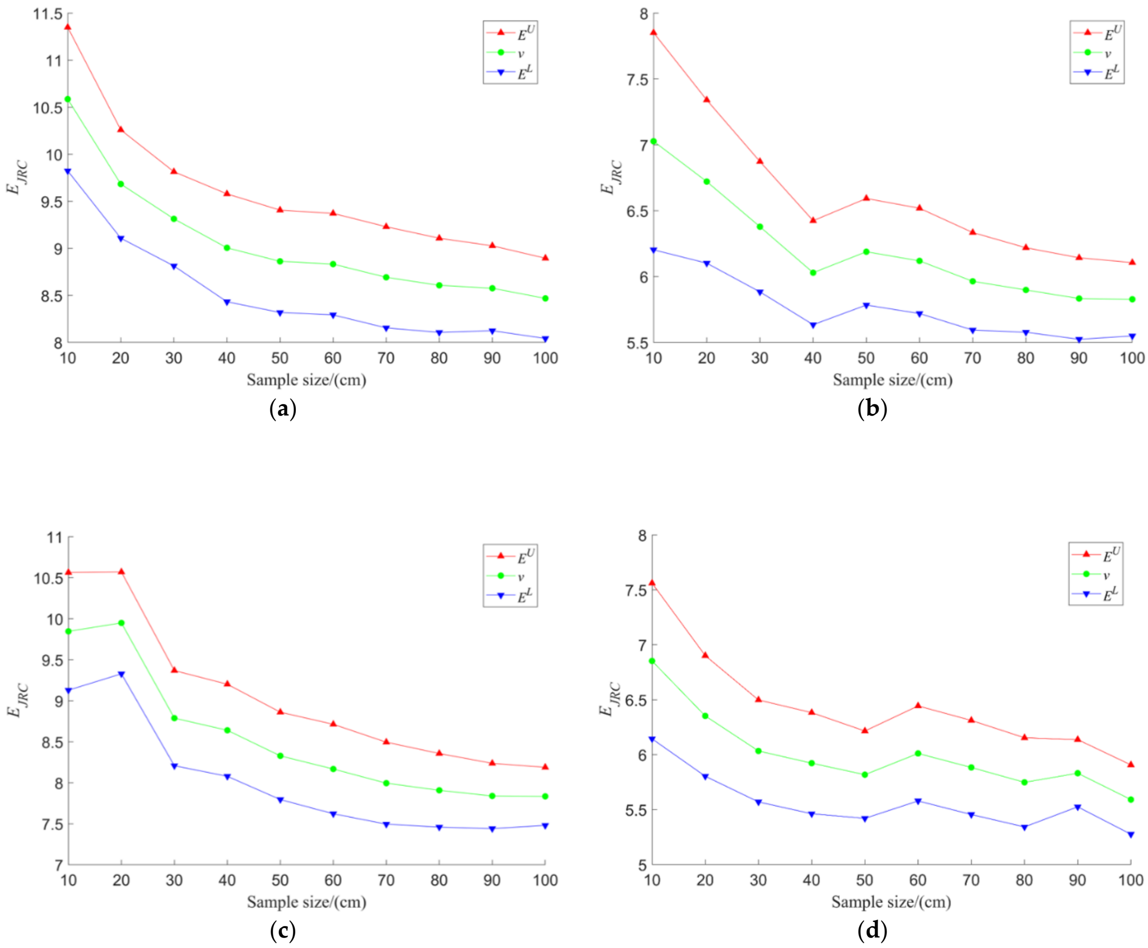


Figure 1. (**a**) E_{JRC} (JRC-CNNCVs) corresponding to JRC values at different sizes in the 0° direction; (**b**) E_{JRC} (JRC-CNNCVs) corresponding to the JRC values at different sizes in the 90° direction; (**c**) E_{JRC} (JRC-CNNCVs) corresponding to the JRC values at different sizes in the 180° direction; (**d**) E_{JRC} (JRC-CNNCVs) corresponding to the JRC values at different sizes in the 270° direction.

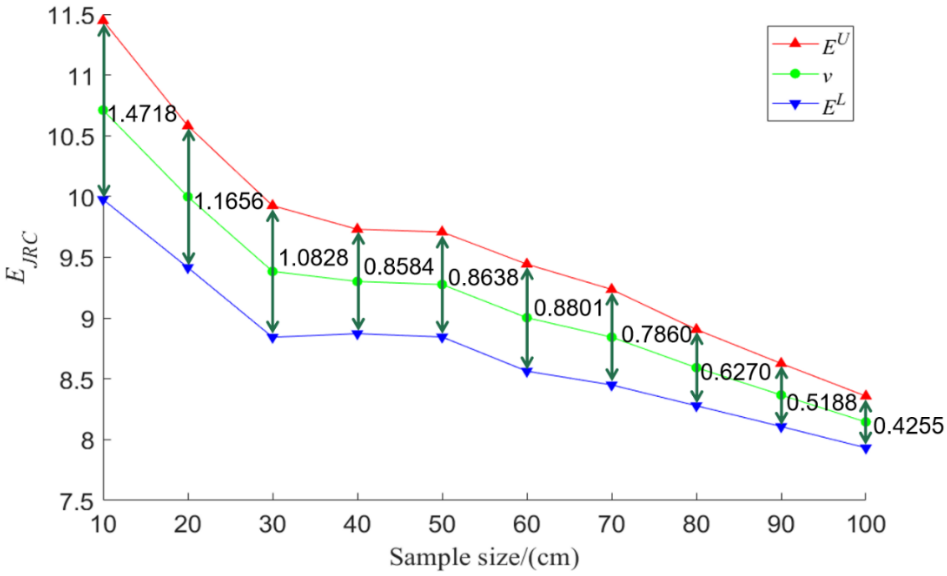
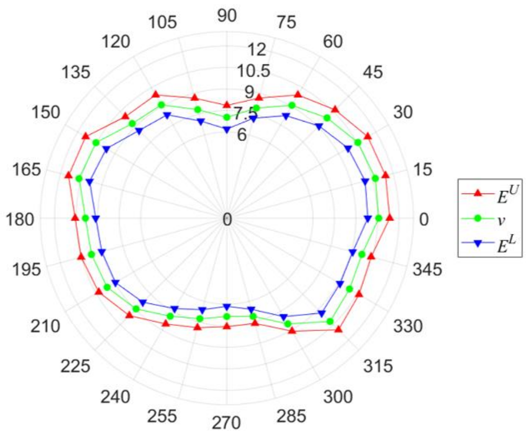
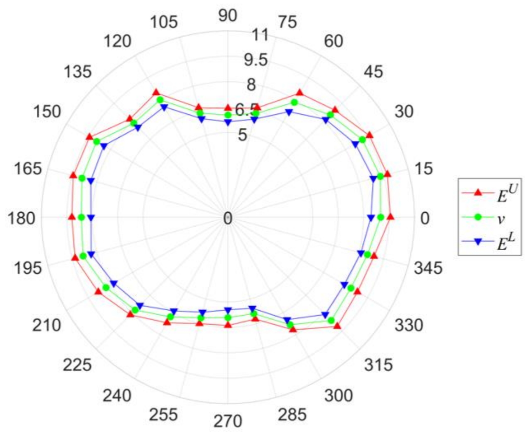


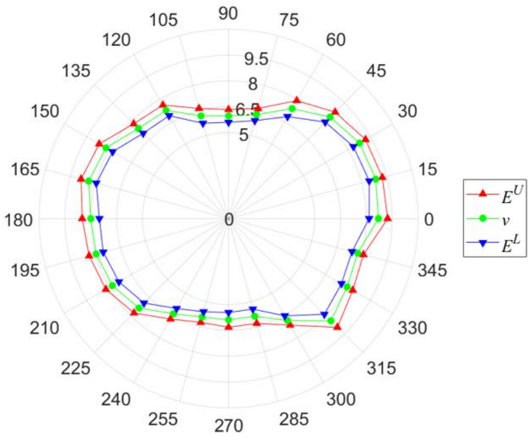
Figure 2. E_{JRC} (JRC-CNNCVs) corresponding to the JRC values of different sizes in the 15° direction



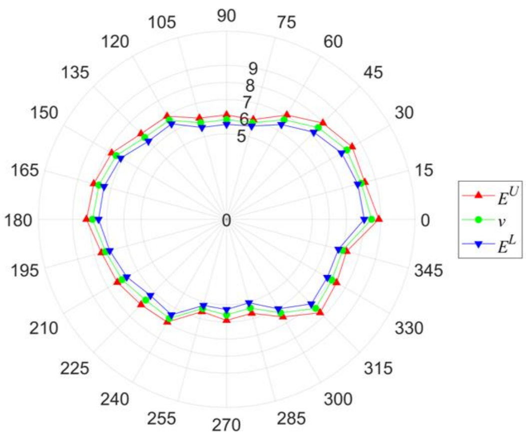
(**a**)



(**b**)



(**c**)



(**d**)

Figure 3. (**a**) E_{JRC} (JRC-CNNCVs) corresponding to the JRC values of different measurement directions under 10 cm sample size; (**b**) E_{JRC} (JRC-CNNCVs) corresponding to the JRC values of different measurement directions under 40 cm sample size; (**c**) E_{JRC} (JRC-CNNCVs) corresponding to the JRC values of different measurement directions under 70 cm sample size; (**d**) E_{JRC} (JRC-CNNCVs) corresponding to the JRC values of different measurement directions under 100 cm sample size

4. Conclusions

Since it is difficult to usually obtain enough large-scale JRC sample data from rock mass joint surfaces due to the limitation of the measurement environment, measurement technology, and other factors, there exists some indeterminate and incomplete information in small-scale JRC sample data. In this case, the existing representation and analysis methods of JRC sample data almost all lack the measures of confidence levels in sample data analysis. Then, the JRC-CNNCV expression obtained from the limited/small-scale JRC sample data can effectively solve the above problems and ensure that the JRC values can fall within CNN with a certain confidence level. Unlike classical statistics which takes the JRC values as crisp values, JRC-CNNCV transformed from the JRC values is composed of the confidence interval and the average value, so the uncertainty and incompleteness contained in the JRC values can be fully reflected by the probabilistic estimation within a confidence interval. As the extension and improvement of the existing JRC-NN expression methods for JRC values, the JRC-CNNCV expression method can effectively ensure the reliability of the small-scale sample data so as to lessen the loss of useful information and simplify the analysis process. In addition, through the expression and analysis method using JRC-CNNCVs for the JRC values of an actual case, this study also revealed the scale effect and anisotropy of the JRC values so as to further verify the effectiveness and convenience of the proposed expression and analysis method. It is clear that the proposed expression and analysis method can further enhance the credibility of the analysis results on the JRC characteristics (the scale effect and anisotropy of the JRC values) from a probabilistic point of view. In the future, CNNCVs combined with other analysis methods will present more in-depth analysis of the scale effect and anisotropy of the JRC values, and the CNNCV expression and analysis method will be further extended to engineering or experiment data processing.

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Analyzing Critical Success Factors of IoT-Enabled Green Supply Chain Management Using Bipolar Neutrosophic-DEMATEL

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Abstract: Recent Supply Chain Management Systems (SCMS) that is based on newer technologies are intelligent enough to lower costs, improve product quality, and speed up the decision-making process in the manufacturing operation. The reduction of overall environmental impact, is a goal of the Green Supply Chain Management (GSCM) systems. It is achieved by integrating environmentally friendly processes into SCMS. The main and most crucial role in achieving the goal of sustainable development is played by the GSCM practices. It is proved that, the adoption of IoT technology into the GSCM systems can increase its performance and productivity. The goal of this paper is to examine the Critical Success Factors (CSFs) for the efficient adoption of IoT and green solutions throughout the supply chain for the manufacturing sector. As a result of pressure from the government and increased customer awareness of environmental issues, manufacturers are currently focusing on GSCM that is enabled by IoT gadgets. The selection and prioritization of IoT-enabled GSCM success variables is performed, in this paper, using bipolar neutrosophic-DEMATEL approach.

Keywords: Green Supply Chains, IoT, Critical Success Factors, DEMATEL Bipolar-Neutrosophic.

1. Introduction

Currently, because of the increased awareness of sustainability and environmental protection, Green Supply Chain Management (GSCM) has gained a lot of popularity [1]. Industries are required to consider eco-friendly strategies to improve the environment and their green reputation [2]. In this context, organizations around the world have implemented more dependable techniques to encourage sustainable and green management at all levels of their supply chain as a result of changes in rules, legislation, lifestyle, and notably customer tastes in society [3]. The major goals of GSCM are to minimize or eliminate the environmental harms caused by supply chain operations in order to accomplish sustainable development goals [4]. Design, buying, production, storage, and logistics processes should thus be restructured by businesses as a result of GSCM efforts [5]. Reverse logistics is also a crucial component of GSCM for recovering value from discarded goods and materials or properly recycling them [6]. The use of GSCM has several advantages for businesses and communities. The environmental performance of GSCM is increased while waste production is minimized. Companies will be guided by GSCM to increase their eco-proficiency. Companies can stay up and increase their level of commercial performance since GCSM leads to the enhancement of activities in the economic and environmental fields [4]. The implementation of Internet of Things (IoT) technology in the GSCM parts can be viewed to add an intelligence and sustainability assets to GSCM systems [7]. In this context, researchers and students have offered numerous definitions for GSCM. For instance, Hervani et al [8], explicitly used GSCM to integrate sustainable design, efficient material handling, green product procurement, environmentally conscious supplier cooperation, and waste management. However, according to Jayant and Tiwari [9], GCSM is a novel idea for determining the right course for developing products that are compliant with environmental laws and pre-established standards, and businesses require it as a tactic to collaborate on environmental challenges.

According to statistical data, GSCM can control 80% of environmental consequences by using ecologically friendly approaches [10]. Therefore, it is an important issue to identify and analyze the Critical Success Factors (CSFs) for the good implementation of the IoT-Enabled GSCM. It is the primary objective of this investigation to analyze the CSFs of GSCM. As a result, the organizations focus only on these critical factors not all the factors included in the implementation process. As the more you focus on smaller number of factors the more you can give your best in all of them. We will help this organizations by using the opinions of three experienced experts to build an integrated strategy of the decision-making trial and evaluation laboratory (DEMATEL) and Bipolar-Neutrosophic sets (BNSs) to remove the vagueness of those opinions by using a wider scale to identify critical success factors by grouping them into cause and effect groups [11], [12]. DEMATEL is a method used to develop and analyze a structural model of relationships and interdependences between success factors into a matrices or digraphs [13]–[16]. It will assist the decision-makers in determining the success factors of greater influence, which will be the critical success factors, by dividing these factor to cause and effect based on their values and their importance.

The following goals are the main emphasis of the research paper:

1. Identifying the critical success factors (CSFs) for modern GSCM systems to provide competitive advantages to organizations.
2. This work also aims at clarifying contextual relationships between the CSFs and prioritizing these CSFs using an integration of DAMTEL and the BNS methods according the opinions of three experts.
3. Considering the modern information technological (IT) paradigms such as IoT [17], [18], Big Data [19], [20], and Big Data Analytics (BDA) [21]–[23] that are now becoming a critical parts for implementing an intelligent and more productive GSCM systems.

The majority of publications in the literature used the fuzzy set, which has limitations because it only considers the membership function and ignores the non-membership function and indeterminacy function [24], [25].Utilizing Smarandache's Neutrosophic sets (NSs), a generalization of intuitionistic fuzzy sets, we were able to overcome this flaw. The focus of NS is on the membership and non-membership functions, and it does take the indeterminacy function into account. This strategy can deal with incomplete knowledge in the actual world because it is a generalization [26].

The following sections are organized as follows: Section 2 discusses the literature review of IoT-enabled GSCM supply chain and its CSFs .Section 3 introduces the basic concepts for the research. Section 4 presents the research methodology which is the integration of bipolar neutrosophic sets and the DEMATEL method. We also introduce Application of BNS-DEMATEL approach for analyzing the CSFs of the IoT-enabled GSCM in section 4. Section 5 discusses the outcomes of the research. In section 6 we conclude the research.

2. Literature review

In academic and professional communities, GSCM is gaining popularity. It is a relatively new idea that's gaining popularity with suppliers and producers centered on improving green processes, reducing waste through reverse logistics, raising the caliber of products across their entire life cycles, and reducing harmful environmental activities [27]. In this section we will focus on the key aspects examined in the related works of experimental GSCM implementation, one of the sustainability's branches [28], to determine the most important elements for its successful implementation. Traditional SCM methods, on the other hand, can have a negative influence on the environment and act as a source of pollution [29]. Examples include the production, distribution, and waste of raw materials. Therefore, it is crucial to incorporate green practices like green manufacturing, green packaging, and reverse logistics into overall SCM activities in order to safeguard the environment [30]. To preserve the environment against unwelcome activities, many nations seek to set environmental standards and regulations for the industry. In order to achieve sustainable environmental, economic, and social development, these standards mandate that enterprises use green and environmentally friendly practices throughout all SCM activities [31]. As a result, numerous researchers have demonstrated in their work how important it is to adopt GSCM in a way that also considers the organization's environment.

2.1 Utilization of MCDM tools in the GSCM implementation

Researchers' interest in employing causal analysis in their studies has grown over the past few years. The primary explanation is that problems arise for a variety of reasons. In order to identify the relative relevance of the components, decision-makers must adopt a technique known as the Multi-Criteria Decision Making (MCDM) approach when evaluating such an issue [10]. MCDM is subfield of operations research methodologies where the multifaceted decision-making problem can be reduced to a smaller problem [32]. The MCDM considerably helps to organise and prioritise the decision-making challenges. It also supports decision-makers in analyzing, choosing, and ranking options based on the assessment of numerous decision problem criteria [33]. There is a need for an efficient technique to assess the various aspects that function as GSCM components. Consequently, the MCDM approaches remain the best choice. The evaluation of green SCM decision problems makes extensive use of the MCDM approaches, including the DEMATEL method, Analytical Hierarchy Process (AHP) , Analytic Hierarchy Process (ANP), Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS), Linear Programming, and Fuzzy Programming [1]. This study use an integrated approach of the DEMATEL method and BNSs to identify the CSFs for the near optimal implementation of the GSCM systems.

2.2 Proposed CSFs for the successful GSCM implementation

For the purpose of implementing IoT-Enabled GSCM practices, this study identifies numerous significant key factors. These green indications are thought of as a supplementary tools for SCM operations. The purpose of this study was to determine the key CSFS from the standpoint of the practices applied by green supply chain enabled by IoT technology using a thorough set of literature reviews. A thorough assessment of the literature led to the identification of the twenty CSFs under two key dimensions. The two main dimensions are: green enablers which include the green drivers for implementing the GSCM. The second dimension is the IoT enablers which include the main drivers for enabling the implantation of IoT in the GSCM system. The full description of the CSFs is presented in table 1. It has been determined that the adoption of green SCM methods frequently uses MCDM approaches. These strategies are thought to be crucial for resolving difficult decision-making issues. This study focus on prioritizing the CSFs for implementing the IoT-enabled GSCM successfully using an integrated approach of the DEMATEL method and BNSs to remove the vagueness of those opinions by using a wider scale to identify critical success factors by grouping them into cause and effect groups.

Table 1: Proposed CSFs for the Successful implementation of IoT-Enabled GSCM

Code	CSF	DESCRIPTION	Source
A	**Green enablers**		
F1	Influence from investors and stakeholders	Investors or stakeholders have an interest in the company. Additionally, they are entitled to collect profits that the business publishes.	[34], [35]
F2	Waste management	Wastes are substances that are not primary products and that the producer wants to dispose of because they are no longer needed for the producer's own purposes of production, transformation, or consumption.	[25], [36]
F3	Environmental regulations	Organizations are required to abide by environmental regulations set forth by the government (such as hazardous and poisonous regulations), and penalties are always a possibility if they do not.	[37]–[40]
F4	Global competitive advantage	Sustainable business practises give organisations a considerable competitive advantage over those that don't, which ultimately helps the organization's bottom line. Global competitiveness is a major force behind an organization's adoption of sustainable practices.	[41], [42]
F5	Management of toxic/ harmful/ hazardous materials and waste and pollution preventative measures	Sustainable business practices help organizations control their toxic waste production, which has a negative impact on both the environment and people, as well as their consumption of hazardous materials.	[43]–[45]
F6	Green packaging and transportation	The rising CO2 gas emissions during the 1990s have put the environment at risk due to freight transportation, which is why green transportation was started. Green packaging is characterized as being constructed entirely of natural plants and being environmentally friendly. It is safe for the environment, human health, and the welfare of cattle.	[46]–[48]
F7	Top management commitment	It occurs when individuals holding top rank positions directly contribute to a specific and critically important area of a business.	[49]–[51]
F8	Greening competition pressures	Competitive advantages associated with going green, better brand perception, and financial gains will all benefit competitors who have environmental management systems.	[52]
F9	power negotiations along the supply chain	Requirements, advantages, and restrictions that the market imposes on the participants of a negotiation.	[53], [54]
F10	Green marketing	Companies can promote their goods based on their "green" reputation, giving them a competitive edge in the global marketing arena. Additionally, because these businesses are	[55]

		adhering to environmental regulations, new markets are now available to them.	
F11	Standards and regulations (ISO 14000)	Certifications encourage businesses to improve their quality while using a green strategy. Environmentally friendly operations between suppliers and customers need the use of ISO 14000.	[56], [57]
F12	Reverse logistics	It addresses the activities involved in product reuse. The reverse logistics also includes actions for refurbishing and remanufacturing.	[58]
F13	creation of highly qualified and competent human labor	SCM thought leaders advise businesses to take a more proactive approach to developing SCM people with the skills and industry-specific competences required to manage supply chain processes that are becoming more complicated and strategically significant. This will help in the management of the green processes.	[59]
F14	green practices, policies, and infrastructure	Companies must make considerable changes to their management policies, operations, infrastructure, and products to successfully implement supply chain greening, frequently by adopting new business models.	[60], [61]
F15	Collaboration with suppliers	Although this CSF doesn't act as a direct main driver, it should be underlined that supply chain collaboration and integration can more effectively advance sustainability. Incorporating the thoughts and suggestions provided by suppliers can be quite beneficial.	[62], [63]
F16	Recycling and lifecycle management	Establish a set of standards for the collection, handling, and recovery of used electronics and electrical equipment, and hold producers financially accountable for these actions.	[64]
B	**IoT enablers**		
F17	Radio Frequency Identification (RFID) and Global positioning system (GPS)	The smart GSCM systems enabled the real-time location of people and resources both indoors and outside thanks to RFID and GPS technologies. They made it possible to manage stock updates, transportation, and item tracking.	[7], [65]
F18	Cloud computing and IoT applications	Through the use of the Internet, cloud computing reduces uncertainty for decision-makers by offering services like infrastructure, platform, and software. It enables decision-makers in GSCM systems of any business to make decisions at the appropriate time, location, product, and quantity. It host the IoT applications the enable the management and tracking of the GSCM entities.	[18], [66]–[69]
F19	Sensor technologies and sensor network	Sensor and sensors network allow for the real-time data collection and transmission in the IoT-Enabled GSCM systems.	[70]–[72]
F20	Big Data and Big Data Analytics (BDA) tools	As a result of large amounts of data collected by the IoT sensors, Big data technologies must be adopted in the GSCM system for managing such volumes of data. BDA tools allow for the real-time analysis of the collected big data to provide GSCM decision makers with accurate and timely data.	[21], [23], [73]

3. Preliminaries

To fully describe our suggested strategy, this section is broken down into three subsections. We shall first give a brief overview of the neutrosophic sets. The DEMATEL approach will then be demonstrated. Finally, we will present the DEMATEL approach that we have proposed using BNS.

3.1 Neutrosophic Sets

In this part, the notion of a neutrosophic set is discussed, along with some of its operations, including the scoring, accuracy, and certainty functions that are used to compare BNSs. BNSs are successor to the neutrosophic sets, fuzzy sets, intuitionistic fuzzy sets, and bipolar fuzzy sets. The fuzzy set was utilised in a bulk of articles in the literature, but it has drawbacks because it only takes the membership function into account while ignoring the function of non-membership and the function of indeterminacy. We overcame this drawback by using the concept of Neutrosophic sets (NSs). The function of indeterminacy is considered, although the both of membership and non-membership methods are the main emphasis of NS [74], [75]. As a generalization, this method can deal with information gaps in the real world.

Definition 1. Let S be a points' space. And s∈S. A neutrosophic set N in S is described by the following three functions:

1. The indeterminacy-membership function $I^N(s)$.
2. The truth-membership function $T^N(s)$.
3. The falsity-membership function $F^N(s)$.

$T^N(s)$, $I^N(s)$, and $F^N(s)$ are actual nonstandard or standard subsets of $(s): S \rightarrow]-0,1+[$ and $F_N(s): S \rightarrow]-0,1+[$. Where the sum of T_N (s), I_N (s) and F_N(s), so 0− ≤ sup (s) + sup s + sup s ≤3+ is not limited.

Definition 2 [12]: A BNS N in ꜫ is characterised as an item with the form N⊚⊚⊚s, $T^p(s)$, $I^p(s)$, $F^p(s)$, $T^n(s)$, $I^n(s)$, $F^n(s)$⊚: s ∈ ꜫ ⊚, where T^p, I^p , F^p : ꜫ → ⊚1, 0⊚and T^n , I^n , F^n : ꜫ → ⊚-1, 0⊚. The positive membership degree $T^p(s)$, $I^p(s)$, $F^p(s)$ of an item ∈ ꜫ pointing to a BNS N and the negative membership degree $T^n(x)$, $I^n(x)$, $F^n(x)$ of an item ∈ ꜫ identifies a counter-property that is implicit and comparable to a BNS A , Assume that Ã⊚⊚ T^p, I^p, F^p, T^n, I^n, F^n ⊚ be a Bipolar Neutrosophic Number (BNN).following that, the score function S (Ã), accuracy function a (Ã), and certainty function c (Ã) of a BNN are described as in the following relations:

$$S(\tilde{A}) ⊚ \frac{1}{6} ⊚⊚ T^p + 1 - I^p + 1 - F^p + 1 + T^n - I^n - F^n ⊚ \quad (1)$$

$$a(\tilde{A}) = T^p - F^p + T^n - F^n \quad (2)$$

$$c(\tilde{A}) = T^p - F^n \quad (3)$$

3.2 DEMATEL

The DEMATEL approach was developed to assess and depict the nature and intensity of the direct and indirect relationships between complex real-world aspects in a study system [76]. DEMATEL is a method for group decision-making that involves gathering ideas and determining the relationship between causes and effects in complex problems [77]. The DEMATEL method helps to uncover the optimal answer in solving problems involving complex systems by assessing the overall relationships between the structural parts of a study system and grouping elements into cause and effect groups [13], [78]. It is constructed upon the foundation of graph theory [2].

4. Proposed BNS-DEMATEL approach for analyzing the CSFs of the IoT-enabled GSCM

In this part we will integrate the DEMATEL method with BNS neutrosophic set to overcome the vagueness in the expert's opinions which will be used in DEMATEL matrices. The steps involved in the suggested approach are shown in Figure 1.

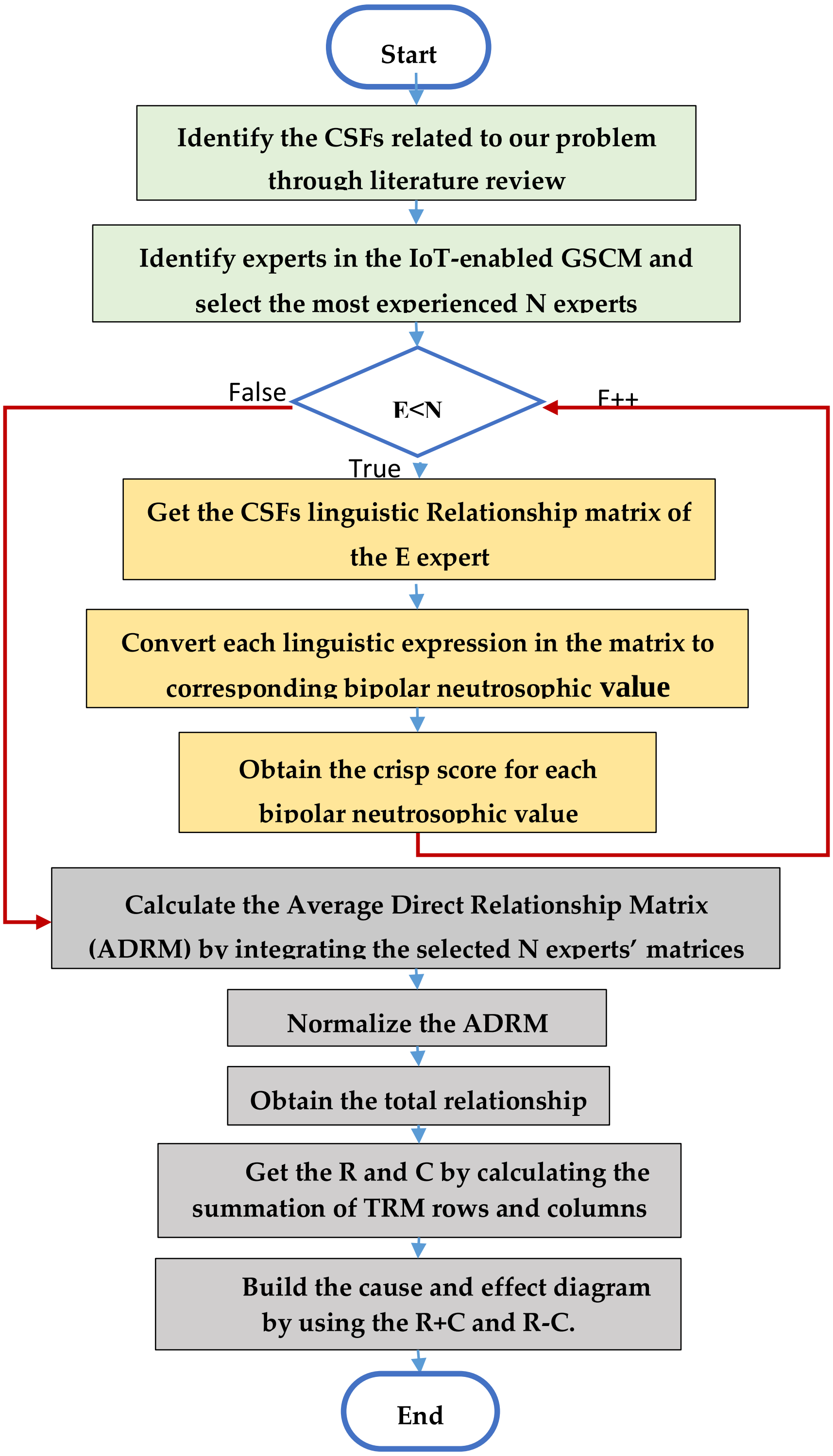

Figure 1: Steps of the Neutrosophic DEMATEL approach

Step 1: Identify the CSFs for the IoT-enabled GSCM

It is the first step in our model to discover the CSFs for implementing the green supply chain that is enabled by IoT gadgets. By surveying the literature, we have identified twenty CSFs for implementing the IoT-enabled GSCM. The identified factors are shown in table 1. We classified the identified CSFs into two groups. The first group is the critical factors for the green manufacturing. Where the second group include the factor required for enabling the GSCM by the IoT gadgets.

Step 2: Identify experts in the IoT-enabled GSCM and select the most experienced N experts:

We searched for experts having experience in Green supply chain management operations and IoT technology. After filtering the experts, we have selected the most experienced three experts in the fields of IoT and GSCM. The metadata about the selected experts is provided in table 2. We then provide our experts with a full description about the selected CSFs. Afterwards, we initiate our request of linguistic Relationship matrix from each expert.

Table 2: Experts' metadata

expert	Experience (years)	expertise	occupation	profession	Gender
E1	13	Very good	Industry	GSCM	Male
E2	10	Good	Industry	IOT-GSCM	Male
E3	9	Medium	Industry	IOT-GSCM	Male

Step 3: Get the CSFs linguistic Relationship matrix of each expert

Here, we make a pairwise comparison matrix between CSFs based on each expert's opinion using the linguistic expressions shown in figure 2. Table 3 show the linguistic relationship matrix for expert 1.

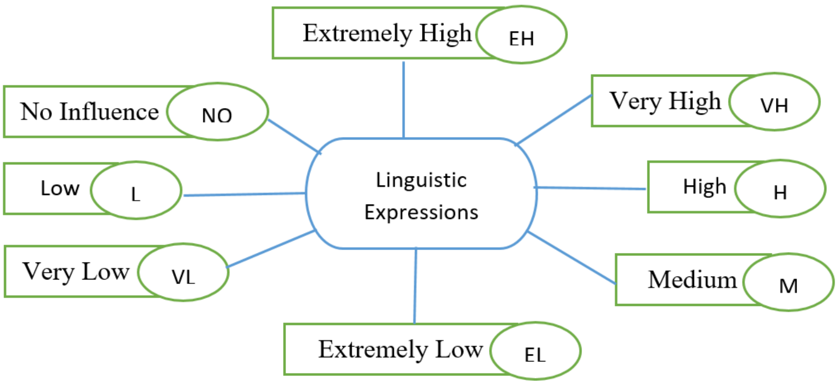

Figure 2: Linguistic expressions

Step4: Convert each linguistic expression in the linguistic Relationship matrix into corresponding bipolar neutrosophic value

Now, we will replace the linguistic expressions into its corresponding bipolar neutrosophic values according to table 4.

Step 5: Obtain the crisp score for each bipolar neutrosophic value

We firstly calculate the crisp score related to each bipolar neutrosophic value according to eq. (1). Table 5 show the calculated crisp scores for the linguistic expressions used in this study. In table 6 and table 7, we present crisp score matrix of expert 1.

Step 6: Calculate the Average Direct Relationship Matrix (ADRM) by integrating the selected N experts' matrices

In this step we will integrate the three collected matrices of the three experts into one matrix that is called average direct relationship matrix which represent an average of the observations collected from the chosen experts. Each value in the ADRM matrix is estimated according to the following equation:

$$ADRM_{i,j} = \frac{\sum_{E=1}^{N} V_{i,j}^{E}}{N} \quad (4)$$

Where $A_{i,j}$ is the ADRM value at row i and column j, it denote the degree to which the factor i affects the factor j, $V_{i,j}^{E}$ is the value of the crisp matrix at row i and column j for expert E, N is the number of experts. Table 8 and 9 show the ADRM of the three experts.

Step 7: Normalize the ADRM

In this step we will normalize the initial direct relationship matrix using the following equations.

$$S = Max\{\max_{1 \le i \le N} \sum_{j=1}^{N} ADRM_{i,j}, \max_{1 \le j \le N} \sum_{i=1}^{N} ADRM_{i,j}\}$$

(5)

$$NADRM = \frac{ADRM}{S} \quad (6)$$

Table 10 and 11 show the normalized ADRM.

Step 8: Obtain the total relationship matrix

Her, we obtain the total relationship Matrix using the following equation

$$TRM = NADRM * (I - NADRM)^{-1} \quad (7)$$

Where I is the identity matrix. Table 12 and 13 show the normalized ADRM.

Step 9: Get the Ri and Cj by calculating the summation of TRM rows and columns

we will calculate Calculate R+C (which indicates the degree of importance), R-C(which divide the CSFs into cause or effect groups, if the result is positive then it's in cause group (which has significant effect on the overall goal an need more attention) and if the result is negative then it's in effect group(which is affected by other factors easily but it's doesn't mean it is not important as every factor has his own influence on other factors as we if it has high important (high R+ C) and negative (R-C) such as F7 we can consider it as cause group) and by using the following equations

$$C = \sum_{j=1}^{N} TRM_{i,j}$$

(8)

$$R = \sum_{i=1}^{N} TRM_{i,j} \quad (9)$$

Table 14 show the summation of TRM rows and columns.

Step 10: Build the cause and effect diagram by using R+C and R-C

In this step we will build the diagram based on the result in the previous step we will use the values of Ri + Cj as the horizontal axes and use the values of Ri − Cj as the vertical axes and CSFs with positive values (above the x-axes) it's in cause group and CSFs with negative values (below the x-axes)) it's in effect group. Figure 2 show the casual diagram

Table 3: Linguistic relationship matrix for expert 1

Code	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13	F14	F15	F16	F17	F18	F19	F20
F1	0	M	H	M	M	H	M	L	H	VH	H	VH	L	EL	H	EH	VH	VH	VH	VH
F2	H	0	EH	H	M	VH	H	M	VL	VL	M	EH	VH	H	M	L	VL	L	EL	L
F3	VH	VH	0	VH	H	EH	H	M	H	H	M	M	VL	H	VH	H	H	H	H	H
F4	M	EH	H	0	L	M	H	L	VL	H	VH	M	H	L	H	VH	H	H	H	H
F5	H	H	M	H	0	M	H	H	H	M	L	H	VH	M	EH	VH	H	VH	M	EH
F6	EH	M	VH	H	H	0	L	H	H	M	H	L	EH	H	H	H	EH	H	M	EH
F7	VH	H	M	M	M	H	0	VH	VH	VH	EH	VH	VH	VH	M	H	VH	VH	VH	EH
F8	L	M	M	M	H	M	H	0	M	EH	M	M	H	H	L	M	EH	EH	EH	EH
F9	M	VL	M	H	H	L	M	H	0	H	VH	H	M	M	H	VL	H	H	H	VH
F10	H	H	M	VH	M	VL	VH	L	H	0	M	M	H	H	M	L	EH	EH	EH	EH
F11	EH	H	VH	M	VH	EH	H	VH	VH	M	0	M	M	VL	EH	VH	VL	M	L	M
F12	H	VH	M	M	VL	VH	VH	VL	M	M	L	0	M	H	VL	H	EH	VH	EH	EH
F13	M	H	H	H	H	H	VH	M	VL	M	M	H	0	VH	L	EL	M	M	M	M
F14	M	M	H	EH	L	L	VL	EH	L	EH	M	VH	M	0	EH	M	VH	H	VH	VH
F15	H	L	M	L	H	M	EL	L	M	H	VH	H	M	L	0	VL	VM	H	H	VH
F16	M	M	VH	H	M	L	H	VL	VH	M	VL	VL	H	L	VH	0	VH	M	VH	H
F17	EH	H	M	VH	M	M	VH	VH	M	EH	M	EH	M	H	H	EH	0	EH	EH	VH
F18	VH	H	M	VH	H	H	H	VH	H	EH	M	VH	M	H	H	H	H	0	EH	EH
F19	H	H	M	VH	M	H	VH	VH	M	EH	M	EH	M	H	H	H	EH	EH	0	M
F20	VH	EH	H	VH	H	VH	EH	EH	VH	EH	H	EH	M	H	EH	H	H	EH	M	0

Table 4: Linguistic expressions with its corresponding bipolar neutrosophic value.

Linguistic Expression	Bipolar Neutrosophic value
EH	(1.00,0.00,0.10,-0.10,-0.90,-1.00)
VH	(0.85,0.15,0.20,-0.20,-0.70,-0.90)
H	(0.75,0.20,0.25,-0.25,-0.60,-0.50)
M	(0.50,0.50,0.50,-0.50,-0.50,-0.50)
L	(0.30,0.40,0.60,-0.30,-0.20,-0.10)
VL	(0.25,0.70,0.80,-0.55,-0.15,-0.30)
EL	(0.15,0.90,0.80,-0.65,-0.10,-0.10)
NO	(0.00,1.00,1.00,-1.00,0.00,0.00)

Table 5: Crisp scores for the Study linguistic expressions

Bipolar Neutrosophic Number Scale	Crisp score
(1.00,0.00,0.10,-0.10,-0.90,-1.00)	0.9500
(0.85,0.15,0.20,-0.20,-0.70,-0.90)	0.8167
(0.75,0.20,0.25,-0.25,-0.60,-0.50)	0.6917
(0.50,0.50,0.50,-0.50,-0.50,-0.50)	0.5000
(0.30,0.40,0.60,-0.30,-0.20,-0.10)	0.3833
(0.25,0.70,0.80,-0.55,-0.15,-0.30)	0.2750
(0.15,0.90,0.80,-0.65,-0.10,-0.10)	0.1667
(0.00,1.00,1.00,-1.00,0.00,0.00)	0.0000

Table 6: Part 1 of the crisp score matrix of expert 1

Code	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10
F1	0	0.5000	0.6917	0.5000	0.5000	0.6917	0.5000	0.3833	0.6917	0.8167
F2	0.6917	0	0.9500	0.6917	0.5000	0.8167	0.6917	0.5000	0.2750	0.2750
F3	0.8167	0.8167	0	0.8167	0.6917	0.9500	0.6917	0.5000	0.6917	0.6917
F4	0.5000	0.9500	0.6917	0	0.3833	0.5000	0.6917	0.3833	0.2750	0.6917
F5	0.6917	0.6917	0.5000	0.6917	0	0.5000	0.6917	0.6917	0.6917	0.5000
F6	0.9500	0.5000	0.8167	0.6917	0.6917	0	0.3833	0.6917	0.6917	0.5000
F7	0.8167	0.6917	0.5000	0.5000	0.5000	0.6917	0	0.8167	0.8167	0.8167
F8	0.3833	0.5000	0.5000	0.5000	0.6917	0.5000	0.6917	0	0.5000	0.9500
F9	0.5000	0.2750	0.5000	0.6917	0.6917	0.3833	0.5000	0.6917	0	0.6917
F10	0.6917	0.6917	0.5000	0.8167	0.5000	0.2750	0.8167	0.3833	0.6917	0
F11	0.9500	0.6917	0.8167	0.5000	0.8167	0.9500	0.6917	0.8167	0.8167	0.5000
F12	0.6917	0.8167	0.5000	0.5000	0.2750	0.8167	0.8167	0.2750	0.5000	0.5000
F13	0.5000	0.6917	0.6917	0.6917	0.6917	0.6917	0.8167	0.5000	0.2750	0.5000
F14	0.5000	0.5000	0.6917	0.9500	0.3833	0.3833	0.2750	0.9500	0.3833	0.9500
F15	0.6917	0.3833	0.5000	0.3833	0.6917	0.5000	0.1667	0.3833	0.5000	0.6917

F16	0.5000	0.5000	0.8167	0.6917	0.5000	0.3833	0.6917	0.2750	0.8167	0.5000
F17	0.9500	0.6917	0.5000	0.8167	0.5000	0.5000	0.8167	0.8167	0.5000	0.9500
F18	0.8167	0.6917	0.5000	0.8167	0.6917	0.6917	0.6917	0.8167	0.6917	0.9500
F19	0.6917	0.6917	0.5000	0.8167	0.5000	0.6917	0.8167	0.8167	0.5000	0.9500
F20	0.8167	0.9500	0.6917	0.8167	0.6917	0.8167	0.9500	0.9500	0.8167	0.9500

Table 7: Part 2 of the crisp score matrix of expert 1

Code	F11	F12	F13	F14	F15	F16	F17	F18	F19	F20
F1	0.6917	0.8167	0.3833	0.1667	0.6917	0.9500	0.8167	0.8167	0.8167	0.8167
F2	0.5000	0.9500	0.8167	0.6917	0.5000	0.3833	0.2750	0.3833	0.1667	0.3833
F3	0.5000	0.5000	0.2750	0.6917	0.8167	0.6917	0.6917	0.6917	0.6917	0.6917
F4	0.8167	0.5000	0.6917	0.3833	0.6917	0.8167	0.6917	0.6917	0.6917	0.6917
F5	0.3833	0.6917	0.8167	0.5000	0.9500	0.8167	0.6917	0.8167	0.5000	0.9500
F6	0.6917	0.3833	0.9500	0.6917	0.6917	0.6917	0.9500	0.6917	0.5000	0.9500
F7	0.9500	0.8167	0.8167	0.8167	0.5000	0.6917	0.8167	0.8167	0.8167	0.9500
F8	0.5000	0.5000	0.6917	0.6917	0.3833	0.5000	0.9500	0.9500	0.9500	0.9500
F9	0.8167	0.6917	0.5000	0.5000	0.6917	0.2750	0.6917	0.6917	0.6917	0.8167
F10	0.5000	0.5000	0.6917	0.6917	0.5000	0.3833	0.9500	0.9500	0.9500	0.9500
F11	0	0.5000	0.5000	0.2750	0.9500	0.8167	0.2750	0.5000	0.3833	0.5000
F12	0.3833	0	0.5000	0.6917	0.2750	0.6917	0.9500	0.8167	0.9500	0.9500
F13	0.5000	0.6917	0	0.8167	0.3833	0.1667	0.5000	0.5000	0.5000	0.5000
F14	0.5000	0.8167	0.5000	0	0.9500	0.5000	0.8167	0.6917	0.8167	0.8167
F15	0.8167	0.6917	0.5000	0.3833	0	0.2750	0.8167	0.6917	0.6917	0.8167
F16	0.2750	0.2750	0.6917	0.3833	0.8167	0	0.8167	0.5000	0.8167	0.6917
F17	0.5000	0.9500	0.5000	0.6917	0.6917	0.9500	0	0.9500	0.9500	0.8167
F18	0.5000	0.8167	0.5000	0.6917	0.6917	0.6917	0.6917	0	0.9500	0.9500
F19	0.5000	0.9500	0.5000	0.6917	0.6917	0.6917	0.9500	0.9500	0	0.5000
F20	0.6917	0.9500	0.5000	0.6917	0.9500	0.6917	0.6917	0.9500	0.5000	0

Table 8: Part 1 of the average direct relationship matrix

Code	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10
F1	0.0000	0.5000	0.6917	0.5000	0.5000	0.6917	0.5000	0.3833	0.6917	0.8167
F2	0.6917	0.0000	0.9500	0.6917	0.5000	0.8167	0.6917	0.5000	0.2750	0.2750
F3	0.8167	0.8167	0.0000	0.8167	0.6917	0.9500	0.6917	0.5000	0.6917	0.6917
F4	0.5000	0.9500	0.6917	0.0000	0.3833	0.5000	0.6917	0.3833	0.2750	0.6917
F5	0.6917	0.6917	0.5000	0.6917	0.0000	0.5000	0.6917	0.6917	0.6917	0.5000
F6	0.9500	0.5000	0.8167	0.6917	0.6917	0.0000	0.3833	0.6917	0.6917	0.5000
F7	0.8167	0.6917	0.5000	0.5000	0.5000	0.6917	0.0000	0.8167	0.8167	0.8167
F8	0.3833	0.5000	0.5000	0.5000	0.6917	0.5000	0.6917	0.0000	0.5000	0.9500
F9	0.5000	0.2750	0.5000	0.6917	0.6917	0.3833	0.5000	0.6917	0.0000	0.6917
F10	0.6917	0.6917	0.5000	0.8167	0.5000	0.2750	0.8167	0.3833	0.6917	0.0000
F11	0.9500	0.6917	0.8167	0.5000	0.8167	0.9500	0.6917	0.8167	0.8167	0.5000
F12	0.6917	0.8167	0.5000	0.5000	0.2750	0.8167	0.8167	0.2750	0.5000	0.5000
F13	0.5000	0.6917	0.6917	0.6917	0.6917	0.6917	0.8167	0.5000	0.2750	0.5000
F14	0.5000	0.5000	0.6917	0.9500	0.3833	0.3833	0.2750	0.9500	0.3833	0.9500
F15	0.6917	0.3833	0.5000	0.3833	0.6917	0.5000	0.1667	0.3833	0.5000	0.6917
F16	0.5000	0.5000	0.8167	0.6917	0.5000	0.3833	0.6917	0.2750	0.8167	0.5000
F17	0.9500	0.6917	0.5000	0.8611	0.5000	0.5000	0.8167	0.8167	0.5000	0.8000
F18	0.8167	0.6917	0.5000	0.8167	0.6917	0.6917	0.6917	0.8167	0.6917	0.9500
F19	0.6917	0.6917	0.5000	0.8167	0.5000	0.6917	0.8167	0.8167	0.5000	0.8000
F20	0.8611	0.9500	0.6917	0.8167	0.6917	0.8167	0.9500	0.9500	0.8167	0.9500

Table 9: Part 2 of the average direct relationship matrix

Code	F11	F12	F13	F14	F15	F16	F17	F18	F19	F20
F1	0.6917	0.8167	0.3833	0.1667	0.6917	0.9500	0.8167	0.8167	0.8167	0.8611
F2	0.5000	0.9500	0.8167	0.6917	0.5000	0.3833	0.2750	0.3833	0.1667	0.3833
F3	0.5000	0.5000	0.2750	0.6917	0.8167	0.6917	0.6917	0.6917	0.6917	0.6917
F4	0.8167	0.5000	0.6917	0.3833	0.6917	0.8167	0.6917	0.6917	0.6917	0.6917

F5	0.3833	0.6917	0.8167	0.5000	0.9500	0.8167	0.6917	0.8167	0.5000	0.9500
F6	0.6917	0.3833	0.9500	0.6917	0.6917	0.6917	0.9500	0.6917	0.5000	0.9500
F7	0.9500	0.8167	0.8167	0.8167	0.5000	0.6917	0.8167	0.8167	0.8167	0.9500
F8	0.5000	0.5000	0.6917	0.6917	0.3833	0.5000	0.9500	0.9500	0.9500	0.9500
F9	0.8167	0.6917	0.5000	0.5000	0.6917	0.2750	0.6917	0.6917	0.6917	0.8167
F10	0.5000	0.5000	0.6917	0.6917	0.5000	0.3833	0.9500	0.9500	0.9500	0.9500
F11	0.0000	0.5000	0.5000	0.2750	0.9500	0.8167	0.2750	0.5000	0.3833	0.5000
F12	0.3833	0.0000	0.5000	0.6917	0.2750	0.6917	0.9500	0.8167	0.9500	0.9500
F13	0.5000	0.6917	0.0000	0.8167	0.3833	0.1667	0.5000	0.5000	0.4611	0.5000
F14	0.5000	0.8167	0.5000	0.0000	0.9500	0.5000	0.7111	0.6917	0.8167	0.8167
F15	0.8167	0.6917	0.5000	0.3833	0.0000	0.2750	0.8167	0.6917	0.6917	0.8611
F16	0.2750	0.2750	0.6917	0.3833	0.8167	0.0000	0.8167	0.5000	0.8167	0.6917
F17	0.5000	0.9500	0.5000	0.6917	0.6917	0.9500	0.0000	0.9500	0.9500	0.8167
F18	0.5000	0.8167	0.5000	0.6917	0.6917	0.6917	0.6917	0.0000	0.8000	0.9500
F19	0.5000	0.9500	0.5000	0.6917	0.6917	0.6917	0.9500	0.9500	0.0000	0.5000
F20	0.6917	0.9500	0.5000	0.6917	0.9500	0.6917	0.6917	0.9500	0.5000	0.0000

Table 10: Part 1 of the normalization matrix

Code	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10
F1	0.0000	0.0331	0.0458	0.0331	0.0331	0.0458	0.0331	0.0254	0.0458	0.0540
F2	0.0458	0.0000	0.0629	0.0458	0.0331	0.0540	0.0458	0.0331	0.0182	0.0182
F3	0.0540	0.0540	0.0000	0.0540	0.0458	0.0629	0.0458	0.0331	0.0458	0.0458
F4	0.0331	0.0629	0.0458	0.0000	0.0254	0.0331	0.0458	0.0254	0.0182	0.0458
F5	0.0458	0.0458	0.0331	0.0458	0.0000	0.0331	0.0458	0.0458	0.0458	0.0331
F6	0.0629	0.0331	0.0540	0.0458	0.0458	0.0000	0.0254	0.0458	0.0458	0.0331
F7	0.0540	0.0458	0.0331	0.0331	0.0331	0.0458	0.0000	0.0540	0.0540	0.0540
F8	0.0254	0.0331	0.0331	0.0331	0.0458	0.0331	0.0458	0.0000	0.0331	0.0629
F9	0.0331	0.0182	0.0331	0.0458	0.0458	0.0254	0.0331	0.0458	0.0000	0.0458

F10	0.0458	0.0458	0.0331	0.0540	0.0331	0.0182	0.0540	0.0254	0.0458	0.0000
F11	0.0629	0.0458	0.0540	0.0331	0.0540	0.0629	0.0458	0.0540	0.0540	0.0331
F12	0.0458	0.0540	0.0331	0.0331	0.0182	0.0540	0.0540	0.0182	0.0331	0.0331
F13	0.0331	0.0458	0.0458	0.0458	0.0458	0.0458	0.0540	0.0331	0.0182	0.0331
F14	0.0331	0.0331	0.0458	0.0629	0.0254	0.0254	0.0182	0.0629	0.0254	0.0629
F15	0.0458	0.0254	0.0331	0.0254	0.0458	0.0331	0.0110	0.0254	0.0331	0.0458
F16	0.0331	0.0331	0.0540	0.0458	0.0331	0.0254	0.0458	0.0182	0.0540	0.0331
F17	0.0629	0.0458	0.0331	0.0570	0.0331	0.0331	0.0540	0.0540	0.0331	0.0529
F18	0.0540	0.0458	0.0331	0.0540	0.0458	0.0458	0.0458	0.0540	0.0458	0.0629
F19	0.0458	0.0458	0.0331	0.0540	0.0331	0.0458	0.0540	0.0540	0.0331	0.0529
F20	0.0570	0.0629	0.0458	0.0540	0.0458	0.0540	0.0629	0.0629	0.0540	0.0629

Table 11: Part 2 of the normalization matrix

Code	F11	F12	F13	F14	F15	F16	F17	F18	F19	F20
F1	0.0458	0.0540	0.0254	0.0110	0.0458	0.0629	0.0540	0.0540	0.0540	0.0570
F2	0.0331	0.0629	0.0540	0.0458	0.0331	0.0254	0.0182	0.0254	0.0110	0.0254
F3	0.0331	0.0331	0.0182	0.0458	0.0540	0.0458	0.0458	0.0458	0.0458	0.0458
F4	0.0540	0.0331	0.0458	0.0254	0.0458	0.0540	0.0458	0.0458	0.0458	0.0458
F5	0.0254	0.0458	0.0540	0.0331	0.0629	0.0540	0.0458	0.0540	0.0331	0.0629
F6	0.0458	0.0254	0.0629	0.0458	0.0458	0.0458	0.0629	0.0458	0.0331	0.0629
F7	0.0629	0.0540	0.0540	0.0540	0.0331	0.0458	0.0540	0.0540	0.0540	0.0629
F8	0.0331	0.0331	0.0458	0.0458	0.0254	0.0331	0.0629	0.0629	0.0629	0.0629
F9	0.0540	0.0458	0.0331	0.0331	0.0458	0.0182	0.0458	0.0458	0.0458	0.0540
F10	0.0331	0.0331	0.0458	0.0458	0.0331	0.0254	0.0629	0.0629	0.0629	0.0629
F11	0.0000	0.0331	0.0331	0.0182	0.0629	0.0540	0.0182	0.0331	0.0254	0.0331
F12	0.0254	0.0000	0.0331	0.0458	0.0182	0.0458	0.0629	0.0540	0.0629	0.0629

F13	0.0331	0.0458	0.0000	0.0540	0.0254	0.0110	0.0331	0.0331	0.0305	0.0331
F14	0.0331	0.0540	0.0331	0.0000	0.0629	0.0331	0.0471	0.0458	0.0540	0.0540
F15	0.0540	0.0458	0.0331	0.0254	0.0000	0.0182	0.0540	0.0458	0.0458	0.0570
F16	0.0182	0.0182	0.0458	0.0254	0.0540	0.0000	0.0540	0.0331	0.0540	0.0458
F17	0.0331	0.0629	0.0331	0.0458	0.0458	0.0629	0.0000	0.0629	0.0629	0.0540
F18	0.0331	0.0540	0.0331	0.0458	0.0458	0.0458	0.0458	0.0000	0.0529	0.0629
F19	0.0331	0.0629	0.0331	0.0458	0.0458	0.0458	0.0629	0.0629	0.0000	0.0331
F20	0.0458	0.0629	0.0331	0.0458	0.0629	0.0458	0.0458	0.0629	0.0331	0.0000

Table 12: Part 1 of the total relationship matrix

Code	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10
F1	0.2061	0.2233	0.2266	0.2324	0.2010	0.2266	0.2264	0.2053	0.2185	0.2548
F2	0.2170	0.1619	0.2151	0.2123	0.1743	0.2071	0.2064	0.1837	0.1641	0.1894
F3	0.2658	0.2503	0.1915	0.2606	0.2198	0.2496	0.2446	0.2209	0.2250	0.2564
F4	0.2274	0.2414	0.2184	0.1900	0.1852	0.2063	0.2279	0.1961	0.1828	0.2361
F5	0.2544	0.2402	0.2204	0.2495	0.1738	0.2198	0.2428	0.2295	0.2220	0.2420
F6	0.2767	0.2343	0.2457	0.2563	0.2233	0.1934	0.2298	0.2357	0.2274	0.2485
F7	0.2856	0.2617	0.2413	0.2610	0.2250	0.2524	0.2214	0.2589	0.2496	0.2844
F8	0.2323	0.2257	0.2159	0.2356	0.2140	0.2158	0.2404	0.1843	0.2071	0.2664
F9	0.2219	0.1955	0.2004	0.2287	0.2001	0.1938	0.2114	0.2118	0.1605	0.2330
F10	0.2524	0.2388	0.2176	0.2558	0.2031	0.2040	0.2489	0.2102	0.2194	0.2083
F11	0.2609	0.2301	0.2318	0.2275	0.2184	0.2391	0.2324	0.2279	0.2229	0.2314
F12	0.2452	0.2386	0.2115	0.2290	0.1825	0.2304	0.2411	0.1965	0.2017	0.2318
F13	0.2097	0.2098	0.2023	0.2171	0.1892	0.2024	0.2184	0.1890	0.1674	0.2080
F14	0.2353	0.2226	0.2249	0.2586	0.1922	0.2058	0.2110	0.2385	0.1962	0.2630

F15	0.2238	0.1921	0.1914	0.1999	0.1915	0.1920	0.1812	0.1834	0.1838	0.2217
F16	0.2147	0.2022	0.2140	0.2226	0.1825	0.1872	0.2160	0.1795	0.2056	0.2139
F17	0.2897	0.2590	0.2378	0.2793	0.2210	0.2372	0.2695	0.2543	0.2272	0.2805
F18	0.2780	0.2554	0.2345	0.2731	0.2301	0.2452	0.2582	0.2515	0.2357	0.2856
F19	0.2635	0.2486	0.2278	0.2659	0.2120	0.2389	0.2587	0.2449	0.2174	0.2692
F20	0.3023	0.2903	0.2653	0.2931	0.2480	0.2722	0.2930	0.2778	0.2614	0.3061

Table 13: Part 2 of the total relationship matrix

Code	F11	F12	F13	F14	F15	F16	F17	F18	F19	F20
F1	0.2152	0.2538	0.1994	0.1852	0.2416	0.2430	0.2676	0.2699	0.2561	0.2817
F2	0.1773	0.2299	0.1994	0.1909	0.1983	0.1794	0.2003	0.2082	0.1838	0.2174
F3	0.2121	0.2438	0.2015	0.2246	0.2584	0.2351	0.2692	0.2715	0.2568	0.2818
F4	0.2142	0.2245	0.2099	0.1899	0.2312	0.2247	0.2471	0.2497	0.2365	0.2583
F5	0.2020	0.2529	0.2315	0.2110	0.2625	0.2387	0.2657	0.2758	0.2424	0.2939
F6	0.2257	0.2399	0.2440	0.2272	0.2541	0.2376	0.2872	0.2748	0.2482	0.3002
F7	0.2554	0.2833	0.2508	0.2500	0.2576	0.2523	0.2973	0.3007	0.2847	0.3190
F8	0.2048	0.2380	0.2203	0.2207	0.2245	0.2166	0.2774	0.2809	0.2663	0.2893
F9	0.2105	0.2316	0.1932	0.1925	0.2267	0.1877	0.2432	0.2465	0.2331	0.2622
F10	0.2072	0.2404	0.2209	0.2210	0.2330	0.2111	0.2780	0.2818	0.2672	0.2901
F11	0.1696	0.2299	0.2047	0.1879	0.2539	0.2306	0.2307	0.2459	0.2250	0.2565
F12	0.1928	0.1998	0.2031	0.2147	0.2116	0.2234	0.2704	0.2649	0.2591	0.2814
F13	0.1808	0.2195	0.1518	0.2026	0.1955	0.1701	0.2182	0.2206	0.2058	0.2291
F14	0.2024	0.2527	0.2051	0.1728	0.2552	0.2133	0.2602	0.2616	0.2556	0.2776
F15	0.2009	0.2214	0.1837	0.1761	0.1731	0.1787	0.2393	0.2349	0.2218	0.2527
F16	0.1721	0.1997	0.1990	0.1799	0.2278	0.1622	0.2434	0.2266	0.2333	0.2459
F17	0.2251	0.2880	0.2288	0.2392	0.2654	0.2657	0.2437	0.3056	0.2907	0.3078

F18	0.2224	0.2761	0.2260	0.2362	0.2622	0.2462	0.2834	0.2425	0.2772	0.3119
F19	0.2160	0.2769	0.2199	0.2305	0.2544	0.2401	0.2914	0.2938	0.2205	0.2770
F20	0.2523	0.3047	0.2444	0.2541	0.2978	0.2647	0.3052	0.3236	0.2801	0.2766

Table 14: Summation of TRM rows and columns

Code	R	C	R⊚C	R-C
F1	4.6345	4.9626	9.5971	-0.3281
F2	3.9161	4.6217	8.5378	-0.7056
F3	4.8394	4.4342	9.2736	0.4052
F4	4.3978	4.8484	9.2462	-0.4506
F5	4.7707	4.0868	8.8575	0.6839
F6	4.9099	4.4193	9.3292	0.4907
F7	5.2925	4.6792	9.9717	0.6133
F8	4.6761	4.3798	9.0558	0.2963
F9	4.2840	4.1954	8.4794	0.0885
F10	4.7089	4.9304	9.6393	-0.2216
F11	4.5571	4.1589	8.7159	0.3982
F12	4.5296	4.9068	9.4364	-0.3772
F13	4.0072	4.2374	8.2446	-0.2302
F14	4.6044	4.2068	8.8112	0.3976
F15	4.0431	4.7850	8.8281	-0.7419
F16	4.1280	4.4211	8.5492	-0.2931
F17	5.2154	5.2186	10.4340	-0.0032
F18	5.1313	5.2797	10.4110	-0.1483

F19	4.9675	4.9441	9.9116	0.0234
F20	5.6130	5.5102	11.1232	0.1027

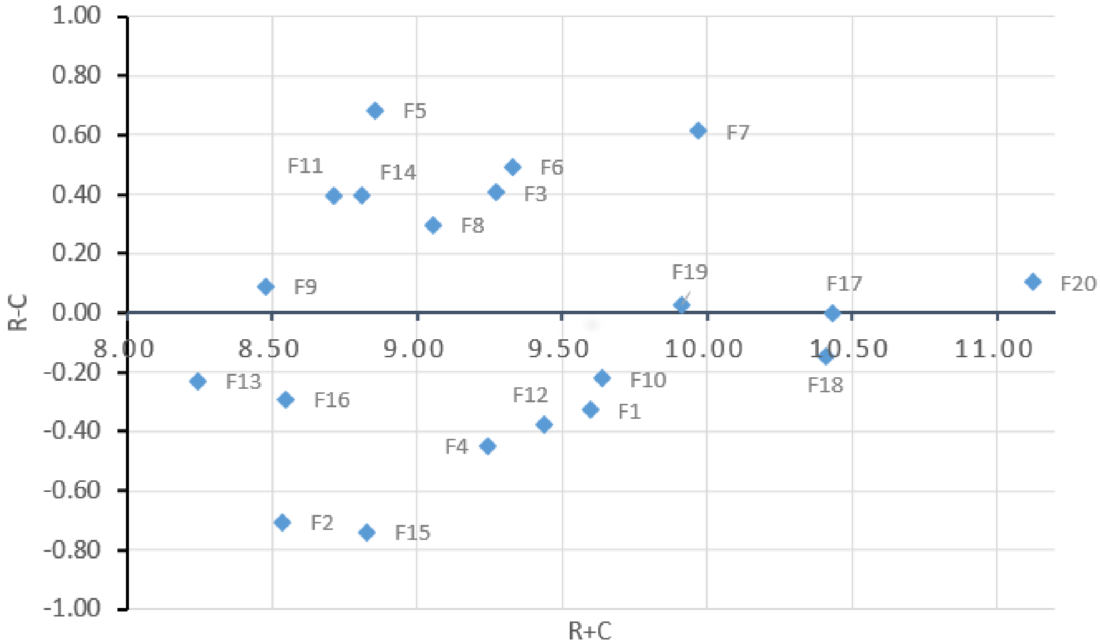

Figure 2: Causal diagram

5. Results and Discussions

This study aims to identify and rank the key elements in GSCM systems that are enabled by the Internet of Things. We have chosen 20 CSFs from the literature review that we felt were pertinent to our issue. Following the identification of these 20 CSFs, the Neutrosophic DEMATEL method—regarded as an effective MCDM tool—was used to organize them into cause- and effect-related categories. It has the capacity to convert the intricate relationships between the requirements of real-world problems into an easily understandable, structured model. The outcome drawn by using the suggested model to analyze data gathered from the chosen experts.

5.1 Ranking of the CSFs

The ranking was carried out based on R+C values presented in table 13. It is clear that the Big Data and BDA tools (F20) was the most critical factor with the highest importance value of 11.1232, while creation of highly qualified and competent human labor (F13) with of value of 8.2446 is the least influent CSF of the 20 selected ones. The influence degree for all CSFs included in the calculation are presented in Table 14. We recommend that the organizations should concentrate on these crucial CSFs, and once IoT-enabled GSCM implementation has reached the appropriate level, the implementation procedure will be adjusted. The degree of implementation will now be raised through a continual improvement process, with the least important CSFs being attended to in accordance with their importance. For more clarification of the results, Figure 3 visualizes the importance ranking of the CSFs.

5.2 Cause/effect grouping of the CSFs

According to the R-C values, the CSFs were classified into cause and effect groups (Table 15). Ten CSFs (F20, F7, F19, F6, F3, F8, F5, F14, F11, and F9) were recognized to be in the cause group and rest ten CSFs (F17, F18, F10, F1, F12, F4, F15, F16, F2, and F13) were recognized to be in the effect group. This analysis showed that F5 is the most influencing CSF, which has the greatest R-C value of 0.6839, while the most influenced CSF is discovered to be F15, which has the lowest R-C value of minus 0.7419.

5.3 CSFs Interactions

Due to the case scenario's consideration of 20 CSFs, it was challenging to depict all CSF interactions on an Impact Relationship Map (IRM). In order to see how each CSF interacts with other CSFs (both influencing and being impacted), the IRM for each CSF has been constructed based on the threshold (▽) that is calculated using the following expression:

$$\triangledown = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} TRM_{i,j}}{N*N} \qquad (10)$$

Despite the fact that IRMs have been created for all CSFs, only the IRM for the F16 AMB is displayed in Figure 4 as an example. Each CSF influences and is influenced by a variety of other CSFs. Table 16 shows the full interactions among all CSFs.

Table 15: CSFs importance ranking with related cause/effect grouping

CSF	Code	$Ri + Cj$	Importance rank	$Ri - Cj$	Cause	Effect
Big Data and Big Data Analytics (BDA) tools	F20	11.1232	1	0.1027	√	
RFID and GPS	F17	10.4340	2	-0.0032		√
Cloud computing and IoT applications	F18	10.4110	3	-0.1483		√
Top management commitment	F7	9.9717	4	0.6133	√	
Sensor technologies and sensor network	F19	9.9116	5	0.0234	√	
Green marketing	F10	9.6393	6	-0.2216		√
Influence from investors and stakeholders	F1	9.5971	7	-0.3281		√
Reverse logistics	F12	9.4364	8	-0.3772		√
Green packaging and transportation	F6	9.3292	9	0.4907	√	
Environmental regulations	F3	9.2736	10	0.4052	√	

Global competitive advantage	F4	9.2462	11	-0.4506		√
Greening competition pressures	F8	9.0558	12	0.2963	√	
Management of toxic/ harmful/ hazardous materials and waste and pollution preventative measures	F5	8.8575	13	0.6839	√	
Collaboration with suppliers	F15	8.8281	14	-0.7419		√
green practices, policies, and infrastructure	F14	8.8112	15	0.3976	√	
Standards and regulations (ISO 14000)	F11	8.7159	16	0.3982	√	
Recycling and lifecycle management	F16	8.5492	17	-0.2931		√
Waste management	F2	8.5378	18	-0.7056		√
power negotiations along the supply chain	F9	8.4794	19	0.0885	√	
creation of highly qualified and competent human labor	F13	8.2446	20	-0.2302		√

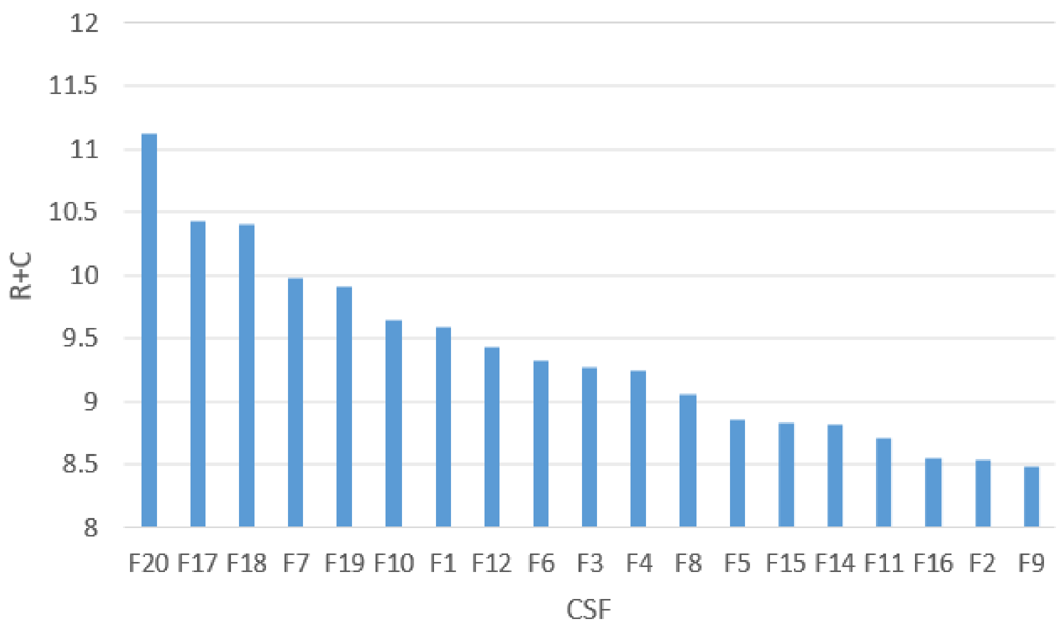

Figure 3: CSFs importance ranking

Table 16: CSFs interactions

CSF	influencing	influenced
F1	F11,F13,F16,F17,F18,F19,F20	F3,F5,F6,F7,F10,F11,F12,F14,F17,F18,F19,F20
F2	--	F3,F4,F5,F6,F7,F10,F12,F17,F18,F19,F20
F3	F1,F2,F4,F6,F7,F10,F11,F13,F16,F17,F18,F19,F20	F6,F7,F17,F18,F20
F4	F2,F11,F18,F19,F20	F3,F5,F6,F7,F8,F14,F17,F18,F19,F20
F5	F1,F2,F4,F7,F10,F11,F13,F16,F17,F18,F19,F20	F20
F6	F1,F2,F3,F4,F8,F10,F11,F13,F14,F16,F17,F18,F19,F20	F3,F7,F11,F17,F18,F19,F20
F7	F1,F2,F3,F4,F6,F8,F9,F10,F11,F12,F13,F14,F15,F16,F17, F18,F19,F20	F3,F5,F8,F10,F12,F17,F18,F19,F20
F8	F4,F7,F11,F13,F18,F19,F20	F6,F7,F14,F17,F18,F19,F20
F9	F18,F19	F7,F18,F20
F10	F1,F2,F4,F7,F10,F13,F18,F19,F20	F3,F5,F6,F7,F10,F11,F12,F14,F17,F18,F19,F20
F11	F1,F6,F10,F16,F19	F1,F3,F4,F5,F6,F7,F8,F14,F17,F18,F19,F20
F12	F1,F2,F7,F10,F18,F19,F20	F7,F20
F13	--	F1,F3,F5,F6,F7,F8,F10,F14,F17,F18,F19,F20
F14	F1,F4,F8,F10,F11,F13,F16,F18,F19,F20	F6,F7,F20
F15	F18,F19	F7,F17,F18,F20
F16	F18,F20	F1,F3,F5,F6,F7,F11,F14,F17,F18,F19,F20
F17	F1,F2,F3,F4,F6,F7,F8,F10,F11,F13,F15,F16,F17,F18,F19, F20	F1,F3,F5,F6,F7,F17,F18,F19,F20
F18	F1,F2,F3,F4,F6,F7,F8,F9,F10,F11,F13,F15,F16,F17,F18, F19,F20	F1,F3,F4,F5,F6,F7,F8,F9,F10,F12,F14,F15,F16, F17,F18,F19,F20
F19	F1,F2,F4,F6,F7,F8,F10,F11,F13,F16,F17,F18,F19	F1,F3,F4,F5,F6,F7,F8,F9,F10,F11,F12,F14,F15, F17,F18,F19,F20
F20	F1,F2,F3,F4,F5,F6,F7,F8,F9,F10,F11,F12,F13,F14,F15, F16,F17,F18,F19,F20	F1,F3,F5,F6,F7,F8,F10,F12,F14,F16,F17,F18,F20

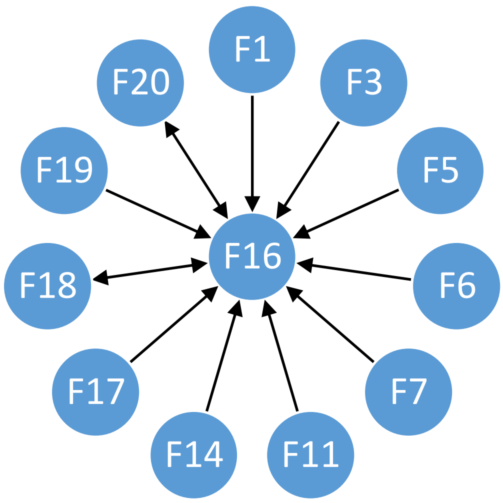

Figure 4: F16 interactions with other CSFs

6. Conclusion

Poor strategic planning, ineffective management, and poor information management all contribute to weak core competencies and poor information awareness in the manufacturing operation. It is evident that by concentrating more on the root causes of an issue and efficiently manage the effects can enhance system performance and productivity. The similar theory was used in IoT-enabled GSCM environments using neutrosophic DEMATEL method to prioritize the CSFs responsible for the company's performance. Green supply chain management has been viewed as a key component of firms' efforts to improve their performance. As a result, the focus of this study was to investigate the crucial success criteria needed for the management of green supply chains that are enabled by IoT technology. A survey of the literature led to the discovery of 20 CSFs categorized into two groups: green enablers and IoT enablers. After applying the neutrosophic DEMATEL approach, we identified the relative importance of each factor, the Cause/effect grouping, and the how each CSF influence/influenced by other CSFs. Finally, we recommended that the organizations should concentrate on selected CSFs, and once IoT-enabled GSCM implementation has reached the appropriate level, the implementation procedure will be adjusted. The degree of implementation will now be raised through a continual improvement process, with the least important CSFs being attended to in accordance with their importance.

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Tangent and Cotangent Similarity Measures of Pentapartitioned Neutrosophic Pythagorean Set in Virtual Education During Covid Pandemic

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Abstract: In this paper, a new tangent and cotangent similarity measures between two Pentapartitioned Neutrosophic Pythagorean [PNP] sets with truth membership, falsity membership, ignorance and contradiction membership as dependent Neutrosophic component is proposed and its properties are investigated. The unknown membership alone will be considered as independent Neutrosophic components. Also, the weighted similarity measures are also studied with a decision making problem.

Keywords: PNP set, Tangent similarity measure, cotangent similarity measure.

1. Introduction

Traditionally, the teaching and learning method uses several exercises fixing, sending and evaluating ideas and information about a subject. Learning is that the method of getting relative permanent changes in understanding, attitude, knowledge, information, capability and skill through expertise. A modification are often set or involuntary, to raised or worse learning. The training method is an enclosed cognitive event. To assist this teaching and learning method, it is necessary the utilization of a laptoop tool ready to stimulate these changes. Also, it is necessary that it will operate as validation and serving tool to the college students.

The COVID-19 pandemic has caused important disruption with in the domain of education, that is considered as essential determinant for economic progress of any country. Even developed countries are waging a battle against COVID-19 for minimizing the impact on their economy because of prolonged lockdown. Education sector isn't an exception, and method of educational delivery has been grossly affected. There has been unforeseen and impetuous transition from real classroom to on-line and virtual teaching methodology across the world. There's an enormous question on the sustainability of online mode of teaching post-pandemic and its percussions on world education market. Impact of lockdown on the teaching—learning method has been studied in present paper with the objective to assess the quality of online classes and challenges associated with them. The paper proposes about the benefits of social media in virtual education among College Students

In order to deal with uncertainties, the thought of fuzzy sets and fuzzy set operations was introduced by Zadeh [17]. The speculation of fuzzy topological space was studied and developed by C.L. Chang [3]. The paper of Chang sealed the approach for the subsequent growth of the various fuzzy topological ideas. Since then a lot of attention has been paid to generalize the fundamental ideas of general topology in fuzzy setting and therefore a contemporary theory of fuzzy topology has been developed. Atanassov and plenty of researchers [1] worked on intuitionistic fuzzy sets within the literature. Florentine Smarandache [15] introduced the idea of Neutrosophic set in 1995 that provides the information of neutral thought by introducing the new issue referred to as uncertainty within the set. Thus neutrosophic set was framed and it includes the parts of truth membership function(T), indeterminacy membership function(I), and falsity membership function(F) severally. Neutrosophic sets deals with non normal interval of]−0 1+[. Pentapartitioned neutrosophic set and its properties were introduced by Rama Malik and Surpati Pramanik [14]. In this case, indeterminacy is divided into three components: contradiction, ignorance, and an unknown membership function. The concept of Pentapartitioned neutrosophic pythagorean sets was initiated by R. Radha and A. Stanis Arul Mary[9].

Similarity measure is an important topic in the current fuzzy, Pythagorean , Neutrosophic and different hybrid environments. Recently, the improved correlation coefficients of Pentapartitioned Neutrosophic Pythagorean sets and Quadripartitioned Neutrosophic Pythagorean sets was introduced by R. Radha and A. Stanis Arul Mary. Pranamik and Mondal [5,6]has also proposed weighted similarity measures based on tangent function and cotangent function and its application on medical diagnosis. In this paper, the weighted similarity measures of Tangent and Cotangent functions has been applied to PNP sets in virtual education during Covid Pandemic.

2. Preliminaries

2.1 Definition [15]

Let X be a universe. A Neutrosophic set A on X can be defined as follows:

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}$$

Where $T_A, I_A, F_A: U \to [0,1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$

2.2 Definition [9]

Let X be a universe. A Pentapartitioned neutrosophic pythagorean [PNP] set A with T, F, C and U as dependent neutrosophic components and I as independent component for A on X is an object of the form

$$A = \{< x, T_A, C_A, I_A, U_A, F_A >: x \in X\}$$

Where $T_A + F_A \leq 1, C_A + U_A \leq 1$ *and*

$$(T_A)^2 + (C_A)^2 + (I_A)^2 + (U_A)^2 + (F_A)^2 \leq 3$$

Here, $T_A(x)$ is the truth membership, $C_A(x)$ is contradiction membership, $U_A(x)$ is ignorance membership, $F_A(x)$ is the false membership and $I_A(x)$ is an unknown membership.

2.3 Definition [14]

Let P be a non-empty set. A Pentapartitioned neutrosophic set A over P characterizes each element p in P a truth -membership function T_A , a contradiction membership function C_A , an ignorance membership function G_A, unknown membership function U_A and a false membership function F_A , such that for each p in P

$$T_A + C_A + G_A + U_A + F_A \leq 5$$

2.4 Definition [9]

The complement of a pentapartitioned neutrosophic pythagorean set A on R Denoted by A^C or A^* and is defined as

$A^C = \{< x, F_A(x), U_A(x), 1 - G_A(x), C_A(x), T_A(x) > : x \in X\}$

2.5 Definition [9]

Let A $= < x, T_A(x), C_A(x), G_A(x), U_A(x), F_A(x) >$ and

B $= < x, T_B(x), C_B(x), G_B(x), U_B(x), F_B(x) >$ are pentapartitioned neutrosophic pythagorean sets. Then

A ∪ B $= < x, max(T_A(x), T_B(x)), max(C_A(x), C_B(x)), \min(G_A(x), G_B(x)),$

$\min(U_A(x), U_B(x)), \min(F_A(x), F_B(x)), >$

A ∩ B $= < x, min(T_A(x), T_B(x)), min(C_A(x), C_B(x)), \max(G_A(x), G_B(x))$

$$, max(U_A(x), U_B(x)), max(F_A(x), F_B(x)) >$$

2.6 Definition[9]

A PNP topology on a nonempty set R is a family of a PNP sets in R satisfying the following axioms

1) $0,1 \in r$
2) $R_1 \cap R_2 \in r$ for any $R_1, R_2 \in r$
3) $\cup R_i \in r$ for any $R_i: i \in I \subseteq r$

The complement R^* of PNP open set (PNPOS, in short) in PNP topological space [PNPTS] (R,r), is called a PNP closed set [PNPCS].

3. Tangent and Cotangent Similarity Measures of PNP Sets

3.1 Definition

Let P = {(r, $B1_P(r), B2_P(r), B3_P(r), B4_P(r), B5_P(r)): r \in R$} and Q = {(r, $B1_Q(r)$, $B2_Q(r)$, $B3_Q(r)$, $B4_Q(r)$, $B5_Q)(r)$: $r \in R$} be two Pentapartitioned Neutrosophic Pythagorean numbers with B1 and B5, B2 and B4 as dependent Neutrosophic components. Now tangent similarity function which measures the similarity between two vectors based only on the direction, ignoring the impact of the distance between them can be presented as follows

$$T_{PNP}(P,Q) = \frac{1}{n}\sum_{i=1}^{n}[1-\tan(\frac{\pi}{20}[|B1^2_P(r_i) - B1^2_Q(r_i)|+|B2^2_P(r_i) - B2^2_Q(r_i)| + |B3^2_P(r_i) - B3^2_Q(r_i)| + |B4^2_P(r_i) - B4^2_Q(r_i)| + |B5^2_P - B5^2_Q|(r_i)])]$$

3.2 Theorem

The defined tangent similarity measure $T_{PNP}(P,Q)$ between PNP set P and Q satisfies the following properties

1. $0 \leq T_{PNP}(P,Q) \leq 1$;
2. $T_{PNP}(P,Q) = 1$ iff P = Q;
3. $T_{PNP}(P,Q) = T_{PNP}(Q,P)$;
4. If T is a PNP set in R and $P \subseteq Q \subseteq T$ then

$T_{PNP}(P,T) \leq T_{PNP}(P,Q)$ and $T_{PNP}(P,T) \leq T_{PNP}(Q,T)$.

Proof

1) As the truth membership, contradiction membership, ignorance membership, falsity membership and the unknown membership function of the PNP sets and the value of the tangent function also is within [0,1].

Hence $0 \leq T_{PNP}(P,Q) \leq 1$.

2) For any two PNP sets P and Q if P = Q, this implies $B1_P(r_i) = B1_Q(r_i)$, $B2_P(r_i) = B2_Q(r_i)$, $B3_P(r_i) = B3_Q(r_i)$, $B4_P(r_i) = B4_Q(r_i)$ and $B5_P(r_i) = B5_Q(r_i)$.

Hence $|B1^2_P(r_i) - B1^2_Q(r_i)| = 0, |B2^2_P(r_i) - B2^2_Q(r_i)| = 0, |B3^2_P(r_i) - B3^2_Q(r_i)| = 0$, $|B4^2_P(r_i) - B4^2_Q(r_i)| = 0$ and $|B5^2_P(r_i) - B5^2_Q(r_i)|$.

Thus $T_{PNP}(P,Q) = 1$.

Conversely, if $T_{PNP}(P,Q) = 1$, then $|B1^2_P(r_i) - B1^2_Q(r_i)| = 0, |B2^2_P(r_i) - B2^2_Q(r_i)| = 0$, $|B3^2_P(r_i) - B3^2_Q(r_i)| = 0$, $|B4^2_P(r_i) - B4^2_Q(r_i)| = 0$ and $|B5^2_P(r_i) - B5^2_Q(r_i)|$ since tan(0) = 0. So we can write $B1_P(r_i) = B1_Q(r_i)$, $B2_P(r_i) = B2_Q(r_i)$, $B3_P(r_i) = B3_Q(r_i)$, $B4_P(r_i) = B4_Q(r_i)$ and $B5_P(r_i) = B5_Q(r_i)$.

Hence P = Q.

3) The Proof is obvious

4) If $P \subseteq Q \subseteq T$ then $B1_P(r_i) \leq B1_Q(r_i) \leq B1_T(r_i)$, $B2_P(r_i) \leq B2_Q(r_i) \leq B2_T(r_i)$, $B3_P(r_i) \leq B3_Q(r_i) \leq B3_T(r_i)$, $B4_P(r_i) \leq B4_Q(r_i) \leq B4_T(r_i)$ and $B5_P(r_i) \leq B5_Q(r_i) \leq B5_T(r_i)$.

$$|B1^2_P(r_i) - B1^2_Q(r_i)| \leq |B1^2_P(r_i) - B1^2_T(r_i)|,$$
$$|B1^2_Q(r_i) - B1^2_T(r_i)| \leq |B1^2_P(r_i) - B1^2_T(r_i)|,$$

$$|B2_P^2(r_i) - B2_Q^2(r_i)| \leq |B2_P^2(r_i) - B2_T^2(r_i)|,$$
$$|B2_Q^2(r_i) - B2_T^2(r_i)| \leq |B2_P^2(r_i) - B2_T^2(r_i)|,$$
$$|B3_P^2(r_i) - B3_Q^2(r_i)| \leq |B3_P^2(r_i) - B3_T^2(r_i)|,$$
$$|B3_Q^2(r_i) - B3_T^2(r_i)| \leq |B3_P^2(r_i) - B3_T^2(r_i)|,$$
$$|B4_P^2(r_i) - B4_Q^2(r_i)| \leq |B4_P^2(r_i) - B4_T^2(r_i)|,$$
$$|B4_Q^2(r_i) - B4_T^2(r_i)| \leq |B4_P^2(r_i) - B4_T^2(r_i)|,$$
$$|B5_P^2(r_i) - B5_Q^2(r_i)| \leq |B5_P^2(r_i) - B5_T^2(r_i)|,$$
$$|B5_Q^2(r_i) - B5_T^2(r_i)| \leq |B5_P^2(r_i) - B5_T^2(r_i)|.$$

Thus,

$T_{PNP}(P,T) \leq T_{PNP}(P,Q)$ and $T_{PNP}(P,T) \leq T_{PNP}(Q,T)$

Since tangent function is increasing in the interval $[0, \frac{\pi}{4}]$.

3.3 Definition

Let P = {(r, $B1_P(r), B2_P(r), B3_P(r), B4_P(r), B5_P(r)): r \in R\}$ and Q = {(r, $B1_Q(r), B2_Q(r), B3_Q(r), B4_Q(r), B5_Q)(r): r \in R\}$ be two Pentapartitioned Neutrosophic Pythagorean numbers with B1 and B5, B2 and B4 as dependent Neutrosophic components. Now weighted tangent similarity function which measures the similarity between two vectors based only on the direction, ignoring the impact of the distance between them can be presented as follows

$$T_{WPNP}(P,Q) = \sum_{i=1}^{n} w_i [1 - \tan(\frac{\pi}{20}[|B1_P^2(r_i) - B1_Q^2(r_i)| + |B2_P^2(r_i) - B2_Q^2(r_i)| + |B3_P^2(r_i) - B3_Q^2(r_i)| + |B4_P^2(r_i) - B4_Q^2(r_i)| + |B5_P^2 - B5_Q^2|(r_i)])]$$

Where $w_i \in [0,1], i = 0,1,2 \dots n$ are the weights and $\sum_{i=1}^{n} w_i = 1$. If we take $w_i = \frac{1}{n}, i = 0,1,2 \dots, n$, then $T_{WPNP}(P,Q) = T_{PNP}(P,Q)$.

3.4 Theorem

The defined weighted tangent similarity measure $T_{WPNP}(P,Q)$ between PNP set P and Q satisfies the following properties

1) $0 \leq T_{WPNP}(P,Q) \leq 1$;
2) $T_{WPNP}(P,Q) = 1$ iff P = Q;
3) $T_{WPNP}(P,Q) = T_{WPNP}(Q,P)$;
4) If T is a PNP set in R and $P \subseteq Q \subseteq T$ then

$T_{WPNP}(P,T) \leq T_{WPNP}(P,Q)$ and $T_{WPNP}(P,T) \leq T_{WPNP}(Q,T)$.

Proof

1) As the truth membership, contradiction membership, ignorance membership, falsity membership and the unknown membership function of the PNP sets and the value of the tangent function also is within [0,1] and Where $w_i \in [0,1], i = 0,1,2 \dots n$ are the weights and $\sum_{i=1}^{n} w_i = 1$.

Hence $0 \leq T_{WPNP}(P,Q) \leq 1$.

2) For any two PNP sets P and Q if P = Q, this implies $B1_P(r_i) = B1_Q(r_i), B2_P(r_i) = B2_Q(r_i)$, $B3_P(r_i) = B3_Q(r_i)$, $B4_P(r_i) = B4_Q(r_i)$ and $B5_P(r_i) = B5_Q(r_i)$.
Hence $|B1_P^2(r_i) - B1_Q^2(r_i)| = 0, |B2_P^2(r_i) - B2_Q^2(r_i)| = 0, |B3_P^2(r_i) - B3_Q^2(r_i)| = 0$, $|B4_P^2(r_i) - B4_Q^2(r_i)| = 0$ and $|B5_P^2(r_i) - B5_Q^2(r_i)|$.

Thus $T_{WPNP}(P,Q) = 1$.
Conversely, if $T_{WPNP}(P,Q) = 1$, then $|B1_P^2(r_i) - B1_Q^2(r_i)| = 0, |B2_P^2(r_i) - B2_Q^2(r_i)| = 0$, $|B3_P^2(r_i) - B3_Q^2(r_i)| = 0$, $|B4_P^2(r_i) - B4_Q^2(r_i)| = 0$ and $|B5_P^2(r_i) - B5_Q^2(r_i)|$ since tan(0) = 0. So we can write

B $1_P(r_i) = B1_Q(r_i)$, B $2_P(r_i) = B2_Q(r_i)$, B $3_P(r_i) = B3_Q(r_i)$, B $4_P(r_i) = B4_Q(r_i)$ and B $5_P(r_i) = B5_Q(r_i)$.

Hence P = Q.

3) The Proof is obvious

4) If $P \subseteq Q \subseteq T$ then $B1_P(r_i) \le B1_Q(r_i) \le B1_T(r_i)$, $B2_P(r_i) \le B2_Q(r_i) \le B2_T(r_i)$, $B3_P(r_i) \le B3_Q(r_i) \le B3_T(r_i)$, $B4_P(r_i) \le B4_Q(r_i) \le B4_T(r_i)$ and $B5_P(r_i) \le B5_Q(r_i) \le B5_T(r_i)$ and $\sum_{i=1}^{n} w_i = 1$.

$$|B1_P^2(r_i) - B1_Q^2(r_i)| \le |B1_P^2(r_i) - B1_T^2(r_i)|,$$
$$|B1_Q^2(r_i) - B1_T^2(r_i)| \le |B1_P^2(r_i) - B1_T^2(r_i)|,$$
$$|B2_P^2(r_i) - B2_Q^2(r_i)| \le |B2_P^2(r_i) - B2_T^2(r_i)|,$$
$$|B2_Q^2(r_i) - B2_T^2(r_i)| \le |B2_P^2(r_i) - B2_T^2(r_i)|,$$
$$|B3_P^2(r_i) - B3_Q^2(r_i)| \le |B3_P^2(r_i) - B3_T^2(r_i)|,$$
$$|B3_Q^2(r_i) - B3_T^2(r_i)| \le |B3_P^2(r_i) - B3_T^2(r_i)|,$$
$$|B4_P^2(r_i) - B4_Q^2(r_i)| \le |B4_P^2(r_i) - B4_T^2(r_i)|,$$
$$|B4_Q^2(r_i) - B4_T^2(r_i)| \le |B4_P^2(r_i) - B4_T^2(r_i)|,$$
$$|B5_P^2(r_i) - B5_Q^2(r_i)| \le |B5_P^2(r_i) - B5_T^2(r_i)|,$$
$$|B5_Q^2(r_i) - B5_T^2(r_i)| \le |B5_P^2(r_i) - B5_T^2(r_i)|.$$

Thus,

$T_{WPNP}(P,T) \le T_{WPNP}(P,Q)$ and $T_{WPNP}(P,T) \le T_{WPNP}(Q,T)$
Since tangent function is increasing in the interval $[0, \frac{\pi}{4}]$.

3.5 Definition

Assume that P = {(r, $B1_P(r), B2_P(r), B3_P(r), B4_P(r), B5_P(r)): r \in R\}$ and
Q = {(r, $B1_Q(r), B2_Q(r), B3_Q(r), B4_Q(r), B5_Q)(r): r \in R\}$ are two Pentapartitioned Neutrosophic Pythagorean numbers with B1 and B5, B2 and B4 as dependent Neutrosophic components. A cotangent similarity measure between two PNP sets P and Q is proposed as follows

$$COT_{PNP}(P,Q) = \frac{1}{n}\sum_{i=1}^{n} [\cot(\frac{\pi}{20}[5 + |B1_P^2(r_i) - B1_Q^2(r_i)| + |B2_P^2(r_i) - B2_Q^2(r_i)| + |B3_P^2(r_i) - B3_Q^2(r_i)| + |B4_P^2(r_i) - B4_Q^2(r_i)| + |B5_P^2 - B5_Q^2|(r_i)])]$$

3.6 Theorem

The cotangent similarity measure $COT_{PNP}(P,Q)$ between PNP set P and Q also satisfies the following properties

1) $0 \leq COT_{PNP}(P,Q) \leq 1$;
2) $COT_{PNP}(P,Q) = 1$ iff P = Q;
3) $COT_{PNP}(P,Q) = COT_{PNP}(Q,P)$;
4) If T is a PNP set in R and $P \subseteq Q \subseteq T$ then

$COT_{PNP}(P,T) \leq COT_{PNP}(P,Q)$ and $COT_{PNP}(P,T) \leq COT_{PNP}(Q,T)$.

3.7 Definition

Assume that P = {(r, $B1_P(r), B2_P(r), B3_P(r), B4_P(r), B5_P(r)): r \in R$} and
Q = {(r, $B1_Q(r), B2_Q(r), B3_Q(r), B4_Q(r), B5_Q)(r): r \in R$} are two Pentapartitioned Neutrosophic Pythagorean numbers with B1 and B5, B2 and B4 as dependent Neutrosophic components. A weighted cotangent similarity measure between two PNP sets P and Q is proposed as follows

$$COT_{WPNP}(P,Q) = \sum_{i=1}^{n} w_i [\cot(\frac{\pi}{20}[5 + |B1^2_P(r_i) - B1^2_Q(r_i)| + |B2^2_P(r_i) - B2^2_Q(r_i)| + |B3^2_P(r_i) - B3^2_Q(r_i)| + |B4^2_P(r_i) - B4^2_Q(r_i)| + |B5^2_P - B5^2_Q|(r_i)])]$$

Where $w_i \in [0,1], i = 0,1,2 \dots n$ are the weights and $\sum_{i=1}^{n} w_i = 1$. If we take $w_i = \frac{1}{n}, i = 0,1,2 \dots, n$, then $COT_{WPNP}(P,Q) = COT_{PNP}(P,Q)$.

3.8 Theorem

The weighted cotangent similarity measure $COT_{PNP}(P,Q)$ between PNP set P and Q also satisfies the following properties

1) $0 \leq COT_{WPNP}(P,Q) \leq 1$;
2) $COT_{WPNP}(P,Q) = 1$ iff P = Q;
3) $COT_{WPNP}(P,Q) = COT_{WPNP}(Q,P)$;
4) If T is a PNP set in R and $P \subseteq Q \subseteq T$ then

$COT_{WPNP}(P,T) \leq COT_{WPNP}(P,Q)$ and $COT_{WPNP}(P,T) \leq COT_{WPNP}(Q,T)$.

Proof

1) As the truth membership, contradiction membership, ignorance membership, falsity membership and the unknown membership function of the PNP sets and the value of the tangent function also is within [0,1] and $\sum_{i=1}^{n} w_i = 1$.
Hence $0 \leq COT_{WPNP}(P,Q) \leq 1$.
2) For any two PNP sets P and Q if P = Q, this implies $B1_P(r_i) = B1_Q(r_i)$, $B2_P(r_i) = B2_Q(r_i)$, $B3_P(r_i) = B3_Q(r_i)$, $B4_P(r_i) = B4_Q(r_i)$ and $B5_P(r_i) = B5_Q(r_i)$.
Hence $|B1^2_P(r_i) - B1^2_Q(r_i)| = 0, |B2^2_P(r_i) - B2^2_Q(r_i)| = 0, |B3^2_P(r_i) - B3^2_Q(r_i)| = 0$,
$|B4^2_P(r_i) - B4^2_Q(r_i)| = 0$ and $|B5^2_P(r_i) - B5^2_Q(r_i)|$.
Thus $COT_{WPNP}(P,Q) = 1$.
Conversely, if $T_{WPNP}(P,Q) = 1$, then $|B1^2_P(r_i) - B1^2_Q(r_i)| = 0, |B2^2_P(r_i) - B2^2_Q(r_i)| = 0$, $|B3^2_P(r_i) - B3^2_Q(r_i)| = 0$, $|B4^2_P(r_i) - B4^2_Q(r_i)| = 0$ and $|B5^2_P(r_i) - B5^2_Q(r_i)|$ since tan(0) = 0. So we can write

$B\,1_P(r_i) = B1_Q(r_i)$, $B\,2_P(r_i) = B2_Q(r_i)$, $B\,3_P(r_i) = B3_Q(r_i)$, $B\,4_P(r_i) = B4_Q(r_i)$ and $B\,5_P(r_i) = B5_Q(r_i)$.

Hence P = Q.

3) The Proof is obvious

4) If $P \subseteq Q \subseteq T$ then $B1_P(r_i) \leq B1_Q(r_i) \leq B1_T(r_i)$, $B2_P(r_i) \leq B2_Q(r_i) \leq B2_T(r_i)$,

$B\,3_P(r_i) \geq B3_Q(r_i) \geq B3_T(r_i)$, $B\,4_P(r_i) \geq B4_Q(r_i) \geq B4_T(r_i)$ and $B\,5_P(r_i) \geq B5_Q(r_i) \geq B5_T(r_i)$ and $\sum_{i=1}^{n} w_i = 1$.

$$|B1_P^2(r_i) - B1_Q^2(r_i)| \leq |B1_P^2(r_i) - B1_T^2(r_i)|,$$
$$|B1_Q^2(r_i) - B1_T^2(r_i)| \leq |B1_P^2(r_i) - B1_T^2(r_i)|,$$
$$|B2_P^2(r_i) - B2_Q^2(r_i)| \leq |B2_P^2(r_i) - B2_T^2(r_i)|,$$
$$|B2_Q^2(r_i) - B2_T^2(r_i)| \leq |B2_P^2(r_i) - B2_T^2(r_i)|,$$
$$|B3_P^2(r_i) - B3_Q^2(r_i)| \leq |B3_P^2(r_i) - B3_T^2(r_i)|,$$
$$|B3_Q^2(r_i) - B3_T^2(r_i)| \leq |B3_P^2(r_i) - B3_T^2(r_i)|,$$
$$|B4_P^2(r_i) - B4_Q^2(r_i)| \leq |B4_P^2(r_i) - B4_T^2(r_i)|,$$
$$|B4_Q^2(r_i) - B4_T^2(r_i)| \leq |B4_P^2(r_i) - B4_T^2(r_i)|,$$
$$|B5_P^2(r_i) - B5_Q^2(r_i)| \leq |B5_P^2(r_i) - B5_T^2(r_i)|,$$
$$|B5_Q^2(r_i) - B5_T^2(r_i)| \leq |B5_P^2(r_i) - B5_T^2(r_i)|.$$

The cotangent function is decreasing function within the interval $[0, \frac{\pi}{4}]$.

Hence $\sum_{i=1}^{n} w_i = 1$.

Hence, we can write

$COT_{WPNP}(P,T) \leq COT_{WPNP}(P,Q)$ and $COT_{WPNP}(P,T) \leq COT_{WPNP}(Q,T)$

4. Decision Making Based on Tangent and Cotangent Similarity Measures

Let $A_1, A_2, \dots A_m$ be a discrete set of candidates, $C_1, C_2, .. C_n$ be the set of criteria for each candidate and D_1, D_2, … . , D_k are the alternatives of each candidate. The decision -maker provides the ranking of alternatives with respect to each candidate. The ranking presents the performance of candidates $A_i (i = 1,2, \dots m)$ against the criteria C_j $(j = 1,2 \dots , n)$.The values associated with the alternatives for MADM problem can be presented in the following decision matrix(see Tab 1 and Tab 2). The relation between candidates and attributes are given in Tab 1. The relation between attributes and alternatives are given in the Tab 2.

Table 1 : The relation between candidates and attributes

R_1	C_1	C_2	…	C_n
A_1	a_{11}	a_{12}	…	a_{1n}
A_2	a_{21}	a_{13}	…	a_{2n}
…	…	…	…	…

A_m	a_{m1}	a_{m2}	…	a_{mn}

Table 2 : The relation between attributes and alternatives

R_2	D_1	D_2	…	D_k
C_1	c_{11}	c_{12}	…	c_{1k}
C_2	c_{21}	c_{22}	…	c_{2k}
…	…	…	…	…
C_n	c_{n1}	c_{n2}	…	c_{nk}

Here a_{ij} and c_{ij} are all Pentapartitioned Neutrosophic Pythagorean Fuzzy numbers.
The steps corresponding to Pentapartitioned Neutrosophic Pythagorean number based on tangent and cotangent functions are presented following steps.

Step 1: Determination of the relation between candidates and attributes
The relation between candidate $A_i(i = 1,2, \dots m)$ and the attribute $C_j\ (j = 1,2 \dots n)$ is presented in Table 3.

Table 3 : The relation between candidates and attributes in terms of PNP sets

R_1	C_1	C_2	…	C_n
A_1	$(b1_{11}, b2_{11}, b3_{11}, b4_{11}, b5_{11})$	$(b1_{12}, b2_{12}, b3_{12}, b4_{12}, b5_{12})$	…	$(b1_{1n}, b2_{1n}, b3_{1n}, b4_{1n}, b5_{1n})$
A_2	$(b1_{21}, b2_{21}, b3_{21}, b4_{21}, b5_{21})$	$(b1_{22}, b2_{22}, b3_{22}, b4_{22}, b5_{22})$	…	$(b1_{2n}, b2_{2n}, b3_{2n}, b4_{2n}, b5_{2n})$
…	…	…	…	…
A_m	$(b1_{m1}, b2_{m1}, b3_{m1}, b4_{m1}, b5_{m1})$	$(b1_{m2}, b2_{m2}, b3_{m2}, b4_{m2}, b5_{m2})$	…	$(b1_{mn}, b2_{mn}, b3_{mn}, b4_{mn}, b5_{mn})$

Table 4 : The relation between attributes and alternatives in terms of PNP sets

R_2	D_1	D_2	…	D_k
C_1	$(c1_{11}, cb2_{11}, c3_{11}, c4_{11}, c5_{11})$	$(c1_{12}, c2_{12}, c3_{12}, c4_{12}, c5_{12})$	…	$(c1_{1k}, c2_{1k}, c3_{1k}, c4_{1k}, c5_{1k})$
C_2	$(b1_{21}, b2_{21}, b3_{21}, b4_{21}, b5_{21})$	$(c1_{22}, c2_{22}, c3_{22}, c4_{22}, c5_{22})$	…	$(c1_{2k}, c2_{2k}, c3_{2k}, c4_{2k}, c5_{2k})$
…	…	…	…	…

C_n	$(c1_{n1}, c2_{n1}, c3_{n1}, c4_{n1}, c5_{n1})$	$(c1_{n2}, c2_{n2}, c3_{n2}, c4_{n2}, c5_{n2})$	…	$(c1_{nk}, c2_{nk}, c3_{nk}, c4_{nk}, c5_{nk})$

Step 3: Determination of the relation between attributes and alternatives

Determine the similarity measure between the Tab 3 and Tab 4 using $T_{PNP}(P,Q)$ $T_{WPNP}(P,Q)$, $COT_{PNP}(P,Q)$ and $COT_{WPNP}(P,Q)$.

Step 4: Ranking the alternatives

Ranking the alternatives is prepared based on the descending order of the similarity measures. Highest value reflects the best alternative.

Step 5: End

5. Example

Higher education institutions have faced various challenges in adapting online education to control the pandemic spread of COVID. The present work aims to apply similaty measures between social media and its benefits of students.Let D ={ R1, R2 ,R3} be a set of College student respondents , E = { YouTube, Facebook, WhatsApp, Blog} be Social medias and H ={ Communication Tool, Online Learning, Connecting with experts, Global exposure} be its benefits.The solution strategy is to determine the student regarding the relation between student respondents and its benefits in virtual education (see Tab 5) and the relation between socal media and its benefits in Table 6 . Further we have calculated Tangent and Cotangent similarity measures can be calculated in Table 7 and 8. Also the weighted similarity measures of the tangent and cotangent functions of PNP sets be calculated in Table 9 and 10.

Table 5 : (P1) The relation between respondents and benefits in Virtual Education

P1	*Online Learning*	*Communication Tool*	*Connecting with Experts*	*Global Exposure*
R1	*(0.7,0.2,0.8,0.3,0.3)*	*(0.1,0.2,0.9,0.3,0.7)*	*(0.4,0.2,0.2,0.3,0.6)*	*(0.2,0.2,0.7,0.3,0.8)*
R2	*(0.3,0.2,0.1,0.3,0.5)*	*(0.6,0.2,0.8,0.3,0.4)*	*(0.6,0.2,0.1,0.3,0.4)*	*(0.2,0.2,0.9,0.3,0.7)*
R3	*(0.1,0.2,0.8,0.3,0.5)*	*(0.6,0.2,0.8,0.3,0.4)*	*(0.6,0.2,0.1,0.3,0.4)*	*(0.7,0.2,0.7,0.3,0.3)*

Table 6: (P2) The relation between Social Media and its benefits

P2	*WhatsApp*	*YouTube*	*Facebook*	*Blog*
Online Learning	*(0.4,0.2,0.6.0.3,0.1)*	*(0.1,0.2,0.5,0.3.0.5)*	*(0.2,0.2,0.5,0.3,0.4)*	*(0.2,0.4,0.6,0.7,0.3)*
Communication Tool	*(0.7,0.2,0.9,0.3,0.3)*	*(0.5,0.2,0.9,0.3,0.5)*	*(0.7,0.2,0.1,0.3,0.2)*	*(0.3,0.5,0.8,0.1,0.2)*
Connecting with Experts	*(0.1,0.2,0.5,0.3,0.7)*	*(0.8,0.2.0.1,0.3,0.2)*	*(0.6,0.2,0.8,0.3,0.4)*	*(0.4,0.6,0.2,0.1,0.1)*

Global Exposure	(0.6,0.2,0.3,0.3,0.4)	(0.5,0.2,0.2,0.3,0.5)	(0.6,0.2,0.9,0.3,0.4)	(0.1,0.2,0.3,0.5,0.3)

Table 7: The Tangent Similarity Measure between P1 and P2

Tangent Similarity Measure	WhatsApp	YouTube	Facebook	Blog
R1	**0.9467**	0.9430	0.9148	0.9463
R2	0.9583	**0.9792**	0.9603	0.9350
R3	0.9595	**0.9791**	0.9504	0.9430

Table 8: The Weighted Tangent Similarity Measure between P1 and P2

Weighted Tangent Similarity Measure	WhatsApp	YouTube	Facebook	Blog
R1	**0.9444**	0.9409	0.919	0.9308
R2	0.9547	**0.977**	0.9597	0.9319
R3	0.9635	**0.9771**	0.9565	0.9416

Table 9: The Cotangent Similarity Measure between P1 and P2

Cotangent Similarity Measure	WhatsApp	YouTube	Facebook	Blog
R1	**0.8995**	0.8927	0.8504	0.8706
R2	0.9583	**0.9599**	0.9195	0.8788
R3	0.9244	**0.9488**	0.9092	0.8927

Table 10: The Weighted Cotangent Similarity Measure between P1 and P2

Weighted Cotangent	WhatsApp	YouTube	Facebook	Blog

Similarity Measure				
R1	**0.8957**	0.889	0.8587	0.871
R2	0.9547	**0.9561**	0.9185	0.8732
R3	0.9308	**0.9425**	0.9207	0.8904

The highest similarity measures reflects the benefits of Social Media among College Students. Therefore Student R2 and R3 gains knowledge more from YouTube and R1 from WhatsApp.

6. Conclusion

In this paper, we have proposed tangent and cotangent similarity measures for Pentapartitioned Neutrosophic Pythagorean set with dependent Neutrosophic components and proved some of its basic properties. Furthermore, we have also investigated about the weighted similarity measures in Decision Making and illustrated with an example. In future, we can study about the improved similarity measure for the above set and can be used in Medical Diagnosis, Data mining. Clustering Analysis etc.

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Neutrosophic Hybrid MCDM Framework to Evaluate the Risks of Excavation System

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Abstract: The building of excavations is an extremely dangerous job that incorporates a variety of different variables. It is possible to significantly lower the likelihood of an accident occurring by first accurately identifying high-risk variables and then taking appropriate preventative steps. Single-valued neutrosophic verbal sets (SVNVS) can effectively represent qualitative and vague information when used in the identification process for high-risk variables of excavation systems. In addition, the identification of high-risk elements associated with an excavation system is a multi-criteria decision-making (MCDM) issue. This issue may be resolved by using the multi-attribute border approximation area comparison (MABAC) technique. The MABAC method operates on the presumption that criteria are compensating. However, the identification process for high-risk variables of excavation systems may include characteristics that are not compensatory. Under conditions of single-valued neutrosophic sets, a MABAC approach is developed. The weights of the criterion are calculated using this approach, which uses the mean-squared deviation weight method. In addition to that, an illustrated example is carried out to demonstrate the process that is involved in the MABAC approach.

Keywords*:* Neutrosophic Sets; MCDM; MABAC; Mean Squared Deviation weight; SVNSs; Excavation System.

1. Introduction

Accidents are more likely to occur during the construction of the excavation if possible high-risk elements are not recognized and mitigated on time. One way to think of the excavation is as a sophisticated construction network for subterranean engineering[1], [2]. Due to the highly disguised nature of the construction process, the processing of construction information connected to excavation construction presents the managers of the project with a particularly difficult problem when compared to the processing of construction information linked to other civil engineering projects[3], [4]. In addition, in geotechnical engineering, the experiences of specialists and engineers are essential, and they may give helpful references for engineering projects at various phases. This is because excavation construction is fraught with a great deal of uncertainty and fuzziness[5], [6].

To acquire correct risk levels, it is necessary to conduct an excavation risk assessment. The multivariable and nonlinear connection that exists among the variables and risk levels is the source of the majority of the challenges that are associated with this procedure[7], [8]. In the most recent decades, a large number of scholars have developed a variety of approaches to anticipate or evaluate the dangers associated with deep excavation. These methods include the fuzzy set theory as well as machine learning techniques like artificial neural networks (ANNs).

Smarandache offered the neutrosophic set for the first time from a philosophical standpoint at the beginning[9]. A neutrosophic set may be summed up using three degrees: the degree of truth membership, the degree of indeterminacy membership, and the degree of falsity membership. It generalizes the idea of classic sets, fuzzy sets, interval-valued fuzzy sets, vague sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, tautological sets, and vague intuitionistic fuzzy sets[10]–[12]. From a scientific standpoint, it is necessary to specify the neutrosophic set as well as the set-theoretic procedures. If this is not the case, then it will be difficult to use in actual scenarios[13], [14]. In light of this, Wang et al. came up with the idea of a single-valued neutrosophic set (SVNS), and they also presented the set-theoretic operators and several features associated with SVNSs[15], [16].

A novel approach has been developed, and it's called the MABAC technique. It demonstrates the basis of decision-making by using a clear calculation approach, a systematic process, and good logic in its operation. Peng and Yang utilized the MABAC to the R&D project choice technique to rate the projects and achieve the one they sought. This was accomplished by integrating the benefits of Pythagorean fuzzy sets with the MABAC[17], [18]. MABAC is a technique that was suggested by Xue et al. for the selection of materials to be used in interval-valued intuitionistic fuzzy environments. However, to the best of our knowledge, the investigation of the MADM issue using the MABAC approach has not been published in the current body of scholarly literature[19]–[21]. As a result, using the MABAC approach in MADM to rank the alternatives and come up with the best one while working in a single-valued neutrosophic system is an exciting study area[22]–[24].

The main contribution in this paper is organized as follows:

I. The identification of the risks in the excavation systems is evaluated under the single-valued neutrosophic sets.
II. This kind of this problem has not been applied under a neutrosophic environment in previous research.
III. The excavation criteria are computed by the mean squared deviation.
IV. The MABAC method is extended by the single-valued neutrosophic sets to rank the risks in the excavation system.
V. A real case study is conducted in this paper in Egypt.
VI. This research uses the cost and profit criteria and the single-valued neutrosophic operations in the normalization process.

The organization of the structure of this work is described below. In Section 2, the MABAC approach is constructed such that it may solve the issue of identifying dangers in excavation systems. In Section 3, we look at an example that illustrates how the excavation system in Egypt worked. The last section of the paper is called Section 4.

2. The MABAC Method

In this section, a MABAC approach for evaluating excavation systems is presented. The MABAC approach is broken down into two distinct stages. Obtaining the weight vector of variables is the primary objective of the initial phase[15]. During the second step, the discrepancies between the excavation system and the appropriate border approximation region are determined and the options are ranked. Fig. 1 is a diagram that illustrates the framework of the MABAC approach. The remainder of this section will go into further depth about its specifics[25], [26].

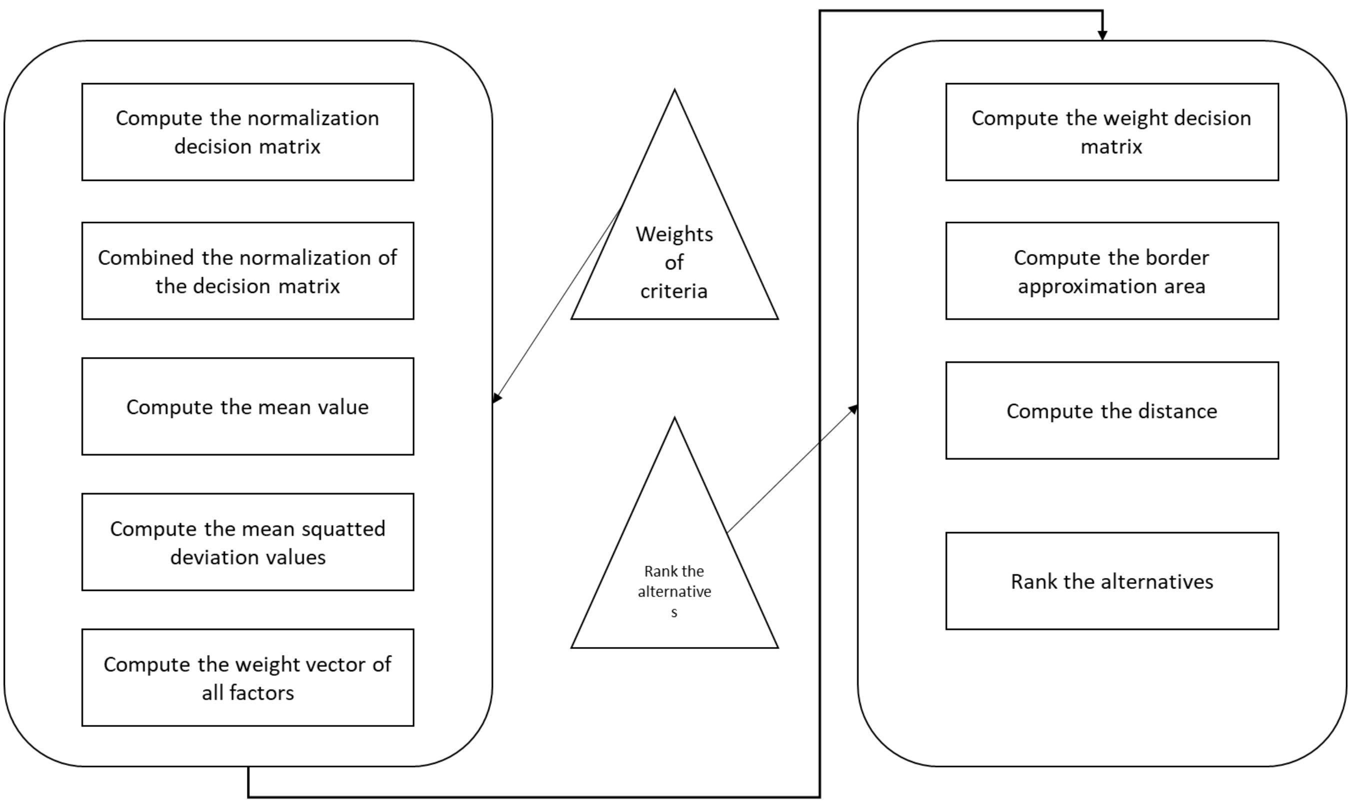

Fig 1. The framework of the MABAC method.

Table 1. The MABAC variables.

Symbols	Description
M	Number of alternatives
n	Number of criteria

EXCSA ($EXCSA_1, EXCSA_2, EXCSA_3 \ldots . EXCSA_m$)	Alternatives
EXCSC ($EXCSC_1, EXCSC_2, EXCSC_3 \ldots . EXCSC_n$)	Criteria
DM_1, DM_2, DM_3	Decision Makers
$w = (w_1, w_2 . w_3, \ldots w_e)^T$	Weight Vector
e	Experts
$DM_g(g = 1,2,3, \ldots \ldots e)$	Decision Makers
$r = 1,2,3 \ldots . . m$	Alternatives
$j = 1,2,3, \ldots . n$	Criteria

Obtained the decision matrix as:

$$H^g = \begin{pmatrix} H_{11}^g & H_{12}^g & \cdots & H_{1n}^g \\ H_{21}^g & H_{22}^g & & H_{2n}^g \\ \vdots & & \ddots & \vdots \\ H_{m1}^g & H_{m2}^g & \cdots & H_{mn}^g \end{pmatrix},$$

Where $H_{rj}^g = (S_{rj}^g, T_{rj}^g, I_{rj}^g, F_{rj}^g)$ is a single-valued neutrosophic verbal number (SVNVN) of $EXCSA_r$ against $EXCSC_j$ donated by experts $DM_g(g = 1,2,3, \ldots \ldots g)$

Phase 1: Compute the weight vector of factors.

At this point in the process, the weight vector of the factors is acquired. A mean-squared deviation weight approach is used to estimate the relative importance of each criterion. The following is an explanation of the particulars of this phase.

Step 1: Compute the normalization decision matrix.

In this step, if the criterion is cost then the criterion should be normalized. The profit criteria are not normalized.

$$Nor_{rj}^g = \begin{cases} neg(H_{rj}^g) & cost\ criteria \\ H_{rj}^g & otherwise \end{cases} \quad (1)$$

Step 2: Combined the normalization of the decision matrix.

There are many decision-makers and experts, so the normalized decision matrices should be combined into one matrix. The combined normalized decision matrix obtained by $Com = (Com_{rj})_{m \times n}$

Step 3: Compute the mean value.

In the future phases, a mean-squared deviation weight approach will be established. The mean value of all the different alternatives is used in this technique to evaluate every criterion. At this stage, the mean value of all the alternatives concerning the criteria is determined.

The mean value donated as $M(Com_j)$

Step 4: Compute the mean squatted deviation values (ϑ)

The mean squared deviation can be computed as:

$$\vartheta(Com_j) = \sqrt{\sum_{r=1}^{m}\left(d(Com_{rj} - M(Com_j))\right)^2} \quad (2)$$

Step 5: Compute the weight vector of all factors.

The weights of factors can be computed as:

$$w_j = \frac{\vartheta_j}{\sum_{j=1}^{n}\vartheta_j} \quad (3)$$

Phase 2: Rank the alternatives by the MABAC method.

Step 6: Compute the weight decision matrix.

The weight decision matrix can be computed by multiplying the weight vector of each criterion by the aggregated normalized decision matrix as:

$$WD\ (wd_{rj}) = w_j * Com_{rj} \quad (4)$$

Step 7: Compute the border approximation area.

The border approximation area can be computed by the MABAC method and donated as $B = (b_j)_{n\times 1}$

$$b_j = \left(\prod_{r=1}^{m} wd_{rj}\right)^{\frac{1}{m}} \quad (5)$$

Step 8: Compute the distance between the weighted normalized decision matrix and the border approximation area.

The distance between b_j and wd_{rj} can be computed as:

$$T = (t_{rj})_{m\times n} = \begin{cases} t(Com_{rj}, b_j)\ if\ Com_{rj} > b\_j \\ -t(Com_{rj}, b_j)\ otherwise \end{cases} \quad (6)$$

Step 9: Rank the alternatives.

The alternatives are ranked according to:

$$F_r = \sum_{j=1}^{n} t_{rj} \quad (7)$$

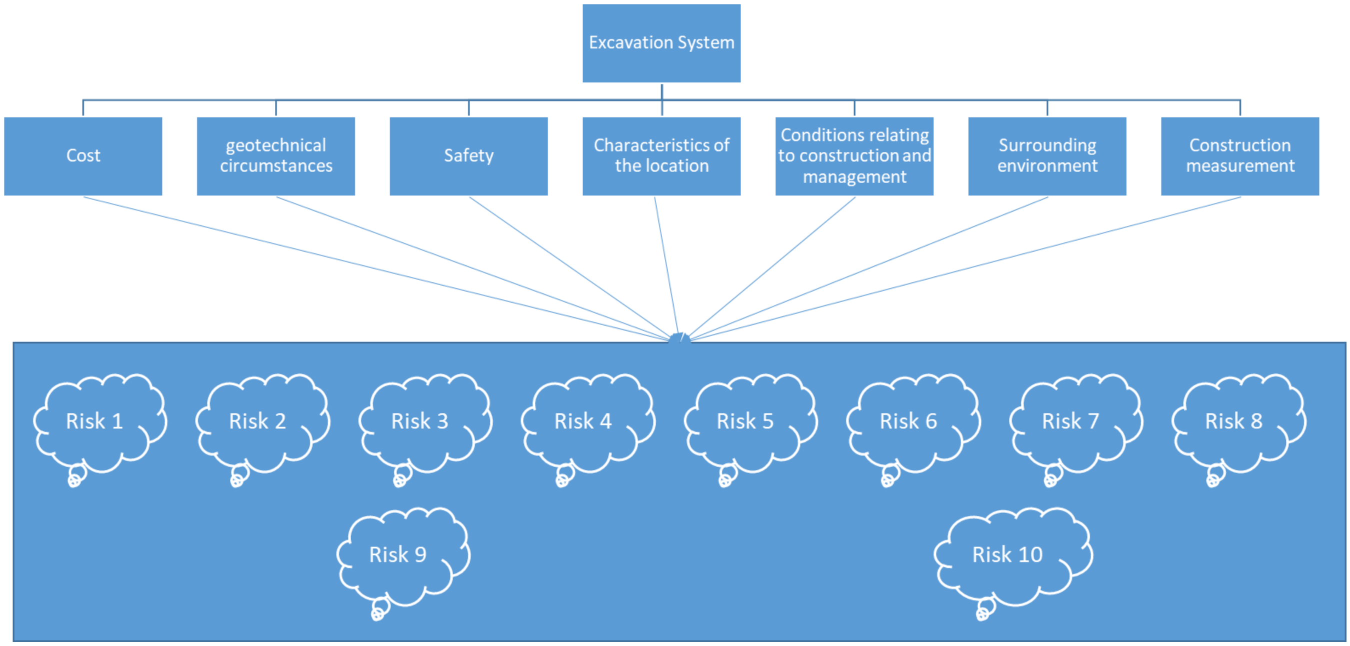

Fig 2. The hierarchy tree of criteria and alternatives.

3. Results

In this part, the MABAC approach is used to evaluate the excavation system. This section's primary objective is to explain how the MABAC should be used.

The building activity of excavation has a greater danger. Because the building of the excavation involves a variety of different aspects. An investigation into the building of an excavation in Zhuhai, China, is used as the case study. Excavation construction carried out in a risk-free way is an extremely important aspect of the construction unit.

Archaeology one hundred years ago was quite different from what it is now. Petrie used enormous teams of Egyptian excavators; nevertheless, their efforts were not acknowledged for the work that they did throughout the excavations that took place on a grander scale and at a quicker speed.

Archaeologists utilize a wide variety of techniques that allow for more exact documentation than ever before, which has resulted in digs that are more specific and concentrated than ever before. Archaeologists in Egypt oversee their digs, and there has been an increase in the number of efforts made to engage local populations in Egypt via various outreach programs. The project will make in Saqqara, Egypt. The Saqqara site is part of a sprawling necropolis at Egypt's ancient capital of Memphis that includes the famed Giza Pyramids as well as smaller pyramids at Abu Sir, Dahshur, and Abu Ruwaysh. The ruins of Memphis were designated a UNESCO World Heritage site in the 1970s.

It is necessary to establish both criteria and risk variables. For this research, five highly knowledgeable specialists in expert systems have been asked to carry out an excavation risk assessment. Fig 2. Shows the hierarchy tree between criteria and alternatives (risks). There are three experts to evaluate the criteria and alternatives. The weights vector of experts is (1/3,1/3,1/3).

In this portion, it was mentioned that SVNVN may be used to characterize assessments. Every specialist evaluates every risk concerning every factor. The verbal aspects are taken into consideration. For instance, the first decision-makers compared $\mathrm{EXCSA}_1$to EXCSC_1 based on its verbal value. In addition, we have asked every supervisor to give the following data: (1) The extent to which the individual thinks that the evaluation is accurate. (2) The extent to which the individual believes that the evaluation is inaccurate. (3) The extent to which he does not have complete confidence in the evaluation. An SVNN can show all three of these different types of information. Let experts evaluate the criteria and alternatives to build the decision matrix. Table 2 shows the decision matrix by the e1.

Table 2. The decision matrix of e1 by the SVNVNs.

	EXCSC_1	EXCSC_2	EXCSC_3	EXCSC_4	EXCSC_5	EXCSC_6	EXCSC_7
EXCSA_1	0.9,0.1,0.1	0.3,0.8,0.2	0.1,0.8,0.6	0.1,0.8,0.6	0.4,0.6,0.2	0.9,0.1,0.1	0.9,0.1,0.1
EXCSA_2	0.7,0.5,0.1	0.6,0.5,0.1	0.9,0.1,0.1	0.7,0.5,0.1	0.4,0.6,0.2	0.8,0.1,0.1	0.4,0.6,0.2
EXCSA_3	0.1,0.8,0.6	0.2,0.8,0.4	0.4,0.6,0.2	0.8,0.1,0.1	0.9,0.1,0.1	0.7,0.5,0.1	0.1,0.8,0.6
EXCSA_4	0.6,0.5,0.1	0.9,0.1,0.1	0.3,0.8,0.2	0.7,0.5,0.1	0.2,0.8,0.4	0.4,0.6,0.2	0.6,0.5,0.1
EXCSA_5	0.4,0.6,0.2	0.8,0.1,0.1	0.3,0.8,0.2	0.4,0.6,0.2	0.1,0.8,0.6	0.2,0.8,0.4	0.8,0.1,0.1
EXCSA_6	0.1,0.8,0.6	0.2,0.8,0.4	0.9,0.1,0.1	0.8,0.1,0.1	0.3,0.8,0.2	0.4,0.6,0.2	0.9,0.1,0.1
EXCSA_7	0.7,0.5,0.1	0.6,0.5,0.1	0.4,0.6,0.2	0.7,0.5,0.1	0.9,0.1,0.1	0.7,0.5,0.1	0.3,0.8,0.2
EXCSA_8	0.4,0.6,0.2	0.9,0.1,0.1	0.3,0.8,0.2	0.8,0.1,0.1	0.3,0.8,0.2	0.8,0.1,0.1	0.1,0.8,0.6
EXCSA_9	0.1,0.8,0.6	0.8,0.1,0.1	0.7,0.5,0.1	0.9,0.1,0.1	0.3,0.8,0.2	0.4,0.6,0.2	0.6,0.5,0.1
EXCSA_{10}	0.9,0.1,0.1	0.6,0.5,0.1	0.4,0.6,0.2	0.1,0.8,0.6	0.7,0.5,0.1	0.6,0.5,0.1	0.9,0.1,0.1

Phase 1: Compute the weights vector of each criterion.

Step 1: Compute the normalization decision matrix.

In this step, we specify the cost and profit criteria to make a normalization matrix on the cost criteria only. Cost criterion is a cost criterion and others are profit criteria. The normalization decision matrix is shown in Table 3.

Table 3. The normalized decision matrix of e2 (Nor^2).

	EXCSC_1	EXCSC_2	EXCSC_3	EXCSC_4	EXCSC_5	EXCSC_6	EXCSC_7
EXCSA_1	0.1,0.5,0.6	0.3,0.8,0.2	0.1,0.8,0.6	0.1,0.8,0.6	0.4,0.6,0.2	0.9,0.1,0.1	0.2,0.8,0.4
EXCSA_2	0.4,0.2,0.2	0.6,0.5,0.1	0.9,0.1,0.1	0.7,0.5,0.1	0.2,0.8,0.4	0.8,0.1,0.1	0.4,0.6,0.2
EXCSA_3	0.6,0.2,0.1	0.2,0.8,0.4	0.4,0.6,0.2	0.2,0.8,0.4	0.9,0.1,0.1	0.6,0.5,0.1	0.1,0.8,0.6
EXCSA_4	0.1,0.5,0.6	0.6,0.5,0.1	0.3,0.8,0.2	0.7,0.5,0.1	0.2,0.8,0.4	0.4,0.6,0.2	0.6,0.5,0.1

$EXCSA_5$	0.1,0.5,0.6	0.8,0.1,0.1	0.3,0.8,0.2	0.6,0.5,0.1	0.1,0.8,0.6	0.2,0.8,0.4	0.8,0.1,0.1
$EXCSA_6$	0.4,0.2,0.2	0.2,0.8,0.4	0.2,0.8,0.4	0.8,0.1,0.1	0.3,0.8,0.2	0.6,0.5,0.1	0.2,0.8,0.4
$EXCSA_7$	0.1,0.5,0.7	0.6,0.5,0.1	0.6,0.5,0.1	0.7,0.5,0.1	0.9,0.1,0.1	0.6,0.5,0.1	0.3,0.8,0.2
$EXCSA_8$	0.1,0.5,0.6	0.2,0.8,0.4	0.3,0.8,0.2	0.8,0.1,0.1	0.2,0.8,0.4	0.8,0.1,0.1	0.6,0.5,0.1
$EXCSA_9$	0.6,0.2,0.1	0.8,0.1,0.1	0.6,0.5,0.1	0.9,0.1,0.1	0.3,0.8,0.2	0.4,0.6,0.2	0.2,0.8,0.4
$EXCSA_{10}$	0.1,0.5,0.6	0.6,0.5,0.1	0.6,0.5,0.1	0.1,0.8,0.6	0.7,0.5,0.1	0.6,0.5,0.1	0.6,0.5,0.1

Step 2: Combined the normalization of the decision matrix.

This step shows the combined decision matrix table 4 shows the integrated decision matrix.

Table 4. The integration decision matrix.

	$EXCSC_1$	$EXCSC_2$	$EXCSC_3$	$EXCSC_4$	$EXCSC_5$	$EXCSC_6$	$EXCSC_7$
$EXCSA_1$	0.1,0.5,0.6	0.3,0.8,0.2	0.1,0.8,0.6	0.1,0.8,0.6	0.4,0.6,0.2	0.9,0.1,0.1	0.2,0.8,0.4
$EXCSA_2$	0.4,0.2,0.2	0.6,0.5,0.1	0.9,0.1,0.1	0.7,0.5,0.1	0.2,0.8,0.4	0.8,0.1,0.1	0.4,0.6,0.2
$EXCSA_3$	0.6,0.2,0.1	0.2,0.8,0.4	0.4,0.6,0.2	0.2,0.8,0.4	0.9,0.1,0.1	0.6,0.5,0.1	0.1,0.8,0.6
$EXCSA_4$	0.1,0.5,0.6	0.6,0.5,0.1	0.3,0.8,0.2	0.7,0.5,0.1	0.2,0.8,0.4	0.4,0.6,0.2	0.6,0.5,0.1
$EXCSA_5$	0.1,0.5,0.6	0.8,0.1,0.1	0.3,0.8,0.2	0.6,0.5,0.1	0.1,0.8,0.6	0.2,0.8,0.4	0.8,0.1,0.1
$EXCSA_6$	0.4,0.2,0.2	0.2,0.8,0.4	0.2,0.8,0.4	0.8,0.1,0.1	0.3,0.8,0.2	0.6,0.5,0.1	0.2,0.8,0.4
$EXCSA_7$	0.1,0.5,0.7	0.6,0.5,0.1	0.6,0.5,0.1	0.7,0.5,0.1	0.9,0.1,0.1	0.6,0.5,0.1	0.3,0.8,0.2
$EXCSA_8$	0.1,0.5,0.6	0.2,0.8,0.4	0.3,0.8,0.2	0.8,0.1,0.1	0.2,0.8,0.4	0.8,0.1,0.1	0.6,0.5,0.1
$EXCSA_9$	0.6,0.2,0.1	0.8,0.1,0.1	0.6,0.5,0.1	0.9,0.1,0.1	0.3,0.8,0.2	0.4,0.6,0.2	0.2,0.8,0.4
$EXCSA_{10}$	0.1,0.5,0.6	0.6,0.5,0.1	0.6,0.5,0.1	0.1,0.8,0.6	0.7,0.5,0.1	0.6,0.5,0.1	0.6,0.5,0.1

Step 3: Compute the mean value.

The values of the mean can be computed in this step.

Step 4: Compute the mean squatted deviation values (ϑ)

The mean squared error of each alternative against criteria computed by using Eq. (2). The results are shown in Table 5.

Table 5. The mean values mean squared deviation values to each criterion and the weight of the criteria.

	Mean values	Mean squared deviation	$Weight$
$EXCSC_1$	0.464444	4.436222	0.139157
$EXCSC_2$	0.608889	4.436222	0.144117
$EXCSC_3$	0.565556	4.436222	0.146621
$EXCSC_4$	0.635556	4.436222	0.14046
$EXCSC_5$	0.515556	4.436222	0.143866
$EXCSC_6$	0.665556	4.436222	0.143415
$EXCSC_7$	0.553333	4.436222	0.142363
Sum	4.008889	4.436222	1

Step 5: Compute the weight vector of all factors.

The weights of the criteria can be computed using Eq. (2). The last column in Table 5 shows the weights of the criteria. The sum of all criteria is 1 as shown in the last row in Table 5.

Phase 2: Rank the alternatives by the MABAC method.

Step 6: Compute the weight decision matrix.

The weighted decision matrix can be computed by using Eq. (4). Table 6 shows the values of multiplying the weights of criteria by the normalization matrix.

Table 6. The weighted decision matrix.

	$EXCSC_1$	$EXCSC_2$	$EXCSC_3$	$EXCSC_4$	$EXCSC_5$	$EXCSC_6$	$EXCSC_7$
$EXCSA_1$	0.018554325502179,0.0834944647598056,0.0881330461353504	0.0432349847217352,0.115293292591294,0.0288233231478235	0.0293242498622452,0.107522249494899,0.068423249678572	0.0140459850723839,0.112367880579071,0.0842759104343034	0.0575464609527626,0.0863196914291439,0.0287732304763813	0.129073786505034,0.0143415318338927,0.0143415318338927	0.0711816861193207,0.0711816861193207,0.0332181201890163
$EXCSA_2$	0.0324700696288133,0.0510243951309923,0.0603015578820819	0.0864699694434704,0.0720583078695587,0.0144116615739117	0.131959124380103,0.0146621249311226,0.0146621249311226	0.0983218955066874,0.0702299253619195,0.0140459850723839	0.0479553841273022,0.0959107682546043,0.0383643073018417	0.114732254671142,0.0143415318338927,0.0143415318338927	0.0569453488954566,0.0854180233431849,0.0284726744477283
$EXCSA_3$	0.0649401392576266,0.0	0.0288233231478235,0.1	0.0586484997244903,0.0	0.0655479303377916,0.0	0.105501845080065,0.03	0.0956102122259513,0.0	0.0142363372238641,0.1

	371086510043581,0.0278314882532685	15293292591294,0.0576466462956469	879727495867355,0.0293242498622452	702299253619195,0.0327739651688958	83643073018417,0.0191821536509209	717076591694635,0.0143415318338927	13890697790913,0.0854180233431849
$EXCSA_4$	0.0139157441266343,0.0695787206331714,0.0834944647598056	0.0912738566347743,0.0576466462956469,0.0192155487652156	0.0488737497704086,0.107522249494899,0.0293242498622452	0.0983218955066874,0.0702299253619195,0.0140459850723839	0.0383643073018417,0.105501845080065,0.0479553841273022	0.0573661273355708,0.0860491910033562,0.0286830636677854	0.0759271318606088,0.0759271318606088,0.0189817829651522
$EXCSA_5$	0.0231929068777238,0.0603015578820819,0.0649401392576266	0.0960777438260782,0.0384310975304313,0.0192155487652156	0.0439863747933677,0.117296999448981,0.0293242498622452	0.0655479303377916,0.0795939154101755,0.0234099751206399	0.0143866152381907,0.115092921905525,0.0863196914291439	0.0286830636677854,0.114732254671142,0.0573661273355708	0.113890697790913,0.0142363372238641,0.0142363372238641
$EXCSA_6$	0.0742173020087161,0.0278314882532685,0.018554325502179	0.0384310975304313,0.105685518208686,0.0480388719130391	0.0977474995408172,0.0488737497704086,0.0293242498622452	0.112367880579071,0.0140459850723839,0.0140459850723839	0.043159845714572,0.115092921905525,0.0287732304763813	0.0669271485581659,0.0812686803920586,0.0239025530564878	0.0711816861193207,0.0711816861193207,0.0332181201890163
$EXCSA_7$	0.018554325502179,0.0649401392576266,0.0834944647598056	0.0864699694434704,0.0720583078695587,0.0144116615739117	0.068423249678572,0.0830853746096946,0.0244368748852043	0.0983218955066874,0.0702299253619195,0.0140459850723839	0.105501845080065,0.0383643073018417,0.0191821536509209	0.0956102122259513,0.0717076591694635,0.0143415318338927	0.0474544574128805,0.104399806308337,0.0284726744477283
$EXCSA_8$	0.0231929068777238,0.0603015578820819,0.0649401392576266	0.0960777438260782,0.0480388719130391,0.0288233231478235	0.0439863747933677,0.117296999448981,0.0293242498622452	0.112367880579071,0.0140459850723839,0.0140459850723839	0.0383643073018417,0.115092921905525,0.0383643073018417	0.114732254671142,0.0143415318338927,0.0143415318338927	0.0379635659303044,0.099654360567049,0.0616907946367446
$EXCSA_9$	0.0649401392576266,0.0371086510043581,0.0278314882532685	0.0960777438260782,0.0384310975304313,0.0192155487652156	0.0977474995408172,0.0733106246556129,0.0146621249311226	0.126413865651455,0.0140459850723839,0.0140459850723839	0.043159845714572,0.115092921905525,0.0287732304763813	0.0573661273355708,0.0860491910033562,0.0286830636677854	0.0664362403780327,0.0854180233431849,0.0284726744477283
$EXCSA_{10}$	0.0510243951309923,0.106687371637529,0.111325953013074	0.0864699694434704,0.0720583078695587,0.0144116615739117	0.068423249678572,0.0830853746096946,0.0244368748852043	0.0140459850723839,0.112367880579071,0.0842759104343034	0.100706306667335,0.0719330761909533,0.0143866152381907	0.0860491910033562,0.0717076591694635,0.0143415318338927	0.0901634690844729,0.0569453488954566,0.0189817829651522

Step 7: Compute the border approximation area.

The border approximation area can be computed by using Eq. (5).

Step 8: Compute the distance between the weighted normalized decision matrix and the border approximation area.

The distance between the weighted decision matrix and border approximation area can be determined by using Eq. (6). The results are shown in Table 7.

Table 7. The distance between the weighted decision matrix and border approximation area.

	$EXCSC_1$	$EXCSC_2$	$EXCSC_3$	$EXCSC_4$	$EXCSC_5$	$EXCSC_6$	$EXCSC_7$
$EXCSA_1$	0.047025	0.026686	0.041417	0.074759	0.007341	0.011038	0.018116
$EXCSA_2$	0.000639	0.012274	-0.00257	0.046667	0.016932	-0.0033	0.01337
$EXCSA_3$	-0.01328	0.041097	0.012093	0.032621	-0.00225	0.034941	0.056079
$EXCSA_4$	0.023832	0.00747	0.021868	0.046667	0.026523	0.02538	0.01337
$EXCSA_5$	0.005278	-0.00694	0.026755	0.032621	0.050501	0.054063	-0.0151
$EXCSA_6$	-0.02255	0.03149	0.012093	0.004529	0.021728	0.02538	0.018116
$EXCSA_7$	0.023832	0.012274	0.012093	0.046667	-0.00225	0.034941	0.022861
$EXCSA_8$	0.005278	0.012274	0.026755	0.004529	0.026523	-0.0033	0.041843
$EXCSA_9$	-0.01328	-0.00694	0.021868	0.018575	0.021728	0.02538	0.022861
$EXCSA_{10}$	0.125881	0.012274	0.012093	0.074759	0.021728	0.02538	0.008625

Step 9: Rank the alternatives.

The sum of each row can be computed using Eq. (7). Then rank the alternatives according to the lowest value of the sum. Table 8 shows the rank of alternatives.

Table 8. The rank of alternatives.

	Sum of distance	Rank
$EXCSA_1$	0.226382	9
$EXCSA_2$	0.08401	1
$EXCSA_3$	0.161305	7

EXCSA_4	0.16511	8
EXCSA_5	0.147174	5
EXCSA_6	0.090781	3
EXCSA_7	0.150418	6
EXCSA_8	0.113899	4
EXCSA_9	0.090193	2
EXCSA_{10}	0.280739	10

4. Conclusion

In engineering practice, a built decision structure for risk analysis of an excavation system provides a useful guide for project supervisors to recognize high-risk aspects. This helps project supervisors to take appropriate measures in time to minimize the occurrence likelihood of risk accidents in the initial building phase of excavation. The method that has been proposed may be used in any other engineering project that calls for the judgments of DMs and the information tracked of variables. Additionally, the proposed framework is adaptable for use in the MCDM process. The last point is that the approach associated with MCDM modeling may be transformed into computer software, which can minimize the amount of time and effort required to gather and analyze the views from a variety of specialists.

A technique for a neutrosophic excavating system has been devised mainly for this work. SVNVNs are used inside the excavation system approach to display qualitative and ambiguous information. MABAC has been upgraded so that it can manage SVNVNs. In addition to this, the excavation system approach presents the central concept of MABAC and considers the non-compensation of requirements. In addition, to acquire criterion weights, the mean-squared deviation weight technique using SVNVNs has been devised. From the MABAC method and neutrosophic sets, alternative 2 is the best, and alternative 10 is the worst.

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Novel Subsethood Measures for Totally Dependent-Neutrosophic Sets and Their Usage in Multiple Attribute Decision-Making

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Abstract: Multiple attribute decision-making (MADM) models are accepted as powerful tools for evaluating alternatives when the decision-analysts should consider more than one attribute while reaching a decision. The decision-makers who are consulted need a scale for expressing their judgments, experiences, or opinions. The fuzzy logic and its contemporary versions can supply different kinds of scales to allow the decision-makers to state their ideas. A recent version is totally dependent-neutrosophic (picture fuzzy) sets which include independently assignable elements: positive, neutral, negative, and refusal membership degrees. In this study, we aim to contribute to the literature of the totally dependent-neutrosophic sets by (i) proposing three new subsethood measures dedicatedly developed for totally dependent-neutrosophic sets for the first time in the literature and (ii) showing their applicability in a decision-making case study including a novel totally dependent-neutrosophic version of EDAS (Evaluation Based on Distance from Average Solution) method which is extended differently from the existing ones. To validate the proposed method, a comparative analysis with the existing totally dependent-neutrosophic MADM methods is provided. As a result, the proposed Subsethood Measure-based Totally Dependent-Neutrosophic Version of EDAS (SM-TDN-EDAS) method involving fewer steps than others gave similar rankings.

Keywords: Totally dependent-neutrosophic sets, subsethood measure, EDAS, multiple criteria evaluation, fuzzy numbers.

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1. Introduction

In multiple attribute decision-making (MADM) problems in which the decision analysts do not have enough or proper cardinal data such as cost of investment, sales profit, market share at a certain time, etc., the linguistic terms are used by the decision-makers while expressing their preferences, opinions, or feelings. Linguistic terms are often defined in different fuzzy environments. Zadeh [1] initiated the concept of fuzzy sets as a symbolization and representation tool for quantifying human judgments. In the traditional definition of fuzzy sets, there is just membership degree (μ_A) which takes a value between 0 and 1. In general, the membership degree is a measure of the optimism or agreement level of judgment. Thus, it has a positive meaning.

For decades, various scholars have defined several fuzzy sets for smoothing the symbolization of the vagueness and ambiguity which are hidden in the human subconscious. Atanassov [2] initiated the concept of intuitionistic fuzzy sets (IFS) and add a new element into set definition: non-membership degree (υ_A). This novel item puts a level of resilience into the representation of judgments since the decision-maker can state his/her pessimistic view or disagreement level. So, non-membership degree exposes a negative meaning. Accordingly, Atanassov [2] also introduced a new measure regarding hesitancy which has a neutral meaning: $\pi_A = 1 - \mu_A - \upsilon_A$. Therefore, IFS can cope with three dimensions of judgments (membership, non-membership, and hesitancy). In real life, we can represent these degrees with yes, no, and abstain. However, the hesitancy degree in IFSs depends on the others so that the decision-maker cannot independently assign any value for that.

After the development of IFS, some other extensions such as Pythagorean fuzzy sets [3], q-Rung orthopair fuzzy sets [4], neutrosophic sets [5], spherical fuzzy sets [6], etc. have been introduced. From a different perspective, Cuong and Kreinovich [7] defined picture fuzzy sets as the generalization of fuzzy sets and IFSs. However, Smarandache, for the first time, renamed it by "totally dependent-neutrosophic set (TDNS)" [8]. In this paper, we use "totally dependent-neutrosophic set (TDNS)" instead of "picture fuzzy set (PFS)".

A TDNS is characterized by three independently assignable degrees expressing the positive membership, the neutral membership (which is equivalent to hesitancy degree), and the negative membership (which means non-membership). The sole constraint regarding these three degrees is that their sum must not exceed 1. The remaining part is called refusal degree and it represents the decision maker's choice of refusing to share his/her preference.

For illustration, Cuong [9] gives the voting process as an example of TDNS for clarifying the elements defined: the voters may be divided into four groups of those who: vote for the candidate, abstain, vote against the candidate, and refusal of the voting, i.e., casting a veto. Garg [10] gives another example. When a decision analyst consults a certain decision-maker regarding a certain topic, then he/she may state that 0.3 is the possibility that statement is true, 0.4 is the possibility that statement is false and 0.2 is the possibility that he/she is not sure of it. This issue cannot be handled by fuzzy sets or IFSs. This declaration of preference can be well-defined by TDNS as $(\mu, \eta, v) = (0.3, 0.2, 0.4)$ where μ is the positive membership degree, η is the neutral membership degree, and v is the negative membership degree. As seen, their sum is 0.9 and the remaining part is called refusal degree which is equal to (1-0.9=) 0.1. Formally, the refusal degree is defined as $\pi = 1 - \mu - \eta - v$. More formal definitions and operations are explained in Section 2. As seen from the examples, TDNS has greater representation power than IFS, neutrosophic sets, or other extensions since it exposes an additional fourth component, namely refusal degree. TDNS is the only fuzzy set definition that can address this issue.

The subsethood measure (or inclusion measure) indicates the degrees of quantitative extensions of the qualitative set inclusion relation. In classical set theory, since either a crisp set A is a subset of a crisp set B or vice versa, subsethood measure should be two-valued: 0 and 1. Fuzzy subsethood measures determine the degree to which a fuzzy set contains another fuzzy set within the range of [0, 1]. This notion fuzzifies classical fuzzy set containment which is a crisp property: a fuzzy set B contains a fuzzy set A if $\mu_A \leq \mu_B$. Kosko [11] argues that if this inequality holds for all but just a few elements, one can still consider A to be a subset of B to some degree. Many researchers such as Kosko [11], Sanchez [12], and Young [13] define several axioms for developing subsethood measures. As seen from Section 2, even though there are attempts to stating subsethood measures for various fuzzy sets, there is no proposition for TDNS. As the first contribution of this study to the existing literature, we have developed subsethood measures for TDNS and we proved that they satisfy the required axiomatic properties.

To show our measures' applicability in real-life MADM problem-solving issues, we have integrated the concept of subsethood measure in a well-known MADM method, namely EDAS (Evaluation Based on Distance from Average Solution). EDAS method was firstly presented by Keshavarz Ghorabaee et al. [14] for searching the distances between each alternative and average solution. EDAS is very similar to TOPSIS and VIKOR, but they take the distances between each alternative and positive/negative ideal alternatives as a decision criterion: the best alternative among the set of alternatives should be as distant as possible from the negative ideal alternative and as close as possible to the positive ideal one [15]. EDAS cancels the phase of obtaining the ideal solutions which might be complex by considering the distance between each alternative and average solution that can be easily found from the current data in the problem.

For enriching the representation power of EDAS, some extensions including TDNS have been proposed in the literature. For example, Zhang et al. [16] developed a TDNS-based EDAS with newly defined operations and illustrated its application in green supplier selection while Liang et al. [17] integrated TDNS-based EDAS and ELECTRE methods for cleaner production evaluation in gold mines. Similarly, Li et al. [18] defined totally dependent-neutrosophic (picture fuzzy) ordered weighted interaction averaging operator and totally dependent-neutrosophic (picture fuzzy) hybrid ordered weighted interaction averaging operator and used them in TDNS-based EDAS. Ping et al. [19] combined TDNS-based EDAS with quality function deployment and showed its application in an illustrative example. Tirmikcioglu Cinar [20] applied TDNS-based EDAS method for team leader selection for an audit firm. To the best of our knowledge, the literature does not have any integration of subsethood measures and EDAS until now. This study's second contribution is this integration proposition to ease the mathematical operations of EDAS/TDNS-based EDAS and smooth the complexity.

As a summarization, it can be stated that this study proposes some subsethood measures for TDNS for the first time in the literature and their usability is shown in a novel TDNS extension of EDAS. The rest of the paper is organized as follows. Section 2 gives the preliminaries of TDNS and its operations, and the extensive literature survey's results on subsethood measure definitions for various fuzzy set environments. In Section 3, the definitions of three novel subsethood measures are detailed and it is proven that the proposed measures satisfy the required properties. In Section 4, novel subsethood measure-based totally dependent-neutrosophic (picture fuzzy) extension of EDAS (SM-TDN-EDAS) is explained step-by-step. To demonstrate the new extension's usability, the results of a case study are shared in Section 5. Section 6 concludes the study with the findings and further research potential.

2. Preliminaries

In this chapter, the details of TDNS and operations defined on it are given. Then, the results of an extensive literature survey on subsethood measures for various fuzzy sets are stated.

2.1. Totally dependent-neutrosophic set

Cuong and Kreinovich [7] presented TDNS theory which is a generalization of Zadeh's fuzzy set theory and Atanassov's IFS theory and gave basic operations on TDNSs. A TDNS is defined with the help of the degree of positive membership, the degree of neutral membership, the degree of negative membership, and the degree of refusal membership mappings such that the sum of these components is equal to 1. Essentially, fundamental structures of TDNS have enough application to carry out situations requiring opinions of humans, which is comprising answer types such as yes, no, abstain, and refusal.

Definition 1. [9,21] Let X be a universal set. Then a TDNS A on X is defined as follows:

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\} \quad (1)$$

where μ_A, η_A, ν_A are mapping from X to $[0,1]$. For all $x \in X$, $\mu_A(x)$ is called positive membership degree of $x \in A$, $\eta_A(x)$ is called neutral membership degree of $x \in A$ and $\nu_A(x)$ is negative membership degree of $x \in A$. Also, μ_A, η_A, ν_A satisfy the following condition:

$$0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1, \forall x \in X \tag{2}$$

and $\pi(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ is called refusal membership degree of x in A.

We denote by $TDNS(X)$ the collection of TDNSs on X. Cuong [9] defined the subsethood, equality, union, intersection, and complement for every two TDNSs A and B as follow:

1. $A \subseteq B$ if $\forall x \in X, \mu_A(x) \le \mu_B(x),\ \eta_A(x) \le \eta_B(x),\ \nu_A(x) \ge \nu_B(x)$;
2. $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
3. $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x))\rangle |\ x \in X\}$
4. $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x))\rangle |\ x \in X\}$
5. $A^c = \{\langle x, \nu_A(x), \eta_A(x), \mu_A(x)\rangle |\ x \in X\}$.

For all $A, B \in TDNS(X)$, Cuong [9] presented normalized Hamming distance measure by extending distance measure for IFS.

$$d_1(A,B) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\left(\mu_A(x_i) - \mu_B(x_i)\right)^2 + \left(\eta_A(x_i) - \eta_B(x_i)\right)^2 + (\nu_A(x_i) - \nu_B(x_i))^2\right)\right]^{\frac{1}{2}} \tag{3}$$

for all $A, B \in TDNS(X)$, Van Dinh et al. [22] introduced some distance measures for TDNSs as follow:

$$d_2(A,B) = \frac{1}{n}\sum_{i=1}^{n}(\max\{|\mu_A(x_i) - \mu_B(x_i)|, |\eta_A(x_i) - \eta_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|\}), \tag{4}$$

$$d_3(A,B) = \left[\sum_{i=1}^{n}\left(\max\left\{\left(\mu_A(x_i) - \mu_B(x_i)\right)^2, \left(\eta_A(x_i) - \eta_B(x_i)\right)^2, (\nu_A(x_i) - \nu_B(x_i))^2\right\}\right)\right]^{\frac{1}{2}}. \tag{5}$$

2.2. Subsethood measures for different fuzzy environments

The subsethood measure (also called inclusion measure or degree) indicates the degrees of quantitative extensions of the qualitative set inclusion relation. In classical set theory, since either a crisp set A is a subset of a crisp set B or vice versa, subsethood measure should be two-valued. Fuzzy subsethood measures determine the degree to which a fuzzy set contains another between 0 and 1. Many researchers have studied different subsethood measures for fuzzy sets, IFS, and neutrosophic sets.

Sinha and Dougherty [23] presented the axiomatic structure of subsethood measure for fuzzy sets. Young [13] introduced different axioms of the definition of subsethood measure for fuzzy sets from axioms of Sinha and Dougherty [23]. Fan et al. [24] and Guoshun and Yunsheng [25] defined new different subsethood measures for fuzzy sets. Bustince et al. [26] defined strong S-subsethood measures for interval-valued fuzzy sets (IVFS). Vlachos and Sergiadis [27] and Takáč [28,29] presented different subsethood measures for IVFS. Rickard et al. [30] introduced subsethood measure for Type-2 fuzzy sets and generalized Type-n fuzzy sets.

Liu and Xiong [31] proposed the definition of subsethood measure for IFS. Cornelis and Kerre [32] introduced a different framework of subsethood measure for IFSs by considering the subsethood degree to be in the unit square $[0,1]^2$. Grzegorzewski and Mrowka [33] presented subsethood measure for IFSs based on the Hamming distance measure. Zhang et al. [34] defined subsethood measure for IFSs and IVFSs. Xie et al. [35] gave a new axiomatic definition and some inclusion measures for IFSs. Zhang et al. [36] introduced another new axiomatic definition and presented inclusion measure for IFSs.

Şahin and Küçük [37] proposed subsethood measure for single-valued neutrosophic sets (SVNSs) while Şahin and Karabacak [38] presented a subsethood measure for interval-valued neutrosophic sets (IVNS). Ji and Zhang [39] introduced a subsethood measure for IVNSs based on the Hausdorff distance measure. Zhang and Wang [40] proposed an inclusion measure for hesitant fuzzy sets (HFSs). Finally, Aydoğdu [41] introduced the very first subsethood measure for TDNSs (PFSs) as a conference proceeding for the first time in the literature.

3. Novel subsethood measures for TDNS

This In this section, we propose axioms of the definition of subsethood measure for TDNSs and some new subsethood measures for TDNSs based on the distance measures of TDNSs. To establish the subsethood degree to which A belongs to B, we use the distance between TDNSs A and $A \cap B$. d_1, d_2, and d_3 distance measures are given in Eqs. (3-5) in Chapter 2.1.

Definition 2. Let X be a universe of discourse. A mapping $S: TDNS(X) \times TDNS(X) \to [0,1]$ is called subsethood measure if it satisfies the following properties. For all $A, B, C \in TDNS(X)$,

1. $S(A,B) = 1$ iff $A \subseteq B$,
2. $S(A,A^c) = 1 \Leftrightarrow \mu_A(x) \le \nu_A(x)$,
3. $S(A,B) = 0$ if $A = \langle x, 1,0,0\rangle$ and $B = \langle x, 0,0,1\rangle$,
4. If $A \subseteq B \subseteq C$, then $S(C,A) \le S(B,A)$ and $S(C,A) \le S(C,B)$.

The following theorem gives the subsethood measures based on distance measures.

Theorem: Let X be a universe of discourse. For $A, B \in TDNS(X)$, the mappings

$$S_1(A,B) = 1 - \frac{1}{\sqrt{2}} d_1(A, A \cap B) \tag{6}$$

$$S_2(A,B) = 1 - d_2(A, A \cap B) \tag{7}$$

$$S_3(A,B) = 1 - \frac{1}{\sqrt{n}} d_3(A, A \cap B) \tag{8}$$

are subsethood measures for TDNSs.

Proof: In order that $S_i(A,B)$ $(i = 1,2,3)$ to be described as a subsethood measure for TDNSs, it must satisfy the properties of Definition 2. For simplicity, we only prove that $S_1(A,B)$ satisfies these properties. $S_2(A,B)$ and $S_3(A,B)$ may also be shown in the same fashion.

Let $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$ and $B = \{(x, \mu_B(x), \eta_B(x), \nu_B(x)) \mid x \in X\}$ be two TDNSs. Since $A^c = \{(x, \nu_A(x), \eta_A(x), \mu_A(x)) \mid x \in X\}$, we have $A \cap A^c = \{(x, \min(\mu_A(x), \mu_{A^c}(x) = \nu_A(x)), \min(\eta_A(x), \eta_{A^c}(x) = \eta_A(x)), \max(\nu_A(x), \nu_{A^c}(x) = \mu_A(x))) \mid x \in X\}$.

1. Let $A \subseteq B$, then $A \cap B = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$.

$$S_1(A,B) = 1 - \frac{1}{\sqrt{2}} d_1(A, A \cap B)$$

$$= 1 - \frac{1}{\sqrt{2}} \left[\frac{1}{n}\sum_{i=1}^{n}\left(\left(\mu_A(x_i) - \mu_{A\cap B}(x_i)\right)^2 + \left(\eta_A(x_i) - \eta_{A\cap B}(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_{A\cap B}(x_i)\right)^2\right)\right]^{\frac{1}{2}}$$

$$= 1.$$

Conversely, suppose that $S_1(A,B) = 1$, then $d_1(A, A \cap B) = 0$. So $\mu_A(x) = \mu_{A\cap B}(x)$, $\eta_A(x) = \eta_{A\cap B}(x)$ and $\nu_A(x) = \nu_{A\cap B}(x)$. Because of the definition of intersection and inclusion of TDNSs, TDNS A is a subset of TDNS B.

2. If $\mu_A(x) \le \nu_A(x)$, then

$$A \cap A^c = \{(x, \min(\mu_A(x), \nu_A(x)), \min(\eta_A(x), \eta_A(x)), \max(\nu_A(x), \mu_A(x))) \mid x \in X\}$$
$$= \{(x, \mu_A(x), \eta_A(x), \nu_A(x))\} \ .$$

Thus,

$$S_1(A, A^c) = 1 - \frac{1}{\sqrt{2}} d_1(A, A \cap A^c)$$

$$= 1 - \frac{1}{\sqrt{2}}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\left(\mu_A(x_i) - \mu_{A\cap A^c}(x_i)\right)^2 + \left(\eta_A(x_i) - \eta_{A\cap A^c}(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_{A\cap A^c}(x_i)\right)^2\right)\right]^{\frac{1}{2}}$$

$$= 1 - \frac{1}{\sqrt{2}}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\left(\mu_A(x_i) - \mu_A(x_i)\right)^2 + \left(\eta_A(x_i) - \eta_A(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_A(x_i)\right)^2\right)\right]^{\frac{1}{2}} = 1$$

3. For $A = \{\langle x, 1,0,0\rangle\}$ and $B = \{\langle x, 0,0,1\rangle\}$, we have $A \cap B = \{\langle x, 0,0,1\rangle\}$. Hence

$$S_1(A, B) = 1 - \frac{1}{\sqrt{2}} d_1(A, A \cap B)$$

$$= 1 - \frac{1}{\sqrt{2}}\left[\frac{1}{n}\sum_{i=1}^{n}((1-0)^2 + (0-0)^2 + (0-1)^2)\right]^{\frac{1}{2}}$$

$$= 0$$

4. To prove that $S_1(C, A) \leq S_1(B, A)$, it suffices to show $d_1(C, C \cap A) \geq d_1(B, B \cap A)$. Since $A \subseteq B \subseteq C$, $\mu_A(x) \leq \mu_B(x) \leq \mu_C(x)$, $\eta_A(x) \leq \eta_B(x) \leq \eta_C(x)$ and $\nu_A(x) \geq \nu_B(x) \geq \nu_C(x)$. We get

$$d_1(C, C \cap A) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\left(\mu_C(x_i) - \mu_{C\cap A}(x_i)\right)^2 + \left(\eta_C(x_i) - \eta_{C\cap A}(x_i)\right)^2 + \left(\nu_C(x_i) - \nu_{C\cap A}(x_i)\right)^2\right)\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{n}\sum_{i=1}^{n}\left(\left(\mu_C(x_i) - \mu_A(x_i)\right)^2 + \left(\eta_C(x_i) - \eta_A(x_i)\right)^2 + \left(\nu_C(x_i) - \nu_A(x_i)\right)^2\right)\right]^{\frac{1}{2}}$$

$$\geq \left[\frac{1}{n}\sum_{i=1}^{n}\left(\left(\mu_B(x_i) - \mu_A(x_i)\right)^2 + \left(\eta_B(x_i) - \eta_A(x_i)\right)^2 + \left(\nu_B(x_i) - \nu_A(x_i)\right)^2\right)\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{n}\sum_{i=1}^{n}\left(\left(\mu_B(x_i) - \mu_{B\cap A}(x_i)\right)^2 + \left(\eta_B(x_i) - \eta_{B\cap A}(x_i)\right)^2 + \left(\nu_B(x_i) - \nu_{B\cap A}(x_i)\right)^2\right)\right]^{\frac{1}{2}}$$

$$= d_1(B, B \cap A).$$

Similarly, it can be shown that $S(C, A) \leq S(C, B)$.

4. Subsethood measure-based totally dependent-neutrosophic set extension of EDAS (SM-TDN-EDAS)

EDAS is a distance-based MADM method like TOPSIS and VIKOR. The distances between each alternative and positive/negative ideal alternatives are computed and operationalized by the mentioned methods and then these distance measures are accepted as a criterion for reaching a decision about the rankings of alternatives. They include steps that are dedicated to obtaining or generating a positive and a negative ideal solution. In EDAS these probably complex and confusing steps are eliminated because the distance between alternative and the average solution is considered. Therefore, decision-analyst does not need to generate positive/negative ideal solutions but to compute the average performance scores of each attribute. Traditional EDAS uses two distinct measures: positive distance from average (PDA) and negative distance from average (NDA). Naturally, the decision reached should be based on higher positive distance and lower negative distance.

TDNS is one of the recent fuzzy concepts that can be used in MADM analysis in representing human judgments, opinions, or expertise. After a brief literature review, studies extending various MADM approaches into TDNS environment are exemplified and summarized in Table A1. In the first column, the studies are given while the second column shows the study's methodology which includes extension(s) of the MADM approach(es) under TDNS and the third column depicts the application of the study. As seen from the table, VIKOR (VlseKriterijumska Optimizacija I Kompromisno Resenje), EDAS, and TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) extensions have the majority. There are also TDNS extensions of ARAS, TODIM, MABAC, MULTIMOORA, and PROMETHEE II in a few studies. Also, it is found that there is no proposition integrating subsethood measure and EDAS under any kind of fuzzy sets as well as TDNSs.

This study has extended EDAS method under the totally dependent-neutrosophic (picture fuzzy) environment in a different manner from existing extensions that are summarized in Table A1 as a contribution to the literature. In this novelty, we propose to use the subsethood measures as decision criteria rather than PDA and NDA. Indeed, our basic aim is to show the applicability of subsethood measures in a MADM problem-solving methodology under TDNS environment. Additionally, it is seen that the EDAS method's mathematical part is smoothed since the calculation complexity is reduced by replacing the idea of measuring distances to the average solution with calculating subsethood degree to the average solution in which includes just one operation. Also, TDNS may provide a higher independence possibility to the decision-makers since it is allowed to express independent degrees for positive, negative, and hesitancy preferences. The refusal degrees can also be calculated as a fourth element.

In this novel extension, there are 5 steps explained below.

Step 1. Decision-makers (e=1,…,k) are asked to express their judgments about alternatives' (i=1,…,m) performances with respect to attributes (j=1,…,n). So, after collecting data from decision-makers, there will be k decision matrices $(X^1, X^2, \dots, X^k)$ in hand. The judgments are aggregated via an aggregation operator defined for TDNS. In this step, the decision-makers can be weighted according to their expertise (ω_e). $\langle \mu_{ij}^e, \eta_{ij}^e, v_{ij}^e \rangle$ depicts the linguistic evaluation of e^{th} decision-maker and $\langle \mu_{ij}, \eta_{ij}, v_{ij} \rangle$ represents the aggregated performance evaluation. For obtaining aggregated decision matrix (Eq. 10), the totally dependent-neutrosophic (picture fuzzy) weighted averaging (TDNWA) operator (Eq. 9) defined by Zhang et al. [16] is utilized.

$$X^{agg} = PFWA_\omega(X^1, X^2, \dots, X^k) = \bigoplus_{e=1}^{k} \omega_e X^e = \langle \mu_{ij}, \eta_{ij}, v_{ij} \rangle$$
$$= \left\{ 1 - \prod_{e=1}^{k} \left(1 - \mu_{ij}^e\right)^{\omega_e} , \prod_{e=1}^{k} \left(\eta_{ij}^e\right)^{\omega_e}, \prod_{e=1}^{k} \left(v_{ij}^e\right)^{\omega_e} \right\} \tag{9}$$

$$X^{agg} = \begin{bmatrix} \langle \mu_{11}, \eta_{11}, v_{11} \rangle & \cdots & \langle \mu_{1n}, \eta_{1n}, v_{1n} \rangle \\ \vdots & \ddots & \vdots \\ \langle \mu_{m1}, \eta_{m1}, v_{m1} \rangle & \cdots & \langle \mu_{mn}, \eta_{mn}, v_{mn} \rangle \end{bmatrix} \tag{10}$$

Step 2. The attributes included in any decision problem can be cost or benefit type. In order to convert any cost attribute to a benefit one, the positive and negative membership degrees should be replaced while the neutral membership degree keeps its value. This is called normalization.

After normalization, the weights of attributes representing the importance and significance of the attribute should be considered. There are 4 possibilities: *(i)* When the weights are already known as prior information, they can be used directly; *(ii)* When the decision-makers' preferences are important for the decision problem in hand, their expertise can be consulted and the subjective weights may be calculated via different approaches such as Analytic Hierarchy Process (AHP) or Analytic Network Process (ANP), etc.; *(iii)* When the subjectivity is not desired with the purpose of eliminating manipulation risk that may be originated from the decision-makers or when there is not enough time for data collection, the objective weights can be computed from the current data by referring to the methods such as entropy-based approaches or maximizing standard deviation method; *(iv)* If required, a mixture of objective and subjective methods can be used.

Independent from the methodology used, the weighted normalized decision matrix is obtained via Eq. (11) where w_j represents the weight of attribute *j*. For this weighting process, we utilized the weighting formula proposed by Jovcic et al. [42].

$$\langle \mu_{ij}^{w}, \eta_{ij}^{w}, v_{ij}^{w} \rangle = w_j * \langle \mu_{ij}, \eta_{ij}, v_{ij} \rangle = \langle 1-(1-\mu_{ij})^{w_j}, \eta_{ij}^{w_j}, (\eta_{ij}+v_{ij})^{w_j} - (\eta_{ij})^{w_j} \rangle \quad (11)$$

Step 3. The basic distinctive feature of EDAS is the consideration of average scores rather than positive or negative ideals. In this step, the TDN average scores of each attribute will be obtained. For this purpose, all the weighted aggregated performance scores depicted in columns are averaged. Firstly, the addition operation is used iteratively as given in Eq. (12).

$$\langle \mu_{1j}^{w}, \eta_{1j}^{w}, v_{1j}^{w} \rangle + \langle \mu_{2j}^{w}, \eta_{2j}^{w}, v_{2j}^{w} \rangle = \langle 1-\left(1-\mu_{1j}^{w}\right)\left(1-\mu_{2j}^{w}\right), \eta_{1j}^{w}\eta_{2j}^{w}, \left(\eta_{1j}^{w}+v_{ij}^{w}\right)\left(\eta_{2j}^{w}+v_{2j}^{w}\right) - \eta_{1j}^{w}\eta_{2j}^{w} \rangle \quad (12)$$

The sum of the overall TDN numbers is represented by $\langle \mu_{ij}^{sum}, \eta_{ij}^{sum}, v_{ij}^{sum} \rangle$ for each attribute *j*. Then, multiplication by a scalar $(\lambda = 1/m > 0)$ operation is used (Eq. 13). The mathematical operations are defined by Jovcic et al [42].

$$\begin{aligned} \widetilde{AV} = \langle \mu_j^{AV}, \eta_j^{AV}, v_j^{AV} \rangle &= \frac{1}{m} * \langle \mu_{ij}^{sum}, \eta_{ij}^{sum}, v_{ij}^{sum} \rangle \\ &= \langle 1-(1-\mu_{ij}^{sum})^{\frac{1}{m}}, (\eta_{ij}^{sum})^{\frac{1}{m}}, (\eta_{ij}^{sum}+v_{ij}^{sum})^{\frac{1}{m}} - (\eta_{ij}^{sum})^{\frac{1}{m}} \rangle \end{aligned} \quad (13)$$

Step 4. Rather than measuring the negative and positive distances from the average solution, this study proposes the usage of subsethood degrees. In this step, each alternative's subsethood degree to the average solution will be measured. For this purpose, one of the subsethood measures proposed in this study can be used alternately. They are rewritten with the appropriate notions in Eqs. (14-16). Suppose $\tilde{A}_i = \langle \mu_{ij}^{w}, \eta_{ij}^{w}, v_{ij}^{w} \rangle$ shows the TDN evaluation scores of alternative *i* and $\widetilde{AV} = \langle \mu_j^{AV}, \eta_j^{AV}, v_j^{AV} \rangle$ represents the average solution,

$$S_1\left(\tilde{A}_i, \widetilde{AV}\right) = 1 - \frac{1}{\sqrt{2}} d_1\left(\tilde{A}_i, \tilde{A}_i \cap \widetilde{AV}\right) \quad (14)$$

$$S_2\left(\tilde{A}_i, \widetilde{AV}\right) = 1 - d_2\left(\tilde{A}_i, \tilde{A}_i \cap \widetilde{AV}\right) \quad (15)$$

$$S_3\left(\tilde{A}_i, \widetilde{AV}\right) = 1 - \frac{1}{\sqrt{n}} d_3\left(\tilde{A}_i, \tilde{A}_i \cap \widetilde{AV}\right) \quad (16)$$

where

$$d_1\left(\tilde{A}_i, \tilde{A}_i \cap \widetilde{AV}\right) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} \left(\mu_{ij}^{w} - \min\left(\mu_{ij}^{w}, \mu_j^{AV}\right)\right)^2 \\ +\left(\eta_{ij}^{w} - \min\left(\eta_{ij}^{w}, \eta_j^{AV}\right)\right)^2 \\ +\left(v_{ij}^{w} - \max\left(v_{ij}^{w}, v_j^{AV}\right)\right)^2 \end{array}\right)\right]^{\frac{1}{2}} \quad (17)$$

$$d_2\left(\tilde{A}_i, \tilde{A}_i \cap \widetilde{AV}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\max\left\{\begin{array}{c} \left|\mu_{ij}^{w} - \min\left(\mu_{ij}^{w}, \mu_j^{AV}\right)\right|, \\ \left|\eta_{ij}^{w} - \min\left(\eta_{ij}^{w}, \eta_j^{AV}\right)\right|, \\ \left|v_{ij}^{w} - \max\left(v_{ij}^{w}, v_j^{AV}\right)\right| \end{array}\right\}\right) \quad (18)$$

$$d_3\left(\tilde{A}_i, \tilde{A}_i \cap \widetilde{AV}\right) = \left[\sum_{i=1}^{n}\left(\max\left\{\begin{array}{c} \left(\mu_{ij}^{w} - \min\left(\mu_{ij}^{w}, \mu_j^{AV}\right)\right)^2, \\ \left(\eta_{ij}^{w} - \min\left(\eta_{ij}^{w}, \eta_j^{AV}\right)\right)^2, \\ \left(v_{ij}^{w} - \max\left(v_{ij}^{w}, v_j^{AV}\right)\right)^2 \end{array}\right\}\right)\right]^{\frac{1}{2}} \quad (19)$$

Step 5. The decision-makers expect that the best alternative should have the lowest possibility of being a subset of the average solution since the average solution does not represent the ideal solution but a mean one. So, it is required that the subsethood measure between the best alternative and the average solution should be the lowest one. Thus, the alternatives are ranked in ascending order of their subsethood measures against average solution and it is decided that the alternative with the minimum subsethood measure is the best one.

5. A hypothetical application

In this study, we have aimed to develop a novel TDNS version of EDAS with the integration of subsethood degree instead of distances between alternatives and average solution. We have also tried to keep the computations totally dependent-neutrosophic (picture fuzzy) until the very end of the steps. The proposed SM-TDN-EDAS is here applied in a real case. This case is taken from Jovcic, et al. [42]. They used a TDNS version of ARAS (Additive Ratio Assessment) method to the freight distribution concept selection problem for a tire manufacturing company in the Czech Republic. They considered 5 experts' evaluations on 3 alternatives with respect to 23 sub-criteria under four main criteria. In order to show the applicability of our method proposition of SM-TDN-EDAS, we chose the environmental main criterion which includes 5 sub-criteria, namely air pollution, noise pollution, the effect on public health, energy consumption, and vehicle utilization. The alternatives are freight distribution by own transport fleet, freight distribution by the 3PL provider, and freight distribution by combining own transport fleet and 3PL services. They collected the data from experts and found the aggregated decision matrix of X^{agg} as given in Table 1. Here we have the aggregated decision matrix so that we did not apply TDNWA operator just for this case.

Table 1. Aggregated decision matrix (X^{agg})

	C_1			C_2			C_3			C_4			C_5		
A_1	0.2	0.4	0.2	0.4	0.2	0.2	0	0.6	0.2	0.8	0.2	0	0	0.2	0.8
A_2	0.4	0.4	0	0.2	0.4	0.2	0.2	0.4	0	0	0.2	0.8	0.8	0.2	0
A_3	0.4	0.4	0	0.2	0.6	0.2	0.2	0.4	0.2	0	0.4	0.6	0.4	0.4	0.2

The weights of attributes (w_j) are provided as 0.2593, 0.0963, 0.1333, 0.1407, 0.3704. There is no need for normalization since all the attributes have benefit features. The weighted matrix is found by operating Eq. (11) and is given in Table 2. For illustration purposes, the weighting of the first alternatives' scores concerning the first criterion is given as follows:

$$\langle \mu_{11}^{w}, \eta_{11}^{w}, v_{11}^{w} \rangle = 0.2593 * \langle 0.2,0.4,0.2 \rangle =$$
$$\langle 1-(1-0.2)^{0.2593}, 0.4^{0.2593}, (0.4+0.2)^{0.2593} - 0.4^{0.2593} \rangle = \langle 0.0562,0.7885,0.0874 \rangle$$

Referring to Eqs. (12-13), the average solution's performance scores with respect to each attribute are obtained. To illustrate, the average solution's performance score for attribute 1 is given:

- $\langle \mu_{11}^{w}, \eta_{11}^{w}, v_{11}^{w} \rangle + \langle \mu_{21}^{w}, \eta_{21}^{w}, v_{21}^{w} \rangle = \langle 0.0562,0.7885,0.0874 \rangle + \langle 0.1241,0.7885,0 \rangle =$ $\langle 1-(1-0.0562)(1-0.1241), 0.7885 * 0.7885, (0.7885+0.0874)(0.7885+0) - 0.7885 * 0.7885 \rangle = \langle 0.1733,0.6218,0.0689 \rangle$.
- $\langle 0.1733,0.6218,0.0689 \rangle + \langle \mu_{31}^{w}, \eta_{31}^{w}, v_{31}^{w} \rangle = \langle 0.1733,0.6218,0.0689 \rangle + \langle 0.1241,0.7885,0 \rangle = \langle 1-(1-0.1733)(1-0.1241), 0.6218 * 0.7885, (0.6218+0.0689)(0.7885+0) - 0.6218 * 0.7885 \rangle = \langle 0.2759,0.4903,0.0544 \rangle$.
- $\langle \mu_{1}^{AV}, \eta_{1}^{AV}, v_{1}^{AV} \rangle = \frac{1}{3} * \langle 0.2759,0.4903,0.0544 \rangle = \langle 1-(1-0.2759)^{\frac{1}{3}}, (0.4903)^{\frac{1}{3}}, (0.4903+0.0544)^{\frac{1}{3}} - (0.4903)^{\frac{1}{3}} \rangle = \langle 0.1020,0.7885,0.0281 \rangle$.

All the TDN values of the average solution are shown in the last row of Table 2. In the next phase, the subsethood measures of each alternative to the average solution are calculated. Eq. (14-16) defines three novel subsethood measures and we use all of them for comparison purposes. To illustrate, the first subsethood measure (Eq. 14) between $\tilde{A}_1$ and $\widetilde{AV}$ is:

$d_1(\tilde{A}_1, \tilde{A}_1 \cap \widetilde{AV}) = \Big[\frac{1}{5}((0.0562 - \min(0.0562, 0.1020))^2 + (0.7885 - \min(0.7885, 0.7885))^2 + (0.0874 - \max(0.0874, 0.0281))^2 + \cdots + (0 - \min(0, 0.2303))^2 + (0.5509 - \min(0.5509, 0.6002))^2 + (0.4491 - \max(0.4491, 0.1695))^2)\Big]^{\frac{1}{2}} = 0.0762.$

$S_1(\tilde{A}_i, \widetilde{AV}) = 1 - \frac{1}{\sqrt{2}} d_1(\tilde{A}_i, \tilde{A}_i \cap \widetilde{AV}) = 1 - \frac{1}{\sqrt{2}} * 0.0762 = 0.9461.$

Table 3 shows all the solutions including alternatives' distance values (please see Eqs.17-19) and subsethood measures for three different definitions (please see Eqs.14-16). In the last step, the alternatives are ranked in ascending order of subsethood measures: S_1, S_2, and S_3. For each measure, similar rankings of alternatives are obtained as seen from the columns of *Ranking* in Table 3. For S_1: $A_2 \succ A_1 \succ A_3$; for S_2 and S_3: $A_2 \succ A_3 \succ A_1$ which is the same ranking obtained by the original methodology. The results of other applications specified by Jovcic et al. [42] are summarized in Table 4 and it is seen that all these methods have given similar rankings. For each ranking, the most convenient alternative is found as A_2. The rankings of the other alternatives are slightly different in the various applications. For instance, in the original application, there are so many consecutive steps while our proposition of SM-TDN-EDAS includes just 5 steps. Our method's contribution to complexity reduction is obvious.

6. Conclusion and future work

Subsethood (inclusion) measures are very important components of fuzzy sets like entropy, distance, or similarity measures. In the literature, there are many subsethood measures developed for fuzzy sets, IFSs, and neutrosophic sets but there is no proposition for TDNSs. TDNS is generally accepted by the MADM field as one of the important fuzzy environments because it gives an extensive representation opportunity to the decision-maker. TDNS is defined by four elements, namely positive, negative, neutral, and refusal membership degrees and the first three elements can be independently assignable. The only rule is that the sum of these four elements should be equal to 1. In order to exploit this feature in the applications of MADM,

- for the first time in the literature, three subsethood measures were developed for TDNSs and it is proven that these definitions satisfy the required axiomatic properties;

Table 2. Weighted aggregated decision matrix ($w_j * X^{agg}$)

	C_1			C_2		
A_1	0.0562	0.7885	0.0874	0.0480	0.8564	0.0591
A_2	0.1241	0.7885	0.0000	0.0213	0.9155	0.0365
A_3	0.1241	0.7885	0.0000	0.0213	0.9520	0.0267
$\widetilde{AV}$	*0.1020*	*0.7885*	*0.0281*	*0.0303*	*0.9071*	*0.0413*
	C_3			C_4		
A_1	0.0000	0.9342	0.0365	0.2026	0.7974	0.0000
A_2	0.0293	0.8850	0.0000	0.0000	0.7974	0.2026
A_3	0.0293	0.8850	0.0492	0.0000	0.8790	0.1210
$\widetilde{AV}$	*0.0196*	*0.9011*	*0.0282*	*0.0727*	*0.8237*	*0.1036*
	C_5					
A_1	0.0000	0.5509	0.4491			
A_2	0.4491	0.5509	0.0000			

A_3	0.1724	0.7122	0.1154
$\widetilde{AV}$	*0.2303*	*0.6002*	*0.1695*

Table 3. Results

	d_1	S_1	Ranking	d_2	S_2	Ranking	d_3	S_3	Ranking	Ranking by Jovcic et al. [42]
A_1	0.0762	0.9461	2	0.0361	0.9639	3	0.1352	0.9395	3	3
A_2	0.1256	0.9112	1	0.0567	0.9433	1	0.2225	0.9005	1	1
A_3	0.0665	0.9529	3	0.0500	0.9500	2	0.1361	0.9392	2	2

Table 4. Comparison of different TDNS (PFS)-based MADM methods [42]

Method	Ranking of Alternatives
TDNS TOPSIS [43]	$A_2 \succ A_3 \succ A_1$
TDNS EDAS [16]	$A_2 \succ A_3 \succ A_1$
TDNS MABAC [44]	$A_2 \succ A_3 \succ A_1$
TDNS VIKOR [45]	$A_2 \succ A_3 \succ A_1$
TDNS Fuzzy TODIM [46]	$A_2 \succ A_1 \succ A_3$

EDAS, a well-known MADM approach is extended into TDNS in a different manner from the existing state-of-the-art propositions, i.e., the traditional and extended versions have focused on the distance between each alternative and average solution while the proposed version called SM-TDN-EDAS considered the subsethood degree of each alternative to the average solution as a decision criterion. So, the number of mathematical operations is significantly reduced in this new version;

To validate the novel SM-TDN-EDAS method, an application is conducted, and the resulting rankings are compared with different applications' rankings. It is found that the existing methods and current study give similar rankings. So, it is clear that the proposition is robust.

The study also needs some improvements. Rather than enforcing the decision-makers to allocate directly positive, neutral, and negative membership degrees, a further study may work on providing appropriate linguistic terms which have TDN number correspondences so that the data collection process is eased and becomes more practical. Also, novel aggregation operators, entropy measures, similarity, and distance measures as well as division and subtraction operators can be defined for the concept of TDNS.

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Appendix A

Table A1. Literature overview of MADM approaches under TDNS (PFS)

Paper	MADM Methods Used	Application
Zhang. et al. [16]	TDN(PF)-EDAS for selection	A numerical example for green supplier selection
Liang, et al. [17]	Extended SWARA for subjective criteria weights; Mean-squared deviation model for objective criteria weights; TDN(PF)-EDAS for establishing difference matrix; ELECTRE III for ranking orders; Extended MABAC, EDAS for comparison	Evaluating the cleaner production performance for gold mines in China
Li, et al. [18]	Maximizing deviation method for criteria weights; TDN(PF)-EDAS for evaluation	A numerical example of selecting an optimal emergency alternative
Ping, et al. [19]	TOPSIS & Maximum Entropy Theory for Expert Weighting; TDN(PF)-EDAS for evaluation; HF-VIKOR, cloud model GRA for comparison	A numerical example of characteristic prioritization in quality function deployment
Jovcic, et al. [42]	TDN(PF)-ARAS for selection; TDN(PF)-TOPSIS, TDN(PF)-EDAS, TDN(PF)-TODIM, TDN(PF)-VIKOR, TDN(PF)-MABAC, TDN(PF)-GRA for comparison	Freight distribution concept selection problem for a tire manufacturing company in the Czech Republic
Torun and Gördebil [43]	Fuzzy TOPSIS, IF-TOPSIS, and TDN(PF)-TOPSIS for comparison	Citizens' satisfaction level from public services in Turkey
Wang, et al. [44]	Modified maximizing deviation method for criteria weighting; prospect theory-based TDN(PF)-MABAC for evaluation; TDN(PF)-MABAC, TDN(PF)-VIKOR for comparison	Risk ranking of energy performance contracting project in Shanghai, China
Wang, et al. [45]	TDN(PF)-entropy-based objective weighting of attributes; TDN(PF)-normalized projection-based VIKOR for evaluation;	Risk evaluation of construction projects in China
Wei [46]	TDN(PF)-TODIM for evaluation	A numerical example of evaluation of emerging technology commercialization
Meksavang, et al. [47]	TDN(PF)-VIKOR for evaluation; fuzzy TOPSIS, IF-VIKOR, IF-GRA for comparison	A numerical example of sustainable supplier selection case in the beef supply chain
Si, et al. [48]	TDN(PF)-VIKOR & TDN(PF)-TOPSIS for evaluation	Ranking of tiger reserve national parks in India
Sindhu, et al. [49]	Linear programming for criteria weighting; TDN(PF)-TOPSIS for evaluation	A numerical example of human resource management

Zeng, et al. [50]	TDN(PF)-TOPSIS for evaluation	A numerical example of selecting Enterprise Resource Planning System
Arya and Kumar [51]	TDN(PF)-entropy-based TDN(PF)-VIKOR and TDN(PF)-TODIM for evaluation	Numerical examples based on election forecast through opinion polls
Joshi [52]	TDN(PF)-entropy-based TDN(PF)-VIKOR for evaluation; TOPSIS, VIKOR for comparison	Numerical examples based on election forecast through opinion polls
Joshi [53]	R-Norm information measure-based TDN(PF)-VIKOR for evaluation; TDN(PF)-TODIM for comparison	A numerical example of election; A numerical example of investment alternative evaluation
Lin, et al. [54]	TDN(PF)-entropy based criteria weighting; TDN(PF)-MULTIMOORA for evaluation; TDN(PF)-TODIM for comparison	Site selection of car-sharing station in Beijing, China
Tian, et al. [55]	Improved AHP for criteria weighting; TDN(PF)-PROMETHEE II for evaluation; TDN(PF)-VIKOR for comparison	Tourism environmental impact assessment in Hubei, China
Tian and Peng [56]	Improved ANP for criteria weighting; TDN(PF)-TODIM for evaluation	Personalized tourism attraction evaluation
Gül and Aydoğdu [57]	TDN(PF)-CODAS for evaluation; CODAS, spherical fuzzy CODAS, and spherical fuzzy TOPSIS for comparison	Selecting the best green supplier in Turkey
Simic, et al. [58]	CODAS, TOPSIS, EDAS, TODIM, VIKOR, MABAC, Cross-entropy, Projection, Grey relational projection, and Grey relational analysis under TDN(PF) environment	Locating a new vehicle shredding facility in the Republic of Serbia

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Theory on Duplicity of Finite Neutrosophic Rings

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Abstract

This article introduces the notion of duplex elements of the finite rings and corresponding neutrosophic rings. The authors establish duplex ring $Dup(R)$ and neutrosophic duplex ring $Dup\left(R(I)\right)$ by way of various illustrations. The tables of different duplicities are constructed to reveal the comparison between rings $Dup(Z_n)$, $Dup\left(Dup(Z_n)\right)$ and $Dup\left(Dup\left(Dup(Z_n)\right)\right)$ for the cyclic ring Z_n. The proposed duplicity structures have several algebraic systems with dissimilar consequences. Author's characterize finite rings with $R+R$ is different from the duplex ring $Dup(R)$. However, this characterization supports that $R+R=Dup(R)$ for some well known rings, namely zero rings and finite fields.

Keywords: Multiplicative function, Duplex form; Duplex ring, neutrosophic duplex element, neutrosophic duplex ring

1. Introduction

In the most general sense, elementary number theory deals with and manages the results and properties of different sets of numbers. In this paper, we will examine and discuss some significant sets of numbers in Z_n, called duplexity. We will briefly present the notion of duplexity of Z_n and enumerate how many number of duplex elements are there in Z_n. For the integer x, the element form $x+x$ is called a *duplex form* of x. The most important problem in the elementary theory of integers is to determine the possible forms of duplexes among the integers. For instance, it is clear to see that any duplex form must be of form $2k$, or $2k+2$ in Z, because every even integer is a multiple of 2. This illustration specifies that the ring of integers Z satisfies the conditions: $x+x=2x$, $x+2x=3x$ and so on, but $Z+Z\neq 2Z$, $Z+2Z\neq 3Z$ and so on, where the operation addition '$+$'defined on Z. In general, a duplex form $x+x$ exists in the ring Z of integers. Now, we shall study the enumeration of duplex elements in the finite commutative ring Z_n, and which are finitely many duplex forms $x+x$ in Z_n, where the operation addition '$+$' defined on Z_n.

First, we can generally describe a ring R is an algebraic structure $(R,+,\cdot)$ as an additive abelian group with a multiplicative binary operation such that the structure $(R,+,\cdot)$ is associative and fulfils distributive axioms $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$. A ring R is finite commutative if $|R|<\infty$ and $ab=ba$ for all a, b in R, see [1]. An element u in a commutative ring with unity 1 is called a unit if there exists an element x in R such that $xu=1=ux$, and specifically x is called a multiplicative inverse of u, and vice versa.

All the elements in R which are not multiplicative inverse elements are said to be zero divisors. Note that set of units and zero divisors of R are usually denoted by $R^{\times}$ and $Z(R)$, respectively, and R can be partitioned by the disjoint sets $Z(R)$ and $R^{\times}$. For any subring R of, the set R/S denotes the quotient ring. Now our attention is shifted to focus on the ring Z_n which is isomorphic to the quotient ring Z/nZ, which are the main tools in this paper for various values of $n \geq 1$. For $a, n \in Z$ with $n > 0$, we represent the congruence class of modulo n by the notion $[a]_n$, and the ring Z_n is the set $\{[a]_n : a \in Z\}$, or equivalently $\{[0],[1],[2],...,[n-1]\}$.

But there is a one-to-one correspondence between the complete residue systems $\{[0],[1],[2],...,[n-1]\}$ and $\{0,1,2,...,n-1\}$, and thus Z_n can be written simply as $Z_n = \{0,1,2,...,n-1\}$ for complete residue systems modulo n [2]. It is worth clarifying that the ring $Z_n = \{0,1,2,...,n-1\}$ is a commutative ring with unity 1 under addition and multiplication modulo n. We are happy to say that the ring Z_n has countably many applications in various fields such as algebraic number theory, algebraic coding theory, Cryptography, algebraic circuit theory, Antenna theory and algebraic design theory [3-8]. Further, the problem of enumeration of various types of elements in Z_n up to countably finite has received considerable attention in recent years; see for examples [9-14].

Now starts the basic notions, definitions and results of classical rings.

Let R be a finite commutative ring with nonzero identity and $R^{\times}$ be the set of group units of R. Given a finite commutative ring R, the ring $R+R = \{r+r : r \in R, r \in R\}$ is known as *duplex form* of R. However, the problem of characterizing finite commutative rings up to isomorphism has established considerable attention in recent years (see 15 and 16) initiating from the research works of Eldridge [17]. In this chapter, authors characterize finite commutative rings in terms of their duplexes. First, write the notion $Char(R)$ to denote a positive integer n such that $na = 0$ for every a in R, where $na = a + a + ... + a$ (n copies). Recall that the ring Z_n is a finite commutative ring with nonzero unity 1 under addition and multiplication modulo n. Also, the number of the form $a + ib$, $a, b \in Z_n$, is called Gaussian integer, and the set of Gaussian integers represented by $Z_n[i]$, and defined as $Z_n[i] = \{a + bi : a, b \in Z_n, i^2 = -1\}$. Further, note that $|Z_n| = n$ and $|Z_n[i]| = n^2$.

Neutrosophic Duplex elements are the solutions of some specific neutrosophic equation, and which are main mathematical tools for studying additive elements and their additive reciprocals of an object and their mutual symmetries, which are logically related to neutrosophic systems and their automorphisms. The characterizations of the duplex elements of any finite commutative ring have not been done in general theory of neutrosophic mathematics. But in recent years, the interplay between additive self inverses and group units of a classical ring and its corresponding neutrosophic ring was studied by Chalapathi and co-authors [18-20].

Now reconsider some notations, preliminaries and results of neutrosophic ring theory.

Let 0, 1 and I be three distinct components of any neutrosophic logical system with $0^2 = 0$, $1^2 = 1$ and $I^2 = I$. Then the component I is called the *indeterminate* of a system with some specific algebraic axioms: $0I = 0$, $1I = I$, $I + I = 2I$, and I^{-1} does not exists under usual neutrosophic addition and neutrosophic Multiplication defined on the required system. The component I is a concrete mathematical tool to deal with inconsistent, incomplete and indeterminate information which exist in the real world systems. A nonempty set

N together with I is denoted by $N(I)$ and defined as $N(I)=\{a+bI : a,b\in N, I^2=I\}$, which is called neutrosophic set. Neutrosophic is an innovative research field of philosophy with the composition of indeterminacy founded by Smarandache to develop and deal indeterminacy of a system in nature and science [21]. In addition, the neutrosophic set and their interactions play an important role in classical and modern algebra, and generate a specific theory in modern mathematics called *neutrosophic algebraic theory*, and it contains many algebraic structures, like neutrosophic groups, neutrosophic rings, neutrosophic Boolean rings, neutrosophic zero rings and neutrosophic field [22-25]. First, classical rings and their useful results are standard and follow those from [26]. Next, the other neutrosophic concepts and further terminology with corresponding notions will be explained in detail as follows. For any finite commutative ring R, the nonempty neutrosophic set $R(I)=\{a+bI : a,b\in R, I^2=I\}$ is called a Neutrosophic ring generated by R and I under the following neutrosophic binary operations:

$$(a+bI)+(c+dI)=(a+c)+(b+d)I,\ (a+bI)(c+dI)=(ac)+(ad+bc+bd)I.$$

Particularly, $0=0+0I$, $1=1+0I$, $I=0+1I$ are main components of the neutrosophic ring $R(I)$ with $R(I)=R+RI=\langle R\cup I\rangle$. Note that, if R is finite, and then $|R|$ denotes the number of elements in R, consequently that $|R(I)|=|R|^2$.

The contributions of this manuscript are three folds.

First, we propose the use of modular arithmetic to determine the duplex elements for the finite ring Z_n. The number of duplex elements $D(n)$ over Z_n is distributed in $Z_m\times Z_n$. Thus, the enumeration of this procedure is suitable for enumerating the number of duplex elements in $Z_m\times Z_n$. *Second*, we thoroughly characterize finite rings over their duplicities. We provide necessary constructive conditions on various finite rings and weights to achieve their related consequences. *Third* and finally, we establish systematic procedure to construct neutrosophic duplex rings over given classical rings. We prove that neutrosophic duplex rings generated by our basic neutrosophic rule $R(I)=R+RI=\langle R\cup I\rangle$ exhibit a specific structure, and maintain the basic neutrosophic properties of $R(I)$.

2. **Enumeration of Duplex Elements in Z_n**

As the heading suggests, the present section has as its goal is another simple contribution of Z_n, called duplex of Z_n. For those who consider the theory of integers and basic number theory. The intrinsic beauty of the duplex of Z_n has a strange fascination for modern mathematicians. Generally speaking, the duplex of Z_n deals with the characterization of Z_n with $Z_n+Z_n\neq 2Z_n$, or $Z_n+Z_n=2Z_n$.

This section enumerates all duplex elements which are in Z_n, and also demonstrate a number-theoretic connection between the finite number of positive integers and duplex elements in Z_n. Also, this section generates the function $D(n)$ which is a multiplicative function but not complete. Additionally, prove that $|D(Z_n)|=\dfrac{n}{(2,n)}$ and $|D(Z_m\times Z_n)|=\dfrac{mn}{(2,m)(2,n)}$.

Before moving on to the other important concepts and results of the duplex of Z_n, let us define duplex elements of Z_n with different illustrations.

First, we prove that $D(n)$ is a multiplicative function but not complete with an illustration.

Definition 2.1.
An element a in Z_n is called a *duplex element* in Z_n if and only if the equation $x + x = a$ has a solution in Z_n.

The set of all duplex elements in Z_n is denoted by $D(Z_n)$, and $D(n)$ denotes the number of duplex elements in Z_n with $D(n) \neq 0$, since $x + x = 0$ is solvable in Z_n. The function $D(n)$ is called the *duplex function* of n.

For any $n > 1$, we have $D(Z_{2n}) \neq Z_{2n}$ but $D(Z_{2n-1}) = Z_{2n-1}$. This means that the units of Z_{2n} are not the duplex elements in Z_{2n}. For example $D(8) = 4$ since the equations $x + x = 0$, $x + x = 2$, $x + x = 4$ and $x + x = 6$ have an individual solution in Z_8, but $x + x = 1$, $x + x = 3$, $x + x = 5$ and $x + x = 7$ do not have a solution in Z_8.

The following table illustrates the number of duplex elements in Z_1, Z_2, Z_3,..., Z_{10}, respectively.

n	1	2	3	4	5	6	7	8	9	10
$D(n)$	1	1	3	2	5	3	7	4	9	5

For any positive integers m and n, the notation $gcd(m,n)$, or (m,n) denotes the greatest common divisor of m and n. Particularly, $gcd(m,n) = 1$ if and only if m and n are called relatively prime. Suppose $gcd(m,n) = 1$. Then the function $f: N \rightarrow \mathbb{R}$ is called a Number-Theoretic function, and it is called *multiplicative* if $f(mn) = f(m)f(n)$. Naturally, many number-theoretic functions exist in the theory of numbers [2] and which are completely characterized by its value of n when $n \geq 1$. Now we show that the duplex function $D(n)$ is a *multiplicative* function.

Theorem 2.2. Let $gcd(m,n) = 1$. Then the number theoretic relation is $D(mn) = D(m)D(n)$.

Proof: First of all we adopt the notation: $D(mn)$ is the number of duplex elements in Z_{mn} and $D(m)D(n)$ is the number of duplex elements in $Z_m \times Z_n$. Because $gcd(m,n) = 1$, the ring Z_{mn} is isomorphic to the ring $Z_m \times Z_n$ by the ring isomorphism ψ: $Z_{mn} \rightarrow Z_m \times Z_n$ related by $\psi(t) = (t \ mod \ m, t \ mod \ n)$ for every element t in Z_{mn} (see [1]).

First, we prove that $D(mn) \leq D(m)D(n)$. For this let a be a duplex element in Z_{mn}, then the equation $x + x = a$ is solvable in Z_{mn}. Consequently, there is an element b in Z_{mn} such that $b + b = a$ is solvable in Z_{mn}. Since ψ is an injective map from Z_{mn} onto $Z_m \times Z_n$, so there exists an element (x,y) in $Z_m \times Z_n$ such that $\psi(b) = (x,y)$. Therefore,

$$\psi(a) = \psi(b + b) = \psi(b) + \psi(b) = (x,y) + (x,y) = (x + x, y + y)$$

is solvable in $Z_m \times Z_n$. This implies that $\psi(a)$ is also a duplex element in $Z_m \times Z_n$. Hence, $D(mn) \leq D(m)D(n)$. On the other hand, we can show that $D(mn) \geq D(m)D(n)$. Suppose c is a duplex element in Z_m and d is a duplex element in Z_n. Then there exists u in Z_m and v in Z_n such that

$$(u + u, v + v) = (c,d) \text{ in } Z_m \times Z_n.$$

So, we have

$\psi^{-1}[(c,d)] = \psi^{-1}[(u + u, v + v)] = \psi^{-1}[(u,v) + (u,v)] = \psi^{-1}[(u,v)] + \psi^{-1}[(u,v)]$ is solvable in Z_{mn}. This implies that the element $\psi^{-1}[(c,d)]$ is also a duplex element in Z_{mn}. This shows that $D(mn) \geq$

$D(m)D(n)$. Combination of inequalities $D(mn) \leq D(m)D(n)$ and $D(mn) \geq D(m)D(n)$ yields that the equality $D(mn) = D(m)D(n)$, and this shows that $D(n)$ is a number-theoretic multiplicative function. ∎

Now we continue our study by verifying other generalizations of duplex function. This requires the following:

Example 2.3. Consider $m = 2, n = 4$, we find that $D(2) = 1, D(4) = 2$ and $D(8) = 4$ with $D(8) \neq D(2)D(4)$.

Corollary 2.4. Prove that $D(1) = 1$.

Proof: Because of $0 + 0 = 0$, the element 0 is a duplex element in Z_n. So there exists an n such that $D(n) \neq 0$. But by the Theorem ◎2.2◎

$$D(n) = D(n1) = D(n)D(1).$$

Being $D(n)$ non-zero, $D(n)$ may be cancelled from both sides of the above equation to give $D(1) = 1$. ∎

The following theorem plays an important role in studying the duplexity of the ring Z_n.

Theorem 2.5. For every $n \geq 1$, the units of Z_{2n} are not the duplex elements in Z_{2n}.

Proof: Suppose $u \in Z_{2n}^{\times}$ be a duplex element in Z_{2n}. Then there exists an element x in Z_{2n} such that $x + x = u$ is solvable in Z_{2n}. By the basic celebrations of Z_{2n}, the number $2n$ divides the element $x + x - u$. So, there exists q in Z such that $x + x - u = 2nq$. But $u \epsilon Z_{2n}^{\times}$ implies that $gcd(u, 2n) = 1$, and it implies that $gcd(x + x - 2nq,\ 2n) = 1$, which is not true because $gcd(x + x - 2nq,\ 2n) > 1$ for every x in Z_{2n}. Hence every unit in Z_{2n} is not a duplex element in Z_{2n}. Particularly, $D(Z_{2n}^{\times}) = \emptyset$. ∎

Our next goal is to establish a formula for enumerating the number of duplex elements in Z_n. Once this is established, enumerating formulas in a simple form for the different values of n will complete our enumerating procedure. We start with the trivial observation that the duplex element in Z_1 is 0, so that $D(1) = 1$ because $x + x = 0$ is solvable in Z_1. We are now ready to prove that $D(2^n) = 2^{n-1}$, where $n \geq 2$. Because $x + x = 2x(mod\ \ 2^n)$ for all x in Z_{2^n}, it follows that the duplex element in Z_2n is a multiple of 2 under multiplication modulo 2^n, but the total number of multiples of 2 in Z_{2^n}, is 2^{n-1} since $2^n + 2^n \equiv 0(mod\ \ 2^n)$ and thus $D(2^n) = 2^{n-1}$.

Further, we start with the simple observation that for every x in Z_{p^n}, where $p > 2$ is a prime. This concludes that every element in Z_{p^n} is a duplex element in Z_{p^n}, and thus $D(p^n) = p^n$. Finally, we aim to establish a formula for enumeration number of duplex elements in Z_n whenever $n \geq 1$. For every x in Z_n, we have

$$(2, n)x = (2x, nx) ◎ a(2x) + b(nx) \text{ for some } a \text{ and } b \text{ in } Z_n$$

$$◎\ 2ax \text{ in } Z_n ◎\ ax + ax \text{ in } Z_n.$$

This observation shows that x is a duplex element in Z_n if and only if $(2, n)x$ is also a duplex element in Z_n.

As we explored duplex elements in Z_n we were led to specify how many there are. We found the answer in the following way.

Theorem 2.6. The number of duplex elements in Z_n is $D(n) = \frac{n}{(2,n)}$.

Proof: Suppose there is an element x in Z_n such that the duplex form $x + x$ can be written as $x + x = nq + (2, n)r$ in Z. By the Bezout's Theorem (ref.[2]),

$$x + x = nq + (2x + ny)r \quad \text{for some } x, y \text{ in } Z.$$

$$◎\ nq + 2xr + nyr ◎\ n(q + yr) + 2xr.$$

Now $x + x < n$, so $x + x = 2xr$ is a duplex in Z_n. Conversely, suppose that there is an element y in Z_n such that $y + y = mn + (2,n)s$ in Z. Then the number $(2,n)$ divides y. Thus there is an element t such that $y = (2,n)t$, and hence an element $(2,n)t$ is a duplex element in Z_n. Therefore the number of duplex elements in Z_n is

$$D(n) = \frac{|Z_n|}{(2,n)} = \frac{n}{(2,n)} \cdot \blacksquare$$

The following example demonstrates the preceding theorem.

Example 2.7. Because $(2,n) = 1$ *or* 2, the number of duplex elements in Z_9 is 9 and the number of duplex elements in Z_{10} is 5.

Our next aim is to enumerate the number of duplex elements in the ring $Z_m \times Z_n$ for every positive integer m and n. We recall that $Z_m \times Z_n \cong Z_{mn}$ if and only if $(m,n) = 1$. This relation explores that $D(mn) = D(m)D(n)$. Further, if $(m,n) \neq 1$, then by the Theorem [2.2] the number of duplex elements in $Z_m \times Z_n$ is

$$D(m)D(n) = \frac{m}{(2,m)} \cdot \frac{n}{(2,n)} = \frac{mn}{(2,m)(2,n)}\cdot$$

However, we observe that

$$D(Z_m \times Z_n) = D(Z_m) \times D(Z_n) \neq D(Z_{mn}) \text{ whenever} (m,n) \neq 1.$$

Subsequently, $D(m)D(n) = \frac{mn}{(2,m)(2,n)}$ is not equal to $D(mn) = \frac{mn}{(2,mn)}$. For instance, $D(2) = \frac{2}{(2,2)} = 1$,

$D(4) = \frac{4}{(2,4)} = 2$, $D(8) = \frac{8}{(2,8)} = 4$ but $D(2 \cdot 4) \neq D(2)D(4)$.

Theorem 2.8. Let $m, n \in N$. Then $D(Z_m \times Z_n) = D(Z_m) \times D(Z_n)$. Particularly, we have $|D(Z_m \times Z_n)| = \frac{mn}{(2,m)(2,n)}$.

Proof: Because of Definition [2.1] the duplex of $Z_m \times Z_n$ is defined as

$D(Z_m \times Z_n) = \{(a,b) \epsilon\ Z_m \times Z_n : (x,y) + (x,y) = (a,b) \text{ is solvable in } Z_m \times Z_n\}$

$= \{(a,b) \epsilon\ Z_m \times Z_n : (x + x, y + y) = (a,b) \text{ is solvable in } Z_m \times Z_n\}$

$= \{(a,b) \epsilon\ Z_m : x + x = a \text{ is solvable in } Z_m\} \times \{(a,b) \epsilon\ Z_n : y + y = b \text{ is solvable in } Z_n\}$

$= D(Z_m) \times D(Z_n)$.

This result has summarized the cardinality of $D(Z_m) \times D(Z_n)$. So, we have

$$|D(Z_m \times Z_n)| = |D(Z_m) \times D(Z_n)| = |D(Z_m)||D(Z_n)| = \frac{m}{(2,m)} \cdot \frac{n}{(2,n)} = \frac{mn}{(2,m)(2,n)}\cdot$$

3. Duplicity of finite Rings

A ring R is *cyclic* if the structure $(R,+)$ is a cyclic group, where the additive operation $+$ is defined over the ring R. In [27] the author Buck introduced a special ring structure, called cyclic ring. This algebraic structure establishes various results and it explore different algebraic concepts. Generally, every cyclic ring is commutative but it is a ring with unity or without unity. For instance, Z_9 is a cyclic ring with unity but $R^0 = \{0,3,6\}$ is also a cyclic ring without unity under addition and multiplication modulo 9. Further, if R is a cyclic ring then obviously the Cartesian product ring $R \times R$ is not a cyclic ring. For instance, $Z_9 \times Z_9$ is not a cyclic ring, in view of the fact that the structure $(Z_9 \times Z_9, +)$ is not a cyclic group under addition modulo 9. Throughout the paper, authors consider the ring Z_n as a cyclic ring of order n.

Recall that the element $x+x$ is called duplex form an element in Z_n under addition and multiplication defined over Z_n. Under this duplex form, we explore the following connections over Z_n : $x+x=2x$, $x+x+x=3x$ and so on, but $Z_n + Z_n \neq 2Z_n$, $Z_n + Z_n + Z_n \neq 3Z_n$, and so on, where the addition '$+$' defined over the ring Z_n. Now summarize these concepts in the following definitions.

Definition 3.1. An element a in a ring R is called duplex element in R if the equation $x+x=a$ has a solution in R.
For instance, the element 0 is a duplex element in every ring R, since $x+x=0$ is solvable in R.

Definition 3.2. The duplex ring of a ring R is denoted by $Dup(R)$ and defined as

$$Dup(R) = \{a : x+x=a \text{ is solvable in } R\}.$$

For instance, the following short table illustrates the duplex rings of the rings $Z_1, Z_2, \ldots, Z_{10}$.

R	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
$Dup(R)$	Z_1	$\{0\}$	Z_3	$\{0,2\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0,2,4,6\}$	Z_9	$\{0,2,4,6,8\}$

With this information available, it is an easy task to prove the following result.

Theorem 3.3. The duplicity of R is a subring of R.

Proof. Let a and b be any two elements in $Dup(R)$. Then there exists x and y in R such that $a = x+x$ and $b = y+y$. It is clear that

$$a+b = (x+x)+(y+y) = (x+y)+(x+y), \quad ab = (x+x)(y+y) = xy+xy+xy+xy = (xy+xy)+(xy+xy),$$

which shows that $a+b$ and ab are both elements in $Dup(R)$, and thus $Dup(R)$ is a subring of R. ■

With the support of the preceding theorem, let us define duplex of duplex.

Definition 3.4. The duplex of duplex of a ring R is denoted by $Dup(Dup(R))$ and defined as

$$Dup(Dup(R)) = \{d \in Dup(R) : x+x=d \text{ is solvable in } Dup(R)\}.$$

Similarly, define $Dup(Dup(Dup(R)))$ as follows.

$$Dup(Dup(Dup(R))) = \{y \in Dup(Dup(R)) : x+x=y \text{ is solvable in } Dup(Dup(R))\}.$$

These notions lead directly to the following tabular information.

R	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
$Dup(R)$	Z_1	$\{0\}$	Z_3	$\{0,2\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0,2,4,6\}$	Z_9	$\{0,2,4,6,8\}$
$Dup(Dup(R))$	Z_1	$\{0\}$	Z_3	$\{0\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0,4\}$	Z_9	$\{0,2,4,6,8\}$
$Dup(Dup(Dup(R)))$	Z_1	$\{0\}$	Z_3	$\{0\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0\}$	Z_9	$\{0,2,4,6,8\}$

In vision of the preceding table, authors conclude the following.

1. $Dup(Dup(Dup(Z_n))) = Z_n \Leftrightarrow n$ is odd.

2. $Dup\left(Dup\left(Dup(Z_n)\right)\right) = Dup\left(Z_n\right) \Leftrightarrow n$ is a perfect square.

3. $Dup\left(Dup\left(Dup(Z_n)\right)\right) = \left(0\right) \Leftrightarrow n = 2^k$ for some positive integer k .

Under this information, the following theorem provides the structure of duplex ring of the ring Z_n .

Theorem 3.5. For any positive integer n , there exists duplex ring $Dup(Z_n)$ of the ring Z_n such that $Dup(Z_n) = (2,n)Z_n$, where $(2,n)$ is the greatest common divisor of the numbers 2 and n .

Proof. It is well known that $Z_n + Z_n \neq 2Z_n$ but the duplex equation $x + x = a$ is solvable in the ring Z_n for every positive integer n . So, the calculations

$$Dup(Z_n) = (2,n)\left(\frac{2}{(2,n)}\right)Z_n \subseteq (2,n)Z_n$$

conform that the first set inclusion $Dup(Z_n) \subseteq (2,n)Z_n$ is true. Before proving another way of this result, consider Bezout's Theorem [2], the number $(2,n)$ can be written as $(2,n) = 2x + ny$ for some integers x and y . Applying this Bezout's result,

$$(2,n)Z_n = (2x+ny)Z_n = 2xZ_n, \text{ since } ny \equiv 0(\text{mod}\, n)$$
$$= (x+x)Z_n \subseteq Dup(Z_n).$$

Two set inclusions $Dup(Z_n) \subseteq (2,n)Z_n$ and $(2,n)Z_n \subseteq Dup(Z_n)$ finalize that the duplicity of the ring Z_n as $Dup(Z_n) = (2,n)Z_n$.■

As an immediate application of preceding theorem, authors deduce the following results.

Corollary 3.6. $Dup(Z_n) = Z_n$ if and only if n is odd.

Proof. Noting that $(2,n) = 1$ if and only if n is odd, so may write $Dup(Z_n) \subseteq (2,n)Z_n = 1Z_n = Z_n$.■

In the same way, the relation $Dup(Z_n) \subseteq (2,n)Z_n$ yields the following corollary, and it is another basic fact regarding the order of the duplex ring $Dup(Z_n)$.

Corollary 3.7. Let $n \in N$. Then the cardinality of the duplex ring $Dup(Z_n)$ is $\left|Dup(Z_n)\right| = \dfrac{n}{(2,n)}$.

Proof. It is clear from the Theorem ◎section2◎ and additionally there is a one to one correspondence $(a+a) \mapsto \dfrac{n}{(2,n)}a$ for every element a in Z_n .■

Theorem 3.8. Let $m,n \in N$. Then $Dup(Z_m \times Z_n) = (2,m)(2,n)(Z_m \times Z_n)$.

Proof. By the Theorem ◎3.5◎ we have $Dup(Z_m) = (2,m)Z_m$ and $Dup(Z_n) = (2,n)Z_n$. So, it is clear from the calculations $Dup(Z_m \times Z_n) = Dup(Z_m) \times Dup(Z_n) = (2,m)Z_m \times (2,n)Z_n = (2,m)(2,n)(Z_m \times Z_n)$.■

There is an attractive illustration of the finite fields. First, notice that $Dup(Z_2) \neq Z_2$. For any odd prime, it is well known that $Dup(Z_p) = Z_p$. Particularly, if $p \equiv 3(\text{mod}\, 4)$ then $Z_p[i]$ is a field of Gaussian integers and $Dup\left(Z_p[i]\right) = Z_p[i]$. Even if R is not a field then there exists R such that $Dup(R) = R$. For instance, $Z_p \times Z_p$

is not a field but $Dup\left(Z_p \times Z_p\right) = Z_p \times Z_p$. Further, there is another attractive ring R with $Dup(R) = R$, those types of rings are called zero rings and it is denoted by $R = R^0$. Now, we show that $Dup\left(R^0\right) = R^0$.

Theorem 3.9. The duplex ring of any zero rings is itself a zero ring.

Proof. The theorem is certainly true for $\left|R^0\right| = 1$, because $\left|R^0\right| = 1$ if and only if $R^0 = (0)$. Thus we may hereafter restrict our attention to nontrivial zero ring $R^0 \neq (0)$. Let $\left|R^0\right| > 1$. Then we have to prove that $Dup\left(R^0\right) = R^0$. By virtue of the first theorem of this section, $Dup\left(R^0\right) \subseteq R^0$.we makes a start by showing that $R^0 \subseteq Dup\left(R^0\right)$. For a proof by contradiction, assume that R^0 Ú $Dup\left(R^0\right)$. Then the element a is in R^0 and the equation $a = x + x$ is not solvable in R^0. Accordingly, $x + x \neq a$ for some x is in R^0. Squaring on both sides of $x + x \neq a$, it gives $(x+x)^2 \neq a^2 \Rightarrow x^2 + x^2 + x^2 + x^2 \neq a^2 \Rightarrow 0 \neq 0$, it is not true in R^0, and thus our assumption is not true. Hence, $R^0 \subseteq Dup\left(R^0\right)$. So, we finish that $Dup\left(R^0\right) = R^0$.

4.Duplicity of Neutrosophic Rings

In this section, we establish duplex rings and their corresponding neutrosophic duplex rings. On the other hand, first we prove some results of this duplicity and which are useful for subsequent results as well as for the next concepts.

Now, this study is going to define duplicity of R and $R(I)$, and study their properties with different illustrations. We notice that $R + R \neq 2R$, $R + R + R \neq 3R$, and so on.

Definition 4.1. Let R be a finite commutative ring. Then the structure $Dup(R)$ is called duplex ring, and it is defined as $Dup(R) = \{a : x + x = a \text{ is solvable in } R\}$.

For any ring R, there is a neutrosophic duplex ring $Dup(R(I))$ of the neutrosophic ring $R(I)$, and it is defined as $Dup(R(I)) = \{\alpha : \beta + \beta = \alpha \text{ is solvable in } R(I)\}$, where $\alpha = a + Ib$ and $\beta = c + Id$ are neutrosophic elements in $R(I)$.

For example, $Dup(Z_2(I)) = \{0 + 0I\}$, $Dup(Z_5(I)) = Z_5(I)$ but $Dup(Z_5(i, I)) \neq Z_5(i, I)$ where $Z_5(i)$ is the ring of Gaussian integers and $Z_5(i, I)$ is the neutrosophic ring of Gaussian integers.

The following is a basic result to the preceding analysis of duplicity.

Theorem 4.2. The duplicity of $R(I)$ is a neutrosophic subring of $R(I)$.

Proof. By the Theorem ⦅3.3⦆ $Dup(R)$ is a subring of R. Further, we have $R(I) = R + RI$, and therefore, $Dup(R(\text{I})) = Dup(R) + Dup(R)I$. This relation explore that $Dup(R(I))$ is generated by $Dup(R)$ and I, and hence $Dup(R(I))$ is a neutrosophic subring of $R(I)$.■

Corollary 4.3. The duplicity of $R(I)$ is a neutrosophic ideal of $R(I)$.

Proof. It is clear from the observation that $Dup(R)$ is an ideal R, and thus $Dup(R(I))$ is a neutrosophic ideal of $R(I)$. ■

The following examples are interesting illustrations of the preceding results. Here note that $Dup(Z_2(I)) = \{0+0I\}$.

Example 4.4. For any odd prime p, the neutrosophic duplex ring of $Z_p(I)$ is again $Z_p(I)$, that is $Dup(Z_p(I)) = Z_p(I)$.

Example 4.5. $Dup(Z_4(I)) = \{0, 2, 2I, 2+2I\}$, $Dup(Z_9(I)) = Z_9(I)$.

In this illustration we observed a connection of duplicity of rings and some fundamental concepts of rings. Under this observation, the following theorem provides a necessary and sufficient condition for the characteristic and duplicity of rings.

Theorem 4.6. Let $Char(R(I))$ be the characteristic of $R(I)$ with $R(I) \neq (0)$. Then, $Dup(R(\mathrm{I})) = (0)$ if and only if $Char(R(I)) = 2$.

Proof. It is well known that $|R| > 1$ if and only if $|R(I)| \geq 4$. So, we have $R \neq (0)$ if and only if $R(I) \neq (0)$, and additionally $Char(R) = Char(R(I))$. Thus we finish that

$Dup(R(\mathrm{I})) = (0) \Leftrightarrow Dup(R) + Dup(R)I = (0) \Leftrightarrow Dup(R) = (0) \Leftrightarrow r + r = 0$ is solvable in the ring R

$\Leftrightarrow 2r = 0$ for every r in R $\Leftrightarrow Char(R) = 2$. ■

The following example explores this theorem.

Example 4.7. $Dup(Z_2) = (0)$; $Dup(Z_2(I)) = (0)$, $Dup(Z_2[i]) = (0)$; $Dup(Z_2(i, I)) = (0)$,

$Dup(Z_2[x]) = (0)$; $Dup(Z_2(x, I)) = (0)$, where $Z_2[i]$ and $Z_2[x]$ are both rings of Gaussian integers and polynomials under addition and multiplication modulo 2, respectively.

The following theorem plays a significant role in characterizing finite neutrosophic rings and neutrosophic fields in terms of their corresponding duplicity of systems. Given a finite field F, there exists a neutrosophic field $F(I)$ with $F(I) = F + FI$. For instance, $Z_2(I)$, $Z_3(I)$, $Z_5(I)$ are all finite neutrosophic fields. Make a note of that $F(I) + F(I) \neq 2F(I)$.

Theorem 4.8. For any finite field F, the system $F(I) + F(I)$ is also equal to itself the neutrosophic field $F(I)$, where $F(I) + F(I)$ is defined as $F(I) + F(I) = \{\alpha + \alpha : \alpha \in F(I), \alpha \in F(I)\}$.

Proof. Because the element $x + 0$ in $F + F$, we have $x + 0 = x$, which is in F. This implies that $F + F \subseteq F$. To go the other way, let us suppose that $F \not\subset F + F$. Then, $x \in F$ implies that $x \notin F + F$. So, there is an element a in F such that $x \neq a + a$. It is not true for any finite field F, because the structure $(F, +)$ is an abelian group and the equation $x = a + a$ is solvable in $(F, +)$. Thus our point of view $F \subseteq F + F$ is also true. Hence, $F + F = F$. Suppose that F has the duplex form. We end up with the computations

$F(I) + F(I) = (F + FI) + (F + FI) = (F + F) + (IF + FI) = (F + F) + (F + F)\mathrm{I} = F + FI = F(I)$,where $IF = FI$. ■

Corollary 4.9. $Dup(F) = F$ and $Dup(F(I)) = F(I)$ whenever $Char(F) \neq 2$.

Proof. It is simply proved from Theorem ◎above◎ and Theorem ◎above◎. ■

The following is an example to the preceding analysis of the duplex of F and $F(I)$.

Example 4.10. $Z_2 + Z_2 = Z_2$ but $Dup(Z_2) \neq Z_2$. However, $Dup(Z_2) = (0)$.

$$Z_2(I) + Z_2(I) = Z_2(I) \text{ but } Dup(Z_2(I)) \neq Z_2(I). \text{ However, } Dup(Z_2(I)) = (0 + 0I).$$

With these results among our tools, we know that the necessary information to now carry out a proof of the fact that duplicity of a zero ring is again itself zero ring. For more information about zero rings, redder refer [24, 27]. A ring $R^0 = (R^0, +, \cdot)$ is called a zero ring if $ab = 0$ for every a and b in R^0. Every finite zero rings is commutative, and also zero ring is a ring without unity. For instance, the ring $R^0 = \{0, 5, 10, 15, 20\}$ is a finite commutative ring without unity under addition and multiplication modulo 25. Additionally, the authors Chalapathi and Madhavi introduced and studied the extended structure of zero rings, called, neutrosophic zero rings [24]. For any zero ring R^0, there exists corresponding neutrosophic zero ring $R^0(I)$, which is also commutative and without unity.

Theorem 4.11. Let R^0 be a finite zero ring. Then, $R^0(I) + R^0(I) = R^0(I)$.

Proof. The theorem is certainly true for $R^0 = (0)$, because $R^0 = (0)$ if and only if $R^0(I) = (0)$. Thus we may here after restrict our attention to nontrivial zero ring $R^0 \neq (0)$. Suppose $|R^0| > 1$ be the positive integer such that $|R^0(I)| \geq 4$. Then, first of all we prove that $R^0 + R^0 = R^0$ for any finite zero ring R^0. By virtue of addition of two rings, $R^0 + R^0 = \{a + a : a \in R^0\}$. The crux of our argument is that $R^0 + R^0$ is a subring of R^0, and this fact fallows that $R^0 + R^0 \subseteq R^0$. For a proof by a contradiction, assume that $R^0 \not\subset R^0 + R^0$. For some a in R^0, there exists $x \in R^0$ such that $x \neq a + a$. Now squaring on both sides of $x \neq a + a$, we calculate

$$x^2 \neq (a+a)^2 \Rightarrow 0 \neq (a+a)^2, \text{ since } x^2 = 0 \Rightarrow a^2 + a^2 + (a+a) \neq 0$$

$$\Rightarrow a + a \neq 0, \text{ since } a^2 = 0$$

$$\Rightarrow a \neq -a.$$

This means that every element in a finite zero ring R^0 has not mutually additive inverse. This violates the basic condition of the zero rings [24], that means that every nonzero element in nontrivial zero ring has mutually additive inverse, giving us our contradiction. Thus, we have $R^0 \subseteq R^0 + R^0$, and hence $R^0 + R^0 = R^0$. Finally, the theorem follows the following calculations.

$$R^0(I) + R^0(I) = (R^0 + R^0 I) + (R^0 + R^0 I) = (R^0 + R^0) + (R^0 I + R^0 I) = (R^0 + R^0) + (R^0 + R^0) I$$

$$= R^0 + R^0 I = R^0(I). \blacksquare$$

Remark 4.12. From Theorem 4.11, $R^0(I)+R^0(I)=R^0(I)$ but $R^0(I)+R^0(I)=Dup\left(R^0(I)\right)$. This explores that the neutrosophic equation $\alpha+\alpha=\beta$ is solvable in $R^0(I)$ for every neutrosophic elements α and β in $R^0(I)$.

Corollary 4.13. For any zero rings R^0, we have $Dup\left(R^0(I)\right)=R^0(I)$.

Proof. It is full fill from the following calculations.

$$Dup\left(R^0(I)\right)=Dup\left(R^0\right)+Dup\left(R^0\right)I=R^0+R^0I=R^0(I).$$

5. Conclusions

In this paper, we have determined and counted all duplex elements in the finite cyclic ring Z_n. We have established that there is a number theoretic connection between the duplex function $D(n)$ and elements in Z_n, and also prove that $D(n)=\frac{n}{(2,n)}$. More importantly, we have shown that $D(Z_m\times Z_n)=\frac{mn}{(2,m)(2,n)}$. We have also discussed duplicity of finite rings and neutrosophic rings. A short discussion about how this duplex ring could be applied to the neutrosophic rings.

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An Integrated Model for Ranking Risk Management in Industrial Internet of Things (IIoT) system

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Abstract

The Industrial Internet of Things (IIoT) was developed based on the technology and applications of the Internet of Things (IoT) in an industrial environment. As it is a sub-set of the IoT, it requires higher levels of safety and security. While increased productivity, better management and high operational efficiency are its main goals, they involve managing many risks, such as conflicting criteria and uncertain information, that need to be assessed and ranked. Therefore, in this paper, the Multi-criteria Decision-making (MCDM) method is used to deal with these criteria and a neutrosophic environment to overcome the uncertainty. Also, the Analytical Hierarchical Process (AHP) and Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) approaches are proposed. The former is used to obtain the weights of the criteria and the latter to rank the management of risk in the IIOT system. Numerical examples are provided and a sensitivity analysis conducted to test the reliability of this model.

Keywords: IIoT; IoT; neutrosophic sets; AHP; TOPSIS, risks; SVNSs

1. Introduction

The novelthe IoT that appeared in recent years is based on the development of wireless technologies. In 1998, its concept was introduced by Kevin Ashton for objects or connected to the internet. It has many advantages in applications such as transportation, healthcare and smart homes as well as industry for reducing costs while effectively controlling operations. The concept of the IIoT was introduced based on the innovations and benefits of the IoT in industry. The large amounts of data collected and analyzed by IIoT in industry are used to enhance the performances of industrial systems , provide many services and reduce operational costs [1].

There are several terms for the IIOT, such as Industry 4.0, smart manufacturing and the IoT in industry. The main reason for the IIoT is its use of advanced technologies and applications, including deep learning, machine learning, cloud computing and 5G, for optimizing industrial processes. In 2011, the German government introduced the term Industry 4.0. Its main goal is to collect and analyze the data and information of any product and enhance the efficiency of its manufacture.

The IIoT is a sub-set of the IoT which requires more safety and security. It will enable Industry 5.0 to reduce the gap between humans and machines. By 2025, 70 billion devices will be connected to it and, in 2023, its share in the global market will be USD14.2 trillion.

The IIoT plays a vital role in many fields and companies, such as providers of healthcare, producers of agriculture and manufacturers, to increase their performances, efficiency and productivity through smart management; for example, hospitals can overcome their limitations by using IIoT technologies to connect medical devices. Although the IIoT helps workers to improve efficiency and safety[2], it has many risks which, as they threaten industrial processes and affect the performances of systems, should be ranked in terms of their significance.

The problem of ranking these risks includes uncertainty and vague information. Although a fuzzy set is used to solve the uncertainty, it cannot deal with the value of indeterminacy [3]. To overcome this problem, a neutrosophic set is introduced. It handles both uncertainty and vague information by representing the indeterminacy value.[4] A single-valued neutrosophic set (SVNS) includes the three values of truth, indeterminacy and falsity (T,I,F). It is a sub-set of a neutrosophic set and represents data using single-valued neutrosophic numbers (SVNNs)[5].

As ranking the risks of the IIoT involves different, multiple and conflicting factors, the concept of the Multi-criteria Decision-making (MCDM) method, which solves complex decision-making problems, is used[6]. In this study, the Analytical Hierarchical Process (AHP) and Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) approaches are employed. The former is used to calculate the weights of the criteria and is a common MCDM method. It depends on a pair-wise comparison of the criteria and alternatives. It helps decision-makers select the best solution and decision given vague and imprecise information. It has been applied to solve medical, engineering, manufacturing and educational problems and is easy to use [7]. In this paper, the TOPSIS method is used to rank the risks in the IIoT. It performs mathematical calculations to compute the best alternatives and is a common MCDM method [8].

The main contributions of this work are as follows.

I. It describes the benefits and risks of the IIoT and ranks these risks to help enterprises, companies, etc. consider them.
II. It uses different units of criteria and alternatives to assess these risks.
III. It employs the MCDM methods AHP and TOPSIS.
IV. It introduces SVNSs to overcome the vagueness and uncertainty of information in the IIoT.

The remainder of this paper is organized as follows: related work is presented in section 2; a hybrid model in section 3; a numerical example in section 4; a sensitivity analysis in section 5; and, finally, conclusions and suggested future work in section 6.

2. Related Work

In this section, a literature review of the IIoT and our model are provided, with the concept of the IIoT and cyber physical systems (CPS) presented in [9, 10] The IIoT is used in many fields, such as healthcare and agriculture, and many companies. It aids farmers in computing their agricultural variables, such as water and nutrients in the soil as well as the fertilizers used to increase productivity [11, 12] Many companies, such as Microsoft [13] and the Climate Group encourage agricultural pursuits [14] Sisinni et al. described the IIoT's challenges, such as energy efficiency, real-time cohabitation and interoperability, privacy and security, as well as its opportunities and directions [2] . Sadeghi et al. considered privacy and security as its main challenges [15] They concluded that cyber-attacks are very critical as they cause physical damage and threats to humans. Boyes et al. proposed a framework for analyzing security and sensitivity threats [16] Younan et al. discussed the IIoT's issues and recommended technologies for them IIoT [17]

As the challenges and risks of the IIoT include a great deal of vague and uncertain information, a fuzzy sets have been used. ElHamdi et al. discussed an agricultural framework using fuzzy sets to compute the best locations for sensors on a shop floor [18] Collotta et al. used a fuzzy model to enhance power management in smart homes [19] However, as these sets have several limitations, such as not considering indeterminacy values, neutrosophics ones were used to overcome this uncertainty by taking these values into account. Abdel-Basset et al. used neutrosophic sets to solve the problem of the IoT's transition difficulties [20] which no previous research had considered. Therefore, in this study, SVNSs are proposed to overcome uncertainty of the risks of the IIoT.

As the risks of the IIoT have many different and conflicting criteria and factors, MCDM approaches have been used to overcome this problem [21] Grida et al. used a MCDM framework to assess the performance of the IoT in a supply chain [22] Durão et al. used the AHP, which is a common MCDM method for computing the weights of the criteria [23], for the selection process in the IoT [24] Zhang et al. used the fuzzy AHP method to assess system security in the IoT [25], with another MCDM method, TOPSIS, used to rank the alternatives. Wang and et al. used the fuzzy AHP and TOPSIS methods to design a framework for assessing security in the IoT [26] Tariq et al. adopted the TOPSIS method to determine the challenges in the medical field using the IoT [27] Also, Çalık employed it to select green suppliers in the IoT [28]

From the review of the literature, it is clear that no study proposed using SVNSs with the AHP and TOPSIS methods to rank risks in the IIoT.

3. Hybrid Model

In this paper, a hybrid model with SVNSs and MCDM with AHP and TOPSIS methods is proposed. The AHP one is used to calculate the weights of the criteria and the TOPSIS one to rank the risks of the IIoT. The first stage in this hybrid model is using the SVNSs to overcome uncertain information. The research framework is shown in Fig. 1.

3.1. Single-valued Neutrosophic Sets (SVNSs)

A SVNS is a sub-set of a neutrosophic set. It deals with the three values of truth, indeterminancy and falsity (T,I,F). It has the function of scoring accuracy and certainty, and handles vague and inconsistent information well.

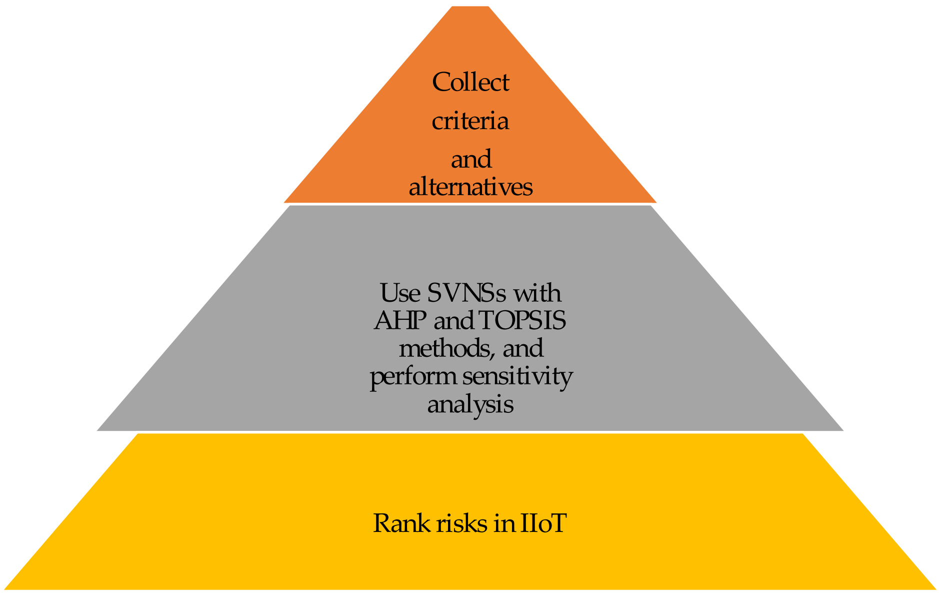

Fig. 1. Research framework

3.2. AHP Method

The AHP method is used to calculate the weights of the criteria. Its steps are illustrated in Fig. 2 and executed as follows [29]

Step 1. Build a pair-wise comparison decision matrix among the criteria using the opinions of experts and decision-makers as

$$A^P = \begin{bmatrix} A_{11}^P & \cdots & A_{1d}^P \\ \vdots & \ddots & \vdots \\ A_{c1}^P & \cdots & A_{cd}^P \end{bmatrix} \quad (1)$$

where P refers to the decision-makers, c the number of criteria and d the number of alternatives.

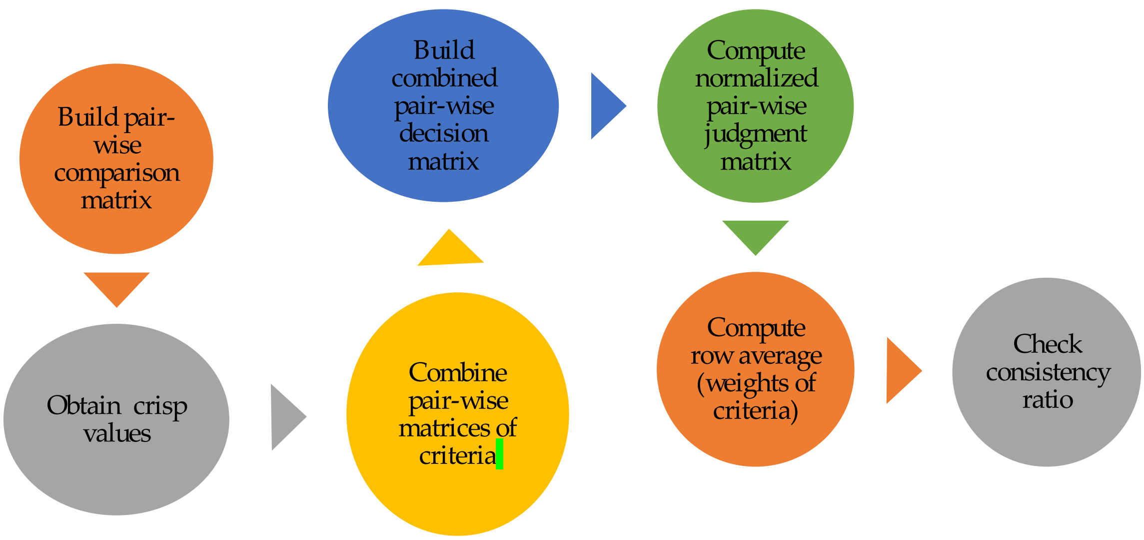

Fig. 2. Steps in AHP method

Step 2. Obtain the crisp values by converting the opinions of the decision-makers to SVNNs according to the values in Table 1. Then, convert these numbers to crisp values to obtain one instead of three values using the score function

$$F(A_{rs}^P) = \frac{2 + T_{rs}^P - I_{rs}^P - F_{rs}^P}{3} \quad (2)$$

where T_{rs}^P , I_{rs}^P, F_{rs}^P refer to the truth, indeterminacy and falsity values of the SVNNs' $c = 1,2, \dots r$, $d = 1,2,3, \dots s$.

Table 1. Scales of SVNSs

Linguistic Variable	SVNNs
Very Corrupt	⊚0.30,0.7,0.75⊚
Corrupt	⊚0.40,0.6,0.65⊚
Equal	⊚0.6,0.5,0.6⊚
Honest	⊚0.85,0.35,0.35⊚
Very Honest	⊚0.95,0.2,0.3⊚

Step 3. Combine the pair-wise matrices of the criteria in one matrix using

$$A_{rs} = \frac{\sum_{P=1}^{P} A_{rs}}{P} \quad (3)$$

Step 4. Build a combined pair-wise decision matrix as

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1s} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rs} \end{bmatrix} \quad (4)$$

Step 5. Compute a normalized pair-wise comparison matrix using the combined pair-wise comparison matrix as

$$Z_r^c = \frac{A_r}{\sum_{r=1}^{c} A_r}; r = 1,2,3, \dots \dots . c \quad (5)$$

Step 6. Compute the row average (weights of the criteria) after building the normalized pair-wise comparison matrix as

$$w_r = \frac{\sum_{s=1}^{d}(Z_{rs})}{s}; r = 1,2,3, \dots \dots . c; s = 1,2,3, \dots d; \quad (6)$$

Step 7. Use the consistency ratio

to check the consistency of the opinions of the decision-makers by

$$CR = \frac{CI}{RI} \quad (7)$$

$$CI = \frac{\lambda_{max} - n}{n-1} \quad (8)$$

where RI refers to a random index, CI the consistency index and n the number of criteria.

3.3. TOPSIS Method

This method is used to rank the risks in the IIoT. Its steps are shown in Fig. 3 and described as follows [29]

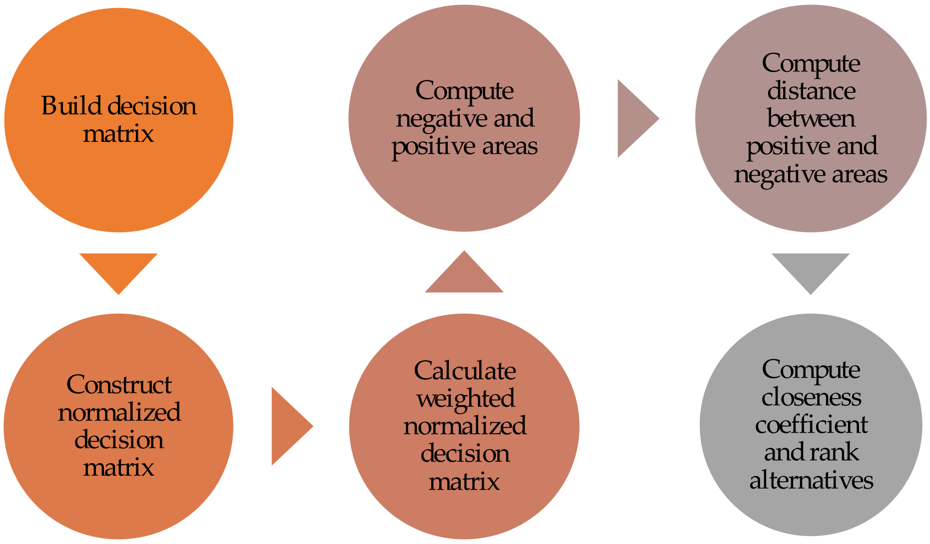

Fig. 3. Steps in TOPSIS method

Step 8. Build a decision matrix of the criteria and alternatives using Eqs. (1), (2), (3) and (4).

Step 9. Construct a normalized decision matrix as

$$Y_{rs} = \frac{A_{rs}}{\sqrt[2]{\sum_{r=1}^{c} A_{rs}^{2}}} \quad r = 1,2,3, \dots \dots, c \text{ and } s = 1,2,3 \dots \dots, d \tag{9}$$

Step 10. Calculate the weighted normalized decision matrix by multiplying the normalized decision matrix by the weights of the criteria as

$$X_{rs} = Y_{rs} * W_s \tag{10}$$

Step 11. Compute the negative and positive areas for the positive and negative criteria, respectively, by

$$E_d{}^{+} = \begin{cases} \max(X_{rs}) & for\ positive\ criteria \\ \min(X_{rs}) & for\ negative\ criteria \end{cases} \quad \text{Positive area} \tag{11}$$

$$E_d{}^{-} = \begin{cases} \min(X_{rs}) & for\ positive\ criteria \\ \max(X_{rs}) & for\ negative\ criteria \end{cases} \quad \text{Negative area} \tag{12}$$

Step 12. Calculate the Euclidean distance between the positive and negative areas for the positive and negative criteria, respectively, as

$$I_r^{+} = \sqrt{\sum_{s=1}^{d} (X_{rs} - E_s^{+})^2} \quad \text{for positive criteria} \tag{13}$$

$$I_r^{-} = \sqrt{\sum_{s=1}^{d} (X_{rs} - E_s^{-})^2} \quad \text{for negative criteria} \tag{14}$$

Step 13. Compute the closeness coefficient using Eq. (15) and then rank the alternatives in descending order of H_r as

$$H_r = \frac{I_r^-}{I_r^+ + I_r^-} \tag{15}$$

4. Results obtained from Hybrid Model

The first step in building the hierarchy tree is to determine the goal for this study (ranking the risks in the IIoT) and collect the criteria and alternatives, that is, four main criteria, fourteen sub-criteria and four alternatives, as shown in Fig. 4. The alternatives are A_1 - catastrophic risk, A_2 –cyber-attack risk, A_3 - environmental risk and A_4 - infrastructure Risk, with all the criteria positive.

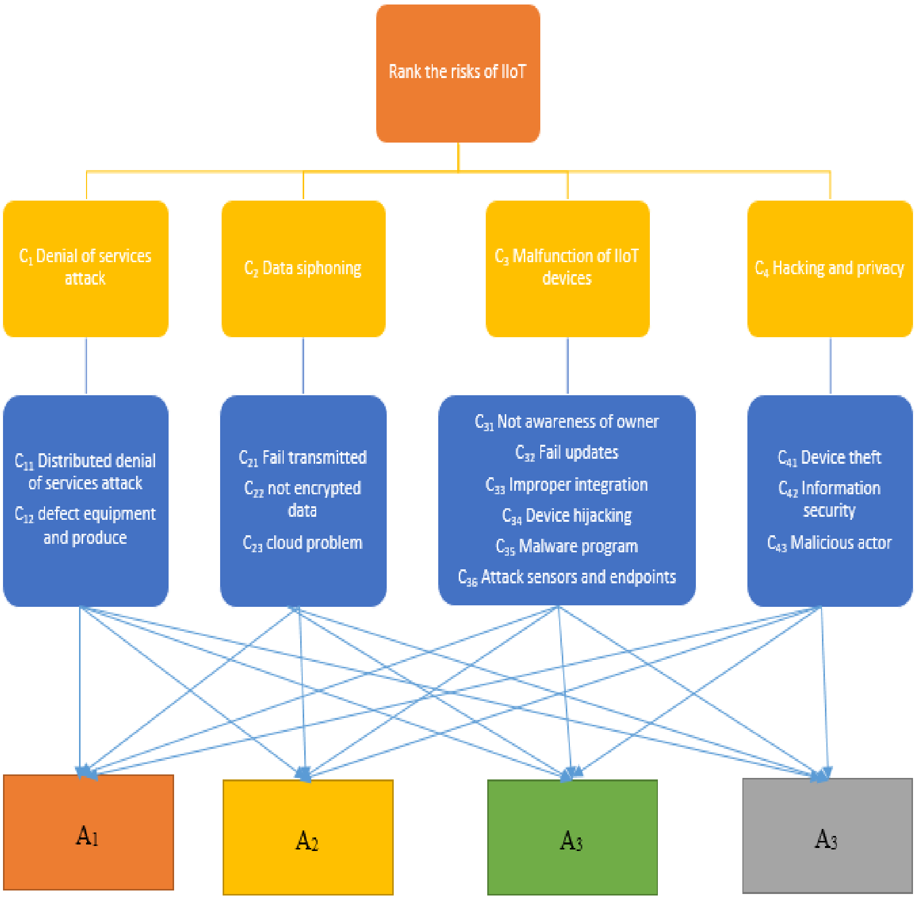

Fig. 4. Goal, criteria and alternatives for this research

Three decision-makers with expertise in the IIoT are proposed. The first has a PhD degree in the IIoT and the others Master's degrees in that field. Beginning with the SVNNs in Table 1, their opinions regarding building the pair-wise comparison matrix using Eq. (1) are obtained. Then, the score function is applied to convert their linguistic terms into values which are converted into three numbers (T,I,F) to obtain one value using Eq. (2). The values of the three pair-wise comparison matrices are then combined in one matrix using Eqs. (3) and(4), and shown in Table 2 for the main criteria and Tables 3.1, 3.2, 3.3, 3.4 for the sub-criteria.

Table 2. Combined pair-wise comparison matrix for main criteria

Criteria	C_1	C_2	C_3	C_4
C_1	0.5	0.49443	0.60557	0.8167
C_2	2.20438	0.5	0.7167	0.75003
C_3	1.79983	1.39528	0.5	0.75003
C_4	1.22444	1.33834	1.33834	0.5

Table 3.1. Combined pair-wise comparison matrix for sub-criteria C_1

Criteria	C_{11}	C_{12}
C_{11}	0.5	0.6389
C_{12}	1.742882	0.5

Table 3.2. Combined pair-wise comparison matrix for sub-criteria C_2

Criteria	C_{21}	C_{22}	C_{23}
C_{21}	0.5	0.527767	0.7167
C_{22}	2.147428	0.5	0.672233
C_{23}	1.395284	1.685934	0.5

Table 3.3. Combined pair-wise comparison matrix for sub-criteria C_3

Criteria	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}
C_{31}	0.5	0.6389	0.6389	0.605567	0.672233	0.605567
C_{32}	1.742882	0.5	0.605567	0.750033	0.6389	0.6389
C_{33}	1.742882	1.79983	0.5	0.672247	0.750033	0.6389
C_{34}	1.79983	1.338336	1.685843	0.5	0.672233	0.750033
C_{35}	1.685934	1.742882	1.338336	1.685934	0.5	0.527767
C_{36}	1.79983	1.742882	1.742882	1.338336	2.147428	0.5

Table 3.4. Combined pair-wise comparison matrix for sub-criteria C_4

Criteria	C_{41}	C_{42}	C_{43}

C_{41}	0.5	0.783367	0.527767
C_{42}	1.281388	0.5	0.6389
C_{43}	2.147428	1.742882	0.5

After building the combined pair-wise comparison matrices, the AHP method is applied to obtain the weights of the criteria. Firstly, Eq. (5) is used to normalize the pair-wise comparison matrix in Table 4 and then the weights of the criteria are computed by Eq. (6). In Table 5. the weights of the main and sub-criteria as well as their local and global weights are shown. The results indicate that the C_4 (hacking and privacy) has the highest weight with a value of 0.2934 and C_1 (denial of service attack) the lowest with a value of 0.1754. In Fig. 5, the weights of the main criteria are illustrated. C_{43} (malicious actor) has the highest weight of the sub-criteria and C_{11} (unaware of owner) the lowest. Then, the consistency ratio is checked to test whether the opinions of the experts are consistent using Eqs. (7) and (8); if it is less than 0.1, they are consistent.

Table 4. Normalized pair-wise comparison matrix for main criteria using AHP method

Criteria	C_1	C_2	C_3	C_4
C_1	0.087281	0.132625	0.191598	0.289942
C_2	0.384799	0.134118	0.226761	0.266275
C_3	0.314181	0.374266	0.158198	0.266275
C_4	0.21374	0.358991	0.423443	0.177508

Table 5. Weights of main and sub-criteria

Criteria	Weights of main criteria	Criteria	Local Weights	Global Weights
C_1	0.175362	C_{11}	0.392	0.068757
		C_{12}	0.608	0.106643
C_2	0.252988	C_{21}	0.233	0.058926
		C_{22}	0.357	0.090285
		C_{23}	0.41	0.103689
C_3	0.27823	C_{31}	0.106	0.0295
		C_{32}	0.129	0.035901
		C_{33}	0.155	0.043137
		C_{34}	0.174	0.048424
		C_{35}	0.192	0.053434
		C_{36}	0.244	0.067905
C_4	0.293421	C_{41}	0.21	0.061614
		C_{42}	0.338	0.099169
		C_{43}	0.452	0.132617

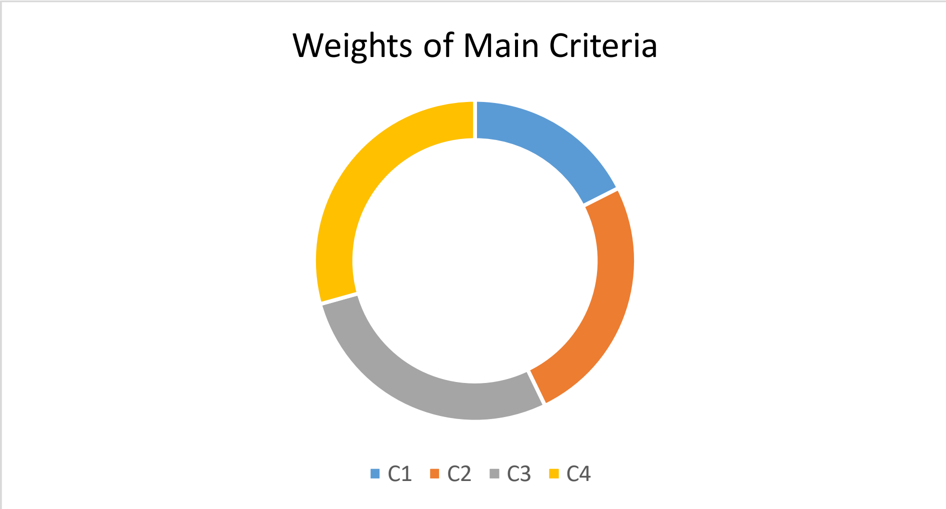

Fig. 5. Weights of main criteria

In applying the TOPSIS method to rank the alternatives, the first step is to build a decision matrix of the criteria and alternatives using Eqs. (1) (2), (3) and (4) (Table 6). Then, the decision matrix is normalized using Eq. (9) (Table 7) and, using Eq. (10), the weighted normalized decision matrix is computed by multiplying the values of the normalized decision matrix by the weights of the criteria (Table 8). As Eqs. (11) and (12) are applied to obtain the positive and negative ideal solutions for the positive and negative criteria, respectively, while all the criteria are positive. The distance from each alternative is computed using Eqs. (13) and (14) for the positive and negative criteria, respectively, and the closeness coefficient using Eq. (15) (Table 9). Finally, the alternatives are ranked in the descending order of the values of the closeness coefficient. Of the risks, the A_2 cyber-attack is the highest and the A_1 catastrophic the lowest. Table 9 shows the ranks of the alternatives and Fig. 6 those of the risks obtained from the TOPSIS method.

Table 6. Combined decision matrixof criteria and alternatives

Criteria@alternatives	C_{11}	C_{12}	C_{21}	C_{22}	C_{23}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{41}	C_{42}	C_{43}
A_1	0.67498	0.51665	0.35833	0.67498	0.67498	0.51665	0.67498	0.67498	0.44165	0.35833	0.67498	0.67498	0.35833	0.7583
A_2	0.7958	0.7583	0.48748	0.67498	0.7583	0.51665	0.6458	0.7583	0.35	0.54583	0.7583	0.3875	0.48748	0.599975

A_3	0.56248	0.43333	0.5	0.51665	0.63748	0.71248	0.6833	0.52498	0.40415	0.7583	0.52498	0.7583	0.6458	0.7208
A_4	0.32915	0.7583	0.7958	0.40415	0.7583	0.67498	0.3875	0.44165	0.62915	0.7958	0.51665	0.32915	0.7958	0.433325

Table 7. Normalized decision matrix using TOPSIS method

Criteria/alternatives	C_{11}	C_{12}	C_{21}	C_{22}	C_{23}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{41}	C_{42}	C_{43}
A_1	0.548632	0.407845	0.320582	0.582775	0.475854	0.422256	0.553016	0.550883	0.47168	0.280263	0.538151	0.594488	0.301083	0.591715
A_2	0.646841	0.598604	0.436128	0.582775	0.534597	0.422256	0.529113	0.618889	0.373798	0.426916	0.604586	0.341293	0.409602	0.468171
A_3	0.45719	0.342068	0.447334	0.446077	0.449416	0.582303	0.559837	0.42846	0.43163	0.593103	0.418558	0.667877	0.542635	0.562453
A_4	0.267539	0.598604	0.711977	0.348944	0.534597	0.551655	0.317484	0.360454	0.671929	0.622434	0.41192	0.289901	0.668672	0.338131

Table 8. Weighted normalized decision matrix using TOPSIS method

Criteria/alternatives	C_{11}	C_{12}	C_{21}	C_{22}	C_{23}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{41}	C_{42}	C_{43}
A_1	0.037722	0.043494	0.018891	0.052616	0.049341	0.012456	0.019854	0.023763	0.022841	0.014975	0.036543	0.036629	0.029858	0.078471
A_2	0.044475	0.063837	0.025699	0.052616	0.055432	0.012456	0.018996	0.026697	0.018101	0.022812	0.041055	0.021028	0.04062	0.062087
A_3	0.031435	0.036479	0.026359	0.040274	0.0466	0.017178	0.020099	0.018482	0.020901	0.031692	0.028422	0.041151	0.053813	0.074590
A_4	0.018395	0.063837	0.041954	0.031505	0.055432	0.016274	0.011398	0.015549	0.032538	0.033259	0.027972	0.017862	0.066312	0.044841

Table 9. Closeness coefficient and ranks of alternatives

Alternative	Closeness Coefficient	Rank
A_1	0.487915	A_2
A_2	0.547426	A_3
A_3	0.541891	A_4
A_4	0.50286	A_1

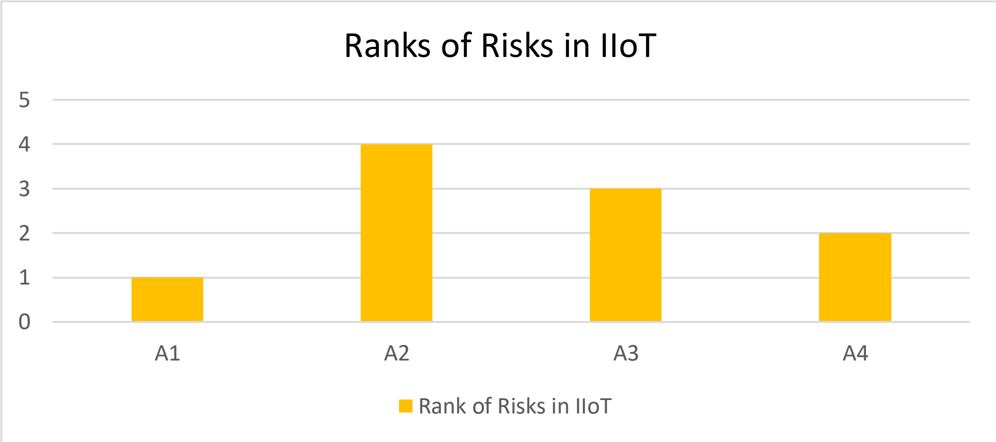

Fig. 6. Ranks of risks in IIoT using TOPSIS method

5. Sensitivity Analysis

When the weights of the criteria change, so do the ranks of the risks. In this sensitivity analysis, five scenarios of changing weights are considered. In the first, all the weights of the criteria are equal and, in the second, that of the first criterion is 0.5 while the others are equal and so on. However, in all the scenarios, the sum of the weights of the criteria must equal 1, as shown in Table 10. When the weights of the main criteria are changed, so are those of the sub-criteria, as shown in Table 11.

Table 10. Five scenarios with different weights of main criteria

Scenario\Criterion	C_1	C_2	C_3	C_4
Scenario 1	0.25	0.25	0.25	0.25
Scenario 2	0.5	0.1667	0.1667	0.1667
Scenario 3	0.1667	0.5	0.1667	0.1667
Scenario 4	0.1667	0.1667	0.5	0.1667
Scenario 5	0.1667	0.1667	0.1667	0.5

Table 11. Sub-criteria for five scenarios with different weights of main criteria

Sub-criterion\weight	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
C_{11}	0.098	0.196	0.065346	0.065346	0.065346
C_{12}	0.152	0.304	0.101354	0.101354	0.101354
C_{21}	0.05825	0.038841	0.1165	0.038841	0.038841
C_{22}	0.08925	0.059512	0.1785	0.059512	0.059512
C_{23}	0.1025	0.068347	0.205	0.068347	0.068347
C_{31}	0.0265	0.01767	0.01767	0.053	0.01767
C_{32}	0.03225	0.021504	0.021504	0.0645	0.021504
C_{33}	0.03875	0.025839	0.025839	0.0775	0.025839
C_{34}	0.0435	0.029006	0.029006	0.087	0.029006
C_{35}	0.048	0.032006	0.032006	0.096	0.032006
C_{36}	0.061	0.040675	0.040675	0.122	0.040675
C_{41}	0.0525	0.035007	0.035007	0.035007	0.105

C_{42}	0.0845	0.056345	0.056345	0.056345	0.169
C_{43}	0.113	0.075348	0.075348	0.075348	0.226

In the next step, the risks are ranked using the TOPSIS method for the different scenarios. In scenarios 1, 2 and 3, A_2 has the highest rank and A_3 the lowest. In scenario 4, A_2 has the highest rank and A_1 the lowest while, in scenario 5, A_3 has the highest rank and A_2 the lowest. The ranks of the risks for the five scenarios are presented in Table 12 and those in the IIoT in Fig. 7.

Table 12. Ranks of risks for five scenarios

Alternative	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
A_1	A_2	A_2	A_2	A_2	A_3
A_2	A_4	A_4	A_4	A_4	A_1
A_3	A_1	A_1	A_1	A_3	A_4
A_4	A_3	A_3	A_3	A_1	A_2

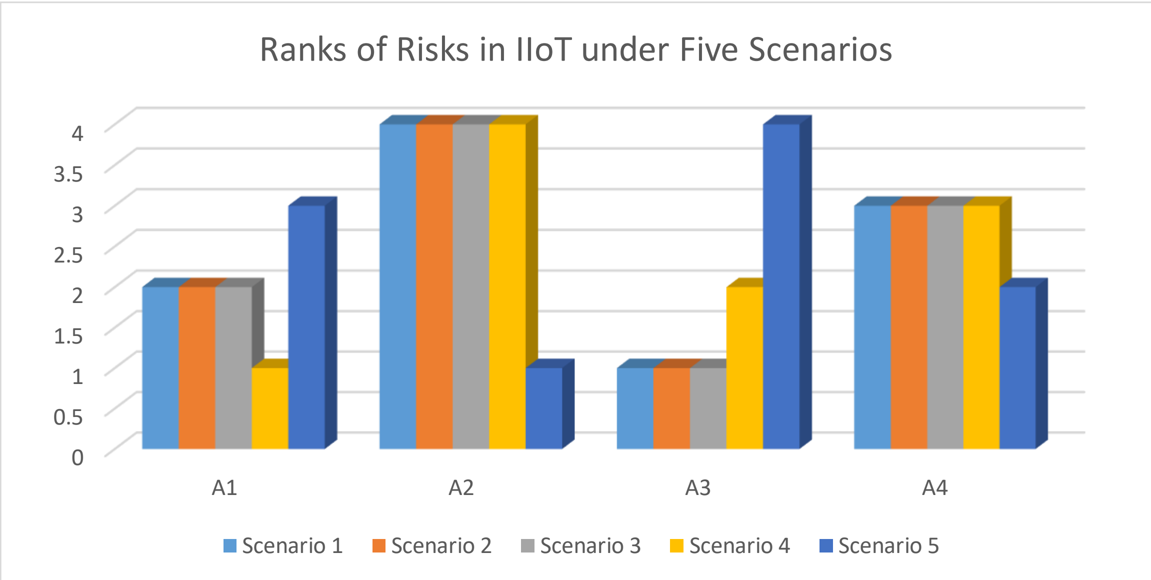

Fig. 7. Ranks of risks in IIoT for five scenarios

6. Conclusions

In this research, SVNSs using MCDM methods rank the risks in the IIoT and the importance of the role the IIoT plays in increasing a system's productivity, efficiency and performance by using the proposed hybrid model. This model includes the AHP and TOPSIS methods, with the former ranking the weights of the criteria and the latter the weights of the criteria. The neutrosophic environment overcomes the vague and uncertain information by considering the indeterminacy value. Four main criteria, fourteen sub-criteria, four alternatives and three decision-makers are adopted in this study.

Future work on this topic will apply other MCDM methods, such as VIKOR, to build a fuzzy model and compare it with the neutrosophic one.

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Interval Pentapartitioned Neutrosophic Sets

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Abstract

Pentapartitioned neutrosophic set is a powerful mathematical tool, which is the extension of neutrosophic set and n-valued neutrosophic refined logic for better designing and modeling real-life problems. A generalization of the notion of pentapartitioned neutrosophic set is introduced. The new notion is called Interval Pentapartitioned Neutrosophic set (IPNS). Pentapartitioned neutrosophic set is developed by combining the pentapartitioned neutrosophic set and interval neutrosophic set. We define several set theoretic operations of IPNSs, namely, inclusion, complement, intersection. We also establish various properties of set-theoretic operators.

Keywords: Neutrosophic set, Single valued neutrosophic set, Interval neutrosophic set, Pentapartitioned neutrosophic set, Interval pentapartitioned neutrosophic set

1. Introduction

Smarandache [1] developed the Neutrosophic Set (NS) by extending fuzzy set [2] and intuitionistic fuzzy set [3] by introducing the degrees of indeterminacy and rejection (falsity or non-membership) as independent components. Wang et al. [4] defined Interval NS (INS) as a subclass of NS by considering that the truth membership degree, indeterminacy membership degree and falsity membership degree are independent and assume values from the subunitary interval of [0, 1]. In 2010, Wang et al. [5] defined the Single Valued NS (SVNS) by restricting the degrees of membership, indeterminacy and falsity in [0, 1]. In 2013, Smarandache [6] presented n- valued neutrosophic refined logic.

Chatterjee et al. [7] defined Quadripartitioned SVNS (QSVNS) that involves degrees of truth, falsity, unknown and contradiction membership based on four valued logics [6].

Mallick and Pramanik [8] developed the theory of Pentapartitioned NS (PNS) by diving indeterminacy into three independent components, namely contradiction, ignorance, unknown. In this paper, we start the investigation of generalization of the notion-the Interval Pentapartitioned Neutrosophic Set (IPNS).We also establish some basic properties of the proposed set. The proposed structure is generalization of existing theories of INS and PNS.

The organization of the paper is as follows: Section 2 presents some preliminary results. Section 3 introduces the concept of IPNS and set-theoretic operations over IPNS. Section 4 concludes the paper by stating the future scope of research.

2. Preliminary

Definition 2.1. Let a set W be fixed. An NS [1] D over W is defined as:

$D = \{w(T_D(w), I_D(w), F_D(w)) : w \in W\}$ where $T_D, I_D, F_D : W \to]^-0, 1^+[$ and

$^-0 \le T_D(w) + I_D(w) + F_D(w) \le 3^+$.

Definition 2.2 Let a set W be fixed. An SVNS D over W is defined as:

$D = \{w(T_D(w), I_D(w), F_D(w)) : w \in W\}$ where $T_D, I_D, F_D : W \to [0, 1]$ and

$0 \le T_D(w) + I_D(w) + F_D(w) \le 3$.

Definition 2.3. Let a set W be fixed. An INS D over W is defined as:

$D = \{(w, (T_D(w), I_D(w), F_D(w)) : w \in W\}$

where for each $w \in W$, $T_D(w), I_D(w), F_D(w) \subseteq [0,1]$ are the degrees of membership functions of truth, indeterminacy, and falsity and

$T_D(w) = [\inf T_D(w), \sup T_D(w)], I_D(w) = [\inf I_D(w), \sup I_D(w)], F_D(w) = [\inf F_D(w), \sup F_D(w)]$ and

$0 \le \sup T_D(w) + \sup I_D(w) + \sup F_D(w) \le 3.$

D can be expressed as:

$D = \{w, ([\inf T_D(w), \sup T_D(w)], [\inf I_D(w), \sup I_D(w)], [\inf F_D(w), \sup F_D(w)]) : w \in W\}$

3. The Basic Theory of IPNSs

Definition 3.1. IPNS

Suppose that *W* be a fixed set. Then D, an IPNS over *W* is denoted as follows:

D = {(w, $T_D(w)$, $C_D(w)$, $G_D(w)$, $U_D(w)$, $F_D(w)$): $w \in W$}, where for each point $w \in W$, $T_D(w)$, $C_D(w)$, $G_D(w)$ U_D,(w) $F_D(w)$ $\subseteq$ [0, 1] are the degrees of membership functions of truth, contradiction, ignorance, unknown, and falsity and $T_D(w)$ = [inf T_D(w, sup $T_D(w)$] , $C_D(w)$ = [inf $C_D(w)$, sup $C_D(w)$] , $G_D(w)$ = [inf $G_D(w)$, sup $G_D(w)$], U_D,(w) = [inf $U_D(w)$, sup $U_D(w)$], $F_D(w)$ = [inf $F_D(w)$, sup $F_D(w)$] $\subseteq$ [0, 1] and $0 \le$ sup $T_D(w)$ + sup $C_D(w)$ + sup $G_D(w)$ + sup $U_D(w)$ + sup $F_D(w)$ ≤ 5.

An IPNS in R^1 is illustrated in Figure 1.

Example 3.1. Assume that W = [w_1, w_2, w_3], where w_1, w_2, and w_3 denote respectively capability, trustworthiness, and price. The values of w_1, w_2, and w_3 are in [0, 1]. They are obtained from the questionnaire of some domain experts, their option could be degree of truth (good), degree of contradiction, degree of ignorance, degree of unknown, and degree of false (poor). D_1 is an IPNS of *W* defined by

D_1 = {[0.4, 0.7], [0.1, 0.2], [0.1, 0.2], [0.2, 0.3], [0.2, 0.4]}/w_1 + {[0.5, 0.8], [0.2, 0.3], [0.1, 0.2], [0.15, 0.25], [0.2, 0.3]}/w2+ [0.6, 0.8], [01, 0.2], [0.2, 0.3], [0.15, 0.25], [0.1, 0.2]}/ w_3

D_2 is an IPNS of *W* defined by

D_2 = {[0.5, 0.9], [0.15, 0.25], [0.15, 0.25], [0.2, 0.3], [0:2, 0:3]}/w_1 + {[0.5, 0.8], [0.25, 0.3], [0.1, 0.2], [0.15, 0.25], [0.1; 0.3]}/w2+ [0.4, 0.7]; [01; 0.2], [0.2; 0.3], [0.15, 0.25], [0.15; 0.2]}/ w_3

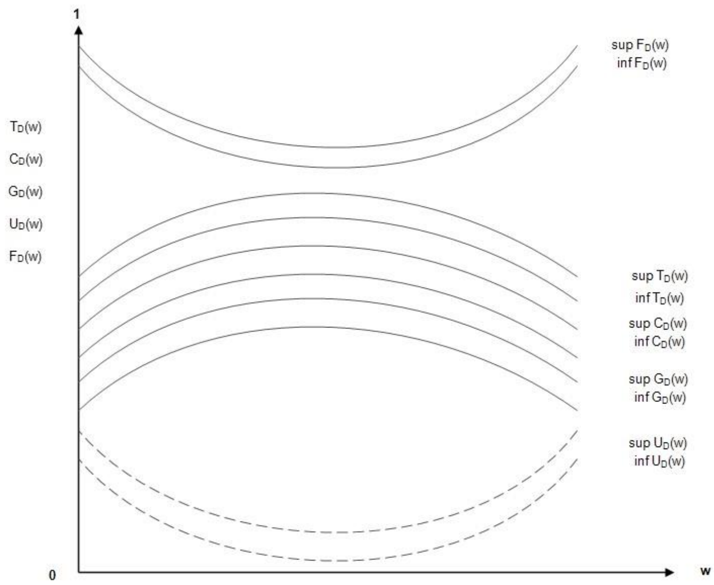

Figure 1: Illustration of an IPNS in R[1]

Definition 3.2 An IPNS is said to be empty (null) denoted by $\hat{0}$ if and only if its truth-membership, contradiction membership, ignorance membership, unknown membership and falsity membership function values are respectively defined as follows:

inf $T_D(w)$= sup $T_D(w)$ = 0, inf $C_D(w)$= sup $C_D(w)$ = 0, inf $G_D(w)$= sup $G_D(w)$= 1, inf $U_D(w)$ = sup $U_D(w)$= 1, inf $F_D(w)$= sup $F_D(w)$ = 1 ,

$$\hat{0} = \langle\langle [0,0],[0,0],[1,1],[1,1],[1,1] \rangle\rangle$$

Definition 3.3 An IPNS is said to be unity *denoted by* $\hat{1}$ if and only if its truth-membership, contradiction membership, ignorance membership, unknown membership and falsity membership function values are respectively defined as follows:

inf $T_D(w)$= sup $T_D(w)$ = 1, inf $C_D(w)$= sup $C_D(w)$ = 1, inf $G_D(w)$= sup $G_D(w)$= 0, inf $U_D(w)$ = sup $U_D(w)$= 0, inf $F_D(w)$= sup $F_D(w)$ = 0 ,

$$\hat{1} = \langle\langle [1,1],[1,1],[0,0],[0,0],[0,0] \rangle\rangle$$

Also, we have $\underline{0} = \langle 0,0,1,1,1 \rangle$ *and* $\underline{1} = \langle 1,1,0,0,0 \rangle$ for pentapartitioned neutrosophic set

Definition 3.4. (Containment) Assume that D_1 and D_2 be any two IPNS over W, D_1 is said to be contained in D_2, denoted by $D_1 \subseteq D_2$ if and only if

$$\begin{aligned}
&\inf T_{D_1}(w) \le \inf T_{D_2}(w), \sup T_{D_1}(w) \le \sup T_{D_2}(w),\\
&\inf C_{D_1}(w) \le \inf C_{D_2}(w), \sup C_{D_1}(w) \le \sup C_{D_2}(w),\\
&\inf G_{D_1}(w) \ge \inf G_{D_2}(w), \sup G_{D_1}(w) \ge \sup G_{D_2}(w),\\
&\inf U_{D_1}(w) \ge \inf U_{D_2}(w), \sup U_{D_1}(w) \ge \sup U_{D_2}(w),\\
&\inf F_{D_1}(w) \ge \inf F_{D_2}(w), \sup F_{D_1}(w) \ge \sup F_{D_2}(w),
\end{aligned}$$

for any $w \in W$.

Definition 3.5. Ay two IPNSs D_1 and D_2 are equal if and only if $D_1 \subseteq D_2$ and $D_1 \supseteq D_2$

Definition 3.6. (Complement) Let $D = ⊚(w, T_D(w), C_D(w), G_D(w), U_D(w), F_D(w)) : w \in W⊚$ be an IPNS. The complement of D is denoted by D' and defined as:

$$\begin{aligned}
&T_{D'}(w) = F_D(w),\ C_{D'}(w) = U_D(w),\\
&\inf G_{D'}(w) = 1 - \sup G_D(w),\\
&\sup G_{D'}(w) = 1 - \inf G_D(w),\\
&U_{D'}(w) = C_D(w), F_{D'}(w) = T_D(w)\\
&D' = ⊚(w, ⊚\inf F_D(w), \sup F_D(w)⊚, ⊚\inf U_D(w), \sup U_D(w)⊚, ⊚1 - \sup G_D(w), 1 - \inf G_D(w)⊚,\\
&\quad ⊚\inf C_D(w), \sup C_D(w)⊚, ⊚\inf T_D(w), \sup T_D(w)⊚) : w \in W⊚
\end{aligned}$$

Example 3.2 . Consider an IPNS D of the form:

D = {[0.4, 0.75], [0.1, 0.25], [0.1, 0.2], [0.2, 0.3], [0.2, 0.4]}/w_1 + {[0.5, 0.8], [0.2, 0.3], [0.1, 0.2], [0.15, 0.25], [0.2, 0.35]}/w2+ [0.75, 0.85], [0.15, 0.25], [0.2, 0.35], [0.15, 0.25], [0.1, 0.25]}/ w_3

Then, complement of D is obtained as:

D'== {[0.2, 0.4] , [0.2, 0.3], [0.8, 0.9], [0.1, 0.25], [0.4, 0.75]}/w_1 + {[0.2, 0.35], [0.15, 0.25], [0.8, 0.9], [0.2, 0.3], [0.5, 0.8]}/w2+ [0.1, 0.25], [0.15, 0.25], [0.65, 0.8], [0.15, 0.25], [0.75, 0.85]}/ w_3

Definition 3.7.(Intersection)

The intersection of any two IPNSs D_1 and D_2 is an IPNS D_3, written as $D_3 = D_1 \cap D_2$, such that

$$\begin{aligned}
&\{(w, [\inf T_{D_3}(w), \sup T_{D_3}(w)], [\inf C_{D_3}(w), \sup C_{D_3}(w)], [\inf G_{D_3}(w), \sup G_{D_3}(w)], [\inf U_{D_3}(w), \sup U_{D_3} w)],\\
&[\inf F_{D_3}(w), \sup F_{D_3}(w)]) : \forall w \in W\}.\\
&= \{(w, [\min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \min(\sup T_{D_1}(w), \sup T_{D_2}(w))],\\
&[\min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \min(\sup C_{D_1}(w), \sup C_{D_2}(w))],\\
&[\max(\inf G_{D_1}(w), \inf G_{D_2}(w)), \max(\sup G_{D_1}(w), \sup G_{D_2}(w))],\\
&[\max(\inf U_{D_1}(w), \inf U_{D_2}(w))], \max(\text{suf } U_{D_1}(w), \text{suf } U_{D_2}(w))],\\
&[\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\sup F_{D_1}(w), \sup F_{D_2}(w))]) : w \in W\}
\end{aligned}$$

Example 3.3. Let D_1 and D_2 be the IPNSs defined in Example 3.1.

Then, $D_1 \cap D_2$ ={[0.4, 0.7], [0.1, 0.2], [0.15, 0.25], [0.2, 0.3], [0.2, 0.4]}/w_1 + {[0.5, 0.8], [0.2, 0.3], [0.1, 0.2], [0.15, 0.25], [0.2, 0.3]}/w2+ [0.6, 0.8], [0.1, 0.2], [0.2, 0.3], [0.15, 0.25], [0.15, 0.2]}/ w_3

Definition 3.8. (**Union**) The union of any two IPNSs D_1 and D_2 is denoted by an IPNS D_3, written as $D_3 = D_1 \cup D_2$ and is defined by

$$\{(w, [\inf T_{D_3}(w), \sup T_{D_3}(w)], [\inf C_{D_3}(w), \sup C_{D_3}(w)],$$
$$[\inf G_{D_3}(w), \sup G_{D_3}(w)), [\inf U_{D_3}(w), \sup U_{D_3}(w)], [\inf F_{D_3}(w), \sup F_{D_3}]) : w \in W\}.$$
$$= \{(w, [\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\sup T_{D_1}(w), \sup T_{D_2}(w))],$$
$$[\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\sup C_{D_1}(w), \sup C_{D_2}(w))],$$
$$[\min(\inf G_{D_1}(w), \inf G_{D_2}(w)), \min(\sup G_{D_1}(w), \sup G_{D_2}(w))],$$
$$[\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\sup U_{D_1}(w), \sup U_{D_2}(w))],$$
$$[\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\sup F_{D_1}(w), \sup F_{D_2}(w))]) : w \in W\}.$$

Example 3. 4. Let D_1 and D_2 be the IPNSs in example 3.1. Then

$D_1 \cup D_2$ = {[0.5, 0.9], [0.15, 0.25], [0.1, 0.2], [0.2, 0.3], [0.2, 0.3]}/w1 + {[0.5, 0.8], [0.25, 0.3], [0.1, 0.2], [0.15, 0.25], [0.1, 0.3]}/w2+ [0.4, 0.7], [0.1, 0.2], [0.2, 0.3], [0.15, 0.25], [0.1, 0.2]}/ w3

Theorem 3.1 For any two IPNSs D_1 and D_2:

$(a) D_1 \cup D_2 = D_2 \cup D_1$

$(b) D_1 \cap D_2 = D_2 \cap D_1$

Proof: (a):

Assume that D_1 and D_2 be any two IPNSs over W *defined by*

$$D_i = \{(w, T_{D_i}(w), C_{D_i}(w), G_{D_i}(w), U_{D_i}(w), F_{Di_2}(w))\} : w \in W\}, i = 1, 2, \text{and}\, T_{D_i}(w), C_{D_i}(w), G_{D_i}(w), U_{D_i}(w), F_{Di_2}(w))\} \subseteq [0,1]$$

We have,

$$D_1 \cup D_2 = \{(w, [\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\sup T_{D_1}(w), \sup T_{D_2}(w))],$$
$$[\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\sup C_{D_1}(w), \sup C_{D_2}(w))],$$
$$[\min(\inf G_{D_1}(w), \inf G_{D_2}(w)), \min(\sup G_{D_1}(w), \sup G_{D_2}(w))],$$
$$[\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\sup U_{D_1}(w), \sup U_{D_2}(w))],$$
$$[\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\sup F_{D_1}(w), \sup F_{D_2}(w))]) : w \in W\}$$
$$= \{w, [\max(\inf T_{D_2}(w), \inf T_{D_1}(w)), \max(\sup T_{D_2}(w), \sup T_{D_1}(w))], [\max(\inf C_{D_2}(w), \inf C_{D_1}(w)), \max(\sup C_{D_2}(w), \sup C_{D_1}(w))],$$
$$[\min(\inf G_{D_2}(w), \inf G_{D_1}(w)), \min(\sup G_{D_2}(w), \sup G_{D_1}(w))], [\min(\inf U_{D_2}(w), \inf U_{D_1}(w)),$$
$$\min(\sup U_{D_2}(w), \sup U_{D_1}(w))], [\min(\inf F_{D_2}(w), \inf F_{D_1}(w)), \min(\sup F_{D_2}(w), \sup F_{D_1}(w))]: w \in W\}$$
$$= D_2 \cup D_1$$

b) $D_1 \cap D_2 = D_2 \cap D_1$

$$D_1 \cap D_2 = \{(w, [\min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \min(\sup T_{D_1}(w), \sup T_{D_2}(w))],$$
$$[\min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \min(\sup C_{D_1}(w), \sup C_{D_2}(w))],$$
$$[\max(\inf G_{D_1}(w), \inf G_{D_2}(w)), \max(\sup G_{D_1}(w), \sup G_{D_2}(w))],$$
$$[\max(\inf U_{D_1}(w), \inf U_{D_2}(w)), \max(\sup U_{D_1}(w), \sup U_{D_2}(w))],$$
$$[\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\sup F_{D_1}(w), \sup F_{D_2}(w))]) : w \in W\}.$$
$$= \{(w, [\min(\inf T_{D_2}(w), \inf T_{D_1}(w)), \min(\sup T_{D_2}(w), \sup T_{D_1}(w))],$$
$$[\min(\inf C_{D_2}(w), \inf C_{D_1}(w)), \min(\sup C_{D_2}(w), \sup C_{D_1}(w))],$$
$$[\max(\inf G_{D_2}(w), \inf G_{D_1}(w)), \max(\sup G_{D_2}(w), \sup G_{D_1}(w))],$$
$$[\max(\inf U_{D_2}(w), \inf U_{D_1}(w)), \max(\sup U_{D_2}(w), \sup U_{D_1}(w))],$$
$$[\max(\inf F_{D_2}(w), \inf F_{D_1}(w)), \max(\sup F_{D_2}(w), \sup F_{D_1}(w))]) : \forall w \in W\}.$$
$$= D_2 \cap D_1$$

Theorem 3. 2. For any three IPNSs, $D_1, D_2, \text{and} D_3$:

$$(a) D_1 \cup (D_2 \cup D_3) = (D_1 \cup D_2) \cup D_3$$
$$(b) D_1 \cap (D_2 \cap D_3) = (D_1 \cap D_2) \cap D_3$$

Proof (a): Assume that D_1, D_2 and D_3 be any three IPNSs over W *defined by*
$D_i = \{(w, T_{D_i}(w), C_{D_i}(w), G_{D_i}(w), U_{D_i}(w), F_{D_i}(w)) : w \in W\}, i = 1, 2, 3,$ and
$T_{D_i}(w), C_{D_i}(w), G_{D_i}(w), U_{D_i}(w), F_{D_i}(w)) \subseteq [0,1], i = 1, 2, 3.$

$$\begin{aligned}
&(D_1 \cup D_2) \cup D_3) = \{(w, [\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\sup T_{D_1}(w), \sup T_{D_2}(w))],\\
&[\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\sup C_{D_1}(w), \sup C_{D_2}(w))],\\
&[\min(\inf G_{D_1}(w), \inf G_{D_2}(w)), \min(\sup G_{D_1}(w), \sup G_{D_2}(w))], [\min(\inf U_{D_1}(w), \inf U_{D_2}(w)),\\
&\min(\sup U_{D_1}(w), \sup U_{D_2}(w))], [\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\sup F_{D_1}(w), \sup F_{D_2}(w))]) : w \in W\} \cup\\
&\{w([\inf G_{D_3}(w), \sup G_{D_3}(w)), [\inf U_{D_3}(w), \sup U_{D_3}(w)], [\inf F_{D_3}(w), \sup F_{D_3}]) : w \in W\}\\
&= \{(w, [\max(\inf T_{D_1}(w), \inf T_{D_2}(w), \inf T_{D_3}(w)), \max(\sup T_{D_1}(w), \sup T_{D_2}(w), \sup T_{D_3}(w))],\\
&[\max(\inf C_{D_1}(w), \inf C_{D_2}(w), \inf C_{D_3}(w)), \max(\sup C_{D_1}(w), \sup C_{D_2}(w), \sup C_{D_3}(w))],\\
&[\min(\inf G_{D_1}(w), \inf G_{D_2}(w), \inf G_{D_3}(w)), \min(\sup G_{D_1}(w), \sup G_{D_2}(w), \sup G_{D_3}(w))],\\
&[(\inf U_{D_1}(w), \inf U_{D_2}(w), \inf U_{D_3}(w)), \min(\sup U_{D_1}(w), \sup U_{D_2}(w), \sup U_{D_3}(w))],\\
&[(\inf F_{D_1}(w), \inf F_{D_2}(w), \inf F_{D_3}(w)), \min(\sup F_{D_1}(w), \sup F_{D_2}(w), \sup F_{D_3}(w))]) : w \in W\}
\end{aligned}$$

$$\begin{aligned}
&= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),\\
&[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \cup\\
&\{(w, [\max(\inf T_{D_2}(w), \inf T_{D_3}(w)), \max(\sup T_{D_2}(w), \sup T_{D_3}(w))],\\
&[\max(\inf C_{D_2}(w), \inf C_{D_3}(w)), \max(\sup C_{D_2}(w), \sup C_{D_3}(w))],\\
&[\min(\inf G_{D_2}(w), \inf G_{D_3}(w)), \min(\sup G_{D_2}(w), \sup G_{D_3}(w))],\\
&[\min(\inf U_{D_2}(w), \inf U_{D_3}(w)), \min(\sup U_{D_2}(w), \sup U_{D_3}(w))],\\
&[\min(\inf F_{D_2}(w), \inf F_{D_3}(w)), \min(\sup F_{D_2}(w), \sup F_{D_3}(w))]) : w \in W\}\\
&= D_1 \cup (D_2 \cup D_3)
\end{aligned}$$

Proof. (b):

$$\begin{aligned}
&D_1 \cap (D_2 \cap D_3)\\
&= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),\\
&[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \cap\\
&\{(w, [\min(\inf T_{D_2}(w), \inf T_{D_3}(w)), \min(\sup T_{D_2}(w), \sup T_{D_3}(w))],\\
&[\min(\inf C_{D_2}(w), \inf C_{D_3}(w)), \min(\sup C_{D_2}(w), \sup C_{D_3}(w))],\\
&[\max(\inf G_{D_2}(w), \inf G_{D_3}(w)), \max(\sup G_{D_2}(w), \sup G_{D_3}(w))],\\
&[\max(\inf U_{D_2}(w), \inf U_{D_3}(w)), \max(\sup U_{D_2}(w), \sup U_{D_3}(w))],\\
&[\max(\inf F_{D_2}(w), \inf F_{D_3}(w)), \max(\sup F_{D_2}(w), \sup F_{D_3}(w))]) : w \in W\}.
\end{aligned}$$

$= \{(w, [\min(\inf T_{D_1}, \inf T_{D_2}(w), \inf T_{D_3}(w)), \min(\sup T_{D_1}(w), \sup T_{D_2}(w), \sup T_{D_3}(w))],$

$[\min(\inf C_{D_1}(w), \inf C_{D_2}(w), \inf C_{D_3}(w)), \min(\sup C_{D_1}(w), \sup C_{D_2}(w), \sup C_{D_3}(w))],$

$[\max(\inf G_{D_1}(w), \inf G_{D_2}(w), \inf G_{D_3}(w)), \max(\sup G_{D_1}(w), \sup G_{D_2}(w), \sup G_{D_3}(w))],$

$[\max(\inf U_{D_1} \inf U_{D_2}(w), \inf U_{D_3}(w)), \max(\sup U_{D1}, \sup U_{D_2}(w), \sup U_{D_3}(w))],$

$[\max(\inf F_{D_1}(w), \inf F_{D_2}(w), \inf F_{D_3}(w)), \max(\sup F_{D_1}(w), \sup F_{D_2}(w), \sup F_{D_3}(w))]) : w \in W\}$

$=$

$\{(w, [\min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \min(\sup T_{D_1}(w), \sup T_{D_2}(w))],$

$[\min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \min(\sup C_{D_1}(w), \sup C_{D_2}(w))],$

$[\max(\inf G_{D_1}(w), \inf G_{D_2}(w)), \max(\sup G_{D_1}(w), \sup G_{D_2}(w))],$

$[\max(\inf U_{D_1}(w), \inf U_{D_2}(w)), \max(\sup U_{D_1}(w), \sup U_{D_2}(w))],$

$[\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\sup F_{D_1}(w), \sup F_{D_2}(w))]) : w \in W\} \cap$

$\{w([\inf G_{D_3}(w), \sup G_{D_3}(w)), [\inf U_{D_3}(w), \sup U_{D_3}(w)], [\inf F_{D_3}(w), \sup F_{D_3}]) : w \in W\}$

$= (D_1 \cap D_2) \cap D_3$

Theorem 3.3. For any two IPNSs, $D_1, \text{and} D_2$:

(a) $D_1 \cup (D_1 \cap D_2) = D_1$

(b) $D_1 \cap (D_1 \cup D_2) = D_1$

Proof .(a):

$D_1 \cup (D_1 \cap D_2) =$

$\{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),$

$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$

$\cup$

$\{(w, [\min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \min(\sup T_{D_1}(w), \sup T_{D_2}(w))],$

$[\min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \min(\sup C_{D_1}(w), \sup C_{D_2}(w))],$

$[\max(\inf G_{D_1}(w), \inf G_{D_2}(w)), \max(\sup G_{D_1}(w), \sup G_{D_2}(w))],$

$[\max(\inf U_{D_1}(w), \inf U_{D_2}(w)), \max(\sup U_{D_1}(w), \sup U_{D_2}(w))],$

$[\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\sup F_{D_1}(w), \sup F_{D_2}(w))]) : w \in W\}.$

$=$

$\{w, ([\max(\inf T_{D_1}(w), \min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\sup T_{D_1}(w), \min(\sup T_{D_1}(w), \sup T_{D_2}(w))],$

$[\max(\inf C_{D_1}(w), \min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max((\sup C_{D_1}(w), \min(\sup C_{D_1}(w), \sup C_{D_2}(w))]),$

$([\min(\inf (G_{D_1}(w), \max(\inf G_{D_1}(w), \inf G_{D_2}(w))), \min(\sup G_{D_1}(w), \max(\sup G_{D_1}(w), \sup G_{D_2}(w))],$

$[\min(\inf (U_{D_1}(w), \max(\inf U_{D_1}(w), \inf U_{D_2}(w))), \min(\sup U_{D_1}(w), \max(\sup U_{D_1}(w), \sup U_{D_2}(w))],$

$[\min(\inf (F_{D_1}(w), \max(\inf F_{D_1}(w), \inf F_{D_2}(w))), \min(\sup F_{D_1}(w), \max(\sup F_{D_1}(w), \sup F_{D_2}(w))]) : w \in W\}$

$=$

$\{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),$

$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$

$= D_1$

Proof (b):

$$D_1 \cap (D_1 \cup D_2)$$
$$= \langle w, (\langle \inf T_{D_1}(w), \sup T_{D_1}(w)\rangle, \langle \inf C_{D_1}(w), \sup C_{D_1}(w)\rangle,$$
$$\langle \inf G_{D_1}(w), \sup G_{D_1}(w)), \langle \inf U_{D_1}(w), \sup U_{D_1}(w)\rangle, \langle \inf F_{D_1}(w), \sup F_{D_1}(w)\rangle\rangle : w \in \rangle \cap$$
$$\langle (w, \langle \max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\sup T_{D_1}(w), \sup T_{D_2}(w))\rangle,$$
$$\langle \max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\sup C_{D_1}(w), \sup C_{D_2}(w))\rangle,$$
$$\langle \min(\inf G_{D_1}(w), \inf G_{D_2}(w)), \min(\sup G_{D_1}(w), \sup G_{D_2}(w))\rangle,$$
$$\langle \min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\sup U_{D_1}(w), \sup U_{D_2}(w))\rangle,$$
$$\langle \min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\sup F_{D_1}(w), \sup F_{D_2}(w))\rangle\rangle : w \in W\rangle$$
$$= \langle w, \langle \min((\inf T_{D_1}(w), \max(\inf T_{D_1}(w), \inf T_{D_2}(w))), \min((\sup T_{D_1}(w), \max(\sup T_{D_1}(w), \sup T_{D_2}(w))\rangle,$$
$$\langle \min((\inf C_{D_1}(w), \max(\inf C_{D_1}(w), \inf C_{D_2}(w))), \min((\sup C_{D_1}(w), \max(\sup C_{D_1}(w), \sup C_{D_2}(w))\rangle,$$
$$\langle \max((\inf G_{D_1}(w), \min(\inf G_{D_1}(w), \inf G_{D_2}(w))), \max((\sup G_{D_1}(w)), \min(\sup G_{D_1}(w), \sup G_{D_2}(w))\rangle,$$
$$\langle \max((\inf U_{D_1}(w), \min(\inf U_{D_1}(w), \inf U_{D_2}(w))), \max((\sup U_{D_1}(w), \min(\sup U_{D_1}(w), \sup U_{D_2}(w))\rangle,$$
$$\langle \max((\inf F_{D_1}(w), \min(\inf F_{D_1}(w), \inf F_{D_2}(w))), \max((\sup F_{D_1}(w), \min(\sup F_{D_1}(w), \sup F_{D_2}(w))\rangle\rangle : w \in W\rangle$$
$$= \langle w, (\langle \inf T_{D_1}(w), \sup T_{D_1}(w)\rangle, \langle \inf C_{D_1}(w), \sup C_{D_1}(w)\rangle,$$
$$\langle \inf G_{D_1}(w), \sup G_{D_1}(w)), \langle \inf U_{D_1}(w), \sup U_{D_1}(w)\rangle, \langle \inf F_{D_1}(w), \sup F_{D_1}(w)\rangle\rangle : w \in \rangle$$
$$= D_1$$

Theorem 3.4. For any IPNS D_1:

(a) $D_1 \cup D_1 = D_1$

(b) $D_1 \cap D_1 = D_1$

Proof. (a):

$$D_1 \cup D_1 = \langle w, (\langle \inf T_{D_1}(w), \sup T_{D_1}(w)\rangle, \langle \inf C_{D_1}(w), \sup C_{D_1}(w)\rangle, \langle \inf G_{D_1}(w), \sup G_{D_1}(w)),$$
$$\langle \inf U_{D_1}(w), \sup U_{D_1}(w)\rangle, \langle \inf F_{D_1}(w), \sup F_{D_1}(w)\rangle\rangle : w \in W\rangle \cup \langle w, (\langle \inf T_{D_1}(w), \sup T_{D_1}(w)\rangle,$$
$$\langle \inf C_{D_1}(w), \sup C_{D_1}(w)\rangle, \langle \inf G_{D_1}(w), \sup G_{D_1}(w)), \langle \inf U_{D_1}(w), \sup U_{D_1}(w)\rangle,$$
$$\langle \inf F_{D_1}(w), \sup F_{D_1}(w)\rangle\rangle : w \in W\rangle$$

$$= \langle w, (\langle \max(\inf T_{D_1}(w), \inf T_{D_1}(w)), \max(\sup T_{D_1}(w), \sup T_{D_1}(w))\rangle,$$
$$\langle \max(\inf C_{D_1}(w), (\inf C_{D_1}(w))), \max(\sup T_{D_1}(w), \sup C_{D_1}(w)\rangle),$$
$$\langle \min(\inf G_{D_1}(w), \sup G_{D_1}(w)), \langle \min(\inf U_{D_1}(w), \inf U_{D_1}(w)), (\min(\sup U_{D_1}(w), (\mathrm{suf}\ U_{D_1}(w))\rangle,$$
$$\langle \min(\inf F_{D_1}(w), \inf F_{D_1}(w)), \min(\sup F_{D_1}(w), \sup F_{D_1}(w))\rangle\rangle : w \in W\rangle$$
$$= \langle w, (\langle \inf T_{D_1}(w), \sup T_{D_1}(w)\rangle, \langle \inf C_{D_1}(w), \sup C_{D_1}(w)\rangle, \langle \inf G_{D_1}(w), \sup G_{D_1}(w)),$$
$$\langle \inf U_{D_1}(w), \sup U_{D_1}(w)\rangle, \langle \inf F_{D_1}(w), \sup F_{D_1}(w)\rangle\rangle : w \in W\rangle$$
$$= D_1$$

Proof. (b):

$$D_1 \cap D_1 = \langle w, (\langle \inf T_{D_1}(w), \sup T_{D_1}(w)\rangle, \langle \inf C_{D_1}(w), \sup C_{D_1}(w)\rangle, \langle \inf G_{D_1}(w), \sup G_{D_1}(w)),$$
$$\langle \inf U_{D_1}(w), \sup U_{D_1}(w)\rangle, \langle \inf F_{D_1}(w), \sup F_{D_1}(w)\rangle\rangle : w \in W\rangle \cup \langle w, (\langle \inf T_{D_1}(w), \sup T_{D_1}(w)\rangle,$$
$$\langle \inf C_{D_1}(w), \sup C_{D_1}(w)\rangle, \langle \inf G_{D_1}(w), \sup G_{D_1}(w)), \langle \inf U_{D_1}(w), \sup U_{D_1}(w)\rangle,$$
$$\langle \inf F_{D_1}(w), \sup F_{D_1}(w)\rangle\rangle : w \in W\rangle$$

$$= \{(w, \langle [\min(\inf T_{D_1}(w), \inf T_{D_1}(w)), \min(\sup T_{D_1}(w), \sup T_{D_1}(w))],$$
$$[\min(\inf C_{D_1}(w), \inf C_{D_1}(w)), \min(\sup C_{D_1}(w), \sup C_{D_1}(w))],$$
$$[\max(\inf G_{D_1}(w), \inf G_{D_1}(w)), \max(\sup G_{D_1}(w), \sup G_{D_1}(w))],$$
$$[\max(\inf U_{D_1}(w), \inf U_{D_1}(w)), \max(\sup U_{D_1}(w), \sup U_{D_1}(w))],$$
$$[\max(\inf F_{D_1}(w), \inf F_{D_1}(w)), \max(\sup F_{D_1}(w), \sup F_{D_1}(w))]\rangle) : \forall w \in W\}'$$

$$= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)),$$
$$[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$$
$$= D_1$$

Theorem 3.5 For any IPNS D_1,

$$\text{(a)} D_1 \cap \hat{0} = \hat{0}$$
$$\text{(b)} D_1 \cup \hat{1} = \hat{1}$$

Proof. (a):

$$D_1 \cap \hat{0}$$
$$= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)],$$
$$[\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \cap \{\langle[0, 0], [0, 0], [1, 1], [1, 1], [1, 1]\rangle\}$$
$$= \{(w, \langle[\min(\inf T_{D_1}(w), 0), \min(\sup T_{D_1}(w), 0)],$$
$$[\min(\inf C_{D_1}(w), 0), \min(\sup C_{D_1}(w), 0)],$$
$$[\max(\inf G_{D_1}(w), 1), \max(\sup G_{D_1}(w), 1)],$$
$$[\max(\inf U_{D_1}(w), 1), \max(\sup U_{D_1}(w), 1)],$$
$$[\max(\inf F_{D_1}(w), 1), \max(\sup F_{D_1}(w), 1]) : w \in W\}$$
$$= \{(w, \langle[0, 0)], [0, 0)], [1, 1], [1, 1], [1, 1]), w \in W\}$$
$$= \hat{0}$$

Proof. (b):

$$D_1 \cup \hat{1}$$
$$= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),$$
$$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \cup \{\langle[1, 1], [1, 1], [0, 0], [0, 0], [0, 0]\rangle\}$$
$$= \{(w, \langle[\max(\inf T_{D_1}(w), 1), \max(\sup T_{D_1}(w), 1))],$$
$$[\max(\inf C_{D_1}(w), 1), \max(\sup C_{D_1}(w), 1)],$$
$$[\min(\inf G_{D_1}(w), 0), \min(\sup G_{D_1}(w), 0)],$$
$$[\min(\inf U_{D_1}(w), 0), \min(\sup U_{D_1}(w), 0)],$$
$$[\min(\inf F_{D_1}(w), 0)), \min(\sup F_{D_1}(w), 0)]) : w \in W\}$$
$$= \{w([1, 1], [1, 1], [0, 0], [0, 0], [0, 0]) : w \in W\}$$
$$= \hat{1}$$

Theorem 3.6 For any IPNS D_1,

$$\text{(a)} D_1 \cup \hat{0} = D_1$$
$$\text{(b)} D_1 \cap \hat{1} = D_1$$

Proof. (a):

$D_1 \cup \hat{0}$

$== \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)],$

$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \cup \{\langle [0,0],[0,0],[1,1],[1,1],[1,1]\rangle\}$

$= \{(w, [\max(\inf T_{D_1}(w), 0), \max(\sup T_{D_1}(w), 0))],$

$[\max(\inf C_{D_1}(w), 0), \max(\sup C_{D_1}(w), 0)],$

$[\min(\inf G_{D_1}(w), 1), \min(\sup G_{D_1}(w), 1)],$

$[\min(\inf U_{D_1}(w), 1), \min(\sup U_{D_1}(w), 1)],$

$[\min(\inf F_{D_1}(w), 1)), \min(\sup F_{D_1}(w), 1)]) : w \in W\}$

$w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)],$

$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$

$= D_1$

$D_1 \cap \hat{1}$

$= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)], [\inf G_{D_1}(w), \sup G_{D_1}(w)),$

$[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \cap \{\langle [1,1],[1,1],[0,0],[0,0],[0,0]\rangle\}$

$= \{(w, [\min(\inf T_{D_1}(w), 1), \min(\sup T_{D_1}(w), 1)], [\min(\inf C_{D_1}(w), 1), \min(\sup C_{D_1}(w), 1)],$

$[\max(\inf G_{D_1}(w), 0), \max(\sup G_{D_1}(w), 0)], [\max(\inf U_{D_1}(w), 0), \max(\sup U_{D_1}(w), 0)],$

$[\max(\inf F_{D_1}(w), 1), \max(\sup F_{D_1}(w), 1]) : \forall w \in W\}$

$= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)], [\inf G_{D_1}(w), \sup G_{D_1}(w)),$

$[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$

$= D_1$

Theorem 3.7 For any IPNS D_1, $(D_1')' = D_1$

Assume that $D_1 = \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)], [\inf G_{D_1}(w), \sup G_{D_1}(w)),$

$[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$

$D_1' = \{w, ([\inf F_{D_1}(w), \sup F_{D_1}(w)], [\inf U_{D_1}(w), \sup U_{D_1}(w)], [1 - \sup G_{D_1}(w), 1 - \inf G_{D_1}(w)],$

$[\inf C_{D_1}(w), \sup C_{D_1}(w)], [\inf T_{D_1}(w), \sup T_{D_1}(w)]) : w \in W\}$

$\therefore (D_1')' = \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)], [\inf G_{D_1}(w), \sup G_{D_1}(w)),$

$[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$

$= D_1$

Theorem 3.8. For any two IPNSs, D_1 and D_2:

$(a)(D_1 \cup D_2)' = D_1' \cap D_2'$

$(b)(D_1 \cap D_2)' = D_1' \cup D_2'$

Proof. (a):

To prove the theorem 3.8, we need some propositions:

i. If $P \subset \mathbb{R}$ and $a \in \mathbb{R}$, then

$aP = \{ y \in \mathbb{R} : y = ax \text{ for some } x \in P \}$.

Proposition 1. *If* $a \geq 0$,

$$\begin{aligned}&\text{then } \sup aP = a\sup P,\ \inf aP = a\inf P,\\ &\text{if } a<0, \text{ then}\\ &\sup aP = a\inf P,\ \text{Inf } aP = a\sup P.\end{aligned}$$

In particular, $\sup(-P) = -\inf P,\ Inf\,(-P) = -\,sup\,P.$

Proposition 2. If P and Q are nonempty set, then

sup(P + Q) = sup P+ sup Q, inf (P + Q) = inf P + inf Q

sup(P - Q) = sup P - inf Q, inf (P - Q) = inf P + inf Q

Now,

$$\begin{aligned}D_1 \cup D_2 = \langle(w, &\langle\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\sup T_{D_1}(w), \sup T_{D_2}(w))\rangle,\\ &\langle\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\sup C_{D_1}(w), \sup C_{D_2}(w))\rangle,\\ &\langle\min(\inf G_{D_1}(w), \inf G_{D_2}(w)), \min(\sup G_{D_1}(w), \sup G_{D_2}(w))\rangle,\\ &\langle\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\sup U_{D_1}(w), \sup U_{D_2}(w))\rangle,\\ &\langle\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\sup F_{D_1}(w), \sup F_{D_2}(w))\rangle) : w \in W\rangle\end{aligned}$$

$$\begin{aligned}(D_1 \cup D_2)' = \langle(w, &\langle\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\sup F_{D_1}(w), \sup F_{D_2}(w))\rangle,\\ &\langle\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\sup U_{D_1}(w), \sup U_{D_2}(w))\rangle,\\ &\langle 1-\sup(\min(\sup G_{D_1}(w), \sup G_{D_2}(w)), 1-\inf((\min(\inf G_{D_1}(w), \inf G_{D_2}(w))\rangle,\\ &\langle\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\sup C_{D_1}(w), \sup C_{D_2}(w))\rangle,\\ &\langle\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\sup T_{D_1}(w), \sup T_{D_2}(w))\rangle) : w \in W\rangle \qquad (1)\end{aligned}$$

$$\begin{aligned}D_1' \cap D_2' = &\langle(w, \langle\inf F_{D_1}(w), \sup F_{D_1}(w)\rangle, \langle\inf U_{D_1}(w), \sup U_{D_1}(w)\rangle,\\ &\langle 1-\operatorname{suf} G_{D_1}(w), 1-\inf G_{D_1}(w)\rangle, \langle\inf C_{D_1}(w), \sup C_{D_1}(w)\rangle,\\ &\langle\inf T_{D_1}(w), \sup T_{D_1}(w)\rangle) : w \in W\rangle \cap \langle w, (\langle\inf F_{D_2}(w), \sup F_{D_2}(w)\rangle,\\ &\langle\inf U_{D_2}(w), \sup U_{D_2}(w)\rangle, \langle 1-\sup G_{D_2}(w), 1-\inf G_{D_2}(w)\rangle,\\ &\langle\inf C_{D_2}(w), \sup C_{D_2}(w)\rangle, \langle\inf T_{D_2}(w), \sup T_{D_2}(w)\rangle) : w \in W\rangle\\ = &\langle(w, \langle\min(\inf F_{D_1}(w), \operatorname{minf} F_{D_2}(w), \min(\sup F_{D_1}(w), \sup F_{D_2}(w)\rangle,\\ &\langle\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\sup U_{D_1}(w), \sup U_{D_2}(w))\rangle,\\ &\langle\max(\inf(1-\sup G_{D_1}(w)), \inf(1-\sup G_{D_2}(w))), \max(\sup(1-\inf G_{D_1}(w)), \sup(1-\inf G_{D_2}(w)))\rangle,\\ &\langle\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\sup C_{D_1}(w), \sup C_{D_2}(w))\rangle,\\ &\langle\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\sup T_{D_1}(w), \sup T_{D_2}(w))\rangle) : w \in W\rangle \qquad (2)\end{aligned}$$

Now, the theorem 3.8. (a) will be proved, if we can prove that

$$1-\sup(\min(\sup G_{D_1}(w), \sup G_{D_2}(w))) = \max(\inf(1-\sup G_{D_1}(w)), \inf(1-\sup G_{D_2}(w)))$$

$$1-\inf(\min(\inf G_{D_1}(w), \inf G_{D_2}(w))) \circledcirc \max(\sup(1-\inf G_{D_1}(w)), \sup(1-\inf G_{D_2}(w)))$$

Now, assume that

$$\begin{aligned}&\inf G_{D_1}(w) = c_1, \operatorname{sur} G_{D_1}(w) = d_1\\ &\inf G_{D_2}(w) = c_2, \operatorname{sur} G_{D_1}(w) = d_2\end{aligned}$$

$$1-\sup(\min(\text{suf } G_{D_1}(w), \text{suf } G_{D_2}(w)))$$
$$=1-\sup(\min(d_1, d_2))$$
$$=\begin{cases}1-\sup d_1, & \text{if } d_1 \geq d_2\\ 1-\sup d_2, & \text{if } d_1 \leq d_2\end{cases}$$
$$=\begin{cases}1-d_1, & \text{if } d_1 \geq d_2\\ 1-d_2, & \text{if } d_1 \leq d_2\end{cases} \qquad (3)$$
$$\max(\inf(1-\sup G_{D_1}(w)), \inf(1-\sup G_{D_2}(w)))$$
$$=\max(\inf(1-d_1), \inf(1-d_2))$$
$$=\max(1-\sup d_1, 1-\sup d_2)) \text{ by proposition 2.}$$
$$=\max(1-d_1, 1-d_2))$$
$$=\begin{cases}1-d_1, & \text{if } d_1 \geq d_2\\ 1-d_2, & \text{if } d_1 \leq d_2\end{cases} \qquad (4)$$

Therefore, from (3) and (4), we have

$$1-\sup(\min(suf\ G_{D_1}(w), suf\ G_{D_2}(w)) = \max(\inf(1-\sup G_{D_1}(w)), \inf(1-\sup G_{D_2}(w))) \qquad (5)$$

Now,

$$1-\inf(\min(\inf G_{D_1}(w), \inf G_{D_2}(w)))$$
$$=1-\inf(\min(c_1, c_2))$$
$$=\begin{cases}1-\inf c_1, & \text{if } c_1 \geq c_2\\ 1-\inf c_2, & \text{if } c_1 \leq c_2\end{cases}$$
$$=\begin{cases}1-c_1, & \text{if } c_1 \geq c_2\\ 1-c_2, & \text{if } c_1 \leq c_2\end{cases} \qquad (6)$$
$$\max(\sup(1-\inf G_{D_1}(w)), \sup(1-\inf G_{D_2}(w)))$$
$$=\max(\sup(1-c_1), \sup(1-c_2))$$
$$=\max(1-\inf c_1, 1-\inf c_2)$$
$$=\begin{cases}1-\inf c_1, & \text{if } c_1 \geq c_2\\ 1-\inf c_2, & \text{if } c_1 \leq c_2\end{cases}$$
$$=\begin{cases}1-c_1, & \text{if } c_1 \geq c_2\\ 1-c_2, & \text{if } c_1 \leq c_2\end{cases} \qquad (7)$$

Therefore, from (6) and (7), we have

$$1-\inf((\min(\inf G_{D_1}(w), \inf G_{D_2}(w)) = \max(\sup(1-\inf G_{D_1}(w)), \sup(1-\inf G_{D_2}(w))) \qquad (8)$$

Therefore from (1), (2), (5) and (8), we prove that

$$(D_1 \cup D_2)' = D_1' \cap D_2'.$$

Proof. (b):

$$(D_1 \cap D_2)' = \circledcirc(w, \circledcirc\min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \min(\sup T_{D_1}(w), \sup T_{D_2}(w))\circledcirc,$$
$$\circledcirc\min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \min(\sup C_{D_1}(w), \sup C_{D_2}(w))\circledcirc,$$
$$\circledcirc\max(\inf G_{D_1}(w), \inf G_{D_2}(w)), \max(\sup G_{D_1}(w), \sup G_{D_2}(w))\circledcirc,$$
$$\circledcirc\max(\inf U_{D_1}(w), \inf U_{D_2}(w)), \max(\sup U_{D_1}(w), \sup U_{D_2}(w))\circledcirc,$$
$$\circledcirc\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\sup F_{D_1}(w), \sup F_{D_2}(w))\circledcirc) : w \in W\circledcirc'$$
$$= \circledcirc(w, \circledcirc\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\sup F_{D_1}(w), \sup F_{D_2}(w))\circledcirc),$$
$$\circledcirc\max(\inf U_{D_1}(w), \inf U_{D_2}(w)), \max(\sup U_{D_1}(w), \sup U_{D_2}(w))\circledcirc,$$
$$\circledcirc 1 - \sup(\max(\sup G_{D_1}(w), \sup G_{D_2}(w)), 1 - \inf\max(\inf U_{D_1}(w), \inf U_{D_2}(w))\circledcirc,$$
$$\circledcirc\min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \min(\sup C_{D_1}(w), \sup C_{D_2}(w))\circledcirc,$$
$$\circledcirc\min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \min(\sup T_{D_1}(w), \sup T_{D_2}(w))\circledcirc) : w \in W\circledcirc \qquad (9)$$

Now $D_1' \cup D_2' \circledcirc$

$$\circledcirc w, (\circledcirc\inf F_{D_1}(w), \sup F_{D_1}(w)\circledcirc, \circledcirc\inf U_{D_1}(w), \sup U_{D_1}(w)\circledcirc, \circledcirc 1 - \sup G_{D_1}(w), 1 - \inf G_{D_1}(w)\circledcirc, \circledcirc\inf C_{D_1}(w), \sup C_{D_1}(w)\circledcirc),$$
$$\circledcirc\inf T_{D_1}(w), \sup T_{D_1}(w)\circledcirc) : w \in W\circledcirc \cup \circledcirc w, (\circledcirc\inf F_{D_2}(w), \sup F_{D_2}(w)\circledcirc, \circledcirc\inf U_{D_2}(w), \sup U_{D_2}(w)\circledcirc,$$
$$\circledcirc 1 - \sup G_{D_2}(w), 1 - \inf G_{D_2}(w)\circledcirc, \circledcirc\inf C_{D_2}(w), \sup C_{D_2}(w)\circledcirc), \circledcirc\inf T_{D_2}(w), \sup T_{D_2}(w)\circledcirc) : w \in W\circledcirc$$
$$= \circledcirc w, \circledcirc\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\sup F_{D_1}(w)\circledcirc, \sup F_{D_2}(w)\circledcirc,$$
$$\circledcirc\max(\inf U_{D_1}(w), \inf U_{D_2}(w)), \max(\sup U_{D_1}(w)\circledcirc, \sup U_{D_2}(w)\circledcirc,$$
$$\circledcirc\min((1 - \mathrm{suf}\, G_{D_1}(w), 1 - \mathrm{suf}\, G_{D_2}(w)), \min((1 - \inf G_{D_1}(w), 1 - \inf G_{D_2}(w))\circledcirc,$$
$$\circledcirc\min((\inf C_{D_1}(w), \inf C_{D_2}(w)), \min((\sup C_{D_1}(w), \sup C_{D_2}(w))\circledcirc) : w \in W\circledcirc \qquad (10)$$

To prove the theorem 3.8. (b), we are to prove

$$1 - \sup(\max(\mathrm{suf}\, G_{D_1}(w), \sup G_{D_2}(w))) = \min(1 - \sup G_{D_1}(w), 1 - \sup G_{D_2}(w))$$

$$1 - \inf\max(\inf U_{D_1}(w), \inf U_{D_2}(w)) \circledcirc \min(1 - \inf G_{D_1}(w), 1 - \inf G_{D_2}(w)) .$$

Now

$$1 - \sup(\max(\sup G_{D_1}(w), \sup G_{D_2}(w))$$
$$= 1 - \sup(\max(d_1, d_2)$$
$$= \begin{cases} \{1 - \sup d_1, \text{if } d_1 \ge d_2 \\ \{1 - \sup d_2, \text{if } d_1 \ge d_2 \end{cases}$$
$$= \begin{cases} \{1 - d_1, \text{if } d_1 \ge d_2 \\ \{1 - d_2, \text{if } d_1 \ge d_2 \end{cases} \qquad (11)$$

$$\min((1 - \sup G_{D_1}(w), 1 - \sup G_{D_2}(w))$$
$$= \min(1 - d_1, 1 - d_2)$$
$$= \begin{cases} 1 - d_1, \text{if } d_1 \ge d_2 \\ 1 - d_2, \text{if } d_1 \le d_2 \end{cases} \qquad (12)$$

Therefore from (11) and (12), we have

$$1 - \sup(\max(\sup G_{D_1}(w), \sup G_{D_2}(w))) = \min(1 - \sup G_{D_1}(w), 1 - \sup G_{D_2}(w)) \qquad (13)$$

Now

$$1-\inf\max(\inf U_{D_1}(w),\inf U_{D_2}(w))$$
$$=1-\inf(\max(c_1,c_2))$$
$$=\begin{cases}1-\inf c_1, \text{if } c_1\geq c_2\\ 1-\inf c_2, \text{if } c_1\leq c_2\end{cases}$$
$$=\begin{cases}1-c_1, \text{if } c_1\geq c_2\\ 1-c_2, \text{if } c_1\leq c_2\end{cases} \quad (14)$$

$$\min((1-\inf G_{D_1}(w),1-\inf G_{D_2}(w))$$
$$=\min((1-c_1,1-c_2)$$
$$=\begin{cases}1-c_1, \text{if } c_1\geq c_2\\ 1-c_2, \text{if } c_1\leq c_2\end{cases} \quad (15)$$

Therefore, from (14) and (15), we have

$$1-\inf\max(\inf U_{D_1}(w),\inf U_{D_2}(w)) = \min(1-\inf G_{D_1}(w),1-\inf G_{D_2}(w)) \quad (16)$$

Therefore, from (9), (10,), (13) and (16),

$(D_1\cap D_2)' = D_1'\cup D_2'$.

Theorem 3.9. For any two IPNSs D_1, D_2,

$$D_1\subseteq D_2 \Leftrightarrow D_2'\subseteq D_1'.$$

Proof.

$$D_1\subseteq D_2 \Leftrightarrow$$
$$\inf T_{D_1}(w)\leq\inf T_{D_2}(w), \sup T_{D_1}(w)\leq \sup T_{D_2}(w),$$
$$\inf C_{D_1}(w)\leq\inf C_{D_2}(w), \sup C_{D_1}(w)\leq \sup C_{D_2}(w),$$
$$\inf G_{D_1}(w)\geq\inf G_{D_2}(w), \sup G_{D_1}(w)\geq \sup G_{D_2}(w),$$
$$\inf U_{D_1}(w)\geq\inf U_{D_2}(w), \sup U_{D_1}(w)\geq \sup U_{D_2}(w),$$
$$\inf F_{D_1}(w)\geq\inf F_{D_2}(w), \sup F_{D_1}(w)\geq \sup F_{D_2}(w),$$
$$\Leftrightarrow$$

$$\inf F_{D_2}(w)\leq\inf F_{D_1}(w), \sup F_{D_2}(w)\leq \sup F_{D_1}(w),$$
$$\inf U_{D_2}(w)\leq\inf U_{D_2}(w), \sup U_{D_2}(w)\leq \sup U_{D_1}(w),$$
$$1-\sup G_{D_2}(w)\geq 1-\sup G_{D_1}(w), 1-\inf G_{D_2}(w)\geq 1-\sup G_{D_1}(w),$$
$$\inf T_{D_2}\geq\inf T_{D_1},\ \sup T_{D_2}(w)\geq \sup T_{D_1}(w)$$
$$\Leftrightarrow$$
$$D_2'\subseteq D_1'$$

Note1: We establish the following properties of IPNSs:

1. Commutativity
2. Associativity
3. Idempotency
4. Absorption
5. De Morgan's laws
6. Involution

Note 2. IPNS may also be called as Interval Pentapartitioned Single Valued Neutrosophic Set (IPSVNS).

4. Conclusion

In this paper, we develop the notion of IPNS by combing the concept of PNS and INS. We define the notion of inclusion, complement, intersection, union of IPNSs. We prove some of the properties of IPNSs, namely, commutativity, associativity, idempotency, absorption, De Morgan's laws and involution. In the future, we shall develop the logic system based on the truth-value based IPNSs and utilize the theory to deal with practical applications in the areas such as information fusion, bioinformatics, military intelligence, web intelligence, etc.

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An Asymmetric Measure of Comparison of Neutrosophic Sets

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Abstract: The single-valued neutrosophic (SVNS) set is a beneficial and significant tool to deal with uncertainty with the neutrality of truth. This article introduces a non-conventional asymmetric measure of comparison in the single-valued neutrosophic framework. Such a measure is applied to the problems where the conventional symmetric measures of comparison do not produce valid computational results and require a directed closeness or discrimination between two abstractions represented by neutrosophic sets. We prove some properties of the proposed neutrosophic comparison measure and empirically justify its utilization in a problem of strategic decision-making, pattern recognition and medical diagnosis. The assessment of the performance of the proposed measure using "Degree of Confidence" shows the advantage of the proposed measure.

Keywords: Single-valued neutrosophic set (SVNS); asymmetric measure; inaccuracy measure; pattern recognition.

1. Introduction

Many problems concerning decision-making, identification of patterns, machine learning, computer vision, data analytics, etc., predominantly utilize some measure of comparisons. Several studies are available regarding comparison measures in various uncertain and vague settings. The prevalently investigated comparison measures in uncertain environments are divergence, distance, dissimilarity, and similarity measures. One common characteristic of divergence measures, distance measures, dissimilarity measures, and similarity measures in fuzzy and non-standard fuzzy settings is that these are symmetric. But, in specific comparisons, the symmetric comparison is not suitable. For instance, "P is like Q" may be preferred over "Q is like P" or vice-versa. Such situations need an asymmetric or directed comparison measure. This study proposes an asymmetric measure of comparison for single-valued neutrosophic sets.

However, in an uncertain environment due to randomness, an asymmetric measure of comparison of two probability distributions was proposed by Kullback-Leibler [1-2]. These measures found vital application in communication theory and economics. Kerridge inaccuracy is a non-parametric generalization of Shannon's entropy [3]. Kerridge [4] termed it an 'inaccuracy measure' for measuring the inaccuracy between two probability distributions. Moreover, numerous probabilistic information measures were put forward during the second half of the twentieth century.

In 1965, Zadeh [5] coined a new form of uncertainty due to vagueness or linguistic imprecision and developed fuzzy theory. As Shannon's entropy [3] quantifies the uncertainty due to randomness, De-Luca and Termini [6] introduced fuzzy entropy to quantify the uncertainty due to vagueness. This pioneered fuzzy entropy is structurally similar to that of Shannon's entropy [3] but practically different. Various extensions of fuzzy theory and information measures in these frameworks have been developed in the last three decades.

Smarandache [7] developed a more advanced notion, "Neutrosophy," to more comprehensively model the vagary of information. The neutrosophic theory reconciles certain pitfalls of the fuzzy approach and its extensions. Wang *et al*. [8] proposed a single-valued neutrosophic set as a subclass of a neutrosophic set. In SVNs, the data information is indicated with 3-tuple, i.e., degree of membership, degree of indeterminacy, and degree of non-membership. Hence, the information evaluation in terms of neutrosophic sets seems more suitable for studies concerning decision-making, pattern recognition, clustering analysis, etc. A single-valued neutrosophic set allows us to choose truth membership, false membership, and indeterminacy in an unrestrictive manner in contrast to other fuzzy extensions.

Several neutrosophic information measures have been proposed, such as entropy, similarity, distance, and divergence measures over the years. Some prominent researches are due to Chai *et al.* [9], Wang [10], Biswas, *et al.* [11], Wu *et al.* [12], Aydogdu [13], Bourmi and Smarandache [14], Bourmi and Smarandache [15], Khan *et al.*, [16], Majumdar and Samanta [17], Ye [18-19], Ye and Fu [20], Chakraborty *et al.* [21], Chakraborty *et al.* [22], Haque *et al.* [23], Haque *et al.* [24], and Chakraborty *et al.* [25], Bonissone [26], Eshragh and Mamdani [27], etc. References [28-30] also report the work on developing new similarity/distance measures for fuzzy and SVNSs.

Motivation and contribution

Recent trends notice that all the existing measures of comparison (distance/similarity/divergence) in the neutrosophic framework are symmetric. But there are practical circumstances where the asymmetric comparison is more suitable. For example, we consider the following two sentences:

I. Saddam Hussain was like Hitler.
II. Hitler was like Saddam Hussain.

Off course, sentence-I would be the apparent preference for the comparison, probably due to the genocide instinct of the latter. In such a situation, one concept is the target, another is the base, and the main focus is the target. In sentence-I and -II, Saddam Hussain is the target, and Hitler is the base.

Further, a problem of medical diagnosis, where the symptoms of the patient are compared with symptoms of certain diseases (as established by medical experts), also seems to be better dealt with using asymmetric comparison measures. In such a problem, the patient's symptoms in single-valued neutrosophic representation (P) must be treated as a target, and the pre-assigned symptoms of the disease (Q) may be treated as a base. The direction of comparison in this problem must be $Q \rightarrow P$ instead of $P \rightarrow Q$. Such asymmetric comparisons are unavoidable in any discipline and can essentially need to be investigated using some asymmetric measures of comparison. In view of these facts, natural question arise what is the concept of asymmetric measure of comparison? How to construct an asymmetric measure of comparison in neutrosophic environment? Does such a comparison measure practically valid and effective? Moreover, to best of our knowledge there is no asymmetric measure of information in the literature concerning neutrosophic information theory. Here-mentioned facts and research gap in neutrosophic information theory motivated us to consider this study.

The novel contribution of this article is as follows.

- We introduce a novel concept of asymmetric comparison measure for SVNSs and term it a "Single-Valued Neutrosophic Inaccuracy Measure."
- We prove some algebraic properties of the proposed comparison measure for SVNSs.
- We also deduce a performance index "Degree of Confidence" to examine the performance of various neutrosophic comparison measures in the classification problems
- We also discuss applications of the proposed measure in pattern recognition and medical diagnosis problem.

The remaining paper is structured as follows. Section 2 presents preliminaries. Section 3 introduces inaccuracy measures/asymmetric measure of comparison between SVNSs. In section 4, we discuss some properties of the proposed measure. Section 5 presents an application of the proposed inaccuracy measure. Section 6 includes the comparative study. Finally, Section 7 concludes the article.

2. Preliminaries

This section considers some notions related to single-valued neutrosophic sets and inaccuracy measures.

Definition 2.1[3]. Let $Y = (y_1, y_2, y_{3,\dots,}y_n)$ be a random variable associated with an experiment. Let $P = (p_1, p_2, p_{3,\dots}p_n)$ be the probability distribution of random variable Y. Shannon's entropy measure is given by

$$H(P) = -\sum_{i=1}^{n} p_i log_2 p_i.$$

Definition 2.2[1][2]. Let $Y = (y_1, y_2, y_{3,\dots,}y_n)$ be a random variable associated with an experiment. Let $P = (p_1, p_2, p_{3,\dots}p_n)$ and $Q = (q_1, q_2, q_{3,\dots}q_n)$ be two probability distributions. Then the divergence measure between P and Q is given by

$$D\,(P, Q) = \sum_{i=1}^{n} p_i log_2 \frac{p_i}{q_i}.$$

Definition 2.3[4]. Let $P = (p_1, p_2, p_{3,\dots}p_n)$ and $Q = (q_1, q_2, q_{3,\dots}q_n)$ be two probability distributions. Then inaccuracy of distribution Q with respect to distribution P is given by

$$I(P, Q) = -\sum_{i=1}^{n} p_i \, log_2 \, q_i.$$

A particular case of a neutrosophic set is a single-valued neutrosophic set which was proposed by Wang *et al*. [8]

Definition 2.5[8]. Let y_i be a generic element of the universal set Y. A truth-membership function characterizes a single-valued neutrosophic set $\rho_A(y_i)$, indeterminacy-membership function $\theta_A(y_i)$ and falsity-membership function$\delta_A(y_i)$. Also, for each $y_i \in Y$, $\rho_A(y_i)$, $\theta_A(y_i)$, $\delta_A(y_i) \in [0, 1]$ with condition $\rho_A(y_i) + \theta_A(y_i) + \delta_A(y_i) \in [0, 3]$.

In other words, a single-valued neutrosophic set A can be denoted by a triplet, *i.e.*,

$$A = \{\langle \rho_A(y_i), \theta_A(y_i), \delta_A(y_i)\rangle | \; y_i \in Y\}$$

Notation: SVNS (Y) denotes the set of all neutrosophic elements in Y.

Some of the basic and useful operations on SVNS are defined as follows:

Definition 2.6[8]. Let $A = \{\langle \rho_A(y_i), \theta_A(y_i), \delta_A(y_i)\rangle | \; y_i \in Y\}$ and

$$B = \{\langle \rho_B(y_i), \theta_B(y_i), \delta_B(y_i)\rangle | \; y_i \in Y\}.$$

be two SVNSs, then the union of A and B is defined as

$$A \cup B = \{< max.(\rho_A(y_i), \rho_B(y_i)), min.(\theta_A(y_i), \theta_B(y_i)), min.(\delta_A(y_i), \delta_B(y_i)) > \; | \; y_i \in Y\}.$$

Definition 2.7 [8] For two SVNSs, A and B, the intersection of A and B is

$$A \cap B = \{< min.(\rho_A(y_i), \rho_B(y_i)), max.(\theta_A(y_i), \theta_B(y_i)), max.(\delta_A(y_i), \; \delta_B(y_i)) > | \; y_i \in Y\}.$$

Definition 2.8[8]. Let $A = \{\langle \rho_A(y_i), \theta_A(y_i), \delta_A(y_i)\rangle | \; y_i \in Y\}$ be an SVNs. Then the complement of A is defined as

$$A^c = \{< 1 - \rho_A(y_i), 1 - \theta_A(y_i), 1 - \delta_A(y_i) > | \; y_i \in Y\}.$$

Definition 2.9[8]. Let $A = \{\langle \rho_A(y_i), \theta_A(y_i), \delta_A(y_i)\rangle | \; y_i \in Y\}$ and $B = \{\langle \rho_B(y_i), \theta_B(y_i), \delta_B(y_i)\rangle | \; y_i \in Y\}$ be two SVNSs, then $A \subseteq B$ if

$$\rho_A(y_i) \le \rho_B(y_i), \; \theta_A(y_i) \ge \theta_B(y_i), \; \delta_A(y_i) \ge \delta_B(y_i), \; \forall \; y_i \in Y.$$

Hatzimichailidis [31] introduced the notion of Degree of confidence (DoC) in intuitionistic fuzzy environment. The definition of DoC in neutrosophic settings is as follows.

Definition 2.10. Let P_i be an unknown pattern classified to some pattern from the class P_j. Degree of confidence of neutrosophic comparison measure *M* estimates the confidence level that comparison measure in classifying a pattern P_i to the pattern P_k (belongs to a class of patterns) and it can be computed as

$$DOC = \sum_{j=1 \; j \neq k}^{n} |M(P_i, P_k) - M(P_i, P_j)|.$$

The greater the degree of confidence (*DOC*) for a comparison measure, the more confident the classification result of the measure is.

3. Inaccuracy Measure of a Single-Valued Neutrosophic Set

In this section, we propose an inaccuracy measure of single-valued neutrosophic sets and discuss their properties. Verma and Sharma [32] presented an inaccuracy measure of fuzzy sets as follows:

$$I(A, B) = -\frac{1}{n}\sum_{i=1}^{n}[\rho_A(y_i)\log(\rho_B(y_i)) + (1 - \rho_A(y_i))\log(1 - \rho_B(y_i))]. \quad (1)$$

where ρ_A and ρ_B are the membership functions associated with fuzzy sets A and B.

We can write

$$I\,(A,B) = -\frac{1}{n}\sum_{i=1}^{n} S(x,y). \tag{2}$$

where, $S\,(x,y) = -xlogy - (1-x)\log(1-y)$ is called Karridge's inaccuracy function for two events.

Since in a single-valued neutrosophic set, the non-membership and indeterminacy are independent of the membership function; therefore, the inaccuracy function utilized in equation (2) can be modified as

$$S\,(x,y) = -x_1 logy_1 - x_2 logy_2 - x_3 logy_3, \tag{3}$$

where $x_i\,, y_i \in [0,1]\,, i = 1,2,3.$

Consequently, the inaccuracy measure for two single-valued neutrosophic sets $A = \{\langle\rho_A(y_i),\theta_A(y_i),\delta_A(y_i)\rangle|\, y_i \in Y\}$ and $B = \{\langle\rho_B(y_i),\theta_B(y_i),\delta_B(y_i)\rangle|\, y_i \in Y\}$, is defined as follows:

$$I_{SVNS}(A,B) = -\frac{1}{n}\sum_{i=1}^{n}[\rho_A(y_i)\log(\rho_B(y_i)) + \theta_A(y_i)\log\,\theta_B(y_i) + \delta_A(y_i)\log\,\delta_B(y_i)]. \tag{4}$$

with convention $0.\log 0 = 0$.

Next, we prove some properties of the proposed inaccuracy measure of SVNSs.

Theorem 3.1. Let $A,B,C \in SVNS(Y)$, the proposed inaccuracy measure satisfies the following properties:

a) $I_{SVNS}(A,B) = 0$ if and only if either $\rho_A(y_i) = \rho_B(y_i) = 0, \theta_A(y_i) = \theta_B(y_i) = 0, \delta_A(y_i) = \delta_B(y_i) = 0$ or $\rho_A(y_i) = \rho_B(y_i) = 1, \theta_A(y_i) = \theta_B(y_i) = 1, \delta_A(y_i) = \delta_B(y_i) = 1\; where\; i = 1,2,3,\ldots,n\,;\ \forall A,B,C \in SVNS(Y)$.

b) $I_{SVNS}(A, B\cup C) + I_{SVNS}(A, B\cap C) = I_{SVNS}(A,B) + I_{SVNS}(A,C)\ \ \forall A,B,C \in SVNS(Y)$.

c) $I_{SVNS}(A\cup B, C) + I_{SVNS}(A\cap B, C) = (\,I_{SVNS}(A,C) + I_{SVNS}(B,C);\ \ \forall A,B,C \in SVNS(Y)$.

d) $I_{SVNS}(A\cup B, A\cap B) + I_{SVNS}(A\cap B, A\cup B) = I_{SVNS}(A,B) + I_{SVNS}(B,A);\ \ \forall A,B \in SVNS(Y)$.

Proof. a)

Let $I_{SVNS}(A,B) = 0$, then, from Eq. (4), we have

$$-\frac{1}{n}\sum_{i=1}^{n}[\rho_A(y_i)\log(\rho_B(y_i)) + \theta_A(y_i)\log\,\theta_B(y_i) + \delta_A(y_i)\log\,\delta_B(y_i)] = 0,$$

$[\rho_A(y_i)\log(\rho_B(y_i)) + \theta_A(y_i)\log\,\theta_B(y_i) + \delta_A(y_i)\log\,\delta_B(y_i)] = 0\ \ \forall\, i = 1,2,3,\ldots,n.$

The above relation holds, if and only if

$\rho_A(y_i) = \rho_B(y_i) = 0, \theta_A(y_i) = \theta_B(y_i) = 0, \delta_A(y_i) = \delta_B(y_i) = 0$

or $\rho_A(y_i) = \rho_B(y_i) = 1, \theta_A(y_i) = \theta_B(y_i) = 1, \delta_A(y_i) = \delta_B(y_i) = 1\ \forall\, i = 1,2,3,\ldots,n$.

Conversely,

Suppose, $\rho_A(y_i) = \rho_B(y_i) = 0, \theta_A(y_i) = \theta_B(y_i) = 0, \delta_A(y_i) = \delta_B(y_i) = 0$

or $\rho_A(y_i) = \rho_B(y_i) = 1, \theta_A(y_i) = \theta_B(y_i) = 1, \delta_A(y_i) = \delta_B(y_i) = 1\ \forall\, i = 1,2,3,\ldots,n.$

i.e., $[\rho_A(y_i)\log(\rho_B(y_i)) + \theta_A(y_i)\log\,\theta_B(y_i) + \delta_A(y_i)\log\,\delta_B(y_i)] = 0$

or,

$$-\frac{1}{n}\sum_{i=1}^{n}[\rho_A(y_i)\log(\rho_B(y_i)) + \theta_A(y_i)\log\ \theta_B(y_i) + \ \delta_A(y_i)\log\ \delta_B(y_i)] \ \ = 0$$

Which implies that $I_{SVNS}(A, B) = 0$. □

Proof. b)

For this, we divide the universal set Y into two disjoint subsets, *i.e.*,

$Y_1 = \{\rho_A(y_i) \geq \rho_B(y_i) \geq \rho_C(y_i);\ \theta_A(y_i) \leq \theta_B(y_i) \leq \theta_C(y_i);\ \delta_A(y_i) \leq \delta_B(y_i) \leq \delta_C(y_i) |\ y_i \in Y\}$ (5)

$Y_2 = \{\rho_A(y_i) \leq \rho_B(y_i) \leq \rho_C(y_i);\ \theta_A(y_i) \geq \theta_B(y_i) \geq \theta_C(y_i);\ \delta_A(y_i) \geq \delta_B(y_i) \geq \delta_C(y_i) |\ y_i \in Y\}$ (6)

Then by taking L.H.S, we have

$$I_{SVNS}(A, B \cup C) = -\tfrac{1}{n}\textstyle\sum_{Y_1}[\rho_A(y_i)\log(\rho_{B\cup C}(y_i)) + \theta_A(y_i)\log\ \theta_{B\cup C}(y_i) + \ \delta_A(y_i)\log\ \delta_{B\cup C}(y_i)] +$$

$$\left(-\frac{1}{n}\sum_{Y_2}[\rho_A(y_i)\log(\rho_{B\cup C}(y_i)) + \theta_A(y_i)\log\ \theta_{B\cup C}(y_i) + \ \delta_A(y_i)\log\ \delta_{B\cup C}(y_i)]\right)$$

Now using (5) and (6), we get

$$I_{SVNS}(A, B \cup C) = -\tfrac{1}{n}\textstyle\sum_{Y_1}[\rho_A(y_i)\log(\rho_B(y_i)) + \theta_A(y_i)\log\ \theta_C(y_i) + \ \delta_A(y_i)\log\ \delta_C(y_i)] -$$

$$\tfrac{1}{n}\textstyle\sum_{Y_2}[\rho_A(y_i)\log(\rho_C(y_i)) + \theta_A(y_i)\log\ \theta_B(y_i) + \ \delta_A(y_i)\log\ \delta_B(y_i)].$$

Which implies that

$I_{SVNS}(A, B \cup C) = \ I_{SVNS}(A, B) + I_{SVNS}(A, C)$. (7)

Now by taking,

$$I_{SVNS}(A, B \cap C) = -\frac{1}{n}\sum_{Y_1}[\rho_A(y_i)\log(\rho_{B\cap C}(y_i)) + \theta_A(y_i)\log\ \theta_{B\cap C}(y_i) + \ \delta_A(y_i)\log\ \delta_{B\cap C}(y_i)]$$

$$-\frac{1}{n}\sum_{Y_2}[\rho_A(y_i)\log(\rho_{B\cap C}(y_i)) + \theta_A(y_i)\log\ \theta_{B\cap C}(y_i) + \ \delta_A(y_i)\log\ \delta_{B\cap C}(y_i)]$$

Now using (5) and (6), we get

$$I_{SVNS}(A, B \cap C) = -\frac{1}{n}\sum_{Y_1}[\rho_A(y_i)\log(\rho_C(y_i)) + \theta_A(y_i)\log\ \theta_B(y_i) + \ \delta_A(y_i)\log\ \delta_B(y_i)]$$
$$-\frac{1}{n}\sum_{Y_2}[\rho_A(y_i)\log(\rho_B(y_i)) + \theta_A(y_i)\log\ \theta_C(y_i) + \ \delta_A(y_i)\log\ \delta_C(y_i)]$$

which implies that

$I_{SVNS}(A, B \cap C) = \ I_{SVNS}(A, B) + I_{SVNS}(A, C)$ (8)

Adding (7) and (8), we get

$$I_{SVNS}(A, B \cup C) + I_{SVNS}(A, B \cap C) \leq \left(I_{SVNS}(A, B) + I_{SVNS}(A, C)\right). \square$$

Proof. c)

By taking L.H.S, we have

$I_{SVNS}(A\cup B, C) = -\frac{1}{n}\sum_{Y_1}[\rho_{A\cup B}(y_i)\log(\rho_C(y_i)) + \theta_{A\cup B}(y_i)\log \theta_C(y_i) + \delta_{A\cup B}(y_i)\log \delta_C(y_i)] +$

$\left(-\frac{1}{n}\sum_{Y_2}[\rho_{A\cup B}(y_i)\log(\rho_C(y_i)) + \theta_{A\cup B}(y_i)\log \theta_C(y_i) + \delta_{A\cup B}(y_i)\log \delta_C(y_i)]\right).$

Now using (5) and (6), we get

$$I_{SVNS}(A\cup B, C) = -\frac{1}{n}\sum_{Y_1}[\rho_A(y_i)\log(\rho_C(y_i)) + \theta_B(y_i)\log \theta_C(y_i) + \delta_B(y_i)\log \delta_C(y_i)]$$

$$-\frac{1}{n}\sum_{Y_2}[\rho_B(y_i)\log(\rho_C(y_i)) + \theta_A(y_i)\log \theta_C(y_i) + \delta_A(y_i)\log \delta_C(y_i)].$$

which implies that

$$I_{SVNS}(A\cup B, C) = I_{SVNS}(A, C) + I_{SVNS}(B, C). \tag{9}$$

Again, by taking L.H.S, we have

$$I_{SVNS}(A\cap B, C) = -\frac{1}{n}\sum_{Y_1}[\rho_{A\cap B}(y_i)\log(\rho_C(y_i)) + \theta_{A\cap B}(y_i)\log \theta_C(y_i) + \delta_{A\cap B}(y_i)\log \delta_C(y_i)]$$
$$-\frac{1}{n}\sum_{Y_2}[\rho_{A\cap B}(y_i)\log(\rho_C(y_i)) + \theta_{A\cap B}(y_i)\log \theta_C(y_i) + \delta_{A\cap B}(y_i)\log \delta_C(y_i)].$$

Now using (5) and (6), we get

$$I_{SVNS}(A\cap B, C) = -\frac{1}{n}\sum_{Y_1}[\rho_B(y_i)\log(\rho_C(y_i)) + \theta_A(y_i)\log \theta_C(y_i) + \delta_A(y_i)\log \delta_C(y_i)]$$
$$-\frac{1}{n}\sum_{Y_2}[\rho_A(y_i)\log(\rho_C(y_i)) + \theta_B(y_i)\log \theta_C(y_i) + \delta_B(y_i)\log \delta_C(y_i)].$$

Which implies that

$$I_{SVNS}(A\cap B, C) = I_{SVNS}(A, C) + I_{SVNS}(B, C). \tag{10}$$

Adding (9) and (10), we get

$$I_{SVNS}(A, B\cup C) + I_{SVNS}(A, B\cap C) \leq \left(I_{SVNS}(A, C) + I_{SVNS}(B, C)\right). \ \square$$

Proof. d)

By taking two disjoint subsets of universal set Y, *i.e.,*

$$Y_1 = \{\rho_A(y_i) \geq \rho_B(y_i);\ \theta_A(y_i) \leq \theta_B(y_i);\ \delta_A(y_i) \leq \delta_B(y_i) |\ y_i \in Y\}, \tag{11}$$

$$Y_2 = \{\rho_A(y_i) \leq \rho_B(y_i);\ \theta_A(y_i) \geq \theta_B(y_i);\ \delta_A(y_i) \geq \delta_B(y_i) |\ y_i \in Y\}. \tag{12}$$

By using equations (5), (6) and by taking L.H.S, we have

$$I_{SVNS}(A\cup B, A\cap B)$$
$$= -\frac{1}{n}\sum_{Y_1}[\rho_{A\cup B}(y_i)\log(\rho_{A\cap B}(y_i)) + \theta_{A\cup B}(y_i)\log \theta_{A\cap B}(y_i) + \delta_{A\cup B}(y_i)\log \delta_{A\cap B}(y_i)]$$
$$-\frac{1}{n}\sum_{Y_2}[\rho_{A\cup B}(y_i)\log(\rho_{A\cap B}(y_i)) + \theta_{A\cup B}(y_i)\log \theta_{A\cap B}(y_i) + \delta_{A\cup B}(y_i)\log \delta_{A\cap B}(y_i)].$$

Now using (5) and (6), we get

$$I_{SVNS}(A\cup B,A\cap B)=-\frac{1}{n}\sum_{Y_1}[\rho_A(y_i)\log(\rho_B(y_i))+\theta_B(y_i)\log\ \theta_A(y_i)+\ \delta_B(y_i)\log\ \delta_A(y_i)]$$

$$-\frac{1}{n}\sum_{Y_2}[\rho_B(y_i)\log(\rho_A(y_i))+\theta_A(y_i)\log\ \theta_B(y_i)+\ \delta_A(y_i)\log\ \delta_B(y_i)].$$

Which implies that

$$I_{SVNS}(A\cup B,A\cap B)$$
$$=-\frac{1}{n}\sum_{Y_1\cup Y_2}[\rho_A(y_i)\log(\rho_B(y_i))+\theta_A(y_i)\log\ \theta_B(y_i)+\ \delta_A(y_i)\log\ \delta_B(y_i)]$$
$$-\frac{1}{n}\sum_{Y_1\cup Y_2}[\rho_B(y_i)\log(\rho_A(y_i))+\theta_B(y_i)\log\ \theta_A(y_i)+\ \delta_B(y_i)\log\ \delta_A(y_i)].$$

or

$$I_{SVNS}(A\cup B,A\cap B)=\ I_{SVNS}(A,B)+I_{SVNS}(B,A). \tag{13}$$

Now,

$$I_{SVNS}(A\cap\ B,A\cup B)$$
$$=\ -\frac{1}{n}\sum_{Y_1}[\rho_{A\cap B}(y_i)\log(\rho_{A\cup B}(y_i))+\theta_{A\cap B}(y_i)\log\ \theta_{A\cup B}(y_i)+\ \delta_{A\cap B}(y_i)\log\ \delta_{A\cup B}(y_i)]$$
$$-\frac{1}{n}\sum_{Y_2}[\rho_{A\cap B}(y_i)\log(\rho_{A\cup B}(y_i))+\theta_{A\cap B}(y_i)\log\ \theta_{A\cup B}(y_i)+\ \delta_{A\cap B}(y_i)\log\ \delta_{A\cup B}(y_i)].$$

Now using (5) and (6), we get

$$I_{SVNS}(A\cap B,A\cup B)$$
$$=-\frac{1}{n}\sum_{Y_1}[\rho_B(y_i)\log(\rho_A(y_i))+\theta_A(y_i)\log\ \theta_B(y_i)+\ \delta_A(y_i)\log\ \delta_B(y_i)]$$
$$-\frac{1}{n}\sum_{Y_2}[\rho_A(y_i)\log(\rho_B(y_i))+\theta_B(y_i)\log\ \theta_A(y_i)+\ \delta_B(y_i)\log\ \delta_A(y_i)].$$

Which implies that

$$I_{SVNS}(A\cup B,A\cap B)$$
$$=-\frac{1}{n}\sum_{Y_1\cup Y_2}[\rho_A(y_i)\log(\rho_B(y_i))+\theta_A(y_i)\log\ \theta_B(y_i)+\ \delta_A(y_i)\log\ \delta_B(y_i)]$$
$$-\frac{1}{n}\sum_{Y_1\cup Y_2}[\rho_B(y_i)\log(\rho_A(y_i))+\theta_B(y_i)\log\ \theta_A(y_i)+\ \delta_B(y_i)\log\ \delta_A(y_i)].$$

or

$$I_{SVNS}(A\cap\ B,A\cup B)\leq\ I_{SVNS}(A,B)+I_{SVNS}(B,A) \tag{14}$$

Adding (13) and (14), we get

$$I_{SVNS}(A\cup B,A\cap B)+I_{SVNS}(A\cap B,A\cup B)\leq\left(I_{SVNS}(A,B)+I_{SVNS}(B,A)\right).\square$$

In the next section, we investigate the application of the proposed inaccuracy measure.

4. Applications

In this section, we empirically illustrate the practical application of our proposed asymmetric measure of comparison in strategic decision-making and medical diagnosis.

4.1 Application to strategic decision-making

Let us consider a very pertinent corporate problem in which corporation Y wants to launch one of its five products using five strategies. Let the set of products be $P = \{C_1, C_2, C_3, C_4, C_5\}$, and the set of strategies be $S = \{Z_1, Z_2, Z_3, Z_4, Z_5\}$. Table 1 represents the weights of strategies of corporation Y in terms of memberships $\rho_y(Z_i)$, indeterminacy $\theta_y(Z_i)$ and non-membership value $\delta_y(Z_i)$ where $i = 1, 2, 3, 4, 5$. The weights have been assigned to these strategies because of their feasibility based on certain factors.

Table 1. Weights of strategies of the corporation as Single-Valued neutrosophic number

$(\rho_y(Z_1), \theta_y(Z_1), \delta_y(Z_1))$	$(\rho_y(Z_2), \theta_y(Z_2), \delta_y(Z_2))$	$(\rho_y(Z_3), \theta_y(Z_3), \delta_y(Z_3))$	$(\rho_y(Z_4), \theta_y(Z_4), \delta_y(Z_4))$	$(\rho_y(Z_5), \theta_y(Z_5), \delta_y(Z_5))$
(0.8, 0.2, 0.3)	(0.4, 0.4, 0.2)	(0.5, 0.4, 0.2)	(0.6, 0.2, 0.1)	(0.7, 0.5, 0.3)

It may be noted that $(\rho_y(Z_i), \theta_y(Z_i), \delta_y(Z_i))$ indicates the degree of importance, degree of inconclusiveness, and degree of the unimportance of strategy Z_i to the corporation in its implementation.

Table 2. represent the degree of importance, degree of inconclusiveness, and degree of the unimportance of products $(\rho_{C_j}(Z_i), \theta_{C_j}(Z_i), \delta_{C_j}(Z_i))$ concerning the strategy Z_i , where $j = 1, 2, 3, 4, 5$.

Table 2. Weights of strategies implementation for product launch as Single-Valued neutrosophic number

	$(\rho_{C_1}(Z_1), \theta_{C_1}(Z_1), \delta_{C_1}(Z_1))$	$(\rho_{C_2}(Z_2), \theta_{C_2}(Z_2), \delta_{C_2}(Z_2))$	$(\rho_{C_3}(Z_3), \theta_{C_3}(Z_3), \delta_{C_3}(Z_3))$	$(\rho_{C_4}(Z_4), \theta_{C_4}(Z_4), \delta_{C_4}(Z_4))$	$(\rho_{C_5}(Z_5), \theta_{C_5}(Z_5), \delta_{C_5}(Z_5))$
C_1	(0.5, 0.4, 0.4)	(0.6, 0.3,0.2)	(0.5, 0.2, 0.1)	(0.6, 0.2, 0.8)	(0.9, 0.2, 0.1)
C_2	(0.9, 0.3, 0.2)	(0.8, 0.7, 0.3)	(0.6, 0.3, 0.2)	(0.3, 0.4, 0.5)	(0.4, 0.2, 0.2)
C_3	(0.6, 0.5, 0.2)	(0.7, 0.6, 0.5)	(0.5, 0.3, 0.3)	(0.2, 0.5, 0.6)	(0.5, 0.4, 0.3)
C_4	(0.9, 0.2, 0.1)	(0.8, 0.8, 0.7)	(0.4, 0.3, 0.2)	(0.8, 0.5, 0.4)	(0.3, 0.4, 0.5)
C_5	(0.7, 0.7, 0.6)	(0.1, 0.5, 0.2)	(0.4, 0.3, 0.7)	(0.6, 0.3, 0.2)	(0.5, 0.4, 0.2)

In an objective of the corporation to launch a suitable product because of the suitability of the five strategies, with the minimum risk (inaccuracy in our case), we use Eq. (4) to compute inaccuracy measures $I_{SVNS}(Y, C_i)$, $i = 1, 2, 3, 4, 5$ using the data of Table 1 and Table 2.

Table 3. Inaccuracy measures between products and strategies

$\boldsymbol{I_{SVNS}(Y, C_1)}$	$\boldsymbol{I_{SVNS}(Y, C_2)}$	$\boldsymbol{I_{SVNS}(Y, C_3)}$	$\boldsymbol{I_{SVNS}(Y, C_4)}$	$\boldsymbol{I_{SVNS}(Y, C_5)}$
1.6421	1.5129	1.4554	**1.2930**	1.4933

According to the inaccuracy measures presented in Table 3, the product C_4 will be more suitable for launch because of the available strategies.

4.2 Application to Medical Diagnosis

First, we state the problem of medical diagnosis and present it in the framework of a neutrosophic environment.

Medical diagnosis: The process of identifying an actual disease of a patient based on their symptoms is termed a medical diagnosis.

Substantial uncertainties occur in most diagnostic decisions and can be handled using fuzzy methodologies. In medical science, several diseases have many symptoms in common. Therefore, identifying the appropriate illness from which a patient is suffering is difficult for physicians/experts. Let $D = \{D_1, D_2, D_3 \ldots . D_n\}$ be the set of diseases with several common symptoms and S $= \{r_1, r_2, r_3, \ldots, r_n\}$ be the set of symptoms of the patient under investigation. In this scenario, S is an SVNS and $D_1, D_2, D_3 \ldots . D_n$ are also SVNSs. We compare each of D_i with S. The patient is diagnosed with a disease D_i with which S is maximum directed closeness.

The flowchart of the process is shown in the figure 1.

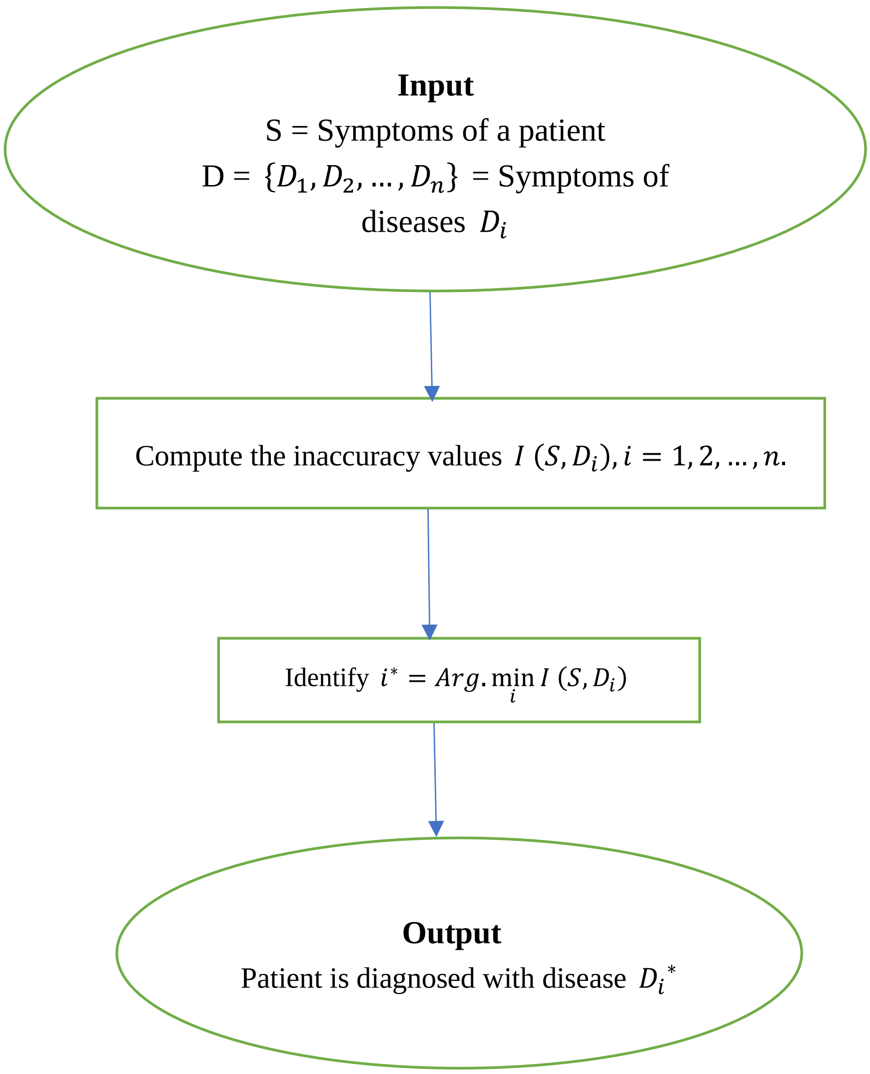

Figure1: Flowchart of Medical Diagnosis using Inaccuracy Measure

We consider the following numerical example to illustrate the procedure.

We present a numerical example to check the impact of the proposed inaccuracy measure.

Numerical Example: We consider a set of three diseases, $D = \{(D_1, viral\ fever)\ (D_2, malaria)\ (D_3, typhoid)\}$, each of which has three common symptoms given in set $R = \{(r_1, fever),\ (r_2, headache)\ \ (r_3, cough)\}$.

An expert team of doctors in the form of SVNSs assesses the characteristic information of the given diseases. The indicative information of symptoms and diagnosis for patients represented in the form of SVNSs as shown in Table 4:

Table 4. Characteristics information of the diseases described in the form of SVNSs

	r_1	r_2	r_3
D_1	(0.3, 0.2, 0.5)	(0.1, 0.3, 0.7)	(0.4, 0.3, 0.3)
D_2	(0.2, 0.2. 0.6)	(0.1, 0.1, 0.8)	(0.2, 0.3, 0.6)
D_3	(0.2, 0.1, 0.7)	(0.6, 0.3, 0.1)	(0.3, 0.4, 0.3)

The set P_1 represents the symptoms of the patient under investigation as an SVNS.

$P_1 = \{(r_1, (0.1, 0.2, 0.7))(r_2, (0.8, 0.2, 0.3))(r_3, (0.2, 0.4, 0.4))\}$.

Our task is to evaluate the closeness of P_1 with D_i using various SVN comparison measures.

To check the effectiveness of the proposed inaccuracy measure, we consider the following similarity/ distance measures for SVNSs.

$$S_1 = 1 - \frac{1}{n}\sum_{i=1}^{n} max.\{|\rho_A(y_i) - \rho_B(y_i)|, |\theta_A(y_i) - \theta_B(y_i)|, |\delta_A(y_i) - \delta_B(y_i)|\}.$$

(Bourmi and Smarandache [33])

$$S_2 = \frac{\sum_{i=1}^{n}\{min.(\rho_A(y_i),\rho_B(y_i)) + min.(\theta_A(y_i),\theta_B(y_i)) + min.(\delta_A(y_i),\delta_B(y_i))\}}{\sum_{i=1}^{n}\{max.(\rho_A(y_i),\rho_B(y_i)) + max.(\theta(y_i),\theta_B(y_i)) + max.(\delta_A(y_i),\delta_B(y_i))\}}.$$ (Majumdar and Samanta [17])

$$S_3 = \frac{1}{3n}\sum_{i=1}^{n}\left[\frac{min.(\rho_A(y_i),\rho_B(y_i))}{max.(\rho_A(y_i),\rho_B(y_i))} + \frac{min.(\theta_A(y_i),\delta_B(y_i))}{max.(\delta_A(y_i),\delta_B(y_i))} + \frac{min.(\delta_A(y_i),\delta_B(y_i))}{max.(\delta_A(y_i),\delta_B(y_i))}\right].$$ (Ye and Zhang [34])

$$S_4 = \frac{1}{n}\sum_{i=1}^{n}\left[1 - \frac{|\rho_A(y_i) - \rho_B(y_i)| + |\theta_A(y_i) - \theta_B(y_i)| + |\delta_A(y_i) - \delta_B(y_i)|}{3}\right].$$ (Ali Aydogdu [13])

$$S_5 = 1 - \frac{1}{3n}\sum_{y\in n}\left|\left(\rho_A{}^2(y_i) - \rho_B{}^2(y_i)\right) - \left(\theta_A{}^2(y_i) - \theta_B{}^2(y_i)\right) - \left(\delta_A{}^2(y_i) - \delta_B{}^2(y_i)\right)\right|.$$ (Chai *et al.* [9])

$$S_6 = \frac{1}{n}\sum_{i=1}^{n}\frac{2(\rho_A(y_i).\rho_B(y_i) + \theta_A(y_i).\theta_A(y_i) + \delta_A(y_i).\delta_A(y_i))}{(\rho_A{}^2(y_i) + \theta_A{}^2(y_i) + \delta_A{}^2(y_i)) + (\rho_B{}^2(y_i) + \theta_B{}^2(y_i) + \delta_B{}^2(y_i))}.$$ (Ye [35])

$$DM_1 = 1 - \frac{1}{n}\sum_{i=1}^{n}\left[1 - \frac{|\rho_A(y_i) - \rho_B(y_i)| + |\theta_A(y_i) - \theta_B(y_i)| + |\delta_A(y_i) - \delta_B(y_i)|}{3}\right].$$

(Ali Aydogdu [13])

$$DM_2 = \frac{1}{3n}\sum_{y\in n}\left|\left(\rho_A{}^2(y_i) - \rho_B{}^2(y_i)\right) - \left(\theta_A{}^2(y_i) - \theta_B{}^2(y_i)\right) - \left(\delta_A{}^2(y_i) - \delta_B{}^2(y_i)\right)\right|.$$ (Chai *et al.* [9])

$$DM_3 = \frac{1}{n}\sum_{y\in n}\left(|\rho_A{}^2(y_i) - \rho_B{}^2(y_i)| \vee |\theta_A{}^2(y_i) - \theta_B{}^2(y_i)| \vee |\delta_A{}^2(y_i) - \delta_B{}^2(y_i)|\right).$$ (Chai *et al.* [9])

Now, compute the similarity/distance measure between patient P_1 and diagnosis D. Similarly, we compute the proposed inaccuracy measure between the patient and the diagnosis. Table 5 shows the result obtained by calculating the different existing and proposed inaccuracy measures.

Table 5: The similarity /distance measures between the symptoms of a patient P_1 and diagnosis D_i

	D_1	D_2	D_3	Ranking
$S_1(P_1, D_i)$	0.5867	0.68	0.8734	$D_3 > D_2 > D_1$
$S_2(P_1, D_i)$	0.5238	0.5609	0.75	$D_3 > D_2 > D_1$
$S_3(P_1, D_i)$	0.0582	0.0623	0.25	$D_3 > D_2 > D_1$
$S_4(P_1, D_i)$	0.3108	0.3108	0.33	$D_3 > D_2 > D_1$
$S_5(P_1, D_i)$	0.8556	0.8389	0.9523	$D_3 > D_1 > D_2$
$S_6(P_1, D_i)$	0.7806	0.7987	0.9598	$D_3 > D_2 > D_1$
$DM_1(P_1, D_i)$	0.6892	0.6892	0.67	$D_3 > D_2 > D_1$
$DM_3(P_1, D_i)$	0.1444	0.1611	0.0477	$D_3 > D_1 > D_2$
$DM_4(P_1, D_i)$	0.33	0.32	0.1266	$D_3 > D_2 > D_1$
$I(D_i, P_1)$	1.7949	1.5818	1.314	$D_3 > D_2 > D_1$
$I(P_1, D_i)$	2.0504	2.0283	1.5870	$D_3 > D_2 > D_1$

Analysis: From the Table 5, we observe that all the comparison measures diagnosing the patient P_1 for Typhoid. Our proposed asymmetric comparison measure from both directions $D_i \rightarrow P_1$ and $P_1 \rightarrow D_i$ also resulting the same diagnosis (refer last two rows of the Table 5). Thus, we conclude that our proposed measure is consistent with existing models. The proposed model is more effective from the following observations.

In the figure 2, the directed comparison $I(D_i, P_1)$ shows the greater discriminating capability within the diseases. Thus, the proposed asymmetric measure is sensitive to the direction of comparison from the view point of the discriminating power. In the considered numerical problem, the diagnostic result due to both directed comparisons ($I(D_i, P_1)$ and $I(P_1, D_i)$) remains same but discriminating power is different.

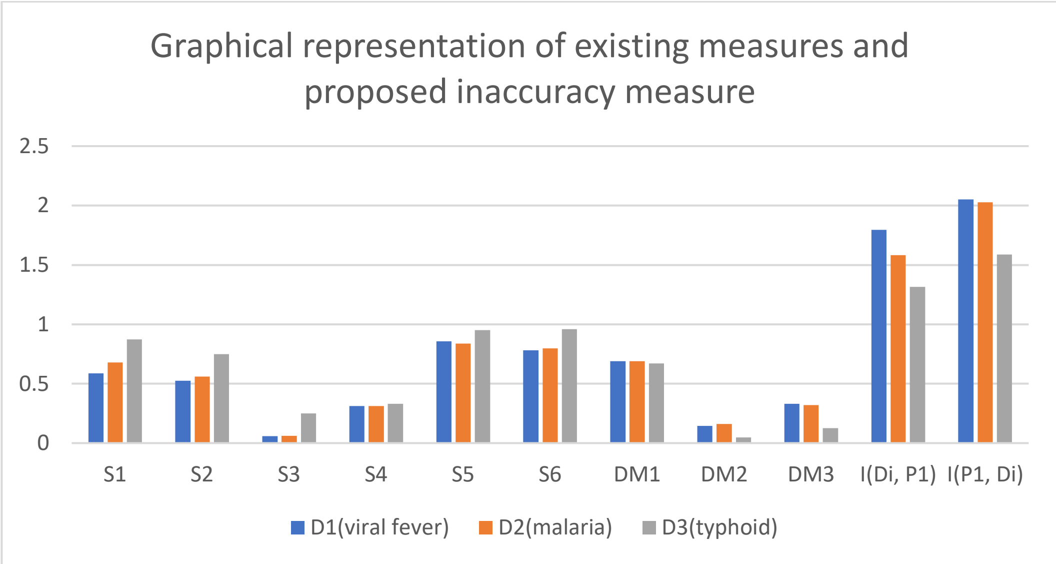

Figure 2: Graphical representation of computed values of closeness due to various measures in Table 5

From the above graph, we see that the diagnosis of patient P_1 is typhoid. It shows that the proposed inaccuracy measure is feasible and effective.

In the next section, we compare some existing divergence measures, similarity measures, and the proposed inaccuracy measure.

5. Comparative Study

To check the superiority of the proposed inaccuracy measure, we consider the numerical example obtained from Thao and Smarandache [36].

Let us suppose, for universal set $U = \{u_1, u_2, u_3, \dots, u_m\}$, there are n patterns in the form of neutrosophic set $\{A_1, A_2, A_3, \dots, A_n\}$. Suppose that we have an unknown sample B. Our goal is to classify sample B into which pattern A_i.

For this, we have to calculate the proposed inaccuracy measure, existing divergence measures, and similarity measures of unknown sample B with each pattern $A_i (n = 1, 2, 3, \dots n)$.

Assume $A_1 = \{(u_1, 0.7, 0.7, 0.2), (u_2, 0.7, 0.8, 0.4), (u_3, 0.6, 0.8, 0.2)\}$.

$A_2 = \{(u_1, 0.5, 0.7, 0.3), (u_2, 0.7, 0.7, 0.5), (u_3, 0.8, 0.6, 0.1)\}$.

$A_3 = \{(u_1, 0.9, 0.5, 0.1), (u_2, 0.7, 0.6, 0.4), (u_3, 0.8, 0.5, 0.2)\}$.

Unknown Sample

B = $\{(u_1, 0.7, 0.8, 0.4), (u_2, 0.8, 0.5, 0.3), (u_3, 0.5, 0.8, 0.5)\}$.

For the comparative study, we consider all measures listed in section 4 along with the following existing divergence measures and similarity measures:

$$S_7 = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{\left(\rho_A{}^2(y_i) \wedge \rho_B{}^2(y_i)\right)}{\left(\rho_A{}^2(y_i) \vee \rho_B{}^2(y_i)\right)} + \frac{\left((1-\theta_A{}^2(y_i)) \wedge (1-\theta_B{}^2(y_i))\right)}{\left((1-\theta_A{}^2(y_i)) \vee (1-\theta_B{}^2(y_i))\right)} + \frac{\left((1-\delta_A{}^2(y_i)) \wedge (1-\delta_B{}^2(y_i))\right)}{\left((1-\delta_A{}^2(y_i)) \vee (1-\delta_B{}^2(y_i))\right)}\right\}. \qquad \text{(Chai et al. [9])}$$

$$DM_4(A, B) = \frac{1}{n}\sum_{i=1}^{n}\left[D_T{}^i(A, B) + D_I{}^i(A, B) + D_F{}^i(A, B)\right]. \qquad \text{(Thao and Smarandache [36])}$$

where, $DM_T{}^i(A, B) = \rho_A(y_i) \ln \frac{2\rho_A(y_i)}{\rho_A(y_i)+\rho_B(y_i)} + \rho_B(y_i) \ln \frac{2\rho_B(y_i)}{\rho_A(y_i)+\rho_B(y_i)}$

$$DM_I{}^i(A,B) = \theta_A(y_i)\ln\frac{2\theta_A(y_i)}{\theta_A(y_i)+\theta_B(y_i)} + \theta_B(y_i)\ln\frac{2\theta_B(y_i)}{\theta_A(y_i)+\theta_B(y_i)}$$

$$D_F{}^i(A,B) = \delta_A(y_i)\ln\frac{2\delta_A(y_i)}{\delta_A(y_i)+\delta_B(y_i)} + \delta_B(y_i)\ln\frac{2\delta_B(y_i)}{\delta_A(y_i)+\delta_B(y_i)}$$

$DM_j(A,B) =$

$$\sum_{i=1}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{\rho_A(y_i)} - \sqrt{\rho_B(y_i)}\right)^{2(\alpha+1)}}{(\rho_A(y_i)+\rho_B(y_i))^{\alpha}} + \frac{\left(\sqrt{1-\rho_A(y_i)} - \sqrt{1-\rho_B(y_i)}\right)^{2(\alpha+1)}}{(2-\rho_A(y_i)+\rho_B(y_i))^{\alpha}}\right] +$$

$$\sum_{i=1}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{\theta_A(y_i)} - \sqrt{\theta_B(y_i)}\right)^{2(\alpha+1)}}{\left(\theta_A(y_i)+\theta_B(y_i)\right)^{\alpha}} + \frac{\left(\sqrt{1-\theta_A(y_i)} - \sqrt{1-\theta_B(y_i)}\right)^{2(\alpha+1)}}{\left(2-\theta_A(y_i)+\theta_B(y_i)\right)^{\alpha}}\right] +$$

$$\sum_{i=1}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{\delta_A(y_i)} - \sqrt{\delta_B(y_i)}\right)^{2(\alpha+1)}}{\left(\delta_A(y_i)+\delta_B(y_i)\right)^{\alpha}} + \frac{\left(\sqrt{1-\delta_A(y_i)} - \sqrt{1-\delta_B(y_i)}\right)^{2(\alpha+1)}}{\left(2-\delta_A(y_i)+\delta_B(y_i)\right)^{\alpha}}\right];\ j = 5,6.$$ (Guleria *et al.* [37])

The result obtained by calculating the proposed inaccuracy measure, existing divergence measure, and similarity measure of unknown sample B with each pattern A_i is shown in Table 7.

Table 7. Result of the Existing Similarity Measure and Proposed Divergence Measure, along with the Degree of Confidence

	(A_1, B)	(A_2, B)	(A_3, B)	DOC
S_1	0.7333	0.7333	**0.7666**	**0.0666**
S_2	**0.7762**	0.7036	0.6745	**0.1743**
S_3	**0.7620**	0.6781	0.64	**0.2059**
S_4	**0.8666**	0.7996	0.7776	**0.156**
S_5	**0.9933**	0.9622	0.9555	**0.0689**
S_6	**0.9844**	0.9319	0.9203	**0.1166**
S7	**0.8021**	0.7063	0.6993	**0.1986**
D_1	**0.1333**	0.2004	0.2224	**0.1562**
D_2	**0.0067**	0.0378	0.0445	**0.0689**
D_3	**0.25**	0.29	0.2966	**0.0866**
DM_4	**0.1537**	0.2674	0.2951	**0.25513**
$DM_5(when\ \alpha = 1)$	**0.0352**	0.1090	0.1161	**0.1547**
$DM_6(when\ \alpha = 4)$	**0.0001**	0.0103	0.0032	**0.0133**
$I_{SVNS}(B_i, A)$	**1.4854**	1.7288	1.8444	**0.6024**
$I_{SVNS}(A_i, B))$	1.2092	1.2554	**1.1457**	**0.1732**

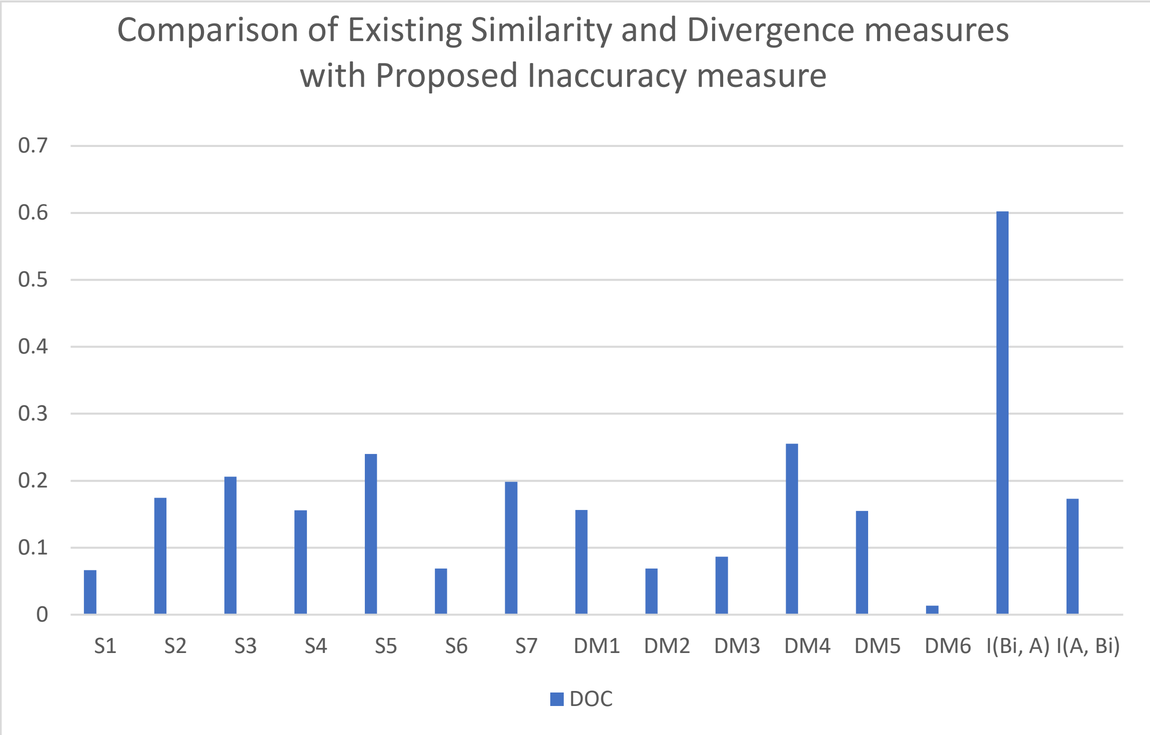

Figure 3: Graphical representation of degree of confidence of various comparison measures

Analysis: The highest value of similarity, the lowest value of divergence/inaccuracy, and the degree of confidence of the existing similarity, divergence measure, and the proposed inaccuracy measure are in bold in Table 7. The computed values of the comparison measures indicate that unknown sample B belongs to the pattern A_1. Only S_1 shows a different result. Our proposed inaccuracy measure's highest value of DOC, when the direction of comparison is $B_i \rightarrow A$. This justifies its effectiveness over other comparison measures, as illustrated graphically in the figure 3.

6. Conclusion

In this work, we have proposed an inaccuracy measure for SVNSs, to find directed discrimination between two SVNSs and studied some of their mathematical properties. The illustrative numerical problem in a corporate crisis of product launch has shown the applicability of the proposed measure. In addition, the advantage of the proposed measure has been justified by using a performance index DOC and in a medical diagnosis problem. The proposed asymmetric comparison measure may be impactful to the various studies in data science, machine learning and computer vision requiring a directed comparative analysis. The limitation of this article is that all the investigations have been done using hypothetical data. In future, we plan to investigate the applications of the suggested asymmetric comparison metrics in cluster analysis, multiple attribute decision-making, and medical diagnosis using real data sets. However, applying the proposed measures to actual data sets needs an efficient method of converting the crisp data to single valued neutrosohic data set without potential loss of information. Thus, formulating a suitable data conversion process because of the given scenario is also a problem for future investigations. Some recent studies [21-25] investigate the applicability of neutrosophic methods in various disciplines like decision-making, pattern recognition, inventory management, pollution in megacities, etc. We also plan to explore the relevance of the proposed approach to these disciplines.

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On Novel Hellinger Divergence Measure of Neutrosophic Hypersoft Sets in Symptomatic Detection of COVID-19

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Abstract: Numerous decision science processes involves the divergence measure is the most suitable information measure for dealing with the vagueness and impreciseness of the factors affecting the decision-making models. In this manuscript, a new kind of Hellinger information measure for a single-valued neutrosophic hypersoft set along with some important results have been presented and studied in detail. Also, we have presented the implementation of the proposed information measure to deal with the symptomatic detection of COVID-19 with a numerical illustration. In view of the existing methods related to divergence measures, some comparative results and remarks along with some important advantages have also been presented.

Keywords: Single-Valued Neutrosophic Hypersoft Set; Hellinger divergence measure; Decision-making; COVID-19.

1. Introduction

The notion of fuzzy set theory [1], which have been devised in the recent decade by the efforts of the various eminent researchers working in this area. Subsequently, the idea of a "multi-criteria decision-making (*MCDM*)" problem gives a wider range and serves as one of the delightful options in the area of uncertainty and decision sciences. Atanassov [2] devised the notion of an intuitionistic fuzzy set in order to deal with the uncertainty components of indeterminacy/hesitation margin in the inexact/incomplete information which is in the grades of membership and the grades of non-membership degree. In order to cover the incomplete, ambiguous and contradictory information, the concept of the environment of "neutrosophic set" [3] was devised by Samarandache. The neutrosophic set mainly captures the three components of uncertainty, i.e., "degrees of truth, indeterminacy and falsity" which covers a greater span of information and which is applicable in various decision sciences problems [4-8].

In order to utilize the notion of neutrosophic setup in various problems of decision-making are very difficult. To overcome this issue, Wang et al. [9] came up with the notion of "single-valued neutrosophic set *(SVNSS)*" which properly covers all the three uncertainty components and manages the contradictory information. Due to the limitations of the information that humans get or perceive from their environment; all features of the objects portrayed by the *SVNSS* are perfectly acceptable for dealing with the uncertainties. The notion of a *SVNSS* has been expanding swiftly because of its broader area of hypothetical distinction [10-14]. Molodtsov [15] came up with the parametrized idea of "soft set theory" which covers the "parameterization" of the criterions with their respective sub-criterions. Soft sets are of great importance in many problems of decision-making, game theory and artificial intelligence. The idea of a soft set has been further generalized by Maji [16] with the incorporation of a neutrosophic soft set. Then, Samrandache [17] introduces the novel concept of a hypersoft set with the inclusion of sub-attributes of the respective attributes. However, when there are several sub-attributes the theory of hypersoft sets cannot handle such situations. For overcoming such limitations, the idea of neutrosophic hypersoft sets *(NHSS)* was devised with the coutsey of Saqlain et al. [18-21].

In literature, numerous researchers have thoroughly examined the different types of “similarity measures, divergence measures, distance measures, and entropy measures” for various types of fuzzy sets and their extensions because of their practical utility in the many fields of engineering and sciences. The notion of directed divergence measure was first presented by Bhandari and Pal [22] which is the modified version of the information measures as stated by Kullback and Leibler [23]. The “divergence measure” based on exponential measures has been presented by Fan and Xie [24]. Ghosh et al. [25] presented divergence nature of fuzzy measures in the recognition problem of automation. Also, some “divergence measures for intuitionistic fuzzy sets” were given by Shang and Jiang [26]. Hung and Yang [27] introduced the set of axioms for the divergence measures of intuitionistic fuzzy sets by utilizing the Hausdorff metric. Next, Montes et al. [28] developed some major relations between the distance, divergence and dissimilarity measures.

From the above discussions, the degree of indeterminacy and hesitancy is missing from the intuitionistic fuzzy sets which restrict the various experts for assessing the uncertainities. In order to deal with these kind of shortcomings, the surroundings of “neutrosophic sets” is more effective in the various applications of sciences and engineering. Broumi and Smarandache [29] presented the different types of “information measures” for neutrosophic environment. The similarity kind of information measures for *SVNSS* by utilizing the distance measures have been proposed by Majumdar and Samanta [30]. Further, similarity measures for interval neutrosophic sets have been given by Ye [31]. Also, Ye [32] studied the trigonometric similarity measures for single-valued neutrosophic sets and apply them to multi-criteria decision-making problems. The relationships between the different types of information measures with their trigonometric axiomatic definitions have been given by Wu et al. [33]. Also, Thao and Smarandache [34] established a novel divergence measure for neutrosophic sets to solve the problems related to medical and recognition problems. Also, different types of information measures concerning the various extensions of fuzzy sets and fuzzy soft set are already existing in the literature [35-38].

However, there are some “distance and similarity measures” related to neutrosophic hypersoft sets utilized in the *TOPSIS* technique to compute the *MCDM* problems given by Saqlain et al. [39]. Also, “trigonometric similarity measures for neutrosophic hypersoft sets” to solve the renewable energy source selection problem given by Jafar et al. [40]. In addition to these, various other researchers [41-45] have proposed different types of similarity measures and utilized them in various types of pattern recognition problems and other decision-making problems. In the literature, there is similarity/distance measures for both “neutrosophic and single-valued neutrosophic hypersoft sets *(SVNHSS)*” but there are no divergence measures for *SVNHSSs* available. Here, based on the generalized “Hellinger” fuzzy divergence information measure for fuzzy environment given by Ohlan et al. [46], we have presented a novel kind of Hellinger divergence measure for the *SVNHSS* to moderate the research gap in this area.

The remaining of the manuscript is being organized as. Section 2 involves some fundamental notions related to *SVNHSSs* and some basic operations which are already existing in the literature. In Section

3, we present some set-theoretic operations on *SVNHSSs* and a novel notion of the Hellinger divergence measure for two *SVNHSSs*. Also, we establish the validity of the proposed Hellinger divergence measure under the standard axioms. Various important properties related to the Hellinger divergence measure have been studied and discussed in Section 4. Further, by utilizing the proposed divergence measure, the methodology for the symptomatic detection of COVID-19 has been presented in Section 5. An associated numerical example for illustrating the proposed methodology has been solved and presented in Section 6. In Section 7, a brief discussion of results and the proposed algorithmic technique containing some important points on comparative advantages, importance and shortcomings have been presented. In the end, the paper has been concluded in Section 8 with some possible scope for future work.

2. Preliminaries and Fundamental Notions

In this section, some of the fundamental notions in context with the extensions of the neutrosophic set and information measures are presented.

Definition 1. *[47] "Let* X *be the universal set and* $P(X)$ *be the power set of* X*. Consider* $k^1, k^2, \dots, k^n$ *for* $n \geq 1$ *be n well-defined attributes whose corresponding attribute values are respectively the sets* $K^1, K^2, \dots, K^n$ *with* $K^i \cap K^j = \emptyset$*, for* $i \neq j$ *and* $i, j \in \{1,2,..,n\}$*, then the pair* $(\mathfrak{R},\ K^1 \times\ K^2 \times \dots \times\ K^n)$ *is said to be Hypersoft set over the set* X*, where* $\mathfrak{R} : K^1 \times\ K^2 \times \dots \times\ K^n \rightarrow P(X)$*."*

Definition 2. *[47] "Let* X *be the universal set and* $P(X)$ *be the power set of* X*. Consider* $k^1, k^2, \dots, k^n$ *for* $n \geq 1$ *be n well-defined attributes whose corresponding attribute values are respectively the sets* $K^1, K^2, \dots, K^n$ *with* $K^i \cap K^j = \emptyset$*, for* $i \neq j$ *and* $i, j \in \{1,2,..,n\}$ *and their relation* $K^1 \times\ K^2 \times \dots \times\ K^n = \Gamma$*, then the pair* $(\mathfrak{R}, \Gamma)$ *is said to be Neutrosophic Hypersoft set (NHSS) over* X*, where,* $\mathfrak{R} : K^1 \times\ K^2 \times \dots \times\ K^n \rightarrow P(X)$ *and* $\mathfrak{R}(K^1 \times\ K^2 \times \dots \times\ K^n) = \{\langle x,\ T(\mathfrak{R}(\Gamma)),\ I(\mathfrak{R}(\Gamma)),\ F(\mathfrak{R}(\Gamma))\rangle,\ x\ \epsilon\ X\}$ *; where T is the degree of truthness, I is the degree of indeterminacy and F is the degree of falsity such that T, I, F :* $V \rightarrow (0^-,\ 1^+)$ *and satisfies the constraint* $0^- \leq T(\mathfrak{R}(\Gamma)) +\ I(\mathfrak{R}(\Gamma)) + F(\mathfrak{R}(\Gamma) \leq 3^+$*.*
While dealing with applications of science and engineering, it becomes very difficult to handle situations under a neutrosophic environment. In order to deal with such situations notion of Single-Valued Neutrosophic HyperSoft sets$(SVNHSS)$ *is very useful and applicable. "*

Definition 3. *[48] "Let* X *be the universal set and* $P(X)$ *be the power set of* X*. Consider* $k^1, k^2, \dots, k^n$ *for* $n \geq 1$ *be n well-defined attributes whose corresponding attribute values are respectively the sets* $K^1, K^2, \dots, K^n$ *with* $K^i \cap K^j = \emptyset$*, for* $i \neq j$ *and* $i, j \in \{1,2,..,n\}$ *and their relation* $K^1 \times\ K^2 \times \dots \times\ K^n = \Gamma$*, then the pair* $(\mathfrak{R}, \Gamma)$ *is said to be a Single-Valued Neutrosophic Hypersoft set* $(SVNHSS)$ *over* X*, where,* $\mathfrak{R} : K^1 \times\ K^2 \times \dots \times\ K^n \rightarrow P(X)$ *and* $\mathfrak{R}(K^1 \times\ K^2 \times \dots \times\ K^n) = \{\langle x,\ T(\mathfrak{R}(\Gamma)),\ I(\mathfrak{R}(\Gamma)),\ F(\mathfrak{R}(\Gamma))\rangle,\ x\ \epsilon\ X\}$ *; where T is the degree of truthness, I is the degree of indeterminacy and F is the degree of falsity such that T, I, F :* $V \rightarrow [0, 1]$ *and satisfies the constraint* $0 \leq T(\mathfrak{R}(\Gamma)) +\ I(\mathfrak{R}(\Gamma)) + F(\mathfrak{R}(\Gamma) \leq 3$*. "*

Definition 4. *[49] "Consider* A *and* B *be two single-valued neutrosophic sets, then the axiomatic definition of divergence measure are as follows:*

i. $\mathbb{I}(A, B) = \mathbb{I}(B, A)$;

ii. $\mathbb{I}(A, B) \geq 0$ and $\mathbb{I}(A, B) = 0$ iff $A = B$.

iii. $\mathbb{I}(A \cap B,\ B \cap A\ ) \leq\ \mathbb{I}(A, B)\ \ \forall\ \ B \in SVNS(X)$.

iv. $\mathbb{I}(A \cup B,\ B \cup A\ ) \leq\ \mathbb{I}(A, B)\ \ \forall\ \ B \in SVNS(X)$. "

3. Binary Operations and Hellinger Divergence Measure of Neutrosophic Hypersoft Sets

In this section, we first propose some binary operations on $\boldsymbol{SVNHSSs}$. We shall denote the collection of $\boldsymbol{SVNHSS}$ on $\boldsymbol{X}$ by $\boldsymbol{SVNHSS(X)}$. Now, for any two $\boldsymbol{SVNHSSs}$ $\boldsymbol{A, B} \in \boldsymbol{SVNHSS(X)}$, analogous to the operations given for the single-valued neutrosophic sets, we define some basic operations as follows:

• "**Union of** A **and** B": $A \cup B = \{x,\ T_{A\cup B}(\mathfrak{R}(\Gamma)),\ I_{A\cup B}(\mathfrak{R}(\Gamma)),\ F_{A\cup B}(\mathfrak{R}(\Gamma))|\ \ x \in X\ \}$

where,

$T_{A\cup B}(\mathfrak{R}(\Gamma))(x) = max\{T_A(\mathfrak{R}(\Gamma))(x),\ T_B(\mathfrak{R}(\Gamma))(x)\}$, $I_{A\cup B}(\mathfrak{R}(\Gamma))(x) = min\{I_A(\mathfrak{R}(\Gamma))(x), I_B(\mathfrak{R}(\Gamma))(x)\}$

and $F_{A\cup B}(\mathfrak{R}(\Gamma))(x) = min\{F_A(\mathfrak{R}(\Gamma))(x),\ F_B(\mathfrak{R}(\Gamma))(x)\}\ \forall\ \ x\ \ \in X$.

• "**Intersection of** A **and** B": $A \cap B = \{x,\ T_{A\cap B}(\mathfrak{R}(\Gamma)),\ I_{A\cap B}(\mathfrak{R}(\Gamma)),\ F_{A\cap B}(\mathfrak{R}(\Gamma))|\ \ x \in X\ \}$

where,

$T_{A\cap B}(\mathfrak{R}(\Gamma))(x) = min\{T_A(\mathfrak{R}(\Gamma))(x),\ T_B(\mathfrak{R}(\Gamma))(x)\}$, $I_{A\cap B}(\mathfrak{R}(\Gamma))(x) = max\{I_A(\mathfrak{R}(\Gamma))(x), I_B(\mathfrak{R}(\Gamma))(x)\}$

and $F_{A\cap B}(\mathfrak{R}(\Gamma))(x) = max\{F_A(\mathfrak{R}(\Gamma))(x),\ F_B(\mathfrak{R}(\Gamma))(x)\}\ \forall\ \ x\ \ \in X$.

•**Containment:** $A \subseteq B$ if and only if

$T_A(\mathfrak{R}(\Gamma))(x) \leq T_B(\mathfrak{R}(\Gamma))(x),\ \ I_A(\mathfrak{R}(\Gamma))(x) \geq I_B(\mathfrak{R}(\Gamma))(x),\ \ F_A(\mathfrak{R}(\Gamma))(x) \geq F_B(\mathfrak{R}(\Gamma))(x)\ \ \forall\ \ x\ \ \in X$.

• "**Complement:** The complement of a neutrosophic hypersoft set A , denoted by $\bar{A}$," defined by

$T_{\bar{A}}(\mathfrak{R}(\Gamma))(x) = 1 -\ \ T_A(\mathfrak{R}(\Gamma))(x),\ I_{\bar{A}}(\mathfrak{R}(\Gamma))(x) = 1 -\ \ I_A(\mathfrak{R}(\Gamma))(x),\ F_{\bar{A}}(\mathfrak{R}(\Gamma))(x) = 1 -\ \ F_A(\mathfrak{R}(\Gamma))(x)$
"

Next, we introduce a novel Hellinger divergence measure for any two $SVNHSS$ with some of its important properties. For any two fuzzy sets, A and B Ohlan et al. [44] presented the generalized form of divergence measure given by Hellinger as follows:

$$d_\gamma(A, B) = \sum_{i=1}^{n}\left(\frac{\left(\sqrt{\mu_A(x_i)} - \sqrt{\mu_B(x_i)}\right)^{2(\gamma+1)}}{\sqrt{\mu_A(x_i)\mu_B(x_i)}} + \frac{\left(\sqrt{\mu_{A^c}(x_i)} - \sqrt{\mu_{B^c}(x_i)}\right)^{2(\gamma+1)}}{\sqrt{\mu_{A^c}(x_i)\mu_{B^c}(x_i)}}\right), \gamma \in\ \ \mathbb{N}. \qquad (1)$$

Now, on similar lines to the above-presented divergence measure given by (1), we introduce the following parameterized divergence measure for single valued neutrosophic hypersoft set:

$$\mathbb{I}_{\gamma}(A,B) = \sum_{i=1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{T_A(\Re(\Gamma))(x_i)} - \sqrt{T_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_A(\Re(\Gamma))(x_i)+T_B(\Re(\Gamma))(x_i)}\right)^{\gamma}} + \frac{\left(\sqrt{1-T_A(\Re(\Gamma))(x_i)} - \sqrt{1-T_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_A(\Re(\Gamma))(x_i)-T_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+ \sum_{i=1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{I_A(\Re(\Gamma))(x_i)} - \sqrt{I_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_A(\Re(\Gamma))(x_i)+I_B(\Re(\Gamma))(x_i)}\right)^{\gamma}} + \frac{\left(\sqrt{1-I_A(\Re(\Gamma))(x_i)} - \sqrt{1-I_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_A(\Re(\Gamma))(x_i)-I_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+ \sum_{i=1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{F_A(\Re(\Gamma))(x_i)} - \sqrt{F_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_A(\Re(\Gamma))(x_i)+F_B(\Re(\Gamma))(x_i)}\right)^{\gamma}} + \frac{\left(\sqrt{1-F_A(\Re(\Gamma))(x_i)} - \sqrt{1-F_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_A(\Re(\Gamma))(x_i)-F_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right], \gamma \in \mathbb{N}. \quad (2)$$

Further, to check the validation of the proposed parameterized divergence measure for *SVNHSSs*, we propose the following theorem as follows:

Theorem 1. *The proposed divergence measure* $\mathbb{I}_{\gamma}(A,B)$ *given by* (2) *is a reliable divergence measure for two* $SVNHSSs$.

Proof: In order to check the validation of the proposed divergence measure, we need to check whether (2) satisfies the axioms of the divergence measures given in Section 2.

i. Since (2) holds symmetry for A and B, therefore it is clear that $\mathbb{I}(A,B) = \mathbb{I}(B,A)$.

ii. Also, we observe that $\mathbb{I}(A,B) = 0$ iff $T_A(\Re(\Gamma))(x) = T_B(\Re(\Gamma))(x)$, $I_A(\Re(\Gamma))(x) = I_B(\Re(\Gamma))(x)$, $F_A(\Re(\Gamma))(x) = F_B(\Re(\Gamma))(x)$ $\forall\ x \in X$.

Now, it remains shown that $\mathbb{I}(A,B) \geq 0$. In order to prove the non-negativity, first we need to prove the convexity of $\mathbb{I}_{\gamma}$. Since, $\mathbb{I}_{\gamma}(A,B)$ is of Csiszar's g-divergence type with generating mapping $g_{\gamma}: (0,\infty) \to \mathbb{R}^{+}$, defined by ,

$$g_{\gamma}(s) = \frac{2^{\gamma}(\sqrt{s}-1)^{2(\gamma+1)}}{(s+1)^{\gamma}} \quad ; \quad g_{\gamma}(1) = 0. \quad (3)$$

Further, differentiate (3) with respect to s two times we get,

$$g_{\gamma}''(s) = \left(\frac{2^{\gamma}}{2}\right)\frac{\left(2s + 2\gamma\sqrt{s} + 2\gamma s^{\frac{3}{2}} + 4\gamma s + s^2 + 1\right)(\gamma+1)(\sqrt{s}-1)^{2\gamma}}{(s+1)^{\gamma+2}s^{3/2}}.$$

Since, $\gamma \in \mathbb{N}$ and $s \in (0,\infty)$, therefore, $g_\gamma''(s) \geq 0$ which shows the convexity of $g_\gamma''(s)$. Hence, $\mathbb{I}(A,B) \geq 0$.

iii. In order to prove this part, we divide the collection X into two disjoint subsets X_1 and X_2 defined as

$$X_1 = \{x_i \in X|\ T_A(\mathfrak{N}(\Gamma))(x) \geq T_B(\mathfrak{N}(\Gamma))(x) \geq T_C(\mathfrak{N}(\Gamma))(x),\ I_A(\mathfrak{N}(\Gamma))(x) \leq I_B(\mathfrak{N}(\Gamma))(x) \leq I_C(\mathfrak{N}(\Gamma))(x),\ F_A(\mathfrak{N}(\Gamma))(x) \leq F_B(\mathfrak{N}(\Gamma))(x) \leq F_C(\mathfrak{N}(\Gamma))(x)\}; \quad (4)$$

and

$$X_2 = \{x_i \in X|\ T_A(\mathfrak{N}(\Gamma))(x) \geq T_B(\mathfrak{N}(\Gamma))(x) \geq T_C(\mathfrak{N}(\Gamma))(x),\ I_A(\mathfrak{N}(\Gamma))(x) \leq I_B(\mathfrak{N}(\Gamma))(x) \leq I_C(\mathfrak{N}(\Gamma))(x),\ F_A(\mathfrak{N}(\Gamma))(x) \leq F_B(\mathfrak{N}(\Gamma))(x) \leq F_C(\mathfrak{N}(\Gamma))(x)\}. \quad (5)$$

Now, by making use of the definition of neutrosophic sets and (2) in association with (4) and (5), the components of X_1 will vanish and the components of X_2 will only remain on the left-hand side. Hence, the left side will remain with only one term and the right side remain with two terms. The detailed steps of calculation can be shown easily. Therefore, axiom *iii* is satisfied.

iv. The proof of this axiom can be done by the union operation accordingly as the proof of the axiom of *iii*. Therefore, $\mathbb{I}_\gamma(A,B)$ is a validated divergence measure between the single-valued neutrosophic hypersoft sets A and B.

4. Properties of Novel Parameterized Neutrosophic Hypersoft Divergence Measure

In this section, we give some of the important properties of the proposed divergence measure in a single-valued neutrosophic environment.

Theorem 2. . *For any A, B and $C \in SVNHSS(X)$, the parametric divergence information measure (2) holds the below mentioned fundamental properties:*

1. "$\mathbb{I}_\gamma(A \cup B, A \cap B) = \mathbb{I}_\gamma(A,B)$
2. $\mathbb{I}_\gamma(A \cup B, A) + \mathbb{I}_\gamma(A \cap B, A) = \mathbb{I}_\gamma(A,B)$
3. $\mathbb{I}_\gamma(A \cup B, C) + \mathbb{I}_\gamma(A \cap B, C) = \mathbb{I}_\gamma(A,C) + \mathbb{I}_\gamma(B,C)$
4. $\mathbb{I}_\gamma(A, A \cup B) = \mathbb{I}_\gamma(B, A \cap B)$
5. $\mathbb{I}_\gamma(A, A \cap B) = \mathbb{I}_\gamma(B, A \cup B)$."

Proof: In order to prove the above-stated properties, we divide the set X between two disjoint subsets X_1 & X_2 defined as

$$X_1 = \{x_i \in X|\ T_A(\mathfrak{N}(\Gamma))(x) \geq T_B(\mathfrak{N}(\Gamma))(x) \geq T_C(\mathfrak{N}(\Gamma))(x),\ I_A(\mathfrak{N}(\Gamma))(x) \leq I_B(\mathfrak{N}(\Gamma))(x) \leq I_C(\mathfrak{N}(\Gamma))(x),\ F_A(\mathfrak{N}(\Gamma))(x) \leq F_B(\mathfrak{N}(\Gamma))(x) \leq F_C(\mathfrak{N}(\Gamma))(x)\}; \quad (6)$$

and

$$X_2 = \{x_i \in X | \ T_A(\Re(\Gamma))(x) \geq T_B(\Re(\Gamma))(x) \geq T_C(\Re(\Gamma))(x), \ I_A(\Re(\Gamma))(x) \leq I_B(\Re(\Gamma))(x) \leq I_C(\Re(\Gamma))(x), \ F_A(\Re(\Gamma))(x) \leq F_B(\Re(\Gamma))(x) \leq F_C(\Re(\Gamma))(x)\}. \quad (7)$$

1. $\mathbb{I}_\gamma(A \cup B, A \cap B)$

$$= \sum_{i=1}^{n} 2^\gamma \left[\frac{\left(\sqrt{T_{A\cup B}(\Re(\Gamma))(x_i)} - \sqrt{T_{A\cap B}(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_{A\cup B}(\Re(\Gamma))(x_i) + T_{A\cap B}(\Re(\Gamma))(x_i)}\right)^\gamma} + \frac{\left(\sqrt{1-T_{A\cup B}(\Re(\Gamma))(x_i)} - \sqrt{1-T_{A\cap B}(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_{A\cup B}(\Re(\Gamma))(x_i) - T_{A\cap B}(\Re(\Gamma))(x_i)}\right)^\gamma} \right]$$

$$+ \sum_{i=1}^{n} 2^\gamma \left[\frac{\left(\sqrt{I_{A\cup B}(\Re(\Gamma))(x_i)} - \sqrt{I_{A\cap B}(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_{A\cup B}(\Re(\Gamma))(x_i) + I_{A\cap B}(\Re(\Gamma))(x_i)}\right)^\gamma} + \frac{\left(\sqrt{1-I_{A\cup B}(\Re(\Gamma))(x_i)} - \sqrt{1-I_{A\cap B}(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_{A\cup B}(\Re(\Gamma))(x_i) - I_{A\cap B}(\Re(\Gamma))(x_i)}\right)^\gamma} \right]$$

$$+ \sum_{i=1}^{n} 2^\gamma \left[\frac{\left(\sqrt{F_{A\cup B}(\Re(\Gamma))(x_i)} - \sqrt{F_{A\cap B}(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_{A\cup B}(\Re(\Gamma))(x_i) + F_{A\cap B}(\Re(\Gamma))(x_i)}\right)^\gamma} + \frac{\left(\sqrt{1-F_{A\cup B}(\Re(\Gamma))(x_i)} - \sqrt{1-F_{A\cap B}(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_{A\cup B}(\Re(\Gamma))(x_i) - F_{A\cap B}(\Re(\Gamma))(x_i)}\right)^\gamma} \right]$$

Now, by making use of Equations (6) and (7), we have

$$\mathbb{I}_\gamma(A \cup B, A \cap B) = \sum_{x_i \in X_1}^{n} 2^\gamma \left[\frac{\left(\sqrt{T_B(\Re(\Gamma))(x_i)} - \sqrt{T_A(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_B(\Re(\Gamma))(x_i) + T_A(\Re(\Gamma))(x_i)}\right)^\gamma} + \frac{\left(\sqrt{1-T_B(\Re(\Gamma))(x_i)} - \sqrt{1-T_A(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_B(\Re(\Gamma))(x_i) - T_A(\Re(\Gamma))(x_i)}\right)^\gamma} \right]$$

$$+\sum_{x_i \in X_1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{I_B(\Re(\Gamma))(x_i)}-\sqrt{I_A(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_B(\Re(\Gamma))(x_i)+I_A(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-I_B(\Re(\Gamma))(x_i)}-\sqrt{1-I_A(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_B(\Re(\Gamma))(x_i)-I_A(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i \in X_1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{F_B(\Re(\Gamma))(x_i)}-\sqrt{F_A(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_B(\Re(\Gamma))(x_i)+F_A(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-F_B(\Re(\Gamma))(x_i)}-\sqrt{1-F_A(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_B(\Re(\Gamma))(x_i)-F_A(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i \in X_2}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{T_A(\Re(\Gamma))(x_i)}-\sqrt{T_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_A(\Re(\Gamma))(x_i)+T_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-T_A(\Re(\Gamma))(x_i)}-\sqrt{1-T_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_A(\Re(\Gamma))(x_i)-T_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i \in X_2}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{I_A(\Re(\Gamma))(x_i)}-\sqrt{I_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_A(\Re(\Gamma))(x_i)+I_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-I_A(\Re(\Gamma))(x_i)}-\sqrt{1-I_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_A(\Re(\Gamma))(x_i)-I_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i \in X_2}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{F_A(\Re(\Gamma))(x_i)}-\sqrt{F_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_A(\Re(\Gamma))(x_i)+F_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-F_A(\Re(\Gamma))(x_i)}-\sqrt{1-F_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_A(\Re(\Gamma))(x_i)-F_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$= \mathbb{I}_{\gamma}(A,B).$$

2. Proof of this can be done on similar lines as of 1.

3. $\mathbb{I}_{\gamma}(A \cup B, C) + \mathbb{I}_{\gamma}(A \cap B, C)$

$$= "\sum_{i=1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{T_{A\cup B}(\Re(\Gamma))(x_i)}-\sqrt{T_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_{A\cup B}(\Re(\Gamma))(x_i)+T_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\right.$$

$$\left.\frac{\left(\sqrt{1-T_{A\cup B}(\Re(\Gamma))(x_i)}-\sqrt{1-T_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_{A\cup B}(\Re(\Gamma))(x_i)-T_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{i=1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{T_{A\cap B}(\Re(\Gamma))(x_i)}-\sqrt{T_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_{A\cap B}(\Re(\Gamma))(x_i)+T_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-T_{A\cap B}(\Re(\Gamma))(x_i)}-\sqrt{1-T_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_{A\cap B}(\Re(\Gamma))(x_i)-T_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{i=1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{I_{A\cup B}(\Re(\Gamma))(x_i)}-\sqrt{I_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_{A\cup B}(\Re(\Gamma))(x_i)+I_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-I_{A\cup B}(\Re(\Gamma))(x_i)}-\sqrt{1-I_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_{A\cup B}(\Re(\Gamma))(x_i)-I_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{i=1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{I_{A\cap B}(\Re(\Gamma))(x_i)}-\sqrt{I_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_{A\cap B}(\Re(\Gamma))(x_i)+I_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-I_{A\cap B}(\Re(\Gamma))(x_i)}-\sqrt{1-I_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_{A\cap B}(\Re(\Gamma))(x_i)-I_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{i=1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{F_{A\cup B}(\Re(\Gamma))(x_i)}-\sqrt{F_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_{A\cup B}(\Re(\Gamma))(x_i)+F_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-F_{A\cup B}(\Re(\Gamma))(x_i)}-\sqrt{1-F_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_{A\cup B}(\Re(\Gamma))(x_i)-F_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{i=1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{F_{A\cap B}(\Re(\Gamma))(x_i)}-\sqrt{F_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_{A\cap B}(\Re(\Gamma))(x_i)+F_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-F_{A\cap B}(\Re(\Gamma))(x_i)}-\sqrt{1-F_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_{A\cap B}(\Re(\Gamma))(x_i)-F_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$= \sum_{x_i\in X_1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{T_B(\Re(\Gamma))(x_i)}-\sqrt{T_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_B(\Re(\Gamma))(x_i)+T_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-T_B(\Re(\Gamma))(x_i)}-\sqrt{1-T_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_B(\Re(\Gamma))(x_i)-T_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i\in X_2}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{T_A(\Re(\Gamma))(x_i)}-\sqrt{T_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_A(\Re(\Gamma))(x_i)+T_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-T_A(\Re(\Gamma))(x_i)}-\sqrt{1-T_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_A(\Re(\Gamma))(x_i)-T_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{T_A(\Re(\Gamma))(x_i)}-\sqrt{T_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_A(\Re(\Gamma))(x_i)+T_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-T_A(\Re(\Gamma))(x_i)}-\sqrt{1-T_C(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_A(\Re(\Gamma))(x_i)-T_C(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+ \sum_{x_i\in X_2}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{T_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{T_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_B(\mathfrak{R}(\Gamma))(x_i)+T_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-T_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-T_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_B(\mathfrak{R}(\Gamma))(x_i)-T_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)+I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-I_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_B(\mathfrak{R}(\Gamma))(x_i)-I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i\in X_2}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)+I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-I_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_A(\mathfrak{R}(\Gamma))(x_i)-I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+ \sum_{x_i\in X_1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)+I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-T_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-T_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_A(\mathfrak{R}(\Gamma))(x_i)-I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i\in X_2}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)+I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-I_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_B(\mathfrak{R}(\Gamma))(x_i)-I_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i)+F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-F_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_B(\mathfrak{R}(\Gamma))(x_i)-F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+ \sum_{x_i\in X_2}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i)+F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-F_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_A(\mathfrak{R}(\Gamma))(x_i)-F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+ \sum_{x_i \in X_1}^{n} 2^{\gamma} \left[\frac{\left(\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i)} - \sqrt{F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i) + F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} + \frac{\left(\sqrt{1-F_A(\mathfrak{R}(\Gamma))(x_i)} - \sqrt{1-F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_A(\mathfrak{R}(\Gamma))(x_i) - F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} \right]$$

$$+ \sum_{x_i \in X_2}^{n} 2^{\gamma} \left[\frac{\left(\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i)} - \sqrt{F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i) + F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} + \frac{\left(\sqrt{1-F_B(\mathfrak{R}(\Gamma))(x_i)} - \sqrt{1-F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_B(\mathfrak{R}(\Gamma))(x_i) - F_C(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} \right]$$

$$= \mathbb{I}_{\gamma}(A, C) + \mathbb{I}_{\gamma}(B, C).\text{"}$$

4. $\mathbb{I}_{\gamma}(\mathrm{A}, \mathrm{A} \cup \mathrm{B})$

$$= \text{"} \sum_{i=1}^{n} 2^{\gamma} \left[\frac{\left(\sqrt{T_A(\mathfrak{R}(\Gamma))(x_i)} - \sqrt{T_{A\cup B}(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_A(\mathfrak{R}(\Gamma))(x_i) + T_{A\cup B}(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} + \frac{\left(\sqrt{1-T_A(\mathfrak{R}(\Gamma))(x_i)} - \sqrt{1-T_{A\cup B}(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_A(\mathfrak{R}(\Gamma))(x_i) - T_{A\cup B}(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} \right]$$

$$+ \sum_{i=1}^{n} 2^{\gamma} \left[\frac{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)} - \sqrt{I_{A\cup B}(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i) + I_{A\cup B}(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} + \frac{\left(\sqrt{1-I_A(\mathfrak{R}(\Gamma))(x_i)} - \sqrt{1-I_{A\cup B}(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_A(\mathfrak{R}(\Gamma))(x_i) - I_{A\cup B}(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} \right]$$

$$+ \sum_{i=1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{F_A(\Re(\Gamma))(x_i)}-\sqrt{F_{A\cup B}(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_A(\Re(\Gamma))(x_i)+F_{A\cup B}(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-F_A(\Re(\Gamma))(x_i)}-\sqrt{1-F_{A\cup B}(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_A(\Re(\Gamma))(x_i)-F_{A\cup B}(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$= \sum_{x_i\in X_1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{T_A(\Re(\Gamma))(x_i)}-\sqrt{T_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_A(\Re(\Gamma))(x_i)+T_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-T_A(\Re(\Gamma))(x_i)}-\sqrt{1-T_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_A(\Re(\Gamma))(x_i)-T_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{I_A(\Re(\Gamma))(x_i)}-\sqrt{I_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_A(\Re(\Gamma))(x_i)+I_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-I_A(\Re(\Gamma))(x_i)}-\sqrt{1-I_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_A(\Re(\Gamma))(x_i)-I_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\gamma}\left[\frac{\left(\sqrt{F_A(\Re(\Gamma))(x_i)}-\sqrt{F_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_A(\Re(\Gamma))(x_i)+F_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\frac{\left(\sqrt{1-F_A(\Re(\Gamma))(x_i)}-\sqrt{1-F_B(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_A(\Re(\Gamma))(x_i)-F_B(\Re(\Gamma))(x_i)}\right)^{\gamma}}\right]$$

$$= \mathbb{I}_{\gamma}(B, A\cap B).\text{"}$$

5. Proof of this can be done on similar lines.

Theorem 3. *For any A and $B \in SVNHSS(X)$, the parametric divergence information measure (2) holds the below mentioned properties:*

1. "$\mathbb{I}_{\gamma}(\bar{A}\ ,\bar{B}) = \mathbb{I}_{\gamma}(A,B)$
2. $\mathbb{I}_{\gamma}(\overline{A\cup B},\overline{A\cap B}) = \mathbb{I}_{\gamma}(\bar{A}\cap\bar{B},\bar{A}\cup\bar{B}) = \mathbb{I}_{\gamma}(A,B)$
3. $\mathbb{I}_{\gamma}(A,\bar{B}) = \mathbb{I}_{\gamma}(\bar{A},B)$
4. $\mathbb{I}_{\gamma}(A,\bar{B}) + \mathbb{I}_{\gamma}(\bar{A}\ ,\bar{B}) = \mathbb{I}_{\gamma}(A,B) + \mathbb{I}_{\gamma}(\bar{A},B)$"

Proof :

1. Proof of (1) can easily be done with the definition of *complement* and the required results hold.
2. Proof of (2) can be done by making use of (6) and (7) as:

$$\mathbb{I}_{\gamma}(\overline{A\cup B},\overline{A\cap B}) =$$

$$\sum_{x_i\in X_1}^{n} 2^{\gamma}\frac{\left(\sqrt{1-T_B(\Re(\Gamma))(x_i)}-\sqrt{1-T_A(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_B(\Re(\Gamma))(x_i)-T_A(\Re(\Gamma))(x_i)}\right)^{\gamma}}+\sum_{x_i\in X_1}^{n} 2^{\gamma}\frac{\left(\sqrt{T_B(\Re(\Gamma))(x_i)}-\sqrt{T_A(\Re(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_B(\Re(\Gamma))(x_i)+T_A(\Re(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_2}^{n} 2^{\gamma}\frac{\left(\sqrt{1-T_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_A(\mathfrak{R}(\Gamma))(x_i)-T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\sum_{x_i\in X_2}^{n} 2^{\gamma}\frac{\left(\sqrt{T_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_A(\mathfrak{R}(\Gamma))(x_i)+T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_1}^{n} 2^{\gamma}\frac{\left(\sqrt{1-I_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-I_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_B(\mathfrak{R}(\Gamma))(x_i)-I_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\sum_{x_i\in X_1}^{n} 2^{\gamma}\frac{\left(\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)+I_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_2}^{n} 2^{\gamma}\frac{\left(\sqrt{1-I_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_A(\mathfrak{R}(\Gamma))(x_i)-I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\sum_{x_i\in X_2}^{n} 2^{\gamma}\frac{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)+I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_1}^{n} 2^{\gamma}\frac{\left(\sqrt{1-F_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-F_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_B(\mathfrak{R}(\Gamma))(x_i)-F_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\sum_{x_i\in X_1}^{n} 2^{\gamma}\frac{\left(\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i)+F_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_2}^{n} 2^{\gamma}\frac{\left(\sqrt{1-F_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_A(\mathfrak{R}(\Gamma))(x_i)-F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\sum_{x_i\in X_2}^{n} 2^{\gamma}\frac{\left(\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i)+F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$=\mathbb{I}_{\gamma}(A,B).$$

Now, $\mathbb{I}_{\gamma}(\bar{A}\cap\bar{B},\bar{A}\cup\bar{B})=$

$$\sum_{x_i\in X_1}^{n} 2^{\gamma}\frac{\left(\sqrt{1-T_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-T_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_B(\mathfrak{R}(\Gamma))(x_i)-T_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\text{"}\sum_{x_i\in X_1}^{n} 2^{\gamma}\frac{\left(\sqrt{T_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{T_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_B(\mathfrak{R}(\Gamma))(x_i)+T_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_2}^{n} 2^{\gamma}\frac{\left(\sqrt{1-T_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_A(\mathfrak{R}(\Gamma))(x_i)-T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\sum_{x_i\in X_2}^{n} 2^{\gamma}\frac{\left(\sqrt{T_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{T_A(\mathfrak{R}(\Gamma))(x_i)+T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_1}^{n} 2^{\gamma}\frac{\left(\sqrt{1-I_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-I_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_B(\mathfrak{R}(\Gamma))(x_i)-I_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\sum_{x_i\in X_1}^{n} 2^{\gamma}\frac{\left(\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)+I_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_2}^{n} 2^{\gamma}\frac{\left(\sqrt{1-I_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_A(\mathfrak{R}(\Gamma))(x_i)-I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}+\sum_{x_i\in X_2}^{n} 2^{\gamma}\frac{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)+I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_1}^{n} 2^{\gamma} \frac{\left(\sqrt{1-F_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-F_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_B(\mathfrak{R}(\Gamma))(x_i)-F_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} + \sum_{x_i\in X_1}^{n} 2^{\gamma} \frac{\left(\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i)+F_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_2}^{n} 2^{\gamma} \frac{\left(\sqrt{1-F_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_A(\mathfrak{R}(\Gamma))(x_i)-F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} + \sum_{x_i\in X_2}^{n} 2^{\gamma} \frac{\left(\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i)+F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}"$$

$$= \mathbb{I}_{\gamma}(A,B).$$

Therefore, $\mathbb{I}_{\gamma}(\overline{A\cup B},\overline{A\cap B}) = \mathbb{I}_{\gamma}(\bar{A}\cap\bar{B},\bar{A}\cup\bar{B}) = \mathbb{I}_{\gamma}(A,B)$.

3. $\mathbb{I}_{\gamma}(A,\bar{B})$

$$=\sum_{x_i\in X_1}^{n} 2^{\gamma} \frac{\left(\sqrt{T_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-T_B(\mathfrak{R}(\Gamma))(x_i)-T_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} + \sum_{x_i\in X_1}^{n} 2^{\gamma} \frac{\left(\sqrt{1-T_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{1-T_A(\mathfrak{R}(\Gamma))(x_i)+T_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+"\sum_{x_i\in X_1}^{n} 2^{\gamma} \frac{\left(\sqrt{I_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-I_B(\mathfrak{R}(\Gamma))(x_i)-I_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} + \sum_{x_i\in X_1}^{n} 2^{\gamma} \frac{\left(\sqrt{1-I_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{1-I_A(\mathfrak{R}(\Gamma))(x_i)+I_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}$$

$$+\sum_{x_i\in X_1}^{n} 2^{\gamma} \frac{\left(\sqrt{F_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{1-F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{2-F_B(\mathfrak{R}(\Gamma))(x_i)-F_A(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}} + \sum_{x_i\in X_1}^{n} 2^{\gamma} \frac{\left(\sqrt{1-F_A(\mathfrak{R}(\Gamma))(x_i)}-\sqrt{F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{2(\gamma+1)}}{\left(\sqrt{1-F_A(\mathfrak{R}(\Gamma))(x_i)+F_B(\mathfrak{R}(\Gamma))(x_i)}\right)^{\gamma}}"$$

$$= \mathbb{I}_{\gamma}(\bar{A},B).$$

4. Proof of (4) can be done by making use of (1) and (3), which satisfies $\mathbb{I}_{\gamma}(A,\bar{B}) + \mathbb{I}_{\gamma}(\bar{A}\ ,\bar{B}) = \mathbb{I}_{\gamma}(A,B) + \mathbb{I}_{\gamma}(\bar{A},B)$.

5. Utilization of the Proposed Parameterized Divergence Measure in the MCDM Problem.

In this section, we propose a methodology for the *MCDM* based on proposed parameterized divergence measures of $SVNHSSs$. The steps of the proposed methodology have been explained with the help of Figure 1 in an abstract way. Consider the set of m alternatives $\{Y_1, Y_2, \ldots, Y_n\}$ and n attributes $k^1, k^2, \ldots, k^n$ and "whose corresponding attribute values are respectively the sets $K^1, K^2, \ldots, K^n$ with $K^i \cap K^j = \emptyset$, for $i \neq j$ and $i,j \in \{1,2,..,n\}$." The set of all possible $SVNHSSs$ are given by $(\mathfrak{R},\Gamma)$, where $\Gamma = K^1 \times\ K^2 \times \ldots \times\ K^n$. The aim of an expert is to choose the best suitable alternative out of the available alternatives which satisfy the n attribute values. The opinions of all the experts have been considered in terms of a matrix representation $\mathrm{H} = \left[\mathrm{h_{ij}}\right]_{\mathrm{m\times n}}$ called single-valued neutrosophic hypersoft matrix where $h_{ij} = \left(T(\mathfrak{R}(\Gamma))_{ij}, I(\mathfrak{R}(\Gamma))_{ij}, F(\mathfrak{R}(\Gamma))_{ij}\right)$. The necessary steps involved in the algorithm of the proposed methodology are outlined as follows:

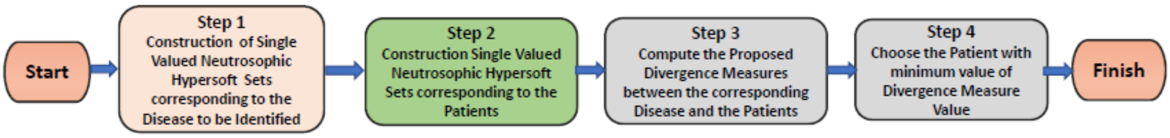

Figure 1: Algorithmic Steps of the proposed Methodology

Step 1: In the first step, construct the single-valued neutrosophic hypersoft decision matrix based on available information.

Step 2: In this step, remove the heterogeneity in the attributes (if any) and convert it into a homogeneous type of attribute. Majorly, there are two types of attributes i.e. cost type and benefit type, we convert the cost type attributes into benefit type. For this, the expert matrix $H = [h_{ij}]_{m\times n}$ is converted into a new expert matrix $H' = [h'_{ij}]_{m\times n}$ where h'_{ij} is given by

$$h'_{ij} = \left(T(\mathfrak{R}(\Gamma))_{ij}, I(\mathfrak{R}(\Gamma))_{ij}, F(\mathfrak{R}(\Gamma))_{ij}\right) = \begin{cases} h_{ij} \ ; \ \text{for benefit criteria} \\ h_{ij}{}^{c} \ ; \ \text{for cost criteria.} \end{cases}$$

Step 3: In this step, compute the values of the proposed divergence measure of the alternatives $Y_i's$ with respect to the sub-attributes individually.

Step 4: In this step, the ordering of alternatives can be done with the least value of the proposed divergence measure.

6. Use of Proposed Divergence Measures in Symptomatic Detection of COVID-19.

In this section, we shall make use of above-stated methodology for the symptomatic detection of COVID-19 on the basis of divergence measures for $SVNHSSs$. Consider a set of four patients $\{Y_1, Y_2, Y_3, Y_4\}$ in a hospital having symptoms of COVID-19. Suppose there are three stages of characterization of the symptoms as $severe(x^1)$, $mild(x^2)$ and $no(x^3)$. The universal set $X = \{x^1, x^2, x^3\}$. Let $K = \{K^1 = sense\ \ of\ \ taste,\ K^2 = temperature,\ K^3 = chest\ \ pain,\ K^4 = flu\ \}$ be the set of symptoms that are further classified into sub-attributes:

$K^1 =$ "$sense\ \ of\ \ taste = \{no\ \ taste, can\ \ taste\}$"

$K^2 =$ "$temperature = \{97.5 - 98.5, 98.6 - 99.5, 99.6 - 101.5, 101.6 - 102.5\}$"

$K^3 =$ "$chest\ \ pain = \{shortness\ \ of\ \ breath,\ no\ \ pain,\ normal\ \ pain\ \ angina\}$"

$K^4 =$ "$flu = \{sore\ \ throat, cough,\ strep\ \ throat\}$"

Now, let us define a relation $\mathfrak{R} : K^1 \times\ K^2 \times \ldots \times\ K^n \to P(X)$ defined as,

$\mathfrak{R}\ (K^1 \times\ K^2 \times \ldots \times\ K^n) = \{\xi = shortness\ \ of\ \ breath, \zeta = 101.3, \varrho = sore\ \ throat, \varsigma = no\ \ taste\}$ is the most prominent sample of the patient for the confirmation of COVID-19.

Step1: Let $(\mathfrak{R}, \Gamma)$ be a $SVNHSS(X)$ for COVID-19 prepared with the help of medical experts as given in Table 1.

Table 1. $SVNHSS(\mathfrak{R}, \Gamma)$ for COVID-19

$(\mathfrak{R}, \Gamma)$	K^1	K^2	K^3	K^4
x^1	$\xi(0.4,0.2,0.3)$	$\zeta(0.3,0.4,0.3)$	$\varrho(0.7,0.1,0.2)$	$\varsigma(0.4,0.2,0.3)$

x^2	$\xi(0.5,0.1,0.3)$	$\zeta(0.1,0.8,0.1)$	$\varrho(0.4,0.3,0.2)$	$\varsigma(0.5,0.2,0.3)$
x^3	$\xi(0.3,0.5,0.1)$	$\zeta(0.1,0.2,0.7)$	$\varrho(0.1,0.6,0.2)$	$\varsigma(0.5,0.4,0.1)$

Next, the $SVNHSSs$ for the patients under consideration are given in Table 2-Table 5.

Table 2. $SVNHSS(\mathfrak{N},\Gamma)$ for the patient Y_1

$(\mathfrak{N},\Gamma)$	K^1	K^2	K^3	K^4
x^1	$\xi(0.5,0.2,0.3)$	$\zeta(0.8,0.1,0.0)$	$\varrho(0.2,0.7,0.1)$	$\varsigma(0.9,0.1,0.0)$
x^2	$\xi(0.3,0.1,0.5)$	$\zeta(0.2,0.8,0.0)$	$\varrho(0.5,0.2,0.2)$	$\varsigma(0.6,0.1,0.2)$
x^3	$\xi(0.4,0.5,0.1)$	$\zeta(0.7,0.2,0.0)$	$\varrho(0.3,0.6,0.1)$	$\varsigma(0.4,0.5,0.1)$

Table 3. $SVNHSS(\mathfrak{N},\Gamma)$ for the patient Y_2

$(\mathfrak{N},\Gamma)$	K^1	K^2	K^3	K^4
x^1	$\xi(0.2,0.6,0.2)$	$\zeta(0.2,0.5,0.3)$	$\varrho(0.6,0.1,0.2)$	$\varsigma(0.7,0.2,0.1)$
x^2	$\xi(0.3,0.4,0.3)$	$\zeta(0.2,0.6,0.2)$	$\varrho(0.4,0.3,0.2)$	$\varsigma(0.5,0.2,0.3)$
x^3	$\xi(0.8,0.1,0.1)$	$\zeta(0.1,0.2,0.7)$	$\varrho(0.1,0.6,0.2)$	$\varsigma(0.8,0.1,0.1)$

Table 4. $SVNHSS(\mathfrak{N},\Gamma)$ for the patient Y_3

$(\mathfrak{N},\Gamma)$	K^1	K^2	K^3	K^4
x^1	$\xi(0.3,0.4,0.3)$	$\zeta(0.2,0.6,0.1)$	$\varrho(0.3,0.6,0.0)$	$\varsigma(0.4,0.2,0.3)$
x^2	$\xi(0.5,0.1,0.3)$	$\zeta(0.1,0.8,0.1)$	$\varrho(0.4,0.3,0.2)$	$\varsigma(0.5,0.2,0.3)$
x^3	$\xi(0.5,0.5,0.0)$	$\zeta(0.2,0.0,0.8)$	$\varrho(0.4,0.5,0.1)$	$\varsigma(0.4,0.4,0.1)$

Table 5. $SVNHSS(\mathfrak{N},\Gamma)$ for the patient Y_4

$(\mathfrak{N},\Gamma)$	K^1	K^2	K^3	K^4
x^1	$\xi(0.9,0.0,0.1)$	$\zeta(0.2,0.6,0.1)$	$\varrho(0.6,0.1,0.2)$	$\varsigma(0.5,0.2,0.2)$
x^2	$\xi(0.3,0.5,0.2)$	$\zeta(0.4,0.0,0.6)$	$\varrho(0.2,0.3,0.5)$	$\varsigma(0.7,0.2,0.1)$
x^3	$\xi(0.4,0.3,0.3)$	$\zeta(0.4,0.2,0.4)$	$\varrho(0.3,0.6,0.1)$	$\varsigma(0.0,0.1,0.9)$

Step 2: Since all the attributes are of benefit type, so there is no need for normalization of attributes.

Step 3: In this step, we shall make use of the proposed divergence measure to compute the values of the divergence measure for different patients. Now, by applying the proposed divergence measure (1), we get $\mathbb{I}_\gamma(\mathfrak{N}, Y_1) = 0.3457$ for the patient Y_1, $\mathbb{I}_\gamma(\mathfrak{N}, Y_2) = 0.6243$ for the patient Y_2, $\mathbb{I}_\gamma(\mathfrak{N}, Y_3) = 0.4892$ for the patient Y_3 and $\mathbb{I}_\gamma(\mathfrak{N}, Y_4) = 0.8657$ for the patient Y_4.

Step 4: Now, the minimum value of the divergence measure is 0.3457 which is for the patient Y_1, hence out of all four patients, Y_1 is suffering from COVID-19 on the basis of symptomatic detection.

7. Discussion on Results and Methodology

In this section, we briefly present a discussion of the proposed methodology and the obtained results by mentioning some remarks on comparative advantages, importance and limitations. The important discussion points on the notions of neutrosophic hypersoft set and its Hellinger divergence measure are as follows:

- The computed value of the divergence measure is more deterministic as compared with other obtained values.

- The utilization of a neutrosophic hypersoft set and its measure have a superiority to dealing incorporating a broader notion of applicability in uncertain situations of a direct parameter and sub-parameterization due to the hypersoft feature.
- The hypersoft sets which are already existing in the literature – "intuitionistic fuzzy hypersoft set, Pythagorean fuzzy Hypersoft set, and Neutrosophic hypersoft set" due to the rejection and abstain components being excluded, each have their own shortcomings.
- The methodology implementing the proposed Hellinger divergence measure be effectively and consistently applied to different group strategic MCDM issues as well as in a broader framework.

The following characteristic comparison table (Table 1) represents the added-on advantages of the proposed methodology over the existing ones:

Table 1: Characteristic Comparison Table

Authors	Divergence Measures	Truthiness	Indeterminacy	Falsity	Sub-Attributes
Ohlan et al. [46]	"Fuzzy Sets"	✓	×	×	×
Kadian et al. [50]	"Intuitionistic Fuzzy Sets"	✓	×	✓	×
Montes et al.[51]	"Picture Fuzzy Sets"	✓	×	✓	×
Proposed	"Single-valued Neutrosophic Hypersoft Sets"	✓	✓	✓	✓

8. Conclusions & Scope for Future Work

The Hellinger divergence measure for $SVNHSSs$ has been successfully presented along with some important deliberations and properties. In literature, the Hellinger divergence measure for $SVNHSSs$ is novel and utilized to propose a new methodology for the symptomatic detection of COVID-19. The necessary steps of the proposed methodology have been illustrated successfully. The obtained results based on the proposed methodology are found to be efficient and consistent. In the future, the utility distribution can further be incorporated in the Hellinger divergence measure to propose a `useful' Hellinger divergence measure for $SVNHSSs$, and eventually, the total ambiguity and hybrid ambiguity can be discussed with due applications. Also, the notion of expert sets can be appended with the proposed Hellinger divergence measure.

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Neutrosophic Fuzzy Ideals in Γ Rings

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Abstract: Fuzzy sets are a major oversimplification and extension of classical sets. Fuzzy sets have become a recognized research topic in many fields. This paper proposes a new type of set theory is neutrosophic set. As a novel study field, new hybrid sets created from neutrosophic sets are gaining prominence. The neutrosophic set is used to describe indeterminacy and uncertainty in any information. The neutrosophic set extension has been explored by many researchers. Here we introduce properties of Neutrosophic Fuzzy (NF) ideals in Γ Rings. Some new neutrosophic operations are explored.

Keywords: Γ Rings; Fuzzy set; Neutrosophic fuzzy set; Neutrosophic fuzzy ideal; Neutrosophic $\Gamma-$endomorphism.

1. Introduction

In 1965, Zadeh proposed the fuzzy set as a method to deal with imprecise data [21]. Many applications have been found for fuzzy sets in various fields of research, these include intuitionistic fuzzy sets, picture fuzzy sets, orthopair fuzzy sets, and neutrosophic sets. Also, various algebraic structures have been discussed in fuzzy versions by many researchers. One of the algebraic structures is the gamma ring. In 1964 Nobusawa [9] first proposed the gamma ring concept. This is rather common when compared to a ring. Barnes [3] weakened the requirements of Nobusawa's gamma ring. As a continuation of his research, researchers are interested in gamma rings with apartness [6,7,10]. Gamma ring structure is used to investigate the number of Generalizations that are identical to the corresponding parts of Kyuno's ring theory [8]. Uddin[19] generalized the results of gamma endomorphism in gamma rings. Ardakani [2] discussed derivations of prime and semi-prime gamma rings. Atanassov created Intuitionistic fuzzy set to address the issue of non-determinacy brought on by a single membership function in the fuzzy set. The intuitionistic fuzzy set is highly helpful in that it offers a flexible model to explain the uncertainty and ambiguity inherent in decision-making. In 2010 Palaniappan et.al [11, 12, 13] proposed the intuitionistic fuzzy ideals and intuitionistic fuzzy prime ideals in Γ-Rings. Neutrosophic logic was introduced by Florentin Smarandache in 1995. Neutrosophic set is a generalization of the intuitionistic fuzzy set discussed by Smarandache[17]. Neutrosophic set is a set where each element of the universe has a degree of truth, indeterminacy, and falsity respectively, and which lies between 0 and 1. There are several applications in various fields. Salama [15] states the characteristic function of a Neutrosophic set. In 2010 Wang introduced the single-valued Neutrosophic sets [20]. Many authors exhibited NF ideals [5,14,16,18]. Agboola primarily focused on neutrosophic canonical hypergroups and neutrosophic hyperrings [1]. Chalapathi stated about neutrosophic rings [4]. During this paper, we introduced the notion of NF ideals in the gamma ring structure.

2. Prerequisites:

The required definitions are incorporated in this section.

Definition 2.1: [9] Consider (N,Γ) is an abelian group where $N=\{p,q,r\}$ and $\Gamma=\{\alpha,\beta,\gamma \ldots.\}$ and for all $p,q,r \in N$ and $\alpha,\beta \epsilon \Gamma$,

(1) $p\alpha q \in N$

(2) $(p+q)\alpha\, r = p\alpha\, r + q\alpha\, r, p(\alpha+\beta)q = p\alpha q + p\beta q, p\alpha(q+r) = p\alpha q + p\alpha r,$

(3) $(p\alpha q)\beta\, r = p\alpha(q\beta r)$. Then N is a Γ Ring.

Later the improved by Barnes [3]

(1') $p\alpha q \in N \;\; \alpha p\beta \epsilon \Gamma,$

(2') $(p+q)\alpha\, r = p\alpha\, r + q\alpha\, r, p(\alpha+\beta)q = p\alpha q + p\beta q, p\alpha(q+r) = p\alpha q + p\alpha r,$

(3') $(p\alpha q)\beta\, r = p(\alpha q\beta)r = p\alpha(q\beta r),$

(4') $p\alpha q = 0 \; for\ all\ p,q\epsilon N\ implies\ \alpha = 0$

Definition 2.2: [16] A fuzzy set φ in a Γ Ring N is called fuzzy ideal of N if x, y ϵ R

(i) $\varphi(x-y) \geq \min\{\varphi(x),\varphi(y)\}$

(ii) $\varphi(x\alpha y) \geq \max\{\varphi(x),\varphi(y)\}$

Definition 2.3: [18] A NF set $\mathcal{A}$ on the universe of discourse X characterized by a truth membership function $\mathcal{U}_{\mathcal{A}}$ (x), a indeterminacy function $\mathcal{V}_{\mathcal{A}}$ (x) and a falsity membership function $\mathcal{W}_{\mathcal{A}}(x)$ is defined as $\mathcal{A}$⊚ ⊚⊚x, $\mathcal{U}_{\mathcal{A}}$ (x), $\mathcal{V}_{\mathcal{A}}$ (x), $\mathcal{W}_{\mathcal{A}}(x) >: x\epsilon X\}$,

Where $\mathcal{U}_{\mathcal{A}}, \mathcal{V}_{\mathcal{A}}, \mathcal{W}_{\mathcal{A}}$: X→ [0,1] and 0≤ $\mathcal{U}_{\mathcal{A}}$ (x)⊚ $\mathcal{V}_{\mathcal{A}}$ (x)⊚ $\mathcal{W}_{\mathcal{A}}(x)$ ≤3

Definition 2.4: [20] Let X be a non-void set and let $\mathcal{A}$ ⊚⊚$\mathcal{U}_{\mathcal{A}}$, $\mathcal{V}_{\mathcal{A}}, \mathcal{W}_{\mathcal{A}} >$ and $\mathcal{B} =< \mathcal{U}_{\mathcal{B}}, \mathcal{V}_{\mathcal{B}}, \mathcal{W}_{\mathcal{B}} >$ be two NS sets in X. Then

Complement: $\mathcal{C}(\mathcal{A})$

$\mathcal{U}_{\mathcal{C}(\mathcal{A})}$(x) ⊚ $1-\mathcal{U}_{\mathcal{A}}$(x), $\mathcal{V}_{\mathcal{C}(\mathcal{A})}$(x) ⊚ $1-\mathcal{V}_{\mathcal{A}}$(x) , $\mathcal{W}_{\mathcal{C}(\mathcal{A})}$(x) ⊚ $1-\mathcal{W}_{\mathcal{A}}$(x).

Containment: $\mathcal{A} \subseteq \mathcal{B}$

inf $\mathcal{U}_{\mathcal{A}}$(x)≤ inf $\mathcal{U}_{\mathcal{B}}$(x), sup$\mathcal{U}_{\mathcal{A}}$(x) ≤ sup $\mathcal{U}_{\mathcal{B}}$(x), inf $\mathcal{W}_{\mathcal{A}}$(x) ≥inf $\mathcal{W}_{\mathcal{B}}$(x), sup $\mathcal{W}_{\mathcal{A}}$(x) ≥ sup $\mathcal{W}_{\mathcal{B}}$(x),

Union: $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$

$\mathcal{U}_{\mathcal{C}}$(x) ⊚ $\mathcal{U}_{\mathcal{A}}$(x) ⊚ $\mathcal{U}_{\mathcal{B}}$(x)- $\mathcal{U}_{\mathcal{A}}(x) * \mathcal{U}_{\mathcal{B}}$(x), $\mathcal{V}_{\mathcal{C}}$(x) ⊚ $\mathcal{V}_{\mathcal{A}}$(x) ⊚ $\mathcal{V}_{\mathcal{B}}$(x)- $\mathcal{V}_{\mathcal{A}}(x) * \mathcal{V}_{\mathcal{B}}$(x),

$\mathcal{W}_{\mathcal{C}}$(x)⊚ $\mathcal{W}_{\mathcal{A}}$(x) ⊚ $\mathcal{W}_{\mathcal{B}}$(x)- $\mathcal{W}_{\mathcal{A}}(x) * \mathcal{W}_{\mathcal{B}}$(x),

Intersection: $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$

$\mathcal{U}_{\mathcal{C}}$(x) ⊚ $\mathcal{U}_{\mathcal{A}}(x) * \mathcal{U}_{\mathcal{B}}$(x), $\mathcal{V}_{\mathcal{C}}$(x) ⊚ $\mathcal{V}_{\mathcal{A}}(x) * \mathcal{V}_{\mathcal{B}}$(x), $\mathcal{W}_{\mathcal{C}}$(x) ⊚ $\mathcal{W}_{\mathcal{A}}(x) * \mathcal{W}_{\mathcal{B}}$(x) for all x in X.

Definition 2.5: A function $\theta: G_1 \to G_2$ where G_1 and G_2 are Γ Rings is said to be a Γ -homomorphism if $\theta(p+q) = \theta(p) + \theta(q), \theta(p\alpha q) = \theta(p)\alpha\theta(q)$ for all $p,q,\epsilon N, \alpha\epsilon\Gamma$.

Definition 2.6: A function $\theta: G_1 \to G_2$ Where θ is a Γ -homomorphism and G_1 and G_2 are Γ Rings is said to be a Γ – endomorphism if $G_2 \subseteq G_1$.

3. NF ideals of Γ Ring:

Definition 3.1: Let N be a Γ Ring. A NF set $\mathcal{A}$ in N is said to be NF ideal of N if

(i)$\mathcal{U}_{\mathcal{A}}(p-q) \geq \{\mathcal{U}_{\mathcal{A}}(p) \wedge \mathcal{U}_{\mathcal{A}}(q)\}$, $\mathcal{V}_{\mathcal{A}}(p-q) \leq \{\mathcal{V}_{\mathcal{A}}(p) \vee \mathcal{V}_{\mathcal{A}}(q)\}$, and
$\mathcal{W}_{\mathcal{A}}(p-q) \leq \{\mathcal{W}_{\mathcal{A}}(p) \vee \mathcal{W}_{\mathcal{A}}(q)\}$
(ii)$\mathcal{U}_{\mathcal{A}}(p\alpha q) \geq \mathcal{U}_{\mathcal{A}}(q)[resp. \mathcal{U}_{\mathcal{A}}(p\alpha q) \geq \mathcal{U}_{\mathcal{A}}(p)]$, $\mathcal{V}_{\mathcal{A}}(p\alpha q) \leq \mathcal{V}_{\mathcal{A}}(q)[resp. \mathcal{V}_{\mathcal{A}}(p\alpha q) \leq \mathcal{V}_{\mathcal{A}}(p)$⊚ and
$\mathcal{W}_{\mathcal{A}}(p\alpha q) \leq \mathcal{W}_{\mathcal{A}}(q)[resp. \mathcal{W}_{\mathcal{A}}(p\alpha q) \leq \mathcal{W}_{\mathcal{A}}(p)]$ for all p, q $\epsilon\ N, \alpha\ \epsilon\ \Gamma$.

Example 3.2: Let N⊚ ⊚0, 1, 2, 3⊚ and $\alpha =$ ⊚0, 1, 2, 3⊚ and define N and α as follows

-	0	1	2	3
0	0	1	2	3
1	1	1	3	2
2	2	3	3	2
3	3	2	2	2

α	0	1	2	3
0	0	1	2	3
1	1	1	3	2
2	2	3	3	2
3	3	2	2	2

$\mathcal{U}_{\mathcal{A}}(x) = \begin{cases} 0.7 & if\ x = 0 \\ 0.8\ if\ x = 1 \\ 0.8\ if\ x = 2,3 \end{cases}$, $\mathcal{V}_{\mathcal{A}}(x)$ ⊚ $\begin{cases} 0.9\ if\ x = 0 \\ 0.7\ if\ x = 1 \\ 0.6\ if\ x = 2,3 \end{cases}$, $\mathcal{W}_{\mathcal{A}}(x)$ ⊚ $\begin{cases} 0.8\ if\ x = 0 \\ 0.5\ if\ x = 1 \\ 0.3\ if\ x = 2,3 \end{cases}$

Clearly N is a NF ideal of N.

Definition 3.3: Consider NF ideal $\varphi =< \mathcal{U}_{\varphi}, \mathcal{V}_{\varphi}, \mathcal{W}_{\varphi} >$ of a Γ Ring N is normal if $\mathcal{U}_{\varphi}(0) = 1, \mathcal{V}_{\varphi}(0) = 0$, and $\mathcal{W}_{\varphi}(0) = 0$.

Theorem 3.4: Let $\varphi =< \mathcal{U}_{\varphi}, \mathcal{V}_{\varphi}, \mathcal{W}_{\varphi} >$ be a NF ideal of a Γ Ring N and let $\mathcal{U}_{\varphi}^{+}(p) = \mathcal{U}_{\varphi}(p) + 1 - \mathcal{U}_{\varphi}(0)$, $\mathcal{V}_{\varphi}^{+}(P) = \mathcal{V}_{\varphi}(x) - \mathcal{V}_{\varphi}(0)$ and $\mathcal{W}_{\varphi}^{+}(p) = \mathcal{W}_{\varphi}(p) - \mathcal{W}_{\varphi}(0)$. If $\mathcal{U}_{\varphi}^{+}(p) + \mathcal{V}_{\varphi}^{+}(P) + \mathcal{W}_{\varphi}^{+}(p) \leq 3$ for all p$\in N$, then $\varphi^{+} =< \mathcal{U}_{\varphi}^{+}, \mathcal{V}_{\varphi}^{+}, \mathcal{W}_{\varphi}^{+} >$ is a normal NF ideal of N.

Proof: First of all, let us note that $\mathcal{U}_{\varphi}^{+}(0) = 1, \mathcal{V}_{\varphi}^{+}(0) = 0$ and $\mathcal{W}_{\varphi}^{+}(0) = 0$ and $\mathcal{U}_{\varphi}^{+}, \mathcal{V}_{\varphi}^{+}, \mathcal{W}_{\varphi}^{+} \in [0,1]$ for every p$\in N$ so $\varphi^{+} =< \mathcal{U}_{\varphi}^{+}, \mathcal{V}_{\varphi}^{+}, \mathcal{W}_{\varphi}^{+} >$ is a normal NF set. To prove φ^{+} is a NF ideal. Let p, $q \in N$ and $\alpha \in \Gamma$ then

$\mathcal{U}_{\varphi}^{+}(p-q) = \mathcal{U}_{\varphi}(p-q) + 1 - \mathcal{U}_{\varphi}(0)$
$\geq \{\mathcal{U}_{\varphi}(p) \wedge \mathcal{U}_{\varphi}(q)\} + 1 - \mathcal{U}_{\varphi}(0)$
⊚$\{\mathcal{U}_{\varphi}(p) + 1 - \mathcal{U}_{\varphi}(0)\} \wedge \{\mathcal{U}_{\varphi}(q) + 1 - \mathcal{U}_{\varphi}(0)\}$
$= \mathcal{U}_{\varphi}^{+}(p) \wedge \mathcal{U}_{\varphi}^{+}(q)$
$\mathcal{V}_{\varphi}^{+}(p-q) = \mathcal{V}_{\varphi}(p-q) - \mathcal{V}_{\varphi}(0)$
$\leq \{\mathcal{V}_{\varphi}(p) \vee \mathcal{V}_{\varphi}(q)\} - \mathcal{V}_{\varphi}(0)$
⊚$\{\mathcal{V}_{\varphi}(p) - \mathcal{V}_{\varphi}(0)\} \vee \{\mathcal{V}_{\varphi}(q) - \mathcal{V}_{\varphi}(0)\}$
$= \mathcal{V}_{\varphi}^{+}(p) \vee \mathcal{V}_{\varphi}^{+}(q)$
$\mathcal{W}_{\varphi}^{+}(p-q) = \mathcal{W}_{\varphi}(p-q) - \mathcal{W}_{\varphi}(0)$
$\leq \{\mathcal{W}_{\varphi}(p) \vee \mathcal{W}_{\varphi}(q)\} - \mathcal{W}_{\varphi}(0)$
⊚$\{\mathcal{W}_{\varphi}(p) - \mathcal{W}_{\varphi}(0)\} \vee \{\mathcal{W}_{\varphi}(q) - \mathcal{W}_{\varphi}(0)\}$
$= \mathcal{W}_{\varphi}^{+}(p) \vee \mathcal{W}_{\varphi}^{+}(q)$ and
$\mathcal{U}_{\varphi}^{+}(p\alpha q) = \mathcal{U}_{\varphi}(p\alpha q) + 1 - \mathcal{U}_{\varphi}(0)$
$\geq \mathcal{U}_{\varphi}(q) + 1 - \mathcal{U}_{\varphi}(0) = \mathcal{U}_{\varphi}^{+}(q)$
$\mathcal{U}_{\varphi}^{+}(p\alpha q) \geq \mathcal{U}_{\varphi}^{+}(q)$
$\mathcal{V}_{\varphi}^{+}(p\alpha q) = \mathcal{V}_{\varphi}(p\alpha q) - \mathcal{V}_{\varphi}(0)$

$\leq \mathcal{V}_{\varphi}(q) - \mathcal{V}_{\varphi}(0) ◎ \mathcal{V}_{\varphi}^{+}(q)$

$\mathcal{V}_{\varphi}^{+}(p\alpha q) \leq \mathcal{V}_{\varphi}^{+}(q)$

$\mathcal{W}_{\varphi}^{+}(p\alpha q) = \mathcal{W}_{\varphi}(p\alpha q) - \mathcal{W}_{\varphi}(0)$

$\leq \mathcal{W}_{\varphi}(q) - \mathcal{W}_{\varphi}(0) = \mathcal{W}_{\varphi}^{+}(q)$

$\mathcal{W}_{\varphi}^{+}(p\alpha q) \leq \mathcal{W}_{\varphi}^{+}(q)$

Hence φ^{+} is a NF ideal of a Γ Ring N.

Definition 3.5: Let $X =< \mathcal{U}_X, \mathcal{V}_X, \mathcal{W}_X >$ and $Y =< \mathcal{U}_Y, \mathcal{V}_Y, \mathcal{W}_Y >$ be two NF subsets of a Γ Ring N. Then the Neutrosophic sum of X and Y is $X \oplus Y =< \mathcal{U}_{X\oplus Y}, \mathcal{V}_{X\oplus Y}, \mathcal{W}_{X\oplus Y} >$ in N given by

$$\mathcal{U}_{X\oplus Y}(P) = \begin{cases} \bigvee_{p=q+r} \{\mathcal{U}_X(q) \wedge \mathcal{U}_Y(r)\} & if\ p = q + r, \\ 0 & otherwise \end{cases}$$

$$\mathcal{V}_{X\oplus Y}(P) = \begin{cases} \bigwedge_{p=q+r} \{\mathcal{V}_X(q) \vee \mathcal{V}_X(r)\} & if\ p = q + r, \\ 1 & otherwise \end{cases}$$

$$\mathcal{W}_{X\oplus Y}(P) = \begin{cases} \bigwedge_{p=q+r} \{\mathcal{W}_X(q) \vee \mathcal{W}_Y(r)\} & if\ p = q + r, \\ 1 & otherwise \end{cases}$$

Theorem 3.6: If $X =< \mathcal{U}_X, \mathcal{V}_X, \mathcal{W}_X >$ and $Y =< \mathcal{U}_Y, \mathcal{V}_Y, \mathcal{W}_Y >$ be two NF subsets of a Γ Ring N then the Neutrosophic sum $X \oplus Y =< \mathcal{U}_{X\oplus Y}, \mathcal{V}_{X\oplus Y}, \mathcal{W}_{X\oplus Y} >$ is a NF ideal of Γ Ring.

Proof: For any p, $q \in N$, *we have*

$\mathcal{U}_{X\oplus Y}(p) \wedge \mathcal{U}_{X\oplus Y}(q)$ ◎ $\bigvee \{\mathcal{U}_X(x) \wedge \mathcal{U}_Y(y): p = x + y\} \wedge \bigvee \{\mathcal{U}_X(c) \wedge \mathcal{U}_Y(d): q = c + d\}$

◎ $\bigvee \{(\mathcal{U}_X(x) \wedge \mathcal{U}_Y(y)) \wedge (\mathcal{U}_X(c) \wedge \mathcal{U}_Y(d)): p = x + y, q = c + d\}$

◎ $\bigvee \{(\mathcal{U}_X(x) \wedge \mathcal{U}_Y(y)) \wedge (\mathcal{U}_X(-c) \wedge \mathcal{U}_Y(-d)): p = x + y, q = -c - d\}$

$= \bigvee \{(\mathcal{U}_X(x) \wedge \mathcal{U}_X(-c)) \wedge (\mathcal{U}_Y(y) \wedge \mathcal{U}_Y(-d)): p = x + y, q = -c - d\}$

$\leq \bigvee \{(\mathcal{U}_X(x - c) \wedge \mathcal{U}_Y(y - d)): p - q = \{(x - c) + (y - d)$ ◎

◎ $\mathcal{U}_{X\oplus Y}(p - q)$

$\mathcal{U}_{X\oplus Y}(p) \wedge \mathcal{U}_{X\oplus Y}(q) \leq \mathcal{U}_{X\oplus Y}(p - q)$

$\mathcal{V}_{X\oplus Y}(p) \vee \mathcal{V}_{X\oplus Y}(q)$ ◎ $\bigwedge \{\mathcal{V}_X(x) \vee \mathcal{V}_Y(y): p = x + y\} \vee \bigwedge \{\mathcal{V}_X(c) \vee \mathcal{V}_Y(d): q = c + d\}$

◎ $\bigwedge \{(\mathcal{V}_X(x) \vee \mathcal{V}_Y(y)) \vee (\mathcal{V}_X(c) \vee \mathcal{V}_Y(d)): p = x + y, q = c + d\}$

◎ $\bigwedge \{(\mathcal{V}_X(x) \vee \mathcal{V}_Y(y)) \vee (\mathcal{V}_X(-c) \vee \mathcal{V}_Y(-d)): p = x + y, q = -c - d\}$

$= \bigwedge \{(\mathcal{V}_X(x) \vee \mathcal{V}_X(-c)) \vee (\mathcal{V}_Y(y) \vee \mathcal{V}_Y(-d)): p = x + y, q = -c - d\}$

$\geq \bigwedge \{(\mathcal{V}_X(x - c) \vee \mathcal{V}_Y(y - d)): p - q = (x - c) + (y - d)\}$

◎ $\mathcal{V}_{X\oplus Y}(p - q)$

$\mathcal{V}_{X\oplus Y}(p) \vee \mathcal{V}_{X\oplus Y}(q) \geq \mathcal{V}_{X\oplus Y}(p - q)$

$\mathcal{W}_{X\oplus Y}(p)\vee\mathcal{W}_{X\oplus Y}(q) ⊚ \bigwedge\{\mathcal{W}_X(x)\bigvee\mathcal{W}_Y(y): p = x + y\}\bigvee\bigwedge \{\mathcal{W}_X(c)\bigvee\mathcal{W}_Y(d): q = c + d\}$

$⊚ \bigwedge \{(\mathcal{W}_X(x)\bigvee\mathcal{W}(y))\bigvee(\mathcal{W}_X(c)\bigvee\mathcal{W}(d)): p = x + y, q = c + d\}$

$⊚ \bigwedge\{(\mathcal{W}_X(x)\bigvee\mathcal{W}_Y(y))\bigvee(\mathcal{W}_X(-c)\bigvee\mathcal{W}_Y(-d)): p = x + y, q = -c - d\}$

$= \bigwedge(\mathcal{W}_X(x)\bigvee\mathcal{W}_X(-c))\bigvee(\mathcal{W}_Y(y)\bigvee\mathcal{W}_Y(-d)): p = x + y, q = -c - d\}$

$\geq \bigwedge\{(\mathcal{W}_X(x - c)\bigvee\mathcal{W}_Y(Y - d)): p - q = \{(x - c) + (y - d)\}$

$⊚ \mathcal{W}_{X\oplus Y}(p - q)$

$\mathcal{W}_{X\oplus Y}(p)\vee\mathcal{W}_{X\oplus Y}(q) \geq \mathcal{W}_{X\oplus Y}(p - q)$

$\mathcal{U}_{X\oplus Y}(p) = \bigvee \{\mathcal{U}_X(x)\bigwedge\mathcal{U}_Y(y): p = x + y\}$

$\leq \bigvee\{\mathcal{U}_X(x\alpha q)\bigwedge\mathcal{U}_Y(Y\alpha q): p\alpha q = x\alpha q + Y\alpha q\}$

$⊚\bigvee \{\mathcal{U}_X(U)\bigwedge\mathcal{U}_Y(V): p\alpha q = U + V\}$

$⊚ \mathcal{U}_{X\oplus Y}(p\alpha q)$

$\mathcal{U}_{X\oplus Y}(p\alpha q) \geq \mathcal{U}_{X\oplus Y}(p)$

$\mathcal{V}_{X\oplus Y}(p) = \bigwedge \{\mathcal{V}_X(x)\bigvee\mathcal{V}_Y(y): p = x + y\}$

$\geq \bigwedge\{\mathcal{V}_X(x\alpha q)\bigvee\mathcal{V}_Y(Y\alpha q): p\alpha q = x\alpha q + Y\alpha q\}$

$⊚\bigwedge \{\mathcal{V}_X(U)\bigvee\mathcal{V}_Y(V): p\alpha q = U + V\}⊚\mathcal{V}_{X\oplus Y}(p\alpha q)$

$\mathcal{V}_{X\oplus Y}(p\alpha q) \leq \mathcal{V}_{X\oplus Y}(p)$

$\mathcal{W}(p) = \bigwedge\{\mathcal{W}_X(x)\bigvee\mathcal{W}(y): p = x + y\}$

$\geq \bigwedge\{\mathcal{W}_X(x\alpha q)\bigvee\mathcal{W}_Y(Y\alpha q): p\alpha q = x\alpha q + Y\alpha q\}$

$⊚ \bigwedge \{\mathcal{W}_X(U)\bigvee\mathcal{W}_Y(V): p\alpha q = U + V\} ⊚\mathcal{W}_{X\oplus Y}(p\alpha q)$

$\mathcal{W}_{X\oplus Y}(p\alpha q) \leq \mathcal{W}_{X\oplus Y}(p)$

We conclude that $X\oplus Y$ is a NF ideal of N.

Definition 3.7: Suppose that $X =< \mathcal{U}_X, \mathcal{V}_X, \mathcal{W}_X >$ and $Y =< \mathcal{U}_Y, \mathcal{V}_Y, \mathcal{W}_Y >$ be two NF subsets of a Γ Ring N. Then $X \circ Y =< \mathcal{U}_{X\circ Y}, \mathcal{V}_{X\circ Y}, \mathcal{W}_{X\circ Y} >$ in N given by

$$\mathcal{U}_{X\circ Y}(p) = \bigvee\begin{cases}\bigwedge_{1\leq i\leq k}\{\mathcal{U}_X(x_i)\bigwedge\mathcal{U}_Y(y_i)\}: p = \sum_1^k x_i\alpha y_i, x_iy_i\epsilon N, \alpha\epsilon\Gamma, k\epsilon Z^+, \\ 0 \qquad otherwise\end{cases}$$

$$\mathcal{V}_{X\circ Y}(p) = \bigwedge\begin{cases}\bigvee_{1\leq i\leq k}\{\mathcal{V}_X(x_i)\bigvee\mathcal{V}_Y(y_i)\}: p = \sum_1^k x_i\alpha y_i, x_iy_i\epsilon N, \alpha\epsilon\Gamma, k\epsilon Z^+ \\ 1 \qquad otherwise\end{cases}$$

$$\mathcal{W}_{X\circ Y}(p) = \bigwedge\begin{cases}\bigvee_{1\leq i\leq k}\{\mathcal{W}_X(x_i)\bigvee\mathcal{W}_Y(y_i)\}: p = \sum_1^k x_i\alpha y_i, x_iy_i\epsilon N, \alpha\epsilon\Gamma, k\epsilon Z^+ \\ 1 \qquad otherwise\end{cases}$$

Theorem 3.8: If $=< \mathcal{U}_X, \mathcal{V}_X, \mathcal{W}_X >$ and $Y =< \mathcal{U}_Y, \mathcal{V}_Y, \mathcal{W}_Y >$ be two NF subsets of a Γ Ring N then the composition $X \circ Y =< \mathcal{U}_{X\circ Y}, \mathcal{V}_{X\circ Y}, \mathcal{W}_{X\circ Y} >$ is a NF ideal of N.

Proof: For any $p, q \in N$ we have

$$\mathcal{U}_{X\circ Y}(p-q) = \bigvee\{\wedge_{1\le i\le k}\mathcal{U}_X(u_i)\bigwedge\mathcal{U}_Y(v_i) : p-q = \sum_1^k u_i\alpha v_i, u_i, v_i\epsilon N, \alpha\epsilon\Gamma, k\epsilon Z^+\} \ge$$

$$\bigvee\{(\wedge_{1\le i\le m}\mathcal{U}_X(x_i)\bigwedge\mathcal{U}_Y(y_i))\bigwedge(\wedge_{1\le i\le n}\mathcal{U}_X(-c_i)\bigwedge\mathcal{U}_Y(d_i))$$

$$: p = \sum_1^m x_i\alpha y_i, -q = \sum_1^n -c_i\alpha d_i, x_i, y, -c_i, d_i\epsilon N, \alpha\epsilon\Gamma \ and\ m.n\epsilon Z^+\}$$

$$= \bigvee\{(\bigwedge_{1\le i\le m}\mathcal{U}_X(x_i)\bigwedge\mathcal{U}_Y(y_i))\bigwedge(\bigwedge_{1\le i\le n}\mathcal{U}_X(-c_i)\bigwedge\mathcal{U}_Y(d_i))$$

$$: p = \sum_1^m x_i\alpha y_i, q = \sum_1^n c_i\alpha d_i, x_i, y, -c_i, d_i\epsilon N, \alpha\epsilon\Gamma \ and\ m.n\epsilon Z^+\}$$

$$= \bigvee\{\bigwedge_{1\le i\le m}\mathcal{U}_X(x_i)\bigwedge\mathcal{U}_Y(y_i) : p = \sum_1^m x_i\alpha y_i x_i, y_i\epsilon N, \alpha\epsilon\Gamma \ and\ m\epsilon Z^+\}\bigwedge$$

$$\bigvee\{\bigwedge_{1\le i\le m}\mathcal{U}_X(c_i)\bigwedge\mathcal{U}_Y(d_i) : q = \sum_1^n c_i\alpha d_i, x_i, y_i, c_i, d_i\epsilon N, \alpha\epsilon\Gamma \ and\ n\epsilon Z^+\}$$

$$\mathcal{U}_{X\circ Y}(p-q) \ge \mathcal{U}_{X\circ Y}(p)\wedge\mathcal{U}_{X\circ Y}(q)$$

$$\mathcal{V}_{X\circ Y}(p-q) = \bigwedge\{\bigvee_{1\le i\le k}\mathcal{V}_X(u_i)\bigvee\mathcal{V}_Y(v_i) : p-q = \sum_1^k u_i\alpha v_i, u_i, v_i\epsilon N, \alpha\epsilon\Gamma, k\epsilon Z^+\}$$

$$\le \bigwedge\{(\bigvee_{1\le i\le k}\{(\mathcal{V}_X(x_i)\bigvee\mathcal{V}_Y(y_i))\bigvee(\bigvee_{1\le i\le n}(\mathcal{V}_X(-c_i)\bigvee\mathcal{V}_Y(d_i))$$

$$: p = \sum_1^m x_i\alpha y_i, -q ⊚ \sum_1^n -c_i\alpha d_i, x_i, y_i, -c_i, d_i\in N, \alpha\in\Gamma \text{ and } m, n \in Z^+ ⊚$$

$$= \bigwedge\{(\bigvee_{1\le i\le k}\{(\mathcal{V}_X(x_i)\bigvee\mathcal{V}_Y(y_i))\bigvee(\bigvee_{1\le i\le n}(\mathcal{V}_X(c_i)\bigvee\mathcal{V}_Y(d_i))$$

$$: p = \sum_1^m x_i\alpha y_i,, q ⊚ \sum_1^n c_i\alpha d_i, x_i, y_i, -c_i, d_i\in N, \alpha\in\Gamma \text{ and } m, n \in Z^+ ⊚$$

$$= \bigwedge\{\bigvee_{1\le i\le k}\{(\mathcal{V}_X(x_i)\bigvee\mathcal{V}_Y(y_i))\} : p = \sum_1^m x_i\alpha y_i, x_i, y_i, \text{ci}, \text{di}\in N, \alpha\in\Gamma \text{ and } m \in Z^+\}\bigvee\bigwedge$$

$$\{\bigvee_{1\le i\le m}\{\mathcal{V}_X(c_i)\bigvee\mathcal{V}_y(d_i)\} : q = \sum_1^n \text{ci}\alpha\text{di}, \text{ci}, \text{di}\in\alpha\in\Gamma \text{ and } m, n \in Z^+\}.$$

$$⊚\mathcal{V}_{X\circ Y}(p)\bigvee\mathcal{V}_{X\circ Y}(q)$$

$\mathcal{V}_{X\circ Y}(p-q) \leq \mathcal{V}_{X\circ Y}(p) \vee \mathcal{V}_{X\circ Y}(q)$

$\mathcal{W}_{X\circ Y}(p-q) = \wedge\{\bigvee_{1\leq i\leq k} \mathcal{W}_X(u_i) \vee \mathcal{W}_Y(v_i) : p-q = \sum_1^k u_i\alpha v_i, u_i, v_i \epsilon N, \alpha\epsilon\Gamma, k\epsilon Z^+\}$

$\leq \wedge\{\bigvee_{1\leq i\leq k} \{(\mathcal{W}_X(x_i) \vee \mathcal{W}_Y(y_i)) \vee (\bigvee_{1\leq i\leq n} (\mathcal{W}_X(-c_i) \vee \mathcal{W}_Y(d_i))$

$: p = \sum_1^m x_i\alpha y_i$, -q ◎ $\sum_1^n -c_i\alpha d_i$, x_i, y_i, $-c_i, d_i \in$ N, $\alpha\in\Gamma$ and m, n $\in Z^+$◎

$= \wedge\{\bigvee_{1\leq i\leq k} \{(\mathcal{W}_X(x_i) \vee \mathcal{W}_Y(y_i))\} \vee (\bigvee_{1\leq i\leq n} (\mathcal{W}_X(c_i) \vee \mathcal{W}_Y(d_i)))$

$: p = \sum_1^m x_i\alpha y_i$, q ◎ $\sum_1^n c_i\alpha d_i$, x_i, y_i, $c_i, d_i \in \alpha \in \Gamma$ and m, n $\in Z^+$◎

$= \wedge\{\bigvee_{1\leq i\leq k} \{(\mathcal{W}_X(x_i) \vee \mathcal{W}_Y(y_i))\}$: p $= \sum_1^m x_i\alpha y_i, x_i, y_i$, ci , di$\in$ N, $\alpha\in\Gamma$ and m, n $\in Z^+\} \vee \wedge$

$\{\bigvee_{1\leq i\leq m} \{\mathcal{W}_X(c_i) \vee \mathcal{W}_y(d_i)\}$: q $= \sum_1^n$ ciαdi, x_i, y_i, ci , di$\in$N, $\alpha\in\Gamma$ and m, n $\in Z^+\}$.

◎$\mathcal{W}_{X\circ Y}(p) \vee \mathcal{W}_{X\circ Y}(q)$

$\mathcal{W}_{X\circ Y}(p-q) \leq \mathcal{W}_{X\circ Y}(p) \vee \mathcal{W}_{X\circ Y}(q)$

$\mathcal{U}_{X\circ Y}(p) = \vee\{\left(\bigwedge_{1\leq i\leq m} \mathcal{U}_X(x_i) \wedge \mathcal{U}_Y(y_i)\right): p = \sum_1^m x_i\alpha y_i x_i, y_i \epsilon N, \alpha\epsilon\Gamma \text{ and } m\epsilon Z^+\}$

$\leq \vee\{\left(\bigwedge_{1\leq i\leq m} \mathcal{U}_X(x_i) \wedge \mathcal{U}_Y(y_i\alpha q)\right): p\alpha q = \sum_1^m x_i\alpha(y_i\alpha q)x_i, y_i\alpha q\epsilon N, \alpha\epsilon\Gamma \text{ and } m\epsilon Z^+\}$

$= \vee\{\left(\wedge_{1\leq i\leq m} \mathcal{U}(u_i) \wedge \mathcal{U}_Y(v_i)\right): p\alpha q = \sum_1^m u_i\alpha v_i, u_i, v_i \epsilon N, \alpha\epsilon\Gamma \text{ and } m\epsilon Z^+\} = \mathcal{U}_{X\circ Y}(p\alpha q)$.

$\mathcal{U}_{X\circ Y}(p) \leq \mathcal{U}_{X\circ Y}(p\alpha q)$ and similiarly we get $\mathcal{U}_{X\circ Y}(q) \leq \mathcal{U}_{X\circ Y}(p\alpha q)$

$\mathcal{V}_{X\circ Y}(p) = \wedge\{\bigvee_{1\leq i\leq m} \{\mathcal{V}_X(x_i) \vee \mathcal{V}_Y(y_i)\}: p = \sum_1^m x_i\alpha y_i, x_i, y_i \epsilon N, \alpha\epsilon\Gamma, k\epsilon Z^+\}$

$\geq \wedge\{\bigvee_{1\leq i\leq m} \{(\mathcal{V}_X(x_i) \vee \mathcal{V}_Y(y_i\alpha q))\}: p\alpha q = \sum_1^m x_i\alpha(y_i\alpha q)x_i, y_i\alpha q\epsilon N, \alpha\epsilon\Gamma \text{ and } m\epsilon Z^+\}$

$= \wedge\{\bigvee_{1\leq i\leq m} \{(\mathcal{V}_X(u_i) \vee \mathcal{V}_Y(v_i))\}$

$: p\alpha q = \sum_1^m u_i\alpha v_i,\ u_i, v_i \epsilon M, \alpha\epsilon\Gamma \text{ and } m\epsilon Z^+\}$ ◎$\mathcal{V}_{X\circ Y}(p\alpha q)$

$\mathcal{V}_{X\circ Y}(p) \geq \mathcal{V}_{X\circ Y}(p\alpha q)$ and similiarly we get $\mathcal{V}_{X\circ Y}(q) \geq \mathcal{V}_{X\circ Y}(p\alpha q)$

$$\mathcal{W}_{X\circ Y}(p) = \bigwedge\{ \bigvee_{1\leq i\leq m} \{\mathcal{W}_X(x_i)\bigvee \mathcal{W}_Y(y_i)\}: p = \sum_1^m x_i\alpha y_i, x_i, y_i\epsilon N, \alpha\epsilon\Gamma, k\epsilon Z^+\}$$

$$\geq \bigwedge\{ \bigvee_{1\leq i\leq m} \{(\mathcal{W}_X(x_i)\bigvee \mathcal{W}_Y(y_i\alpha q))\}:\ p\alpha q = \sum_1^m x_i\alpha(y_i\alpha q)x_i, y_i\alpha q\epsilon N, \alpha\epsilon\Gamma \ and\ m\epsilon Z^+\}$$

$$= \bigwedge\{ \bigvee_{1\leq i\leq m} \{(\mathcal{W}_X(u_i)\bigvee \mathcal{W}_Y(v_i))\}:$$

$$: p\alpha q = \sum_1^m u_i\alpha v_i,\ u_i, v_i\epsilon N, \alpha\epsilon\Gamma\ and\ m\epsilon Z^+\} ⊚ \mathcal{W}_{X\circ Y}(p\alpha q)$$

$\mathcal{W}_{X\circ Y}(p) \geq \mathcal{W}_{X\circ Y}(p\alpha q)$ and similiarly we get $\mathcal{W}_{X\circ Y}(p) \geq \mathcal{W}_{X\circ Y}(p\alpha q)$

Therefore $X \circ Y$ is a NF ideal of N.

Definition 3.9: If $\{\varphi_i\}_{i\epsilon j}$ be an arbitrary family of NF set in X, where $\varphi_i = \langle \wedge \mathcal{U}_{\eta_i}, \vee \mathcal{V}_{\eta_i}, \vee \mathcal{W}_{\eta_i}, \rangle$ for each $i \in J$. The

$(i) \cap\varphi_i = \langle \wedge \mathcal{U}_{\eta_i}, \vee \mathcal{V}_{\eta_i}, \vee \mathcal{W}_{\eta_i} \rangle$ $(ii) \cup\varphi_i = \langle \vee \mathcal{U}_{\eta_i}, \wedge \mathcal{V}_{\eta_i}, \wedge \mathcal{W}_{\eta_i} \rangle$

Theorem 3.10: $If\ \{\varphi_i\}_{i\epsilon j}$ be an arbitrary family of NF set in N, then $\cup\varphi_i = \langle \vee \mathcal{U}_{\varphi_i}, \wedge \mathcal{V}_{\varphi_i}, \wedge \mathcal{W}_{\varphi_i} \rangle$ is a NF ideal of N.

Proof: Let $p, q \in N$ and $\alpha\epsilon\ \Gamma$ then

$(\cup_{i\in J} \mathcal{U}_{\varphi_i})$ (p - q) ⊚ $\vee_{i\in J} \mathcal{U}_{\varphi_i}(p-q)$

$\geq \vee_{i\in J}(\mathcal{U}_{\varphi_i}(p) \wedge \mathcal{U}_{\varphi_i}(q))$ ⊚ $\vee_{i\in J}(\mathcal{U}_{\varphi_i}(p)) \wedge \vee_{i\in J}(\mathcal{U}_{\varphi_i}(q))$

⊚$(\cup_{i\in J} \mathcal{U}_{\varphi_i})(p) \wedge (\cup_{i\in J} \mathcal{U}_{\varphi_i})(q)$

$(\cup_{i\in J} \mathcal{V}_{\varphi_i})$ (p - q) ⊚ $\wedge_{i\in J} \mathcal{V}_{\varphi_i}(p-q)$

$\leq \wedge_{i\in J}(\mathcal{V}_{\varphi_i}(p) \vee \mathcal{V}_{\varphi_i}(q))$⊚$(\wedge_{i\in J} \mathcal{V}_{\varphi_i})(p) \vee (\wedge_{i\in J} \mathcal{V}_{\varphi_i})(q)$

⊚$(\cup_{i\in J} \mathcal{V}_{\varphi_i})(p) \vee (\cup_{i\in J} \mathcal{V}_{\varphi_i})(q)$

$(\cup_{i\in J} \mathcal{W}_{\varphi_i})$ (p - q) ⊚ $\wedge_{i\in J} \mathcal{W}_{\varphi_i}(p-q)$

$\leq \wedge_{i\in J}(\mathcal{W}_{\varphi_i}(p) \vee \mathcal{W}_{\varphi_i}(q))$⊚$(\wedge_{i\in J} \mathcal{W}_{\varphi_i})(p) \vee (\wedge_{i\in J} \mathcal{W}_{\varphi_i})(q)$

⊚$(\cup_{i\in J} \mathcal{W}_{\varphi_i})(p) \vee (\cup_{i\in J} \mathcal{W}_{\varphi_i})(q)$

Also $(\cup_{i\in J} \mathcal{U}_{\varphi_i})$ (p α q) = $\vee_{i\in J} \mathcal{U}_{\varphi_i}(p\alpha q) \geq \vee_{i\in J} \mathcal{U}_{\varphi_i}(q)$⊚$(\cup_{i\in J} \mathcal{U}_{\varphi_i})(q)$

$(\cup_{i\in J} \mathcal{V}_{\varphi_i})$ (p α q) = $\wedge_{i\in J} \mathcal{V}_{\varphi_i}(p\ \alpha q) \leq \wedge_{i\in J} \mathcal{V}_{\varphi_i}$(q)⊚ $(\cup_{i\in J} \mathcal{V}_{\varphi_i})(q)$

$(\cup_{i\in J} \mathcal{W}_{\varphi_i})$ (p α q) = $\wedge_{i\in J} \mathcal{W}_{\varphi_i}(p\ \alpha q) \leq \wedge_{i\in J} \mathcal{W}_{\varphi_i}$(q)⊚ $(\cup_{i\in J} \mathcal{W}_{\varphi_i})(q)$

Similiarly for right ideals

$(\cup_{i\in J} \mathcal{U}_{\varphi_i})$ (p α q) = $\vee_{i\in J} \mathcal{U}_{\varphi_i}(p\alpha q) \geq \vee_{i\in J} \mathcal{U}_{\varphi_i}(p)$⊚$(\cup_{i\in J} \mathcal{U}_{\varphi_i})(p)$

$(\cup_{i\in J} \mathcal{V}_{\varphi_i})$ (p α q) = $\wedge_{i\in J} \mathcal{V}_{\varphi_i}(p\ \alpha q) \leq \wedge_{i\in J} \mathcal{V}_{\varphi_i}$(p)⊚ $(\cup_{i\in J} \mathcal{V}_{\varphi_i})(p)$

$(\cup_{i\in J} \mathcal{W}_{\varphi_i})$ (p α q) = $\wedge_{i\in J} \mathcal{W}_{\varphi_i}(p\ \alpha q) \leq \wedge_{i\in J} \mathcal{W}_{\varphi_i}$(p)⊚ $(\cup_{i\in J} \mathcal{W}_{\varphi_i})(p)$

Hence$\cup_{i\in J} \varphi_i$ is a NF ideal of N.

Definition 3.11: Let $X =< \mathcal{U}_X, \mathcal{V}_X, \mathcal{W}_X >$ and $Y =< \mathcal{U}_Y, \mathcal{V}_Y, \mathcal{W}_Y >$ be two NF subsets of a Γ Ring N then the product of of X and Y is $X\Gamma Y =< \mathcal{U}_{X\Gamma Y}, \mathcal{V}_{X\Gamma Y}, \mathcal{W}_{X\Gamma Y} >$ in N given by

$$\mathcal{U}_{X\Gamma Y}(P) = \begin{cases} \bigvee_{p=q\alpha r} \{\mathcal{U}_X(q) \wedge \mathcal{U}_Y(r)\} & if\ p = q\alpha r \\ 0 & otherwise \end{cases}$$

$$\mathcal{V}_{X\Gamma Y}(P) = \begin{cases} \bigwedge_{p=q\alpha r} \{\mathcal{V}_X(q) \vee \mathcal{V}_Y(r)\} & if\ p = q\alpha r \\ 1 & otherwise \end{cases}$$

$$\mathcal{W}_{X\Gamma Y}(P) = \begin{cases} \bigwedge_{p=q\alpha r} \{\mathcal{W}_X(q) \vee \mathcal{W}_Y(r)\} & if\ p = q\alpha r \\ 1 & otherwise \end{cases}$$

Theorem 3.12: Assume that $X =< \mathcal{U}_X, \mathcal{V}_X, \mathcal{W}_X >$ and $Y =< \mathcal{U}_Y, \mathcal{V}_Y, \mathcal{W}_Y >$ be NF subsets of a Γ Ring N then $X \cap Y$ is a NF left (resp. right) ideal of N. If X is a NF left ideal and Y is a NF right ideal then $X\Gamma Y \subseteq X \cap Y$

Proof: Suppose X and Y are Neutrosophic contained in M and let $p, q \in N, \alpha \in \Gamma$.

$\mathcal{U}_{X\cap Y}(p - q) = \mathcal{U}_{X\cap Y}(p) \wedge \mathcal{U}_{X\cap Y}(q)$

$\geq [\{\mathcal{U}_X(p) \wedge \mathcal{U}_X(q) ⊚\wedge⊚ \mathcal{U}_Y(p) \wedge \mathcal{U}_Y(q)\}]$

⊚ ⊚$\mathcal{U}_X(p) \wedge \mathcal{U}_Y(p)] \wedge [\mathcal{U}_X(q) \wedge \mathcal{U}_Y(q)]$

$= \mathcal{U}_{X\cap Y}(p) \wedge \mathcal{U}_{X\cap Y}(q)$

$\mathcal{V}_{X\cap Y}(p - q) = \mathcal{V}_{X\cap Y}(p) \vee \mathcal{V}_{X\cap Y}(q)$

$\leq [\{\mathcal{V}_X(p) \vee \mathcal{V}_X(q) ⊚\vee⊚ \mathcal{V}(p) \vee \mathcal{V}_Y(q)\}]$

⊚ ⊚$\mathcal{V}_X(p) \vee \mathcal{V}_Y(p)] \wedge [\mathcal{V}_X(q) \vee \mathcal{V}_Y(q)]$

$= \mathcal{V}_{X\cap Y}(p) \vee \mathcal{V}_{X\cap Y}(q)$

$\mathcal{W}_{X\cap Y}(p - q) = \mathcal{W}_{X\cap Y}(p) \vee \mathcal{W}_{X\cap Y}(q)$

$\leq [\{\mathcal{W}_X(p) \vee \mathcal{W}_X(q) ⊚ ⊚ \mathcal{W}(p) \vee \mathcal{W}_Y(q)\}]$

⊚ ⊚$\mathcal{W}_X(p) \vee \mathcal{W}_Y(p)] \wedge [\mathcal{W}_X(q) \vee \mathcal{W}_Y(q)]$

$= \mathcal{W}_{X\cap Y}(p) \vee \mathcal{W}_{X\cap Y}(q)$

$\mathcal{U}_X(p\alpha q) \geq \mathcal{U}_X(q)$, $\mathcal{V}_X(p\alpha q) \leq \mathcal{V}_X(q)$, and $\mathcal{W}_X(p\alpha q) \leq \mathcal{W}_X(q)$,

$\mathcal{U}_Y(p\alpha q) \geq \mathcal{U}_Y(q)$, $\mathcal{V}_Y(p\alpha q) \leq \mathcal{V}_Y(q)$, and $\mathcal{W}_Y(p\alpha q) \leq \mathcal{W}_Y(q)$,

Clearly X and Y are NF ideal of N, we have,

Now,

$\mathcal{U}_{X\cap Y}(p\alpha q)$⊚ $\mathcal{U}_X(p\alpha q) \wedge \mathcal{U}_Y(p\alpha q)$

$\geq \mathcal{U}_X(q) \wedge \mathcal{U}_Y(q)$ ⊚$\mathcal{U}_{X\cap Y}(q)$

$\mathcal{V}_{X\cap Y}(p\alpha q)$⊚ $\mathcal{V}_X(p\alpha q) \vee \mathcal{V}_Y(p\alpha q$

$\leq \mathcal{V}_X(q) \vee \mathcal{V}_Y(q)$ ⊚$\mathcal{V}_{X\cap Y}(p\alpha q)$

$\mathcal{W}_{X\cap Y}(p\alpha q)$⊚ $\mathcal{W}_X(p\alpha q) \vee \mathcal{W}_Y(p\alpha q)$

$\leq \mathcal{W}_X(q) \vee \mathcal{W}_Y(q)$ ⊚$\mathcal{W}_{X\cap Y}(q)$

Therefore $X \cap Y$ is a NF ideal of N.

To Prove $\mathcal{U}_{X\Gamma Y}(p)$ ⊚0 and $\mathcal{V}_{X\Gamma Y}(p)$⊚1, $\mathcal{W}_{X\Gamma Y}(p)$⊚1.

Suppose $X\Gamma Y(p) \neq (0,1)$

The definition of $X\Gamma Y$,

$\mathcal{U}_X(p) = \mathcal{U}_X(q\alpha r) \geq \mathcal{U}_X(q)$,$\mathcal{V}_X(p) = \mathcal{V}_X(q\alpha r) \leq \mathcal{V}_X(q)$and $\mathcal{W}_X(p) = \mathcal{W}_X(q\alpha r) \leq \mathcal{W}_X(q)$

$\mathcal{U}_Y(p) = \mathcal{U}_Y(q\alpha r) \geq \mathcal{U}_Y(q)$,$\mathcal{V}_Y(p) = \mathcal{V}_Y(q\alpha r) \leq \mathcal{V}_Y(q)$and $\mathcal{W}_Y(p) = \mathcal{W}_Y(q\alpha r) \leq \mathcal{W}_Y(q)$

Since X is a NF right ideal and Y is a NF left ideal of N, we have

$\mathcal{U}_X(p) = \mathcal{U}_X(q\alpha r) \geq \mathcal{U}_X(q)$,$\mathcal{V}_X(p) = \mathcal{V}_X(q\alpha r) \leq \mathcal{V}_X(q)$and $\mathcal{W}_X(p) = \mathcal{W}_X(q\alpha r) \leq \mathcal{W}_X(q)$

$\mathcal{U}_Y(p) = \mathcal{U}_Y(q\alpha r) \geq \mathcal{U}_Y(r)$,$\mathcal{V}_Y(p) = \mathcal{V}_Y(q\alpha r) \leq \mathcal{V}_Y(r)$and $\mathcal{W}_Y(p) = \mathcal{W}_Y(q\alpha r) \leq \mathcal{W}_Y(r)$

By the Definition of $X\Gamma Y$

$$\mathcal{U}_{X\Gamma Y}(p) = \bigvee_{\mathrm{p=q\alpha r}} \{\mathcal{U}_{\mathrm{X}}(\mathrm{q}) \wedge \mathcal{U}_{\mathrm{Y}}(\mathrm{r})\} \leq \mathcal{U}_{\mathrm{X}}(p) \wedge \mathcal{U}_{\mathrm{Y}}(p) = \mathcal{U}_{\mathrm{X}\cap Y}(p),$$

$$\mathcal{V}_{X\Gamma Y}(p) = \bigwedge_{p=q\alpha r} \{\mathcal{V}_X(q) \vee \mathcal{V}_Y(r)\} \geq \{\mathcal{V}_X(q) \vee \mathcal{V}_Y(r)\} = \mathcal{V}_{\mathrm{X}\cap Y}(p)$$

$$\mathcal{W}_{X\Gamma Y}(p) = \bigwedge_{p=q\alpha r} \{\mathcal{W}_X(q) \vee \mathcal{W}_Y(r)\} \geq \{\mathcal{W}_X(q) \vee \mathcal{W}_Y(r)\} = \mathcal{W}_{\mathrm{X}\cap Y}(p)$$

Consequently, $X\Gamma Y \subseteq X \cap Y$

Corollary 3.13: If $X =< \mathcal{U}_X, \mathcal{V}_X, \mathcal{W}_X >$ and $Y =< \mathcal{U}_Y, \mathcal{V}_Y, \mathcal{W}_Y >$ be two neutrosophic fuzzy subsets of a Γ Ring N, then X ∪ Y is a NF ideal of N.

Definition 3.14: A Γ Ring N is regular if there exists $p \in N, \forall\, x \in N$ and $\alpha, \beta \in \Gamma$ then x⊚x$\alpha p\beta x$

Result 3.15: A Γ Ring N is said to be regular $\Leftrightarrow$ $if\ I\Gamma J = I \cap J$ for each right ideal I and for each left ideal J of N.

Theorem 3.16: A Γ Ring N is regular if for each NF right ideal X and for each NF left ideal Y of N, $X\Gamma Y = X \cap Y$.

Proof. Suppose that N is regular.

By theorem 3.12, $X\Gamma Y \subseteq X \cap Y$

Therefore, it is sufficient to prove $X \cap Y \subseteq X\Gamma Y$

Let x∈ $N, \alpha, \beta \in \Gamma$

By definition, there exists p∈ N such that x ⊚ x$\alpha p\beta x$

$\mathcal{U}_X(x) = \mathcal{U}_X(\mathrm{x}\alpha p\beta x) \geq \mathcal{U}_X(\mathrm{x}\alpha p) \geq \mathcal{U}_X(x), \mathcal{V}_X(x) = \mathcal{V}_X(\mathrm{x}\alpha p\beta x) \leq \mathcal{V}_X(\mathrm{x}\alpha p) \leq \mathcal{V}_X(x)$.

$\mathcal{W}_X(x) = \mathcal{W}_X(\mathrm{x}\alpha p\beta x) \leq \mathcal{W}_X(\mathrm{x}\alpha p) \leq \mathcal{W}_X(x)$.

So, $\mathcal{U}_X(\mathrm{x}\alpha p) \geq \mathcal{U}_X(x)$, $\mathcal{V}_X(\mathrm{x}\alpha p) \leq \mathcal{V}_X(x)$ and $\mathcal{W}_X(\mathrm{x}\alpha p) \leq \mathcal{W}_X(x)$.

Furthermore,

$$\mathcal{U}_{X\Gamma Y}(x) = \bigvee_{\mathrm{x=x}\alpha p\beta x} \{\mathcal{U}_X(\mathrm{x}\alpha p) \wedge \mathcal{U}_Y(x)\} \geq \{\mathcal{U}_{\mathrm{X}}(x) \wedge \mathcal{U}_{\mathrm{Y}}(x)\} = \mathcal{U}_{\mathrm{X}\cap Y}(x),$$

$$\mathcal{V}_{X\Gamma Y}(x) = \bigwedge_{\mathrm{x=x}\alpha p\beta x} \{\mathcal{V}_X(\mathrm{x}\alpha p) \vee \mathcal{V}(x)\} \leq \{\mathcal{V}_{\mathrm{X}}(x) \vee \mathcal{V}_{\mathrm{Y}}(x)\} = \mathcal{V}_{\mathrm{X}\cap Y}(x),$$

$$\mathcal{W}_{X\Gamma Y}(x) = \bigwedge_{\mathrm{x=x}\alpha p\beta x} \{\mathcal{W}_X(\mathrm{x}\alpha p) \vee \mathcal{W}(x)\} \leq \{\mathcal{W}_{\mathrm{X}}(x) \vee \mathcal{W}_{\mathrm{Y}}(x)\} = \mathcal{W}_{\mathrm{X}\cap Y}(x),$$

Thus X∩$Y \subseteq X\Gamma Y$. Hence $X\Gamma Y$ = X∩Y.

Definition 3.17: An ideal φ of the Γ Ring N is said to be prime if for any ideals X and Y of N, $X\Gamma Y \subseteq \varphi \Longrightarrow X \subseteq \varphi$ or $Y \subseteq \varphi$.

Definition 3.18: Let φ be a NF ideal of a Γ Ring N. Then φ is said to be prime if φ is not a constant mapping and for any neutrosophic X, Y of a Γ Ring N, $X\Gamma Y \subseteq \varphi$ implies $X \subseteq \varphi$ or $Y \subseteq \varphi$.

Theorem 3.19: Let $\mathcal{J}$ be an ideal of a Γ Ring N.∋ $\mathcal{J} \neq N$ Then $\mathcal{J}$ is a prime ideal of N iff $\left(\mathcal{U}_{\chi_{\mathcal{J}}}, \mathcal{V}_{\overline{\chi}_{\mathcal{J}}}, \mathcal{W}_{\overline{\chi}_{\mathcal{J}}}\right)$ is a NF prime ideal of N.

Proof: $(\Rightarrow)$Suppose $\mathcal{J}$ is a prime ideal of N. and let $\varphi = \left(\mathcal{U}_{\chi_{\mathcal{J}}}, \mathcal{V}_{\overline{\chi}_{\mathcal{J}}}, \mathcal{W}_{\overline{\chi}_{\mathcal{J}}}\right)$. Since $\mathcal{J} \neq N$. φ is not a constant mapping on N. Let X and Y be two NF ideal of N such that $X\Gamma Y \subseteq \varphi$ and $X \not\subset \varphi$ or $Y \not\subset \varphi$, then $\exists\, p, q \in N$ such that

$\mathcal{U}_X(p) > \mathcal{U}_\varphi(p) = \mathcal{U}_{\chi_{\mathcal{J}}}(p), \mathcal{V}_X(p) < \mathcal{V}_\varphi(p) = \mathcal{V}_{\overline{\chi}_{\mathcal{J}}}(p)$ and $\mathcal{W}_X(p) < \mathcal{W}_\varphi(p) = \mathcal{W}_{\overline{\chi}_{\mathcal{J}}}(p)$,

$\mathcal{U}_Y(p) > \mathcal{U}_\varphi(p) = \mathcal{U}_{\chi_{\mathcal{J}}}(p), \mathcal{V}_Y(p) < \mathcal{V}_\varphi(p) = \mathcal{V}_{\overline{\chi}_{\mathcal{J}}}(p)$ and $\mathcal{W}_Y(p) < \mathcal{W}_\varphi(p) = \mathcal{W}_{\overline{\chi}_{\mathcal{J}}}(p)$

Thus $\mathcal{U}_X(p) \neq 0, \mathcal{V}_X(p) \neq 1, \mathcal{W}_X(p) \neq 1$ *and* $\mathcal{U}_Y(q) \neq 0, \mathcal{V}_Y(q) \neq 1, \mathcal{W}_Y$. But $\mathcal{U}_{\chi_{\mathcal{J}}}(p) = 0, \mathcal{V}_{\chi_{\mathcal{J}}}(p) = 0$ *and* $\mathcal{W}_{\chi_{\mathcal{J}}}(p) = 0$, so $p \notin \mathcal{J}, q \notin \mathcal{J}$. *Since* $\mathcal{J}$ is a prime ideal of N, by the Theorem 5◎3◎there exists r ∈ N and $\alpha, \beta \in \Gamma$. such that $p\alpha r\beta q \notin \mathcal{J}$. Let c◎ $p\alpha r\beta q$ then $\mathcal{U}_{\chi_{\mathcal{J}}}(c) = 0, \mathcal{V}_{\overline{\chi}_{\mathcal{J}}}(c)$ *and* $\mathcal{W}_{\overline{\chi}_{\mathcal{J}}}(c)$◎1.Thus $X\Gamma Y(c) = (0,1)$. But $\mathcal{U}_{X\Gamma Y}(c) = \vee_{c=m\gamma n}[\ \mathcal{U}_X(m) \wedge \mathcal{U}_Y(n)] \geq \mathcal{U}_X(p\alpha r) \wedge \mathcal{U}_Y(q)]$ (since c◎$p\alpha r\beta q)$ $\geq \mathcal{U}_X(p) \wedge \mathcal{U}_Y(q) > 0$. (since $\mathcal{U}_X(p) \neq 0$ *and* $\mathcal{U}_Y\ (p) \neq 0$)

$\mathcal{V}_{X\Gamma Y}(c) = \ \wedge_{c=m\gamma n}[\mathcal{V}_X(m) \vee \mathcal{V}_Y(n)] \leq [\mathcal{V}_X(p\alpha r) \vee \mathcal{V}_Y(q)] \leq \mathcal{V}_X(p) \vee \mathcal{V}_X(q)$◎1

(since $\mathcal{V}_X(p) \neq 1$ *and* $\mathcal{V}_Y(p) \neq 1$).

$$\mathcal{W}_{X\Gamma Y}(c) = \bigwedge_{c=m\gamma n} [\mathcal{W}_X(m) \vee \mathcal{W}_Y(n)] \leq [\mathcal{W}_X(p\alpha r) \vee \mathcal{W}_Y(q)] \leq \mathcal{W}_X(p) \vee \mathcal{W}_X(q) < 1$$

(since $\mathcal{W}_X(p) \neq 1$ *and* $\mathcal{W}(p) \neq 1$.)Then XΓ$B(c) \neq (0,1)$. This contradicts the result. Then for any two NF ideals Xand Y $X\Gamma Y \subseteq \varphi$. implies $A \subseteq \varphi$ or $B \subseteq \varphi$. Hence φ is a NF ideals of N.

$(\Leftarrow)$ Suppose $\varphi = \left(\mathcal{U}_{\chi_{\mathcal{J}}}, \mathcal{V}_{\overline{\chi}_{\mathcal{J}}}, \mathcal{W}_{\overline{\chi}_{\mathcal{J}}}\right)$ is a NF prime ideal of N. Since φ is not a constant mapping on N, $\varphi \neq N$. Let X, Y be two ideals of N such that $X\Gamma Y \subseteq \mathcal{J}$ and let $\overline{X} = \left(\mathcal{U}_{\chi_X}, \mathcal{V}_{\overline{\chi}_X}, \mathcal{W}_{\overline{\chi}_X}\right)$ and $\overline{Y} = \left(\mathcal{U}_{\chi_Y}, \mathcal{V}_{\overline{\chi}_Y}, \mathcal{W}_{\overline{\chi}_Y}\right)$ be two fuzzy ideals of N. Consider the product $\overline{X}\Gamma\overline{Y}$. let $p \in N$ if $\overline{X}\Gamma\overline{Y}(p) = (0,1)$ then $\overline{X}\Gamma\overline{Y} \subseteq \mathcal{U}$. Suppose $\overline{X}\Gamma\overline{Y} \neq (0,1)$ then $\mathcal{U}_{\overline{X}\Gamma\overline{Y}}(p) = \vee_{p=q\gamma r}[\ \mathcal{U}_{\chi_X}(q) \wedge \mathcal{U}_{\chi_Y}(r)] \neq 0$, $\mathcal{V}_{\overline{X}\Gamma\overline{Y}}(p) = \wedge_{p=q\gamma r}[\mathcal{V}_{\overline{\chi}_X}(q) \vee \mathcal{V}_{\overline{\chi}_Y}(r)] \neq 1$◎ and $\mathcal{W}_{\overline{X}\Gamma\overline{Y}}(p) = \wedge_{p=q\gamma r}\left[\mathcal{W}_{\overline{\chi}_X}(q) \vee \mathcal{W}_{\overline{\chi}_Y}(r)\right] \neq 1$.There exist $q, r \in N$. with p◎qαr such that $\mathcal{U}_{\chi_X}(q) \neq 0$.$\mathcal{V}_{\overline{\chi}_X}(q) \neq 1$*and* $\mathcal{W}_{\overline{\chi}_X}(q) \neq 1$, $\mathcal{U}_Y(r) \neq 0, \mathcal{V}_{\overline{\chi}_Y}(r) \neq 1$. $\mathcal{W}_{\overline{\chi}_Y}(r) \neq 1$. So $\mathcal{U}_{\chi_x}(q) = 1$, $\mathcal{V}_{\overline{\chi}_x}(q) = 0$, $\mathcal{W}_{\overline{\chi}_Y}(q) = 0$ and $\mathcal{U}_{\chi_Y}(r) = 1$, $\mathcal{V}_{\overline{\chi}_Y}(r) = 0, \mathcal{W}_{\overline{\chi}_Y}(r) = 0$. This implies $q \in X$ *and* $r \in Y$. Thus p◎q$\alpha r \in X\Gamma Y \subseteq \mathcal{J}$, So $\mathcal{U}_{\chi_{\mathcal{J}}}(p) = 1, \mathcal{V}_{\overline{\chi}_{\mathcal{J}}}(p) = 0$ *and* $\mathcal{W}_{\overline{\chi}_{\mathcal{J}}}(p)$◎0. It follows that $\overline{X}\Gamma\overline{Y}(p) \subseteq \varphi$ .Since φ is a NF ideal of N, either $\overline{X} \subseteq \varphi$ or $\overline{Y} \subseteq \varphi$. Thus either X ⊆ φ or Y ⊆ φ. Hence $\mathcal{J}$ is a prime ideal of N.

Definition 3.20: (Neutrosophic Γ endomorphism) Mapping $\theta: N \to N$ of the Γ Ring N into itself is called a neutrosophic Γ-endomorphism of N. If for $p, q \in N$, $\alpha \epsilon\, \Gamma$ then

$(i)\mathcal{U}(p+q)\theta = \mathcal{U}(p\theta) + \mathcal{U}(q\theta), \mathcal{V}(p+q)\theta = \mathcal{V}(p\theta) + \mathcal{V}(q\theta)$ *and*

$\mathcal{W}(p+q)\theta = \mathcal{W}(p\theta) + \mathcal{W}(q\theta)$….. (1)

$(ii)\mathcal{U}(p\alpha q)\theta = \mathcal{U}(p\theta\alpha q\theta), \mathcal{V}(p\alpha q)\theta = \mathcal{V}(p\theta\alpha q\theta)$ *and*

$\mathcal{W}(p\alpha q)\theta = \mathcal{W}(p\theta\alpha q\theta)$……………(2)

Let Δ represent the group of Γ -endomorphism of the Γ Ring N. The multiplication and addition on the set as Δ follows, If $x, y \in \Delta$ then

$\mathcal{U}(p(x\alpha y)) = \mathcal{U}((px)\alpha y)\ p \in N, \alpha\epsilon\ \Gamma, \mathcal{V}(p(x\alpha y)) = \mathcal{V}((px)\alpha y)\ p \in N, \alpha\epsilon\ \Gamma$ and

$\mathcal{W}(p(x\alpha y)) = \mathcal{W}((px)\alpha y)\ p \in N,\ \alpha\epsilon\ \Gamma \ldots\ldots\ldots\ldots (3)$

$\mathcal{U}(p(x+y)) = \mathcal{U}(px) + \mathcal{U}(py))\ p \in N,\ \mathcal{V}(p(x+y)) = \mathcal{V}(px) + \mathcal{V}(py) p \in N$

$\mathcal{W}(p(x+y)) = \mathcal{W}(px) + \mathcal{W}(py) p \in N \ldots\ldots\ldots (4)$

Theorem 3.21: If Δ be the group of all neutrosophic Γ -endomorphism of a Γ Ring N. Then Δ is a Γ -endomorphism of a Γ Ring with unity with respect to usual operations**.**

Proof: Given Δ be the set of all Neutrosophic Γ -endomorphism of a Γ -ring M.

To Prove Δ is a Γ Ring with Unity and Let $x, y, z \in \Delta, \alpha\epsilon\ \Gamma$, p $\epsilon\ N$,

$(i)\ \mathcal{U}(x$ ((a + b) α c) = $\mathcal{U}((x(a+b)\alpha c))$

⊚ $\mathcal{U}((xa+xb)\alpha c)$

⊚ $\mathcal{U}((xa)\alpha c + (xb)\alpha c)$

⊚$\mathcal{U}(x(a\alpha c) + x(b\alpha c))$

⊚ $\mathcal{U}(x(a\alpha c + b\alpha c))$

Hence $\mathcal{U}((a+b)\alpha c) = \ \mathcal{U}(a\alpha c + b\alpha c)$

$\mathcal{V}(x($(a+b) α c) = $\mathcal{V}((x(a+b)\alpha c))$

⊚ $\mathcal{V}((xa+xb)\alpha c)$

⊚ $\mathcal{V}((xa)\alpha c + (xb)\alpha c)$

⊚$\mathcal{V}(x(a\alpha c) + x(b\alpha c))$

⊚ $\mathcal{V}(x(a\alpha c + b\alpha c))$

Hence $\mathcal{V}((a+b)\alpha c)) = \ \mathcal{V}(a\alpha c + b\alpha c)$

$\mathcal{W}(x($(a+b) α c) = $\mathcal{W}((x(a+b)\alpha c))$

⊚ $\mathcal{W}((xa+xb)\alpha c)$

⊚ $\mathcal{W}((xa)\alpha c + (xb)\alpha c)$

⊚$\mathcal{W}(x(a\alpha c) + x(b\alpha c))$

⊚ $\mathcal{W}(x(a\alpha c + b\alpha c)$

Hence $\mathcal{W}((a+b)\alpha c)) = \ \mathcal{W}(a\alpha c + b\alpha c)$

Now $\mathcal{U}(x$ (a(α⊚β)c)⊚ $\mathcal{U}((xa)(\alpha+\beta)$c a,c∈△, $\alpha, \beta \in \Gamma$, x $\epsilon\ N$

⊚ $\mathcal{U}((xa)\alpha$c ⊚ $(xa)\beta c)$

⊚ $\mathcal{U}(x(a\alpha$c ⊚$a\beta c))$

$\mathcal{U}(($a(α⊚β)c)⊚ $\mathcal{U}(a\alpha$c ⊚$a\beta c)$

$\mathcal{V}(x$ (a(α⊚β)c)⊚ $\mathcal{V}((xa)(\alpha+\beta)$c a,c∈△, $\alpha, \beta \in \Gamma$, x $\epsilon\ N$

⊚ $\mathcal{V}((xa)\alpha$c ⊚ $(xa)\beta c)$

⊚ $\mathcal{V}(x(a\alpha$c ⊚$a\beta c))$

$\mathcal{V}(($a(α⊚β)c)⊚ $\mathcal{V}(a\alpha$c ⊚$a\beta c)$

$\mathcal{W}(x$ (a(α⊚β)c)⊚ $\mathcal{W}((xa)(\alpha+\beta)$c a,c∈△, $\alpha, \beta \in \Gamma$, x $\epsilon\ N$

⊚ $\mathcal{W}((xa)\alpha$c ⊚ $(xa)\beta c)$

⊚ $\mathcal{W}(x(a\alpha$c ⊚$a\beta c))$

$\mathcal{W}(($a(α⊚β)c)⊚ $\mathcal{W}((a\alpha$c ⊚$a\beta c))$

Again,

$\mathcal{U}$ (x (aα(b⊚c)))⊚ $\mathcal{U}((xa)\alpha(b+c))$ a,b,c∈△, $\alpha \in \Gamma$, x ϵ N

⊚ $\mathcal{U}((xa)\alpha b) + \mathcal{U}((xa)\alpha c)$

⊚ $\mathcal{U}(x(a\alpha b) + (x(a\alpha c))$

⊚ $\mathcal{U}(x(a\alpha c + b\alpha c))$

Hence $\mathcal{U}$(aα(b + c)) = $\mathcal{U}((a\alpha b + a\alpha c))$

$\mathcal{V}$ (x (aα(b⊚c)))⊚ $\mathcal{V}$ $((xa)\alpha(b+c))$ a,b,c∈△, $\alpha \in \Gamma$, x ϵ N

⊚ $\mathcal{V}$ $((xa)\alpha b) + \mathcal{V}$ $((xa)\alpha c)$

⊚ $\mathcal{V}$ $(x(a\alpha b) + (x(a\alpha c))$

⊚ $\mathcal{V}$ $(x(a\alpha c + b\alpha c))$

Hence $\mathcal{V}$ (aα(b + c)) = $\mathcal{V}$ $((a\alpha b + a\alpha c))$

$\mathcal{W}$(x (aα(b⊚c)))⊚ $\mathcal{W}((xa)\alpha(b+c))$ a,b,c∈△, $\alpha \in \Gamma$, x ϵ N

⊚ $\mathcal{W}((xa)\alpha b) + \mathcal{U}((xa)\alpha c)$

⊚ $\mathcal{W}(x(a\alpha b) + (x(a\alpha c))$

⊚ $\mathcal{W}(x(a\alpha c + b\alpha c))$

Hence $\mathcal{W}$(aα(b + c)) = $\mathcal{W}((a\alpha b + a\alpha c))$

$(ii)\mathcal{U}$ ((x(aαb)βc))⊚ $\mathcal{U}((x(a\alpha b))\beta$c), a,b,c∈△, $\alpha, \beta \in \Gamma$, x ϵ N

⊚ $\mathcal{U}(((xa)\alpha b)\beta$c)

⊚ $\mathcal{U}((xa)\alpha(b\beta$c))

⊚ $\mathcal{U}(x(a\alpha(b\beta$c)))

⊚ $\mathcal{U}(x(a\alpha(b\beta$c)))

Hence $\mathcal{U}$((aαb)βc)= $\mathcal{U}(a\alpha(b\beta$c))

$\mathcal{V}$ (x((aαb)βc)))⊚ $\mathcal{V}((x(a\alpha b))\beta$c), a,b,c∈△, $\alpha, \beta \in \Gamma$, x ϵ N

⊚ $\mathcal{V}(((xa)\alpha b)\beta$c)

⊚ $\mathcal{V}((xa)\alpha(b\beta$c))

⊚ $\mathcal{V}(x(a\alpha(b\beta$c)))

⊚ $\mathcal{V}(x(a\alpha(b\beta$c))

Hence$\mathcal{V}$((aαb)βc)= $\mathcal{V}(a\alpha(b\beta$c))

$\mathcal{W}$((x(aαb)βc))) ⊚ $\mathcal{W}((x(a\alpha b))\beta$c), a,b,c ∈ △, $\alpha, \beta \in \Gamma$, x ϵ N

⊚ $\mathcal{W}(((xa)\alpha b)\beta$c)

⊚ $\mathcal{W}(xa)\alpha(b\beta$c))

⊚ $\mathcal{W}(x(a\alpha(b\beta$c)))

⊚ $\mathcal{W}(x(a\alpha(b\beta$c))

Hence $\mathcal{W}$((aαb)βc)= $\mathcal{W}(a\alpha(b\beta$c))

(iii) For all a∈△ then there exists unity element 1∈△ such that

$\mathcal{U}$(x(1αa)) ⊚ $\mathcal{U}$(((x1)α)a) ⊚ $\mathcal{U}(xa)$, $\alpha \in \Gamma$, x ϵ N, $\mathcal{V}$(x(1αa)) ⊚ $\mathcal{V}$(((x1)α)a) ⊚ $\mathcal{V}(xa)$, $\alpha \in \Gamma$, x ϵ N,

And $\mathcal{W}$(x(1αa)) ⊚ $\mathcal{W}$(((x1)α)a) ⊚ $\mathcal{W}(xa)$, $\alpha \in \Gamma$, x ϵ N,

And $\mathcal{U}$(x(aα1))⊚ $\mathcal{U}$((xa)α)1) ⊚ xa, $\mathcal{V}$(x(aα1))⊚ $\mathcal{V}$((xa)α)1) ⊚ xa, and

$\mathcal{W}$(x(aα1)) ⊚ $\mathcal{W}$((xa)α)1) ⊚ xa

Hence $\mathcal{U}$(aα1) ⊚ $\mathcal{U}$(1αa) ⊚ a, $\mathcal{V}$(aα1)⊚ $\mathcal{V}$(1αa) ⊚ a, and $\mathcal{W}$(aα1)⊚ $\mathcal{W}$(1αa) ⊚ a.

Thus △ satisfies all the conditions of Γ Ring. Hence △ is a Γ Ring with unity.

Theorem 3.22: Let Δ be the set of all neutrosophic Γ endomorphism of the Γ Ring N. If $x \epsilon \Delta$ then x has (Multiplicative inverse) in Δ if and only if x is one to one function.

Proof: Assume Δ be the set of all neutrosophic Γ -endomorphism of a Γ -ring M. If $x \epsilon \Delta$ then x has an inverse in Δ. To prove x is one to one function. Let x has an inverse y in Δ. $x\alpha y = y\alpha x = 1, \alpha\epsilon\Gamma$.

Then for each $p\epsilon N \cdot$ we get

$\mathcal{U}((py)\alpha x) = \mathcal{U}(p(y\alpha x)) = \mathcal{U}(p), \mathcal{V}((py)\alpha x) = \mathcal{V}(p(y\alpha x)) = \mathcal{V}(p)$ and

$\mathcal{W}((py)\alpha x) = \mathcal{W}(p(y\alpha x)) = \mathcal{W}(p)$ Clearly x is onto.

Furthermore $p_1, p_2 \epsilon N$ such that

$\mathcal{U}(p_1 x) = \mathcal{U}(p_2 x), \mathcal{V}(p_1 x) = \mathcal{V}(p_2 x)$, and $\mathcal{W}(p_1 x) = \mathcal{W}(p_2 x)$,

$\mathcal{U}(p_1) = \mathcal{U}(p_1.1) = \mathcal{U}(p_1(x\alpha y)) = \mathcal{U}((p_1.x)\alpha y) = \mathcal{U}((p_2.x)\alpha y) = \mathcal{U}(p_2(x\alpha y)) = \mathcal{U}(p_2.1) = \mathcal{U}(p_2)$

$\mathcal{V}(p_1) = \mathcal{V}(p_1.1) = \mathcal{V}(p_1(x\alpha y)) = \mathcal{V}((p_1.x)\alpha y) = \mathcal{V}((p_2.x)\alpha y) = \mathcal{V}(p_2(x\alpha y)) = \mathcal{V}(p_2.1) = \mathcal{V}(p_2)$.

$\mathcal{W}(p_1) = \mathcal{W}(p_1.1) = \mathcal{W}(p_1(x\alpha y)) = \mathcal{W}((p_1.x)\alpha y) = \mathcal{W}((p_2.x)\alpha y) = \mathcal{W}(p_2(x\alpha y)) = \mathcal{W}(p_2)$.

Therefore x is one to one mapping.

Conversely, Let us assume that the Γ -endomorphism x is one to one mapping of N onto N. So that each element of N is of the form $px, p\epsilon N$. We define a mapping y of N into N as follows

$\mathcal{U}\left(((px)\alpha)y\right) = \mathcal{U}(p), p\epsilon N, \alpha\epsilon\Gamma$. If $p, q\epsilon N$ then

$\mathcal{U}\left(((px + qx)\alpha)y\right) ⊚ \mathcal{U}((((p + q)x)\alpha)y) ⊚ \mathcal{U}(p + q) ⊚ \mathcal{U}(((px)\alpha)y) + \mathcal{U}(((qx)\alpha)y) ⊚$

$\mathcal{U}(((px\alpha qx)\alpha)y) = \mathcal{U}(((p\alpha q)x\alpha)y) = \mathcal{U}(p\alpha q) = \mathcal{U}((px)\alpha)yx(((qx)\alpha)y)$

$\mathcal{V}\left(((px)\alpha)y\right) = \mathcal{V}(p), p\epsilon N, \alpha\epsilon\Gamma$. If $p, q\epsilon N$ then

$\mathcal{V}\left(((px + qx)\alpha)y\right) ⊚ \mathcal{V}((((p + q)x)\alpha)y) ⊚ \mathcal{V}((p + q) ⊚ \mathcal{V}(((px)\alpha)y + \mathcal{V}(((qx)\alpha)y) ⊚$

$\mathcal{V}(((px\alpha qx)\alpha)y) = \mathcal{V}(((p\alpha q)x\alpha)y) = \mathcal{V}(p\alpha q) = \mathcal{V}((px)\alpha)yx(((qx)\alpha)y)$

$\mathcal{W}\left(((px)\alpha)y\right) = \mathcal{W}(p), p\epsilon N, \alpha\epsilon\Gamma$. If $p, q\epsilon N$ then

$\mathcal{W}\left(((px + qx)\alpha)y\right) ⊚ \mathcal{W}((((p + q)x)\alpha)y) ⊚ \mathcal{W}((p + q) ⊚ \mathcal{W}(((px)\alpha)y + \mathcal{W}(((qx)\alpha)y) ⊚$

$\mathcal{W}(((px\alpha qx)\alpha)y) = \mathcal{W}(((p\alpha q)x\alpha)y) = \mathcal{W}(p\alpha q) = \mathcal{W}((px)\alpha)yx(((qx)\alpha)y)$

We see that y is a neutrosophic Γ endomorphism of N. Furthermore

$\mathcal{U}((px)\alpha y) = \mathcal{U}(p(x\alpha y)) = \mathcal{U}(p)$ For every p in N and hence $x\alpha y = 1$ finally $p\epsilon N$, $\mathcal{U}(((px)\alpha)(y\alpha x)) = \mathcal{U}\left((p(x\alpha y))\alpha x\right) = \mathcal{U}(p(1)\alpha x) = \mathcal{U}(p(1\alpha x)) = \mathcal{U}(px)$, $\mathcal{V}((px)\alpha y) = \mathcal{V}(p(x\alpha y) = \mathcal{V}(p)$ For every p in N and hence $x\alpha y = 1$ finally $p\epsilon N$, $\mathcal{V}(((px)\alpha)(y\alpha x)) = \mathcal{V}\left((p(x\alpha y))\alpha x\right) = \mathcal{V}(p(1)\alpha x) = \mathcal{V}(p(1\alpha x)) = \mathcal{V}(px)$, and $\mathcal{W}((px)\alpha y) = \mathcal{W}(p(x\alpha y) = \mathcal{W}(p)$ For every p in N and hence $x\alpha y = 1$ finally $p\epsilon N$, $\mathcal{W}(((px)\alpha)(y\alpha x)) = \mathcal{W}\left((p(x\alpha y))\alpha x\right) = \mathcal{W}(p(1)\alpha x) = \mathcal{W}(p(1\alpha x)) = \mathcal{W}(px)$,. That is equivalent to the statement that $\mathcal{U}(q(y\alpha x)) = \mathcal{U}(q), \mathcal{V}(q(y\alpha x)) = \mathcal{V}(q)$ and $\mathcal{W}(q(y\alpha x)) = \mathcal{W}(q)$. For every $q\ \epsilon N$. Hence $y\alpha x = 1$ and y is the inverse of x in Δ.

4. Conclusions

In recent years, many algebraic structures have been considered neutrosophic structures. Using neutrosophic environments, we analyzed gamma rings. NF prime ideals are introduced in this article, along with their basic algebraic properties. In addition, some new neutrosophic operations are discussed.

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Neutrosophic RHO –Ideal with Complete Neutrosophic RHO–Ideal in RHO–Algebras

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Abstract: In real-life structures, indeterminacy is always present. Neuosophic sets theory is a well-known mathematical tool for dealing with indeterminacy. Smarandache proposed the neutrosophic set approach. Neutrosophic sets deal with vague data. In this study, we introduced and investigated several types of ρ −algebra ideals, which we called neutrosophic ρ −subalgebra, complete neutrosophic ρ −subalgebra, neutrosophic ρ −ideal, complete neutrosophic ρ −ideal, neutrosophic $\bar{\rho}$ −ideal, and complete neutrosophic $\bar{\rho}$ −ideal, respectively. We also proposed some hypotheses to explain some of the relationships between these ideal types.

Keywords: Neutrosophic ρ −subalgebra; neutrosophic ρ −ideal; neutrosophic $\bar{\rho}$ −ideal.

1. Introduction

Many different problems in our lives, such as engineering and medical sciences, necessitate uncertainty. Non-classical sets, like fuzzy sets ([19,]-[24]), soft sets ([25]-[31]), and permutation sets ([32]-[37]) are used to solve some problems in decision making. Smarandache [2] investigates neutrosophic sets as a method for dealing with issues involving unreliable, indeterminate, and persistent data. Imai & Iseki [6] introduce the concepts of BCK −algebra and BCI −algebra. The d −algebra was then introduced by Negger & Kim [9] as a generalization of BCK −algebra. In d −algebra, Negger et al. [8] discussed the ideal theory. In 1965, Zadeh proposed the concept of a fuzzy set [12]. Following that, Atanassov introduced the intuitionistic fuzzy set [1], which is a natural generalization of fuzzy set. Jun et al. [7] later applied the intuitionistic fuzzy set concept to d −algebra. Hasan [4] developed the concept of an intuitionistic fuzzy d −ideal of d −algebra in 2017. After that, Hasan [5] in 2020 introduced the concept of intuitionistic fuzzy d −filter. Smarandache [3] proposed the concept of a neutrosophic set. Next, some basic properties of this notion are studied ([13]-[18]). Also, Smarandache and Rezaei studied the neutrosophic triplet of BI-

algebras[38]. In 2021, some notions of neutrosophic ideals in BCK-algebras are discussed [39].The $\rho-$algebra was first introduced by Khalil and Abud Alradha[10]. In this paper, we define neutrosophic $\rho-$subalgebra, complete neutrosophic $\rho-$subalgebra, neutrosophic $\rho-$ideal, full neutrosophic $\rho-$ideal, neutrosophic $\rho-$ideal, neutrosophic $\bar{\rho}-$ideal, and complete neutrosophic $\bar{\rho}-$ideal of $\rho-$algebra, and investigate the relationship between these types.

2. Preliminaries and Some Results.

Here, we will recall basic ideas and results that are necessary in this research.

Definition 2.1.[10] A ρ-algebra is a non-empty set $\mho$ with a constant 0 and a binary operation "$\oint$" satisfying the following axioms:

(1) $\alpha \oint \alpha = 0$,

(2) $0 \oint \alpha = 0$,

(3) $\alpha \oint \beta = 0 = \beta \oint \alpha$ imply that $\alpha = \beta$,

(4) For all $\alpha \neq \beta \in \mho - \{0\}$ imply that $\alpha \oint \beta = \beta \oint \alpha \neq 0$.

Definition 2.2. [10] A non-empty subset Υ of a ρ-algebra $(\mho, \oint, 0)$ is called ρ- subalgebra of $\mho$ if $\alpha \oint \beta \in \Upsilon$ for any $\alpha, \beta \in \Upsilon$.

Definition 2.3.[10] A non- empty subset Υ of a ρ-algebra $\mho$ is called an ρ- ideal of $\mho$ if satisfies:

(1) $\alpha, \beta \in \Upsilon \Rightarrow \alpha \oint \beta \in \Upsilon$,

(2) $\alpha \oint \beta \in \Upsilon$ & $\beta \in \Upsilon \Rightarrow \alpha \in \Upsilon$.

Remark 2.4[10]. If Υ is any a ρ- Ideal, then it is easy to show that Υ is ρ- subalgebra. However, the convers maybe not true.

Definition2.5.[10] A non- empty subset Υ of a ρ-algebra $\mho$ is called an $\bar{\rho}$- ideal of $\mho$ if satisfies:

(1) $0 \in \Upsilon$,

(2) $\alpha \in \Upsilon$ & $\beta \in \mho \Rightarrow \alpha \oint \beta \in \Upsilon$.

Proposition 2.6. [10] Let $\emptyset \neq \Upsilon \subseteq \mho$ where $\mho$ is ρ-algebra. Then Υ is a ρ- subalgebra of $\mho$ if it is $\bar{\rho}$-Ideal.

Definition 2.7. [2] A Neutrosophic set $\mathcal{N}$ (briefly, NS) over the universal $\mho$ is defined by $\mathcal{N} = \{< \alpha, \mathcal{N}_T(\alpha), \mathcal{N}_I(\alpha), \mathcal{N}_F(\alpha) >| \alpha \in \mho\}$, where $\mathrm{N_T}(\alpha), \mathrm{N_I}(\alpha), \mathrm{N_F}(\alpha): \mho \longrightarrow [0,1]$ are maps, with $\mathcal{N}_T(\alpha), \mathcal{N}_I(\alpha)$ and $\mathcal{N}_F(\alpha)$ are real numbers and their values represent the degree of membership, indeterminate and non- membership of α to $\mathcal{N}$ respectively.

Definition 2.8 . [2] A complement neutrosophic set $\mathcal{N}^c$ over the universal $\mho$ is defined by

$\mathcal{N}^c = 1 - \mathcal{N} = 1 - \{< \alpha, \mathcal{N}_T(\alpha), \mathcal{N}_I(\alpha), \mathcal{N}_F(\alpha) >| \alpha \in \mho\} =$

$\{< \alpha, 1 - \mathcal{N}_T(\alpha), 1 - \mathcal{N}_I(\alpha), 1 - \mathcal{N}_F(\alpha) >| \alpha \in \mho\} = \{< \alpha, \mathcal{N}_{T^c}(\alpha), \mathcal{N}_{I^c}(\alpha), \mathcal{N}_{F^c}(\alpha) >| \alpha \in \mho\}$.

Definition 2.9. [2] Let $\mathcal{N}$ be (NS) over the universal $\mho$ and $t \in [0,1]$ then the set $\mathcal{N}_t = \{\alpha \in \mho | \mathcal{N}_T(\alpha) \geq t, \mathcal{N}_I(\alpha) \leq t, \mathcal{N}_F(\alpha) \geq t\}$ is called neutrosophic set t-cut, (briefly, NS-t-cut) .

Definition 2.10. [2] Let $(\mho, *, 0)$ be a ρ-algebra and $\mathcal{N} = \{< \alpha, \mathcal{N}_T(\alpha), \mathcal{N}_I(\alpha), \mathcal{N}_F(\alpha) >| \alpha \in \mho\}$ be a neutrosophic set (NS) of $\mho$. We say $\mathcal{N}$ is a neutrosophic ρ -constant of $\mho$ if all the maps $\mathcal{N}_T$, $\mathcal{N}_I$, $\mathcal{N}_F$: $\mho \rightarrow [0,1]$ are constant maps.

3. Neutrosophic ρ – Subalgebra and Complete Neutrosophic ρ – Subalgebra:

Definition 3.1. A (NS) $\mathcal{N}$ in $\mho$ is called a neutrosophic ρ $-$subalgebra (briefly, NS $-\rho - SA$) of $\mho$ if such that:

(i) $\mathcal{N}_T(\alpha \oint \beta) \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$,

(ii) $\mathcal{N}_I(\alpha \oint \beta) \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$,

(iii) $\mathcal{N}_F(\alpha \oint \beta) \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$, for any $\alpha, \beta \in \mho$.

Example 3.2. Assume $\mho$ = {0,1,2,3} is a set and $\oint$ is defined by table (1). So, we get ($\mho$, $\oint$,0) is a ρ-algebra, We define a (NS) $\mathcal{N}$ in $\mho$ as follows:

$$\mathcal{N}_T = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.5 & 0.4 & 0.4 & 0.4 \end{pmatrix}, \mathcal{N}_I = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.2 & 0.3 & 0.3 & 0.3 \end{pmatrix}, \mathcal{N}_F = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.3 & 0.2 & 0.2 & 0.2 \end{pmatrix}$$

Hence, $\mathcal{N}$ is (NS $-\rho - SA$).

$\oint$	**0**	**1**	**2**	**3**
0	0	0	0	0
1	1	0	1	2
2	2	1	0	2
3	3	2	2	0

Table (1) , $\mathcal{N}$ is (NS $-\rho - SA$)

Lemma 3.3. Let $\mathcal{N}$ be (NS $-\rho - SA$) of $\mho$ then:

(i) $\mathcal{N}_T(0) \geq \mathcal{N}_T(\alpha)$, (ii) $\mathcal{N}_I(0) \leq \mathcal{N}_I(\alpha)$, (iii) $\mathcal{N}_F(0) \geq \mathcal{N}_F(\alpha)$, for any $\alpha \in \mho$.

Proof: Let $\mathcal{N}$ be (NS $-\rho - SA$) then

(i) $\mathcal{N}_T(0) = \mathcal{N}_T(\alpha \oint \alpha) \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\alpha)\} = \mathcal{N}_T(\alpha)$.

(ii) $\mathcal{N}_I(0) = \mathcal{N}_I(\alpha \oint \alpha) \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\alpha)\} = \mathcal{N}_I(\alpha)$.

(iii) $\mathcal{N}_F(0) = \mathcal{N}_F(\alpha \oint \alpha) \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\alpha)\} = \mathcal{N}_F(\alpha)$.

Lemma 3.4. Let $\mathcal{N}$ be (NS $-\rho - SA$) of $\mho$ then:

(i) $\mathcal{N}_{T^c}(\alpha) \geq \mathcal{N}_{T^c}(0)$, (ii) $\mathcal{N}_{I^c}(\alpha) \leq \mathcal{N}_{I^c}(0)$, (iii) $\mathcal{N}_{F^c}(\alpha) \geq \mathcal{N}_{F^c}(0)$,for any $\alpha \in \mho$.

Proof: Let $\mathcal{N}$ be (NS $-\rho - SA$) , then from lemma (3.3) we obtain: $\mathcal{N}_T(0) \geq \mathcal{N}_T(\alpha)$, $\mathcal{N}_I(0) \leq \mathcal{N}_I(\alpha)$, $\mathcal{N}_F(0) \geq \mathcal{N}_F(\alpha)$, for any $\alpha \in \mho$. Since, $\mathcal{N}^c = 1 - \mathcal{N}$, thus

$\mathcal{N}_{T^c}(\alpha) = 1 - \mathcal{N}_T(\alpha) \geq 1 - \mathcal{N}_T(0) = \mathcal{N}_{T^c}(0)$,

$\mathcal{N}_{I^c}(\alpha) = 1 - \mathcal{N}_I(\alpha) \leq 1 - \mathcal{N}_I(0) = \mathcal{N}_{I^c}(0)$,

$\mathcal{N}_{F^c}(\alpha) = 1 - \mathcal{N}_F(\alpha) \geq 1 - \mathcal{N}_F(0) = \mathcal{N}_{F^c}(0)$, This completes proof.

Proposition 3.5: Let $\mathcal{N}$ be (NS) of $\rho -$ algebra ($\mho$, $\oint$,0), then $\mathcal{N}$ is (NS $-\rho - SA$) if it is $\mathcal{N} = \{\prec \alpha, \mathcal{N}_T(\alpha) = \mathcal{N}_T(0), \mathcal{N}_I(\alpha) = \mathcal{N}_I(0), \mathcal{N}_F(\alpha) = \mathcal{N}_F(0) \succ \mid \alpha \in \mho\}$.

Proof: Let $\mathcal{N}$ be (NS) and $\mathcal{N}_T(\alpha) = \mathcal{N}_T(0), \mathcal{N}_I(\alpha) = \mathcal{N}_I(0), \mathcal{N}_F(\alpha) = \mathcal{N}_F(0)$, for any $\alpha \in \mathcal{N}$. Now, Let $\alpha, \beta \in \mho$, then $\mathcal{N}_T(\alpha \oint \beta) = \mathcal{N}_T(0) = min\{\mathcal{N}_T(0), \mathcal{N}_T(0)\} = min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$, thus $\mathcal{N}_T(\alpha \oint \beta) \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$, $\mathcal{N}_I(\alpha \oint \beta) = \mathcal{N}_I(0) = max\{\mathcal{N}_I(0), \mathcal{N}_I(0)\} = max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$,

thus $\mathcal{N}_I(\alpha \oint \beta) \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$,

$\mathcal{N}_F(\alpha \oint \beta) = \mathcal{N}_F(0) = min\{\mathcal{N}_F(0), \mathcal{N}_F(0)\} = min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$, thus $\mathcal{N}_F(\alpha \oint \beta) \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$. Hence $\mathcal{N}$ is (NS $-\rho - SA$).

Proposition 3.6. Let $\mathcal{N}$ be (NS) of $\rho -$ algebra $(\mho, \oint, 0)$, then $\mathcal{N}^c$ is(NS $-\rho - SA$). If $\mathcal{N} = \{< \alpha, \mathcal{N}_{T^c}(\alpha) = \mathcal{N}_{T^c}(0), \mathcal{N}_{I^c}(\alpha) = \mathcal{N}_{I^c}(0), \mathcal{N}_{F^c}(0) = \mathcal{N}_{F^c}(0) > \mid \alpha \in \mho\}$

Proof: Let $\mathcal{N}^c = \{< \alpha, \mathcal{N}_{T^c}(\alpha) = \mathcal{N}_{T^c}(0), \mathcal{N}_{I^c}(\alpha) = \mathcal{N}_{I^c}(0), \mathcal{N}_{F^c}(0) = \mathcal{N}_{F^c}(0) > \mid \alpha \in \mho\}$ and let $\alpha, \beta \in \mho$, then $\mathcal{N}_{T^c}(\alpha \oint \beta) = \mathcal{N}_{T^c}(0) = min\{\mathcal{N}_{T^c}(0), \mathcal{N}_{T^c}(0)\} = min\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\}$, thus $\mathcal{N}_{T^c}(\alpha \oint \beta) \geq min\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\}$, $\mathcal{N}_{I^c}(\alpha \oint \beta) = \mathcal{N}_{I^c}(0) = max\{\mathcal{N}_{I^c}(0), \mathcal{N}_{I^c}(0)\} = max\{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\}$, thus $\mathcal{N}_{I^c}(\alpha \oint \beta) \leq max\{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\}$, $\mathcal{N}_{F^c}(\alpha \oint \beta) = \mathcal{N}_{F^c}(0) = min\{\mathcal{N}_{F^c}(0), \mathcal{N}_{F^c}(0)\} = min\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\beta)\}$, thus $\mathcal{N}_{F^c}(\alpha \oint \beta) \geq min\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\beta)\}$. Hence $\mathcal{N}^c$ is (NS $-\rho - SA$).

Definition 3.7. Let $\mathcal{N}$ be (NS) of $\rho -$ algebra $(\mho, \oint, 0)$, then $K(\mathcal{N}) = \{\alpha \in \mho \mid \mathcal{N}_T(\alpha) = \mathrm{N_T}(0), \mathrm{N_I}(\alpha) = \mathrm{N_I}(0)$ and $\mathrm{N_F}(\alpha) = \mathrm{N_F}(0)\}$ is a subset of $\mho$ and it is called neutrosophic ρ –kernel of $\mathcal{N}$ over $\mho$.

Example 3.8. Let $\mho = \{a, b, c, d\}$ and define $\oint$ on the set $\mho$ as table (2). Then $(\mho, \oint, a)$ is a ρ-algebra, we define a (NS) $\mathcal{N}$ in $\mho$ as follows:

$\mathcal{N}_{\mathrm{T}} = \begin{pmatrix} a & b & c & d \\ 0.1 & 0.2 & 0.5 & 0.1 \end{pmatrix}$, $\mathcal{N}_{\mathrm{I}} = \begin{pmatrix} a & b & c & d \\ 0.1 & 0.3 & 0.4 & 0.1 \end{pmatrix}$,

$\mathcal{N}_{\mathrm{I}} = \begin{pmatrix} a & b & c & d \\ 0.1 & 0.3 & 0.5 & 0.1 \end{pmatrix}$, $K(\mathcal{N}) = \{a, d\}$.

$\oint$	a	b	c	d
$\boldsymbol{a}$	a	a	a	a
$\boldsymbol{b}$	b	a	b	d
$\boldsymbol{c}$	c	b	a	d
$\boldsymbol{d}$	d	d	d	a

Table (2), $K(\mathcal{N}) = \{a, d\}$

Proposition 3.9. If $\mathcal{N}$ is (NS $-\rho - SA$) of $(\mho, \oint, 0)$, then $K(\mathcal{N}^c)$ is a $(\rho - SA)$.

Proof: Let $\alpha, \beta \in K(\mathcal{N}^c)$. Then; $\mathcal{N}_{T^c}(\alpha) = \mathcal{N}_{T^c}(\beta) = \mathcal{N}_{T^c}(0), \mathcal{N}_{I^c}(\alpha) = \mathcal{N}_{I^c}(\beta) = \mathcal{N}_{I^c}(0)$ and $\mathcal{N}_{F^c}(\alpha) = \mathcal{N}_{F^c}(\beta) = \mathcal{N}_{F^c}(0)$.

Also $\mathcal{N}_{T^c}(\alpha \oint \beta) = 1 - \mathcal{N}_T(\alpha \oint \beta) \leq 1 - min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$

[since $\mathcal{N}_T(\alpha \oint \beta) \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$].

$= max\{1 - \mathcal{N}_T(\alpha), 1 - \mathcal{N}_T(\beta)\}$

$= max\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\}$

$= max\{\mathcal{N}_{T^c}(0), \mathcal{N}_{T^c}(0)\} = \mathcal{N}_{T^c}(0)$,

$\mathcal{N}_{I^c}(\alpha \oint \beta) = 1 - \mathcal{N}_I(\alpha \oint \beta) \geq 1 - max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$[since $\mathcal{N}_I(\alpha \oint \beta) \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$].

$= min\{1 - \mathcal{N}_I(\alpha), 1 - \mathcal{N}_I(\beta)\}$

$= min\ \{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\}$

$= min\{\mathcal{N}_{I^c}(0), \mathcal{N}_{I^c}(0)\} = \mathcal{N}_{I^c}(0),$

$\mathcal{N}_{F^c}(\alpha \oint \beta) = 1 - \mathcal{N}_F(\alpha \oint \beta) \leq 1 - min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$

[since $\mathcal{N}_F(\alpha \oint \beta) \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$].

$= max\{1 - \mathcal{N}_F(\alpha), 1 - \mathcal{N}_F(\beta$

$= max\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\beta)\}$

$= max\{\mathcal{N}_{F^c}(0), \mathcal{N}_{F^c}(0)\} = \mathcal{N}_{F^c}(0),$

and from lemma (3.4), W obtain:

$\mathcal{N}_{T^c}(\alpha \oint \beta) \geq \mathcal{N}_{T^c}(0)$, $\mathcal{N}_{I^c}(\alpha \oint \beta) \leq \mathcal{N}_{I^c}(0)$, $\mathcal{N}_{F^c}(\alpha \oint \beta) \geq \mathcal{N}_{F^c}(0)$

thus $\mathcal{N}_{T^c}(\alpha \oint \beta) = \mathcal{N}_{T^c}(0)$, $\mathcal{N}_{I^c}(\alpha \oint \beta) = \mathcal{N}_{I^c}(0)$, $\mathcal{N}_{F^c}(\alpha \oint \beta) = \mathcal{N}_{F^c}(0)$,

this implies $\alpha \oint \beta \in K(\mathcal{N}^c)$, hence $K(\mathcal{N}^c)$ is ρ −subalgebra.

Proposition 3.10. Let $\mathcal{N}$ be (NS $-\rho - SA$) then $\mathcal{N}_t$ is ρ −subalgebra.

Proof: Assume that $\mathcal{N}$ is (NS $-\rho - SA$) and $\alpha, \beta \in \mathcal{N}_t$, then $(\mathcal{N}_T(\alpha) \geq t, \mathcal{N}_I(\alpha) \leq t, \mathcal{N}_F(\alpha) \geq t)$ and $(\mathcal{N}_T(\beta) \geq t, \mathcal{N}_I(\beta) \leq t, \mathcal{N}_F(\beta) \geq t)$. Also, $\mathcal{N}_T(\alpha \oint \beta) \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\} \geq t$, $\mathcal{N}_I(\alpha \oint \beta) \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\} \leq t$, $\mathcal{N}_F(\alpha \oint \beta) \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\} \geq t$, this implies $\alpha \oint \beta \in \mathcal{N}_t$, hence $\mathcal{N}_t$ subalgebra.

Proposition 3.11. Let $(\mho, \oint, 0)$ be a ρ-algebra and $\mathcal{N}$ be a (NS) of $\mho$. Then $\mathcal{N}$ is (NS $-\rho - SA$)if it is neutrosophic ρ −constant.

Proof: Assume that $\mathcal{N}$ is constant. Then for all $\alpha \in \mho$, $\mathcal{N}_T(\alpha) = \mathcal{N}_T(0)$, $\mathcal{N}_I(\alpha) = \mathcal{N}_I(0)$ and $\mathcal{N}_F(\alpha) = \mathcal{N}_F(0)$, and so $\mathcal{N}_T(0) \geq \mathcal{N}_T(\alpha)$, $\mathcal{N}_I(0) \leq \mathcal{N}_I(\alpha)$and $\mathcal{N}_F(0) \geq \mathcal{N}_F(\alpha)$. Next, for all $\alpha, \beta \in \mho$, $\mathcal{N}_T(\alpha \oint \beta) = \mathcal{N}_T(0) = min\{\mathcal{N}_T(0), \mathcal{N}_T(0)\} \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$, $\mathcal{N}_I(\alpha \oint \beta) = \mathcal{N}_I(0) = max\{\mathcal{N}_I(0), \mathcal{N}_I(0)\} \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$, $\mathcal{N}_F(\alpha \oint \beta) = \mathcal{N}_F(0) = min\{\mathcal{N}_F(0), \mathcal{N}_F(0)\} \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$, hence $\mathcal{N}$ is (NS $-\rho - SA$).

Proposition 3.12. Let $\mathcal{N}$ be (NS $-\rho - SA$). Then $0 \in \mathcal{N}_t$, if $\mathcal{N}_t \neq \emptyset$.

Proof: Assume that $\mathcal{N}$ is (NS $-\rho - SA$) and $\mathcal{N}_t \neq \emptyset$ then there is at least $\alpha \in \mathcal{N}_t$, also from Lemma (3.3) and Definition (2.9) we obtain, $\mathcal{N}_T(0) \geq \mathcal{N}_T(\alpha) \geq t$, $\mathcal{N}_I(0) \leq \mathcal{N}_I(\alpha) \leq t$, $\mathcal{N}_F(0) \geq \mathcal{N}_F(\alpha) \geq t$, this means $0 \in \mathcal{N}_t$.

Corollary 3.13. If $\mathcal{N}$ neutrosophic ρ -constant then $\mathcal{N}_t$ is ρ −subalgebra.

Proof: From proposition (3.11) and proposition (3.10).

Definition 3.14. Let $\mathcal{N}$ be (NS) in $\mho$ then it is called a complete neutrosophic ρ −subalgebra (briefly, CNS $-\rho - SA$) of $\mho$ if it satisfies the following conditions:

(i) $\mathcal{N}_T(\alpha \oint \beta) \leq max\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$,

(ii) $\mathcal{N}_I(\alpha \oint \beta) \geq min\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$,

(iii) $\mathcal{N}_F(\alpha \oint \beta) \leq max\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$, for any $\alpha, \beta \in \mho$.

Example 3.15 .Assume Ʊ = {0,1,2,3} is a set and $\oint$ is defined by table (3). So, we get (Ʊ, $\oint$,0) is a ρ-algebra, we define a (NS) $\mathcal{N}$ in Ʊ as follows:

$\mathcal{N}_T = \begin{pmatrix} 0 & 1 & 2 \\ 0.1 & 0.3 & 0.3 \end{pmatrix}$, $\mathcal{N}_I = \begin{pmatrix} 0 & 1 & 2 \\ 0.6 & 0.3 & 0.3 \end{pmatrix}$, $\mathcal{N}_F = \begin{pmatrix} 0 & 1 & 2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}$

Hence, $\mathcal{N}$ is (NS $-\rho - SA$).

$\oint$	**0**	**1**	**2**
0	0	0	0
1	1	0	1
2	2	1	0

Table (3) , $\mathcal{N}$ is (NS $-\rho - SA$)

Lemma 3.16. Let $\mathcal{N}$ be (CNS $-\rho - SA$) of Ʊ then:

(i) $\mathcal{N}_T(0) \leq \mathcal{N}_T(\alpha)$, (ii) $\mathcal{N}_I(0) \geq \mathcal{N}_I(\alpha)$, (iii) $\mathcal{N}_F(0) \leq \mathcal{N}_F(\alpha)$, for any $\alpha \in$ Ʊ.

Proof: Let $\mathcal{N}$ be (CNS $-\rho - SA$) then,

(i) $\mathcal{N}_T(0) = \mathcal{N}_T(\alpha \oint \alpha) \leq max\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\alpha)\} = \mathcal{N}_T(\alpha)$.

(ii) $\mathcal{N}_I(0) = \mathcal{N}_I(\alpha \oint \alpha) \geq min\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\alpha)\} = \mathcal{N}_I(\alpha)$.

(iii) $\mathcal{N}_F(0) = \mathcal{N}_F(\alpha \oint \alpha) \leq max\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\alpha)\} = \mathcal{N}_F(\alpha)$.This completes proof.

Proposition 3.17. If $\mathcal{N}$ is a (CNS $-\rho - SA$), then $K(\mathcal{N})$ is $\rho -$ subalgebra.

Proof: Let $\alpha, \beta \in K(\mathcal{N})$, then $\mathcal{N}_T(\alpha) = \mathcal{N}_T(\beta) = \mathcal{N}_T(0)$, $\mathcal{N}_I(\alpha) = \mathrm{N_I}(\beta) = \mathrm{N_I}(0)$ and $\mathcal{N}_F(\alpha) = \mathcal{N}_F(\beta) = \mathcal{N}_F(0)$. Also, $\mathcal{N}_T(\alpha \oint \beta) \leq max\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\} = max\{\mathcal{N}_T(0), \mathcal{N}_T(0)\} = \mathcal{N}_T(0)$, $\mathcal{N}_I(\alpha \oint \beta) \geq min\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\} = min\{\mathcal{N}_I(0), \mathcal{N}_I(0)\} = \mathcal{N}_I(0)$, $\mathcal{N}_F(\alpha \oint \beta) \leq max\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\} = max\{\mathcal{N}_F(0), \mathcal{N}_F(0)\} = \mathcal{N}_F(0)$, and from lemma (3.16) $\mathcal{N}_T(0) \leq \mathcal{N}_T(\alpha \oint \beta)$, $\mathcal{N}_I(0) \geq \mathcal{N}_I(\alpha \oint \beta)$, $\mathcal{N}_F(0) \leq \mathcal{N}_F(\alpha \oint \beta)$, thus $\mathcal{N}_T(\alpha \oint \beta) = \mathcal{N}_T(0)$, $\mathcal{N}_I(\alpha \oint \beta) = \mathcal{N}_I(0)$, $\mathcal{N}_F(\alpha \oint \beta) = \mathcal{N}_F(0)$, and $\alpha \oint \beta \in K(\mathcal{N})$ hence $K(\mathcal{N})$ is $\rho -$ subalgebra.

Proposition 3.18. Let $\mathcal{N}$ be (NS) then $\mathcal{N}$ is (NS $-\rho - SA$) if and only if $\mathcal{N}^c$ is (CNS $-\rho - SA$).

Proof: Let $\mathcal{N}$ be (NS $-\rho - SA$) then $\mathcal{N}_T(\alpha \oint \beta) \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$, $\mathcal{N}_I(\alpha \oint \beta) \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$, $\mathcal{N}_F(\alpha \oint \beta) \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$, for any $\alpha, \beta \in$ Ʊ .

Now, $\mathcal{N}_{T^c}(\alpha \oint \beta) = 1 - \mathcal{N}_T(\alpha \oint \beta) \leq 1 - min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$

$$= max\{1 - \mathcal{N}_T(\alpha), 1 - \mathcal{N}_T(\beta)\}$$

$$= max\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\},$$

$\mathcal{N}_{I^c}(\alpha \oint \beta) = 1 - \mathcal{N}_I(\alpha \oint \beta) \geq 1 - max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\} = min\{1 - \mathcal{N}_I(\alpha), 1 - \mathcal{N}_I(\beta)\}$

$$= min\{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\},$$

$\mathcal{N}_{F^c}(\alpha \oint \beta) = 1 - \mathcal{N}_F(\alpha \oint \beta) \leq 1 - min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$

$$= max\{1 - \mathcal{N}_F(\alpha), 1 - \mathcal{N}_F(\beta)\}$$

$$= max\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\beta)\},$$

Hence $\mathcal{N}^c$ is (CNS $-\rho - SA$).

Conversely: Let $\mathcal{N}^c$ be (CNS $-\rho-SA$) then $\mathcal{N}_{T^c}(\alpha \oint \beta) \le max\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\}$, $\mathcal{N}_{I^c}(\alpha \oint \beta) \ge min\{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\}$, $\mathcal{N}_{F^c}(\alpha \oint \beta) \le max\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\beta)\}$, for any $\alpha, \beta \in \mho$.

Now, $\mathcal{N}_T(\alpha \oint \beta) = 1 - \mathcal{N}_{T^c}(\alpha \oint \beta) \ge 1 - max\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\}$

$= min\{1 - \mathcal{N}_{T^c}(\alpha), 1 - \mathcal{N}_{T^c}(\beta)\}$

$= min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$,

$\mathcal{N}_I(\alpha \oint \beta) = 1 - \mathcal{N}_{I^c}(\alpha \oint \beta) \le 1 - min\{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\}$

$= max\{1 - \mathcal{N}_{I^c}(\alpha), 1 - \mathcal{N}_{I^c}(\beta)\}$

$= max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$

$\mathcal{N}_F(\alpha \oint \beta) = 1 - \mathcal{N}_{F^c}(\alpha \oint \beta) \ge 1 - max\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\beta)\}$

$= min\{1 - \mathcal{N}_{F^c}(\alpha), 1 - \mathcal{N}_{F^c}(\beta)\}$

$= min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$, Hence $\mathcal{N}$ is (NS $-\rho-SA$).

Corollary 3.19.

1- Let $\mathcal{N}^c$ be is (CNS $-\rho-SA$) then $\mathcal{N}_t$ is $\rho-$ subalgebra.

2- Let $\mathcal{N}$ be a neutrosophic ρ -constant then $\mathcal{N}_t$ is $\rho-$ subalgebra.

Proof (1): From Proposition (3.18) and Proposition (3.10).

Proof (2): From Proposition (3.11) and Proposition (3.10).

4. Neutrosophic ρ -Ideal and Complete Neutrosophic ρ -Ideal:

Definition 4.1. Assume $(\mho, \oint, 0)$ is a ρ-algebra and $\mathcal{N}$ is (NS) of $\mho$. We say $\mathcal{N}$ is a neutrosophic ρ -ideal of $\mho$ (briefly, NS $-\rho-I$))if such that:

(i) $\mathcal{N}_T(\alpha \oint \beta) \ge min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$,

(ii) $\mathcal{N}_I(\alpha \oint \beta) \le max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$,

(iii) $\mathcal{N}_F(\alpha \oint \beta) \ge min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$,

(iv) $\mathcal{N}_T(\alpha) \ge min\{\mathcal{N}_T(\alpha \oint \beta), \mathcal{N}_T(\beta)\}$,

(v) $\mathcal{N}_I(\alpha) \le max\{\mathcal{N}_I(\alpha \oint \beta), \mathcal{N}_I(\beta)\}$,

(vi) $\mathcal{N}_F(\alpha) \ge min\{\mathcal{N}_F(\alpha \oint \beta), \mathcal{N}_F(\beta)\}$, for any $\alpha, \beta \in \mho$.

Example 4.2. Let $\mho = \{\alpha, \gamma, \beta, \delta\}$ be a set with the following table (4), it is clear that $(\mho, \oint, \alpha)$ is a ρ-algebra, We define a (NS) $\mathcal{N}$ in $\mho$ as follows:

$\mathcal{N}_T = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0.8 & 0.7 & 0.7 & 0.7 \end{pmatrix}$, $\mathcal{N}_I = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0.4 & 0.6 & 0.6 & 0.6 \end{pmatrix}$, $\mathcal{N}_F = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0.6 & 0.5 & 0.5 & 0.5 \end{pmatrix}$

Hence, $\mathcal{N}$ is (NS $-\rho-I$).

$\oint$	α	β	γ	δ
α	α	α	α	α
β	β	α	β	γ
γ	γ	β	α	γ
δ	δ	γ	γ	α

Table (4): $\mathcal{N}$ is (NS $-\rho - I$).

Lemma 4.3: Every (NS $-\rho - I$)is (NS $-\rho - SA$).

Proof: Let $\mathcal{N}$ be (NS $-\rho - I$). Then from Definition (4.1)-[(i),(ii),(iii)], we get $\mathcal{N}$ is (NS $-\rho - SA$).

Corollary 4.4.

1- Let $\mathcal{N}$ be (NS $-\rho - I$) then $\mathcal{N}_t$ is ρ −subalgebra.

2-Let $\mathcal{N}$ be (NS $-\rho - I$) then $\mathcal{N}^c$is (CNS $-\rho - SA$).

Proof 1: From Lemma (4.3**)** and Proposition (3.10)**.**

Proof 2: From Lemma (4.3**)** and Proposition (3.18)**.**

Lemma 4.5. Let $\mathcal{N}$ be (NS $-\rho - I$) of Ʊ. Then;

(i) $\mathcal{N}_T(0) \geq \mathcal{N}_T(\alpha)$, (ii) $\mathcal{N}_I(0) \leq \mathcal{N}_I(\alpha)$, (iii) $\mathcal{N}_F(0) \geq \mathcal{N}_F(\alpha)$,for any $\alpha \in$ Ʊ.

Proposition 4.6. If $\mathcal{N}$ is (NS $-\rho - I$), then $K(\mathcal{N})$ is $\rho -$ ideal.

Proof: Let $\mathcal{N}$ be (NS $-\rho - I$) and let $\alpha, \beta \in K(\mathcal{N})$, then $\mathcal{N}_T(\alpha) = \mathcal{N}_T(\beta) = \mathcal{N}_T(0)$, $\mathcal{N}_I(\alpha) = \mathcal{N}_I(\beta) = \mathcal{N}_I(0)$ and $\mathcal{N}_F(\alpha) = \mathcal{N}_F(\beta) = \mathcal{N}_F(0)$. Also, $\mathcal{N}_T(\alpha \oint \beta) \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\} = min\{\mathcal{N}_T(0), \mathcal{N}_T(0)\} = \mathcal{N}_T(0)$,

$\mathcal{N}_I(\alpha \oint \beta) \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\} = max\{\mathcal{N}_I(0), \mathcal{N}_I(0)\} = \mathcal{N}_I(0)$,

$\mathcal{N}_F(\alpha \oint \beta) \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\} = min\{\mathcal{N}_F(0), \mathcal{N}_F(0)\} = \mathcal{N}_F(0)$, and from lemma (4.5) we obtain $\mathcal{N}_T(0) \geq \mathcal{N}_T(\alpha \oint \beta)$, $\mathcal{N}_I(0) \leq \mathcal{N}_I(\alpha \oint \beta)$, $\mathcal{N}_F(0) \geq \mathcal{N}_F(\alpha \oint \beta)$. Hence, $\alpha \oint \beta \in K(\mathcal{N})$. Now, assume that, $\alpha \oint \beta \in K(\mathcal{N})$ & $\beta \in K(\mathcal{N})$ then $\mathcal{N}_T(\alpha \oint \beta) = \mathcal{N}_T(0)$, $\mathcal{N}_I(\alpha \oint \beta) = \mathcal{N}_I(0)$, $\mathcal{N}_F(\alpha \oint \beta) = \mathcal{N}_F(0)$, and $\mathcal{N}_T(\beta) = \mathcal{N}_T(0)$, $\mathcal{N}_I(\beta) = \mathcal{N}_I(0)$, $\mathcal{N}_F(\beta) = \mathcal{N}_F(0)$, thus $\mathcal{N}_T(\alpha) \geq min\{\mathcal{N}_T(\alpha \oint \beta), \mathcal{N}_T(\beta)\} = \mathcal{N}_T(0)$, $\mathcal{N}_I(\alpha) \leq max\{\mathcal{N}_I(\alpha \oint \beta), \mathcal{N}_I(\beta)\} = \mathcal{N}_I(0)$, $\mathcal{N}_F(\alpha) \geq min\{\mathcal{N}_F(\alpha \oint \beta), \mathcal{N}_F(\beta)\} = \mathcal{N}_F(0)$, and from lemma (4.5) We obtain $\mathcal{N}_T(0) = \mathcal{N}_T(\alpha)$, $\mathcal{N}_I(0) = \mathcal{N}_I(\alpha)$, $\mathcal{N}_F(0) = \mathcal{N}_F(\alpha)$, thus $\alpha \in K(\mathcal{N})$, hence $K(\mathcal{N})$ is $\rho -$ ideal.

Proposition 4.7. If $\mathcal{N}$ is (NS $-\rho - I$), then $K(\mathcal{N}^c)$ is $\rho -$ ideal.

Proof: Let $\mathcal{N}$ be (NS $-\rho - I$) and let $\alpha, \beta \in K(\mathcal{N}^c)$, then

$\mathcal{N}_{T^c}(\alpha) = \mathcal{N}_{T^c}(\beta) = \mathcal{N}_{T^c}(0)$, $\mathcal{N}_{I^c}(\alpha) = \mathcal{N}_{I^c}(\beta) = \mathcal{N}_{I^c}(0)$ and $\mathcal{N}_{F^c}(\alpha) = \mathcal{N}_{F^c}(\beta) = \mathcal{N}_{F^c}(0)$.

Also, $\mathcal{N}_{T^c}(\alpha \oint \beta) = 1 - \mathcal{N}_T(\alpha \oint \beta) \leq 1 - min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$

$$= max\{1 - \mathcal{N}_T(\alpha), 1 - \mathcal{N}_T(\beta)\}$$

$$= max\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\}$$

$$= max\{\mathcal{N}_{T^c}(0), \mathcal{N}_{T^c}(0)\} = \mathcal{N}_{T^c}(0),$$

$\mathcal{N}_{I^c}(\alpha \oint \beta) = 1 - \mathcal{N}_I(\alpha \oint \beta) \geq 1 - max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$

$$= min\{1 - \mathcal{N}_I(\alpha), 1 - \mathcal{N}_I(\beta)\}$$

$$= min\{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\}$$

$$= min\{\mathcal{N}_{I^c}(0), \mathcal{N}_{I^c}(0)\} = \mathcal{N}_{I^c}(0),$$

$\mathcal{N}_{F^c}(\alpha \oint \beta) = 1 - \mathcal{N}_F(\alpha \oint \beta) \leq 1 - min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$

$$= max\{1 - \mathcal{N}_F(\alpha), 1 - \mathcal{N}_F(\beta)\}$$

$$= max\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\alpha)\} = \mathcal{N}_{F^c}(0)\}$$

$$= max\{\mathcal{N}_{F^c}(0), \mathcal{N}_{F^c}(0)\} = \mathcal{N}_{F^c}(0)\},$$

and from lemma (3.4),we obtain

$\mathcal{N}_{T^c}(\alpha \oint \beta) \geq \mathcal{N}_{T^c}(0)$, $\mathcal{N}_{I^c}(\alpha \oint \beta) \leq \mathcal{N}_{I^c}(0)$, $\mathcal{N}_{F^c}(\alpha \oint \beta) \geq \mathcal{N}_{F^c}(0)$

thus $\mathcal{N}_{T^c}(\alpha \oint \beta) = \mathcal{N}_{T^c}(0), \mathcal{N}_{I^c}(\alpha \oint \beta) = \mathcal{N}_{I^c}(0), \mathcal{N}_{F^c}(\alpha \oint \beta) = \mathcal{N}_{F^c}(0)$,
this implies $\alpha \oint \beta \in K(\mathcal{N}^c)$. Now, let $\alpha \oint \beta, \beta \in K(\mathcal{N}^c)$, then $\mathcal{N}_{T^c}(\alpha \oint \beta) = \mathcal{N}_{T^c}(0)$, $\mathcal{N}_{I^c}(\alpha \oint \beta) = \mathcal{N}_{I^c}(0)$, $\mathcal{N}_{F^c}(\alpha \oint \beta) = \mathcal{N}_{F^c}(0)$.
And $\mathcal{N}_{T^c}(\beta) = \mathcal{N}_{T^c}(0), \mathcal{N}_{I^c}(\beta) = \mathcal{N}_{I^c}(0), \mathcal{N}_{F^c}(\beta) = \mathcal{N}_{F^c}(0)$. Since $\mathcal{N}$ is (NS $-\rho - I$) then,

$$\mathcal{N}_{T^c}(\alpha) = 1 - \mathcal{N}_T(\alpha) \leq 1 - min\{\mathcal{N}_T(\alpha \oint \beta), \mathcal{N}_T(\beta)\}$$
$$= max\{1 - \mathcal{N}_T(\alpha \oint \beta), 1 - \mathcal{N}_T(\beta)\}$$
$$= max\{\mathcal{N}_{T^c}(\alpha \oint \beta), \mathcal{N}_{T^c}(\beta)\}$$
$$= max\{\mathcal{N}_{T^c}(0), \mathcal{N}_{T^c}(0)\} = \mathcal{N}_{T^c}(0),$$
$$\mathcal{N}_{I^c}(\alpha) = 1 - \mathcal{N}_I(\alpha) \geq 1 - max\{\mathcal{N}_I(\alpha \oint \beta), \mathcal{N}_I(\beta)\}$$
$$= min\{1 - \mathcal{N}_I(\alpha \oint \beta), 1 - \mathcal{N}_I(\beta)\}$$
$$= min\{\mathcal{N}_{I^c}(\alpha \oint \beta), \mathcal{N}_{I^c}(\beta)\}$$
$$= min\{\mathcal{N}_{I^c}(0), \mathcal{N}_{I^c}(0)\} = \mathcal{N}_{I^c}(0),$$
$$\mathcal{N}_{F^c}(\alpha) = 1 - \mathcal{N}_F(\alpha) \leq 1 - min\{\mathcal{N}_F(\alpha \oint \beta), \mathcal{N}_F(\beta)\}$$
$$= max\{1 - \mathcal{N}_F(\alpha \oint \beta), 1 - \mathcal{N}_F(\beta)\}$$
$$= max\{\mathcal{N}_{F^c}(\alpha \oint \beta), \mathcal{N}_{F^c}(\alpha)\} = \mathcal{N}_{F^c}(0)\}$$
$$= max\{\mathcal{N}_{F^c}(0), \mathcal{N}_{F^c}(0)\} = \mathcal{N}_{F^c}(0),$$

and from lemma (3.4),we obtain
$\mathcal{N}_{T^c}(\alpha) \geq \mathcal{N}_{T^c}(0), \mathcal{N}_{I^c}(\alpha) \leq \mathcal{N}_{I^c}(0), \mathcal{N}_{F^c}(\alpha) \geq \mathcal{N}_{F^c}(0)$
thus $\mathcal{N}_{T^c}(\alpha) = \mathcal{N}_{T^c}(0), \mathcal{N}_{I^c}(\alpha) = \mathcal{N}_{I^c}(0), \mathcal{N}_{F^c}(\alpha) = \mathcal{N}_{F^c}(0)$,
this implies $\alpha \in K(\mathcal{N}^c)$, hence $K(\mathcal{N}^c)$is $\rho -$ ideal.

Proposition 4.8. If $\mathcal{N}$ is (NS $-\rho - I$), then $\mathcal{N}_t$ is ρ −ideal.

Proof: Assume that $\mathcal{N}$ is (NS $-\rho - I$) and $\alpha, \beta \in \mathcal{N}_t$, then $\mathcal{N}_T(\alpha \oint \beta) \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\} \geq t, \mathcal{N}_I(\alpha \oint \beta) \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\} \leq t$, $\mathcal{N}_F(\alpha \oint \beta) \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\} \geq t$, this implies $\alpha \oint \beta \in \mathcal{N}_t$. Now, assume that $\alpha \oint \beta \in \mathcal{N}_t$ & $\beta \in \mathcal{N}_t$, and [since $\mathcal{N}$ is (NS $-\rho - I$)]. We obtain $\mathcal{N}_T(\alpha) \geq min\{\mathcal{N}_T(\alpha \oint \beta), \mathcal{N}_T(\beta)\} \geq t$, $\mathcal{N}_I(\alpha) \leq max\{\mathcal{N}_I(\alpha \oint \beta), \mathcal{N}_I(\beta) \leq t$, $\mathcal{N}_F(\alpha) \geq min\{\mathcal{N}_F(\alpha \oint \beta), \mathcal{N}_F(\beta)\} \geq t$, thus $\alpha \in \mathcal{N}_t$, hence $\mathcal{N}_t$ is ρ −ideal.

Definition4.9. Assume $(\mho, \oint, 0)$ is a ρ-algebra and $\mathcal{N}$ is (NS) of $\mho$. We say $\mathcal{N}$ is a complete neutrosophic ρ -ideal of $\mho$ (briefly, CNS$-\rho - I$) if such that:
(i) $\mathcal{N}_T(\alpha \oint \beta) \leq max\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$,
(ii) $\mathcal{N}_I(\alpha \oint \beta) \geq min\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$,
(iii) $\mathcal{N}_F(\alpha \oint \beta) \leq max\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$,
(iv) $\mathcal{N}_T(\alpha) \leq max\{\mathcal{N}_T(\alpha \oint \beta), \mathcal{N}_T(\beta)\}$,
(v) $\mathcal{N}_I(\alpha) \geq min\{\mathcal{N}_I(\alpha \oint \beta), \mathcal{N}_I(\beta)\}$,
(vi) $\mathcal{N}_F(\alpha) \leq max\{\mathcal{N}_F(\alpha \oint \beta), \mathcal{N}_F(\beta)\}$, for any $\alpha, \beta \in \mho$.

Example 4.10. Let $\mho = \{0, 1, 2, 3, 4\}$ be a set with the following table (5), it is clear that $(\mho, \oint, 0)$ is a ρ-algebra, We define a (NS) $\mathcal{N}$ in $\mho$ as follows:

$\mathcal{N}_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.4 & 0.5 & 0.5 & 0.5 & 0.5 \end{pmatrix}$, $\mathcal{N}_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.7 & 0.3 & 0.3 & 0.3 & 0.3 \end{pmatrix}$, and

$\mathcal{N}_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.2 & 0.4 & 0.4 & 0.4 & 0.4 \end{pmatrix}$. Hence, $\mathcal{N}$ is (CNS $-\rho - I$).

$\oint$	**0**	**1**	**2**	**3**	4
0	0	0	0	0	0
1	1	0	1	2	4
2	2	1	0	2	2
3	3	2	2	0	3
4	4	4	2	3	0

Table (5): $\mathcal{N}$ is (CNS $-\rho - I$).

Proposition 4.11: Let $\mathcal{N}$ be NS then $\mathcal{N}$ is (NS $-\rho - I$) if and only if $\mathcal{N}^c$ is (CNS $-\rho - I$).

Proof: Let $\mathcal{N}$ be (NS $-\rho - I$) ,From proof proposition (3.18) we obtain $\mathcal{N}_{T^c}(\alpha \oint \beta) \le max\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\}$, $\mathcal{N}_{I^c}(\alpha \oint \beta) \ge min\{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\}$, $\mathcal{N}_{F^c}(\alpha \oint \beta) \le max\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\beta)\}$.

Now, $\mathcal{N}_{T^c}(\alpha) = 1 - \mathcal{N}_T(\alpha) \le 1 - min\{\mathcal{N}_T(\alpha \oint \beta), \mathcal{N}_T(\beta)\}$

$$= max\{1 - \mathcal{N}_T(\alpha \oint \beta), 1 - \mathcal{N}_T(\beta)\}$$
$$= max\{\mathcal{N}_{T^c}(\alpha \oint \beta), \mathcal{N}_{T^c}(\beta)\},$$

$\mathcal{N}_{I^c}(\alpha) = 1 - \mathcal{N}_I(\alpha) \ge 1 - max\{\mathcal{N}_I(\alpha \oint \beta), \mathcal{N}_I(\beta)\}$

$$= min\{1 - \mathcal{N}_I(\alpha), 1 - \mathcal{N}_I(\beta)\}$$
$$= min\{\mathcal{N}_{I^c}(\alpha \oint \beta), \mathcal{N}_{I^c}(\beta)\},$$

$\mathcal{N}_{F^c}(\alpha) = 1 - \mathcal{N}_F(\alpha) \le 1 - min\{\mathcal{N}_F(\alpha \oint \beta), \mathcal{N}_F(\beta)\}$

$$= max\{1 - \mathcal{N}_F(\alpha \oint \beta), 1 - \mathcal{N}_F(\beta)\}$$
$$= max\{\mathcal{N}_{F^c}(\alpha \oint \beta), \mathcal{N}_{F^c}(\beta)\}.$$

Hence $\mathcal{N}^c$ is (CNS $-\rho - I$).

Conversely: Let $\mathcal{N}^c$ be (CNS $-\rho - I$) then from proof proposition, (3.18) we obtain,

$\mathcal{N}_T(\alpha \oint \beta) \ge min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$, $\mathcal{N}_I(\alpha \oint \beta) \le max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$, $\mathcal{N}_F(\alpha \oint \beta) \ge min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$.

Now, $\mathcal{N}_T(\alpha) = 1 - \mathcal{N}_{T^c}(\alpha) \ge 1 - max\{\mathcal{N}_{T^c}(\alpha \oint \beta), \mathcal{N}_{F^c}(\beta)\}$

$$= min\{1 - \mathcal{N}_{T^c}(\alpha \oint \beta), 1 - \mathcal{N}_{T^c}(\beta)\}$$
$$= min\{\mathcal{N}_T(\alpha \oint \beta), \mathcal{N}_T(\beta)\},$$

$\mathcal{N}_I(\alpha) = 1 - \mathcal{N}_{T^c}(\alpha) \le 1 - min\{\mathcal{N}_{I^c}(\alpha \oint \beta), \mathcal{N}_{I^c}(\beta)\}$

$$= max\{1 - \mathcal{N}_{I^c}(\alpha \oint \beta), 1 - \mathcal{N}_{I^c}(\beta)\}$$
$$= max\{\mathcal{N}_I(\alpha \oint \beta), \mathcal{N}_I(\beta)\}$$

$\mathcal{N}_F(\alpha) = 1 - \mathcal{N}_{F^c}(\alpha) \ge 1 - max\{\mathcal{N}_{F^c}(\alpha \oint \beta), \mathcal{N}_{F^c}(\beta)\}$

$$= min\{1 - \mathcal{N}_{F^c}(\alpha \oint \beta), 1 - \mathcal{N}_{F^c}(\beta)\}$$

$= min\{\mathcal{N}_F(\alpha \oint \beta), \mathcal{N}_F(\beta)\}$.

Hence $\mathcal{N}$ is $(\text{NS} - \rho - I)$

Corollary 4.12: If $\mathcal{N}^c$ is $(\text{CNS} - \rho - I)$. Then;

1- $\mathcal{N}_t$ is ρ −subalgebra,

2- $\mathcal{N}^c$ is $(\text{CNS} - \rho - SA)$,

3- $\mathcal{N}_t$ is ρ −ideal.

Proof 1: From proposition (4.11) and corollary (4.4)-1.

Proof 2: From proposition (4.11) and corollary (4.4)-2.

Proof 3: From Proposition (4.11) and Proposition (4.8).

5. Neutrosophic $\bar{\rho}$ -Ideal and Complete Neutrosophic $\bar{\rho}$ -Ideal

Definition 5.1. Assume $(\mho, \oint, 0)$ is a ρ-algebra and $\mathcal{N}$ is (NS) of $\mho$. We say $\mathcal{N}$ is a neutrosophic $\bar{\rho}$ -ideal of $\mho$ (briefly, $\text{NS} - \bar{\rho} - I$) if such that:

(i) $\mathcal{N}_T(0) \geq \mathcal{N}_T(\alpha)$,

(ii) $\mathcal{N}_I(0) \leq \mathcal{N}_I(\alpha)$,

(iii) $\mathcal{N}_F(0) \geq \mathcal{N}_F(\alpha)$,

(iv) $\mathcal{N}_T(\alpha \oint \beta) \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$,

(v) $\mathcal{N}_I(\alpha \oint \beta) \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$,

(vi) $\mathcal{N}_F(\alpha \oint \beta) \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$, for any $\alpha, \beta \in \mho$.

Example 5.2. Let $\mho = \{x, y, z, w\}$ be a set with the following table(6), it is clear that $(\mho, \oint, x)$ is a ρ-algebra. We define a (NS) $\mathcal{N}$ in $\mho$ as follows:

$\mathcal{N}_T = \begin{pmatrix} x & y & z & w \\ 0.6 & 0.4 & 0.4 & 0.4 \end{pmatrix}$, $\mathcal{N}_I = \begin{pmatrix} x & y & z & w \\ 0.1 & 0.2 & 0.2 & 0.2 \end{pmatrix}$, $\mathcal{N}_F = \begin{pmatrix} x & y & z & w \\ 0.2 & 0.1 & 0.1 & 0.1 \end{pmatrix}$.

Hence, $\mathcal{N}$ is $(\text{NS} - \bar{\rho} - I)$.

$\oint$	x	y	z	w
x	x	x	x	x
y	y	x	z	w
z	z	z	x	z
w	w	w	z	x

Table (6): $\mathcal{N}$ is $(\text{NS} - \bar{\rho} - I)$.

Lemma 5.3. If $\mathcal{N}$ is $(\text{NS} - \bar{\rho} - I)$, then $\mathcal{N}$ is $(\text{NS} - \rho - SA)$.

Corollary 5.4. Let $\mathcal{N}$ be $(\text{NS} - \bar{\rho} - I)$. Then;

1- $\mathcal{N}_t$ is ρ −subalgebra,

2- $\mathcal{N}^c$is $(\text{CNS} - \rho - S)$.

Proof (1): From lemma (5.3) and proposition (3.10).

Proof (2): From lemma (5.3) and proposition (3.18).

Proposition 5.5. If $\mathcal{N}$ is (NS $-\bar{\rho}-I$), then $\mathcal{N}_t$ is $\bar{\rho}$ −ideal.

Proof: Assume that $\mathcal{N}$ is (NS $-\bar{\rho}-I$) and $\alpha, \beta \in \mathcal{N}_t$, then $\mathcal{N}_T(\alpha \oint \beta) \geq min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\} \geq t, \mathcal{N}_I(\alpha \oint \beta) \leq max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\} \leq t$, $\mathcal{N}_F(\alpha \oint \beta) \geq min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\} \geq t$, this implies $\alpha \oint \beta \in \mathcal{N}_t$. Since [$\mathcal{N}$ is (NS $-\bar{\rho}-I$)], and $\mathcal{N}$ ={< $\alpha, \mathcal{N}_T(\alpha) \geq$ t , $\mathcal{N}_I(\alpha) \leq$ t , $\mathcal{N}_F(\alpha) \geq$ t >| $\alpha \in \mho$}. We obtain $\mathcal{N}_T(0) \geq \mathcal{N}_T(\alpha) \geq$ t, $\mathcal{N}_I(0) \leq \mathcal{N}_I(\alpha) \leq t$, $\mathcal{N}_F(0) \geq \mathcal{N}_F(\alpha) \geq$ t, thus $0 \in \mathcal{N}_t$, hence $\mathcal{N}_t$ is $\bar{\rho}$ −ideal.

Definition5.6. Assume ($\mho$,$\oint$, 0) is a ρ-algebra and $\mathcal{N}$ is (NS) of $\mho$. We say $\mathcal{N}$ is a complete neutrosophic $\bar{\rho}$ -ideal of $\mho$ (briefly, CNS $-\bar{\rho}-I$). If such that:

(i)$\mathcal{N}_T(0) \leq \mathcal{N}_T(\alpha)$,

(ii) $\mathcal{N}_I(0) \geq \mathcal{N}_I(\alpha)$,

(iii) $\mathcal{N}_F(0) \leq \mathcal{N}_F(\alpha)$,

(iv) $\mathcal{N}_T(\alpha \oint \beta) \leq max\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$,

(v) $\mathcal{N}_I(\alpha \oint \beta) \geq min\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$,

(vi) $\mathcal{N}_F(\alpha \oint \beta) \leq max\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$,for any $\alpha, \beta \in \mho$.

Example 5.7. Let $\mho$ = { 0 ,1 ,2,3 } be a set with the following table(7), it is clear that ($\mho$, $\oint$, 0) is a ρ-algebra. We define a (NS) $\mathcal{N}$ in $\mho$ as follows:

$$\mathcal{N}_T = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.4 & 0.5 & 0.5 & 0.5 \end{pmatrix}, \mathcal{N}_I = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.3 & 0.2 & 0.2 & 0.2 \end{pmatrix}, \mathcal{N}_F = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.1 & 0.2 & 0.2 & 0.2 \end{pmatrix}.$$

Hence, $\mathcal{N}$ is (CNS $-\bar{\rho}-I$).

$\oint$	**0**	**1**	**2**	**3**
0	0	0	0	0
1	1	0	2	3
2	2	2	0	2
3	3	3	2	0

Table (7): $\mathcal{N}$ is (CNS $-\bar{\rho}-I$).

Lemma 5.8. If $\mathcal{N}$ is (CNS $-\bar{\rho}-I$), then $\mathcal{N}$ is (CNS $-\rho-SA$).

Proposition 5.9: Let $\mathcal{N}$ be (CNS $-\bar{\rho}-I$), then $K(\mathcal{N})$ is $\rho-$ subalgebra.

Proof: Assume $\alpha, \beta \in K(N)$, then $N_T(\alpha) = N_T(\beta) = N_T(0), N_I(\alpha) = N_I(\beta) = N_I(0)$ and $N_F(\alpha) = N_F(\beta) = N_F(0)$. Also, $N_T(\alpha \oint \beta) \leq \max\{N_T(\alpha), N_T(\beta)\} = \max\{N_T(0), \mathcal{N}_T(0)\} = \mathcal{N}_T(0)$, $\mathcal{N}_I(\alpha \oint \beta) \geq min\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\} = min\{\mathcal{N}_I(0), \mathcal{N}_I(0)\} = \mathcal{N}_I(0)$, $\mathcal{N}_F(\alpha \oint \beta) \leq max\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\} = max\{\mathcal{N}_F(0), \mathcal{N}_F(0)\} = \mathcal{N}_F(0)$, and $\mathcal{N}_T(0) \leq \mathcal{N}_T(\alpha \oint \beta)$, $\mathcal{N}_I(0) \geq \mathcal{N}_I(\alpha \oint \beta)$, $\mathcal{N}_F(0) \leq \mathcal{N}_F(\alpha \oint \beta)$, thus $\mathcal{N}_T(\alpha \oint \beta) = \mathcal{N}_T(0)$, $\mathcal{N}_I(\alpha \oint \beta) = \mathcal{N}_I(0)$, $\mathcal{N}_F(\alpha \oint \beta) = \mathcal{N}_F(0)$, and $\alpha \oint \beta \in K(\mathcal{N})$ hence $K(\mathcal{N})$ is ρ −subalgebra.

Proposition 5.10. Let $\mathcal{N}$ be (NS) then $\mathcal{N}$ is (NS $-\bar{\rho}-I$) if and only if $\mathcal{N}^c$ is (CNS $-\bar{\rho}-I$).

Proof: Let $\mathcal{N}$ be (NS $-\bar{\rho}-I$), we obtain $\mathcal{N}_T(0) \geq \mathcal{N}_T(\alpha)$, $\mathcal{N}_I(0) \leq \mathcal{N}_I(\alpha)$, $\mathcal{N}_F(0) \geq \mathcal{N}_F(\alpha)$, thus $\mathcal{N}_{T^c}(\alpha) = 1 - \mathcal{N}_T(\alpha) \geq 1 - \mathcal{N}_T(0) = \mathcal{N}_{T^c}(0)$, $\mathcal{N}_{I^c}(\alpha) = 1 - \mathcal{N}_I(\alpha) \leq 1 - \mathcal{N}_I(0) = \mathcal{N}_{I^c}(0)$, $\mathcal{N}_{F^c}(\alpha) = 1 - \mathcal{N}_F(\alpha) \geq 1 - \mathcal{N}_F(0) = \mathcal{N}_{F^c}(0)$.

Now, $\mathcal{N}_{T^c}(\alpha \oint \beta) = 1 - \mathcal{N}_T(\alpha \oint \beta) \leq 1 - min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$

$= max\{1 - \mathcal{N}_T(\alpha), 1 - \mathcal{N}_T(\beta)\}$

$= max\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\}$,

$\mathcal{N}_{I^c}(\alpha \oint \beta) = 1 - \mathcal{N}_I(\alpha \oint \beta) \geq 1 - max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$

$= min\{1 - \mathcal{N}_I(\alpha), 1 - \mathcal{N}_I(\beta)\} = min\{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\}$,

$\mathcal{N}_{F^c}(\alpha \oint \beta) = 1 - \mathcal{N}_F(\alpha \oint \beta) \leq 1 - min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$

$= max\{1 - \mathcal{N}_F(\alpha), 1 - \mathcal{N}_F(\beta)\} = max\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\beta)\}$,

Hence $\mathcal{N}^c$ is $(\text{CNS} - \bar{\rho} - I)$.

Conversely: Let $\mathcal{N}^c$ be $(\text{CNS} - \bar{\rho} - I)$, then $\mathcal{N}_{T^c}(0) \leq \mathcal{N}_{T^c}(\alpha)$, $\mathcal{N}_{I^c}(0) \geq \mathcal{N}_{I^c}(\alpha)$, $\mathcal{N}_{F^c}(0) \leq \mathcal{N}_{F^c}(\alpha)$, $\mathcal{N}_T(0) = 1 - \mathcal{N}_{T^c}(0) \geq 1 - \mathcal{N}_{T^c}(\alpha) = \mathcal{N}_T(\alpha)$,

$\mathcal{N}_I(0) = 1 - \mathcal{N}_{I^c}(0) \leq 1 - \mathcal{N}_{I^c}(\alpha) = \mathcal{N}_I(\alpha)$,

$\mathcal{N}_F(0) = 1 - \mathcal{N}_{F^c}(0) \geq 1 - \mathcal{N}_{F^c}(\alpha) = \mathcal{N}_F(\alpha)$,

and from the following

$\mathcal{N}_{T^c}(\alpha \oint \beta) \leq max\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\}$, $\mathcal{N}_{I^c}(\alpha \oint \beta) \geq min\{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\}$,

$\mathcal{N}_{F^c}(\alpha \oint \beta) \leq max\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\beta)\}$, for any $\alpha, \beta \in \mho$,

We obtain,

$\mathcal{N}_T(\alpha \oint \beta) = 1 - \mathcal{N}_{T^c}(\alpha \oint \beta) \geq 1 - max\{\mathcal{N}_{T^c}(\alpha), \mathcal{N}_{T^c}(\beta)\}$

$= min\{1 - \mathcal{N}_{T^c}(\alpha), 1 - \mathcal{N}_{T^c}(\beta)\} = min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}$,

$\mathcal{N}_I(\alpha \oint \beta) = 1 - \mathcal{N}_{I^c}(\alpha \oint \beta) \leq 1 - min\{\mathcal{N}_{I^c}(\alpha), \mathcal{N}_{I^c}(\beta)\}$

$= max\{1 - \mathcal{N}_{I^c}(\alpha), 1 - \mathcal{N}_{I^c}(\beta)\} = max\{\mathcal{N}_I(\alpha), \mathcal{N}_I(\beta)\}$

$\mathcal{N}_F(\alpha \oint \beta) = 1 - \mathcal{N}_{F^c}(\alpha \oint \beta) \geq 1 - max\{\mathcal{N}_{F^c}(\alpha), \mathcal{N}_{F^c}(\beta)\}$

$= min\{1 - \mathcal{N}_{F^c}(\alpha), 1 - \mathcal{N}_{F^c}(\beta)\} = min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\}$.

Hence $\mathcal{N}$ is $(\text{NS} - \bar{\rho} - I)$.

Corollary 5.11. If $\mathcal{N}^c$ is $(\text{CNS} - \bar{\rho} - I)$. Then;

1- $\mathcal{N}_t$ is ρ −subalgebra,

2- $\mathcal{N}^c$ is $(\text{CNS} - \rho - SA)$,

3- $\mathcal{N}_t$ is $\bar{\rho}$ −ideal.

Proof 1: From proposition (5.10) and corollary (5.4)-1.

Proof 2: From Proposition (5.10) and Corollary (5.4)-2.

Proof 3: From Proposition (5.10) and Proposition (5.5).

6. Conclusion

We presented and examined several kinds of ρ −algebra ideals in this research, which we called neutrosophic ρ −subalgebra, complete neutrosophic ρ −subalgebra, neutrosophic ρ −ideal, complete neutrosophic ρ −ideal, neutrosophic ρ −ideal, neutrosophic ρ −ideal, neutrosophic $\bar{\rho}$ −ideal, and complete neutrosophic $\bar{\rho}$ −ideal, respectively. We also suggested some theories try to explain some of these ideal type relationships. In future work, we will use soft set theory to study our notions and results in neutrosophic soft sets.

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Neutrosophic n-Valued Refined Sets and Topologies

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Abstract. In n-Valued refined logic truth value T can be split into many types of truths: $T_1, T_2, ..., T_p$ and I into many types of indeterminacies: $I_1, I_2, ..., I_r$ and F into many types of falsities: $F_1, F_2, ..., F_s$, where p, r and s are integers greater than 1, and $p + r + s = n$. Importance of n-valued refined logic and sets appeared in different applications specially in medical diagnosis. In this paper we post a condition on neutrosophic n-valued refined sets to make them functional to be applied in different mathematical branches. We define and study n-valued refined topological spaces. We defined neutrosophic n-valued refined α-open, β-open, pre-open and semi-open sets and studied their properties. We constructed different counter examples to clarify the relations between these different types of neutrosophic n-valued refined generalized open sets.

Keywords: n-valued refined topology; refined logic; refined sets; n-valued refined α-open; semi-open sets; n-valued refined generalized open sets.)

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1. Introduction

Neutrosophic sets are, first, introduced in 2005 by [26,27] as a generalization of intuitionistic fuzzy sets [13], where any element $x \in X$ we have three degrees; the degree of membership(T), indeterminacy(I), and non-membership(F). Neurosophic vague sets are introduced in 2015 by [30]. Neutrosophic vague topological spaces introduced in [21] we are many different notations are introduced and studied such as neurosophic vague continuity and compactness.

Neutrosophic topologies are defined and studied by Smarandache [27], Lupianez [19,20] and Salama [?]. Open and closed neutrosophic sets, interior, exterior, closure and boundary of neutrosophic sets can be found in [29].

Neutrosophic sets applied to generalize many notaions about soft topology and applications [18], [23], [16], generalized open and closed sets [31] , fixed point theorems [18] , graph theory

[17]and rough topology and applications [22]. Neutrosophy has many applications especially in decision making, for more details about new trends of neutrosophic applications one can consult [1]- [7].

Generalized topology and continuity introduced in 2002 in [**?**] which is a generalization of topological spaces and has different properties than general topology, see for example [8], [11] and [12]. Neutrosophic generalized sets and topologies are introduced and studies by Murad M. Arar in 2020 see [9] and [10]. In n-valued refined logic truth value T can be split into many types of truths: $T_1, T_2, ..., T_p$ and I into many types of indeterminacies: $I_1, I_2, ..., I_r$ and F into many types of falsities: $F_1, F_2, ..., F_s$, where p, r and s are integers greater than 1, and $p + r + s = n$ see [28]. Importance of n-valued refined logic and sets appeared in different applications specially in medial diagnosis see [25] and [14], where a strong assumption is assumed to make them functional; that is $p = r = s$.

Definition 1.1. [26]: We say that the set A is *neutrosophic* on X if
$A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle; x \in X\}; \mu, \sigma, \nu : X \to]^-0, 1^+[$ and $^-0 \leq \mu(x)+\sigma(x)+\nu(x) \leq 3^+$.

The class of all neutrosophic sets on the universe X will be denoted by $\mathcal{N}(X)$. The basic neutrosophic operations (inclusion, union, and intersection) where first introduced by [24].

Definition 1.2 (*Neutrosophic sets operations*)**.** Let $A, A_\alpha, B \in \mathcal{N}(X)$ such that $\alpha \in \Delta$. Then we define the neutrsophic:

(1) (*Inclusion*): $A \sqsubseteq B$ If $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
(2) (*Equality*): $A = B \Leftrightarrow A \sqsubseteq B$ and $B \sqsubseteq A$.
(3) (*Intersection*) $\underset{\alpha\in\Delta}{\sqcap} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\wedge}\mu_{A_\alpha}(x), \underset{\alpha\in\Delta}{\vee}\sigma_{A_\alpha}(x), \underset{\alpha\in\Delta}{\vee}\nu_{A_\alpha}(x)\rangle; x \in X\}$.
(4) (*Union*) $\underset{\alpha\in\Delta}{\sqcup} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\vee}\mu_{A_\alpha}(x), \underset{\alpha\in\Delta}{\wedge}\sigma_{A_\alpha}(x), \underset{\alpha\in\Delta}{\wedge}\nu_{A_\alpha}(x)\rangle; x \in X\}$.
(5) (*Complement*) $A^c = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle; x \in X\}$
(6) (*Universal set*) $1_X = \{\langle x, 1, 0, 0\rangle; x \in X\}$; called the *neutrosophic universal set.*
(7) (*Empty set*) $0_X = \{\langle x, 0, 1, 1\rangle; x \in X\}$; called the *neutrosophic empty set.*

Proposition 1.3. *[24] For $A, A_\alpha \in \mathcal{N}(X)$ for every $\alpha \in \Delta$ we have:*

(1) $A \sqcap (\underset{\alpha\in\Delta}{\sqcup} A_\alpha) = \underset{\alpha\in\Delta}{\sqcup}(A \sqcap A_\alpha)$.
(2) $A \sqcup (\underset{\alpha\in\Delta}{\sqcap} A_\alpha) = \underset{\alpha\in\Delta}{\sqcap}(A \sqcup A_\alpha)$.

Definition 1.4. [24] [*Neutrosophic Topology*] $\tau \subseteq \mathcal{N}(X)$ is called a *neutrosophic topology* for X if

(1) $0_X, 1_X \in \tau$.
(2) If $A_\alpha \in \tau$ for every $\alpha \in \Delta$, then $\underset{\alpha\in\Delta}{\sqcup} A_\alpha \in \tau$,
(3) For every $A, B \in \tau$, we have $A \sqcap B \in \tau$.

The ordered pair (X, τ) will be said a *neutrosophic space* over X. The elements of τ will be called *neutrosophic open sets*. For any $A \in \mathcal{N}(X)$, If $A^c \in \tau$, then we say A is *neutrosophic closed*.

2. Neutrosophic n-valued refined sets and topology

In neutrosophic n-valued refined logic (see [28]) the membership degree refined (split) into r values $\mu_1, \mu_2, ..., \mu_r$, the indetermancy refined into s values $\sigma_1, \sigma_2, ..., \sigma_s$ and the nonmebership refined into t values $\nu_1, \nu_2, ..., \nu_t$ such that $n = r + s + t$ and

$$^{-}0 \leq \sum_{i=1}^{r} \mu_i + \sum_{i=1}^{s} \sigma_i + \sum_{i=1}^{t} \nu_i \leq n^{+}$$

Some authors assumes that $r = s = t$ see for example [14]. Actually, there is no guarntee that the membership, intedermancy and nonmembership degrees refined or split into the same number of values, and we will not get a functional system of Neutrosophic n-valued refined sets if no more restrictions are assumed on r, s and t. This accurse when we define the basic set operations on the neutrosophic n-valued refined sets, especially when we try to define the neutrosophic n-valued refined complement of a given neutrosophic n-valued refined set; where r plays the role of t and vice versa. We will be back to this discussion after stating some definitions and theorems.

Definition 2.1. [26]: A is called a *neutrosophic n-valued refined set* on a universe X if $A = \{\langle x, \mu_A^1(x), \mu_A^2(x), ..., \mu_A^r(x); \sigma_A^1(x), \sigma_A^2(x), ..., \sigma_A^s(x); \nu_A^1(x), \nu_A^2(x), ..., \nu_A^t(x)\rangle; x \in X\}$; $\mu_A^i, \sigma_A^j, \nu_A^k : X \to]^{-}0, 1^{+}[$ for every $i = 1, ..., r$, $j = 1, ..., s$, $k = 1, ..., t$ such that $r + s + t = n$ and

$$^{-}0 \leq \sum_{i=1}^{r} \mu_A^i(x) + \sum_{j=1}^{s} \sigma_A^j + \sum_{k=1}^{t} \nu_A^k \leq n^{+}.$$

The class of all neutrosophic n-valued refined sets on the universe X will be denoted by $\mathcal{R}_n(X)$.

The following is the definition of the basic operations (inclusion, union, intersection and complement) on neutrosophic n-valued refined sets.

Definition 2.2. [*Neutrosophic n-valued refined sets operations*] Let $A, A_\alpha, B \in \mathcal{R}_n(X)$ such that $\alpha \in \Delta$. Then we define the neutrsophic n-valued refined:

(1) (*Inclusion*): $A \sqsubseteq_R B$ If $\mu_A^i(x) \leq \mu_B^i(x)$, $\sigma_A^j(x) \geq \sigma_B^j(x)$ and $\nu_A^k(x) \geq \nu_B^k(x)$ for every $i = 1, ..., r$, $j = 1, ..., s$, $k = 1, ..., t$.

(2) (*Equality*): $A = B \Leftrightarrow A \sqsubseteq_R B$ and $B \sqsubseteq_R A$.

(3) (*Intersection*) $\underset{\alpha \in \Delta}{\sqcap_R} A_\alpha(x) = \{\langle x, \underset{\alpha \in \Delta}{\wedge} \mu_{A_\alpha}^1(x), ..., \underset{\alpha \in \Delta}{\wedge} \mu_{A_\alpha}^r(x); \underset{\alpha \in \Delta}{\vee} \sigma_A^1(x), ..., \underset{\alpha \in \Delta}{\vee} \sigma_A^s(x); \underset{\alpha \in \Delta}{\vee} \nu_A^1(x), ..., \underset{\alpha \in \Delta}{\vee} \nu_A^t(x)\rangle; x \in X\}$.

(4) (*Union*) $\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\vee}\mu^1_{A_\alpha}(x), ..., \underset{\alpha\in\Delta}{\vee}\mu^r_{A_\alpha}(x); \underset{\alpha\in\Delta}{\wedge}\sigma^1_A(x), ..., \underset{\alpha\in\Delta}{\wedge}\sigma^s_A(x); \underset{\alpha\in\Delta}{\wedge}\nu^1_A(x), ..., \underset{\alpha\in\Delta}{\wedge}\nu^t_A(x)\rangle; x \in X\}$.

(5) (*Complement*) $A^c = \{\langle x, \nu^1_A(x), ..., \nu^t_A(x); 1-\sigma^1_A(x), ..., 1-\sigma^s_A(x); \mu^1_A(x), ..., \mu^r_A(x)\rangle; x \in X\}$

(6) (*Universal set*) $1_X = \{\langle x, 1, ..., 1; 0, ..., 0; 0, ..., 0\rangle; x \in X\}$; called the *neutrosophic n-valued refined universal set.*

(7) (*Empty set*) $0_X = \{\langle x, 0, ..., 0; 1, ..., 1; 1, ..., 1\rangle; x \in X\}$; called the *neutrosophic n-valued refined empty set.*

Theorem 2.3. *Let $A_\alpha, A, B \in \mathcal{R}_n(X)$ such that $\alpha \in \Delta$. Then we have*

(1) *If $A \sqsubseteq_R B \sqsubseteq_R C$, then $A \sqsubseteq_R C$.*

(2) *If $A \sqsubseteq_R B$, then $B^c \sqsubseteq_R A^c$.*

(3) $(\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha) \sqcap_R A = \underset{\alpha\in\Delta}{\sqcup_R}(A_\alpha \sqcap_R A)$

(4) $(\underset{\alpha\in\Delta}{\sqcap_R} A_\alpha) \sqcup_R A = \underset{\alpha\in\Delta}{\sqcap_R}(A_\alpha \sqcup_R A)$

[Demorgan's Laws]

(5) $(A \sqcup_R B)^c = A^c \sqcap_R B^c$

(6) $(A \sqcap_R B)^c = A^c \sqcup_R B^c$

Proof. (1) and (2) are Straight forward! (3) and (4) can be proved using the following two propositions:

$-\ (\underset{\alpha\in\Delta}{\vee} a_\alpha) \wedge b = \underset{\alpha\in\Delta}{\vee}(a_\alpha \wedge b)$

$-\ (\underset{\alpha\in\Delta}{\wedge} a_\alpha) \vee b = \underset{\alpha\in\Delta}{\wedge}(a_\alpha \vee b)$

Now, we prove (3) and (4) can be proved by duality:

$(A \sqcup_R B)^c = (\{\langle x, \mu^1_A(x) \vee \mu^1_B(x), ..., \mu^r_A(x) \vee \mu^r_B(x); \sigma^1_A(x) \wedge \sigma^1_B(x), ..., \sigma^s_A(x) \wedge \sigma^s_B(x); \nu^1_A(x) \wedge \nu^1_B(x), ..., \nu^t_A(x) \wedge \nu^t_A(x)\rangle; x \in X\})^c$

$= \{\langle x, \nu^1_A(x) \wedge \nu^1_B(x), ..., \nu^t_A(x) \wedge \nu^t_A(x); 1-(\sigma^1_A(x) \wedge \sigma^1_B(x)), ..., 1-(\sigma^s_A(x) \wedge \sigma^s_B(x)); \mu^1_A(x) \vee \mu^1_B(x), ..., \mu^r_A(x) \vee \mu^r_B(x)\rangle; x \in X\}$

$= \{\langle x, \nu^1_A(x) \wedge \nu^1_B(x), ..., \nu^t_A(x) \wedge \nu^t_A(x); (1-\sigma^1_A(x)) \vee (1-\sigma^1_B(x)), ..., (1-\sigma^s_A(x)) \vee (1-\sigma^s_B(x)); \mu^1_A(x) \vee \mu^1_B(x), ..., \mu^r_A(x) \vee \mu^r_B(x)\rangle; x \in X\}$

$= \{\langle x, \nu^1_A(x), ..., \nu^t_A(x); 1-\sigma^1_A(x), ..., 1-\sigma^s_A(x); \mu^1_A(x), ..., \mu^r_A(x)\rangle; x \in X\} \sqcap_R$
$\{\langle x, \nu^1_B(x), ..., \nu^t_A(x); 1-\sigma^1_B(x), ..., 1-\sigma^s_B(x); \mu^1_B(x), ..., \mu^r_B(x)\rangle; x \in X\} = A^c \sqcap_R B^c$ □

So, as the above theorem shows, the system defined in Definition 2.2 is rich to a certain extent, but it still needs to be stronger to deal with some situations: for example $A \sqcap_R A^c$ is not well-defined if $r \neq t$. The concept $True$ (membership) and $False$ (nonmembership) are related, it is reasonable to discuss them in any world simultaneously, so we can assume $r = t$, and this is what F. Smarandache did in [28] when he discussed the relative (absolute) truth and falsity simultaneously. The condition $r = s = t$ mentioned in [14] is very strong and will not add any value to us, actually it implies that n is divisible by 3, since $n = r + s + t$, so it does not include some worlds, for example a world of seven and five-valued logic which discussed in [28]. On the other hand if we, only, assume $r = t$, then n can be any value since we have not assumed any condition on s and worlds of any n-valued logic will be included.

Definition 2.4. : Let A be a *neutrosophic n-valued refined set* on a universe X. If $r = s$, then we call A a homogeneous neutrosophic n-valued refined set. n will be called the dimension of A, and r, s will be called the sub-dimensions of A. The class of all *homogeneous neutrosophic n-valued refined sets* on the universe X with sub-dimensions r, s will be denoted by $\mathcal{R}_{(n,r,s)}(X)$.

The following is obvious:

Proposition 2.5. *Let $A, B \in \mathcal{R}_{(n,r,s)}(X)$. Then*

(1) $A \sqcap_R B \in \mathcal{R}_{(n,r,s)}(X)$.
(2) $A \sqcup_R B \in \mathcal{R}_{(n,r,s)}(X)$.
(3) $A^c \in \mathcal{R}_{(n,r,s)}(X)$.

Example 2.6. Let $X = \{a, b\}$, and let $A, B \in \mathcal{R}_{(5,2,1)}(X)$ such that
$A = \{\langle a, 0.2, 0.1; 0.7; 0.1, 0.4\rangle, \langle b, 0.5, 0.3; 0.2; 0.9, 0.5\rangle\}$ and
$B = \{\langle a, 0.4, 0.01; 0.3; 0.4, 0.3\rangle, \langle b, 0.4, 0.2; 0.1; 0.7, 0.7\rangle\}$. Then we have:
$A \sqcap_R B = \{\langle a, 0.2, 0.01; 0.7; 0.4, 0.4\rangle, \langle b, 0.4, 0.2; 0.2; 0.9, 0.7\rangle\} \in \mathcal{R}_{(5,2,1)}$
$A \sqcup_R B = \{\langle a, 0.4, 0.1; 0.3; 0.1, 0.3\rangle, \langle b, 0.5, 0.3; 0.1; 0.7, 0.5\rangle\} \in \mathcal{R}_{(5,2,1)}$
$A^c = \{\langle a, 0.1, 0.4; 0.3; 0.2, 0.1\rangle, \langle b, 0.9, 0.5; 0.8; 0.5, 0.3\rangle\} \in \mathcal{R}_{(5,2,1)}$

Definition 2.7 (*Neutrosophic n-valued Refined Topology*)**.** $\tau \subset \mathcal{R}_{(n,r,s)}(X)$ is called a *neutrosophic n-valued refined topology* on X if

(1) $0_X, 1_X \in \tau$.
(2) For every $A, B \in \tau$, we have $A \sqcap_R B \in \tau$.
(3) If $A_\alpha \in \tau$ for every $\alpha \in \Delta$, then $\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha \in \tau$,

Elements of τ are called *neutrosophic n-valued refined open sets.* $A \in \mathcal{R}_{(n,r,s)}(X)$ is said *neutrosophic n-valued refined closed set* if $A^c \in \tau$.
The class of all neutrosophic n-valued refined topologies on X with sub-dimensions r, s will be denoted by $TOP_{(n,r,s)}(X)$.

Definition 2.8. Let $\tau \subseteq \mathcal{R}_{(n,r,s)}(X)$ be a neutrosophic n-valued refined topology on X and let $A \in \mathcal{R}_{(n,r,s)}(X)$. Then:

(1) The neutrosophic n-valued refined interior of A is defined to be
$Int_R(A) = \sqcup_R\{O \in \tau; O \sqsubseteq_R A\}$.

(2) The neutrosophic n-valued refined closure of A is defined to be
$Cl_R(A) = \sqcap_R\{C \in \mathcal{R}_{(n,r,s)}(X); C^c \in \tau \quad \text{and} \quad A \sqsubseteq_R C\}$

Example 2.9. Let $X = \{a, b\}$, and let $\tau = \{0_X, 1_X, A, B, C, D\} \subset \mathcal{R}_{(5,2,1)}(X)$ where
$A = \{\langle a, 0.2, 0.1; 0.7; 0.1, 0.4\rangle, \langle b, 0.5, 0.3; 0.2; 0.9, 0.5\rangle\}$,
$B = \{\langle a, 0.4, 0.01; 0.3; 0.4, 0.3\rangle, \langle b, 0.4, 0.2; 0.1; 0.7, 0.7\rangle\}$,
$C = \{\langle a, 0.2, 0.01; 0.7; 0.4, 0.4\rangle, \langle b, 0.4, 0.2; 0.2; 0.9, 0.7\rangle\}$
$D = \{\langle a, 0.4, 0.1; 0.3; 0.1, 0.3\rangle, \langle b, 0.5, 0.3; 0.1; 0.7, 0.5\rangle\}$
Then τ is a Neutrosophic 5-valued refined topology on X. All closed set are: $0_X, 1_X, A^c, B^c, C^c, D^c$ where
$A^c = \{\langle a, 0.1, 0.4; 0.3; 0.2, 0.1\rangle, \langle b, 0.9, 0.5; 0.8; 0.5, 0.3\rangle\}$
$B^c = \{\langle a, 0.4, 0.3; 0.7; 0.4, 0.01\rangle, \langle b, 0.7, 0.7; 0.9; 0.4, 0.2\rangle\}$,
$C^c = \{\langle a, 0.4, 0.4; 0.3; 0.2, 0.01\rangle, \langle b, 0.9, 0.7; 0.8; 0.4, 0.2\rangle\}$
$D^c = \{\langle a, 0.1, 0.3; 0.7; 0.4, 0.1\rangle, \langle b, 0.7, 0.5; 0.9; 0.5, 0.3\rangle\}$
Let $K = \{\langle a, 0.43, 0.09; 0.2; 0.1, 0.2\rangle, \langle b, 0.5, 0.25; 0.1; 0.5, 0.6\rangle\}$. Then the open sets in τ contained in K are only $0_X, B, C$, so that $Int_R(K) = 0_X \sqcup_R B \sqcup_R C = B$. Now; we consider the set $K^c = \{\langle a, 0.1, 0.2; 0.8; 0.43, 0.09\rangle, \langle b, 0.5, 0.6; 0.9; 0.5, 0.25\rangle\}$ and compute $Cl_R(K^c)$; the only closed sets containing K^c are 1_X,B^c and C^c, so that $Cl_R(K^c) = 1_X \sqcap_R B^c \sqcap_R C^c = B^c$. Which means $Cl_R(K^c) = B^c$ and so $(Cl_R(K^c))^c = B = Int_R(K)$; that is $Int_R(K) = (Cl_R(K^c))^c$ and this leads us to the following theorem:

Theorem 2.10. *Let (X, τ) be an n-valued refined topological space with sub-dimensions r, s and let $A \in \mathcal{R}_{(n,r,s)}(X)$. Then we have:*

(1) $Int_R(A) = (Cl_R(A^c))^c$
(2) $Cl_R(K) = (Int_R(K^c))^c$

Proof. Since $\vee$ and $\wedge$ has duality, we will, only, proof part (1).
Let $A = \{\langle x, \mu_A^1(x), ..., \mu_A^r(x); \sigma_A^1(x), ..., \sigma_A^s(x); \nu_A^1(x), ..., \nu_A^r(x)\rangle; x \in X\}$. Then
$A^c = \{\langle x, \nu_A^1(x), ..., \nu_A^r(x); 1 - \sigma_A^1(x), ..., 1 - \sigma_A^s(x); \mu_A^1(x), ..., \mu_A^r(x)\rangle; x \in X\}$, so
$Cl_R(A^c) = \sqcap_R\{C \in \mathcal{R}_{(n,r,s)}(X); C^c \in \tau \quad \text{and} \quad A^c \sqsubseteq_R C\}$. We apply Demorgan's Laws in Theorem 2.3 to get: $(Cl_R(A^c))^c = \sqcup_R\{C^c \in \mathcal{R}_{(n,r,s)}(X); C^c \in \tau \quad \text{and} \quad C^c \sqsubseteq_R A\} = \sqcup_R\{O \in \mathcal{R}_{(n,r,s)}(X); O \in \tau \quad \text{and} \quad O \sqsubseteq_R A\} = Int_R(A)$.

□

Theorem 2.11. *Let (X, τ) be an n-valued refined topological space with sub-dimensions r, s and let $A, B \in \mathcal{R}_{(n,r,s)}(X)$. Then we have:*

(1) *$Int_R(A) \sqsubseteq_R A$.*
(2) *If A is a neutrosophic n-valued refined open set, then $Int_R(A) = A$.*

(3) $Int_R(Int_R(A)) = Int_R(A)$.

(4) *If* $A \sqsubseteq_R B$, *then* $Int_R(A) \sqsubseteq_R Int_R(B)$.

(5) $Int_R(A \sqcap_R B) = Int_R(A) \sqcap_R Int_R(B)$

(6) $Int_R(A \sqcup_R B) \sqsupseteq_R Int_R(A) \sqcup_R Int_R(B)$

(7) $Int_R(\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha) \sqsupseteq_R \underset{\alpha\in\Delta}{\sqcup_R} Int_R(A_\alpha)$

(8) $A \sqsubseteq_R Cl_R(A)$.

(9) *If* A *is a neutrosophic* n*-valued refined closed set, then* $Cl_R(A) = A$.

(10) $Cl_R(Cl_R(A)) = Cl_R(A)$.

(11) *If* $A \sqsubseteq_R B$, *then* $Int_R(A) \sqsubseteq_R Int_R(B)$.

(12) $Cl_R(A \sqcup_R B) = Cl_R(A) \sqcup_R Cl_R(B)$

(13) $Cl_R(A \sqcap_R B) \sqsubseteq_R Cl_R(A) \sqcap_R Cl_R(B)$

(14) $Cl_R(\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha) \sqsupseteq_R \underset{\alpha\in\Delta}{\sqcup_R} Cl_R(A_\alpha)$

Proof. (1) Let $O \in \tau$ such that $O \sqsubseteq_R A$. Then for every $x \in X$ we have $\mu^i_O(x) \leq \mu^i_A(x)$ for every $i = 1, .., r$, $\sigma^i_O(x) \geq \sigma^i_A(x)$ for every $i = 1, .., s$ and $\nu^i_O(x) \geq \nu^i_A(x)$ for every $i = 1, .., r$, which implies that $\bigvee_{O\in\tau, O\sqsubseteq_R A} \mu^i_O(x) \leq \mu^i_A(x)$ for every $i = 1, .., r$, $\bigwedge_{O\in\tau, O\sqsubseteq_R A} \sigma^i_O(x) \geq \sigma^i_O(x)$ for every $i = 1, .., s$ and $\bigwedge_{O\in\tau, O\sqsubseteq A} \nu^i_O(x) \geq \nu^i_A(x)$ for every $i = 1, .., r$; that is $Int_R(A) \sqsubseteq A$.

(2) Since A is open, then, from the definition of $Int_R(A)$, we have $A \sqsubseteq_R Int_R(A)$, and from part (1) we have the converse, and we done.

(3) Since $Int_R(A)$ is a neutrosophic n-valued refined open set, we have (from part (2)) $Int_R(Int_R(A)) = Int_R(A)$.

(4) Let O be a *neutrosophic* n*-valued refined open* set such that $O \sqsubseteq_R A$. Then since $A \sqsubseteq_R B$, we have $O \sqsubseteq_R B$, that is $Int_R(A) \sqsubseteq_R Int_R(B)$

(5) From part (4) we have $Int_R(A \sqcap_R B) \sqsubseteq_R Int_R(A) \sqcap_R Int_R(B)$. On the other hand, $Int_R(A) \sqcap_R Int_R(B)$ is a neutrosophic n-valued refined open set contained in A and B, so that $Int_R(A) \sqcap_R Int_R(B) \sqsubseteq_R Int_R(A \sqcap_R B)$, and we done.

(6) Since $Int_R(A) \sqsubseteq_R A$ and $Int_R(B) \sqsubseteq_R B$, we have $Int_R(A) \sqcup_R Int_R(B)$ is a neutrosophic n-valued refined open set contained in $A \sqcup_R B$, which implies that $Int_R(A) \sqcup_R Int_R(B) \sqsubseteq_R Int_R(A \sqcup_R B)$.

(7) Since $A_\alpha \sqsubseteq_R \underset{\alpha\in\Delta}{\sqcup_R} A_\alpha$ for every $\alpha \in \Delta$, $Int_R(A_\alpha) \sqsubseteq_R Int_R(\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha)$ for every $\alpha \in \Delta$, that is $\underset{\alpha\in\Delta}{\sqcup_R} Int_R(A_\alpha) \sqsubseteq_R Int_R(\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha)$.

The remaining 5 parts can be proved by duality. □

Equality in parts (7) and (13) of Theorem 2.11 does not hold.

Example 2.12. Consider the neutrosophic 5-valued refined topological space (X, τ) defined in Example 2.9 and let $K = \{\langle a, 0, 1; 0; 1, 1\rangle, \langle b, 1, 1; 0; 0, 1\rangle\}$, and $L = \{\langle a, 1, 0; 1; 0, 0\rangle, \langle b, 0, 0; 1; 1, 0\rangle\}$. Then $K \sqcup_R L = \{\langle a, 1, 1; 0; 0, 0\rangle, \langle b, 1, 1; 0; 0, 0\rangle\} = 1_X$. So we have $Int_R(K \sqcup_R L) = 1_X$, and since K and L contains no neutrosophic n-valued refined open set except 0_X we have $Int_R(K) = Int_R(L) = 0_X$, which means $Int_R(K) \sqcup_R Int_R(L) = 0_X$, hence equality in parts (7) and (8) of Theorem 2.11 does not hold. For part (13) let $K = \{\langle a, 0.1, 0.4; 0.6; 0.5, 0.1\rangle, \langle b, 0.7, 0.5; 0.9; 0.5, 0.3\rangle\}$, $L = \{\langle a, 0.1, 0.3; 0.7; 0.3, 0.1\rangle, \langle b, 0.7, 0.5; 0.9; 0.5, 0.3\rangle\}$. Then
$K \sqcap_R L = \{\langle a, 0.1, 0.3; 0.7; 0.5, 0.1\rangle, \langle b, 0.7, 0.5; 0.9; 0.5, 0.3\rangle\}$. The only neutrosophic 5-valued Refined closed sets containing K are: $1_X, A^c$ and C^c, so that we have $Cl_R(K) = 1_X \sqcap_R A^c \sqcap_R C^c = A^c$. Again the only neutrosophic 5-valued Refined closed sets containing L are:1_X, A^c and C^c, so that we have $Cl_R(L) = 1_X \sqcap_R A^c \sqcap_R C^c = A^c$, and $Cl_R(K) \sqcap_R Cl_R(L) = A^c \sqcap_R A^c = A^c$, on the other hand the only neutrosophic 5-valued Refined closed sets containing $K \sqcap_R L$ are: $1_X, A^c, B^c$ and D^c, so that we have $Cl_R(K \sqcap_R L) = 1_X \sqcap_R A^c \sqcap_R B^c \sqcap_R D^c = D^c$. Note that D^c is a proper subset of A^c, so equality in Theorem 2.11 part (13) does not hold.

Question 2.13. *Is there a neutrosophic n-valued refined topological space (X, τ) shows that equality in part (14) of Theorem 2.11 does not hold.*

Definition 2.14 (*Neutrosophic n-valued refined pre-open and pre-closed sets*)**.** Let $\tau \in TOP_{(n,r,s)}(X)$ and $A \in \mathcal{R}_{(n,r,s)}(X)$. Then A is said to be:

(1) A *neutrosophic n-valued refined semi-open set*, if $A \sqsubseteq_R Cl_R(Int_R(A))$. The complement of a neutrosophic n-valued refined semi-open set is called a *neutrosophic n-valued refined semi-closed set.*

(2) A *neutrosophic n-valued refined pre-open set*, if $A \sqsubseteq_R Int_R(Cl_R(A))$. The complement of a neutrosophic n-valued refined pre-open set is called a *neutrosophic n-valued refined pre-closed set.*

(3) A *neutrosophic n-valued refined α-open set*, if $A \sqsubseteq_R Int_R(Cl_R(Int_R(A)))$. The complement of a neutrosophic n-valued refined α-open set is called a *neutrosophic n-valued refined α-closed set.*

(4) A *neutrosophic n-valued refined β-open set*, if $A \sqsubseteq_R Cl_R(Int_R(Cl_R(A)))$. The complement of a neutrosophic n-valued refined β-open set is called a *neutrosophic n-valued refined β-closed set.*

Theorem 2.15. *Let $\tau \in TOP_{(n,r,s)}(X)$ and $A \in \mathcal{R}_{(n,r,s)}(X)$. Then:*

(1) *Every* Neutrosophic n-valued refined open (closed) set*, is* neutrosophic n-valued refined α-open (closed) set.

(2) *Every* Neutrosophic n-valued refined α-open (α-closed) set*, is* neutrosophic n-valued refined pre-open (pre-closed) set *and* neutrosophic n-valued refined semi-open (semi-closed) set.

(3) *Every* Neutrosophic n-valued refined pre-open (pre-closed) or semi-open (semi-closed) set*, is a* neutrosophic n-valued refined β-open (β-closed) set.

Proof. (1) Let A be a *Neutrosophic n-valued refined open set*. Then, from Theorem 2.11 part (2) and (8), we have $Int_R(A) = A$ and $A \sqsubseteq_R Cl_R(A)$. So $Int_R(Cl_R(int_R(A))) \sqsupseteq_R Int_R(Cl_R(A)) \sqsupseteq_R Int_R(A) = A$. That is A is a *neutrosophic n-valued refined α-open set*. Now, suppose that A is a *Neutrosophic n-valued refined closed set*. Then A^c is a *Neutrosophic n-valued refined open set*, which implies A^c is a *neutrosophic n-valued refined α-open set*, and so A is a is a *neutrosophic n-valued refined α-closed set*.

(2) Obvious! we only use Theorem 2.11 part (1).

(3) Obvious! we only use Theorem 2.11 part (8) .

□

None of the above implications reverse. The following is an example of a neutrosophic 5-valued refined α-open set which is not open, and another example of a neutrosophic 5-valued refined *pre*-open (so it is β-open) set which is neither *semi*-open nor α-open.

Example 2.16. Consider $\tau = \{0_X, 1_X, A, B, C, D\}$ in Example 2.9 and let
$H = \{\langle a, 0.5, 0.1; 0.3; 0.1, 0.3\rangle, \langle b, 0.5, 0.3; 0.1; 0.7, 0.5\rangle\}$.
Then the neutrosophic 5-valued refined open sets contained in H are $0_X, A, B, C, D$; so we have $Int_R(H) = 0_X \sqcup_R A \sqcup_R B \sqcup_R C \sqcup_R D = D$, and since the only neutrosophic 5-valued refined close set containing D is 1_X, we have $Cl_R(Int_R(H)) = 1_X$, which implies $Int_R(Cl_R(int_R(H))) = 1_X$, hence $A \sqsubseteq_R Int_R(Cl_R(int_R(A)))$ and H is a neutrosophic 5-valued refined α-open set but not a neutrosophic 5-valued refined open set.
Consider, again, the set $K = \{\langle a, 0.1, 0.4; 0.6; 0.1, 0.3\rangle, \langle b, 0.9, 0.2; 0.4; 0.1, 0.5\rangle\}$. Since $\mu^1_K(a) < \mu^1_O(a)$ for every $O \in \tau - \{0_X\}$, we have the only Neutrosophic 5-valued refined open set contained in K is 0_X and $Int_R(K) = 0_X$, which implies $Cl_R(Int_R(K)) = 0_X$ and $Int_R(Cl_R(Int_R(K))) = 0_X$, so K is not a neutrosophic 5-valued refined semi-open nor α-open set; on the other hand, $\mu^1_K(b) > \mu^1_D(b)$ for every neutrosophic 5-valued refined closed set D in τ except for 1_X, that means $Cl_R(K) = 1_X$ and $int_R(Cl_R(A)) = 1_X$, hence $K \sqsubseteq_R Int_R(Cl_R(A))$ and K is a neutrosophic 5-valued refined pre-open set but not α-open. Since every neutrosophic 5-valued refined pre-open set is a neutrosophic 5-valued refined β-open set, K is, also, and example of a neutrosophic 5-valued refined β-open set which is not neutrosophic 5-valued refined semi-open.

Here we give an example of a a neutrosophic 5-valued refined *semi*-open (so it is β-open) set which is neither *pre*-open nor α-open.

Example 2.17. Let $X = \{a\}$, and let $\tau = \{0_X, 1_X, A, B\} \subset \mathcal{R}_{(5,2,1)}(X)$ where $A = \{\langle a, 0.2, 0.1; 0.7; 0.3, 0.4\rangle\}$, $B = \{\langle a, 0.3, 0.2; 0.5; 0.2, 0.3\rangle\}$. Since $A \sqcap_R B = A$ and $A \sqcup_R B = B$, τ is a neutrosophic 5-valued refined topology on X. The 5-valued refined closed sets in (X, τ) are: $0_X, 1_X, A^c, B^c$ where $A^c = \{\langle a, 0.3, 0.4; 0.3; 0.2, 0.1\rangle\}$ and $B^c = \{\langle a, 0.2, 0.3; 0.5; 0.3, 0.2\rangle\}$. Consider the neutrosophic 5-valued refined set $L = \{\langle a, 0.2, 0.2; 0.5; 0.3, 0.3\rangle\}$. Then the only neutrosophic 5-valued refined open sets contained in K are $0_X, A$, so that $Int_R(L) = 0_X \sqcup_R A = A$. To find $Cl_R(Int_R(L))$ we note that the neutrosophic 5-valued refined closed sets containing $Int_R(L)$ are $1_X, A^c, B^c$, so $Cl_R(Int_R(L)) = 1_x \sqcap_R A^c \sqcap_R B^c = B^c$, and since $L \sqsubseteq_R B^c$, L is a neutrosophic 5-valued refined semi-open sets. Now, we will show that L is not α-open. First note that the neutrosophic 5-valued refined open sets contained in $Cl_R(Int_R(K)) = B^c$ are 0_X and A, so we have $Int_R(Cl_R(Int_R(L))) = A$, and since L is not contained in A, L is not a neutrosophic α-open set.
We will show L is not a neutrosophic 5-valued refined *pre*-open set. The only neutrosophic 5-valued refined closed sets containing L are $1_X, A^c$ and B^c, so $Cl_R(L) = 1_X \sqcap_R A^c \sqcap_R B^c = B^c$, and since the neutrosophic 5-valued refined open sets contained in B^c are 0_X and A, we have $Int_R(Cl_R(L)) = A$ which not containing L, that is L is not a neutrosophic 5-valued refined pre-open set. So L is, also, an example of a neutrosophic 5-valued refined *semi*-open set which is not pre-open. And since every neutrosophic 5-valued refined *semi*-open set is β-open set, K is an example of a neutrosophic 5-valued refined β-open set which is not *pre*-open.

Finally we will give an example of a a neutrosophic 5-valued refined β-open set which is neither *pre*-open nor *semi*-open.

Example 2.18. Let (X, τ) as in Example 2.17 and consider the neutrosophic 5-valued refined set $M = \{\langle a, 0.2, 0.1; 0.9; 0.3, 0.5\rangle\}$. Then the only neutrosophic 5-valued refined open sets in τ contained in K is 0_X, so $Int_R(M) = 0_X$, which implies $Cl_R(Int_R(M)) = 0_X$, and since M is not contained in 0_X, we have M is not neutrosophic 5-valued refined semi-open set; on the other hand the neutrosophic 5-valued refined closed sets containing M are $1_X, A^c$ and B^c, so that $Cl_R(M) = B^c$, and since the only neutrosophic 5-valued refined open sets contained in B^c are 0_X and A we have $Int_R(Cl_R(M)) = A$. Since $Int_R(Cl_R(M)) = A$ and A does not contain M, we have M is not a neutrosophic 5-valued refined *pre*-open set. Now, to find $Cl_R(Int_R(Cl_R(M)))$ we note that the only neutrosophic 5-valued refined closed sets in τ

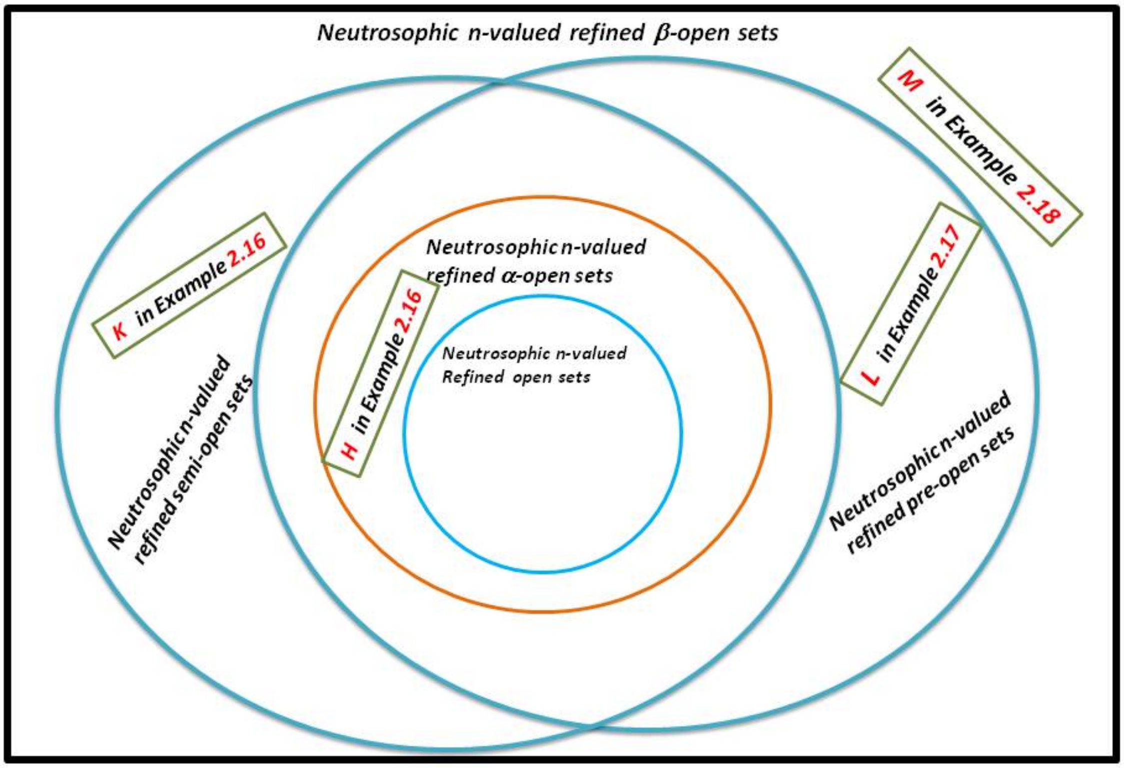

FIGURE 1. Relations between differen types of generalized neutrosophic n-valued refined open sets.

containing A are $1_X, A^c$ and B^c, so $Cl_R(Int_R(Cl_R(M))) = B^c$ which contains M, so M is a neutrosophic 5-valued refined β-open set but not *semi*-open nor *pre*-open.

The following diagram shows the relations between different types of generalized neutrosophic n-valued refined sets:

Theorem 2.19. *Let $\tau \in TOP_{(n,r,s)}(X)$ and $K \in \mathcal{R}_{(n,r,s)}(X)$. Then*

(1) *If there is a neutrosophic n-valued refined open set U such that $K \sqsubseteq_R U \sqsubseteq_R Cl_R(K)$, then K is a neutrosophic n-valued refined pre-open set.*

(2) *If there is a neutrosophic n-valued refined open set U such that $U \sqsubseteq_R K \sqsubseteq_R Cl_R(U)$, then K is a neutrosophic n-valued refined semi-open set.*

Proof. (1) $K \sqsubseteq_R U \sqsubseteq_R Int_R(Cl_R(U)) \sqsubseteq_R Int_R(Cl_R(Cl_R(K))) = Int_R(Cl_R(K))$.

(2) Since $Cl_R(Int_R(U)) = Cl_R(U)$ we have

$Cl_R(Int_R(K)) \sqsupseteq_R Cl_R(Int_R(U)) = Cl_R(U) \sqsupseteq_R K)$.

□

Theorem 2.20. *Let $\tau \in TOP_{(n,r,s)}(X)$ and $K \in \mathcal{R}_{(n,r,s)}(X)$. Then the union of any collection of neutrosophic n-valued refined α-open, β-open, pre-open or semi-open sets is a neutrosophic n-valued refined α-open, β-open, pre-open or semi-open set respectively.*

Proof. We will prove it for neutrosophic n-valued refined β-open sets, and the remaining parts can be proved in the same manner. Let A_γ be a neutrosophic n-valued refined β-open set for every $\gamma \in \Delta$. Then $A_\gamma \sqsubseteq_R Cl_R(int_R(Cl_R(A_\gamma)))$ for every $\gamma \in \Delta$. Then from parts (7) and (14) of Theorem 2.11 we have:

$$Cl_R(int_R(Cl_R(\sqcup_{R\,\gamma\in\Delta} A_\gamma))) \sqsupseteq_R Cl_R(int_R(\sqcup_{R\,\gamma\in\Delta} Cl_R(A_\gamma))) \sqsupseteq_R Cl_R(\sqcup_{R\,\gamma\in\Delta} int_R(Cl_R(A_\gamma))) \sqsupseteq_R \sqcup_{R\,\gamma\in\Delta} Cl_R(int_R(Cl_R(A_\gamma))) \sqsupseteq_R \sqcup_{R\,\gamma\in\Delta} A_\gamma$$ □

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Conflicts of Interest: The author declare that he has no conflict of interest.

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A Comparative Study of Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps on Covid Variants

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Abstract: The pandemic situation created by COVID'19 is ridiculous. It has made even the blood relations hide themselves from the infected person. The whole world was stunned by this situation. This is because of the uncertainty in the way in which this disease is spread. As an advancement of this disease, a few other variants like delta, omicron etc. also got spread. It is essential to find a solution to this situation . The variants Omicron and Delta are taken into consideration here. Though both the vibrant colours look alike, the symptoms and prevention methods changes for each of these vibrants. This work aims to make a study of the parameters responsible for these variants. As a result of this study, the parameters involved in the spread of these diseases are identified, and the prevention parameters are concluded. The major benefit of this comparatively study is to identify the parameters that are inconclusive, applying the concepts of fuzzy cognitive maps and neutrosophic cognitive maps is applied to bring out the result.

Keywords: Fuzzy set, Fuzzy graph, Neutrosophic Cognitive Map, Fuzzy Cognitive Map.

1 Introduction

A new viral infection, COVID-19 (Coronavirus Disease 2019), emerged in early 2020 and attracted widespread attention. The virus spread around the world at a very high speed, and many studies have been carried out. Examining different epidemic patterns of COVID-19 based on official data.

Nowadays, the development of the Corona virus forms a lot of variants, such as beta, gamma, delta, and omicron, etc.,. In the Corona variants, especially the Delta variant causes, the more deaths among the population which have different symptoms when compared with the initial form of Corona. Recently, the Corona variant, Omicron spreads all over the world and has a different symptoms, prevention methods etc.,. In the medical field, the experts have a different opinions on the diseases with respect to prevention, symptoms, causes etc., even though the vaccinated people are getting affected, which leads to fear among the population.

The applications of FCM and NCM in the medical field are with respect to the knowledge base, and data base of patient, diagnosis which is to recognise symptoms and signs, the other method of diagnosing gallstones is through ultrasound and radiation, knowledge acquisition and to find the the possibility of problems with indeterminate cases in which fuzzy logic plays an important role. In that it played a great role in the invention of the Doctor Moon.

Al-Subhi et al. have suggested a decision-making model in project management. In this process, FCM and NCM technique in bringing out the decision on effective implementation of projects, Bertolini, M. has used a FCM algorithm in finding the important factors that affects human reliability. A food-processing An industrial plants have been considered for this decision-making algorithm. Jantzen, J. et al. dedicated their work in the process industry. Fuzzy controllers have been applied in identifying the predictive control in the cement industry. Kalaichelvi. A. and Gomathy, L. have studied the problems faced by girl students who got married during the period of study using NCM. The study is based on the responses received from the graduate students of Coimbatore city.

-Khatua, D. et al. have presented fuzzy dynamical system-based granular differentiability in identifying an optimal control model for COVID-19. The fuzzy SEIAHRD model described by them proposes a disease control procedure for the disease specified. Martin, N. et al. have developed a methodology that helps to risk factors of Lifestyle Diseases. Decagonal Linguistic Neutrosophic Fuzzy cognitive map is applied in the analysis. Mary. M.F.J. et al. aim to identify the factors affecting the quality of the training of elementary education teachers in Tamil Nadu. Various factors like techno-pedagogic skills, the students' academic skills, teaching competencies, etc. are analyzed applying FCM and NCM. Montazemi, A.R. et al. utilised cognitive maps in the design and development of intelligent information systems. Causal mapping is used to investigate the cognition of decision-makers. Papageorgiou. E.I. et al introduced the concept reduction approach in decision making and management. FCM is applied in modelling solid waste management systems.

Pramanik, S et al made an analysis of the problems faced by the construction workers with the help of NCM. The analysis has been performed with the list of issues given by the workers of West Bengal. Raich.V.V., et al performed their study by pointing out the qualities of an effective teacher. Fuzzy relational maps the concept of the Teacher Quality Index has been put in a place to bring the results. Ramalingam, S. et al. made an mathematical analysis of COVID-19 based on the symptoms of the disease. FCM and NCM concepts are applied in finding out the conclusion. Schuh. C introduced fuzzy set theory in medical sciences on three concrete medical fuzzy applications. Stylios, C.D. et al. discussed knowledge sharing, modelling methodology, knowledge-based reasoning with the help of FCM. Their study has provided effective results in identifying the knowledge-based methodologies. Vasantha, et al. in performed a search in in order to overcome the hindrance posed by complicated nature of psychological or social data. The search is based on imaginative play in children, applying the concepts of NCM. Visalakshi, V. et al. performed a survey on women to identify their entrepreneurial mindset. Combined Effective Time Dependent Data Matrix, and Average Time Dependent Data Matrix concepts applied in extracting the suggestions on pointing out the factors that affect entrepreneurship. William, M.A. et al. analysed the risk factors on women getting affected by breast cancer, making use of the NCM and FCM.

Kumaravel,S.K. et al. and Murugesan, R. et al. discussed the effectiveness of online classes considering the opinion of faculty and students during the COVID pandemic. The fuzzy models, like combined effective time-dependent matrix (CETD), average time-dependent data matrix (ATD), and refined time-dependent data matrix (RTD) are applied in their work using the fuzzy matrix theory. Devi, R.N. and Muthumari, G. have expressed a view on the properties of distance measure in P-F graph and applied Neutrosophic overset in real life scenarios for a decision making problem. They also introduced neutrosophic over topologized dominance graphs in their work. Recently, they have discussed various types of energy in Nover Top Graphs.

In this paper, a comparative study is made on different parameters related to omicron and delta such as travelling history, Prevention measures for the disease, Blood pressure, Cancer patient, Loss of taste and smell, Brain fog, etc. The parameters are analysed by taking any one as ON state by the concept of FCM. Simultaneously, those parameters are analysed through the NCM, as like FCM, by considering the same state as ON state. The aim of this paper is to analyze the COVID through comparison between FCM and NCM among the COVID variant parameters.

2 Parameters of omicron & delta virus

In order to analyse the parameters of The Omicron and Delta viruses, data were collected from the medical experts. Based on their opinions the following factors were identified and collected .

- C_1-Travelling history
- C_2-Prevention measures for the disease
- C_3-Maintain social distance, Wearing mask and Continuous hand wash
- C_4-Fever, cough and difficulty breathing
- C_5-Brain fog
- C_6- the possibility of delta variant
- C_7-Blood pressure, Cancer patient, diabetes, older age who violate precautions
- C_8-High risk of getting omicron and delta variant
- C_9- the possibility of omicron
- C_{10}-Loss of taste and smell
- C_{11}-No symptoms
- C_{12}-Spread more easily

3 Fuzzy cognitive map

The experts opinions were collected and based on their opinion, they formed a graph by mapping between the parameters. In which, based on the fuzzy cognitive map concept, weight age was assigned by the casual relation between the nodes. i.e., the edge weight was assigned as 1 (positive causality between the nodes), if the relationship between the nodes had a majority of respondents, but at the same time the majority of respondents was uncertain then it is denoted by 0. The corresponding fuzzy cognitive maps for the parameters are given below in figure 1

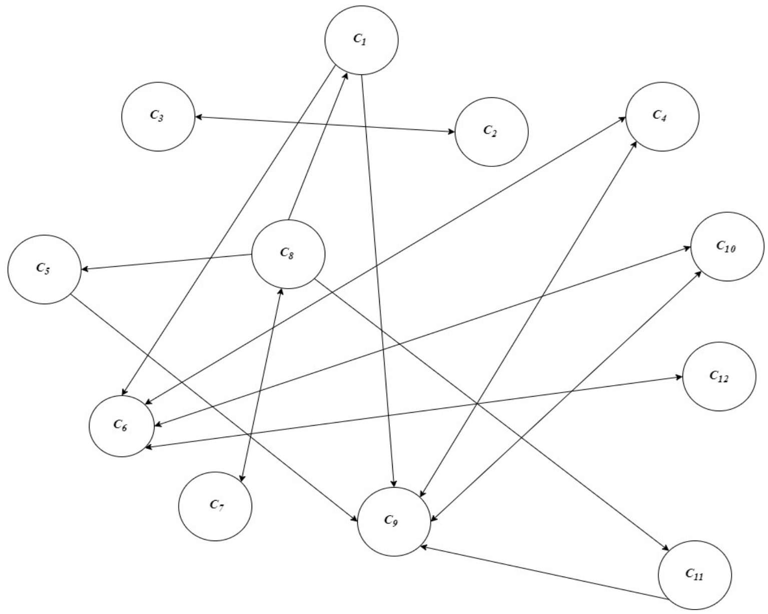

Figure 1: Fuzzy Cognitive Maps on COVID VARIANTS

3.1 Matrix Representation of FCM

The matrix representation of the fuzzy cognitive map is designed based on the connectives between the nodes, which are the possibilities among the parameters. The entries of the matrix are noted as either 0 or 1. The number 0 denoted an unconnected node, and 1 represented the connection between the nodes. The adjacent matrix of figure 1 is given below.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} \mathrm{C}_1 & \mathrm{C}_2 & \mathrm{C}_3 & \mathrm{C}_4 & \mathrm{C}_5 & \mathrm{C}_6 & \mathrm{C}_7 & \mathrm{C}_8 & \mathrm{C}_9 & \mathrm{C}_{10} & \mathrm{C}_{11} & \mathrm{C}_{12} \end{array} \\
\mathbf{E} = \begin{array}{c} \mathrm{C}_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

3.2 Iteration Process of FCM

Case-1: the possibility of delta variant - ON **state**

Let us consider the C_6 parameter as ON state. i.e., the possibility of delta variant for the iteration process.

The initial matrix required for the process is taken as below, which has the entries as 0 for the OFF state and 1 for ON state.

$$
\begin{aligned}
A_1 &= [\,0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\,] \\
A_1 * E &= [\,0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\,] \\
&\rightarrow [\,0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\,] \\
&= A_2 \\
A_2 * E &= [\,0\ 0\ 0\ 1\ 0\ 3\ 0\ 0\ 2\ 1\ 0\ 1\,] \\
&\rightarrow [\,0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\,] \\
&= A_3 \\
A_3 * E &= [\,0\ 0\ 0\ 2\ 0\ 3\ 0\ 0\ 2\ 2\ 0\ 1\,] \\
&\rightarrow [\,0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\,] \\
&= A_4 \\
\therefore\ A_3 &= A_4
\end{aligned}
$$

The last two iterations' values are obtained as the same, so that the iteration process may stop and it shows that when the C_6 parameter is taken as ON state then the parameters $C_4,\ C_9,\ C_{10}$, and C_{12} are obtained as ON state. It concludes that the parameters fever, cough, difficulty breathing, loss of taste, and smell are the risk factor for the parameter C_6, i.e., the the possibility of a delta variant as well as for omicron variant. In general, delta variant's spread more easily and faster. Its symptoms are loss of smell and taste, and which might cause omicron.

Case-2:High risk of getting omicron and delta variant- ON state

Let us take the C_8 parameter as ON state. i.e., high risk of getting Omicron and Delta for the iteration process. The initial matrix required for the process is taken as below, which have the entries as 0 for the OFF state and 1 for ON state.

$$
\begin{aligned}
A_1 &= [\,0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\,] \\
A_1 * E &= [\,1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\,] \\
&\rightarrow [\,1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\,] \\
&= A_2 \\
A_2 * E &= [\,1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\,] \\
&\rightarrow [\,1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\,] \\
&= A_3 \\
A_3 * E &= [\,1\ 0\ 0\ 2\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 1\,] \\
&\rightarrow [\,1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\,] \\
&= A_4
\end{aligned}
$$

$$\begin{aligned} A_4 * E &= \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 1 & 1 & 4 & 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right] \\ &\rightarrow \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \\ &= A_5 \\ \therefore\ A_4 &= A_5 \end{aligned}$$

The last two iteration values are obtained as the same, so that the iteration process may stop, and it shows that when the C_8 parameter is taken as ON state then the parameters C_2 and C_3 are obtained as ON states. It concludes that the parameters Loss of taste and smell, No symptoms, spreads more easily are the factors related to the parameter C_8. Also, the persons who are violating C_2 and C_3 parameters, then it causes a the possibility of omicron and delta. Those parameters are treated as important factors to prevent from the COVID Variants. .

Case-3: Prevention measures for the disease -ON state

For the iteration process, the parameter C_2 is considered in the ON state. i.e., prevention measures of the diseases. The initial matrix required for the process is taken as below, which has the entries as 0 for the OFF state and 1 for ON state.

$$\begin{aligned} A_1 &= \left[\begin{array}{cccccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ A_1 * E &= \left[\begin{array}{cccccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ &\rightarrow \left[\begin{array}{cccccccccccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ &= A_2 \\ A_2 * E &= \left[\begin{array}{cccccccccccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ &\rightarrow \left[\begin{array}{cccccccccccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ &= A_3 \\ \therefore\ A_2 &= A_3 \end{aligned}$$

The iteration process may be stopped when the current and previous iteration seems as same. The above iteration process shows that when the C_2 parameter is taken as ON state then expect the parameters C_3 are obtained as OFF state. It concludes that the parameter travelling history is more related to the parameter C_2, so that avoiding travelling from one place to another via public transport or independently is one of the main prevention measures from the effects of COVID variants.

Case-4: the possibility of having Omicron Virus- ON state

For the iteration process, the parameter C_9 i.e., the possibility of having Omicron Virus is considered as ON state and the rest of the parameters are taken as OFF state which is denotes as initial matrix A_1. The initial matrix required for the process is taken as below, which has the entries of 0 for the OFF state and 1 for the ON state.

$$A_1 = \left[\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$A_1 * E = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$
$$= A_2$$
$$A_2 * E = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$
$$= A_3$$
$$A_3 * E = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 1 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$
$$= A_4$$
$$A_4 * E = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 2 & 2 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$
$$= A_5$$
$$\therefore A_4 = A_5$$

The above iteration process shows that when the C_9 parameter is taken as ON state then the parameters $C_4,\ C_6,\ C_{10}$, and C_{12} are obtained as ON states. It concludes that the possibility of the Omicron virus having major symptoms such as loss of taste and smell, Cough, fever, difficulty breathing, etc., leads to the spread pf Omicron virus more easily. So that based on the clarity of factors, one can prevent themselves from diseases.

Case-5: Spread more easily -ON state

Let us take the C_{12} parameter, i.e., spread more easily as ON state. For the iteration process and rest of the parameters are taken as OFF state which denotes initial matrix A_1.The initial matrix required for the process is taken as below, which has the entries as 0 for the OFF state and 1 for ON state.

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 * E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= A_2$$
$$A_2 * E = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$
$$= A_3$$

$$A_3 * E = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 2 & 1 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$
$$= A_4$$
$$A_4 * E = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 2 & 2 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$
$$= A_5$$
$$\therefore\ A_4 = A_5$$

The above iteration process shows that when the C_{12} parameter is taken as ON state then the parameters $C_4,\ C_6,\ C_9$, and C_{10} are obtained as ON state. It concludes that the parameter C_4,is the main factor of possibility of omicron delta also those factors spread the CORONA variants more easily from one person to another and it causes loss of smell and taste.

4 Neutrosophic cognitive map

The experts opinions were collected, and based on their opinions, a graph was formed by mapping between the parameters. In which, based on the Neutrosophic cognitive map concept, weight age was assigned by the casual relation between the nodes. i.e., the edge weight was assigned as 1 (positive causality between the nodes), if the relationship between the nodes had majority of respondents, at the same time the respondents which are uncertain or indeterminate then it is denoted by I. The number zero is assigned, when there is no relationship between the parameters based on experts opinion. In the neutrosophic graph, for the indeterminate case the edges between the nodes is drawn by the dotted lines. The corresponding neutrosophic cognitive maps for the parameters is given below in figure 2

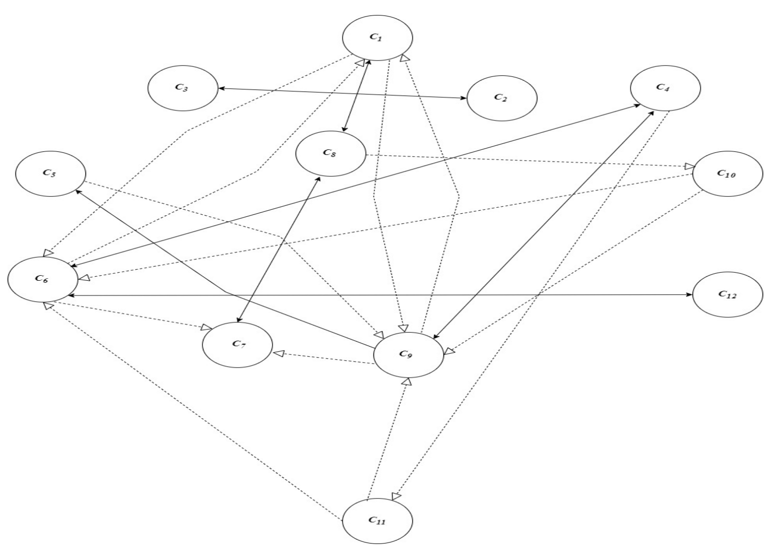

Figure 2: Neutrosophic Cognitive Maps on COVID VARIANTS

4.1 Matrix representation of NCM

The matrix representation of the neutrosophic cognitive map is designed based on the connectives between the nodes, which are possibilities among the parameters. In addition to the FCM concept, here one more case is occur, when there is an inconclusive possibility of relationship between the parameters which is denoted as indeterminate I case. The entries of the matrix are noted as 0 or 1 or I. The number 0 denotes for unconnected, 1 represents the connection between the nodes and I noted for the indeterminate case between the nodes which is connected by the dotted lines. The adjacent matrix of figure 2 is given below.

$$\mathbf{E} = \begin{array}{c} \\ \mathrm{C}_1 \\ \mathrm{C}_2 \\ \mathrm{C}_3 \\ \mathrm{C}_4 \\ \mathrm{C}_5 \\ \mathrm{C}_6 \\ \mathrm{C}_7 \\ \mathrm{C}_8 \\ \mathrm{C}_9 \\ \mathrm{C}_{10} \\ \mathrm{C}_{11} \\ \mathrm{C}_{12} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} \mathrm{C}_1 & \mathrm{C}_2 & \mathrm{C}_3 & \mathrm{C}_4 & \mathrm{C}_5 & \mathrm{C}_6 & \mathrm{C}_7 & \mathrm{C}_8 & \mathrm{C}_9 & \mathrm{C}_{10} & \mathrm{C}_{11} & \mathrm{C}_{12} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 & \mathbf{1} & \mathbf{I} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 \\ \mathbf{I} & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{I} & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{I} & 0 & 0 \\ \mathbf{I} & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 & 0 & \mathbf{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 & 0 & \mathbf{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

4.2 Iteration Process of NCM

Case-1: the possibility of delta variant -ON state

Let us consider the C_6 parameter as being in the ON state. i.e., the possibility of a delta variant for the iteration process. The initial matrix required for the process is taken as below, which has the entries as 0 for the OFF state and 1 for ON state. While comparing with the FCM iteration process, here in the each step of iteration I may observe based on choosing of parameter as ON or OFF state.

$$\begin{aligned} B_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ B_1 * E &= \begin{bmatrix} I & 0 & 0 & 1 & 0 & 0 & I & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} I & 0 & 0 & 1 & 0 & 1 & I & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= B_2 \\ B_2 * E &= \begin{bmatrix} I & 0 & 0 & 1 & 0 & I^2+1 & I & 2I & I^2+1 & 0 & I & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} I & 0 & 0 & 1 & 0 & 1 & I & I & 1 & 0 & I & 1 \end{bmatrix} \\ &= B_3 \\ B_3 * E &= \begin{bmatrix} 3I & 0 & 0 & 2 & 1 & 2I^2+1 & 3I & 2I & 2I^2+1 & I^2 & I & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} I & 0 & 0 & 1 & 1 & 1 & I & I & 1 & I & I & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= B_4 \\
B_4 * E &= \begin{bmatrix} 3I & 0 & 0 & 2 & 1 & 3I^2+1 & 3I & 2I & 3I^2 = I+1 & I^2 & I & 1 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} I & 0 & 0 & 1 & 1 & 1 & I & I & 1 & I & I & 1 \end{bmatrix} \\
&= B_5 \\
\therefore\ B_4 &= B_5
\end{aligned}
$$

The last two iterations values are obtained as the same, so that the iteration process may be stopped and it shows that when the C_6 parameter is taken as ON state then the parameters $C_4,\ C_5,\ C_6,\ C_9$, and C_{12} are obtained as ON state and the states $C_1,\ C_7,\ C_8,\ C_{10}$, and C_{11} are obtained as I. It concludes that the parameters spread easily, the patient with the symptoms of fever,cough, breathing problem, but it may be cause of omicron if the person have the symptom of brain fog except prevention.

Case-2: High risk of getting omicron and delta - ON state

Let us take the C_8 parameter as ON state. i.e., high risk of getting Omicron and Delta for the iteration process. The initial matrix required for the process is taken as below, which has the entries as 0 for the OFF state and 1 for ON state.

$$
\begin{aligned}
B_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
B_1 * E &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & I & 0 & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & I & 0 & 0 \end{bmatrix} \\
&= B_2 \\
B_2 * E &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & I+I^2 & 1 & 2 & I+I^2 & I & 0 & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & I & 1 & 1 & I & I & 0 & 0 \end{bmatrix} \\
&= B_3 \\
B_3 * E &= \begin{bmatrix} 1+I & 0 & 0 & I & I & 2I+I^2 & 1+I & 3 & 2I+I^2 & I & 0 & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 0 & I & I & I & 1 & 1 & I & I & 0 & 0 \end{bmatrix} \\
&= B_4 \\
B_4 * E &= \begin{bmatrix} 1+I & 0 & 0 & I & I & 3I+I^2 & 1+I & 3 & 3I+2I^2 & I & I^2 & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 0 & I & I & I & 1 & 1 & I & I & I & 0 \end{bmatrix} \\
&= B_5 \\
B_5 * E &= \begin{bmatrix} 1+I & 0 & 0 & I & I & 4I+I^2 & 1+I & 3 & 3I+2I^2 & I & I^2 & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 0 & I & I & I & 1 & 1 & I & I & I & 0 \end{bmatrix} \\
&= B_6 \\
\therefore B_5 &= B_6
\end{aligned}
$$

The iteration process may be stopped when the current and previous iterations seem as the same. The above iteration process shows that when the C_7 parameter is taken as ON state then expect the parameters C_2, C_3, and C_{12} to be obtained as ON state, and the rest of parameters shows as I or ON state. It concludes that the parameter C_7 i.e., persons with blood pressure, cancer, diabetes, and the person who travelled from one country to another country have a high risk factor for the diseases. Also, a few of the parameters are indeterminate cases.

Case-3: Prevention measures of the diseases - ON state

For the iteration process, the parameter C_2 i.e., prevention measures for the diseases is considered as ON state. The initial matrix required for the process is taken as below, which has the entries as 0 for the OFF state and 1 for ON state. While comparing with the FCM iteration process, here in each step of iteration I may occur based on the choice of parameter as ON or OFF state.

$$\begin{aligned}
B_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
B_1 * E &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&\to \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= B_2 \\
B_2 * E &= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&\to \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= B_3 \\
\therefore B_2 &= B_3
\end{aligned}$$

The iteration process may be stopped when the current and previous iterations seem as the same. The above iteration process shows that when the C_2 parameter is taken as ON state, then expect the parameters C_3 to be obtained as OFF state. It concludes that maintaining social distance and usuage of hand sanitizer and wearing mask are the prevention measures for the COVID variants.

Case-4: the possibility of Omicron - ON state.

For the iteration process, the parameter C_9 i.e., the possibility of Omicron is considered as ON state. The initial matrix required for the process is shown below.

$$\begin{aligned}
B_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\
B_1 * E &= \begin{bmatrix} I & 0 & 0 & 1 & 1 & 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&\to \begin{bmatrix} I & 0 & 0 & 1 & 1 & 0 & I & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\
&= B_2 \\
B_2 * E &= \begin{bmatrix} I & 0 & 0 & 1 & 1 & I^2+1 & I & 2I & I^2+I+1 & 0 & I & 0 \end{bmatrix} \\
&\to \begin{bmatrix} I & 0 & 0 & 1 & 1 & 1 & I & I & 1 & 0 & I & 0 \end{bmatrix} \\
&= B_3 \\
B_3 * E &= \begin{bmatrix} 3I & 0 & 0 & 2 & 1 & 2I^2+1 & 3I & 2I & I^2+2I+1 & I^2 & I & 1 \end{bmatrix} \\
&\to \begin{bmatrix} I & 0 & 0 & 1 & 1 & 1 & I & I & 1 & I & I & 1 \end{bmatrix} \\
&= B_4 \\
B_4 * E &= \begin{bmatrix} 3I & 0 & 0 & 2 & 1 & 3I^2+2 & 3I & 2I & I^2+3I+1 & I^2 & I & 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&\rightarrow \begin{bmatrix} I & 0 & 0 & 1 & 1 & 1 & I & I & 1 & I & I & 1 \end{bmatrix} \\
&= B_5 \\
\therefore\ B_4 &= B_5
\end{aligned}$$

The iteration process may be stopped when the current and previous iterations seem as the same. The above iteration process shows that the person affected by omicron has brain fog and common symptoms like fever, cough, and breathing problems. It also spread easily from one infected person to another. It may cause delta variant because the symptoms are more similar. The parameter C_1, i.e., travelling history, seems to be as indeterminate case.

Case-5: Spread more easily - ON state

For the iteration process, the parameter C_{12} i.e., spread more easily, is considered as ON state. The initial matrix required for the process is shown below.

$$\begin{aligned}
B_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
B_1 * E &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= B_2 \\
B_2 * E &= \begin{bmatrix} I & 0 & 0 & 1 & 0 & 1 & I & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} I & 0 & 0 & 1 & 0 & 1 & I & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= B_3 \\
B_3 * E &= \begin{bmatrix} I & 0 & 0 & 1 & 0 & I^2+2 & I & 2I & I^2+1 & 0 & I & 1 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} I & 0 & 0 & 1 & 0 & 1 & I & I & 1 & 0 & I & 1 \end{bmatrix} \\
&= B_4 \\
B_4 * E &= \begin{bmatrix} 3I & 0 & 0 & 2 & 1 & I^2+3 & 2I & 2I & 2I^2+1 & I^2 & I & 1 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} I & 0 & 0 & 1 & 1 & 1 & I & I & 1 & I & I & 1 \end{bmatrix} \\
&= B_5 \\
\therefore B_5 &= B_6
\end{aligned}$$

The above iteration process shows that when the C_{12} parameter is taken as ON state then expect that the parameters C_2 and C_3 are obtained as ON or I state. It means that Omicron and Delta variants are spread more easily and faster, when the persons affected by fever, brain fog, cough, and difficulty breathing.

5 Comparison and discussion

Table 1: Comparison Results of FCM and NCM on COVID Variants

On State	FCM Iteration	NCM Iteration	Comparison Remarks
C_6	$A_3 = A_4$; [0 0 0 1 0 1 0 0 1 1 0 1]	$B_4 = B_5$; [I 0 0 1 1 1 I I 1 I I 1]	Changes in Parameters
C_8	$A_4 = A_5$; [1 0 0 1 1 1 1 1 1 1 1 1]	$B_5 = B_6$; [1 0 0 I I I 1 1 I I I 0]	Changes in Parameters
C_2	$A_2 = A_3$; [0 1 1 0 0 0 0 0 0 0 0 0]	$B_2 = B_3$; [0 1 1 0 0 0 0 0 0 0 0 0]	No Changes
C_9	$A_4 = A_5$; [0 0 0 1 0 1 0 0 1 1 0 1]	$B_4 = B_5$; [I 0 0 1 1 1 I I 1 I I 1]	Changes in Parameters
C_{12}	$A_4 = A_5$; [0 0 0 1 0 1 0 0 1 1 0 1]	$B_5 = B_6$; [I 0 0 1 1 1 I I 1 I I 1]	Changes in Parameters

The parameters related to the COVID variants such as Omicron and Delta are considered as C_1, C_2, C_3, $\cdots$, C_{12}. The FCM and NCM among the parameters are designed based on the experts opinions. The adjacent matrix of fuzzy cognitive maps and neutrosophic cognitive maps is evaluated and it is used for the iteration process. The comparison is made between the FCM and NCM, by considering any one state as ON state commonly.From which we have obtained the following results.

First we are taking 6^{th} parameter as on state (i.e.,) Using the possibility of delta variant (C_6) as on state,In our comparison of FCM and NCM we are getting that it spreads more easily and the symptoms are fever, cough, difficulty breathing, but it may be the the possibility of getting omicron with brain fog.In next case, we are analyzing the high risk factors of diseases, the result shows that persons with blood pressure, cancer patient, diabetes, older age who are violating C_3.Prevention measures include maintaining social distance,wearing mask, often wash our hand. The possibility of omicron FCM shows that one of the important symptoms of omicron is brain fog in off state but while we are analysing by the NCM method, brain fog in on state.while we are taking C_{12} in state of on, In FCM it shows some less parameter in on state but in NCM there are some indeterminate state like depending on our travelling history and also this diseases spreads more easily for the persons in the 7th parameter.

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A study on n-HyperSpherical Neutrosophic matrices

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Abstract. The n-HyperSpherical Neutrosophic matrices (n-HSNMs), an extension of the Spherical Neutrosophic matrices, are proposed in this study. We investigate the basic properties of n-HSNMs and compare the idea n-HSNMs with spherical fuzzy matrices. Then, it looks at the characteristics of specific mathematical operations, like max-min, algebraic product, min-max, algebraic sum, and complement. Additionally, scalar multiplication ($n\mathcal{S}$)and exponentiation ($\mathcal{S}^n$)operations of an n-HSNM $\mathcal{S}$ are created and their advantageous properties are illustrated using algebraic operations. Then, we present a new operation (@) on n-HyperSpherical Neutrosophic matrices and look at the distributional rules that result from combining the operations (⊕, ⊗, ∧, and ∨).

Keywords: Neutrosophic sets; Spherical fuzzy matrix; n-HyperSpherical Neutrosophic matrix; Algebraic sum and product.

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1. Introduction

Khan et al. [5] and Im et al. [4] both established the notion of an intuitionistic fuzzy matrix (IFM) to broaden the idea of Thomason's [11] fuzzy matrix. Every element in an IFM is represented by $\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle$ along with $\mu_{a_{ij}}, \nu_{a_{ij}} \in [0, 1]$ and also $0 \leq \mu_{a_{ij}} + \nu_{a_{ij}} \leq 1$. As the introduction of IFM, a limited number of analysts have made major contributions to the advancement of the IFM hypothesis [3, 7, 8]. In such a situation, IFM fails to produce a reasonable solution. In order to deal with this situation, we invented the notion of Pythagorean fuzzy matrices (PyFM) by assigning membership degrees such as $\zeta_{a_{ij}}$ and non-membership degrees such as $\zeta_{a_{ij}}$, with the requirement that $0 \leq \zeta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$.

The design of picture fuzzy matrices (PFM) by Dogra and Pal cite1 is well-known, however decision makers are limited in assigning values because to the conditions on $\eta_{a_{ij}}$, $\zeta_{a_{ij}}$, and $\delta_{a_{ij}}$. In [9], certain algebraic procedures on Picture fuzzy matrices are defined, as well as their desired features. Occasionally, the total of their membership degrees is more than 1. In such a case, PFM fails to produce a plausible result. To illustrate this dilemma, we'll use an example that is both provisional and contradictory to membership degrees. $0.2, 0.6$ and 0.6, respectively, are the choices. This is satisfying in the circumstance where their total is more than 1, and PFM fails to handle such data. In order to deal with such situations, the authors [10] developed a new structure of Spherical fuzzy matrices (SFMs), which increase the degree memberships $\eta_{a_{ij}}$, $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$ to a size that is somewhat larger than image fuzzy matrices. In SFM, the degree memberships are fulfilling the follows: $0 \leq \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$ $(n \geq 1)$.

Matrixes have a vital role in science and technology, as we all know. However, in some cases, the conventional matrix theory fails to answer problems with uncertainties that arise in an uncertain environment. In [6], fuzzy and neutrosophic relational maps were presented. Square Neutrosophic Fuzzy Matrices with elements of $a + Ib$ type, where a and b are fuzzy numbers from $[0, 1]$, are characterized by Dhar, Broumi, and Smarandache [2].

In this work, we extend the ideas of Spherical Neutrosophic matrices to n-Hyper Spherical Neutrosophic matrix by assigning neutral membership degree say $\eta_{a_{ij}}$ together with positive and negative participation measures say $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$ with condition that $0 \leq \zeta^n_{a_{ij}} + \eta^n_{a_{ij}} + \delta^n_{a_{ij}} \leq 3$ $(n \geq 1)$.

The following is structure of this work. In Section 2, n-Hyper Spherical Neutrosophic matrices are characterized, as well as their algebraic operations and desired features. In Section 3, we define and study the algebraic characteristics of a new operation(@) on n-Hyper Spherical Neutrosophic matrices. The results are relevant in Section 4, n-Hyper Spherical Neutrosophic matrix and algebraic structure on this matrix. In Section 5, where we compose the paper's conclusion.

Definition 1.1. A Pythagorean fuzzy matrix (PFM) of order $m \times n$ is characterized as $\mathcal{S} = (\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \rangle)$ where $\zeta_{a_{ij}} \in [0, 1]$ and $\delta_{a_{ij}} \in [0, 1]$ whether the membership and non-membership values of the ij^{th} element in $\mathcal{S}$ fulfilling the requirement

$$0 \leq \zeta_{a_{ij}}^n + \delta_{a_{ij}}^n \leq 1, \ \forall \ i, j.$$

Definition 1.2. [9] A Picture fuzzy matrix (PFM) $\mathcal{S}$ with the formula, $\mathcal{S} = (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$ and non-negative real integers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ fulfilling the requirement

$$0 \leq \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leq 1, \ \forall \ i, j,$$

where $\zeta_{a_{ij}} \in [0,1]$, $\eta_{a_{ij}} \in [0,1]$ and $\delta_{a_{ij}} \in [0,1]$ represent the degree of membership, degree of neutral membership, and degree of non-membership respectively.

Definition 1.3. [10] A Spherical fuzzy matrix $\mathcal{S}$ with the formula, $\mathcal{S} = (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$ of a non-negative real integers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ fulfilling the requirement

$$0 \leq \zeta_{a_{ij}}^2 + \eta_{a_{ij}}^2 + \delta_{a_{ij}}^2 \leq 1, \ \forall \ i, j,$$

where $\zeta_{a_{ij}} \in [0,1]$, $\eta_{a_{ij}} \in [0,1]$ and $\delta_{a_{ij}} \in [0,1]$ represent the degree of membership, the degree of non-membership.

2. n-Hyper Spherical Neutrosophic matrices and their basic operations

The n-HyperSpherical Neutrosophic matrix and its algebraic operations are characterized in this section and also demonstrated De Morgan's rules over complement, commutativity, Idempotency, absorption law, distributivity, and associativity.

Now we'll describe Algebraic operations of n-HyperSpherical Neutrosophic matrices by limiting the measure of negative membership, neutral membership, and positive membership while retaining their total in the range $[0, \sqrt[n]{3}]$.

In $[5, 6, 9, 10]$, we employ some basic notations to arrive at our main findings.

Definition 2.1. A n-Hyper Spherical Neutrosophic matrix (n-HSNM) M of the form, $M = (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ fulfilling the requirement

$$0 \leq \zeta_{a_{ij}}^n + \eta_{a_{ij}}^n + \delta_{a_{ij}}^n \leq 3, \ \forall \ i, j,$$

where $\zeta_{a_{ij}} \in [0,1]$, $\eta_{a_{ij}} \in [0,1]$, and $\delta_{a_{ij}} \in [0,1]$ represent the degree of membership, the degree of neutral membership, and the degree of non-membership.

The n-HyperSpherical Fuzzy matrix (n-HSFM) is a specific instance of the Neutrosophic matrix (NFM). Because, $\zeta_{a_{ij}}, \eta_{a_{ij}}$, and $\delta_{a_{ij}} \in [0,1]$ imply that one also has $\zeta_{a_{ij}}^n, \eta_{a_{ij}}^n$, and $\delta_{a_{ij}}^n \in [0,1]$ for $n \geq 1$, they are neutrosophic components and each n-HSFS is a NM. The reciprocal, however, is false because if at least one component has a value of 1 and at least one of the other two components has a value of > 0, as in the case of $\zeta_{a_{ij}} = 1$ and $\eta_{a_{ij}} > 0, \delta_{a_{ij}} \in [0,1]$, then $\zeta_{a_{ij}}^n + \eta_{a_{ij}}^n + \delta_{a_{ij}}^n > 1$ for $(n \geq 1)$. The number of triplets ζ, η, δ that are NFM components, but not n-HSFM components, is infinite.

When the neutrosophic components $\zeta_{a_{ij}} = 0.9$, $\eta_{a_{ij}} = 0.4$, $\delta_{a_{ij}} = 0.5$, for some given ij^{th} element are used, they are not considered to be spherical fuzzy matrix components since $(0.9)^2 + (0.4)^2 + (0.5)^2 = 1.22 > 1$. For $\zeta_{a_{ij}}, \eta_{a_{ij}}$ and $\delta_{a_{ij}} \in [0, 1]$, there exist infinitely many values whose sum of squares is strictly bigger than 1, hence they are neutrosophic components rather than spherical fuzzy matrix components.

Let $N_{m\times n}$ represents the collection of all the n-Hyper Spherical Neutrosophic matrices.

Definition 2.2. The n-Hyper Spherical Neutrosophic matrices $\mathcal{S}$ and $\mathcal{T}$ are of the form, $\mathcal{S} = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ and $\mathcal{T} = \left(\left\langle \zeta_{b_{ij}}, \eta_{b_{ij}}, \delta_{b_{ij}} \right\rangle\right)$. Then

- $\mathcal{S} < \mathcal{T}$ iff $\forall i, j, \zeta_{a_{ij}} \leq \zeta_{b_{ij}}, \eta_{a_{ij}} \leq \eta_{b_{ij}}$ or $\eta_{a_{ij}} \geq \eta_{b_{ij}}, \delta_{a_{ij}} \geq \delta_{b_{ij}}$.
- $\mathcal{S}^C = \left(\left\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}} \right\rangle\right)$.
- $\mathcal{S} \wedge \mathcal{T} = \left(\left\langle \min\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right)\right\rangle\right)$.
- $\mathcal{S} \vee \mathcal{T} = \left(\left\langle \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right)\right\rangle\right)$.
- $\mathcal{S} \otimes \mathcal{T} = \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}}, \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} \right\rangle\right)$.
- $\mathcal{S} \oplus \mathcal{T} = \left(\left\langle \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$.

Definition 2.3. The scalar multiplication operation over n-HSNM $\mathcal{S}$ and is characterized by

$$n\mathcal{S} = \left(\left\langle \sqrt[n]{1 - [1 - \zeta^n_{a_{ij}}]^n}, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n \right\rangle\right).$$

Definition 2.4. The exponentiation operation over n-HSNM $\mathcal{S}$ and is characterized by

$$\mathcal{S}^n = \left(\left\langle [\zeta_{a_{ij}}]^n, \sqrt[n]{1 - [1 - \eta^n_{a_{ij}}]^n}, \sqrt[n]{1 - [1 - \delta^n_{a_{ij}}]^n} \right\rangle\right).$$

Let $N_{m\times n}$ represents the collection of all the n-Hyper Spherical Neutrosophic matrices.

The algebraic product and algebraic sum of n-HSNMs' are connected by the embracing theorem.

Theorem 2.5. *For $\mathcal{S}, \mathcal{T} \in N_{m\times n}$, then $\mathcal{S} \otimes \mathcal{T} \leq \mathcal{S} \oplus \mathcal{T}$.*

Proof. Let

$$\begin{aligned} \mathcal{S} \oplus \mathcal{T} &= \left(\left\langle \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right) \\ \text{and} & \\ \mathcal{S} \otimes \mathcal{T} &= \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}}, \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} \right\rangle\right). \end{aligned}$$

Assume that,

$$\begin{aligned} \zeta_{a_{ij}}\zeta_{b_{ij}} &\leq \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}} \\ (i.e)\zeta_{a_{ij}}\zeta_{b_{ij}} - \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}} &\geq 0 \\ (i.e)\zeta^n_{a_{ij}}(1 - \zeta^n_{b_{ij}}) + \zeta^n_{b_{ij}}(1 - \zeta^n_{a_{ij}}) &\geq 0 \end{aligned}$$

which is true as $0 \leq \zeta^n_{a_{ij}} \leq 1$ and $0 \leq \zeta^n_{b_{ij}} \leq 1$ and

$$\begin{aligned} \eta_{a_{ij}}\eta_{b_{ij}} &\leq \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}} \\ (i.e)\eta_{a_{ij}}\eta_{b_{ij}} - \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}} &\geq 0 \\ (i.e)\eta^n_{a_{ij}}(1-\eta^n_{b_{ij}}) + \eta^n_{b_{ij}}(1-\eta^n_{a_{ij}}) &\geq 0 \end{aligned}$$

which is true as $0 \leq \eta^n_{a_{ij}} \leq 1$ and $0 \leq \eta^n_{b_{ij}} \leq 1$, and

$$\begin{aligned} \delta_{a_{ij}}\delta_{b_{ij}} &\leq \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} \\ (i.e)\delta_{a_{ij}}\delta_{b_{ij}} - \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} &\geq 0 \\ (i.e)\delta^n_{a_{ij}}(1-\delta^n_{b_{ij}}) + \delta^n_{b_{ij}}(1-\delta^n_{a_{ij}}) &\geq 0, \end{aligned}$$

which is true as

$$0 \leq \delta^n_{a_{ij}} \leq 1$$

and

$$0 \leq \delta^n_{b_{ij}} \leq 1.$$

Hence, $\mathcal{S} \otimes \mathcal{T} \leq \mathcal{S} \oplus \mathcal{T}$. □

Theorem 2.6. *For any n-Hyper Spherical Neutrosophic matrix p, then*

(i) $\mathcal{S} \oplus \mathcal{S} \geq \mathcal{S}$.

(ii) $\mathcal{S} \otimes \mathcal{S} \leq \mathcal{S}$.

Proof. (i) Let

$$\begin{aligned} \mathcal{S} \oplus \mathcal{S} &= (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle) \oplus (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle) \\ \mathcal{S} \oplus \mathcal{S} &= \left(\left\langle \sqrt[n]{2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^n}, (\eta_{a_{ij}})^n, (\delta_{a_{ij}})^n \right\rangle\right) \\ \sqrt[n]{2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^n} &= \sqrt[n]{\zeta_{a_{ij}} + \zeta_{a_{ij}}(1-\zeta_{a_{ij}})} \geq \zeta_{a_{ij}}, \quad \forall\, i,j \end{aligned}$$

and

$$\begin{aligned} (\eta_{a_{ij}})^n &\leq \eta_{a_{ij}}, \quad \forall\, i,j \\ (\delta_{a_{ij}})^n &\leq \delta_{a_{ij}}, \quad \forall\, i,j. \end{aligned}$$

Thus, $\mathcal{S} \oplus \mathcal{S} \geq \mathcal{S}$. Similarly, we can also demonstrate that (ii) $\mathcal{S} \otimes \mathcal{S} \leq \mathcal{S}$. □

Theorem 2.7. *For* $\mathcal{S}, \mathcal{T}, \mathcal{U} \in N_{m\times n}$, *then*

(i) $\mathcal{S} \oplus \mathcal{T} = \mathcal{T} \oplus \mathcal{S}$.

(ii) $\mathcal{S} \otimes \mathcal{T} = \mathcal{T} \otimes \mathcal{S}$.

(iii) $(\mathcal{S} \oplus \mathcal{T}) \oplus \mathcal{U} = \mathcal{S} \oplus (\mathcal{T} \oplus \mathcal{U})$.

(iv) $(\mathcal{S} \otimes \mathcal{T}) \otimes \mathcal{U} = \mathcal{S} \otimes (\mathcal{T} \otimes \mathcal{U})$.

Proof. (i) Let

$$\begin{aligned} \mathcal{S} \oplus \mathcal{T} &= \left(\left\langle \sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \eta_{a_{ij}} \eta_{b_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \right\rangle\right) \\ &= \left(\left\langle \sqrt[n]{\zeta_{b_{ij}}^n + \zeta_{a_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{a_{ij}}^n}, \eta_{b_{ij}} \eta_{a_{ij}}, \delta_{b_{ij}} \delta_{a_{ij}} \right\rangle\right) \\ &= \mathcal{T} \oplus \mathcal{S}. \end{aligned}$$

(ii) Let

$$\begin{aligned} \mathcal{S} \otimes \mathcal{T} &= \left(\left\langle \zeta_{a_{ij}} \zeta_{b_{ij}}, \sqrt[n]{\eta_{a_{ij}}^n + \eta_{b_{ij}}^n - \eta_{a_{ij}}^n \eta_{b_{ij}}^n}, \sqrt[n]{\delta_{a_{ij}}^n + \delta_{b_{ij}}^n - \delta_{a_{ij}}^n \delta_{b_{ij}}^n} \right\rangle\right) \\ &= \left(\left\langle \zeta_{b_{ij}} \zeta_{a_{ij}}, \sqrt[n]{\eta_{b_{ij}}^n + \eta_{a_{ij}}^n - \eta_{b_{ij}}^n \eta_{a_{ij}}^n}, \sqrt[n]{\delta_{b_{ij}}^n + \delta_{a_{ij}}^n - \delta_{b_{ij}}^n \delta_{a_{ij}}^n} \right\rangle\right) \\ &= \mathcal{T} \otimes \mathcal{S}. \end{aligned}$$

(iii) Let

$$\begin{aligned} (\mathcal{S} \oplus \mathcal{T}) \oplus \mathcal{U} &= \left(\left\langle \left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \eta_{a_{ij}} \eta_{b_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \right) \oplus \left(\zeta_{c_{ij}}, \eta_{c_{ij}}, \delta_{c_{ij}} \right) \right\rangle\right) \\ &= \Bigg[\sqrt[n]{\left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n} \right)^n + \zeta_{c_{ij}}^n - \left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n} \right)^n \zeta_{c_{ij}}^n}, \\ &\quad \eta_{a_{ij}} \eta_{b_{ij}} \eta_{c_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \delta_{c_{ij}} \Bigg] \\ &= \Bigg[\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n + \zeta_{c_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{c_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n + \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n}, \\ &\quad \eta_{a_{ij}} \eta_{b_{ij}} \eta_{c_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \delta_{c_{ij}} \Bigg] \\ &= \Bigg[\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n + \zeta_{c_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{c_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n + \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n}, \\ &\quad \eta_{a_{ij}} \eta_{b_{ij}} \eta_{c_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \delta_{c_{ij}} \Bigg]. \end{aligned}$$

Let us assume that

$$\begin{aligned} \mathcal{S} \oplus (\mathcal{T} \oplus \mathcal{U}) &= \Bigg[\sqrt[n]{\zeta_{a_{ij}}^n + \left(\sqrt[n]{\zeta_{b_{ij}}^n + \zeta_{c_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n} \right)^n - \zeta_{a_{ij}}^n \left(\sqrt[n]{\zeta_{b_{ij}}^n + \zeta_{c_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n} \right)^n}, \\ &\quad \eta_{a_{ij}} \eta_{b_{ij}} \eta_{c_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \delta_{c_{ij}} \Bigg] \\ &= \Bigg[\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n + \zeta_{c_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{c_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n + \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n}, \\ &\quad \eta_{a_{ij}} \eta_{b_{ij}} \eta_{c_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \delta_{c_{ij}} \Bigg]. \end{aligned}$$

Thus, $(\mathcal{S} \oplus \mathcal{T}) \oplus \mathcal{U} = \mathcal{S} \oplus (\mathcal{T} \oplus \mathcal{U})$. Similarly, we can also demonstrate that (iv) $(\mathcal{S} \otimes \mathcal{T}) \otimes \mathcal{U} = \mathcal{S} \otimes (\mathcal{T} \otimes \mathcal{U})$. □

Theorem 2.8. *For $\mathcal{S}, \mathcal{T} \in N_{m \times n}$, then*

(i) $\mathcal{S} \oplus (\mathcal{S} \otimes \mathcal{T}) \geq \mathcal{S}$.

(ii) $\mathcal{S} \otimes (\mathcal{S} \oplus \mathcal{T}) \leq \mathcal{S}$.

Proof. (i) Let

$$\begin{aligned}
\mathcal{S} \oplus (\mathcal{S} \otimes \mathcal{T}) &= (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle) \oplus \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}}, \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} \right\rangle \right) \\
&= \Big[\sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{a_{ij}}\zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}[\zeta^n_{a_{ij}}\zeta^n_{b_{ij}}]}, \eta_{a_{ij}}[\sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}}], \\
&\quad \delta_{a_{ij}}[\sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}}] \Big] \\
&= \Big[\sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}[1 - \zeta^n_{a_{ij}}]}, \eta_{a_{ij}} \left(\sqrt[n]{1 - [1 - \eta^n_{a_{ij}}][1 - \eta^n_{b_{ij}}]} \right), \\
&\quad \delta_{a_{ij}} \left(\sqrt[n]{1 - [1 - \delta^n_{a_{ij}}][1 - \delta^n_{b_{ij}}]} \right) \Big] \\
&\geq \mathcal{S}.
\end{aligned}$$

Hence $\mathcal{S} \oplus (\mathcal{S} \otimes \mathcal{T}) \geq \mathcal{S}$. Similarly, we can also demonstrate that $(ii)\mathcal{S} \otimes (\mathcal{S} \oplus \mathcal{T}) \leq \mathcal{S}$. □

The theorem that follows is self-evident.

Theorem 2.9. *For $\mathcal{S}, \mathcal{T} \in N_{m\times n}$, then*

(i) $\mathcal{S} \vee \mathcal{T} = \mathcal{T} \vee \mathcal{S}$.

(ii) $\mathcal{S} \wedge \mathcal{T} = \mathcal{T} \wedge \mathcal{S}$.

Theorem 2.10. *For $\mathcal{S}, \mathcal{T}, \mathcal{U} \in N_{m\times n}$, then*

(i) $\mathcal{S} \oplus (\mathcal{T} \vee \mathcal{U}) = (\mathcal{S} \oplus \mathcal{T}) \vee (\mathcal{S} \oplus \mathcal{U})$.

(ii) $\mathcal{S} \otimes (\mathcal{T} \vee \mathcal{U}) = (\mathcal{S} \otimes \mathcal{T}) \vee (\mathcal{S} \otimes \mathcal{U})$.

(iii) $\mathcal{S} \oplus (\mathcal{T} \wedge \mathcal{U}) = (\mathcal{S} \oplus \mathcal{T}) \wedge (\mathcal{S} \oplus \mathcal{U})$.

(iv) $\mathcal{S} \otimes (\mathcal{T} \wedge \mathcal{U}) = (\mathcal{S} \otimes \mathcal{T}) \wedge (\mathcal{S} \otimes \mathcal{U})$.

Proof. We'll start by proving (i) and$(ii) - (iv)$ may be demonstrated similarly.
(i) Let

$$\begin{aligned}
\mathcal{S} \oplus (\mathcal{T} \vee \mathcal{U}) &= \Big[\sqrt[n]{\zeta^n_{a_{ij}} + \max\left(\zeta^n_{b_{ij}}, \zeta^n_{c_{ij}}\right) - \zeta^n_{a_{ij}}.\max\left(\zeta^n_{b_{ij}}, \zeta^n_{c_{ij}}\right)}, \\
&\quad \eta_{a_{ij}}.\max\left(\eta_{b_{ij}}, \eta_{c_{ij}}\right), \delta_{a_{ij}}.\max\left(\delta_{b_{ij}}, \delta_{c_{ij}}\right) \Big] \\
&= \Big[\sqrt[n]{\max\left(\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}}, \zeta_{a^n_{ij}} + \zeta^n_{c_{ij}}\right) - \max\left(\zeta^n_{a_{ij}}\zeta^n_{b_{ij}}, \zeta^n_{a_{ij}}\zeta^n_{c_{ij}}\right)}, \\
&\quad \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \eta_{a_{ij}}\eta_{c_{ij}}\right), \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \delta_{a_{ij}}\delta_{c_{ij}}\right) \Big] \\
&= \Big[\sqrt[n]{\max\left(\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}, \zeta^n_{a_{ij}} + \zeta^n_{c_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{c_{ij}}\right)}, \\
&\quad \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \eta_{a_{ij}}\eta_{c_{ij}}\right), \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \delta_{a_{ij}}\delta_{c_{ij}}\right) \Big] \\
&= (\mathcal{S} \oplus \mathcal{T}) \vee (\mathcal{S} \oplus \mathcal{U}).
\end{aligned}$$

□

Theorem 2.11. *For $\mathcal{S}, \mathcal{T} \in N_{m\times n}$, then*

(i) $(\mathcal{S} \wedge \mathcal{T}) \oplus (\mathcal{S} \vee \mathcal{T}) = \mathcal{S} \oplus \mathcal{T}$.

(ii) $(\mathcal{S} \wedge \mathcal{T}) \otimes (\mathcal{S} \vee \mathcal{T}) = \mathcal{S} \otimes \mathcal{T}$.

(iii) $(\mathcal{S} \oplus \mathcal{T}) \wedge (\mathcal{S} \otimes \mathcal{T}) = \mathcal{S} \otimes \mathcal{T}$.

(iv) $(\mathcal{S} \oplus \mathcal{T}) \vee (\mathcal{S} \otimes \mathcal{T}) = \mathcal{S} \oplus \mathcal{T}$.

Proof. We'll start by demonstrating (i), and $(ii) - (iv)$ may be demonstrated similarly.
(i) Let

$$\begin{aligned}(\mathcal{S} \wedge \mathcal{T}) \oplus (\mathcal{S} \vee \mathcal{T}) &= \Bigg[\sqrt[n]{\min\left(\zeta^n_{a_{ij}}, \zeta^n_{b_{ij}}\right) + \max\left(\zeta^n_{a_{ij}}, \zeta^n_{b_{ij}}\right) - \min\left(\zeta^n_{a_{ij}}, \zeta^n_{b_{ij}}\right).\max\left(\zeta^n_{a_{ij}}, \zeta^n_{b_{ij}}\right)}, \\ &\quad \max\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right).\min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \quad \max\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right).\min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right)\Bigg] \\ &= \left(\left\langle \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right) \\ &= \mathcal{S} \oplus \mathcal{T}.\end{aligned}$$

□

The operator complement obeys De Morgan's principles in the following theorems for the operation $\oplus, \otimes, \vee, \wedge$.

Theorem 2.12. *For $\mathcal{S}, \mathcal{T} \in N_{m\times n}$, then*

(i) $(\mathcal{S} \oplus \mathcal{T})^C = \mathcal{S}^C \otimes \mathcal{T}^C$.

(ii) $(\mathcal{S} \otimes \mathcal{T})^C = \mathcal{S}^C \oplus \mathcal{T}^C$.

(iii) $(\mathcal{S} \oplus \mathcal{T})^C \leq \mathcal{S}^C \oplus \mathcal{T}^C$.

(iv) $(\mathcal{S} \otimes \mathcal{T})^C \geq \mathcal{S}^C \otimes \mathcal{T}^C$.

Proof. We'll show that $(iii), (iv)$, and $(i), (ii)$ are simple.
(iii) Let

$$\begin{aligned}(\mathcal{S} \oplus \mathcal{T})^C &= \left(\left\langle \delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}}, \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}} \right\rangle\right) \\ \mathcal{S}^C \oplus \mathcal{T}^C &= \left(\left\langle \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}}\zeta_{b_{ij}} \right\rangle\right).\end{aligned}$$

Since

$$\begin{aligned}\delta_{a_{ij}}\delta_{b_{ij}} &\leq \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} \\ \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}} &\geq \eta_{a_{ij}}\eta_{b_{ij}} \\ \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}} &\geq \zeta_{a_{ij}}\zeta_{b_{ij}}.\end{aligned}$$

Hence $(\mathcal{S} \oplus \mathcal{T})^C \leq \mathcal{S}^C \oplus \mathcal{T}^C$.

(iv) Let

$$\begin{aligned} (\mathcal{S} \otimes \mathcal{T})^C &= \left(\left\langle \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}}\zeta_{b_{ij}} \right\rangle\right) \\ \mathcal{S}^C \otimes \mathcal{T}^C &= \left(\left\langle \delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}}, \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}} \right\rangle\right). \end{aligned}$$

Since

$$\begin{aligned} \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} &\geq \delta_{a_{ij}}\delta_{b_{ij}} \\ \eta_{a_{ij}}\eta_{b_{ij}} &\leq \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}} \\ \zeta_{a_{ij}}\zeta_{b_{ij}} &\leq \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}}. \end{aligned}$$

Hence $(\mathcal{S} \otimes \mathcal{T})^C \geq \mathcal{S}^C \otimes \mathcal{T}^C$. □

Theorem 2.13. *For $\mathcal{S}, \mathcal{T} \in N_{m\times n}$, then*

(i) $(\mathcal{S}^C)^C = \mathcal{S}$.
(ii) $(\mathcal{S} \vee \mathcal{T})^C = \mathcal{S}^C \wedge \mathcal{T}^C$.
(iii) $(\mathcal{S} \wedge \mathcal{T})^C = \mathcal{S}^C \vee \mathcal{T}^C$.

Proof. We'll prove only (ii), (i) is self-evident.

$$\begin{aligned} \mathcal{S} \vee \mathcal{T} &= \left(\left\langle \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right)\right\rangle\right) \\ (\mathcal{S} \vee \mathcal{T})^C &= \left(\left\langle \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right)\right\rangle\right) \\ \Rightarrow \mathcal{S}^C &= \left(\left\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}}\right\rangle\right) \\ \mathcal{T}^C &= \left(\left\langle \delta_{b_{ij}}, \eta_{b_{ij}}, \zeta_{b_{ij}}\right\rangle\right) \\ \Rightarrow \mathcal{S}^C \wedge \mathcal{T}^C &= \left(\left\langle \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right)\right\rangle\right). \end{aligned}$$

Hence $(\mathcal{S} \vee \mathcal{T})^C = \mathcal{S}^C \wedge \mathcal{T}^C$. Similarly, we can also demonstrate that $(iii)(\mathcal{S} \wedge \mathcal{T})^C = \mathcal{S}^C \vee \mathcal{T}^C$.

□

We'll show the algebraic characteristics of n-Hyper Spherical Neutrosophic matrices under scalar multiplication and exponentiation using the definitions 1.1, 1.2 and 1.3.

Theorem 2.14. *If $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then $n > 0$,*

(i) $n(\mathcal{S} \oplus \mathcal{T}) = n\mathcal{S} \oplus n\mathcal{T},\ n > 0.$
(ii) $n_1\mathcal{S} \oplus n_2\mathcal{S} = (n_1 + n_2)\mathcal{S},\ n_1, n_2 > 0.$
(iii) $(\mathcal{S} \otimes \mathcal{T})^n = \mathcal{S}^n \otimes \mathcal{T}^n,\ n > 0.$
(iv) $\mathcal{S}^n_1 \otimes \mathcal{S}^n_2 = \mathcal{S}^{(n_1+n_2)},\ n_1, n_2 > 0.$

Proof. According to the concept, for the two n-HSNMs $\mathcal{S}$ and $\mathcal{T}$, and $n, n_1, n_2 > 0$, we have

(i) Let

$$\begin{aligned} n(\mathcal{S} \oplus \mathcal{T}) &= n\left(\left\langle \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right) \\ &= \left(\left\langle \sqrt[n]{1-[1-\zeta^n_{a_{ij}}]^n[1-\zeta^n_{a_{ij}}]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right) \\ &= \left(\left\langle \sqrt[n]{1-[1-\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right) \\ n\mathcal{S} \oplus n\mathcal{T} &= \left(\left\langle \left(\sqrt[n]{1-[1-\zeta^n_{a_{ij}}]^n}, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n \right) \oplus \left(\sqrt[n]{1-[1-\zeta^n_{b_{ij}}]^n}, [\eta_{b_{ij}}]^n, [\delta_{b_{ij}}]^n \right) \right\rangle\right) \\ &= \left(\left\langle \sqrt[n]{1-[1-\zeta^n_{a_{ij}}]^n[1-\zeta^n_{b_{ij}}]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right) \\ &= \left(\left\langle \sqrt[n]{1-[1-\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right) \\ &= n(\mathcal{S} \oplus \mathcal{T}). \end{aligned}$$

(ii)Let

$$\begin{aligned} n_1\mathcal{S} \oplus n_2\mathcal{T} &= \left(\left\langle \left(\sqrt[n]{1-[1-\zeta^n_{a_{ij}}]^{n_1}}, [\eta_{a_{ij}}]^{n_1}, [\delta_{a_{ij}}]^{n_1} \right) \oplus \left(\sqrt[n]{1-[1-\zeta^n_{a_{ij}}]^{n_2}}, [\eta_{a_{ij}}]^{n_2}, [\delta_{a_{ij}}]^{n_2} \right) \right\rangle\right) \\ &= \left(\left\langle \sqrt[n]{1-[1-\zeta^n_{a_{ij}}]^{n_1+n_2}}, [\eta_{a_{ij}}]^{n_1+n_2}, [\delta_{a_{ij}}]^{n_1+n_2} \right\rangle\right) \\ &= (n_1+n_2)\mathcal{S}. \end{aligned}$$

(iii) Let

$$\begin{aligned} (\mathcal{S} \otimes \mathcal{T})^n &= \left[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[n]{1-[1-\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}]^n}, \sqrt[n]{1-[1-\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}]^n} \right] \\ &= \left[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[n]{1-[1-\eta^n_{a_{ij}}]^n[1-\eta^n_{b_{ij}}]^n, 1-[1-\delta^n_{a_{ij}}]^n[1-\delta^n_{b_{ij}}]^n} \right] \\ \mathcal{S}^n \otimes \mathcal{T}^n &= \left[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[n]{1-[1-\eta^n_{a_{ij}}]^n + 1-[1-\eta^n_{b_{ij}}]^n - \left(1-[1-\eta^n_{a_{ij}}]^n\right)\left(1-[1-\eta^n_{b_{ij}}]^n\right)} \right. \end{aligned}$$

$$\left. \sqrt[n]{1-[1-\delta^n_{a_{ij}}]^n + 1-[1-\delta^n_{b_{ij}}]^n - \left(1-[1-\delta^n_{a_{ij}}]^n\right)\left(1-[1-\delta^n_{b_{ij}}]^n\right)} \right]$$

$$\begin{aligned} &= \left(\left\langle \left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[n]{1-[1-\eta^n_{a_{ij}}]^n[1-\eta^n_{b_{ij}}]^n}, \sqrt[n]{1-[1-\delta^n_{a_{ij}}]^n[1-\delta^n_{b_{ij}}]^n} \right\rangle\right) \\ &= (P \otimes Q)^n. \end{aligned}$$

(iv) Let

$$\begin{aligned} \mathcal{S}^{n_1} \otimes \mathcal{S}^{n_2} &= \left[\left(\zeta_{a_{ij}}\right)^{n_1+n_2}, \sqrt[n]{1-[1-\eta^n_{a_{ij}}]^{n_1} + 1-[1-\eta^n_{a_{ij}}]^{n_2} - \left(1-[1-\eta^n_{a_{ij}}]^{n_1}\right)\left(1-[1-\eta^n_{a_{ij}}]^{n-2}\right)}, \right. \\ &\quad \left. \sqrt[n]{1-[1-\delta^n_{a_{ij}}]^{n_1} + 1-[1-\delta^n_{a_{ij}}]^{n-2} - \left(1-[1-\delta^n_{a_{ij}}]^{n_1}\right)\left(1-[1-\delta^n_{a_{ij}}]^{n_2}\right)} \right] \\ &= \left(\left\langle \left(\zeta_{a_{ij}}\right)^{n_1+n_2}, \sqrt[n]{1-[1-\eta^n_{a_{ij}}]^{n_1+n_2}}, \sqrt[n]{1-[1-\delta^n_{a_{ij}}]^{n_1+n_2}} \right\rangle\right) \\ &= \mathcal{S}^{(n_1+n_2)}. \end{aligned}$$

Hence proved. □

Theorem 2.15. *Suppose $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then $n > 0$,*

(i) $n\mathcal{S} \leq n\mathcal{T}$.

(ii) $\mathcal{S}^n \leq \mathcal{T}^n$.

Proof. (i) Let $\mathcal{S} \leq \mathcal{T}$. Then, we have

$$\zeta_{a_{ij}} \leq \zeta_{b_{ij}} \quad \eta_{a_{ij}} \geq \eta_{b_{ij}} \quad \text{and} \quad \delta_{a_{ij}} \geq \delta_{b_{ij}} \quad \forall\, i, j.$$

Now,

$$\begin{aligned} \sqrt[n]{1-[1-\zeta^n_{a_{ij}}]^n} &\leq \sqrt[n]{1-[1-\zeta^n_{b_{ij}}]^n} \\ \left[\eta_{a_{ij}}\right]^n &\geq \left[\eta_{b_{ij}}\right]^n \\ \left[\delta_{a_{ij}}\right]^n &\geq \left[\delta_{b_{ij}}\right]^n \quad \forall\, i, j. \end{aligned}$$

(ii) Also,

$$\begin{aligned} [\zeta_{a_{ij}}]^n &\geq [\zeta_{b_{ij}}]^n \\ \sqrt[n]{1-[1-\eta^n_{a_{ij}}]^n} &\leq \sqrt[n]{1-[1-\eta^n_{b_{ij}}]^n} \\ \sqrt[n]{1-[1-\delta^n_{a_{ij}}]^n} &\leq \sqrt[n]{1-[1-\delta^n_{b_{ij}}]^n} \quad \forall\, i, j. \end{aligned}$$

□

Similarly, we can prove the following theorems.

Theorem 2.16. *For $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then $n > 0$,*

(i) $n(\mathcal{S} \wedge \mathcal{T}) = n\mathcal{S} \wedge n\mathcal{T}$.

(ii) $n(\mathcal{S} \vee \mathcal{T}) = n\mathcal{S} \vee n\mathcal{T}$.

Theorem 2.17. *Suppose $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then $n > 0$,*

(i) $(\mathcal{S} \wedge \mathcal{T})^n = \mathcal{S}^n \wedge \mathcal{T}^n$.

(ii) $(\mathcal{S} \vee \mathcal{T})^n = \mathcal{S}^n \vee \mathcal{T}^n$.

Theorem 2.18. *Suppose $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then $n > 0$,*
$(\mathcal{S} \oplus \mathcal{T})^n \neq \mathcal{S}^n \oplus \mathcal{T}^n$.

Proof. Let

$$\begin{aligned} (\mathcal{S} \oplus \mathcal{T})^n &= \left[\left(\sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}} \right)^n, \sqrt[n]{1-[1-\eta^n_{a_{ij}}\eta^n_{b_{ij}}]^n}, \sqrt[n]{1-[1-\delta^n_{a_{ij}}\delta^n_{b_{ij}}]^n} \right] \\ \mathcal{S}^n \oplus \mathcal{T}^n &= \left[\sqrt[n]{[\zeta^n_{a_{ij}}]^n + [\zeta^n_{b_{ij}}]^n - [\zeta^n_{a_{ij}}]^n[\zeta^n_{b_{ij}}]^n}, \left(\sqrt[n]{1-[1-\eta^n_{a_{ij}}]^n} \right)^n . \left(\sqrt[n]{1-[1-\eta^n_{b_{ij}}]^n} \right)^n, \right. \\ &\left. \left(\sqrt[n]{1-[1-\delta^n_{a_{ij}}]^n} \right)^n . \left(\sqrt[n]{1-[1-\delta^n_{a_{ij}}]^n} \right)^n \right]. \end{aligned}$$

Hence $(\mathcal{S} \oplus \mathcal{T})^n \neq \mathcal{S}^n \oplus \mathcal{T}^n$. □

3. New operation (@) on n-HyperSpherical Neutrosophic matrices

In this part, we describe and show the algebraic properties of a new operation(@) on n-HyperSpherical Neutrosophic matrices. We also go through the Disstributivity rules in the situation when the $\oplus, \otimes, \vee$ and $\wedge$ operations are combined.

Definition 3.1. A n-HyperSpherical Neutrosophic matrices $\mathcal{S}$ and $\mathcal{T}$ are of the form, $\mathcal{S} = (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$ and $\mathcal{T} = (\langle \zeta_{b_{ij}}, \eta_{b_{ij}}, \delta_{b_{ij}} \rangle)$. Then

$$\mathcal{S}@\mathcal{T} = \left(\left\langle \sqrt[n]{\frac{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}}}{2}}, \sqrt[n]{\frac{\eta^n_{a_{ij}} + \eta^n_{b_{ij}}}{2}}, \sqrt[n]{\frac{\delta^n_{a_{ij}} + \delta^n_{b_{ij}}}{2}} \right\rangle \right).$$

Remark 3.2. Obviously, for every two n-HyperSpherical Neutrosophic matrices $\mathcal{S}$ and $\mathcal{T}$, then $\mathcal{S}@\mathcal{T}$ is a n-HyperSpherical Neutrosophic matrix.

Simple illustration given: For $\mathcal{S}@\mathcal{T}$,

$$\begin{aligned} 0 &\leq \frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2} + \frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2} + \frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2} \\ &\leq \frac{\zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}}}{2} + \frac{\zeta_{b_{ij}} + \eta_{b_{ij}} + \delta_{b_{ij}}}{2} \\ &\leq \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Theorem 3.3. *For any n-HyperSpherical Neutrosophic matrix $\mathcal{S}$, then $\mathcal{S}@\mathcal{S} = \mathcal{S}$.*

Proof. Let

$$\begin{aligned} \mathcal{S}@\mathcal{S} &= \left(\left\langle \sqrt[n]{\frac{\zeta^n_{a_{ij}} + \zeta^n_{a_{ij}}}{2}}, \sqrt[n]{\frac{\eta^n_{a_{ij}} + \eta^n_{a_{ij}}}{2}}, \sqrt[n]{\frac{\delta^n_{a_{ij}} + \delta^n_{a_{ij}}}{2}} \right\rangle \right) \\ &= \left(\left\langle \left(\sqrt[n]{\frac{\zeta^n_{a_{ij}} + \zeta^n_{a_{ij}}}{2}} \right)^n, \left(\sqrt[n]{\frac{\eta^n_{a_{ij}} + \eta^n_{a_{ij}}}{2}} \right)^n, \left(\sqrt[n]{\frac{\delta^n_{a_{ij}} + \delta^n_{a_{ij}}}{2}} \right)^n \right\rangle \right) \\ &= \left(\left\langle \frac{2\zeta^n_{a_{ij}}}{2}, \frac{2\eta^n_{a_{ij}}}{2}, \frac{2\delta^n_{a_{ij}}}{2} \right\rangle \right) \\ &= (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle). \end{aligned}$$

Since $\zeta^n_{a_{ij}} \leq \zeta_{a_{ij}}, \eta^n_{a_{ij}} \leq \eta_{a_{ij}}, \delta^n_{a_{ij}} \leq \delta_{a_{ij}}$

$$\mathcal{S}@\mathcal{S} = \mathcal{S}.$$ □

Remark 3.4. If $v, w \in [0, 1]$, then $vw \leq \frac{v+w}{2}, \frac{v+w}{2} \leq v + w - vw$.

Theorem 3.5. *Suppose $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then*

(i) $(\mathcal{S} \oplus \mathcal{T}) \vee (\mathcal{S} @ \mathcal{T}) = \mathcal{S} \oplus \mathcal{T}$.

(ii) $(\mathcal{S} \otimes \mathcal{T}) \wedge (\mathcal{S} @ \mathcal{T}) = \mathcal{S} \otimes \mathcal{T}$.

(iii) $(\mathcal{S} \oplus \mathcal{T}) \wedge (\mathcal{S} @ \mathcal{T}) = \mathcal{S} @ \mathcal{T}$.

(iv) $(\mathcal{S} \otimes \mathcal{T}) \vee (\mathcal{S} @ \mathcal{T}) = \mathcal{S} @ \mathcal{T}$.

Proof. We'll show that (i) and (iii), as well as (ii) and (iv), may be demonstrated in the similar manner. (i) Let

$$\begin{aligned}(\mathcal{S} \oplus \mathcal{T}) \vee (\mathcal{S} @ \mathcal{T}) &= \Bigg[\max\left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \sqrt[n]{\frac{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n}{2}}\right), \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \sqrt[n]{\frac{\eta_{a_{ij}}^n + \eta_{b_{ij}}^n}{2}}\right), \\ &\qquad \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[n]{\frac{\delta_{a_{ij}}^n + \delta_{b_{ij}}^n}{2}}\right)\Bigg] \\ &= \left(\left\langle \sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right) \\ &= \mathcal{S} \oplus \mathcal{T}.\end{aligned}$$

(iii) Let

$$\begin{aligned}(\mathcal{S} \oplus \mathcal{T}) \wedge (\mathcal{S} @ \mathcal{T}) &= \Bigg[\min\left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \sqrt[n]{\frac{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n}{2}}\right), \max\left(\eta_{a_{ij}}\eta_{b_{ij}}, \sqrt[n]{\frac{\eta_{a_{ij}}^n + \eta_{b_{ij}}^n}{2}}\right), \\ &\qquad \max\left(\delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[n]{\frac{\delta_{a_{ij}}^n + \delta_{b_{ij}}^n}{2}}\right)\Bigg] \\ &= \left(\left\langle \sqrt[n]{\frac{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n}{2}}, \sqrt[n]{\frac{\eta_{a_{ij}}^n + \eta_{b_{ij}}^n}{2}}, \sqrt[n]{\frac{\delta_{a_{ij}}^n + \delta_{b_{ij}}^n}{2}} \right\rangle\right) \\ &= \mathcal{S} @ \mathcal{T}.\end{aligned}$$

Hence proved. □

Remark 3.6. Under the n-Hyper Spherical Neutrosophic matrix operations of algebraic sum and algebraic product, the n-Hyper Spherical Neutrosophic matrix forms a semi-lattice, associativity, commutativity, and idempotency. When $\oplus, \otimes$ and $\wedge, \vee, @$ are combined, the distributive law also holds.

4. **Applications**

The results are relevant to the development of n-Hyper Spherical Neutrosophic semi-lattice structure, n-Hyper Spherical Neutrosophic matrix, and algebraic structure on this matrix.

5. Conclusion

n-Hyper Spherical Neutrosophic matrices and their algebraic operations are characterized in this study. Then various qualities are demonstrated, including associativity, idempotency, distributivity, commutativity, absorption law, and De Morgan's laws over complement. Lastly, we established a new operation(@) on n-Hyper Spherical Neutrosophic matrices and studied distributive laws in the situation of combining the operations of $\oplus, \otimes, \wedge$, and $\vee$. This finding can be used to the n-Hyper Spherical Neutrosophic matrix theory in the future. The conclusions of this work will be useful in the creation of the n-Hyper Spherical Neutrosophic semilattice and its algebraic property. The applicability of the suggested aggregating operators of n-HSNMs in risk analysis, decision making and many other fuzzy environments will need to be studied in the future.

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Decision-making techniques based on similarity measures of possibility neutrosophic soft expert sets

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Abstract. A neutrosophic set (NS) is a novel computing technique that accesses uncertain information by using three memberships. The main goal of this study is to come up with a novel approach called the "possibility neutrosophic soft expert set" (PNSE-set), which is based on the idea that each element of the universe of discourse has a certain level of possibility. Based on this new approach, the set-theoretical operations on PNSE-set (i.e complement, subset, equality, union, intersection, DeMorgans laws, AND-product, and OR-product operations) are introduced, along with illustrative examples and relevant laws. A generalized algorithm is proposed and applied to decision-making problems. Meanwhile, a similarity measure of two PNSE-sets is offered, and it's tested in real-life applications involving medical diagnosis applications. Finally, this work is supported by a comparative analysis of three recent methods.

Keywords: neutrosophic set; similarity measure; decision making; possibility neutrosophic soft expert sets.

1. Introduction

With the rapid development that our world is witnessing in all areas of our daily life, we face several practical problems that include uncertain, inconsistent, and incomplete information, and this requires a new and effective mathematical tools to deal with problems. Smarandache managed to overcome the weaknesses that appeared in both [3] and [4] by establishing an idea of a neutrosophic set (NS). An NS is considered a more comprehensive mathematical tool for human thinking, as it covers the aspects of right and wrong and the indeterminacy between them through its mathematical structure, which contains three functions, namely $T(u)$ true function, $I(u)$ indeterminacy function, and $F(u)$ falsity function, such that image of all of them belong $]^{-}0, ^{+}1[$

However, NS and its extensions [5], [6] have their own intrinsic difficulties and weaknesses in precisely expressing their preferences. To overcome this drawback, Molodtsov [7] established a new parameterization tool named soft set (SS). After Molodtsov, a lot of researchers combined the SS with the NS and its extension; for instance,Maji [8] introduced a neutrosophic soft set (NSS), which can be considered a new track of thinking that opens the horizons for researchers in engineering, computer science, and others.Peng [9] proposed similarity measures on neutrosophic soft sets to measure level soft sets based on some algorithms. Broumi [10] tested the notions of relation between NS-sets in decision-making applications. Some techniques of MAGDM and MADM tested on neutrosophic environment by [11]- [13]. Naeem et al. [14]- [17] discussed fuzzy, soft, and m-polar neutrosophic environments with decision-making. Al-Sharqi et al. [18]- [22] merged all of NS and SS into a complex environment and applied it in some real-life applications. In addition, researchers have applied these mathematical tools in various fields [23]- [32]. Alkhazaleh pointed out all these theories have their own shortcomings. One of these shortcomings is the soft set's inability to absorb users' opinions (experts) simultaneously. To overcome these difficulties, Alkhazaleh et al. [33] created a new technique for modelling uncertainty called a "soft expert set" (SES) based on the merged concept of a "soft set" with an expert system. This approach has now been applied in many fields, such as intelligent systems, game theory, measurement theory, cybernetics, probability theory, and so on. Research on SES is progressing rapidly up to now. This concept has been studied and combined with fuzzy set theory and its extensions by researchers. Alkhazaleh et al. were the first to introduce the model fuzzy-ESSs [34] and neutrosophic-SESs [35]. Alhazaymeh and Hassan merged a soft expert set with a vague set and gave some new hybrid notions [36].Ihsan et al. [37] have developed m-polar fuzzy SESs with the same properties. Hassan et al. [38], [39] demonstrated the properties of the Q-NSE-set. Pramanik et al. [40]compiend SNS and SES and they proposed the idea SNSES. Subsequently, more general properties and applications of soft expert set theory have been investigated by Hassan and others, for instance, see [41]- [44]. From a scientific point of view, an element's probability degree will significantly influence modelling some applications under multiple attribute decision-making problems. Therefore, several researchers studied this idea in fuzzy set theory and its extinctions. For instance, Alkhazaleh et al. [45]first established the possibility setting on fuzzy soft sets and defined similarity measures for two possibility fuzzy soft sets. Alhazaymeh and Hassan then presented the concepts of possibility vague soft set [46] and possibility interval-valued vague soft set [47]. Al-Quran and Hassan [48] proposed the possibility neutrosophic vague soft set and employed it in medical diagnosis applications. Karaaslan [49] suggested the theory of possibility of NSSs

as an extension of [50] and illustrated its application in decision-making. Selvachandran and Salleh [51] established the idea of possibility intuitionistic fuzzy-SESs by a develops the structures in [45]- [47]. But there are some limitations in [49]- [51]. In the first one can only be used by one user, while more than one user can use the second, but it lacks an important tool, which is the indefiniteness found in NS. To overcome these limitations, we will organize in this work a new hybrid concept called possibility neutrosophic soft expert sets (PNSE-sets) by assigning a possibility degree to each approximate member of an NSE-set. This model keeps the advantages of the SESs by allowing users to understand the experts' opinions without the requirement for further operations. Also, Similarity measures [52], [53] are layered extensively in the fuzzy environment . Therefore, based on this model, we define the measure of similarity between two PNSE-sets and show how this measure can be used in medical diagnosis.
This article is divided into eight parts, which are as follows: we review some important definitions and properties in Section 2. The general framework of the proposed concept, some properties, and numerical examples in Section 3.. Then, in section 4, we present basic operations on the PNSE-set together with some propositions and numerical examples. Some applications in decision-making are solved by PNSE-setting in Section 5. In Section 6, we define the similarity measure between two PNSE-sets and show the importance of this measure by one application in medical diagnosis. Finally, Section 7 contains a brief comparison between PNSE-set and some other methods to show the reader the importance of this work. In addition, conclusions of this work showed in Section 8.

2. **Preliminaries**

In this part, we give the most important definitions and properties of $[1, 7]$ a that will be used in later parts of this work.

Definition 2.1. Neutrosophic Set (N-set) $[1, 2]$ An N-set $\ddot{\mathfrak{N}}$ is characterized by $\ddot{\mathfrak{N}} = \left\{\left\langle v, \dot{\mathfrak{T}}_{\ddot{\mathfrak{N}}}(v), \dot{\mathfrak{I}}_{\ddot{\mathfrak{N}}}(v), \dot{\mathfrak{F}}_{\ddot{\mathfrak{N}}}(v), \forall v \in \mathfrak{V}\right\rangle\right\}$ such that $\dot{\mathfrak{T}}_{\ddot{\mathfrak{N}}}(v), \dot{\mathfrak{I}}_{\ddot{\mathfrak{N}}}(v), \dot{\mathfrak{F}}_{\ddot{\mathfrak{N}}}(v) : \mathfrak{V} \to [0, 1]$ are real-valued truth-membership, indeterminacy-membership, and non-membership, respectively

Definition 2.2. (Properties of N-set) [1,2] If $\ddot{\mathfrak{N}}$ and $\ddot{\mathfrak{M}}$ are two N-sets on $\mathfrak{V}$ then for $v \in \mathfrak{V}$, we have:

(i) $\ddot{\mathfrak{N}} \subseteq \ddot{\mathfrak{M}}$ if $\dot{\mathfrak{T}}_{\ddot{\mathfrak{N}}}(v) \leq \dot{\mathfrak{T}}_{\ddot{\mathfrak{M}}}(v)$, $\dot{\mathfrak{I}}_{\ddot{\mathfrak{N}}}(v) \geq \dot{\mathfrak{I}}_{\ddot{\mathfrak{M}}}(v)$ and $\dot{\mathfrak{F}}_{\ddot{\mathfrak{N}}}(v) \geq \dot{\mathfrak{F}}_{\ddot{\mathfrak{M}}}(v)$ for all $v \in \mathfrak{V}$.

(ii) $\ddot{\mathfrak{N}}^c = \left\{\left\langle v, \dot{\mathfrak{T}}_{\ddot{\mathfrak{N}}^c}(v), \dot{\mathfrak{I}}_{\ddot{\mathfrak{N}}^c}(v), \dot{\mathfrak{F}}_{\ddot{\mathfrak{N}}^c}(v)\right\rangle\right\} = \left\{\left\langle v, \dot{\mathfrak{F}}_{\ddot{\mathfrak{N}}}(v), 1 - \dot{\mathfrak{I}}_{\ddot{\mathfrak{N}}}(v), \dot{\mathfrak{T}}_{\ddot{\mathfrak{N}}}(v)\right\rangle\right\}$.

(iii) If $\ddot{\mathfrak{N}} \cup (\cap) \ddot{\mathfrak{M}} = \ddot{\mathfrak{D}}$ and defined as follows
$\ddot{\mathfrak{D}} = \left\{\left\langle v, \dot{\mathfrak{T}}_{\ddot{\mathfrak{D}}}(v), \dot{}_{\ddot{\mathfrak{D}}}(v), \dot{\mathfrak{F}}_{\ddot{\mathfrak{D}}}(v)\right\rangle\right\}$

where
$\dot{\mathfrak{T}}_{\ddot{\mathfrak{D}}}(v) = max\,(min)\left[\dot{\mathfrak{T}}_{\ddot{\mathfrak{N}}}(v), \dot{\mathfrak{T}}_{\ddot{\mathfrak{M}}}(v)\right],$
$\dot{\mathfrak{I}}_{\ddot{\mathfrak{D}}}(v) = min\,(max)\left[\dot{\mathfrak{I}}_{\ddot{\mathfrak{N}}}(v), \ddot{\mathfrak{I}}_{\ddot{\mathfrak{M}}}(v)\right],$
$\dot{\mathfrak{F}}_{\ddot{\mathfrak{D}}}(v) = min\,(max)\left[\mathfrak{F}_{\ddot{\mathfrak{N}}}(v), \mathfrak{F}_{\ddot{\mathfrak{M}}}(v)\right].$

Definition 2.3. Soft Set (SS) [7] A pair $\left(\tilde{\mathfrak{F}}, \mathfrak{E}\right)$ is a SS on fixed set $\mathfrak{V}$, where $\tilde{\mathfrak{F}} : \mathfrak{E} \to \mathfrak{P}(\mathfrak{V})$ such that $\mathfrak{A}$ is a subset of attributes set $\mathfrak{E}$.

3. Possibility Neutrosophic Soft Expert Sets(PNSE-set)

In the current section, we will establish the main definition of possibility neutrosophic soft expert sets (PNSE-sets) and the elementary properties of PNSE-sets are conceptualized with some numerical examples.

Definition 3.1. The pair$(\mathfrak{F}_\mu, \mathfrak{Z})$is called the possibility neutrosophic soft expert set (PNSE-set) over a nonempty soft universe $(\mathfrak{V}, \mathfrak{Z})$ if

$$\mathfrak{F}_\mu : A \to N^V \times I^V$$

defined by

$$\mathfrak{F}_\mu(z_i) = \{\mathfrak{F}(z_i)(v_n), \mu(z_i)(v_n)\}$$

with

$$\mathfrak{F}(z_i)(v_n) = \langle \rho(z_i)(v_n), \eta(z_i)(v_n), \psi(z_i)(v_n)\rangle\, \forall z_i \in \mathfrak{P} \subseteq Z, v_n \in V.$$

Where,

(1) For $\mathfrak{V} = \{v_1, v_2, v_3, ..., v_n\}$be a non-empty initial universe, $\mathfrak{P} = \{p_1, p_2, p_3, ..., p_j\}$be a parameters set, $\mathfrak{M} = \{m_1, m_2, m_3, ..., m_k\}$be a set of experts, $\mathfrak{O} = \{1 = agree, 0 = disagree\}$be a set of opinions, and $\mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{O}\}$.
(2) $\Im : \mathfrak{Z} \to N^{\mathfrak{V}}$and $\mu : \mathfrak{Z} \to I^{\mathfrak{V}}$,$N^{\mathfrak{V}}$ and $I^{\mathfrak{V}}$ indicates the collection of all neutrosophic and fuzzy subset of $\mathfrak{V}$ respectively.
(3) $\mathfrak{F}(z)(v_n)$ is the degree of neutrosophic membership of $v \in \mathfrak{V}$ in $\mathfrak{F}(z)$, i.e$(\rho(z)(v_n), \eta(z)(v_n), \psi(z)(v_n))$ denotes to three neutrosophic memberships receptively.
(4) $\mu(z)(v_n)$ is a degree of possibility membership of $v \in \mathfrak{V}$ in $\mathfrak{F}(z)$.

so $\mathfrak{F}_\mu(z_i)$ can be written as below:
$\left\{\left(\frac{v_1}{F(z)(v_1)}, \mu(z)(v_1)\right), \left(\frac{v_2}{F(z)(v_2)}, \mu(z)(v_2)\right), \left(\frac{v_3}{F(z)(v_3)}, \mu(z)(v_3)\right) ..., \left(\frac{v_n}{F(z)(v_n)}, \mu(z)(v_n)\right)\right\}$
for $i = 1, 2, 3, ..., n$

Remark 3.2.

1. If we have $\mathfrak{P} \subseteq \mathfrak{Z}$ it is also possible to write a PNSE-set as $(\mathfrak{F}_\mu, \mathfrak{P})$ and to essay way the PNSE-set can be write as $\mathfrak{F}_\mu$.
2. Here in this work, we suppose that the set of opinions consists of only two values (i,e agree and disagree), but it is possible to include other options that match the nature of the problem.

Example 3.3. Let $\mathfrak{V} = \{v_1, v_2, v_3\}$be the universal set of elements, let $\mathfrak{P} = \{p_1, p_2\}$be a parameters set, whee p_1 =cheap, p_2 =beautiful and let $\mathfrak{M} = \{m_1, m_2\}$ be a set containing two experts. Assume that $\mathfrak{F}_\mu : A \to N^V \times I^V$ is a function represented as follows:

$$\mathfrak{F}_\mu (p_1, m_1, 1) = \left\{ \left(\frac{v_1}{\langle 0.5,0.3,0.1\rangle}, 0.2 \right), \left(\frac{v_2}{\langle 0.4,0.2,0.7\rangle}, 0.3 \right), \left(\frac{v_3}{\langle 0.6,0.1,0.6\rangle}, 0.5 \right) \right\}$$

$$\mathfrak{F}_\mu (p_2, m_1, 1) = \left\{ \left(\frac{v_1}{\langle 0.6,0.3,0\rangle}, 0.5 \right), \left(\frac{v_2}{\langle 0.1,0.3,0.9\rangle}, 0.7 \right), \left(\frac{v_3}{\langle 0.9,0.4,0.2\rangle}, 0.6 \right) \right\}$$

$$\mathfrak{F}_\mu (p_1, m_2, 1) = \left\{ \left(\frac{v_1}{\langle 0.3,0.4,0.6\rangle}, 0.1 \right), \left(\frac{v_2}{\langle 0.4,0.5,0.3\rangle}, 0.5 \right), \left(\frac{v_3}{\langle 0.2,0.4,0.4\rangle}, 0.8 \right) \right\}$$

$$\mathfrak{F}_\mu (p_2, m_2, 1) = \left\{ \left(\frac{v_1}{\langle 0.1,0.1,0.4\rangle}, 0.3 \right), \left(\frac{v_2}{\langle 0.6,0.2,0.4\rangle}, 0.8 \right), \left(\frac{v_3}{\langle 0.3,0.2,0.5\rangle}, 0.6 \right) \right\}$$

$$\mathfrak{F}_\mu (p_1, m_1, 0) = \left\{ \left(\frac{v_1}{\langle 0.2,0.8,0.3\rangle}, 0.5 \right), \left(\frac{v_2}{\langle 0.3,0.4,0.2\rangle}, 0.5 \right), \left(\frac{v_3}{\langle 0.3,0.2,0.6\rangle}, 0.8 \right) \right\}$$

$$\mathfrak{F}_\mu (p_2, m_1, 0) = \left\{ \left(\frac{v_1}{\langle 0.4,0.9,2\rangle}, 0.9 \right), \left(\frac{v_2}{\langle 0.4,0.3,0.2\rangle}, 0.7 \right), \left(\frac{v_3}{\langle 0.3,0.4,0.7\rangle}, 0.2 \right) \right\}$$

$$\mathfrak{F}_\mu (p_1, m_2, 0) = \left\{ \left(\frac{v_1}{\langle 0.4,0.3,0,3\rangle}, 0.4 \right), \left(\frac{v_2}{\langle 0.2,0.6,0.6\rangle}, 0.7 \right), \left(\frac{v_3}{\langle 0.7,0.3,0.5\rangle}, 0.5 \right) \right\}$$

$$\mathfrak{F}_\mu (p_2, m_2, 0) = \left\{ \left(\frac{v_1}{\langle 0.2,0.4,0.8\rangle}, 0.3 \right), \left(\frac{v_2}{\langle 0.5,0.5,0.2\rangle}, 0.9 \right), \left(\frac{v_3}{\langle 0.1,0.1,0.7\rangle}, 0.1 \right) \right\}$$

Now, we can present PNSE-set $(\mathfrak{F}_\mu, \mathfrak{Z})$ as be formed of the following aggregate of approximations:

$$(\mathfrak{F}_\mu, \mathfrak{Z}) = \Big\{ (p_1, m_1, 1) = \left\{ \left(\frac{v_1}{\langle 0.5,0.3,0.1\rangle}, 0.2 \right), \left(\frac{v_2}{\langle 0.4,0.2,0.7\rangle}, 0.3 \right), \left(\frac{v_2}{\langle 0.6,0.1,0.6\rangle}, 0.5 \right) \right\},$$

$$(p_2, m_1, 1) = \left\{ \left(\frac{v_1}{\langle 0.6,0.3,0\rangle}, 0.5 \right), \left(\frac{v_2}{\langle 0.1,0.3,0.9\rangle}, 0.7 \right), \left(\frac{v_3}{\langle 0.9,0.4,0.2\rangle}, 0.6 \right) \right\},$$

$$(p_1, m_2, 1) = \left\{ \left(\frac{v_1}{\langle 0.3,0.4,0.6\rangle}, 0.1 \right), \left(\frac{v_2}{\langle 0.4,0.5,0.3\rangle}, 0.5 \right), \left(\frac{v_3}{\langle 0.2,0.4,0.4\rangle}, 0.8 \right) \right\},$$

$$(p_2, m_2, 1) = \left\{ \left(\frac{v_1}{\langle 0.1,0.1,0.4\rangle}, 0.3 \right), \left(\frac{v_2}{\langle 0.6,0.2,0.4\rangle}, 0.8 \right), \left(\frac{v_3}{\langle 0.3,0.2,0.5\rangle}, 0.6 \right) \right\},$$

$$(p_1, m_1, 0) = \left\{ \left(\frac{v_1}{\langle 0.2,0.8,0.3\rangle}, 0.5 \right), \left(\frac{v_2}{\langle 0.3,0.4,0.2\rangle}, 0.5 \right), \left(\frac{v_3}{\langle 0.3,0.2,0.6\rangle}, 0.8 \right) \right\},$$

$$(p_2, m_1, 0) = \left\{ \left(\frac{v_1}{\langle 0.4,0.9,0.2\rangle}, 0.9 \right), \left(\frac{v_2}{\langle 0.4,0.3,0.2\rangle}, 0.7 \right), \left(\frac{v_3}{\langle 0.3,0.4,0.7\rangle}, 0.2 \right) \right\},$$

$$(p_1, m_2, 0) = \left\{ \left(\frac{v_1}{\langle 0.4,0.3,0.3\rangle}, 0.4 \right), \left(\frac{v_2}{\langle 0.2,0.6,0.6\rangle}, 0.7 \right), \left(\frac{v_3}{\langle 0.7,0.3,0.5\rangle}, 0.5 \right) \right\},$$

$$(p_2, m_2, 0) = \left\{ \left(\frac{v_1}{\langle 0.2,0.4,0.8\rangle}, 0.3 \right), \left(\frac{v_2}{\langle 0.5,0.5,0.2\rangle}, 0.9 \right), \left(\frac{v_3}{\langle 0.1,0.1,0.7\rangle}, 0.1 \right) \right\} \Big\}$$

Then we say that $(\mathfrak{F}_\mu, \mathfrak{Z})$ is a is said to be possibility neutrosophic soft expert set (PNSE-set) over soft universe $(\mathfrak{V}, \mathfrak{Z})$

Definition 3.4. For two PNSE-sets $(\mathfrak{F}_\mu, A)$ and $(\mathfrak{G}_\varphi, B)$ over $(\mathfrak{V}, \mathfrak{Z})$.Then $(\mathfrak{F}_\mu, A)$ is said to be be a PNSE-subset of $(\mathfrak{G}_\varphi, B)$ if $A \subseteq B$, and $\forall z \in A \subseteq \mathfrak{Z}$ the next conditions are fulfilled:
1. $\mu(z)$ is fuzzy subset of $\varphi(z)$.
2. $\mathfrak{F}_\mu(z)$ is neutrosophic subset of $\mathfrak{G}_\varphi(z)$.
And we denoted this relation as $(\mathfrak{F}_\mu, A) \subseteq (\mathfrak{G}_\varphi, B)$.In this issue, $(\mathfrak{G}_\varphi, B)$ is named a PNSE-superset of $(\mathfrak{F}_\mu, A)$.

Definition 3.5. If $(\mathfrak{F}_\mu, A)$ and $(\mathfrak{G}_\varphi, B)$ be two PNSE-sets over $(\mathfrak{V}, \mathfrak{Z})$.Then $(\mathfrak{F}_\mu, A)$ is equal to $(\mathfrak{G}_\varphi, B)$ if $\forall z \in A \subseteq \mathfrak{Z}$ the next conditions are fulfilled:
1. $\mu(z)$ is equal of $\varphi(z)$.
2. $\mathfrak{F}_\mu(z)$ is equal of $\mathfrak{G}_\varphi(z)$.
And we denoted this relation as $(\mathfrak{F}_\mu, A) = (\mathfrak{G}_\varphi, B)$.In this words, $(\mathfrak{G}_\varphi, B)$ is equal of $(\mathfrak{F}_\mu, A)$ if $(\mathfrak{G}_\varphi, B)$ is PNSE-subset of $(\mathfrak{F}_\mu, A)$ and $(\mathfrak{F}_\mu, A)$ is PNSE-subset of $(\mathfrak{G}_\varphi, B)$.

Definition 3.6. A PNSE-set $(\mathfrak{F}_\mu, A)$ is named null-PNSE-set, indicated by $\left(\ddot{\Phi}_\mu, A\right)$ and given as follows

$$\ddot{\Phi}_\mu(z_i) = \{\mathfrak{F}(z_i)(v_n), \mu(z_i)(v_n)\}, \forall z_i \in A \subseteq \mathfrak{Z}$$

where $\mathfrak{F}(z_i)(v_n) = \langle 0, 1, 1\rangle$ such that $\forall z_i \in A \subseteq \mathfrak{Z}, v \in \mathfrak{V}$ we have $\rho(z_i)(v_n) = 0$,$\eta(z_i)(v_n) = 1$,$\psi(z_i)(v_n) = 1$ and $\mu(z_i)(v_n) = 0$.

Definition 3.7. A PNSE-set $(\mathfrak{F}_\mu, A)$ is named to be absolute-PNSE-set, indicate by $(\mathfrak{F}_\mu, A)_{Abso}$ and given as follows

$$\mathfrak{F}_\mu(z_i) = \{\mathfrak{F}(z_i)(v_n), \mu(z_i)(v_n)\}, \forall z_i \in A \subseteq \mathfrak{Z}$$

where $\mathfrak{F}(z_i)(v_n) = \langle 1, 0, 0\rangle$ such that $\forall z_i \in A \subseteq \mathfrak{Z}, v \in \mathfrak{V}$ we have $\rho(z_i)(v_n) = 1$,$\eta(z_i)(v_n) = 0$,$\psi(z_i)(v_n) = 0$ and $\mu(z_i)(v_n) = 1$.

Definition 3.8. Let $(\mathfrak{F}_\mu, A)$ be a PNSE-set over $(\mathfrak{V}, \mathfrak{Z})$. Then an agree-PNSE-set over non-empty universal $\mathfrak{V}$ denoted $(\mathfrak{F}_\mu, A)_1$ is a PNSE-subset of $(\mathfrak{V}, \mathfrak{Z})$ and its given as follows:

$$\mathfrak{F}_\mu(z_i)_1 = \{\mathfrak{F}(z_i)(v_n), \mu(z_i)(v_n)\}, \forall z_i \in A \subseteq \mathfrak{Z} = \mathfrak{P} \times \mathfrak{M} \times 1.$$

Definition 3.9. Let $(\mathfrak{F}_\mu, A)$ be a PNSE-set over $(\mathfrak{V}, \mathfrak{Z})$. Then a disagree-PNSE-set over non-empty universal $\mathfrak{V}$ denoted $(\mathfrak{F}_\mu, A)_0$ is a PNSE-subset of $(\mathfrak{V}, \mathfrak{Z})$ and its given as follows:

$$\mathfrak{F}_\mu (z_i)_0 = \{\mathfrak{F}(z_i)(v_n), \mu(z_i)(v_n)\}, \forall z_i \in A \subseteq \mathfrak{Z} = \mathfrak{P} \times \mathfrak{M} \times 0.$$

4. Fundamental set theoretic operations of PNSE-set

In the next part, we offer some fundamental mathematical operations on PNSE-set, namely complement on one set of PNSE-set, union, and intersection on two or more sets of PNSE-set, followed by AND, OR operations on two or more sets of PNSE-set. Finally, we offer some properties related to these operations with suitable examples.

Definition 4.1. Let $\left(\mathfrak{F}_\mu, \dot{A}\right)$ be a PNSE-set over fixed set (soft universe) $(\mathfrak{V}, \mathfrak{Z})$. Then the complement of a PNSE-set $\left(\mathfrak{F}_\mu, \dot{A}\right)$ indicated by $\left(\mathfrak{F}_\mu, \dot{A}\right)^c$ is given as follows:

$$\left(\mathfrak{F}_\mu, \dot{A}\right)^c = \mathfrak{F}^c_\mu (z_i) = \{\ddot{c}(\mathfrak{F}(z)(v_n)), \dot{c}(\mu(z)(v_n))\}$$

where $\ddot{c}$ indicates a neutrosophic complement and $\dot{c}$ indicates a fuzzy complement.

Example 4.2. Take the part given in Example 3.3 where,
$\mathfrak{F}_\mu (p_1, m_1, 1)$
$= \left\{ \left(\frac{v_1}{\langle 0.5,0.3,0.1\rangle}, 0.2\right), \left(\frac{v_2}{\langle 0.4,0.2,0.7\rangle}, 0.3\right), \left(\frac{v_3}{\langle 0.6,0.1,0.6\rangle}, 0.8\right) \right\}$
Now, by employing the neutrosophic complement and fuzzy complement, we get the complement of the part that is given by $\mathfrak{F}^c_\mu (p_1, m_1, 1)$
$= \left\{ \left(\frac{v_1}{\langle 0.1,0.7,0.5\rangle}, 0.8\right), \left(\frac{v_2}{\langle 0.7,0.8,0.4\rangle}, 0.7\right), \left(\frac{v_3}{\langle 0.6,0.9,0.6\rangle}, 0.2\right) \right\}$

Proposition 4.3. *Let $\left(\mathfrak{F}_\mu, \dot{A}\right)$ be a PNSE-set over fixed set $(\mathfrak{V}, \mathfrak{Z})$. Then the following property applies:*

$$\left(\left(\mathfrak{F}_\mu, \dot{A}\right)^c\right)^c = \left(\mathfrak{F}_\mu, \dot{A}\right)$$

Proof. Assume that $\left(\mathfrak{F}_\mu, \dot{A}\right)$ be a PNSE-set over fixed set $(\mathfrak{V}, \mathfrak{Z})$ and defined as $\left(\mathfrak{F}_\mu, \dot{A}\right) = \mathfrak{F}_\mu(z_i) = (\mathfrak{F}(z_i), \mu(z_i))$.
Now, let $\left(\mathfrak{F}_\mu, \dot{A}\right)^c = \left(\mathfrak{G}_\varphi, \dot{B}\right)$.
Then based on definition 4.1 $\left(\mathfrak{G}_\mu, \dot{B}\right) = \mathfrak{G}_\varphi(z_i) = (\mathfrak{G}(z_i), \varphi(z_i))$.Such that $\mathfrak{G}(z_i) = \ddot{c}(\mathfrak{F}(z_i))$ and $\varphi(z_i) = \dot{c}(\mu(z_i))$. Thus it leads us to
$\left(\mathfrak{G}_\mu, \dot{B}\right)^c = \mathfrak{G}^c_\varphi(z_i) = (\ddot{c}(\mathfrak{G}(z_i)), \dot{c}(\varphi(z_i))) = (\ddot{c}(\ddot{c}(\mathfrak{F}(z_i))), \dot{c}(\dot{c}(\mu(z_i)))) = (\mathfrak{F}(z_i), \mu(z_i)) = \left(\mathfrak{F}_\mu, \dot{A}\right)$.
Thus $\left(\left(\mathfrak{F}_\mu, \dot{A}\right)^c\right)^c = (\mathfrak{G}_\varphi, B)^c = \left(\mathfrak{F}_\mu, \dot{A}\right)$. Hence we get $\left(\left(\mathfrak{F}_\mu, \dot{A}\right)^c\right)^c = \left(\mathfrak{F}_\mu, \dot{A}\right)$. □

Definition 4.4. If $\left(\mathfrak{F}_\mu, \dot{A}\right)$ and $\left(\mathfrak{G}_\varphi, \dot{B}\right)$ two PNSE-sets on fixed set (soft universe) $(\mathfrak{V}, \mathfrak{Z})$. Then the union operation of these sets is also PNSE-set $\left(\mathfrak{H}_\Psi, \dot{C}\right)$ and denoted by $\left(\mathfrak{H}_\Psi, \dot{C}\right) = \left(\mathfrak{F}_\mu, \dot{A}\right) = \ddot{\cup}\,(\mathfrak{G}_\varphi, B)$. Where $\dot{C} = \dot{A} \cup \dot{B}$ and

$\Psi(z_i) = max(\mu(z_i), \varphi(z_i)), \qquad \forall z_i \in \dot{C} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}.$

$\mathfrak{H}(z_i) = \mathfrak{F}(z_i)\ddot{\cup}\mathfrak{G}(z_i), \qquad \forall z_i \in \dot{C} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}.$

where

$$\mathfrak{H}(z_i) = \begin{cases} \mathfrak{F}(z_i) & ,\text{if} z_i \in \dot{A} - \dot{B} \\ \mathfrak{G}(z_i) & ,\text{if} z_i \in \dot{B} - \dot{A} \\ max(\mathfrak{F}(z_i), \mathfrak{G}(z_i)) & ,\text{if} z_i \in \dot{A} \cap \dot{B} \end{cases}$$

Proposition 4.5. *Let* $\left(\mathfrak{F}_\mu, \dot{A}\right)$, $\left(\mathfrak{G}_\varphi, \dot{B}\right)$ *and* $\left(\mathfrak{H}_\Psi, \dot{C}\right)$ *be any three optional PNSE-sets over* $(\mathfrak{V}, \mathfrak{Z})$.*Then the following results are achieved:*

(i). $\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cup} \left(\mathfrak{G}_\varphi, \dot{B}\right) = \left(\mathfrak{G}_\varphi, \dot{B}\right) \ddot{\cup} \left(\mathfrak{F}_\mu, \dot{A}\right)$.*(Aommutative Condition)*

(ii) $\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cup} \left(\left(\mathfrak{G}_\varphi, \dot{B}\right) \ddot{\cup} \left(\mathfrak{H}_\Psi, \dot{C}\right)\right) = \left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cup} \left(\mathfrak{G}_\varphi, \dot{B}\right)\right) \ddot{\cup} \left(\mathfrak{H}_\Psi, \dot{C}\right)$.*(Associative Condition)*

Proof. Assume that $\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cup} \left(\mathfrak{G}_\varphi, \dot{B}\right) = \left(\mathfrak{H}_\Psi, \dot{C}\right)$.Then based on Definition 4.4, $\forall z_i \in \dot{C} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}$. we have

$$\left(\mathfrak{H}_\Psi, \dot{C}\right) = \mathfrak{H}_\Psi(z_i) = (\mathfrak{H}(z_i), \Psi(z_i))$$

where $\mathfrak{H}(z_i) = \mathfrak{F}(z_i)\ddot{\cup}\mathfrak{G}(z_i)$ and $\Psi(z_i) = max(\mu(z_i), \varphi(z_i))$. So,$\mathfrak{H}(z_i) = \mathfrak{F}(z_i)\ddot{\cup}\mathfrak{G}(z_i) = \mathfrak{G}(z_i)\ddot{\cup}\mathfrak{F}(z_i)$ and $\Psi(z_i) = max(\mu(z_i), \varphi(z_i)) = max(\varphi(z_i), \mu(z_i))$. we have the union of these sets is commutative by Definition 4.4.

Therefore,$\left(\mathfrak{H}_\Psi, \dot{C}\right) = \left(\mathfrak{G}_\varphi, \dot{B}\right) \ddot{\cup} \left(\mathfrak{F}_\varphi, \dot{A}\right)$.

Then we get the union of two PNSE-sets is commutative, such that $\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cup} \left(\mathfrak{G}_\varphi, \dot{B}\right) = \left(\mathfrak{G}_\varphi, \dot{B}\right) \ddot{\cup} \left(\mathfrak{F}_\mu, \dot{A}\right)$.

(ii) The proof of this part is equivalent to (i) and is therefore overlooked. □

Definition 4.6. If $\left(\mathfrak{F}_\mu, \dot{A}\right)$ and $\left(\mathfrak{G}_\varphi, \dot{B}\right)$ two PNSE-sets on fixed set (soft universe) $(\mathfrak{V}, \mathfrak{Z})$. Then the intersection operation of these sets is also PNSE-set $\left(\mathfrak{H}_\Psi, \dot{C}\right)$ and denoted by $\left(\mathfrak{H}_\Psi, \dot{C}\right) = \left(\mathfrak{F}_\mu, \dot{A}\right) = \ddot{\cap} \left(\mathfrak{G}_\varphi, \dot{B}\right)$. Where $\dot{C} = \dot{A} \cup \dot{B}$ and

$\Psi(z_i) = min(\mu(z_i), \varphi(z_i)), \qquad \forall z_i \in \dot{C} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}.$

$\mathfrak{H}(z_i) = \mathfrak{F}(z_i)\ddot{\cap}\mathfrak{G}(z_i), \qquad \forall z_i \in \dot{C} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}.$

where

$$\mathfrak{H}(z_i) = \begin{cases} \mathfrak{F}(z_i) & ,\text{if} z_i \in \dot{A} - dotB \\ \mathfrak{G}(z_i) & ,\text{if} z_i \in \dot{B} - \dot{A} \\ min(\mathfrak{F}(z_i), \mathfrak{G}(z_i)) & ,\text{if} z_i \in \dot{A} \cap \dot{B} \end{cases}$$

Proposition 4.7. *Let $\left(\mathfrak{F}_\mu, \dot{A}\right)$, $\left(\mathfrak{G}_\varphi, \dot{B}\right)$ and $\left(\mathfrak{H}_\Psi, \dot{A}\right)$ be any three optional PNSE-sets over $(\mathfrak{V}, \mathfrak{Z})$. Then the following results are achieved:*

(i). $\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cap} (\mathfrak{G}_\varphi, B) = \left(\mathfrak{G}_\varphi, \dot{B}\right) \ddot{\cap} \left(\mathfrak{F}_\mu, \dot{B}\right)$*.(Aommutative Condition)*
(ii) $\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cap} \left(\left(\mathfrak{G}_\varphi, \dot{B}\right) \ddot{\cap} \left(\mathfrak{H}_\Psi, \dot{C}\right)\right) = \left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cap} \left(\mathfrak{G}_\varphi, \dot{B}\right)\right) \ddot{\cap} \left(\mathfrak{H}_\Psi, \dot{C}\right)$*.(Associative Condition)*

Proof. The proof of these two parts (i, ii) is equivalent to (i, ii) in proposition 4. 5 and are and are overlooked. □

Proposition 4.8. *Let $\left(\mathfrak{F}_\mu, \dot{A}\right)$, $\left(\mathfrak{G}_\varphi, \dot{B}\right)$ and $\left(\mathfrak{H}_\Psi, \dot{C}\right)$ be any three optional PNSE-sets over $(\mathfrak{V}, \mathfrak{Z})$. Then the following results are satisfying:*
(i). $\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cup} \left(\left(\mathfrak{G}_\varphi, \dot{B}\right) \ddot{\cap} \left(\mathfrak{H}_\Psi, \dot{C}\right)\right) = \left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cup} \left(\mathfrak{G}_\varphi, \dot{B}\right)\right) \ddot{\cap} \left((\mathfrak{F}_\mu, A) \ddot{\cup} \left(\mathfrak{H}_\Psi, \dot{C}\right)\right)$
(ii). $\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cap} \left(\left(\mathfrak{G}_\varphi, \dot{B}\right) \ddot{\cup} \left(\mathfrak{H}_\Psi, \dot{C}\right)\right) = \left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cap} \left(\mathfrak{G}_\varphi, \dot{B}\right)\right) \ddot{\cup} \left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cap} \left(\mathfrak{H}_\Psi, \dot{C}\right)\right)$

Proof. The proof of these propositions clear dependency Definitions 4.4 and 4.6 and is therefore overlooked. □

Proposition 4.9. *Let $\left(\mathfrak{F}_\mu, \dot{A}\right)$ and $\left(\mathfrak{G}_\varphi, \dot{A}\right)$ be any two optional PNSE-sets over $(\mathfrak{V}, \mathfrak{Z})$. Then De Morgans laws satisfying:*
(i). $\left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cup} \left(\mathfrak{G}_\varphi, \dot{A}\right)\right)^c = \left(\left(\mathfrak{F}_\mu, \dot{A}\right)^c \ddot{\cap} \left(\mathfrak{G}_\varphi, \dot{A}\right)^c\right)$.
(ii). $\left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cap} \left(\mathfrak{G}_\varphi, \dot{A}\right)\right)^c = \left(\left(\mathfrak{F}_\mu, \dot{A}\right)^c \ddot{\cup} \left(\mathfrak{G}_\varphi, \dot{A}\right)^c\right)$.

Proof. (i) Assume that $\left(\mathfrak{F}_\mu, \dot{A}\right)$ and $\left(\mathfrak{G}_\varphi, \dot{A}\right)$ be any two optional PNSE-sets over $(\mathfrak{V}, \mathfrak{Z})$ defined as following:

$$\left(\mathfrak{F}_\Psi, \dot{A}\right) = \mathfrak{F}_\mu(z_i) = (\mathfrak{F}(z_i), \mu(z_i)), \qquad \forall z_i \in \dot{C} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}.$$
$$\left(\mathfrak{G}_\varphi, \dot{B}\right) = \mathfrak{G}_\varphi(z_i) = (\mathfrak{G}(z_i), \varphi(z_i)), \qquad \forall z_i \in \dot{C} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}.$$

Now, since the commutative and associative properties are fulfilled with PNSE-set, it follows that

$$\left(\mathfrak{F}_\mu, \dot{A}\right)^c \ddot{\cup} \left(\mathfrak{G}_\varphi, \dot{B}\right)^c$$
$$= (\mathfrak{F}(z_i), \mu(z_i))^c \ddot{\cup} (\mathfrak{G}(z_i), \varphi(z_i))^c$$
$$= (\ddot{c}(\mathfrak{F}(z_i)), \dot{c}(\mu(z_i))) \ddot{\cup} (\ddot{c}(\mathfrak{G}(z_i)), \dot{c}(\varphi(z_i)))$$
$$= (\ddot{c}(\mathfrak{F}(z_i)), \ddot{\cup}\ddot{c}(\mathfrak{G}(z_i))) \max(\dot{c}(\mu(z_i)), \dot{c}(\varphi(z_i)))$$

$= (\ddot{c}(\mathfrak{F}(z_i)\ddot{\cap}\mathfrak{G}(z_i))), \dot{c}(\min(\mu(z_i), \varphi(z_i)))$
$= \left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\cap} \left(\mathfrak{G}_\varphi, \dot{B}\right)\right)^c.$

(ii) (ii)The proof of the(ii) is comparable to the proof of the (i) and therefore overlooked. □

Definition 4.10. Let $\left(\mathfrak{F}_\mu, \dot{A}\right)$and $\left(\mathfrak{G}_\varphi, \dot{B}\right)$ be any two optional PNSE-sets over $(\mathfrak{V}, \mathfrak{Z})$.Then $\left(\mathfrak{F}_\mu, \dot{A}\right) AND \left(\mathfrak{G}_\varphi, \dot{B}\right)$ indicated by $\left(\mathfrak{F}_\mu, \dot{A}\right)\ddot{\wedge} \left(\mathfrak{G}_\varphi, \dot{B}\right)$ is a PNSE-set and defined as:

$$\left(\mathfrak{F}_\mu, \dot{A}\right)\ddot{\wedge} \left(\mathfrak{G}_\varphi, \dot{B}\right) = \left(\mathfrak{H}_\Psi, \dot{A} \times \dot{B}\right)$$

where $\left(\mathfrak{H}_\Psi, \dot{A} \times \dot{B}\right) = (\mathfrak{H}(z_i, z_j), \Psi(z_i, z_j))$, such that $\mathfrak{H}(z_i, z_j) = \mathfrak{F}(z_i)\ddot{\cap}\mathfrak{G}(z_j)$ and $\Psi(z_i, z_j) = min(\mu(z_i), \varphi(z_j))$, $\forall (z_i, z_j) \in A \times \dot{B} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}$ $and\ddot{\cap}$ depicts the basic intersection operation.

Definition 4.11. Let $\left(\mathfrak{F}_\mu, \dot{A}\right)$and $\left(\mathfrak{G}_\varphi, \dot{B}\right)$ be any two optional PNSE-sets over $(\mathfrak{V}, \mathfrak{Z})$.Then $\left(\mathfrak{F}_\mu, \dot{A}\right) OR \left(\mathfrak{G}_\varphi, \dot{B}\right)$ indicated by $\left(\mathfrak{F}_\mu, \dot{A}\right)\ddot{\vee} \left(\mathfrak{G}_\varphi, \dot{B}\right)$ is a PNSE-set and defined as:

$$\left(\mathfrak{F}_\mu, \dot{A}\right)\ddot{\vee} \left(\mathfrak{G}_\varphi, \dot{B}\right) = \left(\mathfrak{H}_\Psi, \dot{A} \times \dot{B}\right)$$

where $\left(\mathfrak{H}_\Psi, \dot{A} \times \dot{B}\right) = (\mathfrak{H}(z_i, z_j), \Psi(z_i, z_j))$, such that $\mathfrak{H}(z_i, z_j) = \mathfrak{F}(z_i)\ddot{\cup}\mathfrak{G}(z_j)$ and $\Psi(z_i, z_j) = max(\mu(z_i), \varphi(z_j))$, $\forall (z_i, z_j) \in \dot{A} \times \dot{B} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}$ $and\ddot{\cup}$ depicts the basic union.

Proposition 4.12. *Let* $\left(\mathfrak{F}_\mu, \dot{A}\right)$ *and* $\left(\mathfrak{G}_\varphi, \dot{B}\right)$ *be any two optional PNSE-sets over* $(\mathfrak{V}, \mathfrak{Z})$*.Then De Morgans laws satisfying:*
(i). $\left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\vee} \left(\mathfrak{G}_\varphi, \dot{B}\right)\right)^c = \left(\left(\mathfrak{F}_\mu, \dot{A}\right)^c \ddot{\wedge} \left(\mathfrak{G}_\varphi, \dot{B}\right)^c\right).$
(ii). $\left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\wedge} \left(\mathfrak{G}_\varphi, \dot{B}\right)\right)^c = \left(\left(\mathfrak{F}_\mu, \dot{A}\right)^c \ddot{\vee} (\mathfrak{G}_\varphi, B)^c\right).$

Proof. (i) Assume that $\left(\mathfrak{F}_\mu, \dot{A}\right)$ and $\left(\mathfrak{G}_\varphi, \dot{B}\right)$ be any two optional PNSE-sets over $(\mathfrak{V}, \mathfrak{Z})$ defined as following:

$$\left(\mathfrak{F}_\Psi, \dot{A}\right) = \mathfrak{F}_\mu(z_i) = (\mathfrak{F}(z_i), \mu(z_i)), \qquad \forall z_i \in \dot{C} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}.$$
$$\left(\mathfrak{G}_\varphi, \dot{B}\right) = \mathfrak{G}_\varphi(z_i) = (\mathfrak{G}(z_i), \varphi(z_i)), \qquad \forall z_i \in \dot{C} \subseteq \mathfrak{Z} = \{\mathfrak{P} \times \mathfrak{M} \times \mathfrak{Q}\}.$$

Now, since the commutative and associative properties are fulfilled with PNSE-set, it follows that
$\left(\mathfrak{F}_\mu, \dot{A}\right)^c \ddot{\vee} \left(\mathfrak{G}_\varphi, \dot{B}\right)^c$
$= (\mathfrak{F}(z_i), \mu(z_i))^c \ddot{\vee} (\mathfrak{G}(z_i), \varphi(z_i))^c$
$= (\ddot{c}(\mathfrak{F}(z_i)), \dot{c}(\mu(z_i))) \ddot{\vee} (\ddot{c}(\mathfrak{G}(z_i)), \dot{c}(\varphi(z_i)))$
$= (\ddot{c}(\mathfrak{F}(z_i)), \ddot{\vee}\ddot{c}(\mathfrak{G}(z_i))) \max(\dot{c}(\mu(z_i)), \dot{c}(\varphi(z_i)))$
$= (\ddot{c}(\mathfrak{F}(z_i)\ddot{\wedge}\mathfrak{G}(z_i))), \dot{c}(\min(\mu(z_i), \varphi(z_i)))$
$= \left(\left(\mathfrak{F}_\mu, \dot{A}\right) \ddot{\wedge} \left(\mathfrak{G}_\varphi, \dot{B}\right)\right)^c.$

(ii)The proof of the second part is similar to the proof of the first part therefore omitted. □

Proposition 4.13. *Let* $(\mathfrak{F}_\mu, A)$, $(\mathfrak{G}_\varphi, B)$ *and* $(\mathfrak{H}_\Psi, C)$ *be any three optional PNSE-sets over* $(\mathfrak{V}, \mathfrak{Z})$. *Then the following results are achieved:*

(i). $(\mathfrak{F}_\mu, A)\ddot{\vee}\ ((\mathfrak{G}_\varphi, B)\,\ddot{\vee}\,(\mathfrak{H}_\Psi, C)) =((\mathfrak{F}_\mu, A)\,\ddot{\vee}\,(\mathfrak{G}_\varphi, B))\ddot{\vee}(\mathfrak{H}_\Psi, C)$.
(ii). $(\mathfrak{F}_\mu, A)\ddot{\wedge}\ ((\mathfrak{G}_\varphi, B)\,\ddot{\wedge}\,(\mathfrak{H}_\Psi, C)) =((\mathfrak{F}_\mu, A)\,\ddot{\wedge}\,(\mathfrak{G}_\varphi, B))\ddot{\wedge}(\mathfrak{H}_\Psi, C)$.
(iii). $(\mathfrak{F}_\mu, A)\ddot{\vee}\ ((\mathfrak{G}_\varphi, B)\,\ddot{\wedge}\,(\mathfrak{H}_\Psi, C)) =((\mathfrak{F}_\mu, A)\,\ddot{\vee}\,(\mathfrak{G}_\varphi, B))\ddot{\wedge}\,((\mathfrak{F}_\mu, A)\,\ddot{\vee}\,(\mathfrak{H}_\Psi, C))$.
(iV). $(\mathfrak{F}_\mu, A)\ddot{\wedge}\ ((\mathfrak{G}_\varphi, B)\,\ddot{\vee}\,(\mathfrak{H}_\Psi, C)) =((\mathfrak{F}_\mu, A)\,\ddot{\wedge}\,(\mathfrak{G}_\varphi, B))\ddot{\vee}\,((\mathfrak{F}_\mu, A)\,\ddot{\wedge}\,(\mathfrak{H}_\Psi, C))$.

Proof. The proof of these propositions are clear by Definitions 4.10 and 4.11 and therefore omitted. □

Remark 4.14. Due $A \times B \neq B \times A$, therefore AND operation and OR operation don't satisfy commutative law.

Example 4.15. Let $(\mathfrak{F}_\mu, A)$ and $(\mathfrak{G}_\varphi, B)$ be any two optional PNSE-sets over $(\mathfrak{V}, \mathfrak{Z})$ and let $A = \{(p_1, m_1, 1), (p_2, m_2, 1)\}$,$B = \{(p_2, m_2, 1), (p_1, m_1, 0)\}$. Then the PNSE-set defined as bellow:

$(\mathfrak{F}_\mu, A) =$
$\Big\{(p_1, m_1, 1) = \Big\{\Big(\frac{v_1}{\langle 0.5,0.3,0.1\rangle}, 0.2\Big), \Big(\frac{v_2}{\langle 0.4,0.2,0.7\rangle}, 0.3\Big), \Big(\frac{v_3}{\langle 0.6,0.1,0.6\rangle}, 0.5\Big)\Big\},$
$(p_2, m_2, 1) = \Big\{\Big(\frac{v_1}{\langle 0.6,0.3,0\rangle}, 0.5\Big), \Big(\frac{v_2}{\langle 0.5,0.3,0.8\rangle}, 0.4\Big), \Big(\frac{v_3}{\langle 0.1,0.5,0.2\rangle}, 0.9\Big)\Big\}\Big\}$
and
$(\mathfrak{G}_\mu, B) =$
$\Big\{(p_2, m_2, 1) = \Big\{\Big(\frac{v_1}{\langle 0.3,0.4,0\rangle}, 0.7\Big), \Big(\frac{v_2}{\langle 0.3,0.7,0.2\rangle}, 0.4\Big), \Big(\frac{v_3}{\langle 0.1,0.4,0.8\rangle}, 0.6\Big)\Big\},$
$(p_1, m_1, 0) = \Big\{\Big(\frac{v_1}{\langle 0.3,0.7,0.5\rangle}, 0.8\Big), \Big(\frac{v_2}{\langle 0.6,0.3,0.2\rangle}, 0.7\Big), \Big(\frac{v_3}{\langle 0.3,0.4,0.8\rangle}, 1\Big)\Big\}\Big\}$
Then,
$(\mathfrak{F}_\mu, A)\,\ddot{\cup}\,(\mathfrak{G}_\mu, B) =$
$\Big\{(p_1, m_1, 1) = \Big\{\Big(\frac{v_1}{\langle 0.5,0.3,0.1\rangle}, 0.2\Big), \Big(\frac{v_2}{\langle 0.4,0.2,0.7\rangle}, 0.3\Big), \Big(\frac{v_3}{\langle 0.6,0.1,0.6\rangle}, 0.5\Big)\Big\},$
$(p_2, m_2, 1) = \Big\{\Big(\frac{v_1}{\langle 0.6,0.3,0\rangle}, 0.7\Big), \Big(\frac{v_2}{\langle 0.5,0.3,0.2\rangle}, 0.4\Big), \Big(\frac{v_3}{\langle 0.1,0.4,0.2\rangle}, 0.9\Big)\Big\},$
$(p_1, m_1, 0) = \Big\{\Big(\frac{v_1}{\langle 0.3,0.7,0.5\rangle}, 0.8\Big), \Big(\frac{v_2}{\langle 0.6,0.3,0.2\rangle}, 0.7\Big), \Big(\frac{v_3}{\langle 0.3,0.4,0.8\rangle}, 1\Big)\Big\}\Big\}.$
$(\mathfrak{F}_\mu, A)\,\ddot{\cap}\,(\mathfrak{G}_\mu, B) =$
$\Big\{(p_1, m_1, 1) = \Big\{\Big(\frac{v_1}{\langle 0.5,0.3,0.1\rangle}, 0.2\Big), \Big(\frac{v_2}{\langle 0.4,0.2,0.7\rangle}, 0.3\Big), \Big(\frac{v_3}{\langle 0.6,0.1,0.6\rangle}, 0.5\Big)\Big\},$
$(p_2, m_2, 1) = \Big\{\Big(\frac{v_1}{\langle 0.3,0.4,0\rangle}, 0.5\Big), \Big(\frac{v_2}{\langle 0.3,0.7,0.8\rangle}, 0.4\Big), \Big(\frac{v_3}{\langle 0.1,0.5,0.8\rangle}, 0.6\Big)\Big\},$
$(p_1, m_1, 0) = \Big\{\Big(\frac{v_1}{\langle 0.3,0.7,0.5\rangle}, 0.8\Big), \Big(\frac{v_2}{\langle 0.6,0.3,0.2\rangle}, 0.7\Big), \Big(\frac{v_3}{\langle 0.3,0.4,0.8\rangle}, 1\Big)\Big\}\Big\}.$
$(\mathfrak{F}_\mu, A)\,\ddot{\vee}\,(\mathfrak{G}_\mu, B) = (\mathfrak{H}_\Psi, C = A \times B) =$

$$\Big\{(p_1,m_1,1),(p_1,m_2,1)=\Big\{\Big(\tfrac{v_1}{\langle 0.5,0.3,0.1\rangle},0.7\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.4\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.6\Big)\Big\},$$
$$(p_1,m_1,1),(p_1,m_1,0)=\Big\{\Big(\tfrac{v_1}{\langle 0.5,0.3,0.1\rangle},0.8\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.7\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.1\Big)\Big\},$$
$$(p_2,m_2,1),(p_2,m_2,1)=\Big\{\Big(\tfrac{v_1}{\langle 0.5,0.3,0.1\rangle},0.7\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.4\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.9\Big)\Big\},$$
$$(p_2,m_2,1),(p_1,m_1,0)=\Big\{\Big(\tfrac{v_1}{\langle 0.5,0.3,0.1\rangle},0.8\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.7\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},1\Big)\Big\}\Big\}$$

and

$$(\mathfrak{F}_\mu,A)\,\ddot{\wedge}\,(\mathfrak{G}_\mu,B)=(\mathfrak{H}_\Psi,C=A\times B)=$$
$$\Big\{(p_1,m_1,1),(p_1,m_2,1)=\Big\{\Big(\tfrac{v_1}{\langle 0.5,0.3,0.1\rangle},0.2\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.3\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.3\Big)\Big\},$$
$$(p_1,m_1,1),(p_1,m_1,0)=\Big\{\Big(\tfrac{v_1}{\langle 0.5,0.3,0.1\rangle},0.2\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.3\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.3\Big)\Big\},$$
$$(p_2,m_2,1),(p_2,m_2,1)=\Big\{\Big(\tfrac{v_1}{\langle 0.5,0.3,0.1\rangle},0.2\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.3\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.3\Big)\Big\},$$
$$(p_2,m_2,1),(p_1,m_1,0)=\Big\{\Big(\tfrac{v_1}{\langle 0.5,0.3,0.1\rangle},0.2\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.3\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.3\Big)\Big\}\Big\}$$

5. **Decision-Making Application on PNSE-sets**

In this part, we introduce a new generalized algorithm to show the efficiency of the proposed model to help the decision maker (user) make the right decision from available alternatives based on hypothetical data, as in the following example.

Example 5.1. Suppose Mr. Xu wants to choose a primary school for his daughter out of three schools available in the universe, $\mathfrak{V}=\{v_1,v_2,v_3\}$. Mr. Xu asked for the opinion of three of his friends (experts) and could represent his friends (experts) by the set $\mathfrak{M}=\{m_1,m_2,m_3\}$ and the opines set $\mathfrak{Q}=\{1=agree,0=disagree\}$ describes the opinions set of Mr. Xu friends. Mr. Xu friends consider a set of attributes $\mathfrak{P}=\{p_1,p_2,p_3\}$ where the attributes represent the characteristics that depend on selecting the suitable school namely, $p_1=teachingquality$, $p_2=cost$, and $p_3=environment$, respectively. According to the evaluation of experts, the PNSE-set $(\mathfrak{F}_\mu,\mathfrak{Z}=\mathfrak{P})$ is obtained.

$$(\mathfrak{F}_\mu,\mathfrak{P})=$$
$$\Big\{(p_1,m_1,1)=\Big\{\Big(\tfrac{v_1}{\langle 0.5,0.3,0.1\rangle},0.2\Big),\Big(\tfrac{v_2}{\langle 0.4,0.2,0.7\rangle},0.3\Big),\Big(\tfrac{v_2}{\langle 0.6,0.1,0.6\rangle},0.5\Big)\Big\},$$

$$(p_2,m_1,1)=\Big\{\Big(\tfrac{v_1}{\langle 0.6,0.3,0\rangle},0.5\Big),\Big(\tfrac{v_2}{\langle 0.1,0.3,0.9\rangle},0.3\Big),\Big(\tfrac{v_3}{\langle 0.9,0.4,0.2\rangle},0.6\Big)\Big\},$$

$$(p_3,m_1,1)=\Big\{\Big(\tfrac{v_1}{\langle 0.4,0.1,0.2\rangle},0.3\Big),\Big(\tfrac{v_2}{\langle 0,0.1,0.7\rangle},0.5\Big),\Big(\tfrac{v_3}{\langle 0.7,0.2,0.4\rangle},0.5\Big)\Big\},$$

$$(p_1,m_2,1)=\Big\{\Big(\tfrac{v_1}{\langle 0.3,0.4,0.6\rangle},0.1\Big),\Big(\tfrac{v_2}{\langle 0.4,0.5,0.3\rangle},0.5\Big),\Big(\tfrac{v_3}{\langle 0.2,0.4,0.4\rangle},0.8\Big)\Big\},$$

$$(p_2, m_2, 1) = \left\{ \left(\frac{v_1}{\langle 0.3,0.3,0.6 \rangle}, 0.6 \right), \left(\frac{v_2}{\langle 0.8,0.4,0.6 \rangle}, 0.9 \right), \left(\frac{v_3}{\langle 0.5,0.5,0.7 \rangle}, 0.8 \right) \right\},$$

$$(p_3, m_2, 1) = \left\{ \left(\frac{v_1}{\langle 0.6,0.3,0 \rangle}, 0.5 \right), \left(\frac{v_2}{\langle 0.1,0.3,0.9 \rangle}, 0.7 \right), \left(\frac{v_3}{\langle 0.9,0.4,0.2 \rangle}, 0.6 \right) \right\},$$

$$(p_1, m_3, 1) = \left\{ \left(\frac{v_1}{\langle 0.3,0.4,0.6 \rangle}, 0.1 \right), \left(\frac{v_2}{\langle 0.4,0.5,0.3 \rangle}, 0.5 \right), \left(\frac{v_3}{\langle 0.2,0.4,0.4 \rangle}, 0.8 \right) \right\},$$

$$(p_2, m_3, 1) = \left\{ \left(\frac{v_1}{\langle 0.1,0.1,0.4 \rangle}, 0.3 \right), \left(\frac{v_2}{\langle 0.6,0.2,0.4 \rangle}, 0.8 \right), \left(\frac{v_3}{\langle 0.3,0.2,0.5 \rangle}, 0.6 \right) \right\},$$

$$(p_3, m_3, 1) = \left\{ \left(\frac{v_1}{\langle 0.5,0.4,0.7 \rangle}, 0.2 \right), \left(\frac{v_2}{\langle 0.3,0.5,0.6 \rangle}, 0.5 \right), \left(\frac{v_3}{\langle 0,0.3,0.6 \rangle}, 0.7 \right) \right\},$$

$$(p_1, m_1, 0) = \left\{ \left(\frac{v_1}{\langle 0.2,0.8,0.3 \rangle}, 0.5 \right), \left(\frac{v_2}{\langle 0.3,0.4,0.2 \rangle}, 0.5 \right), \left(\frac{v_3}{\langle 0.3,0.2,0.6 \rangle}, 0.8 \right) \right\},$$

$$(p_2, m_1, 0) = \left\{ \left(\frac{v_1}{\langle 0.4,0.9,0.2 \rangle}, 0.9 \right), \left(\frac{v_2}{\langle 0.4,0.3,0.2 \rangle}, 0.7 \right), \left(\frac{v_3}{\langle 0.3,0.4,0.7 \rangle}, 0.2 \right) \right\},$$

$$(p_3, m_1, 0) = \left\{ \left(\frac{v_1}{\langle 0.6,0.4,0.1 \rangle}, 0.6 \right), \left(\frac{v_2}{\langle 0.5,0.4,0.3 \rangle}, 0.8 \right), \left(\frac{v_3}{\langle 0.4,0.6,0.5 \rangle}, 0.6 \right) \right\},$$

$$(p_1, m_2, 0) = \left\{ \left(\frac{v_1}{\langle 0.4,0.3,0.3 \rangle}, 0.7 \right), \left(\frac{v_2}{\langle 0.2,0.6,0.6 \rangle}, 0.3 \right), \left(\frac{v_3}{\langle 0.7,0.3,0.5 \rangle}, 0.9 \right) \right\},$$

$$(p_2, m_2, 0) = \left\{ \left(\frac{v_1}{\langle 0.7,0.5,0.4 \rangle}, 0.7 \right), \left(\frac{v_2}{\langle 0.4,0.3,0.7 \rangle}, 0.8 \right), \left(\frac{v_3}{\langle 0.1,0.3,0.6 \rangle}, 0.4 \right) \right\},$$

$$(p_3, m_2, 0) = \left\{ \left(\frac{v_1}{\langle 0.5,0.3,0.6 \rangle}, 0.5 \right), \left(\frac{v_2}{\langle 0.1,0.5,0.8 \rangle}, 0.4 \right), \left(\frac{v_3}{\langle 0.5,0.3,0.7 \rangle}, 0.8 \right) \right\},$$

$$(p_1, m_3, 0) = \left\{ \left(\frac{v_1}{\langle 0.2,0.5,0.6 \rangle}, 0.5 \right), \left(\frac{v_2}{\langle 0.6,0.9,0.5 \rangle}, 0.2 \right), \left(\frac{v_3}{\langle 0.4,0,0.7 \rangle}, 0.8 \right) \right\},$$

$$(p_2, m_3, 0) = \left\{ \left(\frac{v_1}{\langle 0.5,0.4,0.2 \rangle}, 0.6 \right), \left(\frac{v_2}{\langle 0.3,0.7,0.3 \rangle}, 0.4 \right), \left(\frac{v_3}{\langle 0.8,0.3,0.6 \rangle}, 0.3 \right) \right\},$$

$$(p_3, m_3, 0) = \left\{ \left(\frac{v_1}{\langle 0.2,0.4,0.8 \rangle}, 0.3 \right), \left(\frac{v_2}{\langle 0.5,0.5,0.2 \rangle}, 0.9 \right), \left(\frac{v_3}{\langle 0.1,0.1,0.7 \rangle}, 0.1 \right) \right\} \right\}$$

Next, by using the proposed algorithm given below together with the PNSE-set model $(\mathfrak{F}_\mu, \mathfrak{P})$, we will solve the problem noted at the beginning of this part to help Mr. Xu choose the appropriate school. The generalised algorithm is shown below.

Algorithm 1

Step 1: Build a PNSE-set model $(\mathfrak{F}_\mu, \mathfrak{P})$ depending on opinion of Experts.

Step 2: Find the values of $\rho(z)(v_n) - \eta(z)(v_n) + \psi(z)(v_n)$ $\forall v_n \in \mathfrak{V}$., where

$\rho(z)(v_n), \eta(z)(v_n)$ and $\psi(z)(v_n)$ are the three neutrosophic membership functions(truth, indeterminacy and falsehood) $\forall v \in \mathfrak{V}$, respectively and $\mu(z)(v_n)$ indicated to possibility grade of $v \in \mathfrak{V}$.

Step 3: For both agree-PNSES and disagree-PNSES values, take the greatest numerical degree.

Step 4: Calculate values of the score $\mathfrak{R}_i = \mathfrak{M}_i - \mathfrak{N}_i$, where $\mathfrak{M}_i, \mathfrak{N}_i$ are degree for agree-PNSES and disagree-PNSES $\forall v_i \in \mathfrak{V}$

Step 5: Choose the value of the highest score in $\mathfrak{Z}_i = \max_{v_i \in \mathfrak{V}} \{\mathfrak{R}_i\}$.Then the decision is to choose an alternative v_i as the optimal or most suitable solution to the problem.

Now, from Table 1, we get the values $\rho(z)(v_n) - \eta(z)(v_n) + \psi(z)(v_n)$ $\forall v_n \in \mathfrak{V}$. It is to be noted that the first column and second column in Table 1 symbolize the values of $\rho(z)(v_n) - \eta(z)(v_n) + \psi(z)(v_n)$ and the degree of PNSE-set for all $v_n \in \mathfrak{V}$ respectively.
Tables 2 and 3 present the highest numerical degree for the elements in the agree-PNSE-set and disagree-PNSE-set, respectively.
The values of $\mathfrak{M}_i$ and $\mathfrak{N}_i$ are gaven in Table 4 and represent numerical grades for both the agree-PNSE-set and disagree-PNSE-set, respectively.
Then $\mathfrak{D}_i = \max_{v_i \in \mathfrak{V}} \{\mathfrak{R}_i\} = \{\mathfrak{R}_3\}$. Therefore, based on the opinions of experts, the appropriate school is v_3.

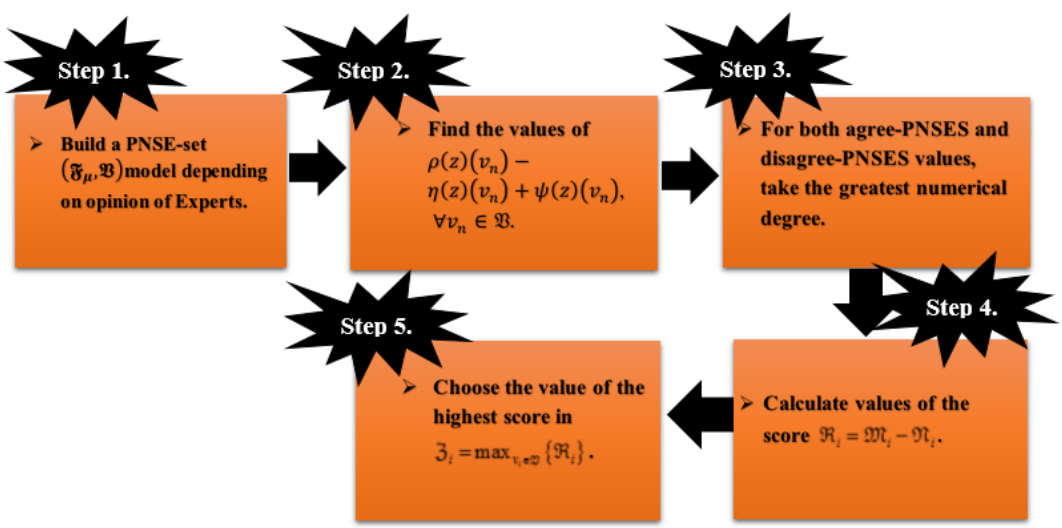

Figure 1: Representation of algorithm 1.

Remark 5.2. If we have more than one alternative with the highest $\mathbb{R}_i$ grade, then any of those alternatives can be selected as the best solution to the problem.

TABLE 1. Values of $\rho(z)(v_n) - \eta(z)(v_n) + \psi(z)(v_n)\ \forall v_n \in \mathfrak{V}$.

V_n	v_1	v_2	v_3
$(p_1, m_1, 1)$	0.3,0.2	0.9,0.3	-0.1,0.5
$(p_2, m_1, 1)$	0.3,0.5	0.7,0.3	0.7,0.6
$(p_3, m_1, 1)$	0.5,0.3	0.6,0.5	0.9,0.5
$(p_1, m_2, 1)$	0.5,0.1	0.2,0.5	0.2,0.8
$(p_2, m_2, 1)$	0.6,0.6	-0.2,0.9	0.7,0.8
$(p_3, m_2, 1)$	0.3,0.5	0.7,0.7	0.7,0.6
$(p_1, m_3, 1)$	0.5,0.1	0.2,0.5	0.2,0.8
$(p_2, m_3, 1)$	0.4,0.3	0,0.8	0.6,0.6
$(p_3, m_3, 1)$	0.8,0.2	0.4,0.5	0.3,0.7
$(p_1, m_1, 0)$	-0.3,0.5	0.1,0.5	0.7,0.8
$(p_2, m_1, 0)$	-0.3,0.9	0.1,0.7	0.6,0.2
$(p_3, m_1, 0)$	0.1,0.6	0.4,0.8	0.3,0.6
$(p_1, m_2, 0)$	0.4,0.7	0.2,0.3	-0.1,0.9
$(p_2, m_2, 0)$	0.6,0.7	0.8,0.8	0.4,0.4
$(p_3, m_2, 0)$	0.8,0.5	0.4,0.4	0.9,0.8
$(p_1, m_3, 0)$	0.3,0.5	0.2,0.2	0.1,0.4
$(p_2, m_3, 0)$	0.3,0.6	0.7,0.4	0.2,0.3
$(p_3, m_3, 0)$	0.6,0.3	0.2,0.9	0.3,0.1

TABLE 2. Numerical grade for agree-PNSES.

	V_n	**Highest numerical grade**	**Degree of possibility**
(p_1, m_1)	v_2	0.9	0.3
(p_2, m_1)	v_3	0.7	0.6
(p_3, m_1)	v_3	0.9	0.5
(p_1, m_2)	v_1	0.5	0.1
(p_2, m_2)	v_3	0.7	0.8
(p_3, m_2)	v_2	0.7	0.7
(p_1, m_3)	v_1	0.5	0.1
(p_2, m_3)	v_3	0.6	0.6
(p_3, m_3)	v_1	0.8	0.2
Score(v_1)=0.26		Score(v_2)=0.76	Score(v_3)=1.79

6. Similarity Measure on PNSE-Sets

Similarity measures are considered essential tools in fuzzy set theory and its extensions, where numerous researchers have extensively studied it and employed it in many areas of our daily life, such as medical diagnosis, decision making, pattern recognition, and so forth. In

TABLE 3. Numerical grade for disagree-PNSES.

	V_n	**Highest numerical grade**	**Degree of possibility**
(p_1, m_1)	v_3	0.7	0.8
(p_2, m_1)	v_3	0.6	0.2
(p_3, m_1)	v_2	0.4	0.8
(p_1, m_2)	v_1	0.4	0.7
(p_2, m_2)	v_2	0.8	0.8
(p_3, m_2)	v_3	0.9	0.2
(p_1, m_3)	v_3	0.1	0.4
(p_2, m_3)	v_2	0.7	0.4
(p_3, m_3)	v_1	0.6	0.3
Score(v_1)=0.46		Score(v_2)=1.24	Score(v_3)=1.44

TABLE 4. The score of $\mathfrak{R}_i = \mathfrak{M}_i - \mathfrak{N}_i$

$\mathfrak{M}_i$	$\mathfrak{N}_i$	$\mathfrak{R}_i$
Score(v_1)=0.26	Score(v_1)=0.46	-0.2
Score(v_2)= 0.76	Score(v_2)=1.24	-0.48
Score(v_3)=1.79	Score(v_3)=1.44	0.35

this part, we illustrate the similarity measure between two PNSE-sets and use a medical diagnosis example to demonstrate the importance of the proposed similarity measures in solving real-world problems.

Definition 6.1. Let $\mathfrak{F}_\mu$ and $\mathfrak{G}_\varphi$ be two PNSE-sets over $(\mathfrak{V}, \mathfrak{Z})$. Similarity measure between $\mathfrak{F}_\mu$ and $\mathfrak{G}_\varphi$ indicated by $\hat{S}\left(\mathfrak{F}_\mu, \mathfrak{G}_\varphi\right)$ is defined as follows:

$$\hat{S}\left(\mathfrak{F}_\mu, \mathfrak{G}_\varphi\right) = \ddot{M}\left(\mathfrak{F}\left(z\right), \mathfrak{G}\left(z\right)\right) \times \ddot{M}\left(\mu\left(z\right), \varphi\left(z\right)\right),$$

such that

$$\ddot{M}\left(\mathfrak{F}\left(z\right), \mathfrak{G}\left(z\right)\right) = max \ddot{M}_i\left(\mathfrak{F}\left(z\right), \mathfrak{G}\left(z\right)\right),$$
$$\ddot{M}\left(\mu\left(z\right), \varphi\left(z\right)\right) = max \ddot{M}_i\left(\mu\left(z\right), \varphi\left(z\right)\right),$$

where

$$\ddot{M}_i\left(\mathfrak{F}\left(z\right), \mathfrak{G}\left(z\right)\right) = 1 - \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}\left(\dot{\phi}_{\mathfrak{F}(z_i)}\left(v_j\right) - \dot{\phi}_{\mathfrak{G}(z_i)}\left(v_j\right)\right)^2},$$

such that and,
$\dot{\phi}_{\mathfrak{F}_\mu(z)}\left(v_j\right) = \frac{\rho_{\mathfrak{F}_\mu(z_i)}(v_j)+\eta_{\mathfrak{F}_\mu(z_i)}(v_j)+\psi_{\mathfrak{F}_\mu(z_i)}(v_j)}{3}$, $\dot{\phi}_{\mathfrak{G}_\mu(z)}\left(v_j\right) = \frac{\rho_{\mathfrak{G}_\mu(z_i)}(v_j)+\eta_{\mathfrak{G}_\mu(z_i)}(v_j)+\psi_{\mathfrak{G}_\mu(z_i)}(v_j)}{3}$.

$$\ddot{M}\left(\mu\left(z_i\right),\varphi\left(z_i\right)\right)=1-\frac{\sum\limits_{j=1}^{n}\left|\mu_j(z_i)-\varphi_j(z_i)\right|}{\sum\limits_{j=1}^{n}\left|\mu_j(z_i)+\varphi_j(z_i)\right|}$$

Definition 6.2. Let $\mathfrak{F}_\mu$ and $\mathfrak{G}_\varphi$ be two PNSE-sets over $(\mathfrak{V},\mathfrak{Z})$.We say that $\mathfrak{F}_\mu$ and $\mathfrak{G}_\varphi$ are signicantly similar if $\hat{S}\left(\mathfrak{F}_\mu,\mathfrak{G}_\varphi\right)\geq\frac{1}{2}$.

Proposition 6.3. *Let $\mathfrak{F}_\mu$, $\mathfrak{G}_\varphi$ and $\mathfrak{H}_\lambda$ be three PNSE-sets over $(\mathfrak{V},\mathfrak{Z})$.Then the following results are achieved:*

(i). $\hat{S}\left(\mathfrak{F}_\mu,\mathfrak{G}_\varphi\right)=\hat{S}\left(\mathfrak{G}_\mu,\mathfrak{F}_\varphi\right)$.
(ii). $0\leq\hat{S}\left(\mathfrak{F}_\mu,\mathfrak{G}_\varphi\right)\leq 1$.
(iii).If $\mathfrak{F}_\mu=\mathfrak{G}_\varphi$ *then* $\hat{S}\left(\mathfrak{F}_\mu,\mathfrak{G}_\varphi\right)=1$.
(iv).$\mathfrak{F}_\mu\subseteq\mathfrak{G}_\varphi\subseteq\mathfrak{H}_\lambda$ *then* $\hat{S}\left(\mathfrak{F}_\mu,\mathfrak{G}_\varphi\right)\leq\hat{S}\left(\mathfrak{G}_\varphi,\mathfrak{H}_\lambda\right)$.
(v).If $\mathfrak{F}_\mu\cap\mathfrak{G}_\varphi=\Phi\Leftrightarrow\hat{S}\left(\mathfrak{F}_\mu,\mathfrak{G}_\varphi\right)=0$.

Proof. The proof of these propositions are clear by Definitions 6.1 and therefore omitted. □

6.1. *Application in Medical Diagnosis based on Similarity Measure of PNSE-set*

In this subsection, we crete an algorithm works to measure similarity ratio of two PNSE-sets. This proposed algorithm employ to estimate whether a sick person has dengue fever based on the accompanying symptoms. To run this algorithm, we created two models of PNSE-sets depends on the assistance of physicians (experts) such that the first PNSE-set represent illness stat and the second PNSE-set represent the ill person state. Based on similarity degree, if it is ≥ 0.5, then the ill person may have dengue fever.

Algorithm 2

Step 1: Create a PNSE-set $\mathfrak{F}_\mu$ for the disease (dengue fever), based on assistance of physicians (experts).

Step 2: Build PNSE-set $\mathfrak{G}_\varphi$ for the patient person describes the severity of the symptoms experienced by the sick person by helping a medical expert person.

Step 3: Calculate similarity measure between a PNSE-set $\mathfrak{F}_\mu$ for illness and a PNSE-set $\mathfrak{G}_\varphi$ for the patient person, and if the similarity ratio is ≥ 0.5, then the person might have dengue fever. Meanwhile, if the similarity ratio is $\prec 0.5$, the person might not have dengue fever.

Now, to test this proposed algorithm, we present an applied example to ascertain whether a person has dengue fever or not.

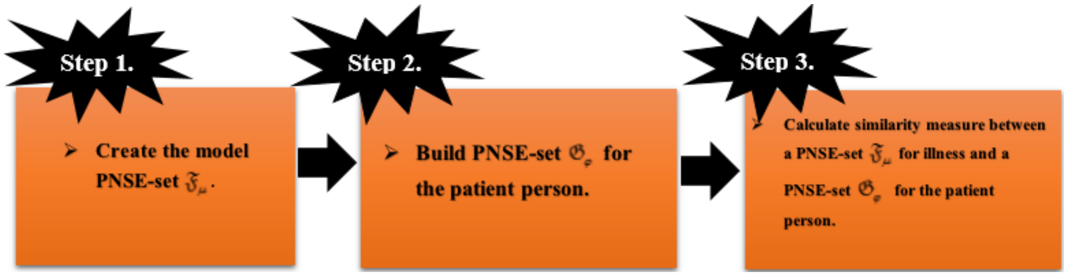

Figure 2: Representation of algorithm 2.

Example 6.4. Consider our universal set include only two alternatives, Yes and No, that is, $\mathfrak{V} = \{v_1 = Yes, v_2 = No\}$ and attributes set that includes a set of symptoms $\mathfrak{P} = \{p_1, p_2, p_3\}$ where p_1 =body temperature, p_2 =cough with chest congestion, and p_3 =headache.
Now, we apply our proposed algorithm.
Step 1: Create the model PNSE-set $\mathfrak{F}_{\mu}$ for dengue fever by the assistance of two physicians (experts), can be expressed with $\mathfrak{M} = \{m_1, m_2\}$ while the set $\mathfrak{Q} = \{1 = agree, 0 = disagree\}$ describes the set of opinions of two physicians (experts). :

$$\mathfrak{F}_{\mu} =$$

$$\Big\{(p_1, m_1, 1) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\}, (p_2, m_1, 1) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\},$$

$$(p_3, m_1, 1) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\}, (p_1, m_2, 1) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\},$$

$$(p_2, m_2, 1) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\}, (p_3, m_2, 1) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\},$$

$$(p_1, m_1, 0) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\}, (p_2, m_1, 0) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\},$$

$$(p_3, m_1, 0) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\}, (p_1, m_2, 0) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\},$$

$$(p_2, m_2, 0) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\}, (p_3, m_2, 0) = \Big\{\Big(\tfrac{v_1}{\langle 1,0,0\rangle}, 1\Big), \Big(\tfrac{v_2}{\langle 0,1,1\rangle}, 1\Big)\Big\}\Big\}$$

Step 2: Create a model of PNSE-set $\mathfrak{G}_{\varphi}$ for sick person X as following:

$$\mathfrak{G}_{\varphi} =$$

$$\Big\{(p_1, m_1, 1) = \Big\{\Big(\tfrac{v_1}{\langle 0.5,0.3,0.1\rangle}, 0.2\Big), \Big(\tfrac{v_2}{\langle 0.6,0.1,0.6\rangle}, 0.5\Big)\Big\},$$

$$(p_2, m_1, 1) = \Big\{\Big(\tfrac{v_1}{\langle 0.6,0.3,0\rangle}, 0.5\Big), \Big(\tfrac{v_3}{\langle 0.9,0.4,0.2\rangle}, 0.6\Big)\Big\},$$

$$(p_3, m_1, 1) = \Big\{\Big(\tfrac{v_1}{\langle 0.3,0.4,0.6\rangle}, 0.1\Big), \Big(\tfrac{v_3}{\langle 0.2,0.4,0.4\rangle}, 0.8\Big)\Big\},$$

$(p_1, m_2, 1) = \left\{ \left(\frac{v_1}{\langle 0.1,0.1,0.4\rangle}, 0.3 \right), \left(\frac{v_3}{\langle 0.3,0.2,0.5\rangle}, 0.6 \right) \right\},$

$(p_2, m_2, 1) = \left\{ \left(\frac{v_1}{\langle 0.2,0.8,0.3\rangle}, 0.5 \right), \left(\frac{v_3}{\langle 0.3,0.2,0.6\rangle}, 0.8 \right) \right\},$

$(p_3, m_2, 1) = \left\{ \left(\frac{v_1}{\langle 0.4,0.9,0.2\rangle}, 0.9 \right), \left(\frac{v_3}{\langle 0.3,0.4,0.7\rangle}, 0.2 \right) \right\},$

$(p_1, m_1, 0) = \left\{ \left(\frac{v_1}{\langle 0.4,0.3,0.3\rangle}, 0.4 \right), \left(\frac{v_3}{\langle 0.7,0.3,0.5\rangle}, 0.5 \right) \right\},$

$(p_2, m_1, 0) = \left\{ \left(\frac{v_1}{\langle 0.8,0.1,0.3\rangle}, 0.1 \right), \left(\frac{v_3}{\langle 0.2,0.1,0.5\rangle}, 0.7 \right) \right\},$

$(p_3, m_1, 0) = \left\{ \left(\frac{v_1}{\langle 0.3,0.4,0.3\rangle}, 0.4 \right), \left(\frac{v_3}{\langle 0.9,0.3,0.5\rangle}, 0.9 \right) \right\},$

$(p_1, m_2, 0) = \left\{ \left(\frac{v_1}{\langle 0.4,0.3,0.3\rangle}, 0.6 \right), \left(\frac{v_3}{\langle 0.7,0.3,0.5\rangle}, 0.6 \right) \right\},$

$(p_2, m_2, 0) = \left\{ \left(\frac{v_1}{\langle 0.5,0.5,0.7\rangle}, 0.4 \right), \left(\frac{v_3}{\langle 0.8,0.4,0.5\rangle}, 0.5 \right) \right\},$

$(p_3, m_2, 0) = \left\{ \left(\frac{v_1}{\langle 0.2,0.4,0.8\rangle}, 0.3 \right), \left(\frac{v_3}{\langle 0.1,0.1,0.7\rangle}, 0.1 \right) \right\}$

Step 3: Calculate similarity between $\mathfrak{F}_\varphi$ and $\mathfrak{G}_\varphi$ according to Definition 6.1 given above. Then,

$$\ddot{M}\left(\mu\left(z_1 = (p_1, m_1, 1)\right), \varphi\left(z_1 = (p_1, m_1, 1)\right)\right) = 1 - \frac{\sum_{j=1}^{2} |\mu_1(z_1) - \varphi_1(z_1)|}{\sum_{j=1}^{2} |\mu_1(z_1) + \varphi_1(z_1)|}$$
$$= 1 - \frac{|1-0.2| + |1-0.5|}{|1+0.2| + |1+0.5|} = 0.52$$

Similarly we get, $\ddot{M}(\mu(z_2), \varphi(z_2)) = 0.71$, $\ddot{M}(\mu(z_3), \varphi(z_3)) = 0.62$, $\ddot{M}(\mu(z_4), \varphi(z_4)) = 0.62$, $\ddot{M}(\mu(z_5), \varphi(z_5)) = 0.62$, $\ddot{M}(\mu(z_6), \varphi(z_6)) = 0.79$, $\ddot{M}(\mu(z_7), \varphi(z_7)) = 0.62$, $\ddot{M}(\mu(z_8), \varphi(z_8)) = 0.62$, $\ddot{M}(\mu(z_9), \varphi(z_9)) = 0.58$, $\ddot{M}(\mu(z_{10}), \varphi(z_{10})) = 0.85$, $\ddot{M}(\mu(z_{11}), \varphi(z_{11})) = 0.75$, $\ddot{M}(\mu(z_{12}), \varphi(z_{12})) = 0.34$,then
$\ddot{M}(\mu(z), \varphi(z)) = max\ddot{M}_i(\mu(z), \varphi(z))$,

$$\ddot{M}_1(\mathfrak{F}(z_1), \mathfrak{G}(z_1)) = 1 - \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} \left(\dot{\phi}_{\mathfrak{F}(z_1)}(v_j) - \dot{\phi}_{\mathfrak{G}(z_1)}(v_j) \right)^2},$$
$$= 1 - \frac{1}{\sqrt{2}} \sqrt{(1-0.3)^2 + (1-0.43)^2} = 0.36$$

Similarly, we get the rest of the values in Table 5

TABLE 5. Valudes of $\ddot{M}_1(\mathfrak{F}(z_i),\mathfrak{G}(z_i))$ and $\ddot{M}(\mu(z_i),\varphi(z_i))$

$\ddot{M}_1(\mathfrak{F}(z_i),\mathfrak{G}(z_i))$	**Degree**	$\ddot{M}(\mu(z_i),\varphi(z_i))$	**Degree**
$\ddot{M}_1(\mathfrak{F}(z_1),\mathfrak{G}(z_1))$	0.36	$\ddot{M}(\mu(z_2),\varphi(z_2))$	0.52
$\ddot{M}_1(\mathfrak{F}(z_2),\mathfrak{G}(z_2))$	0.39	$\ddot{M}(\mu(z_2),\varphi(z_2))$	0.71
$\ddot{M}_1(\mathfrak{F}(z_3),\mathfrak{G}(z_3))$	0.38	$\ddot{M}(\mu(z_3),\varphi(z_3))$	0.62
$\ddot{M}_1(\mathfrak{F}(z_4),\mathfrak{G}(z_4))$	0.26	$\ddot{M}(\mu(z_4),\varphi(z_4))$	0.62
$\ddot{M}_1(\mathfrak{F}(z_5),\mathfrak{G}(z_5))$	0.40	$\ddot{M}(\mu(z_5),\varphi(z_5))$	0.62
$\ddot{M}_1(\mathfrak{F}(z_6),\mathfrak{G}(z_6))$	0.48	$\ddot{M}(\mu(z_6),\varphi(z_6))$	0.79
$\ddot{M}_1(\mathfrak{F}(z_7),\mathfrak{G}(z_7))$	0.41	$\ddot{M}(\mu(z_7),\varphi(z_7))$	0.62
$\ddot{M}_1(\mathfrak{F}(z_8),\mathfrak{G}(z_8))$	0.41	$\ddot{M}(\mu(z_8),\varphi(z_8))$	0.62
$\ddot{M}_1(\mathfrak{F}(z_9),\mathfrak{G}(z_9))$	0.48	$\ddot{M}(\mu(z_9),\varphi(z_9))$	0.58
$\ddot{M}_1(\mathfrak{F}(z_{10}),\mathfrak{G}(z_{10}))$	0.56	$\ddot{M}(\mu(z_{10}),\varphi(z_{10}))$	0.85
$\ddot{M}_1(\mathfrak{F}(z_{11}),\mathfrak{G}(z_{11}))$	0.49	$\ddot{M}(\mu(z_{11}),\varphi(z_{11}))$	0.75
$\ddot{M}_1(\mathfrak{F}(z_{12}),\mathfrak{G}(z_{12}))$	0.64	$\ddot{M}(\mu(z_{12}),\varphi(z_{12}))$	0.34
$\ddot{M}(\mathfrak{F}(z),\mathfrak{G}(z))=0.64$		$\ddot{M}(\mu(z),\varphi(z))=0.85.$	

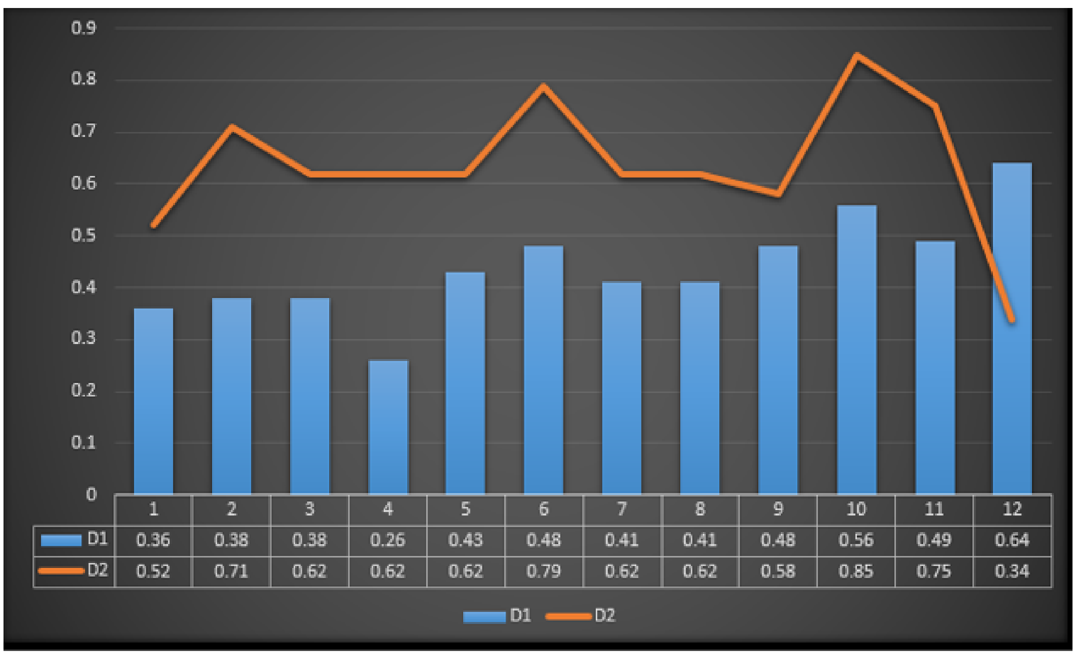

Figure 3: Statistical chart.

Then, the similarity measure between a PNSE-set $\mathfrak{F}_\mu$ for illness and a PNSE-set $\mathfrak{G}_\varphi$ for the patient person:

$\hat{S}(\mathfrak{F}_\mu,\mathfrak{G}_\varphi)=0.64\times 0.85=0.54$ (The patient has dengue fever).

From Table 5, we give the following statistical chart (Figure 3), which shows the differing opinions of experts (physicians) about the condition of patients based on the strength of symptoms. Where we will point to $\ddot{M}_1\left(\mathfrak{F}\left(z_i\right), \mathfrak{G}\left(z_i\right)\right)$ by symbol D_1 and $\ddot{M}\left(\mu\left(z_i\right), \varphi\left(z_i\right)\right)$ by symbol D_2.

7. Comparison with Some Methods in Literature

In the literature section, we mentioned that there are many contributions discussed based on fuzzy-like, intuitionistic fuzzy-like and neutrosophic-like. As a result, in this section, we will compare our proposed PNSE-set to other existing models that aim to find the relationship between the degree of probability and the fuzzy environment. First of all, the PNSE-set is an extension of PIFSE-set and PFSESet. With three neutrosopic membership functions, the PNSE-set can deal with alternatives and attributes in an alternatives set V and a set of attributes E in greater detail, whereas the PIFSE-set appears to have some weaknesses in dealing with alternatives and attributes that exist in an alternatives set V and an attributes set E. It can only get a handle on the uncertainty issues considering both the membership and non-membership values, whereas PNSE-set can get a handle on these issues as well as the issues containing indeterminacy and inconsistent data. These tools makes it more flexible and practical than the PIFSE-set. On the other hand, it is worthwhile to note that the PNSE-set was created to overcome one of the main shortcomings of the PNS-set so that it is more advantageous to deal with expert set opinions about alternatives and attributes that exist in an alternatives set V and a attributes set E.

To further clarify the usefulness and difference of our concept with other methods, we present Figure 4, which contains some basic criteria to back up this comparison.

Where the symbols (TM,FM,IM,PT,DOP,and ES) indicate to true membership, false membership, indeterminate membership, Parameterization tools, Degree of Possibility, and Expert set respectively. Finally, based on all that has been mentioned above, it can be said that our proposed concept is a generalization of all the concepts mentioned above.

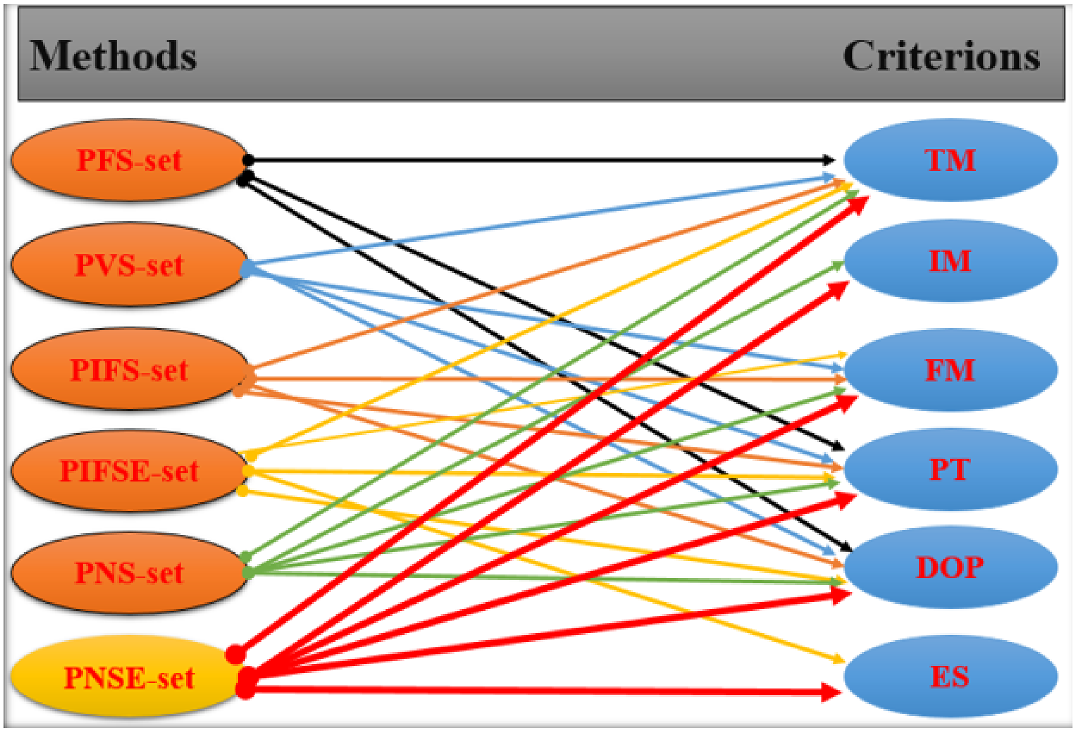

Figure 4: Comparison with current models under suitable criteria.

8. Conclusion

In this work, in the first part, the possibility neutrosophic soft expert set (PNSE-set) is developed in order to fix some weaknesses in [49]- [51]. Some properties and some fundamental set-theory were set up on PNSE-set. Also, using this method, we proposed an algorithm to solve the assumed problem in the decision-making problem. In the second part, we succeeded in applying similarity measures to this method by computing the similarity ratio between PNSE-sets. Then, these measures are applied to medical diagnosis to discover if the patient has dengue fever or not. In addition, a comparison between the existing methods and the PNSE-set was given. Finally, for further work on these topics, We recommend developing these tools by integrating them with some other mathematical structures, such as the hypersoft set [54]- [56], algebraic structures, topological structures, and other ideas [57]- [63].

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A Total Order on Single Valued and Interval Valued Neutrosophic Triplets

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Abstract. L.A.Zadeh (1965) proposed the concept of fuzzy subsets, which was later expanded to include intuitionistic fuzzy subsets by K.Atanassov (1983). We have come across several generalisations of sets since the birth of fuzzy sets theory, one of which is Florentine Smarandache [15] introduced the neutrosophic sets as a major category. Many real-life decision-making problems have been studied in [10], [13], [16]. In multi-criteria decision making (MCDM) situations [1], [2], [6], the ordering of neutrosophic triplets (T; I; F) is crucial. In this study, we define and analyse new membership, non-membership, and average score functions on single-valued neutrosophic triplets (T; I; F). We create a technique for ordering single valued neutrosophic triplets (SVNT) using these three functions, with the goal of achieving a total ordering on neutrosophic triplets. The total ordering on IVNT is then provided by extending these score functions and ranking mechanism to interval valued neutrosophic triplets (IVNT). A comparison is also made between the suggested method and the present ranking method in the literature.

Keywords: Neutrosophic Sets; Interval Valued Neutrosophic Triplets; MCDM

1. Introduction

Our daily life is filled with uncertain situations that require us to make the best decisions possible given the volatility. Despite this, L.A.Zadeh established the concept of fuzzy sets [18] in 1965 to handle such ambiguity. This idea of fuzzy sets, which claims that available data is not necessarily an accurate value but always contains the hand of uncertainty, was reluctantly acknowledged at the time and that analyzing this uncertainty or vagueness might bring a tremendous revolution in the future with real-life MCDM problems. Later, a great progress has been made in the research of fuzzy set generalisations resulting in numerous forms of fuzzy sets such as intuitionistic fuzzy sets, neutrosophic sets, picture fuzzy sets, bi-polar fuzzy sets, and so on [3], [4], [5], [15], [20]. These different versions of fuzzy sets were widely used in a variety of real-world problems.

More specifically, theory of neutrosophic sets is one of the most growing research areas due to its needfulness in various real life situations like Medical diagnosis [27], Supply chain management [25]. Neutrosophic sets are later expanded to many other research areas like Graph theory [28], Optimization [29], Goal Programming [23]. The MCDM is a rising topic of research due to its importance in most real-world challenges [12], [14], [17], [19]. Some MCDM problems have been studied in real-world scenarios using neutrosophic sets. To solve such MCDM problems, we need total ordering on neutrosophic triplets. For each fuzzy MCDM problem, there are several techniques of total ordering on fuzzy numbers available in the literature [7], [8], [9], [11]. Furthermore, the decision maker selects the total ordering strategy that best suits his needs. The total order does not have to be unique in fuzzy MCDM. Various kinds of MCDM and MADM problems have been studied based on neutrosophic sets in literature [21], [22], [26], [30].

Florentine Smarandache [16] defined three score functions on single-valued neutrosophic triplets based on which a total ordering on single-valued neutrosophic triplets has been proposed and the proposed score functions have been extended to interval valued neutrosophic triplets. But the proposed ordering methods in the literature give total ordering only on Single valued neutrosophic triplets but only neutrosophically total ordering / a partial ordering on Interval valued neutrosophic triplets. There is no total ordering method exists in the literature. To overcome this research gap, we construct a new total ordering on neutrosophic triplets in this study which can be extended to a total ordering method on Interval valued neutrosophic triplets, which contributes a lot to IVNT based MCDM problems.

In section 2, we define some key terms that will help us to comprehend the rest of the work. The proposed ranking method's motivation is presented in section 3. In sections 4 and 5, we introduce new scoring functions and suggest a complete total ordering technique for single-valued neutrosophic triplets based on those functions (T, I, F). The scoring functions and ranking approach in sections 4 and 5 are generalised to interval valued neutrosophic triplets in sections 7 and 8. In section 9, we detail our ranking method's algorithm as well as its comparison to other ranking method in literature. To achieve this we have considered an MCDM problem from [21]. In [21] the Similarity measure based MCDM ranking method have been studied, we inherits the method and modified the given data to our SVNT based MCDM data to compute the ranking throughwhich we compare our method with existing methods. In this section 9 the limitations of existing methods and advantages of our proposed method are discussed.

2. Preliminaries

This section contains all of the necessary definitions to move deeper into the concept of total ordering on neutrosophic triplets.

Definition 2.1. [16] Let $\mathcal{M} = \{(T, I, F), \text{where } T, I, F \in [0, 1], 0 \leq T + I + F \leq 3\}$ be the set of single valued neutrosophic triplet (SVNT) numbers. Let $N = (T, I, F) \in \mathcal{M}$ be a SVNT number, where T denotes grade of membership ; I denotes indeterminacy grade ; F denotes grade of non-membership .

Definition 2.2. [16] A SVNT score $s : \mathcal{M} \to [0, 1]$ is given by

$$s(T, I, F) = \frac{T + (1 - I) + (1 - F)}{3}.$$

A SVNT accuracy score $a : \mathcal{M} \to [-1, 1]$ is given by

$$a(T, I, F) = T - F.$$

A SVNT certainty score $c : \mathcal{M} \to [0, 1]$ is given by

$$c(T, I, F) = T.$$

With the foregoing functions, Smarandache created a total ordering in SVNT [16].

3. Motivation

Let us use the ranking method of [16] for following three neutrosophic triplets $n_1 = (1, 0, 0)$ where $t = 1, i = 0, f = 0$; $n_2 = (0, 1, 0)$ where $t = 0, i = 1, f = 0$ and $n_3 = (0, 0, 1)$ where $t = 0, i = 0, f = 1$.

It is natural to assume that the ranking order is $n_1 > n_2 > n_3$. We normally put full membership first and full non-membership last, since n_1, n_2 and n_3 signify absolute membership, hesitant (which is somewhat of membership and somewhat of non-membership), and absolute non-membership, respectively.

But, according to the ranking method of [16], we get $s(1, 0, 0) = 1$, $s(0, 1, 0) = \frac{1}{3}$, $s(0, 0, 1) = \frac{1}{3}$. Therefore, we get $R(n_1) > R(n_2) = R(n_3)$. So, we go to next step to find ordering between n_2 and n_3. Since $a(0, 1, 0) = 0$, $a(0, 0, 1) = -1$, $R(n_2) > R(n_3)$. Finally, we get the ranking $R(n_1) > R(n_2) > R(n_3)$. In this case, when we intuitively discovered the ranking order, we are unable to rank them using the score function (step 1) alone in the present technique and must rely on the accuracy function (step 2).

We intended to rank these types of triplets using the score function alone, rather than having to move on to the next function. The score function was defined in [16] by summing all the positive quantities ($T, (1 - I)$, and $(1 - F)$) of the triplet (T, I, F), with $1 - I$ and $1 - F$ representing positive triplet quantities. However, various portions of non-indeterminacy

$((1-I)T)$ and $((1-I)F)$ should be recognised. Positive and negative amounts of (T, I, F) may be represented as $1-I$, which is based on positive and negative quantities of neutrosophic information. As a result, we created a new membership score function by combining membership (T) and positive membership quantity from indeterminacy $((1-I)T)$, then subtracting negative membership (F), indeterminacy (I), and positive non-membership quantity from indeterminacy $((1-I)F)$. The proposed new score functions are based on this basic idea and reasoning.

4. Membership, Non-membership and Average score functions on SVNT

Let $\mathcal{M} = \{(T, I, F), \text{where } T, I, F \in [0, 1], 0 \leq T + I + F \leq 3\}$. where $T,\ I,\ F$ are single valued. Based on the motivation given in the last paragraph, the following score functions are defined.

Definition 4.1. A SVNT membership score $S^+ : \mathcal{M} \to [0, 1]$ is given by

$$S^+(T, I, F) = \frac{2 + T + (1-I)T - F - I - (1-I)F}{4} = \frac{2 + (T-F)(2-I) - I}{4}.$$

Definition 4.2. A SVNT non-membership score $S^- : \mathcal{M} \to [0, 1]$ is given by

$$S^-(T, I, F) = \frac{2 + F + (1-I)F - T - I - (1-I)T}{4} = \frac{2 + (F-T)(2-I) - I}{4}.$$

Definition 4.3. A SVNT average score $C : \mathcal{M} \to [0, 1]$ is given by

$$C(T, I, F) = \frac{T + F}{2}$$

Definition 4.4. A SVNT indeterminacy score $H : \mathcal{M} \to [0, \frac{1}{2}]$ given by

$$H(T, I, F) = \frac{I}{2}$$

Remark 4.5. We note that $0 \leq S^+ + S^- \leq 1$ because of $S^+ + S^- = 1 - \frac{I}{2}$(which is ≤ 1).

Remark 4.6. We note that $S^+ + S^- + H = 1$, which shows the sum of all membership, non-membership and indeterminacy scores equals to 1.

From the above remark, we note that S^+ and S^- form membership and non-membership functions of IFS with indeterminacy H. So, any neutrosophic set A can be viewed as intuitionistic fuzzy set $IF(A) = (S^+(A), S^-(A))$.

Remark 4.7. When there is no indeterminacy (i.e I=0), we get $S^+ + S^- = 1$, which is the fuzzy form of neutrosophic triplets.

Remark 4.8. As we mentioned in earlier, let us try to rank the following three triplets $n_1 = (1, 0, 0)$, $n_2 = (0, 1, 0)$ and $n_3 = (0, 0, 1)$ When we use membership score, we get $S^+(1, 0, 0) = 1$, $S^+(0, 1, 0) = \frac{1}{4}$ and $S^+(0, 0, 1) = 0$. Thus, we got the ranking as $R(n_1) > R(n_2) > R(n_3)$. As we mentioned in section 3, we have ranked these triplets by using score function itself.

Remark 4.9. We note that we can rank these triplets by using non-membership score as follows $S^-(1,0,0) = 0$, $S^-(0,1,0) = \frac{1}{4}$ and $S^-(0,0,1) = 1$ which gives us $R(1,0,0) > R(0,1,0) > R(0,0,1)$. Thus, again we get $R(n_1) > R(n_2) > R(n_3)$ by using non-membership score only.

5. A total order on SVNT

In this section, we present a new ranking technique for SVNT that preserves total ordering.

5.1. *New Ranking Algorithm for SVNT*

Let $A = (a,b,c)$ and $B = (d,e,f)$ be two SVNT of $\mathcal{M}$, where $T(A) = a$, $I(A) = b$, $F(A) = c$; $T(B) = d$, $I(B) = e$, $F(B) = f$ and $a,b,c,d,e,f \in [0,1]$.

Step 1: Apply proposed new neutrosophic membership score function S^+.

(1) If $S^+(a,b,c) > S^+(d,e,f)$ $(S^+(a,b,c) < S^+(d,e,f))$, then $(a,b,c) > (d,e,f)$ $((a,b,c) < (d,e,f))$.

(2) Suppose $S^+(a,b,c) = S^+(d,e,f)$, go to step 2.

Step 2: Apply proposed new neutrosophic non-membership score function S^-.

(1) If $S^-(a,b,c) > S^-(d,e,f)$ $(S^-(a,b,c) < S^-(d,e,f))$, then $(a,b,c) < (d,e,f)$ $((a,b,c) > (d,e,f))$.

(2) Suppose $S^-(a,b,c) = S^-(d,e,f)$, go to step 3.

Step 3: Apply proposed new neutrosophic average function C.

(1) If $C(a,b,c) > C(d,e,f)$ $(C(a,b,c) < C(d,e,f))$, then $(a,b,c) > (d,e,f)$ $((a,b,c) < (d,e,f))$.

(2) Suppose $C(a,b,c) = C(d,e,f)$, then conclude that $(a,b,c) \equiv (d,e,f)$.

Theorem 5.1. *A total order on $\mathcal{M}$ is formed by the single-valued neutrosophic membership, non-membership, and average score functions.*

Proof. Let $n_1 = (t_1, i_1, f_1)$ and $n_2 = (t_2, i_2, f_2)$ be two SVNT of $\mathcal{M}$. We show that for any two SVNT n_1 and n_2 in $\mathcal{M}$, either $n_1 < n_2$ or $n_1 > n_2$ or $n_1 = n_2$. First we apply membership score function S^+. Suppose $S^+ (n_1) > S^+ (n_2)($ or $S^+(n_1) < S^+(n_2))$, then we have $n_1 > n_2($ or $n_1 < n_2)$, which is done. When $S^+ (n_1) = S^+ (n_2)$, we have to go to step 2. So, Suppose $\frac{2+(t_1-f_1)(2-i_1)-i_1}{4} = \frac{2+(t_2-f_2)(2-i_2)-i_2}{4}$, equivalently, if $(t_1-f_1)(2-i_1)-i_1 = (t_2-f_2)(2-i_2)-i_2$, we apply step 2 using non-membership score. Hence, if $S^- (n_1) > S^- (n_2)$ $(S^- (n_1) < S^- (n_2))$, then $n_1 < n_2(n_1 > n_2)$, which is done. When $S^- (n_1) = S^- (n_2)$, equivalently, if $(f_1 - t_1)(2 - i_1) - i_1 = (f_2 - t_2)(2 - i_2) - i_2$, we have to go to step 3 using average score function. Hence, suppose $C (n_1) > C (n_2)$ (or $C (n_1) < C (n_2))$, then we have $n_1 > n_2$ (or $n_1 < n_2)$, which is done. When $C (n_1) = C (n_2)$, we have $t_1 + f_1 = t_2 + f_2$. At this stage, we have triplets n_1 and n_2 satisfying following system of 3 equations.

$$(t_1 - f_1)(2 - i_1) - i_1 = (t_2 - f_2)(2 - i_2) - i_2 \tag{1}$$

$$(f_1 - t_1)(2 - i_1) - i_1 = (f_2 - t_2)(2 - i_2) - i_2 \tag{2}$$

$$t_1 + f_1 = t_2 + f_2 \tag{3}$$

Now, we solve this system of equations. By adding equations 1 and 2, we get $i_1 = i_2$ which makes equation 1 into

$$t_1 - f_1 = t_2 - f_2$$

now, by adding the above equation with equation 3, we get $f_1 = f_2$ and $t_1 = t_2$.
Thus, we get

$$(t_1, i_1, f_1) = (t_2, i_2, f_2).$$

As a result, we infer that any two SVNT are either bigger than the other or identical. As a result, we have established a total ordering on $\mathcal{M}$. □

The following statement's proofs are direct applications of definitions, hence proofs are omitted.

Proposition 5.2. *Let $n_1 = (t_1, i_1, f_1)$ and $n_2 = (t_1, i_2, f_1)$*

(1) *If $i_1 > i_2$, then $R(n_1) < R(n_2)$.*
(2) *If $i_1 < i_2$, then $R(n_1) > R(n_2)$.*

Proposition 5.3. *Let $n_1 = (t_1, i_1, f_1)$ and $n_2 = (t_1, i_1, f_2)$*

(1) *If $f_1 > f_2$, then $R(n_1) < R(n_2)$.*
(2) *If $f_1 < f_2$, then $R(n_1) > R(n_2)$.*

Proposition 5.4. *Let $n_1 = (t_1, i_1, f_1)$ and $n_2 = (t_2, i_1, f_1)$*

(1) *If $t_1 > t_2$, then $R(n_1) > R(n_2)$.*
(2) *If $t_1 < t_2$, then $R(n_1) < R(n_2)$.*

Remark 5.5. Let $n_1 = (t, 0, f)$ and $n_2 = (t, 1, f)$ in which n_1 and n_2 have same membership and non-membership grades with n_1 has no indeterminacy and n_2 has full indeterminacy. Then $R(n_1) > R(n_2)$ which favors our intuition.

Remark 5.6. Let $n_1 = (t_1, i_1, f_1)$ and $n_2 = (t_2, i_1, f_2)$ i.e., indeterminacy of n_1 = indeterminacy of n_2. If $S^+(n_1) > S^+(n_2)$, then $S^-(n_1) < S^-(n_2)$ which is more logical.

6. Equivalence of proposed method over existing ranking method

In this section, we examine the new approach's equivalency to the existing ranking algorithm [16].

Remark 6.1. Our proposed Algorithm and Florentin Smarandache [16] Algorithm for ranking Neutrosophic Triplets are same when the triplets have same indeterminacy value.

Proof. Let $n_1 = (t_1, i_1, f_1)$ and $n_2 = (t_2, i_1, f_2)$ be two SVNT of $\mathcal{M}$, where n_1 and n_2 have same indeterminacy value i_1. In [16], if $s(n_1) > s(n_2)$, then $(n_1) > (n_2)$.
Now, $s(n_1) > s(n_2) \Leftrightarrow \frac{2+t_1-f_1-i_1}{3} > \frac{2+t_2-f_2-i_1}{3} \Leftrightarrow t_1 - f_1 > t_2 - f_2 \Leftrightarrow \frac{2+(t_1-f_1)(2-i_1)-i_1}{4} > \frac{2+(t_2-f_2)(2-i_1)-i_1}{4} \Leftrightarrow S^+(n_1) > S^+(n_2)$. Similarly, $s(n_1) < s(n_2) \Leftrightarrow S^+(n_1) < S^+(n_2)$. Further, $s(n_1) = s(n_2) \Leftrightarrow S^+(n_1) = S^(n_2)$.
Hence, ranking by membership score function by s in [16] is same as ranking by proposed neutrosophic membership score S^+.

By similar argument, we have $a(n_1) > a(n_2) \Leftrightarrow S^-(n_1) < S^-(n_2)$ and $a(n_1) = a(n_2) \Leftrightarrow S^-(n_1) = S^-(n_2)$. Hence, ranking by membership score function by a in [16] is same as ranking by proposed neutrosophic membership score S^-.

Now, we prove that $c(n_1) > c(n_2) \Leftrightarrow C(n_1) > C(n_2)$, $c(n_1) < c(n_2) \Leftrightarrow C(n_1) < C(n_2)$ and $c(n_1) = c(n_2) \Leftrightarrow C(n_1) = C(n_2)$ if $s(n_1) = s(n_2)$ and $a(n_1) = a(n_2)$ (and hence $S^+(n_1) = S^+(n_2)$, $S^-(n_1) = S^-(n_2)$). If $s(n_1) = s(n_2)$ and $a(n_1) = a(n_2)$, then $t_1 - f_1 = t_2 - f_2$. Now, $c(n_1) > c(n_2) \Leftrightarrow t_1 > t_2 \Leftrightarrow f_1 > f_2$ using $t_1 - f_1 = t_2 - f_2) \Leftrightarrow \frac{t_1+f_1}{2} > \frac{t_2+f_2}{2} \Leftrightarrow C(n_1) > C(n_2)$. Similarly, $c(n_1) < c(n_2) \Leftrightarrow C(n_1) < C(n_2)$. Further, $c(n_1) = c(n_2) \Leftrightarrow C(n_1) = C(n_2)$ if $s(n_1) = s(n_2)$ and $a(n_1) = a(n_2)$ (and hence $S^+(n_1) = S^+(n_2)$, $S^-(n_1) = S^-(n_2)$). As a result, if triplets share the same indeterminacy, ranking by membership score function in [16] is the same as ranking by proposed neutrosophic membership score function. □

The proof of the following remarks are immediate applications of definitions, hence they are omitted.

Remark 6.2. Let $n_1 = (t_1, i_1, f_1)$ and $n_2 = (t_2, i_2, f_2)$ be two SVNT. When $(t_1-f_1) > (t_2-f_2)$ and $i_1 < i_2$, our suggested Algorithm for ranking Neutrosophic Triplets (T, I, F) and Florentin Smarandache's [16] Algorithm for ranking Neutrosophic Triplets are the same.

Remark 6.3. Our proposed Algorithm and Florentin Smarandache [16] Algorithm for ranking of SVNT (T, I, F) are ranking in a same manner when the difference between membership and non-membership values $(T - F)$ of triplets (T, I, F) have same value.

7. Membership, Non-Membership and Average score functions on IVNT

The algorithm for ranking SVNT is expanded to IVNT in this section. We begin by discussing score functions for neutrosophic triplets with interval values.

Definition 7.1. Let $\mathcal{M}_{int} = \{(T, I, F), \text{where } T, I, F \text{ are closed subsets of } [0,\ 1]\}$ be the set of IVNT. Let $N = (T, I, F) \in \mathcal{M}_{int}$ be a IVNT number. Here $T^L = inf\ T$ and $T^U = sup\ T$; $I^L = inf\ I$ and $I^U = sup\ I$; $F^L = inf\ F$ and $F^U = sup\ F$; where $T^L, T^U, I^L, I^U, F^L, F^U \in [0, 1]$ with $T^L < T^U, I^L < I^U, F^L < F^U$. Then neutrosophic triplet N is of the form $([T^L, T^U], [I^L, I^U], [F^L, F^U])$

Definition 7.2. If two intervals $[a, b]$ and $[c, d]$ have same midpoint, then they are said to be neutrosophically equal and are indicated as $[a, b] =_N [c, d]$.

Definition 7.3. An IVNT membership score function $S^+ : \mathcal{M}_{int} \to [0, 1]$ is defined by

$$S^+(T, I, F) = \frac{8 + (T^L + T^U - F^L - F^U)(4 - I^L - I^U) - 2(I^L + I^U)}{12}.$$

Definition 7.4. An IVNT non-membership score function $S^- : \mathcal{M}_{int} \to [0, 1]$ is defined by

$$S^-(T, I, F) = \frac{8 + (F^L + F^U - T^L - T^U)(4 - I^L - I^U) - 2(I^L + I^U)}{12}.$$

Definition 7.5. An IVNT average score function $C : \mathcal{M}_{int} \to [0, 1]$ is defined by

$$C(T, I, F) = \frac{T^L + T^U + F^L + F^U}{4}.$$

We now provide a new technique for ranking neutrosophic triplets with interval values.

8. A total order on IVNT

In this section, we introduce score functions through which a new algorithm for total ordering on interval valued neutrosophic triplets is aimed.

8.1. *Ranking algorithm on IVNT*

Let $A = ([T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U])$ and $B = ([T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U])$ be two interval valued neutrosophic triplets of $\mathcal{M}_{int}$. Now, by applying the following algorithm, we can rank any two numbers as either one is bigger than other or both are neutrosophically equal.

Step 1: Apply our New Neutrosophic Membership score function S^+.

(1) If $S^+(A) > S^+(B)$ $(S^+(A) < S^+(B))$, then $R(A) > R(B)$ $(R(A) < R(B))$.

(2) Suppose $S^+(A) = S^+(B)$, we go to step 2.

Step 2: Apply our New Neutrosophic non-membership score function S^-.

(1) If $S^-(A) > S^-(B)$ $(S^-(A) < S^-(B))$, then $R(A) < R(B)$ $(R(A) > R(B))$.

(2) Suppose $S^-(A) = S^-(B)$, we go to step 3.

Step 3: Apply our New Neutrosophic Average function C.
(1) If $C(A) > C(B)$ $(C(A) < C(B))$, then $R(A) > R(B)(R(A) < R(B))$.
(2) Suppose $C(A) = C(B)$, then conclude that $([T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U]) =_N ([T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U])$. So A and B are neutrosophically equal.

Theorem 8.1. *We prove that the interval valued neutrosophic membership, interval valued neutrosophic non-membership and average score functions together form a neutrosophically total ordering on M, that is either they are greater(lesser) than other numbers or neutrosophically equal.*

Proof. Now, we prove for any two interval valued neutrosophic triplets $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ and $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$,
either $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$
or $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) < ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$
or $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) =_N ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$.

Let$([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ and $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ be two interval valued neutrosophic triplets. First, we apply membership score function S^+.
If $S^+ ([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > S^+ ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is done.
If $S^+ ([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) < S^+ ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) < ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is also done.
But, when we get the equality

$$S^+([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) = S^+([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U]),$$

we have,

$$\frac{8 + (t_1^L + t_1^U - f_1^L - f_1^U)(4 - i_1^L - i_1^U) - 2(i_1^L + i_1^U)}{12} = \frac{8 + (t_2^L + t_2^U - f_2^L - f_2^U)(4 - i_2^L - i_2^U) - 2(i_2^L + i_2^U)}{12}$$

$\Leftrightarrow$

$$(t_1^L + t_1^U - f_1^L - f_1^U)(4 - i_1^L - i_1^U) - 2(i_1^L + i_1^U) = (t_2^L + t_2^U - f_2^L - f_2^U)(4 - i_2^L - i_2^U) - 2(i_2^L + i_2^U). \tag{4}$$

So, next we go for non-membership score function S^-.
If $S^- ([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > S^- ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) < ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is done. If $S^- ([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) < S^- ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is also done.

But, when we get the equality

$$S^{-}([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) = S^{-}([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U]),$$

we have,

$$\frac{8 + (f_1^L + f_1^U - t_1^L - t_1^U)(4 - i_1^L - i_1^U) - 2(i_1^L + i_1^U)}{12} = \frac{8 + (f_2^L + f_2^U - t_2^L - t_2^U)(4 - i_2^L - i_2^U) - 2(i_2^L + i_2^U)}{12}$$

$\Leftrightarrow$

$$(f_1^L + f_1^U - t_1^L - t_1^U)(4 - i_1^L - i_1^U) - 2(i_1^L + i_1^U) = (f_2^L + f_2^U - t_2^L - t_2^U)(4 - i_2^L - i_2^U) - 2(i_2^L + i_2^U). \tag{5}$$

So, we next go for average score function. If C $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > C$ $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is done.
If C $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) < C$ $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ $< ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is also done.
But, when we get the equality C $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) = C$ $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, we have

$$t_1^L + t_1^U + f_1^L + f_1^U = t_2^L + t_2^U + f_2^L + f_2^U. \tag{6}$$

If these triplets would have not ranked till now, then we have triplets $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ and $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ satisfying following system of 3 equations (from the equations 4, 5 and 6).

$$\begin{cases} (t_1^L + t_1^U - f_1^L - f_1^U)(4 - i_1^L - i_1^U) - 2(i_1^L + i_1^U) & = (t_2^L + t_2^U - f_2^L - f_2^U)(4 - i_2^L - i_2^U) - 2(i_2^L + i_2^U) \\ (f_1^L + f_1^U - t_1^L - t_1^U)(4 - i_1^L - i_1^U) - 2(i_1^L + i_1^U) & = (f_2^L + f_2^U - t_2^L - t_2^U)(4 - i_2^L - i_2^U) - 2(i_2^L + i_2^U) \\ t_1^L + t_1^U + f_1^L + f_1^U & = t_2^L + t_2^U + f_2^L + f_2^U \end{cases}$$

By adding equations 4 and 5, we get $i_1^L + i_1^U = i_2^L + i_2^U$ which makes equation 4 into

$$t_1^L + t_1^U - f_1^L - f_1^U = t_2^L + t_2^U - f_2^L - f_2^U.$$

Now, by adding the above equation with equation 6, we get $t_1^L + t_1^U = t_2^L + t_2^U$ and hence we get $f_1^L + f_1^U = f_2^L + f_2^U$.
Thus, the system of 3 equations become

$$\begin{cases} t_1^L + t_1^U & = t_2^L + t_2^U \\ i_1^L + i_1^U & = i_2^L + i_2^U \\ f_1^L + f_1^U & = f_2^L + f_2^U \end{cases}$$

$\Longrightarrow$

$$\begin{cases} \frac{t_1^L+t_1^U}{2} & = \frac{t_2^L+t_2^U}{2} \\ \frac{i_1^L+i_1^U}{2} & = \frac{i_2^L+i_2^U}{2} \\ \frac{f_1^L+f_1^U}{2} & = \frac{f_2^L+f_2^U}{2} \end{cases}$$

Hence intervals $[t_1^L, t_1^U]$ and $[t_2^L, t_2^U]$, $[i_1^L, i_1^U]$ and $[i_2^L, i_2^U]$, $[f_1^L, f_1^U]$ and $[f_2^L, f_2^U]$ are neutrosophically equal.

Therefore

$$([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) =_N ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U]).$$

We can therefore conclude that for any interval-valued neutrosophic triplets, one is larger than the other or both are neutrosophically equal..

Note: We did not extract the total ordering between interval valued neutrosophic triplets using our ranking algorithm. Take $A = ([0.2, 0.8], [0.1, 0.3], [0.2, 0.4])$ and $B = ([0.4, 0.6], [0, 0.4], [0.1, 0.5])$ as examples. We find A and B are neutrosophically equal using the preceding procedure, but they are not the same interval valued neutrosophic triplets. As a result, we will add furthermore three score functions along with membership, non-membership, and average score functions, to produce total ordering on neutrosophic interval valued numbers. □

8.2. *Total ordering on IVNT*

As we mentioned, we derive three new score functions through which the total ordering on interval valued neutrosophic triplets is achieved.

Definition 8.2. An IVNT positive range score function $S'^{+} : \mathcal{M}_{int} \to [0, 1]$ is defined by

$$S'^{+}(T, I, F) = \frac{8 + (T^U - T^L - F^U + F^L)(4 - I^U + I^L) - 2(I^U - I^L)}{12}.$$

Definition 8.3. An IVNT negative range score function $S'^{-} : \mathcal{M}_{int} \to [0, 1]$ is defined by

$$S'^{-}(T, I, F) = \frac{8 + (F^U - F^L - T^U + T^L)(4 - I^U + I^L) - 2(I^U - I^L)}{12}.$$

Definition 8.4. An IVNT average range score function $C' : \mathcal{M}_{int} \to [0, 1]$ is defined by

$$C'(T, I, F) = \frac{T^U - T^L + F^U - F^L}{4}.$$

Now, we introduce new algorithm for total ordering the interval valued neutrosophic triplets.

8.3. *Total ordering algorithm on IVNT*

Let $A = ([T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U])$ and $B = ([T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U])$ be two interval valued neutrosophic triplets of $\mathcal{M}_{int}$. Now, by applying the following algorithm, we derive a total ordering.

Step 1: Apply our interval valued neutrosophic membership score function S^+.
(1) If $S^+(A) > S^+(B)$ $(S^+(A) < S^+(B))$, then $R(A) > R(B)$ $(R(A) < R(B))$.
(2) Suppose $S^+(A) = S^+(B)$, we go to step 2.
Step 2: Apply our interval valued neutrosophic non-membership score function S^-.
(1) If $S^-(A) > S^-(B)$ $(S^-(A) < S^-(B))$, then $R(A) < R(B)$ $(R(A) > R(B))$.
(2) Suppose $S^-(A) = S^-(B)$, we go to step 3.
Step 3: Apply our interval valued neutrosophic average function C.
(1) If $C(A) > C(B)$ $(C(A) < C(B))$, then $R(A) > R(B)$ $(R(A) < R(B))$.
(2) Suppose $C(A) = C(B)$, then we go to step 4.
Step 4: Apply our interval valued neutrosophic positive range score function S'^+.
(1) If $S'^+(A) > S'^+\ (B)$ $(S'^+(A) < S'^+(B))$, then $R(A) > R(B)$ $(R(A) < R(B))$.
(2) Suppose $S'^+(A) = S'^+(B)$, we go to step 5.
Step 5: Apply our interval valued neutrosophic negative range score function S'^-.
(1) If $S'^-(A) > S'^-(B)$ $(S'^-(A) < S'^-(B))$, then $R(A) < R(B)$ $(R(A) > R(B))$.
(2) Suppose $S'^-(A) = S'^-(B)$, we go to step 6.
Step 6: Apply our interval valued neutrosophic average range score function C'.
(1) If $C'(A) > C'(B)$ $(C'(A) < C'(B))$, then $R(A) > R(B)$ $(R(A) < R(B))$.
(2) Suppose $C'(A) = C'(B)$, then we can conclude that $A = B$.

Theorem 8.5. *We prove that given algorithm preserves total ordering on interval valued neutrosophic triplets*

Proof. We prove for any two interval valued neutrosophic triplets $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ and $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, either $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ or $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) < ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ or $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ and $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ are same.

Let $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ and $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ be two interval valued neutrosophic triplets. By applying step 1, step 2 and step 3, if we would get either $R(A) > R(B)$ or $R(A) < R(B)$, then we are done. Suppose, we get $S^+([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])) = S^+([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, $S^-([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])) = S^-([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ and $C([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])) = C([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then we have $t_1^L + t_1^U = t_2^L + t_2^U$,

$i_1^L + i_1^U = i_2^L + i_2^U$ and $f_1^L + f_1^U = f_2^L + f_2^U$ by 8.1. Now we go to step 4.

Further we apply interval valued neutrosophic positive range score function S'^{+}. If S'^{+} $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > S'^{+}$ $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ $>$ $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is done. If S'^{+} $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ $<$ S'^{+} $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) < ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is also done.
But, when we get the equality

$$S'^{+}([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) = S'^{+}([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U]),$$

we have,

$$\frac{8 + (t_1^U - t_1^L - f_1^U + f_1^L)(4 - i_1^U + i_1^L) - 2(i_1^U - i_1^L)}{12} = \frac{8 + (t_2^U - t_2^L - f_2^U + f_2^L)(4 - i_2^U + i_2^L) - 2(i_2^U - i_2^L)}{12}$$

$\Leftrightarrow$

$$(t_1^U - t_1^L - f_1^U + f_1^L)(4 - i_1^U + i_1^L) - 2(i_1^U - i_1^L) = (t_2^U - t_2^L - f_2^U + f_2^L)(4 - i_2^U - i_2^L) - 2(i_2^U - i_2^L). \tag{7}$$

So, next, we go for interval valued neutrosophic negative range score function S'^{-}. If S'^{-} $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > S'^{-}$ $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ $<$ $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is done. If S'^{-} $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ $<$ S'^{-} $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is also done.
But, when we get the equality

$$S'^{-}([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) = S'^{-}([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U]),$$

we have,

$$\frac{8 + (f_1^U - f_1^L - t_1^U + t_1^L)(4 - i_1^U + i_1^L) - 2(i_1^U - i_1^L)}{12} = \frac{8 + (f_2^U - f_2^L - t_2^U + t_2^L)(4 - i_2^U + i_2^L) - 2(i_2^U - i_2^L)}{12}$$

$\Leftrightarrow$

$$(f_1^U - f_1^L - t_1^U + t_1^L)(4 - i_1^U + i_1^L) - 2(i_1^U - i_1^L) = (f_2^U - f_2^L - t_2^U + t_2^L)(4 - i_2^U + i_2^L) - 2(i_2^U - i_2^L). \tag{8}$$

So we next go for average range score function. If C' $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ $>$ C' $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) > ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is done. If C' $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ $<$ C' $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, then

$([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) < ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ which is also done. But, when we get the equality $C'\ ([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) = C'\ ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$, we have

$$t_1^U - t_1^L + f_1^U - f_1^L = t_2^U - t_2^L + f_2^U - f_2^L. \tag{9}$$

If these triplets would have not ranked till now, then we have triplets $([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U])$ and $([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U])$ satisfying following system of 6 equations (From the equations 4, 5, 6, 7, 8 and 9)

$$\begin{cases} (t_1^L + t_1^U - f_1^L - f_1^U)(4 - i_1^L - i_1^U) - 2(i_1^L + i_1^U) & = (t_2^L + t_2^U - f_2^L - f_2^U)(4 - i_2^L - i_2^U) - 2(i_2^L + i_2^U) \\ (f_1^L + f_1^U - t_1^L - t_1^U)(4 - i_1^L - i_1^U) - 2(i_1^L + i_1^U) & = (f_2^L + f_2^U - t_2^L - t_2^U)(4 - i_2^L - i_2^U) - 2(i_2^L + i_2^U) \\ t_1^L + t_1^U + f_1^L + f_1^U & = t_2^L + t_2^U + f_2^L + f_2^U \\ (t_1^U - t_1^L - f_1^U + f_1^L)(4 - i_1^U + i_1^L) - 2(i_1^U - i_1^L) & = (t_2^U - t_2^L - f_2^U + f_2^L)(4 - i_2^U - i_2^L) - 2(i_2^U - i_2^L) \\ (f_1^U - f_1^L - t_1^U + t_1^L)(4 - i_1^U + i_1^L) - 2(i_1^U - i_1^L) & = (f_2^U - f_2^L - t_2^U + t_2^L)(4 - i_2^U + i_2^L) - 2(i_2^U - i_2^L) \\ t_1^U - t_1^L + f_1^U - f_1^L & = t_2^U - t_2^L + f_2^U - f_2^L \end{cases}$$

Now we solve these system of equations. By adding equations 7 and 8, we get $i_1^U - i_1^L = i_2^U - i_2^L$, which makes equation 7 into

$$t_1^U - t_1^L - f_1^U + f_1^L = t_2^U - t_2^L - f_2^U + f_2^L.$$

Now, by adding the above equation with equation 9, we get $t_1^U - t_1^L = t_2^U - t_2^L$ and by substituting in the above equation, we get $f_1^U - f_1^L = f_2^U - f_2^L$.
Thus the system of 6 equations become

$$\begin{cases} t_1^L + t_1^U & = t_2^L + t_2^U \\ i_1^L + i_1^U & = i_2^L + i_2^U \\ f_1^L + f_1^U & = f_2^L + f_2^U \\ i_1^U - i_1^L & = i_2^U - i_2^L \\ t_1^U - t_1^L & = t_2^U - t_2^L \\ f_1^U - f_1^L & = f_2^U - f_2^L \end{cases}$$

By solving the above system of equations, we get $t_1^L = t_2^L; t_1^U = t_2^U; i_1^L = i_2^L; i_1^U = i_2^U; f_1^L = f_2^L; f_1^U = f_2^U$. Therefore

$$([t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U]) = ([t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U]).$$

We can therefore conclude that for any interval-valued neutrosophic triplet, either one is larger than the other or both are equal. Alternatively, we have demonstrated that our technique achieves total ordering on neutrosophic triplets with interval values. □

Table 1. Assessment of companies corresponding to the criteria. [21]

	Risk	**Availability of raw material**	**Availability of labor**	**Market demand**	**Production quantity**
Automobile company(I_1)	(0.7,0.4,0.3)	(0.27,0.4,0.7)	(0.5,0.1,0.2)	(0.5,0.9,0.4)	(0.1,0.3,0.8)
Food company(I_2)	(0.5,0.8,0.2)	(0.15,0.36,0.78)	(0.9,0,0.2)	(0.6,0.96,0.45)	(0.3,0.5,0.75)
Electronics company(I_3)	(0.9,0.1,0.1)	(0.3,0.6,0.9)	(0.25,0.4,0.5)	(0.72,0.85,0.3)	(0.3,0.45,0.87)
Oil company (I_4)	(0.8,0.6,0.3)	(0.1,0.8,0.2)	(1,0.5,0)	(0.57,0.8,0.35)	(0.1,0.8,0.6)
Parmacutical company(I_5)	(0.65,0.2,0.8)	(0.2,0.45,0.65)	(0.7,0.4,0.6)	(0.4,0.7,0.6)	(0.7,0.2,0.3)

9. Results and Discussions

9.1. *Comparative study with existing methods*

In this section, we consider the same Neutrosophic set based MCDM problem given in [21]. The total ordering methods given in [16], [21] and our proposed method are applied for this MCDM problem. Then, we analyze and compare the ranking order obtained according to each method.

MCDM Problem [21]: Consider an investor who wants to invest into a business. The investor has initially chosen five companies from which one is chosen based on a number of variables which include risk (c_1), raw material availability (c_2), labor availability (c_3), market demand (c_4), and production quantity (c_5). Let us denote those five businesses as automobile company (I_1), food manufacturing (I_2), electronic manufacturing (I_3), oil (I_4), and pharmaceutical (I_5). Table 1 shows the single-valued neutrosophic fuzzy values of each company with respect to each criterion. Since SVNT are denoted as (μ, T, I, F) in [21] and we denote SVNT as (T, I, F), the table given in [21] and the similarity measure method used in [21] have been modified accordingly by applying $\mu = 1$,

When we use first similarity method based ordering method for SVNT in [21], we get $S_1(a^*, I_1) = 0.82, S_1(a^*, I_2) = 0.80, S_1(a^*, I_3) = 0.80, S_1(a^*, I_4) = 0.81, S_1(a^*, I_5) = 0.82$ which gives a ranking $I_5 = I_1 > I_4 > I_2 = I_3$, that leads to a state that not able to make a concrete decision.

When we use second similarity method based ordering method for SVNT in [21], we get $S_1(a^*, I_1) = 0.71, S_1(a^*, I_2) = 0.70, S_1(a^*, I_3) = 0.67, S_1(a^*, I_4) = 0.69, S_1(a^*, I_5) = 0.71$ which gives a ranking $I_5 = I_1 > I_2 > I_4 > I_3$, that again leads to a same problem.

Table 2. Score values of companies corresponding to the criteria. [16]

	Risk	Availability of raw material	Availability of labor	Market demand	Production quantity
Automobile company(I_1)	0.66	0.39	0.73	0.4	0.33
Food company(I_2)	0.5	0.34	0.9	0.4	0.35
Electronics company(I_3)	0.9	0.27	0.45	0.52	0.33
Oil company (I_4)	0.63	0.37	0.83	0.47	0.23
Parmacutical company(I_5)	0.55	0.37	0.56	0.37	0.73

Table 3. Score values(S^+) of companies corresponding to the criteria.

	Risk	Availability of raw material	Availability of labor	Market demand	Production quantity
Automobile company(I_1)	0.55	0.28	0.63	0.24	0.24
Food company(I_2)	0.32	0.24	0.86	0.25	0.25
Electronics company(I_3)	0.86	0.19	0.43	0.37	0.46
Oil company (I_4)	0.51	0.14	0.75	0.32	0.11
Parmacutical company(I_5)	0.44	0.25	0.46	0.25	0.64

Now we are going to compute this problem by method in [16] .

From table 2, the aggregated score values for $I_1 = 0.50, I_2 = 0.50, I_3 = 0.49, I_4 = 0.51, I_5 = 0.51$. Hence, $I_5 = I_4 > I_1 = I_2 > I_3$. To differentiate I_5 and I_4, we go for accuracy scores of $I_5 = 0.14, I_4 = -0.186$, which gives $I_5 > I_4$. Similarly accuracy score of $I_1 = -0.06, I_2 = -0.034$, which gives $I_2 > I_1$. Now we get an ordering $I_5 > I_4 > I_2 > I_1 > I_3$.

Now we compute the same problem by the proposed total ordering method.

From Table 3, we get scores of $I_1 = 0.39, I_2 = 0.38, I_3 = 0.46, I_4 = 0.36, I_5 = 0.41$. Therefore the ranking order will be as $I_3 > I_5 > I_1 > I_2 > I_4$.

By the above results, we came to know that the existing method in [21] may not be helpful in some MCDM situations and the existing method [16] and our proposed method give ranking order which may not be coincide. From the above discussions, we can conclude that our proposed method and total ordering method in [16] are better than the existing similarity based ranking method [21]. Now, in the next subsection, we are analyzing the limitations of existing total ordering [16] and advantages of our proposed total ordering method.

9.2. *Limitations of existing method and advantage of our proposed method*

Let us compare existing method [16] and proposed method with an basic example as follows. Consider two neutrosophic triplets $A = (0.9, 0.5, 0.3)$ and $B = (0.8, 0.6, 0.1)$. By Smarandache method [16], we have $s(A) = 0.7$ and $s(B) = 0.7$. Thus, we get $s(A) = s(B)$. So, we go to next score $a(A) = 0.6$ and $a(B) = 0.7$ so we get $A < B$. But, in proposed method, we have $S^+(A) = 0.65$ and $S^+(B) = 0.595$. Thus, we get $A > B$. Here our ranking is different from existing ranking and we can find ranking in less steps compared to existing method. Consider $A = (0.5, 0.3, 0.2)$ and $B = (0, 0, 0)$, by Smarandache method [16], we have $s(A) = 0.667$ and $s(B) = 0.667$. Thus, we get $s(A) = s(B)$. So, we go to next score $a(A) = 0.3$ and $a(B) = 0$ and therefore we get $A > B$. But, in proposed method, we have $S^+(A) = 0.54$ and $S^+(B) = 0.5$. Thus we get $A > B$. Now proposed ranking is same from existing ranking and we can find ranking in less steps compared to existing method.

Consider $A = (0.5, 0.2, 0.3)$ and $B = (0.4, 0.2, 0.2)$. By Smarandache method [16], we have $s(A) = 0.667$ and $s(B) = 0.667$. Thus we get $s(A) = s(B)$. So, we go to next score $a(A) = 0.2$ and $a(B) = 0.2$ and therefore we get $a(A) = a(B)$. Since still we are unable to rank A and B, so we are going to next score $c(A) = 0.5$ and $c(B) = 0.4$. so we get $A > B$.
But, in proposed method, we have $S^+(A) = 0.54$ and $S^+(B) = 0.54$, Thus we get $S^+(A) = S^+(B)$. So, we go to next score $S^-(A) = 0.36$ and $S^-(B) = 0.36$. We go for next score $C(A) = 0.4$ and $C(B) = 0.3$. Thus, we get $A > B$, in this example both the method needs all three score functions and both ranking were same. These are some of examples to understand that both the ranking method may need not to be similar for single valued neutrosophic triplets. Consider the example given in the previous section. Let $A = ([0.2, 0.8], [0.1, 0.3], [0.2, 0.4])$ and $B = ([0.4, 0.6], [0, 0.4], [0.1, 0.5])$. By using the existing algorithm, we get A and B are neutrosophically equal. But when we apply our method, we get $S'^+(A) = 0.76$, $S'^+(B) = 0.536$. Hence, we get $R(A) > R(B)$. This is one of the example that the existing method failed to rank as it will conclude both of them were neutrosophically equal, but our method rank them in a better way.

Our proposed ranking method involves not only membership, non-membership and indeterminacy values alone, it also consider the part of membership and non-membership value which lying inside the hesitance value. So the formation of our score functions were different from existing score functions, which results there are some difference in the ranking between our method and existing method. Proposed method will be very useful and easy to rank within step 1 itself in many cases, whereas in existing method we may need to go for further steps. Further, in interval valued neutrosophic ranking method, the existing method gives only neutrosophically total ordering, but our method gives total ordering and as well as neutrosophically total ordering.

10. Conclusion and future scope

The proposed ranking approach takes into account not only membership, non-membership, and indeterminacy values, but also the portion of membership and non-membership value that is contained within the hesitance value. As a result, the development of our score functions differed from that of current score functions resulting in some differences in ranking between our approach and that of existing methods. In many circumstances, the proposed method will be very beneficial and straightforward to rank within step 1, whereas the old method may require additional stages. Furthermore, the existing interval valued neutrosophic ranking approach delivers neutrosophically total ordering only, but our method gives both total ordering and neutrosophically total ordering. Thus a new algorithm for total ordering both single and interval valued neutrosophic triplets has been derived which will be a beneficial tool for decision makers in MCDM problems.

In this paper, a ranking approach for IVNT is developed as a generalisation of ranking approach for SVNT. By comparing our proposed work with the existing work, we have come to a conclusion that our proposed method involves less steps compared to previous method in some stages. Further, proposed method gives a reliable ordering on alternatives of the MCDM problems due its total ordering nature compared to other methods. In near future, total ordering to triangular and trapezoidal neutrosophic numbers will be studied and hence this opens a new study in the field of neutrosophic sets.

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On Neutrosophic Homeomorphisms via Neutrosophic Functions

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Abstract. By using neutrosophic m-alpha closed sets in neutrosophic topological spaces, we introduce the space known as neutrosophic t-alpha space in this paper. We also introduce the mappings referred as neutrosophic m-alpha continuous functions, homeomorphisms, and connectedness, and we research the characterizations and their properties.

Keywords: neutrosophic t-alpha space, neutrosophic m-alpha closed sets, neutrosophic m-alpha continuous functions, neutrosophic m-alpha homeomorphisms and neutrosophic m-alpha connectedness.

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1. Introduction

L. A. Zadeh [14] first put forward the idea of fuzzy sets in 1965, C. L. Chang [5] created fuzzy topological spaces in 1968, built around the idea of fuzzy sets, In 1986 [2] K. Atanassov derived the intuitionistic fuzzy sets, In 1997 [6] D.Coker have introduced the intuitionistic fuzzy topological spaces, F. Smarandache [9] proposed A unified field approach in neutrosophic logic in 1999 and analyzed some of its characteristics, F. Smarandache [10] started researching neutrosophy and neutrosophic logic in 2002. A. A. Salama and S. A. Alblowi [8] examined the neutrosophic set and neutrosophic topological spaces in 2012 and mentioned some of their findings. Broumi Said and Florentin Smarandache [4] proposed the intuitionistic neutrosophic soft set concept and derived some results in 2013. Smarandache, Florentin, Said Broumi, Mamoni Dhar, and Pinaki Majumdar [11] brought new intuitionistic fuzzy soft set results and derived some results in 2014. Wadel Faris Al-omeri and Florentin Smarandache [13] suggested a new neutrosophic sets using neutrosophic topological spaces in Wadel Faris Al-omeri and

Florentin Smarandache's article.

In 2017 [1] I.Arokiarani, R.Dhavaseelan, S. Jafari and M.Parimala have derived some new notions and functions in neutrosophic topological spaces, In 2021 [3] P. Basker, Broumi Said have introduced $N\psi_{\alpha}^{\#0}$ and $N\psi_{\alpha}^{\#1}$-spaces in neutrosophic topological spaces, In 2021 [7] D. Nagarajan, S. Broumi, F. Smarandache, and J. Kavikumar derived the analysis of neutrosophic multiple regression and have given some properties, In 2021 [12] A. Vadivel1, C. John Sundar derived the neutrosophic δ-open maps and neutrosophic δ-closed maps

The abbreviations NS and NTS refer to the neutrosophic set and neutrosophic toplogical spaces, respectively, throughout this study.

2. **Preliminaries**

We should review and analyze definitions before we begin our study.

Definition 2.1. A NS, A in a NTS is referred to as a neutrosophic set, $N\alpha$-open set ($N\alpha OS$), if A is a subset of $Nint(Ncl(Nint(A)))$. The complement of $N\alpha OS$ is called $N\alpha CS$.

Definition 2.2. (a) Assume N is an NTS and $n \in N$. N_1 is a subset of N is called as $N\alpha$-nbhd of n $\exists$ $N\alpha$-open set N_2 such that $n \in N_2 \subset N_1$.
The collection of all $N\alpha$-nbhd of $n \in N$ is called $N\alpha$-neighbourhood system at n and shall be denoted by $NBH_{N\alpha}(n)$.

(b) Let N be a NTS and N_1 be a subset of N, A subset N_2 of N is supposed to be $N\alpha$-nbhd of N_1 $\exists$ $N\alpha$-open set M such that $N_1 \in M \subseteq N_2$.

(c) Let N_1 be a subset of N. A point $n_1 \in N_1$ is supposed to be $N\alpha$-interior point of N_1, if N_1 is an $NBH_{N\alpha}(n_1)$. The entirety of everything $N\alpha$-interior points of N_1 is referred to as an $N\alpha$-interior of N_1 and is denoted by $NBH_{N\alpha}(n_1)$.

(d) $N\alpha$-interior of N_1 is the union of all $N\alpha OS \subset N_1$ and it is denoted by $INT_N\alpha(N_1)$. $INT_N\alpha(N_1) = \bigcup\{M : M\ is\ N\alpha OS,\quad M \subseteq N_1\}$.

(e) $N\alpha$-closure of N_1 is the intersection of all $N\alpha CS \supset N_1$ and it is denoted by $CL_N\alpha(N_1)$. $CL_N\alpha(N_1) = \bigcap\{M : M\ is\ a\ N\alpha\text{-}closed\ set\ and\ N_1 \subseteq M\}$.

(f) $\bigcap$ of all $N\alpha$-open subsets of $(N,\ \tau_N)$ containing N_1 is called the $N\alpha$-kernel of N_1 (briefly, $nk_{\#}^{N\alpha}(N_1)$). $nk_{\#}^{N\alpha}(N_1) = \cap\{M \in N\alpha(N,\ \tau_N) : N_1 \subseteq M\}$.

(g) Let $n \in N_1$. Then $N\alpha$-kernel of n is meant to refer to as $nk_{\#}^{N\alpha}(\{n\}) = \cap\{M \in N\alpha(N,\ \tau_N) : n \in M\}$. $CL_N\alpha(N_1) = \bigcap\{M : N_1 \subset M \in N\alpha(N,\ \tau_N)\}$.

3. On $t_{\#}^{N\alpha}$-space via $N\alpha OS$

Definition 3.1. L is NS in a NTS, $N\alpha^{M_\#}CS$ if $Nint(Ncl(L))$ is a subset of Q, only when L is a $\subset$ of Q and Q is $N\alpha OS$. The opponent of $N\alpha^{M_\#}CS$ is called an $N\alpha^{M_\#}OS$.

Example 3.2. Here $N = \{n_1, n_2, n_3\}$ with $\tau_N = \{0_N, 1_N, O_1, O_2\}$ where
$O_1 = \langle(\frac{7}{10}, \frac{7}{10}, \frac{5}{10}), (\frac{3}{10}, \frac{8}{10}, 1), (1, \frac{8}{10}, \frac{6}{10})\rangle$,
$O_2 = \langle(\frac{2}{10}, \frac{5}{10}, \frac{9}{10}), (\frac{3}{10}, \frac{7}{10}, 1), (\frac{7}{10}, \frac{6}{10}, 1)\rangle$,
$O_3 = \langle(\frac{3}{10}, \frac{3}{10}, \frac{5}{10}), (\frac{7}{10}, \frac{2}{10}, 0), (0, \frac{2}{10}, \frac{4}{10})\rangle$,
$O_4 = \langle(\frac{8}{10}, \frac{5}{10}, \frac{1}{10}), (\frac{7}{10}, \frac{3}{10}, 0), (\frac{3}{10}, \frac{4}{10}, 0)\rangle$,
$O_5 = \langle(\frac{4}{10}, \frac{5}{10}, 1), (\frac{2}{10}, \frac{3}{10}, 1), (\frac{5}{10}, \frac{3}{10}, 1)\rangle$. Here the sets $O_3,\ O_4\ and\ O_5$ are the $N\alpha^{M_\#}CS$.

Definition 3.3. A NTS is neutrosophic in nature which is $t_{\#}^{N\alpha}$-space if every $N\alpha^{M_\#}CS$ is CS.

Theorem 3.4. *For a TS that is neutrosophic $(N,\ \tau_N)$ The criteria listed below are equivalent.*

(a) $(N,\ \tau_N)$ *is* $t_{\#}^{N\alpha}$*-space.*

(b) *Every singleton* $\{n_1\}$ *is either* $N\alpha CS$ *(or)* $NclNopen$.

Proof. $(a) \Rightarrow (b)$ Let $n_1 \in N$. Suppose $\{n_1\}$ is not an $N\alpha CS$ of $(N,\ \tau_N)$. Then $N - \{n_1\}$ is not an $N\alpha OS$. Thus $N - \{n_1\}$ is an $N\alpha CS$ of $(N,\ \tau_N)$. Since $(N,\ \tau_N)$ is a $t_{\#}^{N\alpha}$-space, $N - \{n_1\}$ is a $N\alpha CS$ of $(N,\ \tau_N)$, i.e., $\{n_1\}$ is $N\alpha OS$ of $(N,\ \tau_N)$.

$(b) \Rightarrow (a)$ Let N_1 be an $N\alpha^{M_\#}CS$ of $(N,\ \tau_N)$. Let $n_1 \in Nint(Ncl(N_1))$ by (b), $\{n_1\}$ is either $N\alpha CS$ (or) $NclNopen$.

$Case(i)$: Let $\{n_1\}$ be an $N\alpha CS$. If we take the presumption that $n_1 \notin N_1$, we would now have $n_1 \in Nint(Ncl(N_1)) - N_1$ which isn't possible. Hence $n_1 \in N_1$.

$Case(ii)$: Let $\{n_1\}$ be a $NclNopen$. Since $n_1 \in Nint(Ncl(N_1))$, then $\{n_1\} \bigcap N_1 \neq \phi_N$. This demonstrates that $n_1 \in N_1$. As a result, in both circumstances, we have $Nint(Ncl(A)) \subseteq N_1$.

Trivially $N_1 \subseteq Nint(Ncl(N_1))$. Therefore $N_1 = Nint(Ncl(N_1))$ (or) equivalently N_1 is $NclNopen$. Hence $(N,\ \tau_N)$ is a $t_{\#}^{N\alpha}$-space.

□

Definition 3.5. A function $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is called

(a) an $N\alpha^{M\#}$-continuous if $D^{-1}(Y)$ is $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$ for every closed set Y of $(N^{II},\ \tau_N^{ii})$.

(b) an $N\alpha^{M\#}$-irresolute if $D^{-1}(Y)$ is $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$ for every $N\alpha^{M\#}CS$ Y of $(N^{II},\ \tau_N^{ii})$.

Example 3.6. Let $N = \{n_1, n_2, n_3\}$ with $\tau_N = \{0_N, 1_N, \eta_1^{\#}, \eta_2^{\#}, \eta_3^{\#}, \eta_4^{\#}\}$ and $\delta_N = \{0_N, 1_N, \eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*\}$ where
$\eta_1^{\#} = \langle(\frac{4}{10}, \frac{4}{10}, \frac{6}{10}), (\frac{5}{10}, \frac{4}{10}, \frac{6}{10}), (\frac{5}{10}, \frac{8}{10}, \frac{7}{10})\rangle$,
$\eta_2^{\#} = \langle(\frac{5}{10}, \frac{7}{10}, \frac{7}{10}), (\frac{6}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{4}{10})\rangle$,
$\eta_3^{\#} = \langle(\frac{5}{10}, \frac{7}{10}, \frac{7}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{4}{10})\rangle$,
$\eta_4^{\#} = \langle(\frac{4}{10}, \frac{4}{10}, \frac{3}{10}), (\frac{6}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{5}{10}, \frac{7}{10}, \frac{7}{10})\rangle$,
$\eta_1^* = \langle(\frac{5}{10}, \frac{7}{10}, \frac{6}{10}), (\frac{4}{10}, \frac{5}{10}, \frac{4}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{6}{10})\rangle$,
$\eta_2^* = \langle(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{4}{10}, \frac{4}{10}, \frac{3}{10}), (\frac{5}{10}, \frac{7}{10}, \frac{6}{10})\rangle$,
$\eta_3^* = \langle(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{4}{10}, \frac{5}{10}, \frac{4}{10}), (\frac{5}{10}, \frac{7}{10}, \frac{6}{10})\rangle$,
$\eta_4^* = \langle(\frac{5}{10}, \frac{7}{10}, \frac{6}{10}), (\frac{4}{10}, \frac{4}{10}, \frac{3}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{6}{10})\rangle$. Thus, $(N,\ \tau_N)$ and $(N,\ \delta_N)$ are Nutrosophic Topologies. Define $\Lambda : (N,\ \tau_N) \longrightarrow (N,\ \delta_N)$ as $\Lambda(n_1) = n_1$, $\Lambda(n_2) = n_3$, $\Lambda(n_3) = n_2$. Then Λ is $N\alpha^{M\#}$-continuous, since $\Lambda^{-1}(L_{\#})$ is $N\alpha^{M\#}CS$ in $(N,\ \tau_N)$ for every closed set $L_{\#}$ of $(N,\ \delta_N)$ where $L_{\#} = \langle(\frac{3}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{4}{10}, \frac{4}{10}, \frac{4}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{6}{10})\rangle$.

Proposition 3.7. *If $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be an $N\alpha^{M\#}$-continuous function and $(N^I,\ \tau_N^i)$ be a $t_{\#}^{N\alpha}$-space, D is continuous.*

Proof. Assume Y to be closed in $(N^{II},\ \tau_N^{ii})$. As such D is an $N\alpha^{M\#}$-continuous function, $D^{-1}(Y)$ is an $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$. Since $(N^I,\ \tau_N^i)$ is a $t_{\#}^{N\alpha}$-space, $D^{-1}(Y)$ is closed set in $(N^I,\ \tau_N^i)$. Hence D is continuous.

□

Remark 3.8. Let $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be a mapping and $(N^I,\ \tau_N^i)$ be a $t_{\#}^{N\alpha}$-space, then D is continuous if one of the following conditions is satisfied.

(a) f is $N\alpha^{M\#}$-continuous.

(b) f is $N\alpha^{M\#}$-irresolute.

Theorem 3.9. *A map $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is an $N\alpha^{M\#}$-continuous function $\Longleftrightarrow$ every open set's inverted image in $(N^{II},\ \tau_N^{ii})$ are the $N\alpha^{M\#}OS$ in the $(N^I,\ \tau_N^i)$.*

Proof. $Necessity$: Assume $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be an $N\alpha^{M\#}$-continuous function and Z be a collection that is open in $(N^{II},\ \tau_N^{ii})$, $N^{II} - Z$ is closed $(N^{II},\ \tau_N^{ii})$. As such D is an $N\alpha^{M\#}$-continuous function, $f^{-1}(N^{II} - Z) = N^I - D^{-1}(Z)$ is an $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$ and hence $D^{-1}(Z)$ is an $N\alpha^{M\#}OS$ in $(N^I,\ \tau_N^i)$.

$Sufficiency$: Assume that $D^{-1}(Y)$ is an $N\alpha^{M\#}OS$ in $(N^I,\ \tau_N^i)$ for each open set N^{II} in $(N^{II},\ \tau_N^{ii})$. Assume Y is a closed set in $(N^{II},\ \tau_N^{ii})$, $N^{II} - Y$ is a set that is open in $(N^{II},\ \tau_N^{ii})$. By assumption, $D^{-1}(N^{II} - Y) = N^I - D^{-1}(Y)$ is an $N\alpha^{M\#}OS$ in $(N^I,\ \tau_N^i)$, which implies that $D^{-1}(Y)$ is an $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$. Hence D is an $N\alpha^{M\#}$-continuous.

□

Proposition 3.10. *Let $D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be any topological space that is neutrosophic $(N^{II},\ \tau_N^{ii})$ is a $t_{\#}^{N\alpha}$-space. If $D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ and $D_2 : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^{III},\ \tau_N^{iii})$ are $N\alpha^{M\#}$-continuous functions, then their composition $D_2 \circ D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{III},\ \tau_N^{iii})$ is an $N\alpha^{M\#}$-continuous.*

Proof. Assume Y is a closed set in $(N^{III},\ \tau_N^{iii})$. As such $D_2 : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^{III},\ \tau_N^{iii})$ is an $N\alpha^{M\#}$-continuous function, $D_2^{-1}(Y)$ is an $N\alpha^{M\#}CS$ in $(N^{II},\ \tau_N^{ii})$. Since $(N^{II},\ \tau_N^{ii})$ is a $t_{\#}^{N\alpha}$-space, $D_2^{-1}(Y)$ is a closed set in $(N^{II},\ \tau_N^{ii})$. Since $D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is an $N\alpha^{M\#}$-continuous function, $D_1^{-1}(D_2^{-1}(Y)) = (D_2 \circ D_1)^{-1}(Y)$ is an $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$. Hence $D_2 \circ D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{III},\ \tau_N^{iii})$ is an $N\alpha^{M\#}$-continuous function.

□

Definition 3.11. A map $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is said to be

(p) $N\alpha^{M\#}$-closed map if $D(Y)$ is $N\alpha^{M\#}$-closed in $(N^{II},\ \tau_N^{ii})$ for every NCS Y of $(N^I,\ \tau_N^i)$.

(q) $N\alpha^{M\#}$-open map if $D(Y)$ is $N\alpha^{M\#}$-open in $(N^{II},\ \tau_N^{ii})$ for every NOS Y of $(N^I,\ \tau_N^i)$.

Theorem 3.12. *Let $D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ and $D_2 : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^{III},\ \tau_N^{iii})$ be two mappings and $(N^{II},\ \tau_N^{ii})$ be a $t_{\#}^{N\alpha}$-space, then*

(a) $D_2 \circ D_1$ is $N\alpha^{M\#}$-continuous, if D_1 and D_2 are $N\alpha^{M\#}$-continuous.

(b) $D_2 \circ D_1$ is $N\alpha^{M\#}$-closed, if D_1 and D_2 are $N\alpha^{M\#}$-closed.

Proof. (a) Let Y be a NCS of $(N^{III},\ \tau_N^{iii})$, then $D_2^{-1}(Y)$ is $N\alpha^{M\#}$-closed set in $(N^{II},\ \tau_N^{ii})$. Since $(N^{II},\ \tau_N^{ii})$ is a $t_{\#}^{N\alpha}$-space, then $D_2^{-1}(Y)$ is a NCS in $(N^{II},\ \tau_N^{ii})$. But D_1 is $N\alpha^{M\#}$-continuous, then $(D_2 \circ D_1)^{-1}(Y) = D_1^{-1}(D_2^{-1}(Y))$ is $N\alpha^{M\#}$-closed in $(N^I,\ \tau_N^i)$ this implies that $(D_2 \circ D_1)$ is $N\alpha^{M\#}$-continuous mappings.

(b) The proof is similar.

□

Remark 3.13. Let $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be a mapping from a $t_{\#}^{N\alpha}$-space $(N^I,\ \tau_N^i)$ into a space $(N^{II},\ \tau_N^{ii})$, then

(p) D_1 is continuous mapping if, D_1 is $N\alpha^{M\#}$-continuous.

(q) D_1 is closed mapping if, D_1 is $N\alpha^{M\#}$-closed.

Theorem 3.14. *Let $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is surjective closed and $N\alpha^{M\#}$-irresolute, then $(N^{II},\ \tau_N^{ii})$ $t_{\#}^{N\alpha}$-space if $(N^I,\ \tau_N^i)$ is also $t_{\#}^{N\alpha}$-space.*

Proof. Let Y be an $N\alpha^{M\#}$-closed subset of $(N^{II},\ \tau_N^{ii})$. Then $D_1^{-1}(Y)$ is $N\alpha^{M\#}$-closed set in $(N^I,\ \tau_N^i)$. Since, $(N^I,\ \tau_N^i)$ is a $t_{\#}^{N\alpha}$-space, then $D_1^{-1}(Y)$ is closed set in $(N^I,\ \tau_N^i)$. Hence, Y is closed set in $(N^{II},\ \tau_N^{ii})$ and so, $(N^{II},\ \tau_N^{ii})$ is $t_{\#}^{N\alpha}$-space.

□

Proposition 3.15. *If $D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is $N\alpha^{M\#}$-closed, $D_2 : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^{III},\ \tau_N^{iii})$ is an $N\alpha^{M\#}$-closed, and $(N^{II},\ \tau_N^{ii})$ is a $t_{\#}^{N\alpha}$-space, then their composition*

$D_2 \circ D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{III},\ \tau_N^{iii})$ *is* $N\alpha^{M_\#}$*-closed.*

Proof. Let N_1 be a NCS of $(N^I,\ \tau_N^i)$. Then by assumption $D_1(N_1)$ is $N\alpha^{M_\#}$-closed in $(N^{II},\ \tau_N^{ii})$. Since $(N^{II},\ \tau_N^{ii})$ is a $t_\#^{N\alpha}$-space, $D_1(N_1)$ is NCS in $(N^{II},\ \tau_N^{ii})$ and again by assumption $D_2(D_1(N_1))$ is $N\alpha^{M_\#}$-closed in $(N^{III},\ \tau_N^{iii})$. i.e., $(D_2 \circ D_1)(N_1)$ is $N\alpha^{M_\#}$-closed in $(N^{III},\ \tau_N^{iii})$ and so $D_2 \circ D_1$ is $N\alpha^{M_\#}$-closed.

□

Proposition 3.16. *For any bijection* $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ *the following statements are equivalent:*

(p) $D^{-1} : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^I,\ \tau_N^i)$ *is* $N\alpha^{M_\#}$*-continuous.*

(q) D *is* $N\alpha^{M_\#}$*-open map.*

(r) D *is* $N\alpha^{M_\#}$*-closed map.*

Proof. $(p) \Longrightarrow (q)$ Let U be a NOS of $(N^I,\ \tau_N^i)$. By assumption, $(D^{-1})^{-1}(U) = D(U)$ is $N\alpha^{M_\#}$-open in $(N^{II},\ \tau_N^{ii})$ and so D is $N\alpha^{M_\#}$-open.

$(q) \Longrightarrow (r)$ Let F be a NCS of $(N^I,\ \tau_N^i)$. Then F^c is NOS in $(N^I,\ \tau_N^i)$. By assumption, $D(F^c)$ is $N\alpha^{M_\#}$-open in $(N^{II},\ \tau_N^{ii})$. That is $D(F^c) = (D(F))^c$ is $N\alpha^{M_\#}$-open in $(N^{II},\ \tau_N^{ii})$ and therefore $D(F)$ is $N\alpha^{M_\#}$-closed in $(N^{II},\ \tau_N^{ii})$. Hence D is $N\alpha^{M_\#}$-closed.

$(r) \Longrightarrow (p)$ Let F be a NCS of $(N^I,\ \tau_N^i)$. By assumption, $D(F)$ is $N\alpha^{M_\#}$-closed in $(N^{II},\ \tau_N^{ii})$. But $D(F) = (D^{-1})^{-1}(F)$ and therefore D^{-1} is $N\alpha^{M_\#}$-continuous.

□

4. On $N\alpha^{M_\#}$-homeomorphisms

Definition 4.1. A function $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is supposed to be an $N\alpha^{M_\#}$-homeomorphism $[(hmpm(N,\ \tau_N))_{N\alpha^{M_\#}}]$ if both D and D^{-1} are $N\alpha^{M_\#}$-irresolute.

We are using the entire family of all $N\alpha^{M\#}$-homeomorphisms of a NTS $(N^I,\ \tau_N^i)$ onto itself by $N\alpha^{M\#}$-$H(N,\ \tau_N)$.

Example 4.2. Let $M^{N^1} = \{\alpha, \beta\}$, $M^{N^2} = \{\gamma, \delta\}$, $O_1^{\#} = \langle(\frac{2}{10}, \frac{6}{10}, \frac{3}{10}), (\frac{3}{10}, \frac{6}{10}, \frac{4}{10}), (\frac{3}{10}, \frac{7}{10}, \frac{4}{10})\rangle$, $O_2^{\#} = \langle(\frac{4}{10}, \frac{6}{10}, \frac{5}{10}), (\frac{5}{10}, \frac{6}{10}, \frac{6}{10}), (\frac{5}{10}, \frac{7}{10}, \frac{6}{10})\rangle$. Then $\tau_{E1} = \{0_E, 1_E, O_1^{\#}\}$ and $\tau_{E2} = \{0_E, 1_E, O_2^{\#}\}$ are neutrosophic topologies on M^{N^1} and M^{N^2} respectively. Define a bijective mapping $F_{Nf_\#} = (M^{N^1},\ \tau_{E1}) \longrightarrow (M^{N^2},\ \tau_{E2})$ by $F_{Nf_\#}(\alpha) = \gamma$ and $F_{Nf_\#}(\beta) = \delta$. Then $F_{Nf_\#}$ is a $N\alpha^{M\#}$-irresolute $F_{Nf_\#}^{-1}$ is also a $N\alpha^{M\#}$-irresolute. Therefore the bijection function $F_{Nf_\#}$ is a $(hmpm(N,\ \tau_N))_{N\alpha^{M\#}}$.

Proposition 4.3. *Let $D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ and $D_2 : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^{III},\ \tau_N^{iii})$ are $(hmpm(N,\ \tau_N))_{N\alpha^{M\#}}$, then their composition*
$D_2 \circ D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{III},\ \tau_N^{iii})$ is also $(hmpm(N,\ \tau_N))_{N\alpha^{M\#}}$.

Proof. Let J be an $N\alpha^{M\#}OS$ in $(N^{III},\ \tau_N^{iii})$. Since D_2 is $N\alpha^{M\#}$-irresolute, $D_2^{-1}(J)$ is $N\alpha^{M\#}OS$ in $(N^{II},\ \tau_N^{ii})$. Since D_1 is $N\alpha^{M\#}$-irresolute, $D_1^{-1}(D_2^{-1}(Y)) = (D_2 \circ D_1)^{-1}(Y)$ is $N\alpha^{M\#}OS$ in $(N^I,\ \tau_N^i)$. Therefore $D_2 \circ D_1$ is $N\alpha^{M\#}$-irresolute.

Also for an $N\alpha^{M\#}OS$, G in $(N^I,\ \tau_N^i)$, we have $(D_2 \circ D_1)(G) = D_2(D_1(G)) = D_2(W)$, where $W = D_1(G)$. By hypothesis, $D_1(G)$ is $N\alpha^{M\#}OS$ in $(N^{II},\ \tau_N^{ii})$ and so again by hypothesis, $D_2(D_1(G))$ is an $N\alpha^{M\#}OS$ in $(N^{III},\ \tau_N^{iii})$. That is $(D_2 \circ D_1)(G)$ is an $N\alpha^{M\#}OS$ in $(N^{III},\ \tau_N^{iii})$ and therefore $(D_2 \circ D_1)^{-1}$ is $N\alpha^{M\#}$-irresolute. Also $D_2 \circ D_1$ is a bijection. Hence $D_2 \circ D_1$ is $(hmpm(N,\ \tau_N))_{N\alpha^{M\#}}$.

□

Theorem 4.4. *The set $N\alpha^{M\#}$-$H(N,\ \tau_N)$ is a subset of the map composition.*

Proof. Establish a binary operation $* : N\alpha^{M\#}\text{-}H(N,\ \tau_N) \times N\alpha^{M\#}\text{-}H(N,\ \tau_N) \longrightarrow N\alpha^{M\#}$-$H(N,\ \tau_N)$ by $D_1 * D_2 = D_2 \circ D_1$ for all $D_1, D_2 \in N\alpha^{M\#}$-$H(N,\ \tau_N)$ and *circ* is the standard map composition operation. $D_2 \circ D_1 \in N\alpha^{M\#}$-$H(N,\ \tau_N)$.

We notice that maps are made up of associative elements, and the identity map is no exception $I : (N,\ \tau_N) \longrightarrow (N,\ \tau_N)$ belonging to $N\alpha^{M\#}$-$H(N,\ \tau_N)$ identity element as a distinguishing feature. If $D_1 \in N\alpha^{M\#}$-$H(N,\ \tau_N)$, then $D_1^{-1} \in N\alpha^{M\#}$-$H(N,\ \tau_N)$ such that $D_1 \circ D_1^{-1} = D_1^{-1} \circ D_1 = I$. As a result, there is an inverse for each element of $N\alpha^{M\#}$-$H(N,\ \tau_N)$.

Consequently $N\alpha^{M\#}$-$H(N,\ \tau_N),\ \circ)$ is a network of under the operation on map composition.

□

Proposition 4.5. *Let $J : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be an $N\alpha^{M\#}$-homeomorphism, J causes the group to become isomorphic $N\alpha^{M\#}$-$H(N^I,\ \tau_N^i)$ onto $N\alpha^{M\#}$-$H(N^{II},\ \tau_N^{ii})$.*

Proof. Making use of the map J, We construct a map $\Psi_J : N\alpha^{M\#}$-$H(N^I,\ \tau_N^i) \longrightarrow N\alpha^{M\#}$-$H(N^{II},\ \tau_N^{ii})$ by $\Psi_J(F) = J \circ F \circ J^{-1}$ for every $F \in N\alpha^{M\#}$-$H(N^I,\ \tau_N^i)$. Then Ψ_J is a bijection. Further, for all $h_1, h_2 \in N\alpha^{M\#}$-$H(N^I,\ \tau_N^i)$, $\Psi_J(F_1 \circ F_2) = J \circ (F_1 \circ F_2) \circ J^{-1} = (J \circ F_1 \circ J^{-1}) \circ (J \circ F_2 \circ J^{-1}) = \Psi_J(F_1) \circ \Psi_J(F_2)$.

Therefore, Ψ_J It is an isomorphism caused by a homeomorphism by J.

□

5. **On $N\alpha^{M\#}$-connectedness**

Definition 5.1. A $NTS(N,\ \tau_N)$ is noted to be $N\alpha^{M\#}$-connected if N can't be characterized as a non-empty union of two distinct elements $N\alpha^{M\#}OS$. A subset of N is $N\alpha^{M\#}$-connected if any of this $N\alpha^{M\#}$-connected as a subspace.

Theorem 5.2. *For a $NTS(N,\ \tau_N)$, the following are better compared.*

(a) $(N,\ \tau_N)$ is $N\alpha^{M\#}$-connected.

(b) $(N,\ \tau_N)$ and ϕ_N seem to be the only subsets of $(N,\ \tau_N)$ both of which are $N\alpha^{M\#}$-open and $N\alpha^{M\#}$-closed.

(c) Each $N\alpha^{M\#}$-continuous map of $(N^I,\ \tau_N^i)$ into a discrete space $(N^{II},\ \tau_N^{ii})$ the map is constant if there are at least two points.

Proof. $(a) \Longrightarrow (b)$: Suppose $(N^I,\ \tau_N^i)$ is $N\alpha^{M\#}$-connected. Let S be both a valid subset $N\alpha^{M\#}OS$ and $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$. Its complement N/S is also $N\alpha^{M\#}$-open and $N\alpha^{M\#}$-closed. $N = S \cup (N/S)$, a non-empty union that is disjointed $N\alpha^{M\#}$-open sets that are incompatible (a). Therefore $S = \phi$ *or* N.

$(b) \implies (a)$: Suppose that $N = I_1 \cup I_2$ where I_1 and I_2 are disjoint non-empty $N\alpha^{M\#}$-open subsets of $(N^I,\ \tau_N^i)$. Then I_1 is both $N\alpha^{M\#}$-open and $N\alpha^{M\#}$-closed. By assumption $I_1 = \phi\ or\ N$. Therefore N is $N\alpha^{M\#}$-connected.

$(b) \implies (c)$: Let $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be an $N\alpha^{M\#}$-continuous map. Then $(N^I,\ \tau_N^i)$ is covered by $N\alpha^{M\#}$-open and $N\alpha^{M\#}$-closed covering $\{D^{-1}(n_{ii}) : n_{ii} \in N_{ii}\}$. By assumption $D^{-1}(n_{ii}) = \phi_N\ or\ N$ for each $n_{ii} \in N_{ii}$. If $D^{-1}(n_{ii}) = \phi$ for all $n_{ii} \in N_{ii}$, then D a map that isn't a map. Then $\exists$ a point $n_{ii} \in N_{ii}$ such that $D^{-1}(n_{ii}) \neq \phi_N$ and hence $D^{-1}(n_{ii}) = N$. This shows that D is a constant map.

$(c) \implies (b)$: Let S be both $N\alpha^{M\#}$-open and $N\alpha^{M\#}$-closed in N. Suppose $S \neq \phi$. Let $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be an $N\alpha^{M\#}$-continuous map defined by $D(S) = n_{ii}$ and $D(S^c) = \{\omega\}$ for a few key reasons n_{ii} and ω in $(N^{II},\ \tau_N^{ii})$. By assumption D is a constant map. Therefore we have $S = N$.

□

Theorem 5.3. *Every $N\alpha^{M\#}$-Space that is linked is connected.*

Proof. Let $(N^I,\ \tau_N^i)$ be $N\alpha^{M\#}$-linked(connected). Suppose N is not connected. There is then a suitable non-empty subset. B of $(N^I,\ \tau_N^i)$ which has both an open and a closed sets in $(N^I,\ \tau_N^i)$. Since every closed set is $N\alpha^{M\#}$-closed, B is a proper non empty subset of $(N^I,\ \tau_N^i)$ as well as $N\alpha^{M\#}OS$ and $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$, $(N^I,\ \tau_N^i)$ is not $N\alpha^{M\#}$-connected. This proves the theorem.

□

Theorem 5.4. *If $J : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is an $N\alpha^{M\#}$-continuous and N is $N\alpha^{M\#}$-connected, then $(N^{II},\ \tau_N^{ii})$ is linked.*

Proof. Presume that $(N^{II},\ \tau_N^{ii})$ is not linked. Let $N^{ii} = V_1 \cup V_2$ where V_1 and V_2 are disjoint non-empty OS in $(N^{II},\ \tau_N^{ii})$. As such J is $N\alpha^{M\#}$-continuous and onto, $N = J^{-1}(V_1) \cup J^{-1}(V_2)$ where $J^{-1}(V_1)$ and $J^{-1}(V_2)$ are disjoint non-empty $N\alpha^{M\#}$-open sets in $(N^I,\ \tau_N^i)$.

This is diametrically opposed to the fact that $(N^I,\ \tau_N^i)$ is $N\alpha^{M\#}$-connected. Furthermore N^{ii} is connected.

□

Theorem 5.5. *If $J : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is an $N\alpha^{M\#}$-irresolute and $(N^I,\ \tau_N^i)$ is $N\alpha^{M\#}$-connected, then $(N^{II},\ \tau_N^{ii})$ is $N\alpha^{M\#}$-connected.*

Proof. Suppose that $(N^{II},\ \tau_N^{ii})$ is not $N\alpha^{M\#}$-connected. Let $N^{ii} = V_1 \cup V_2$ where V_1 and V_2 are disjoint non-empty $N\alpha^{M\#}$-open sets in $(N^{II},\ \tau_N^{ii})$. Since J is $N\alpha^{M\#}$-irresolute and onto, $N = j^{-1}(V_1) \cup j^{-1}(V_2)$ where $J^{-1}(V_1)$ and $J^{-1}(V_2)$ are disjoint non-empty $N\alpha^{M\#}$-open sets in $(N^I,\ \tau_N^i)$.

This contradicts the fact that $(N^I,\ \tau_N^i)$ is $N\alpha^{M\#}$-connected. Hence $(N^{II},\ \tau_N^{ii})$ is $N\alpha^{M\#}$-connected.

□

Theorem 5.6. *Suppose that $(N^I,\ \tau_N^i)$ is $t_{\#}^{N\alpha}$-space then $(N^I,\ \tau_N^i)$ is connected $\Longleftrightarrow$ $N\alpha^{M\#}$-connected.*

Proof. Suppose that $(N^I,\ \tau_N^i)$ is connected. Then $(N^I,\ \tau_N^i)$ disjoint union of two non-empty proper subsets of the set cannot be expressed in $(N^I,\ \tau_N^i)$. Suppose $(N^I,\ \tau_N^i)$ is not a $N\alpha^{M\#}$-connected space. Let V_1 and V_2 be any two $N\alpha^{M\#}$-open subsets of $(N^I,\ \tau_N^i)$ such that $N^{ii} = V_1 \cup V_2$, where $V_1 \cap V_2 = \phi_N$ and $V_1 \subset N$, $V_2 \subset N$ Since $(N^I,\ \tau_N^i)$ is $t_{\#}^{N\alpha}$-space and V_1, V_2 are $N\alpha^{M\#}$-open. V_1, V_2 are open subsets of $(N^I,\ \tau_N^i)$, which contradicts that $(N^I,\ \tau_N^i)$ is connected. Therefore $(N^I,\ \tau_N^i)$ is $N\alpha^{M\#}$-connected.

Conversely, every open set is $N\alpha^{M\#}$-open. Therefore every $N\alpha^{M\#}$-connected space is connected.

□

Theorem 5.7. *If the $N\alpha^{M\#}$-open sets Z_1 and Z_2 form a separation of $(N^I,\ \tau_N^i)$ and if $(N^{II},\ \tau_N^{ii})$ is $N\alpha^{M\#}$-connected subspace of $(N^I,\ \tau_N^i)$, then $(N^{II},\ \tau_N^{ii})$ lies entirely within Z_1 or Z_2.*

Proof. Since Z_1 and Z_2 are both $N\alpha^{M\#}$-open in $(N^I,\ \tau_N^i)$, the sets $Z_1 \cap N^{ii}$ and $Z_2 \cap N^{ii}$ are $N\alpha^{M\#}$-open in $(N^{II},\ \tau_N^{ii})$. These two sets are incompatible, thus their union is impossible is $(N^{II},\ \tau_N^{ii})$. They would represent a separation if they were both non-empty $(N^{II},\ \tau_N^{ii})$.

Therefore, one of them is empty. Hence $(N^{II},\ \tau_N^{ii})$ must lie entirely in Z_1 or in Z_2.

□

6. Conclusion

The notions of $N\alpha^{M\#}CS$ in neutrosophic topological spaces have been discussed in this research study. We have also introduced the neutrosophic $t_{\#}^{N\alpha}$-space in this paper. The mappings known as neutrosophic $N\alpha^{M\#}$-continuous functions, $N\alpha^{M\#}$-irresolute functions, homeomorphisms and connectedness have also been introduced and investigate their characterizations and distinguishing features.

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Neutrosophic 2−normed spaces and generalized summability

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Abstract. The aim of present paper is to introduce the concept of neutrosophic 2−norm space (briefly abbreviated as $N-2-NS$) and study statistical summability in these spaces. We construct examples to demonstrate that statistical convergence is stronger method than usual convergence. Finally, we define statistically Cauchy sequence, statistical completeness and obtain the Cauchy convergence criteria.

Keywords: Neutrosophic norm spaces, statistical convergence, statistical completeness, neutrosophic 2-norm spaces.

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1. Introduction

Summability method is primarily concerned with the assignment of a limit in some generalized form to those sequences which do not converge in the usual sense. Over the years, many summability methods have been developed. One among these is developed by Henry Fast[6] and Schoenberg [20] independently by use of the natural density δ of subsets of $\mathbb{N}$ and called it as statistical convergence. For any set $K \subseteq \mathbb{N}$, the natural density of K is denoted by $\delta(K)$ and is defined by $\lim_n \frac{1}{n}|\{k \leq n : k \in K\}|$ provided the limit exists. Using δ, statistical convergent can be defined as follows.

"A sequence $x = (x_k)$ of numbers is said to be statistical convergent to L if for each $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n}\Big|\{k \leq n : |x_k - L| \geq \epsilon\}\Big| = 0;$$

or equivalently $\delta\big(\{k \leq n : |x_k - L| \geq \epsilon\}\big) = 0$. In this case, we write $S - \lim_k x_k = L$". Over the years, statistical convergence and related concepts have been further explored by numerous authors in different directions. For some interesting works on statistical convergence in this concern, we refer [5], [8], [11]-[13], and [19].

Apart from this, fuzzy sets were introduced by Zadeh [22] in 1965 as a generalization of crisp sets and turned out to be a very effective tool to deal with those situations which can not be fit in the framework of classical sets. These sets have wide applications in many areas of science and technology, especially in control engineering, artifical intelegence, robotics and many more to achieve better solutions. During development phase of fuzzy sets, many interesting generalizations of these sets have been appeared in the literature. For instance: intuitionistic fuzzy sets [1], vague fuzzy sets [4], interval-valued fuzzy sets [21], neutrosophic sets [15], etc. These sets have been further used to define some new kind of spaces such as fuzzy normed spaces [7], intuitionistic fuzzy metric spaces [17], intuitionistic fuzzy 2-normed spaces [16], intuitionistic fuzzy topological spaces [18] and neutrosophic normed spaces ([2], [3]). Recently, these spaces have been explored from sequence spaces point of view and linked with summability theory. Many summability method such as statistical convergence, ideal convergence, and lacunary statistical convergence have been developed. For an extensive view in this direction, we refer to the reader [10], [12]-[14]. In present work, we define a generalized neutrosophic normed space which we call neutrosophic $2-$norm space and introduce the convergence structure in these spaces. Later, we define statistical convergence, statistical Cauchy sequences in a neutrosophic $2-$norm space and develop some of their properties.

2. **Prelimanaries**

This section record a few definitions and outcomes that will be required in present study. Through out this work, $\mathbb{R}^+$ will denote the open interval $(0,\infty)$ and $\mathbb{N}$, the set of positive integers.

Definition 2.1 [11] Let $I=[0,1]$. A function $\circ : I\times I \to I$ is said to be a $t-$norm for all $f,g,h,i\in I$ we have:

(i) $f\circ g = g\circ f$;

(ii)$f\circ(g\circ h) = (f\circ g)\circ h$;

(iii) $\circ$ is continuous;

(iv) $f\circ 1 = f$ for every $f\in[0,\ 1]$ and

(v) $f\circ g\leq h\circ i$ whenever $f\leq h$ and $g\leq i$.

Definition 2.2 [11] Let $I=[0,1]$. A function $\diamond : I\times I\to I$ is said to be a continuous triangular conorm or $t-$conorm for all $f,g,h,i\in I$ we have:

(i) $f\diamond g = g\diamond f$;

(ii)$f\diamond(g\diamond h) = (f\diamond g)\circ h$;

(iii) $\diamond$ is continuous;

(iv) $f \diamond 0 = f$ for every $f \in [0, 1]$

(v) $f \diamond g \leq h \diamond i$ whenever $f \leq h$ and $g \leq i$.

Kirişci and Şimşek [13] recently defined NNS as follows.

Definition 2.3 [13] Let F is a vector space, $N = \{\langle \vartheta, \mathcal{H}(\vartheta), \mathcal{I}(\vartheta), \mathcal{J}(\vartheta) \rangle : \vartheta \in F\}$ be a normed space in which $N : F \times \mathbb{R}^+ \to [0,\ 1]$ and $\circ, \diamond$ respectively are $t-$norm and $t-$conorm. The four touple $V = (F, N,\ \circ, \diamond\ )$ is called a neutrosophic normed spaces (NNS) briefly it for every $p, q \in F$, $\rho\ \mu > 0$ and for every $\varsigma \neq 0$ we have

(i) $0 \leq \mathcal{H}(p,\ \rho) \leq 1$, $0 \leq \mathcal{I}(p,\ \rho) \leq 1$, $0 \leq \mathcal{J}(p,\ \rho) \leq 1$ for every $\rho \in \mathbb{R}^+$;

(ii) $\mathcal{H}(p,\ \rho) + \ \mathcal{I}(p,\ \rho) + \mathcal{J}(p,\ \rho) \leq 3$ for $\rho \in \mathbb{R}^+$;

(iii) $\mathcal{H}(p,\ \rho) = 1 \quad (\text{for } \rho > 0)$ iff $p = 0$;

(iv) $\mathcal{H}(\varsigma p, \rho) = \mathcal{H}\left(p,\ \frac{\rho}{|\varsigma|}\right)$;

(v) $\mathcal{H}(p,\ \mu) \circ \mathcal{H}(q,\ \rho) \leq \mathcal{H}(p + q,\ \mu + \rho)$;

(vi) $\mathcal{H}(p,\ .)$ is a non-decreasing function that runs continuously;

(vii) $\lim_{\rho \to \infty} \mathcal{H}(p,\ \rho) = 1$;

(viii) $\mathcal{I}(p, \rho) = 0\ \ (\text{for} \rho > 0)$ iff $p = 0$;

(ix) $\mathcal{I}(\varsigma p, \rho) = \mathcal{I}\left(p,\ \frac{\rho}{|\varsigma|}\right)$;

(x) $\mathcal{I}(p,\ \mu) \diamond \mathcal{I}(q,\ \rho) \geq \mathcal{I}(p + q, \rho + \mu)$;

(xi) $\mathcal{I}(p,\ .)$ is a non-decreasing function that runs continuously;

(xii) $\lim_{\lambda \to \infty} \mathcal{I}(p, \rho) = 0$;

(xiii) $\mathcal{J}(p, \rho) = 0 \quad (\text{for } \rho > 0)$ iff $p = 0$;

(xiv) $\mathcal{J}(\varsigma p, \rho) = \mathcal{J}\left(p,\ \frac{\rho}{|\varsigma|}\right)$;

(xv) $\mathcal{J}(p,\ \mu) \diamond \mathcal{J}(q,\ \rho) \geq \mathcal{J}(p + q, \rho + \mu)$;

(xvi) $\mathcal{J}(p,\ .)$ is a non-decreasing function that runs continuously;

(xvii) $\lim_{\lambda \to \infty} \mathcal{J}(p, \rho) = 0$;

(xviii) If $\rho \leq 0$, then $\mathcal{H}(p, \rho) = 0$, $\mathcal{I}(p, \rho) = 1$ and $\mathcal{J}(p, \rho) = 1$.

We call $N(\mathcal{H}, \mathcal{I}, \mathcal{J})$ the neutrosophic norm.

We next give the notions of statistical convergence and statistical Cauchy sequences in neutrosophic norm spaces as introduced in [13].

Definition 2.4 [13] Let V be a NNS. Choose $0 < \epsilon < 1$ and $\rho > 0$. A sequence (v_k) in V is said to be statistical convergent if $\exists\ v_0 \in F$ s.t. $\lim_n \frac{1}{n}|\{k \leq n : \mathcal{H}(v_k - v_0, \rho) \leq 1 - \varepsilon$ or $\mathcal{I}(v_k - v_0, \rho) \geq \varepsilon$ and $\mathcal{J}(v_k - v_0, \rho) \geq \varepsilon\}| = 0$; or equivalently, the set's natural density $A(\varepsilon, \rho) = \{k \leq n :\ \mathcal{H}(v_k - v_0; \rho) \leq 1 - \varepsilon$ or $\mathcal{I}(v_k - v_0; \rho) \geq \varepsilon$ and $\mathcal{J}(v_k - v_0, \rho) \geq \varepsilon\}$ is zero, i.e., $\delta(A(\varepsilon, \rho)) = 0$. we can write it as $S(N) - \lim_{k \to \infty} v_k = v_0$.

Definition 2.5 [13] Let V be a NNS. Choose $0 < \epsilon < 1$ and $\rho > 0$. A sequence (v_k) in V is said to be statistical Cauchy if $\exists \ p \in \mathbb{N}$ s.t. $\lim_n \frac{1}{n}|\{k \leq n : \mathcal{H}(v_k - v_p, \rho) \leq 1 - \varepsilon$ or $\mathcal{I}(v_k - v_p, \rho) \geq \varepsilon$ and $\mathcal{J}(v_k - v_p, \rho) \geq \varepsilon\}| = 0$; or equivalently, the natural density of the set $A(\varepsilon, \rho) = \{k \leq n : \ \mathcal{H}(v_k - v_p, \rho) \leq 1 - \varepsilon$ or $\mathcal{I}(v_k - v_p, \rho) \geq \varepsilon$ and $\mathcal{J}\ (v_k - v_p, \rho) \geq \varepsilon\}$ is zero, i.e., $\delta\left(A\left(\varepsilon, \rho\right)\right) = 0$.

We now turn towards the paper [9] and would like to quote the idea of two norm.

Definition 2.6 [9] Let V be a $d-$dimensional real vector space, where $2 \leq d < \infty$. A $2-$norm on V is a function $\|.,.\| : V \times V \to \mathbb{R}$ fulfilling the below listed requirements:

For all $p, q \in V$, and scalar α, we have

(i) $||p, q|| = 0$ iff p and q are linearly dependent;

(ii) $||p, q|| = ||p, q||$;

(iii)$||\alpha p, q|| = |\alpha|||p, q||$ and

(iv) $||p, q + r|| \leq ||p, q|| + ||p, r||$.

The pair $(V, ||.,.||)$ is known as $2-$normed space in this case.

Let $V = \mathbb{R}^2$ and for $p = (p_1, p_2)$ and $q = (q_1, q_2)$ we define $||p, q|| = |p_1q_2 - p_2q_1|$, then $||p, q||$ is a $2-$ norm on $V = \mathbb{R}^2$.

We now proceed with our main results.

3. Neutrosophic-2-norm spaces $(N - 2 - NS)$

This section starts with the following definition of neutrosophic$-2-$norm spaces.

Definition 3.1 Let F is a vector space, $N_2 = (\{(p, q), \mathcal{H}(p, q), \mathcal{I}(p, q), \mathcal{J}(p, q)\} : (p, q) \in F \times F)$ be a $2-$norm space s.t. $N_2 : F \times F \times \mathbb{R}^+ \to [0, 1]$. If $\circ, \diamond$ respectively denotes $t - norm$ and $t - conorm$, then four-tuple $V = (F, N_2, \circ, \diamond)$ is known as neutrosophic $2-$norm spaces (briefly $N - 2 - NS$) if for every $p, q, w \in V,\ \rho, \mu \geq 0$ and $\varsigma \neq 0$:

(i) $0 \leq \mathcal{H}(p, q; \rho) \leq 1$, $0 \leq \mathcal{I}(p, q; \rho) \leq 1$ and $0 \leq \mathcal{J}(p, q; \rho) \leq 1$ for every $\rho \in R^+$;

(ii) $\mathcal{H}(p, q; \rho) + \mathcal{I}(p, q; \rho) + \mathcal{J}(p, q; \rho) \leq 3$;

(iii) $\mathcal{H}(p, q; \rho) = 1$ iff p, q are linearly dependent;

(iv) $\mathcal{H}(\varsigma p,\ q;\ \rho) = \mathcal{H}(p, q; \frac{\rho}{|\varsigma|})$ for each $\varsigma \neq 0$;

(v) $\mathcal{H}(p, q; \rho) \circ \mathcal{H}(p, w; \mu) \leq \mathcal{H}(p, q + w; \rho + \mu)$;

(vi) $\mathcal{H}(p, q; .) : (0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;

(vii) $\lim\limits_{\rho \to \infty} \mathcal{H}(p, q; \rho) = 1$;

(viii) $\mathcal{H}(p, q; \rho) = \mathcal{H}(q, p; \rho)$

(ix) $\mathcal{I}(p, q; \rho) = 0$ iff p, q are linearly dependent;

(x) $\mathcal{I}(\varsigma p, q; \rho) = \mathcal{I}(p, q; \frac{\rho}{|\varsigma|})$ for each $\varsigma \neq 0$;

(xi) $\mathcal{I}(p, q; \rho) \diamond \mathcal{I}(p, w; \mu) \geq \mathcal{I}(p, q + w; \rho + \mu)$;

(xii) $\mathcal{I}(p, q; .) : (0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;

(xiii) $\lim_{\rho \to \infty} \mathcal{I}(p, q; \rho) = 0$;

(xiv) $\mathcal{I}(p, q; \rho) = \mathcal{I}(q, p; \rho)$

(xvi) $\mathcal{J}(p, q; \rho) = 0$ iff p, q are linearly dependent;

(xv)$\mathcal{J}(\varsigma p,\ q;\ \rho) = \mathcal{J}(p,\ q; \frac{\rho}{|\varsigma|})$ for each $\varsigma \neq 0$;

(xvi) $\mathcal{J}(p, q; \rho) \diamond \mathcal{J}(p, w; \mu) \geq \mathcal{J}(p, q + w; \rho + \mu)$;

(xvii) $\mathcal{J}(p, q; .) : (0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;

(xviii) $\lim_{\lambda \to \infty} \mathcal{J}(p, q; \rho) = 0$;

(xix) $\mathcal{J}(p, q; \rho) = \mathcal{J}(q, p; \rho)$

(xx) if $\rho \leq 0$, then $\mathcal{H}(p, q; \rho) = 0$, $\mathcal{I}(p, q; \rho) = 1$, $\mathcal{J}(p, q\rho) = 1$.

In this case, we call $N_2(\mathcal{H}, \mathcal{I}, \mathcal{J})$ a neutrosophic $2-$norm on F and is denoted by N_2.

Example 3.1 Let $(F, ||., .||)$ be a $N - 2 - NS$. We define the continuous $t - norm$ and $t - conorm$ by

$p \circ q = pq$ and $p \diamond q = p + q - pq$.

For $p, q \in F$, $\rho > 0$ with $\rho > \|p, q\|$, we define

$\mathcal{H}(p, q; \rho) = \frac{\rho}{\rho + \|p,q\|}$, $\mathcal{I}(p, q; \rho) = \frac{\|p,q\|}{\rho + \|p,q\|}$, and $\mathcal{J}(p, q; \rho) = \frac{\|p,q\|}{\rho}$.

If we take $||p, q|| \geq \rho$, then $\mathcal{H}(p, q; \rho) = 0$, $\mathcal{I}(p, q; \rho) = 1$, $\mathcal{J}(p, q; \rho) = 1$ and $(F, N_2, \circ, \diamond)$ is a $N - 2 - NS$ where $N_2 : F \times F \times R^+ \to [0, 1]$.

We now define convergence structure and Cauchy sequences in $N - 2 - NS$.

Definition 3.2 Let V be a $N - 2 - NS$. Choose $0 < \epsilon < 1$ and $\rho > 0$. A sequence (v_k) in a V is said to be convergent if $\exists$ a positive integer m and $v_0 \in F$ s.t. $\mathcal{H}(v_k - v_0, w; \rho) > 1 - \epsilon$, $\mathcal{I}(v_k - v_0, w; \rho) < \epsilon$ and $\mathcal{J}(v_k - v_0, w; \rho) < \epsilon$ for all $k \geq m$ and $w \in V$ which is equivalently to say $\lim_{k\to\infty} \mathcal{H}(v_k - v_0, w; \rho) = 1$, $\lim_{k\to\infty} \mathcal{I}(v_k - v_0, w; \rho) = 0$ and $\lim_{k\to\infty} \mathcal{J}(v_k - v_0, w; \rho) = 0$. In this case, we write $N_2 - \lim_{k\to\infty} v_k = v_0$.

Theorem 3.1 Let V be a $N - 2 - NS$ (u_k) and (v_k) be two sequences in V and α being any scalar.

(i) If (u_k) is convergent w.r.t. N_2, then its limit is unique.

(ii) If $N_2 - \lim_{k\to\infty} u_k = u_0$, then $N_2 - \lim_{k\to\infty} \alpha u_k = \alpha u_0$.

(iii) If $N_2 - \lim_{k\to\infty} u_k = u_0$ and $N_2 - \lim_{k\to\infty} v_k = v_0$, then $N_2 - \lim_{k\to\infty}(u_k + v_k) = (u_0 + v_0)$.

Proof. Omitted. □

Definition 3.3 Let V be a $N-2-NS$. Choose $0<\epsilon<1$ and $\rho>0$. A sequence (v_k) in a V if $\exists$ a positive integer m is said to be Cauchy s.t. $\mathcal{H}(v_k-v_n,w;\rho)>1-\epsilon$, $\mathcal{I}(v_k-v_n,w;\rho)<\epsilon$ and $\mathcal{J}(v_k-v_n,w;\rho)<\epsilon\ \forall\ k,n\geq m$ and $\forall\ w\in V$.

Definition 3.4 A $N-2-NS$ V is said to be complete if and only if each Cauchy sequence in V is converget in V.

Theorem 3.2 Every convergent sequence in a $N-2-NS$, V is Cauchy however converse is not true.

Proof. Let $\epsilon>0$ and choose $r>0$ s.t. $(1-\epsilon)\circ(1-\epsilon)>1-r$ and $\epsilon\diamond\epsilon<r$. For $\rho>0$, if we take (v_k) be any convergent in V with $N_2-\lim_{k\to\infty}v_k=v_0$. There is an integer m s.t. $\mathcal{H}(v_k-v_0,w;\rho)>1-\epsilon$, $\mathcal{I}(v_k-v_0,w;\rho)<\epsilon$ and $\mathcal{J}(v_k-v_0,w;\rho)<\epsilon$ for all $k\geq m$ and $w\in V$. Now, for all $k,n\geq m$ we have $\mathcal{H}(v_k-v_n,w;\rho)\geq\mathcal{H}(v_k-v_0,w;\frac{\rho}{2})\circ\ \mathcal{H}(v_n-v_0,w;\frac{\rho}{2})$ $>(1-\epsilon)\circ(1-\epsilon)>r$. Similarly one can easily get $\mathcal{I}(v_k-v_n,w;\rho)<r$ and $\mathcal{J}(v_k-v_n,w;\rho)<r$ for every $k,n\geq m$. This prove that the (v_k) sequence is Cauchy. □

Example 3.2 Let $F=\{z_{mn}=(\frac{1}{m},\frac{1}{n}):m,n\in\mathbb{N}\}\subseteq\mathbb{R}^2$ be a 2−normed space with $\|(m,n)\|=|\frac{1}{m}-\frac{1}{n}|$. If we define the neutrosophic norm N_2 as in Example 3.1 then $V=(F,N_2,\circ,\diamond)$ is a $N-2-NS$. Further, the sequence z_{mn} is Cauchy but not convergent as $\lim_{k\to\infty}\mathcal{I}\,(v_k-v_0,w;\ \rho)\neq 0$. □

4. Statistical Convergence in $N-2-NS$

This section explore the statistical convergence and its properties in a $N-2-NS$.

Definition 4.1 Let V be a $N-2-NS$. Choose $0<\epsilon<1$ and $\rho>0$. A sequence (v_k) in V is said to be statistical convergent to v_0 provided that $\lim_n\frac{1}{n}|\{k\leq n:\mathcal{H}(v_k-v_0,w;\rho)\leq 1-\varepsilon$ or $\mathcal{I}(v_k-v_0,w;\rho)\geq\varepsilon$ and $\mathcal{J}(v_k-v_0,w;\rho)\geq\varepsilon\}|=0$ for every $w\in V$ or equivalently, $\delta\left(A\left(\varepsilon,\rho\right)\right)=0$. where $A(\varepsilon,\rho)=\{k\leq n:\ \mathcal{H}(v_k-v_0,w;\rho)\leq 1-\varepsilon$ or $\mathcal{I}(v_k-v_0,w;\rho)\geq\varepsilon$ and $\mathcal{J}(v_k-v_0,w;\rho)\geq\varepsilon\}$ and we write $S(N_2)-\lim_{k\to\infty}v_k=v_0$.

Theorem 4.1 Let V be a $N-2-NS$ and (v_k) be any sequence in V. If $N_2-\lim_{k\to\infty}v_k=v_0$, then $S(N_2)-\lim_{k\to\infty}v_k=v_0$.

Proof According to the hypothesis, for every $\epsilon>0$ and $\rho>0$, there is an integer $k_0\in\mathbb{N}$ s.t. $\mathcal{H}(v_k-v_0,w;\rho)>1-\epsilon$ and $\mathcal{I}(v_k-v_0,w;\rho)<\epsilon$, $\mathcal{J}(v_k-v_0,w;\rho)<\epsilon$ for all $k\geq k_0$ and every $w\in V$. This guarantees that the set $\{k\in\mathbb{N}:\mathcal{H}(v_k-v_0,w;\rho)\leq 1-\epsilon$ or $\mathcal{I}(v_k-v_0,w;\rho)<\epsilon$,

$\mathcal{J}(v_k - v_0, w; \rho) < \epsilon\}$ has a finite number of terms whose density is zero. This immediately shows that $S(N_2) - \lim_{k\to\infty} v_k = v_0$. □

In general, The converse of the theorem is false.

Example 4.1 Let $(\mathbb{R}, |.|)$ be the real space with the usual norm. For $f, g \in [0,1]$. Let the $t - norm$ and $t - conorm$ are defined by $f \circ g = fg$ and $f \diamond g = \min\{f+g, 1\}$. Choose $p, q \in F$ and $\rho > 0$ with $\rho > \|p,q\|$. If we define $\mathcal{H}(p,q;\rho) = \frac{\rho}{\rho+\|p,q\|}$, $\mathcal{I}(p,q;\rho) = \frac{\|p,q\|}{\rho+\|p,q\|}$ and $\mathcal{J}(p,q;\rho) = \frac{\|p,q\|}{\rho}$, then $N_2(\mathcal{H},\mathcal{I},\mathcal{J})$ is a neutrosophic-2-norm and $V = (F, N_2, \circ, \diamond)$ is a $N - 2 - NS$. Define a sequence (v_k) by

$$v_k = \begin{cases} (0,\ 1), & \text{if } k = m^2,\ m \in \mathbb{N}; \\ (0,\ 0), & \text{otherwise.} \end{cases} \tag{1}$$

Let, $A_n(\varepsilon,\rho) = \{k \le n : \ \mathcal{H}(v_k, w;\ \rho) \le 1 - \varepsilon$ or $\mathcal{I}(v_k, w; \rho) \ge \varepsilon,\ \mathcal{J}(v_k, w; \rho) \ge \varepsilon\}$, then $A_n(\varepsilon,\rho) = \{k \le n : \ \frac{\rho}{\rho+||p,q||} \le 1 - \varepsilon$ or $\frac{||p,q||}{\rho+||p,q||} \ge \varepsilon,\ \frac{||p,q||}{\rho} \ge \varepsilon\} = \{k \le n : ||p,q|| \ge \frac{\rho\epsilon}{1-\epsilon}$ or $||p,q|| \ge \rho\epsilon\} = \{k \le n : v_k = (0,1)\} = \{k \le n : k = m^2\}$ and therefore we have $\lim_n \frac{1}{n}|A_n(\varepsilon,\rho)| = \{k \le n : k = m^2\} \le \frac{\sqrt{n}}{n} = \frac{1}{n} \to 0$. Thus $S(N_2) - \lim_{k\to\infty} v_k = 0$. However, the sequence (v_k) is not usual convergent. □

Lemma 4.1 Let V be a $N - 2 - NS$. Then for every $0 < \epsilon < 1$, $\rho > 0$ and for every $w \in V$, The statements below are equivalents:

(i) $S(N_2) - \lim_{k\to\infty} v_k = v_0$;

(ii) $\delta\{k \in \mathbb{N} : \mathcal{H}(v_k - v_0, w; \rho) \le 1 - \epsilon\} = \delta\{k \in \mathbb{N} : \mathcal{I}(v_k - v_0, w; \rho) \ge \epsilon\} = \delta\{k \in \mathbb{N} : \mathcal{J}(v_k - v_0, w; \rho) \ge \epsilon\} = 0$;

(iii) $\delta\{k \in \mathbb{N} : \mathcal{H}(v_k - v_0, w; \rho) > 1 - \epsilon$ and $\mathcal{I}(v_k - v_0, w; \rho) < \epsilon$, $\mathcal{J}(v_k - v_0, w; \rho) < \epsilon\} = 1$;

(iv) $\delta\{k \in \mathbb{N} : \mathcal{H}(v_k - v_0, w; \rho) > 1 - \epsilon\} = \delta\{k \in \mathbb{N} : \mathcal{I}(v_k - v_0, w; \rho) < \epsilon\} = \delta\{k \in \mathbb{N} : \mathcal{J}(v_k - v_0, w; \rho) < \epsilon\} = 1$ and

(v) $S(N_2) - \lim_{k\to\infty} \mathcal{H}(v_k - v_0, w; \rho) = 1$ or $S(N_2) - \lim_{k\to\infty} \mathcal{I}(v_k - v_0, w; \rho) = 0$ and $S(N_2) - \lim_{k\to\infty} \mathcal{J}(v_k - v_0, w; \rho) = 0$.

Proof. Omitted. □

Theorem 4.2 Let V be a $N - 2 - NS$. For any sequence (v_k), if $S(N_2) - \lim_{k\to\infty} v_k$ exists then it must be unique.

Proof Assume that $S(N_2) - \lim_{k\to\infty} v_k = v_1$ and $S(N_2) - \lim_{k\to\infty} v_k = v_2$. For a given $\epsilon > 0$, choose $l > 0$ s.t. $(1 - l) \circ (1 - l) > 1 - \epsilon$ and $q \diamond q < \epsilon$. For any $\rho > 0$ and any $w \in V$ The following sets are defined: $K_{\mathcal{H},1}(l,\rho) = \{k \in \mathbb{N} : \mathcal{H}(v_k - v_1, w; \rho) \le 1 - l\}$, $K_{\mathcal{H},2}(l,\rho) = \{k \in \mathbb{N} : \mathcal{H}(v_k - v_2, w; \rho) \le 1 - l\}$; $K_{\mathcal{I},1}(l,\rho) = \{k \in \mathbb{N} : \mathcal{I}(v_k - v_1, w; \rho) \ge l\}$, $K_{\mathcal{I},2}(l,\rho) = \{k \in \mathbb{N} : \mathcal{I}(v_k - v_2, w; \rho) \ge l\}$; $K_{\mathcal{J},1}(l,\rho) = \{k \in \mathbb{N} : \mathcal{Y}(v_k - v_1, w; \rho) \ge l\}$, $K_{\mathcal{J},2}(l,\rho) =$

$\{k \in \mathbb{N} : \mathcal{J}(v_k - v_2, w; \rho) \geq l\}$. Since $S(N_2) - \lim_{k\to\infty} v_k = v_1$, so by Lemma 4.1, we have $\delta\{K_{\mathcal{H},1}(l,\rho)\} = \delta\{K_{\mathcal{I},1}(l,\rho)\} = \delta\{K_{\mathcal{J},1}(l,\rho)\} = 0$. Furthermore, using $S(N_2) - \lim_{k\to\infty} v_k = v_2$, we get, $\delta\{K_{\mathcal{H},2}(l,\lambda)\} = \delta\{K_{\mathcal{I},2}(l,\rho)\} = \delta\{K_{\mathcal{J},2}(l,\rho)\} = 0$. Now let $K_{\mathcal{H}}, \mathcal{I}, \mathcal{J}(\epsilon,\rho) = \{K_{\mathcal{H},1}(\epsilon,\rho) \cup K_{\mathcal{H},2}(\epsilon,\rho)\} \cap \{K_{\mathcal{I},1}(\epsilon,\rho) \cup K_{\mathcal{I},2}(\epsilon,\rho)\} \cap \{K_{\mathcal{J},1}(\epsilon,\rho) \cup K_{\mathcal{J},2}(\epsilon,\rho)\}$. Then observe that $\delta(\{K_{\mathcal{H},\mathcal{I},\mathcal{J}}(\epsilon,\rho)\}) = 0$ which implies $\delta(\{\mathbb{N}/K_{\mathcal{H},\mathcal{I},\mathcal{J}}(\epsilon,\rho)\}) = 1$. If $k \in \mathbb{N}/K_{\mathcal{H},\mathcal{I},\mathcal{J}}(\epsilon,\rho)$, then we have the following possibilities.

Case 1 $k \in \mathbb{N}/\{K_{\mathcal{H},1}(\epsilon,\rho) \cup K_{\mathcal{H},2}(\epsilon,\rho)\}$,

Case 2 $k \in \mathbb{N}/\{K_{\mathcal{I},1}(\epsilon,\rho) \cup K_{\mathcal{I},2}(\epsilon,\rho)\}$,

Case 3 $k \in \mathbb{N}/\{K_{\mathcal{J},1}(\epsilon,\rho) \cup K_{\mathcal{J},2}(\epsilon,\rho)\}$.

We prove the result only for case 1 as other cases can be obtain similarly. Assume, $k \in \mathbb{N}/\{K_{\mathcal{H},1}(\epsilon,\rho) \cup K_{\mathcal{H},2}(\epsilon,\rho)\}$. Then for any $w \in V$ we have $\mathcal{H}(v_k - v_1, w; \rho) > 1 - l$ and $\mathcal{H}(v_k - v_2, w; \rho) > 1 - l$. Now $\mathcal{H}(v_1 - v_2, w; \rho) \geq \mathcal{H}(v_k - v_1, w, ; \frac{\rho}{2}) \circ \mathcal{H}(v_k - v_2, w; \frac{\rho}{2}) > (1-l) \circ (1-l) > 1 - \epsilon$ (by choice of q). i.e., $\mathcal{H}(v_1 - v_2, w; \rho) > 1 - \epsilon$. Since $\epsilon > 0$ is arbitrary so we have $\mathcal{H}(v_1 - v_2, w; \rho) = 1$, and therefore $v_1 - v_2 = 0$. This shows that $v_1 = v_2$. Similarly in case 2 and case 3, we obtain $\mathcal{I}(v_1 - v_2, w; \rho) < \epsilon$ and $\mathcal{J}(v_1 - v_2, w; \rho) < \epsilon$ which gives $\mathcal{I}(v_1 - v_2, w; \rho) = 0$ and $\mathcal{J}(v_1 - v_2, w; \rho) = 0$. The complete proof of the theorem. □

Theorem 4.3 Let V be a $N-2-NS$; (u_k) and (v_k) be two sequences in V and α being any scalar.

(i) If $S(N_2) - \lim_{k\to\infty} u_k = u_0$, then $S(N_2) - \lim_{k\to\infty} \alpha u_k = \alpha u_0$.

(ii) If $S(N_2) - \lim_{k\to\infty} u_k = u_0$ and $S(N_2) - \lim_{k\to\infty} v_k = v_0$, then $S(N_2) - \lim_{k\to\infty}(u_k + v_k) = (u_0 + v_0)$.

Proof. Omitted. □

Theorem 4.4 Let V be a $N-2-NS$ and (v_k) be any sequence in V, then $S(N_2) - \lim_{k\to\infty} v_k = v_0$ iff an ascending index sequence of natural numbers $K = \{k_n : n \in \mathbb{N}\}$ exists with $\delta\{K\} = 1$ and $N_2 - \lim\limits_{n\to\infty} v_{k_n} = v_0$.

Proof Necessity: Assume that $S(N_2) - \lim_{k\to\infty} v_k = v_0$. For any $\rho > 0$, $j \in \mathbb{N}$ and $w \in V$, let, $K_{N_2}(j,\rho) = \{n \in \mathbb{N} : \mathcal{H}(v_n - v_0, w; \rho) > 1 - \frac{1}{j}$ and $\mathcal{I}(v_n - v_0, w; \rho) < \frac{1}{j}$, $\mathcal{J}(v_n - v_0, w; \rho) < \frac{1}{j}\}$. Then it is clear that $K_{N_2}(j+1,\rho) \subset K_{N_2}(j,\rho)$. Since $S(N_2) - \lim_{k\to\infty} v_k = v_0$, so we have $\delta\{K_{N_2}(j,\rho)\} = 1$. Let m_1 be an arbitrary number in $K_{N_2}(1,\rho)$. Then, $\exists$ a number $m_2 \in K_{N_2}(2,\rho), (m_2 > m_1)$, such that for all $n \geq m_2$, $\frac{1}{n}|\{k \leq n : \mathcal{H}(v_k - v_0, w; \rho) > 1 - \frac{1}{2}$ and $\mathcal{I}(v_k - v_0, w; \rho) < \frac{1}{2}$, $\mathcal{J}(v_k - v_0, w; \rho) < \frac{1}{2}\}| > \frac{1}{2}$. Again on the similar lines there is another number $m_3 \in K_{N_2}(3,\rho), (m_3 > m_2)$, such that for all $n \geq m_3$, $\frac{1}{n}|\{k \leq n : \mathcal{H}(v_k - v_0, w; \rho) > 1 - \frac{1}{3}$ and $\mathcal{I}(v_k - v_0, w; \rho) < \frac{1}{3}$, $\mathcal{J}(v_k - v_0, w; \rho) < \frac{1}{3}\}| > \frac{2}{3}$ and so on. Thus we can set a sequence $\{m_j\}_{j\in\mathbb{N}}$ of positive integers satisfying $m_j \in K_{N_2}(j,\rho)$ and for all $n \geq m_j (j \in \mathbb{N})$: $\frac{1}{n}|\{k \leq n : \mathcal{H}(v_k - v_0, w; \rho) > 1 - \frac{1}{j}$ and $\mathcal{I}(v_k - v_0, w; \rho) < \frac{1}{j}$, $\mathcal{J}(v_k - v_0, w; \rho) < \frac{1}{j}\}| > \frac{j-1}{j}$.

Define $K = \{n \in \mathbb{N} : 1 < n < m_1\} \cup \{\underset{j\in\mathbb{N}}{\cup}\{n \in K_{N_2}(j,\rho) : m_j \leq n < m_{j+1}\}\}$, Then it is obvious that, for all n satisfying $(m_j \leq n < m_j + 1)$, we have $\frac{1}{n}|\{k \leq n : k \in K\}| \geq \frac{1}{n}|\{k \leq n : \mathcal{H}(v_k - v_0, w; \rho) > 1 - \frac{1}{j}$ and $\mathcal{I}(v_k - v_0, w; \rho) < \frac{1}{j}$, $\mathcal{J}(v_k - v_0, w; \rho) < \frac{1}{j}\}| > \frac{j-1}{j}$. By taking limit on both side, we have $\delta(K) = 1$. It remains to prove that the subsequence of the sequence (v_k) over K is N_2-convergent to v_0. For this, let $\epsilon > 0$ be any number and select a number $j \in \mathbb{N}$ with $\frac{1}{j} < \epsilon$. Moreover, let $n \geq m_j$ as well as $n \in K$ Then, according to the definition of K, $\exists$ a number $l \geq j$ s.t, $m_l \leq n < m_{l+1}$ and $n \in K_{N_2}(j,\rho)$. Thus, for every $\epsilon > 0$, and for every $w \in V$ we have $\mathcal{H}(v_n - v_0, w; \rho) > 1 - \frac{1}{j} > 1 - \epsilon$ and $\mathcal{I}(v_n - v_0, w; \rho) < \frac{1}{j} < \epsilon$, $\mathcal{J}(v_n - v_0, w; \rho) < \frac{1}{j} < \epsilon$ for all $n \geq h_w$ and $n \in K$. This shows that $N_2 - \lim_{n\in K} v_n = v_0$.

Sufficiency: In second part, we assume that there is a set $K = \{k_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with $\delta\{K\} = 1$ and $N_2 - \lim_{n\in K} v_n = v_0$. We shall show that $S(N_2) - \lim_{k\to\infty} v_k = v_0$. Let $\epsilon > 0$ and $\rho > 0$. Since, $N_2 - \lim_{n\in K} v_n = v_0$ so there exist positive integer n_0 such that $\mathcal{H}(v_{k_n} - v_0, w; \rho) > 1 - \epsilon$ and $\mathcal{I}(v_{k_n} - v_0, w; \rho) < \epsilon$, $\mathcal{J}(v_{k_n} - v_0, w; \rho) < \epsilon$ for every $k_n \geq k_{n_0}$ and every $w \in V$. This implies the containment: $T_{N_2}(\epsilon, \rho) = \{n \in \mathbb{N} : \mathcal{H}(v_n - v_0, w; \rho) \leq 1 - \epsilon$ and $\mathcal{I}(v_n - v_0, w; \rho) \geq \epsilon$, $\mathcal{J}(v_n - v_0, w; \rho) \geq \epsilon\} \subseteq \mathbb{N} - \{v_{n_0}, v_{n_0+1}, v_{n_0+2}, ...\}$. and therefore $\delta\{T_{N_2}(\epsilon,\rho)\} \leq \delta\{\mathbb{N} - \{v_{n_0}, v_{n_0+1}, v_{n_0+2}, ...\}$. As $\delta\{K\} = 1$, so $\delta\{T_{N_2}(\epsilon,\rho)\} = 0$. This shows that $S(N_2) - \lim_{k\to\infty} v_k = v_0$ and therefore the complete proof of the Theorem. □

Finally we define statistical Cauchy sequence in $N - 2 - NS$ and obtain the Cauchy convergence criteria in these spaces.

Definition 4.2 Let V be a $N - 2 - NS$, $\epsilon > 0$ and $\lambda > 0$. A sequence (v_k) in V is said to be statistical Cauchy if $\exists\ p \in \mathbb{N}$ s.t. $\lim_n \frac{1}{n}|\{k \leq n : \mathcal{H}(v_k - v_p, w; \rho) \leq 1 - \varepsilon$ or $\mathcal{I}(v_k - v_p, w; \rho) \geq \varepsilon$ and $\mathcal{J}(v_k - v_p, w; \rho) \geq \varepsilon\}| = 0$ for every $w \in V$ or equivalently, the natural density of the set $A(\varepsilon, \rho) = \{k \leq n :\ \mathcal{H}(v_k - v_p, w; \rho) \leq 1 - \varepsilon$ or $\mathcal{I}(v_k - v_p, w; \rho) \geq \varepsilon$ and $\mathcal{J}\ (v_k - v_p, w; \rho) \geq \varepsilon\}$ is zero, i.e., $\delta\left(A\left(\varepsilon,\ \rho\right)\right) = 0$.

Theorem 4.5 Let V be a $N - 2 - NS$, then every statistical convergent sequence in V is statistical Cauchy.

Proof Let (v_k) be a statistical convergent to v_0 and $\epsilon > 0$ be given. Chose $\mu > 0$ s.t. $(1-\epsilon)\circ(1-\epsilon) > 1-\mu$ and $\epsilon\diamond\epsilon < \mu$. For $\rho > 0$, if we define $A(\varepsilon,\ \rho) = \{k \leq n :\ \mathcal{H}(v_k - v_0, w;\ \frac{\rho}{2}) \leq 1 - \varepsilon$ or $\mathcal{I}(v_k - v_0, w; \frac{\rho}{2}) \geq \varepsilon$ and $\mathcal{J}(v_k - v_0, w; \frac{\rho}{2}) \geq \varepsilon\}$, then $\delta(A(\varepsilon,\rho)) = 0$ and therefore $\delta(A^C(\varepsilon,\rho)) = 1$. Let $p \in A^c(\epsilon,\rho)$ then for any $w \in V$ we have $\mathcal{H}(v_p - v_0, w; \frac{\rho}{2}) > 1 - \epsilon$ and $\mathcal{I}(v_p - v_0, w; \frac{\rho}{2}) < \epsilon, \mathcal{J}(v_p - v_0, w; \frac{\rho}{2}) < \epsilon$.

Define $\mathcal{B}(\mu,\rho)=\{k\le n:\mathcal{H}(v_k-v_p,w;\rho)\le 1-\mu$ or $\mathcal{I}(v_k-v_p,w;\rho)\ge\mu,\mathcal{J}(v_k-v_p,w;\rho)\ge\mu\}$. We claim that $\mathcal{B}(\mu,\rho)\subset A(\epsilon,\rho)$. Let $q\in\mathcal{B}(\mu,\rho)$. Then we have $\mathcal{H}(v_q-v_p,w;\rho)\le 1-\mu$ or $\mathcal{I}(v_q-v_p,w;\rho)\ge\mu,\mathcal{J}(v_q-v_p,w;\rho)\ge\mu$.

Case (i): Suppose $\mathcal{H}(v_q-v_p,w;\rho)\le 1-\mu$, Then we have $\mathcal{H}(v_q-v_0,w;\frac{\rho}{2})\le 1-\epsilon$ and therefore $q\in A(\epsilon,\rho)$ (as otherwise, i.e, if $\mathcal{H}(v_q-v_0,w;\frac{\rho}{2})>1-\epsilon$, then $1-\mu\ge\mathcal{H}(v_q-v_p,w;\rho)\ge\mathcal{H}(v_q-v_0,w;\frac{\rho}{2})\circ\mathcal{H}(v_p-v_0,w;\frac{\rho}{2})>(1-\epsilon)\circ(1-\epsilon)>1-\mu$ which is not possible). Hence $\mathcal{B}(\mu,\rho)\subset A(\epsilon,\rho)$.

Case (ii): Suppose $\mathcal{I}(v_q-v_p,w;\rho)\ge\mu,\mathcal{J}(v_q-v_p,w;\rho)\ge\mu$. We first consider $\mathcal{I}(v_q-v_p,w;\rho)\ge\mu$, then we have $\mathcal{I}(v_q-v_0,w;\frac{\rho}{2})\ge\epsilon$ as otherwise, i.e, if $\mathcal{I}(v_q-v_0,w;\frac{\rho}{2})<\epsilon$, then $\mu\le\mathcal{I}(v_q-v_p,w;\rho)\le\mathcal{I}(v_q-v_0,w;\frac{\rho}{2})\diamond\mathcal{I}(v_p-v_0,w;\frac{\rho}{2})<\epsilon\diamond\epsilon<\mu$ which is not possible. On the same lines we have $\mathcal{J}(v_q-v_0,w;\frac{\rho}{2})\ge\epsilon$. Hence $\mathcal{B}(\mu,\rho)\subset A(\epsilon,\rho)$ and therefore the Theorem is proved. □

Definition 4.3 A neutrosophic $2-$normed space V is said to be statistically complete if every statistical Cauchy sequence in V is statistical convergent in V.

Theorem 4.6 Every neutrosophic $2-$normed space V is statistically complete.

Proof Let (v_k) be statistical Cauchy sequence in V. To prove the Theorem, we have to show that (v_k) is statistical convergent in V. Suppose that (v_k) is not statistical convergent. Let $\epsilon>0$ and $\rho>0$. Then $\exists\, p\in\mathbb{N}$ such that $w\in V$ if we take $A(\epsilon,\rho)=\{k\le n:\mathcal{H}(v_k-v_p,w;\rho)\le 1-\epsilon$ or $\mathcal{I}(v_k-v_p,w;\rho)\ge\epsilon$, $\mathcal{J}(v_k-v_p,w;\rho)\ge\epsilon\}$ and $\mathcal{B}(\epsilon,\rho)=\{k\le n:\mathcal{H}(v_k-v_0,w;\frac{\rho}{2})>1-\epsilon$ or $\mathcal{I}(v_k-v_0,w;\frac{\rho}{2})<\epsilon$, $\mathcal{J}(v_k-v_0,w;\frac{\rho}{2})<\epsilon\}$, then $\delta(A(\epsilon,\rho))=\delta(B(\epsilon,\rho))=0$ and therefore we have $\delta(A^C(\epsilon,\rho))=\delta(B^C(\epsilon,\rho))=1$.

Since $\mathcal{H}(v_k-v_p,w;\rho)\ge 2\mathcal{H}(v_k-v_0,w;\frac{\rho}{2})>1-\epsilon$ and $\mathcal{I}(v_k-v_p,w;\rho)\le 2\mathcal{I}(v_k-v_0,w;\frac{\rho}{2})<\epsilon$, $\mathcal{J}(v_k-v_p,w;\rho)\le 2\mathcal{J}(v_k-v_0,w;\frac{\rho}{2})<\epsilon$ if $\mathcal{H}(v_k-v_0,w;\frac{\rho}{2})>\frac{1-\epsilon}{2}$ and $\mathcal{I}(v_k-v_0,w;\frac{\rho}{2})<\frac{\epsilon}{2}$, $\mathcal{J}(v_k-v_0,w;\frac{\rho}{2})<\frac{\epsilon}{2}$. We have $\delta(\{k\le n:\mathcal{H}(v_k-v_p,w;\rho)>1-\epsilon$ and $\mathcal{I}(v_k-v_p,w;\rho)<\epsilon$, $\mathcal{J}(v_k-v_p,w;\rho)<\epsilon\})=0$. i.e.,$\delta(A^C(\epsilon,\rho))=0$. In this way we obtain a contradiction as $\delta(A^C(\epsilon,\rho))=1$. Hence, (v_k) is statistically convergent w.r.t. $2-$norm N_2. □

Theorem 4.7 Let V be a $N-2-NS$ and (v_k) be a sequence in V, then the following statements are equivalents.

(i) (v_k) is a statistically cauchy sequence w.r.t. N_2.

(ii) There is a set $K=\{k_n\}\subseteq\mathbb{N}$ with $\delta\{K\}=1$ and the associated subsequence $\{v_{k_n}\}_{n\in\mathbb{N}}$ is a cauchy sequence w.r.t. N_2.

5. Conclusion

Fuzzy sets and its generalizations have been frequently used in many branches of science, engineering and technology, especially, in control theory and mathematical modeling of various systems. In present work, we define a neutrosophic $2-$normed space as a generalization of fuzzy normed space and study a generalized limit in a more general setting. The results and definitions presented here will provide a new framework to resolve divergence related problems in these spaces.

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New approach to bisemiring theory via the bipolar valued neutrosophic normal sets

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Abstract. In this paper, we introduce the notion of bipolar-valued neutrosophic subbisemiring (BVNSBS), level sets of BVNSBS, and bipolar valued neutrosophic normal subbisemiring (BVNNSBS) of a bisemiring. The concept of BVNSBS is a new generalization of subbisemiring over bisemirings. We discussed the theory of (ξ, τ)-BVNSBS and (ξ, τ)-BVNNSBS over bisemirings and presented several illustrative examples to demonstrate the sufficiency and validity of the proposed theorems, lemmas, and propositions.

Keywords: Fuzzy set; Bipolar valued neutrosophic subbisemiring; Bipolar valued neutrosophic bisemiring; Homomorphism; Normal.

1. Introduction

Classical mathematics may not always be the solution for practical situations in economics, medical sciences, engineering, social sciences, and environmental sciences, which involves various uncertainties, imprecise and incomplete information. The limitation of classical mathematics that is unable to deal with uncertainties and fuzziness motivated the introduction of mathematical theory such as probability theory, fuzzy set theory [1], rough set theory [2], vague set theory [3], interval mathematics [4], and soft set theory [5]. However, these theories were insufficient and have limitations in dealing with uncertainties. Probability theory can only deal with stochastically stable problems, which may not apply to many problems in the field of economic, environmental, and social sciences. Interval mathematics takes calculation errors into account by constructing an interval estimate for the solution that is useful in many areas, but it is not appropriately adaptable for problems that arise from unreliable, inadequate, and change of information. On the other hand, the fuzzy set theory introduced by Zadeh [1] is most appropriate for dealing with uncertainties and vagueness. Membership of an element in a fuzzy set is a single value between the interval, but in real-life problems, the degree of non-membership may not always be equal to 1 minus the degree of membership as there may be some degree of hesitation. Works on fuzzy set theory are progressing rapidly and have resulted in the conception of many hybrid fuzzy models. In 1983, Atanasov [6] proposed intuitionistic fuzzy sets as a generalization of the notion of fuzzy set, which incorporated the degree of hesitation. Later, Zhang [7] introduced bipolar fuzzy sets in which the membership function is mapped to intervals, thereby allowing it to deal with complex problems in both positive and negative aspects. Later, Zhang [8] proposed that bipolar fuzzy logic should combine both fuzziness and polarity by introducing the (Yin) (Yang) bipolar fuzzy sets. Lee [9] introduced the operation in bipolar-valued fuzzy sets, whereas Lee [10] discovered that bipolar-valued fuzzy sets can represent the degree of satisfaction to counter property but fail to express uncertainties in assigning membership degree. These concepts have been widely applied to handle incomplete information arising from practical situations. However, these were still unable to address uncertainties such as indeterminate and inconsistent information.

In 1999, Smarandache [11] proposed the neutrosophic theory that deals with "the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra". The idea of neutrosophic logic is a logic that states that each proposition is estimated to have a degree of trust, degree of indeterminacy, and degree of falsity. Smarandache [12] further generalized the theory of intuitionistic fuzzy sets to the neutrosophic model, and introduced the truth, indeterminacy, and falsity components that represent the membership, indeterminacy, and non-membership values of a neutrosophic set, respectively. In contrast to intuitionistic fuzzy sets, neutrosophic sets used indeterminacy as a completely independent measure of the membership and non-membership information, and thus it can effectively describe uncertain and inconsistent information and overcome the limitation of the existing approaches in handling uncertain information.

The original neutrosophic theory was introduced from a philosophical standpoint. Hence, it may be difficult to be applied in practical problems. Subsequently, Wang et al. [13] generalized the neutrosophic set from a technical point of view and specified the set-theoretic operators on an instance of a neutrosophic set, called the single-valued neutrosophic set, which takes values from the subset of [0, 1], thereby enabling it to be used feasibly for real-world problems. Over the years, subsequent developments and extensions of the neutrosophic set were proposed. Deli et al. [14] proposed bipolar neutrosophic sets as an extension of bipolar fuzzy sets [7]. Ye [15] introduced the concept of simplified neutrosophic sets. Peng et al. [16] introduced multi-valued neutrosophic sets that allow the truth, indeterminacy, and falsity membership degrees to have a set of crisp values between zero and one, respectively. Das et al. [17] introduced the notion of neutrosophic fuzzy sets by combining fuzzy sets with neutrosophic fuzzy sets to overcome the difficulties in handling the non-standard interval of neutrosophic components.

On the other hand, the fuzzy set theory had been applied and contributed to the generalization of many fundamental concepts in algebra. Extensive research has been done on the fuzzy algebraic structure of semirings introduced by Vandiver [18], which is a generalization of a ring by relaxing the conditions on the additive structure requiring just a monoid rather than a group and have been proven useful for dealing with problems in various areas. The application of semirings had been studied extensively by Golan [19] and Glazek [20].

Ahsan, Saifullah and Farid Khan [21] initiated the study of fuzzy semirings, while Feng, Jun and Zhao [22], and Yousafzai et al. [23] studied semigroups and semirings using fuzzy set and soft sets, respectively. Furthermore, Mockor [24] introduced the notion of a semiring-valued fuzzy set for special commutative partially pre-ordered semiring and introduced F-transform and inverse F-transform for these fuzzy-type structures. Other than that, palanikumar et al. [25–30] studied the algebraic structure of various semirings that constitute a natural generalization of semirings.

Recently, many studies applied bipolar fuzzy information in various algebraic structures, for instance, semigroups [31–33] and BCK/BCI-algebras [34–37]. Zararsz et al. [38] discussed the notion of bipolar fuzzy metric spaces with application. Selvachandran and Salleh [39] introduced vague soft hyperrings and vague soft hyperideals. Jun, Kim and Lee [40] introduced bipolar fuzzy translation in BCK/BCI-algebra and investigated its properties, whereas Jun and Park [41] introduced bipolar fuzzy regularity, bipolar fuzzy regular subalgebra, bipolar fuzzy filter, and bipolar fuzzy closed quasi filter in BCH-algebras. Apart from that, Sen, Ghosh and Ghosh [42] extended the study of semirings and proposed the concept of bisemiring in 2004. Later, Hussain [43] defined the congruence relation between bisemiring and bisemiring homomorphisms, followed by the factor bisemiring. Hussain et al. [44] further generalized bisemiring to a new algebraic structure called -semiring and congruence relations on homomorphisms and n-semirings.

To the best of our knowledge, studies on bisemiring theory using bipolar valued neutrosophic sets have not been studied extensively, and further generalization for bisemiring is still needed for various practical problems. In this paper, we introduce the notion of bipolar valued neutrosophic subbisemiring (BVNSBS), level sets of BVNSBS, and bipolar valued neutrosophic normal subbisemiring (BVNNSBS) of a bisemiring. The concept of BVNSBS is a new generalization of subbisemiring over bisemirings. We discussed the theory for (ξ, τ)-BVNSBS and (ξ, τ)-BVNNSBS over bisemiring theory and presented several illustrative examples. The rest of the paper is organized as follows: Section 2 outlines the preliminary definitions and results, Section 3 introduces the notion of BVNSBS, Section 4 discusses the (ξ, τ)-BVNSBS and Section 5 discusses the (ξ, τ)-BVNNSBS.

2. Preliminaries

Definition 2.1. [9] Let U be the universe set. A bipolar valued fuzzy set ϑ in U is an object having the form $\vartheta = \{(u, \vartheta^+(u), \vartheta^-(u))|u \in U\}$, where $\vartheta^- : U \to [-1, 0]$ and $\vartheta^+ : U \to [0, 1]$ are mappings. The positive membership degree $\vartheta^+(u)$ denoted the satisfaction degree of an element u to the property corresponding to a bipolar valued fuzzy set $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$, and the negative membership degree $\vartheta^-(u)$ denotes the satisfaction degree of u to some implicit counter-property of $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$. If $\vartheta^+(u) \neq 0$ and $\vartheta^-(u) = 0$, it is the situation that u is regarded as having only positive satisfaction for $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$. If $\vartheta^+(u) = 0$ and $\vartheta^-(u) \neq 0$, it is the situation that u does not satisfy the property of $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$ but somewhat satisfies the counter property of $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$. It is possible for an element u to be $\vartheta^+(u) \neq 0$ and $\vartheta^-(u) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain. For the sake of simplicity, we shall use the symbol $\vartheta = \langle U; \vartheta^-, \vartheta^+\rangle$ for the bipolar valued fuzzy set $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar valued fuzzy sets.

Definition 2.2. [11] A neutrosophic set K in a universe set U is an object having the structure $K = \left\{\left\langle m, \vartheta^T_K(m), \vartheta^I_K(m), \vartheta^F_K(m)\right\rangle |m \in U\right\}$, where $\vartheta^T_K(m), \vartheta^I_K(m), \vartheta^F_K(m) : U \to [0, 1]$ represents the truth-membership function , the indeterminacy membership function and the falsity-membership function respectively. There is no restriction on the sum of $\vartheta^T_K, \vartheta^I_K, \vartheta^F_K$ and so $0 \leq \vartheta^T_K + \vartheta^I_K + \vartheta^F_K \leq 3$.

Definition 2.3. [11] Let $K = \left\{\left\langle m, \vartheta^T_K(m), \vartheta^I_K(m), \vartheta^F_K(m)\right\rangle |m \in U\right\}$ and $L = \left\{\left\langle m, \vartheta^T_L(m), \vartheta^I_L(m), \vartheta^F_L(m)\right\rangle |m \in U\right\}$ be any two neutrosophic sets of a set U. Then
$K \cap L = \left\{ \left\langle m, \min\{\vartheta^T_K(m), \vartheta^T_L(m)\}, \min\{\vartheta^I_K(m), \vartheta^I_L(m)\}, \max\{\vartheta^F_K(m), \vartheta^F_L(m)\}\right\rangle \Big| m \in U\right\}$,
$K \cup L = \left\{ (\left\langle m, \max\{\vartheta^T_K(m), \vartheta^T_L(m)\}, \max\{\vartheta^I_K(m), \vartheta^I_L(m)\}, \min\{\vartheta^F_K(m), \vartheta^F_L(m)\}\right\rangle \Big| m \in U\right\}$.

Definition 2.4. [11] For any neutrosophic set $K = \left\{\left\langle m, \vartheta^T_K(m), \vartheta^I_K(m), \vartheta^F_K(m)\right\rangle |m \in U\right\}$ of a set U, we defined a (ξ, τ)-cut of as the crisp subset $\{\vartheta^T_K(m) \geq \xi, \vartheta^I_K(m) \geq \xi, \vartheta^F_K(m) \leq \tau|m \in U\}$ of U.

Definition 2.5. [11] Let K and L be any two neutrosophic set of U. Then $K \times L = \{\vartheta^T_{K\times L}(m, n), \vartheta^I_{K\times L}(m, n), \vartheta^F_{K\times L}(m, n)|\forall m, n \in U\}$, where $\vartheta^T_{K\times L}(m, n) = \min\{\vartheta^T_K(m), \vartheta^T_L(n)\}, \vartheta^I_{K\times L}(m, n) = \frac{\vartheta^I_K(m)+\vartheta^I_L(n)}{2}, \vartheta^F_{K\times L}(m, n) = \max\{\vartheta^F_K(m), \vartheta^F_L(n)\}$.

Definition 2.6. [44] A fuzzy subset K of a bisemiring $(S, \uplus_1, \uplus_2, \uplus_3)$ is said to be a fuzzy subbisemiring of S if $\vartheta_K(m \uplus_1 n) \geq \min\{\vartheta_K(m), \vartheta_K(n)\}, \vartheta_K(m \uplus_2 n) \geq \min\{\vartheta_K(m), \vartheta_K(n)\}, \vartheta_K(m \uplus_3 n) \geq \min\{\vartheta_K(m), \vartheta_K(n)\}$, for all $m, n \in S$.

Definition 2.7. [44] Let $(S_1, +, \cdot, \times)$ and $(S_2, \boxplus, \circ, \otimes)$ be any two bisemirings. A function $\phi : S_1 \to S_2$ is said to be a homomorphism if $\phi(m + n) = \phi(m) \boxplus \phi(n), \phi(m \cdot n) = \phi(m) \circ \phi(n), \phi(m \times n) = \phi(m) \otimes \phi(n)$, for all $m, n \in S_1$.

3. Bipolar Valued Neutrosophic Subbisemiring (BVNSBS)

In what follows, let $\mathcal{S}$ denote a bisemiring unless otherwise noted. In this section, we communication the concept of bipolar valued neutrosophic subbisemiring, strongest neutrosophic relation on $\mathcal{S}$. Furthermore, we introduce the arbitrary intersection bipolar valued neutrosophic subbisemiring and list some properties.

Definition 3.1. A bipolar valued neutrosophic subset K of $\mathcal{S}$ is said to be BVNSBS of $\mathcal{S}$ if it satisfies the following conditions:

$$\left\{\begin{pmatrix} \vartheta_K^{T+}(m \uplus_1 n) \geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\}, \\ \vartheta_K^{T-}(m \uplus_1 n) \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} \end{pmatrix} \begin{pmatrix} \vartheta_K^{T+}(m \uplus_2 n) \geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\}, \\ \vartheta_K^{T-}(m \uplus_2 n) \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} \end{pmatrix} \begin{pmatrix} \vartheta_K^{T+}(m \uplus_3 n) \geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\}, \\ \vartheta_K^{T-}(m \uplus_3 n) \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} \vartheta_K^{I+}(m \uplus_1 n) \geq \frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}, \\ \vartheta_K^{I-}(m \uplus_1 n) \leq \frac{\vartheta_K^{I-}(m)-\vartheta_K^{I-}(n)}{2} \end{pmatrix} OR \begin{pmatrix} \vartheta_K^{I+}(m \uplus_2 n) \geq \frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}, \\ \vartheta_K^{I-}(m \uplus_2 n) \leq \frac{\vartheta_K^{I-}(m)-\vartheta_K^{I-}(n)}{2} \end{pmatrix} OR \begin{pmatrix} \vartheta_K^{I+}(m \uplus_3 n) \geq \frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}, \\ \vartheta_K^{I-}(m \uplus_3 n) \leq \frac{\vartheta_K^{I-}(m)-\vartheta_K^{I-}(n)}{2} \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} \vartheta_K^{F+}(m \uplus_1 n) \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}, \\ \vartheta_K^{F-}(m \uplus_1 n) \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\} \end{pmatrix} \begin{pmatrix} \vartheta_K^{F+}(m \uplus_2 n) \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}, \\ \vartheta_K^{F-}(m \uplus_2 n) \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\} \end{pmatrix} \begin{pmatrix} \vartheta_K^{F+}(m \uplus_3 n) \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}, \\ \vartheta_K^{F-}(m \uplus_3 n) \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\} \end{pmatrix}\right\}$$

for all $m, n \in \mathcal{S}$.

Example 3.2. Let $\mathcal{S} = \{l_1, l_2, l_3, l_4\}$ be the bisemiring with the following Cayley table:

$\uplus_1$	l_1	l_2	l_3	l_4
l_1	l_1	l_1	l_1	l_1
l_2	l_1	l_2	l_1	l_2
l_3	l_1	l_1	l_3	l_3
l_4	l_1	l_2	l_3	l_4

$\uplus_2$	l_1	l_2	l_3	l_4
l_1	l_1	l_2	l_3	l_4
l_2	l_2	l_2	l_4	l_4
l_3	l_3	l_4	l_3	l_4
l_4	l_4	l_4	l_4	l_4

$\uplus_3$	l_1	l_2	l_3	l_4
l_1	l_1	l_1	l_1	l_1
l_2	l_1	l_2	l_3	l_4
l_3	l_4	l_4	l_4	l_4
l_4	l_4	l_4	l_4	l_4

$(\vartheta_K^+(l), \vartheta_K^-(l))$	$l = l_1$	$l = l_2$	$l = l_3$	$l = l_4$
$\left(\vartheta_K^{T+}(l), \vartheta_K^{T-}(l)\right)$	$(0.55, -0.7)$	$(0.35, -0.6)$	$(0.15, -0.3)$	$(0.25, -0.4)$
$\left(\vartheta_K^{I+}(l), \vartheta_K^{I-}(l)\right)$	$(0.65, -0.8)$	$(0.5, -0.5)$	$(0.3, -0.1)$	$(0.4, -0.2)$
$\left(\vartheta_K^{F+}(l), \vartheta_K^{F-}(l)\right)$	$(0.25, -0.15)$	$(0.35, -0.25)$	$(0.65, -0.65)$	$(0.55, -0.45)$

Clearly, K is an BVNSBS of $\mathcal{S}$.

Theorem 3.3. *The intersection of a family of $BVNSBS^s$ of $\mathcal{S}$ is a $BVNSBS$ of $\mathcal{S}$.*

Proof. Let $\{O_i | i \in I\}$ be a family of $BVNSBS^s$ of $\mathcal{S}$ and $K = \bigcap_{i\in I} O_i$.
Let m and n in $\mathcal{S}$. Now,

$$\begin{aligned}
\vartheta_K^{T+}(m \uplus_1 n) &= \inf_{i\in I} \vartheta_{O_i}^{T+}(m \uplus_1 n) \\
&\geq \inf_{i\in I} \min\{\vartheta_{O_i}^{T+}(m), \vartheta_{O_i}^{T+}(n)\} \\
&= \min\left\{\inf_{i\in I} \vartheta_{O_i}^{T+}(m), \inf_{i\in I} \vartheta_{O_i}^{T+}(n)\right\} \\
&= \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\} \\
\vartheta_K^{T-}(m \uplus_1 n) &= \sup_{i\in I} \vartheta_{O_i}^{T-}(m \uplus_1 n) \\
&\leq \sup_{i\in I} \max\{\vartheta_{O_i}^{T-}(m), \vartheta_{O_i}^{T-}(n)\} \\
&= \max\left\{\sup_{i\in I} \vartheta_{O_i}^{T-}(m), \sup_{i\in I} \vartheta_{O_i}^{T-}(n)\right\} \\
&= \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\}.
\end{aligned}$$

Now,

$$\begin{aligned}
\vartheta_K^{I+}(m \uplus_1 n) &= \inf_{i\in I} \vartheta_{O_i}^{I+}(m \uplus_1 n) \\
&\geq \inf_{i\in I} \frac{\vartheta_{O_i}^{I+}(m) + \vartheta_{O_i}^{I+}(n)}{2} \\
&= \frac{\inf_{i\in I} \vartheta_{O_i}^{I+}(m) + \inf_{i\in I} \vartheta_{O_i}^{I+}(n)}{2} \\
&= \frac{\vartheta_K^{I+}(m) + \vartheta_K^{I+}(n)}{2}
\end{aligned}$$

$$\begin{aligned}\vartheta_K^{I-}(m \uplus_1 n) &= \sup_{i\in I} \vartheta_{O_i}^{I-}(m \uplus_1 n)\\ &\leq \sup_{i\in I} \frac{\vartheta_{O_i}^{I-}(m) + \vartheta_{O_i}^{I-}(n)}{2}\\ &= \frac{\sup_{i\in I} \vartheta_{O_i}^{I-}(m) + \sup_{i\in I} \vartheta_{O_i}^{I-}(n)}{2}\\ &= \frac{\vartheta_K^{I-}(m) + \vartheta_K^{I-}(n)}{2}.\end{aligned}$$

Now,

$$\begin{aligned}\vartheta_K^{F+}(m \uplus_1 n) &= \sup_{i\in I} \vartheta_{O_i}^{F+}(m \uplus_1 n)\\ &\leq \sup_{i\in I} \max\{\vartheta_{O_i}^{F+}(m), \vartheta_{O_i}^{F+}(n)\}\\ &= \max\left\{\sup_{i\in I} \vartheta_{O_i}^{F+}(m), \sup_{i\in I} \vartheta_{O_i}^{F+}(n)\right\}\\ &= \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}\end{aligned}$$

$$\begin{aligned}\vartheta_K^{F-}(m \uplus_1 n) &= \inf_{i\in I} \vartheta_{O_i}^{F-}(m \uplus_1 n)\\ &\geq \inf_{i\in I} \min\{\vartheta_{O_i}^{F-}(m), \vartheta_{O_i}^{F-}(n)\}\\ &= \min\left\{\inf_{i\in I} \vartheta_{O_i}^{F-}(m), \inf_{i\in I} \vartheta_{O_i}^{F-}(n)\right\}\\ &= \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\}.\end{aligned}$$

Similarly, we can prove that other two operations. Hence K is an BVNSBS of $\mathcal{S}$.

Theorem 3.4. *If K and L are any two $BVNSBS^s$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, then $K \times L$ is a BVNSBS of $\mathcal{S}_1 \times \mathcal{S}_2$.*

Proof. Let K and L be two $BVNSBS^s$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Let $m_1, m_2 \in \mathcal{S}_1$ and $n_1, n_2 \in \mathcal{S}_2$. Then (m_1, n_1) and (m_2, n_2) are in $\mathcal{S}_1 \times \mathcal{S}_2$. Now,

$$\begin{aligned}\vartheta_{K\times L}^{T+}[(m_1, n_1) \uplus_1 (m_2, n_2)] &= \vartheta_{K\times L}^{T+}(m_1 \uplus_1 m_2, n_1 \uplus_1 n_2)\\ &= \min\{\vartheta_K^{T+}(m_1 \uplus_1 m_2), \vartheta_L^{T+}(n_1 \uplus_1 n_2)\}\\ &\geq \min\{\min\{\vartheta_K^{T+}(m_1), \vartheta_K^{T+}(m_2)\}, \min\{\vartheta_L^{T+}(n_1), \vartheta_L^{T+}(n_2)\}\}\\ &= \min\{\min\{\vartheta_K^{T+}(m_1), \vartheta_L^{T+}(n_1)\}, \min\{\vartheta_K^{T+}(m_2), \vartheta_L^{T+}(n_2)\}\}\\ &= \min\{\vartheta_{K\times L}^{T+}(m_1, n_1), \vartheta_{K\times L}^{T+}(m_2, n_2)\}.\end{aligned}$$

Similarly, $\vartheta^{T-}_{K\times L}[(m_1,n_1) \uplus_1 (m_2,n_2)] \leq \max\{\vartheta^{T-}_{K\times L}(m_1,n_1), \vartheta^{T-}_{K\times L}(m_2,n_2)\}$.
Now,

$$\begin{aligned}
\vartheta^{I+}_{K\times L}[(m_1,n_1) \uplus_1 (m_2,n_2)] &= \vartheta^{I+}_{K\times L}(m_1 \uplus_1 m_2, n_1 \uplus_1 n_2)\\
&= \frac{\vartheta^{I+}_{K}(m_1 \uplus_1 m_2) + \vartheta^{I+}_{L}(n_1 \uplus_1 n_2)}{2}\\
&\geq \frac{1}{2}\left[\frac{\vartheta^{I+}_{K}(m_1) + \vartheta^{I+}_{K}(m_2)}{2} + \frac{\vartheta^{I+}_{L}(n_1) + \vartheta^{I+}_{L}(n_2)}{2}\right]\\
&= \frac{1}{2}\left[\frac{\vartheta^{I+}_{K}(m_1) + \vartheta^{I+}_{L}(n_1)}{2} + \frac{\vartheta^{I+}_{K}(m_2) + \vartheta^{I+}_{L}(n_2)}{2}\right]\\
&= \frac{1}{2}\Big[\vartheta^{I+}_{K\times L}(m_1,n_1) + \vartheta^{I+}_{K\times L}(m_2,n_2)\Big].
\end{aligned}$$

Similarly, $\vartheta^{I-}_{K\times L}[(m_1,n_1) \uplus_1 (m_2,n_2)] \leq \frac{1}{2}\Big[\vartheta^{I-}_{K\times L}(m_1,n_1) + \vartheta^{I-}_{K\times L}(m_2,n_2)\Big]$.
Now,

$$\begin{aligned}
\vartheta^{F+}_{K\times L}[(m_1,n_1) \uplus_1 (m_2,n_2)] &= \vartheta^{F+}_{K\times L}(m_1 \uplus_1 m_2, n_1 \uplus_1 n_2)\\
&= \max\{\vartheta^{F+}_{K}(m_1 \uplus_1 m_2), \vartheta^{F+}_{L}(n_1 \uplus_1 n_2)\}\\
&\leq \max\{\max\{\vartheta^{F+}_{K}(m_1), \vartheta^{F+}_{K}(m_2)\}, \max\{\vartheta^{F+}_{L}(n_1), \vartheta^{F+}_{L}(n_2)\}\}\\
&= \max\{\max\{\vartheta^{F+}_{K}(m_1), \vartheta^{F+}_{L}(n_1)\}, \max\{\vartheta^{F+}_{K}(m_2), \vartheta^{F+}_{L}(n_2)\}\}\\
&= \max\{\vartheta^{F+}_{K\times L}(m_1,n_1), \vartheta^{F+}_{K\times L}(m_2,n_2)\}.
\end{aligned}$$

Similarly, $\vartheta^{F-}_{K\times L}[(m_1,n_1) \uplus_1 (m_2,n_2)] \geq \min\{\vartheta^{F-}_{K\times L}(m_1,n_1), \vartheta^{F-}_{K\times L}(m_2,n_2)\}$.
Similarly, we can prove other two operations. Hence, $K \times L$ is an BVNSBS of $\mathcal{S}$.

Corollary 3.5. *If $K_1, K_2, ..., K_n$ are the family of $BVNSBS^s$ of $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n$ respectively, then $K_1 \times K_2 \times ... \times K_n$ is an BVNSBS of $\mathcal{S}_1 \times \mathcal{S}_2 \times ... \times \mathcal{S}_n$.*

Definition 3.6. Let K be a bipolar valued neutrosophic subset in $\mathcal{S}$, the strongest neutrosophic relation on $\mathcal{S}$, that is a bipolar valued neutrosophic relation on K is O such that

$$\left\{\begin{array}{l}
\begin{pmatrix}\vartheta^{T+}_{O}(m,n) = \min\{\vartheta^{T+}_{K}(m), \vartheta^{T+}_{K}(n)\},\\ \vartheta^{T-}_{O}(m,n) = \max\{\vartheta^{T-}_{K}(m), \vartheta^{T-}_{K}(n)\}\end{pmatrix}, \begin{pmatrix}\vartheta^{I+}_{O}(m,n) = \frac{\vartheta^{I+}_{K}(m)+\vartheta^{I+}_{K}(n)}{2},\\ \vartheta^{I-}_{O}(m,n) = \frac{\vartheta^{I-}_{K}(m)+\vartheta^{I-}_{K}(n)}{2}\end{pmatrix},\\
\begin{pmatrix}\vartheta^{F+}_{O}(m,n) = \max\{\vartheta^{F+}_{K}(m), \vartheta^{F+}_{K}(n)\},\\ \vartheta^{F-}_{O}(m,n) = \min\{\vartheta^{F-}_{K}(m), \vartheta^{F-}_{K}(n)\}\end{pmatrix}
\end{array}\right\}.$$

Theorem 3.7. *Let K be the BVNSBS of $\mathcal{S}$ and O be the strongest bipolar valued neutrosophic relation of $\mathcal{S}$. Then K is an BVNSBS of $\mathcal{S}$ if and only if O is an BVNSBS of $\mathcal{S} \times \mathcal{S}$.*

Proof. Let K be the BVNSBS of $\mathcal{S}$ and O be the strongest bipolar valued neutrosophic relation of $\mathcal{S}$. Then for any $m = (m_1, m_2)$ and $n = (n_1, n_2)$ are in $\mathcal{S} \times \mathcal{S}$. Now,

$$\begin{aligned}
\vartheta_O^{T+}(m \uplus_1 n) &= \vartheta_O^{T+}[((m_1, m_2) \uplus_1 (n_1, n_2)]\\
&= \vartheta_O^{T+}(m_1 \uplus_1 n_1, m_2 \uplus_1 n_2)\\
&= \min\{\vartheta_K^{T+}(m_1 \uplus_1 n_1), \vartheta_K^{T+}(m_2 \uplus_1 n_2)\}\\
&\geq \min\{\min\{\vartheta_K^{T+}(m_1), \vartheta_K^{T+}(n_1)\}, \min\{\vartheta_K^{T+}(m_2), \vartheta_K^{T+}(n_2)\}\}\\
&= \min\{\min\{\vartheta_K^{T+}(m_1), \vartheta_K^{T+}(m_2)\}, \min\{\vartheta_K^{T+}(n_1), \vartheta_K^{T+}(n_2)\}\}\\
&= \min\{\vartheta_O^{T+}(m_1, m_2), \vartheta_O^{T+}(n_1, n_2)\}\\
&= \min\{\vartheta_O^{T+}(m), \vartheta_O^{T+}(n)\}.
\end{aligned}$$

Similarly, $\vartheta_O^{T-}(m \uplus_2 n) \leq \max\{\vartheta_O^{T-}(m), \vartheta_O^{T-}(n)\}$.
Now,

$$\begin{aligned}
\vartheta_O^{I+}(m \uplus_1 n) &= \vartheta_O^{I+}[((m_1, m_2) \uplus_1 (n_1, n_2)]\\
&= \vartheta_O^{I+}(m_1 \uplus_1 n_1, m_2 \uplus_1 n_2)\\
&= \frac{\vartheta_K^{I+}(m_1 \uplus_1 n_1) + \vartheta_K^{I+}(m_2 \uplus_1 n_2)}{2}\\
&\geq \frac{1}{2}\left[\frac{\vartheta_K^{I+}(m_1) + \vartheta_K^{I+}(n_1)}{2} + \frac{\vartheta_K^{I+}(m_2) + \vartheta_K^{I+}(n_2)}{2}\right]\\
&= \frac{1}{2}\left[\frac{\vartheta_K^{I+}(m_1) + \vartheta_K^{I+}(m_2)}{2} + \frac{\vartheta_K^{I+}(n_1) + \vartheta_K^{I+}(n_2)}{2}\right]\\
&= \frac{\vartheta_O^{I+}(m_1, m_2) + \vartheta_O^{I+}(n_1, n_2)}{2}\\
&= \frac{\vartheta_O^{I+}(m) + \vartheta_O^{I+}(n)}{2}.
\end{aligned}$$

Similarly, $\vartheta_O^{I-}(m \uplus_1 n) \leq \frac{\vartheta_O^{I-}(m)+\vartheta_O^{I-}(n)}{2}$.
Similarly, $\vartheta_O^{F+}(m \uplus_1 n) \leq \max\{\vartheta_O^{F+}(m), \vartheta_O^{F+}(n)\}$ and $\vartheta_O^{F-}(m \uplus_1 n) \geq \min\{\vartheta_O^{F-}(m), \vartheta_O^{F-}(n)\}$.
Similarly to prove other two operations. Hence O is an BVNSBS of $\mathcal{S} \times \mathcal{S}$.

Conversely assume that O is an BVNSBS of $\mathcal{S} \times \mathcal{S}$, then for any $m = (m_1, m_2)$ and $n = (n_1, n_2)$ are in $\mathcal{S} \times \mathcal{S}$. Now,

$$\begin{aligned}
\min\{\vartheta_K^{T+}(m_1 \uplus_1 n_1), \vartheta_K^{T+}(m_2 \uplus_1 n_2)\} &= \vartheta_O^{T+}(m_1 \uplus_1 n_1, m_2 \uplus_1 n_2)\\
&= \vartheta_O^{T+}[(m_1, m_2) \uplus_1 (n_1, n_2)]\\
&= \vartheta_O^{T+}(m \uplus_1 n)
\end{aligned}$$

$$\geq \min\{\vartheta_O^{T+}(m), \vartheta_O^{T+}(n)\}$$
$$= \min\{\vartheta_O^{T+}(m_1, m_2), \vartheta_O^{T+}(n_1, n_2)\}$$
$$= \min\{\min\{\vartheta_K^{T+}(m_1), \vartheta_K^{T+}(m_2)\}, \min\{\vartheta_K^{T+}(n_1), \vartheta_K^{T+}(n_2)\}\}.$$

If $\vartheta_K^{T+}(m_1 \uplus_1 n_1) \leq \vartheta_K^{T+}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{T+}(m_1) \leq \vartheta_K^{T+}(m_2)$ and $\vartheta_K^{T+}(n_1) \leq \vartheta_K^{T+}(n_2)$.
We get $\vartheta_K^{T+}(m_1 \uplus_1 n_1) \geq \min\{\vartheta_K^{T+}(m_1), \vartheta_K^{T+}(n_1)\}$.

$$\begin{aligned}\max\{\vartheta_K^{T-}(m_1 \uplus_1 n_1), \vartheta_K^{T-}(m_2 \uplus_1 n_2)\} &= \vartheta_O^{T-}(m_1 \uplus_1 n_1, m_2 \uplus_1 n_2)\\ &= \vartheta_O^{T-}[(m_1, m_2) \uplus_1 (n_1, n_2)]\\ &= \vartheta_O^{T-}(m \uplus_1 n)\\ &\leq \max\{\vartheta_O^{T-}(m), \vartheta_O^{T-}(n)\}\\ &= \max\{\vartheta_O^{T-}(m_1, m_2), \vartheta_O^{T-}(n_1, n_2)\}\\ &= \max\{\max\{\vartheta_K^{T-}(m_1), \vartheta_K^{T-}(m_2)\}, \max\{\vartheta_K^{T-}(n_1), \vartheta_K^{T-}(n_2)\}\}.\end{aligned}$$

If $\vartheta_K^{T-}(m_1 \uplus_1 n_1) \geq \vartheta_K^{T-}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{T-}(m_1) \geq \vartheta_K^{T-}(m_2)$ and $\vartheta_K^{T-}(n_1) \geq \vartheta_K^{T-}(n_2)$.
We get $\vartheta_K^{T-}(m_1 \uplus_1 n_1) \leq \max\{\vartheta_K^{T-}(m_1), \vartheta_K^{T-}(n_1)\}$ for all $m_1, n_1 \in \mathcal{S}$. Now,

$$\begin{aligned}\frac{1}{2}\Big[\vartheta_K^{I+}(m_1 \uplus_1 n_1) + \vartheta_K^{I+}(m_2 \uplus_1 n_2)\Big] &= \vartheta_O^{I+}(m_1 \uplus_1 n_1, m_2 \uplus_1 n_2)\\ &= \vartheta_O^{I+}[(m_1, m_2) \uplus_1 (n_1, n_2)]\\ &= \vartheta_O^{I+}(m \uplus_1 n)\\ &\geq \frac{\vartheta_O^{I+}(m) + \vartheta_O^{I+}(n)}{2}\\ &= \frac{\vartheta_O^{I+}(m_1, m_2) + \vartheta_O^{I+}(n_1, n_2)}{2}\\ &= \frac{1}{2}\Bigg[\frac{\vartheta_K^{I+}(m_1) + \vartheta_K^{I+}(m_2)}{2} + \frac{\vartheta_K^{I+}(n_1) + \vartheta_K^{I+}(n_2)}{2}\Bigg].\end{aligned}$$

If $\vartheta_K^{I+}(m_1 \uplus_1 n_1) \leq \vartheta_K^{I+}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{I+}(m_1) \leq \vartheta_K^{I+}(m_2)$ and $\vartheta_K^{I+}(n_1) \leq \vartheta_K^{I+}(n_2)$.
We get, $\vartheta_K^{I+}(m_1 \uplus_1 n_1) \geq \frac{\vartheta_K^{I+}(m_1)+\vartheta_K^{I+}(n_1)}{2}$.
Similarly, $\frac{1}{2}\Big[\vartheta_K^{I-}(m_1 \uplus_1 n_1) + \vartheta_K^{I-}(m_2 \uplus_1 n_2)\Big] \leq \frac{1}{2}\Bigg[\frac{\vartheta_K^{I-}(m_1)+\vartheta_K^{I-}(m_2)}{2} + \frac{\vartheta_K^{I-}(n_1)+\vartheta_K^{I-}(n_2)}{2}\Bigg]$.
If $\vartheta_K^{I-}(m_1 \uplus_1 n_1) \geq \vartheta_K^{I-}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{I-}(m_1) \geq \vartheta_K^{I-}(m_2)$ and $\vartheta_K^{I-}(n_1) \geq \vartheta_K^{I-}(n_2)$.
We get, $\vartheta_K^{I-}(m_1 \uplus_1 n_1) \leq \frac{\vartheta_K^{I-}(m_1)+\vartheta_K^{I-}(n_1)}{2}$.
Similarly, $\max\{\vartheta_K^{F+}(m_1 \uplus_1 n_1), \vartheta_K^{F+}(m_2 \uplus_1 n_2)\} \leq \max\{\max\{\vartheta_K^{F+}(m_1), \vartheta_K^{F+}(m_2)\},$
$\max\{\vartheta_K^{F+}(n_1), \vartheta_K^{F+}(n_2)\}\}$.
If $\vartheta_K^{F+}(m_1 \uplus_1 n_1) \geq \vartheta_K^{F+}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{F+}(m_1) \geq \vartheta_K^{F+}(m_2)$ and $\vartheta_K^{F+}(n_1) \geq \vartheta_K^{F+}(n_2)$.
We get, $\vartheta_K^{F+}(m_1 \uplus_1 n_1) \leq \max\{\vartheta_K^{F+}(m_1), \vartheta_K^{F+}(n_1)\}$.
Similarly, $\min\{\vartheta_K^{F-}(m_1 \uplus_1 n_1), \vartheta_K^{F-}(m_2 \uplus_1 n_2)\} \geq \min\{\min\{\vartheta_K^{F-}(m_1), \vartheta_K^{F-}(m_2)\},$
$\min\{\vartheta_K^{F-}(n_1), \vartheta_K^{F-}(n_2)\}\}$.

If $\vartheta_K^{F-}(m_1 \uplus_1 n_1) \leq \vartheta_K^{F-}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{F-}(m_1) \leq \vartheta_K^{F-}(m_2)$ and $\vartheta_K^{F-}(n_1) \leq \vartheta_K^{F-}(n_2)$.
We get, $\vartheta_K^{F-}(m_1 \uplus_1 n_1) \geq \min\{\vartheta_K^{F-}(m_1), \vartheta_K^{F-}(n_1)\}$.
Similarly to prove other two operations. Hence K is an BVNSBS of $\mathcal{S}$.

Theorem 3.8. *Let K be bipolar valued neutrosophic subset in $\mathcal{S}$. Then $\vartheta = \{(\vartheta_K^{T+}, \vartheta_K^{T-}), (\vartheta_K^{I+}, \vartheta_K^{I-}), (\vartheta_K^{F+}, \vartheta_K^{F-})\}$ is an BVNSBS of $\mathcal{S}$ if and only if all non empty level set $\vartheta^{(t,s)}$ is a subbisemiring of $\mathcal{S}$ for $t, s \in [-1, 0] \times [0, 1]$.*

Proof. Assume that ϑ is an BVNSBS of $\mathcal{S}$. For each $t, s \in [-1, 0] \times [0, 1]$ and $a_1, a_2 \in \vartheta^{(t,s)}$. We have $\vartheta_K^{T+}(a_1) \geq t, \vartheta_K^{T+}(a_2) \geq t$ and $\vartheta_K^{I+}(a_1) \geq t, \vartheta_K^{I+}(a_2) \geq t$ and $\vartheta_K^{F+}(a_1) \leq s, \vartheta_K^{F+}(a_2) \leq s$. Now, $\vartheta_K^{T+}(a_1 \uplus_1 a_2) \geq \min\{\vartheta_K^{T+}(a_1), \vartheta_K^{T+}(a_2)\} \geq t$ and $\vartheta_K^{I+}(a_1 \uplus_1 a_2) \geq \frac{\vartheta_K^{I+}(a_1)+\vartheta_K^{I+}(a_2)}{2} \geq \frac{t+t}{2} = t$ and $\vartheta_K^{F+}(a_1 \uplus_1 a_2) \leq \max\{\vartheta_K^{F+}(a_1), \vartheta_K^{F+}(a_2)\} \leq s$. Since, $t, s \in [-1, 0] \times [0, 1]$, we have $\vartheta_K^{T-}(a_1) \leq t, \vartheta_K^{T-}(a_2) \leq t$ and $\vartheta_K^{I-}(a_1) \leq t, \vartheta_K^{I-}(a_2) \leq t$ and $\vartheta_K^{F-}(a_1) \geq s, \vartheta_K^{F-}(a_2) \geq s$. Now, $\vartheta_K^{T-}(a_1 \uplus_1 a_2) \leq \max\{\vartheta_K^{T-}(a_1), \vartheta_K^{T-}(a_2)\} \leq t$ and $\vartheta_K^{I-}(a_1 \uplus_1 a_2) \leq \frac{\vartheta_K^{I-}(a_1)+\vartheta_K^{I-}(a_2)}{2} \leq \frac{t+t}{2} = t$ and $\vartheta_K^{F-}(a_1 \uplus_1 a_2) \geq \min\{\vartheta_K^{F-}(a_1), \vartheta_K^{F-}(a_2)\} \geq s$. This implies that $a_1 \uplus_1 a_2 \in \vartheta^{(t,s)}$. Similarly, to prove other two operations. Hence, $\vartheta^{(t,s)}$ is a subbisemiring of $\mathcal{S}$ for each $t, s \in [-1, 0] \times [0, 1]$.

Conversely, assume that $\vartheta^{(t,s)}$ is a subbisemiring of $\mathcal{S}$ for each $t, s \in [-1, 0] \times [0, 1]$. Suppose if there exist $a_1, a_2 \in \mathcal{S}$ such that $\vartheta_K^{T+}(a_1 \uplus_1 a_2) < \min\{\vartheta_K^{T+}(a_1), \vartheta_K^{T+}(a_2)\}$, $\vartheta_K^{I+}(a_1 \uplus_1 a_2) < \frac{\vartheta_K^{I+}(a_1)+\vartheta_K^{I+}(a_2)}{2}$ and $\vartheta_K^{F+}(a_1 \uplus_1 a_2) > \max\{\vartheta_K^{F+}(a_1), \vartheta_K^{F+}(a_2)\}$. Select $t, s \in [0, 1]$ such that $\vartheta_K^{T+}(a_1 \uplus_1 a_2) < t \leq \min\{\vartheta_K^{T+}(a_1), \vartheta_K^{T+}(a_2)\}$ and $\vartheta_K^{I+}(a_1 \uplus_1 a_2) < t \leq \frac{\vartheta_K^{I+}(a_1)+\vartheta_K^{I+}(a_2)}{2}$ and $\vartheta_K^{F+}(a_1 \uplus_1 a_2) > s \geq \max\{\vartheta_K^{F+}(a_1), \vartheta_K^{F+}(a_2)\}$. Then $a_1, a_2 \in \vartheta^{(t,s)}$, but $a_1 \uplus_1 a_2 \notin \vartheta^{(t,s)}$. Suppose if there exist $a_1, a_2 \in \mathcal{S}$ such that $\vartheta_K^{T-}(a_1 \uplus_1 a_2) > \max\{\vartheta_K^{T-}(a_1), \vartheta_K^{T-}(a_2)\}$, $\vartheta_K^{I-}(a_1 \uplus_1 a_2) > \frac{\vartheta_K^{I-}(a_1)+\vartheta_K^{I-}(a_2)}{2}$ and $\vartheta_K^{F-}(a_1 \uplus_1 a_2) < \min\{\vartheta_K^{F-}(a_1), \vartheta_K^{F-}(a_2)\}$. Select $t, s \in [-1, 0]$ such that $\vartheta_K^{T-}(a_1 \uplus_1 a_2) > t \geq \max\{\vartheta_K^{T-}(a_1), \vartheta_K^{T-}(a_2)\}$ and $\vartheta_K^{I-}(a_1 \uplus_1 a_2) > t \geq \frac{\vartheta_K^{I-}(a_1)+\vartheta_K^{I-}(a_2)}{2}$ and $\vartheta_K^{F-}(a_1 \uplus_1 a_2) < s \leq \min\{\vartheta_K^{F-}(a_1), \vartheta_K^{F-}(a_2)\}$. Then $a_1, a_2 \in \vartheta^{(t,s)}$, but $a_1 \uplus_1 a_2 \notin \vartheta^{(t,s)}$. This contradicts to that $\vartheta^{(t,s)}$ is a subbisemiring of $\mathcal{S}$. Hence $\vartheta_K^{T+}(a_1 \uplus_1 a_2) \geq \min\{\vartheta_K^{T+}(a_1), \vartheta_K^{T+}(a_2)\}$, $\vartheta_K^{T-}(a_1 \uplus_1 a_2) \leq \max\{\vartheta_K^{T-}(a_1), \vartheta_K^{T-}(a_2)\}$, $\vartheta_K^{I+}(a_1 \uplus_1 a_2) \geq \frac{\vartheta_K^{I+}(a_1)+\vartheta_K^{I+}(a_2)}{2}$, $\vartheta_K^{I-}(a_1 \uplus_1 a_2) \leq \frac{\vartheta_K^{I-}(a_1)+\vartheta_K^{I-}(a_2)}{2}$ and $\vartheta_K^{F+}(a_1 \uplus_1 a_2) \leq \max\{\vartheta_K^{F+}(a_1), \vartheta_K^{F+}(a_2)\}$, $\vartheta_K^{F-}(a_1 \uplus_1 a_2) \geq \min\{\vartheta_K^{F-}(a_1), \vartheta_K^{F-}(a_2)\}$. Similarly to prove other two operations such as $\uplus_2$ and $\uplus_3$. Hence $\tilde{\vartheta} = \{(\vartheta_K^{T+}, \vartheta_K^{T-}), (\vartheta_K^{I+}, \vartheta_K^{I-}), (\vartheta_K^{F+}, \vartheta_K^{F-})\}$ is an BVNSBS of $\mathcal{S}$.

Definition 3.9. Let K be any BVNSBS of $\mathcal{S}$ and $a \in \mathcal{S}$. Then the pseudo bipolar valued neutrosophic coset $(aA)^z$ is defined by

$$\left\{\begin{array}{l} \begin{pmatrix} ((a\vartheta_K^{T+})^z)(m) = z(a)\vartheta_K^{T+}(m), \\ ((a\vartheta_K^{T-})^z)(m) = z(a)\vartheta_K^{T-}(m) \end{pmatrix}, \begin{pmatrix} ((a\vartheta_K^{I+})^z)(m) = z(a)\vartheta_K^{I+}(m), \\ ((a\vartheta_K^{I-})^z)(m) = z(a)\vartheta_K^{I-}(m) \end{pmatrix}, \\ \begin{pmatrix} ((a\vartheta_K^{F+})^z)(m) = z(a)\vartheta_K^{F+}(m), \\ ((a\vartheta_K^{F-})^z)(m) = z(a)\vartheta_K^{F-}(m) \end{pmatrix} \end{array}\right\}.$$

for every $m \in \mathcal{S}$ and for some $z \in P$, where P is a any non-empty set.

Theorem 3.10. *Let K be any BVNSBS of $\mathcal{S}$, then the pseudo bipolar valued neutrosophic coset $(aA)^z$ is an BVNSBS of $\mathcal{S}$, for every $a \in \mathcal{S}$.*

Proof. Now, $((a\vartheta_K^{T+})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{T+}(m \uplus_1 n) \geq z(a)\ \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\} = \min\{z(a)\ \vartheta_K^{T+}(m), z(a)\ \vartheta_K^{T+}(n)\} = \min\{((a\vartheta_K^{T+})^z)(m), ((a\vartheta_K^{T+})^z)(n)\}$. Thus, $((a\vartheta_K^{T+})^z)(m \uplus_1 n) \geq \min\{((a\vartheta_K^{T+})^z)(m), ((a\vartheta_K^{T+})^z)(n)\}$. Now, $((a\vartheta_K^{I+})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{I+}(m \uplus_1 n) \geq z(a)\left[\frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}\right] = \frac{z(a)\ \vartheta_K^{I+}(m)+z(a)\ \vartheta_K^{I+}(n)}{2} = \frac{((a\vartheta_K^{I+})^z)(m)+((a\vartheta_K^{I+})^z)(n)}{2}$. Thus, $((a\vartheta_K^{I+})^z)(m \uplus_1 n) \geq \frac{((a\vartheta_K^{I+})^z)(m)+((a\vartheta_K^{I+})^z)(n)}{2}$. Now, $((a\vartheta_K^{F+})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{F+}(m \uplus_1 n) \leq z(a)\ \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\} = \max\{z(a)\ \vartheta_K^{F+}(m), z(a)\ \vartheta_K^{F+}(n)\} = \max\{((a\vartheta_K^{F+})^z)(m), ((a\vartheta_K^{F+})^z)(n)\}$. Thus, $((a\vartheta_K^{F+})^z)(m \uplus_1 n) \leq \max\{((a\vartheta_K^{F+})^z)(m), ((a\vartheta_K^{F+})^z)(n)\}$. Also, $((a\vartheta_K^{T-})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{T-}(m \uplus_1 n) \leq z(a)\ \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} = \max\{z(a)\ \vartheta_K^{T-}(m), z(a)\ \vartheta_K^{T-}(n)\} = \max\{((a\vartheta_K^{T-})^z)(m), ((a\vartheta_K^{T-})^z)(n)\}$.
Thus, $((a\vartheta_K^{T-})^z)(m \uplus_1 n) \leq \max\{((a\vartheta_K^{T-})^z)(m), ((a\vartheta_K^{T-})^z)(n)\}$. Now, $((a\vartheta_K^{I-})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{I-}(m \uplus_1 n) \leq z(a)\left[\frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2}\right] = \frac{z(a)\ \vartheta_K^{I-}(m)+z(a)\ \vartheta_K^{I-}(n)}{2} = \frac{((a\vartheta_K^{I-})^z)(m)+((a\vartheta_K^{I-})^z)(n)}{2}$. Thus, $((a\vartheta_K^{I-})^z)(m \uplus_1 n) \leq \frac{((a\vartheta_K^{I-})^z)(m)+((a\vartheta_K^{I-})^z)(n)}{2}$. Now, $((a\vartheta_K^{F-})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{F-}(m \uplus_1 n) \geq z(a)\ \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\} = \min\{z(a)\ \vartheta_K^{F-}(m), z(a)\ \vartheta_K^{F-}(n)\} = \min\{((a\vartheta_K^{F-})^z)(m), ((a\vartheta_K^{F-})^z)(n)\}$. Thus, $((a\vartheta_K^{F-})^z)(m \uplus_1 n) \geq \min\{((a\vartheta_K^{F-})^z)(m), ((a\vartheta_K^{F-})^z)(n)\}$. Similarly to prove other two operations such as $\uplus_2$ and $\uplus_3$. Hence $(aA)^z$ is an BVNSBS of $\mathcal{S}$.

Definition 3.11. Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings. Let $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ be any function and K be any BVNSBS in $\mathcal{S}_1$, O be any BVNSBS in $\Lambda(\mathcal{S}_1) = \mathcal{S}_2$. If $\vartheta_K = \{(\vartheta_K^{T+}, \vartheta_K^{T-}), (\vartheta_K^{I+}, \vartheta_K^{I-}), (\vartheta_K^{F+}, \vartheta_K^{F-})\}$ is a bipolar valued neutrosophic set in $\mathcal{S}_1$, then ϑ_O is a bipolar valued neutrosophic set in $\mathcal{S}_2$, defined by

$$\vartheta_O^{T+}(n) = \begin{cases} \sup \vartheta_K^{T+}(m) & if\ \ m \in \Lambda^{-1}(n) \\ 0 & otherwise \end{cases} ; \vartheta_O^{T-}(n) = \begin{cases} \inf \vartheta_K^{T-}(m) & if\ \ m \in \Lambda^{-1}(n) \\ -1 & otherwise \end{cases}$$

$$\vartheta_O^{I+}(n) = \begin{cases} \sup \vartheta_K^{I+}(m) & if\ \ m \in \Lambda^{-1}(n) \\ 0 & otherwise \end{cases} ; \vartheta_O^{I-}(n) = \begin{cases} \inf \vartheta_K^{I-}(m) & if\ \ m \in \Lambda^{-1}(n) \\ -1 & otherwise \end{cases}$$

$$\vartheta_O^{F+}(n) = \begin{cases} \inf \vartheta_K^{F+}(m) & if\ \ m \in \Lambda^{-1}(n) \\ 1 & otherwise \end{cases} ; \vartheta_O^{F-}(n) = \begin{cases} \sup \vartheta_K^{F-}(m) & if\ \ m \in \Lambda^{-1}(n) \\ 0 & otherwise \end{cases}.$$

for all $m \in \mathcal{S}_1$ and $n \in \mathcal{S}_2$ is called the image of ϑ_K under Λ.
If $\vartheta_O = \{(\vartheta_O^{T+}, \vartheta_O^{T-}), (\vartheta_O^{I+}, \vartheta_O^{I-}), (\vartheta_O^{F+}, \vartheta_O^{F-})\}$ is a bipolar valued neutrosophic set in $\mathcal{S}_2$, then neutrosophic set $\vartheta_K = \Lambda \circ \vartheta_O$ in $\mathcal{S}_1$ [ie, the bipolar valued neutrosophic set defined by $\vartheta_K(m) = \vartheta_O(\Lambda(m))$] is called the preimage of ϑ_O under Λ.

Theorem 3.12. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings. The homomorphic image of BVNSBS of $\mathcal{S}_1$ is an BVNSBS of $\mathcal{S}_2$.*

Proof. Let $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ be any homomorphism. Then $\Lambda(m \vee_1 n) = \Lambda(m) \sqcup_1 \Lambda(n), \Lambda(m \vee_2 n) = \Lambda(m) \sqcup_2 \Lambda(n)$ and $\Lambda(m \vee_3 n) = \Lambda(m) \sqcup_3 \Lambda(n)$ for all $m, n \in \mathcal{S}_1$. Let $O = \Lambda(K)$, K is any BVNSBS of $\mathcal{S}_1$. Let $\Lambda(m), \Lambda(n) \in \mathcal{S}_2$. Let $m \in \Lambda^{-1}(\Lambda(m))$ and $n \in \Lambda^{-1}(\Lambda(n))$ be such that $\vartheta_K^{T+}(m) = \sup\limits_{z \in \Lambda^{-1}(\Lambda(m))} \vartheta_K^{T+}(z)$, $\vartheta_K^{T+}(n) = \sup\limits_{z \in \Lambda^{-1}(\Lambda(n))} \vartheta_K^{T+}(z)$ and $\vartheta_K^{T-}(m) = \inf\limits_{z \in \Lambda^{-1}(\Lambda(m))} \vartheta_K^{T-}(z)$, $\vartheta_K^{T-}(n) = \inf\limits_{z \in \Lambda^{-1}(\Lambda(n))} \vartheta_K^{T-}(z)$. Now,

$$\begin{aligned} \vartheta_O^{T+}(\Lambda(m) \sqcup_1 \Lambda(n)) &= \sup_{z' \in \Lambda^{-1}(\Lambda(m) \sqcup_1 \Lambda(n))} \vartheta_K^{T+}(z') \\ &= \sup_{z' \in \Lambda^{-1}(\Lambda(m \vee_1 n)} \vartheta_K^{T+}(z') \\ &= \vartheta_K^{T+}(m \vee_1 n) \\ &\geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\} \\ &= \min\{\vartheta_O^{T+}\Lambda(m), \vartheta_O^{T+}\Lambda(n)\}. \end{aligned}$$

Thus, $\vartheta_O^{T+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \min\{\vartheta_O^{T+}\Lambda(m), \vartheta_O^{T+}\Lambda(n)\}$.

$$\begin{aligned} \vartheta_O^{T-}(\Lambda(m) \sqcup_1 \Lambda(n)) &= \inf_{z' \in \Lambda^{-1}(\Lambda(m) \sqcup_1 \Lambda(n))} \vartheta_K^{T-}(z') \\ &= \inf_{z' \in \Lambda^{-1}(\Lambda(m \vee_1 n)} \vartheta_K^{T-}(z') \\ &= \vartheta_K^{T-}(m \vee_1 n) \\ &\leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} \\ &= \max\{\vartheta_O^{T-}\Lambda(m), \vartheta_O^{T-}\Lambda(n)\}. \end{aligned}$$

Thus, $\vartheta_O^{T-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \max\{\vartheta_O^{T-}\Lambda(m), \vartheta_O^{T-}\Lambda(n)\}$.
Let $m \in \Lambda^{-1}(\Lambda(m))$ and $n \in \Lambda^{-1}(\Lambda(n))$ be such that $\vartheta_K^{I+}(m) = \sup\limits_{z \in \Lambda^{-1}(\Lambda(m))} \vartheta_K^{I+}(z)$, $\vartheta_K^{I+}(n) = \sup\limits_{z \in \Lambda^{-1}(\Lambda(n))} \vartheta_K^{I+}(z)$, $\vartheta_K^{I-}(m) = \inf\limits_{z \in \Lambda^{-1}(\Lambda(m))} \vartheta_K^{I-}(z)$, $\vartheta_K^{I-}(n) = \inf\limits_{z \in \Lambda^{-1}(\Lambda(n))} \vartheta_K^{I-}(z)$.
Now,

$$\begin{aligned} \vartheta_O^{I+}(\Lambda(m) \sqcup_1 \Lambda(n)) &= \sup_{z' \in \Lambda^{-1}(\Lambda(m) \sqcup_1 \Lambda(n))} \vartheta_K^{I+}(z') \\ &= \sup_{z' \in \Lambda^{-1}(\Lambda(m \vee_1 n)} \vartheta_K^{I+}(z') \\ &= \vartheta_K^{I+}(m \vee_1 n) \\ &\geq \frac{\vartheta_K^{I+}(m) + \vartheta_K^{I+}(n)}{2} \\ &= \frac{\vartheta_O^{I+}\Lambda(m) + \vartheta_O^{I+}\Lambda(n)}{2}. \end{aligned}$$

Thus, $\vartheta_O^{I+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \frac{\vartheta_O^{I+}\Lambda(m)+\vartheta_O^{I+}\Lambda(n)}{2}$.
Similarly, $\vartheta_O^{I-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \frac{\vartheta_O^{I-}\Lambda(m)+\vartheta_O^{I-}\Lambda(n)}{2}$.
Let $\Lambda(m), \Lambda(n) \in \mathcal{S}_2$. Let $m \in \Lambda^{-1}(\Lambda(m))$ and $n \in \Lambda^{-1}(\Lambda(n))$ be such that $\vartheta_K^{F+}(m) = \inf_{z\in\Lambda^{-1}(\Lambda(m))} \vartheta_K^{F+}(z)$, $\vartheta_K^{F+}(n) = \inf_{z\in\Lambda^{-1}(\Lambda(n))} \vartheta_K^{F+}(z)$, $\vartheta_K^{F-}(m) = \sup_{z\in\Lambda^{-1}(\Lambda(m))} \vartheta_K^{F-}(z)$ and $\vartheta_K^{F-}(n) = \sup_{z\in\Lambda^{-1}(\Lambda(n))} \vartheta_K^{F-}(z)$. Now,

$$\begin{aligned}\vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)) &= \inf_{z'\in\Lambda^{-1}(\Lambda(m)\sqcup_1\Lambda(n))} \vartheta_K^{F+}(z') \\ &= \inf_{z'\in\Lambda^{-1}(\Lambda(m\vee_1 n)} \vartheta_K^{F+}(z') \\ &= \vartheta_K^{F+}(m \vee_1 n) \\ &\leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\} \\ &= \max\{\vartheta_O^{F+}\Lambda(m), \vartheta_O^{F+}\Lambda(n)\}.\end{aligned}$$

Thus, $\vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \max\{\vartheta_O^{F+}\Lambda(m), \vartheta_O^{F+}\Lambda(n)\}$.
Similarly, $\vartheta_O^{F-}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \min\{\vartheta_O^{F-}\Lambda(m), \vartheta_O^{F-}\Lambda(n)\}$.
Similarly, to prove other two operations. Hence O is an BVNSBS of $\mathcal{S}_2$.

Theorem 3.13. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings. The homomorphic preimage of BVNSBS of $\mathcal{S}_2$ is an BVNSBS of $\mathcal{S}_1$.*

Proof. Let $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ be any homomorphism. Then $\Lambda(m \vee_1 n) = \Lambda(m) \sqcup_1 \Lambda(n), \Lambda(m \vee_2 n) = \Lambda(m) \sqcup_2 \Lambda(n)$ and $\Lambda(m \vee_3 n) = \Lambda(m) \sqcup_3 \Lambda(n)$ for all $m, n \in \mathcal{S}_1$. Let $O = \Lambda(K)$, where O is any BVNSBS of $\mathcal{S}_2$. Let $m, n \in \mathcal{S}_1$. Now, $\vartheta_K^{T+}(m \vee_1 n) = \vartheta_O^{T+}(\Lambda(m \vee_1 n)) = \vartheta_O^{T+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \min\{\vartheta_O^{T+}\Lambda(m), \vartheta_O^{T+}\Lambda(n)\} = \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\}$. Thus, $\vartheta_K^{T+}(m \vee_1 n) \geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\}$. Now, $\vartheta_K^{I+}(m \vee_1 n) = \vartheta_O^{I+}(\Lambda(m \vee_1 n)) = \vartheta_O^{I+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \frac{\vartheta_O^{I+}\Lambda(m)+\vartheta_O^{I+}\Lambda(n)}{2} = \frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}$. Thus, $\vartheta_K^{I+}(m \vee_1 n) \geq \frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}$. Now, $\vartheta_K^{F+}(m \vee_1 n) = \vartheta_O^{F+}(\Lambda(m\vee_1 n)) = \vartheta_O^{F+}(\Lambda(m)\sqcup_1\Lambda(n)) \leq \max\{\vartheta_O^{F+}\Lambda(m), \vartheta_O^{F+}\Lambda(n)\} = \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}$. Thus, $\vartheta_K^{F+}(m \vee_1 n) \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}$. Also, $\vartheta_K^{T-}(m \vee_1 n) = \vartheta_O^{T-}(\Lambda(m \vee_1 n)) = \vartheta_O^{T-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \max\{\vartheta_O^{T-}\Lambda(m), \vartheta_O^{T-}\Lambda(n)\} = \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\}$. Thus, $\vartheta_K^{T-}(m \vee_1 n) \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\}$. We have, $\vartheta_K^{I-}(m\vee_1 n) = \vartheta_O^{I-}(\Lambda(m\vee_1 n)) = \vartheta_O^{I-}(\Lambda(m)\sqcup_1\Lambda(n)) \leq \frac{\vartheta_O^{I-}\Lambda(m)+\vartheta_O^{I-}\Lambda(n)}{2} = \frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2}$. Thus, $\vartheta_K^{I-}(m \vee_1 n) \leq \frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2}$. Now, $\vartheta_K^{F-}(m \vee_1 n) = \vartheta_O^{F-}(\Lambda(m \vee_1 n)) = \vartheta_O^{F-}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \min\{\vartheta_O^{F-}\Lambda(m), \vartheta_O^{F-}\Lambda(n)\} = \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\}$. Thus, $\vartheta_K^{F-}(m \vee_1 n) \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\}$. Similarly to prove other two operations, hence K is an BVNSBS of $\mathcal{S}_1$.

Theorem 3.14. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings. If $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ is a homomorphism, then $\Lambda(K_{(t,s)})$ is a level subbisemiring of BVNSBS O of $\mathcal{S}_2$.*

Proof. Let $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ be any homomorphism. Then $\Lambda(m \vee_1 n) = \Lambda(m) \sqcup_1 \Lambda(n), \Lambda(m \vee_2 n) = \Lambda(m) \sqcup_2 \Lambda(n)$ and $\Lambda(m \vee_3 n) = \Lambda(m) \sqcup_3 \Lambda(n)$ for all $m, n \in \mathcal{S}_1$. Let $O = \Lambda(K)$, K is an BVNSBS of $\mathcal{S}_1$. By Theorem 3.12, O is an BVNSBS of $\mathcal{S}_2$. Let $K_{(t,s)}$ be any level subbisemiring of K. Suppose that $m, n \in K_{(t,s)}$. Then $\Lambda(m \vee_1 n), \Lambda(m \vee_2 n)$ and $\Lambda(m \vee_3 n) \in K_{(t,s)}$. Now, $\vartheta_O^{T+}(\Lambda(m)) = \vartheta_K^{T+}(m) \geq t, \vartheta_O^{T+}(\Lambda(n)) = \vartheta_K^{T+}(n) \geq t$. Thus, $\vartheta_O^{T+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \vartheta_K^{T+}(m \vee_1 n) \geq t$. Now, $\vartheta_O^{I+}(\Lambda(m)) = \vartheta_K^{I+}(m) \geq t, \vartheta_O^{I+}(\Lambda(n)) = \vartheta_K^{I+}(n) \geq t$. Thus, $\vartheta_O^{I+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \vartheta_K^{I+}(m \vee_1 n) \geq t$. Now, $\vartheta_O^{F+}(\Lambda(m)) = \vartheta_K^{F+}(m) \leq s, \vartheta_O^{F+}(\Lambda(n)) = \vartheta_K^{F+}(n) \leq s$. Thus, $\vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \vartheta_K^{F+}(m \vee_1 n) \leq s$, for all $\Lambda(m), \Lambda(n) \in \mathcal{S}_2$. Also, $\vartheta_O^{T-}(\Lambda(m)) = \vartheta_K^{T-}(m) \leq t, \vartheta_O^{T-}(\Lambda(n)) = \vartheta_K^{T-}(n) \leq t$. Thus, $\vartheta_O^{T-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \vartheta_K^{T-}(m \vee_1 n) \leq t$. Now, $\vartheta_O^{I-}(\Lambda(m)) = \vartheta_K^{I-}(m) \leq t, \vartheta_O^{I-}(\Lambda(n)) = \vartheta_K^{I-}(n) \leq t$. Thus, $\vartheta_O^{I-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \vartheta_K^{I-}(m \vee_1 n) \leq t$. Now, $\vartheta_O^{F-}(\Lambda(m)) = \vartheta_K^{F-}(m) \geq s, \vartheta_O^{F-}(\Lambda(n)) = \vartheta_K^{F-}(n) \geq s$. Thus, $\vartheta_O^{F-}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \vartheta_K^{F-}(m \vee_1 n) \geq s$, for all $\Lambda(m), \Lambda(n) \in \mathcal{S}_2$. Similarly to prove other operations, hence $\Lambda(K_{(t,s)})$ is a level subbisemiring of BVNSBS O of $\mathcal{S}_2$.

Theorem 3.15. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings. If $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ is any homomorphism, then $K_{(t,s)}$ is a level subbisemiring of BVNSBS K of $\mathcal{S}_1$.*

Proof. Let $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ be any homomorphism. Then $\Lambda(m \vee_1 n) = \Lambda(m) \sqcup_1 \Lambda(n), \Lambda(m \vee_2 n) = \Lambda(m) \sqcup_2 \Lambda(n)$ and $\Lambda(m \vee_3 n) = \Lambda(m) \sqcup_3 \Lambda(n)$ for all $m, n \in \mathcal{S}_1$. Let $O = \Lambda(K)$, O is an BVNSBS of $\mathcal{S}_2$. By Theorem 3.13, K is an BVNSBS of $\mathcal{S}_1$. Let $\Lambda(K_{(t,s)})$ be a level subbisemiring of O. Suppose that $\Lambda(m), \Lambda(n) \in \Lambda(K_{(t,s)})$. Then $\Lambda(m \vee_1 n), \Lambda(m \vee_2 n)$ and $\Lambda(m \vee_3 n) \in \Lambda(K_{(t,s)})$. Now, $\vartheta_K^{T+}(m) = \vartheta_O^{T+}(\Lambda(m)) \geq t, \vartheta_K^{T+}(n) = \vartheta_O^{T+}(\Lambda(n)) \geq t$. Thus, $\vartheta_K^{T+}(m \vee_1 n) \geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\} \geq t$. Now, $\vartheta_K^{I+}(m) = \vartheta_O^{I+}(\Lambda(m)) \geq t, \vartheta_K^{I+}(n) = \vartheta_O^{I+}(\Lambda(n)) \geq t$. Thus, $\vartheta_K^{I+}(m \vee_1 n) \geq \frac{\vartheta_K^{I+}(m) + \vartheta_K^{I+}(n)}{2} \geq t$. Now, $\vartheta_K^{F+}(m) = \vartheta_O^{F+}(\Lambda(m)) \leq s, \vartheta_K^{F+}(n) = \vartheta_O^{F+}(\Lambda(n)) \leq s$. Thus, $\vartheta_K^{F+}(m \vee_1 n) = \vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\} \leq s$, for all $m, n \in \mathcal{S}_1$. Also, $\vartheta_K^{T-}(m) = \vartheta_O^{T-}(\Lambda(m)) \leq t, \vartheta_K^{T-}(n) = \vartheta_O^{T-}(\Lambda(n)) \leq t$. Thus, $\vartheta_K^{T-}(m \vee_1 n) \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} \leq t$. Now, $\vartheta_K^{I-}(m) = \vartheta_O^{I-}(\Lambda(m)) \leq t, \vartheta_K^{I-}(n) = \vartheta_O^{I-}(\Lambda(n)) \leq t$. Thus, $\vartheta_K^{I-}(m \vee_1 n) \leq \frac{\vartheta_K^{I-}(m) + \vartheta_K^{I-}(n)}{2} \leq t$. Now, $\vartheta_K^{F-}(m) = \vartheta_O^{F-}(\Lambda(m)) \geq s, \vartheta_K^{F-}(n) = \vartheta_O^{F-}(\Lambda(n)) \geq s$. Thus, $\vartheta_K^{F-}(m \vee_1 n) = \vartheta_O^{F-}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\} \geq s$, for all $m, n \in \mathcal{S}_1$. In the same way, prove the other two operations, hence $K_{(t,s)}$ is a level subbisemiring of BVNSBS K of $\mathcal{S}_1$.

4. (ξ, τ)–**Bipolar Valued Neutrosophic Subbisemiring**

In this section, we discuss (ξ, τ)–bipolar valued neutrosophic subbisemiring. In what follows that, $(\xi^+, \tau^+) \in [0, 1]$ and $(\xi^-, \tau^-) \in [-1, 0]$ be such that $0 \leq \xi^+ < \tau^+ \leq 1$ and $-1 \leq \tau^- < \xi^- \leq 0$, both $(\xi, \tau) \in [0, 1]$ are arbitrary but fixed.

Definition 4.1. Let K be any bipolar valued neutrosophic subset of $\mathcal{S}$ is called a (ξ,τ)-BVNSBS of $\mathcal{S}$ if it satisfies the following conditions:

$$\left\{\begin{pmatrix}\max\{\vartheta_K^{T+}(m\uplus_1 n),\xi^+\}\geq\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\},\\ \min\{\vartheta_K^{T-}(m\uplus_1 n),\xi^-\}\leq\max\{\vartheta_K^{T-}(m),\vartheta_K^{T-}(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\max\{\vartheta_K^{T+}(m\uplus_2 n),\xi^+\}\geq\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\},\\ \min\{\vartheta_K^{T-}(m\uplus_2 n),\xi^-\}\leq\max\{\vartheta_K^{T-}(m),\vartheta_K^{T-}(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\max\{\vartheta_K^{T+}(m\uplus_3 n),\xi^+\}\geq\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\},\\ \min\{\vartheta_K^{T-}(m\uplus_3 n),\xi^-\}\leq\max\{\vartheta_K^{T-}(m),\vartheta_K^{T-}(n),\tau^-\}\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}\max\{\vartheta_K^{I+}(m\uplus_1 n),\xi^+\}\geq\min\left\{\frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2},\tau^+\right\}\\ \min\{\vartheta_K^{I-}(m\uplus_1 n),\xi^-\}\leq\max\left\{\frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2},\tau^-\right\}\end{pmatrix}\\ \text{OR}\\ \begin{pmatrix}\max\{\vartheta_K^{I+}(m\uplus_2 n),\xi^+\}\geq\min\left\{\frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2},\tau^+\right\}\\ \min\{\vartheta_K^{I-}(m\uplus_2 n),\xi^-\}\leq\max\left\{\frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2},\tau^-\right\}\end{pmatrix}\\ \text{OR}\\ \begin{pmatrix}\max\{\vartheta_K^{I+}(m\uplus_3 n),\xi^+\}\geq\min\left\{\frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2},\tau^+\right\}\\ \min\{\vartheta_K^{I-}(m\uplus_3 n),\xi^-\}\leq\max\left\{\frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2},\tau^-\right\}\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}\min\{\vartheta_K^{F+}(m\uplus_1 n),\xi^+\}\leq\max\{\vartheta_K^{F+}(m),\vartheta_K^{F+}(n),\tau^+\},\\ \max\{\vartheta_K^{F-}(m\uplus_1 n),\xi^-\}\geq\min\{\vartheta_K^{F-}(m),\vartheta_K^{F-}(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\min\{\vartheta_K^{F+}(m\uplus_2 n),\xi^+\}\leq\max\{\vartheta_K^{F+}(m),\vartheta_K^{F+}(n),\tau^+\},\\ \max\{\vartheta_K^{F-}(m\uplus_2 n),\xi^-\}\geq\min\{\vartheta_K^{F-}(m),\vartheta_K^{F-}(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\min\{\vartheta_K^{F+}(m\uplus_3 n),\xi^+\}\leq\max\{\vartheta_K^{F+}(m),\vartheta_K^{F+}(n),\tau^+\},\\ \max\{\vartheta_K^{F-}(m\uplus_3 n),\xi^-\}\geq\min\{\vartheta_K^{F-}(m),\vartheta_K^{F-}(n),\tau^-\}\end{pmatrix}\right\}$$

for all $m,n\in\mathcal{S}$.

Example 4.2. By the Example 3.2,

$\left(\vartheta_K^{+}(l),\vartheta_K^{-}(l)\right)$	$l=l_1$	$l=l_2$	$l=l_3$	$l=l_4$
$\left(\vartheta_K^{T+}(l),\vartheta_K^{T-}(l)\right)$	$(0.85,-0.95)$	$(0.8,-0.75)$	$(0.7,-0.55)$	$(0.75,-0.65)$
$\left(\vartheta_K^{I+}(l),\vartheta_K^{I-}(l)\right)$	$(0.95,-0.8)$	$(0.9,-0.7)$	$(0.8,-0.5)$	$(0.85,-0.55)$
$\left(\vartheta_K^{F+}(l),\vartheta_K^{F-}(l)\right)$	$(0.65,-0.25)$	$(0.85,-0.35)$	$(0.95,-0.45)$	$(0.90,-0.40)$

Clearly, K is a $(0.60,0.70)$-BVNSBS of $\mathcal{S}$.

Theorem 4.3. *The intersection of family of (ξ,τ)- $BVNSBS^s$ of $\mathcal{S}$ is a (ξ,τ)- $BVNSBS$ of $\mathcal{S}$.*

Proof. Let $\{O_i|i\in I\}$ be any family of (ξ,τ)- $BVNSBS^s$ of $\mathcal{S}$ and $K=\bigcap_{i\in I}O_i$.

Let m and n in $\mathcal{S}$. Now,

$$\begin{aligned}\max\{\vartheta_K^{T+}(m\uplus_1 n),\xi^+\}&=\inf_{i\in I}\ \max\{\vartheta_{O_i}^{T+}(m\uplus_1 n),\xi^+\}\\ &\geq\inf_{i\in I}\ \min\{\vartheta_{O_i}^{T+}(m),\vartheta_{O_i}^{T+}(n),\tau^+\}\\ &=\min\left\{\inf_{i\in I}\ \vartheta_{O_i}^{T+}(m),\inf_{i\in I}\ \vartheta_{O_i}^{T+}(n),\tau^+\right\}\\ &=\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\}\end{aligned}$$

$$\begin{aligned}
\min\{\vartheta_K^{T-}(m \uplus_1 n), \xi^-\} &= \sup_{i\in I}\ \min\{\vartheta_{O_i}^{T-}(m \uplus_1 n), \xi^-\} \\
&\leq \sup_{i\in I}\ \max\{\vartheta_{O_i}^{T-}(m), \vartheta_{O_i}^{T-}(n), \tau^-\} \\
&= \max\left\{\sup_{i\in I}\ \vartheta_{O_i}^{T-}(m), \sup_{i\in I}\ \vartheta_{O_i}^{T-}(n), \tau^-\right\} \\
&= \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n), \tau^-\}.
\end{aligned}$$

Now,

$$\begin{aligned}
\max\{\vartheta_K^{I+}(m \uplus_1 n), \xi^+\} &= \inf_{i\in I}\ \max\{\vartheta_{O_i}^{I+}(m \uplus_1 n), \xi^+\} \\
&\geq \inf_{i\in I}\ \min\left\{\frac{\vartheta_{O_i}^{I+}(m) + \vartheta_{O_i}^{I+}(n)}{2}, \tau^+\right\} \\
&= \min\left\{\frac{\inf_{i\in I}\ \vartheta_{O_i}^{I+}(m) + \inf_{i\in I}\ \vartheta_{O_i}^{I+}(n)}{2}, \tau^+\right\} \\
&= \min\left\{\frac{\vartheta_K^{I+}(m) + \vartheta_K^{I+}(n)}{2}, \tau^+\right\}.
\end{aligned}$$

$$\begin{aligned}
\min\{\vartheta_K^{I-}(m \uplus_1 n), \xi^-\} &= \sup_{i\in I}\ \min\{\vartheta_{O_i}^{I-}(m \uplus_1 n), \xi^-\} \\
&\leq \sup_{i\in I}\ \max\left\{\frac{\vartheta_{O_i}^{I-}(m) + \vartheta_{O_i}^{I-}(n)}{2}, \tau^-\right\} \\
&= \max\left\{\frac{\sup_{i\in I}\ \vartheta_{O_i}^{I-}(m) + \sup_{i\in I}\ \vartheta_{O_i}^{I-}(n)}{2}, \tau^-\right\} \\
&= \max\left\{\frac{\vartheta_K^{I-}(m) + \vartheta_K^{I-}(n)}{2}, \tau^-\right\}.
\end{aligned}$$

Now,

$$\begin{aligned}
\min\{\vartheta_K^{F+}(m \uplus_1 n), \xi^+\} &= \sup_{i\in I}\ \min\{\vartheta_{O_i}^{F+}(m \uplus_1 n), \xi^+\} \\
&\leq \sup_{i\in I}\ \max\{\vartheta_{O_i}^{F+}(m), \vartheta_{O_i}^{F+}(n), \tau^+\} \\
&= \max\left\{\sup_{i\in I}\ \vartheta_{O_i}^{F+}(m), \sup_{i\in I}\ \vartheta_{O_i}^{F+}(n), \tau^+\right\} \\
&= \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n), \tau^+\} \\
\max\{\vartheta_K^{F-}(m \uplus_1 n), \xi^-\} &= \inf_{i\in I}\ \max\{\vartheta_{O_i}^{F-}(m \uplus_1 n), \xi^-\} \\
&\geq \inf_{i\in I}\ \min\{\vartheta_{O_i}^{F-}(m), \vartheta_{O_i}^{F-}(n), \tau^-\} \\
&= \min\left\{\inf_{i\in I}\ \vartheta_{O_i}^{F-}(m), \inf_{i\in I}\ \vartheta_{O_i}^{F-}(n), \tau^-\right\} \\
&= \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n), \tau^-\}.
\end{aligned}$$

Similarly to prove other operations. Hence, K is a (ξ, τ)- BVNSBS of $\mathcal{S}$.

Theorem 4.4. *If K and L are any two $(\xi,\tau)-BVNSBS^s$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, then $K\times L$ is a $(\xi,\tau)-BVNSBS$ of $\mathcal{S}_1\times\mathcal{S}_2$.*

Proof. Let K and L be two $(\xi,\tau)-BVNSBS^s$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Let $m_1,m_2\in\mathcal{S}_1$ and $n_1,n_2\in\mathcal{S}_2$. Then (m_1,n_1) and (m_2,n_2) are in $\mathcal{S}_1\times\mathcal{S}_2$. Now
$\max\left\{\vartheta^{T+}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)],\xi^+\right\}$

$$\begin{aligned}
&=\max\left\{\vartheta^{T+}_{K\times L}(m_1\uplus_1 m_2,n_1\uplus_1 n_2),\xi^+\right\}\\
&=\min\left\{\max\{\vartheta^{T+}_{K}(m_1\uplus_1 m_2),\xi^+\},\max\{\vartheta^{T+}_{L}(n_1\uplus_1 n_2),\xi^+\}\right\}\\
&\geq\min\left\{\min\{\vartheta^{T+}_{K}(m_1),\vartheta^{T+}_{K}(m_2),\tau^+\},\min\{\vartheta^{T+}_{L}(n_1),\vartheta^{T+}_{L}(n_2),\tau^+\}\right\}\\
&=\min\left\{\{\min\{\vartheta^{T+}_{K}(m_1),\vartheta^{T+}_{L}(n_1)\},\min\{\vartheta^{T+}_{K}(m_2),\vartheta^{T+}_{L}(n_2)\}\},\tau^+\right\}\\
&=\min\left\{\vartheta^{T+}_{K\times L}(m_1,n_1),\vartheta^{T+}_{K\times L}(m_2,n_2),\tau^+\right\}.
\end{aligned}$$

Also, $\min\left\{\vartheta^{T-}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)],\xi^-\right\}\leq\max\left\{\vartheta^{T-}_{K\times L}(m_1,n_1),\vartheta^{T-}_{K\times L}(m_2,n_2),\tau^-\right\}$.
Now, $\max\left\{\vartheta^{I+}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)],\xi^+\right\}$

$$\begin{aligned}
&=\max\left\{\vartheta^{I+}_{K\times L}(m_1\uplus_1 m_2,n_1\uplus_1 n_2),\xi^+\right\}\\
&=\min\left\{\frac{1}{2}\left[\max\left\{\vartheta^{I+}_{K}(m_1\uplus_1 m_2),\xi^+\right\}+\max\left\{\vartheta^{I+}_{L}(n_1\uplus_1 n_2),\xi^+\right\}\right]\right\}\\
&\geq\min\left\{\frac{1}{2}\left[\min\left\{\frac{\vartheta^{I+}_{K}(m_1)+\vartheta^{I+}_{K}(m_2)}{2},\tau^+\right\}+\min\left\{\frac{\vartheta^{I+}_{L}(n_1)+\vartheta^{I+}_{L}(n_2)}{2},\tau^+\right\}\right]\right\}\\
&=\min\left\{\frac{1}{2}\left[\frac{\vartheta^{I+}_{K}(m_1)+\vartheta^{I+}_{L}(n_1)}{2}+\frac{\vartheta^{I+}_{K}(m_2)+\vartheta^{I+}_{L}(n_2)}{2}\right],\tau^+\right\}\\
&=\min\left\{\frac{\vartheta^{I+}_{K\times L}(m_1,n_1)+\vartheta^{I+}_{K\times L}(m_2,n_2)}{2},\tau^+\right\}.
\end{aligned}$$

Also, $\min\left\{\vartheta^{I-}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)],\xi^-\right\}\leq\max\left\{\frac{\vartheta^{I-}_{K\times L}(m_1,n_1)+\vartheta^{I-}_{K\times L}(m_2,n_2)}{2},\tau^-\right\}$.
Similarly, $\min\left\{\vartheta^{F+}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)],\xi^+\right\}$

$$\begin{aligned}
&=\min\left\{\vartheta^{F+}_{K\times L}(m_1\uplus_1 m_2,n_1\uplus_1 n_2),\xi^+\right\}\\
&=\max\left\{\min\{\vartheta^{F+}_{K}(m_1\uplus_1 m_2),\xi^+\},\min\{\vartheta^{F+}_{L}(n_1\uplus_1 n_2),\xi^+\}\right\}\\
&\leq\max\left\{\max\{\vartheta^{F+}_{K}(m_1),\vartheta^{F+}_{K}(m_2),\tau^+\},\max\{\vartheta^{F+}_{L}(n_1),\vartheta^{F+}_{L}(n_2),\tau^+\}\right\}\\
&=\max\left\{\{\max\{\vartheta^{F+}_{K}(m_1),\vartheta^{F+}_{L}(n_1)\},\max\{\vartheta^{F+}_{K}(m_2),\vartheta^{F+}_{L}(n_2)\}\},\tau^+\right\}\\
&=\max\left\{\vartheta^{F+}_{K\times L}(m_1,n_1),\vartheta^{F+}_{K\times L}(m_2,n_2),\tau^+\right\}.
\end{aligned}$$

Also, $\max\left\{\vartheta^{F-}_{K\times L}[(m_1,n_1)\uplus_1 (m_2,n_2)],\xi^-\right\} \geq \min\left\{\vartheta^{F-}_{K\times L}(m_1,n_1),\vartheta^{F-}_{K\times L}(m_2,n_2),\tau^-\right\}$.
In the same way, prove the other two operations. Hence $K\times L$ is a (ξ,τ)- BVNSBS of $\mathcal{S}_1\times\mathcal{S}_2$.

Corollary 4.5. *If $K_1,K_2,...,K_n$ are the family of (ξ,τ)- $BVNSBS^s$ of $\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n$ respectively, then $K_1\times K_2\times...\times K_n$ is a (ξ,τ)- BVNSBS of $\mathcal{S}_1\times\mathcal{S}_2\times...\times\mathcal{S}_n$.*

Definition 4.6. Let K be any (ξ,τ)- bipolar valued neutrosophic subset in $\mathcal{S}$, the strongest (ξ,τ)- bipolar valued neutrosophic relation on $\mathcal{S}$, that is a (ξ,τ)- bipolar valued neutrosophic relation on K is O such that

$$\left\{\begin{array}{l}\begin{pmatrix}\max\{\vartheta^{T+}_O(m,n),\xi^+\}=\min\{\vartheta^{T+}_K(m),\vartheta^{T+}_K(n),\tau^+\},\\ \min\{\vartheta^{T-}_O(m,n),\xi^-\}=\max\{\vartheta^{T-}_K(m),\vartheta^{T-}_K(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\max\{\vartheta^{I+}_O(m,n),\xi^+\}=\min\{\vartheta^{I+}_K(m),\vartheta^{I+}_K(n),\tau^+\},\\ \min\{\vartheta^{I-}_O(m,n),\xi^-\}=\max\{\vartheta^{I-}_K(m),\vartheta^{I-}_K(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\min\{\vartheta^{F+}_O(m,n),\xi^+\}=\max\{\vartheta^{F+}_K(m),\vartheta^{F+}_K(n),\tau^+\},\\ \max\{\vartheta^{F-}_O(m,n),\xi^-\}=\min\{\vartheta^{F-}_K(m),\vartheta^{F-}_K(n),\tau^-\}\end{pmatrix}\end{array}\right\}.$$

Theorem 4.7. *Let K be any $(\xi,\tau)-BVNSBS$ of $\mathcal{S}$ and O be the strongest (ξ,τ)- bipolar valued neutrosophic relation of $\mathcal{S}$. Then K is a $(\xi,\tau)-BVNSBS$ of $\mathcal{S}$ if and only if O is a $(\xi,\tau)-BVNSBS$ of $\mathcal{S}\times\mathcal{S}$.*

Theorem 4.8. *Let $(\mathcal{S}_1,\vee_1,\vee_2,\vee_3)$ and $(\mathcal{S}_2,\sqcup_1,\sqcup_2,\sqcup_3)$ be any two bisemirings. The homomorphic image of $(\xi,\tau)-BVNSBS$ of $\mathcal{S}_1$ is a $(\xi,\tau)-BVNSBS$ of $\mathcal{S}_2$.*

Proof. Let $\Lambda:\mathcal{S}_1\to\mathcal{S}_2$ be any homomorphism. Then $\Lambda(m\vee_1 n)=\Lambda(m)\sqcup_1\Lambda(n),\Lambda(m\vee_2 n)=\Lambda(m)\sqcup_2\Lambda(n)$ and $\Lambda(m\vee_3 n)=\Lambda(m)\sqcup_3\Lambda(n)$ for all $m,n\in\mathcal{S}_1$. Let $O=\Lambda(K)$, K is any (ξ,τ)-BVNSBS of $\mathcal{S}_1$. Let $\Lambda(m),\Lambda(n)\in\mathcal{S}_2$. Let $m\in\Lambda^{-1}(\Lambda(m))$ and $n\in\Lambda^{-1}(\Lambda(n))$ be such that $\vartheta^{T+}_K(m)=\sup_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta^{T+}_K(z)$, $\vartheta^{T+}_K(n)=\sup_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta^{T+}_K(z)$, $\vartheta^{T-}_K(m)=\inf_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta^{T-}_K(z)$ and $\vartheta^{T-}_K(n)=\inf_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta^{T-}_K(z)$. Now,

$$\begin{aligned}\max\left[\vartheta^{T+}_O(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^+\right]&=\max\left[\sup_{z'\in\Lambda^{-1}(\Lambda(m)\sqcup_1\Lambda(n))}\vartheta^{T+}_K(z')\ ,\xi^+\right]\\&=\max\left[\sup_{z'\in\Lambda^{-1}(\Lambda(m\vee_1 n)}\vartheta^{T+}_K(z')\ ,\xi^+\right]\\&=\max\left[\vartheta^{T+}_K(m\vee_1 n),\xi^+\right]\\&\geq\min\left\{\vartheta^{T+}_K(m),\vartheta^{T+}_K(n),\tau^+\right\}\\&=\min\left\{\vartheta^{T+}_O\Lambda(m),\vartheta^{T+}_O\Lambda(n),\tau^+\right\}.\end{aligned}$$

Similarly, $\min\left[\vartheta_O^{T-}(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^-\right]\le\max\left\{\vartheta_O^{T-}\Lambda(m),\vartheta_O^{T-}\Lambda(n),\tau^-\right\}$.

Let $\Lambda(m),\Lambda(n)\in\mathcal{S}_2$. Let $m\in\Lambda^{-1}(\Lambda(m))$ and $n\in\Lambda^{-1}(\Lambda(n))$ be such that $\vartheta_K^{I+}(m)=\sup\limits_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta_K^{I+}(z)$ and $\vartheta_K^{I+}(n)=\sup\limits_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta_K^{I+}(z)$, $\vartheta_K^{I-}(m)=\inf\limits_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta_K^{I-}(z)$ and $\vartheta_K^{I-}(n)=\inf\limits_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta_K^{I-}(z)$. Now,

$$\begin{aligned}\max\left[\vartheta_O^{I+}(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^+\right]&=\max\left[\sup_{z'\in\Lambda^{-1}(\Lambda(m)\sqcup_1\Lambda(n))}\vartheta_K^{I+}(z')\ ,\xi^+\right]\\&=\max\left[\sup_{z'\in\Lambda^{-1}(\Lambda(m\vee_1 n)}\vartheta_K^{I+}(z')\ ,\xi^+\right]\\&=\max\left[\vartheta_K^{I+}(m\vee_1 n),\xi^+\right]\\&\ge\min\left\{\frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2},\tau^+\right\}\\&=\min\left\{\frac{\vartheta_O^{I+}\Lambda(m)+\vartheta_O^{I+}\Lambda(n)}{2},\tau^+\right\}\end{aligned}$$

Similarly, $\min\left[\vartheta_O^{I-}(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^-\right]\le\max\left\{\frac{\vartheta_O^{I-}\Lambda(m)+\vartheta_O^{I-}\Lambda(n)}{2},\tau^-\right\}$.

Let $m\in\Lambda^{-1}(\Lambda(m))$ and $n\in\Lambda^{-1}(\Lambda(n))$ be such that $\vartheta_K^{F+}(m)=\inf\limits_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta_K^{F+}(z)$,$\vartheta_K^{F+}(n)=\inf\limits_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta_K^{F+}(z)$, $\vartheta_K^{F-}(m)=\sup\limits_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta_K^{F-}(z)$ and $\vartheta_K^{F-}(n)=\sup\limits_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta_K^{F-}(z)$. Now,

$$\begin{aligned}\min\left[\vartheta_O^{F+}(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^+\right]&=\min\left[\inf_{z'\in\Lambda^{-1}(\Lambda(m)\sqcup_1\Lambda(n))}\vartheta_K^{F+}(z')\ ,\xi^+\right]\\&=\min\left[\inf_{z'\in\Lambda^{-1}(\Lambda(m\vee_1 n)}\vartheta_K^{F+}(z')\ ,\xi^+\right]\\&=\min\left[\vartheta_K^{F+}(m\vee_1 n),\xi^+\right]\\&\le\max\left\{\vartheta_K^{F+}(m),\vartheta_K^{F+}(n),\tau^+\right\}\\&=\max\left\{\vartheta_O^{F+}\Lambda(m),\vartheta_O^{F+}\Lambda(n),\tau^+\right\}.\end{aligned}$$

Similarly, $\max\left[\vartheta_O^{F-}(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^-\right]\ge\min\left\{\vartheta_O^{F-}\Lambda(m),\vartheta_O^{F-}\Lambda(n),\tau^-\right\}$. In the same way, prove the other two operations. Hence O is a (ξ,τ)-BVNSBS of $\mathcal{S}_2$.

Theorem 4.9. *Let $(\mathcal{S}_1,\vee_1,\vee_2,\vee_3)$ and $(\mathcal{S}_2,\sqcup_1,\sqcup_2,\sqcup_3)$ be any two bisemirings. The homomorphic preimage of (ξ,τ)-BVNSBS of $\mathcal{S}_2$ is a (ξ,τ)-BVNSBS of $\mathcal{S}_1$.*

Proof. Let $\Lambda:\mathcal{S}_1\to\mathcal{S}_2$ be any homomrphism. Then $\Lambda(m\vee_1 n)=\Lambda(m)\sqcup_1\Lambda(n),\Lambda(m\vee_2 n)=\Lambda(m)\sqcup_2\Lambda(n)$ and $\Lambda(m\vee_3 n)=\Lambda(m)\sqcup_3\Lambda(n)$ for all $m,n\in\mathcal{S}_1$. Let $O=\Lambda(K)$, where O is any (ξ,τ)-BVNSBS of $\mathcal{S}_2$. Let $m,n\in\mathcal{S}_1$. Then $\max\{\vartheta_K^{T+}(m\vee_1 n),\xi^+\}=\max\{\vartheta_O^{T+}(\Lambda(m\vee_1 n)),\xi^+\}=\max\{\vartheta_O^{T+}(\Lambda(m)\sqcup_1\Lambda(n)),\xi^+\}\ge\min\{\vartheta_O^{T+}\Lambda(m),\vartheta_O^{T+}\Lambda(n),\tau^+\}=\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\}$. Thus, $\max\{\vartheta_K^{T+}(m\vee_1 n),\xi^+\}\ge\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\}$. Also,

$\min\{\vartheta_K^{T-}(m \vee_1 n), \xi^-\} = \min\{\vartheta_O^{T-}(\Lambda(m \vee_1 n)), \xi^-\} = \min\{\vartheta_O^{T-}(\Lambda(m) \sqcup_1 \Lambda(n)), \xi^-\} \leq \max\{\vartheta_O^{T-}\Lambda(m), \vartheta_O^{T-}\Lambda(n), \tau^-\} = \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n), \tau^-\}$. Thus, $\min\{\vartheta_K^{T-}(m \vee_1 n), \xi^-\} \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n), \tau^-\}$. Now, $\max\{\vartheta_K^{I+}(m \vee_1 n), \xi^+\} = \max\{\vartheta_O^{I+}(\Lambda(m \vee_1 n)), \xi^+\} = \max\{\vartheta_O^{I+}(\Lambda(m) \sqcup_1 \Lambda(n)), \xi^+\} \geq \min\{\vartheta_O^{I+}\Lambda(m), \vartheta_O^{I+}\Lambda(n), \tau^+\} = \min\{\vartheta_K^{I+}(m), \vartheta_K^{I+}(n), \tau^+\}$. Thus, $\max\{\vartheta_K^{I+}(m \vee_1 n), \xi^+\} \geq \min\{\vartheta_K^{I+}(m), \vartheta_K^{I+}(n), \tau^+\}$. Also, $\min\{\vartheta_K^{I-}(m \vee_1 n), \xi^-\} = \min\{\vartheta_O^{I-}(\Lambda(m \vee_1 n)), \xi^-\} = \min\{\vartheta_O^{I-}(\Lambda(m) \sqcup_1 \Lambda(n)), \xi^-\} \leq \max\{\vartheta_O^{I-}\Lambda(m), \vartheta_O^{I-}\Lambda(n), \tau^-\} = \max\{\vartheta_K^{I-}(m), \vartheta_K^{I-}(n), \tau^-\}$. Thus, $\min\{\vartheta_K^{I-}(m \vee_1 n), \xi^-\} \leq \max\{\vartheta_K^{I-}(m), \vartheta_K^{I-}(n), \tau^-\}$. Now, $\min\{\vartheta_K^{F+}(m \vee_1 n), \xi^+\} = \min\{\vartheta_O^{F+}(\Lambda(m \vee_1 n)), \xi^+\} = \min\{\vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)), \xi^+\} \leq \max\{\vartheta_O^{F+}\Lambda(m), \vartheta_O^{F+}\Lambda(n), \tau^+\} = \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n), \tau^+\}$. Thus, $\min\{\vartheta_K^{F+}(m \vee_1 n), \xi^+\} \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n), \tau^+\}$. Also, $\max\{\vartheta_K^{F-}(m \vee_1 n), \xi^-\} = \max\{\vartheta_O^{F-}(\Lambda(m \vee_1 n)), \xi^-\} = \max\{\vartheta_O^{F-}(\Lambda(m) \sqcup_1 \Lambda(n)), \xi^-\} \geq \min\{\vartheta_O^{F-}\Lambda(m), \vartheta_O^{F-}\Lambda(n), \tau^-\} = \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n), \tau^-\}$. Thus, $\max\{\vartheta_K^{F-}(m \vee_1 n), \xi^-\} \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n), \tau^-\}$. In the same way, prove the other two operations, hence K is a (ξ, τ)-BVNSBS of $\mathcal{S}_1$.

5. (ξ, τ)-Bipolar Valued Neutrosophic Normal Subbisemiring

In this section, we interact the theory for (ξ, τ)-bipolar valued neutrosophic normal subbisemiring. Here $BVNNSBS$ stands for bipolar valued neutrosophic normal subbisemiring.

Definition 5.1. Let K be any bipolar valued neutrosophic subset of $\mathcal{S}$ is said to be a $BVNNSBS$ of $\mathcal{S}$ if it satisfies the following conditions:

$$\left\{\begin{array}{l} \begin{pmatrix} \vartheta_K^{T+}(m \uplus_1 n) = \vartheta_K^{T+}(n \uplus_1 m), \\ \vartheta_K^{T-}(m \uplus_1 n) = \vartheta_K^{T-}(n \uplus_1 m) \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{T+}(m \uplus_2 n) = \vartheta_K^{T+}(n \uplus_2 m), \\ \vartheta_K^{T-}(m \uplus_2 n) = \vartheta_K^{T-}(n \uplus_2 m) \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{T+}(m \uplus_3 n) = \vartheta_K^{T+}(n \uplus_3 m), \\ \vartheta_K^{T-}(m \uplus_3 n) = \vartheta_K^{T-}(n \uplus_3 m) \end{pmatrix} \end{array}\right\} \quad \left\{\begin{array}{c} \begin{pmatrix} \vartheta_K^{I+}(m \uplus_1 n) = \vartheta_K^{I+}(n \uplus_1 m), \\ \vartheta_K^{I-}(m \uplus_1 n) = \vartheta_K^{I-}(n \uplus_1 m) \end{pmatrix} \\ \text{OR} \\ \begin{pmatrix} \vartheta_K^{I+}(m \uplus_2 n) = \vartheta_K^{I+}(n \uplus_2 m), \\ \vartheta_K^{I-}(m \uplus_2 n) = \vartheta_K^{I-}(n \uplus_2 m) \end{pmatrix} \\ \text{OR} \\ \begin{pmatrix} \vartheta_K^{I+}(m \uplus_3 n) = \vartheta_K^{I+}(n \uplus_3 m), \\ \vartheta_K^{I-}(m \uplus_3 n) = \vartheta_K^{I-}(n \uplus_3 m) \end{pmatrix} \end{array}\right\}$$

$$\left\{\begin{array}{l} \begin{pmatrix} \vartheta_K^{F+}(m \uplus_1 n) = \vartheta_K^{F+}(n \uplus_1 m), \\ \vartheta_K^{F-}(m \uplus_1 n) = \vartheta_K^{F-}(n \uplus_1 m) \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{F+}(m \uplus_2 n) = \vartheta_K^{F+}(n \uplus_2 m), \\ \vartheta_K^{F-}(m \uplus_2 n) = \vartheta_K^{F-}(n \uplus_2 m) \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{F+}(m \uplus_3 n) = \vartheta_K^{F+}(n \uplus_3 m), \\ \vartheta_K^{F-}(m \uplus_3 n) = \vartheta_K^{F-}(n \uplus_3 m) \end{pmatrix} \end{array}\right\}$$

for all $m, n \in \mathcal{S}$.

Theorem 5.2. *(a) The intersection of a family of $BVNNSBS^s$ of $\mathcal{S}$ is a $BVNNSBS$ of $\mathcal{S}$.*
(b) The intersection of a family of $(\xi,\tau)-BVNNSBS^s$ of $\mathcal{S}$ is a $(\xi,\tau)-BVNNSBS$ of $\mathcal{S}$.

Proof. Proof follows from Theorem 3.3 and Theorem 4.3.

Theorem 5.3. *(a) If $K_1, K_2, ..., K_n$ are the family of $BVNNSBS^s$ of $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n$ respectively, then $K_1 \times K_2 \times ... \times K_n$ is a $BVNNSBS$ of $\mathcal{S}_1 \times \mathcal{S}_2 \times ... \times \mathcal{S}_n$.*
(b) If $K_1, K_2, ..., K_n$ are the family of $(\xi,\tau)-BVNNSBS^s$ of $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n$ respectively, then $K_1 \times K_2 \times ... \times K_n$ is a $(\xi,\tau)-BVNNSBS$ of $\mathcal{S}_1 \times \mathcal{S}_2 \times ... \times \mathcal{S}_n$.

Proof. Proof follows from Theorem 3.4 and Theorem 4.4.

Theorem 5.4. *(a) Let K be any $BVNNSBS$ of $\mathcal{S}$ and O be the strongest bipolar valued neutrosophic relation of $\mathcal{S}$. Then K is a $BVNNSBS$ of $\mathcal{S}$ if and only if O is a $BVNNSBS$ of $\mathcal{S} \times \mathcal{S}$.*
(b) Let K be any $(\xi,\tau)-BVNNSBS$ of $\mathcal{S}$ and O be the strongest (ξ,τ) bipolar valued neutrosophic relation of $\mathcal{S}$. Then K is a $(\xi,\tau)-BVNNSBS$ of $\mathcal{S}$ if and only if O is a $(\xi,\tau)-BVNNSBS$ of $\mathcal{S} \times \mathcal{S}$.

Proof. Proof follows from Theorem 3.7.

Theorem 5.5. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings.*
(a) The homomorphic image of any $BVNNSBS$ of $\mathcal{S}_1$ is a $BVNNSBS$ of $\mathcal{S}_2$.
(b) The homomorphic image of any $(\xi,\tau)-BVNNSBS$ of $\mathcal{S}_1$ is a $(\xi,\tau)-BVNNSBS$ of $\mathcal{S}_2$.

Proof. Proof follows from Theorem 3.12 and Theorem 4.8.

Theorem 5.6. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings.*
(a) The homomorphic preimage of any $BVNNSBS$ of $\mathcal{S}_2$ is a $BVNNSBS$ of $\mathcal{S}_1$.
(b) The homomorphic preimage of any $(\xi,\tau)-BVNNSBS$ of $\mathcal{S}_2$ is a $(\xi,\tau)-BVNNSBS$ of $\mathcal{S}_1$.

Proof. Proof follows from Theorem 3.13 and Theorem 4.9.

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Application Of Some Topological Indices In Nover Topologized Graphs

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Abstract: In this paper Nover top Z_1 and Z_2 $index$ and Nover top $H-index$, $Rd-index$, $GA-index$, $CON-index$ are investigated and some of the related theorems are discussed.These indices are also calculated for some specific types of Nover top graphs such as $2-regular$, K_2 and $K_{2,2}$ Nover Top graphs.

Keywords: Nover top Z_1 and Z_2 $index$, $H-index$, $Rd-index$, $GA-index$, $CON-index$

1 Introduction

There are many applications of graph theory to a wide variety of subjects which include operation Research, Physics, chemistry, Economics, Genetics, Engineering, computer Science etc.,In a classical graph for each vertex or edge there are two possibilities arises that is either in the graph or not in the graph and this will not classical graph model for uncertain problems. Fuzzy set [14] is a generalized version of the classical set in which objects have different membership degrees between zero and one. More work has already been done on fuzzy graphs. Zadeh introduced the degree of membership/Truth(T) in 1965 and defined the fuzzy set. Atanassov [2] introduced the degree of non-memebership/Falsehood(F) in 1983 and defined the intuitionstic fuzzy set. Smarandache [13,14,15,16,18,19] introduced the degree of Indeterminancy/Neutrality(I) as an independent component in 1995 and defined the neutrosophic set on there compenents (T,I,F).Smarandache has introduced in 2020 the n-SuperHyperGraph, with super-vertices [that are groups of vertices] and hyper-edges defined on power-set of power-set... that is the most general form of graph as today, and n-HyperAlgebra.A SuperHyperGraph, is a HyperGraph (where a group of Edges form a HyperEdge) such that a group of vertices are united all together into a SuperVertex like a group of people (=vertices) that are united all together into an organization (=SuperVertex) ;and further on the n-SuperHyperGraph where many groups (=SuperVertices) are united all together to form a group-of-groups (called 2-SuperVertex, or Type-2 SuperVertex), then a group of Type-2 SuperVertices forms a Type-3 SuperVertex, . . . , and so on up to Type-n SuperVertex, for any n 1, which better reflects our reality. Later Narmada Devi[5,6,7,8,9,10] worked on new type of neutrosophic over,off graph and minimal domination via neutrosophic over graph and neutrosophic over topologized graph. [20,21,23] A lot of topological indices are available in chemical-graph theory and H. Wiener proposed the first index to estimate the boiling point of alkanes called 'Wiener index'. Many topological indices exist only in the crisp but it's new to the Nover graph environment. The main aim of this paper is to define the topological indices in Nover graphs.The various topology indices such as Zagreb index, Randic index, Geometric-arithmetic, Hormonic are discussed them. Neutrosophic over graphs in addition to the degree of accuracy of each membership function, the degree of its membership is uncertain, as well as its inaccuracy. so in many cases, it may be more logical to use this model than graphs in real-world problems. Since that neutrosophic over graphs are more efficient than fuzzy graphs for modelling real problems. In this paper , we try to calculate some Neutrosophic over topological indices for this type of graphs.

2 Preliminiaries

Definition 2.1. [4] A set $\mathscr{D}$ of vertices of $\mathscr{G}$ is said to be a *topologized domination set* $\mathscr{D}$ if $\mathscr{G}$ is a *topologized graph* and every vertex in $\mathscr{V} - \mathscr{D}$ is adjacent to atleast one vertex of in $\mathscr{D}$.

Definition 2.2. [6,7,9] A Neutrosophic Over set $\mathscr{D}$ is defined as
$\mathscr{D} = (x, <(T(x), I(x), F(x)>), x \in X$ such that there exist some element in $\mathscr{D}$ that have atleast one neutrosophic component that is > 1 and no element has neutrosophic component that are < 0 and $((T(x), I(x), F(x)) \in [0, \Omega]$ where Ω is called Overlimit such that $0 < 1 < \Omega$.

Definition 2.3. [5,6,7] A Nover graph is a pair $\mathscr{G} = (A, B)$ of a crisp graph $\mathscr{G}^* = (\mathscr{V}, \mathscr{E})$ where A is Nvertex over set on $\mathscr{V}$ and B is a Nedge over set on $\mathscr{E}$ such that $\mathscr{T}_{\mathscr{B}}(xy) \leq (T_A(x) \wedge T_A(y))$, $I_B(xy) \leq (I_A(x) \wedge I_A(y))$, $F_B(xy) \geq (F_A(x) \vee F_A(y))$.

Definition 2.4. [9,10] A topologized graph is a topological space $\mathscr{H}$ such that

(i) every singleton is open or closed

(ii) $\forall h \in \mathscr{H}$, $|\eth(h)| \leq 2$, since $\eth(h)$ is denoted by the boundary of a point h.

Definition 2.5. [9] A *Nover graph* $\mathscr{G} = (A, B)$ is called *NOver Top graph* if $\mathscr{G}^*$ satisfy the following condition

(i) every singleton is open or closed in $\mathscr{V}$.

(ii) $\forall f \in \mathscr{F}$, $|\partial(f)| \leq 2$ where $\partial(f)$ is denoted by the boundary of a point x

Definition 2.6. [9] Let $\mathscr{G}$ be a Nover top graph. Let $x, y \in \mathscr{V}$. Then x dominate y in $\mathscr{G}$ if edge xy is effective edge $T_B(xy) = (T_A(x) \wedge T_A(y))$, $I_B(xy) = (I_A(x) \wedge I_A(y))$, $F_B(xy) = (F_A(x) \vee F_A(y))$.
A subset $\mathscr{D}_{\mathscr{N}}$ of $\mathscr{V}$ is called a Nover top dominating set in $\mathscr{G}$ if every vertex $\mathscr{V} \notin \mathscr{D}_{\mathscr{N}}$ there exists $u \in \mathscr{D}_{\mathscr{N}}$ such that u dominates $\mathscr{V}$.

3 Z_1 and Z_2 index in Nover top graphs

Definition 3.1. Let $\mathscr{G} = (A, B)$ be the Nover top graph with non-empty vertex set. The Z_1 index is denoted by $\mathscr{B}_{\text{Nov}}(\mathscr{G})$ and defined as

$$\mathscr{B}_{\text{Nov}}(\mathscr{G}) = \sum_{i=1}^{n} (T_A(u_i), I_A(u_i), F_A(u_i)) d_2(u_i), \forall u_i \in \mathscr{V}$$

Definition 3.2. The Z_2 index is denoted by by $\mathscr{B}^*{}_{\text{Nov}}(\mathscr{G})$ and defined as

$$\mathscr{B}^*{}_{\text{Nov}}(\mathscr{G}) = \frac{1}{2}\sum_{i=1}^{n}\left[(T_A(u_i), I_A(u_i), F_A(u_i))d(u_i)\right]\left[(T_A(v_j), I_A(v_j), F_A(u_i)d(v_j)\right],$$

$\forall i \neq j$, and $(u_i, v_j) \in \mathscr{E}$

Example 3.1. Consider the Nover top graph $\mathscr{G} = (A, B)$ as shown in Figure 1.

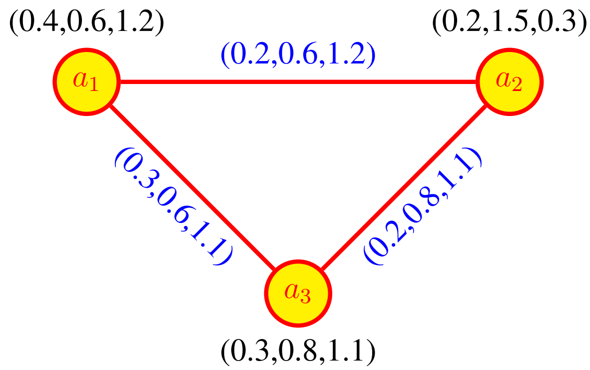


Figure 1: A Nover top graph $\mathscr{G}$

Let a_1, a_2 and a_3 denote the vertices and $(0.2, 0.6, 1.2), (0.2, 0.8, 1.1), (0.3, 0.6, 1.1)$ denote the edges which are labelled $f_u(0.2, 0.6, 1.2) = (a_1, a_2)$, $f_u(0.2, 0.8, 1.1) = (a_2, a_3)$, $f_u(0.3, 0.6, 1.1) = (a_1, a_3)$.

Let $\mathscr{X} = \Big\{a_1, a_2, a_3, (0.2, 0.6, 1.2), (0.2, 0.8, 1.1), (0.3, 0.6, 1.1)\Big\}$ be a topological space defined by the topology

$$\tau = \Big\{\emptyset, \mathscr{X}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\Big\}$$

Here for every $x \in \mathscr{X}$, $\{x\}$ is open.

By the definition of Nover top graph, we have $|\partial(A)| \leq 2$ and $\partial(a_1) = \{a_2, a_3\}$, $\partial(a_2) = \{a_1, a_3\}$, $\partial(a_3) = \{a_1, a_2\}$ with $\partial(a_i) = 2$. Hence this graph is Nover top graph.

The Z_1 index is

$$d(a_1) = (.2 + .3, .6 + .6, 1.2 + 1.1) = (0.5, 1.2, 2.3)$$
$$d(a_2) = (.2 + .2, .6 + .8, 1.2 + 1.1) = (0.4, 1.4, 2.3)$$
$$d(a_3) = (.2 + .3, .8 + .6, 1.1 + 1.1) = (0.5, 1.4, 2.2)$$

Now, we have

$$d_2(a_1) = (0.04 + 0.09, 0.36 + 0.36, 1.44 + 1.21) = (0.13, 0.69, 2.65)$$
$$d_2(a_2) = (0.04 + 0.04, 0.36 + 0.64, 1.44 + 1.21) = (0.08, 1, 2.65)$$
$$d_2(a_3) = (0.04 + 0.09, 0.64 + 0.36, 1.21 + 1.21) = (0.13, 1, 1.42)$$

$$\mathscr{B}_{\text{Nov}}(\mathscr{G}) = \sum_{i=1}^{n}(T_A(u_i), I_A(u_i), F_A(u_i))d_2(u_i)$$

$$= (.4, .6, 1.2)(0.13, 0.69, 2.65) + (.2, 1.5, 0.3)(0.08, 1, 2.65) + (.3, .8, 1.1)(0.13, 1, 1.42)$$
$$= (.052 + 0.414 + 3.18) + (.016 + 1.5 + 0.795) + (0.039 + .8 + 1.562)$$
$$= 8.358$$

Example 3.2. Let $\mathscr{G}$ be a same Nover top graph as defined in example 3.1 Then Z_2 index is

$$\begin{aligned}\mathscr{B}^{*}{}_{\text{Nov}}(\mathscr{G}) &= \frac{1}{2}[(0.4, 0.6, 1.2)(0.5, 1.2, 2.3) \times (0.2, 1.5, 0.3)(0.4, 1.4, 2.3)+\\ &\quad (0.4, 0.6, 1.2)(0.5, 1.2, 2.3) \times (0.3, 0.8, 1.1)(0.5, 1.4, 2.2)+\\ &\quad (0.2, 1.5, 0.3)(0.4, 1.4, 2.3) \times (0.3, 0.8, 1.1)(0.5, 1.4, 2.2)]\\ &= \frac{1}{2}[(0.2 + 0.72 + 2.76) \times (0.08 + 2.1 + 0.69)+\\ &\quad (0.2 + 0.72 + 2.76) \times (0.15 + 1.12 + 2.42)+\\ &\quad (0.08 + 2.1 + 0.69) \times (0.15 + 1.12 + 2.42)]\\ &= \frac{1}{2}[3.68 \times 2.87 + 3.68 \times 3.69 + 2.87 \times 3.69]\\ &= \frac{1}{2}[10.5616 + 13.5792 + 10.5903]\\ &= \frac{1}{2}[34.7311] = 17.3656\end{aligned}$$

Example 3.3. Consider the Nover top graph $\mathscr{G} = (A, B)$ as shown in Fig. 2

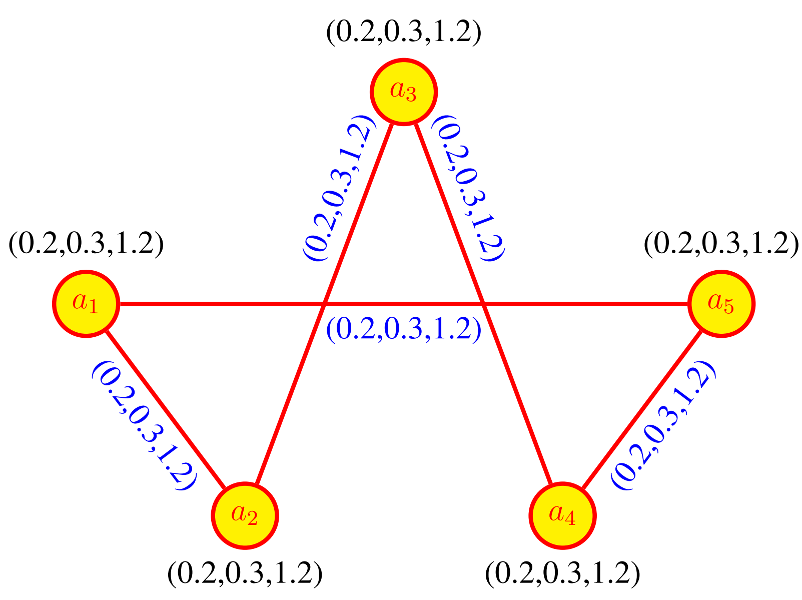


Figure 2: Storng Nover top graph $\mathscr{G}$

Let a_1, a_2, a_3, a_4 and a_5 denote the vertices and $(0.2, 0.3, 1.2)$, $(0.2, 0.3, 1.2)$, $(0.2, 0.3, 1.2)$, $(0.2, 0.3, 1.2)$ and $(0.2, 0.3, 1.2)$ denote the edges which are labelled $f_u(0.2, 0.3, 1.2) = (a_1, a_2)$, $f_u(0.2, 0.3, 1.2) = (a_2, a_3)$, $f_u(0.2, 0.3, 1.2) = (a_3, a_4)$, $f_u(0.2, 0.3, 1.2) = (a_4, a_5)$, $f_u(0.2, 0.3, 1.2) = (a_1, a_5)$.

Let $\mathscr{X} = \Big\{a_1, a_2, a_3, , a_4, a_5, (0.2, 0.3, 1.2), (0.2, 0.3, 1.2), (0.2, 0.3, 1.2), (0.2, 0.3, 1.2), (0.2, 0.3, 1.2)\Big\}$ be a topological space defined by the topology

$$\tau = \Big\{\emptyset, \mathscr{X}, \{a_1\}, \{a_2, a_3\}, \{a_4\}, \{a_5\}, \{a_1, a_2, a_3\}, \{a_1, a_4\}, \{a_1, a_5\}, \{a_1, a_2, a_3, a_4\}, \{a_1, a_2, a_3, a_5\}, \{a_2, a_3, a_4\}, \{a_2, a_3, a_5\}, \{a_2, a_3, a_4, a_5\}, \{a_4, a_5\}, \{a_1, a_4, a_5\}, \{a_1, a_2, a_4, a_5\}, \{a_1, a_3, a_4, a_5\}\Big\}$$

Here for every $x \in \mathscr{X}$, $\{x\}$ is open or closed.

By the definition of Nover top graph, we have $|\partial(A)| \leq 2$ and $\partial(a_1) = \{a_2, a_5\}$, $\partial(a_2) = \{a_1, a_3\}$, $\partial(a_3) = \{a_2, a_4\}$, $\partial(a_4) = \{a_3, a_5\}$, $\partial(a_5) = \{a_1, a_4\}$ with $\partial(a_i) = 2$. Hence this graph is Nover top graph.

The Z_1 index is

$$\begin{aligned}
d(a_1) &= (0.2 + 0.3, 0.3 + 0.3, 1.2 + 1.2) = (0.4, 0.6, 2.4)\\
d(a_2) &= (0.4, 0.6, 2.4)\\
d(a_3) &= (0.4, 0.6, 2.4)\\
d(a_4) &= (0.4, 0.6, 2.4)\\
d(a_5) &= (0.4, 0.6, 2.4)
\end{aligned}$$

Now, we have

$$\begin{aligned}
d_2(a_1) &= (0.04 + 0.04, 0.36 + 0.36, 1.44 + 1.44) = (0.08, 0.72, 2.88)\\
d_2(a_2) &= (0.08, 0.72, 2.88)\\
d_2(a_3) &= (0.08, 0.72, 2.88)\\
d_2(a_4) &= (0.08, 0.72, 2.88)\\
d_2(a_5) &= (0.08, 0.72, 2.88)
\end{aligned}$$

$$\begin{aligned}
\mathscr{B}_{\text{Nov}}(\mathscr{G}) &= \sum_{i=1}^{5} (T_A(u_i), I_A(u_i), F_A(u_i)) d_2(u_i)\\
&= (.2, .3, 1.2)(0.08, 0.72, 2.88) + (.2, .3, 1.2)(0.08, 0.72, 2.88)\\
&\quad + (.2, .3, 1.2)(0.08, 0.72, 2.88) + (.2, .3, 1.2)(0.08, 0.72, 2.88) + (.2, .3, 1.2)(0.08, 0.72, 2.88)\\
&= (.016 + 0.216 + 3.456) + (.016 + 0.216 + 3.456) + (.016 + 0.216 + 3.456)\\
&\quad + (.016 + 0.216 + 3.456) + (.016 + 0.216 + 3.456)\\
&= 3.688 + 3.688 + 3.688 + 3.688 + 3.688\\
&= 18.44
\end{aligned}$$

$$\begin{aligned}
\mathscr{B}^*{}_{\text{Nov}}(\mathscr{G}) = \frac{1}{2}[&(.2, .3, 1.2)(0.4, 0.6, 2.4) \times (.2, .3, 1.2)(0.4, 0.6, 2.4)\\
&+ (.2, .3, 1.2)(0.4, 0.6, 2.4) \times (.2, .3, 1.2)(0.4, 0.6, 2.4)\\
&+ (.2, .3, 1.2)(0.4, 0.6, 2.4) \times (.2, .3, 1.2)(0.4, 0.6, 2.4)\\
&+ (.2, .3, 1.2)(0.4, 0.6, 2.4) \times (.2, .3, 1.2)(0.4, 0.6, 2.4)
\end{aligned}$$

$$
\begin{aligned}
&\quad + (.2, .3, 1.2)(0.4, 0.6, 2.4) \times (.2, .3, 1.2)(0.4, 0.6, 2.4)] \\
&= \frac{1}{2}[(0.08 + 0.18 + 2.88) \times (0.08 + 0.18 + 2.88) \\
&\quad + (0.08 + 0.18 + 2.88) \times (0.08 + 0.18 + 2.88) \\
&\quad + (0.08 + 0.18 + 2.88) \times (0.08 + 0.18 + 2.88) \\
&\quad + (0.08 + 0.18 + 2.88) \times (0.08 + 0.18 + 2.88) \\
&\quad + (0.08 + 0.18 + 2.88) \times (0.08 + 0.18 + 2.88) \\
&= \frac{1}{2}[3.14 \times 3.14 + 3.14 \times 3.14 + 3.14 \times 3.14 + 3.14 \times 3.14 + 3.14 \times 3.14] \\
&= \frac{1}{2}[9.8596 + 9.8596 + 9.8596 + 9.8596 + 9.8596] \\
&= \frac{1}{2}[49.298] = 24.649
\end{aligned}
$$

Definition 3.3. Let $\mathscr{G} = (A, B)$ be an neutrosophic over top graph. $\mathscr{G}$ is a regular strong neutrosophic over top graph if it satisfies the following conditions.

$$T_B(a,b) = \min(T_A(a), T_A(b)), I_B(a,b) = \min(I_A(a), I_A(b)), F_B(a,b) = \max(F_A(a), F_A(b))$$

Theorem 3.1. Let $\mathscr{G}$ be the regular Nover top graph. Then, we have

$$\mathscr{B}_{\text{Nov}}(\mathscr{G}) = c^2 \times \sum_{i=1}^{n}[T_A(u_i) + I_A(u_i)] + c_1^2 \times \sum_{i=1}^{n} F_A(u_i), \forall u_i \in \mathscr{V}$$

where $\sum_{v \neq u} T_B(v,u) = c$, $\sum_{v \neq u} I_B(v,u) = c$, $\sum_{v \neq u} F_B(v,u) = c_1$.

Proof:

Given the degree of definition of each vertex

$$
\begin{aligned}
d(v) &= (d_T(v), d_I(v), d_F(v)) \\
&= \left(\sum_{\substack{v \in \mathscr{V} \\ v \neq u}} T_B(v,u), \sum_{\substack{v \in \mathscr{V} \\ v \neq u}} I_B(v,u), \sum_{\substack{v \in \mathscr{V} \\ v \neq u}} F_B(v,u) \right)
\end{aligned}
$$

On the other hand, for regular Nover top graphs, we know that

$$\sum_{v \neq u} T_B(v,u) = c, \sum_{v \neq u} I_B(v,u) = c, \sum_{v \neq u} F_B(v,u) = c_1$$

Therefore,

$$
\begin{aligned}
d(v) &= (d_T(v), d_I(v), d_F(v)) \\
&= (c, c, c_1)
\end{aligned}
$$

Now, by embedding the formula in the Z_1 index, we will get the desired result. The proof is complete.

Theorem 3.2. Let $\mathscr{G}$ be the regular Nover top graph. Then, we have

$$\mathscr{B}^*{}_{\text{Nov}}(\mathscr{G}) = \frac{1}{2}c^2\sum_{i=1}^{n}[T_A(u_i)+I_A(u_i)][T_A(v_j)+I_A(v_j)] + \frac{1}{2}c_1^2\sum_{i=1}^{n}[F_A(u_i)F_A(v_j)],$$

$\forall i \neq j$ and $(u_i, v_j) \in \mathscr{E}$
where $\sum\limits_{v\neq u} T_B(u,v) = c$, $\sum\limits_{v\neq u} I_B(u,v) = c$, $\sum\limits_{v\neq u} F_B(u,v) = c_1$.

Proof: Assume $\mathscr{G}$ is regular Nover top graph, using the Z_2 index formula for $\mathscr{G}$, we have $\forall i \neq j$ and $(u_i, v_j) \in \mathscr{E}$.

$$\begin{aligned}
\mathscr{B}^*{}_{\text{Nov}}(\mathscr{G}) &= \frac{1}{2}\sum_{i=1}^{n}\left[(T_A(u_i), I_A(u_i), I_A(u_i))d(u_i)\right]\left[(T_A(v_j), I_A(v_j), F_A(v_J)d(v_j)\right] \\
&= \frac{1}{2}\sum\left[(T_A(u_i), I_A(u_i), I_A(u_i))d(d_T(u_i), d_I(u_i), d_F(u_i)\right] \\
&\qquad \left[(T_A(v_j), I_A(v_j), F_A(v_J)d(d_T(v_j), d_I(v_j), d_F(v_j)\right] \\
&= \frac{1}{2}\sum\left[(T_A(u_i), I_A(u_i), I_A(u_i))(c, c, c_1)\right]\left[(T_A(v_j), I_A(v_j), F_A(v_J)(c, c, c_1)\right] \\
&= \frac{1}{2}\sum\left[cT_A(u_i) + cI_A(u_i) + c_1 I_A(u_i)\right] \times \left[cT_A(v_j) + cI_A(v_j) + c_1 F_A(v_j)\right] \\
&= \frac{1}{2}\sum c\left[T_A(u_i) + I_A(u_i)\right] c_1[I_A(u_i)]c\left[T_A(v_j) + I_A(v_j)\right] c_1[F_A(v_j)] \\
&= \frac{1}{2}c^2\sum\left[T_A(u_i) + I_A(u_i)\right]\left[T_A(v_j) + I_A(v_j)\right] + \frac{1}{2}c_1^2\sum[I_A(u_i)F_A(v_j)]
\end{aligned}$$

The desired result was obtained.
These above two theorems are illustrated the following example.

Example 3.4. Consider the Nover top graph $G = (A, B)$ as shown in Fig

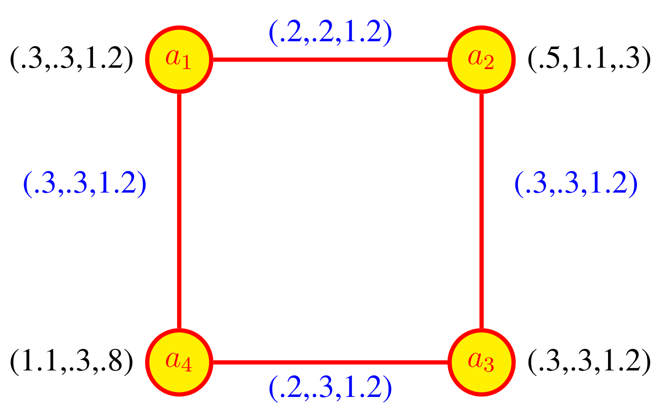


Figure 3: Regular Nover top graph

Let a_1, a_2, a_3 and a_4 denote the vertices and $(.2, .2, 1.2)$, $(.3, .3, 1.2)$, $(.2, .2, 1.2)$ and $(.3, .3, 1.2)$ denote the edges which are labelled $f_u(.2, .2, 1.2) = (a_1, a_2)$, $f_u(.3, .3, 1.2) = (a_2, a_3)$, $f_u(.2, .2, 1.2) = (a_3, a_4)$, $f_u(.3, .3, 1.2) = (a_4, a_1)$.

Let $X = \Big\{a_1, a_2, a_3, , a_4, (.2, .2, 1.2), (.3, .3, 1.2), (.2, .2, 1.2), (.3, .3, 1.2)\Big\}$ be a topological space defined by the topology

$$\tau = \Big\{\emptyset, \mathscr{X}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_1, a_2\}, \{a_1, a_4\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_3, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_2, a_3, a_4\}, \{a_1, a_3, a_4\}\Big\}$$

Here for every $x \in \mathscr{X}$, $\{x\}$ is open.

By the definition of Nover top graph, we have $|\partial(A)| \leq 2$ and $\partial(a_1) = \{a_2, a_4\}$, $\partial(a_2) = \{a_1, a_3\}$, $\partial(a_3) = \{a_2, a_4\}$, $\partial(a_4) = \{a_1, a_3\}$ with $\partial(a_i) = 2$. Hence this graph is Nover top graph.

The Z_1 index is

$$\begin{aligned} d(a_1) &= (0.5, 0.5, 2.4) \\ d(a_2) &= (0.5, 0.5, 2.4) \\ d(a_3) &= (0.5, 0.5, 2.4) \\ d(a_4) &= (0.5, 0.5, 2.4) \end{aligned}$$

$$\begin{aligned} \mathscr{B}_{\text{Nov}}(\mathscr{G}) &= c^2 \sum_{i=1}^{n} [T_A(u_i) + I_A(u_i)] + c_1^2 \sum_{i=1}^{n} [I_A(u_i)] \\ &= (.5)^2[(.3 + .3) + (.5 + 1.1) + (.3 + .3) + (1.1 + .3)] + (2.4)^2[1.2 + .3 + 1.2 + .8] \\ &= (.5)^2[.6 + 1.6 + .6 + 1.4] + (2.4)^2[3.5] \\ &= (.5)^2 \times 4.2 + (2.4)^2 \times 3.5 \\ &= 1.05 + 20.16 \\ &= 21.21 \end{aligned}$$

The Z_2 index is

$$\begin{aligned} \mathscr{B}^*{}_{\text{Nov}}(\mathscr{G}) &= \frac{1}{2}(.5)^2[.6 \times 1.6 + .6 \times 1.4 + 1.6 \times .6 + .6 \times 1.4] + \\ &\quad \frac{1}{2}(2.4)^2[1.2 \times .3 + 1.2 \times .8 + .3 \times 1.2 + 1.2 \times .8] \\ &= \frac{1}{2}(.5)^2(3.6) + \frac{1}{2}(2.4)^2(2.64) \\ &= \frac{1}{2} \times 0.9 + \frac{1}{2} \times 15.2064 \\ &= 0.45 + 7.6034 \\ &= 8.0532 \end{aligned}$$

4 H-index and Rd-index in Nover top graphs

Definition 4.1. The H-index of Nover top graph $\mathscr{G}$ is defined as

$$\text{H}_{\text{Nov}}(\mathscr{G}) = \sum \frac{1}{[A(u_i) \cdot d(u_i)][A(v_j) \cdot d(v_j)]}, u_i, v_j \in \mathscr{E}, i \neq j$$

Definition 4.2. R-index in Nover top graph $\mathscr{G}$ is defined as

$$\text{Rd}_{\text{Nov}}(\mathscr{G}) = \frac{1}{\sum\{[A(u_i)d(u_i)][A(v_j)d(v_j)]\}^{\frac{1}{2}}}, u_i, v_j \in \mathscr{E}, \forall i \neq j$$

Example 4.1. For example (3.1), the H-index of Nover top graph $\mathscr{G}$ is

$$\begin{aligned}
\text{H}_{\text{Nov}}(\mathscr{G}) = & \frac{1}{(.4, .6, 1.2)(.5, 1.2, 2.3) + (.2, 1.5, .3)(.4, 1.4, 2.3)} + \\
& \frac{1}{(.4, .6, 1.2)(.5, 1.2, 2.3) + (.3, .8, 1.1)(.5, 1.4, 2.2)} + \\
& \frac{1}{(.2, 1.5, .3)(.4, 1.4, 2.3) + (.3, .8, 1.1)(.5, 1.4, 2.2)} \\
& = \frac{1}{3.68 + 2.87} + \frac{1}{3.68 + 3.69} + \frac{1}{2.87 + 3.69} \\
& = \frac{1}{6.55} + \frac{1}{7.37} + \frac{1}{6.56} \\
& = 0.153 + 0.136 + 0.152 \\
& = 0.441
\end{aligned}$$

Example 4.2. For example (3.1), the Rd-index of Nover top graph $\mathscr{G}$ is

$$\begin{aligned}
\text{Rd}_{\text{Nov}}(\mathscr{G}) = & \frac{1}{\sqrt{(.4..6, 1.2)(.5, 1.2, 2.3) \times (.2, 1.5, .3)(.4, 1.4, 2.3)}} + \\
& \frac{1}{\sqrt{(.4..6, 1.2)(.5, 1.2, 2.3) \times (.3, .8, 1.1)(.5, 1.4, 2.2)}} + \\
& \frac{1}{\sqrt{(.2.1.5, .3)(.4, 1.4, 2.3) \times (.3, .8, 1.1)(.5, 1.4, 2.2)}} \\
& = \frac{1}{\sqrt{10.566}} + \frac{1}{\sqrt{13.5792}} + \frac{1}{\sqrt{10.5903}} \\
& = \frac{1}{3.24986} + \frac{1}{3.68499} + \frac{1}{3.2543} \\
& = 0.0300 + 0.27137 + 0.3073 \\
& = 0.6086
\end{aligned}$$

Definition 4.3. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be any neutrosophic over graphs isomorphism $f : \mathscr{G}_1 \to \mathscr{G}_2$ is bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions

(a) $T_{A_1}(x_1) = T_{A_2}(f(x_1))$, $I_{A_1}(x_1) = I_{A_2}(f(x_1))$ and $F_{A_1}(x_1) = F_{A_2}(f(x_1))$

(b) $T_{B_1}(x_1, y_1) = T_{B_2}(f(x_1), f(y_1))$, $I_{B_1}(x_1, y_1) = I_{B_2}(f(x_1), f(y_1))$ and $F_{B_1}(x_1, y_1) = F_{B_2}(f(x_1), f(y_1))$ for all $x_1 \in V_1$, $x_1, y_1 \in E_1$

Example 4.3.

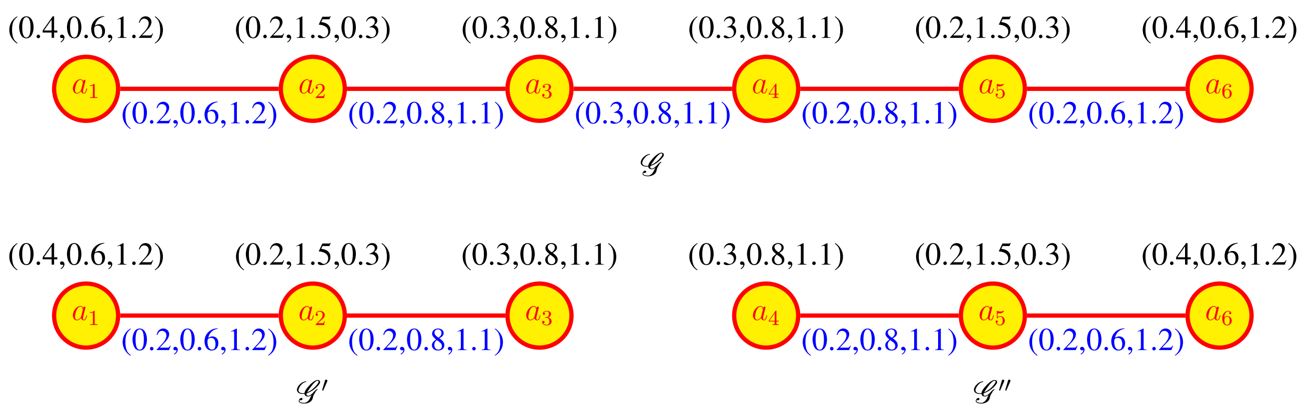


Figure 4: $\mathscr{G}'$ and $\mathscr{G}$" are isomorphic graphs

Theorem 4.1. Let $\mathscr{G}$ be the connected Nover top $T(u,v)$, $I(u,v)$, $F(u,v)$ graph and (T,I,F) be the true membership, indeterminancy membership and falsity membership value of the chosen edge of $\mathscr{G}$ such that removal of (μ,γ,σ) from $\mathscr{G}$ splits into two Nover top graphs such that whose vertex set satisfies $|V_{\mathscr{G}'}| < |V_{\mathscr{G}''}|$ and therefore its disconnected. Then

(i) $\mathscr{B}_{Nov}(\mathscr{G}') < \mathscr{B}_{Nov}(\mathscr{G}'')$

(ii) ${\mathscr{B}^*}_{Nov}(\mathscr{G}') < {\mathscr{B}^*}_{Nov}(\mathscr{G}'')$

(iii) $H_{Nov}(\mathscr{G}') < H_{Nov}(\mathscr{G}'')$

(iv) $Rd_{Nov}(\mathscr{G}') < Rd_{Nov}(\mathscr{G}'')$

where $|V_{\mathscr{G}'}| < |V_{\mathscr{G}''}|$ denote the cardinality of $\mathscr{G}'$ & $\mathscr{G}''$ respectively.

Proof:

Let $\mathscr{G}$ be a connected Nover top graph where splitted into two Nover top graph $\mathscr{G}'$ and $\mathscr{G}''$ by removing the chosen membership value of the edge in $\mathscr{G}$. We know that $\mathscr{G}$ be the Nover top graph and H be the Nover top sub graph of $\mathscr{G}$ such that $H = \mathscr{G} - u$ then $\mathscr{B}_{Nov}(\mathscr{G} - u) < \mathscr{B}_{Nov}(\mathscr{G})$ and ${\mathscr{B}^*}_{Nov}(\mathscr{G} - u) < {\mathscr{B}^*}_{Nov}(\mathscr{G})$.

Therefore, we get, $\mathscr{B}_{Nov}(\mathscr{G}') < \mathscr{B}_{Nov}(\mathscr{G}'') < \mathscr{B}_{Nov}(\mathscr{G})$ which implies that $\mathscr{B}_{Nov}(\mathscr{G}') < \mathscr{B}_{Nov}(\mathscr{G}'')$.

Hence the theorem proved.

The remaining cases are trivially true by the following above method.

5 GA- index in Nover top graphs

Definition 5.1. The GA-index of Nover top graph $\mathscr{G}$ is defined as

$$\mathrm{GA}_{\mathrm{Nov}}(\mathscr{G}) = \frac{2\{[A(u_i)d(u_i)][A(v_j)d(v_j)]\}^{\frac{1}{2}}}{[A(u_i)d(v_j) + A(v_j)d(v_j)]^{\frac{1}{2}}}$$

Theorem 5.1. Let $\mathscr{G}$ be a Nover top graph with m edges. Then $\mathrm{GA}_{\mathrm{Nov}}(\mathscr{G}) \leq m$ with equality if and only if every component of $\mathscr{G}$ is regular.

Example 5.1. We have the previous example (3.1)

$$\mathrm{GA}_{\mathrm{Nov}}(\mathscr{G}) = \frac{2\begin{bmatrix} (.4,.6,1.2)(.5,1.2,2.3)+(.2,1.5,.3)(.4,1.4,2.3)+ \\ (.4,.6,1.2)(.5,1.2,2.3)+(.3,.8,1.1)(.5,1.4,2.3)+ \\ (.2,1.5,.3)(.4,1.4,2.3)+(.3,.8,1.1)(.5,1.4,2.2) \end{bmatrix}^{\frac{1}{2}}}{\begin{bmatrix} (.4,.6,1.2)+(.5,1.2,2.3)+(.2,1.5,.3)+(.4,1.4,2.3)+ \\ (.4,.6,1.2)+(.5,1.2,2.3)+(.3,.8,1.1)+(.5,1.4,2.2)+ \\ (.2,1.5,.3)+(.4,1.4,2.3)+(.3,.8,1.1)+(.5,1.4,2.2) \end{bmatrix}^{\frac{1}{2}}}$$

$$= \frac{2[(3.68\times 2.87)+(3.68\times 3.69)+(2.87\times 3.69)]^{\frac{1}{2}}}{[2.2+4+2+4.1+2.2+4+2.2+4.1+2+4.1+2.2+4.1]^{\frac{1}{2}}}$$

$$= \frac{2[10.562+13.579+10.5903]^{\frac{1}{2}}}{[37.2]^{\frac{1}{2}}}$$

$$= \frac{2[34.7313]^{\frac{1}{2}}}{6.099}$$

$$= \frac{2\times 5.8933}{6.099}$$

$$= 1.9325$$

6 Connectivity Index in Nover top graphs

Definition 6.1. The strength of connectedness between u_i and v_j is defined as

$$\mathrm{CONN}_P(u_i,v_j) = \left(\min_{e\in P_{u_iv_j}} T_B(e), \min_{e\in P_{u_iv_j}} I_B(e), \max_{e\in P_{u_iv_j}} F_B(e)\right)$$

where $P_{u_iv_j}$ is the path between u_i and v_j

$$|\mathrm{CONN}_P(u_i,v_j)| = 2\left(\min_{e\in P_{u_iv_j}} T_B(e)\right) - \left(\min_{e\in P_{u_iv_j}} I_B(e)\right) - \left(\max_{e\in P_{u_iv_j}} F_B(e)\right)$$

Then $\mathrm{CONN}_P(u_i,v_j) = \max_p\{|\mathrm{CONN}_P(u_i,v_j)|\}$.

Definition 6.2. The Connectivity index (CI) of $\mathscr{G}$ is defined by

$$\mathrm{CI}_{\mathrm{Nov}}(\mathscr{G}) = \sum_{u_iv_j\in\mathscr{V}} A(u_i)\cdot A(v_j)\times \mathrm{CONN}_{\mathscr{G}}(u_i,v_j)$$

Here $\mathrm{CONN}_{\mathscr{G}}(u_i,v_j)$ is the strength of connectedness between u_i and v_j.

Example 6.1. For example (3.1), the strength of connectedness between a_1 and a_2 from the direct path $p_1 = a_1a_2$ is $\mathrm{CONN}_{p_1}(a_1,a_2) = (0.2, 0.6, 1.2)$.

From path $p_2 = a_1a_3a_2$

$$\mathrm{CONN}_{p_2}(a_1,a_2) = (\min\{0.3,0.2\}, \min\{0.6,0.8\}, \max\{1.1,1.1\})$$

$$= (0.2, 0.6, 1.1)$$

a_1 and a_3 from the direct path $p_1 = a_1a_3$ is

$$\text{CONN}_{p_1}(a_1, a_3) = (0.3, 0.6, 1.1)$$

From path $p_2 = a_1a_2a_3$

$$\begin{aligned}\text{CONN}_{p_2}(a_1, a_3) &= (\min\{0.2, 0.2\}, \min\{0.6, 0.8\}, \max\{1.2, 1.1\}) \\ &= (0.2, 0.6, 1.2)\end{aligned}$$

a_2 and a_3 from the direct path $p_1 = a_2a_3$ is

$$\text{CONN}_{p_1}(a_2, a_3) = (0.2, 0.8, 1.1)$$

From path $p_2 = a_2a_1a_3$

$$\begin{aligned}\text{CONN}_{p_2}(a_2, a_3) &= (\min\{0.2, 0.3\}, \min\{0.6, 0.6\}, \max\{1.2, 1.1\}) \\ &= (0.2, 0.6, 1.2)\end{aligned}$$

Then, we have for a_1 and a_2

$$\begin{aligned}|\text{CONN}_{p_1}(a_1, a_2)| &= 2 \times (0.2) - 0.6 - 1.2 = -1.4 \\ |\text{CONN}_{p_2}(a_1, a_2)| &= 2 \times (0.2) - 0.6 - 1.1 = -1.3\end{aligned}$$

For a_1 and a_3

$$\begin{aligned}|\text{CONN}_{p_1}(a_1, a_3)| &= 2 \times (0.3) - 0.6 - 1.1 = -1.3 \\ |\text{CONN}_{p_2}(a_1, a_3)| &= 2 \times (0.2) - 0.6 - 1.2 = -1.4\end{aligned}$$

For a_2 and a_3

$$\begin{aligned}|\text{CONN}_{p_1}(a_2, a_3)| &= 2 \times (0.2) - 0.8 - 1.1 = -1.5 \\ |\text{CONN}_{p_2}(a_2, a_3)| &= 2 \times (0.2) - 0.6 - 1.2 = -1.4\end{aligned}$$

Since we have

$$\begin{aligned}\text{CONN}_{\mathscr{G}}(a_1, a_2) &= -1.3 \\ \text{CONN}_{\mathscr{G}}(a_1, a_3) &= -1.3 \\ \text{CONN}_{\mathscr{G}}(a_2, a_3) &= -1.4\end{aligned}$$

Then $\text{CI}_{\text{Nov}}(\mathscr{G})$ is calculated as follows.

$$\begin{aligned}\text{CI}_{\text{Nov}}(\mathscr{G}) &= \sum_{u_iv_j \in \mathscr{V}} (T_A(u_i), I_A(u_i), F_A(u_i))(T_A(v_j), I_A(v_j), F_A(v_J) \times \text{CONN}_{\mathscr{G}}(u_i, v_j) \\ &= (.4, .6, 1.2)(.2, 1.5, .3) \times (-1.3) + (.4, .6, 1.2)(.3, .8, 1.1) \times (-1.3) + \\ &\quad (.2, 1.5, .3)(.3, .8, 1.1) \times (-1.4)\end{aligned}$$

$$= (.08 + .9 + .36)(-1.3) + (.12 + .48 + 1.32)(-1.3) + (.06 + 1.2 + .33)(-1.4)$$
$$= (1.34)(-1.3) + (1.92)(-1.3) + (1.59)(-1.4)$$
$$= -1.742 - 2.496 - 2.226$$
$$= -6.464$$

Then CI of $\mathscr{G}$ is equal -6.464, which the negative sign indicates the high level of false and indeterminacy information in the problem.

Theorem 6.1. Let $\mathscr{G}$ and $\mathscr{G}_1$ be the two Nover top graphs are isomorphic, then the topological indices values of two Nover top graphs are equal.

Proof: Let $\mathscr{G} = (\mathscr{V}_{\mathscr{G}}, A_{\mathscr{G}}, B_{\mathscr{G}})$ and $\mathscr{G}_1 = (\mathscr{V}_{\mathscr{G}_1}, A_{\mathscr{G}_1}, B_{\mathscr{G}_1})$ be isomorphic Nover top graphs.

Hence there is an identity function

$$\mu_A : A_{\mathscr{G}}(u) \to A_{\mathscr{G}_1}(u^*) \text{ for all } u \in \mathscr{V}_{\mathscr{G}}, \exists u^* \in \mathscr{V}_{\mathscr{G}_1}$$

as well as

$$\mu_B : B_{\mathscr{G}}(u, v) \to B_{\mathscr{G}_1}(u^*, v^*),$$

then each vertex of $\mathscr{G}$ corresponds to an vertex in $\mathscr{G}_1$, with the same membership value and the same edges.

Hence, the Neutrosophic over top graph structure may differ but collection of vertices and edges are same gives the equal topological indices value.

Theorem 6.2. Let $G = (\mathscr{V}_G, A_G, B_G)$ is a Nover top graph and H is the NOver top subgraph of G, such that H is made by removing edge $uv \in B_G$ from G. Then, we have $\text{CI}_{\text{Nov}(H)} < \text{CI}_{\text{Nov}(\mathscr{G})}$ iff uv is a bridge.

Proof: Now suppose that uv is an edge that has maximum (or) minimum components, so they will have an effect on $\text{CONN}_{\mathscr{G}}(u, v)$.

Therefore, by removing edge uv, the value of $\text{CONN}_{\mathscr{G}}(u, v)$ will decrease, then we have $\text{CI}_{\text{Nov}(H)} < \text{CI}_{\text{Nov}(\mathscr{G})}$.

Since the bridge is called the edge that has its deletion reducing the $\text{CONN}_{\mathscr{G}}(u, v)$, however, uv is a bridge.

Conversely, given that uv is a bridge. By the definition of bridge we have, for the edge uv, $\text{CONN}_{\mathscr{G}}(u, v) > \text{CONN}_{G-uv}(u, v)$, so we conclude that, $\text{CI}_{\text{Nov}(H)} < \text{CI}_{\text{Nov}(\mathscr{G})}$.

Example 6.2.

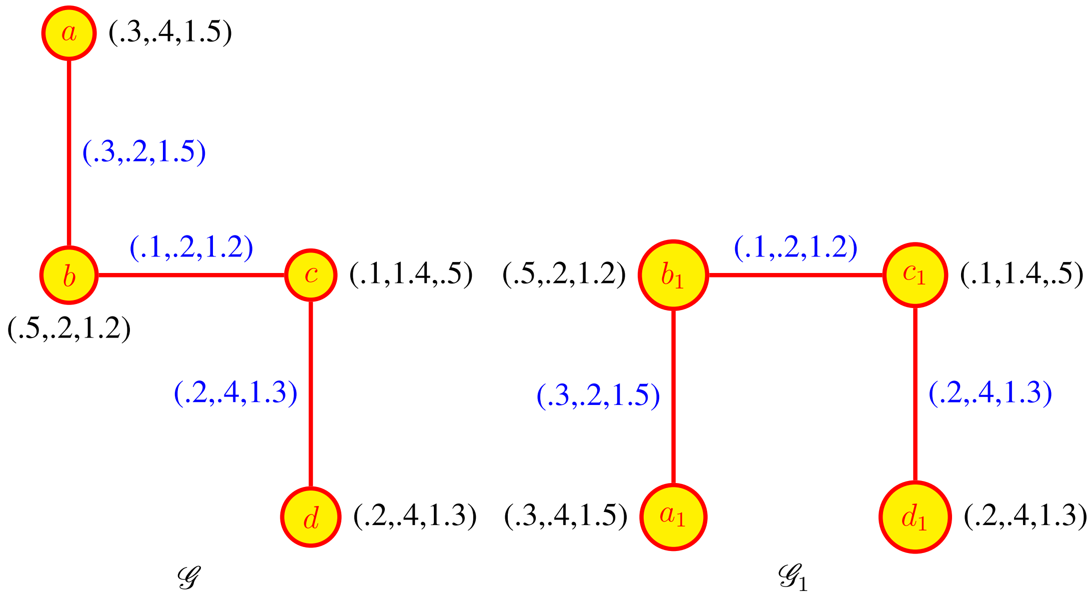
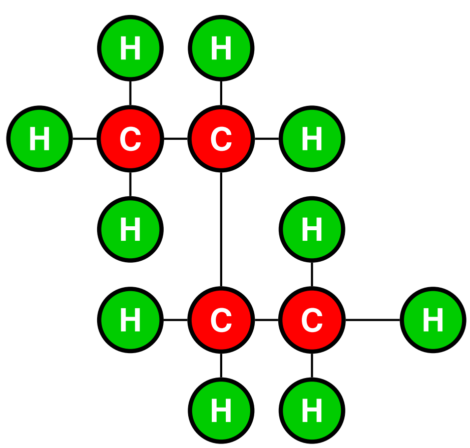


Figure 5: $\mathscr{G}$ and $\mathscr{G}_1$ are isomorphic Nover top graphs

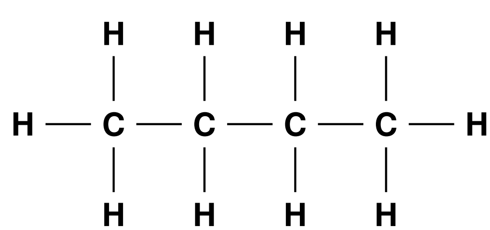
7 Illustration

Wiener was introduced two parameters for the specific purpose of correlating the boiling points of members of the alkane series of molecular structure which satisfied a linear formula $t_B = aW + bP + c$ where t_B is the boiling point of a given alkane,W is the wiener number, P is the polarity number and a, b, c are constants. A topological representing of a molecule structure is called molecule graph which is a collection of points representing the atoms in the molecule and set of lines representing the covalent bonds. A hydrogen-detected graph is a molecular graph in which hydrogen atoms are not considered.

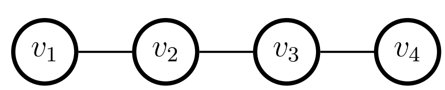
Example 7.1. Consider an alkane series butane where molecular and its hydrogen detected graph are given there.



Butane Structure



Butane



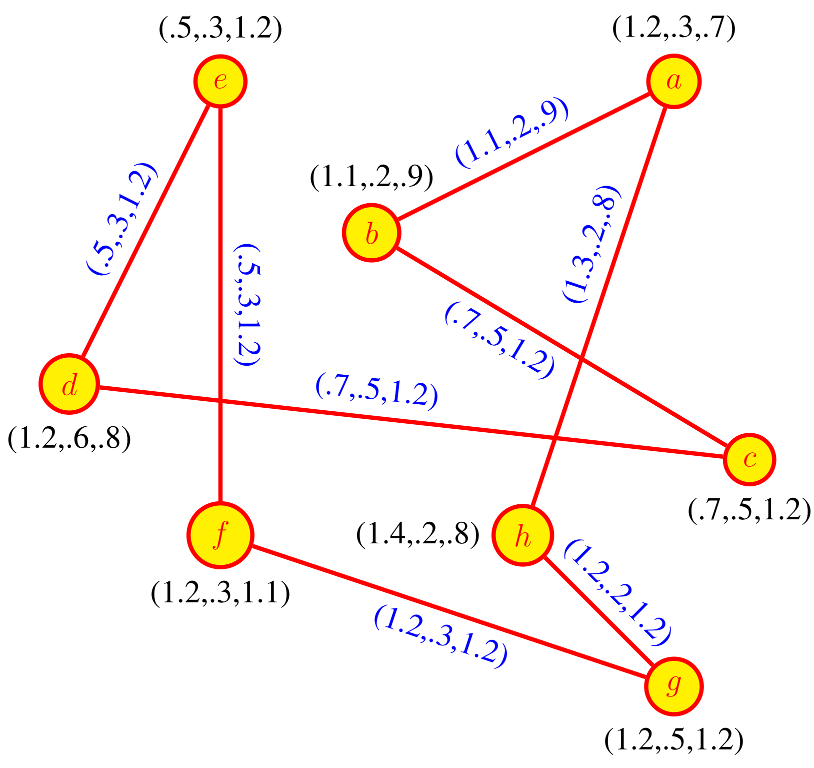
The path graph P_4

Suppose we can consider hydrogen detected graph P_4 as in Nover top graph structure, we can calculate the Nover top Z_1 index for P_4. Hence the boiling point of butane in Nover top graph is satisfies linear equation as $t_B = aW + bP + c$. Since butane is non-polar, $b = 0$. so,$t_B = aW + c$.If we can take for particular value of a, b, c and get the boiling point of butane both of them is same.

8 Application

Most of the administration in the society, they depends the supporters than on themself. The application of dominating set and connecting index of Nover Top graph is expressed as a following example.An office decide to appoint a head under 8 employees.The employess are assumed to be a,b,c,d,e,f,g,h. In this Nover Top graph, the employees and strength between them are considered as vertex and edges.The true membership,inderterminancy and false membership is taken as a vertex.The true membership value is based on employees talent, work experience and salary basis.The inderterminancy membership function are considered as the persons with ability and skill but do not work in the suitable task.The false membership function is considred as be the lack of compatibility between educational major and occupation, lack of ability, non-skill and health issue.

Consider the following graphical structure



Let a, b, c, d, e, f, g and h denote the vertices and $(1.1, 0.2, 0.9)$, $(0.7, 0.5, 1.2)$, $(0.5, 0.3, 1.2)$, $(0.5, 0.3, 1.2)$,$(1.2, 0.3, 1.2)$,$(1.2, 0.2, 0.8)$, and $(1.3, 0.2, 0.8)$ denote the edges and there is a strong relationship between them.

Let $\mathscr{V} = \{a, b, c, d, e, f, g, h, (1.1, 0.2, 0.9), (0.7, 0.5, 1.2), (0.5, 0.3, 1.2), (0.5, 0.3, 1.2), (1.2, 0.3, 1.2), (1.2, 0.2, 0.8), (1.3, 0.2, 0.8)\}$ be a topology

$\tau = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{a,b\}, \{a,d\}\{b,c\}, \{e,f\}, \{f,g\}, \{g,h\}....$
$\{a,b,c,d\}, \{a,b,d,f\}, \{e,f,g,h\}, ...\{a,b,c,d,e,f,g\}, \}$.

Here for every $x \in X$, $\{x\}$ is open or closed. By the definition of Nover top graph, we have $|\partial(A)| \leq 2$ and $\partial(a) = (b,h)$, $\partial(b) = (a,c)$, $\partial(c) = (b,d)$, $\partial(d) = (e,c)$, $\partial(e) = (d,f)$, $\partial(f) = (e,g)$, $\partial(g) = (f,h)$, $\partial(h) = (a,g)$ with $\partial(a_i) = 2$. Hence this graph is Nover top graph.

$$T_A(a) = \text{min}[T_B(a,b), T_B(a,h)] = \text{min}[1.1, 1.1] = 1.1$$
$$I_A(a) = \text{min}[I_B(a,b), I_B(a,h)] = \text{min}[0.2, 0.2] = 0.2$$
$$F_A(a) = \text{max}[F_B(a,b), F_B(a,h5)] = \text{max}[0.9, 0.8] = 0.9$$
$$T_A(b) = \text{min}[T_B(b,a), T_B(b,c)] = \text{min}[1.1, 0.7] = 0.7$$
$$I_A(b) = \text{min}[I_B(b,a), I_B(b,c)] = \text{min}[0.2, 0.5] = 0.2$$
$$F_A(b) = \text{max}[F_B(b,a), F_B(b,c)] = \text{max}[0.9, 1.2] = 1.2$$
$$T_A(c) = \text{min}[T_B(c,b), T_B(c,d)] = \text{min}[0.7, 0.7] = 0.7$$
$$I_A(c) = \text{min}[I_B(c,b), I_B(c,d)] = \text{min}[0.2, 0.5] = 0.2$$
$$F_A(c) = \text{max}[F_B(c,b), F_B(c,d)] = \text{max}[1.3, 1.3] = 1.3$$
$$T_A(d) = \text{min}[T_B(d,c), T_B(d,e)] = \text{min}[0.1, 0.1] = 0.1$$
$$I_A(d) = \text{min}[I_B(d,c), T_B(d,e)] = \text{min}[0.7, 0.7] = 0.7$$
$$F_A(d) = \text{max}[F_B(d,c), T_B(d,e)] = \text{max}[1.2, 1.2] = 1.2$$
$$T_A(e) = \text{min}[T_B(e,d), T_B(e,f)] = \text{min}[0.5, 0.5] = 0.5$$
$$I_A(e) = \text{min}[I_B(e,d), I_B(e,f)] = \text{min}[0.3, 0.3] = 0.3$$
$$F_A(e) = \text{max}[F_B(e,d), F_B(e,f)] = \text{max}[1.2, 1.2] = 1.2$$
$$T_A(f) = \text{min}[T_B(f,e), T_B(f,g)] = \text{min}[0.5, 1.2] = 0.5$$
$$I_A(f) = \text{min}[I_B(f,e), I_B(f,g)] = \text{min}[0.3, 0.3] = 0.3$$
$$F_A(f) = \text{max}[F_B(f,e), F_B(f,g)] = \text{max}[1.3, 1.3] = 1.3$$
$$T_A(g) = \text{min}[T_B(g,h), T_B(g,f)] = \text{min}[1.2, 1.2] = 1.2$$
$$I_A(g) = \text{min}[I_B(g,h), I_B(g,f)] = \text{min}[0.2, 0.3] = 0.2$$
$$F_A(g) = \text{max}[F_B(g,h), F_B(g,f)] = \text{max}[1.2, 1.2] = 1.2$$
$$T_A(h) = \text{min}[T_B(h,a), T_B(h,g)] = \text{min}[1.3, 1.2] = 0.1$$
$$I_A(h) = \text{min}[I_B((h,a), I_B(h,g)] = \text{min}[0.2, 0.2] = 0.2$$
$$F_A(h) = \text{max}[F_B((h,a), F_B(h,g)] = \text{max}[0.8, 1.2] = 1.2$$

Here a dominates b because

$$T_B(ab) \leq T_A(a) \wedge T_A(b), 1.1 \leq 1.1 \wedge 0.7$$
$$I_B(ab) \leq I_B(a) \wedge T_B(b), 0.2 \leq 0.2 \wedge 0.2$$
$$F_B(ab) \geq F_A(a) \vee T_B(b), 0.9 \geq 0.9 \vee 0.9$$

Here b dominates c because

$$T_B(bc) \leq T_A(b) \wedge T_A(c), 0.7 \leq 0.7 \wedge 0.7$$
$$I_B(bc) \leq I_B(b) \wedge T_B(c), 0.2 \leq 0.2 \wedge 0.2$$

$$F_B(bc) \geq F_A(b) \vee T_B(c), 1.2 \geq 0.9 \vee 1.2$$

Here c dominates d because

$$T_B(cd) \leq T_A(c) \wedge T_A(d), 0.7 \leq 0.7 \wedge 0.5$$
$$I_B(cd) \leq I_B(c) \wedge T_B(d), 0.5 \leq 0.2 \wedge 0.3$$
$$F_B(cd) \geq F_A(c) \vee T_B(d), 1.2 \geq 1.2 \vee 1.2$$

Here d dominates e because

$$T_B(de) \leq T_A(d) \wedge T_A(e), 0.5 \leq 0.5 \wedge 0.0.5$$
$$I_B(de) \leq I_B(d) \wedge T_B(e), 0.3 \leq 0.3 \wedge 0.3$$
$$F_B(de) \geq F_A(d) \vee T_B(e), 1.2 \geq 1.2 \vee 1.2$$

Here e dominates f because

$$T_B(ef) \leq T_A(e) \wedge T_A(f), 0.5 \leq 0.5 \wedge 0.5$$
$$I_B(ef) \leq I_B(e) \wedge I_B(f), 0.3 \leq 0.3 \wedge 0.3$$
$$F_B(ef) \geq F_A(e) \vee F_B(f), 1.2 \geq 1.2 \vee 1.2$$

Here f dominates g because

$$T_B(fg) \leq T_A(f) \wedge T_A(g), 1.2 \leq 0.5 \wedge 1.2$$
$$I_B(fg) \leq I_B(f) \wedge I_B(g), 0.3 \leq 0.3 \wedge 0.2$$
$$F_B(fg) \geq F_A(f) \vee F_B(g), 1.2 \geq 1.2 \vee 1.2$$

Here g dominates h because

$$T_B(gh) \leq T_A(g) \wedge T_A(h), 1.2 \geq 1.2 \vee 1.2$$
$$I_B(gh) \leq I_B(g) \wedge I_B(h), 0.2 \leq 0.2 \wedge 0.2$$
$$F_B(gh) \geq F_A(g) \vee F_B(h), 1.2 \geq 1.2 \vee 1.2$$

Here h dominates a because

$$T_B(ha) \leq T_A(a) \wedge T_A(h), 1.3 \geq 1.1 \vee 1.2$$
$$I_B(ha) \leq I_B(a) \wedge I_B(h), 0.2 \leq 0.2 \wedge 0.2$$
$$F_B(ha) \geq F_A(a) \vee F_B(h), 0.8 \geq 0.9 \vee 1.2$$

$V = \{a, b, c, d, e, f, g, h\}$, $D_N = \{c, e, h\}$ and $V - D_n = \{a, b, d, f, g\}$, $|D_N| = 3$.
Now we obtain the connectivity index for all paths

$$|\text{CONN}_{p_1}(a, b)| = 2 \times (1.1) - 0.5 - 0.9 = 0.8$$
$$|\text{CONN}_{p_1}(b, c)| = 2 \times (0.7) - 0.2 - 1.2 = 0$$
$$|\text{CONN}_{p_1}(c, d)| = 2 \times (0.7) - 0.5 - 1.2 = -0.3$$

$$|\text{CONN}_{p_1}(d,e)| = 2\times(0.5) - 0.3 - 1.2 = -0.5$$
$$|\text{CONN}_{p_1}(e,f)| = 2\times(0.5) - 0.3 - 1.2 = -0.5$$
$$|\text{CONN}_{p_1}(f,g)| = 2\times(1.2) - 0.3 - 1.1 = 1$$
$$|\text{CONN}_{p_1}(g,h)| = 2\times(1.2) - 0.2 - 1.1 = 1.1$$
$$|\text{CONN}_{p_1}(h,a)| = 2\times(1.3) - 0.2 - 0.8 = 1.6$$

Also, If needed, we can calculate the connectivity index for indirect relationship for Novertop graph.

Then $\text{CI}_{\text{Nov}}(\mathscr{G})$ we have,

$$\begin{aligned}
\text{CI}_{\text{Nov}}(\mathscr{G}) &= \sum_{u_iv_j\in\mathscr{V}} (T_A(u_i), I_A(u_i), F_A(u_i))(T_A(v_j), I_A(v_j), F_A(v_J) \times \text{CONN}_G(u_i, v_j) \\
&= (1.3,0.3,0.7)(1.1,0.2,0.9)\times(0.8) + (1.3,0.3,0.2)(1.4,0.2,0.8)\times(1.6)+ \\
&\quad (1.1,0.2,0.9)(0.7,0.5,1.2)\times(-0.3)(0.7,0.5,1.2)(1.2,0.6,0.8)\times(-0.3)+ \\
&\quad (1.2,0.6,0.8)(0.5,0.3,1.2)\times(-0.4)(0.5,0.3,1.2)(1.2,0.3,1.1)\times(-0.4)+ \\
&\quad (1.2,0.5,1.1)(1.4,0.2,0.8)\times(1) \\
&= (2.12)(0.8) + (2.23)(1.6) + (2.03)(-0.3) + (2.1)(-0.3) + (1.74)(-0.4)+ \\
&\quad (2.82)(-0.4) + (2.94)(1) \\
&= 4.478
\end{aligned}$$

Then connectivity index of G is equal 4.478, which the positive sign indicates the high level of true information in the problem.

Hence the employee (h) a relationship is high and good. so "h" is the most dominating person.so we can select h is the head of the office.

9 Conclusion

In this paper has focussed an some topological indices for Neutrosophic over topologized graphs by using strong domination. It is an easy way to calculate the connectivity index.so we use the new method that is strong domination to find and calculate the connectivity index. The some topological indices for some standard neutrosophic over topologized graphs such as $2-regular$, K_2 and $K_{2,2}$ are given.

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Neutrosophic Generalized Exponential Distribution with Application

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Abstract. The objective of this article is to create a Neutrosophic Generalized Exponential (NGE) distribution in the presence of uncertainty. It is possible to calculate the mean, variance, moments, and reliability expression of the NGE distribution. With the help of graphs, the nature of the distribution and the reliability and hazard functions are studied. To determine the NGE distribution's parameters, a maximum likelihood estimation technique is used. The performance of estimated parameters is further tested using simulations. Finally, an actual data set is examined to show how the NGE distribution works. According to a model validity test, the NGE distribution is superior to the existing neutrosophic distributions that can be found in the literature.

Keywords: Generalized exponential distribution; Neutrosophic; Indeterminacy; Maximum likelihood estimation; Simulation; Reliability.

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1. Introduction

Numerous researchers have started developing various studies based on Neutrosophic statistics in recent years. The original research on neutrosophic statistics was initiated by Smarandache [1]. This new area of research is a generalization of the fuzzy logic environment, and it is used in an uncertain environment. Due to its ability to administer sets of values in an interval form, neutrosophic statistics play a crucial role in statistics and other research fields. For more details about Neutrosophic statistics and its related works, please refer to [2–11].

The neutrosophic theory of probability is indispensable and has practical applications. This area of study has not received a great deal of attention. Some authors have focused more on the neutrosophic statistics approach and its applications in various fields in recent years. For more information about neutrosophic probability, see [12,13]. Patro and Smarandache [14] presented the neutrosophic statistical distribution, more problems, and more solutions. Alhabib et al. [9] studied some neutrosophic probability distributions by generalizing some classical probability distributions such as the Poisson distribution, exponential distribution, and uniform distribution to the neutrosophic type. Nayana et al. [15] created a new neutrosophic model using the DUS-Weibull transformation, while Alhasan and Smarandache [16] studied the neutrosophic Weibull distribution. Zeina and Hatip [17] developed the neutrosophic random variables. They studied various statistical properties and examples. Sherwani [18] studied neutrosophic beta distribution with properties and applications. The other application of neutrosophic statistics in various field like quality control, sampling plans, process capability analysis and social science indeterminacy environment studied by [19–22]. The neutrosophic theory has many applications in a variety of fields, such as the neutrosophic treatment of the static model, the integration of renewable energy using a variety of resources, such as photovoltaic panels and wind turbines, and COVID-19 and its Omicron mutation. In traditional mathematics, crispness is the most crucial prerequisite; however, in actual problems, ambiguous data are present. In order to solve these issues, mathematical concepts based on uncertainty must be used. Uncertainty modeling is something that many scientists and engineers are interested in because it helps them define and explain the useful information that is hidden in uncertain data. Although it is one of the most crucial tools and has practical applications, the neutrosophic probability theory has not gotten much attention. It has, however, been the subject of some studies. More studies have focused in recent years on various areas of neutrosophic statistics, including correlation, regression analysis, test procedures, probability distributions, etc.

The mentioned studies and literature reviews have motivated us to develop a neutrosophic generalized exponential distribution and its properties.

1.1. *Neutrosophic Approach*

Neutrosophic statistics is the generalization of classical statistics. We administer with specific or crumple values in classical statistics, but in neutrosophic statistics, the sample values are chosen from a population with an uncertain environment. In neutrosophic statistics, the information can be vague, imprecise, ambiguous, uncertain, incomplete, or even unknown. Neutrosophic numbers have a standard form based on classical statistics, which is given below.

$$X_N = E + I$$

Data is broken down into two parts, E and I, where E is the exact or determined data and I is the uncertain, inexact, or indeterminate part of the data. It is equivalent to $X_N \in [X_L, X_U]$. A subscript $_N$ is used to distinguish the neutrosophic random variable, for example, X_N.

1.2. *Generalized Exponential distribution*

The generalized exponential (GE) distribution is one of the most widely used and flexible distributions compared to the exponential, gamma, and Weibull distributions; see [23] for more details. The GE distribution has more applications in reliability analysis, hydrology, quality control and medical field etc, please refer [24–30].

If a continuous random variable X_i; $i = 1, 2, \ldots, n$ is followed by the generalized exponential distribution with shape parameter δ and scale parameter υ then its probability density function (p.d.f.) and cumulative distribution function are respectively given as follows:

$$f(x) = \frac{\delta}{\upsilon}\left(1 - \exp\{-\frac{x}{\upsilon}\}\right)^{\delta-1}\exp\{-\frac{x}{\upsilon}\}; \qquad x > 0,\ \delta > 0,\ \upsilon > 0, \tag{1}$$

and

$$F(x) = \left(1 - \exp\{-\frac{x}{\upsilon}\}\right)^{\delta}; \qquad x > 0,\ \delta > 0,\ \upsilon > 0. \tag{2}$$

2. **Neutrosophic Generalized Exponential distribution**

Let us assume that $X_{N_i} \in [X_L, X_U]$, $i = 1, 2, \ldots, n_N$ is neutrosophic random variable following the neutrosophic generalized exponential (NGE) distribution with neutrosophic shape parameter $\delta_N \in [\delta_L, \delta_U]$ and neutrosophic scale parameter $\upsilon_N \in [\upsilon_L, \upsilon_U]$. The neutrosophic probability density function (n.p.d.f.) of NGE distribution is given as follows:

$$f(x_N) = \frac{\delta_N}{\upsilon_N}\left(1 - \exp\{-\frac{x_N}{\upsilon_N}\}\right)^{\delta_N-1}\exp\{-\frac{x_N}{\upsilon_N}\}; \qquad x_N > 0,\ \delta_N > 0,\ \upsilon_N > 0 \tag{3}$$

Where $X_N \in [X_L, X_U]$, $\delta_N \in [\delta_L, \delta_U]$, $\upsilon_N \in [\upsilon_L, \upsilon_U]$. NGE distribution with neutrosophic shape parameter δ_N and neutrosophic scale parameter υ_N is denoted as NGED (δ_N, υ_N). NGE distribution is transformed into a neutrosophic exponential distribution with neutrosophic scale parameter $\upsilon_N \in [\upsilon_L, \upsilon_U]$ when NGED $(1, \upsilon_N)$. Figure 1 display the p.d.f. plots for different parametric values of NGE distribution.

The developed NGE distribution is more flexible on account of the different shapes of the density function. From Figure 1, The curves of p.d.f. show that the behavior of the curves exponentially diminishes and starts from the infinite point for $\delta_N < 1$. For $\delta_N = 1$, its behavior exponentially diminishes but starts from a specific point on the y-axis. The density curves show unimodal behavior for $\delta_N > 1$.

The cumulative distribution function (c.d.f.) of NGE distribution is

$$F(x_N) = \left(1 - \exp\{-\frac{x_N}{\upsilon_N}\}\right)^{\delta_N}; \qquad x_N > 0,\ \delta_N > 0,\ \upsilon_N > 0. \tag{4}$$

The survival function and hazard function of NGE distribution respectively expressed as

$$s(x_N) = 1 - \left(1 - \exp\{-\frac{x_N}{\upsilon_N}\}\right)^{\delta_N}, \tag{5}$$

and

$$h(x_N) = \frac{\frac{\delta_N}{\upsilon_N}\left(1 - \exp\{-\frac{x_N}{\upsilon_N}\}\right)^{\delta_N - 1}\exp\{-\frac{x_N}{\upsilon_N}\}}{1 - \left(1 - \exp\{-\frac{x_N}{\upsilon_N}\right)^{\delta_N}} \tag{6}$$

From Figure 2, it is interesting to note that the NGE distribution has variable shapes. The survival function and failure rate curves for various neutrosophic parametric values are presented in Figures 3 and 4. Form Figure 4, the failure rate of NGE distribution is a bathtub and increasing behavior, which is very important for analyzing data sets in various fields.

3. Statistical Properties

In this section, we reviewed some statistical characteristics of the NGE distribution.

The mean and variance values are respectively expressed as

$$\mu_N = \frac{1}{\upsilon_N}\left[\psi\left(\delta_N + 1\right) - \psi\left(1\right)\right], \tag{7}$$

and

$$\sigma_N^2 = \frac{1}{\upsilon_N^2}\left[\psi'\left(1\right) - \psi'\left(\delta_N + 1\right)\right]. \tag{8}$$

The expressions $\psi(\cdot)$ denotes the digamma function while $\psi'(\cdot)$ denotes a derivative of $\psi(\cdot)$. For details about classical GED moments, refer to [23]. The q^{th} quantile of NGE distribution is obtained as follows:

$$x_{Nq} = -\upsilon_N \ln\left(1 - \mathrm{q}_N^{\frac{1}{\delta_N}}\right). \tag{9}$$

Consequently, the median value is $x_{N(0.5)} = -\upsilon_N \ln\left(1 - 2^{\frac{-1}{\delta_N}}\right)$.

4. Estimation of parameters

In this section, using the method of maximum likelihood estimation (MLE) the parameters of NGE distribution are estimated. Let $X_{N1}, X_{N2}, \ldots, X_{Nn}$ be a neutrosophic random sample of size n taken from NGE distribution. The log-likelihood equation is given by

$$l\left(\delta_N, \upsilon_N\right) = \ln\left(L\right) = n\ln\left(\delta_N\right) - n\ln\left(\upsilon_N\right) - \sum_{i=1}^{n}\frac{x_{Ni}}{\upsilon_N} + \left(\delta_N - 1\right)\sum_{i=1}^{n}\ln\left(1 - \exp\{-\frac{x_{Ni}}{\upsilon_N}\}\right) \tag{10}$$

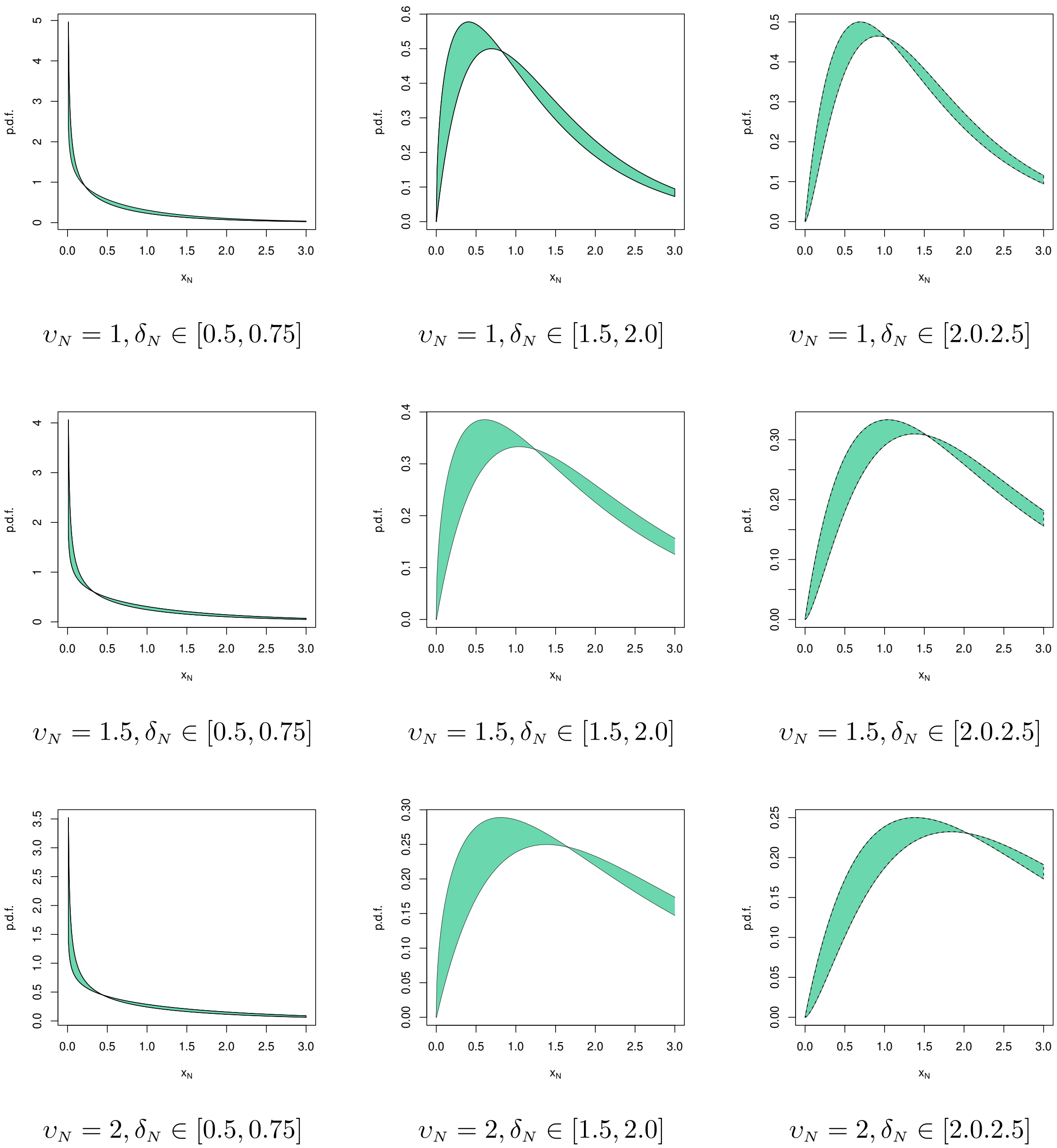
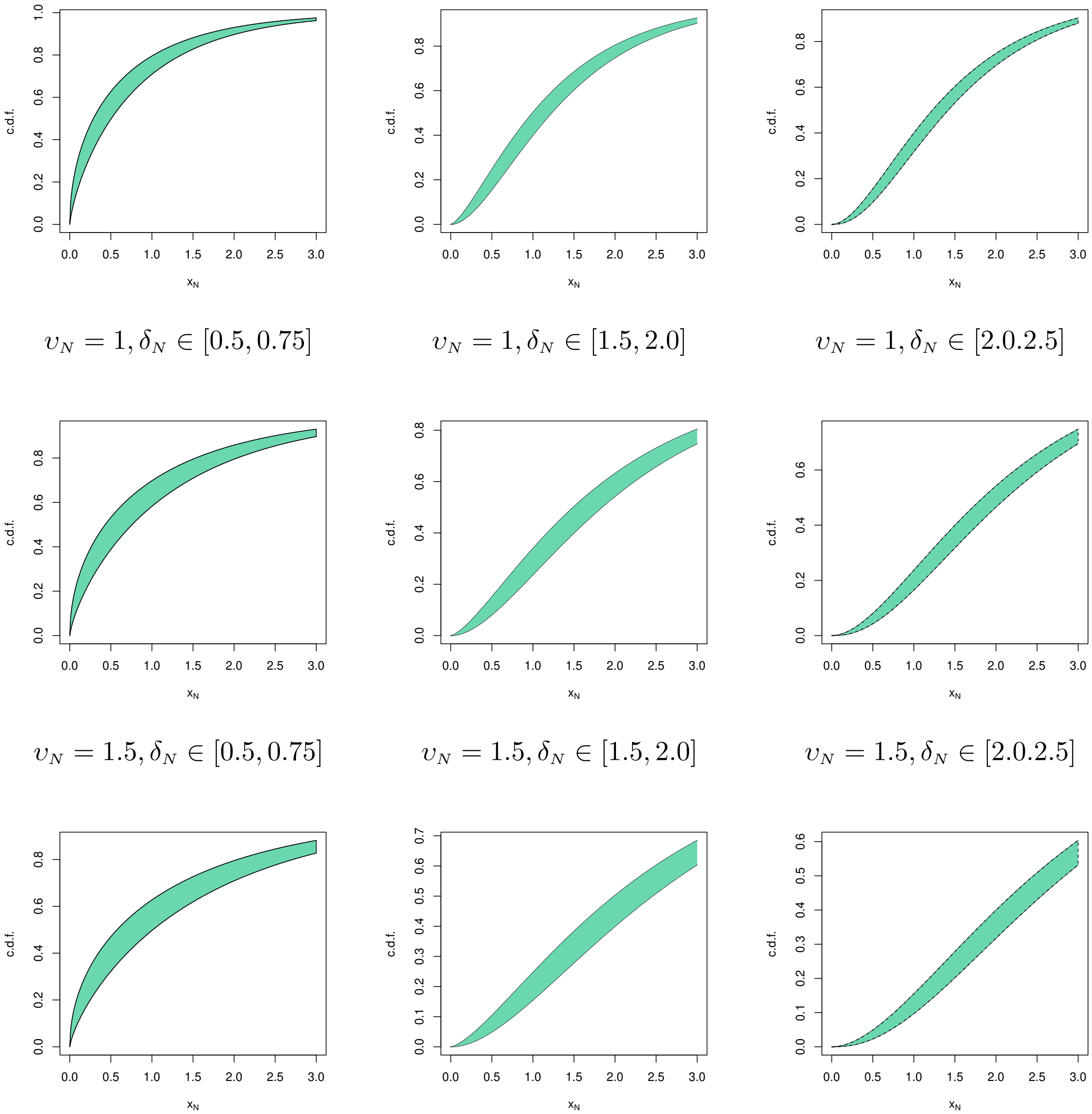


FIGURE 1. The p.d.f. plots of NGE distribution

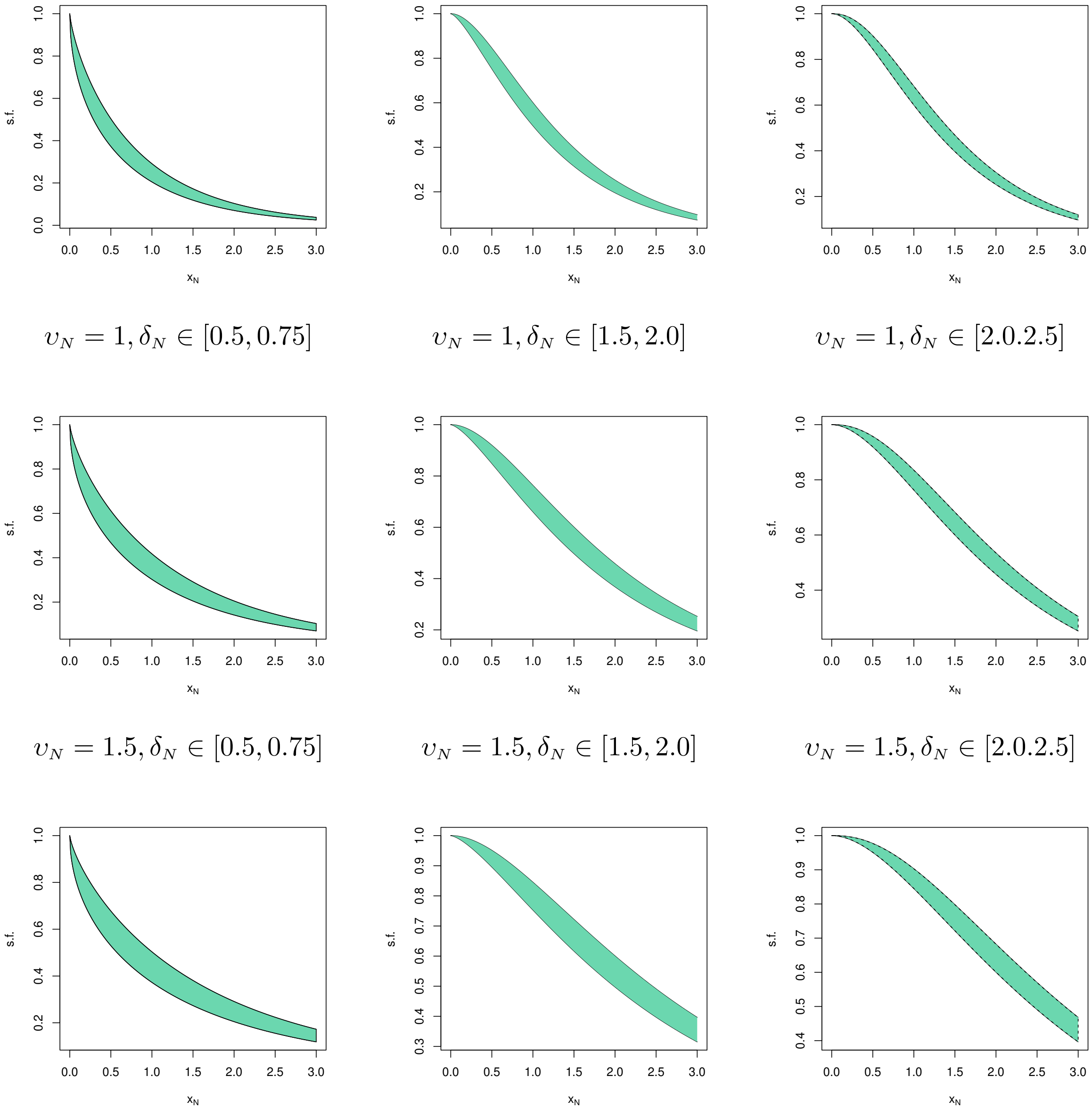


$\upsilon_N = 1, \delta_N \in [0.5, 0.75]$ $\upsilon_N = 1, \delta_N \in [1.5, 2.0]$ $\upsilon_N = 1, \delta_N \in [2.0.2.5]$

$\upsilon_N = 1.5, \delta_N \in [0.5, 0.75]$ $\upsilon_N = 1.5, \delta_N \in [1.5, 2.0]$ $\upsilon_N = 1.5, \delta_N \in [2.0.2.5]$

$\upsilon_N = 2, \delta_N \in [0.5, 0.75]$ $\upsilon_N = 2, \delta_N \in [1.5, 2.0]$ $\upsilon_N = 2, \delta_N \in [2.0.2.5]$

FIGURE 2. The c.d.f. plots of NGE distribution

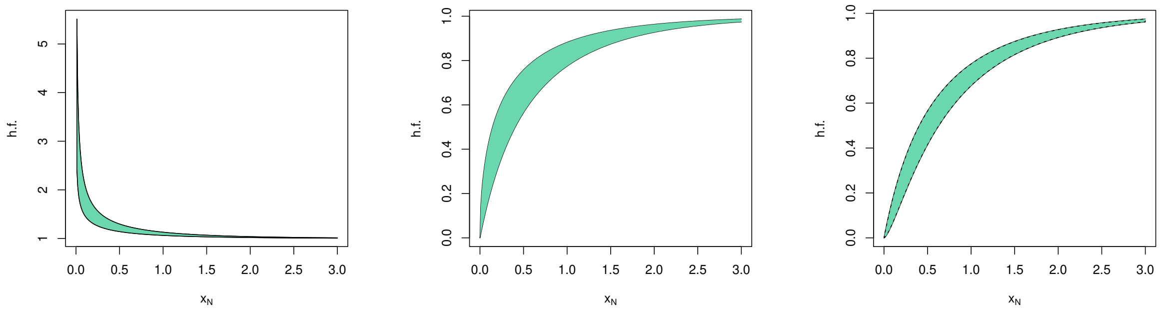


$\upsilon_N = 1, \delta_N \in [0.5, 0.75]$ $\upsilon_N = 1, \delta_N \in [1.5, 2.0]$ $\upsilon_N = 1, \delta_N \in [2.0.2.5]$

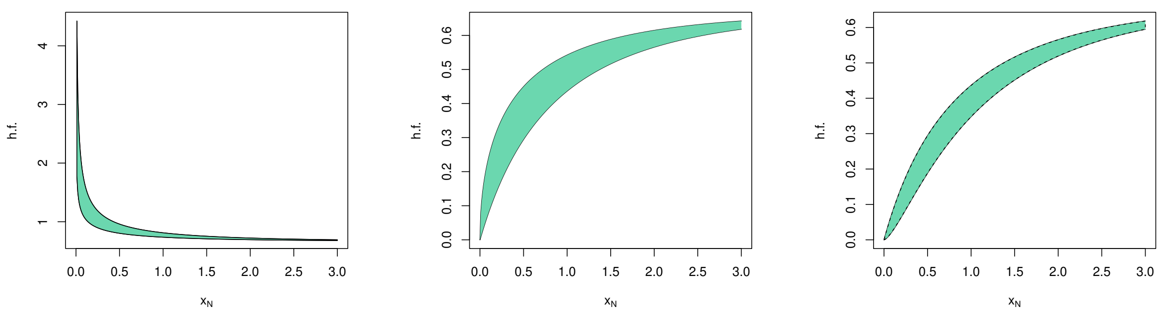
$\upsilon_N = 1.5, \delta_N \in [0.5, 0.75]$ $\upsilon_N = 1.5, \delta_N \in [1.5, 2.0]$ $\upsilon_N = 1.5, \delta_N \in [2.0.2.5]$

$\upsilon_N = 2, \delta_N \in [0.5, 0.75]$ $\upsilon_N = 2, \delta_N \in [1.5, 2.0]$ $\upsilon_N = 2, \delta_N \in [2.0.2.5]$

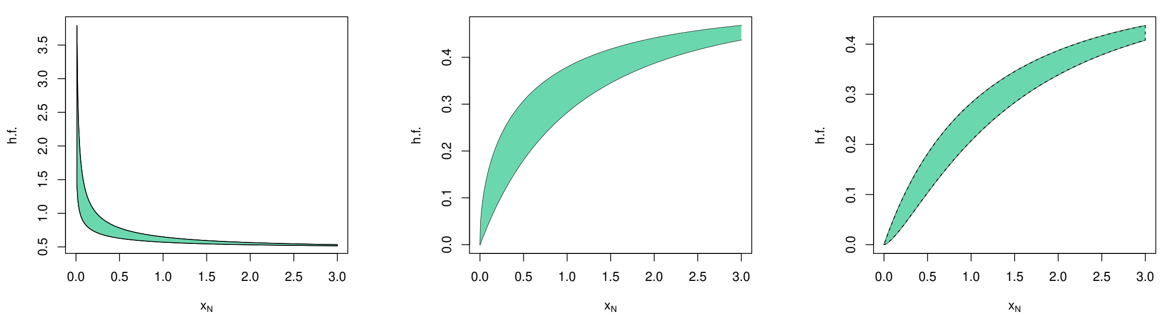
FIGURE 3. The survival function plots of NGE distribution



$\upsilon_N = 1, \delta_N \in [0.5, 0.75]$ $\upsilon_N = 1, \delta_N \in [1.5, 2.0]$ $\upsilon_N = 1, \delta_N \in [2.0.2.5]$



$\upsilon_N = 1.5, \delta_N \in [0.5, 0.75]$ $\upsilon_N = 1.5, \delta_N \in [1.5, 2.0]$ $\upsilon_N = 1.5, \delta_N \in [2.0.2.5]$



$\upsilon_N = 2, \delta_N \in [0.5, 0.75]$ $\upsilon_N = 2, \delta_N \in [1.5, 2.0]$ $\upsilon_N = 2, \delta_N \in [2.0.2.5]$

FIGURE 4. The hazard function plots of NGE distribution

The MLEs of δ_N and υ_N are denoted as $\hat{\delta}_N \in \left[\hat{\delta}_L, \hat{\delta}_U\right]$ and $\hat{\upsilon}_N \in [\hat{\upsilon}_L, \hat{\upsilon}_U]$ respectively, and are obtained by maximizing the equation (10). Thus $\hat{\delta}_N$ and $\hat{\upsilon}_N$ are the solutions of the following two derivative equations

$$\frac{\partial l\left(\delta_N, \upsilon_N\right)}{\partial \delta_N} = \frac{n}{\delta_N} + \sum_{i=1}^{n} \ln\left(1 - \exp\{-\frac{x_{Ni}}{\upsilon_N}\}\right) = 0 \tag{11}$$

and

$$\frac{\partial l\left(\delta_N, \upsilon_N\right)}{\partial \upsilon_N} = \frac{-n}{\upsilon_N} + \sum_{i=1}^{n} \frac{x_{Ni}}{\upsilon_N^2} - \frac{(\delta_N - 1)}{\upsilon_N^2} \sum_{i=1}^{n} \frac{x_{Ni} \exp\{-\frac{x_{Ni}}{\upsilon_N}\}}{\left(1 - \exp\{-\frac{x_{Ni}}{\upsilon_N}\}\right)} = 0 \tag{12}$$

or simply,

$$\frac{\partial l\left(\delta_N, \upsilon_N\right)}{\partial \upsilon_N} = -n\upsilon_N + \sum_{i=1}^{n} x_{Ni} - (\delta_N - 1) \sum_{i=1}^{n} \frac{x_{Ni} \exp\{-\frac{x_{Ni}}{\upsilon_N}\}}{\left(1 - \exp\{-\frac{x_{Ni}}{\upsilon_N}\}\right)} = 0. \tag{13}$$

Solving Eq. (11) results in

$$\hat{\delta}_N\left(\upsilon_N\right) = \frac{-n}{\sum\limits_{i=1}^{n} \ln\left(1 - \exp\{-\frac{x_{Ni}}{\upsilon_N}\}\right)}. \tag{14}$$

The estimator $\hat{\nu}_N$ is calculated by substituting $\hat{\delta}_N$ value in Eq. (12), which results in an expression in terms of υ_N as

$$-n\upsilon_N + \sum_{i=1}^{n} x_{Ni} + \frac{n}{\sum\limits_{i=1}^{n} \ln\left(1 - \exp\{-\frac{x_{Ni}}{\upsilon_N}\}\right)} \left[\sum_{i=1}^{n} \frac{x_{Ni} \exp\{-\frac{x_{Ni}}{\upsilon_N}\}}{\left(1 - \exp\{-\frac{x_{Ni}}{\upsilon_N}\}\right)}\right] + \sum_{i=1}^{n} \frac{x_{Ni} \exp\{-\frac{x_{Ni}}{\upsilon_N}\}}{\left(1 - \exp\{-\frac{x_{Ni}}{\upsilon_N}\}\right)} = 0 \tag{15}$$

Hence, MLE of υ_N say $\hat{\upsilon}_N$ is an iterative solution of equation (15). After finding $\hat{\upsilon}_N$ by iterative solution, we can substitute in Eq. (14) to get the MLE of $\hat{\delta}_N$.

5. Justification of Estimation with Simulation

To study the performance of the proposed NGE distribution model, a simulation study is carried out. The accomplishment of NGE distribution estimated parameters and their performance are expressed as neutrosophic average estimates (AEs), neutrosophic average biased (Avg. Biases), and neutrosophic measure square error (MSEs) using simulation investigation. The simulation results of average Bias and MSE are summarized in Tables 1-4. It is noticed from the tables that the average Bias and MSE decrease when the size of the sample increases, as expected. According to Tables 1-4, Bias of shape parameters is negative and the scale parameter is positive at different values of shape parametric and scale parametric values.

TABLE 1. $\upsilon_N = [1, 1]$, $\delta_N = [1, 3]$

	AEs		Avg. Biases		MSEs	
	$\hat{\upsilon}_N$	$\hat{\delta}_N$	$\hat{\upsilon}_N$	$\hat{\delta}_N$	$\hat{\upsilon}_N$	$\hat{\upsilon}_N$
30	0.9781	[1.1008,3.4551]	-0.0219	[0.1008,0.4551]	0.2137	[0.3258,1.3166]
50	0.9866	[1.0558,3.2537]	-0.0133	[0.0558,0.2537]	0.1645	[0.2149,0.8673]
100	0.9929	[1.0267,3.1265]	-0.0071	[0.0267,0.1265]	0.1168	[0.1410,0.5542]
200	0.9958	[1.0131,3.0610]	-0.0041	[0.0131,0.0610]	0.0825	[0.0948,0.3640]
500	0.9987	[1.0048,3.0231]	-0.0012	[0.0048,0.0231]	0.0524	[0.0589,0.2294]
1000	0.9995	[1.0024,3.0112]	-0.0005	[0.0024,0.0112]	0.0367	[0.0419,0.1572]

TABLE 2. $\upsilon_N = [1, 1]$, $\delta_N = [0.5, 0.75]$

	AEs		Avg. Biases		MSEs	
	$\hat{\upsilon}_N$	$\hat{\delta}_N$	$\hat{\upsilon}_N$	$\hat{\delta}_N$	$\hat{\upsilon}_N$	$\hat{\delta}_N$
30	0.9775	[0.5397,0.8185]	-0.0225	[0.0397,0.0685]	0.2761	[0.1363,0.2205]
50	0.9862	[0.5225,0.7888]	-0.0138	[0.0225,0.0388]	0.2131	[0.0942,0.1524]
100	0.9925	[0.5108,0.7692]	-0.0075	[0.0108,0.0192]	0.1516	[0.0625,0.1008]
200	0.9955	[0.5053,0.7591]	-0.0045	[0.0053,0.0091]	0.1071	[0.0424,0.0676]
500	0.9987	[0.5018,0.7538]	-0.0012	[0.0018,0.0038]	0.0682	[0.0264,0.0434]
1000	1.0000	[0.5011,0.7514]	0.0000	[0.0011,0.0014]	0.0477	[0.0189,0.0301]

TABLE 3. $\upsilon_N = [0.5, 0.75]$, $\delta_N = [1, 1]$

	AEs		Avg. Biases		MSEs	
	$\hat{\alpha}_N$	$\hat{\delta}_N$	$\hat{\upsilon}_N$	$\hat{\delta}_N$	$\hat{\upsilon}_N$	$\hat{\delta}_N$
30	[0.4877,0.7362]	1.1005	[-0.0123,-0.0138]	0.1005	[0.1196,0.1783]	0.3204
50	[0.4928,0.7412]	1.0562	[-0.0072,-0.0088]	0.0562	[0.0927,0.1364]	0.2154
100	[0.497,0.7436]	1.0274	[-0.003,-0.0064]	0.0274	[0.0652,0.0982]	0.1415
200	[0.4981,0.7465]	1.0132	[-0.0019,-0.0035]	0.0132	[0.0461,0.0692]	0.0948
500	[0.4992,0.7495]	1.0051	[-0.0008,-0.0005]	0.0051	[0.0292,0.0443]	0.0598
1000	[0.4997,0.7497]	1.0025	[-0.0003,-0.0003]	0.0025	[0.0205,0.031]	0.0419

6. **Application**

A realistic attempt of NGE distribution model is studied with help a real data in this section. The Parameter estimates along with the values of AIC (Akaike's Information criteria), BIC (Bayesian Information criteria) and KS (Kolmogorov–Smirnov) statistic are provided for comparision neutrosophic normal distribution (NND), neutrosophic gamma distribution (NGD), neutrosophic Weibull distribution (NWD), neutrosophic Rayleigh distribution (NRD),

TABLE 4. $\upsilon_N = [0.5, 0.75]$, $\delta_N = [1, 3]$

	AEs		Avg. Biases		MSEs	
	$\hat{\alpha}_N$	$\hat{\delta}_N$	$\hat{\upsilon}_N$	$\hat{\delta}_N$	$\hat{\upsilon}_N$	$\hat{\delta}_N$
30	[0.4877,0.7356]	[1.1008,3.4551]	[-0.0123,-0.0144]	[0.1008,0.4551]	[0.1196,0.1411]	[0.3258,1.3166]
50	[0.4928,0.7408]	[1.0558,3.2537]	[-0.0072,-0.0092]	[0.0558,0.2537]	[0.0927,0.1079]	[0.2149,0.8673]
100	[0.497,0.7439]	[1.0267,3.1264]	[-0.0030,-0.0061]	[0.0267,0.1264]	[0.0652,0.0776]	[0.1410,0.5542]
200	[0.4981,0.7467]	[1.0131,3.061]	[-0.0019,-0.0033]	[0.0131,0.061]	[0.0461,0.0546]	[0.0948,0.3640]
500	[0.4992,0.7494]	[1.0048,3.0231]	[-0.0008,-0.0006]	[0.0048,0.0231]	[0.0292,0.0349]	[0.0589,0.2294]
1000	[0.4997,0.7497]	[1.0024,3.0112]	[-0.0003,-0.0003]	[0.0024,0.0112]	[0.0205,0.0244]	[0.0419,0.1572]

neutrosophic exponential distribution (NED) and neutrosophic generalized exponential distribution (NGED).

6.1. *Example 1*

The data set reported in Table 5 attempted is related to remission time in months of 128 cancer patients. The remission times data was originally studied and reported in [31] from bladder cancer research. Under a neutrosophic environment, the remission periods data set is used by [32] to model the neutrosophic exponential distribution.

Based on their study remission periods of cancer patients is well fitted to NED. We use the same data set for the illustration of NGE distribution. Actively, data are the crumple observations, whereas to demonstrate the model, consider them as ambiguous sample observations for specified cancer patients. The developed NGE distribution parameters are estimated based on uncertainties of remission periods of cancer patients. The results in Table 6 shows that NGED is more effective to investigate the properties of uncertainties of remission periods of cancer patients neutrosophic data than the NED.

6.2. *Example 2*

To demonstrate a real example here we considered an rough population compactness of few villages in rural USA. This data is taken from [33] and they studied for neutrosophic W/S test based on the data follows to neutrosophic normal distribution. This data consists of the population of 17 villages in USA and their neutrosophic data, which is reproduced in Table 7 for ready reference. The results in Table 8 also shows that NGED is more suitable to fit the data than the NED.

7. **Conclusions**

In this article, a generalization exponential distribution is developed under neutrosophic statistics environment. Very few researchers have studied probability distributions based on

Table 5. Remission periods of 128 cancer patients.

Remission times								
0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.2
2.23	3.52	4.98	6.97	9.02	13.29	0.4	2.26	3.57
5.06	7.09	9.22	13.8	25.74	0.5	2.46	3.64	5.09
$[7.26, 8.2]$	9.47	14.24	25.82	0.51	2.54	3.7	5.17	7.28
9.74	14.76	$[5.3, 7.1]$	0.81	2.62	3.82	5.32	7.32	10.06
$[12, 14.77]$	32.15	2.64	3.88	5.32	7.39	10.34	14.83	34.26
0.9	2.69	4.18	5.34	7.59	10.66	15.96	36.66	1.05
2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	$[15, 17.2]$	46.12	1.26	2.83	4.33	5.49
7.66	11.25	17.14	$[75.02, 81]$	1.35	2.87	5.62	7.87	11.64
17.36	1.4	3.02	4.34	5.71	7.93	11.79	18.1	1.46
4.4	5.85	8.26	11.98	19.13	1.76	3.25	4.5	6.25
8.37	12.02	$[1.5, 3.2]$	3.31	4.51	6.54	$[7.5, 8.2]$	12.03	20.28
2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93	8.65
12.63	22.69							

Table 6. Estimates and Goodness-of-fit statistics for data set 1.

Model	Parameter	Estimates	LogLikelihood	AIC	BIC	KS
NND	μ	[9.1196,9.2453]	[-478.1315,-482.0838]	[960.263,968.1676]	[975.6711,983.5758]	[0.1899,0.1941]
	σ	[10.1397,10.4577]				
NGD	shape	[1.1896,1.1884]	[-409.7832,-411.5487]	[823.5665,827.0975]	[838.9746,842.5056]	[0.0757,0.0769]
	scale	[7.6658,7.7796]				
NWD	shape	[1.0553,1.0519]	[-410.5979,-412.3855]	[825.1958,828.7710]	[840.6039,844.1791]	[0.0716,0.0737]
	scale	[9.3370,9.4544]				
NRD	υ	[9.6432,9.8702]	[-486.1404,-490.4138]	[976.2808,984.8275]	[991.6890,1000.2360]	[0.3544,0.3542]
NED	υ	[0.1096,0.1081]	[-410.9358,-412.6880]	[825.8715,829.3760]	[841.2796,844.7842]	[0.0815,0.0869]
NGED	υ	[7.9506,8.0568]	[-409.4565,-411.2037]	[822.9129,826.4074]	[838.3210,841.8155]	[0.0752,0.0759]
	δ	[1.2390,1.2397]				

neutrosophic statistics. The mathematical properties of the developed neutrosophic generalization exponential distribution are studied. The nature of the distribution is studied through various neutrosophic parametric combinations. Using the maximum likelihood method the parameters are estimated. A simulation study is carried out under neutrosophic environment. The average Bias and MSE decrease as the sample size increases, as expected. Finally, the application of the proposed NGE distribution is presented through real data sets. A comparative study with other distributions is also done based real data sets. Based on real data examples, we conclude that the NGE distribution furnishes better performance over existing

Table 7. Neutrosophic population density of some villages in the USA

Villages	Population density	Villages	Population density
Aranza	[4.13,4.14]	Charapan	[5.10,5.12]
Corupo	[4.53,4.55]	Comachuen	[5.25,5.27]
San Lorenzo	[4.69,4.70]	Pichataro	[5.36,5.38]
Cheranatzicurin	[4.76,4.78]	Quinceo	[5.94,5.96]
Nahuatzen	[4.77,4.79]	Nurio	[6.06,6.08]
Pomacuaran	[4.96,4.98]	Turicuaro	[6.19,6.21]
Servina	[4.97,4.99]	Urapicho	[6.30,6.32]
Arantepacua	[5.00,5.06]	Capacuaro	[7.73,7.98]
Cocucho	[5.04,5.06]		

Table 8. Estimates and Goodness-of-fit statistics for data set 2.

Model	Parameter	Estimates	LogLikelihood	AIC	BIC	KS
NND	μ	[5.3400,5.3723]	[-21.1554,-21.9577]	[46.3107,47.9155]	[53.6436,55.2483]	[0.2007,0.2024]
	σ	[0.8398,0.8804]				
NGD	shape	[40.4254,37.2310]	[-20.2481,-20.9136]	[44.4962,45.8272]	[51.8290,53.1600]	[0.1821,0.1816]
	scale	[0.1320,0.1442]				
NWD	shape	[5.8980,5.5773]	[-23.0417,-23.9417]	[50.0834,51.8834]	[57.4162,59.2162]	[0.2097,0.2115]
	scale	[5.7143,5.7621]				
NRD	υ	[3.8223,3.8495]	[-34.3024,-34.4533]	[72.6049,72.9067]	[79.9377,80.2395]	[0.4457,0.4439]
NED	υ	[0.1873,0.1861]	[-45.47884,-45.5815]	[94.9577,95.1630]	[102.2905,102.4959]	[0.5386,0.5373]
NGED	υ	[0.6067,0.6187]	[-18.7673,-19.1981]	[41.5345,42.3961]	[48.8674,49.7290]	[0.1443,0.1466]
	δ	[3630.608,3209.943]				

distributions. This article develops a generalized exponential distribution inside a neutrosophic statistical framework. The study of probability distributions based on neutrosophic statistics is quite uncommon. The generated neutrosophic generalization exponential distribution's mathematical characteristics are investigated. The distribution's nature is investigated using a variety of neutrosophic parametric combinations. The parameters are computed using the maximum likelihood approach. Simulation research is conducted in a neutrosophic setting. When expected, as the sample size grows, the average bias and MSE drop. The use of the suggested NGE distribution is then shown using actual data sets. Based on actual data sets, a comparison study with different distributions is also conducted. We draw the conclusion that the NGE distribution offers superior performance over current distributions based on studied instances.

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Hyperbolic Sine Similarity Measure of SVNSs for Open-Pit Mine Slope Stability Classification and Assessment

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Abstract: Slope instability is a common geological hazard in open-pit mines, which may cause huge economic losses and casualties. Thus, it is important to cluster and evaluate the stability of slopes effectively. This article proposes a hyperbolic sine similarity measure of single-valued neutrosophic sets (SVNSs) for a netting clustering method and a slope stability evaluation method to cluster and assess the stability of open-pit mine slopes. This study contains the following main content. First, we present a hyperbolic sine similarity measure between SVNSs. Second, slope stability impact factors are fuzzified into SVNSs by the utilization of true, indeterminate, and false membership functions, and then a netting clustering method using the proposed similarity measure is proposed to cluster the stability of open-pit mine slopes. Third, we propose a slope stability evaluation method based on the proposed similarity measure, where we give the SVNS knowledge of risk grades/patterns based on the clustering results of slope stability and then present the similarity measure values between the risk grades/patterns and the slope samples to assess that each slope sample with the larger measure value belongs to the corresponding slope risk grade. Finally, the proposed netting clustering and evaluation methods are applied to the clustering analysis and assessment of 20 open-pit mine slope samples to verify the rationality and effectivity of the proposed approaches in the scenario of SVNSs.

Keywords: single-valued neutrosophic set; netting clustering method; similarity measure; open-pit mine slope; slope stability assessment.

1. Introduction

Slope instability is a typical geological hazard in open-pit mines, so the disasters and losses caused by slope instability cannot be ignored. Thus, it is important to give some reasonable classification and evaluation methods for slope stability. Traditional qualitative classification methods for the slope stability grades include rock mass strength grading method, geological strength index method, slope failure probability grading method, and so on [1]. However, there are many factors that will affect the analysis of slope stability. Since the slope impact factors include a lot of uncertain and incomplete information, the traditional methods cannot effectively express the uncertain and incomplete information. Therefore, some indeterminate classification methods have been proposed, such as rainfall-induced landslides using ANN (artificial neural network) and fuzzy clustering methods [2], K-means and fuzzy c-means clustering algorithms [3], and a neuro-fuzzy inference system-based clustering methods [4]. But the existing indeterminate clustering methods difficultly express the true, indeterminate, and false information in the evaluation problems of slope stability.

In order to represent indeterminate and inconsistent information in the real world, Smarandache first proposed the concept of neutrosophic sets (NSs) [5] as a conceptual extension of fuzzy sets (FSs) [6] and (interval-valued) intuitionistic FSs (IFSs/IVFSs) [7, 8]. NS is characterized by a true membership function, an indeterminate membership function, and a false membership function independently. However, it is difficult to apply NSs in practical engineering fields because the values of their membership functions fall in the non-standard interval $]^{-}0, 1^{+}[$. As the subsets of NSs, Wang et al. [9, 10] introduced single-valued and interval-valued NSs (SVNSs and IVNSs) when the values of the three membership functions fall in the standard interval [0, 1] to describe indeterminate and inconsistent information in practical engineering issues. Recently, some researchers have applied SVNSs to the assessment of slope stability. Qin [11] proposed a SVNS adaptive neuro fuzzy inference system (SVNS-ANFIS) and applied it to the evaluation of open-pit mine slope stability. Then, Qin [12] further proposed a SVNS Gaussian process regression (SVNS-GPR) approach to predict the stability of open-pit mine slopes. However, SVNSs have not been applied to the clustering analysis of slope stability so far.

As another subclass of neutrosophic theory, a neutrosophic number ($N = a + bI$ for $I \in$ [inf I, sup I]) (NNs) [5, 13, 14] consists of a certain part a and an uncertain part bI, which is also called an uncertain number. Since similarity measures are one of the important research topics in neutrosophic theory, some similarity measures have been proposed and applied in slope stability evaluation problems in the environment of NNs [15]. Li et al. [15] proposed a slope stability evaluation approach based on the tangent and arctangent similarity measure of NNs. Li et al. [16] developed the vector similarity measures of NNs for the assessment of rock slope stability. However, these similarity measures lack the information of true, false, and indeterminate membership degrees. They cannot deal with indeterminate and inconsistent decision-making/evaluation problems in neutrosophic environments. Therefore, some researchers [17-19] presented various similarity measures of SVNSs/IVNSs to perform decision-making problems.

Regarding the current studies, the similarity measures of NNs cannot handle the actual clustering and evaluation problems of slope stability with SVNS information because NN cannot contain the true, false, indeterminate membership degrees. Then, the SVNS-ANFIS and SVNS-GPR methods [11, 12] require large amounts of learning data to train them, leading to complex learning operations and difficult update problems. Therefore, they are difficultly applied to actual clustering and evaluation problems of slope stability. Ye [15] also proposed a clustering method by the similarity measure of SVNSs, but it was not applied to actual clustering analysis and evaluation problems of slope stability because it is difficult to obtain true, false, indeterminate membership degrees from the data of slope samples. However, similarity metrics for clustering analysis and evaluation problems of slope stability have shown obvious superiority over neural networks in terms of data requirements, algorithms, and updated applications. Unfortunately, to date, the similarity measures of SVNS have not been applied to the clustering analysis and evaluation problems of slope stability. How to solve the clustering analysis and evaluation problems of slope stability by the similarity measure of SVNSs is a challenging problem in practical applications. Therefore, this paper will resolve this issue.

In this study, we present a hyperbolic sine similarity measure (HSSM) of SVNSs and its netting clustering analysis and evaluation methods of slope stability. Then, the proposed methods are applied to the clustering analysis and assessment of 20 open-pit mine slope samples. Through the comparative analysis with existing related methods, the proposed approaches reveal their rationality and effectivity in the clustering and evaluation application of the 20 open-pit mine slope samples in the scenario of SVNSs.

The remaining structure of this paper is arranged as follows. In section 2, some basic concepts of SVNSs are introduced. Section 3 proposes HSSM of SVNSs and netting clustering analysis and slope stability evaluation methods using the proposed HSSM. Section 4 applies the proposed clustering method to the slope stability clustering analysis of the 20 open-pit mine slope samples, and then the proposed evaluation method is applied to the stability evaluation of the 20 slope samples. Conclusions and further research are presented in Section 5.

2. Some Basic Concepts of SVNS

Smarandache first introduced NSs as the generalization of FSs, IVFSs, and IFSs. Then, Wang et al. [10] introduced SVNS as a subclass of NS to be applied in real scientific and engineering applications. The definition and operations of SVNSs are introduced below.

Definition 1 [10]. Let X be a universal set. A SVNS D in X can be denoted as $D = \{<x, D_T(x), D_I(x), D_F(x)> | x \in X\}$, where $D_T(x)$, $D_I(x)$, $D_F(x)$ are the true, indeterminate, and false membership functions for any $x \in X$, $D_T(x), D_I(x), D_F(x) \in [0, 1]$, and $0 \le D_T(x) + D_I(x) + D_F(x) \le 3$.

Then, the basic element of SVNS $d = <x, D_T(x), D_I(x), D_F(x)>$ is simply denoted as the single-valued neutrosophic number (SVNN) $d = <D_T, D_I, D_F>$ for the convenient representation.

Definition 2 [10]. Set two SVNSs as $d_1 = <D_{T1}, D_{I1}, D_{F1}>$ and $d_2 = <D_{T2}, D_{I2}, D_{F2}>$, then they follow the following operations.

(1) $d_1 \subseteq d_2$ if and only if $D_{T1} \le D_{T2}$, $D_{I1} \ge D_{I2}$, $D_{F1} \ge D_{F2}$;

(2) $d_1 = d_2$ if and only if $d_1 \subseteq d_2$ and $d_2 \subseteq d_1$;

(3) $d_1^c = <D_{F1}, 1 - D_{I1}, D_{T1}>$ (Complement of d_1);

(4) $d_1 \cup d_2 = <D_{T1} \vee D_{T2}, D_{I1} \wedge D_{I2}, D_{F1} \wedge D_{F2}>$;

(5) $d_1 \cap d_2 = <D_{T1} \wedge D_{T2}, D_{I1} \vee D_{I2}, D_{F1} \vee D_{F2}>$.

Definition 3 [20]. Let $D_1 = \{d_{11}, d_{12}, \ldots, d_{1n}\}$ and $D_2 = \{d_{21}, d_{22}, \ldots, d_{2n}\}$ be two SVNSs, where $d_{1i} = <D_{T1i}, D_{I1i}, D_{F1i}>$ and $d_{2i} = <D_{T2i}, D_{I2i}, D_{F2i}>$ ($i = 1, 2, \ldots, n$) are SVNNs. If the weight of d_{1i} and d_{2i} is specified by $g_i \in [0, 1]$ with $\sum_{i=1}^{n} g_i = 1$, the weighted generalized distance between D_1 and D_2 is defined as

$$G_\varphi(D_1, D_2) = \left\{ \frac{1}{3} \sum_{i=1}^{n} g_i \left[\left|D_{T1i} - D_{T2i}\right|^\varphi + \left|D_{I1i} - D_{I2i}\right|^\varphi + \left|D_{F1i} - D_{F2i}\right|^\varphi \right] \right\}^{1/\varphi} \text{ for } \varphi > 0. \quad (1)$$

Then, the above distance $G_\varphi(D_1, D_2)$ satisfies the following properties [20]:

(A1) $0 \le G_\varphi(D_1, D_2) \le 1$;

(A2) $G_\varphi(D_1, D_2) = 0$ if and only if $D_1 = D_2$;

(A3) $G_\varphi(D_1, D_2) = G_\varphi(D_2, D_1)$;

(A4) If $D_1 \subseteq D_2 \subseteq D_3$ for the SVNS D_3, then $G_\varphi(D_1, D_3) \ge G_\varphi(D_1, D_2)$ and $G_\varphi(D_1, D_3) \ge G_\varphi(D_2, D_3)$.

In view of the complementary relationship between the similarity measure and the distance, the weighted generalized distance-based similarity measure of SVNSs is presented as bellows [20]:

$$S_\varphi(D_1, D_2) = 1 - G_\varphi(D_1, D_2) = 1 - \left\{ \frac{1}{3} \sum_{i=1}^{n} g_i \left[\left|D_{T1i} - D_{T2i}\right|^\varphi + \left|D_{I1i} - D_{I2i}\right|^\varphi + \left|D_{F1i} - D_{F2i}\right|^\varphi \right] \right\}^{1/\varphi}. \quad (2)$$

Then, the weighted generalized distance-based similarity measure of SVNSs also implies the following properties [20]:

(B1) $0 \le S_\varphi(D_1, D_2) \le 1$;

(B2) $S_\varphi(D_1, D_2) = 1$ if and only if $D_1 = D_2$;

(B3) $S_\varphi(D_1, D_2) = S_\varphi(D_2, D_1)$;

(B4) If $D_1 \subseteq D_2 \subseteq D_3$ for the SVNS D_3, then $S_\varphi(D_1, D_2) \ge S_\varphi(D_1, D_3)$ and $S_\varphi(D_2, D_3) \ge S_\varphi(D_1, D_3)$.

3. Netting Clustering and Slope Stability Evaluation Methods Using HSSM of SVNSs

3.1. Netting Clustering Method Using HSSM of SVNSs

Considering the weighted generalized distance of SVNSs, this section further proposes HSSM between SVNSs and its netting clustering method for SVNSs.

First, we propose HSSM of SVNSs.

Definition 4. Let $D_1 = \{d_{11}, d_{12}, \ldots, d_{1n}\}$ and $D_2 = \{d_{21}, d_{22}, \ldots, d_{2n}\}$ be two SVNSs, where $d_{1i} = \langle D_{T1i}, D_{I1i}, D_{F1i}\rangle$ and $d_{2i} = \langle D_{T2i}, D_{I2i}, D_{F2i}\rangle$ ($i = 1, 2, \ldots, n$) are SVNNs. If the weight of d_{1i} and d_{2i} is specified by $g_i \in [0, 1]$ with $\sum_{i=1}^{n} g_i = 1$, the weighted HSSM between D_1 and D_2 is defined by

$$H_\varphi(D_1, D_2) = 1 - \sinh\left[\ln(1+\sqrt{2})G_\varphi(D_1, D_2)\right]$$
$$= 1 - \left\{\sinh\left(\frac{\ln(1+\sqrt{2})}{3}\sum_{i=1}^{n} g_i\left[\left|D_{T1i} - D_{T2i}\right|^\varphi + \left|D_{I1i} - D_{I2i}\right|^\varphi + \left|D_{F1i} - D_{F2i}\right|^\varphi\right]\right)\right\}^{1/\varphi} \quad \text{for } \varphi > 0. \quad (3)$$

Then, HSSM also contains the following properties:

(C1) $0 \le H_\varphi(D_1, D_2) \le 1$;

(C2) $H_\varphi(D_1, D_2) = 1$ if and only if $D_1 = D_2$;

(C3) $H_\varphi(D_1, D_2) = H_\varphi(D_2, D_1)$;

(C4) If $D_1 \subseteq D_2 \subseteq D_3$ for the SVNS D_3, then $H_\varphi(D_1, D_2) \ge H_\varphi(D_1, D_3)$ and $H_\varphi(D_2, D_3) \ge H_\varphi(D_1, D_3)$.

Proof: The properties (C1)-(C3) are obviously true. Therefore, we only prove the property (C4).

For $D_1 \subseteq D_2 \subseteq D_3$, in view of the above properties of the distance measure $G_\varphi(D_1, D_2)$ for SVNSs, there are $G_\varphi(D_1, D_3) \ge G_\varphi(D_1, D_2)$ and $G_\varphi(D_1, D_3) \ge G_\varphi(D_2, D_3)$. Since the sinh($x$) for $x \in [0, 1]$ is an increasing function, based on the compensatory relationship between the distance and the similarity measure, there are also $H_\varphi(D_1, D_2) \ge H_\varphi(D_1, D_3)$ and $H_\varphi(D_2, D_3) \ge H_\varphi(D_1, D_3)$.

Hence, the proof is completed.

In light of the proposed HSSM of SVNSs, we introduce a netting clustering method to cluster open-pit mine slopes in the environment of SVNSs.

In a clustering problem of open-pit mine slops, $D = \{D_1, D_2, \ldots, D_m\}$ is a set of m slopes and $Q = \{q_1, q_2, \ldots, q_n\}$ is a set of n impact factors (indices) of slope stability. The weight of each impact factor q_i is g_i subject to $g_i \in [0, 1]$ and $\sum_{i=1}^{n} g_i = 1$.

Using the suitable true, indeterminate, and false membership functions (MFs) (see Table 2), the measurement values of the slope stability impact indices for each slope sample are fuzzed as the true, indeterminate, and false fuzzy values, which is constructed as the SVNS $D_j = \{d_{j1}, d_{j2}, \ldots, d_{jn}\}$, where $d_{ji} = \langle D_{Tji}, D_{Iji}, D_{Fji}\rangle$ are SVNNs for $D_{Tji}, D_{Iji}, D_{Fji} \in [0, 1]$, $j = 1, 2, \ldots, m$, and $i = 1, 2, \ldots, n$.

In the clustering problem, the netting clustering method is used to cluster the open-pit mine slopes in the environment of SVNSs by the following steps:

Step 1: Establish the hyperbolic sine similarity matrix $H = (h_{ji})_{m\times m}$ ($i, j = 1, 2, \ldots, m$) through the similarity operations of Eq. (3) (usually taking φ = 2 as a typical parameter value) subject to $h_{ji} = H_\varphi(D_j, D_i)$, $h_{jj} = 1$, and $h_{ji} = h_{ij}$.

Step 2: Use the open-pit mine slope samples for replacing all the diagonal elements of the similarity matrix Y.

Step 3: Construct the β-cutting matrices $H^\beta = (h_{ji}^\beta)_{m\times m}$ corresponding to different confidence levels of β by the following formula:

$$h_{ji}^\beta = \begin{cases} 0, h_{ji} < \beta \\ 1, h_{ji} \ge \beta \end{cases} \quad (i, j = 1, 2, \ldots, m). \quad (4)$$

All "0" is deleted in the β-cutting matrixes and "1" is replaced by "*", and then draw the vertical and horizontal lines from "*" to the diagonal elements. The slope samples connected by the same "*" are constructed as a type corresponding to the confidence level β. Update different confidence levels of β from big to small until the slope samples are clustered into the expected types.

3.2. Slope Stability Evaluation Method

In terms of the above clustering results of slope samples with SVNS information, we don't know which type belongs to which risk grade/pattern. Therefore, we must give the stability evaluation of the slope samples to recognize the corresponding risk patterns/grades of the slope stability. To do so, this subsection needs to give a slope stability evaluation method in the setting of SVNSs.

Based on the slope stability classification knowledge/experience, we can establish the expected slope stability patterns/risk grades expressed by their SVNSs $R_k = \{d_{k1}, d_{k2}, \ldots, d_{kn}\}$ that are composed of the SVNNs $d_{ki} = <D_{Tki}, D_{Iki}, D_{Fki}>$ for $D_{Tki}, D_{Iki}, D_{Fki} \in [0, 1]$ ($k = 1, 2, \ldots, p$; $i = 1, 2, \ldots, n$). Suppose that there is a set of m slope samples $A = \{A_1, A_2, \ldots, A_m\}$ to require the risk evaluation of slope stability. Then, the slope samples can be represented by the SVNSs $D_j = \{d_{j1}, d_{j2}, \ldots, d_{jn}\}$ ($j = 1, 2, \ldots, m$) that are composed of the SVNNs $d_{ji} = <D_{Tji}, D_{Iji}, D_{Fji}>$ for $D_{Tji}, D_{Iji}, D_{Fji} \in [0, 1]$ ($i = 1, 2, \ldots, n$).

Regarding the risk evaluation issue of slope stability, the similarity measure between each slope sample D_j ($j = 1, 2, \ldots, m$) and each slope stability pattern R_k ($k = 1, 2, \ldots, p$) is given by the following formula:

$$H_{\varphi}(D_j, R_k) = 1 - \left\{ \sinh\left(\frac{\ln(1+\sqrt{2})}{3} \sum_{i=1}^{n} g_i \left[\left| D_{Tji} - D_{Tki} \right|^{\varphi} + \left| D_{Iji} - D_{Iki} \right|^{\varphi} + \left| D_{Fji} - D_{Fki} \right|^{\varphi} \right] \right) \right\}^{1/\varphi} \text{ for } \varphi > 0. \quad (5)$$

Based on the HSSM values of Eq. (5), we can utilize $H_{\varphi}(D_j, R_{k^*}) = \underset{1 \le k \le m}{\mathrm{Max}} \left(H_{\varphi}(D_j, R_k) \right)$ to recognize that the stability grade of the slope sample D_j belongs to R_{k^*}.

4. Clustering Analysis and Stability Evaluation of Actual Open-Pit Mine Slopes

4.1. Clustering Analysis of Actual Cases

Table 1. Original data of 20 open-pit mine slope samples

D_j	q_1	q_2	q_3	q_4	q_5	q_6
D_1	62.0	47.0	32.0	0.115	43.6	29.1
D_2	40.0	55.0	31.0	0.0321	40.8	28.8
D_3	36.5	55.0	39.0	0.045	43.6	28.7
D_4	35.5	58.0	31.0	0.0273	39.2	28.9
D_5	66.0	57.0	40.0	0.0796	43.0	29.1
D_6	42.0	55.0	30.0	0.0157	37.6	29.1
D_7	43.5	54.0	33.0	0.0291	38.4	29.0
D_8	48.5	60.0	38.0	0.0522	40.5	29.2
D_9	46.5	57.0	40.0	0.0354	40.0	28.8
D_{10}	23.5	64.0	43.0	0.0285	39.8	29.0
D_{11}	59.5	71.0	37.0	0.0576	34.7	29.1
D_{12}	23.5	57.0	34.0	0.0125	31.4	28.9
D_{13}	25.0	65.0	48.0	0.0218	40.8	29.1
D_{14}	23.0	65.0	49.0	0.0141	43.7	28.9
D_{15}	18.0	70.0	41.0	0.0122	35.5	28.8
D_{16}	15.0	80.0	47.0	0.0074	37.8	29.2
D_{17}	16.5	70.0	60.0	0.0122	39.7	28.7
D_{18}	19.0	68.0	51.0	0.0103	37.1	28.9
D_{19}	17.0	70.0	60.0	0.0079	43.1	29.2
D_{20}	10.0	70.0	50.0	0.0044	34.7	29.1

In Zhejiang Province, China, many open-pit mines have slope instability problems, which will lead to a large number of economic losses and casualties. In order to reasonably classify and evaluate the slope stability, we collected 20 slope samples from field survey in Zhejiang Province. The slope height (q_1), slope angle (q_2), potential slip plane angle (q_3), cohesion (q_4), internal friction angle (q_5), and rock density (q_6) are considered as the 6 main impact factors of slope stability. The weight vector of the 6 impact factors is specified as $\boldsymbol{g}$ = (0.33, 0.22, 0.12, 0.1, 0.07, 0.16). In addition to the impact factors, we also collected the safety factor of each slope as the known knowledge/experience. The original data (six impact factors) of the 20 slope samples D_j (j = 1, 2, …, 20) is shown in Table 1.

Table 2. True, indeterminate, and false MFs for impact factors

Impact factor	MF		
	D_T	D_I	D_F
q_1	trapmf[0 0 15 80]	trimf[15 30 80]	trapmf[15 80 100 100]
q_2	trapmf[0 0 40 80]	trimf[40 60 80]	trapmf[40 80 100 100]
q_3	trapmf[0 0 20 60]	trimf[20 40 60]	trapmf[20 60 80 80]
q_4	trapmf[0.01 0.045 0.060 0.060]	trimf[0.01 0.0265 0.045]	trapmf[0 0 0.01 0.045]
q_5	trapmf[30 50 60 60]	trimf[30 40 50]	trapmf[0 0 30 50]
q_6	trimf[28.7 28.7 29.2]	trimf[28.8 28.9 29]	trapmf[28.7 29.2 29.5 29.5]

Table 3. SVNSs of 20 open-pit mine slope samples

D_j	q_1	q_2	q_3	q_4	q_5	q_6
D_1	<0.28,0.4,0.723>	(0.825,0.35,0.175)	(0.7,0.6,0.3)	(0,0,0)	(0.68,0.64,0.32)	(0.2,0,0.8)
D_2	(0.615,0.889,0.385)	(0.625,0.75,0.375)	(0.725,0.55,0.275)	(0.631,0.697,0.369)	(0.54,0.92,0.46)	(0.8,0,0.2)
D_3	(0.669,0.967,0.331)	(0.625,0.75,0.375)	(0.525,0.95,0.475)	(1,0,0)	(0.68,0.64,0.32)	(1,0,0)
D_4	(0.685,0.989,0.315)	(0.55,0.9,0.45)	(0.725,0.55,0.275)	(0.494,0.957,0.506)	(0.46,0.92,0.54)	(0.6,1,0.4)
D_5	(0.215,0.311,0.785)	(0.575,0.85,0.425)	(0.5,1,0.5)	(0,0,0)	(0.65,0.7,0.35)	(0.2,0,0.8)
D_6	(0.585,0.844,0.415)	(0.625,0.75,0.375)	(0.75,0.5,0.25)	(0.163,0.345,0.837)	(0.38,0.76,0.62)	(0.2,0,0.8)
D_7	(0.562,0.811,0.438)	(0.65,0.7,0.35)	(0.675,0.65,0.325)	(0.546,0.859,0.454)	(0.42,0.84,0.58)	(0.4,0,0.6)
D_8	(0.485,0.7,0.515)	(0.5,1,0.5)	(0.55,0.9,0.45)	(1,0,0)	(0.525,0.95,0.475)	(0,0,1)
D_9	(0.515,0.744,0.485)	(0.575,0.85,0.425)	(0.5,1,0.5)	(0.726,0.52,0.274)	(0.5,1,0.5)	(0.8,0,0.2)
D_{10}	(0.869,0.425,0.131)	(0.4,0.8,0.6)	(0.425,0.85,0.575)	(0.529,0.892,0.471)	(0.49,0.98,0.51)	(0.4,0,0.6)
D_{11}	(0.315,0.456,0.685)	(0.225,0.45,0.775)	(0.575,0.85,0.425)	(1,0,0)	(0.235,0.47,0.765)	(0.2,0,0.8)
D_{12}	(0.869,0.425,0.131)	(0.575,0.85,0.425)	(0.65,0.7,0.35)	(0.071,0.152,0.929)	(0.07,0.14,0.93)	(0.6,1,0.4)
D_{13}	(0.846,0.5,0.154)	(0.375,0.75,0.625)	(0.3,0.6,0.7)	(0.337,0.715,0.663)	(0.54,0.92,0.46)	(0.2,0,0.8)
D_{14}	(0.877,0.4,0.123)	(0.375,0.75,0.625)	(0.275,0.55,0.725)	(0.117,0.248,0.883)	(0.685,0.63,0.315)	(0.6,1,0.4)
D_{15}	(0.954,0.15,0.046)	(0.25,0.5,0.75)	(0.475,0.95,0.525)	(0.063,0.133,0.937)	(0.275,0.55,0.725)	(0.8,0,0.2)
D_{16}	(1,0,0)	(0,0,1)	(0.325,0.65,0.675)	(0,0,1)	(0.39,0.78,0.61)	(0,0,1)
D_{17}	(0.977,0.075,0.023)	(0.25,0.5,0.75)	(0,0,1)	(0.063,0.133,0.937)	(0.485,0.97,0.515)	(1,0,0)
D_{18}	(0.938,0.2,0.062)	(0.3,0.6,0.7)	(0.225,0.45,0.775)	(0.009,0.018,0.991)	(0.355,0.71,0.645)	(0.6,1,0.4)
D_{19}	(0.969,0.1,0.031)	(0.25,0.5,0.75)	(0,0,1)	(0,0,1)	(0.655,0.69,0.345)	(0,0,1)
D_{20}	(1,0,0)	(0.25,0.5,0.75)	(0.25,0.5,0.75)	(0,0,1)	(0.235,0.47,0.765)	(0.2,0,0.8)

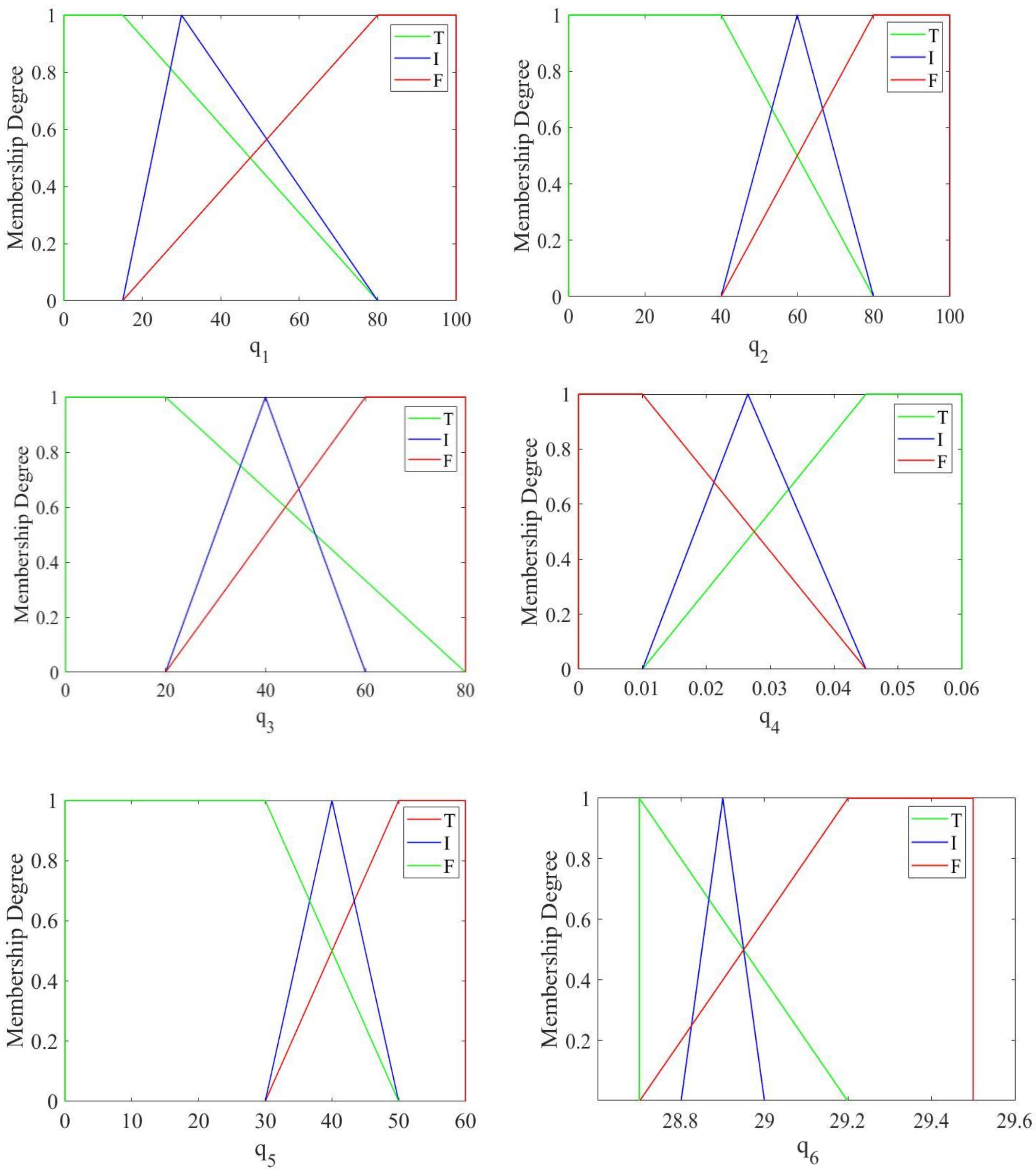

Figure 1. MFs of 6 impact factors

First, we chose appropriate true, indeterminate, and false MFs to fuzzify each impact factor in Table 1 into the form of SVNN. The different MFs for impact factors are shown in Table 2, then Figure 1 shows the curves of 18 MFs for six impact factors. Thus, the data (six impact factors) of the 20 slope samples D_k (k = 1, 2, …, 20) are fuzzified into SVNNs, which are given in Table 3.

Then, we use the proposed netting clustering method to classify the 20 slope samples with SVNS information. By the clustering analysis based on Eqs. (3) and (4) for $\varphi = 2$, the similarity matrix is obtained and shown in Figure 2, and then the slope samples are classified into 4 types when we specify the interval range $0.88899 \le \beta \le 1$, which are shown in Figure 3. Obviously, the set of slope samples {D_1, D_5, D_{11}} is classified into the same type; the set of slope samples {D_2, D_3, D_4, D_6, D_7, D_8, D_9} is classified into the same type; the set of slope samples {D_{10}, D_{12}, D_{13}, D_{14}} is classified into the same type; the set of slope samples {D_{15}, D_{16}, D_{17}, D_{18}, D_{19}, D_{20}} is classified into the same type.

H=

1	0.644762	0.585407	0.568223	0.936656	0.678179	0.703028	0.784727	0.752237	0.56052	0.960426	0.56052	0.575388	0.554675	0.474981	0.411722	0.444608	0.493716	0.455023	0.411722
0.644762	1	0.944582	0.928725	0.576828	0.968344	0.944582	0.865077	0.896953	0.697063	0.686481	0.697063	0.738776	0.682964	0.536873	0.444509	0.491166	0.566843	0.506505	0.444509
0.585407	0.944582	1	0.984174	0.516003	0.91285	0.888995	0.80895	0.841084	0.684017	0.627923	0.684017	0.72728	0.669471	0.519943	0.425927	0.473394	0.550505	0.489009	0.425927
0.568223	0.928725	0.984174	1	0.49836	0.896953	0.873057	0.792814	0.825039	0.678498	0.61099	0.678498	0.721939	0.663898	0.51385	0.41951	0.467142	0.544516	0.48281	0.41951
0.936656	0.576828	0.516003	0.49836	1	0.61099	0.636354	0.7195	0.686481	0.507952	0.896953	0.507952	0.519181	0.503351	0.435999	0.379395	0.409051	0.452354	0.418345	0.379395
0.678179	0.968344	0.91285	0.896953	0.61099	1	0.97626	0.896953	0.928725	0.69985	0.7195	0.69985	0.739968	0.6862	0.543326	0.452312	0.498325	0.572779	0.513436	0.452312
0.703028	0.944582	0.888995	0.873057	0.636354	0.97626	1	0.92079	0.952506	0.699628	0.744078	0.699628	0.738228	0.686407	0.546566	0.456774	0.502209	0.575541	0.517114	0.456774
0.784727	0.865077	0.80895	0.792814	0.7195	0.896953	0.92079	1	0.968344	0.684856	0.825039	0.684856	0.716978	0.673511	0.547221	0.462797	0.505716	0.574059	0.519712	0.462797
0.752237	0.896953	0.841084	0.825039	0.686481	0.928725	0.952506	0.968344	1	0.693282	0.792814	0.693282	0.728184	0.68112	0.548841	0.462037	0.50607	0.57663	0.520469	0.462037
0.56052	0.697063	0.684017	0.678498	0.507952	0.69985	0.699628	0.684856	0.693282	1	0.590467	1	0.958367	0.986126	0.846796	0.761953	0.804544	0.874811	0.818662	0.761953
0.960426	0.686481	0.627923	0.61099	0.896953	0.7195	0.744078	0.825039	0.792814	0.590467	1	0.590467	0.607964	0.583737	0.495697	0.428206	0.463126	0.515987	0.474255	0.428206
0.56052	0.697063	0.684017	0.678498	0.507952	0.69985	0.699628	0.684856	0.693282	1	0.590467	1	0.958367	0.986126	0.846796	0.761953	0.804544	0.874811	0.818662	0.761953
0.575388	0.738776	0.72728	0.721939	0.519181	0.739968	0.738228	0.716978	0.728184	0.958367	0.607964	0.958367	1	0.944476	0.804544	0.718951	0.761953	0.832745	0.776192	0.718951
0.554675	0.682964	0.669471	0.663898	0.503351	0.6862	0.686407	0.673511	0.68112	0.986126	0.583737	0.986126	0.944476	1	0.860817	0.776192	0.818662	0.888782	0.832745	0.776192
0.474981	0.536873	0.519943	0.51385	0.435999	0.543326	0.546566	0.547221	0.548841	0.846796	0.495697	0.846796	0.804544	0.860817	1	0.916661	0.958367	0.972249	0.972249	0.916661
0.411722	0.444509	0.425927	0.41951	0.379395	0.452312	0.456774	0.462797	0.462037	0.761953	0.428206	0.761953	0.718951	0.776192	0.916661	1	0.958367	0.888782	0.944476	1
0.444608	0.491166	0.473394	0.467142	0.409051	0.498325	0.502209	0.505716	0.50607	0.804544	0.463126	0.804544	0.761953	0.818662	0.958367	0.958367	1	0.930576	0.986126	0.958367
0.493716	0.566843	0.550505	0.544516	0.452354	0.572779	0.575541	0.574059	0.57663	0.874811	0.515987	0.874811	0.832745	0.888782	0.972249	0.888782	0.930576	1	0.944476	0.888782
0.455023	0.506505	0.489009	0.48281	0.418345	0.513436	0.517114	0.519712	0.520469	0.818662	0.474255	0.818662	0.776192	0.832745	0.972249	0.944476	0.986126	0.944476	1	0.944476
0.411722	0.444509	0.425927	0.41951	0.379395	0.452312	0.456774	0.462797	0.462037	0.761953	0.428206	0.761953	0.718951	0.776192	0.916661	1	0.958367	0.888782	0.944476	1

Figure 2. The 20×20 similarity matrix *H*

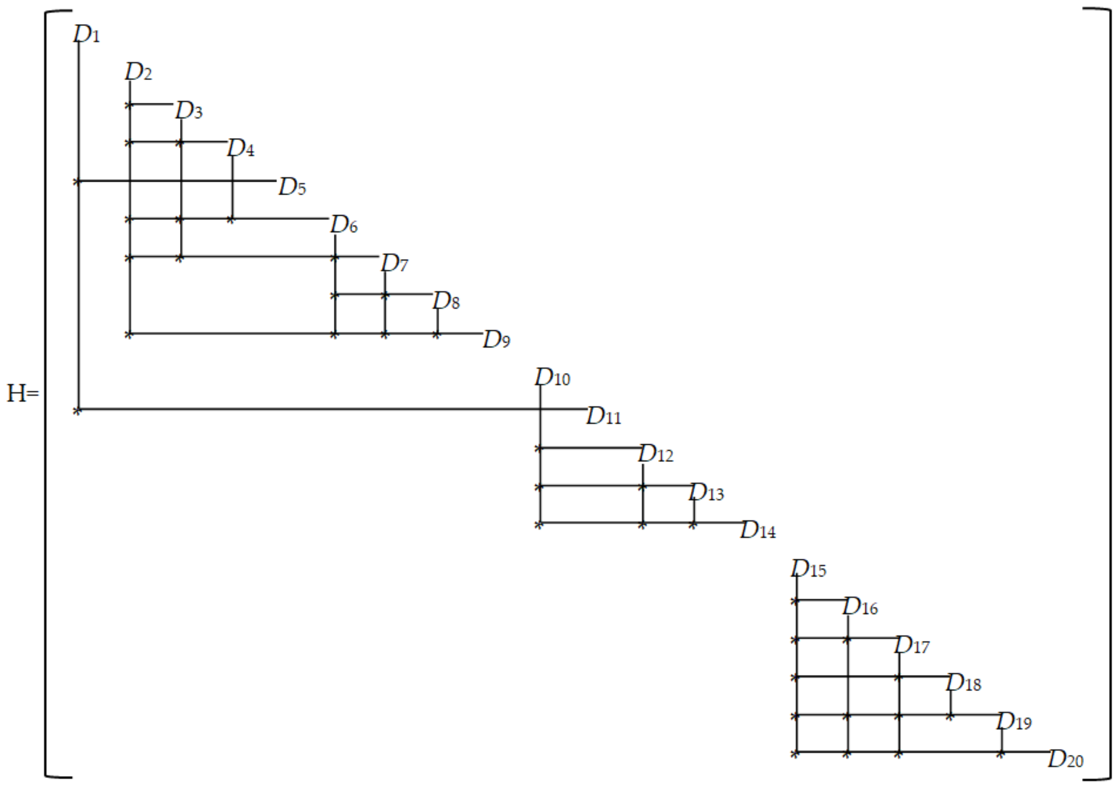

Figure 3. Netting clustering analysis results based on the proposed HSSM

Although the above 20 slope samples are clustered into the four types of slope stability, we don't know which type belongs to which risk grade/pattern. In this case, we must give the stability evaluation of the 20 slope samples to recognize the corresponding risk patterns/grades of the slope stability.

4.2. Clustering Analysis of Actual Cases

According to the above clustering results of the 20 slope samples, there are the four risk patterns/grades. Based on the risk knowledge/experience of the open-pit mine slope stability, we can establish the slope stability four risk patterns/grades: stability (R_1), basic stability (R_2), relative stability (R_3), and instability (R_4), which are expressed by SVNNs in Table 4.

In view of SVNNs in Table 4, we can give the following SVNSs of the four risk patterns/grades:

$R_1 = \{d_{11}, d_{12}, d_{13}, d_{14}, d_{15}, d_{16}\} = \{\langle 0.25, 0.36, 0.75\rangle, \langle 0.81, 0.38, 0.19\rangle, \langle 0.88, 0.25, 0.13\rangle, \langle 0.78, 0.42, 0.15\rangle, \langle 0.88, 0.25, 0.13\rangle, \langle 0.2, 0, 0.8\rangle\}$;

Table 4. Risk patterns of slope stability in the setting of SVNNs

q_i	R_1			R_2			R_3			R_4		
	D_{T1i}	D_{I1i}	D_{F1i}	D_{T2i}	D_{I2i}	D_{F2i}	D_{T3i}	D_{I3i}	D_{F3i}	D_{T4i}	D_{I4i}	D_{F4i}
q_1	0.25	0.36	0.75	0.56	0.81	0.44	0.74	0.83	0.26	0.9	0.34	0.1
q_2	0.81	0.38	0.19	0.59	0.83	0.41	0.38	0.75	0.63	0.13	0.25	0.88
q_3	0.88	0.25	0.13	0.63	0.75	0.38	0.43	0.85	0.58	0.18	0.35	0.83
q_4	0.78	0.42	0.15	0.56	0.84	0.39	0.33	0.7	0.64	0.09	0.2	0.9
q_5	0.88	0.25	0.13	0.63	0.75	0.38	0.38	0.75	0.63	0.13	0.25	0.88
q_6	0.2	0	0.8	0.5	0.5	0.38	0.7	0.5	0.3	0.9	0	0.1

$R_2 = \{d_{21}, d_{22}, d_{23}, d_{24}, d_{25}, d_{26}\} = \{\langle 0.56, 0.81, 0.44\rangle, \langle 0.59, 0.83, 0.41\rangle, \langle 0.63, 0.75, 0.38\rangle, \langle 0.56, 0.84, 0.39\rangle, \langle 0.63, 0.75, 0.38\rangle, \langle 0.5, 0.5, 0.38\rangle\}$;

$R_3 = \{d_{31}, d_{32}, d_{33}, d_{34}, d_{35}, d_{36}\} = \{\langle 0.74, 0.83, 0.26\rangle, \langle 0.38, 0.75, 0.63\rangle, \langle 0.43, 0.85, 0.58\rangle, \langle 0.33, 0.7, 0.64\rangle, \langle 0.38, 0.75, 0.63\rangle, \langle 0.7, 0.5, 0.3\rangle\}$;

$R_4 = \{ d_{41}, d_{42}, d_{43}, d_{44}, d_{45}, d_{46}\} = \{\langle 0.9, 0.334, 0.1\rangle, \langle 0.13, 0.25, 0.88\rangle, \langle 0.18, 0.35, 0. 83\rangle, \langle 0.09, 0.2, 0.9\rangle, \langle 0.13, 0.25, 0.88\rangle, \langle 0.9, 0, 0.1\rangle\}$.

Table 5. Results of the proposed HSSM

D_j	$H_\varphi(D_j, R_1)$	$H_\varphi(D_j, R_2)$	$H_\varphi(D_j, R_3)$	$H_\varphi(D_j, R_4)$	Risk grade
D_1	**0.896665**	0.703826	0.608403	0.517216	R_1
D_2	0.677753	**0.891382**	0.823401	0.640984	R_2
D_3	0.607045	**0.812724**	0.788762	0.615625	R_2
D_4	0.583422	**0.869008**	0.838668	0.550995	R_2
D_5	**0.796918**	0.734908	0.657685	0.515831	R_1
D_6	0.732703	**0.851046**	0.792102	0.617296	R_2
D_7	0.732119	**0.916111**	0.813484	0.60338	R_2
D_8	0.70132	**0.793529**	0.72165	0.513288	R_2
D_9	0.670875	**0.87848**	0.816791	0.637876	R_2
D_{10}	0.620938	0.787465	**0.836347**	0.7285	R_3
D_{11}	**0.775396**	0.696152	0.676804	0.617519	R_1
D_{12}	0.547174	0.758627	**0.764204**	0.734565	R_3
D_{13}	0.644711	0.759293	**0.834798**	0.73285	R_3
D_{14}	0.537107	0.731683	**0.809379**	0.775532	R_3
D_{15}	0.508192	0.640297	0.742197	**0.856468**	R_4
D_{16}	0.480595	0.496986	0.578814	**0.734005**	R_4
D_{17}	0.446757	0.560614	0.655778	**0.842855**	R_4
D_{18}	0.483972	0.646724	0.742483	**0.794429**	R_4
D_{19}	0.530704	0.551393	0.606451	**0.740304**	R_4
D_{20}	0.549504	0.570076	0.651012	**0.786229**	R_4

Then, the slope stability of the 20 slope samples D_j (j = 1, 2, …, 20) is assessed by Eq. (5) for φ = 2, and the HSSM values between the slope samples D_j and the slope stability risk patterns R_k (k = 1, 2, 3, 4) are given in Table 5. The maximum measure value between D_j and R_k reflects the corresponding slope stability risk pattern/grade. From the evaluation results, it can be found that the four types of the 20 slope samples obtained by the proposed clustering method are consistent with the four risk patterns/levels:

(i) The set of slope samples {D_1, D_5, D_{11}} is the risk grade R_1;
(ii) The set of slope samples {D_2, D_3, D_4, D_6, D_7, D_8, D_9} is the risk grade R_2;
(iii) The set of slope samples {D_{10}, D_{12}, D_{13}, D_{14}} is the risk grade R_3;
(iv) The set of slope samples {D_{15}, D_{16}, D_{17}, D_{18}, D_{19}, D_{20}} is the risk grade R_4.

The above results prove the accuracy and validity of the proposed netting clustering method and the proposed evaluation method for the 20 slope samples.

4.3. Comparative Analysis

Regarding comparative analysis, we use the weighted generalized distance-based similarity measure of Eq. (2) [20] to assess the stability risk grades of the 20 slope samples. All the evaluation results are given in Table 6. It is obvious that the risk grade of each slope sample assessed by Eq. (2) for φ = 2 [20] is the same as that evaluated by the proposed HSSM of SVNSs. Therefore, the slope stability evaluation method using the proposed HSSM of SVNSs verifies its effectiveness and accuracy in the open-pit mine slope stability evaluation problems.

Table 6. Evaluation results based on Eq. (2)

D_j	$S_\varphi(D_j, R_1)$	$S_\varphi(D_j, R_2)$	$S_\varphi(D_j, R_3)$	$S_\varphi(D_j, R_4)$	Risk grade
D_1	**0.884997**	0.671241	0.56761	0.471864	R_1
D_2	0.641967	**0.87775**	0.801116	0.603543	R_2
D_3	0.568019	**0.792424**	0.765862	0.578501	R_2
D_4	0.54286	**0.852145**	0.818222	0.507299	R_2
D_5	**0.77512**	0.706827	0.62318	0.470381	R_1
D_6	0.703056	**0.833838**	0.76785	0.577951	R_2
D_7	0.70058	**0.905584**	0.790263	0.561491	R_2
D_8	0.667514	**0.771982**	0.694343	0.471249	R_2
D_9	0.635115	**0.863307**	0.793656	0.60041	R_2
D_{10}	0.581658	0.761448	**0.816313**	0.699008	R_3
D_{11}	**0.751333**	0.662475	0.643164	0.581591	R_1
D_{12}	0.505493	0.732561	**0.735527**	0.709053	R_3
D_{13}	0.608234	0.730184	**0.815707**	0.704047	R_3
D_{14}	0.493782	0.700375	**0.785899**	0.753015	R_3
D_{15}	0.463014	0.602347	0.712527	**0.838671**	R_4
D_{16}	0.440282	0.455418	0.539813	**0.707829**	R_4
D_{17}	0.402051	0.519151	0.619739	**0.823512**	R_4
D_{18}	0.439269	0.609585	0.712951	**0.772755**	R_4
D_{19}	0.491309	0.510861	0.568517	**0.715473**	R_4
D_{20}	0.511998	0.528949	0.614943	**0.76203**	R_4

5. Conclusions

The paper proposed HSSM of SVNSs and established its netting clustering analysis and risk evaluation methods for open-pit mine slopes in the scenario of SVNSs. Then, the proposed netting clustering analysis and risk evaluation methods were used for the clustering analysis and risk evaluation of open-pit mine slopes. In the applications of the clustering analysis and risk evaluation methods of slope samples, they contain the following techniques. First, appropriate true, indeterminate, and false membership functions for the impact factors of slope stability were fuzzified into the true, indeterminate, and false fuzzy values, which are constructed as the form of SVNNs. Then, the proposed netting clustering method based on the proposed HSSM was used to cluster the slope samples. Further, based on the clustering results and risk knowledge of slope stability, we gave the corresponding risk patterns/grades to evaluate the risk grades of slope stability by the HSSM values between the slope samples and the slope stability patterns in the scenario of SVNSs. Finally, the proposed netting clustering analysis and risk evaluation methods were applied to the clustering analysis and risk evaluation of 20 slope samples. The comparative results proved the accuracy, validity and rationality of the proposed netting clustering analysis and risk evaluation methods.

The main advantage of this study is that the proposed clustering method and the slope stability assessment approach can simply and effectively process the clustering analysis and evaluation problems of open-pit mine slopes; while the existing evaluation methods using ANN, ANFIS, and SVNANFIS [2, 4, 9] imply the defects of both the complex learning algorithms and the requirement of larger-scale sample data. It is obvious that the proposed methods effectively overcome the defects of the existing evaluation methods [2, 4, 9] and are more convenient and more reasonable than the existing clustering analysis and evaluation methods [1-4].

Regarding future research, more slope samples and more impact factors will be considered to further verify the accuracy and efficiency of the proposed clustering and evaluation methods. Then, new similarity measures and clustering and evaluation methods will be further proposed to make their clustering and evaluation methods more effective and reasonable in the setting of SVNSs.

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Neutrosophic Logic-based DIANA Clustering algorithm

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Abstract. On the one hand, the most extensively used Hierarchical Clustering techniques are the Hierarchical Divisive Clustering (HDC) algorithms such as DIANA. Its primary goal is to build the tree of Hierarchical Agglomerative Clustering (HAC) in reverse order. On the other hand, Neutrosophy is an extension of fuzzy logic and serves as a model of uncertainty. In addition to the truth (T) and falsity (F) elements of fuzzy logic, single-valued Neutrosophic sets (SVNs) logic estimates the proportion of indeterminacy (I) for a given proposition. In this work, we propose a Neutrosophic Logic-based DIANA Clustering algorithm. Indeterminacy is added to the DIANA hierarchical clustering algorithm using single-valued Neutrosophic sets (SVNs). The suggested algorithm is named Neutro-DIANA (Neutrosophic DIANA) and is broken down into numerous steps. The experimental findings show that the suggested technique for dealing with indeterminacy is effective.

Keywords: Hierarchical Divisive Clustering (HDC), DIANA, Neutrosophic, Indeterminacy, Neutro-DIANA.

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1. Introduction

Clustering is a subsection of unsupervised learning, which is one of the four basic subcategories of machine learning techniques. Clustering, as a learning method, is useful in numerous domains, including market segmentation [1], customer regrouping [2], Big data analysis [3], image processing [4], and so on. Clustering algorithms are classified into four types [5]: (1) K-means and K-medoids for partitioning. (2) Density-based approaches such as DBSCAN and OPTICS. (3) Model-based approaches like SOM and EM. (4) AGNES (AGglomerative NESting) and DIANA (DIvisive ANAlysis) are hierarchical approaches.

The method of organizing data points inside clusters is known as hierarchical clustering. Hierarchical clustering may be done in two ways: agglomerative (bottom-up) and divisive (top-down). In contrast to the Agglomerative method, the Divisive method Hierarchical Clustering starts with a single cluster that contains all entities, then divides the instances into a hierarchy of smaller and smaller clusters until each cluster contains just one entity or a predefined amount of entities.

Numerous hierarchical clustering techniques that use the divisive approach include TWINSPAN (Two-Way Indicator Species Analysis), MONA (divisive hierarchical MONothetic Analysis), and DIANA (DIvisive hierarchical ANAlysis). The most well-known and effective algorithm is DIANA. It is a polythetic divisive method that works with any matrix of dissimilarities. It attempts to combine a collection of data items that are comparable to one another into a single cluster, while dissimilar data objects are connected with other clusters [6].

The DIANA method creates a hierarchy of sub-clusters starting with a single cluster containing all n items. The greatest diameter cluster is split at each stage until there is just one element in each cluster. To achieve this, the algorithm looks for the element in the chosen cluster that differs from other elements on average by the most. The algorithm then reassigns items that are more closely related to the "splinter group" than to the "old group" in succeeding phases once the "splinter group" has been chosen. Two new clusters are the outcome. The greatest distance between items in the two sub-clusters determines the distance between the clusters. The average of all $1-d(i)$, or $d(i)$, is the diameter of the final group including element i divided by the diameter of the whole dataset. This is known as the divisive coefficient (DC).

As previously stated, the DIANA algorithm was designed to cope with crisp numbers, and any data issues should be addressed during the data preparation phase. However, many real-world situations are imprecise and unclear, and their data contains impurities such as imprecision, uncertainty, and so on. As an extension of fuzzy logic, Neutrosophic [7], [8]. is proposed to cope with these information flaws. To achieve this, the Neutrosophic provides a new parameter termed indeterminacy membership (I) in addition to the two values of the fuzzy logic, degree of truth-membership (T) and falsity-membership (F).

In this study, we develop the Neutrosophic set (SVNs)-based Clustering approach to address the shortcoming of fuzzy logic (sets, IFSs, and IVIFSs)-based clustering algorithms, which are unable to capture inconsistent information that corresponds to the real-world data. In a Neutrosophic setting, each element's truth, falsity, and indeterminacy (T, F, I) values are computed to identify whether or not it belongs to any given cluster. Considered to be a neutrosophic component, e is expressed as $e(T, F, I)$. The input data is initially subjected to a neutrosophic real problem formulation. This step's output is sent into the neutrosophic-based DIANA clustering method.

As you can see from this introduction, we get right into the research issue without debating the rationale for the method's selection, the necessity of the hybridization, etc. We would like to let you know that this study is a successor to a paper we wrote on a cutting-edge topic called neutrosophic and machine learning [9]. In that paper, we documented all hybrid machine learning algorithms that used Single-valued Neutrosophic Sets (SVNs) approaches, and we subsequently created a taxonomy of Neutrosophic Machine Learning algorithms. In other words, we have a list of the algorithms that have previously been used, and more details are provided in [9]. In this paper, we will concentrate on the Neutrosophic-based hierarchical clustering approach.

The rest of this paper is structured as follows: Section II delves into the background and preliminaries. The proposed Neutro-DIANA algorithm is explained in detail in Section III, while the experiments and insightful discussion of the findings are provided in Section VI. Lastly, Section VI brings this study to a close and proposes some future research areas.

1.1. *Related works*

The image segmentation technique was enhanced by Qureshi et al. [6] utilizing K-Means Clustering with Neutrosophic Logic. The technique entails converting an image into a neutrosophic collection. The neutrosophic-based k-means approach is used to segment neutrosophic images, and SVNs are used to quantify the indeterminacy in pixels of an image. To tackle the ambiguous and inconsistent information that the fuzzy is unable to handle, Vandhana et al. [10] adopted neutrosophic fuzzy hierarchical clustering. The method is used to analyze and pinpoint regions where illnesses like dengue fever are influenced by environmental and climatic factors. As an extension of the hierarchical clustering method, Sahin [11] presented a single-valued neutrosophic hierarchical clustering technique for clustering SVNSs. The technique was further expanded to categorize interval neutrosophic data. Ye [12] presented the single-valued neutrosophic minimum spanning tree (SVNMST) clustering technique as an extension of the intuitionistic fuzzy minimum spanning tree (IFMST) clustering algorithm. The approach is based on the generalized distance measure of SVNSs. H2D-FCM is a Fuzzy-based divisive hierarchical clustering technique introduced by Bordogna and Pasi [13]. It automatically estimates the number of clusters to produce and then divides the node into sub-clusters using the probabilistic Fuzzy C Means method. In Ding's study [14], Ding et al. addressed the most important topic in Hierarchical clustering algorithms: choosing the appropriate next cluster(s) to divide or merge. They determined that the average similarity approach is the best for divisive clustering and MinMaxmis the best for agglomerative clustering. In another study, Ye [15] introduced clustering algorithms for SVNs using distance-based similarity metrics in another study (Single-Valued Neutrosophic Sets). To meet the aforementioned goals, we present a novel Neutrosophic Hierarchical Divisive Clustering algorithm (n-DIANA), based on a divisive approach.

2. **Background**

2.1. *Single valued neutrosophic set (SVNS)*

Smarandache's notion of the neutrosophic set [8] is challenging to transfer in a genuine application and engineering challenge. As a result, Wang et al. [16, 17] established the neutrosophic set notions of SVNS (single-valued neutrosophic set) and INS (interval neutrosophic set). To execute the necessary calculus, various mathematical operations in a neutrosophic context, such as euclidean distance, average, minimum maximum, and so on, must be defined.

Definition 2.1. Consider X to be a universe discourse and $A1$ to be a single valued neutrosophic set over X. $A1$ takes the following form:

$$A_1 = \{\langle x, \mu_{A_1}(x), \omega_{A_1}(x), \nu_{A_1}(x)\rangle : x \in X\}. \tag{1}$$

where $\mu_{A_1} : X \to [0,1]$, $\omega_{A_1} : X \to [0,1]$, and $\nu_{A_1} : X \to [0,1]$, with the constraint $0 \leq \mu_{A_1}(x) + \omega_{A_1}(x) + \nu_A(x) \leq 3$, $\forall x \in X$

The values $\mu_{A_1}(x)$, $\omega_{A_1}(x)$, and $\nu_{A_1}(x)$ represent the degree of truth-membership, indeterminacy-membership and falsity-membership of x to X respectively.

Definition 2.2. For below, consider tow SVN measurements $A1$ and $A2$, where $A_1 = \{\langle x, \mu_{A_1}(x), \omega_{A_1}(x), \nu_{A_1}(x)\rangle : x \in X\}$, $A_2 = \{\langle x, \mu_{A_2}(x), \omega_{A_2}(x), \nu_{A_2}(x)\rangle : x \in X\}$

The fundamental arithmetic operations are as follows:

$$A_1 + A_2 = \{\langle x, \mu_{A_1}(x) + \mu_{A_2}(x) - \mu_{A_1}(x)\mu_{A_2}(x), \omega_{A_1}(x)\omega_{A_2}(x), \nu_{A_1}(x)\nu_{A_2}(x)\rangle : x \in X\} \tag{2}$$

$$\lambda A_1 = \{\langle x, 1 - (1 - \mu_{A_1}(x))^{\lambda}, (\omega_{A_1}(x))^{\lambda}, (\nu_{A_1}(x))^{\lambda}\rangle : x \in X \text{ and } \lambda \geq 0\}. \tag{3}$$

2.2. *DIANA (DIvisive ANAlysis)*

DIANA [18–20] is a hierarchical clustering strategy that groups items into multiple clusters, each of which contains elements that are similar to one another. The clustering method DIANA utilized in this study may be summed up as follows:

- Step 1: At first, DIANA assumes that all n observations are contained within a single cluster.
- Step 2: Divide the Clusters again and again until each cluster has just one observation.
  - Choose the pair of clusters with the greatest dissimilarity in the current cluster, which is $\{\zeta_r\}$, and $\{\zeta_s\}$, in which $d(\{\zeta_r\}, \{\zeta_s\}) = \max\{d(\zeta_i, \zeta_j)_{0 \leq i,j \leq n}\}$.
  - The cluster is divided into $(zetas)$ and $(zetar)$ clusters to generate the following clusters.
- Step 3: If all clusters are made up of a single element, break; otherwise, continue to step 2.

In comparison to Agglomerative Hierarchical Clustering, divisions in the DIANA technique are based on average distance and cophenetic distance, which are equivalent to average linkage and full linkage, respectively. The mean distance between the cluster centroid and the other objects is computed by taking the average of the Euclidean distances between the cluster centroid and each item.

Consider $\Theta = \{A_i, i = 1 \ldots n\}$ as the space of n observations, and ζ as the cluster's center; Eq. 4 gives the average distance between ζ and other objects.

$$\text{Mean}(d(\zeta, \Theta\backslash\zeta)) = \frac{1}{|\Theta\backslash\zeta|} \sum_{\forall A_i \in \Theta\backslash\zeta} d(\zeta, A_i) \tag{4}$$

3. Neutro-DIANA proposed method

Neutrosophic Clustering is based on the Single-valued Neutrosophic sets (SVNs) technique, in which data points belong to several clusters with membership degrees in the range $[0, 1]$.

Definition . The neutrosophic DIANA algorithm is a clustering algorithm that uses neutrosophic logic principles and neutrosophic sets. It uses SVNs-based operations in the calculation of its clustering algorithm.

3.1. *Neutrosophic Set Formation*

Assume a dataset comprises a collection of n SVNs denoted Θ, where $\Theta = \{A_i/1 \leq i \leq n\}$ is defined in a universe of discourse X in the SVNs environment, and each object is expressed as : Let x be a vector in an n-dimensional real space Rn (the fea- ture space) and let C=c1, c2, ..., cc, be a set of class labels. A neutrosophic classifier is mapping of the type:

$$\psi \colon \mathbb{R}^n \longrightarrow \{\mathbb{TC}(x), \mathbb{IC}(x), \mathbb{FC}(x) | x \in \mathbb{R}^n\} \tag{5}$$

$$\psi \colon \mathbb{R}^n \longrightarrow \{\mathbb{TC}(x), \mathbb{IC}(x), \mathbb{FC}(x) | x \in \mathbb{R}^n\} \tag{6}$$

Let x be a vector in the n-dimensional features space $\mathbb{R}^n$, and $C = \{c_1, c_2, \cdots, c_c\}$, be a collection of class labels. A neutrosophic classifier is a sort of mapping:

$$A_i = \{\langle x_j, \mu_{A_i}(x_j), \omega_{A_i}(x_j), \nu_{A_i}(x_j)\rangle : x_j \in X\}. \tag{7}$$

We generate the Neutrosophic Distance matrix-nD0 using SVNS similarity and/or dissimilarity measurements ((8)), as indicated in the table (1) below.

3.2. *The similarity in Neutrosophic environment*

Definition 3. Euclidean Neutrosophic distance. In the Neutrosophic environment, the mapping form of euclidian distance applied to A_1 and A_2 (two SVNSs) is as follows:

TABLE 1. Neutrosophic Distance matrix-nD0

	x_1	$\cdots$	x_n
A_1	$\langle \mu_{A_1}(x_1), \omega_{A_1}(x_1), \nu_{A_1}(x_1) \rangle$	$\cdots$	$\langle \mu_{A_1}(x_n), \omega_{A_1}(x_n), \nu_{A_1}(x_n) \rangle$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
A_m	$\langle \mu_{A_m}(x_1), \omega_{A_m}(x_1), \nu_{A_m}(x_1) \rangle$	$\cdots$	$\langle \mu_{A_m}(x_n), \omega_{A_m}(x_n), \nu_{A_m}(x_n) \rangle$

$$d_{eucl} = \sqrt{\frac{1}{3} \sum_{i=1}^{n} \sum_{f=(\mu,\omega,\nu)} (f_{A_1}(x) - f_{A_2}(x))^2} \tag{8}$$

where μ, ω and ν are Neutrosophic membership functions.

Definition 4. Similarity and/or dissimilarity of SVNSs measurements. The similarity measure S_{mes} between A_1 and A_2 based on max and min operators, as described by [21], is defined as follows :

$$S_{mes} = \frac{1}{3} \sum_{i=1}^{n} \frac{\sum_{f=(\mu,\omega,\nu)} \min(f_{A_1}(x), f_{A_2}(x))}{\sum_{f=(\mu,\omega,\nu)} \max(f_{A_1}(x), f_{A_2}(x))} \tag{9}$$

The following is the definition of the dissimilarity measure:

$$DIS_{mes} = 1 - S_{mes} \tag{10}$$

3.3. *nDIANA algorithm*

Let $\{A_i // i = 1 \dots n\}$ be a collection of n SVNs nDIANA consists on three main steps.

The nDIANA method starts by treating all n objects as a single cluster level$L(m_c = 0) = \Theta$ object. Using the dissimilarity measures (9), elements A_i are then pairwise compared among themselves, and then separated into two sub-clusters with sub-levels $L(m_{c+1} = 0)$, and $L(m_{c+2} = 0)$, respectively, based on the clusters' furthest (with maximum mean distance) sub-clusters. The subdividing operation is repeated until all clusters have a single-single item. That is, each obtained cluster has a size of 1. In each stage, we reapply the treatment on each sub-cluster recursively, and the distance between the object and the sub-cluster is taken as the average distance between the object and all components of the sub-cluster.

Step 1 : Calculate the similarity and/or dissimilarity measurements of SVNs using equations Eq.9 and/or Eq.10, and then create the Neutrosophic Distance matrix-nD0 (Table 1).

Step 2 : Each stage of the divisive algorithm requires a decision on which cluster to split. To do this, we compute the diameter as indicated in

$$\text{diam}(Q) = max_{j \in Q, h \in Q} d(A_j, A_h) \tag{11}$$

In a loop, choose just the element A_j with the greatest mean dissimilarity to all other elements in the same cluster.

$$d(A_i, \Theta \backslash A_i) = \frac{1}{|\Theta| - 1} \sum_{j \neq i} d(A_i, A_j) \tag{12}$$

$$\Theta_{new} = \Theta_{old} \backslash A_i$$

$$\bar{\Theta}_{new} = \bar{\Theta}_{,old} \cup A_i$$

$$d(A_i, \Theta \backslash A_i) - d(A_i, \bar{\Theta}) = \frac{1}{|\Theta| - 1} \sum_{j \in \Theta, j \neq i} d(A_i, A_j) - \frac{1}{|\bar{\Theta}|} \sum_{h \in \bar{\Theta}} d(A_i, A_h) \tag{13}$$

$\bar{\Theta}$ is the complement of Θ.

Step 3 : If all clusters contain only one observation, the procedure is complete; otherwise, go to step 2 using the sub-clusters formed in the previous iteration.

4. **Results and Discussion**

To demonstrate the usefulness of the proposed Neutrosophic DIANA method, an experiment was conducted on both the simulated and real-world datasets. For the purpose of comparison, we use the numeric example introduced by Sahin in [11]. In this case, dataset consists on five objects A_i with $1 \leq i \leq$, universe of discourse is $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.

TABLE 2. Neutrosophic Set Formation Example

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
A1	0.2 0.05 0.5	0.1 0.15 0.8	0.5 0.05 0.3	0.9 0.55 0.0	0.4 0.4 0.35	0.1 0.4 0.9	0.3 0.15 0.5	1.0 0.6 0.0
A2	0.5 0.6 0.4	0.6 0.3 0.15	1.0 0.6 0.0	0.15 0.05 0.65	0.0 0.25 0.8	0.7 0.65 0.15	0.5 0.5 0.5	0.65 0.05 0.2
A3	0.45 0.05 0.35	0.6 0.5 0.3	0.9 0.05 0.0	0.1 0.6 0.8	0.2 0.35 0.70	0.6 0.4 0.2	0.15 0.05 0.8	0.2 0.6 0.65
A4	1.0 0.65 0.0	1.0 0.25 0.0	0.85 0.65 0.1	0.2 0.05 0.8	0.15 0.3 0.85	0.1 0.6 0.7	0.3 0.6 0.7	0.5 0.35 0.7
A5	0.9 0.2 0.0	0.9 0.4 0.0	0.8 0.05 0.1	0.7 0.45 0.2	0.5 0.25 0.15	0.3 0.3 0.65	0.15 0.1 0.75	0.65 0.5 0.8

The nDIANA algorithm begins with all observations as a single cluster, $L(m_c = 0) = \{A_i / 1 \leq i \leq 5\}$, with 5 is the number of observations.

Utilize (Eq. (9), Eq.(10) to calculate the similarity and dissimilarity measures of SVNSs, and then construct Neutrosophic Distance matrix-nD0 (table 3).

To determine the similarity and dissimilarity measurements of SVNSs, use the equations ((Eq. (9), Eq.(10)), and then create the Neutrosophic Distance matrix-nD0 (table 3).

Next, use Eq. (12), Eq.(13) to compute the average distance between each element and every other element.

TABLE 3. Distance matrix–nD0

	A_1	A_2	A_3	A_4	A_5	Mean
A_1	0.000	0.661	0.563	0.636	0.508	**0.474**
A_2	0.661	0.000	0.438	0.357	0.596	0.410
A_3	0.563	0.438	0.000	0.469	0.433	0.381
A_4	0.636	0.357	0.469	0.000	0.416	0.376
A_5	0.508	0.596	0.433	0.416	0.000	0.390

Select the element with the highest distance mean, in this case (A_1), which is 0.474. As a result, the cluster $\{A_1\}$'s maximum distance from other data points is 0.474. However, before deciding to split, we must first determine which elements are closest to each new cluster. To do this, we must compute the mean distance between each element using the formulas$L(m_c = 1) = \{A_1\}$ and $L(m_c = 2) = \{\Theta\backslash A_1\}$ as given in the table 4.

TABLE 4. Distance matrix-nD01 of each observation with each cluster

	$\{A_1\}$	$\{A_2, A_3, A_4, A_5\}$
A_1	0.000	0.592
A_2	0.661	0.348
A_3	0.563	0.335
A_4	0.636	0.310
A_5	0.508	0.361

Thus, the single cluster $L(m_c = 0) = \Theta$ is split into tow clusters $L(m_c = 1) = \{A_1\}$ and $L(m_c = 2) = \Theta\backslash A_1$. With a new sub-cluster, $L(m_c = 2)$, we carry out the identical processes once more to get a new distance matrix-nD2 (able 5).

TABLE 5. Distance matrix – nD02

	A_2	A_3	A_4	A_5	Mean
A_2	0.000	0.438	0.357	0.596	0.348
A_3	0.438	0.000	0.469	0.433	0.341
A_4	0.357	0.469	0.000	0.416	0.310
A_5	0.596	0.433	0.416	0.000	0.361

From table 5 the maximum of mean distances is between A_5 and the rest at distance 0.361. Then, $L(m_c = 2) = \{A_2, A_3, A_4, A_5\}$ is split into clusters, $L(m_c = 3) = \{A_2, A_3, A_4\}$, and $L(m_c = 4) = \{A_5\}$ at a distance 0.361.

To cross check the stability of each gotten cluster $L(m_c = 3)$ and $L(m_c = 4)$, we examine the closeness of each element to both obtained cluster (table 6).

TABLE 6. Distance matrix-nD02 of each observation with each cluster

	$\{A_2, A_3, A_4\}$	$\{A_5\}$
A_2	0.265	0.596
A_3	0.302	0.433
A_4	0.275	0.416
A_5	0.482	0.000

The sub-cluster $L(m_c = 3) = \{A_2, A_3, A_4\}$ obtained from previous splitting need to be treated, and its Distance matrix–nD03 (see table 7).

TABLE 7. Distance matrix–nD03

	A_2	A_3	A_4	Mean
A_2	0.000	0.438	0.357	0.265
A_3	0.438	0.000	0.469	**0.302**
A_4	0.357	0.469	0.000	0.275

The maximum of mean distances is between A_3 and the rest of elements at distance 0.302. Then, the $L(m_c = 3) = \{A_2, A_3, A_4\}$ is split into clusters, $L(m_c = 5) = \{A_2, A_4\}$, and $L(m_c = 6) = \{A_3\}$ at a distance 0.302.

TABLE 8. Distance matrix-nD03 of each observation with each clusters

	$\{A_2, A_4\}$	$\{A_3\}$
A_2	0.179	0.438
A_3	0.454	0.000
A_4	0.179	0.469

Finally, because the remain cluster $L(m_c = 5) = \{A_2, A_4\}$ contains only two elements, it is divided into $L(m_c = 7) = \{A_2\}$ and $L(m_c = 8) = \{A_4\}$ at distance 0.357 and creates the single–single object in all clusters. As a result, the Neutrosophic Divisive Analysis Clustering (nDIANA) with Neutrosophic computation is terminated.

Here, we outline the specifics of the entire splitting process as implemented by our nDIANA suggested method.

At the beginning $\Theta = \{A_i/1 \leq i \leq 5\}$, all elements are in the same cluster $\{A_1, A_2, A_3, A_4, A_5\}$.

The farthest dissimilarity measure is of A_1 (Eq.14).

$$d(A_1, \Theta \backslash A_1) = \max\{d_{1 \leq i \leq 5}(A_i, \Theta \backslash A_i)\} \quad (14)$$

which is 0.474 terminates $\{A_1, A_2, A_3, A_4, A_5\}$ is split into tow clusters : $\{A_1\}$and $\{A_2, A_3, A_4, A_5\}$.

The farthest dissimilarity measure in gotten sub-cluster is of A_5 (Eq.15).

$$d(A_5, \Theta \backslash \{A_1, A_5\}) = \max\{d_{2 \leq i \leq 5}(A_i, \Theta \backslash \{A_i, A_1\})\} \tag{15}$$

which is 0.361, then $\{A_2, A_3, A_4, A_5\}$ are split into tow clusters : $\{A_2, A_3, A_4\}$ and $\{A_5\}$.

The farthest dissimilarity measure in gotten sub-cluster is of A_3 (Eq.16).

$$d(A_3, \Theta \backslash \{A_1, A_5, A_3\}) = \max\{d_{i=2,4}(A_i, \Theta \backslash \{A_i, A_1, A_5\})\} \tag{16}$$

which is 0.302, then $\{A_2, A_3, A_4\}$ are split into tow clusters : $\{A_2, A_4\}$ and $\{A_3\}$.

There are only two elements left $\{A_2, A_4\}$, the distance between them is 0.357, and in this case the subdivision is automatic to two clusters which are: $\{A_2\}$ and $\{A_4\}$.

By the end, put all the results together we get :

first $((A_1), A_2, A_3, A_4, A_5)$,

next $((A_1), (A_2, A_3, A_4, (A_5)))$,

then $((A_1), (((A_2, A_4), (A_3)), (A_5)))$,

finally $((A_1), ((((A_2), (A_4)), (A_3)), (A_5)))$.

Each and every machine learning method is designed to tackle learning issues involving crisp numbers. However, all data sources produce inaccurate, imprecise, and ambiguous data that has numerous other flaws. A broad framework is provided by the single-valued Neutrosophic set (SVNs), an extension of the fuzzy logic set, to describe and model uncertain, imperfect, and imprecise data with missing and mistakes. By using machine learning algorithms designed for precise numbers, it is possible to build tidy data that is purported to be clean but really goes through a lot of creation and destruction processes simultaneously. Hence, clustering learning in a single-valued Neutrosophic environment is another way to capture and manage data noise and take it into account as an additional source of information. To handle data noise and take it into account as an extra factor, clustering learning in a single-valued Neutrosophic environment is a different technique.

5. **Conclusion**

In conclusion, we obtained the same outcomes when comparing the Agglomerative Hierarchical Clustering Technique and the DIANA with Neutrosophic findings on the simulated data set. And from there, we may conclude that (1) the DIANA with Neutrosophic algorithm can aggregate SVNs on a wide scale, and (2) the uncertainty information acquired by SVNs is crucial for the accomplishment of some aggregation tasks. We have created a useful approach for grouping SVNs using divisive hierarchical clustering.

To reduce data indeterminacy, the hierarchical clustering divisive DIANA method based on Neutrosophic logic is used. The suggested method's findings show that it may be utilized to produce superior outcomes on real-world data. Based on the crisp hierarchical clustering technique, we suggested a hierarchical single-value neutrosophic algorithm for SVN clustering.

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Closed neutrosophic dominating set in neutrosophic graphs

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Abstract: The aim of this article is to concentrate on the notion of closed neutrosophic domination (CND) number $\gamma cl\ (G)$ of a neutrosophic graph (NG) with using effective edge, furthermore we gain a few outcomes on this notion, the relation between $\gamma cl\ (G)$ and some other notions is acquired, eventually the notion of (CND) number of (join neutrosophic graphs) is came in.

Keywords: fuzzy graph, neutrosophic graph, domination set, domination number

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1. Introduction

A graph is a nonempty set whose elements are called vertices or points. It also contains a set of elements consisting of unordered pairs of vertices; these elements are called edges or lines [1]. There are many relations between graph theory and other branches of mathematics such as Topology, Algebra, Probability, Fuzzy and Numerical Analysis. In addition, there are relations with other sciences such as Engineering, Computer Science, Chemistry, Physics, and Biology[2]. The concept of graph domination is one of the topics in graph theory, in which it is used in all the above sciences. The first one who initiated this concept is Claude Berge in 1962[3] .

Ore [4] is the one who introduced the concepts of domination number and dominating sets. After that, this notion started to appear in different kinds and forms. In mathematics, this concept appeared in many fields including fuzzy graph, topological indices of graphs, etc. Additionally, many new definitions in this concept have been used, depending on putting some conditions on the dominating set. The concept of dominance which introduced by V.T.Chandrasekaran and Nagoorgani, and all the concepts of dominant sets, independent set, dominant number, The total dominant number in the fuzzy graph was developed by R.Parvathi and G.Thamizhenthi [5]. A. Somasundaram introduced dominance in fuzzy graph using effective edges, relying on fuzzy graph concept which introduced by Rosenfeld in 1975, which is consequently built on the basis of the fuzzy sets proposed by Zadeh[6] in 1965 as a new mathematical framework for the visualization of unreliability phenomena in a real-life situation[7].

The use of the intuitionistic fuzzy set also played an important role in the transition from mathematics to computer, information science, and communication systems. Use combinatorial optimization, physics, and statistical problem solving to see the graphs[8]. In 1998, Florentin Smarandache [9]introduced the concept of Neutrosophic set which is a powerful general formal framework that generalizes the concept of fuzzy set and intuitionistic fuzzy set by treating with indeterminate membership furthermore of truth and false memberships and then domination in neutrosophic graphs was introduced by M.Mullai [10] . The variety of applications for graphs and their domination sets are increased by appearing of SVNG since its domain is larger than that of FGs and IFGs. SVNGs model the relationships much like any other type of graph. Hence, it is used to address a variety of relationship-based issues. Where FGs and IFGs fail, it may mimic issues with fluctuating and ambiguous information in the actual world[11].

This work aims to introduce the concept of Closed neutrosophic dominating set in neutrosophic graphs which possess more properties than the traditional domination set concept and some other related concepts was provided.

2.preliminaries

Definition 2.1 [12]. Let V be a non-empty set, a fuzzy graph G = (σ, μ) is a couple of functions $\sigma: V \rightarrow [0,1]$ and $\mu: VxV \rightarrow [0,1]$ such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where xy denotes the edge between the vertices x and y furthermore σ and μ represent the fuzzy vertices and fuzzy edges sets on V and E respectively. See figure (1A)

Definition 2.2 [11]. The form G =(V, E) is called an (IFG) where

i) $V = \{v_1, v_2, \ldots, v_n\}$,where $\mu_1: V \rightarrow [0,1]$ and γ_1 : V→ [0,1] such that $\mu_1$is a membership grade and $\gamma_1$is a non-membership respectively of every $v_i \in V(\mathrm{i} = 1, \ldots, \mathrm{n})$,

and $0 \leq \mu_1(v_i) + \gamma_1(v_i) \leq 1$

ii. $E \subseteq V \times V$ where $\mu_2: VxV \rightarrow [0,1]$, and $\gamma_2: VxV \rightarrow [0,1]$, are functions and

$\mu_2: (v_i, v_j) \leq \mu_1(v_i) \wedge \mu_1(v_j)$, $\gamma_2(v_i, v_j) \geq \gamma_1(v_i) \vee \gamma_1(v_j)$

and $0 \leq \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \leq 1, \forall (v_i, v_j) \in E, (i, j = 1,2,3, \ldots, n)$ see figure (1B)

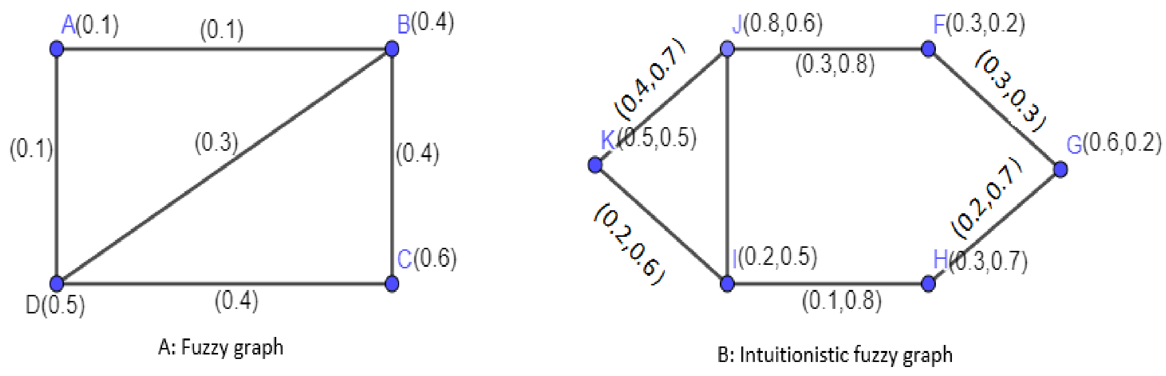

Figure :1

3. Single Valued Neutrosophic Graph (SVNG)[13].

Let $G^* = (V, E)$ refers to a traditional graph, and G = (A, B) to a (SVNG) On G^*

Definition 3.1. A single valued neutrosophic graph (SVNG) on vertices set V is a couple $G = (A, B)$ where $A = (T_A, I_A, F_A)$ and B=(T_B, I_B, F_B) as follows:

1.The T_A , I_A , F_A :V →[0, 1] are functions represent (truth, indeterminacy and falsity) membership degrees respectively, for all v_i ∈V, (i=1,...,n), and $0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$

2. The functions T_B, I_B, F_B: E ⊆V x V → [0, 1], are satisfied the followings:
$T_B(v_i, v_j) \leq T_A(v_i) \wedge T_A(v_j)$
$I_B(v_i, v_j) \leq I_A(v_i) \wedge I_A(v_j)$ and

$F_B(v_i, v_j) \geq F_A(v_i) \vee F_A(v_j)$ for all v_j, v_j ∈V

whereT_B, I_B, F_B *semilarly represent the three types of* membership degrees of each edge respectively, and

$0 \leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$ for all $(v_i, v_J) \in E$, $(i, j = 1,2,3, \dots, n)$

Where A is denote the (SVN) vertex set of V, and B the (SVN) edge set of E, respectively. See figure 2.

Notes;

i) B is symmetric (SVN) relation on A.

ii) When $T_{Bij} = I_{Bij} = F_{Bij}$ =0 for some i and j, then V_i and V_j are not adjacent vertices otherwise there exists an edge $v_i v_j \in E$

iii) If at least one of the conditions in (1) and (2) is not satisfied, then G is not a (SVN) graph

Definition.3.2.[14]. Let $G = (A, B)$ be a (SVNG).

1) $\forall\, x \in V$ the neutrosophic degree $d(x)$ of x is

$\sum T_B(xy), \sum I_B(xy), \sum F_B(xy), \; \forall\, y \in V \textit{ adjacent to } x$

2) For every $v_i \in V$ $|v_i| = \left|\frac{1+T_A(v_i)+I_A(v_i)-F_A(v_i)}{3}\right|$ is called cardinality of the vertex v_i,

Then $|A| = \left|\sum_{x \in A} \frac{1+T_A(x)+I_A(x)-F_A(x)}{3}\right|$ is called vertex cardinality of G,

Similarly, $|e = xy| = \left|\sum_{x,y \in A} \frac{1+T_B(x,y)+I_B(x,y)-F_B(x,y)}{3}\right|$ is known as edge cardinality of G.

Definition. 3.3.[15]. Let G = (A; B) be a (NG) on V. Then An edge $v_1 v_2 \in E$ in G is said to be an effective edge, if
$T_B(v_1, v_2) = T_A(v_1) \wedge T_A(v_2)$
$I_B(v_1, v_2) = I_A(v_1) \wedge I_A(v_1)$ and
$F_B(v_1, v_2) = F_A(v_1) \vee F_A(v_1)$ for v_1, v_2 ∈V

Definition. 3.4. [15]: take G = (A, B) as a (NG), then

1) G is renowned as strong neutrosophic graph if $\forall\, v_i v_j \in E$ is an effective edge.

2) G is renowned a complete neutrosophic graph if $\forall v_i, v_j \in V,\ \exists e = v_i v_j$ is an effective edge.

Definition. 3.5. [8]: A non-empty set $S \subseteq V(G)$ is called an independent neutrosophic set (INS) if

$$T_B(xy) = I_B(xy) = F_B(xy) = 0 \text{ , for all } x, y \in S$$

Definition. 3.6. $N(x)$ refers to open neighborhood of $x \in V$ (G) is define as
$N(x) = \{y \in V / (x, y)$ *is* an *effective edge*$\}$and
$N[x] = N(x) \cup \{x\}$ is closed neighborhood of x.

Definition .3.7. $A \subseteq V$(G) is called a neutrosophic vertex cover (NVC) of G if for each effective

Edge $e = (x, y)$, at least one of x *or* y *belong to* A .

The minimum neutrosophic cardinality (MNC) of all (MNVC) is called a **neutrosophic vertex covering number** of G which denoted by $\alpha_o(G)$.

Definition 3.8 [8]: Let G^* be underline graph of a neutrosophic graph G. The size m of G^* is a set of all edges in G^* and denoted by $m=|E(G^*)|$. Similarly, the order $n=|V(G^*)|$ of $G^*$is the number of vertices in in G^* .

Definition 3.9. Let G = (A, B) be NG then the neutrosophic size $Ș_N$ and neutrosophic order O_N of G are define as
$Ș_N = (\sum T_B(u,v), \sum I_B(u,v), \sum F_B(u,v)), \forall\, uv \in E$ and O_N =
$(\sum T_A(u), \sum I_A(u), \sum F_A(u)), \forall\, u \in V$

Definition. 3.10. [16]. Let G = (A, B) be NG then the set $D \neq \emptyset, D \in V(G)$ is known as a neutrosophic dominating set (NDS) of G if $\forall\, y \in V - D$, $\exists$ *a vertex* $x \in D$ such that $T_B(x,y) = T_A(x) \wedge T_A(y), I_B(x,y) = I_A(x) \wedge I_A(y)$, and $F_B(x,y) = F_A(x) \vee F_A(y)$. The (MNC) for all minimum neutrosophic dominating set in G is called the neutrosophic domination number (NDN) of G which is denoted by γ_N.

4. Closed neutrosophic domination number (CNDN) in neutrosophic graph.

Definition. 4.1 Let G be (NG) with a vertex set V, and $D_k \subseteq V$ for some $k \in Z^+$, then Dk is called **closed neutrosophic dominating (*CND*)** set of G if the followings satisfied:

1) $\forall x \notin Dk$, $\exists$ $y \in D_k$ such that x dominate y and
2) If Dk contains more than two vertices then the two vertices have not been adjacent to the third one.
3) $N[Dk]$=$V(G)$

Algorithm for finding closed neutrosophic dominating set *Dk* can as follows:

Let V=$\{x1,x2,\ldots,xn\}$, and $D_k = \{x_1, x_2, \ldots, x_k\} \subseteq V$

1) Choose $x1 \in V(G)$, assume $D1$=$\{x1\}$, if $N[D_1] = V(G)$ then $D1$ is said to be closed neutrosophic dominating set, otherwise

2) Choose $x2 \in V - D1$(may $x1$ and $x2$ are adjacent) put $D2=\{x1,x2\}$ if $N[D2]=V(G)$, then $D2$ is (CND) set, otherwise,
3) Choose $x_k \in V-N[Dk-1]$, $k \geq 3$, $k \in Z^+$ such that $N[DK] = V(G)$.

Definition. 4.2. A (CND) set Dk in a neutrosophic graph G is known as **minimum closed neutrosophic dominating (*MCND*) set** if the number of Dk elements less than or equal of number of vertices of each of other closed neutrosophic dominating set.

Definition. 4.3. Let $G = (A, B)$ be a NG The minimum neutrosophic cardinality of all (MCND) sets is known as **closed neutrosophic domination number (CNDN)** and denoted by $\boldsymbol{\gamma cl(G)}$ were

$\gamma cl(G)$= {(*min* {$\sum_{x \in D_{ki}} |T_A(\text{x}), I_A(\text{x}), F_A(\text{x})|$,)$x \in D_{\text{ki}}$, *Dki is MCND stes*}.

Definition. 4.4. The (MCND) set with **minimum neutrosophic cardinality** is said to be $\boldsymbol{\gamma cl}$**–*set*.**

Example. 4.5. Consider the given neutrosophic graph G, in figure 2.

We note two ($MCND$) sets: $Dk1$= {A, C, D, H} ={H}∪N(B) and $Dk2$= {I, J, B, D} = N(H)∪{B} such that $N[Dk1]$ =$V(G)$ and $N[Dk2]$ =$V(G)$.

$||Dk1||= |(1.7,2.4,1.7)| = 1.13333$, $||Dk2||= |(2.1,2.1,1.6)| = 1.2$

The closed neutrosophic domination number γcl=*min* {$||Dk1||$, $||Dk2||$}

=min {1.13333,1.2} = 1.1.333

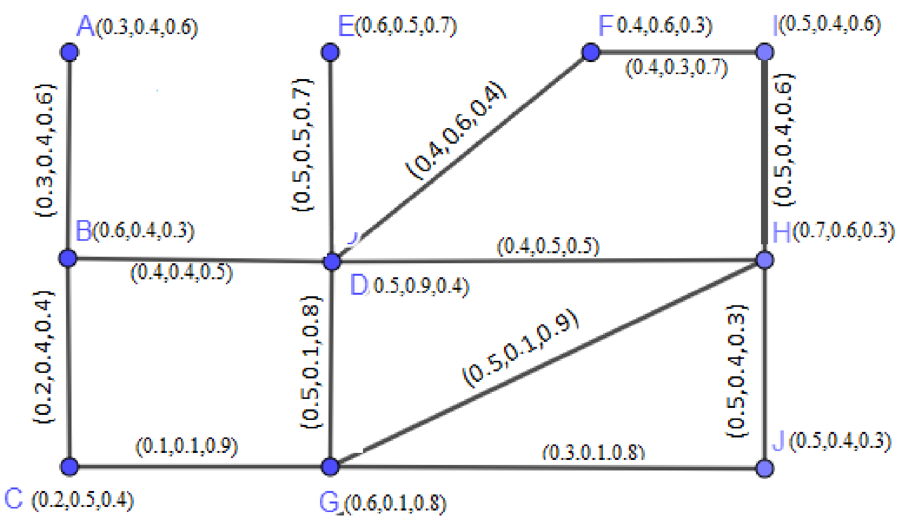

Figure 2: single value neutrosophic graph

Preposition. 4.6. Let G be a neutrosophic graph, then every closed neutrosophic dominating set of G is a neutrosophic dominating set of G.

Proof: It is clear a vertex set $D_k \subseteq V(G)$ in a neutrosophic graph G is closed neutrosophic dominating set if the following provisions satisfied:

1) $\forall x \notin Dk$, $\exists$ $y \in D_k$ such that x dominate y and also if Dk contains more than two vertices then the two vertices have not been adjacent to the third one.

2) $N[Dk]$=$V(G)$, it is obviously $\forall x \notin Dk$ dominated by vertex $y \in D_k$,thus Dk is a neutrosophic dominating set of G.

Remark. 4.7.

i) The converse of proposition 4.6 is not always right, for instance in figure (2) where {B, D, H} is a neutrosophic dominating set but not closed neutrosophic dominating set.

ii) Let G be a neutrosophic graph with (CND) set, then $\gamma N \leq \gamma cl$ is not always true.

Preposition. 4.8. Let G=(A,B) be any (NG) , Where A=(T_A, I_A, F_A), B=(T_B, I_B, F_B) and H=(C,D) be any maximal spanning tree of G, then every closed neutrosophic dominating set of H, is closed neutrosophic dominating set of G and $\gamma_{cl}(G) = \gamma_{cl}(H)$ if $for\ each\ non-adjacent\ pair\ x, y \in D_K$ in H are non-adjacent in G

Proof: Let D_K be closed neutrosophic dominating set of H. Since H is a maximum spanning tree of G, we have A=C. thus, the vertices in $V - D_K$ are dominated by at least one vertex in D_K, then if D_K contains three or more vertices, then the third one has not be adjacent to other vertices and $N_H[D_K] = V(H) = V(G) = N[D_K]$.Therefore $\gamma_{cl}(G) = \gamma_{cl}(H)$.

Example: A graph H in a figure 3) below which is a maximal spanning tree of G, the sets $D1$= {A, C, D, H} and $D2$= {I, J, B, D} are closed neutrosophic domains in H and also in G

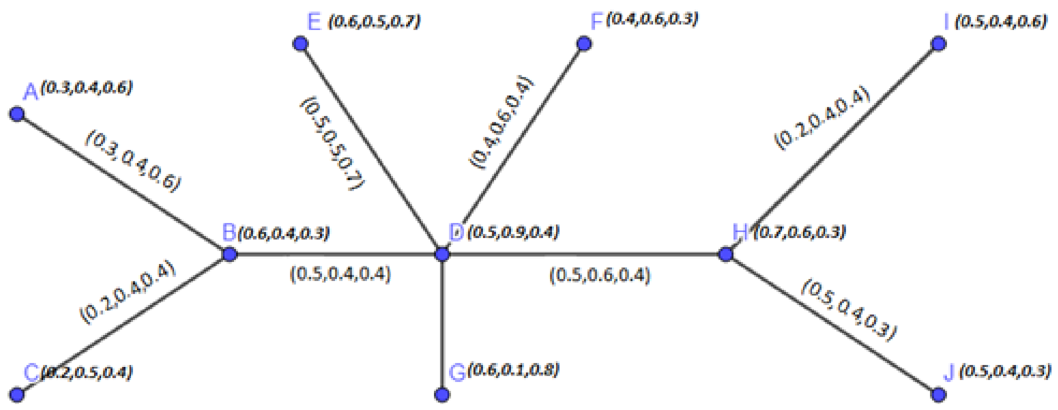

Figure3: spanning tree H of single value neutrosophic graph in figure 2

Note: If the two vertices A and D were adjacent in the figure 2, the theorem would not be true

Preposition. 4.9. Let $G \cong K_n^N$ be a complete neutrosophic graph and D_K is closed neutrosophic dominating set of Then, $V - D_K$ has a closed neutrosophic dominating set.

Proof: Given $G \cong K_n^N$ then every edge $e \in E(G)$ is an effective edge and each vertex $v \in V(G)$ is dominating all others. Thus, a closed neutrosophic dominating set is contains only one vertex then any singleton set of $V - D_K$ is closed neutrosophic dominating set. consequently $V - D_K$ has closed neutrosophic dominating set

Preposition. 4.10. For any neutrosophic graph G= (A, B), Where $A = (T_A, I_A, F_A)$, B= (T_B, I_B, F_B), $min|T_A(x), I_A(x), F_A(x)| \leq \gamma cl(G) \leq$more upper bound equality holds if

$T_B(x, y) < T_A(x) \wedge T_A(y)$, $I_B(x, y) < I_A(x) \wedge I_A(y)$, $F_B(x, y) < F_A(x) \vee F_A$, $\forall x, y \in V$

Proof: Straight forward from the definition of a closed neutrosophic dominating set

Preposition. 4.11. If G be a neutrosophic graph, every (IND) set of G is closed neutrosophic dominating set.

Proof: Assume that S_k be (IND) set of G then it has two probabilities:

Case1: If S_k be a singleton, then it is obviously SK is closed neutrosophic dominating set.

Case2: If the vertices in SK are more than two and since

$T_B(\mathrm{x,y}) \leq T_A(\mathrm{x}) \wedge T_A(\mathrm{Y}), I_B(\mathrm{x,y}) \leq I_A(\mathrm{x}) \wedge I_A(\mathrm{Y}), F_B(\mathrm{x,y}) \geq F_A(\mathrm{x}) \wedge F_A(\mathrm{Y})$, for any $x, y \in S_K$,that's why $\forall\, z \in V - S_k$ has at least an effective edge e in S_k. Thus $N\,[S_k] = V(G)$ and for each $x \in S_k$ belongs to $V - N\,[S_k]$. consequentlyS_k is closed neutrosophic dominating set.

Preposition. 4.12. Let G= (A, B) be neutrosophic graph without isolated vertex and $(T_A(x) = I_A(x) = F_A(x)) = c\ \forall\ x \in V\ and\ c \in [0,1]$ then $\gamma cl\,(G) \leq \mathrm{P} - \alpha_N$ where α_N is the neutrosophic covering number of G.

Proof: Let G= (A, B) be neutrosophic graph with no isolated vertex and V_C be a neutrosophic covering of G, then $V - V_C$ is independent neutrosophic set of G. Thus, $V - V_C$ is closed neutrosophic dominating set of G by proposition (4.5). Hence, $\gamma cl\,(G) \leq \mid\mid V - V_C \mid\mid \leq P - \alpha_N$.

Proposition. 4.13. For any neutrosophic graph G= (A, B), $\gamma_{cl}(G) + \gamma_{cl}\overline{(G)} \leq 2(O_N)$.

$Further, equality\ hold\ if$

$0 < T_B(x,y) < T_A(x) \wedge T_A(y), 0 < I_B(x,y) < I_A(x) \wedge I_A(y), 0 < F_B(x,y) > F(x) \vee F_A(y),$

$\forall\ x, y \in E(G).$

Proof: *i)* Since, $\gamma_{cl}(G) = O_N$ it is trivial the inequality hold.

Since

$$0 < T_B(x,y) \neq T_A(x) \wedge T_A(y), 0 < I_B(x,y) \neq I_A(x) \wedge I_A(y), 0 < F_B(x,y) \neq F(x) \vee F_A(y), \forall\, x, y \in E(G).$$

i.e. $T_B(x,y) < T_A(x) \wedge T_A(y), I_B(x,y) < I_A(x) \wedge I_A(y), F_B(x,y) > F(x) \vee F_A(y), \forall\, x, y \in E(G)$

then $T_A(x) \wedge T_A(y) - T_B(x,y) < T_A(x) \wedge T_A(y),$

$I_A(x) \wedge I_A(y) - I_B(x,y) < I_A(x) \wedge I_A(y),$

$F_A(x) \wedge F_A(y) - F_B(x,y) < F(x) \vee F_A(y), \forall\, x, y \in \overline{E(G)}$

Then, $\gamma_{cl}\overline{(G)} = O_N$ .then , $\gamma_{cl}(G) + \gamma_{cl}\overline{(G)} = O_N + O_N = 2O_N$.

Preposition. 4.14. Let $G \cong K_n^N$ then $\gamma cl\,(K_n^N)$ = min$|(T_A(x_i), I_A(x_i), F_A(x_i))|$, $x_i \in V(G)$,i=1,2,…n

Proof: Let $G \cong K_n^N$ be complete (NG), then for each edge in G is an effective edge and each vertex in G dominates to all others of G. thus, the closed neutrosophic dominating set is contains a single vertex say DK={x}

such that $N[DK] = V\,(G)$, and x has minimum neutrosophic value. Hence, the outcome is gained.

Preposition 4.15. Let G = (A, B) be a strong neutrosophic star, then $\gamma cl\,(G)$ =$|(T_A(\mathrm{x}), I_A(\mathrm{x}), F_A(\mathrm{x}))|$, where x is a root vertex.

Proof: Let G = (A, B) be strong neutrosophic star and $V\,(G) = \{x, x_1, x_2, \ldots, x_n | x \text{ is a root of G}\}$, since all edges of G are effectives and x a dominating $x_i, i = 1,2, \ldots, n$. thus, a closed neutrosophic dominating set contains only one vertex such that $N[x] = V(G)$. Hence, $\gamma cl\,(G)$ =$|(T_A(\mathrm{x}), I_A(\mathrm{x}), F_A(\mathrm{x}))|$.

Preposition. 4.16. Let $G \cong K^{N}_{n,m}$ be complete bipartite (NG) with n, m vertices, then

$\gamma cl\ (K^{N}_{n,m}) =$

$$\begin{cases} |(T_A(x),\ I_A(x), F_A(x))| & \text{if either } n = 1 \text{ or } m = 1 \text{ where } x \in X \text{ or } x \in Y \\ \min|(T_A(x_i), I_A(x_i), F_A(x_i))| + \min|(T_A(y_i), I_A(y_i), F_A(y_i))| & \text{if } n, m \geq 2, \text{where } x_i \in X, i = 1,2,\dots n, y_j \in Y, j = 1,2,\dots m \end{cases}$$

Proof: Assume $G \cong K_{n,m}$ on $V(K_{n,m}) = X \cup Y$,where $X = \{x1, x2, \dots, xn\}$ and $Y = \{y1, y2, \dots, ym\}$. Then there are couple of cases:

case 1: If either n=1 or m=1 then the graph is a star and the prove is hold by Preposition 4.9

case 2: If neither n=1 nor m=1. since for each $x \in X$ dominates to every $y \in Y$ and the contrariwise is true. Then, a $(MCND)$ set of $K_{n,m}$ contains pair of vertices.

Hence, $\gamma cl\ (Kn, m) = \min|(T_A(x_i), I_A(x_i), F_A(x_i))| + \min|(T_A(y_i), I_A(y_i), F_A(y_i))|$ where where $x_i \in X, i = 1,2,\dots n, y_j \in Y, j = 1,2,\dots m$. The prove complete.

Preposition. 4.17. Let $G = (A, B)$ be a strong neutrosophic graph G=C_n^N ,then

$\gamma cl(C_n^N)$=

$$\begin{cases} Min \sum_{i=0}^{\left\lceil \frac{n}{3} \right\rceil - 1} |T_A(x_{j+3i}), I_A(x_{j+3i}), F_A(x_{j+3i})| & j = 1,2,\dots,n \text{ and } n \equiv 0,2 (mod\ 3) \\ min \begin{Bmatrix} |(T_A(x), I(x), F_A(x)| + \sum_{i=0}^{\left\lceil \frac{n}{3} \right\rceil - 1} |(T_A(x_{j+3i+1}), I(x_{j+3i+1}), F_A(x_{j+3i+1})| \\ |(T_A(x), I(x), F_A(x)| + \sum_{i=0}^{\left\lceil \frac{n}{3} \right\rceil - 1} |(T_A(x_{j+3i+2}), I(x_{j+3i+2}), F_A(x_{j+3i+2})| \end{Bmatrix} & j = 1,2,\dots,n \text{ and } n \equiv 1 (mod\ 3) \end{cases}$$

Where taken $j + 3i$, $j + 3i + 1$, $j + 3i + 2$ $modulo\ n$.

Proof: There are two cases depend on n as follows.

Case 1. If $n \equiv 0,2\ (mode3)$, then let $D_j = \left\{x_{j+3i}; i = 0,1,\dots, \left\lceil \frac{n}{3} \right\rceil - 1\right\}$,j=1, 2…, n, one can concluded that each one of the sets D_j is minimum dominating set and it independent. thus, according to proposition (4.5), it is closed neutrosophic dominating set.

Case 2. If $n \equiv 1(mode3)$, and let D_K be minimum closed neutrosophic dominating set of C_n^N,

Then there are two subcases:

i) If $x_i \in D_k$ then $(x_{i+1}\ or\ x_{i-1})$ vertex must be not belonged to D_k, then

$$D_j = \left\{x_{j+1+3i}; i = 0,1,\dots, \left\lceil \frac{n}{3} \right\rceil - 1\right\}, j=1, 2\dots, n,$$

ii) If any two vertices in D_k are not adjacent, then

$$D_j = \left\{x_{j+2+3i}; i = 0,1,\dots, \left\lceil \frac{n}{3} \right\rceil - 1\right\}, j=1, 2\dots, n,$$

From the above cases the prove is done.

Preposition. 4.18. Let W^N_{n+1} be strong neutrosophic wheel with x as a center, then

$\gamma cl\ (W^N_{n+1}) = (T_A(x),\ I_A(x),\ F_A(x))$.

Proof: Let W^N_{n+1} be a strong neutrosophic wheel, since all its edges are effective edge then x is dominating to xi, $i = 1,2, \dots, n$. Then, the (CND) set $DK = \{x\}$ such that $N[DK] = V\ (W^N_{n+1})$.

Hence, $\gamma cl\ (W^{N}_{n+1}) = |(T_A(\mathrm{x}), I_A(\mathrm{x}), F_A(\mathrm{x}))|$

Preposition 4.19. For any strong neutrosophic graph G = (A, B), and $x \in v, d(x) = \Delta(g)$

$then\ \gamma cl \leq |O_N| - \sum_{y \in N(x)} |T_A(\mathrm{y}), I_A(\mathrm{y}).F_A(\mathrm{y})|$.

Proof: Let DK be a $\gamma cl - set$ of G and x be a vertex of G such that $d(x) = \Delta(G)$. Then, $V - N(x)$ is(CND) set, thus

$|DK| \leq |V - N(x)| = |n - \Delta(G)|$, take neutrosophic cardinality to both sides hence,

$\gamma cl \leq |O_N| - \sum_{y \in N(x)} |T_A(\mathrm{y}), I_A(\mathrm{y}).F_A(\mathrm{y})|$

5. Closed neutrosophic dominating set in some operation on neutrosophic graphs.

Definition. 5.1. [14]: Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two neutrosophic graphs on $V1$and $V2$ respectively

then $G1 \cup G2$ is neutrosophic graph on $(V_1 \cup V_2).defined\ as$ $G = G1 \cup G2$ = ((A1 ∪ A2), (B1 ∪ B2)) where:

$$(A1 \cup A2)(x) = \begin{cases} (T_{A1}(x), I_{A1}(x), F_{A1}(x)) & If\ x \in V_1\ and\ x \notin V_2 \\ (T_{A2}(x), I_{A2}(x), F_{A2}(x)) & If\ x \in V_2\ and\ x \notin V_1 \\ (\max(T_{A1}(x), T_{A2}(x)), \max(I_{A1}(x), I_{A2}(x)), \min(F_{A1}(x), F_{A1}(x)) & If\ x \in V_1 \cap V_2 \end{cases}$$

(B1 ∪ B2) (x, y) =

$$\begin{cases} (T_{B1}(xy), I_{B1}(xy), F_{B1}(xy)) & If\ xy \in E_1\ and\ xy \notin E_2 \\ (T_{B1}(xy), I_{B1}(xy), F_{B1}(xy)) & If\ xy \in E_2\ and\ xy \notin E_1 \\ (\max(T_{B1}(xy), T_{B2}(xy)), \max(I_{B1}(xy), I_{B2}(xy))), \min(F_{B1}(xy), F_{B1}(xy) & If\ xy \in E_1 \cap E_2 \end{cases}$$

Example. 5.2. Consider $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two neutrosophic graphs shown in 4 below

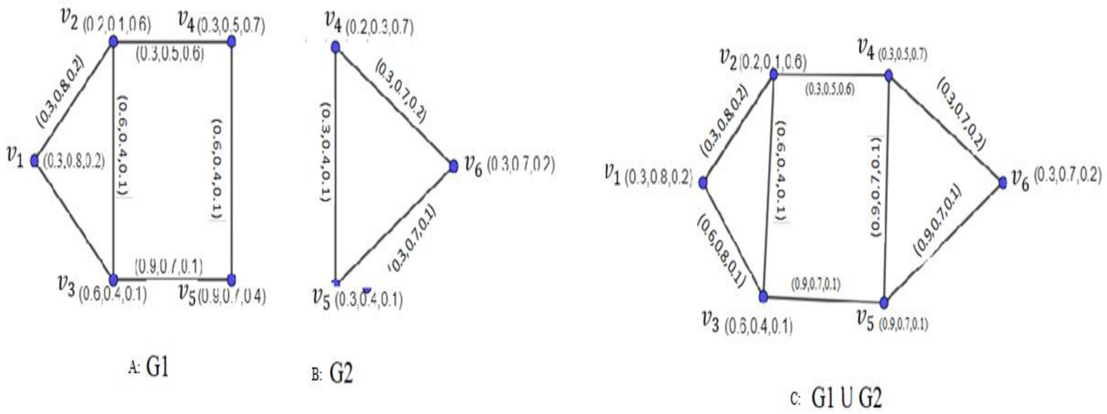

Figure4: A, B, C represent graph G1, G2, G1 U G2 respectively

Proposition. 5.3. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two strong neutrosophic graphs then

$\gamma cl\ (G1 \cup G2) =$

$$\begin{cases} \gamma cl\,(G1) + \gamma cl\,(G2) & \text{if } V1 \cap V2 = \emptyset \\ Min\{\gamma cl\,(G_1^U) + \gamma cl\,(G2 - (V1 \cap V2)), \gamma cl\,(G_2^U) + \gamma cl\,(G1 - (V1 \cap V2))\} & \text{if } V1 \cap V2 \neq \emptyset, \text{and } \forall v \in V1 \cap V2, v \notin D_k(G1 \cup G2) \end{cases}$$

Where G_1^U, G_2^U *the* G1, G2 after changing the (true, indeterminate, false) memberships of $V_1 \cap V_2$ under the union operation

Proof: Let $Dk1$ and $Dk2$ be a $\gamma cl - sets$ of $G1$ and $G2$ respectively.

Case 1. If $V1 \cap V2 = \emptyset$, then $Dk1 \cap Dk2 = \emptyset$. Therefore, $Dk = Dk1 \cup Dk2$ is (CND) set of $G = G1 \cup G2$. Hence, $\gamma cl\ (G) = \gamma cl\ (G1 \cup G2) = ||Dk1 \cup Dk2|| = ||Dk1 + Dk2|| = \gamma cl\ (G1) + \gamma cl\ (G2)$.

Case 2. If $V1 \cap V2 \neq \emptyset$, either $Dk = Dk1 \cup D\ (G_2 - (V_1 \cap V_2))$ or $Dk = Dk2 \cup D\ (G_1 - (V_1 \cap V_2))$

Then $\gamma cl\ (G1 \cup G2) Min\{\gamma cl\,(G_1) + \gamma cl\,(G_2 - (V_1 \cap V_2)), \gamma cl\,(G_2) + \gamma cl\,(G_1 - (V_1 \cap V_2))\}$

Definition. 5.4. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two (NG)s on $V1$ and $V2$ respectively, the join of $G1$ and $G2$ is a neutrosophic graph

$G = G1 + G2 = (A1 + A2, B1 + B2)$ where:

$(A_1 + A_2)(x, y) =$

$$\begin{cases} (T_{A1}(x), I_{A1}(x), F_{A1}(x)) & If\ x \in V_1\ and\ x \notin V_2 \\ (T_{A2}(x), I_{A2}(x), F_{A2}(x)) & If\ x \in V_2\ and\ x \notin V_1 \\ (\max(T_{A1}(x), T_{A2}(x)), \max(I_{A1}(x), I_{A2}(x)), \min(F_{A1}(x), F_{A1}(x)) & If\ x \in V_1 \cap V_2 \end{cases}$$

and

$(B1+B2)\ (x, y)$

$=$

$$\begin{cases} \left(T_{B1}(xy), I_{B1}(xy), F_{B1}(xy)\right) & If\ xy \in E_1\ and\ xy \notin E_2 \\ \left(T_{B1}(xy), I_{B1}(xy), F_{B1}(xy)\right) & If\ xy \in E_2\ and\ xy \notin E_1 \\ \left(\max\left(T_{B1}(xy), T_{B2}(xy)\right), \max\left(I_{B1}(xy), I_{B2}(xy)\right), \min\left(F_{B1}(xy), F_{B1}(xy)\right)\right. & If\ xy \in E_1 \cap E_2 \\ \left(\min\left(T_{A1}(x), T_{A2}(y)\right), \min\left(T_{A1}(x), T_{A2}(y)\right)\right), \max\left(F_{B1}(x), F_{B1}(y)\right. & If\ xy \in E' \end{cases}$$

where $E' = \{x_i y_j\ edges | x_i \in V1 \text{ and } y_i \in V2\}$

Example 5.5. In figure (5) below. Consider the graphs $G1 = (A1, B1)$ and $G2 = (A2, B2)$ be any two (NG)s on $V1$and $V2$ then $G1 + G2$ is given in figure 3.3

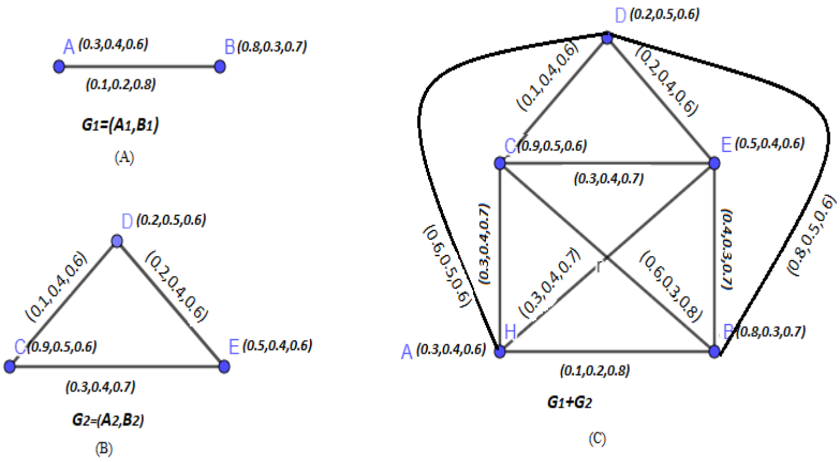

Figure5: A, B, C represent graph G1, G2, G1 + G2 respectively

Observation. 5.6. Consider $G1$ and $G2$ be two strong neutrosophic graphs, and $G = G1 + G2$. if $V1 \cap V2 = \emptyset$,For any $x \in V(G1)$ and $y \in V(G2)$ such that x and y have minimum neutrosophic cardinality values, the set $\{x, y\}$ is a closed neutrosophic dominating set in $G1 + G2$.Thus,

$\gamma cl\ (G) = \gamma cl\ (G1 + G2) = min|(T_A(x_i), I_A(x_i), I_A(x_i))\ | + min\left|\left(T_A(y_j), I_A(y_j), I_A(y_j)\right)\right|, x_i \in V(G1)$, i=1,2,..$|V(G1)|$, $and\ \ y_i \in V(G2)$,j=1,2,..$|V(G2)|$

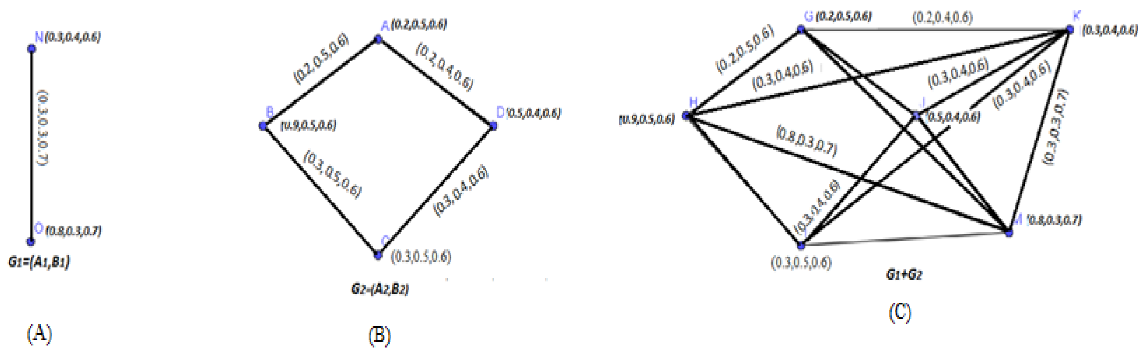

Figure6: A, B, C represent graph G1, G2, G1 + G2 respectively

Theorem. 5.7. Let $G1 = (A_1, B_1)$ and $G2 = (A_2, B_2)$ be any two strong neutrosophic graphs on $V1$and $V2$ respectively. Then, $\gamma cl\ (G) = \gamma cl\ (G1 + G2) \leq min\ \{\gamma cl(G1), \gamma cl\ (G2)\}$.

Proof: Let S1 and S2 be a γcl − sets of $G1$ and $G2$ respectively, by definition of join two neutrosophic graphs, we infer that S1 and S2 are (CND)sets of G. Hence $\gamma cl\ (G) = \gamma cl\ (G1 + G2) \leq min\ \{|S1|, |S2|\}$ = min $\{\gamma cl\ (G1), \gamma cl\ (G2)\}$.

Theorem. 5.8. Let G = (A, B) be a strong (NG) on V with $|V| = n$, then:

i) $\gamma cl\ (G) = |A(x)|$ if and only if $G = K_1^N$ or $G = K_1^N + \cup_{i=1}^{k} Hi$ for Some $k \geq 1$,and strong neutrosophic connected graph $H1, H2, \ldots, Hk$.

ii) $\gamma cl\ (G) = \min|A(x_i)| \ x_i \in V(G)$ if and only if $G = K_2^N$

iii) $\gamma cl\ (G) = |O_n^N|$ if and only if $G = \overline{K_n^N}$;

iv) $\gamma cl\ (G) = |O_n^N| - \min|A(x_i)| \ x_i \in V(K_2^N)$ if and only if $G = K_2^N \cup \overline{K_{n-2}^N}$.

Proof:

i) Suppose that $G = K_1^N + \cup_{i=1}^{k} Hi$ for some k ≥ 1, and strong neutrosophic connected graph $H1, H2, \ldots, Hk$, select $x \in V\ (K_1^N)$, since $V\ (G) = N[x]$. then $\gamma cl\ (G) = |A(x)|$.

Conversely, assume that $\gamma cl\ (G) = A(x)$ and let $x \in V\ (G)$ such that$\{x\}$ is a closed neutrosophic dominating set of G. If $G \neq K_1^N$,then $V\ (G) - \{x\} = N\ (x)$.

Consequently, $G = K_1^N + \cup_{i=1}^{k} Hi$ for some $k \geq$ 1and strong connected neutrosophic graph $H1, H2, \ldots, Hk$. Hence, (i) is satisfied.

ii) When n = 2, the (CND)set is a singleton, thus by (i)
$\gamma cl\ (G) = A(x)$ If and only if $G = K_n^N$.

iii) If $G = \overline{K_n^N}$; it is obviously $D_K = V(G)$ $i.e.$ $\gamma cl\ (G) = O_n^N$. Suppose that $G \neq \overline{K_n^N}$; . If $G =$ K_n^N,n=2, then $\gamma cl\ (G) = \min|A(x_i)| \neq O_n^N$. contradiction, suppose that $G \neq K_n^N$,n=2 and let the vertex x adjacent the vertex y in G construct a closed neutrosophic dominating set $\{x_1, x_2, x_3, \ldots, x_k\}$ in G such that $x_1 = x$ and $x_2 \neq y$. Then $K \leq n - 1$ vertices, thus $\gamma cl\ (G) < O_n^N$, a contradiction. then, (iii) is proved.

v) Now if $n \geq 3$. Suppose that $\gamma cl\ (G) = |O_n^N| - |A(x)|$ then $\Delta_E(G) \geq 1$.

assume that $\Delta_E\ (G) >$ 1and let $x \in V\ (G)$ such that $d_E\ (x) = \Delta_E\ (G)$ construct closed neutrosophic dominating set $\{x1, x2, \ldots, xk\}$ in G such that $x_1 = x$ and $x_2 \in V(G) - N[x]$. Then $k \leq n - 2$, then $\gamma cl\ (G) \neq |O_n^N| - |A(x)|$, a contradiction. Thus, $\Delta_E\ (G) = 1$ therefore$G = K_2^N \cup K_{n-2}^N$ .The converse is directly.

6. Inverse Closed Neutrosophic Domination (ICND) in Neutrosophic Graphs

In this section, the notion of invers closed neutrosophic domination (ICND) γ_{cl}^{-1} in neutrosophic graph is introduced. some interesting relationships are known between closed neutrosophic domination and inverse closed neutrosophic domination.

in addition inverse closed neutrosophic domination in the join of new graphs discussed.

Definition. 6.1. Let D_K be a minimum closed neutrosophic dominating set in G. If $V - D_K$ contains a (CND) set ${D_K}^{-1}$ of G then ${D_K}^{-1}$ is said to be inverse closed neutrosophic dominating set according to D_K. An inverse closed neutrosophic domination number${\gamma_{cl}}^{-1}$ of G which is defined as ${\gamma_{cl}}^{-1} = (\min\{\sum_{x\in {D_K}^{-1}} |T_A(x), I_A(x), F_A(x)|\}, {D_K}^{-1}$ is minimum inverse closed neutrosophic dominating set of G. and minimum invers closed neutrosophic dominating set has minimum neutrosophic cardinality is called ${\gamma_{cl}}^{-1} -$ set of G

Example. 6.2. Let G=C_6^N as in the figure

Observation. 6.3. Let $\boldsymbol{G} = (\boldsymbol{A}, \boldsymbol{B})$ be neutrosophic graph of $n \geq 2$ vertices. if there is inverse closed neutrosophic dominating set in G. then

i) $\min|(T_A(x_i), I_A(x_i), F_A(x_i))| \leq {\gamma_{cl}}^{-1} < O_N,\ x_i \in V - D_K$ where D_K is minimum closed neutrosophic dominating set of G

ii) Not necessary $\gamma_{cl} \leq {\gamma_{cl}}^{-1}$

Example 6.4. Consider the following graph $G = (A, B)$

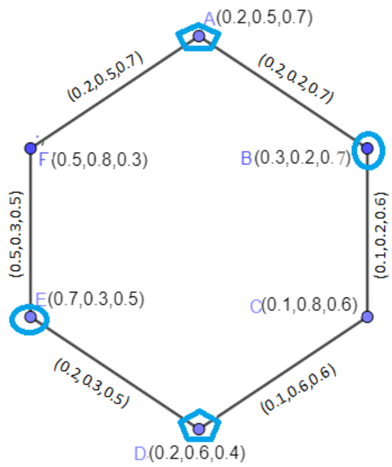

Figure7: Inverse closed neutrosophic dominating set

A minimum closed neutrosophic dominating sets are:

$D_K = \{A, D\}$ *and* $D_{K1}^{-1} = \{B, E\}$ *OR* $D_{K2}^{-1} = \{C, F\}$ *then*

$\gamma_{cl} D_K = |0.2,0.6,0.4| + |0.2,0.6,0.4| = 0.8$ *and* ${\gamma_{cl}}^{-1} D_{K1} = |(0.3,0.2,0.7)| + |(0.7,0.3,0.5)| = 0.76667$

${\gamma_{cl}}^{-1} D_{K2} = |(0.1,0.8,0.6)| + |(0.5,0.8,0.3)|$ =1.1 then ${\gamma_{cl}}^{-1} = \min(0.76667, 1.1) = 0.76667$

then $\gamma_{cl} > {\gamma_{cl}}^{-1}$

Proposition 6.5. Let G= (A, B) be a strong neutrosophic graph of $n \geq 2$ vertices. then,

${\gamma_{cl}}^{-1}(G) = \min|\ (T_A(x), (I_A(x), (F_A(x))|, x \in V - D_K$ where D_K is minimum closed neutrosophic dominating set of G, if and only if either G=K_2^N OR G=$K_2^N + H$ for some strong neutrosophic graph H.

Proof: Let D_K be a $\gamma_{cl} -$ set of G.

Case 1. If G= G=K_2^N Then V(G) =2 then of the two vertices belong to D_K and the other belong to D_K^{-1} i.e., $\gamma_{cl}{}^{-1}(G) = \min|\,(T_A(x), (I_A(x), (F_A(x))|, x \in D_K^{-1}$

Case 2. If G=$K_2^N + H$, since G is strong neutrosophic graph then each of the vertices of K_2^N is adjacent with the all vertices of H, then obviously $\gamma_{cl}{}^{-1}(G) = \min|\,(T_A(x), (I_A(x), (F_A(x))|, x \in D_K^{-1}$

Conversely Suppose that $\gamma_{cl}{}^{-1}(\mathrm{G}) = \min|\,(T_A(x), (I_A(x), (F_A(x))|, x \in V - D_K$, i.e. a minimum inverse closed neutrosophic dominating set contains exactly one vertex say, $D_K{}^{-1} = \{x\}$, *then a* minimum closed neutrosophic dominating set D_K *of* G has only one vertex ,if $G \neq K_2^N$, then

$V - \{x\} = N(X)$.Hence, G=$K_2^N + H$, for some strong neutrosophic graph H. see figure 6

Theorem. 6.6. Let G= (A, B) be a strong neutrosophic graph of $n \geq 2$ vertices. then,

$\gamma_{cl}{}^{-1}(\mathrm{G}) = |O_n^N| - |A(x)|, x \in D_K$ if and only if G≅ strong neutrosophic star

Proof: Let $D_K{}^{-1}$ be a $\gamma_{cl}{}^{-1}$ − set of G and $\gamma_{cl}{}^{-1}(\mathrm{G}) = |O_n^N| - |A(x)|$. Let $x \in D_K \subseteq V(G)$. Then $N[x] = V(G)$,that is (x, y) effective edge for all $y \in V(G) - \{x\}$,

we claim that $(T_B(y,z), I_B(y,z), F_B(y,z)) = (0,0,0), \forall\, y, z \in V(G) - \{x\}$.suppose that

$\exists\, y, z \in V(G) - \{x\}$ *such that*

$T_B(y,z) = (T_A(y) \wedge (T_A(z)$, $I_B(y,z) = (I_A(y) \wedge (I_A(z)$ and $(F_B(y,z) = (F_A(y) \vee (F_A(z)$ i.e. (y,z) is effective edge, thus $x, z \in N[y] \subseteq N[D_K{}^{-1} - \{z\}]$ then $D_K{}^{-1} - \{z\}$ *is* $\gamma_{cl}{}^{-1}$ − *set of G* , a contradiction. Hence G=$K_{1,n-1}^N$. Conversely, consider G=$K_{1,n-1}^N$it is clear that {x} is $\gamma_{cl}{}^{-1}$ − set of G then $D_K{}^{-1} = V(G) - \{x\}$.therefore, $\gamma_{cl}{}^{-1}(\mathrm{G}) = p^N - A(x).x \in D_K$.

Example. 6.7. Consider a strong neutrosophic graph G=$K_{1,4}^N$ in figure 6.2. a minimum closed neutrosophic dominating set $D_K = \{x\}$ and a minimum inverse closed neutrosophic dominating set $V - \{x\}$, then $\gamma_{cl}(K_{1,n-1}^N)$ =(1.2,1.8,2.5) -(0.2,0.3,0.7) = (1,1.5,1.8)

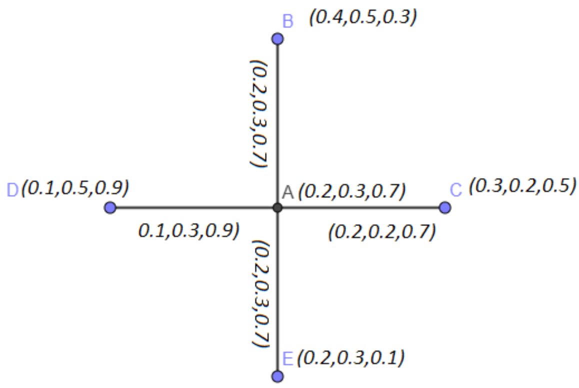

Figure8: Illustration theorem 6.2 ($K_{1,4}^N$)

Theorem. 6.8. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two strong neutrosophic graphs, then a minimum inverse closed neutrosophic dominating set ${D_K}^{-1}$ of $G_1 + G_2$ contains at most two vertices

Proof: Let D_{K1} and D_{K2} are minimum closed neutrosophic dominating set of G_1 and G_2 respectively, we know that a minimum closed neutrosophic dominating set D_K of join any two strong neutrosophic graphs $G_1 + G_2$ contains at most couple of vertices. Then, there exist two cases:
Case 1. If $D_K = \{x\}$ (contains a single vertex) is closed neutrosophic dominating set of $G_1 + G_2$. If $x \in V(G_1)$ then has n-1 neighborhood in G_1. Thus, assume that if there are S_1 and S_2 be the sets contains all vertices have $n-1$ neighborhood in G_1 and G_2 respectively
$S_1 = \{x_i : \deg(x_i) = n-1, x_i \in G_1\}$ and $S_2 = \{y_i : \deg(y_i) = m-1, y_i \in G_2\}$, therefore ${D_K}^{-1} = \{x_i, x_i \in G_1 - \{x\} \; or \; {D_K}^{-1} = \{y_i, y_i \in G_2\}$. Hence , a minimum inverse closed neutrosophic dominating set ${D_K}^{-1}$ contains one vertex .Similarly if $\{x\} \in G_2$ if not, then it is clearly ${D_K}^{-1}$contains two vertices.
Case 2. If $D_K = \{x, y\}$(contains two vertices) is minimum closed neutrosophic dominating set of $G_1 + G_2$.If $x \in V(G_1)$ and $y \in V(G_2)$. Since $D_K = \{x, y\}$ is minimum closed neutrosophic dominating set of $G_1 + G_2$. Then, for any vertex $x_1 \in V(G_1) - D_K$ and $y_1 \in V(G_2) - D_K$, then the set $A = \{x_1, y_1\} \subseteq V(G_1 + G_2) - D_K$ is minimum closed neutrosophic dominating set of $G_1 + G_2$ which is inverse closed neutrosophic dominating set of $G_1 + G_2$. Hence ,from above cases the result is obtained.

Theorem. 6.9. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two strong neutrosophic graphs. If ${\gamma_{cl}}^{-1}(G_1 + G_2) = |(T_A(x), I_A(x), F_A(x))|$,
then $\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ or $\gamma_{cl}(G_2) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$ where $x_1 \in V(G_1 + G_2)$, $x_1 \in V(G_1) \; and \; y_1 \in V(G_2)$ with minimum neutrosophic value

proof: Given $G_1 \; and \; G_2$ two strong neutrosophic graphs. Let ${D_K}^{-1} = \{x\}$ be minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ then ,a minimum closed neutrosophic dominating set D_K of $G_1 + G_2$ also contained one vertex , therefore D_{K1} or D_{K2} contains one vertex, i.e. $\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ or $\gamma_{cl}(G_2) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$.hence ,the result obtain.

Remark 6.10. The propositions converse of theorem 6.4 is not true in general.

Example 6.11. Consider two strong neutrosophic graphs G_1=$K^N_{1,4}$ and G_2=p^N_7 note that $\gamma_{cl}(G_1) = |(T_A(x), I_A(x), F_A(x)|, x$ is rote vertex but a minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ contains two vertices, i.e. ${\gamma_{cl}}^{-1}(G_1 + G_2) \neq |(T_A(x), I_A(x), F_A(x)|$

Proposition 6.12. Let $G_1 \; and G_2$ be two strong neutrosophic graphs such that $(T_{A1}(x), I_{A1}(x), F_{A1}(x) = (c, c, c)$, $c \in [0,1]$ and $(T_{A2}(y), I_{A2}(y), F_{A2}(y)) = (k, k, k)$, $k \in [0,1]$, then ${\gamma_{cl}}^{-1}(G_1 + G_2) = |(T_A(x), I_A(x), F_A(x)|$ if and only if one of the following is hold:

i) $\gamma_{cl}(G_1) = |T_{A1}(x), I_{A1}(x), F_{A1}(x)|$ or $\gamma_{cl}(G_2) = |(T_{A2}(y), I_{A2}(y), F_{A2}(y))|$
ii) $\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ and G_1 has at least two minimums $\gamma_{cl} - sets$;
iii) $\gamma_{cl}(G_2) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$and G_2 has at least two minimums $\gamma_{cl} - sets$;

Proof: Assume that (i) holds and $D_{K1} = \{x_1\} \subseteq V(G_1)$, $D_{K2} = \{y_1\} \subseteq V(G_2)$ are minimum closed neutrosophic dominating sets in $G_1 \, and \, G_2$ respectively ,then $D_{K1} \, and \, D_{K2}$ are minimum closed neutrosophic dominating set in $G_1 + G_2$, Since $D_{K1} \subseteq V(G_1 + G_2) - D_{K2}$, thus $D_{K1} \, is$

$\gamma cl^{-1} - set \, of(G_1 + G_2)$. Now suppose that (ii) hold. Let D_{K1} =$\{x_1\} \, and \, D'_{K2} = \{x_2\}$ are minimum closed neutrosophic dominating set of G_1, then $D_{K1} \, and \, D'_{K1}$ are minimum closed neutrosophic dominating set of $G_1 + G_2$

Since $D_{K1} \subseteq V(G_1 + G_2) - D'_{K1}$ therefore, D_{K1} is minimum inverse closed neutrosophic dominating set of $G_1 + G_2$. Hence, $D_{K1} \, is \, \gamma cl^{-1} - set \, of(G_1 + G_2)$. Similarly, if (iii) holds.

Conversely, suppose that $D_{K1}^{-1} = \{x\}$ be a $\gamma cl^{-1} - set \, of(G_1 + G_2)$. i.e.,
$\gamma cl^{-1}(G_1 + G_2) = |(T_A(x), I_A(x), F_A(x)|$ then by proposition 6.4
$\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ or
$\gamma_{cl}(G_2) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$, if $\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)$ |then D_{K1} =$\{x_1\}$ is minimum closed neutrosophic dominating set of $G_1 + G_2$, since D_{K1}^{-1} has only one vertex thus a minimum closed neutrosophic dominating set (D_{K2}) of G_2contains one vertex then (i) is done.

Suppose that D_{K2} contains at least two vertices, then $\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$, let $D_{K1}^{-1} = \{x\}$ be a minimum inverse closed neutrosophic dominating set of $(G_1 + G_2)$, since V$(G_2) \subseteq N_{G_1+G_2}[x]$ and D_{K2} at least two vertices, $x \notin V(G_2)$.thus , $x \in V(G_1)$,necessarily $\{x\}$

is a $\gamma cl - set \, in \, G_1$, therefore G_1 has at least two a

$\gamma cl - sets \, and \, (ii) holds. by \, the \, same \, way \, we \, prove \, (iii)$

Corollary. 6.13. Let $G = (A, B)$ be any neutrosophic graphs. then
$\gamma_{cl}^{-1}(G_1 + H) = |(T_A(x), I_A(x), F_A(x))|$, x has minimum neutrosophic value if and only if $G \cong K_n^N, n \geq 2 \, or \, G = H_1 + H$ for some strong neutrosophic graphs $H_1 \, and \, H$ satisfying one of the following:

i) $\gamma_{cl}(H) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ and $\gamma_{cl}(H_1) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$.
ii) $\gamma_{cl}(H) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ and H has at least two $\gamma cl - sets$.
iii) $\gamma_{cl}(H_1) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$ and H_1 has at least two $\gamma cl - sets$.

Proof: Suppose that $\gamma_{cl}^{-1}(G) = |(T_A(x), I_A(x), F_A(x))|$, since
$\gamma_{cl}^{-1}(k_n^N) = (\min|(T_A(x), (I_A(x), (F_A(x))|)$, $x \in V(k_n^N) - D_{K1}$, where D_{K1} is minimum closed neutrosophic dominating set of k_n^N and $n \geq 2$.

Assume that $G \neq k_n^N$, suppose , $\gamma_{cl}^{-1}(k_n^N) = |(T_A(x), I_A(x), F_A(x))|$,.then, there exist two distinct vertices x_1 and x_2 of G such that $\{x_1\}$ and $\{x_2\}$ are $\gamma cl - sets$ of G.
Moreover, (x_1, x_2) *is effective edge, put* $H =< \{x_1, x_2\} >$ *and* $H_1 = G - \{x_1, x_2\}$. *Then* $G = H_1 + H$. *Furthermore,* $\{x_1\}$ and $\{x_2\}$ are distinct $\gamma cl - sets$ in H. Consequently, (ii)holds.

iv) Similarly, the converse follows immediately from theorem 6.9

Theorem. 6.1.4. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two strong neutrosophic graphs. Then, a minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ contains two vertices. $|D_{K1}^{-1}| = 2$ if and only if any of the following is hold:

(i) $|D_{K1}| \geq 2$ and $|D_{K2}| \geq 2$ vertices, where D_{K1} and D_{K2} are minimum closed neutrosophic dominating sets of G_1 *and* G_2 respectively.

(ii) $|D_{K1}| = 1$ and $|D_{K2}| \geq 2$, $G \neq k_1^N + (k_1^N + \cup_j H_j)$ for some components H_j of G_j

Proof: Suppose that a minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ contains two vertices, i.e., $|D_{K1}^{-1}| = 2.$ Then, a (MCND) set D_K of $G_1 + G_2$ either has one vertex or two vertices, then there are couple of cases:

Case1: If $|D_K| = 2$ then it is clear that $|D_{K1}| \geq 2$ and $|D_{K2}| \geq 2$, where D_{K1} and D_{K2} are minimum closed neutrosophic dominating set of G_1 *and* G_2 respectively

Case 2: If $|D_K| = 1$ *then* $|D_{K1}| = 1$ or $|D_{K2}| = 1$, i.e.

$\gamma_{cl}(G_1) = min|T_A(x_1), I_A(x_1), F_A(x_1)|$ or $\gamma_{cl}(G_2) = min|T_A(y_1), I_A(y_1), F_A(y_1)|$

Suppose that $\gamma_{cl}(G_1) = min|T_A(x_1), I_A(x_1), F_A(x_1)|$, then $G_1 = \{x_1\} + \cup_j H_j$, then for some component H_j of G_1. Thus, a minimum invers closed neutrosophic dominating set of $(\{x_1\} + \cup_j H_j + G_2)$ Contains two vertices with minimum neutrosophic value, i.e.,

$\gamma_{cl}^{-1}(G_1 + G_2) = \gamma_{cl}^{-1}\ (\{x_1\} + \cup_j H_j + G_2)$=$\sum_{j=1}^{2} x_j$, $x_j \in V(G_1 + G_2) - D_K$. Necessarily, D_{K2} and a minimum closed neutrosophic dominating of $[\cup_j H_j]$ contains two vertices. This, means that, in particular

$$G_1 \neq k_1 + (k_1 + \cup_j H_j).$$

Conversely, assume the first condition is true then a minimum closed neutrosophic dominating set of $G_1 + G_2$ contains two vertices say $D_K = \{x_1, y_1\}$, $x_1 \in V(G_1)$ *and* $y_1 \in V(G_2)$.

Let $x_2 \in V(G_1) - \{x_1\}$, $y_2 \in V(G_2) - \{y_1\}$. Then, $D_K^{-1} = \{x_2, y_2\}$ is minimum closed neutrosophic dominating set of $V(G_1 + G_2) - D_K$, *thus* $D_K^{-1}$is minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ Contains a couple of vertices.

$|D_{K1}^{-1}| = 2.$ hence the result is done.

Now if (ii) hold, let $D_{K1} = \{x_1\} \subseteq V(G_1)$ be a closed neutrosophic dominating set of G_1.

then D_{K1} is closed neutrosophic dominating set in $G_1 + G_2$, consider

$(G_1 + G_2) - \{x_1\}$=$(G_1 - \{x_1\}) + G_2)$, by our imposition $(G_1 - \{x_1\}) \neq k_1^N + \cup_j H_j$

for some components H_j *of* G_1,thus a minimum closed neutrosophic dominating set of

$(G_1 - \{x_1\})$ contains at least a pair of vertices say $|D_K^*| \geq 2$. Now if $|D_K^{'}| \geq 2$ and $|D_{K2}| \geq 2$, then a minimum closed neutrosophic dominating set of $(G_1 - \{x_1\}) + G_2)$ contains two vertices, thus a minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ also contain two vertices. Hence the prove is done.

Conclusion

Dominating sets can be used to model many other problems, including many relating to computer communication networks, social network theory, land surveying, and other similar issues. Determining the domination number for graphs and finding minimum dominating sets could thus prove very useful. Therefore, this study focused on the closed dominant sets, which are more in control of the network graphs, and theorems related to this concept presented and reinforced with necessary examples and graphics.

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Introduction to anti-bitopological spaces

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Abstract. The main aim of this study is to introduce the notion of anti-bitopological space and to reintroduce some basic concepts of topology in this novel framework. We point out that there are at least three possible and not necessarily equivalent methods of defining *openness* (and thus, the notion of *open set*) with respect to two anti-topologies simultaneously. We choose one of these approaches and concentrate on it. This allows us to define anti-bitopological interior and closure in some specific manner. Moreover, we prove some initial lemmas on anti-bitopological boundary. Finally, we study the problem of subspaces.

Keywords: Anti-Bitopological space; anti-topological space; anti-interior; anti-closure; anti-boundary.

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1. **Introduction**

From the purely historical point of view, it would be fair to mention that the whole concept of *topological space* arose from the observation that some very natural structures (like open intervals on real line or open balls on real plane) can be analyzed in the light of more general definitions and properties. This allowed mathematicians to introduce and study many basic topological notions, e.g. interior, closure, density, compactness or connectedness.

Modern times reversed this initial approach (at least to certain extent). Many authors started to redefine and, in particular, to generalize the very concept of topological space. It seems that the main idea is to check what happens when some natural assumptions (like closure of the family under finite intersection or openness of the whole universal set) are dropped. This led to the development of generalized and supra-topologies (see [5] and [11]), infra-topologies (known as infi-topologies too; see [3] and [12]), minimal structures ([15]), weak structures ([6]) and generalized weak structures ([1], [7]). The latter are the weakest: they are just arbitrary collections of subsets of X. In fact, this concept was introduced already in 1966 by Kim-Leong Lim in [10].

One should be aware that all these studies are more or less important. They allow us to recognize which assumptions are necessary to achieve some expected results (and which are superfluous). In fact, they confront us with some questions from the area of philosophy of mathematics. For example: what does it really mean that a function is *continuous*? Should the interior of a set always be open? Should we always assume that empty set is open? Finally, should there be any "authentic" connection between the abstract notion of *openness* and "natural" openness of open interval on real line? Or maybe open sets in various generalized structures are just some *arbitrarily chosen* (or *distinguished*) sets?

Weakening of the initial definition of topological space is not the only possible direction. Recently, some authors started to investigate anti-topological structures (see [2], [17], [22] and [23]). These families are characterized by the fact that any finite intersections or any unions of their elements *does not* belong to such a family. Moreover, $\emptyset$ and X are never open in this specific sense. Clearly, each element of anti-topological space (that is, each anti-open set in a given space) is maximal and minimal at the same time. It cannot have proper anti-open subsets or supersets. However, non-empty intersections of anti-open sets are possible.

As for the bitopological spaces, it seems that their study was started by Kelly in 1963 (see [8]). In 1967 Pervin (see [14]) analyzed the notion of connectedness in this setting. Later there were many papers on bitopologies and this concept has been reintroduced in some generalized frameworks. For example, biminimal (see [4]) and biweak structures (see [9]) have been already studied. Moreover, some authors analyze spaces equipped with three, four, five or even six topologies.

In this paper we would like to introduce anti-bitopological spaces. We define some basic notions and we point out several important subtleties. Some of them are not obvious at the first glance (even if they are not necessarily technically complicated). Finally, we obtain some kind of general framework which may be used in further research.

In 2019 Smarandache (see [18]) generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or NeutroAlgebras) whose operations and axioms are partially true, partially indeterminate, and partially false as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebras) whose operations and axioms are totally false and on 2020 he continued to develop them e.g. in [19], [20] and [21].

The NeutroAlgebras and AntiAlgebras are a new field of research, which is inspired from our real world.

In classical algebraic structures, all operations are 100 % well-defined, and all axioms are 100 % true, but in real life and in many cases these restrictions are too harsh, since in our world we have things that only partially verify some operations or some laws.

Using the process of NeutroSophication of a classical algebraic structure we produce a NeutroAlgebra, while the process of AntiSophication of a classical algebraic structure produces an AntiAlgebra. NeutroTopology is a particular case of NeutroAlgebra and AntiTopology is a particular case of the AntiAlgebra.

2. Preliminaries

Let us recall the definition of anti-topological space.

Definition 2.1. [22] Assume that X is a non-empty universe and $\mathcal{T}$ be a collection of subsets of X. We say that $(X, \mathcal{T})$ is an anti-topological space if and only if the following conditions are satisfied:

(1) $\emptyset, X \notin \mathcal{T}$.

(2) For any $n \in \mathbb{N}$, if $A_1, A_2, ..., A_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} A_i \notin \mathcal{T}$. Here we assume that this intersection is *non-trivial*, i.e. that the sets in question are not all identical.

(3) For any $\{A_i\}_{i \in J \neq \emptyset}$ such that $A_i \in \mathcal{T}$ for each $i \in J$, we have $\bigcup_{i \in J} A_i \notin \mathcal{T}$. We assume that this union is non-trivial, i.e. that is, the sets not all identical.

The elements of $\mathcal{T}$ are called *anti-open sets* and their complements are *anti-closed sets*. The set of all anti-closed sets with respect to a given $\mathcal{T}$ is denoted by $\mathcal{T}_{Cl}$. We say that $\mathcal{T}$ is *anti-closed* under finite intersections and arbitrary unions. In fact, one can prove stronger result:

Lemma 2.2. *(see Lemma 2.3 in [22]).*

If $(X, \mathcal{T})$ is an anti-topological space, then it is anti-closed under arbitrary non-trivial intersections.

Moreover, we have:

Theorem 2.3. *(compare with Lemma 2.5 and Lemma 2.7 in [22]).*

If $(X, \mathcal{T})$ is an anti-topological space, then $\mathcal{T}_{Cl}$ is also an anti-topology on X.

We may also define anti-interior and anti-closure.

Definition 2.4. (see Def. 3.1 in [22] and Def. 3.1. and Def. 3.3 in [2]).

Let $(X, \mathcal{T})$ be an anti-topological space and $A \subseteq X$. Then we define *anti-interior* of A and its *anti-closure* as:

(1) $AntiInt(A) = \bigcup\{U; U \subseteq A \text{ and } U \in \mathcal{T}\}$

(2) $AntiCl(A) = \bigcap\{F; A \subseteq F \text{ and } F \in \mathcal{T}_{Cl}\}$

One can easily prove (again, see [2] and [22, 23]) that $AntiInt(A)$ need not be the biggest anti-open set contained in A and $AntiCl(A)$ may not be the smallest anti-closed set containing A. Clearly, the reason is that anti-interior (resp. anti-closure) need not to be anti-open (resp. anti-closed) at all.

The following two lemmas and remark appearing after them will be important in the next section.

Lemma 2.5. *(see Lemma 8 in [22]).*

The intersection of two anti-topologies (established on the same universe X) is an anti-topological space too.

Lemma 2.6. *(see Lemma 9 in [23]). The union of two anti-topologies (on the same universe) need not to be an anti-topological space.*

Remark 2.7. Clearly, the lemma above does not imply that the union of two different anti-topologies *cannot* be an anti-topology too. Take for example $X = \mathbb{Z}^+ \cup \mathbb{Z}^-$. Suppose $\mathcal{T}_1$ consists of the finite subsets of $\mathbb{Z}^+$ with cardinality 3 (e.g. $\{2, 5, 7\}$, $\{30, 300, 3000\}$, while $\mathcal{T}_2$ consists of the finite subsets of $\mathbb{Z}^-$ with cardinality 3 (e.g. $\{-1, -2, -3\}$). Both these structures are anti-topologies and their union is an anti-topology too. Another example: $X = \{a, b, c\}$, $\mathcal{T}_1 = \{\{a\}, \{b\}\}$ and $\mathcal{T}_2 = \{\{c\}\}$.

On the other hand, it is possible that the union of two anti-topologies:

(1) is closed under non-empty intersections. Take $X = \{a, b, c, d\}$, $\mathcal{T}_1 = \{\{a, b\}, \{b, c\}, \{c, d\}$ and $\mathcal{T}_2 = \{\{b\}, \{c, \}\}$.
(2) is closed under unions. Take $X = \{a, b, c, d\}$, $\mathcal{T}_1 = \{\{a, b\}, \{b, c\}\}$ and $\mathcal{T}_2 = \{\{a, b, c\}\}$.
(3) is a neutro-topology. Take neutro-topological space presented in [17] (Example 9). It is $X = \{a, b, c, d\}$ with neutro-topology $\tau = \{\{a\}, \{a, b\}, \{c, d\}, \{b, c\}, \emptyset\}$. Assume now that we remove empty set from our collection. We stay with $\tau_1 = \{\{a\}, \{a, b\}, \{c, d\}, \{b, c\}\}$. This structure can be easily presented as a union of two anti-topologies, namely $\mathcal{T}_1 = \{\{a\}, \{c, d\}\}$ and $\mathcal{T}_2 = \{\{a, b\}, \{b, c\}\}$ (and this decomposition is not necessarily unique).

However, it is clear that X and $\emptyset$ never belong to $\mathcal{T}_1 \cup \mathcal{T}_2$.

Remark 2.8. Anti-topologies can be considered as a special subclass of anti-minimal spaces which could be defined in the following way: let X be a non-empty universe and $\mathcal{M} \subseteq P(X)$. If $\emptyset, X \notin \mathcal{M}$, then we say that $\mathcal{M}$ is anti-minimal structure on X.

3. Anti-bitopological spaces: introductory notes

In this section we introduce anti-bitopological spaces.

Definition 3.1. If X is a non-empty set endowed with two anti-topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called an anti-bitopological space.

Remark 3.2. It is always an interesting question: if we have two or more structures (e.g. topologies, infra-topologies, minimal structures, weak structures, anti-topologies) on X, then how should we define *open* sets. We mean those sets which will be considered as open with respect to the whole structure, that is: "bi-open" ("tri-open" and in general, n-open). There are several approaches in the literature. For example:

(1) In case of biweak structures some authors (see [9]) assume that A is open if and only if $Int_{w_1}(Int_{w_2}(A)) = A$, where w_1, w_2 are weak structures on X and Int_{w_1}, Int_{w_2} are interior operations relying on these structures. The same approach has been presented in [4] but in the context of biminimal spaces.

(2) In case of tri-topological spaces Palaniammal defined (in [13]) A as tri-open if and only if $A \in \tau_1 \cap \tau_2 \cap \tau_3$. Hence, in his opinion tri-open set should be open in *each* of these three topologies. Alternatively, we could say that A should be open in so-called induced topology $\tau_1 \cap \tau_2 \cap \tau_3$.

(3) However, Priyadharsini and Parvathi assumed in [16] that A is tri-open if and only if $A \in \tau_1 \cup \tau_2 \cup \tau_3$. Hence, A should be open in *at least one* topology.

These approaches are not necessarily equivalent. In particular, the first one need not to be equivalent with the third one. This will be shown in the context of anti-bitopological spaces. The third approach will be fundamental for us. However, we will introduce a convenient notation that will allow us to avoid any confusion.

Definition 3.3. If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A \in \mathcal{T}_1 \cup \mathcal{T}_2$, then we say that A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-*anti-open* and its complement is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-*anti-closed*.

Now we have the following idea:

Definition 3.4. If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A \in \mathcal{T}_1 \cap \mathcal{T}_2$, then we will say that A is $(\mathcal{T}_1 \cap \mathcal{T}_2)$-*anti-open* and its complement is $(\mathcal{T}_1 \cap \mathcal{T}_2)$-*anti-closed*.

We define two basic but natural and important notions:

Definition 3.5. If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A \subseteq X$, then the anti-interior and anti-closure of A (with respect to $\mathcal{T}_1 \cup \mathcal{T}_2$) are defined as follows:

(1) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \bigcup\{B; B \subseteq A, B \in \mathcal{T}_1 \cup \mathcal{T}_2\}$.

(2) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = \bigcap\{C; A \subseteq C, C \in (\mathcal{T}_1 \cup \mathcal{T}_2)_{Cl}\}$.

Clearly, analogous definition can be derived with respect to $(\mathcal{T}_1 \cap \mathcal{T}_2)$-anti-open (anti-closed) sets.

However, we shall not use it in this paper.

Example 3.6. Let $X = \{1,2,3,4,5\}$, $\mathcal{T}_1 = \{\{3\},\{1,2\},\{1,4\}\}$ and $\mathcal{T}_2 = \{\{2\},\{1,4\},\{3,4\}\}$. It can be observed that both $\mathcal{T}_1$ and $\mathcal{T}_2$ are anti-topologies on X. Now, $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\{2\},\{3\},\{1,2\},\{1,4\},\{3,4\}\}$. This is not an anti-topology: note that $\{2\} \cap \{2,4\} = \{2\} \in \mathcal{T}_1 \cup \mathcal{T}_2$. However, it is not closed under intersections (note that $\{1,2\} \cap \{1,4\} = \{1\} \notin (\mathcal{T}_1 \cup \mathcal{T}_2)$ nor unions (observe that $\{1,2\} \cup \{1,4\} = \{1,2,4\} \notin (\mathcal{T}_1 \cup \mathcal{T}_2)$.

Now take $A = \{3,4\}$. Clearly, according to our definition this set is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open. However, $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(A)) = AntiInt_{\mathcal{T}_1}(\{3,4\}) = \{3\} \neq A$. Moreover, $A \notin \mathcal{T}_1 \cap \mathcal{T}_2$.

On the other hand, let $B = \{1,4,5\}$. Now $B \notin \mathcal{T}_1 \cup \mathcal{T}_2$ but $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(B)) = AntiInt_{\mathcal{T}_1}(\{1,4\}) = \{1,4\}$.

However, $C = \{1,4\}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$ and $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(C)) = AntiInt_{\mathcal{T}_1}(\{1,4\}) = \{1,4\} = C$.

The example above suggests that the following definition can be useful:

Definition 3.7. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Then, we say that $A \subseteq X$ is $\mathcal{T}_1\mathcal{T}_2$-*anti-open* if and only if $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(A)) = A$.

One can prove the following theorem.

Theorem 3.8. *If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space and $\{A_i\}_{i \in J \neq \emptyset}$ is a collection of $\mathcal{T}_1\mathcal{T}_2$-anti-open sets. Then $\bigcup_{i \in J} A_i$ is $\mathcal{T}_1\mathcal{T}_2$-anti-open too.*

Proof. ($\subseteq$). Let $x \in AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i))$. It means that there exists some $B \in \mathcal{T}_1$ such that $x \in B$ and $B \subseteq AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i))$. Hence, $x \in AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i)$. But then there is some $C \in \mathcal{T}_2$ such that $x \in C \subseteq \bigcup_{i \in J} A_i$. Then $x \in \bigcup_{i \in J} A_i$.

($\supseteq$). Let $x \in \bigcup_{i \in J} A_i$. Assume that $x \notin AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i))$. Then there is some $k \in J$ such that $x \in A_k$ but for any $\mathcal{T}_1$-anti-open $B \subseteq AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i)$, $x \notin B$. But $A_k = AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(A_k))$, so there is some $C \in \mathcal{T}_1$ such that $x \in C \subseteq AntiInt_{\mathcal{T}_2}(A_k)$. However, $AntiInt_{\mathcal{T}_2}(A_k) \subseteq AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i)$. Assume the contrary. Then there is some $y \in AntiInt_{\mathcal{T}_2}(A_k)$ such that $y \notin AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i)$. Hence there is $D \in \mathcal{T}_2$ such that $y \in D \subseteq A_k$ but for any $G \in \mathcal{T}_2$ such that $G \subseteq \bigcup_{i \in J} A_i$, $y \notin G$. But $D \subseteq A_k \subseteq \bigcup_{i \in J} A_i$. This is contradiction. □

Example 3.6 shows us that $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open set need not to be $\mathcal{T}_1\mathcal{T}_2$-anti-open. At first glance, this result is strange. One could say that the very definition of $\mathcal{T}_1\mathcal{T}_2$-anti-open sets resembles the concept of pseudo-anti-open sets in anti-topological spaces: A is pseudo-anti-open if and only if $AntiInt(A) = A$. And we observed in [22] that each anti-open set is pseudo-anti-open too. However, both anti-open and pseudo-anti-open sets rely on the same interior. Now the situation is different. This is because $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$ need not to be identical with $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(A))$. Of course, we can define the following class:

Definition 3.9. Assume that $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space. Let $A \subseteq X$. We say that A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-*pseudo-anti-open* if and only if $A = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.

Now the following lemma is clear:

Lemma 3.10. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Every $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open set is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-pseudo-anti-open too.*

Remark 3.11. The converse need not to be true. Let us go back to Example 3.6. Now $D = \{1, 2, 3\}$ is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-pseudo-anti-open (its $(\mathcal{T}_1 \cup \mathcal{T}_2)$-interior is $\{1, 2\} \cup \{3\} = D$) but it does not belong to $\mathcal{T}_1 \cup \mathcal{T}_2$. Besides, note that $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(D)) = AntiInt_{\mathcal{T}_1}(\{2\}) = \emptyset$.

The last thing in this section is another one example of anti-bitopological space.

Example 3.12. Let $X = \mathbb{R}$ and assume that $\mathcal{T}_1$ consists of all those open intervals (a, b) of the length 1 such that $a \geq 0$ (e.g. $(0, 1)$, $(2, 3)$ or $(\pi, \pi + 1)$). In fact, b must be $a + 1$. Assume that $\mathcal{T}_2$ consists of all those open intervals (a, b) of the length 1 such that $a < 0$ (e.g. $(-10, -9)$ or $(-0.50, 0.50)$. Again, $b = a + 1$.

Consider $(\mathcal{T}_1 \cup \mathcal{T}_2)$. This is an anti-topology on X and it consists of all open intervals of the length exactly 1. Each $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed set is a complement of some $(a, a + 1)$ (where $a \in \mathbb{R}$). Hence, it is of the form $(-\infty, a] \cup [a+1, +\infty)$. Now take $A = (0, 1)$. Each $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed set B such that $A \subseteq B$ is in one of the two forms. First, it can be $(-\infty, k] \cup [k + 1, +\infty)$ for some $k \leq -1$ (e.g. $B = (-\infty, -2] \cup [-1, +\infty)$). Second, it can be $(-\infty, m] \cup [m + 1, +\infty)$ where $m \geq 1$ (e.g. $B = (-\infty, 3] \cup [4, +\infty)$). As for the intersection of the first subfamily, it is $[0, +\infty)$. An intersection of the second subfamily is $(-\infty, 1]$. Hence, we may calculate an intersection of intersections of these families to find $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((0, 1))$. This will be $[0, +\infty) \cap (-\infty, 1] = [0, 1]$. Besides, we see that this last set is not $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed. This will be later generalized in a separate lemma.

4. Further investigation of $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open sets

In this section we shall investigate some properties of our sets. Moreover, we will introduce the notion of boundary in anti-bitopological context. Some of the results are more general and they are true even for arbitrary generalized weak structures.

4.1. *About closure and interior*

Lemma 4.1. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-topological space. Then the following observations are true:*

(1) *The union (intersection) of two $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open sets may not be $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open.*
(2) *The union (intersection) of two $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed sets may not be $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open.*

Proof. Proof is simple albeit we shall present two cases. Take the same universe and anti-topologies as in Example 3.6. Now both $\{2\}$ and $\{3\}$ are $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open but their union, namely $\{2, 3\}$ does not belong to $\mathcal{T}_1 \cup \mathcal{T}_2$. Moreover, their intersection (which is $\emptyset$) is beyond $\mathcal{T}_1 \cup \mathcal{T}_2$. The reader is encouraged to find appropriate $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed counterexamples. □

Lemma 4.2. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \subseteq A$.*

Proof. Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. Then there is some $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open $B \subseteq X$ such that $x \in B \subseteq A$. Then $x \in A$. □

Remark 4.3. But the converse of the lemma above is not true in general as shown in the example below: let $X = \{1, 2, 3, 4\}$, $\mathcal{T}_1 = \{\{2\}, \{1, 3\}\}$ and $\mathcal{T}_2 = \{\{1\}, \{2, 4\}\}$. Then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space. Let $A = \{2, 3\}$ and then we have $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \{2\} \neq A$.

On the other hand, it is always true that if A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open, then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = A$.

Lemma 4.4. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A \subseteq X$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$ need not to be the largest $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open set contained in A and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$ need not to be the smallest $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed set contained in A.*

Proof. Let us think about $X = \{a, b, c, d\}$ with $\mathcal{T}_1 = \{\{a\}, \{c\}\}$ and $\mathcal{T}_2 = \{\{b\}, \{c, d\}\}$. Then, clearly, $\mathcal{T}_1$ and $\mathcal{T}_2$ are anti-topologies on X and $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space. Let $A = \{b, c, d\}$. Then we have $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \{c\} \cup \{b\} \cup \{c, d\} = \{b, c, d\}$. But this set is not $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open. Now, we see that $(\mathcal{T}_1 \cup \mathcal{T}_2)_{Cl} = \{\{b, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b\}\}$. Take $B = \{b, d\}$. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) = \{b, c, d\} \cap \{a, b, d\} = B$. But B is not $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed at all.

Hence, the proposition above. □

Lemma 4.5. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A \subseteq B$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B)$.*

Proof. Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. Then there is some $C \in \mathcal{T}_1 \cup \mathcal{T}_2$ such that $x \in C \subseteq A$. But $A \subseteq B$, hence $C \subseteq B$. Hence, $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. □

Remark 4.6. The converse of the lemma above need not to be true. Take $X = \mathcal{N}^+$, $\mathcal{T}_1 = \{\{3\}, \{5\}, \{7\}, \{9\}, ...\}$ and $\mathcal{T}_2 = \{\{4\}, \{6\}, \{8\}, \{10\}, ...\}$. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)$ is just a collection of all singletons of X without $\{1\}$ and $\{2\}$. Take $A = \{1, 3, 4\}$ and $B = \{2, 3, 4\}$. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \{3, 4\} = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B)$ (so, in particular, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B)$). But $A \nsubseteq B$.

Lemma 4.7. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A, B \subseteq X$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B)$.*

Proof. We see that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$ and analogously $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B)$. Hence our lemma is true. □

Remark 4.8. As for the converse of the lemma above, it does not need to be true. Take $X = \{a, b, c, d\}$, $\mathcal{T}_1 = \{\{a\}, \{b\}\}$ and $\mathcal{T}_2 = \{\{b, c\}, \{c, d\}\}$. Now let $A = \{a, b, c, e\}$ and $B = \{c, d\}$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \{a, b, c\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B) = \{c, d\}$. Clearly, the intersection of those $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-interiors is $\{c\}$. On the other hand, $A \cap B = \{c\}$ but $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(\{c\}) = \emptyset$ and $\{c\} \nsubseteq \emptyset$.

Lemma 4.9. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A, B \subseteq X$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cup B)$.*

Proof. We have $A \subseteq A \cup B$, so $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cup B)$. Also, $B \subseteq A \cup B$, so $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cup B)$. Therefore, the conclusion holds. □

Lemma 4.10. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.*

Proof. $(\subseteq)$. This is clear in the light of Lemma 4.2.

$(\supseteq)$. Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. Hence there is some $B \in \mathcal{T}_1 \cup \mathcal{T}_2$ such that $x \in B \subseteq A$. But by the very definition of $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-interior we can say that $B \subseteq \bigcup\{C; C \subseteq A, C \in \mathcal{T}_1 \cup \mathcal{T}_2\} = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. Hence $x \in \bigcup\{D; D \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A), D \in \mathcal{T}_1 \cup \mathcal{T}_2\}$ (as we could see, B is an example of such D). But this means that $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A))$. □

Now we would like to prove some theorems about $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure. Some of them seem to be elementary but they are necessary to establish our general framework.

Lemma 4.11. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space with $A \subseteq X$. Then $A \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$.*

Proof. This is clear as a result of the definition of $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure. □

Lemma 4.12. *Assume that $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A \subseteq X$. Suppose that A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = A$.*

Proof. (⊇). This is obvious.

(⊆). Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$. Hence for any $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed B such that $A \subseteq B$, we have that $x \in B$. In particular, $A \subseteq A$ and A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed. Hence $x \in A$. □

Lemma 4.13. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A, B \subseteq X$. Let $A \subseteq B$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B)$.*

Proof. Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$. Hence for any $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed D such that $A \subseteq D$, $x \in D$. Assume that there is some E that is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed and $B \subseteq E$ but $x \notin E$. But $A \subseteq B \subseteq E$. This is contradiction. □

Lemma 4.14. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Let $A, B \subseteq X$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cup B)$.*

Proof. We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Therefore $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cup B)$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cup B)$. Hence our conclusion is clear. □

Remark 4.15. Note that the converse is not necessarily true (albeit analogous converse would be true e.g. in topological spaces). Take $X = \{a, b, c\}$, $\mathcal{T}_1 = \{\{b, c\}, \{a, c\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{\{b, c\}\}$. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)_{Cl} = \{\{a\}, \{b\}, \{c\}\}$. Take $A = \{a\}$ and $B = \{b\}$. Both are identical with their $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closures. On the other hand, $A \cup B = \{a, b\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cup B) = \bigcap \emptyset = X \nsubseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) = \{a, b\}$.

Lemma 4.16. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B)$.*

Proof. We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Therefore, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B)$. □

In the next lemma certain relationships between $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure and $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-interior have been proven.

Lemma 4.17. *If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A \subseteq X$. Then:*

(1) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c$.
(2) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A))^c$.
(3) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A^c))^c$.

Proof:

(1) ($\subseteq$). Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. Then $x \in A$ and there is a $B \in (\mathcal{T}_2)$ such that $x \in B \subseteq A$. Then $x \notin B^c$. But $A^c \subseteq B^c$ and $B^c \in (\mathcal{T}_1 \cup \mathcal{T}_2)_{Cl}$. Thus, we can say that $x \notin (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c)$. Hence, $x \in ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c$.
($\supseteq$). Let $x \in ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c$. Then, there is some B such that B is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed, $A^c \subseteq B$ and $x \notin B$. But B^c is a $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open and $B^c \subseteq (A^c)^c = A$. Moreover, $x \in B^c$, so $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.

(2) The whole thing to do is to take complements of both sides in (1).
(3) The whole thing to do is to take A^c instead of A in (2).

4.2. *About boundary*

In this section we shall investigate the notion of boundary in our anti-bitopological context.

Definition 4.18. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A \subseteq X$. We define $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-boundary of A as the set of these points which belong to the intersection of the $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure of A with the $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure of the complement of A.

It means that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c)$.

Example 4.19. Let $X = \{1, 2, 3, 4\}$, $\mathcal{T}_1 = \{\{2\}, \{1, 3\}\}$ and $\mathcal{T}_2 = \{\{1\}, \{2, 4\}\}$. Then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space. Then all the $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed sets are $\{1, 3, 4\}$, $\{2, 4\}$, $\{2, 3, 4\}$ and $\{1, 3\}$. Let $A = \{2, 3\}$. Then $A^c = \{1, 4\}$. Now, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = \{2, 3, 4\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = \{1, 3, 4\}$.

Hence, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = \{2, 3, 4\} \cap \{1, 3, 4\} = \{3, 4\}$.

Example 4.20. Recall Example 3.12. Consider the same space and the same $A = (0, 1)$. We already know that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = [0, 1]$. Now think about $A^c = (-\infty, 0] \cup [1, +\infty)$. This set is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed hence it is identical with its own $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = [0, 1] \cap ((-\infty, 0] \cup [1, +\infty)) = \{0, 1\}$.

In case of $B = (0, 2)$ we would obtain $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = [0, 2]$, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = (-\infty, 0] \cup [2, +\infty)$ and finally $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = \{0, 2\}$.

In the next proposition we have some fundamental properties of the operation introduced above.

Lemma 4.21. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A \subseteq X$. Then the following results are true:*

(1) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.
(2) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = A \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.
(3) $((\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A))^c = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A^c)$.

(4) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

Proof:

(1) We may show the following sequence of equivalences: $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) \Leftrightarrow x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) \Leftrightarrow x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$ and $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) \Leftrightarrow x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$ and $x \notin (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \Leftrightarrow x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.

(2) We have $A \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = A \setminus ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c)) = A \cap ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c = A \cap (((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c \cup ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c) = (A \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c) \cup (A \cap ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c) = \emptyset \cup (A \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.

(3) Using Lemma 4.17 we may write:
$((\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A))^c = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cap \mathcal{T}_2)AntiCl(A^c))^c = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c \cup ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A^c) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$, as expected.

(4) This can be proved in a similar manner.

Remark 4.22. Note that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$ need not to be equal with $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap A^c$. This condition is too weak. Let $X = \{1, 2, 3, 4\}$, $\mathcal{T}_1 = \{\{1, 3\}, \{2, 4\}, \{5\}\}$ and $\mathcal{T}_2 = \{\{1\}, \{2, 4\}, \{3, 5\}\}$. Consider $A = \{1, 2, 3\}$. Then $A^c = \{4, 5\}$. As for the $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed sets, these are $\{2, 4, 5\}$, $\{1, 3, 5\}$, $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$ and $\{1, 2, 4\}$. Now, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \{1, 3\}$, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = \{1, 2, 3, 4\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = \{2, 4, 5\} \cap \{2, 3, 4, 5\} = \{2, 4, 5\}$. So $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = \{1, 2, 3, 4\} \cap \{2, 4, 5\} = \{2, 4\}$. But this set is different than $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap A^c = \{1, 2, 3, 4\} \cap \{4, 5\} = \{4\}$.

Moreover, it is clear (in the light of the example above) that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$ need not to be equal with $((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap A^c) \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. In our case this last set would be equal to $(\{1, 2, 3, 4\} \cap \{4, 5\}) \setminus \{1, 3\} = \{4\}$.

The next theorem gives us some additional information about boundary.

Lemma 4.23. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A \subseteq X$. Then the following results hold:*

(1) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

(2) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

Proof:

(1) We have: $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A))^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c)^c)^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((\mathcal{T}_1 \cup$

$\mathcal{T}_2)AntiInt(A)) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

(2) We have: $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c)$.

However, we know that $A \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$. Hence $((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c \subseteq A^c$. Thus $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

Remark 4.24. Note that it is not necessarily true that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A \cup B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(B)$.

Counterexample: take $X = \{a, b, c\}$, $\mathcal{T}_1 = \{\{b, c\}, \{c, a\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{\{b, c\}\}$. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)_{Cl} = \{\{a\}, \{b\}, \{c\}\}$. Take $A = \{a\}$ and $B = \{b\}$. Clearly, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = A$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) = B$. Moreover, $A^c = \{b, c\}$ and $B^c = \{a, c\}$. Thus, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = \bigcap \emptyset = X$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B^c) = \bigcap \emptyset = X$.

Then we see that $A \cup B = \{a, b\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{a, b\}) = \bigcap \emptyset = X$. Moreover, $(A \cup B)^c = \{c\}$.

Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = \{a\} \cap X = \{a\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(B) = \{b\} \cap X = \{b\}$. The union of these two sets is $\{a, b\}$. However, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A \cup B) = X \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{c\}) = X \cap \{c\} = \{c\}$. But $\{c\} \nsubseteq \{a, b\}$.

However, analogous property is true in topological spaces.

Remark 4.25. Note that it is not true in general that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(B)$.

Take the same space as in Remark 4.22. Consider $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 5\}$. Calculate $A \cap B = \{2, 3, 4\}$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cap B) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{2, 3, 4\}) = \{1, 2, 3, 4\} \cap \{2, 3, 4, 5\} = \{2, 3, 4\}$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((A \cap B)^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{1, 5\}) = \{1, 3, 5\}$.

Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A \cap B) = \{2, 3, 4\} \cap \{1, 3, 5\} = \{3\}$.

Then we calculate $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = \{1, 2, 3, 4\}$, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{5\}) = \{2, 4, 5\} \cap \{1, 3, 5\} \cap \{2, 3, 4, 5\} = \{5\}$. Thus $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = \{1, 2, 3, 4\} \cap \{5\} = \emptyset$.

Moreover, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) = \{2, 3, 4, 5\}$, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{1\}) = \{1, 3, 5\} \cap \{1, 2, 3, 4\} \cap \{1, 2, 4\} = \{1\}$ and thus $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(B) = \{2, 3, 4, 5\} \cap \{1\} = \emptyset$.

If so, then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(B) = \emptyset \cup \emptyset = \emptyset$. But clearly, $\{3\} \nsubseteq \emptyset$.

However, analogous property is true in topological spaces.

Theorem 4.26. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-topological space and $A \subseteq X$. Then the following results have been found:*

(1) *If A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open, then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus A = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.*

(2) *If A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) \subseteq A$.*

Proof:

Since A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open, therefore $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = A$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c$. Then:

(1) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus A = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c)^c = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

(2) If A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = A$. So, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = A \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) \subseteq A$.

5. **Conclusions**

Here, we have introduced the basics of anti-bitopological space. The whole study of anti-topological spaces is still in its seminal form. This applies even more to anti-bitopologies. Thus, it is important to analyze many standard notions in both these novel frameworks. Some typical properties of interior, closure or boundary which are true in topological spaces are not necessarily true in weaker or just different structures. For example, in topological structures we can prove that $Int(A \cap B) = Int(A) \cap Int(B)$ but in anti-topologies (and anti-bitopologies) only left-to-right inclusion (that is, $(\subseteq)$) holds. Analogously, in topology we can say that $Cl(A \cup B) = Cl(A) \cup Cl(B)$, but in our structures only right-to-left inclusion is true.

In Remark 3.2 we have shown that there at least three possible approaches to the notion of "bi-anti-open" set. We ourselves focused on the family $(\mathcal{T}_1 \cup \mathcal{T}_2)$ but we presented some additional remarks on $(\mathcal{T}_1 \cap \mathcal{T}_2)$ and on those sets which satisfy the condition $A = AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(A))$. In general, these three classes are not identical.

Now we would like to investigate other topological notions in anti-bitopological framework. For example, it would be reasonable to analyze density, nowhere density and connectedness.

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A Novel Approach Towards Parameter Reduction Based on Bipolar Hypersoft Set and Its Application to Decision-Making

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Abstract. For a mathematical model to describe vague (uncertain) problems effectively, it must have the ability to explain the links between the objects and parameters in the problem in the most precise way. There is no suitable model that can handle such scenarios in the literature. This deficiency serves as motivation for this study. In this article, the bipolar hypersoft set (abbreviated, BHSS) is considered since the parameters and their opposite play a symmetrical role. We present a novel theoretical technique for solving decision-making problems using BHSS and investigate parameter reductions for these sets. Algorithms for parameter reduction are provided and explained with examples. The findings demonstrate that our suggested parameter reduction strategies remove unnecessary parameters and still retain the same decision-making options.

Keywords: bipolar hypersoft set; hypersoft set; soft set; parameter reduction; decision-making; algorithm

1. Introduction

Many real-world challenges in disciplines such as engineering, environmental sciences, information knowledge, medical sciences, and social sciences include varying degrees of uncertainty. It is well known that this type of uncertainty cannot be represented using conventional analytical methods. Despite this, they are effectively handled by theories ranging from the fuzzy set [43] to the intuitionistic fuzzy set [4] and the rough set [28, 29], as well as probability theory. However, all of these ideas have inherent problems, some of which were pointed out in Molodtsov (1999). Molodtsov [18, 19] offered a unique theory to address these problems, and the central idea of it is known as a soft set. From this starting point, a novel formal method for modeling uncertainties was created, and because of its adaptability, it has been discussed in the context of intelligent systems, operations research, information science, the theory of probability and measurement theory.

The study of decision-making procedures related to soft sets is an additional topic of interest in this subject. To address problems with decision-making, Maji et al. [17] suggested parameter reduction of soft sets. Subsequently, the Maji et al [17] approach was criticized by Chen et al. [10], who also gave a different idea for parameter reduction of soft sets. Ali [1] studied another point of view on parameter reduction in soft sets. Kong et al. [13] first proposed the idea of normal parameter reduction of soft sets in [10], which was intended to address the problem of suboptimal selection. However, the idea is too abstract and the procedure is difficult to understand and takes a long time. An improved approach is provided in [13], while Ma et al. [14] studied the normal parameter reduction. Xie [42] investigated parameter reduction by attribute reduction in information systems. Maharana and Mohanty [15] focused on the application of parametric reduction of soft set in decision-making problem. Zhan and Alcantud [44] discussed a variety of parameter reduction techniques based on soft (fuzzy) set types. Furthermore, they contrasted the algorithms to highlight their various benefits and drawbacks and provided examples to explain their differences. Using the notion of σ-algebraic soft sets, Khan et al. [11] have developed a novel approach for the normal parameter reduction. Applications in decision-making based on soft set and its extensions can be seen at [2, 3, 5, 6, 12, 16, 36].

In 2018, Smarandache [37, 38] extended the soft set to the hypersoft set. Then, in [39, 40], he extended the soft set and hypersoft set to IndetermSoft Set and IndetermHyperSoft Set, respectively. In addition, he introduced TreeSoft Set [41] as an extension to MultiSoft Set. The authors in [7, 9, 20, 21, 30–35] presented the principles of the hypersoft set and its application. Recently, Musa and Asaad [22] introduced the notion of BHSS as a combination of hypersoft set with bipolarity setting and investigated some of its fundamental operations. They also discussed some topological notions in the frame of bipolar hypersoft setting [8, 23–27].

1.1. *Motivation*

In many real-world decision-making challenges, we come into situations where each attribute needs to be further categorized into its appropriate attribute-valued set. In order to deal with such eventualities, a hypersoft set is projected, using the cartesian product of disjoint attribute-valued sets as the approximate function's domain. To handle uncertainties with this form of approximate function, the current models are insufficient. Therefore, in this paper, the BHSS is taken into consideration because the parameters and their opposite play a symmetric role.

1.2. *Main Contributions*

The following list highlights the study's main contributions:

(1) Some basic definitions are reviewed from the literature.
(2) Theory of BHSSs is used to present a novel theoretical technique for solving decision-making problems and investigate parameter reductions for these sets.
(3) Suggested algorithm is then tested by using it to solve a problem from daily life that involves decision-making.
(4) The scope and future directions of the paper are summarized in order to inspire the reader to pursue further extensions.

1.3. *Paper Layout*

Following is the structure for this paper: The second section, before getting into this article, some background information and ideas are given. In the third section, by utilizing BHSSs, we suggest a new technique for parameter reduction, which will then be followed by an example. The last section, section 4, provides the paper's conclusion and future directions.

2. **Preliminaries**

This part will show a few results that will be useful in the subsequent section. Let $\Re$ represent a finite universe of objects, $2^{\Re}$ the power set of $\Re$, and $\Sigma = \Sigma_1 \times \Sigma_2 \times ... \times \Sigma_n$ the set of parameters. Let $\Lambda = \Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$ and $\Delta = \Delta_1 \times \Delta_2 \times ... \times \Delta_n$ with $\Lambda_i, \Delta_i \subseteq \Sigma_i$ for each $i = 1, 2, ..., n$.

Definition 2.1. [22] A triple $(g, \widehat{g}, \Sigma)$ is called a BHSS over $\Re$, where g and $\widehat{g}$ are mappings given by $g : \Sigma \to 2^{\Re}$ and $\widehat{g} : \neg\Sigma \to 2^{\Re}$ with $g(s) \cap \widehat{g}(\neg s) = \phi$ for all $s \in \Sigma$.

In other words, a BHSS $(g, \widehat{g}, \Sigma)$ over $\Re$ provides two parametrized families of subsets of $\Re$, with the consistency requirement $g(s) \cap \widehat{g}(\neg s) = \phi$ for all $s \in \Sigma$. From now on, a BHSS $(g, \widehat{g}, \Sigma)$ will be represented as follows:

$$(g, \widehat{g}, \Sigma) = \{(s, g(s), \widehat{g}(\neg s)) : s \in \Sigma \text{ and } g(s) \cap \widehat{g}(\neg s) = \phi\}.$$

Example 2.2. Suppose $\Re = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}$ is the set of seven applicants that applying for a job in a company. Let, The Director $= \Sigma_1 = \{s_1 =$goal-oriented,$s_2 =$ risk-taking,$s_3 =$ good under stress $\}$, The Thinker $= \Sigma_2 = \{s_4 =$ logical,$s_5 =$ prepared $\}$, and The Supporter $= \Sigma_3 = \{s_7 =$ stabilizing, $s_8 =$ cautious $\}$ be the set of three personality types and $\Sigma = \Sigma_1 \times \Sigma_2 \times \Sigma_3 = \{\ell_1 = (s_1, s_4, s_7), \ell_2 = (s_1, s_4, s_8), \ell_3 = (s_1, s_5, s_7), \ell_4 = (s_1, s_5, s_8), \ell_5 = (s_2, s_4, s_7), \ell_6 = (s_2, s_4, s_8), \ell_7 = (s_2, s_5, s_7), \ell_8 = (s_2, s_5, s_8), \ell_9 = (s_3, s_4, s_7), \ell_{10} = (s_3, s_4, s_8), \ell_{11} = (s_3, s_5, s_7), \ell_{12} = (s_3, s_5, s_8)\}$. A BHSS

$(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$ can be defined to describe "Analysis of Applicants' Personality" as: $(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma) = \{(\ell_1, \Re, \phi), (\ell_2, \{r_2, r_3, r_4, r_6\}, \{r_5, r_7\}), (\ell_3, \phi, \{r_4, r_5, r_6, r_7\}), (\ell_4, \{r_1, r_2, r_3, r_4\}, \{r_5, r_6, r_7\}), (\ell_5, \{r_1, r_2, r_3\}, \phi), (\ell_6, \{r_1, r_4, r_6\}, \{r_2, r_7\}), (\ell_7, \Re, \phi), (\ell_8, \{r_1, r_7\}, \{r_2, r_6\}), (\ell_9, \{r_1, r_6\}, \{r_2, r_4\}), (\ell_{10}, \phi, \Re), (\ell_{11}, \Re, \phi), (\ell_{12}, \phi, \Re)\}$.

Musa and Asaad [22] represented a BHSS by a binary table to store it in computer memory. The (i, j)-th entry in table is:

$$m_{ij} = \begin{cases} 1 & \text{if } r_i \in \mathcal{g}(\ell_j) \\ 0 & \text{if } r_i \in \Re \setminus \{\mathcal{g}(\ell_j) \cup \widehat{\mathcal{g}}(\neg \ell_j)\} \\ -1 & \text{if } r_i \in \widehat{\mathcal{g}}(\neg \ell_j) \end{cases}$$

Table 1 provides a tabular representation of the BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$ with referring to Example 2.2.

TABLE 1. Tabular form of $(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$

$(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}
r_1	1	0	0	1	1	1	1	1	1	−1	1	−1
r_2	1	1	0	1	1	−1	1	−1	−1	−1	1	−1
r_3	1	1	0	1	1	0	1	0	0	−1	1	−1
r_4	1	1	−1	1	0	1	1	0	−1	−1	1	−1
r_5	1	−1	−1	−1	0	0	1	0	0	−1	1	−1
r_6	1	1	−1	−1	0	1	1	−1	1	−1	1	−1
r_7	1	−1	−1	−1	0	−1	1	1	0	−1	1	−1

Definition 2.3. [22] Let $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda)$ and $(\mathcal{g}_2, \widehat{\mathcal{g}}_2, \Delta)$ be two BHSSs. Then

(1) $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda)$ is a bipolar hypersoft subset of $(\mathcal{g}_2, \widehat{\mathcal{g}}_2, \Delta)$, denoted by $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\mathcal{g}_2, \widehat{\mathcal{g}}_2, \Delta)$, if $\Lambda \subseteq \Delta$ and $\mathcal{g}_1(s) \subseteq \mathcal{g}_2(s)$, $\widehat{\mathcal{g}}_2(\neg s) \subseteq \widehat{\mathcal{g}}_1(\neg s)$ for all $s \in \Lambda$.

(2) $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda)$ and $(\mathcal{g}_2, \widehat{\mathcal{g}}_2, \Delta)$ are bipolar hypersoft equal, if $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\mathcal{g}_2, \widehat{\mathcal{g}}_2, \Delta)$ and $(\mathcal{g}_2, \widehat{\mathcal{g}}_2, \Delta) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda)$.

(3) If $\mathcal{g}_1(s) = \phi$ and $\widehat{\mathcal{g}}_1(\neg s) = \Re$ for all $s \in \Lambda$, then $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda)$ is called a relative null BHSS and denoted by $(\widetilde{\phi}, \widetilde{\Re}, \Lambda)$.

(4) If $\mathcal{g}_1(s) = \Re$ and $\widehat{\mathcal{g}}_1(\neg s) = \phi$ for all $s \in \Lambda$, then $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda)$ is called a relative whole BHSS and denoted by $(\widetilde{\Re}, \widetilde{\phi}, \Lambda)$.

(5) The complement of $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda)$ is a BHSS $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda)^c = (\mathcal{g}_1^c, \widehat{\mathcal{g}}_1^c, \Lambda)$ where $\mathcal{g}_1^c(s) = \widehat{\mathcal{g}}_1(\neg s)$ and $\widehat{\mathcal{g}}_1^c(\neg s) = \mathcal{g}_1(s)$ for all $s \in \Lambda$.

(6) The union of $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda)$ and $(\mathcal{g}_2, \widehat{\mathcal{g}}_2, \Delta)$, denoted by $(\mathcal{g}_1, \widehat{\mathcal{g}}_1, \Lambda) \mathrel{\tilde{\tilde{\sqcup}}} (\mathcal{g}_2, \widehat{\mathcal{g}}_2, \Delta)$, is a BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Gamma)$, where $\Gamma = \Lambda \cap \Delta$ and for all $s \in \Gamma$: $\mathcal{g}(s) = \mathcal{g}_1(s) \cup \mathcal{g}_2(s)$ and $\widehat{\mathcal{g}}(\neg s) = \widehat{\mathcal{g}}_1(\neg s) \cap \widehat{\mathcal{g}}_2(\neg s)$.

(7) The intersection of $(\mathscr{g}_1, \widehat{\mathscr{g}}_1, \Lambda)$ and $(\mathscr{g}_2, \widehat{\mathscr{g}}_2, \Delta)$, denoted by $(\mathscr{g}_1, \widehat{\mathscr{g}}_1, \Lambda) \ \widetilde{\widetilde{\sqcap}} \ (\mathscr{g}_2, \widehat{\mathscr{g}}_2, \Delta)$, is a BHSS $(\mathscr{g}, \widehat{\mathscr{g}}, \Gamma)$, where $\Gamma = \Lambda \cap \Delta$ and for all $s \in \Gamma$: $\mathscr{g}(s) = \mathscr{g}_1(s) \cap \mathscr{g}_2(s)$ and $\widehat{\mathscr{g}}(\neg s) = \widehat{\mathscr{g}}_1(\neg s) \cup \widehat{\mathscr{g}}_2(\neg s)$.

Definition 2.4. [28]

(1) Suppose B is a set of attributes with $B \subseteq A$. We identify a binary relation $\mathcal{IND}(B)$, known as indiscernibility, with expression $\mathcal{IND}(B) = \{(x, y) \in \Re \times \Re : b(x) = b(y), \forall b \in B\}$. Also, $\mathcal{IND}(B) = \cap_{b \in B} \mathcal{IND}(b)$.

(2) We call an element $b \in B$ dispensable if $\mathcal{IND}(B) = \mathcal{IND}(B - \{b\})$. Otherwise, b is called indispensable in B.

3. Parameter Reduction and Decision-Making Problem

In this section, we discuss the idea of reduction of parameters and decision-making problem in case of BHSS. Examples are provided to assist readers comprehend the key findings.

Definition 3.1. Let $\pi : \Sigma \to 2^{\Re \times \Re}$ be mapping. Then a hypersoft binary relation over $\Re$ is the hypersoft set (π, Σ) over $\Re \times \Re$.

In fact, (π, Σ) is a parametrized subsets of binary relations on $\Re$, i.e., there is a binary relation $\pi(s)$ on $\Re$ for each parameter $s \in \Sigma$.

Definition 3.2. If $\pi(s) \neq \phi$ is an equivalence relation over $\Re$ for all $s \in \Sigma$. Then a hypersoft binary relation (π, Σ) over a set $\Re$ is called a hypersoft equivalence relation over $\Re$.

Definition 3.3. The decision value of an object $r_i \in \Re$, denoted by $\mathscr{d}_i$, is defined as:

$$\mathscr{d}_i = \sum_j m_{ij}$$

where m_{ij} is the (i, j)-th element in the BHSS table. The decision table is constructed by joining the column of decision parameter $\mathscr{d}$ with values $\mathscr{d}_i$ to the table of the BHSS $(\mathscr{g}, \widehat{\mathscr{g}}, \Sigma)$.

Now, we provide a definition for the concept of indiscernibility relations related to a BHSS.

Definition 3.4. Let $(\mathscr{g}, \widehat{\mathscr{g}}, \Sigma)$ be a BHSS over $\Re$, then:

(1) If $\phi \neq \mathscr{g}(s) \subset \Re$ and $\phi \neq \widehat{\mathscr{g}}(\neg s) \subset \Re$ with $\mathscr{g}(s) \cup \widehat{\mathscr{g}}(\neg s) \neq \Re$, then $(\mathscr{g}, \widehat{\mathscr{g}}, \Sigma)$ divides $\Re$ into three classes.

(2) If $\mathscr{g}(s) = \phi$, $\widehat{\mathscr{g}}(\neg s) \subset \Re$ or $\widehat{\mathscr{g}}(\neg s) = \phi$, $\mathscr{g}(s) \subset \Re$, then $(\mathscr{g}, \widehat{\mathscr{g}}, \Sigma)$ divides $\Re$ into two classes.

(3) If $\mathscr{g}(s) = \Re$ or $\widehat{\mathscr{g}}(\neg s) = \Re$, then it provides the universal equivalence relation $\Re \times \Re$.

In any of the foregoing three cases, these classes represent an equivalence relation on $\Re$. As a result, we can note that we have an equivalence relation on $\Re$ for each parameter $s \in \Sigma$. If we denote this equivalence relation as $\lambda(s)$ for all $s \in \Sigma$, then (λ, Σ) is a hypersoft equivalence relation over $\Re$. We write

$$\mathcal{IND}(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma) = \cap_{s\in\Sigma}\lambda(s).$$

It is obvious that $\mathcal{IND}(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$ is an equivalence relation over $\Re$. The classes of $\mathcal{IND}(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$ are fundamental types of knowledge that are shown by a BHSS over $\Re$. We could also consider that, $\mathcal{IND}(\Sigma) = \mathcal{IND}(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$.

Definition 3.5. We call a decision table of $(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$ consistent if and only if $\mathcal{IND}(\Sigma) \subseteq \mathcal{IND}(\mathcal{D})$, where $\mathcal{IND}(\mathcal{D})$ is the equivalence relation that divides $\Re$ into categories with similar decision values.

Definition 3.6. Suppose that $\mathcal{T} = (\Re, \Sigma, \Lambda, \mathcal{D})$ is a consistent decision table of BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$ and $\mathcal{T}_\epsilon = (\Re, \Sigma, \Lambda - \epsilon, \mathcal{D}_\epsilon)$ is a decision table genrated from $\mathcal{T}$ by removing some column $\epsilon \in \Lambda$. Then ϵ is dispensable in $\mathcal{T}$ if

(1) $\mathcal{T}_\epsilon$ is consistent, that is, $\Lambda - \epsilon \Rightarrow \mathcal{D}_\epsilon$.
(2) $\mathcal{IND}(\mathcal{D}) = \mathcal{IND}(\mathcal{D}_\epsilon)$.

Otherwise, ϵ is indispensable or core parameter. The set of all core parameters of Λ is denoted by $\mathcal{CORE}(\Lambda)$.

Now, we suggest the following algorithm based on a BHSS.

Algorithm 1.

(1) Identify the BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$.
(2) Identify $\Lambda \subseteq \Sigma$ as the set of choice parameters.
(3) Input $d \in \mathcal{D}$, $d_i = \sum_j m_{ij}$ and place it in the last column of the obtained choice parameters table.
(4) Place the objects that share the same value for d next to each other to rearrange the input.
(5) Specify core parameters as defined in Definition 3.6. Remove each dispensable parameter individually to get a table having a minimum number of condition parameters that has the same classification ability for d as the original table.
(6) Find k such that $d_k = \max d_i$. Then r_k is the best choice object. Any one of r_k's can be chosen if k has multiple values.

Below, we illustrate the proposed algorithm according to Example 2.2:

(1) Identify the BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$ given by Table 1.
(2) Let $\Lambda = \{\ell_1, \ell_3, \ell_5, \ell_7, \ell_9, \ell_{11}\}$ where $\Lambda = \Lambda_1 \times \Lambda_2 \times \Lambda_3$ and $\Lambda_1 = \{s_1, s_2, s_3\}$, $\Lambda_2 = \{s_4, s_5\}$, and $\Lambda_3 = \{s_7\}$.
(3) Table 2 gives the decision table of BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$. We observe that

TABLE 2. Tabular form of $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$

$(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$	ℓ_1	ℓ_3	ℓ_5	ℓ_7	ℓ_9	ℓ_{11}	$\mathcal{d}$
r_1	1	0	1	1	1	−1	3
r_2	1	1	1	−1	−1	−1	0
r_3	1	1	1	0	0	−1	2
r_4	1	1	0	1	−1	−1	1
r_5	1	−1	0	0	0	−1	−1
r_6	1	1	0	1	1	−1	3
r_7	1	−1	0	−1	0	−1	−2

$$\begin{aligned}\mathcal{IND}(\Lambda) &= \{(r_1,r_1),(r_2,r_2),(r_3,r_3),(r_4,r_4),(r_5,r_5),(r_6,r_6),(r_7,r_7)\} \\ &\subset \{(r_1,r_1),(r_2,r_2),(r_3,r_3),(r_4,r_4),(r_5,r_5),(r_6,r_6),(r_7,r_7),(r_1,r_6),(r_6,r_1)\} \\ &= \mathcal{IND}(\mathcal{D}).\end{aligned}$$

Therefore, the decision table is consistent.

(4) Table 3 is obtained by rearranging Table 2 using the same values for $\mathcal{d}$.

TABLE 3. Rearrangement of Table 2

$(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$	ℓ_1	ℓ_3	ℓ_5	ℓ_7	ℓ_9	ℓ_{11}	$\mathcal{d}$
r_1	1	0	1	1	1	−1	3
r_6	1	1	0	1	1	−1	3
r_3	1	1	1	0	0	−1	2
r_4	1	1	0	1	−1	−1	1
r_2	1	1	1	−1	−1	−1	0
r_5	1	−1	0	0	0	−1	−1
r_7	1	−1	0	−1	0	−1	−2

(5) In order to determine $\mathcal{CORE}(\Lambda)$. First, we remove ℓ_1 from Table 3, we obtain Table 4. We observe that removing ℓ_1 has no effect on the classification ability of the decision parameter $\mathcal{d}$, thus ℓ_1 is dispensable in Table 3. Then, if we remove ℓ_3 from Table 3, we are left with Table 5. Due to the removal of ℓ_3, $\mathcal{d}$'s classification is different from that in Table 3. Eliminating ℓ_3 therefore disturbs $\mathcal{d}$'s ability for classification; as a result, ℓ_3 is a core parameter. Continuing in the same manner we determine set of core parameters:

$$\mathcal{CORE}(\Lambda) = \{\ell_3, \ell_5, \ell_7, \ell_9\} \tag{1}$$

Hence, we can conclude that the removal of ℓ_1 and ℓ_{11} has no effect on $\mathcal{d}$'s classification ability, as seen in Table 3. The same classification is presented in Table 6 with minimum condition parameters.

TABLE 4. Tabular form of $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$ after eliminating ℓ_1

$(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$	ℓ_3	ℓ_5	ℓ_7	ℓ_9	ℓ_{11}	d_{ℓ_1}
r_1	0	1	1	1	-1	2
r_6	1	0	1	1	-1	2
r_3	1	1	0	0	-1	1
r_4	1	0	1	-1	-1	0
r_2	1	1	-1	-1	-1	-1
r_5	-1	0	0	0	-1	-2
r_7	-1	0	-1	0	-1	-3

TABLE 5. Tabular form of $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$ after eliminating ℓ_3

$(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$	ℓ_1	ℓ_5	ℓ_7	ℓ_9	ℓ_{11}	d_{ℓ_3}
r_1	1	1	1	1	-1	3
r_6	1	0	1	1	-1	2
r_3	1	1	0	0	-1	1
r_4	1	0	1	-1	-1	0
r_5	1	1	-1	-1	-1	0
r_2	1	0	0	0	-1	-1
r_7	1	0	-1	0	-1	-1

TABLE 6. Tabular form of $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$ after eliminating ℓ_1 and ℓ_{11}

$(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$	ℓ_3	ℓ_5	ℓ_7	ℓ_9	$d_{(\ell_1,\ell_{11})}$
r_1	0	1	1	1	3
r_6	1	0	1	1	3
r_3	1	1	0	0	2
r_4	1	0	1	-1	1
r_2	1	1	-1	-1	0
r_5	-1	0	0	0	-1
r_7	-1	0	-1	0	-2

Now, we define weighted table of the BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$. The reason is that some of the parameters are less important than others, so they must be prioritized lower. So, we propose that the column of that parameter have the following entries:

$$n_{ij} = \begin{cases} m_{ij} \times \varpi_j & \text{if } m_{ij} = 1 \\ 0 & \text{if } m_{ij} = 0 \\ m_{ij} \times (1 - \varpi_j) & \text{if } m_{ij} = -1 \end{cases}$$

instead of 0 and 1 and -1 only, where m_{ij} are the entries in the table of the BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$.

Definition 3.7. The weighted decision value of $r_i \in \Re$ is defined as:

$$d_i = \sum_j n_{ij}$$

Next, we present the revised algorithm:

Algorithm 2.

(a) Identify the BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Sigma)$.

(b) Identify $\Lambda \subseteq \Sigma$ as the set of choice parameters.

(c) Determine the weighted table of the BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$ based on the chosen weights.

(d) Input $d \in \mathcal{D}$, $d_i = \sum_j n_{ij}$ and place it in the last column of the weighted table $\mathcal{T}_\varpi$.

(e) Place the objects that share the same value for d next to each other to rearrange the input.

(f) Specify core parameters. Remove each dispensable parameter individually to get a table having a minimum number of condition parameters that has the same classification ability for d as the original table.

(g) Find k such that $d_k = \max d_i$. Then r_k is the optimal choice object. Any one of r_k's can be chosen if k has multiple values.

Now, the original problem is resolved utilizing the new algorithm. Assume that the selection committee assigns the following weights to the parameters of Λ, beginning with the third step:

$$\ell_3\colon \varpi_3 = 0.8$$
$$\ell_5\colon \varpi_5 = 0.5$$
$$\ell_7\colon \varpi_7 = 0.9$$
$$\ell_9\colon \varpi_9 = 0.9$$

Table 7 gives the weighted decision table of BHSS $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$. We note that

TABLE 7. Weighted Decision Table for $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$

$(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)_\varpi$	ℓ_3	ℓ_5	ℓ_7	ℓ_9	d
r_1	0	0.5	0.9	0.9	2.3
r_2	0.8	0.5	−0.1	−0.1	1.1
r_3	0.8	0.5	0	0	1.3
r_4	0.8	0	0.9	−0.1	1.6
r_5	−0.2	0	0	0	−0.2
r_6	0.8	0	0.9	0.9	2.6
r_7	−0.2	0	−0.1	0	−0.3

$$\mathcal{IND}(\Lambda) = \{(r_1, r_1), (r_2, r_2), (r_3, r_3), (r_4, r_4), (r_5, r_5), (r_6, r_6), (r_7, r_7)\}$$
$$= \mathcal{IND}(\mathcal{D}).$$

Therefore, the decision table is consistent. Table 8 is obtained by rearranging Table 7 based on the descending values for d. We find that

Table 8. Table of weighted BHSS $(g, \widehat{g}, \Lambda)$ after rearrangement

$(g, \widehat{g}, \Lambda)_\varpi$	ℓ_3	ℓ_5	ℓ_7	ℓ_9	d
r_6	0.8	0	0.9	0.9	2.6
r_1	0	0.5	0.9	0.9	2.3
r_4	0.8	0	0.9	−0.1	1.6
r_3	0.8	0.5	0	0	1.3
r_2	0.8	0.5	−0.1	−0.1	1.1
r_5	−0.2	0	0	0	−0.2
r_7	−0.2	0	−0.1	0	−0.3

$$\mathcal{CORE}(\Lambda) = \Lambda. \tag{2}$$

The values of d indicate that $d_6 = \max d_i = 2.6$ and hence $k = 6$. Thus r_6 is the best candidate to choose since it is the best choice object. We observe that the change occurs in the place of r_1. In the first case, r_1 ranked 1st out of 7, however under the weighted criterion r_1 ranks 2nd overall. Similarly, we can identify the position of each object based on the weighted criteria.

4. Conclusions and Discussion

BHSS theory is a valuable mathematical model for expressing uncertainty concerns since it takes into consideration both NOT parameters and parameters sets. In this study, a novel method to decision-making using BHSS was presented, and a decision-making problem was solved to show the technique's validity. Parameter reduction techniques were created and described through examples. The study demonstrated that our recommended parameter reduction procedures minimize the unneeded parameters while keeping the same decision-making choices. The novelty of this work is that this is the first work that employed BHSS to reduce the parameters in decision-making problems. Although the proposed work are flexible and reliable as the findings demonstrated that, this model has limitations regarding some situations that deal with operators having some degree of indeterminacy of our world. Therefore, the future work may include the extension of this study (i.e. IndetermSoft Set, IndetermHyperSoft Set, etc.).

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Extension of G-Algebras to SuperHyper G-Algebras

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Abstract. The theory of superhyperalgebras is a new concept in the study of all branches of algebra structures. In this paper, we introduce a novel concept of (m, n)-superhyper G-algebra and present several results from the study of certain properties of (m, n)-superhyper G-algebras. The purpose of this paper is the study an extension of G-algebras to (m, n)-superhyper G-algebras, as a generalization of a logic algebra. The main motivation of this work was obtained based on an extension of G-algebra to superhyper G-algebra based on the n^{th}-power set of a set.

Keywords: (m, n)-superhyperoperation, (m, n)-superhyperalgebra, (m, n)-superhyper G-algebra.

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1. **Introduction**

The concept of superhyperalgebra has been introduced by Smarandache in [12]. Smarandache presented the n^{th}-power set of a set, superhyper operation, superhyper axiom, superhyper algebra, their corresponding neutrosophic superhyper operation, neutrosophic superhyper axiom, and neutrosophic superhyper algebra. In general, in any field of knowledge, he analyzes to encounter superhyper structures (or more accurately (m, n)-SuperHyperStructures). He studied related concepts, for example, the concepts of superhyperoperation, superhyperaxiom, superhyperstructure, superhyperalgebra, superhyperfunction, superhypergroup, superhypertopology, superhypergraph, and their corresponding neutrosophic superhyperoperation, neutrosophic superhyperaxiom, and neutrosophic superhyperalgebra in [10–14] between 2016-2022. Recently Hamidi et al. investigated some research in this scope such as the spectrum of superhypergraphs via flows [3], on neutro-d-subalgebras [4], neutro-BCK-algebra [5], on neutro G-subalgebra [7], single-valued neutro hyper BCK-subalgebras [6] and superhyper BCK-algebra [8]. The superhyperalgebra theory both extends some well-known algebra results and introduces new topics. The notion of superhyperalgebra is a natural generalization of the notion of algebra and the development of its fundamental properties. In 2012, the concept of G-algebra was introduced by Bandaru and Rafi [2]. They proved that QS-algebras are G-algebras, but the opposite is not necessarily true. The concept of G-algebra is a generalization of Q-algebra, which has many applications in algebra. We can read more about G-algebras in [1, 9]. In this paper, (m, n)-superhyper G-algebras is defined and considered. Examples of (m, n)-superhyper G-algebras are given and some of their properties are described. The concept of (m, n)-superhyper G-algebra is a generalization of G-algebra. The purpose of this paper is the study an extension of G-algebras to (m, n)-superhyper G-algebras, as a generalization of a logic algebra. The main motivation of this work was obtained based on an extension of G-algebra to superhyper G-algebra based on the powerset. In this regard, the notation of n^{th}-power set of a set, superhyper operation, superhyper axiom play the main role in the construction of (m, n)-superhyper G-algebras.

2. Preliminaries

In this section, we recall some concepts that need for our work.

Definition 2.1. [2] Let $X \neq \emptyset$ and $0 \in X$ be a constant. Then a universal algebra $(X, *, 0)$ of type $(2, 0)$ is called a *G-algebra*, if for all $x, y \in X$:
$(G$-$1)$ $x * x = 0$,
$(G$-$2)$ $x * (x * y) = y$.

Proposition 2.2. [2] *If $(X, *, 0)$ is a G-algebra. Then, for all $x, y \in X$, the following conditions hold:*
(i) $x * 0 = x$,
(ii) $0 * (0 * x) = x$,
(iii) $(x * (x * y))y = 0$,
(iv) $x * y = 0$ *implies*$x = y$,
(v) $0 * x = 0 * y$ *implies* $x = y$.

Theorem 2.3. [2] *Let $(X, *, 0)$ be a G-algebra. Then the following are equivalent.*
(i) $(x * y) * z = (x * z) * y$ *for all* $x, y \in X$,
(ii) $(x * y) * (x * z) = z * y$ *for all* $x, y \in X$.

Theorem 2.4. [2]*Let $(X, *, 0)$ be a G-algebra.*
(i) If $(x * y) * (0 * y) = x$ *for all* $x, y \in X$, *then* $x * z = y * z$ *implies* $x = y$.
(ii) $a * x = a * y$ *implies* $x = y$ *for all* $a, x, y \in X$.

Definition 2.5. [14] Let X be a nonempty set. Then $(X, \circ^*_{(m,n)})$ is called an (m, n)-super hyperalgebra, where $\circ^*_{(m,n)} : X^m \to P^n_*(X)$ is called an (m, n)-super hyperoperation, $P^n_*(X)$

TABLE 1. G-algebra $(X, *, 0)$

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	3	2	5	4
2	2	3	0	1	5	4
3	3	2	1	0	4	5
4	4	5	3	2	0	1
5	5	4	2	3	1	0

is the n^{th}-powerset of the set X, $\emptyset \notin P^n_*(X)$, for any subset A of $P^n_*(X)$, we identify $\{A\}$ with A, $m, n \geq 1$ and $X^m = \underbrace{X \times X \times \ldots \times X}_{m\ times}$.

Let $\circ^*_{(m,n)}$ be an (m, n)-super hyperoperation on X and $A_1, \ldots, A_m$ subsets of X. We define

$$\circ^*_{(m,n)}(A_1, \ldots, A_m) = \bigcup_{x_i \in A_i} \circ^*_{(m,n)}(x_1, \ldots, x_m).$$

3. Superhyper G-Algebras

At the beginning of this section, we construct a G-algebra on every nonempty set. Then we give an example of G-algebra.

Theorem 3.1. *Let X be a nonempty set and $0 \in X$ be a constant. Then there exists $*$ on X such that $(X, *, 0)$ is a G-Algebra.*

$$x * y = \begin{cases} 0 & x = y \\ y & o.w. \end{cases}$$

Proof. (G-1) is true because $x * x = 0$. According to the definition $x * y = y$, therefore $x * (x * y) = x * y = y$, and (G-2) also hold. So $(X, *, 0)$ is a G-algebra. □

Example 3.2. Let $X = \{0, 1, 2, 3, 4, 5\}$ which $*$ is defined in Table 1. Then $(X, *, 0)$ is a G-algebra.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ which $*$ is defined in Table 2. Then $(X, *, 0)$ is not a G-algebra, , since $0 * (0 * 2) = 0 * 0 \neq 2$.

In this section, we introduce the concept of (m, n)-superhyper G-algebra based on the n^{th}-power set of a set. Also, investigate the properties of this concept.

Definition 3.4. Let X be a nonempty set and $0 \in X$ be a constant. Then $(X, \circ^*_{(m,n)}, 0)$ is called an (m, n)-superhyper G-algebra, if for all $x, y \in X$:

Table 2

$*$	0	1	2	3
0	0	0	0	3
1	1	0	3	0
2	2	2	0	1
3	3	3	3	0

Table 3. superhyper G-algebra $(X, \circ^*_{(2,1)}, x)$

$\circ^*_{(2,1)}$	x	y	z
x	x	$\{x, y\}$	$\{x, z\}$
y	y	x	$\{y, z\}$
z	$\{x, z\}$	$\{x, y, z\}$	x

Table 4. superhyper G-algebra $(X, \circ^*_{(2,2)}, a)$

$\circ^*_{(2,2)}$	$\{a\}$	$\{b\}$
$\{a\}$	$\{\{a\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$
$\{b\}$	$\{a, b\}$	a

(G_{sh}-1) $0 \in \circ^*(\underbrace{x, x, \ldots, x}_{m})$,

(G_{sh}-2) $y \in \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y))$.

Example 3.5. (i) Let $X = \{x, y, z\}$ and x be a constant. $P_*(X) = \{x, y, z, \{x, y, z\}, \{x, y\}, \{x, z\}, \{y, z\}\}$. Then $(X, \circ^*_{(2,1)}, x)$ is called a $(2, 1)$-superhyper G-algebra as shown in Table 3.

(ii) Let $X = \{a, b\}$, a be a constant and
$P^2_*(X) = \{\{a\}, \{b\}, \{a, b\}, \{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\}\}$. Then $(X, \circ^*_{(2,2)}, a)$ is called a $(2, 2)$-superhyper G-algebra as shown in Table 4.

(iii) Let $X = \{0, 1, 2\}$ and $P_*(X) = \{0, 1, 2, \{0, 1, 2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$. Then $(X, \circ^*_{(3,1)}, 0)$ is called a $(3, 1)$-superhyper G-algebra as shown in Table 5.

We see that two axioms (G_{sh}-1) and (G_{sh}-2) are independent. Let $X = \{0, 1, 2\}$ be a set with Table 6 and Table 7. In Table 6, the axiom (G_{sh}-1) is valid but (G_{sh}-2) does not, because $1 \notin \circ^*(2, \circ^*(2, 1))$, and in Table 7, the axiom (G_{sh}-2) is valid, but the axiom (G_{sh}-1) is not, because $0 \notin \circ^*(1, 1)$.

Table 5. superhyper G-algebra $(X, \circ^*_{(3,1)}, 0)$

$\circ^*_{(3,1)}$	0	1	2
$(0,0)$	0	1	2
$(0,1)$	1	$\{0,2\}$	$\{1,2\}$
$(0,2)$	2	$\{1,2\}$	$\{0,1\}$
$(1,0)$	1	$\{0,2\}$	$\{1,2\}$
$(2,0)$	2	$\{1,2\}$	$\{0,1\}$
$(1,1)$	$\{0,2\}$	$\{0,1\}$	$\{0,1,2\}$
$(1,2)$	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
$(2,1)$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
$(2,2)$	$\{0,1\}$	$\{0,1,2\}$	$\{0,2\}$

Table 6

$\circ^*_{(2,1)}$	0	1	2
0	0	$\{0,1,2\}$	$\{0,2\}$
1	$\{0,1\}$	$\{0,1,2\}$	2
2	$\{0,2\}$	$\{0,2\}$	0

Table 7

$\circ^*_{(2,1)}$	0	1	2
0	0	$\{0,1\}$	$\{0,2\}$
1	$\{0,1\}$	1	2
2	$\{0,2\}$	$\{0,1\}$	$\{0,1,2\}$

The following theore, we construct an (m, n)-superhyper G-algebra on each nonempty set.

Theorem 3.6. *Let X be a nonempty set and $0 \in X$ be a constant. Then there exists $\circ^*_{(m,n)}$ on X such that $(X, \circ^*_{(m,n)}, 0)$ is an (m, n)-superhyper G-algebra.*

$$\circ^*(x_1, x_2, \ldots, x_m) = \begin{cases} \{0\} & \forall i \neq j;\ x_i = x_j \\ \{0, y\} & o.w. \end{cases}$$

Proof. $(G_{sh}\text{-}1)$ is true because $0 \in \circ^*(\underbrace{x, x, \ldots, x}_{m})$. According to the definition $y \in \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y)$, therefore $y \in \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y))$ and $(G_{sh}\text{-}2)$ also hold. So, the proof is complete. □

Proposition 3.7. *Let $(X, \circ^*_{(m,n)}, 0)$ be an (m,n)-superhyper G-algebra. Then for any $x \in X$, the following conditions hold:*

(i) $\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, 0) \subseteq \circ^*\big(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{x, x, \ldots, x}_{m})\big)$,

(ii) $x \in \circ^*\big(\underbrace{0, 0, \ldots, 0}_{m-1}, \circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, x)\big)$.

Proof. *(i)* By $(G_{sh}$-1), $0 \in \circ^*(\underbrace{x, x, \ldots, x}_{m})$. Then we get $\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, 0) \subseteq \circ^*\big(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{x, x, \ldots, x}_{m})\big)$.

(ii) If we put $x = 0$ and $y = x$ in $(G_{sh}$-2), then we get *(ii)*. □

Proposition 3.8. *Let $(X, \circ^*_{(m,n)}, 0)$ be an (m,n)-superhyper G-algebra. Then for any $x, y \in X$,* $0 \in \circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y)), \underbrace{y, y, \ldots, y}_{m-1}\big)$.

Proof. According to $(G_{sh}$-2), $y \in \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y))$. Now we have according to $(G_{sh}$-1), $0 \in \circ^*(\underbrace{y, y, \ldots, y}_{m}) \subseteq \circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y)), \underbrace{y, y, \ldots, y}_{m-1}\big)$ and therefore the proof is complete. □

Theorem 3.9. *Let $(X, \circ^*_{(m,n)}, 0)$ be an (m,n)-superhyper G-algebra. If for any $x, y, z \in X$,* $\circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y), \underbrace{z, z, \ldots, z}_{m-1}\big) = \circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, z), \underbrace{y, y, \ldots, y}_{m-1}\big)$. *Then* $\circ^*(\underbrace{z, z, \ldots, z}_{m-1}, y) \subseteq \circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y), \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, z), \underbrace{x, x, \ldots, x}_{m-2}\big)$.

Proof. By $(G_{sh}$-2), $z \in \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, z))$. Now we have $\circ^*(\underbrace{z, z, \ldots, z}_{m-1}, y) \subseteq \circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, z)), \underbrace{y, y, \ldots, y}_{m-1}\big)$. According to the assumption

$\circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, z)), \underbrace{y, y, \ldots, y}_{m-1}\big) =$

$\circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y), \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, z), \underbrace{x, x, \ldots, x}_{m-2}\big)$. Thus it is obtained. □

Theorem 3.10. *Let $(X, \circ^*_{(m,n)}, 0)$ be an (m,n)-superhyper G-algebra. If for any $x, y, z \in X$,*

$\circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y), \circ^*(\underbrace{x, x, \ldots, x}_{m-1}, z), \underbrace{x, x, \ldots, x}_{m-2}\big) = \circ^*(\underbrace{z, z, \ldots, z}_{m-1}, y)$.

Then $\circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, y), \underbrace{z, z, \ldots, z}_{m-1}\big) = \circ^*\big(\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, z), \underbrace{y, y, \ldots, y}_{m-1}\big)$.

Proof. By (G_{sh}-2), $z \in \circ^*(\underbrace{x,x,\ldots,x}_{m-1}, \circ^*(\underbrace{x,x,\ldots,x}_{m-1},z))$. Now by the assumption, we have
$\circ^*(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\underbrace{z,z,\ldots,z}_{m-1}) \subseteq$
$\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{x,x,\ldots,x}_{m-1},\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z)),\underbrace{x,x,\ldots,x}_{m-2}\big) =$
$\circ^*(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\underbrace{y,y,\ldots,y}_{m-1})$.
Conversely, by (G_{sh}-2), $y \in \circ^*(\underbrace{x,x,\ldots,x}_{m-1}, \circ^*(\underbrace{x,x,\ldots,x}_{m-1},y))$. Therefore by the assumption,
$\circ^*(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\underbrace{y,y,\ldots,y}_{m-1}) \subseteq$
$\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\circ^*(\underbrace{x,x,\ldots,x}_{m-1},\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y)),\underbrace{x,x,\ldots,x}_{m-2}\big)$
$= \circ^*(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\underbrace{z,z,\ldots,z}_{m-1})$. □

Theorem 3.11. *Let $(X,\circ^*_{(m,n)},0)$ be an (m,n)-superhyper G-algebra. If for any $x,y,z \in X$, $\circ^*(\underbrace{z,z,\ldots,z}_{m-1},y) = \circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}\big)$. Then $\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z) \subseteq \circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{z,z,\ldots,z}_{m-1},y),\underbrace{x,x,\ldots,x}_{m-2}\big)$.*

Proof. According to (G_{sh}-2), $\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z) \subseteq$
$\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}),\underbrace{x,x,\ldots,x}_{m-2}\big)$. By the assumption, we have $\circ^*(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}) = \circ^*(\underbrace{z,z,\ldots,z}_{m-1},y)$. Therefore it is obtained. □

Theorem 3.12. *Let $(X,\circ^*_{(m,n)},0)$ be an (m,n)-superhyper G-algebra. If for any $x,y,z \in X$, $\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{z,z,\ldots,z}_{m-1},y),\underbrace{x,x,\ldots,x}_{m-2}\big) = \circ^*(\underbrace{x,x,\ldots,x}_{m-1},z)$. Then $\circ^*(\underbrace{z,z,\ldots,z}_{m-1},y) \subseteq \circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}\big)$.*

Proof. According to (G_{sh}-2), $\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z) \subseteq$
$\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}),\underbrace{x,x,\ldots,x}_{m-2}\big)$ and by the assumption, $\circ^*(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{z,z,\ldots,z}_{m-1},y),\underbrace{x,x,\ldots,x}_{m-2}) = \circ^*(\underbrace{x,x,\ldots,x}_{m-1},z)$. Therefore
$\circ^*(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{z,z,\ldots,z}_{m-1},y),\underbrace{x,x,\ldots,x}_{m-2}) \subseteq$

$\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}),\underbrace{x,x,\ldots,x}_{m-2}\big)$. Thus $\circ^*(\underbrace{z,z,\ldots,z}_{m-1},y)\subseteq\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}\big)$. □

Definition 3.13. Let $(X,\circ^*_{(m,n)},0)$ be an (m,n)-superhyper G-algebra and $A,B\in\circ^*_{(m,n)}(x_1,\ldots,x_m)$. Then A and B are called adjacent.

Proposition 3.14. *Let $(X,\circ^*_{(m,n)},0)$ be an (m,n)-superhyper G-algebra. Then for any $a,x,y\in X$, $\circ^*(\underbrace{a,a,\ldots,a}_{m-1},x)=\circ^*(\underbrace{a,a,\ldots,a}_{m-1},y)$ implies x and y are adjacent.*

Proof. Let $a,x,y\in X$ and $\circ^*(\underbrace{a,a,\ldots,a}_{m-1},x)=\circ^*(\underbrace{a,a,\ldots,a}_{m-1},y)$. It follows that $\circ^*\big(\underbrace{a,a,\ldots,a}_{m-1},\circ^*(\underbrace{a,a,\ldots,a}_{m-1},x)\big)=\circ^*\big(\underbrace{a,a,\ldots,a}_{m-1},\circ^*(\underbrace{a,a,\ldots,a}_{m-1},y)\big)$. Thus according to $(G_{sh}$-2), x and y are adjacent. □

Theorem 3.15. *Let $(X,\circ^*_{(m,n)},0)$ be an (m,n)-superhyper G-algebra. Then for any $x,y\in X$, $\circ^*(\underbrace{0,0,\ldots,0}_{m-1},x)=\circ^*(\underbrace{0,0,\ldots,0}_{m-1},y)$ implies x and y are adjacent.*

Proof. According to the assumption, $\circ^*(\underbrace{0,0,\ldots,0}_{m-1},x)=\circ^*(\underbrace{0,0,\ldots,0}_{m-1},y)$, So we have $\circ^*\big(\underbrace{0,0,\ldots,0}_{m-1},\circ^*(\underbrace{0,0,\ldots,0}_{m-1},x)\big)=\circ^*\big(\underbrace{0,0,\ldots,0}_{m-1},\circ^*(\underbrace{0,0,\ldots,0}_{m-1},y)\big)$. Therefore by Theorem 3.7 (ii), $x\in\circ^*\big(\underbrace{0,0,\ldots,0}_{m-1},\circ^*(\underbrace{0,0,\ldots,0}_{m-1},x)\big)$ and $y\in\circ^*\big(\underbrace{0,0,\ldots,0}_{m-1},\circ^*(\underbrace{0,0,\ldots,0}_{m-1},y)\big)$. By definition x and y are adjacent. □

Theorem 3.16. *Let $(X,\circ^*_{(m,n)},0)$ be an (m,n)-superhyper G-algebra. Then for any $x,y\in X$, $x\in\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},y),\circ^*(\underbrace{0,0,\ldots,0}_{m-1},y),\underbrace{y,y,\ldots,y}_{m-2}\big)$ and $\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z)=\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z)$, implies x and y are adjacent.*

Proof. If $\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z)=\circ^*(\underbrace{y,y,\ldots,y}_{m-1},z)$, then

$\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\circ^*(\underbrace{0,0,\ldots,0}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}\big)=$

$\circ^*\big(\circ^*(\underbrace{y,y,\ldots,y}_{m-1},z),\circ^*(\underbrace{0,0,\ldots,0}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}\big)$. By the assumption

$x\in\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1},z),\circ^*(\underbrace{0,0,\ldots,0}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}\big)$ and

$y\in\circ^*\big(\circ^*(\underbrace{y,y,\ldots,y}_{m-1},z),\circ^*(\underbrace{0,0,\ldots,0}_{m-1},z),\underbrace{x,x,\ldots,x}_{m-2}\big)$. It follows that x and y are adjacent. □

Definition 3.17. A non-empty subset Y of an (m,n)-superhyper G-algebra X is called an (m,n)-superhyper G-subalgebra if for all $a_1, a_2, \ldots, a_m \in Y$, implies $\circ^*_{(m,n)}(a_1, a_2, \ldots, a_m) \in P^n_*(Y)$.

Definition 3.18. Let $(X, \circ^*_{(m,n)}, 0_X)$ and $(X', \circ'^*_{(m,n)}, 0_{X'})$ be (m,n)-superhyper G-algebras. A mapping $\phi : X \longrightarrow X'$ is called a homomorphism if

(i) $\phi(\circ^*(x_1, x_2, \ldots, x_m)) = \circ'^*(\phi(x_1), \phi(x_2), \ldots, \phi(x_m))$, for $x_1, x_2, \ldots, x_m \in X$.

(ii) $0_{X'} \in \phi(0_X)$.

The homomorphism ϕ is said to be a monomorphism (resp., an epimorphism) if it is injective (resp., surjective). If the map ϕ is both injective and surjective then X and X' are said to be isomorphic, written $X \cong X'$. For any homomorphism $\phi : X \longrightarrow X'$, the set $\{x \in X | 0_{X'} \in \phi(x)\}$ is called the kernel of ϕ and is denoted by $Ker\phi$.

Lemma 3.19. *Let $\phi : (X, \circ^*_{(m,n)}, 0_X) \longrightarrow (X', \circ'^*_{(m,n)}, 0_{X'})$ be a homomorphism of (m,n)-superhyper G-algebras, then we have the following:*

(i) *$Ker\phi$ is an (m,n)-superhyper G-algebra of X,*

(ii) *$Im\phi = \{y \in X' | y = \phi(x), for\ some\ x \in X\}$ is an (m,n)-superhyper G-subalgebra of X.*

Proof. (i) Since $0_X \in Ker\phi$, then $Ker\phi \neq \emptyset$. Suppose $x_1, x_2, \ldots, x_m \in Ker\phi$. So $0_{X'} \in \phi(x_i)$ for $i = 1, \ldots, m$. From $\phi(\circ^*(x_1, x_2, \ldots, x_m)) = \circ'^*(\phi(x_1), \phi(x_2), \ldots, \phi(x_m))$. Because $0_{X'} \in \circ'^*(\phi(x_1), \phi(x_2), \ldots, \phi(x_m))$, Implies that $0_{X'} \in \phi(\circ^*(x_1, x_2, \ldots, x_m))$. It follows that, $\circ^*(x_1, x_2, \ldots, x_m) \in Ker\phi$.
(ii) Direct to prove. □

Definition 3.20. An (m,n)-superhyper G-algebra $(X, \circ^*_{(m,n)}, 0)$ is said to be 0-commutative if for any $x, y \in X$, $\circ^*(\underbrace{x, x, \ldots, x}_{m-1}, \circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, y)) = \circ^*(\underbrace{y, y, \ldots, y}_{m-1}, \circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, x))$.

Theorem 3.21. *Let $(X, \circ^*_{(m,n)}, 0)$ be an 0-commutative (m,n)-superhyper G-algebra. Then for any $x, y \in X$, $\circ^*(\underbrace{y, y, \ldots, y}_{m-1}, x) \subseteq \circ^*\big(\circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, x), \circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, y), \underbrace{x, x, \ldots, x}_{m-2}\big)$.*

Proof. Because
X be a 0-commutative, implies that $\circ^*\big(\circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, x), \circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, y), \underbrace{x, x, \ldots, x}_{m-2}\big) = \circ^*\big(\underbrace{y, y, \ldots, y}_{m-1}, \circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, \circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, x))\big)$. By Theorem 3.7 (ii), $\circ^*(\underbrace{y, y, \ldots, y}_{m-1}, x) \subseteq \circ^*\big(\underbrace{y, y, \ldots, y}_{m-1}, \circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, \circ^*(\underbrace{0, 0, \ldots, 0}_{m-1}, x))\big)$ and the result is obtained. □

Theorem 3.22. *Let* $(X, \circ^*_{(m,n)}, 0)$ *be a* 0*-commutative* (m,n)*-superhyper* G*-algebra satisfying* $\circ^*\big(\underbrace{0,0,\ldots,0}_{m-1}, \circ^*(\underbrace{x,x,\ldots,x}_{m-1}, y)\big) = \circ^*(\underbrace{y,y,\ldots,y}_{m-1}, x)$. *Then for any* $x, y \in X$,
$x \in \circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1}, y), \circ^*(\underbrace{0,0,\ldots,0}_{m-1}, y), \underbrace{x,x,\ldots,x}_{m-2}\big)$.

Proof. Because X be a 0-commutative, implies that
$\circ^*\big(\circ^*(\underbrace{x,x,\ldots,x}_{m-1}, y), \circ^*(\underbrace{0,0,\ldots,0}_{m-1}, y), \underbrace{x,x,\ldots,x}_{m-2}\big) =$
$\circ^*\big(\underbrace{y,y,\ldots,y}_{m-1}, \circ^*(\underbrace{0,0,\ldots,0}_{m-1}, \circ^*(\underbrace{x,x,\ldots,x}_{m-1}, y))\big)$. By the assumption and $(G_{sh}$-2),
$x \in \circ^*(\underbrace{y,y,\ldots,y}_{m-1}, \circ^*(\underbrace{y,y,\ldots,y}_{m-1}, x)) = \circ^*\big(\underbrace{y,y,\ldots,y}_{m-1}, \circ^*(\underbrace{0,0,\ldots,0}_{m-1}, \circ^*(\underbrace{x,x,\ldots,x}_{m-1}, y))\big)$. Thus it is obtained. □

4. **Conclusions**

In this paper, we have introduced the novel concept of (m,n)-superhyper G-algebras based on a powerset and studied their properties. We have presented some basic results and examples of this superhyperalgebra. The basis of our work is the extension of G-algebras to superhyper G-algebras using a powerset. We wish that these results are helpful for further studies in the theory of superhyperalgebra. For future work, we hope to investigate the idea of neutrosophic superhyper G-algebras, fuzzy superhyper G-algebras, and soft superhyper G-algebras and obtain some results in this regard and their applications.

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Real Examples of NeutroGeometry & AntiGeometry

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Abstract: For the classical *Geometry*, in a geometrical space, all items (concepts, axioms, theorems, etc.) are totally (100%) true. But, in the real world, many items are not totally true. The *NeutroGeometry* is a geometrical space that has some items that are only partially true (and partially indeterminate, and partially false), and no item that is totally false. The *AntiGeometry* is a geometrical space that has some item that are totally (100%) false. While the Non-Euclidean Geometries [hyperbolic and elliptic geometries] resulted from the total negation of only one specific axiom (Euclid's Fifth Postulate), the AntiGeometry results from the total negation of any axiom [and in general: theorem, concept, idea etc.] and even of more axioms [theorem, concept, idea, etc.] and in general from any geometric axiomatic system (Euclid's five postulates, Hilbert's 20 axioms, etc.), and the NeutroAxiom results from the partial negation of any axiom (or concept, theorem, idea, etc.). Clearly, the AntiGeometry is a generalization of Non-Euclidean Geometries. [5]

Keywords: Non-Euclidean Geometries, Euclidean Geometry, Lobachevski-Bolyai-Gauss Geometry, Riemannian Geometry, NeutroManifold, AntiManifold, NeutroAlgebra, AntiAlgebra, NeutroGeometry, AntiGeometry, Hybrid (Smarandache) Geometry, NeutroAxiom, AntiAxiom, NeutroTheorem, AntiTheorem, Partial Function, NeutroFunction, AntiFunction, NeutroOperation, AntiOperation, NeutroAttribute, AntiAttribute, NeutroRelation, AntiRelation, NeutroStructure, AntiStructure.

1. Introduction

This is a review paper on the newly emerging field of NeutroStructures and AntiStructures, introduced by Smarandache [1] since 2019 and developed [2, 3, 4] in 2020-2021, inspired from our real world since the laws and regulations do not equally apply to all citizens, but in different degrees.

Let T = true, I = indeterminacy, F = false,
where $T, I, F \in [0, 1]$ and $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.

The following neutrosophic triplets occur in our real world:

2. <Structure(1, 0, 0), NeutroStructure(T, I, F), AntiStructure(0, 0, 1)>

In any theoretical field of knowledge, the classical **Structures** have all items (concepts, axioms, theorems, properties, ideas, relationships, etc.) totally (100%) true.

But in the real world, most structures have items that are only partially true (and partially indeterminate, or partially false) and no item that is totally false (as in NeutroStructure), we call them **NeutroStructures**.

And structures that have some items that are totally (100%) false, we call them **AntiStructures**.

3. <Algebra(1, 0, 0), NeutroAlgebra(T, I, F), AntiAlgebra(0, 0, 1)>

As particular cases, when the structures are algebras or geometries, one gets the above neutrosophic triplets.

The Classical Algebraic Structures [Algebra] have all operations totally (100%) well-defined, and all axioms [theorems, concepts, ideas, etc.] totally (100%) true.

The NeutroAlgebraic Structures have operations or axioms (and in general: theorems, concepts, ideas, etc.) that are not totally (100%) well-defined or respectively totally (100%) true, but only partially well-defined or partially true [and none of them is 0% well-defined or respectively 0% true as in AntiAlgebraic Structures]. The NeutroAlgebraic Structures are in between Classical Algebraic Structures and AntiAlgebraic Structures.

And the AntiAlgebraic Structures have at least one operation or one axiom that is 0% well-defined or respectively 0% true.

4. <Geometry(1, 0, 0), NeutroGeometry(T, I, F), AntiGeometry(0, 0, 1)>

1) A geometric structure whose all axioms (and theorems, propositions, etc.) are totally true is called a classical Geometric Structure (or **Geometry**).

2) A geometric structure that has at least one NeutroOperation or one NeutroAxiom (and no AntiOperation and no AntiAxiom) is called a NeutroAlgebraic Structure (or **NeutroGeometry**).

3) A geometric structure that has at least one AntiOperation or one Anti Axiom is called an AntiAlgebraic Structure (or **AntiGeometry**).

Therefore, a neutrosophic triplet is formed: *<Geometry, NeutroGeometry, AntiGeometry>*, where "Geometry" can be any classical Euclidean, Projective, Affine, Discrete, Differential, etc. geometric structure.

*

Similarly, for any field of knowledge, the axioms (and theorems, propositions, concepts, ideas etc.) are categorized in three groups [1 – 4]:

5. <Axiom(1, 0, 0), NeutroAxiom(T, I, F), AntiAxiom(0, 0, 1)>

An axiom, defined on a given set, endowed with some operation(s). When we define an axiom on a given set, it does not automatically mean that the axiom is true for all set's elements. We have three possibilities again:

i) The axiom is true for all set's elements (totally true) [degree of truth T = 1] (as in classical algebraic structures; this is a classical **Axiom**). Neutrosophically we write: Axiom(1,0,0).

ii) The axiom if true for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and false for other elements [degree of falsehood F], where (T,I,F) is different from (1,0,0) and from (0,0,1) (this is **NeutroAxiom**). Neutrosophically we write NeutroAxiom(T,I,F).

iii) The axiom is false for all set's elements [degree of falsehood F = 1](this is **AntiAxiom**). Neutrosophically we write AntiAxiom(0,0,1).

And, of course, the Axiom may be replaced by Theorem, Property, Concept, etc.

6. Examples of AntiGeometry

6.1. The Hyperbolic (Non-Euclidean) Geometry [or Lobachevski-Bolyai-Gauss Geometry] resulted from the total negation of the axiom called Euclid's Fifth Postulate [through a point exterior to a line only one parallel can be drawn to that line] by the AntiAxiom: through a point exterior to a line many parallels can be drawn to that line.

6.2. The Elliptic (Non-Euclidean) Geometry [or Riemannian Geometry] resulted from the total negation of the axiom called Euclid's Fifth Postulate [through a point exterior to a line only one parallel can be drawn to that line] by another AntiAxiom: through a point exterior to a line no parallel can be drawn to that line.

6.3. The second class of the Hybrid (Smarandache) Geometry (or SG)[1] where an axiom is totally denied but in multiple different ways in the same geometric space, which combined the Hyperbolic and Elliptic Geometries into the same geometric space, by totally denying Euclid's Fifth Postulate in two different ways:

a) there are lines and points exteriors to them such that through a point exterior to a line many parallels can be drawn to that line; and

b) there are other lines and points exteriors to them such that through a point exterior to a line no parallel can be drawn to that line.

6.4. New example of AntiGeometry that is not a Non-Euclidean Geometry

Let us have on a plane (π) all circles of radius $r > 0$ and centered into the origin (0, 0).

For example, the below drawn circles (Fig. 1).

By "point" we understand any classical point, and by "line" we understand the circumference of a circle.

Let's take any three distinct points on the circumference of the small circle (similarly it will be for all other circles).

Clearly, the points A, B, C lie on the same line (circumference), and:
the point B lies between the point A and point C;
the point C lies between the point B and point A;
and the point A lies between the point C and point B.

Therefore, Hilbert's Postulate B.3 of the Axioms of Betweenness, stated as follows:

> "If A, B and C are three distinct points lying on the same line, then one and only one of the points lies between the other two."

is totally denied, because for any three distinct points lying on a line one has any point lies between the other two.

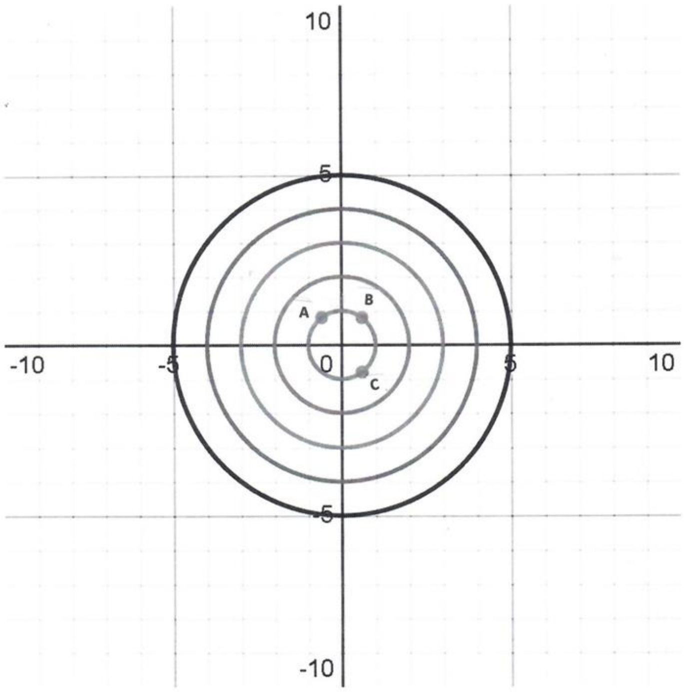

Figure 1. New Example of AntiGeometry that is not a Non-Euclidean Geometry.

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[1] Linfan Mao, *Pseudo-Manifold Geometries with Applications*, Cornell University, New York City, USA, 2006. Abstract: https://arxiv.org/abs/math/0610307, Full paper: https://arxiv.org/pdf/math/0610307. "A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom (1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways."

This geometrical model does not represent a Non-Euclidean Geometry because the Euclid's Fifth Postulate is 100% true. Two lines are considered parallel if they do not intersect.

See the proof below:

Giving a line (circumference C_1 centered in the origin) and a point P_1 that does not lie on it, there exists a unique line (circumference C_2 centered in the origin) that passes though the point P_1 and does not intersect C_1.

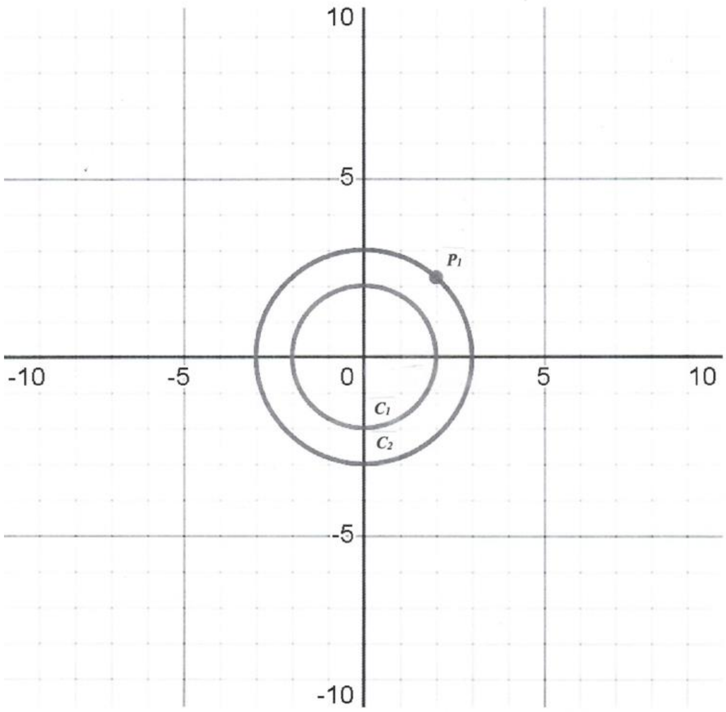

Figure 2. Euclid's Fifth Postulate is totally true.

7. Examples of NeutroGeometry

7.1. The first class of the Hybrid (or Smarandache) Geometry[1] where an axiom is partially true and partially false in the same geometric space.

For example, there are two distinct points that determine a single line, and other two distinct points that determine no line in the same geometric space.

Thus, *Hilbert's Postulate I.1.* of the Axioms of Incidence, announced as follows:

> "For every point P and every point Q not equal to P, there exists a unique line incident with the points P and Q"

becomes partially true and partially false.

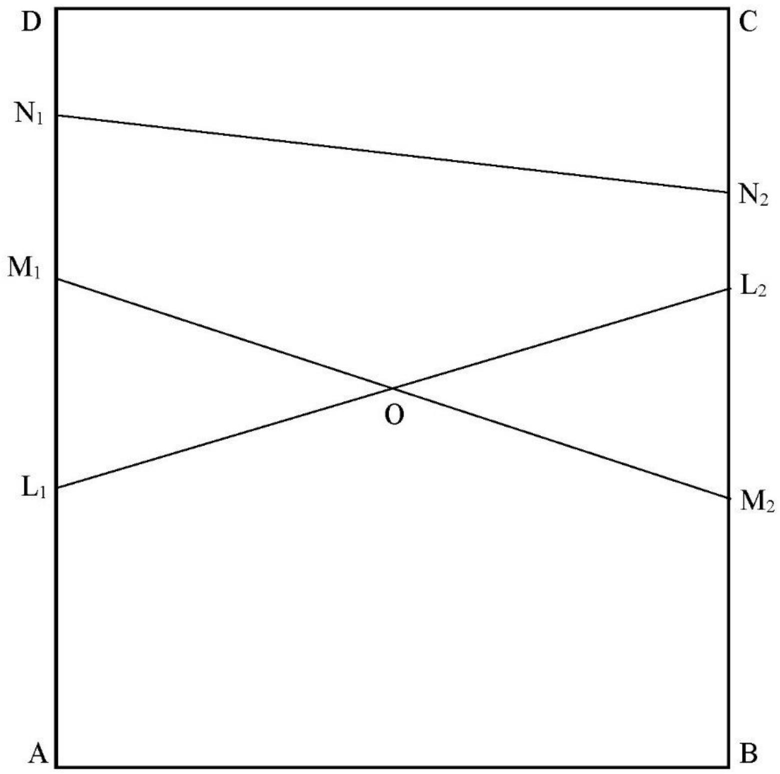

Figure 3: *Example of a NeutroGeometry that is an SG*

Assume the rectangle ABCD is a geometric space, where "point" means any classical point on the sides AB and CD or interior to this rectangle, and "line" is any segment of line connecting a point from AB with a point of CD and passing through the center O of the rectangle.

For example, L_1L_2 is a line since it connects the point L_1 lying on AB, and point L_2 lying on CD, and passes through the center O. Similarly for the line M_1M_2.

But N_1N_2 is not a line, since it does not pass through the center O.

7.2. Example of NeutroGeometry that is not an SG

We consider the previous model of the rectangular geometric space ABCD, but adding some indeterminacy (I), as in our everyday life, i.e. the dark spot below, which represents some marsh area, so M_1M_2 although it is a line since it passes through the origin O, but it has also some degree of indeterminacy when crossing through the indeterminate zone (I).

While L_1L_2 is a totally determinate line, M_1M_2 is partially determinate and partially indeterminate (as in neutrosophy).

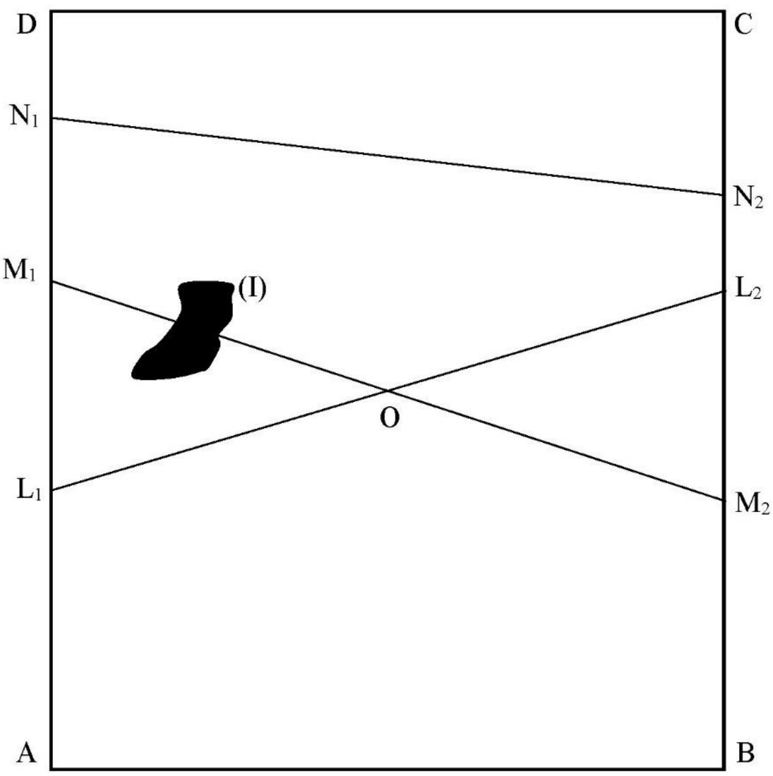

Figure 34: *Example of a NeutroGeometry that is not an SG*

5. Conclusions

In this paper we presented simple examples of NeutroGeometry, AntiGeometry, SG, and Non-Euclidean Geometries..

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A Study of NeutroAlgebra and AntiAlgebra of Ideals in a Factor Ring

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Abstract. If I is an ideal in a ring R and M is the collection of all nontrivial ideals in the factor ring R/I, we find in this paper conditions under which $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ are NeutroAlgebras and AntiAlgebras where $\oplus$, $\otimes$ and $\cap$ are the usual sum, product and intersection of ideals in R/I.

Keywords: ClassicalAlgebra; PartialAlgebra; NeutroAlgebra; AntiAlgebra; NeutrosubAlgebra; sum of ideals, product of ideals; intersection of ideals.

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1. **Introduction and Preliminaries**

In this section, we provide brief introduction to the concepts of NeutroAlgebraic structure and AntiAlgebraic structure. For completeness, basic definitions and results that will be used later in the paper are provided.

The concept of NeutroAlgebraic Structure was introduced by Smarandache in [16]. In [14], Smarandache introduced NeutroAlgebra as a generalization of Partial Algebra. Using the methods of NeutroSophication and AntiSophication, Smarandache in [15] presented and studied NeutroAlgebraic Structures and AntiAlgebraic Structures respectively. Since the presentation of seminal papers [[16], [14] and [15]] by Smarandache, many Neutrosophic Researchers have further studied and published papers on NeutroAlgebraic and AntiAlgebraic Structures as well as NeutroAlgebraic and AntiAlgebraic

Hyper Structures. For full details, see [[1], [2], [3], [4], [5], [6], [7], [9], [10] and [12]]. Kandasamy et.al. in [11] studied NeutroAlgebra of ideals in a ring under the usual sum and product of ideals. They proved that the set of nontrivial ideals in the ring $\mathbb{Z}$ is a NeutroAlgebra under the usual sum of ideals and not a NeutroAlgebra under the usual product of ideals. They also proved that the set of nontrivial ideals in the ring $\mathbb{Z}_n$ is a NeutroAlgebra under the usual sum and product of ideals. They equally showed that the set of nontrivial ideals in polynomial rings $\mathbb{Z}[x]$, $\mathbb{Q}[x]$ and $\mathbb{R}[x]$ are NeutroAlgebras under the usual sum of ideals and not NeutroAlgebras under the product of ideals. They finally showed that the set of nontrivial ideals under the usual product of ideals in the polynomial ring $\mathbb{Z}_n[x]$ is a NeutroAlgebra. The aim of the present paper is to extend the work done in [11] by studying NeutroAlgebra and AntiAlgebra of ideals in a factor ring.

Definition 1.1.

(a) (i) A ClassicalOperation is an operation that is well defined for all the set's elements.

(ii) A NeutroOperation is an operation that is partially well defined, partially indeterminate, and partially outer defined on the given set.

(iii) An AntiOperation is an operation that is outer defined for all set's elements.

(b) (i) A ClassicalLaw/Axiom defined on a nonempty set is a law/axiom that is totally true for all the set's elements.

(ii) A NeutroLaw/Axiom defined on a nonempty set is a law/axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0, 1]$, with $(T, I, F) = (1, 0, 0)$ that represents the ClassicalAxiom/Law, and $(T, I, F) = (0, 0, 1)$ that represents the AntiAxiom.

(iii) An AntiLaw/Axiom defined on a nonempty set is a law/axiom that is false for all the set's elements.

(c) (i) A PartialOperation on a set is an operation that is well defined for some elements of the set and undefined for all the other elements of the set.

(ii) A PartialAlgebra is an algebra that has at least one PartialOperation, and all its other axioms are classical.

Definition 1.2.

(a) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom and no AntiOperation or AntiAxiom.

(b) An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

(c) When a NeutroAlgebra has no NeutroAxiom, then it coincides with the PartialAlgebra.

Theorem 1.3. *[14] The NeutroAlgebra is a generalization of PartialAlgebra.*

Theorem 1.4. *[12] Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of $\mathbb{U}$. If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.*

Example 1.5. (i) Let $X = \mathbb{Z}^+$ and let $f : X \times X \to \mathbb{N}$ be a function defined $\forall x, y \in X$ by $f(x, y) = \sqrt{xy}$. Then (X, f) is a PartialAlgebra with respect to the ClassicalAxiom of commutativity.

(ii) Let $X = \{1, 2, 3\} \subseteq \mathbb{Z}_4$ and let $*$ be a binary operation defined in the Cayley table below.

$*$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

Then $(X, *)$ is not a PartialAlgebra since $2 * 2$ is outer defined. However, $(X, *)$ is a NeutroAlgebra.

(iii) $(\mathbb{N}, \div)$ is not a PartialAlgebra eventhough $\div$ is a PartialOperation over $\mathbb{N}$. Axioms of commutativity and associativity are NeutroAxioms and not ClassicalAxioms.

(iv) $(\mathbb{Z}, \div)$ is a NeutroAlgebra.

(v) Let $X = \mathbb{Z} - \{0\}$ and let $f : X \times X \to X$ be a function defined $\forall x, y \in X$ by $f(x, y) = e^{xy}$. Then (X, f) is an AntiAlgebra.

Definition 1.6. Let I and J be two ideals in a ring R.

(i) The sum of I and J denoted by $I + J$ is defined by

$$I + J = \{x + y : x \in I, y \in J\}.$$

(ii) The product of I and J denoted by $I \times J$ is defined by

$$I \times J = \{xy : x \in I, y \in J\}.$$

(iii) The intersection of I and J denoted by $I \cap J$ is defined by

$$I \cap J = \{x : x \in J \text{ and } x \in J\}.$$

Lemma 1.7. *If $I =< m >$ and $J =< n >$ are ideals in a ring R, then:*

(i) $I + J = \langle GCD(m, n)\rangle$.

(ii) $IJ =< mn >$.

(iii) $I \cap J = \langle LCM[m, n]\rangle$.

Theorem 1.8. *[11] Let $\mathbb{Z}$ be the ring of integers and let J be the collection of all nontrivial ideals in $\mathbb{Z}$. Then $(J, +)$ is an infinite NeutroAlgebra.*

Example 1.9. Let $I =< 2 >, J =< 3 >, K =< 4 >, L =< 5 >, M =< 6 >, N =< 7 >$ be ideals in $\mathbb{Z}$. If $X = \{I, K, M\}$ and $Y = \{J, L, N\}$, then:

(i) $(X, +)$ is a ClassicalAlgebra,

(ii) $(Y, +)$ is a NeutroAlgebra.

(iii) $(X, \cap)$ is a NeutrolAlgebra,

(iv) $(Y, \cap)$ is a NeutrolAlgebra,

Definition 1.10. Let N be a NeutroAlgebra and let M be a nonempty subset of N. M is said to be a NeutrosubAlgebra of N if M is also a NeutroAlgebra under the same operation(s) inherited from N.

Theorem 1.11. *[11] Let $\mathbb{Z}$ be the ring of integers. Let J be the collection of nontrivial ideals in $\mathbb{Z}$ generated by singleton element $n \in \mathbb{Z} - \{1\}$ and let S be the collection of ideals in $\mathbb{Z}$ generated by the primes $p \in \mathbb{Z} - \{1\}$. Then:*

(i) *$(J, +)$ is a NeutroAlgebra which is not a PartialAlgebra.*

(ii) *$(J, \times)$ is not a NeutroAlgebra.*

(iii) *$(S, +)$ is a NeutrosubAlgebra.*

(iv) *$(S, \times)$ is not a NeutrosubAlgebra, in fact, it is an AntiAlgebra.*

Theorem 1.12. *[11] Let $R = \mathbb{Z}_n$ be the ring of integers modulo n where n is a composite such that $6 \leq n < \infty$. Let B be the collection of nontrivial ideals in R. Then:*

(i) *$(B, +)$ is a NeutroAlgebra which is neither a PartialAlgebra nor an AntiAlgebra.*

(ii) *$(B, \times)$ is a NeutroAlgebra which is neither a PartialAlgebra nor an AntiAlgebra.*

Theorem 1.13. *[11] Let $S = R[x]$ be a polynomial ring where $R = \mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{Z}$ or $\mathbb{Z}_p$ with p a prime. Let B be the collection of all proper ideals in S. Then*

(i) *$(B, +)$ is a NeutroAlgebra.*

(ii) *$(B, \times)$ is not a NeutroAlgebra.*

Theorem 1.14. *[11] Let $S = \mathbb{Z}_n[x]$ be a polynomial ring where n is a composite. Let B be the collection of all proper ideals in S. Then*

(i) *$(B, +)$ is a NeutroAlgebra.*

(ii) *$(B, \times)$ is a NeutroAlgebra.*

2. Main Results

In this section, we are going to study NeutroAlgebra and AntiAlgebra of ideals in a factor ring. If I is an ideal in a ring R and M is the collection of all nontrivial ideals in the factor ring R/I, we want to find conditions under which $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ are NeutroAlgebras and AntiAlgebras where $\oplus$, $\otimes$ and $\cap$ are the usual sum, product and intersection of ideals in R/I.

Theorem 2.1. *Let I be an ideal in a ring R. Then each ideal in R/I is of the form J/I where J is an ideal in R containing I.*

Example 2.2. Let $R = \mathbb{Z}$ be the ring of integers and let $I =< 24 >$ be an ideal in $\mathbb{Z}$ generated by 24. By Theorem 2.1, $M_1 =< 2 > /I, M_2 =< 4 > /I, M_3 =< 6 > /I, M_4 =< 8 > /I$ are nontrivial ideals in the factor ring R/I. If $M = \{M_1, M_2, M_3, M_4\}$, and $\oplus$ is the binary operation of addition of ideals in M, then we can generate the following Cayley table:

$\oplus$	M_1	M_2	M_3	M_4
M_1	M_1	M_1	M_1	M_1
M_2	M_1	M_2	M_1	M_1
M_3	M_1	M_1	M_3	M_1
M_4	M_1	M_2	M_1	M_4

It is clear from the table that $\oplus$ is a ClassicalOperation and therefore, $(M, \oplus)$ is a ClassicalAlgebra and not a NeutroAlgebra.

Theorem 2.3. *Let $I =< m >$ be an ideal in $R = \mathbb{Z}$ and let $J =< n >$ be an ideal in $\mathbb{Z}$ containing I where $m \in 2\mathbb{Z}$ with $m \geq 8$ and $n \in 2\mathbb{Z}$ with $n \geq 2$. If M is the collection of all nontrivial ideals in the factor ring R/I of the form J/I and $\oplus$ is the binary operation of addition of ideals in M, then:*

(i) *$\oplus$ is a ClassicalOperation.*
(ii) *$(M, \oplus)$ is a ClassicalAlgebra and not a NeutroAlgebra.*

Proof. (i) Suppose that $A, B \in M$ are arbitrary. Then $A \oplus B$ is nontrivial and $A \oplus B \in M$ $\forall A, B \in M$. Hence, $\oplus$ is a ClassicalOperation.
(ii) Since $\oplus$ is a ClassicalOperation over M, it follows that $(M, \oplus)$ is a ClassicalAlgebra and not a NeutroAlgebra. □

Example 2.4. Let $M = \{M_1, M_2, M_3, M_4\}$ be as defined in Example 2.2. If $\otimes$ is the binary operation of multiplication of ideals in M, then we can generate the following

Cayley table:

$\otimes$	M_1	M_2	M_3	M_4
M_1	M_2	M_4	outer defined	outer defined
M_2	M_4	outer defined	outer defined	outer defined
M_3	outer defined	outer defined	outer defined	outer defined
M_4	outer defined	outer defined	outer defined	outer defined

It is clear from the table that $\otimes$ is a NeutroOperation and therefore, $(M, \otimes)$ is a NeutroAlgebra.

Theorem 2.5. *Let $I =< m >$ be an ideal in $R = \mathbb{Z}$ and let $J =< n >$ be an ideal in $\mathbb{Z}$ containing I where $m \in 2\mathbb{Z}$ with $m \geq 8$ and $n \in 2\mathbb{Z}$ with $n \geq 2$. If M is the collection of all nontrivial ideals in the factor ring R/I of the form J/I and $\otimes$ is the binary operation of multiplication of ideals in M, then:*

(i) *$\otimes$ is a NeutroOperation.*
(ii) *$(M, \otimes)$ is a NeutrolAlgebra.*

Proof. (i) Without any loss of generality, there exists at least one duplet $(A, A) \in M$ and at least one duplet $(A, B) \in M$ such that $A \otimes A \in M$ and $A \otimes B \in M$ with the degree of truth (T) and there exists at least one duplet $(C, D) \in M$ such that $C \otimes D \notin M$ with the degree of falsehood (F). Hence, $\otimes$ is a NeutroOperation.
(ii) Since $\otimes$ is a NeutroOperation over M, it follows that $(M, \otimes)$ is a NeutroAlgebra. □

Example 2.6. Let $X = \{M_1, M_2\}$ and $Y = \{M_3, M_4\}$ be subsets of M where M is the NeutroAlgebra of Example 2.4. Consider the following Cayley tables:

$\otimes$	M_1	M_2
M_1	outer defined	outer defined
M_2	outer defined	outer defined

$\otimes$	M_3	M_4
M_3	outer defined	outer defined
M_4	outer defined	outer defined

It is clear from the tables that both $(X, \otimes)$ and $(Y, \otimes)$ are AntisubAlgebras of M.

Remark 2.7. Every NeutroAlgebra $(M, \otimes)$ of Theorem 2.5 has at least one AntisubAlgebra.

Example 2.8. Let $M = \{M_1, M_2, M_3, M_4\}$ be as defined in Example 2.2. If $\cap$ is the binary operation of intersection of ideals in M, then we can generate the following

Cayley table:

$\cap$	M_1	M_2	M_3	M_4
M_1	M_1	M_2	M_3	M_4
M_2	M_2	M_2	outer defined	M_4
M_3	M_3	outer defined	M_3	outer defined
M_4	M_4	M_4	outer defined	M_4

It is clear from the table that $\cap$ is a NeutroOperation and therefore, $(M, \otimes)$ is a NeutroAlgebra.

Theorem 2.9. *Let $I =< m >$ be an ideal in $R = \mathbb{Z}$ and let $J =< n >$ be an ideal in $\mathbb{Z}$ containing I where $m \in 2\mathbb{Z}$ with $m \geq 8$ and $n \in 2\mathbb{Z}$ with $n \geq 2$. If M is the collection of all nontrivial ideals in the factor ring R/I of the form J/I and $\cap$ is the binary operation of intersection of ideals in M, then:*

(i) $\cap$ *is a NeutroOperation.*
(ii) $(M, \cap)$ *is a NeutroAlgebra.*

Proof. (i) Let $A =< a > /I \in M$ be arbitrary with $a \in 2\mathbb{Z}$. Then $A \cap A = \langle \text{LCM}[a, a] \rangle /I =< a > /I \in M$. This shows that there exists at least a duplet $(A, A) \in M$ with 100% degree of truth (T). Without any loss of generality, there exists at least a duplet $(B, C) \in M$ such that $B \cap C \in M$ with degree of truth (T) and there exists a duplet $(D, E) \in M$ such that $D \cap E \in M$ with degree of falsehood (F). These show that $\cap$ is a NeutroOperation.
(ii) Since $\cap$ is a NeutroOperation, it follows that $(M, \cap)$ is a NeutroAlgebra. □

Example 2.10. Let $X = \{M_1, M_2\}$ and $Y = \{M_3, M_4\}$ be subsets of M where M is the NeutroAlgebra of Example 2.8. Consider the following Cayley tables:

$\cap$	M_1	M_2
M_1	M_1	M_2
M_2	M_2	M_2

$\cap$	M_3	M_4
M_3	M_3	outer defined
M_4	outer defined	M_4

It is clear from the tables that $(X, \cap)$ is a ClassicalsubAlgebra of $(M, \cap)$ while $(Y, \cap)$ is a NeutrosubAlgebra of $(M, \cap)$.

Remark 2.11. Every NeutroAlgebra $(M, \cap)$ of Theorem 2.9 has at least one ClassicalsubAlgebra and at least one NeutrosubAlgebra.

Example 2.12. Let $R = \mathbb{Z}$ be the ring of integers and let $I =< 1155 >$ be an ideal in $\mathbb{Z}$ generated by 1155. By Theorem 2.1, $M_1 =< 3 > /I, M_2 =< 5 > /I, M_3 =< 7 > /I, M_4 =< 11 > /I$ are nontrivial ideals in the factor ring R/I. If $M = \{M_1, M_2, M_3, M_4\}$, and $\oplus$ is the binary operation of addition of ideals in M, then we can generate the following Cayley table:

$\oplus$	M_1	M_2	M_3	M_4
M_1	M_1	outer defined	outer defined	outer defined
M_2	outer defined	M_2	outer defined	outer defined
M_3	outer defined	outer defined	M_3	outer defined
M_4	outer defined	outer defined	outer defined	M_4

It is clear from the table that $\oplus$ is a NeutroOperation and therefore, $(M, \oplus)$ is a NeutroAlgebra.

Example 2.13. Let $X = \{M_1, M_2\}$ and $Y = \{M_3, M_4\}$ be subsets of M where M is the NeutroAlgebra of Example 2.12. Consider the following Cayley tables:

$\oplus$	M_1	M_2
M_1	M_1	outer defined
M_2	outer defined	M_2

$\oplus$	M_3	M_4
M_3	M_3	outer defined
M_4	outer defined	M_4

It is clear from the tables that both $(X, \oplus)$ and $(Y, \oplus)$ are NeutrosubAlgebras of $(M, \oplus)$.

Theorem 2.14. *Let $I =< p >$ be an ideal in $R = \mathbb{Z}$ and let $J =< q >$ be an ideal in $\mathbb{Z}$ containing I where p and q are distinct prime numbers different from 1. If M is the collection of all nontrivial ideals in the factor ring R/I of the form J/I and $\oplus$ is the binary operation of addition of ideals in M, then:*

(i) *$\oplus$ is a NeutroOperation.*
(ii) *$(M, \oplus)$ is a NeutrolAlgebra.*

Proof. (i) Let $A =< a >$ and $B =< b >$ be arbitrary elements of M with a and b distinct primes different from 1. Then $A \oplus A = \langle \text{GCD}(a, a) \rangle /I =< a > /I \in M$. Also, $A \oplus B = \langle \text{GCD}(a, b) \rangle /I =< 1 > /I = R/I \notin M$. These show that there exists at least one duplet $(A, A) \in M$ such that $A \oplus A \in M$ with the degree of truth (T) and there exists at least one duplet $(A, B) \in M$ such that $A \oplus B \notin M$ with the degree of falsehood (F). Hence, $\oplus$ is a NeutroOperation.
(ii) Since $\oplus$ is a NeutroOperation over M, it follows that $(M, \oplus)$ is a NeutroAlgebra. □

Remark 2.15. Every NeutroAlgebra $(M, \oplus)$ of Theorem 2.14 has at least one NeutrosubAlgebra.

Example 2.16. Let $M = \{M_1, M_2, M_3, M_4\}$ be as defined in Example 2.12. If $\otimes$ is the binary operation of multiplication of ideals in M, then we can generate the following Cayley table:

$\otimes$	M_1	M_2	M_3	M_4
M_1	outer defined	outer defined	outer defined	outer defined
M_2	outer defined	outer defined	outer defined	outer defined
M_3	outer defined	outer defined	outer defined	outer defined
M_4	outer defined	outer defined	outer defined	outer defined

It is clear from the table that $\otimes$ is an AntiOperation and therefore, $(M, \otimes)$ is an AntiAlgebra.

Theorem 2.17. *Let $I =< p >$ be an ideal in $R = \mathbb{Z}$ and let $J =< q >$ be an ideal in $\mathbb{Z}$ containing I where p and q are distinct prime numbers different from 1. If M is the collection of all nontrivial ideals in the factor ring R/I of the form J/I and $\otimes$ is the binary operation of multiplication of ideals in M, then:*

(i) $\otimes$ *is an AntiOperation.*
(ii) $(M, \otimes)$ *is an AntiAlgebra.*

Proof. (i) Let $A =< a >$ and $B =< b >$ be arbitrary elements of M with a and b distinct primes different from 1. Then $A \otimes A =< aa > /I \notin M$. This shows that $\forall A \in M$, the duplet $(A, A) \notin M$ with the degree of falsehood (F). Also, $A \otimes B =< ab > /I \notin M$. This shows that $\forall A, B \in M$, the duplet $(A, B) \notin M$ with the degree of falsehood (F). Hence, $\otimes$ is an AntiOperation.
(ii) Since $\otimes$ is an AntiOperation over M, it follows that $(M, \otimes)$ is an AntiAlgebra. □

Remark 2.18. All subAlgebras of AntiAlgebra $(M, \otimes)$ of Theorem 2.17 are all AntisubAlgebras.

Example 2.19. Let $M = \{M_1, M_2, M_3, M_4\}$ be as defined in Example 2.12. If $\cap$ is the binary operation of intersection of ideals in M, then we can generate the following

Cayley table:

$\cap$	M_1	M_2	M_3	M_4
M_1	M_1	outer defined	outer defined	outer defined
M_2	outer defined	M_2	outer defined	outer defined
M_3	outer defined	outer defined	M_3	outer defined
M_4	outer defined	outer defined	outer defined	M_4

It is clear from the table that $\cap$ is a NeutroOperation and therefore, $(M, \cap)$ is a NeutroAlgebra.

Theorem 2.20. *Let $I =< p >$ be an ideal in $R = \mathbb{Z}$ and let $J =< q >$ be an ideal in $\mathbb{Z}$ containing I where p and q are distinct prime numbers different from 1. If M is the collection of all nontrivial ideals in the factor ring R/I of the form J/I and $\cap$ is the binary operation of intersection of ideals in M, then:*

(i) *$\cap$ is a NeutroOperation.*
(ii) *$(M, \cap)$ is a NeutroAlgebra.*

Proof. (i) Let $A =< a >$ and $B =< b >$ be arbitrary elements of M with a and b distinct primes different from 1. Then $A \cap A = \langle \text{LCM}[a, a] \rangle /I =< a > /I \in M$. This shows that $\forall A \in M$, the duplet $(A, A) \in M$ with 100% degree of truth (T). Also, $A \cap B = \langle \text{LCM}[a, b] \rangle /I \notin M$. This shows that for $A \neq B$, there exists at least a duplet $(A, B) \notin M$ with the degree of falsehood (F). Hence, $\cap$ is a NeutroOperation.
(ii) Since $\cap$ is a NeutroOperation over M, it follows that $(M, \cap)$ is a NeutroAlgebra. □

Example 2.21. Let $R = \mathbb{Z}_{12}$ be the ring of integers modulo 12 and let $I =< 6 >$ be an ideal in R generated by 6. By Theorem 2.1, $M_1 =< 2 > / < 6 >, M_2 =< 3 > / < 6 >$ are nontrivial ideals in the factor ring R/I. Let $M = \{M_1, M_2\}$ and let $\oplus$, $\otimes$ and $\cap$ be the binary operations of addition, multiplication and intersection of ideals in M respectively. Consider the following Cayley tables:

$\oplus$	M_1	M_2
M_1	M_1	outer defined
M_2	outer defined	M_2

$\otimes$	M_1	M_2
M_1	outer defined	outer defined
M_2	outer defined	M_2

$\cap$	M_1	M_2
M_1	M_1	outer defined
M_2	outer defined	M_2

It is clear from the tables that $\oplus$, $\otimes$ and $\cap$ are NeutroOperations and thus, $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ are NeutroAlgebras.

Example 2.22. Let $R = \mathbb{Z}_{24}$ be the ring of integers modulo 24 and let $I =< 12 >$ be an ideal in R generated by 12. By Theorem 2.1, $M_1 =< 2 > / < 12 >, M_2 =< 3 > / < 12 >, M_3 =< 4 > / < 12 >, M_4 =< 6 > / < 12 >$ are nontrivial ideals in the factor ring R/I. Let $M = \{M_1, M_2, M_3, M_4\}$ and let $\oplus$, $\otimes$ and $\cap$ be the binary operations of addition, multiplication and intersection of ideals in M respectively. Consider the following Cayley tables:

$\oplus$	M_1	M_2	M_3	M_4
M_1	M_1	outer defined	M_1	M_1
M_2	outer defined	M_2	outer defined	M_2
M_3	M_1	outer defined	M_3	M_1
M_4	M_1	M_2	M_1	M_4

$\otimes$	M_1	M_2	M_3	M_4
M_1	M_3	M_4	outer defined	outer defined
M_2	M_4	outer defined	outer defined	outer defined
M_3	outer defined	outer defined	outer defined	outer defined
M_4	outer defined	outer defined	outer defined	outer defined

$\cap$	M_1	M_2	M_3	M_4
M_1	M_1	M_4	M_3	M_4
M_2	M_4	M_2	outer defined	M_4
M_3	M_3	outer defined	M_3	outer defined
M_4	M_4	M_4	outer defined	M_4

It is clear from the tables that $\oplus$, $\otimes$ and $\cap$ are NeutroOperations and thus, $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ are NeutroAlgebras.

Theorem 2.23. *Let $R = \mathbb{Z}_n$ be the ring of integers modulo n where n is a composite such that $12 \leq n < \infty$, let $I =< p >$ be an ideal in R and let $J =< q >$ be an ideal in R containing I where $p, q \notin \{0, 1\}$. If M is the collection of all nontrivial ideals in the factor ring R/I of the form J/I, and $\oplus$, $\otimes$ and $\cap$ are respectively the binary operations of addition, multiplication and intersection of ideals in M, then:*

(i) *$(M, \oplus)$ is a NeutroAlgebra.*
(ii) *$(M, \otimes)$ is a NeutroAlgebra.*
(iii) *$(M, \cap)$ is a NeutroAlgebra.*

Proof. Similar to the proofs of Theorems 2.14 and 2.20 and so omitted. □

Example 2.24. Let $R = \mathbb{Z}[x]$ be the ring of polynomials in $\mathbb{Z}$ and let $I =< x^2+1 >$ be an ideal in R generated by $x^2 + 1$. By Theorem 2.1, $J =< x^3 + x^2 + x + 1 > /I, K =< x^4 + x^2 > /I$ are nontrivial ideals in the factor ring R/I. Let $M = \{J, K\}$ and let $\oplus$, $\otimes$ and $\cap$ be the binary operations of addition, multiplication and intersection of ideals in M respectively. Consider the following Cayley tables:

$\oplus$	J	K
J	J	outer defined
K	outer defined	K

$\otimes$	J	K
J	inner defined	inner defined
K	inner defined	inner defined

$\cap$	J	K
J	inner defined	inner defined
K	inner defined	inner defined

It can be seen from the tables that $(M, \oplus)$ is a NeutroAlgebra whereas $(M, \otimes)$ and $(M, \cap)$ are not NeutroAlgebras but ClassicalAlgebras.

Theorem 2.25. *Let I be an ideal in the polynomial ring $R = \mathbb{Z}[x]$ or $\mathbb{Q}[x]$ or $\mathbb{R}[x]$ or $\mathbb{Z}_p[x]$ where p is a prime number and let J be an ideal in R containing I. If M is the collection of all nontrivial ideals in the factor ring R/I of the form J/I and $\oplus$, $\otimes$ and $\cap$ are the binary operations of addition, multiplication and intersection of ideals in M respectively. then:*

(i) $(M, \oplus)$ *is a NeutroAlgebra.*
(ii) $(M, \otimes)$ *is a ClassicalAlgebra.*
(iii) $(M, \cap)$ *is a ClassicalAlgebra.*

Theorem 2.26. *Let I be an ideal in the polynomial ring $R = \mathbb{Z}_n[x]$ where n is a composite and let J be an ideal in R containing I. If M is the collection of all nontrivial ideals in the factor ring R/I of the form J/I and $\oplus$, $\otimes$ and $\cap$ are the binary operations of addition, multiplication and intersection of ideals in M respectively. then:*

(i) $(M, \oplus)$ *is a NeutroAlgebra.*
(ii) $(M, \otimes)$ *is a NeutroAlgebra.*
(iii) $(M, \cap)$ *is a NeutroAlgebra.*

Example 2.27. Let $R = \mathbb{Z}_{10}[x]$ be the ring of polynomials in $\mathbb{Z}_{10}$ and let $I =< x+1 >$ be an ideal in R generated by $x + 1$. By Theorem 2.1, $J =< 2x^2 - 2) > /I, K =< 5x^2 + 5x > /I$ are nontrivial ideals in the factor ring R/I. Let $M = \{J, K\}$ and let $\oplus$,

$\otimes$ and $\cap$ be the binary operations of addition, multiplication and intersection of ideals in M respectively. Consider the following Cayley tables:

$\oplus$	J	K
J	J	outer defined
K	outer defined	K

$\oplus$	J	K
J	J	outer defined
K	outer defined	K

$\oplus$	J	K
J	J	outer defined
K	outer defined	K

It can be seen from the tables that $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ are NeutroAlgebras.

3. Conclusion

In this paper, we have extended the work done by Kandasamy et al. in [11]. If I is an ideal in a ring R and M is the collection of all nontrivial ideals in the factor ring R/I, we have provided conditions under which $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ can be NeutroAlgebras and AntiAlgebras where $\oplus$, $\otimes$ and $\cap$ are the usual sum, product and intersection of ideals in R/I. Several examples were provided to illustrate the conditions.

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