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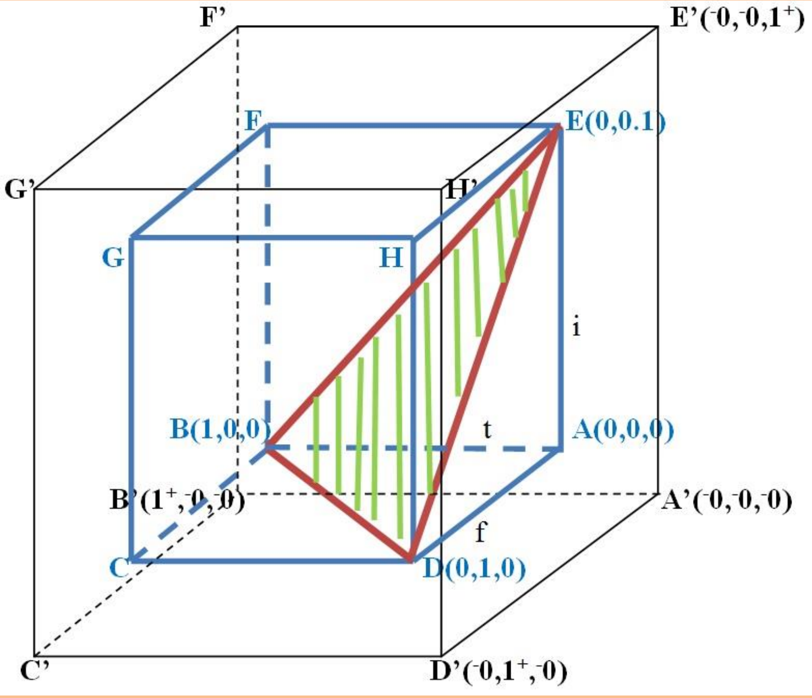

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Contents

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*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeter
minacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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NSS

(t, i, f)-Neutrosophic Structures & *I*-Neutrosophic Structures (Revisited)

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Abstract. This paper is an improvement of our paper "(t, i, f)-Neutrosophic Structures" [1], where we introduced for the first time a new type of structures, called (t, i, f)-Neutrosophic Structures, presented from a neutrosophic logic perspective, and we showed particular cases of such structures in geometry and in algebra.

In any field of knowledge, each structure is composed from two parts: a space, and a set of axioms (or laws) acting (governing) on it. If the space, or at least one of its axioms (laws), has some indeterminacy of the form (t, i, f) $\neq$ (1, 0, 0), that structure is a (t, i, f)-Neutrosophic Structure.

The (t, i, f)-Neutrosophic Structures [based on the components t = truth, i = numerical indeterminacy, f = falsehood] are different from the Neutrosophic Algebraic Structures [based on neutrosophic numbers of the form a + bI, where I = literal indeterminacy and In = I], that we rename as I-Neutrosophic Algebraic Structures (meaning algebraic structures based on indeterminacy "I" only). But we can combine both and obtain the (t, i, f)-I-Neutrosophic Algebraic Structures, i.e. algebraic structures based on neutrosophic numbers of the form a+bI, but also having indeterminacy of the form (t, i, f) $\neq$ (1, 0, 0) related to the structure space (elements which only partially belong to the space, or elements we know nothing if they belong to the space or not) or indeterminacy of the form (t, i, f) $\neq$ (1, 0, 0) related to at least one axiom (or law) acting on the structure space. Then we extend them to Refined (t, i, f)- Refined I-Neutrosophic Algebraic Structures.

Keywords: (t, i, f)-neutrosophic structure, truth-indeterminacy-falsehood, neutrosophic axiom, indeterminacy, degree of indeterminacy, neutrosophic algebraic structures, neutrosophic groupoid, neutrosophic semigroup, neutrosophic group, neutrosophic linear algebras, neutrosophic bi-algebraic structures, neutrosophic N-algebraic structures, (t, i, f)-Neutrosophic Geometry

1 Classification of Indeterminacies

1.1 Numerical Indeterminacy

Numerical Indeterminacy (or Degree of Indeterminacy), which has the form (t, i, f) $\neq$ (1, 0, 0), where t, i, f are numbers, intervals, or subsets included in the unit interval [0, 1], and it is the base for the (t, i, f)-Neutrosophic Structures.

1.1 Non-numerical Indeterminacy

Non-numerical Indeterminacy (or Literal Indeterminacy), which is the letter "I" standing for unknown (non-determinate), such that $I^2 = I$, and used in the composition of the neutrosophic number N = a + bI, where a and b are real or complex numbers, and a is the determinate part of number N, while bI is the indeterminate part of N. The neutrosophic numbers are the base for the I-Neutrosophic Structures.

2 Neutrosophic Algebraic Structures [or *I*-Neutrosophic Algebraic Structures]

A previous type of neutrosophic structures was introduced in algebra by W. B. Vasantha Kandasamy and Florentin Smarandache [2-57], since 2003, and it was called Neutrosophic Algebraic Structures. Later on, more researchers joined the neutrosophic research, such as: Mumtaz Ali, Said Broumi, Jun Ye, A. A. Salama, Muhammad Shabir, K. Ilanthenral, Meena Kandasamy, H. Wang, Y.-Q. Zhang, R. Sunderraman, Andrew Schumann, Salah Osman, D. Rabounski, V. Christianto, Jiang Zhengjie, Tudor Paroiu, Stefan Vladutescu, Mirela Teodorescu, Daniela Gifu, Alina Tenescu, Fu Yuhua, Francisco Gallego Lupiañez, etc.

The neutrosophic algebraic structures are algebraic structures based on sets of neutrosophic numbers of the form N = a + bI, where a, b are real (or complex) numbers, and a is called the determinate part on N and bI is called the indeterminate part of N, with mI + nI = (m + n)I, 0·I = 0, I^n = I for integer n $\geq$ 1, and I / I = undefined.

When a, b are real numbers, then a + bI is called a neutrosophic real number. While if at least one of a, b is a complex number, then a + bI is called a neutrosophic complex number.

We may say "literal indeterminacy" for "I" from a+bI, and "numerical indeterminacy" for "i" from (t, i, f) in order to distinguish them.

The neutrosophic algebraic structures studied by Vasantha-Smarandache in the period 2003-2015 are: neutrosophic groupoid, neutrosophic semigroup, neutrosophic group, neutrosophic ring, neutrosophic field, neutrosophic vector space, neutrosophic linear algebras etc., which later (between 2006-2011) were generalized by the same researchers to neutrosophic bi-algebraic structures, and more general to neutrosophic N-algebraic structures.

Afterwards, the neutrosophic structures were further extended to neutrosophic soft algebraic structures by Florentin Smarandache, Mumtaz Ali, Muhammad Shabir, and Munazza Naz in 2013-2014.

In 2015 Smarandache refined the literal indeterminacy I into different types of literal indeterminacies (depending on the problem to solve) such as $I_1, I_2, \ldots, I_p$ with integer $p \geq 1$, and obtained the refined neutrosophic numbers of the form $N_p = a+b_1I_1+b_2I_2+\ldots+b_pI_p$ where a, $b_1, b_2, \ldots, b_p$ are real or complex numbers, and a is called the determinate part of N_p, while for each $k\epsilon\{1, 2, \ldots, p\}$ b_kI_k is called the k-th indeterminate part of N_p,

and for each $k\epsilon\{1, 2, \ldots, p\}$, one similarly has:

$mI_k + nI_k = (m + n)I_k$, $0 \cdot I_k = 0$, $I_k^n = I_k$ for integer $n \geq 1$, and I_k / I_k = undefined.

The relationships and operations between I_j and I_k, for $j \neq k$, depend on each particular problem we need to solve.

Then consequently Smarandache [2015] extended the neutrosophic algebraic structures to Refined Neutrosophic Algebraic Structures [or Refined I-Neutrosophic Algebraic Structures], which are algebraic structures based on the sets of the refined neutrosophic numbers $a+b_1I_1+b_2I_2+\ldots+b_pI_p$.

3 *(t, i, f)*-Neutrosophic Structures

We now introduce for the first time another type of neutrosophic structures.

These structures, in any field of knowledge, are considered from a neutrosophic logic point of view, i.e. from the truth-indeterminacy-falsehood (t, i, f) values. In neutrosophic logic every proposition has a degree of truth (t), a degree of indeterminacy (i), and a degree of falsehood (f), where t, i, f are standard or non-standard subsets of the non-standard unit interval $]^-0, 1^+[$. In technical applications t, i, and f are only standard subsets of the standard unit interval [0, 1] with:

$^-0 \leq \sup(T) + \sup(I) + \sup(F) \leq 3^+$

where sup(X) means supremum of the subset X.

In general, each structure is composed from: a space, endowed with a set of axioms (or laws) acting (governing) on it. If the space, or at least one of its axioms, has some numerical indeterminacy of the form $(t, i, f) \neq (1, 0, 0)$, we consider it as a (t, i, f)-Neutrosophic Structure.

Indeterminacy with respect to the space is referred to some elements that partially belong [i.e. with a neutrosophic value $(t, i, f) \neq (1, 0, 0)$] to the space, or their appurtenance to the space is unknown.

An axiom (or law) which deals with numerical indeterminacy is called neutrosophic axiom (or law).

We introduce these new structures because in the real world we do not always know exactly or completely the space we work in; and because the axioms (or laws) are not always well defined on this space, or may have indeterminacies when applying them.

4 Refined *(t, i, f)*-Neutrosophic Structures [or *(t_j, i_k, f_l)*-Neutrosophic Structures]

In 2013 Smarandache [76] refined the numerical neutrosophic components (t, i, f) into $(t_1, t_2, \ldots, t_m;\ i_1, i_2, \ldots, i_p;\ f_1, f_2, \ldots, f_r)$, where m, p, r are integers ≥ 1.

Consequently, we now [2015] extend the (t, i, f)-Neutrosophic Structures to $(t_1, t_2, \ldots, t_m;\ i_1, i_2, \ldots, i_p;\ f_1, f_2, \ldots, f_r)$-Neutrosophic Structures, that we called Refined (t, i, f)-Neutrosophic Structures [or (t_j, i_k, f_l)-Neutrosophic Structures].

These are structures whose elements have a refined neutrosophic value of the form $(t_1, t_2, \ldots, t_m;\ i_1, i_2, \ldots, i_p;\ f_1, f_2, \ldots, f_r)$ or the space has some indeterminacy of this form.

5 *(t, i, f)-I*-Neutrosophic Algebraic Structures

The (t, i, f)-Neutrosophic Structures [based on the numerical components t = truth, i = indeterminacy, f = falsehood] are different from the Neutrosophic Algebraic Structures [based on neutrosophic numbers of the form a + bI]. We may rename the last ones as I-Neutrosophic Algebraic Structures (meaning: algebraic structures based on literal indeterminacy "I" only). But we can combine both of them and obtain a (t, i, f)-I-Neutrosophic Algebraic Structures, i.e. algebraic structures based on neutrosophic numbers of the form a + bI, but this structure also having indeterminacy of the form $(t, i, f) \neq (1, 0, 0)$ related to the structure space (elements which only partially belong to the space, or elements we know nothing if they belong to the space or not) or indeterminacy related to at least an axiom (or law) acting on the structure space. Even more, we can generalize them to Refined (t, i, f)- Refined I-Neutrosophic Algebraic Structures, or (t_j, i_k, f_l)-I_s-Neutrosophic Algebraic Structures.

6 Example of Refined *I*-Neutrosophic Algebraic Structure

Let the indeterminacy I be split into I_1 = contradiction (i.e. truth and falsehood simultaneously), I_2 = ignorance (i.e. truth or falsehood), and I_3 = vagueness, and the corresponding 3-refined neutrosophic numbers of the form $a+b_1I_1+b_2I_2+b_3I_3$.

Let (G, *) be a groupoid. Then the 3-refined I-neutrosophic groupoid is generated by I_1, I_2, I_3 and G under * and it is denoted by $N_3(G) = \{(G \cup I_1 \cup I_2 \cup I_3), *\} = \{ a+b_1I_1+b_2I_2+b_3I_3 \,/\, a, b_1, b_2, b_3 \in G \}$.

7 Example of Refined *(t, i, f)*-Neutrosophic Structure

Let (t, i, f) be split as (t_1, t_2; i_1, i_2; f_1, f_2, f_3). Let H = ({h_1, h_2, h_3}, #) be a groupoid, where h_1, h_2, and h_3 are real numbers. Since the elements h_1, h_2, h_3 only partially belong to H in a refined way, we define a refined (t, i, f)-neutrosophic groupoid { or refined (2; 2; 3)-neutrosophic groupoid, since t was split into 2 parts, I into 2 parts, and t into 3 parts } as H = {h_1(0.1, 0.1; 0.3, 0.0; 0.2, 0.4, 0.1), h_2(0.0, 0.1; 0.2, 0.1; 0.2, 0.0, 0.1), h_3(0.1, 0.0; 0.3, 0.2; 0.1, 0.4, 0.0)}.

8 Examples of *(t, i, f)-I*-Neutrosophic Algebraic Structures

8.1 Indeterminate Space (due to Unknown Element); with Neutrosophic Number included

Let B = {2+5I, -I, -4, b(0, 0.9, 0)} a neutrosophic set, which contains two neutrosophic numbers, 2+5I and -I, and we know about the element b that its appurtenance to the neutrosophic set is 90% indeterminate.

8.2 Indeterminate Space (due to Partially Known Element); with Neutrosophic Number included

Let C = {-7, 0, 2+I(0.5, 0.4, 0.1), 11(0.9, 0, 0) }, which contains a neutrosophic number 2+I, and this neutrosophic number is actually only partially in C; the element 11 is also partially in C.

8.3 Indeterminacy Axiom (Law)

Let D = [0+0I, 1+1I] = {c+dI, where c, d ϵ [0, 1]}. One defines the binary law # in the following way:

$\# : D\times D \rightarrow D$

$x \# y = (x_1 + x_2 I) \# (y_1 + y_2 I) = [(x_1 + x_2)/y_1] + y_2 I$,

but this neutrosophic law is undefined (indeterminate) when $y_1 = 0$.

8.4 Little Known or Completely Unknown Axiom (Law)

Let us reconsider the same neutrosophic set D as above. But, about the binary neutrosophic law Θ that D is endowed with, we only know that it associates the neutrosophic numbers 1+I and 0.2+0.3I with the neutrosophic number 0.5+0.4I, i.e.

(1+I)Θ(0.2+0.3I) = 0.5+0.4I.

There are many cases in our world when we barely know some axioms (laws).

9 Examples of Refined *(t, i, f)- Refined I*-Neutrosophic Algebraic Structures

We combine the ideas from Examples 5 and 6 and we construct the following example.

Let's consider, from Example 5, the groupoid (G, *), where G is a subset of positive real numbers, and its extension to a 3-refined I-neutrosophic groupoid, which was generated by I_1, I_2, I_3 and G under the law * that was denoted by $N_3(G) = \{ a+b_1I_1+b_2I_2+b_3I_3 / a, b_1, b_2, b_3 \in G \}$.

We then endow each element from $N_3(G)$ with some (2; 2; 3)-refined degrees of membership/ indeterminacy/ nonmembership, as in Example 6, of the form (T_1, T_2; I_1, I_2; F_1, F_2, F_3), and we obtain a $N_3(G)_{(2;2;3)}$ = $\{ a+b_1I_1+b_2I_2+b_3I_3(T_1, T_2; I_1, I_2; F_1, F_2, F_3) / a, b_1, b_2, b_3 \in G \}$, where

$$T_1 = \frac{a}{a+b_1+b_2+b_3}, T_2 = \frac{0.5a}{a+b_1+b_2+b_3};$$

$$I_1 = \frac{b_1}{a+b_1+b_2+b_3}, I_2 = \frac{b_2}{a+b_1+b_2+b_3};$$

$$F_1 = \frac{0.1b_3}{a+b_1+b_2+b_3}, F_2 = \frac{0.2b_1}{a+b_1+b_2+b_3}, F3 = \frac{b_2+b_3}{a+b_1+b_2+b_3}.$$

Therefore, $N_3(G)_{(2;2;3)}$ is a refined (2; 2; 3)-neutrosophic groupoid and a 3-refined I-neutrosophic groupoid.

10 Neutrosophic Geometric Examples

10.1 Indeterminate Space

We might not know if a point P belongs or not to a space S [we write P(0, 1, 0), meaning that P's indeterminacy is 1, or completely unknown, with respect to S].

Or we might know that a point Q only partially belongs to the space S and partially does not belong to the space S [for example Q(0.3, 0.4, 0.5), which means that with respect to S, Q's membership is 0.3, Q's indeterminacy is 0.4, and Q's non-membership is 0.5].

Such situations occur when the space has vague or unknown frontiers, or the space contains ambiguous (not well defined) regions.

10.2 Indeterminate Axiom

Also, an axiom (α) might not be well defined on the space S, i.e. for some elements of the space the axiom (α) may be valid, for other elements of the space the axiom (α) may be indeterminate (meaning neither valid, nor invalid), while for the remaining elements the axiom (α) may be invalid.

As a concrete example, let's say that the neutrosophic values of the axiom (α) are (0.6, 0.1, 0.2) = (degree of validity, degree of indeterminacy, degree of invalidity).

11 *(t, i, f)*-Neutrosophic Geometry as a Particular Case of *(t, i, f)*-Neutrosophic Structures

As a particular case of (t, i, f)-neutrosophic structures in geometry, one considers a (t, i, f)-Neutrosophic Geometry as a geometry which is defined either on a space with some indeterminacy (i.e. a portion of the space is not known, or is vague, confused, unclear, imprecise), or at least one of its axioms has some indeterminacy of the form (t, i, f) $\neq$ (1, 0, 0) (i.e. one does not know if the axiom is verified or not in the given space, or for some elements the axiom is verified and for others it is not verified).

This is a generalization of the Smarandache Geometry (SG) [57-75], where an axiom is validated and invalidated in the same space, or only invalidated, but in multiple ways. Yet the SG has no degree of indeterminacy related to the space or related to the axiom.

A simple Example of a SG is the following – that unites Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries altogether, in the same space, considering the Fifth Postulate of Euclid: in one region of the SG space the postulate is validated (only one parallel trough a point to a given line), in a second region of SG the postulate is invalidated (no parallel through a point to a given line – elliptical geometry), and in a third region of SG the postulate is invalidated but in a different way (many parallels through a point to a given line – hyperbolic geometry). This simple example shows a hybrid geometry which is partially Euclidean, partially Non-Euclidean Elliptic, and partially Non-Euclidean Hyperbolic. Therefore, the fifth postulate (axiom) of Euclid is true for some regions, and false for others, but it is not indeterminate for any region (i.e. not knowing how many parallels can be drawn through a point to a given line).

We can extend this hybrid geometry adding a new space region where one does not know if there are or there are not parallels through some given points to the given lines (i.e. the Indeterminate component) and we form a more complex (t, i, f)-Neutrosophic Geometry.

12 Neutrosophic Algebraic Examples

12.1 Indeterminate Space (due to Unknown Element)

Let the set (space) be NH = {4, 6, 7, 9, a}, where the set NH has an unknown element "a", therefore the whole space has some degree of indeterminacy. Neutrosophically, we write a(0, 1, 0), which means the element a is 100% unknown.

12.2 Indeterminate Space (due to Partially Known Element)

Given the set M = {3, 4, 9(0.7, 0.1, 0.3)}, we have two elements 3 and 4 which surely belong to M, and one writes them neutrosophically as 3(1, 0, 0) and 4(1, 0, 0), while the third element 9 belongs only partially (70%) to M, its appurtenance to M is indeterminate (10%), and does not belong to M (in a percentage of 30%).

Suppose the above neutrosophic set M is endowed with a neutrosophic law * defined in the following way:

$x_1(t_1, i_1, f_1)* x_2(t_2, i_2, f_2) = \max\{x_1, x_2\}(\min\{t_1, t_2\}, \max\{i_1, i_2\}, \max\{f_1, f_2\})$,

which is a neutrosophic commutative semigroup with unit element 3(1, 0 ,0).

Clearly, if x, y ϵ M, then x*y ϵ M. Hence the neutrosophic law * is well defined.

Since max and min operators are commutative and associative, then * is also commutative and associative.

If x ϵ M, then x*x = x.

Below, examples of applying this neutrosophic law *:

3*9(0.7, 0.1, 0.3) = 3(1, 0, 0)*9(0.7, 0.1, 0.3) = max{3, 9}(min{1, 0.7}, max{0, 0.1}, max{0, 0.3}) = 9(0.7, 0.1, 0.3).

3*4 = 3(1, 0, 0)*4(1, 0, 0) = max{3, 4}(min{1, 1}, max{0, 0}, max{0, 0}) = 4(1, 0, 0).

12.3 Indeterminate Law (Operation)

For example, let the set (space) be NG = ({0, 1, 2}, /), where "/" means division.

NG is a (t, i, f)-neutrosophic groupoid, because the operation "/" (division) is partially defined, partially indeterminate (undefined), and partially not defined. Undefined is different from not defined. Let's see:

2/1 = 1, which belongs to NG; {defined}.

1/0 = undefined; {indeterminate}.

1/2 = 0.5, which does not belongs to NG; {not defined}.

So the law defined on the set NG has the properties that:

- applying this law to some elements, the results are in NG [well defined law];
- applying this law to other elements, the results are not in NG [not well defined law];
- applying this law to again other elements, the results are undefined [indeterminate law].

We can construct many such algebraic structures where at least one axiom has such behavior (such indeterminacy in principal).

Websites at UNM for Neutrosophic Algebraic Structures and respectively Neutrosophic Geometries

http://fs.gallup.unm.edu/neutrosophy.htm, and http://fs.gallup.unm.edu/geometries.htm respectively.

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Expanding Uncertainty Principle to Certainty-Uncertainty Principles with Neutrosophy and Quad-stage Method

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Abstract. The most famous contribution of Heisenberg is uncertainty principle. But the original uncertainty principle is improper. Considering all the possible situations (including the case that people can create laws) and applying Neutrosophy and Quad-stage Method, this paper presents "certainty-uncertainty principles" with general form and variable dimension fractal form. According to the classification of Neutrosophy, "certainty-uncertainty principles" can be divided into three principles in different conditions: "certainty principle", namely a particle's position and momentum can be known simultaneously; "uncertainty principle", namely a particle's position and momentum cannot be known simultaneously; and neutral (fuzzy) "indeterminacy principle", namely whether or not a particle's position and momentum can be known simultaneously is undetermined. The special cases of "certainty-uncertainty principles" include the original uncertainty principle and Ozawa inequality. In addition, in accordance with the original uncertainty principle, discussing high-speed particle's speed and track with Newton mechanics is unreasonable; but according to "certainty-uncertainty principles", Newton mechanics can be used to discuss the problem of gravitational defection of a photon orbit around the Sun (it gives the same result of deflection angle as given by general relativity). Finally, for the reason that in physics the principles, laws and the like that are regardless of the principle (law) of conservation of energy may be invalid; therefore "certainty-uncertainty principles" should be restricted (or constrained) by principle (law) of conservation of energy, and thus it can satisfy the principle (law) of conservation of energy.

Keywords: Neutrosophy, quad-stage method, uncertainty principle, certainty-uncertainty principles, fractal, variable dimension fractal, Ozawa inequality, principle (law) of conservation of energy.

1 Introduction

In quantum mechanics, the uncertainty principle refers to the position and momentum of a particle cannot be determined simultaneously, the uncertainty of position (Δx) and uncertainty of momentum (Δp) obey the following inequality

$$\Delta x \Delta p \geq h / 4\pi \qquad (1)$$

where, h is the Planck constant.

As well-known, the most famous contribution of Heisenberg is uncertainty principle. But the original uncertainty principle is improper.

As a new branch of philosophy, Neutrosophy studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. According to Neutrosophy that there is a 3D Neutrosophic Space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration. More information about Neutrosophy may be found in references [1,2]. Quad-stage is introduced in reference [3], it is the expansion of Hegel's triad-stage (triad thesis, antithesis, synthesis of development). The four stages are "general theses", "general antitheses", "the most important and the most complicated universal relations", and "general syntheses". In quad-stage method, "general theses" may be considered as the notion or idea <A> in neutrosophy; "general antitheses" may be considered as the notion or idea <Anti-A> in neutrosophy; "the most important and the most complicated universal relations" may be considered as the notion or idea <Neut-A> in neutrosophy; and "general syntheses" are the final results. The different kinds of results in the above mentioned four stages can also be classified and induced with the viewpoints of neutrosophy. Thus, the theory and achievement of neutrosophy can be applied as many as possible, and the method of quad-stage will be more effective. The combination of Neutrosophy and quad-stage will be a powerful method to realize many innovations in areas of science, technology, literature and art. Therefore, this paper expands uncertainty principle with Neutrosophy and Quad-stage Method and presents certainty-uncertainty principles.

As expanding uncertainty principle with neutrosophy and quad-stage, the whole process can be divided into the following four stages.

The first stage (stage of “general theses”), for the beginning of development, the thesis (namely uncertainty principle) should be widely, deeply, carefully and repeatedly contacted, explored, analyzed, perfected and so on.

Regarding the advantages of uncertainty principle, that will not be repeated here, while we should stress the deficiencies of uncertainty principle.

For other perspectives on uncertainty principle, we will discuss in detail below, in order to avoid duplication.

The second stage, for the appearance of opposite (antithesis), the antithesis should be also widely, deeply, carefully and repeatedly contacted, explored, analyzed, perfected and so on.

There are many opposites (antitheses) to uncertainty principle. For example: certainty principle, law of conservation of energy, and so on, this paper discusses the problems related to law of conservation of energy in the last part.

The third stage is the one that the most important and the most complicated universal relations. The purpose of this provision stage is to establish the universal relations in the widest scope.

To link and combine uncertainty principle with Neutrosophy and law of conservation of energy, it can be expanded and developed effectively and successfully in the maximum area.

The fourth stage, to carry on the unification and synthesis regarding various opposites and the suitable pieces of information, factors, and so on; and reach one or more results to expand uncertainty principle, and these are the best or agreed with some conditions.

2 Heisenberg inequality, Ozawa inequality and their forms of equality in first stage and second stage

In first stage, we discuss the problems related to Heisenberg inequality firstly.

Heisenberg inequality (Eq.1) can be changed into the following form of equality

$$\Delta x \Delta p = kh/4\pi \qquad (2)$$

where, k is a real number and $k \geq 1$.

For other contents of the first stage (such as Heisenberg inequality cannot consider law of conservation of energy), we will discuss them below.

In second stage, we discuss the problems related to Ozawa inequality (the opposites of Heisenberg inequality) firstly.

Ozawa inequality [4] can be written as follows

$$\Delta Q \Delta P + \Delta Q \sigma(P) + \sigma(Q) \Delta P \geq h/4\pi \qquad (3)$$

It can be changed into the following form of equality

$$\Delta Q \Delta P + \Delta Q \sigma(P) + \sigma(Q) \Delta P = kh/4\pi \qquad (4)$$

where, k is a real number and $k \geq 1$.

For other contents of the second stage (such as Ozawa inequality cannot consider law of conservation of energy), we will also discuss them below.

3 "Certainty-uncertainty principles" with general form

Now we link the viewpoints of Neutrosophy and enter the fourth stage.

According to Neutrosophy, any proposition has three situations of truth, falsehood and indeterminacy respectively. Thus, the original uncertainty principle can be extended into the following "certainty-uncertainty principles" with general form

$$\Delta x \Delta p = Kh \qquad (5)$$

where, K is a real number and $K > 0$.

Eq.(5) can be divided into three principles:

The first one is the “uncertainty principle” ($K \geq K_1$): a particle’s position and momentum cannot be known simultaneously.

Obviously, if $K_1 = 1/4\pi$, then it is the original uncertainty principle.

The second one is the “certainty principle” ($K \leq K_2$): a particle’s position and momentum can be known simultaneously.

Referring to the experiments for establishing Ozawa inequality, the value of K_2 can be decided by related experiments.

The third one is the neutral (fuzzy) “indeterminacy principle" ($K_2 < K < K_1$): whether or not a particle’s position and momentum can be known simultaneously is undetermined.

Similarly, the original Ozawa inequality can be extended into the following Ozawa type’s "certainty-uncertainty principles" with general form

$$\Delta Q \Delta P + \Delta Q \sigma(P) + \sigma(Q) \Delta P = Kh \qquad (6)$$

where, K is a real number and $K > 0$.

Eq.(6) can be divided into three principles:

The first one is the “certainty principle” ($K \geq K_1$): a particle’s position and momentum can be known (namely can be measured with zero-error) simultaneously (here σ(P) or σ(Q) is equal to infinity).

Obviously, if $K_1 = 1/4\pi$, then it is the original Ozawa inequality (with equality form).

It should be noted that here the first one is not the uncertainty principle, but certainty principle.

The second one is the “uncertainty principle” ($K \le K_2$): a particle’s position and momentum cannot be known simultaneously.

The third one is the neutral (fuzzy) “indeterminacy principle" ($K_2 < K < K_1$): whether or not a particle’s position and momentum can be known simultaneously is undetermined.

4 "Certainty-uncertainty principles" with variable dimension fractal form

In order to process Eq. (5) and Eq.(6), as well as other equalities and inequalities that may arise in the future with unified manner, we will link variable dimension fractal to discuss the "certainty-uncertainty principles" with variable dimension fractal form.

The general form of variable dimension fractal is as follows

$$N = \frac{C}{r^D} \quad (7)$$

where, $D = f(r)$, instead of a constant.

For the sake of convenience, we only discuss the situation of $C = 1$, that is

$$N = \frac{1}{r^D} \quad (8)$$

Thus, Eq.(5) can be written as the following variable dimension fractal form

$$\Delta x \Delta p = \frac{1}{h^D} \quad (9)$$

Solving this equation, it gives

$$D = -\frac{\ln(Kh)}{\ln h} \quad (10)$$

Then, the values of D_1 and D_2 corresponding to K_1 and K_2 can be calculated by Eq.(10), for example

$$D_1 = -\frac{\ln(K_1 h)}{\ln h} \quad (11)$$

Similarly, Eq.(6) can be written as the following variable dimension fractal form

$$\Delta Q \Delta P = \frac{1}{h^D} \quad (12)$$

Solving this equation, it gives

$$D = -\frac{\ln(Kh - \Delta Q \sigma(P) - \sigma(Q) \Delta P)}{\ln h} \quad (13)$$

Then, the values of D_1 and D_2 corresponding to K_1 and K_2 can be calculated by Eq.(13), for example

$$D_1 = -\frac{\ln(K_1 h - \Delta Q \sigma(P) - \sigma(Q) \Delta P)}{\ln h} \quad (14)$$

5 Solving the problem of light speed with Newton mechanics

Now we link the problem related to Newton mechanics.

In accordance with the original uncertainty principle, discussing high-speed particle’s speed and track with Newton mechanics is unreasonable; but according to "certainty-uncertainty principles", Newton mechanics can be used to discuss the problem of gravitational defection of a photon orbit around the Sun (it presents the same result of deflection angle as given by general relativity). The solving method can be found in reference [4]; in which, for problem of gravitational defection of a photon orbit around the Sun, the improved formula of gravitation between Sun and photon is as follows:

$$F = -\frac{GMm}{r^2} - \frac{1.5GMmr_0^2}{r^4} \quad (15)$$

where : r_0 is the shortest distance between the light and the Sun, if the light and the Sun are tangent, it is equal to the radius of the Sun.

The funny thing is that, for this problem, the maximum gravitational force given by the improved formula is 2.5 times of that given by the original Newton’s law of gravity.

6 To be restricted (or constrained) by principle (law) of conservation of energy

In this part we will link principle (law) of conservation of energy to discuss further.

For the reason that in physics the principles, laws and the like that are regardless of the principle (law) of conservation of energy may be invalid; therefore "certainty-uncertainty principles" should be restricted (or constrained) by principle (law) of conservation of energy, and thus it can satisfy the principle (law) of conservation of energy.

The general form of the principle (law) of conservation of energy is as follows

$$E(t) = E(0) = const$$

Or

$$1 - \frac{E(t)}{E(0)} = 0$$

Thus, referring to reference [3] for applying least square method to establish "partial and temporary unified theory of natural science so far" including all the equations of natural science so far (in which, the theory of everything to express all of natural laws, described by Hawking that a single equation could be written on a T-shirt, is partially and temporarily realized in the form of "partial and temporary unified variational principle of natural science so far"), Eq.(5) (one kind of "certainty-uncertainty principles" with general form) can be restricted (or constrained) by principle (law) of conservation of energy as follows

$$(\Delta x \Delta p - Kh)^2 + w(1 - \frac{E(t}{E(0)}) = 0 \qquad (16)$$

where, K is a real number and $K > 0$, w is a suitable positive weighted number.

Similarly, Eq.(6) (one kind of Ozawa type's "certainty-uncertainty principles" with general form) can be restricted (or constrained) by principle (law) of conservation of energy as follows

$$(\Delta Q \Delta P + \Delta Q \sigma(P) + \sigma(Q)\Delta P - Kh)^2 + w(1 - \frac{E(t}{E(0)}) = 0$$
(17)

For Eq.(9) (the variable dimension fractal form of Eq.(5)), it can be restricted (or constrained) by principle (law) of conservation of energy as follows

$$(\Delta x \Delta p - \frac{1}{})^2 + w(1 - \frac{(t)}{E(0)}) = 0 \qquad (18)$$

For Eq.(12) (the variable dimension fractal form of Eq.(6)), it can be restricted (or constrained) by principle (law) of conservation of energy as follows

$$(\Delta Q \Delta P - \frac{1}{D})^2 + w(1 - \frac{(t)}{E(0)}) = 0 \qquad (19)$$

As the cases that "certainty-uncertainty principles" should be restricted (or constrained) by other principles (laws) and the like, similar method can be used.

Conclusion

The original uncertainty principle is improper. Considering all the possible situations (including the case that people can create laws), and applying Neutrosophy and Quad-stage Method, this paper presents "certainty-uncertainty principles" with general form and variable dimension fractal form. According to the classification of Neutrosophy, "certainty-uncertainty principles" can be divided into three principles in different conditions: "certainty principle", namely a particle's position and momentum can be known simultaneously; "uncertainty principle", namely a particle's position and momentum cannot be known simultaneously; and neutral (fuzzy) "indeterminacy principle", namely whether or not a particle's position and momentum can be known simultaneously is undetermined.

Referring to the "certainty-uncertainty principles" for a particle's position and momentum, the "certainty-uncertainty principles" for other physical quantities can also be presented with the similar method.

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Rough Neutrosophic Multi-Attribute Decision-Making Based on Rough Accuracy Score Function

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Abstract. This paper presents multi-attribute decision making based on rough accuracy score function with rough neutrosophic attribute values. While the concept of neutrosophic sets is a powerful logic to handle indeterminate and inconsistent information, the theory of rough neutrosophic sets is also a powerful mathematical tool to deal with incompleteness. The rating of all alternatives is expressed with the upper and lower approximation operator and the pair of neutrosophic sets which are characterized by truth-membership degree, indeterminacy-membership degree, and falsity-membership degree. Weight of each attribute is partially known to decision maker. We introduce a multi attribute decision making method in rough neutrosophic environment based on rough accuracy score function. Information entropy method is used to obtain the unknown attribute weights. Rough accuracy score function is defined to determine rough accuracy score values. Then weighted rough accuracy score value is defined to determine the ranking order of all alternatives. Finally, a numerical example is provided to illustrate the applicability and effectiveness of the proposed approach.

Keywords: Neutrosophic set, Rough neutrosophic set, Single-valued neutrosophic set, Grey relational analysis, Information Entropy, Multi-attribute decision making.

Introduction

The concept of rough neutrosophic set is very recently proposed by Broumi et al. [1], [2]. It seems to be very interesting and applicable in realistic problems. It is a new hybrid intelligent structure. The concept of rough set was proposed by Pawlak [3] in 1982 and the concept of neutrosophic set was proposed by Smarandache [4], [5] in 1998. Wang et al. [6] introduced single valued netrosophic sets in 2010. Neutrosophic sets and rough sets are both capable of dealing with uncertainty and incomplete information. The theory of neutrosophic set has achieved success in various areas of research such as medical diagnosis [7], educational problems [8], [9], social problems [10], [11], conflict resolution [12], [13], image processing [14], [15], [16], decision making [17], [18], [19], [20], [21], [22], etc. On the other hand, rough set theory has been successfully applied in the different fields such as artificial intelligence [23], pattern recognition [24], [25], medical diagnosis [26], [27], [28], data mining [29], [30], [31], image processing [32], conflict analysis [33], decision support systems [34], [35], intelligent control [36], etc. It appears that the computational techniques based on any one of neutrosophic sets or rough sets alone will not always offer the best results but a fusion of two or more can often offer better results [2].

Rough neutrosophic set is the generalization of rough fuzzy sets [37], [38] and rough intuitionistic fuzzy sets [39]. Mondal and Pramanik [40] applied the concept of rough neutrosophic set in multi-attribte decision making based on grey relational analysis. Mondal and Pramanik [41] also studied cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis. Literature review reflects that no studies have been made on multi-attribute decision making using rough neutrosophic score function.

In this paper, we develop rough neutrosophic multi-attribute decision making (MADM) based on rough accuracy score function (RASF).

Rest of the paper is organized in the following way. Section 2 presents preliminaries of neutrosophic sets and rough neutrosophic sets. Section 3 is devoted to present multi attribute decision-making method based on rough accuracy score function. Section 4 presents a numerical example of the proposed method. Finally section 5 presents concluding remarks.

2 Mathematical Preliminaries

2.1 Definitions on neutrosophic Set:

The concept of neutrosophy set [4] is derived from the new branch of philosophy, namely, neutrosophy [5]. Neutrosophy succeeds in creating different fields of studies because of its capability to deal with the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

Definition 2.1.1

Let *G* be a space of points (objects) with generic element in *E* denoted by *y*. Then a neutrosophic set *N1* in *G* is characterized by a truth membership function T_{N1} , an indeterminacy membership function I_{N1} and a falsity membership function F_{N1}. The functions T_{N1}, I_{N1} and F_{N1} are real standard or non-standard subsets of $]^{-}0, 1^{+}[$ that is T_{N1}: $G \to\ ]^{-}0, 1^{+}[$; I_{N1}: $G \to\ ]^{-}0, 1^{+}[$; F_{N1}: $G \to\ ]^{-}0, 1^{+}[$.

The sum of $T_{N1}(y),\ I_{N1}(y),\ F_{N1}(y)$ is given by

$$^{-}0 \le \sup T_{N1}(y) + \sup I_{N1}(y) + \sup F_{N1}(y) \le 3^{+}$$

Definition 2.1.2 The complement of a neutrosophic set [5] *A* is denoted by $N1^c$ and is defined as follows: $T_{N1^c}(y) = \{1^+\} - T_{N1}(y)$; $I_{N1^c}(y) = \{1^+\} - I_{N1}(y)$
$F_{N1^c}(y) = \{1^+\} - F_{N1}(y)$

Definition 2.1.3 A neutrosophic set [5] *N1* is contained in the other neutrosophic set *N2*, $N1 \subseteq N2$ if and only if the following results hold.

$$\inf T_{N1}(y) \le \inf T_{N2}(y),\ \sup T_{N1}(y) \le \sup T_{N2}(y)$$
$$\inf I_{N1}(y) \ge \inf I_{N2}(y),\ \sup I_{N1}(y) \ge \sup I_{N2}(y)$$
$$\inf F_{N1}(y) \ge \inf F_{N2}(y),\ \sup F_{N1}(y) \ge \sup F_{N2}(y)$$

for all *y* in *G*.

Definition 2.1.4 Let *G* be a universal space of points (objects) with a generic element of *G* denoted by *y*.
A single valued neutrosophic set [6] *S* is characterized by a truth membership function $T_N(y)$, a falsity membership function $F_N(y)$ and indeterminacy function $I_N(y)$ with $T_N(y),\ F_N(y),\ I_N(y) \in [0, 1]$ for all *y* in *G*.

When *G* is continuous, a SNVS *S* can be written as follows:

$$S = \int_y \langle T_S(y), F_S(y), I_S(y) \rangle / y, \quad \forall y \in G$$

and when *G* is discrete, a SVNS *S* can be written as follows:

$$S = \sum \langle T_S(y), F_S(y), I_S(y) \rangle / y, \quad \forall y \in G$$

It should be observed that for a SVNS *S*,

$$0 \le \sup T_S(y) + \sup F_S(y) + \sup I_S(y) \le 3, \quad \forall y \in G$$

Definition 2.1.5 The complement of a single valued neutrosophic set [6] *S* is denoted by S^c and is defined as follows:

$$T_S^c(y) = F_S(y)\ ;\ I_S^c(y) = 1 - I_S(y)\ ;\ F_S^c(y) = T_S(y)$$

Definition 2.1.6 A SVNS [6] S_{N1} is contained in the other SVNS S_{N2} denoted by $S_{N1} \subseteq S_{N2}$, iff $T_{S_{N1}}(y) \le T_{S_{N2}}(y)$; $I_{S_{N1}}(y) \ge I_{S_{N2}}(y)$; $F_{S_{N1}}(y) \ge F_{S_{N2}}(y)$, $\forall y \in G$.

Definition 2.1.7 Two single valued neutrosophic sets [6] S_{N1} and S_{N2} are equal, i.e. $S_{N1} = S_{N2}$, iff $S_{N1} \subseteq S_{N2}$ and $S_{N1} \supseteq S_{N2}$

Definition 2.1.8 The union of two SVNSs [6] S_{N1} and S_{N2} is a SVNS S_{N3} , written as $S_{N3} = S_{N1} \cup S_{N2}$.

Its truth membership, indeterminacy-membership and falsity membership functions are related to S_{N1} and S_{N2} by the following equations

$$T_{S_{N3}}(y) = \max\left(T_{S_{N1}}(y), T_{S_{N2}}(y)\right);$$
$$I_{S_{N3}}(y) = \max\left(I_{S_{N1}}(y), I_{S_{N2}}(y)\right);$$
$$F_{S_{N3}}(y) = \min\left(F_{S_{N1}}(y), F_{S_{N2}}(y)\right) \text{ for all } y \text{ in } G$$

Definition 2.1.9 The intersection of two SVNSs [6] *N1* and *N2* is a SVNS *N3*, written as $N3 = N1 \cap N2$. Its truth membership, indeterminacy membership and falsity membership functions are related to *N1* an *N2* by the following equations:

$$T_{S_{N3}}(y) = \min\left(T_{S_{N1}}(y), T_{S_{N2}}(y)\right);$$
$$I_{S_{N3}}(y) = \max\left(I_{S_{N1}}(y), I_{S_{N2}}(y)\right);$$
$$F_{S_{N3}}(y) = \max\left(F_{S_{N1}}(y), F_{S_{N2}}(y)\right), \forall y \in G$$

Definition 2.1.1.10 The general SVNS can be presented in the following form as follows:

$$S = \{(y/(T_S(y), I_S(y), F_S(y))) : y \in G\}$$

Finite SVNSs can be represented as follows:

$$S = \begin{Bmatrix} (y_1/(T_S(y_1), I_S(y_1), F_S(y_1))), \cdots, \\ (y_m/(T_S(y_m), I_S(y_m), F_S(y_m))) \end{Bmatrix}, \forall y \in G \quad (1)$$

Let

$$S_{N1} = \begin{Bmatrix} (y_1/(T_{S\,N1}(y_1),\ I_{S\,N1}(y_1),\ F_{S\,N1}(y_1))), \cdots, \\ (y_n/(T_{S\,N1}(y_n),\ I_{S\,N1}(y_n),\ F_{S\,N1}(y_n))) \end{Bmatrix} \quad (2)$$

$$S_{N2} = \begin{Bmatrix} (x_1/(T_{S\,N2}(y_1),\ I_{S\,N2}(y_1),\ F_{S\,N2}(y_1))), \cdots, \\ (x_n/(T_{S\,N2}(y_n),\ I_{S\,N2}(y_n),\ F_{S\,N2}(y_n))) \end{Bmatrix} \quad (3)$$

be two single-valued neutrosophic sets, then the Hamming distance [42] between two SNVS *N1*and *N2* is defined as follows:

$$d_S(S_{N1}, S_{N2}) = \sum_{i=1}^{n} \left\langle \begin{matrix} \left| T_{S_{N1}}(y) - T_{S_{N2}}(y) \right| + \\ \left| I_{S_{N1}}(y) - I_{S_{N2}}(y) \right| + \\ \left| F_{S_{N1}}(y) - F_{S_{N2}}(y) \right| \end{matrix} \right\rangle \quad (4)$$

and normalized Hamming distance [42] between two SNVSs S_{N1} and S_{N2} is defined as follows:

$$ {}^{N}d_S\left(S_{N1}, S_{N2}\right) = \frac{1}{3n}\sum_{i=1}^{n}\left\langle \begin{array}{l} \left|T_{S_{N1}}(y) - T_{S_{N2}}(y)\right| + \\ \left|I_{S_{N1}}(y) - I_{S_{N2}}(y)\right| + \\ \left|F_{S_{N1}}(y) - F_{S_{N2}}(y)\right| \end{array} \right\rangle \quad (5) $$

with the following properties

1. $0 \le d_S\left(S_{N1}, S_{N2}\right) \le 3n$ (6)
2. $0 \le {}^{N}d_S\left(S_{N1}, S_{N2}\right) \le 1$ (7)

2.2 Definitions on rough neutrosophic set

Definition 2.2.1

Let Z be a non-null set and R be an equivalence relation on Z. Let P be neutrosophic set in Z with the membership function T_P, indeterminacy function I_P and non-membership function F_P. The lower and the upper approximations of P in the approximation (Z, R) denoted by $\underline{N}(P)$ and $\overline{N}(P)$ are respectively defined as follows:

$$ \underline{N}(P) = \left\langle \begin{array}{l} < x, T_{\underline{N}(P)}(x), I_{\underline{N}(P)}(x), F_{\underline{N}(P)}(x) > / \\ z \in [x]_R, x \in Z \end{array} \right\rangle, \quad (8) $$

$$ \overline{N}(P) = \left\langle \begin{array}{l} < x, T_{\overline{N}(P)}(x), I_{\overline{N}(P)}(x), F_{\overline{N}(P)}(x) > / \\ z \in [x]_R, x \in Z \end{array} \right\rangle \quad (9) $$

Where, $T_{\underline{N}(P)}(x) = \wedge_z \in [x]_R\, T_P(z)$,

$I_{\underline{N}(P)}(x) = \wedge_z \in [x]_R\, I_P(z)$, $F_{\underline{N}(P)}(x) = \wedge_z \in [x]_R\, F_P(z)$,

$T_{\overline{N}(P)}(x) = \vee_z \in [x]_R\, T_P(z)$, $I_{\overline{N}(P)}(x) = \vee_z \in [x]_R\, T_P(z)$,

$F_{\overline{N}(P)}(x) = \vee_z \in [x]_R\, I_P(z)$

So, $0 \le \sup T_{\underline{N}(P)}(x) + \sup I_{\underline{N}(P)}(x) + \sup F_{\underline{N}(P)}(x) \le 3$

$0 \le \sup T_{\overline{N}(P)}(x) + \sup I_{\overline{N}(P)}(x) + \sup F_{\overline{N}(P)}(x) \le 3$

Here $\vee$ and $\wedge$ denote "max" and "min'' operators respectively, $T_P(z)$, $I_P(z)$ and $F_P(z)$ are the membership, indeterminacy and non-membership function of z with respect to P. It is easy to see that $\underline{N}(P)$ and $\overline{N}(P)$ are two neutrosophic sets in Z.

Thus NS mapping $\underline{N}$, $\overline{N}$: $N(Z) \rightarrow N(Z)$ are, respectively, referred to as the lower and upper rough NS approximation operators, and the pair $(\underline{N}(P), \overline{N}(P))$ is called the rough neutrosophic set [1], [2] in (Z, R).

From the above definition, it is seen that $\underline{N}(P)$ and $\overline{N}(P)$ have constant membership on the equivalence clases of R if $\underline{N}(P) = \overline{N}(P)$; .e. $T_{\underline{N}(P)}(x) = T_{\overline{N}(P)}(x)$, $I_{\underline{N}(P)}(x) = I_{\overline{N}(P)}(x)$, $F_{\underline{N}(P)}(x) = F_{\overline{N}(P)}(x)$ for any x belongs to Z.

P is said to be a definable neutrosophic set in the approximation (Z, R). It can be easily proved that zero neutrosophic set (0_N) and unit neutrosophic sets (1_N) are definable neutrosophic sets.

Definition 2.2.2

If $N(P) = (\underline{N}(P), \overline{N}(P))$ is a rough neutrosophic set in (Z, R) , the rough complement [1], [2] of $N(P)$ is the rough neutrosophic set denoted by $\sim N(P) = (\underline{N}(P)^c, \overline{N}(P)^c)$, where $\underline{N}(P)^c$, $\overline{N}(P)^c$ are the complements of neutrosophic sets of $\underline{N}(P), \overline{N}(P)$ respectively.

$$ \underline{N}(P)^c = \left\langle \begin{array}{l} < x, T_{\underline{N}(P)}(x), 1 - I_{\underline{N}(P)}(x), F_{\underline{N}(P)}(x) > / \\ , x \in Z \end{array} \right\rangle, \text{ and} $$

$$ \overline{N}(P)^c = \left\langle \begin{array}{l} < x, T_{\overline{N}(P)}(x), 1 - I_{\overline{N}(P)}(x), F_{\overline{N}(P)}(x) > / \\ , x \in Z \end{array} \right\rangle \quad (10) $$

Definition 2.2.3

If $N(P_1)$ *and* $N(P_2)$ are the two rough neutrosophic sets of the neutrosophic set P respectively in Z, then the following definitions [1], [2] hold good:

$N(P_1) = N(P_2) \Leftrightarrow \underline{N}(P_1) = \underline{N}(P_2) \wedge \overline{N}(P_1) = \overline{N}(P_2)$

$N(P_1) \subseteq N(P_2) \Leftrightarrow \underline{N}(P_1) \subseteq \underline{N}(P_2) \wedge \overline{N}(P_1) \subseteq \overline{N}(P_2)$

$N(P_1) \cup N(P_2) = < \underline{N}(P_1) \cup \underline{N}(P_2), \overline{N}(P_1) \cup \overline{N}(P_2) >$

$N(P_1) \cap N(P_2) = < \underline{N}(P_1) \cap \underline{N}(P_2), \overline{N}(P_1) \cap \overline{N}(P_2) >$

$N(P_1) + N(P_2) = < \underline{N}(P_1) + \underline{N}(P_2), \overline{N}(P_1) + \overline{N}(P_2) >$

$N(P_1) \,.\, N(P_2) = < \underline{N}(P_1) \,.\, \underline{N}(P_2), \overline{N}(P_1) \,.\, \overline{N}(P_2) >$

If N, M, L are the rough neutrosophic sets in (Z, R), then the following propositions are stated from definitions

Proposition 1 [1], [2]

1. $\sim N(\sim N) = N$
2. $N \cup M = M \cup N, M \cup N = N \cup M$
3. $(L \cup M) \cup N = L \cup (M \cup N)$,
   $(L \cap M) \cap N = L \cap (M \cap N)$
4. $(L \cup M) \cap N = (L \cup M) \cap (L \cup N)$,
   $(L \cap M) \cup N = (L \cap M) \cup (L \cap N)$

Proposition 2 [1], [2]

De Morgan's Laws are satisfied for rough neutrosophic sets

1. $\sim (N(P_1) \cup N(P_2)) = (\sim N(P_1)) \cap (\sim N(P_2))$
2. $\sim (N(P_1) \cap N(P_2)) = (\sim N(P_1)) \cup (\sim N(P_2))$

Proposition 3[1], [2]

If P_1 and P_2 are two neutrosophic sets in U such that $P_1 \subseteq P_2$, *then* $N(P_1) \subseteq N(P_2)$

1. $N(P_1 \cap P_2) \subseteq N(P_2) \cap N(P_2)$
2. $N(P_1 \cup P_2) \supseteq N(P_2) \cup N(P_2)$

Proposition 4 [1], [2]

1. $\underline{N}(P) = \sim \overline{N}(\sim P)$
2. $\overline{N}(P) = \sim \underline{N}(\sim P)$
3. $\underline{N}(P) \subseteq \overline{N}(P)$

Definition 2.2.4

Let $N_{ij}(P) = \left(\underline{N}_{ij}(P), \overline{N}_{ij}(P)\right)$ is a rough neutrosophic set in (*Z*, *R*), where $\underline{N}_{ij}(P) = \left(\underline{T}_{ij}, \underline{I}_{ij}, \underline{F}_{ij}\right)$, $\overline{N}_{ij}(P) = \left(\overline{T}_{ij}, \overline{I}_{ij}, \overline{F}_{ij}\right)$ i = 1, 2, ..., m and j = 1, 2, ..., n. We define the rough accuracy score function (RASF) of $N_{ij}(P)$ as follows:

$$S[N_{ij}(P)] = \frac{\left(2 + \left(\frac{\underline{T}_{ij} + \overline{T}_{ij}}{2}\right) - \left(\frac{\underline{I}_{ij} + \overline{I}_{ij}}{2}\right) - \left(\frac{\underline{F}_{ij} + \overline{F}_{ij}}{2}\right)\right)}{3} \quad (11)$$

Proposition 5:

1. For any values of $N_{ij}(P)$, $0 \le S[N_{ij}(P)] \le 1$

Proof: Since both lower and upper approximations are neutrosophic sets, so the proof of the statement is obvious.

2. $S[N_{ij}(P)] = 0$ when $\underline{T}_{ij} = \overline{T}_{ij} = 0$, $\underline{I}_{ij} = \overline{I}_{ij} = \underline{F}_{ij} = \overline{F}_{ij} = 1$

Proof: This proof is obvious.

3. $S[N_{ij}(P)] = 1$ when $\underline{T}_{ij} = \overline{T}_{ij} = 1$, $\underline{I}_{ij} = \overline{I}_{ij} = \underline{F}_{ij} = \overline{F}_{ij} = 0$

4. For any two rough neutrosophic set $N_{ij}(P_1)$ and $N_{ij}(P_2)$, if $N_{ij}(P_1) \subseteq N_{ij}(P_2)$ then $S[N_{ij}(P_1)] \le S[N_{ij}(P_2)]$.

Proof: Since $N_{ij}(P_1) \subseteq N_{ij}(P_2)$ we have

$\overline{T}^{P_1}_{ij} \le \overline{T}^{P_2}_{ij}$, $\underline{T}^{P_1}_{ij} \le \underline{T}^{P_2}_{ij}$, $\overline{I}^{P_1}_{ij} \ge \overline{I}^{P_2}_{ij}$, and

$\underline{I}^{P_1}_{ij} \ge \underline{I}^{P_2}_{ij}$, $\overline{F}^{P_1}_{ij} \ge \overline{F}^{P_2}_{ij}$, $\underline{F}^{P_1}_{ij} \ge \underline{F}^{P_2}_{ij}$.

$\Rightarrow S[N_{ij}(P_1)] - S[N_{ij}(P_2)] \le 0$.

This proves the proposition.

5. For any two rough neutrosophic set $N_{ij}(P_1)$ and $N_{ij}(P_2)$, if $N_{ij}(P_1) = N_{ij}(P_2)$, then $S[N_{ij}(P_1)] = S[N_{ij}(P_2)]$.

Proof: Since $N_{ij}(P_1) = N_{ij}(P_2)$ we have

$\overline{T}^{P_1}_{ij} = \overline{T}^{P_2}_{ij}$, $\underline{T}^{P_1}_{ij} = \underline{T}^{P_2}_{ij}$, $\overline{I}^{P_1}_{ij} = \overline{I}^{P_2}_{ij}$, $\underline{I}^{P_1}_{ij} = \underline{I}^{P_2}_{ij}$, $\overline{F}^{P_1}_{ij} = \overline{F}^{P_2}_{ij}$,

$\underline{F}^{P_1}_{ij} = \underline{F}^{P_2}_{ij}$

$\Rightarrow S[N_{ij}(P_1)] - S[N_{ij}(P_2)] = 0$.

This completes the proof.

Definition 2.2.5: Let $N_{ij}(P_1)$ and $N_{ij}(P_2)$ be two rough neutrosophic sets. Then the ranking method is defined as follows:

If $S[N_{ij}(P_1)] > S[N_{ij}(P_2)]$ then $N_{ij}(P_1) > N_{ij}(P_2)$.

3. Multi-attribute decision making methods based on rough accuracy score function

Consider a multi-attribute decision making problem with m alternatives and n attributes. Let $A_1, A_2, ..., A_m$ and $C_1, C_2, ..., C_n$ denote the alternatives and attributes respectively.

The rating describes the performance of alternative A_i against attribute C_j. For MADM weight vector $W = \{w_1, w_2,...,w_n\}$ is assigned to the attributes. The weight w_j (j = 1, 2, ..., n) reflects the relative importance of attributes C_j (j = 1, 2, ..., m) to the decision making process. The weights of the attributes are usually determined on subjective basis. They represent the opinion of a single decision maker or accumulate the opinions of a group of experts using a group decision technique. The values associated with the alternatives for MADM problem are presented in the table 1.

Table1: Rough neutrosophic decision matrix

$$D = \left\langle \underline{d}_{ij}, \overline{d}_{ij} \right\rangle_{m\times n} =$$

	C_1	C_2	$\cdots$	C_n
A_1	$\langle \underline{d}_{11}, \overline{d}_{11} \rangle$	$\langle \underline{d}_{12}, \overline{d}_{12} \rangle$	...	$\langle \underline{d}_{1n}, \overline{d}_{1n} \rangle$
A_2	$\langle \underline{d}_{21}, \overline{d}_{21} \rangle$	$\langle \underline{d}_{22}, \overline{d}_{22} \rangle$	...	$\langle \underline{d}_{2n}, \overline{d}_{2n} \rangle$
.	...	...	...	...
.	...	...	...	...
A_m	$\langle \underline{d}_{m1}, \overline{d}_{m1} \rangle$	$\langle \underline{d}_{m2}, \overline{d}_{m2} \rangle$	...	$\langle \underline{d}_{mn}, \overline{d}_{mn} \rangle$

(12)

Here $\left\langle \underline{d}_{ij}, \overline{d}_{ij} \right\rangle$ is the rough neutrosophic number according to the *i*-th alternative and the *j*-th attribute.

In real life situation, the decision makers may have personal biases and some indiguals may give unduly low or unduly high preferences with respect to their preferences. In this case it is necessary to assign very low weights to these baised options. The steps of RASF method under rough neutrosophic environment are described as follows:

Step 1: Construction of the decision matrix with rough neutrosophic form

For multi-attribute decision making problem, the rating of alternative A_i (i = 1, 2,…m) with respect to attribute C_j (j = 1, 2,…n) is assumed as rough neutrosophic set. It can be represented with the following forms:

$$A_i = \left[\begin{array}{l} \frac{C_1}{\left\langle \underline{N}_1(\underline{T}_{i1}\,\underline{I}_{i1}\,\underline{F}_{i1}), \overline{N}_1(\overline{T}_{i1}\,\overline{I}_{i1}\,\overline{F}_{i1}) \right\rangle}, \\ \frac{C_2}{\left\langle \underline{N}_2(\underline{T}_{i2}\,\underline{I}_{i2}\,\underline{F}_{i2}), \overline{N}_2(\overline{T}_{i2}\,\overline{I}_{i2}\,\overline{F}_{i2}) \right\rangle}, \cdots, \\ \frac{C_n}{\left\langle \underline{N}_n(\underline{T}_{in}\,\underline{I}_{in}\,\underline{F}_{in}), \overline{N}_n(\overline{T}_{in}\,\overline{I}_{in}\,\overline{F}_{in}) \right\rangle} : C_j \in C \end{array} \right]$$

$$= \left[\frac{C_j}{\left\langle \underline{N}_j(\underline{T}_{ij}\,\underline{I}_{ij}\,\underline{F}_{ij}), \overline{N}_j(\overline{T}_{ij}\,\overline{I}_{ij}\,\overline{F}_{ij}) \right\rangle} : C_j \in C \right] for \quad (13)$$

$j = 1, 2, \cdots, n$

Here $\overline{N}$ and $\underline{N}$ are neutrosophic sets, and $\left\langle \overline{T}_{ij}, \overline{I}_{ij}, \overline{F}_{ij} \right\rangle$ *and* $\left\langle \underline{T}_{ij}, \underline{I}_{ij}, \underline{F}_{ij} \right\rangle$ are the degrees of truth membership, degree of indeterminacy and degree of falsity membership of the alternative A_i satisfying the attribute C_j, respectively where

$0 \le \underline{T}_{ij}, \overline{T}_{ij} \le 1, \quad 0 \le \underline{I}_{ij}, \overline{I}_{ij} \le 1, \quad 0 \le \underline{F}_{ij}, \overline{F}_{ij} \le 1,$

$0 \le \underline{T}_{ij} + \underline{I}_{ij} + \underline{F}_{ij} \le 3, \; 0 \le \overline{T}_{ij} + \overline{I}_{ij} + \overline{F}_{ij} \le 3$

The rough neutrosophic decision matrix can be presented in the following form (See the table 2):

Table 2: Rough neutrosophic decision matrix

$$d = \left\langle \underline{N}_{ij}(F), \overline{N}_{ij}(F) \right\rangle_{m \times n} = \begin{array}{c|cccc} & C_1 & C_2 & \dots & C_n \\ \hline A_1 & \langle \underline{N}_{11}, \overline{N}_{11} \rangle & \langle \underline{N}_{12}, \overline{N}_{12} \rangle & \dots & \langle \underline{N}_{1n}, \overline{N}_{1n} \rangle \\ A_2 & \langle \underline{N}_{21}, \overline{N}_{21} \rangle & \langle \underline{N}_{22}, \overline{N}_{22} \rangle & \dots & \langle \underline{N}_{2n}, \overline{N}_{2n} \rangle \\ . & \dots & \dots & \dots & \dots \\ . & \dots & \dots & \dots & \dots \\ A_m & \langle \underline{N}_{n1}, \overline{N}_{n1} \rangle & \langle \underline{N}_{n2}, \overline{N}_{n2} \rangle & \dots & \langle \underline{N}_{mn}, \overline{N}_{mn} \rangle \end{array} \quad (14)$$

Here $\underline{N}_{ij}$ *and* $\overline{N}_{ij}$ are lower and upper approximations of the neutrosophic set *P*.

Step 2: Determination of the rough accuracy score matrix

Let us consider a rough neutrosophic set in the form: $N_{ij}(P) = \left\langle \left(\underline{T}_{ij}, \underline{I}_{ij}, \underline{F}_{ij} \right), \left(\overline{T}_{ij}, \overline{I}_{ij}, \overline{F}_{ij} \right) \right\rangle$

The rough accuracy score matrix is formed by using equation (11) and it is presented in the table 3.

Table3: The rough accuracy score matrix

$$RASF_{m \times n} = \begin{array}{c|cccc} & C_1 & C_2 & \dots & C_n \\ \hline A_1 & \langle S[N_{11}(P)] \rangle & \langle S[N_{12}(P)] \rangle & \dots & \langle S[N_{1n}(P)] \rangle \\ A_2 & \langle S[N_{21}(P)] \rangle & \langle S[N_{22}(P)] \rangle & \dots & \langle S[N_{2n}(P)] \rangle \\ . & \dots & \dots & \dots & \dots \\ . & \dots & \dots & \dots & \dots \\ A_m & \langle S[N_{m1}(P)] \rangle & \langle S[N_{m2}(P)] \rangle & \dots & \langle S[N_{mn}(P)] \rangle \end{array} \quad (15)$$

Step 3: Determination of the weights of attribute

During decision-making process, decision makers may enconter unknown attribute weights. In many cases, the importance of the decision makers are not equal. So, it is necessary to determine attribute weight for making a proper decision.

In this paper, we have adopted the entropy method proposed by Majumder and Samanta [42], in rough neutrosophic environment for determining attribute weight as follows.

Let us consider $\left[T_{N(P)}\right]_{ij}(x_i) = \left(\dfrac{\underline{T}_{ij} + \overline{T}_{ij}}{2} \right)$,

$\left[I_{N(P)}\right]_{ij}(x_i) = \left(\dfrac{\underline{I}_{ij} + \overline{I}_{ij}}{2} \right)$, $\left[F_{N(P)}\right]_{ij}(x_i) = \left(\dfrac{\underline{F}_{ij} + \overline{F}_{ij}}{2} \right)$

Now,

$S_N = \left\langle T_{N(P)}(x_i), I_{N(P)}(x_i), F_{N(P)}(x_i) \right\rangle$,

$$E_i(S_N) = 1 - \frac{1}{n} \sum_{i=1}^{m} \left(T_{N(P)}(x_i) + F_{N(p)}(x_i) \right) \left| I_{N(P)}(x_i) - I^c{}_{N(P)}(x_i) \right| \quad (16)$$

which has the following properties:

1. $E_i(S_N) = 0 \Rightarrow S_N$ *is a crisp set and* $I_{S_N}(x_i) = 0 \; \forall x \in E.$
2. $E_i(S_N) = 1 \Rightarrow \left\langle T_{N(P)}(x_i), I_{N(P)}(x_i), F_{N(P)}(x_i) \right\rangle = \langle 0.5, 0.5, 0.5 \rangle \; \forall x \in E.$
3. $E_i(S_{N1}) \ge E_i(S_{N2}) \Rightarrow$ $(T_{N1(P)}(x_i) + F_{N1(P)}(x_i) \le (T_{N2(P)}(x_i) + F_{N2(P)}(x_i)$ *and* $\left| I_{N1(P)}(x_i) - I^c{}_{N1(P)}(x_i) \right| \le \left| I_{N2(P)}(x_i) - I^c{}_{N2(P)}(x_i) \right|$
4. $E_i(S_N) = E_i(S_{N^c}) \; \forall x \in E.$

In order to obtain the entropy value E_j of the *j*-th attribute C_j (j = 1, 2,…, n), equation (16) can be written as:

$$E_j = 1 - \frac{1}{n} \sum_{i=1}^{m} \left(\left[T_{NP}\right]_{ij}(x_i) + \left[F_{NP}\right]_{ij}(x_i) \right) . \left| \left[I_{NP}\right]_{ij}(x_i) - \left[I^c{}_{NP}\right]_{ij}(x_i) \right| \quad (17)$$

For i = 1, 2, …, n; j = 1, 2, …, m

It is observed that $E_j \in [0,1]$. Due to Hwang and Yoon [43], and Wang and Zhang [44], the entropy weight of the *j*-th attibute C_j is presented as follows:

$$w_j = \frac{1 - E_j}{\sum_{j=1}^{n} \left(1 - E_j\right)} \quad (18)$$

We have weight vector W = $(w_1, w_2, \dots, w_n)^T$ of attributes C_j (j = 1, 2, …, n) with $w_j \ge 0$ and $\sum_{i=1}^{n} w_j = 1$

Step 4: Determination of the over all weighted rough accuracy score values of the alternatives

To rank alternatives, we can sum all values in each row of the rough accuracy score matrix corresponding to the attribute weights by the over all weighted rough accuracy score value (*WRASV*) of each alternative A_i (i = 1, 2, ..., n). It is defined as follows:

$$WRASV(A_i) = \sum_{i=1}^{n} w_j \left\langle S[N_{ij}(P)] \right\rangle \quad (19)$$

Step 5: Ranking the alternatives

According to the over all weighted rough accuracy score values *WRASV*(A_i) (i = 1, 2, ..., n), we can rank alternatives A_i (i = 1, 2, ..., n). The highest value of *WRASV*(A_i) (i = 1, 2, ..., n) reflects the best alternative.

4 Numerical example

In this section, rough neutrosophic MADM is considered to demonstrate the applicability and the effectiveness of the proposed approach. Let us consider a decision-making problem stated as follows. A person wants to purchase a SIM card for mobile connection. Now it is necessary to select suitable SIM card for his/her mobile connection. After initial screening there is a panel with three possible alternatives (SIM cards) for mobile connection. The alternatives (SIM cards) are presented as follows:

A_1: Airtel,

A_2: Vodafone and

A_3: BSNL.

The person must take a decision based on the following four attributes of SIM cards:

(1) C_1 is service quality of the corresponding company;

(2) C_2 is the cost and innitial talktime;

(3) C_3 is the call rate per second; and

(4) C_4 is the internet and other facilities.

Step 1: **Construction of the decision matrix with rough neutrosophic form**

We construct the following rough neutrosophic decision matrix (see the table 4)based on the experts' assessment.

Table 4. Decision matrix with rough neutrosophic number

$d_S = \langle \underline{N}(P), \overline{N}(P) \rangle_{3\times 4} =$

	C_1	C_2	C_3	C_4
A_1	$\langle (.7,.3,.3), (.8,.2,.2) \rangle$	$\langle (.6,.4,.4), (.8,.3,.2) \rangle$	$\langle (.6,.3,.3), (.8,.2,.2) \rangle$	$\langle (.7,.4,.4), (.8,.2,.2) \rangle$
A_2	$\langle (.7,.3,.3), (.8,.1,.2) \rangle$	$\langle (.6,.3,.3), (.8,.1,.2) \rangle$	$\langle (.7,.2,.2), (.8,.4,.1) \rangle$	$\langle (.7,.3,.3), (.8,.3,.3) \rangle$
A_3	$\langle (.7,.2,.2), (.8,.1,.1) \rangle$	$\langle (.7,.3,.3), (.8,.1,.1) \rangle$	$\langle (.7,.2,.2), (.9,.2,.2) \rangle$	$\langle (.8,.3,.2), (.9,.1,.1) \rangle$

(23)

The selection process using proposed approach is done based on the following steps:

Step 2: Calculation of the rough accuracy score matrix

Using the rough accuracy score function of $N_{ij}(P)$ from equation (11), the rough accuracy score matrix is presented in the table 5.

Step 3: Determination of the weights of attribute

Rough entropy value E_j of the j-th (j = 1, 2, 3) attributes can be determined from the decision matrix d_S (23) and equation (17) as: E_1= 0.4233, E_2 = 0.5200, E_3 = 0.5150, E_4 = 0.5200.

Table 5. Rough accuracy score matrix

	C_1	C_2	C_3	C_4
A_1	$\langle 0.7500 \rangle$	$\langle 0.6833 \rangle$	$\langle 0.7333 \rangle$	$\langle 0.7167 \rangle$
A_2	$\langle 0.7667 \rangle$	$\langle 0.7500 \rangle$	$\langle 0.7667 \rangle$	$\langle 0.7333 \rangle$
A_3	$\langle 0.8167 \rangle$	$\langle 0.7833 \rangle$	$\langle 0.8000 \rangle$	$\langle 0.8333 \rangle$

(24)

Then the corresponding rough entropy weights w_1, w_2, w_3, w_4 of all attributes according to equation (18) are obtained as follows: w_1 = 0.2853, w_2 = 0.2374, w_3 = 0.2399, w_4 = 0.2374 such that $\sum_{j=1}^{n} w_j = 1$.

Step 4: Determination of the over all weighted rough accuracy score values of the alternatives

Using equation (19), the over all weighted rough accuracy score value (*WRASV*) of each alternative A_i (i = 1, 2, 3) is presented as follows:

$WRASV(A_1)$ = 0.72225, $WRASV(A_2)$ = 0.754806,

$WRASV(A_3)$ = 0.808705.

Step 5: Ranking the alternatives.

According to the over all weighted rough accuracy score values $WRASV(A_i)$ (i = 1, 2, 3), we can rank alternatives A_i (i = 1, 2, 3) as follows:

$WRASV(A_3) > WRASV(A_2) > WRASV(A_1)$

Therefore A_3 (BSNL) is the best SIM card.

Conclusion

In this paper, we have defined rough accuracy score function and studied some of it's properties. Entropy based weighted rough accuracy score value is proposed. We have introduced rough neutrosophic multi-attribute decision-making problem with incompletely known or completely unknown attribute weight information based on rough accuracy score function. Finally, an illustrative example is provided to show the effectiveness of the proposed approach.

However, we hope that the concept presented here will open new avenue of research in current rough neutrosophic decision-making arena. In future the proposed approach can be used for other practical MADM problems in hybrid environment.

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An Extended TOPSIS Method for Multiple Attribute Decision Making based on Interval Neutrosophic Uncertain Linguistic Variables

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Abstract: The interval neutrosophic uncertain linguistic variables can easily express the indeterminate and inconsistent information in real world, and TOPSIS is a very effective decision making method more and more extensive applications. In this paper, we will extend the TOPSIS method to deal with the interval neutrosophic uncertain linguistic information, and propose an extended TOPSIS method to solve the multiple attribute decision making problems in which the attribute value takes the form of the interval neutrosophic uncertain linguistic variables and attribute weight is unknown. Firstly, the operational rules and properties for the interval neutrosophic variables are introduced. Then the distance between two interval neutrosophic uncertain linguistic variables is proposed and the attribute weight is calculated by the maximizing deviation method, and the closeness coefficients to the ideal solution for each alternatives. Finally, an illustrative example is given to illustrate the decision making steps and the effectiveness of the proposed method.

Keywords: The interval neutrosophic linguistic, multiple attribute decision making, TOPSIS, maximizing deviation method

I-Introduction

F. Smarandache [7] proposed the neutrosophic set (NS) by adding an independent indeterminacy-membership function. The concept of neutrosophic set is generalization of classic set, fuzzy set [25], intuitionistic fuzzy set [22], interval intuitionistic fuzzy set [23,24] and so on. In NS, the indeterminacy is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are completely independent. From scientific or engineering point of view, the neutrosophic set and set-theoretic view, operators need to be specified .Otherwise, it will be difficult to apply in the real applications. Therefore, H. Wang et al [8] defined a single valued neutrosophic set (SVNS) and then provided the set theoretic operations and various properties of single valued neutrosophic sets. Furthermore, H. Wang et al.[9] proposed the set theoretic operations on an instance of neutrosophic set called interval valued neutrosophic set (IVNS) which is more flexible and practical than NS. The works on neutrosophic set (NS) and interval valued neutrosophic set (IVNS), in theories and application have been progressing rapidly (e.g, [1,2,4,6,7,8,9,10,11,12,13,14,15,16,17, ,18,19,20,21,27,28,29,30,31,32,33,35,36,37,38,39,40,41,42 ,43,44,45,46,47,48,53].

Multiple attribute decision making (MADM) problem are of importance in most kinds of fields such as engineering, economics, and management. In many situations decision makers have incomplete , indeterminate and inconsistent information about alternatives with respect to attributes. It is well known that the conventional and fuzzy or intuitionistic fuzzy decision making analysis [26, 50, 51,] using different techniques tools have been found to be inadequate to handle indeterminate an inconsistent data. So, Recently, neutrosophic multicriteria decision making problems have been proposed to deal with such situation.

TOPSIS (Technique for Order Performance by Similarity to Ideal Solution) method, initially introduced by C. L. Hwang and Yoon [3], is a widely used method for dealing with MADM problems, which focuses on choosing the alternative with the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). The traditional TOPSIS is only used to solve the decision making problems with crisp numbers, and many extended TOPSIS were proposed to deal with fuzzy information. Z. Yue [55] extended TOPSIS to deal with interval numbers, G. Lee et al.[5] extend TOPSIS to deal wit fuzzy numbers, P. D. Liu and Su [34], Y. Q. Wei and Liu [49] extended TOPSIS to linguistic information environments, Recently, Z. Zhang and C. Wu [53] proposed the single valued neutrosophic or interval neutrosophic TOPSIS method to calculate the relative closeness coefficient of each alternative to the single valued neutrosophic or interval neutrosophic positive ideal solution, based on which the considered alternatives are ranked and then the most desirable one is selected. P. Biswas et al. [32] introduced single –valued neutrosophic multiple attribute decision making problem with incompletely known or completely unknown attribute weight information based on modified GRA.

Based on the linguistic variable and the concept of interval neutrosophic sets, J. Ye [19] defined interval neutrosophic linguistic variable, as well as its operation principles, and developed some new aggregation operators for the interval neutrosophic linguistic information, including interval neutrosophic linguistic arithmetic weighted average (INLAWA) operator, linguistic geometric weighted average(INLGWA) operator and discuss some properties. Furthermore, he proposed the decision making method for multiple attribute decision making (MADM) problems with an illustrated example to show the process of decision making and the effectiveness of the proposed method. In order to process incomplete, indeterminate and inconsistent information more efficiency and precisely J. Ye [20] further proposed the interval neutrosophic uncertain linguistic variables by combining uncertain linguistic variables and interval neutrosophic sets, and proposed the operational rules, score function , accuracy functions ,and certainty function of interval neutrosophic uncertain linguistic variables. Then the interval neutrosophic uncertain linguistic weighted arithmetic averaging (INULWAA) and the interval neutrosophic uncertain linguistic weighted arithmetic averaging (INULWGA) operator are developed, and a multiple attribute decision method with interval neutrosphic uncertain linguistic information was developed.

To do so, the remainder of this paper is set out as follows. Section 2 briefly recall some basic concepts of neutrosphic sets, single valued neutrosophic sets (SVNSs), interval neutrosophic sets(INSs), interval neutrosophic linguistic variables and interval neutrosophic uncertain linguistic variables. In section 3, we develop an extended TOPSIS method for the interval neutrosophic uncertain linguistic variables, In section 4, we give an application example to show the decision making steps, In section 5, a comparison with existing methods are presented. Finally, section 6 concludes the paper.

II-Preliminaries

In the following, we shall introduce some basic concepts related to uncertain linguistic variables, single valued neutrosophic set, interval neutrosophic sets, interval neutrosophic uncertain linguistic sets, and interval neutrosophic uncertain linguistic set.

2.1 Neutrosophic sets

Definition 2.1 [7]

Let U be a universe of discourse then the neutrosophic set A is an object having the form

A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$ >, x ∈ X },

Where the functions $T_A(x)$, $I_A(x)$, $F_A(x)$: U→]$^-$0,1+[define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element x ∈ X to the set A with the condition.

$$^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+. \quad \textbf{(1)}$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of]$^-$0,1$^+$[. So instead of]$^-$0,1$^+$[we need to take the interval [0,1] for technical applications, because]$^-$0,1$^+$[will be difficult to apply in the real applications such as in scientific and engineering problems.

2.2 Single valued Neutrosophic Sets

Definition 2.2 [8]

Let X be an universe of discourse, then the neutrosophic set A is an object having the form

A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$ >, x ∈ X },

where the functions $T_A(x)$,$I_A(x)$, $F_A(x)$: U→[0,1]define respectively the degree of membership , the degree of indeterminacy, and the degree of non-membership of the element x ∈ X to the set A with the condition.

$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ **(2)**

Definition 2.3 [8]

A single valued neutrosophic set A is contained in another single valued neutrosophic set B i.e. A ⊆ B if ∀x ∈ U, $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$. **(3)**

2.3 Interval Neutrosophic Sets

Definition 2.4[9]

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function $T_A(x)$, indeteminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, we have that $T_A(x)$, $I_A(x)$, $F_A(x) \subseteq [0 ,1]$.

For two IVNS, A_{IVNS}= {<x, $[T_A^L(x),T_A^U(x)]$, $[I_A^L(x), I_A^U(x)]$, $[F_A^L(x), F_A^U(x)]$ > | x ∈ X **}** **(4)**

And B_{IVNS}= {<x, $[T_B^L(x),T_B^U(x)]$, $[I_B^L(x), I_B^U(x)]$, $[F_B^L(x), F_B^U(x)]$> | x ∈ X } the two relations are defined as follows:

(1) $A_{IVNS} \subseteq B_{IVNS}$ If and only if $T_A^L(x) \le T_B^L(x)$,$T_A^U(x) \le T_B^U(x)$, $I_A^L(x) \ge I_B^L(x)$,$I_A^U(x) \ge I_B^U(x)$, $F_A^L(x) \ge F_B^L(x)$,$F_A^U(x) \ge F_B^U(x)$

(2)$A_{IVNS} = B_{IVNS}$ if and only if , $T_A(x)$ =$T_B(x)$,$I_A(x)$ =$I_B(x)$,$F_A(x)$ =$F_B(x)$ for any x ∈ X

The complement of A_{IVNS} is denoted by A_{IVNS}^o and is defined by

A_{IVNS}^o= {<x, $[F_A^L(x), F_A^U(x)]$>, $[1 - I_A^U(x), 1 - I_A^L(x)]$,$[T_A^L(x),T_A^U(x)]$ | x ∈ X }

A∩B ={ <x , $[\min(T_A^L(x),T_B^L(x)), \min(T_A^U(x),T_B^U(x))]$, $[\max(I_A^L(x),I_B^L(x)), \max(I_A^U(x),I_B^U(x)]$, $[\max(F_A^L(x),F_B^L(x)), \max(F(x),F_B^U(x))]$ >: x ∈ X }

A∪B ={ <x , $[\max(T_A^L(x),T_B^L(x)), \max(T_A^U(x),T_B^U(x))]$, $[\min(I_A^L(x),I_B^L(x)), \min(I_A^U(x),I_B^U(x)]$, $[\min(F_A^L(x),F_B^L(x)), \min(F_A^U(x),F_B^U(x))]$ >: x ∈ X }

2.4 Uncertain linguistic variable.

A linguistic set is defined as a finite and completely ordered discreet term set,

S=(s_0, s_1,…, s_{l-1}), where l is the odd value. For example, when l=7, the linguistic term set S can be defined as follows: S={s_0(extremely low); s_1(very low); s_2(low); s_3(medium); s_4(high); s_5(very high); s_6(extermley high)}

Definition 2.5. Suppose $\tilde{s}$ = [s_a, s_b], where s_a, $s_b \in \tilde{S}$ with a ≤ b are the lower limit and the upper limit of S, respectively. Then $\tilde{s}$ is called an uncertain linguitic varaible.

Definition 2.6. Suppose $\tilde{s}_1$ = [s_{a_1}, s_{b_1}] and $\tilde{s}_2$ = [s_{a_2}, s_{b_2}] are two uncertain linguistic variable ,then the distance between $\tilde{s}_1$ and $\tilde{s}_2$ is defined as follows.

$d\ (\tilde{s}_1, \tilde{s}_2) = \frac{1}{2(l-1)}\ (|a_2 - a_1|+|b_2 - b_1|)$ **(5)**

2.5 Interval neutrosophic linguistic set

Based on interval neutrosophic set and linguistic variables, J. Ye [18] presented the extension form of the linguistic set, i.e, interval neutroosphic linguistic set, which is shown as follows:

Definition 2.7 :[19] An interval neutrosophic linguistic set A in X can be defined as

A ={<x, $s_{\theta(x)}$, (T_A(x), I_A(x), F_A(x))>| x ∈ X} **(6)**

Where $s_{\theta(x)} \in \hat{s}$, T_A(x) = [T_A^L(x), T_A^U(x)] ⊆ [0.1], I_A(x) = [I_A^L(x), I_A^U(x)] ⊆ [0.1], and F_A(x) = [F_A^L(x), F_A^U(x)] ⊆ [0.1] with the condition 0 ≤ T_A^U(x)+ I_A^U(x)+ F_A^U(x) ≤3 for any x ∈ X. The function T_A(x), I_A(x) and F_A(x) express, respectively, the truth-membership degree, the indeterminacy –membership degree, and the falsity-membership degree with interval values of the element x in X to the linguistic variable $s_{\theta(x)}$.

2.6 Interval neutrosophic uncertain linguistic set.

Based on interval neutrosophic set and uncertain linguistic variables, J.Ye [20] presented the extension form of the uncertain linguistic set, i.e, interval neutrosphic uncertain linguistic set, which is shown as follows:

Definition 2.8 :[20] An interval neutrosophic uncertain linguistic set A in X can be defined as

A ={<x,[$s_{\theta(x)}$, $s_{\rho(x)}$], (T_A(x), I_A(x), F_A(x))>| x ∈ X} **(7)**

Where $s_{\theta(x)} \in \hat{s}$, T_A(x) = [T_A^L(x), T_A^U(x)] ⊆ [0.1], I_A(x) = [I_A^L(x), I_A^U(x)] ⊆ [0.1], and F_A(x) = [F_A^L(x), F_A^U(x)] ⊆ [0.1] with the condition 0 ≤ T_A^U(x)+ I_A^U(x)+ F_A^U(x) ≤3 for any x ∈ X. The function T_A(x), I_A(x) and F_A(x) express, respectively, the truth-membership degree, the indeterminacy–membership degree, and the falsity-membership degree with interval values of the element x in X to the uncertain linguistic variable [$s_{\theta(x)}$, $s_{\rho(x)}$].

Definition 2.9 Let $\tilde{a}_1$=< [$s_{\theta(\tilde{a}_1)}$, $s_{\rho(\tilde{a}_1)}$], ([$T^L(\tilde{a}_1)$,$T^U(\tilde{a}_1)$], [$I^L(\tilde{a}_1)$,$I^U(\tilde{a}_1)$], [$F^L(\tilde{a}_1)$,$F^U(\tilde{a}_1)$])> and $\tilde{a}_2$={<x, [$s_{\theta(\tilde{a}_2)}$, $s_{\rho(\tilde{a}_2)}$], ([$T^L(\tilde{a}_2)$,$T^U(\tilde{a}_2)$], [$I^L(\tilde{a}_2)$,$I^U(\tilde{a}_2)$], [$F^L(\tilde{a}_2)$,$F^U(\tilde{a}_2)$])>

be two INULVs and λ ≥ 0, then the operational laws of INULVs are defined as follows:

$\tilde{a}_1 \oplus \tilde{a}_2$ =< [$s_{\theta(\tilde{a}_1)+\theta(\tilde{a}_2)}$, $s_{\rho(\tilde{a}_1)+\rho(\tilde{a}_2)}$], ([$T^L(\tilde{a}_1)$+ $T^L(\tilde{a}_2)$- $T^L(\tilde{a}_1)$ $T^L(\tilde{a}_2)$,$T^U(\tilde{a}_1)$+ $T^U(\tilde{a}_2)$- $T^U(\tilde{a}_1)$ $T^U(\tilde{a}_2)$], [$I^L(\tilde{a}_1)$ $I^L(\tilde{a}_2)$,$I^U(\tilde{a}_1)$ $I^U(\tilde{a}_2)$], [$F^L(\tilde{a}_1)$ $F(\tilde{a}_2)$,$F^U(\tilde{a}_1)$ $F^L(\tilde{a}_2)$]])> **(8)**

$\tilde{a}_1 \otimes \tilde{a}_2$ =< [$s_{\theta(\tilde{a}_1)\times\theta(\tilde{a}_2)}$], ([$T^L(\tilde{a}_1)$ $T^L(\tilde{a}_2)$, $T^U(\tilde{a}_1)$ $T^U(\tilde{a}_2)$], [$I^L(\tilde{a}_1)$+ $I^L(\tilde{a}_2)$ - $I^L(\tilde{a}_1)$ $I^L(\tilde{a}_2)$, $I^U(\tilde{a}_1)$+ $I^U(\tilde{a}_2)$- $I^U(\tilde{a}_1)$ $I^U(\tilde{a}_2)$], [$F^L(\tilde{a}_1)$+ $F^L(\tilde{a}_2)$ - $F^L(\tilde{a}_1)$ $F(\tilde{a}_2)$, $F^U(\tilde{a}_1)$+ $F^U(\tilde{a}_2)$ - $F^U(\tilde{a}_1)$ $F^U(\tilde{a}_2)$]])> **(9)**

$\lambda\tilde{a}_1$=<[$s_{\lambda\theta(\tilde{a}_1)}$, $s_{\lambda\rho(\tilde{a}_1)}$],([1-$(1-T^L(\tilde{a}_1))^\lambda$,1-$(1-T^U(\tilde{a}_1))^\lambda$], [$(I^L(\tilde{a}_1))^\lambda$,$(I^U(\tilde{a}_1))^\lambda$], [$(F^L(\tilde{a}_1))^\lambda$,$(F^U(\tilde{a}_1))^\lambda$]>

(10)

$\tilde{a}_1^\lambda$=< [$s_{\theta^\lambda(\tilde{a}_1)}$, $s_{\rho^\lambda(\tilde{a}_1)}$], ([$(T^L(\tilde{a}_1))^\lambda$,$(T^U(\tilde{a}_1))^\lambda$], [1-$(1-I^L(\tilde{a}_1))^\lambda$, 1-$(1-I^U(\tilde{a}_1))^\lambda$], [1-$(1-F^L(\tilde{a}_1))^\lambda$, 1-$(1-F^U(\tilde{a}_1))^\lambda$]> **(11)**

Obviously, the above operational results are still INULVs.

III. The Extended TOPSIS for the Interval Neutrosophic Uncertain Linguistic Variables

A. The description of decision making problems with interval neutrosphic uncertain linguistic information.

For the MADM problems with interval neutrosophic uncertain variables, there are m alternatives A= $(A_1, A_2,\dots, A_m)$ which can be evaluated by n attributes C=$(C_1, C_2,\dots, C_n)$ and the weight of attributes A_i is w_i, and meets the conditions $0 \le w_i \le 1$, $\sum_{j=1}^{n} w_j$=1.Suppose z_{ij} (i=1, 2,…, n; j=1, 2,…, m) is the evaluation values of alternative A_i with respect to attribute C_j

And it can be represented by interval neutrosophic uncertain linguistic variable z_{ij}= <[x_{ij}^L, x_{ij}^U],([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U])>, where [x_{ij}^L, x_{ij}^U] is the uncertain linguistic variable, and $x_{ij}^L, x_{ij}^U \in$ S, S =$(s_0, s_1,\dots, s_{l-1})$, T_{ij}^L, T_{ij}^U, I_{ij}^L, I_{ij}^U and F_{ij}^L, $F_{ij}^U \in$ [0, 1] and $0 \le T_{ij}^U + I_{ij}^U + F_{ij}^U \le 3$. Suppose attribute weight vector W=$(w_1, w_2,\dots w_n)$ is completely unknown, according to these condition, we can rank the alternatives $(A_1, A_2,\dots, A_m)$

B. Obtain the attribute weight vector by the maximizing deviation.

In order to obtain the attribute weight vector, we firstly define the distance between two interval neutrosophic uncertain variables.

Definition 3.1

Let $\tilde{s}_1$ = <[s_{a_1}, s_{b_1}],([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U])>, $\tilde{s}_2$ = <[s_{a_2}, s_{b_2}],([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U])> and $\tilde{s}_3$ = <[s_{a_3}, s_{b_3}],([T_C^L, T_C^U], [I_C^L, I_C^U], [F_C^L, F_C^U])>, be any three interval neutrosophic uncertain linguistic variables, and $\tilde{S}$ be the set of linguistic variables, f is a map, and $f: \tilde{S} \times \tilde{S} \longrightarrow$ R. If d$(\tilde{s}_1, \tilde{s}_2)$ meets the following conditions

(1) $0 \le d_{INULV}(\tilde{s}_1, \tilde{s}_2) \le 1$, $d_{INULV}(\tilde{s}_1, \tilde{s}_1)$= 0

(2) $d_{INULV}(\tilde{s}_1, \tilde{s}_2) = d_{INULV}(\tilde{s}_2, \tilde{s}_1)$

(3) $d_{IVNS}(\tilde{s}_1, \tilde{s}_2) + d_{INULV}(\tilde{s}_2, \tilde{s}_3) \ge d_{INULV}(\tilde{s}_1, \tilde{s}_3)$

then $d_{INULV}(\tilde{s}_1, \tilde{s}_2)$ is called the distance between two interval neutrosophic uncertain linguistic variables $\tilde{s}_1$

Definition 3.2:

Let $\tilde{s}_1$ = <[s_{a_1}, s_{b_1}],([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U])>, and $\tilde{s}_2$ = <[s_{a_2}, s_{b_2}],([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U])>, be any two interval neutrosophic uncertain linguistic variables, then the Hamming distance between $\tilde{s}_1$ and $\tilde{s}_2$ can be defined as follows.

$$d_{INULV}(\tilde{s}_1, \tilde{s}_2) = \frac{1}{12(l-1)} (|a_1 \times T_A^L - a_2 \times T_B^L| + |a_1 \times T_A^U - a_2 \times T_B^U| + |a_1 \times I_A^L - a_2 \times I_B^L| + |a_1 \times I_A^U - a_2 \times I_B^U| + |a_1 \times F_A^L - a_2 \times F_B^L| + |a_1 \times F_A^U - a_2 \times F_B^U| + |b_1 \times T_A^L - b_2 \times T_B^L| + |b_1 \times T_A^U - b_2 \times T_B^U| + |b_1 \times I_A^L - b_2 \times I_B^L| + |b_1 \times I_A^U - b_2 \times I_B^U| + |b_1 \times F_A^L - b_2 \times F_B^L| + |b_1 \times F_A^U - b_2 \times F_B^U|) \quad \textbf{(12)}$$

In order to illustrate the effectiveness of definition 3.2, the distance defined above must meet the three conditions in definition 3.1

Proof

Obviously, the distance defined in (12) can meets the conditions (1) and (2) in definition 3.1

In the following, we will prove that the distance defined in (12) can also meet the condition (3) in definition 3.1

For any one interval neutrosophic uncertain linguistic variable $\tilde{s}_3$ = <[s_{a_3}, s_{b_3}],([T_C^L, T_C^U], [I_C^L, I_C^U], [F_C^L, F_C^U])>,

$$d_{IVNS}(\tilde{s}_1, \tilde{s}_3) = \frac{1}{12(l-1)} (|a_1 \times T_A^L - a_3 \times T_C^L| + |a_1 \times T_A^U - a_3 \times T_C^U| + |a_1 \times I_A^L - a_3 \times I_C^L| + |a_1 \times I_A^U - a_3 \times I_C^U| + |a_1 \times F_A^L - a_3 \times F_C^L| + |a_1 \times F_A^U - a_3 \times F_C^U| + |b_1 \times T_A^L - b_3 \times T_C^L| + |b_1 \times T_A^U - b_3 \times T_C^U| + |b_1 \times I_A^L - b_3 \times I_C^L| + |b_1 \times I_A^U - b_3 \times I_C^U| + |b_1 \times F_A^L - b_3 \times F_C^L| + |b_1 \times F_A^U - b_3 \times F_C^U|)$$

$=\frac{1}{12(l-1)}$ $(|a_1 \times T_A^L - a_2 \times T_B^L + a_2 \times T_B^L - a_3 \times T_C^L|$+$|a_1 \times T_A^U - a_2 \times T_B^U + a_2 \times T_B^U - a_3 \times T_C^U|$+$|a_1 \times I_A^L - a_2 \times I_B^L + a_2 \times I_B^L - a_3 \times I_C^L|$+$|a_1 \times I_A^U - a_2 \times I_B^U + a_2 \times I_B^U - a_3 \times I_C^U|$
+$|a_1 \times F_A^L - a_2 \times F_B^L + a_2 \times F_B^L - a_3 \times F_C^L|$+$|a_1 \times F_A^U - a_2 \times F_B^U + a_2 \times F_B^U - a_3 \times F_C^U|$

+$|b_1 \times T_A^L - b_2 \times T_B^L + b_2 \times T_B^L - b_3 \times T_C^L|$+$|b_1 \times T_A^U - b_2 \times T_B^U + b_2 \times T_B^U - b_3 \times T_C^U|$+$|b_1 \times I_A^L - b_2 \times I_B^L + b_2 \times I_B^L - b_3 \times I_C^L|$+$|b_1 \times I_A^U - b_2 \times I_B^U + b_2 \times I_B^U - b_3 \times I_C^U|$
+$|b_1 \times F_A^L - b_2 \times F_B^L + b_2 \times F_B^L - a_3 \times F_C^L|$+$|b_1 \times F_A^U - b_2 \times F_B^U + b_2 \times F_B^U - b_3 \times F_C^U|$

And
$\frac{1}{12(l-1)}$ $(|a_1 \times T_A^L - a_2 \times T_B^L|$+$|a_2 \times T_B^L - a_3 \times T_C^L|$+$|a_1 \times T_A^U - a_2 \times T_B^U|$+$|a_2 \times T_B^U - a_3 \times T_C^U|$+$|a_1 \times I_A^L - a_2 \times I_B^L|$+$|a_2 \times I_B^L - a_3 \times I_C^L|$+$|a_1 \times I_A^U - a_2 \times I_B^U|$+$|a_2 \times I_B^U - a_3 \times I_C^U|$+

$|a_1 \times F_A^L - a_2 \times F_B^L|$+$|a_2 \times F_B^L - a_3 \times F_C^L|$+$|a_1 \times F_A^U - a_2 \times F_B^U|$+$|a_2 \times F_B^U - a_3 \times F_C^U|$+

+$|b_2 \times T_B^L - b_3 \times T_C^L|$+$|b_1 \times T_A^U - b_2 \times T_B^U|$+$|b_2 \times T_B^U - b_3 \times T_C^U|$+$|b_1 \times I_A^L - b_2 \times I_B^L|$+$|b_2 \times I_B^L - b_3 \times I_C^L|$+$|b_1 \times I_A^U - b_2 \times I_B^U|$+$|b_2 \times I_B^U - b_3 \times I_C^U|$+$|b_1 \times F_A^L - b_2 \times F_B^L|$+$|b_2 \times F_B^L - b_3 \times F_C^L|$+$|b_1 \times F_A^U - b_2 \times F_B^U|$+$|b_2 \times F_B^U - b_3 \times F_C^U|)$

$=\frac{1}{12(l-1)}$ $(|a_1 \times T_A^L - a_2 \times T_B^L|$+$|a_1 \times T_A^U - a_2 \times T_B^U|$+$|a_1 \times I_A^L - a_2 \times I_B^L|$+$|a_1 \times I_A^U - a_2 \times I_B^U|$+$|a_1 \times F_A^L - a_2 \times F_B^L|$
+$|a_1 \times F_A^U - a_2 \times F_B^U|$+$|b_1 \times T_A^L - b_2 \times T_B^L|$+$|b_1 \times T_A^U - b_2 \times T_B^U|$+$|b_1 \times I_A^L - b_2 \times I_B^L|$+$|b_1 \times I_A^U - b_2 \times I_B^U|$ +$|b_1 \times F_A^L - b_2 \times F_B^L|$+$|b_1 \times F_A^U - b_2 \times F_B^U|$+
$|a_2 \times T_B^L - a_3 \times T_C^L|$+$|a_2 \times T_B^U - a_3 \times T_C^U|$+$|a_2 \times I_B^L - a_3 \times I_C^L|$+$|a_2 \times I_B^U - a_3 \times I_C^U|$+$|a_2 \times F_B^L - a_3 \times F_C^L|$+$|a_2 \times F_B^U - a_3 \times F_C^U|$+$|b_2 \times T_B^L - b_3 \times T_C^L|$+$|b_2 \times T_B^U - b_3 \times T_C^U|$+$|b_2 \times I_B^L - b_3 \times I_C^L|$+$|b_2 \times I_B^U - b_3 \times I_C^U|$+$|b_2 \times F_B^L - b_3 \times F_C^L|$+$|b_2 \times F_B^U - b_3 \times F_C^U|)$

$=\frac{1}{12(l-1)}$ $(|a_1 \times T_A^L - a_2 \times T_B^L|$+$|a_1 \times T_A^U - a_2 \times T_B^U|$+$|a_1 \times I_A^L - a_2 \times I_B^L|$+$|a_1 \times I_A^U - a_2 \times I_B^U|$+$|a_1 \times F_A^L - a_2 \times F_B^L|$
+$|a_1 \times F_A^U - a_2 \times F_B^U|$+$|b_1 \times T_A^L - b_2 \times T_B^L|$+$|b_1 \times T_A^U - b_2 \times T_B^U|$+$|b_1 \times I_A^L - b_2 \times I_B^L|$+$|b_1 \times I_A^U - b_2 \times I_B^U|$ +$|b_1 \times F_A^L - b_2 \times F_B^L|$+$|b_1 \times F_A^U - b_2 \times F_B^U|)$+
$\frac{1}{12(l-1)}$ $(|a_2 \times T_B^L - a_3 \times T_C^L|$+$|a_2 \times T_B^U - a_3 \times T_C^U|$+$|a_2 \times I_B^L - a_3 \times I_C^L|$+$|a_2 \times I_B^U - a_3 \times I_C^U|$+$|a_2 \times F_B^L - a_3 \times F_C^L|$+$|a_2 \times F_B^U - a_3 \times F_C^U|$+$|b_2 \times T_B^L - b_3 \times T_C^L|$+$|b_2 \times T_B^U - b_3 \times T_C^U|$+$|b_2 \times I_B^L - b_3 \times I_C^L|$+$|b_2 \times I_B^U - b_3 \times I_C^U|$+$|b_2 \times F_B^L - b_3 \times F_C^L|$+$|b_2 \times F_B^U - b_3 \times F_C^U|)$

=$d_{INULV}(\tilde{s}_1, \tilde{s}_2) + d_{INULV}(\tilde{s}_2, \tilde{s}_3)$
So , $d_{INULV}(\tilde{s}_1, \tilde{s}_2) + d_{INULV}(\tilde{s}_2, \tilde{s}_3) \geq d_{INULV}(\tilde{s}_1, \tilde{s}_3)$

Especially, when T_A^L=T_A^U, I_A^L=I_A^U, F_A^L=F_A^U,and T_B^L=T_B^U, I_B^L=I_B^U, and F_B^L=F_B^U the interval neutrosophic uncertain linguistic variables $\tilde{s}_1, \tilde{s}_2$ can be reduced to single valued uncertain linguistic variables. So the single valued neutrosophic uncertain linguistic variables are the special case of the interval neutrosophic uncertain linguistic variables.

Because the attribute weight is fully unknown, we can obtain the attribute weight vector by the maximizing deviation method. Its main idea can be described as follows. If all attribute values z_{ij} (j=1, 2,…, n) in the attribute C_j have a small difference for all alternatives, it shows that the attribute C_j has a small importance in ranking all alternatives, and it can be assigned a small attribute weight, especially, if all attribute values z_{ij} (j=1, 2,…,n) in the attribute C_j are equal, then the attribute C_j has no effect on sorting, and we can set zero to the weight of attribute C_j. On the contrary, if all attribute values z_{ij} (j=1, 2,…, n) in the attribute C_j have a big difference, the attribute C_j will have a big importance in ranking all alternatives, and its weight can be assigned a big value.
Here, based on the maximizing deviation method, we construct an optimization model to determine the optimal relative weights of criteria under interval neutrosophic uncertain linguistic environment. For the criterion $C_i \in$ C, we can use the distance $d(z_{ij}, z_{kj})$ to represent the deviation between attribute values z_{ij} and z_{kj}, and D_{ij} =$\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j$ can present the weighted deviation sum for the alternative A_i to all alternatives, then

$D_j\ (w_j)=\sum_{i=1}^{m} D_{ij}(w_j)=\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j$ presents the weighted deviation sum for all alternatives, $D(w_j)=\sum_{j=1}^{n} D_j(w_j)=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j$, presents total weighted deviations for all alternatives with respect to all attributes.

Based on the above analysis, we can construct a non linear programming model to select the weight vector w by maximizing D (w),as follow:

$$\begin{cases} \text{Max } D(w_j) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j \\ s.t\ \sum_{j=1}^{n} w_j^2,\ w_j \in [0,1],\ j = 1,2,\dots,n \end{cases} \quad (13)$$

Then we can build Lagrange multiplier function, and get

$$L(w_j,\lambda)= \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j + \lambda\,(\sum_{j=1}^{n} w_j^2 - 1)$$

$$\text{Let} \begin{cases} \frac{\partial L(w_j,\lambda)}{\partial w_j} = \sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j + 2\lambda w_j = 0 \\ \frac{\partial L(w_j,\lambda)}{\partial w_j} = \sum_{j=1}^{n} w_j^2 - 1 = 0 \end{cases}$$

We can get

$$\begin{cases} 2\lambda = \sqrt{\sum_{j}^{n}(\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj}))^2} \\ w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})}{\sqrt{\sum_{j}^{n}(\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj}))^2}} \end{cases} \quad \textbf{(14)}$$

Then we can get the normalized attribute weight, and have

$$w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})} \quad \textbf{(15)}$$

C. The Extended TOPSIS Method for the Interval Neutrosophic Uncertain linguistic Information.

The standard TOPSIS method can only process the real numbers, and cannot deal with the interval neutrosophic uncertain linguistic information. In the following, we will extend TOPSIS to process the interval neutrosophic uncertain linguistic variables. The steps are shown as follows

(1) Normalize the decision matrix

Considering the benefit or cost type of the attribute values, we can give the normalized matrix R=(r_{ij}), where r_{ij}=<[r_{ij}^L, r_{ij}^U],],([$\dot{T}_{ij}^L$, $\dot{T}_{ij}^U$], [$\dot{I}_{ij}^L$, $\dot{I}_{ij}^U$], [$\dot{F}_{ij}^L$, $\dot{F}_{ij}^U$])>,The normalization can be made shown as follows.

(i) For benefit type,

$$\begin{cases} r_{ij}^L = x_{ij}^L, r_{ij}^U = x_{ij}^U \quad \text{for } (1 \le i \le m,\ 1 \le j \le n) \\ \dot{T}_{ij}^L = T_{ij}^L,\ \dot{T}_{ij}^U = T_{ij}^U,\ \dot{I}_{ij}^L = I_{ij}^L,\ \dot{I}_{ij}^U = I_{ij}^U,\ \ \dot{F}_{ij}^L = F_{ij}^L, \dot{F}_{ij}^U = F_{ij}^U \end{cases} \quad \textbf{(16)}$$

(ii) For cost type,

$$\begin{cases} r_{ij}^L = \text{neg}(x_{ij}^U), r_{ij}^U = \text{neg}(x_{ij}^L) \quad \text{for } (1 \le i \le m,\ 1 \le j \le n) \\ \dot{T}_{ij}^L = T_{ij}^L,\ \dot{T}_{ij}^U = T_{ij}^U,\ \dot{I}_{ij}^L = I_{ij}^L,\ \dot{I}_{ij}^U = I_{ij}^U,\ \ \dot{F}_{ij}^L = F_{ij}^L, \dot{F}_{ij}^U = F_{ij}^U \end{cases} \quad \textbf{(17)}$$

(2) Construct the weighted normalize matrix

$Y=[y_{ij}]_{m\times n}$

$$\begin{bmatrix} <[y_{11}^L, y_{11}^U],],([\ddot{T}_{11}^L,\ddot{T}_{11}^U],[\ddot{I}_{11}^L,\ddot{I}_{11}^U],[\ddot{F}_{11}^L,\ddot{F}_{11}^U])> & \dots & <[y_{11}^L, y_{11}^U],],([\ddot{T}_{1n}^L,\ddot{T}_{1n}^U],[\ddot{I}_{1n}^L,\ddot{I}_{1n}^U],[\ddot{F}_{1n}^L,\ddot{F}_{1n}^U])> \\ <[y_{21}^L, y_{21}^U],],([\ddot{T}_{21}^L,\ddot{T}_{21}^U],[\ddot{I}_{21}^L,\ddot{I}_{21}^U],[\ddot{F}_{21}^L,\ddot{F}_{21}^U])> & \dots. & <[y_{2n}^L, y_{2n}^U],],([\ddot{T}_{2n}^L,\ddot{T}_{2n}^U],[\ddot{I}_{2n}^L,\ddot{I}_{2n}^U],[\ddot{F}_{2n}^L,\ddot{F}_{2n}^U])> \\ \dots & & \dots\ \dots \\ <[y_{mn}^L, y_{mn}^U],],([\ddot{T}_{mn}^L,\ddot{T}_{mn}^U],[\ddot{I}_{mn}^L,\ddot{I}_{mn}^U],[\ddot{F}_{mn}^L,\ddot{F}_{mn}^U])> & \cdots & <[y_{mn}^L, y_{mn}^U],],([\ddot{T}_{mn}^L,\ddot{T}_{mn}^U],[\ddot{I}_{mn}^L,\ddot{I}_{mn}^U],[\ddot{F}_{mn}^L,\ddot{F}_{mn}^U])> \end{bmatrix}$$

Where

$$\begin{cases} y_{ij}^L = w_j r_{ij}^L, y_{ij}^U = w_j r_{ij}^U \\ \ddot{T}_{ij}^L = 1-(1-\dot{T}_{ij}^L)^{w_j}, \ddot{T}_{ij}^U = 1-(1-\dot{T}_{ij}^U)^{w_j}, \ddot{I}_{ij}^L = (\dot{I}_{ij}^L)^{w_j}, \ddot{I}_{ij}^U = (\dot{I}_{ij}^U)^{w_j},\ \ddot{F}_{ij}^L = (\dot{F}_{ij}^L)^{w_j}, \ddot{F}_{ij}^U = (\dot{F}_{ij}^U)^{w_j} \end{cases} \quad \textbf{(18)}$$

(3) Identify, the sets of the positive ideal solution $Y^+ = (y_1^+, y_2^+, \dots, y_m^+)$ and the negative ideal solution $Y^- = (y_1^-, y_2^-, \dots, y_m^-)$, then we can get

$$Y^+ = (y_1^+, y_2^+, \dots, y_m^+) = (<[y_1^{L+}, y_1^{U+}],([\ddot{T}_1^{L+},\ddot{T}_1^{U+}],[\ddot{I}_1^{L+},\ddot{I}_1^{U+}],[\ddot{F}_1^{L+},\ddot{F}_1^{U+}])>, <[y_2^{L+}, y_2^{U+}],([\ddot{T}_2^{L+},\ddot{T}_2^{U+}],[\ddot{I}_2^{L+},\ddot{I}_2^{U+}],[\ddot{F}_2^{L+},\ddot{F}_2^{U+}])>, \dots, <[y_n^{L+}, y_n^{U+}],([\ddot{T}_n^{L+},\ddot{T}_n^{U+}],[\ddot{I}_n^{L+},\ddot{I}_n^{U+}],[\ddot{F}_n^{L+},\ddot{F}_n^{U+}])> \quad (19)$$

$Y^{-}= (y_1^{-}, y_2^{-},\ldots, y_m^{-})=$

$$)=(<[\boldsymbol{y_1^{L-}}, \boldsymbol{y_1^{U-}}], ([\ddot{\boldsymbol{T}}_1^{L-}, \ddot{\boldsymbol{T}}_1^{U-}], [\ddot{\boldsymbol{I}}_1^{L-}, \ddot{\boldsymbol{I}}_1^{U-}], [\ddot{\boldsymbol{F}}_1^{L-}, \ddot{\boldsymbol{F}}_1^{U-}])>, <[\boldsymbol{y_2^{L-}}, \boldsymbol{y_2^{U-}}], ([\ddot{\boldsymbol{T}}_2^{L-}, \ddot{\boldsymbol{T}}_2^{U-}], [\ddot{\boldsymbol{I}}_2^{L-}, \ddot{\boldsymbol{I}}_2^{U-}], [\ddot{\boldsymbol{F}}_2^{L-}, \ddot{\boldsymbol{F}}_2^{U-}])>,\ldots, <[\boldsymbol{y_n^{L-}}, \boldsymbol{y_n^{U-}}], ([\ddot{\boldsymbol{T}}_n^{L-}, \ddot{\boldsymbol{T}}_n^{U-}], [\ddot{\boldsymbol{I}}_n^{L-}, \ddot{\boldsymbol{I}}_n^{U-}], [\ddot{\boldsymbol{F}}_n^{L-}, \ddot{\boldsymbol{F}}_n^{U-}])> \quad \textbf{(20)}$$

Where

$$\begin{cases} \boldsymbol{y_j^{L+}} = \mathbf{max}_i(\boldsymbol{y_{ij}^{L}}), \boldsymbol{y_j^{U+}} = \mathbf{max}_i(\boldsymbol{y_{ij}^{U}}), \\ \ddot{\boldsymbol{T}}_j^{L+} = \mathbf{max}_i(\ddot{\boldsymbol{T}}_{ij}^{L}), \ddot{\boldsymbol{T}}_j^{U+} = \mathbf{max}_i(\ddot{\boldsymbol{T}}_{ij}^{U}), \ddot{\boldsymbol{I}}_j^{L+} = \mathbf{min}_i(\ddot{\boldsymbol{I}}_{ij}^{L}), \ddot{\boldsymbol{I}}_j^{U+} = \mathbf{min}_i(\ddot{\boldsymbol{I}}_{ij}^{U}), \ddot{\boldsymbol{F}}_j^{L+} = \mathbf{min}_i(\ddot{\boldsymbol{F}}_{ij}^{L}), \ddot{\boldsymbol{F}}_j^{U+} = \mathbf{min}_i(\ddot{\boldsymbol{F}}_{ij}^{U}), \\ \boldsymbol{y_j^{L-}} = \mathbf{min}_i(\boldsymbol{y_{ij}^{L}}), \boldsymbol{y_j^{U-}} = \mathbf{min}_i(\boldsymbol{y_{ij}^{U}}), \\ \ddot{\boldsymbol{T}}_j^{L-} = \mathbf{min}_i(\ddot{\boldsymbol{T}}_{ij}^{L}), \ddot{\boldsymbol{T}}_j^{U-} = \mathbf{min}_i(\ddot{\boldsymbol{T}}_{ij}^{U}), \ddot{\boldsymbol{I}}_j^{L-} = \mathbf{max}_i(\ddot{\boldsymbol{I}}_{ij}^{L}), \ddot{\boldsymbol{I}}_j^{U-} = \mathbf{max}_i(\ddot{\boldsymbol{I}}_{ij}^{U}), \ddot{\boldsymbol{F}}_j^{L-} = \mathbf{max}_i(\ddot{\boldsymbol{F}}_{ij}^{L}), \ddot{\boldsymbol{F}}_j^{U-} = \mathbf{max}_i(\ddot{\boldsymbol{F}}_{ij}^{U}), \end{cases}$$

(21)

(4) Obtain the distance between each alternative and the positive ideal solution, and between each alternative and the negative ideal solution, then we can get

$$D^{+}= (d_1^{+}, d_2^{+},\ldots, d_m^{+})$$
$$D^{-}= (d_1^{-}, d_2^{-},\ldots, d_m^{-}) \qquad \textbf{(22)}$$

Where,

$$\begin{cases} d_i^{+} = \left[\sum_{j=1}^{n}\left(d(y_{ij}, y_j^{+})\right)^2\right]^{\frac{1}{2}} \\ d_i^{-} = \left[\sum_{j=1}^{n}\left(d(y_{ij}, y_j^{-})\right)^2\right]^{\frac{1}{2}} \end{cases} \qquad \textbf{(23)}$$

Where , $d(y_{ij}, y_j^{+})$is the distance between the interval valued neutrosophic uncertain linguistic variables y_{ij} and y_j^{+} and $d(y_{ij}, y_j^{-})$ is the distance between the interval valued neutrosophic uncertain linguistic variables y_{ij} and y_j^{-} which can be calculated by (12)

(5) Obtain the closeness coefficients of each alternative to the ideal solution, and then we can get

$$cc_i=\frac{d_i^{+}}{d_i^{+}+d_i^{-}} \quad \text{(i=1,2,…,m)} \qquad \textbf{(24)}$$

(6) Rank the alternatives

According to the closeness coefficient above, we can choose an alternative with minimum cc_i or rank alternatives according to cc_i in ascending order

IV. An illustrative example

In this part, we give an illustrative example adapted from J. Ye [20] for the extended TOPSIS method to multiple attribute decision making problems in which the attribute values are the interval neutrosophic uncertain linguistic variables.

Suppose that an investment company, wants to invest a sum of money in the best option. To invest the money, there is a panel with four possible alternatives: (1) A_1 is car company; (2) A_2 is food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investement company must take a decision according to the three attributes: (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is a the environmental impact. The weight vector of the attributes is ω= $(0.35, 0.25, 0.4)^{\mathrm{T}}$.The expert evaluates the four possible alternatives of $\mathrm{A_i}$ (i=1,2,3,4) with respect to the three attributes of $\mathrm{C_j}$ (i=1,2,3), where the evaluation information is expressed by the form of INULV values under the linguistic term set S={s_0=extremely poor, s_1=very poor, s_2= poor, s_3= medium, s_4= good, s_5= very good, s_6= extermely good}.

The evaluation information of an alternative $\mathrm{A_i}$ (i=1, 2, 3) with respect to an attribute $\mathrm{C_j}$ (j=1, 2, 3) can be given by the expert. For example, the INUL value of an alternative $\mathrm{A_1}$ with respect to an attribute $\mathrm{C_1}$ is given as <[s_4, s_5], ([0.4, 0.5],[0.2, 0.3], [0.3, 0.4])> by the expert, which indicates that the mark of the alternative $\mathrm{A_1}$ with respect to the attribute $\mathrm{C_1}$ is about the uncertain linguistic value [s_4, s_5,] with the satisfaction degree interval [0.4 ,0.5], indeterminacy degree interval [0.2, 0.3], and dissatisfaction degree interval [0.3, 0.4]. similarly, the four possible alternatives with respect to the three attributes can be evaluated by the expert, thus we can obtain the following interval neutrosophic uncertain linguistic decision matrix:

$(R)_{\mathrm{m\times n}}=$

$$\begin{bmatrix} <([s_4,s_5],([0.4,0.5],[0.2,0.3],[0.3,0.4])> & <([s_5,s_6],([0.4,0.6],[0.1,0.2],[0.2,0.4])> & <([s_4,s_5],([0.2,0.3],[0.1,0.2],[0.5,0.6])> \\ <([s_5,s_6],([0.5,0.7],[0.1,0.2],[0.2,0.3])> & <([s_4,s_5],([0.6,0.7],[0.1,0.2],[0.2,0.3])> & <([s_4,s_5],([0.5,0.7],[0.2,0.2],[0.1,0.2])> \\ <([s_5,s_6],([0.3,0.5],[0.1,0.2],[0.3,0.4])> & <([s_5,s_6],([0.5,0.6],[0.1,0.3],[0.3,0.4])> & <([s_4,s_4],([0.5,0.6],[0.1,0.3],[0.1,0.3])> \\ <([s_3,s_4],([0.7,0.8],[0.0,0.1],[0.1,0.2])> & <([s_3,s_4],([0.5,0.7],[0.1,0.2],[0.2,0.3])> & <([s_5,s_6],([0.3,0.4],[0.1,0.2],[0.1,0.2])> \end{bmatrix}$$

A. Decision steps

To get the best an alternatives, the following steps are involved:

Step 1: Normalization
Because the attributes are all the benefit types, we don’t need the normalization of the decision matrix X
Step 2: Determine the attribute weight vector W, by formula (24), we can get

w_1= 0.337 , w_2= 0.244 , w_3=0.379
Step 3: Construct the weighted normalized matrix, by formula (18), we can get

$$Y = \begin{bmatrix} <([s_{1.508}, s_{1.885}], ([0.175, 0.229], [0.545, 0.635], [0.635, 0.708])> & <([s_{1.225}, s_{1.467}], ([0.117, 0.201], [0.570, 0.675], [0.675, 0.800])> & <([s_{1.508}, s_{1.885}], ([0.081, 0.126], [0.420, 0.545], [0.77, 0.825])> \\ <([s_{1.885}, s_{2.262}], ([0.229, 0.365], [0.42, 0.545], [0.545, 0.635])> & <([s_{0.98},, s_{1.225}], ([0.201, 0.255], [0.570, 0.675], [0.675, 0.745])> & <([s_{1.508}, s_{1.885}], ([0.231, 0.365], [0.545, 0.545], [0.420, 0.545])> \\ <([s_{1.885}, s_{2.262}], ([0.125, 0.23], [0.42, 0.545], [0.635, 0.708])> & <([s_{0.98}, s_{1.225}], ([0.156, 0.201], [0.570, 0.745], [0.745, 0.800])> & <([s_{1.508}, s_{1.508}], ([0.231, 0.292], [0.420, 0.635], [0.420, 0.635])> \\ <([s_{1.131}, s_{1.508}], ([0.364, 0.455], [0.0, 0.42], [0.42, 0.545])> & <([s_{0.735}, s_{0.98}], ([0.156, 0.255], [0.570, 0.674], [0.675, 0.745])> & <([s_{1.885}, s_{2.262}], ([0.126, 0.175], [0.420, 0.545], [0.420, 0.545])> \end{bmatrix}$$

Step 4: Identify the sets of the positive ideal solution $Y^{+}= (y_1^{+}, y_2^{+}, y_3^{+})$ and the negative ideal solution $Y^{-}= (y_1^{-}, y_2^{-}, y_3^{-})$, by formulas (19)- (21), we can get then we can get

$$Y^{+}= (<([s_{1.885}, s_{2.262}], ([0.365, 0.455], [0, 0.42], [0.42, 0.545])>, <([s_{1.225}, s_{1.47}], ([0.201, 0.255], [0.569, 0.674], [0.674, 0.745])>, <([s_{1.885}, s_{2.262}], ([0.230, 0.365], [0.420, 0.545], [0.420, 0.545])>)$$

$$Y^{-}=(<([s_{1.131}, s_{1.508}], ([0.126, 0.230], [0.545, 0.635], [0.635, 0.708])>, <([s_{0.735}, s_{0.98}], ([0.117, 0.201], [0.569, 0.745], [0.745, 0.799])>, <([s_{1.508}, s_{1.508}], ([0.081, 0.126], [0.545, 0.635], [0.770, 0.825])>)$$

Step 5: Obtain the distance between each alternative and the positive ideal solution, and between each alternative and the negative ideal solution, by formulas (22)-(23), we can get
D^{+}= (0.402, 0.065, 0.089, 0.066)

D^{-}= (0.052, 0.073, 0.080, 0.065)

Step 6: Calculate the closeness coefficients of each alternative to the ideal solution, by formula (24) and then we can get

cc_i = (0.885, 0.472, 0.527, 0.503)
Step 7: Rank the alternatives
According to the closeness coefficient above, we can choose an alternative with minimum to cc_i in ascending order. We can get

$$A_2 \geq A_4 \geq A_3 \geq A_1$$

So, the most desirable alternative is A_2

V-Comparison analysis with the existing interval neutrosophic uncertain linguistic multicriteria decision making method.

Recently, J. Ye [20] developed a new method for solving the MCDM problems with interval neutrosophic uncertain linguistic information. In this section, we will perform a comparison analysis between our new method and the existing method, and then highlight the advantages of the new method over the existing method.

(1) Compared with method proposed proposed by J. Ye [20], the method in this paper can solve the MADM problems with unknown weight, and rank the alternatives by the closeness coefficients. However, the method proposed by J. Ye [20] cannot deal with the unknown weight It can be seen that the result of the proposed method is same to the method proposed in [20].

(2) Compared with other extended TOPSIS method
Because the interval neutrosophic uncertain linguistic variables are the generalization of interval neutrosophic linguistic variables (INLV), interval neutrosophic variables (INV),and intuitionistic uncertain linguistic variable. Obviously, the extended TOPSIS method proposed by J. Ye [19], Z. Wei [54], Z. Zhang and C. Wu [3], are the special cases of the proposed method in this paper.
In a word, the method proposed in this paper is more generalized. At the same time, it is also simple and easy to use.

VI-**Conclusion**

In real decision making, there is great deal of qualitative information which can be expressed by uncertain linguistic variables. The interval neutrosophic uncertain linguistic variables were produced by combining the uncertain linguistic variables and interval neutrosophic set, and could easily express the indeterminate and inconsistent information in real world. TOPSIS had been proved to be a very effective decision making method and has been achieved more and more extensive applications. However, the standard TOPSIS method can only process the real numbers. In this paper, we extended TOPSIS method to deal with the interval neutrosophic uncertain linguistic variables information, and proposed an extended TOPSIS method with respect to the MADM problems in which the attribute values take the form of the interval neutrosophic and attribute weight unknown. Firstly, the operational rules and properties for the interval neutrosophic uncertain linguistic variables were presented. Then the distance between two interval neutrosophic uncertain linguistic variables was proposed and the attribute weight was calculated by the maximizing deviation method, and the closeness coefficient to the ideal solution for each alternative used to rank the alternatives. Finally, an illustrative example was given to illustrate the decision making steps, and compared with the existing method and proved the effectiveness of the proposed method. However, we hope that the concept presented here will create new avenue of research in current neutrosophic decision making area.

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Review of Recommender Systems Algorithms Utilized in Social Networks based e-Learning Systems & Neutrosophic System

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Abstract. In this paper, we present a review of different recommender system algorithms that are utilized in social networks based e-Learning systems. Future research will include our proposed our e-Learning system that utilizes Recommender System and Social Network. Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. The fundamental concepts of neutrosophic set, introduced by Smarandache in [21, 22, 23] and Salama et al. in [24-66].The purpose of this paper is to utilize a neutrosophic set to analyze social networks data conducted through learning activities.

Keywords: e-Learning , Social Networks , Recommender System , Neutrosophic System.

1 Introduction

The Internet shows great potential for enhancing collaboration between people and the role of social software has become increasingly relevant in recent years. A vast array of systems exist which employ users' stored profile data, identifying matches for collaboration. Social interaction within an online framework can help university students share experiences and collaborate on relevant topics. As such, social networks can act as a pedagogical agent, for example, with problem-based learning [1].Social networking websites are virtual communities which allow people to connect and interact with each other on a particular subject or to just ‘‘hang out" together online. Membership of online social networks has recently exploded at an exponential rate [2]. Recommender systems cover an important field within collaborative services that are developed in the Web 2.0 environment and enable user-generated opinions to be exploited in a sophisticated and powerful way. Recommender Systems can be considered as social networking tools that provide dynamic and collaborative communication, interaction and knowledge [3].

Course management systems (CMSs) can offer a great variety of channels and workspaces to facilitate information sharing and communication among participants in a course. They let educators distribute information to students, produce content material, prepare assignments and tests, engage in discussions, manage distance classes and enable collaborative learning with forums, chats, file storage areas, news services, etc. Some examples of commercial systems are Blackboard, WebCT and Top Class while some examples of free systems are Moodle, Ilias and Claroline. Nowadays, one of the most commonly used is Moodle (modular object oriented developmental learning environment), a free learning management system enabling the creation of powerful, flexible and engaging online courses and experiences [4,30].

The new era of e-Learning services is mainly based on ubiquitous learning, mobile technologies, social networks (communities) and personalized knowledge management. “The convergence of e-Learning and knowledge management fosters a constructive, open, dynamic, interconnected, distributed, adaptive, user friendly, socially concerned, and accessible wealth of knowledge”. The

knowledge management tools such as community, social software, peer-to-peer and personalized knowledge management and are now commonly are being used in ubiquitous learning. Learners use these tools to generate and share ideas, explore their thinking, and acquire knowledge from other learners. Learners search and navigate the learning objects in this knowledge filled environment. However, due to the failure of indexing methods to provide the anticipated, ubiquitous learning grid for them, learners often fail to reach their desired learning objects [5]. The fundamental concepts of neutrosophic set, introduced by Smarandache [21, 22, 23] and Salama et al. in [24-66], provides a natural foundation for treating mathematically the
neutrosophic phenomena which exist pervasively in our real world and for building new branches of neutrosophic mathematics and computer applications.

This paper goes as follows: Section Two presents different Recommender Systems algorithms that can be utilized in e-Learning. Section three presents the C4.5 algorithm. Section four presents the K-means algorithm. Section five introduces the Support Vector Machines algorithm. Section six highlights the Apriori algorithm. Section seven presents the conclusion and future work.the notion of neutrosophics crisp set.

2 Recommender Systems

There is a need for Personal Recommender Systems in Learning Networks in order to provide learners with advice on the suitable learning activities to follow. Learning Networks target lifelong learners in any learning situation, at all educational levels and in all national contexts. They are community-driven because every member is able to contribute to the learning material. Existing Recommender Systems and recommendation techniques used for consumer products and other contexts are assessed on their suitability for providing navigational support in a Learner Networks..

3 C4.5

Systems that construct classifiers are one of the commonly used tools in data mining. Such systems take as input a collection of cases, each belonging to one of a small number of classes and described by its values for a fixed set of attributes, and output a classifier that can accurately predict the class to which a new case belongs.Like CLS and ID3, C4.5 generates classifiers expressed as decision trees, but it can also construct classifiers in more comprehensible ruleset form.

A. *Decision Trees*

Given a set S of cases, C4.5 first grows an initial tree using the divide-and-conquer algorithms follows:

- If all the cases in S belong to the same class or S is small, the tree is a leaf labeled with the most frequent class in S.
- Otherwise, choose a test based on a single attribute with two or more outcomes. Make this test the root of the tree with one branch for each outcome of the test, partition S into corresponding subsets according to the outcome for each case, and apply the same procedure recursively to each subset.

There are usually many tests that could be chosen in this last step. C4.5 uses two heuristic criteria to rank possible tests: information gain, which minimizes the total entropy of the subsets (but is heavily biased towards tests with numerous outcomes), and the default gain ratio that divides information gain by the information provided by the test outcomes.

B. *Ruleset Classifier*

Complex decision trees can be difficult to understand, for instance because information about one class is usually distributed throughout the tree. C4.5 introduced an alternative formalism consisting of a list of rules of the form “if A and B and C and ... then class X”, where rules for each class are grouped together. A case is classified by finding the first rule whose conditions are satisfied by the case; if no rule is satisfied, the case is assigned to a default class.C4.5 rulesets are formed from the initial (unpruned) decision tree. Each path from the root of the tree to a leaf becomes a prototype rule whose conditions are the outcomes along the path and whose class is the label of the leaf. This rule is then simplified by determining the effect of discarding each condition in turn. Dropping a condition may increase the number N of cases covered by the rule, and also the number E of cases that do not belong to the class nominated by the rule, and may lower the pessimistic error rate determined as above. A hill-climbing algorithm is used to drop conditions until the lowest pessimistic error rate is found.To complete the process, a subset of simplified rules is selected for each class in turn. These class subsets are ordered to minimize the error on the training cases and a default class is chosen. The final ruleset usually has far fewer rules than the number of leaves on the pruned decision tree.The principal disadvantage of C4.5’s rulesets is the amount of CPU time and memory that they require.

4 K-Means Algorithm

The k-means algorithm is a simple iterative method to partition a given dataset into a user specified number of clusters, k. This algorithm has been discovered by several researchers across different disciplines, most notably Lloyd [6], Forgey, Friedman and Rubin, and McQueen. A detailed history of k-means along with descriptions of several variations are given in [7]. Gray and Neuhoff [8] provide a nice historical background for k-means placed in the larger context of hill-climbing algorithms.The algorithm is initialized by picking k points in as the initial k cluster representatives or "centroids". Techniques for selecting these initial seeds include sampling at random from the dataset, setting them as the solution of clustering a small subset of the data or perturbing the global mean of the data k times. Then the algorithm iterates between two steps till convergence:

- Step 1: Data Assignment. Each data point is assigned to its closest centroid, with ties broken arbitrarily. This results in a partitioning of the data.
- Step 2: Relocation of "means". Each cluster representative is relocated to the center (mean) of all data points assigned to it. If the data points come with a probability measure (weights), then the relocation is to the expectations (weighted mean) of the data partitions.

The algorithm converges when the assignments (and hence the values) no longer change.One issue to resolve is how to quantify "closest" in the assignment step. The default measure of closeness is the Euclidean distance, in which case one can readily show that the non-negative cost function

$$\sum_{i=1}^{n}\left(\arg\min_{j}\left\|x_i - c_j\right\|_2^2\right)$$

,will decrease whenever there is a change in the assignment or the relocation steps, and hence convergence is guaranteed in a finite number of iterations. The greedy-descent nature of k-means on a non-convex cost also implies that the convergence is only to a local optimum, and indeed the algorithm is typically quite sensitive to the initial centroid locations.

A. *Limitations*

In addition to being sensitive to initialization, the k-means algorithm suffers from several other problems. First, observe that k-means is a limiting case of fitting data by a mixture of k Gaussians with identical, isotropic covariance matrices, when the soft assignments of data points to mixture components are hardened to allocate each data point solely to the most likely component. So, it will falter whenever the data is not well described by reasonably separated spherical balls, for example, if there are non-covex shaped clusters in the data. This problem may be alleviated by rescaling the data to "whiten" it before clustering, or by using a different distance measure that is more appropriate for the dataset. For example, information-theoretic clustering uses the KL-divergence to measure the distance between two data points representing two discrete probability distributions. It has been recently shown that if one measures distance by selecting any member of a very large class of divergences called Bregman divergences during the assignment step and makes no other changes, the essential properties of k-means, including guaranteed convergence, linear separation boundaries and scalability, are retained [9]. This result makes k-means effective for a much larger class of datasets so long as an appropriate divergence is used.

k-means can be paired with another algorithm to describe non-convex clusters. One first clusters the data into a large number of groups using k-means. These groups are then agglomerated into larger clusters using single link hierarchical clustering, which can detect complex shapes. This approach also makes the solution less sensitive to initialization, and since the hierarchical method provides results at multiple resolutions, one does not need to prespecify k either.The cost of the optimal solution decreases with increasing k till it hits zero when the number of clusters equals the number of distinct data-points. This makes it more difficult to (a) directly compare solutions with different numbers of clusters and (b) to find the optimum value of k. If the desired k is not known in advance, one will typically run k-means with different values of k, and then use a suitable criterion to select one of the results. For example, SAS uses the cube-clustering-criterion, while X-means adds a complexity term (which increases with k) to the original cost function (Eq. 1) and then identifies the k which minimizes this adjusted cost. Alternatively, one can progressively increase the number of clusters, in conjunction with a suitable stopping criterion. Bisecting k-means [10] achieves this by first putting all the data into a single cluster, and then recursively splitting the least compact cluster into two using 2-means. The celebrated LBG algorithm [8] used for vector quantization doubles the number of clusters till a suitable code-book size is obtained. Both these approaches thus alleviate the need to know k before-hand.The algorithm is also sensitive to the presence of outliers, since “mean” is not a robust statistic. A preprocessing step to remove outliers can be helpful. Post-processing the results, for example to eliminate small clusters, or to merge close clusters into a large cluster, is also desirable. Ball and Hall’s ISODATA algorithm from 1967 effectively used both pre- and post-processing on k-means.

5 Support Vector Machines

In today’s machine learning applications, support vector machines (SVM) [11] are considered amust try—it offers one of the most robust and accurate methods among all well-known algorithms. It has a sound theoretical foundation, requires only a dozen examples for training, and is insensitive to the number of dimensions. In addition, efficient methods for training SVM are also being developed at a fast pace. In a two-class learning task, the aim of SVM is to find the best classification function to distinguish between members of the two classes in the training data. The metric for the concept of the “best” classification function can be realized geometrically.

Because there are many such linear hyperplanes, what SVM additionally guarantee is that the best such function is found by maximizing the margin between the two classes. Intuitively, the margin is defined as the amount of space, or separation between the two classes as defined by the hyperplane. Geometrically, the margin corresponds to the shortest distance between the closest data points to a point on the hyperplane. Having this geometric definition allows us to explore how to maximize the margin, so that even though there are an infinite number of hyperplanes, only a few qualify as the solution to SVM.The reason why SVM insists on finding the maximum margin hyperplanes is that it offers the best generalization ability. It allows not only the best classification performance (e.g., accuracy) on the training data, but also leaves much room for the correct classification of the future data.

There are several important questions and related extensions on the above basic formulation of support vector machines. SVM can be easily extended to perform numerical calculations. The first is to extend SVM to perform regression analysis, where the goal is to produce a linear function that can approximate that target function. Careful consideration goes into the choice of the error models; in support vector regression, or SVR, the error is defined to be zero when the difference between actual and predicted values is within an epsilon amount. Otherwise, the epsilon insensitive error will grow linearly. The support vectors can then be learned through the minimization of the Lagrangian. An advantage of support vector regression is reported to be its insensitivity to outliers.

Another extension is to learn to rank elements rather than producing a classification for individual elements [12]. Ranking can be reduced to comparing pairs of instances and producing a +1 estimate if the pair is in the correct ranking order, and −1 otherwise. Thus, a way to reduce this task to SVM learning is to construct new instances for each pair of ranked instance in the training data, and to learn a hyperplane on this new training data. This method can be applied to many areas where ranking is important, such as in document ranking in information retrieval areas.

6 The Apriori algorithm

One of the most popular data mining approaches is to find frequent itemsets from a transaction dataset and derive association rules. Finding frequent itemsets (itemsets with frequency larger than or equal to a user specified minimum support) is not trivial because of its combinatorial explosion. Once frequent itemsets are obtained, it is straightforward to generate association rules with confidence larger than or equal to a user specified minimum confidence.Apriori is a seminal algorithm for finding frequent itemsets using candidate generation [13]. It is characterized as a level-wise complete search algorithm using anti-monotonicity of itemsets, “if an itemset is not frequent, any of its superset is never frequent”. By convention, Apriori assumes that items within a transaction or itemset are sorted in lexicographic order. Apriori first scans the database and searches for frequent itemsets of size 1 by accumulating the count for each item and collecting those that satisfy the minimum support requirement. It then iterates on the following three steps and extracts all the frequent itemsets.

Many of the pattern finding algorithms such as decision tree, classification rules and clustering techniques that are frequently used in data mining have been developed in machine learning research community. Frequent pattern and association rule mining is one of the few exceptions to this tradition. The introduction of this technique boosted data mining research and its impact is tremendous. The algorithm is quite simple and easy to implement. Experimenting with Apriori-like algorithm is the first thing that data miners try to do.

Since Apriori algorithm was first introduced and as experience was accumulated, there have been many attempts to devise more efficient algorithms of frequent itemset mining. Many of them share the same idea with Apriori in that they generate candidates. These include hash-based technique, partitioning, sampling and using vertical data format. Hash-based technique can reduce the size of candidate itemsets. Each itemset is hashed into a corresponding bucket by using an appropriate hash function. Since a bucket can contain different itemsets, if its count is less than a minimum support, these itemsets in the bucket can be removed from the candidate sets. A partitioning can be used to divide the entire mining problem into n smaller problems. The dataset is divided into n non-overlapping partitions such that each partition fits into main memory and each partition is mined separately. Since any itemset that is potentially frequent with respect to the entire dataset must occur as a frequent itemset in at least one of the partitions, all the frequent itemsets found this way are candidates, which can be checked by accessing the entire dataset only once. Sampling is simply to mine a random sampled small subset of the entire data. Since there is no guarantee that we can find all the frequent itemsets, normal practice is to use a lower support threshold. Trade off has to be made between accuracy and efficiency. Apriori uses a horizontal data format, i.e. frequent itemsets are associated with each transaction. Using vertical data format is to use a different format in which transaction IDs (TIDs) are associated with each itemset. With this format,taking the intersection of TIDs can perform mining. The support count is simply the length of the TID set for the itemset. There is no need to scan the database because TID set carries the complete information required for computing support.

The most outstanding improvement over Apriori would be a method called FP-growth (frequent pattern growth) that succeeded in eliminating candidate generation [14]. It adopts a divide and conquer strategy by (1) compressing the database representing frequent items into a structure called FP-tree (frequent pattern tree) that retains all the essential information and (2) dividing the compressed database into a set of conditional databases, each associated with one frequent itemset and mining each one separately. It scans the database only twice. In the first scan, all the frequent items and their support counts (frequencies) are derived and they are sorted in the order of descending support count in each transaction. In the second scan, items in each transaction are merged into a prefix tree and items (nodes) that appear in common in different transactions are counted. Each node is associated with an item and its count. Nodes with the same label are linked by a pointer called node-link. Since items are sorted in the descending order of frequency, nodes closer to the root of the prefix tree are shared by more transactions, thus resulting in a very compact representation that stores all the necessary information. Pattern growth algorithm works on FP-tree by choosing an item in the order of increasing frequency and extracting frequent itemsets that contain the chosen item by recursively calling itself on the conditional FP-tree. FP-growth is an order of magnitude faster than the original Apriori algorithm.There are several other dimensions regarding the extensions of frequent pattern mining. The major ones include the followings:

(1) incorporating taxonomy in items [15]: Use of Taxonomy makes it possible to extract frequent itemsets that are expressed by higher concepts even when use of the base level concepts produces only infrequent itemsets.
(2) incremental mining: In this setting, it is assumed that the database is not stationary and a new instance of transaction keeps added. The algorithm in [16] updates the frequent itemsets without restarting from scratch.
(3) using numeric valuable for item: When the item corresponds to a continuous numeric value, current frequent itemset mining algorithm is not applicable unless the values are discretized. A method of subspace clustering can be used to obtain an optimal value interval for each item in each itemset [17].
(4) using other measures than frequency, such as information gain or value: These measures are useful in finding discriminative patterns but unfortunately do not satisfy anti-monotonicity property. However, these measures have a nice property of being convex with respect to their arguments and it is possible to estimate their upperbound for supersets of a pattern and thus prune unpromising patterns efficiently. Apriori SMP uses this principle [18].
(5) using richer expressions than itemset:Many algorithms have been proposed for sequences, tree and graphs to enable mining from more complex data structure [19].

Closed itemsets: A frequent itemset is closed if it is not included in any other frequent itemsets. Thus, once the closed itemsets are found, all the frequent itemsets can be derived from them. LCM is the most efficient algorithm to find the closed itemsets [20].

7 Neutrosophic System

The first input parameter to the neutrosophic variable “the number Clusters” has three membership function (Y) , non-membership(N) and indeterminacy(I) of n is illustrated in Figure 1.

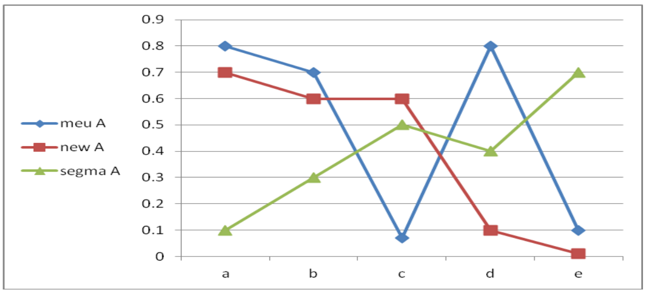

Figure 1: Membership function, non-membership and indeterminacy of neutrosophic set with variable n.

The input neutrosophic variable "the frequency of Subject Items" has the membership functions , non-membership and indeterminacy of f is showed in formulation (1)

$$f = \begin{cases} \text{yes} & \text{the key item is constant} \\ \text{no} & \text{the subject item is variable} \\ \text{Indetermin acy} & \text{non(yes, no)} \end{cases}$$

Formula 1

The output neutrosophic variable "Subject Items" has neutrosophic sets. It should be noted that modifying the membership functions, non-membership and indeterminacy will change the sensitivity of the neutrosophic logic system's output to its inputs. Also increasing the number of neutrosophic sets of the variables will provide better sensitivity control but also increases computational complexity of the system. Table 1 and Figure 2 show the rules used in the neutrosophic system.

Table 1: The neutrosophic system rules

Cluster	membership functions	non-membership	indeterminacy
1	Y	N	I
2	N	I	Y
3	I	Y	N
4	Y	N	I

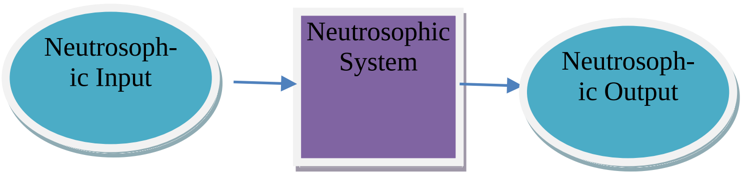

Figure 2: show the graph neutrosophic system.

The Neutrosophic System Equation Given by :

$$A \circ R = B \quad \text{Such That}$$

A : Represent Neutrosophic Data input for e-Learning System .

R : Represent Processing Neutrosophic System Data .

A :Represent Recommendation Subject for Students .

The output of that system determines the number of Subject Items Recommended. This determination is based on the NS analysis whish passes the three parameters of $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where $0^- \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1^+$ and $0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+$, then based on that analysis the system decides the accurate key size in each situation.

8 Conclusion and Future Work

In this paper, we presented the importance of social networks in e-Learning systems. Recommender systems play important roles in e-Learning as they help students to chose among different learning objects to study and activities to participate in. Among the different objects and activities available, recommender systems can chose between different algorithms. Presented algorithms in this paper are: C4.5, K-Means, Support Vector Machine, and Apriori algorithms. Each of those algorithms fit into a certain functionality of the recommender system. Future work will include comparison between other important machine learning algorithms, and our proposed e-Learning model that utilizes different machine learning algorithms for social network supported e-Learning. We have presented a proposed effective e-Learning system that utilizes a newly presented neutrosophic data set in analyzing social network data integrated in e-Learning. Identifying relationships between students is important for learning. Future work include incorporating the results we have achieved in customizing course contents to students, and recommending new learning objects more suitable for personalized learning.

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Misfire Fault Diagnosis Method of Gasoline Engines Using the Cosine Similarity Measure of Neutrosophic Numbers

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Abstract. This paper proposes a distance measure of neutrosophic numbers and a similarity measure based on cosine function, and then develops the misfire fault diagnosis method of gasoline engines by using the cosine similarity measure of neutrosophic numbers. In the fault diagnosis, by the cosine similarity measure between the fault knowledge (fault patterns) and required diagnosis-testing sample with neutrosophic number information and its relation indices, the proposed fault diagnosis method can indicate the main fault type and fault trends. Then, the misfire fault diagnosis results of gasoline engines demonstrate the effectiveness and rationality of the proposed fault diagnosis method. The proposed misfire fault diagnosis method not only gives the main fault types of the engine, but also provides useful information for future fault trends. The proposed method is effective and reasonable in the misfire fault diagnosis of gasoline engines.

Keywords: Neutrosophic number, distance measure, cosine similarity measure, misfire fault diagnosis, gasoline engine.

1 Introduction

Misfire fault is usually produced in gasoline engines [1]. However, it can descend its power, increase its fuel consumption and aggravate its pollution of exhaust emission when the burning quality of mixture gases descends in the combustion chamber of gasoline engines. Therefore, to keep better operating performance of the engine, we have to find out and eliminate the affected factors of low burning quality in the engine. Then, the exhaust emission in gasoline engines mainly contains the components of HC, NO_x, CO, CO_2, O_2, water vapor etc, which can affect the burning quality of mixture gases in the engine. Under different burning conditions in the engine, the content of the components can be changed in some range as the change of operating status or the occurrences of various mechanical and electronic faults in the engine. Hence, one can indicate the operating status of the engine by analyzing the change of exhaust emission content [1].

However, fault diagnosis is an important topic in engineering areas. In many real situations, the fault data cannot provide deterministic values because the fault testing data obtained by experts are usually imprecise or uncertain due to a lack of data, time pressure, measurement errors, or the experts' limited attention and knowledge. In real situations, the fault testing data usually contain the determinate information and the indeterminate information. While neutrosophic numbers proposed originally by Smarandache [2-4] may express it since a neutrosophic number consists of its determinate part and its indeterminate part. Therefore, it is a better tool for expressing incomplete and indeterminate information. The neutrosophic number can be represented as $N = a + bI$, which consists of its determinate part a and its indeterminate part bI. In the worst scenario, N can be unknown, i.e. $N = bI$. When there is no indeterminacy related to N, in the best scenario, there is only its determinate part $N = a$. Obviously, it is very suitable for the expression of incomplete and indeterminate information in fault diagnosis problems. Therefore, the neutrosophic number can effectively represent the fault data with incomplete and indeterminate information. Although the neutrosophic numbers have been defined in neutrosophic probability since 1996 [2], since then, little progress has been make for processing indeterminate problems by neutrosophic numbers in scientific and engineering applications. In order to break through the applied predicament, this paper proposes a distance measure of neutrosophic numbers and a similarity measure of neutrosophic numbers based on cosine function (so-called cosine similarity measure) for handling the misfire fault diagnosis problems of gasoline engines under neutrosophic number environment.

The remainder of the paper is organized as follows. In Section 2, we introduce some basic concepts related to neutrosophic numbers and some basic operational relations of neutrosophic numbers. Section 3 proposes a distance measure and a cosine similarity measure for neutrosophic numbers. Section 4 develops a fault diagnosis method using the cosine similarity measure for the misfire fault diagnosis problems of gasoline engines under neutrosophic number environment and demonstrates the effectiveness and rationality of the misfire fault diagnosis method. Section 5 gives the conclusions and future directions of research.

2 Neutrosophic numbers and their basic operational relations

Smarandache [2-4] firstly proposed a concept of a neutrosophic number, which consists of the determinate part and the indeterminate part. It is usually denoted as $N = a + bI$, where a and b are real numbers, and I is indeterminacy, such that $I^2 = I$, $0 \cdot I = 0$, and I/I = undefined.

For example, a neutrosophic number is $N = 3 + 2I$, If $I \in [0, 0.2]$, it is equivalent to $N \in [3, 3.4]$ for sure $N \geq 3$, this means that the determinate part of N is 3, while the indeterminate part of N is $2I$ and $I \in [0, 0.2]$, which means the possibility for number "N" to be a little bigger than 3.

Let $N_1 = a_1 + b_1I$ and $N_2 = a_2 + b_2I$ be two neutrosophic numbers. Then, Smarandache [2-4] gave the following operational relations of neutrosophic numbers:

(1) $N_1 + N_2 = a_1 + a_2 + (b_1 + b_2)I$;

(2) $N_1 - N_2 = a_1 - a_2 + (b_1 - b_2)I$;;

(3) $N_1 \times N_2 = a_1a_2 + (a_1b_2 + b_1a_2 + b_1b_2)I$;

(4) $N_1^2 = (a_1 + b_1I)^2 = a_1^2 + (2a_1b_1 + b_1^2)I$;

(5) $\frac{N_1}{N_2} = \frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} \cdot I$ for $a_2 \neq 0$ and $a_2 \neq -b_2$;

(6) $$\sqrt{N_1} = \sqrt{a_1 + b_1 I} = \begin{cases} \sqrt{a_1} - (\sqrt{a_1} + \sqrt{a_1 + b_1})I \\ \sqrt{a_1} - (\sqrt{a_1} - \sqrt{a_1 + b_1})I \\ -\sqrt{a_1} + (\sqrt{a_1} + \sqrt{a_1 + b_1})I \\ -\sqrt{a_1} + (\sqrt{a_1} - \sqrt{a_1 + b_1})I \end{cases}.$$

3 Distance measure and cosine similarity measure between neutrosophic numbers

In this section, we propose a distance measure of neutrosophic numbers and a similarity measure between neutrosophic numbers based on cosine function.

Definition 1. Let $A = \{N_{A1}, N_{A2}, \ldots, N_{An}\}$ and $B = \{N_{B1}, N_{B2}, \ldots, N_{Bn}\}$ be two sets of neutrosophic numbers, where $N_{Aj} = a_{Aj} + b_{Aj}I$ and $N_{Bj} = a_{Bj} + b_{Bj}I$ ($j = 1, 2, \ldots, n$) for a_{Aj}, b_{Aj}, a_{Bj}, $b_{Bj} \in R$ (R is all real numbers). Then, a distance measure between A and B is defined as

$$D(A,B) = \frac{1}{2n}\sum_{j=1}^{n}\begin{pmatrix} \left|a_{Aj} + \inf(b_{Aj}I) - a_{Bj} - \inf(b_{Bj}I)\right| \\ + \left|a_{Aj} + \sup(b_{Aj}I) - a_{Bj} - \sup(b_{Bj}I)\right| \end{pmatrix}. \quad (1)$$

Obviously, the distance measure should satisfy the following properties (D1-D3):

(D1) $D(A, B) \geq 0$;

(D2) $D(A, B) = 0$ if $A = B$;

(D3) $D(A, B) = D(B, A)$.

However, when we considers the importance of each element in the set of neutrosophic numbers, the weight of each element w_j ($j = 1, 2, \ldots, n$) can be introduced with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Thus, we have the following weighted distance measure between A and B:

$$D_w(A,B) = \frac{1}{2}\sum_{j=1}^{n} w_j \begin{pmatrix} \left|a_{Aj} + \inf(b_{Aj}I) - a_{Bj} - \inf(b_{Bj}I)\right| \\ + \left|a_{Aj} + \sup(b_{Aj}I) - a_{Bj} - \sup(b_{Bj}I)\right| \end{pmatrix}, \quad (2)$$

Obviously, the weighted distance measure also satisfies the above properties (D1-D3).

To easily apply neutrosophic numbers to fault diagnosis problems in this paper, we propose the similarity measure of neutrosophic numbers based on cosine function.

Definition 2. Let $A = \{N_{A1}, N_{A2}, \ldots, N_{An}\}$ and $B = \{N_{B1}, N_{B2}, \ldots, N_{Bn}\}$ be two sets of neutrosophic numbers, where $N_{Aj} = a_{Aj} + b_{Aj}I \subseteq [0, 1]$ and $N_{Bj} = a_{Bj} + b_{Bj}I \subseteq [0, 1]$ ($j = 1, 2, \ldots, n$) for a_{Aj}, b_{Aj}, a_{Bj}, $b_{Bj} \geq 0$. Then, a cosine similarity measure between A and B is defined as follows:

$$C(A,B) = \sum_{j=1}^{n} w_j \cos\left\{\frac{\pi}{4}\begin{pmatrix} \left|a_{Aj} + \inf(b_{Aj}I) - a_{Bj} - \inf(b_{Bj}I)\right| \\ + \left|a_{Aj} + \sup(b_{Aj}I) - a_{Bj} - \sup(b_{Bj}I)\right| \end{pmatrix}\right\}, \quad (3)$$

where $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Obviously, the cosine similarity measure should satisfy the following properties (P1-P3):

(P1) $0 \leq C(A, B) \leq 1$;

(P2) $C(A, B) = 1$ if $A = B$;

(P3) $C(A, B) = C(B, A)$.

4 Misfire fault diagnosis method of gasoline engines using the cosine similarity measure

4.1 Fault diagnosis method

For a fault diagnosis problem, assume that there are a set of m fault patterns (fault knowledge) $P = \{P_1, P_2, \ldots, P_m\}$ and a set of n characteristics (attributes) $Q = \{Q_1, Q_2, \ldots, Q_n\}$. Then the fault information of a fault pattern P_k ($k = 1, 2, \ldots, m$) with respect to a characteristic Q_j ($j = 1,$

2, …, n) is represented by a set of neutrosophic numbers $P_k = \{N_{k1}, N_{k2}, …, N_{kn}\}$, where $N_{kj} = a_{kj} + b_{kj}I \subseteq [0, 1]$ for a_{kj}, $b_{kj} \geq 0$ (k = 1, 2, …, m; j = 1, 2, …, n). Then, the information of a testing sample is represented by a set of neutrosophic numbers $P_t = \{N_{t1}, N_{t2}, …, N_{tn}\}$, where $N_{tj} = a_{tj} + b_{tj}I \subseteq [0, 1]$ for a_{tj}, $b_{tj} \geq 0$ (t = 1, 2, …, s; j = 1, 2, …, n).

The similarity measure value v_k (k = 1, 2, …, m) can be obtained by the following cosine similarity measure between P_t and P_k:

$$v_k = C(P_t, P_k) = \sum_{j=1}^{n} w_j \cos\left\{\frac{\pi}{4}\left(\begin{array}{l}|a_{tj} + \inf(b_{tj}I) - a_{kj} - \inf(b_{kj}I)| \\ + |a_{tj} + \sup(b_{tj}I) - a_{kj} - \sup(b_{kj}I)|\end{array}\right)\right\}. \quad (4)$$

For convenient fault diagnosis, the cosine values of v_k (k = 1, 2, …, m) are normalized into the relation indices within the interval [–1, 1] by the following formula:

$$\delta_k = \frac{2v_k - v_{\min} - v_{\max}}{v_{\max} - v_{\min}}, \quad (5)$$

where $v_{\max} = \max_{1\leq k\leq m}\{v_k\}$, $v_{\min} = \min_{1\leq k\leq m}\{v_k\}$ and $\delta_k \in [-1, 1]$.

Then, we can rank the relation indices and determine the fault type or predict possible fault trends for the tested equipment. If there is the maximum relation index $\delta_k = 1$, then we can determine that the testing sample P_t should belong to the fault pattern P_k.

4.2 Misfire fault diagnosis of gasoline engines

We apply the fault diagnosis method based on the cosine similarity measure to the misfire fault diagnosis of gasoline engines.

Let us investigate the misfire fault diagnosis problem of the gasoline engine EQ6102. Generally speaking, the misfire faults of the engine can be classified into three fault types: no misfire (normal work), slight misfire and severe misfire to indicate the operating status of the engine. Here, the slight misfire indicates the decline in the performance of ignition capacitance or the ignition delay, or the spark plug misfire in a cylinder of six cylinders, and then the severe misfire indicates the spark plug misfire in two cylinders of six cylinders. According to field-testing data [1], we can obtain the fault knowledge of the three fault types, i.e. a set of three fault patterns $P = \{P_1, P_2, P_3\}$ with respect to a set of five characteristics (five components) $Q = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$, as shown in Table 1.

Table 1 Three fault patterns of misfire faults for the engine EQ6102

P_k (Fault patterns)	Q_1 ($\varphi_{HC}\times10^{-2}$)	Q_2 (φ_{CO2})	Q_3 ($\varphi_{NOx}\times10$)	Q_4 ($\varphi_{CO}\times10^{-1}$)	Q_5 (φ_{O2})
P_1 (Normal work)	[0.03, 0.08]	[0.51, 0.93]	[0.03, 0.08]	[0.3, 0.5]	[0.062, 0.09]
P_2 (Slight misfire)	[0.01, 0.046]	[0.426, 0.84]	[0.04, 0.12]	[0.29, 0.5]	[0.04, 0.11]
P_3 (Severe misfire)	[0.2, 0. 5]	[0.3, 0.7]	[0.1, 0. 3]	[0.1, 0.3]	[0.07, 0.15]

Table 2 Fault knowledge expressed by neutrosophic numbers

P_k (Fault knowledge)	Q_1 ($\varphi_{HC}\times10^{-2}$)	Q_2 (φ_{CO2})	Q_3 ($\varphi_{NOx}\times10$)	Q_4 ($\varphi_{CO}\times10^{-1}$)	Q_5 (φ_{O2})
P_1 (Normal work)	0.03+1.7857I	0.51+15I	0.03+1.7857I	0.3+7.1429I	0.062+I
P_2 (Slight misfire)	0.01+1.2857I	0.426+14.7857I	0.04+2.8571I	0.29+7.5I	0.04+2.5I
P_3 (Severe misfire)	0.2+10.7143I	0.3+14.2857I	0.1+7.1429I	0.1+7.1429I	0.07+2.8571I

Table 3 Tasting samples of exhaust emission

Number of tasting samples (P_t)	Q_1 ($\varphi_{HC}\times10^{-2}$)	Q_2 (φ_{CO2})	Q_3 ($\varphi_{NOx}\times10$)	Q_4 ($\varphi_{CO}\times10^{-1}$)	Q_5 (φ_{O2})	Actual fault types
1	0.0455	0.047	0.033	0.48	0.0527	P_2
2	0.0572	0.075	0.062	0.42	0.0751	P_1
3	0.0261	0.065	0.086	0.453	0.0431	P_2
4	0.0312	0.062	0.051	0.287	0.1064	P_2
5	0.3761	0.045	0.139	0.179	0.1025	P_3
6	0.4220	0.052	0.188	0.194	0.0931	P_3
7	0.0189	0.081	0.091	0.459	0.0377	P_2
8	0.0555	0.086	0.057	0.39	0.0736	P_1
9	0.0551	0.085	0.050	0.386	0.0789	P_1

In Table 1, $\varphi_{HC}\times10^{-2}$, φ_{CO2}, $\varphi_{NOx}\times10$, $\varphi_{CO}\times10^{-1}$ and φ_{O2} in the characteristic set $Q = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$ indicate the exhaust emission concentration of the five components HC, CO_2, NO_x, CO and O_2 expressed by volume percentage, and also we can consider the characteristic values of Q_j (j = 1, 2, 3, 4, 5) as interval values. Then, the interval values can be transformed into the neutrosophic numbers by the indeterminacy $I \in [0, 0.028]$, which express the characteristics of Q_j ($j = 1, 2, \ldots, n$), as shown in Table 2.

To illustrate the effectiveness of the misfire fault diagnosis of the engine, we introduce the nine sets of field-testing samples for the engine EQ6102 from [1], which are shown in Table 3.

Then, the importance of the five characteristics (five components) is considered by the weight vector $\boldsymbol{W} = (w_1, w_2, w_3, w_4, w_5)^T = (0.05, 0.35, 0.3, 0.2, 0.1)^T$ [1]. By using Eqs. (4) and (5), the diagnosis results are shown in Table 4. From Tables 3 and 4, the fault diagnosis results are in accordance with all the actual fault types.

Meanwhile, it is very easy to diagnose or predict fault types of the engine EQ6102 from Table 4. For example, for Number 9, since the relation index of P_1 is equal to 1, it indicates the fault type P_1 (no misfire). Then one can predict that the engine has the slight misfire trend since the relation index of P_2 is 0.923 and the fault type P_3 has a very low possibility of severe misfire due to the negative relation index (–1). Similarly, one can also diagnose and predict fault types according to the relation indices for other testing samples in Table 4. Therefore, the proposed fault diagnosis method for the engine can not only diagnose the main fault type but also predict the future fault trend by the relation indices. Compared with the fault diagnosis method for the engine in [1], the fault diagnosis method proposed in this paper is simpler and easier than the fault diagnosis method by using extension set theory [1].

Table 4 Results of the relation indices and fault diagnoses

Number of tasting samples (P_t)	Relation indices (δ_k)			Fault diagnosis results
	P_1	P_2	P_3	
1	0.5135	1.0000	–1.0000	P_2
2	1.0000	0.9850	–1.0000	P_1
3	0.9957	1.0000	–1.0000	P_2
4	0.9291	1.0000	–1.0000	P_2
5	–1.0000	–0.0077	1.0000	P_3
6	–1.0000	–0.7964	1.0000	P_3
7	0.9903	1.0000	–1.0000	P_2
8	1.0000	0.8578	–1.0000	P_1
9	1.0000	0.9230	–1.0000	P_1

5 Conclusion

This paper proposed a distance measure and a cosine similarity measure between neutrosophic numbers. Then, the fault diagnosis method based on the cosine similarity measure was proposed and was applied to the misfire fault diagnosis of gasoline engines under neutrosophic number environment. The fault diagnosis results of the engine demonstrated the effectiveness and rationality of the proposed fault diagnosis method. This fault diagnosis method can not only determinate the main fault type of engines but also predict future fault trends according to the relation indices, and then it is simpler and easier than the fault diagnosis method based on extension theory. The method proposed in this paper extends existing fault diagnosis methods and provides a useful way for fault diagnoses of gasoline engines. In the future, the developed diagnosis method will be extended to other fault diagnoses, such as vibration faults of turbines, aircraft engines and gearboxes.

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Cosine Similarity Measure Based Multi-attribute Decision-making with Trapezoidal Fuzzy Neutrosophic Numbers

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Abstract. The objective of the study is to present cosine similarity measure based multi-attribute decision making under neutrosophic environment. The assesments of alternatives over the attributes are expressed with trapezoidal fuzzy neutrosophic numbers in which the three independent components namely, truth-membership degree (T), indeterminacy-membership degree (I) and falsity-membership degree (F) are expressed by trapezoidal fuzzy numbers. Cosine similarity measure between two trapezoidal fuzzy neutrosophic numbers and its properties are introduced. Expected value of trapezoidal fuzzy neutrosophic number is defined to determine the attribute weight. With these attribute weights, weighted cosine similarity measure between relative positive ideal alternative and each alternative is determined to find out the best alternative in multi-attribute decision-making problem. Finally, a numerical example is provided to illustrate the proposed approach.

Keywords: Neutrosophic set,Single-valued neutrosophic set,Trapezoidal fuzzy neutrosophic number, Expected value, Cosine similarity measure, Multi-attribute decision making

1 Introduction

Multiple attribute decision-making (MADM) is a process of finding the best option from all the feasible alternatives. In classical MADM methods [1, 2, 3, 4], the ratings and the weights of the attributes are described by crisp values. However, under many conditions, crisp data are inadequate to model real-life situations since human judgments including preferences are often vague and cannot be estimated with an exact numerical value. A more realistic approach may lead to use linguistic assessments instead of exact numerical values i.e. the ratings and weights of the criteria in the problem may be presented by means of linguistic variables. These characteristics indicate the applicability of fuzzy set introduced by Zadeh [5], intuitionistic fuzzy set studied by Atanassov [6] and neutrosophic set pioneered by Smarandache [7] in capturing the decision makers' judgement. However, neutrosophic set [8, 9] generlizes the crisp set [10, 11], fuzzy set [5], intuitionistic fuzzy set [6] and other extension of fuzzy sets. Wang et al. [12] introduced the concept of single valued neutrosophic set from practical point of view. The single valued neutrosophic set consists of three independent membership functions, namely, truth-membership function, indeterminacy-membership function, and falsity-membership function. It is capable of dealing with incomplete, indeterminate, and inconsistent information. The concept of single valued neutrosophic set has been studied and applied in different fields including decision making problems [13, 14, 15, 16, 17, 18, 19, 20, 21].

Several similarity measures in neutrosophic environment have been studied by researchers in the literature. Broumi and Smarandache [22] proposed the Hausdorff distance between neutrosophic sets and some similarity measures based on the Hausdorff distance, set theoretic approach, and matching function to determine the similarity degree between neutrosophic sets. Based on the study of Bhattacharya's distance [23], Broumi and Smarandache [24] proposed cosine similarity measure and established that their proposed similarity measure is more efficient and robust than the existing similarity measures. Pramanik and Mondal [25] proposed cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis.

Majumdar and Samanta [26] developed several similarity measures of single valued neutrosophic sets (SVNSs) based on distances, a maching function, memebership grades, and then proposed an entropy measure for a SVNS. Ye and Zhang [27] proposed three new similarity measures between SVNSs based on the minimum and maximum operators and developed a multiple attribute decision making method based on the weighted similarity measure of SVNSs under single valued neutrosophic environment.
Ye [28] defined generalized distance measure between SVNSs and proposed two distance-based similarity

measures of SVNSs. In the same study, Ye [28] presented a clustering algorithm based on the similarity measures of SVNSs to cluster single-valued neutrosophic data.

Ye [29] also presented the Hamming and Euclidean distances between interval neutrosophic sets (INSs) and their similarity measures and applied them to multiple attribute decision–making problems with interval neutrosophic information. Ye [30] developed three vector similarity measure for SNSs, interval valued neutrosophic sets including the Jaccard [31], Dice [32], and cosine similarity measures [33] for SVNS and INSs and applied them to multicriteria decision-making problems with simplified neutrosophic information. Ye [34] further proposed improved cosine similarity measure of SVNSs and applied it to medical diagnosis with single valued neutrosophic information. Recently, Ye [35] proposed trapezoidal fuzzy neutrosophic number weighted arithmetic averaging (TFNNWAA) operator and a trapezoidal fuzzy neutrosophic number weighted geometric averaging (TFNNWGA) operator to aggregate the trapezoidal fuzzy neutrosophic information. Based on the TFNNWAA and TFNNWGA operators and the score and accuracy functions of a trapezoidal fuzzy neutrosophic numbers, Ye [35] proposed multiple attribute decision making in which the evaluated values of alternatives on the attributes are represented by the form of trapezoidal fuzzy neutrosophic numbers. However, cosine similarity based multiattribute decision making with trapezoidal fuzzy neutrosophic information is yet to appear in the literature.

In this paper, we propose a new approach called “Cosine similarity based multi-attribute decision making with trapezoidal fuzzy neutrosophic numbers”. The expected interval and the expected value theorem for trapezoidal fuzzy neutrosophic numbers are established. Cosine similarity measure of trapezoidal fuzzy neutrosophic numbers is also established.

The rest of the paper is organized as follows: Section 2 briefly presents some preliminaries regarding neutrosophic set and single-valued neutrosophic set. In Section 3, definitions of trapezoidal fuzzy neutrosophic number and some operational laws are studied. Section 4 is confined to define the cosine similarity measure between two trapezoidal fuzzy neutrosophic numbers and its properties. Section 5 is devoted to present the cosine similarity measure based multi-attribute decision making with trapezoidal fuzzy neutrosophic numbers. Section 6 represents an illustrative example that shows the effectiveness and applicability of the proposed approach. Finally, section 7 presents the direction of future research and concluding remarks.

2 Some Preliminaries

In this section, we review some basic definitions and concepts that are used to develop the paper.

Definition 1 Let $\boldsymbol{X} = (x_1, x_2, ..., x_n)$ and $\boldsymbol{Y} = (y_1, y_2, ..., y_n)$ be two n-dimensional vectors with positive components. The cosine [33] of two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ is the inner product of $\boldsymbol{X}$ and $\boldsymbol{Y}$ divided by the products of their lengths and it can be defined as

$$Cos(X,Y) = \frac{X.Y}{\|X\|_2\|Y\|_2} \qquad (1)$$

satisfying the following properties

i. $0 \le Cos(X,Y) \le 1$;

ii. $Cos(X,Y) = Cos(Y,X)$;

iii. $Cos(X,Y) = 1$, if $X = Y$ i.e. $x_i = y_i$ for i = 1, 2, ..., n.

Definition 2 A fuzzy set [5] $\tilde{A}$ in a universe of discourse X is defined by $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle \mid x \in X\}$, where $\mu_{\tilde{A}}(x): X \to [0, 1]$ is called the membership function of $\tilde{A}$ and $\mu_{\tilde{A}}(x)$ is the degree of membership to which $x \in \tilde{A}$.

Definition 3 A fuzzy set [5] $\tilde{A}$ defined on the universal set of real number R is said to be a fuzzy number if its membership function has the following characteristics.

i. $\tilde{A}$ is convex i.e. for any $x_1, x_2 \in X$ the membership function $\mu_{\tilde{A}}(x)$ satisfies the inequality $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$ for $0 \le \lambda \le 1$.

ii. $\tilde{A}$ is normal i.e., if there exists at least one point $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$

iii. $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Definition 4 A trapezoidal fuzzy number [36] $\tilde{A}$ is denoted by (a_1, a_2, a_3, a_4), where, a_1, a_2, a_3, a_4 are real numbers and its membership function $\mu_{\tilde{A}}(x)$ is defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} f(x) = \dfrac{x - a_1}{a_2 - a_1} & a_1 \le x \le a_2; \\ 1 & a_2 \le x \le a_3; \\ g(x) = \dfrac{a_4 - x}{a_4 - a_3} & a_3 \le x \le a_4; \\ 0 & otherwise. \end{cases}$$

Then, $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

1. $\mu_{\tilde{A}}(x)$ is a continuous mapping from R to closed interval [0,1],
2. $\mu_{\tilde{A}}(x) = 0$ for every $x \in (-\infty, a_1]$,
3. $\mu_{\tilde{A}}(x)$ is strictly increasing and continuous on $[a_1, a_2]$,
4. $\mu_{\tilde{A}}(x) = 1$ for every $x \in [a_2, a_3]$,

5. $\mu_{\tilde{A}}(x)$ is strictly decreasing and continuous on [a_3, a_4],
6. $\mu_{\tilde{A}}(x) = 0$ for every x ∈ [a_4, ∞).

The trapezoidal fuzzy number reduces to a triangular fuzzy number if $a_2 = a_3$.

Definition 5 The expected interval and the expected value of fuzzy number [37] $\tilde{A}$ are expressed as follows:

$$\mathrm{EI}(\tilde{A}) = [E(\tilde{A}^L), E(\tilde{A}^U)] \quad (2)$$

$$\mathrm{EV}(\tilde{A}) = \left(E(\tilde{A}^L), E(\tilde{A}^U)\right)/2 \quad (3)$$

where $E(\tilde{A}^L) = a_2 - \int_{a_1}^{a_2} f(x)dx$ and

$$E(\tilde{A}^U) = a_3 + \int_{a_3}^{a_4} g(x)dx.$$

In case of the trapezoidal fuzzy number the expected interval and the expected value of $\tilde{A}$ = (a_1, a_2, a_3, a_4) can be obtained by using the equations (2) and (3) as follows:

$$\mathrm{EI}(\tilde{A}) = \left[\frac{(a_1 + a_2)}{2}, \frac{(a_3 + a_4)}{2}\right] \quad (4)$$

$$\mathrm{EV}(\tilde{A}) = \frac{(a_1 + a_2 + a_3 + a_4)}{4} \quad (5)$$

Definition 6 Cosine similarity measure [33] is defined as the inner product of two vectors divided by the product of their lengths. It is the cosine of the angle between the vector representations of the two fuzzy sets.

Let us assume that A = $\left(\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n)\right)$ and B = $\left(\mu_B(x_1), \mu_B(x_2), ..., \mu_B(x_n)\right)$ are two fuzzy sets in the universe of discourse X = {x_1, x_2,..., x_n}. Then the cosine similarity measure of $\mu_A(x_i)$ and $\mu_B(x_i)$ is

$$C_{Fuzz}(\tilde{A}, \tilde{B}) = \frac{\sum_{i=1}^{n}\left(\mu_A(x_i)\,\mu_B(x_i)\right)}{\sqrt{\sum_{i=1}^{n}\left(\mu_A^2(x_i)\right)}\sqrt{\sum_{i=1}^{n}\left(\mu_B^2(x_i)\right)}} \quad (6)$$

It satisfies the following properties:

i) $0 \le C_{Fuzz}(\tilde{A}, \tilde{B}) \le 1$

ii) $C_{Fuzz}(\tilde{A}, \tilde{B}) = C_{Fuzz}(\tilde{B}, \tilde{A})$

iii) $C_{Fuzz}(\tilde{A}, \tilde{B}) = 1$ if $\tilde{A} = \tilde{B}$.

The value of $C_{Fuzz}(\tilde{A}, \tilde{B})$ is considered zero if $\mu_{\tilde{A}}(x) = 0$ and $\mu_{\tilde{B}}(x) = 0$.

Definition 7 Cosine similarity measure of trapezoidal fuzzy numbers [38]

Let $\tilde{A}$ = (a_1, a_2, a_3, a_4) and $\tilde{B}$ = (b_1, b_2, b_3, b_4) be two trapezoidal fuzzy numbers in the set of real numbers ℝ. The four parameters presented in two numbers $\tilde{A}$ and $\tilde{B}$ can be considered as the vector representations of four elements. Thus the cosine similarity measure of $\tilde{A}$ and $\tilde{B}$ can be defined as the extension of the cosine similarity measure of fuzzy sets as follows:

$$C_{TRFN}(\tilde{A}, \tilde{B}) = \frac{\sum_{i=1}^{n} a_i . b_i}{\sqrt{\sum_{i=1}^{n}(a_i)^2}\sqrt{\sum_{i=1}^{n}(b_i)^2}} \quad (7)$$

It satisfies the following properties:

i) $0 \le C_{TRFN}(\tilde{A}, \tilde{B}) \le 1$

ii) $C_{TRFN}(\tilde{A}, \tilde{B}) = C_{TRFN}(\tilde{B}, \tilde{A})$

iii) $C_{TRFN}(\tilde{A}, \tilde{B}) = 1$, if $\tilde{A} = \tilde{B}$ i.e. $a_i = b_i$ for i = 1, 2, 3, 4.

2.1 Some basic concepts of neutrosophic set

Definition 8

Let *X* be a space of points (objects) with generic element x. Then a neutrosophic set [7] A in *X* is characterized by a truth membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A. The functions T_A, I_A and F_A are real standard or non-standard subsets of $]^-0, 1^+[$ that is $T_A : X \to]^-0, 1^+[$; $I_A : X \to]^-0, 1^+[$; $F_A : X \to]^-0, 1^+[$

$T_A(x)$, $I_A(x)$, and $F_A(x)$ satisfy the relation

i.e. $^-0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$

Definition 9 The complement [7] A^c of a neutrosophic set *A* is defined as follows:

$T_{A^c}(x) = \{1^+\} - T_A(x)$; $I_{A^c}(x) = \{1^+\} - I_A(x)$;

$F_{A^c}(x) = \{1^+\} - F_A(x)$.

Definition 10 A neutrosophic set [7] *A* is contained in other neutrosophic set *B* i.e., $A \subseteq B$ if and only if the following results hold good.

$\inf T_A(x) \le \inf T_B(x)$, $\sup T_A(x) \le \sup T_B(x)$

$\inf I_A(x) \ge \inf I_B(x)$, $\sup I_A(x) \ge \sup I_B(x)$

$\inf F_A(x) \ge \inf F_B(x)$, $\sup F_A(x) \ge \sup F_B(x)$

for all x in *X*.

Definition 11. Let *X* be a universal space of points (objects), with a generic element $x \in X$. A single-valued neutrosophic set [12] $\tilde{\mathcal{N}} \subset X$ is characterized by a true membership function $T_{\tilde{\mathcal{N}}}(x)$, a falsity membership function

$F_{\tilde{N}}(x)$ and an indeterminacy membership function $I_{\tilde{N}}(x)$ with $T_{\tilde{N}}(x)$, $I_{\tilde{N}}(x)$, $F_{\tilde{N}}(x) \in [0, 1]$ for all $x \in X$. For a SVNS $\tilde{\mathcal{N}}$, the relation

$$0 \le \sup T_{\tilde{N}}(x) + \sup I_{\tilde{N}}(x) + \sup F_{\tilde{N}}(x) \le 3 \quad (8)$$

holds for $\forall x \in X$.

When X is continuous SVNSs, $\tilde{\mathcal{N}}$ can be written as follows:

$$\tilde{\mathcal{N}} = \int_x \left\langle T_{\tilde{\mathcal{N}}}(x), I_{\tilde{\mathcal{N}}}(x), F_{\tilde{\mathcal{N}}}(x) \right\rangle \big/ x, \ \ \forall x \in X.$$

and when X is discrete a SVNSs $\tilde{\mathcal{N}}$ can be written as follows:

$$\tilde{\mathcal{N}} = \sum_{i=1}^{m} \left\langle T_{\tilde{\mathcal{N}}}(x), I_{\tilde{\mathcal{N}}}(x), F_{\tilde{\mathcal{N}}}(x) \right\rangle / x, \ \ \forall x \in X.$$

$T_{\tilde{\mathcal{N}}}(x)$, $I_{\tilde{\mathcal{N}}}(x)$, $F_{\tilde{\mathcal{N}}}(x) \in [0, 1]$

Definition 12 The complement $\tilde{\mathcal{N}}^c$ of a single-valued neutrosophic set [12] is defined as follows:

$T_{\tilde{\mathcal{N}}^c}(x) = F_{\tilde{\mathcal{N}}}(x)$; $I_{\tilde{\mathcal{N}}^c}(x) = 1 - I_{\tilde{\mathcal{N}}}(x)$; $F_{\tilde{\mathcal{N}}^c}(x) = T_{\tilde{\mathcal{N}}}(x)$

Definition 13 A single-valued neutrosophic set [12] $\tilde{\mathcal{N}}_A$ is contained in $\tilde{\mathcal{N}}_B$ i.e., $\tilde{\mathcal{N}}_A \subseteq \tilde{\mathcal{N}}_B$, if and only if

$T_{\tilde{\mathcal{N}}_A}(x) \le T_{\tilde{\mathcal{N}}_B}(x)$; $I_{\tilde{\mathcal{N}}_A}(x) \ge I_{\tilde{\mathcal{N}}_B}(x)$; $F_{\tilde{\mathcal{N}}_A}(x) \ge F_{\tilde{\mathcal{N}}_B}(x)$ for $\forall x \in X$.

Definition 14 Two SVNSs [12] $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ are equal, i.e. $\tilde{\mathcal{N}}_A = \tilde{\mathcal{N}}_B$, if and only if $\tilde{\mathcal{N}}_A \subseteq \tilde{\mathcal{N}}_B$ and $\tilde{\mathcal{N}}_A \supseteq \tilde{\mathcal{N}}_B$.

Definition 15 The union of two SVNSs [12] $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ is a SVNS $\tilde{\mathcal{N}}_C$, denoted as $\tilde{\mathcal{N}}_C = \tilde{\mathcal{N}}_A \cup \tilde{\mathcal{N}}_B$. Its truth membership, indeterminacy-membership and falsity membership functions are related to those of $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ as follows:

$T_{\tilde{\mathcal{N}}_C}(x) = \max(T_{\tilde{\mathcal{N}}_A}(x), T_{\tilde{\mathcal{N}}_B}(x))$;

$I_{\tilde{\mathcal{N}}_C}(x) = \max(I_{\tilde{\mathcal{N}}_A}(x), I_{\tilde{\mathcal{N}}_B}(x))$;

$F_{\tilde{\mathcal{N}}_C}(x) = \min(F_{\tilde{\mathcal{N}}_A}(x), F_{\tilde{\mathcal{N}}_B}(x))$ $\forall x \in X$.

Definition 16 The intersection of two SVNSs [12] $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ is denoted as a SVNS $\tilde{\mathcal{N}}_C = \tilde{\mathcal{N}}_A \cap \tilde{\mathcal{N}}_B$, where truth membership, indeterminacy-membership and falsity membership functions are defined as follows:

$T_{\tilde{\mathcal{N}}_C}(x) = \min(T_{\tilde{\mathcal{N}}_A}(x), T_{\tilde{\mathcal{N}}_B}(x))$;

$I_{\tilde{\mathcal{N}}_C}(x) = \min(I_{\tilde{\mathcal{N}}_A}(x), I_{\tilde{\mathcal{N}}_B}(x))$;

$F_{\tilde{\mathcal{N}}_C}(x) = \max(F_{\tilde{\mathcal{N}}_A}(x), F_{\tilde{\mathcal{N}}_B}(x))$ for all x in X.

Definition 17 The addition of two SVNSs [12] $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ is a SVNS $\tilde{\mathcal{N}}_C = \tilde{\mathcal{N}}_A \oplus \tilde{\mathcal{N}}_B$, whose three membership degrees related to $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ are defined as follows:

$T_{\tilde{\mathcal{N}}_C}(x) = T_{\tilde{\mathcal{N}}_A}(x) + T_{\tilde{\mathcal{N}}_B}(x) - T_{\tilde{\mathcal{N}}_A}(x).T_{\tilde{\mathcal{N}}_B}(x)$;

$I_{\tilde{\mathcal{N}}_C}(x) = I_{\tilde{\mathcal{N}}_A}(x).I_{\tilde{\mathcal{N}}_B}(x)$; $F_{\tilde{\mathcal{N}}_C}(x) = F_{\tilde{\mathcal{N}}_A}(x).F_{\tilde{\mathcal{N}}_B}(x)$ $\forall x \in X$.

Definition 18 The multiplication of two SVNSs [12] $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ is a SVNS $\tilde{\mathcal{N}}_C = \tilde{\mathcal{N}}_A \otimes \tilde{\mathcal{N}}_B$, whose three membership degrees related to $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ are defined as follows:

$T_{\tilde{\mathcal{N}}_C}(x) = T_{\tilde{\mathcal{N}}_A}(x).T_{\tilde{\mathcal{N}}_B}(x)$;

$I_{\tilde{\mathcal{N}}_C}(x) = I_{\tilde{\mathcal{N}}_A}(x) + I_{\tilde{\mathcal{N}}_B}(x) - I_{\tilde{\mathcal{N}}_A}(x).I_{\tilde{\mathcal{N}}_B}(x)$;

$F_{\tilde{\mathcal{N}}_C}(x) = F_{\tilde{\mathcal{N}}_A}(x) + F_{\tilde{\mathcal{N}}_B}(x) - F_{\tilde{\mathcal{N}}_A}(x).F_{\tilde{\mathcal{N}}_B}(x)$ $\forall x \in X$.

3 Trapezoidal Fuzzy Neutrosophic Number

Definition 19 A neutrosophic set $\tilde{\mathcal{N}}_A$ in a universe of discourse X is defined in the following form:

$\tilde{\mathcal{N}}_A = \left\{ \left\langle T_{\tilde{\mathcal{N}}_A}(x), I_{\tilde{\mathcal{N}}_A}(x), F_{\tilde{\mathcal{N}}_A}(x) \right\rangle \mid x \in X \right\}$ where, truth membership degree $T_{\tilde{\mathcal{N}}_A}(x): X \in [0, 1]$, indeterminacy membership degree $I_{\tilde{\mathcal{N}}_A}(x): X \in [0, 1]$ and falsity membership degree $F_{\tilde{\mathcal{N}}_A}(x): X \in [0, 1]$. Fuzzy neutrosophic number can be defined by extending a discrete set to a contious set.

Let $\tilde{\mathcal{N}}_A$ be a fuzzy neutrosophic number in the set of real numbers R. Then its truth membership function can be defined as follows:

$$T_{\tilde{\mathcal{N}}_A}(x) = \left\langle \begin{matrix} T^L_{\tilde{\mathcal{N}}_A}(x) & a_{11} \le x \le a_{21} \\ 1 & a_{21} \le x \le a_{31} \\ T^U_{\tilde{\mathcal{N}}_A}(x) & a_{31} \le x \le a_{41} \\ 0 & otherwise \end{matrix} \right\rangle \quad (9)$$

The indeterminacy membership function can be defined as follows:

$$I_{\tilde{\mathcal{N}}_A}(x) = \left\langle \begin{matrix} I^L_{\tilde{\mathcal{N}}_A}(x) & b_{11} \le x \le b_{21} \\ 0 & b_{21} \le x \le b_{31} \\ I^U_{\tilde{\mathcal{N}}_A}(x) & b_{31} \le x \le b_{41} \\ 1 & otherwise \end{matrix} \right\rangle \quad (10)$$

and the falsity membership function can be defined as follows:

$$F_{\tilde{\mathcal{N}}_A}(x) = \left\langle \begin{matrix} F^L_{\tilde{\mathcal{N}}_A}(x) & c_{11} \le x \le c_{21} \\ 0 & c_{21} \le x \le c_{31} \\ F^U_{\tilde{\mathcal{N}}_A}(x) & c_{31} \le x \le c_{41} \\ 1 & otherwise \end{matrix} \right\rangle \quad (11)$$

where $0 \le \sup T_{\tilde{\mathcal{N}}_A}(x) + \sup I_{\tilde{\mathcal{N}}_A}(x) + \sup F_{\tilde{\mathcal{N}}_A}(x) \le 3$, $\forall x \in X$ and $a_{11}, a_{21}, a_{31}, a_{41}, b_{11}, b_{21}, b_{31}, b_{41}, c_{11}, c_{21}, c_{31}, c_{41} \in \mathbf{R}$ such that $a_{11} \le a_{21} \le a_{31} \le a_{41}$, $b_{11} \le b_{21} \le b_{31} \le b_{41}$ and $c_{11} \le c_{21} \le c_{31} \le c_{41}$. $T^L_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$, $I^U_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$, and $F^U_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$ are continuous monotonic increasing functions and $T^U_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$, $I^L_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$, and $I^L_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$ are continuous monotonic decreasing functions.

Definition 20 (Trapezoidal Fuzzy Neutrosophic Number) A trapezoidal fuzzy neutrosophic number (TrFNN) [35] $\tilde{\mathcal{N}}_A$ is denoted by $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ in a universe of discourse *X*. The parameters satisfy the following relations $a_1 \le a_2 \le a_3 \le a_4$, $b_1 \le b_2 \le b_3 \le b_4$ and $c_1 \le c_2 \le c_3 \le c_4$. Its truth membership function is defined as follows:

$$T_{\tilde{\mathcal{N}}_A}(x) = \left\langle \begin{matrix} \dfrac{x - a_1}{a_2 - a_1} & a_1 \le x \le a_2 \\ 1 & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & a_3 \le x \le a_4 \\ 0 & otherwise \end{matrix} \right\rangle \quad (12)$$

Its indeterminacy membership function is defined as follows:

$$I_{\tilde{\mathcal{N}}_A}(x) = \left\langle \begin{matrix} \dfrac{b_2 - x}{b_2 - b_1} & b_1 \le x \le b_2 \\ 0 & b_2 \le x \le b_3 \\ \dfrac{x - b_3}{b_4 - b_3} & b_3 \le x \le b_4 \\ 1 & otherwise \end{matrix} \right\rangle \quad (13)$$

and its falsity membership function is defined as follows:

$$F_{\tilde{\mathcal{N}}_A}(x) = \left\langle \begin{matrix} \dfrac{c_2 - x}{c_2 - c_1} & c_1 \le x < c_2 \\ 0 & c_2 \le x \le c_3 \\ \dfrac{x - c_3}{c_4 - c_3} & c_3 < x \le c_4 \\ 1 & otherwise \end{matrix} \right\rangle \quad (14)$$

3.1 Some operational rules of trapezoidal fuzzy neutrosophic numbers.

Let $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ and $\tilde{\mathcal{N}}_B = \langle (e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4), (g_1, g_2, g_3, g_4) \rangle$ be two TrFNNs in the set of real numbers **R**. Then the operation rules [35] for $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ are presented as follows:

1. $$\tilde{\mathcal{N}}_A \oplus \tilde{\mathcal{N}}_B = \left\langle \begin{matrix} \begin{pmatrix} a_1 + e_1 - a_1 e_1, a_2 + e_2 - a_2 e_2, \\ a_3 + e_3 - a_3 e_3, a_4 + e_4 - a_4 e_4 \end{pmatrix}, \\ (b_1 f_1, b_2 f_2, b_3 f_3, b_4 f_4), \\ (c_1 g_1, c_2 g_2, c_3 g_3, c_4 g_4) \end{matrix} \right\rangle \quad (15)$$

2. $$\tilde{\mathcal{N}}_A \otimes \tilde{\mathcal{N}}_B = \left\langle \begin{matrix} (a_1 e_1, a_2 e_2, a_3 e_3, a_4 e_4), \\ \begin{pmatrix} b_1 + f_1 - b_1 f_1, b_2 + f_2 - b_2 f_2, \\ b_3 + f_3 - b_3 f_3, b_4 + f_4 - b_4 f_4 \end{pmatrix}, \\ \begin{pmatrix} c_1 + g_1 - c_1 g_1, c_2 + g_2 - c_2 g_2, \\ c_3 + g_3 - c_3 g_3, c_4 + g_4 - c_4 g_4 \end{pmatrix} \end{matrix} \right\rangle \quad (16)$$

3. $$\lambda \tilde{\mathcal{N}}_A = \left\langle \begin{matrix} \begin{pmatrix} 1 - (1 - a_1)^\lambda, 1 - (1 - a_2)^\lambda, \\ 1 - (1 - a_3)^\lambda, 1 - (1 - a_4)^\lambda \end{pmatrix}, \\ (b_1^\lambda, b_2^\lambda, b_3^\lambda, b_4^\lambda), (c_1^\lambda, c_2^\lambda, c_3^\lambda, c_4^\lambda) \end{matrix} \right\rangle \quad (17)$$
for $\lambda > 0$;

4. $$(\tilde{\mathcal{N}}_A)^\lambda = \left\langle \begin{matrix} (a_1^\lambda, a_2^\lambda, a_3^\lambda, a_4^\lambda) \\ \begin{pmatrix} 1 - (1 - b_1)^\lambda, 1 - (1 - b_2)^\lambda, \\ 1 - (1 - b_3)^\lambda, 1 - (1 - b_4)^\lambda \end{pmatrix} \\ \begin{pmatrix} 1 - (1 - c_1)^\lambda, 1 - (1 - c_2)^\lambda, \\ 1 - (1 - c_3)^\lambda, 1 - (1 - c_4)^\lambda \end{pmatrix}, \end{matrix} \right\rangle \quad (18)$$
for $\lambda > 0$.

5. $\tilde{\mathcal{N}}_A = \tilde{\mathcal{N}}_B$ if $a_i = e_i$, $b_i = f_i$ and $c_i = g_i$ hold for i = 1, 2, 3, 4 i.e. $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$, $(b_1, b_2, b_3, b_4) = (f_1, f_2, f_3, f_4)$ and $(c_1, c_2, c_3, c_4) = (g_1, g_2, g_3, g_4)$.

3.2 Expected value of trapezoidal fuzzy neutrosophic number

Let $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ be the TrFNN characterized by three independent membership degrees in the set of real numbers **R** where, $T_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$ be the truth membership degree, $I_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$ be the indeterminacy degree and $F_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$ be the falsity membership degree such that the following relation holds.

$0 \le \sup T_{\tilde{\mathcal{N}}_A}(x) + \sup I_{\tilde{\mathcal{N}}_A}(x) + \sup F_{\tilde{\mathcal{N}}_A}(x) \le 3.$

It is also assumed that

$T^L_{\tilde{\mathcal{N}}_A}(x) = \dfrac{x - a_1}{a_2 - a_1}$, $T^U_{\tilde{\mathcal{N}}_A}(x) = \dfrac{a_4 - x}{a_4 - a_3}$ are the two sides of $T_{\tilde{\mathcal{N}}_A}(x)$. Similarly, $I^L_{\tilde{\mathcal{N}}_A}(x) = \dfrac{b_2 - x}{b_2 - b_1}$, $I^U_{\tilde{\mathcal{N}}_A}(x) = \dfrac{x - b_3}{b_4 - b_3}$ are the two sides of $I_{\tilde{\mathcal{N}}_A}(x)$ and $F^L_{\tilde{\mathcal{N}}_A}(x) = \dfrac{c_2 - x}{c_2 - c_1}$, $F^U_{\tilde{\mathcal{N}}_A}(x) = \dfrac{x - c_3}{c_4 - c_3}$ are the two sides of $F_{\tilde{\mathcal{N}}_A}(x)$.

Each of three membership degrees of TrFNN can be taken as the three independent components like fuzzy numbers. Thus similar to fuzzy set, the expected interval or expected value of each membership degree can be determined separately.

Definition 21 We define the expected interval and the expected value of truth membership function

$T_{\tilde{\mathcal{N}}_A}(x) = (a_1, a_2, a_3, a_4)$ of TrFNN $\tilde{\mathcal{N}}_A$ as follows:

$$EI(T_{\tilde{\mathcal{N}}_A}(x)) = \left[\frac{(a_1 + a_2)}{2}, \frac{(a_3 + a_4)}{2}\right] \quad (19)$$

$$EV(T_{\tilde{\mathcal{N}}_A}(x)) = \frac{(a_1 + a_2 + a_3 + a_4)}{4} \quad (20)$$

Similarly, we define the expected interval and the expected value of the indeterminacy membership function of TrFNN as follows:

$$EI(I_{\tilde{\mathcal{N}}_A}(x)) = \left[\frac{(b_1 + b_2)}{2}, \frac{(b_3 + b_4)}{2}\right] \quad (21)$$

$$EV(I_{\tilde{\mathcal{N}}_A}(x)) = \frac{(b_1 + b_2 + b_3 + b_4)}{4} \quad (22)$$

We define the expected interval and the expected value of the falsity membership function of TrFNN as follows:

$$EI(F_{\tilde{\mathcal{N}}_A}(x)) = \left[\frac{(c_1 + c_2)}{2}, \frac{(c_3 + c_4)}{2}\right] \quad (23)$$

$$EV(F_{\tilde{\mathcal{N}}_A}(x)) = \frac{(c_1 + c_2 + c_3 + c_4)}{4} \quad (24)$$

Definition 22 (Truth favorite relative expected value of TrFNN)

Let $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ be the TrFNN in the set of real numbers **R.** Suppose $EV(T_{\tilde{\mathcal{N}}_A}(x))$, $EI(I_{\tilde{\mathcal{N}}_A}(x))$ and $EV(F_{\tilde{\mathcal{N}}_A}(x))$ are the expected values of truth membership, indeterminacy membership and falsity membership component of SVNN $\tilde{\mathcal{N}}_A$. If

$$EV^T(\tilde{\mathcal{N}}_A) = \frac{3EV(T_{\tilde{\mathcal{N}}_A}(x))}{EV(T_{\tilde{\mathcal{N}}_A}(x)) + EV(I_{\tilde{\mathcal{N}}_A}(x)) + EV(F_{\tilde{\mathcal{N}}_A}(x))} \quad (25)$$

then we define $EV(\tilde{\mathcal{N}}_A)$ as the truth favorite relative expected value (TFREV) of $\tilde{\mathcal{N}}_A$.

Theorem 1(Expected value theorem)

Let $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ be the TrFNN in the set of real numbers **R,** then the truth favorite relative expected value (TFREV) of $\tilde{\mathcal{N}}_A$ is defined by

$$EV^T(\tilde{\mathcal{N}}_A) = \frac{3\sum_{i=1}^4 a_i}{\left(\sum_{i=1}^4 a_i + \sum_{i=1}^4 b_i + \sum_{i=1}^4 c_i\right)} \quad (26)$$

Proof: Given that $T_{\tilde{\mathcal{N}}_A}(x)$ is the truth membership, $I_{\tilde{\mathcal{N}}_A}(x)$ is the indeterminacy membership and $F_{\tilde{\mathcal{N}}_A}(x)$ is the falsity membership component of TrFNN $\tilde{\mathcal{N}}_A$. Treating each component of $\tilde{\mathcal{N}}_A$ as the trapezoidal fuzzy number, the combined expected value of the $\tilde{\mathcal{N}}_A$ can be obtained by considering the centroid of three expected values of $T_{\tilde{\mathcal{N}}_A}(x)$, $I_{\tilde{\mathcal{N}}_A}(x)$ and $F_{\tilde{\mathcal{N}}_A}(x)$.

Then, the combined expected value of three membership components can be defined by

$$EV(\tilde{\mathcal{N}}_A) = \frac{1}{3}\left(EV(T_{\tilde{\mathcal{N}}_A}(x)) + EV(I_{\tilde{\mathcal{N}}_A}(x)) + EV(F_{\tilde{\mathcal{N}}_A}(x))\right) \quad (27)$$

Combining Eqs. (20), (22), (24), and (27) we obtain

$$EV(\tilde{\mathcal{N}}_A) = \frac{1}{3}\left(\frac{(a_1 + a_2 + a_3 + a_4)}{4} + \frac{(b_1 + b_2 + b_3 + b_4)}{4} + \frac{(c_1 + c_2 + c_3 + c_4)}{4}\right)$$

$$= \frac{\left(\sum_{i=1}^4 a_i + \sum_{i=1}^4 b_i + \sum_{i=1}^4 c_i\right)}{12} \quad (28)$$

Now, the TFREV of $\tilde{\mathcal{N}}_A$ can be determined by

$$EV^T(\tilde{\mathcal{N}}_A) = \frac{EV(T_{\tilde{\mathcal{N}}_A}(x))}{EV(\tilde{\mathcal{N}}_A)}. \quad (29)$$

Using Eqs.(20) (28) and (29), we obtain the desired TFREV of $\tilde{\mathcal{N}}_A$ as follows:

$$EV^T(\tilde{\mathcal{N}}_A) = \frac{3\sum_{i=1}^4 a_i}{\left(\sum_{i=1}^4 a_i + \sum_{i=1}^4 b_i + \sum_{i=1}^4 c_i\right)} \quad (30)$$

This completes the proof.

Now, if the corresponding elements of three membership degrees of TrFNN $\tilde{\mathcal{N}}_A$ coincide with each other i.e., when $(a_1, a_2, a_3, a_4) = (b_1, b_2, b_3, b_4) = (c_1, c_2, c_3, c_4)$ then combined expected value of $\tilde{\mathcal{N}}_A$ would be

$$EV(\tilde{\mathcal{N}}_A) = \frac{(a_1 + a_2 + a_3 + a_4)}{4} \quad (31)$$

and TFREV of $\tilde{\mathcal{N}}_A$ would be $EV^T(\tilde{\mathcal{N}}_A) = 1$.

It is to be noted that if $a_2 = a_3$, $b_2 = b_3$ and $c_2 = c_3$ of a TrFNN $\tilde{\mathcal{N}}_A = \langle(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4)\rangle$ then $\tilde{\mathcal{N}}_A$ reduces to triangular fuzzy neutrosophic number (TFNN) $\tilde{\mathcal{N}}_A = \langle(a_1, a_2, a_4), (b_1, b_2, b_4), (c_1, c_2, c_4)\rangle$. Then according to Eq.(28), the expected value of TFNN $\tilde{\mathcal{N}}_{Tri} = \langle(l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3)\rangle$ can be defined as follows:

$$EV(\tilde{\mathcal{N}}_{Tri}) = \frac{(l_1 + 2l_2 + l_3 + m_1 + 2m_2 + m_3 + n_1 + 2n_2 + n_3)}{12} \quad (32)$$

and TFREV of $\tilde{\mathcal{N}}_{Tri}$ can be defined as follows:

$$EV^T(\tilde{\mathcal{N}}_{Tri}) = \frac{3(l_1 + 2l_2 + l_3)}{(l_1 + 2l_2 + l_3 + m_1 + 2m_2 + m_3 + n_1 + 2n_2 + n_3)} \quad (33)$$

4 Cosine Similarity Measure of Trapezoidal Fuzzy Neutrosophic Numbers

Definition 23

Let $\tilde{\mathcal{N}}_A = \langle(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4)\rangle$ and $\tilde{\mathcal{N}}_B = \langle(e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4), (g_1, g_2, g_3, g_4)\rangle$ be two TrFNNs in the set of real numbers **R**. The twelve parameters considered in $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ can be taken as two vector representations with twelve elements. Thus, a cosine similarity measure between $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ can be determined in a similar manner to the cosine similarity measure between two trapezoidal fuzzy numbers. Then,

$$Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B) = \frac{\sum_{i=1}^4 a_i e_i + \sum_{i=1}^4 b_i f_i + \sum_{i=1}^4 c_i g_i}{\left[\begin{array}{l}\left(\sqrt{\sum_{i=1}^4 (a_i)^2 + \sum_{i=1}^4 (b_i)^2 + \sum_{i=1}^4 (c_i)^2}\right) \times \\ \left(\sqrt{\sum_{i=1}^4 (e_i)^2 + \sum_{i=1}^4 (f_i)^2 + \sum_{i=1}^4 (g_i)^2}\right)\end{array}\right]}. \quad (34)$$

The cosine similarity measure $Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B)$ of $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ satisfies the following properties:

P1. $0 \le Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B) \le 1$

P2 $Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B) = Cos_{TrFNN}(\tilde{\mathcal{N}}_B, \tilde{\mathcal{N}}_A)$

P3 $Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B) = 1$ for $\tilde{\mathcal{N}}_A = \tilde{\mathcal{N}}_B$

i.e., $a_i = e_i$, $b_i = f_i$ and $c_i = g_i$ for i = 1, 2, 3, 4.

Proof: P1 is shown to be true from the basic definition of cosine value.

P2: Symmetry of Eq. (34) validates the property P2.

P3: By putting $a_i = e_i$, $b_i = f_i$ and $c_i = g_i$ for i = 1, 2, 3, 4 in Eq. (34), the denominator and numerator reduce to $\left(\sum_{i=1}^4 (a_i)^2 + \sum_{i=1}^4 (b_i)^2 + \sum_{i=1}^4 (c_i)^2\right)$ and therefore $Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B) = 1$.

5 Cosine Similarity Based Multiple Attribute Decision-Making Problems with Trapezoidal Fuzzy Neutrosophic Numbers

Let $A_1, A_2, ..., A_m$ be a discrete set of *m* alternatives, and $C_1, C_2, ..., C_n$ be the set of *n* attributes for a multi-attribute decision-making problem. The rating d_{ij} provided by the decision maker describes the performance of the alternative A_i against the attribute C_j. Then the assessment values of the alternatives can be presented in the following decision matrix form.

Table 1. Decision matrix of attribute values

$$D = \langle d_{ij} \rangle_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ . \\ . \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & ... & C_n \end{array} \\ \begin{bmatrix} d_{11} & d_{12} & ... & d_{1n} \\ d_{21} & d_{22} & ... & d_{2n} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ d_{m1} & d_{m2} & ... & d_{mn} \end{bmatrix} \end{array} \quad (35)$$

Step 1. Determination of the most important attributes

In a decision making process, a set of criteria or attributes are to be required to evaluate the best alternative. All attributes are not equal important in the decision making situation. Therefore it is important to choose the set of proper attributes based on experts' opinions.

Step 2. Construction of the decision matrix with TrFNNs

Let us assume that the ratings of alternative A_i (i = 1, 2,…, m) with respect to the attribute C_j (j = 1, 2, …, n) are expressed with TrFNNs. The TrFNN based rating values of the m-th alternative over the n-th attribute can be presented in the following decision matrix.

Table 2. Decision matrix with *TrFNNs*

$$D_{\tilde{N}} = \langle \tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij} \rangle_{m\times n} =$$

$$\begin{bmatrix} \langle \tilde{a}_{11}, \tilde{b}_{11}, \tilde{c}_{11} \rangle & \langle \tilde{a}_{12}, \tilde{b}_{12}, \tilde{c}_{12} \rangle & \dots & \langle \tilde{a}_{1n}, \tilde{b}_{1n}, \tilde{c}_{1n} \rangle \\ \langle \tilde{a}_{21}, \tilde{b}_{21}, \tilde{c}_{21} \rangle & \langle \tilde{a}_{22}, \tilde{b}_{22}, \tilde{c}_{22} \rangle & \dots & \langle \tilde{a}_{2n}, \tilde{b}_{2n}, \tilde{c}_{2n} \rangle \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \langle \tilde{a}_{m1}, \tilde{b}_{m1}, \tilde{c}_{m1} \rangle & \langle \tilde{a}_{m2}, \tilde{b}_{m2}, \tilde{c}_{m2} \rangle & \dots & \langle \tilde{a}_{mn}, \tilde{b}_{mn}, \tilde{c}_{mn} \rangle \end{bmatrix} \quad (36)$$

In the decision matrix $D_{\tilde{N}} = \langle \tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij} \rangle_{m\times n}$, $\tilde{a}_{ij}$ denotes the degree that the alternative A_i (i = 1, 2, ..., m) satisfies the attribute C_j (j = 1, 2, ..., n), $\tilde{b}_{ij}$ denotes the degree of indeterminacy of the alternative A_i over the attribute C_j and $\tilde{c}_{ij}$ denotes the degree that the alternative A_i does not satisfy the attribute C_j. These three membership components $\tilde{a}_{ij}$, $\tilde{b}_{ij}$ and $\tilde{c}_{ij}$ are expressed by the trapezoidal fuzzy numbers with the following properties:

1. a. $\tilde{a}_{ij} = (a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4) \in [0, 1]$;
   b. $\tilde{b}_{ij} = (b_{ij}^1, b_{ij}^2, b_{ij}^3, b_{ij}^4) \in [0, 1]$;
   c. $\tilde{c}_{ij} = (c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4) \in [0, 1]$;

2. $0 \le a_{ij}^4 + b_{ij}^4 + c_{ij}^4 \le 3$ for i = 1, 2, .., m and j = 1, 2, ..., n.

Step 3. Determination of the weights of attributes

The importance of attributes may not be always same to decision maker in decision-making situation. The information available of the attribute weights is often vague or incomplete in the decision making situation. Let $W = (w_1, w_2, ..., w_n)^T$ be the vaguely expressed weight vector assigned to the different attributes. In this case the weight of the attribute C_j for j = 1, 2, ..., n can be presented by the TrFNNs. Let us assume that $w_j = \langle (a_{1j}, a_{2j}, a_{3j}, a_{4j}), (b_{1j}, b_{2j}, b_{3j}, b_{4j}), (c_{1j}, c_{2j}, c_{3j}, c_{4j}) \rangle$ be the TrFNN based weight of attribute C_j. The expected value of w_j (j = 1, 2, ..., n) is determined by using the Eq.(30). These values are to be normalized by the following formula to make dimensionless

$$w_i^N = \frac{EV^T(w_i)}{\sum_{i=1}^{n} EV^T(w_i)} \text{ for i = 1, 2, ..., n.} \quad (37)$$

Step 4. Determination of the positive ideal neutrosophic solution (PINS) and the relative positive ideal neutrosophic solution (RPINS) for TrFNNs based neutrosophic decision matrix

Definition 24 Let H be the collection of two types of attributes namely benifit type attribute (P) and cost type attribute (L) in the MADM problems.

The positive ideal neutrosophic solution (PINS) $Q_{\tilde{N}}^{+} = [q_{\tilde{N}_1}^{+}, q_{\tilde{N}_2}^{+}, ..., q_{\tilde{N}_n}^{+}]$ is the solution of the decision matrix $D_{\tilde{N}} = \langle T_{ij}, I_{ij}, F_{ij} \rangle_{m\times n}$ where, every component of $Q_{\tilde{N}}^{+}$ has the following form:

$$q_{\tilde{N}j}^{+} = \left\langle \begin{matrix} (a_j^{1+}, a_j^{2+}, a_j^{3+}, a_j^{4+}), (b_j^{1+}, b_j^{2+}, b_j^{3+}, b_j^{4+}), \\ (c_j^{1+}, c_j^{2+}, c_j^{3+}, c_j^{4+}) \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \left(\max_i\{a_{ij}^1\}, \max_i\{a_{ij}^2\}, \max_i\{a_{ij}^3\}, \max_i\{a_{ij}^4\}\right), \\ \left(\max_i\{b_{ij}^1\}, \max_i\{b_{ij}^2\}, \max_i\{b_{ij}^3\}, \max_i\{b_{ij}^4\}\right), \\ \left(\max_i\{c_{ij}^1\}, \max_i\{c_{ij}^2\}, \max_i\{c_{ij}^3\}, \max_i\{c_{ij}^4\}\right) \end{matrix} \right\rangle \quad (38)$$

for the benefit type attribute and

$$q_{\tilde{N}j}^{+} = \left\langle \begin{matrix} \left(\min_i\{a_{ij}^1\}, \min_i\{a_{ij}^2\}, \min_i\{a_{ij}^3\}, \min_i\{a_{ij}^4\}\right), \\ \left(\min_i\{b_{ij}^1\}, \min_i\{b_{ij}^2\}, \min_i\{b_{ij}^3\}, \min_i\{b_{ij}^4\}\right), \\ \left(\min_i\{c_{ij}^1\}, \min_i\{c_{ij}^2\}, \min_i\{c_{ij}^3\}, \min_i\{c_{ij}^4\}\right) \end{matrix} \right\rangle \quad (39)$$

for the cost type attribute.

Definition 25 The negative ideal neutrosophic solution (PINS) $Q_{\tilde{N}}^{-} = [q_{\tilde{N}_1}^{-}, q_{\tilde{N}_2}^{-}, ..., q_{\tilde{N}_n}^{-}]$ is the solution of the decision matrix $D_{\tilde{N}} = \langle T_{ij}, I_{ij}, F_{ij} \rangle_{m\times n}$ where, every component of $Q_{\tilde{N}}^{-}$ has the following form:

$$q_{\tilde{N}j}^{-} = \left\langle \begin{matrix} (a_j^{1-}, a_j^{2-}, a_j^{3-}, a_j^{4-}), (b_j^{1-}, b_j^{2-}, b_j^{3-}, b_j^{4-}), \\ (c_j^{1-}, c_j^{2-}, c_j^{3-}, c_j^{4-}) \end{matrix} \right\rangle$$

$$=\left\langle \begin{array}{l} \left(\min\limits_i\{a_{ij}^1\}, \min\limits_i\{a_{ij}^2\}, \min\limits_i\{a_{ij}^3\}, \min\limits_i\{a_{ij}^4\}\right), \\ \left(\min\limits_i\{b_{ij}^1\}, \min\limits_i\{b_{ij}^2\}, \min\limits_i\{b_{ij}^3\}, \min\limits_i\{b_{ij}^4\}\right), \\ \left(\min\limits_i\{c_{ij}^1\}, \min\limits_i\{c_{ij}^2\}, \min\limits_i\{c_{ij}^3\}, \min\limits_i\{c_{ij}^4\}\right) \end{array} \right\rangle \qquad (40)$$

for the benefit type attribute.

$$q_{\tilde{\mathcal{N}}j}^{-}=\left\langle \begin{array}{l} \left(\max\limits_i\{a_{ij}^1\}, \max\limits_i\{a_{ij}^2\}, \max\limits_i\{a_{ij}^3\}, \max\limits_i\{a_{ij}^4\}\right), \\ \left(\max\limits_i\{b_{ij}^1\}, \max\limits_i\{b_{ij}^2\}, \max\limits_i\{b_{ij}^3\}, \max\limits_i\{b_{ij}^4\}\right), \\ \left(\max\limits_i\{c_{ij}^1\}, \max\limits_i\{c_{ij}^2\}, \max\limits_i\{c_{ij}^3\}, \max\limits_i\{c_{ij}^4\}\right) \end{array} \right\rangle \qquad (41)$$

for the cost type attribute.

Step 5. Determination of the weighted cosine similarity measure between each alternative and the ideal alternative

Let w_j be the weight of the attribute C_j for j = 1, 2, ..., n. The weighted cosine similarity measure between the alternative A_i for i = 1, 2, ..., m and the positive ideal alternative $Q_{\tilde{\mathcal{N}}}^{+}$ is

$$Cos_{TrFNN}^{W+}\left(Q_{\tilde{\mathcal{N}}}^{+}, A_i\right)=$$

$$\sum_{j=1}^{n} w_j \frac{\sum_{s=1}^4 a_j^{s+} a_{ij}^s+\sum_{s=1}^4 b_j^{s+} b_{ij}^s+\sum_{s=1}^4 c_j^{s+} c_{ij}^s}{\left\{\begin{array}{l} \left(\sqrt{\sum_{s=1}^4\left(a_j^{s+}\right)^2+\sum_{s=1}^4\left(b_j^{s+}\right)^2+\sum_{s=1}^4\left(c_j^{s+}\right)^2}\right) \times \\ \left(\sqrt{\sum_{s=1}^4\left(a_{ij}^{s}\right)^2+\sum_{s=1}^4\left(b_{ij}^{s}\right)^2+\sum_{s=1}^4\left(c_{ij}^{s}\right)^2}\right) \end{array}\right\}} \qquad (42)$$

Step 6. Ranking the alternatives

The ranking order of all alternatives can be determined by using the weighted cosine similarity measure between the alternative and the positive ideal alternative defined in Eq. (42). The highest value of $Cos_{TrFNN}^{W+}\left(Q_{\tilde{\mathcal{N}}}^{+}, A_i\right)$ reflects the most desired alternative for i = 1, 2, ..., n.

6. Illustrative Example

In this section, multi attribute decision making problem under a trapezoidal fuzzy neutrosophic environment is considered to demonstrate the applicability and the effectiveness of the proposed approach. Let us consider the decision-making problem in which a customer intends to buy a tablet from the set of primarily chosen five alternatives A = (A_1, A_2, A_3, A_4, A_5). The customer takes into account the following four attributes:

1. features (C_1);
2. hardware(C_2);
3. affordable price (C_3);
4. customer care (C_4).

Assume that the weight vector of the four attributes provided by the decision maker is expressed by the trapezoidal fuzzy neutrosophic numbers

$$W=[w_1, w_2, w_3, w_4]=$$

$$\left\{\begin{array}{l} \langle(0.4, 0.5, 0.6, 0.7),(0.0, 0.1, 0.2, 0.3),(0.1, 0.2, 0.3, 0.4)\rangle, \\ \langle(0.2, 0.3, 0.4, 0.5),(0.1, 0.2, 0.2, 0.2),(0.2, 0.2, 0.3, 0.4)\rangle, \\ \langle(0.6, 0.7, 0.8, 0.9),(0.0, 0.1, 0.2, 0.3),(0.1, 0.1, 0.2, 0.3)\rangle, \\ \langle(0.4, 0.5, 0.6, 0.7),(0.0, 0.1, 0.2, 0.3),(0.1, 0.2, 0.3, 0.4)\rangle \end{array}\right\} \qquad (43)$$

Given that the following trapezoidal fuzzy neutrosophic number based decision matrix according to the experts' assessment of the five alternatives with respect to the four attributes:

Table3. Decision matrix with SVNS

$$D_{\tilde{\mathcal{N}}}=\left\langle \tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij}\right\rangle_{5\times 4}=$$

$$\left[\begin{array}{l} \langle(0.5, 0.6, 0.7, 0.8),(0.1, 0.1, 0.2, 0.3),(0.1, 0.2, 0.2, 0.3)\rangle \\ \langle(0.3, 0.4, 0.5, 0.5),(0.1, 0.2, 0.2, 0.4),(0.1, 0.1, 0.2, 0.3)\rangle \\ \langle(0.3, 0.3, 0.3, 0.3),(0.2, 0.3, 0.4, 0.4),(0.6, 0.7, 0.8, 0.9)\rangle \\ \langle(0.7, 0.8, 0.8, 0.9),(0.1, 0.2, 0.3, 0.3),(0.2, 0.2, 0.2, 0.2)\rangle \\ \langle(0.1, 0.2, 0.2, 0.3),(0.2, 0.2, 0.3, 0.4),(0.6, 0.6, 0.7, 0.8)\rangle \end{array}\right.$$

$$\begin{array}{l} \langle(0.1, 0.1, 0.2, 0.3),(0.2, 0.2, 0.3, 0.4),(0.4, 0.5, 0.6, 0.7)\rangle \\ \langle(0.2, 0.3, 0.4, 0.5),(0.1, 0.1, 0.2, 0.3),(0.2, 0.2, 0.3, 0.3)\rangle \\ \langle(0.1, 0.2, 0.2, 0.3),(0.2, 0.3, 0.3, 0.4),(0.4, 0.5, 0.6, 0.6)\rangle \\ \langle(0.5, 0.6, 0.7, 0.7),(0.2, 0.2, 0.2, 0.2),(0.1, 0.1, 0.2, 0.2)\rangle \\ \langle(0.5, 0.6, 0.6, 0.7),(0.1, 0.2, 0.3, 0.4),(0.2, 0.2, 0.3, 0.4)\rangle \end{array}$$

$$\langle(0.3,0.4,0.4,0.5),(0.1,0.2,0.2,0.3),(0.2,0.2,0.3,0.4)\rangle$$
$$\langle(0.2,0.2,0.2,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.8)\rangle$$
$$\langle(0.2,0.3,0.4,0.5),(0.2,0.3,0.3,0.4),(0.3,0.4,0.4,0.5)\rangle$$
$$\langle(0.3,0.4,0.4,0.5),(0.1,0.2,0.2,0.3),(0.1,0.2,0.3,0.4)\rangle$$
$$\langle(0.6,0.7,0.8,0.8),(0.2,0.2,0.3,0.3),(0.1,0.1,0.2,0.3)\rangle$$

$$\left.\begin{matrix}\langle(0.4,0.5,0.6,0.7),(0.2,0.2,0.3,0.4),(0.1,0.2,0.3,0.4)\rangle\\ \langle(0.4,0.5,0.6,0.6),(0.2,0.2,0.3,0.3),(0.2,0.3,0.4,0.4)\rangle\\ \langle(0.2,0.2,0.3,0.4),(0.3,0.3,0.3,0.3),(0.3,0.4,0.5,0.6)\rangle\\ \langle(0.1,0.2,0.3,0.4),(0.2,0.2,0.3,0.3),(0.5,0.6,0.7,0.8)\rangle\\ \langle(0.2,0.3,0.4,0.4),(0.1,0.2,0.3,0.4),(0.3,0.4,0.4,0.5)\rangle\end{matrix}\right] \quad (44)$$

Step 1. Determination of the weight of attributes

The truth favorite relative expected values (TFREVs) of the assessment of four attributes expressed with TrFNNs can be determined by the Eq. (30) as follows: $EV^T(w_1) = 1.737$, $EV^T(w_2) = 1.31$, $EV^T(w_3) = 2.093$ and $EV^T(w_4) = 1.737$. The normalized expected value of the assessment of four attributes is obtained by using the Eq. (37) as $EV^{TN}(w_1) = 0.2525$; $EV^{TN}(w_2) = 0.1907$; $EV^{TN}(w_3) = 0.3042$ and $EV^{TN}(w_4) = 0.2525$.

Step 2. Determination of the relative positive ideal neutrosophic solution (PINS) for the TrFNNs based neutrosophic decision matrix

The positive ideal solution of the decision matrix $D_{\tilde{N}} = \langle \tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij} \rangle_{5\times 4}$ is $Q^+_{\tilde{N}} = [q^+_{\tilde{N}_1}, q^+_{\tilde{N}_2}, q^+_{\tilde{N}_3}, q^+_{\tilde{N}_4}]$ where,

$$q^+_{\tilde{N}_1} = \left\langle \begin{matrix}(0.7,0.8,0.8,0.9),(0.2,0.3,0.4,0.4),\\(0.6,0.7,0.8,0.9)\end{matrix} \right\rangle \quad (45)$$

$$q^+_{\tilde{N}_2} = \left\langle \begin{matrix}(0.5,0.6,0.7,0.7),(0.2,0.2,0.3,0.4),\\(0.4,0.5,0.6,0.7)\end{matrix} \right\rangle \quad (46)$$

$$q^+_{\tilde{N}_3} = \left\langle \begin{matrix}(0.6,0.7,0.8,0.8),(0.2,0.3,0.3,0.4),\\(0.6,0.7,0.8,0.8)\end{matrix} \right\rangle \quad (47)$$

$$q^+_{\tilde{N}_4} = \left\langle \begin{matrix}(0.4,0.5,0.6,0.7),(0.3,0.3,0.3,0.4),\\(0.5,0.6,0.7,0.8)\end{matrix} \right\rangle \quad (48)$$

Step 3. Calculation of the weighted cosine similarity measure between each alternative and the ideal alternative

The weighted cosine similarity measures between positive ideal alternative and each alternative are determined by using the Eq. (42) and the results are shown in the table 4.

Table 4. Decision results of weighted cosine similarity measures

Alternative (A_i)	Weighted cosine similarity measure
Alternative (A_1)	0.910296
Alternative (A_2)	0.918177
Alternative (A_3)	**0.928833**
Alternative (A_4)	0.915722
Alternative (A_5)	0.904869
Ranking Order	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$

Step 4. Ranking of the alternatives

According to the values of weighted cosine similarity measure Table 4 shows that A_3 is the best alternative.

6 Conclusion

In this paper, we have presented cosine similarity measure based multiple attribute decision-making with trapezoidal fuzzy neutrosophic numbers. Expected value theorem and cosine similarity measure of trapezoidal fuzzy neutrosophic numbers are developed. The assessments of alternatives and attribute weights provided by the decision maker are considered with the trapezoidal fuzzy neutrosophic numbers. Ranking order of all alternatives is determined using the proposed cosine similarity measure between positive ideal alternative and each of alternatives. Finally, an illustrative example is provided to show the feasibility of the proposed approach and to demonstrate its practicality and effectiveness. However, the authors hope that the proposed approach will be applicable in medical diagnosis, pattern recognition, and other neutrosophic decision making problems.

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Thesis-Antithesis-Neutrothesis, and Neutrosynthesis

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Abstract. In this short paper we extend the dialectical triad thesis-antithesis-synthesis (dynamics of <A> and <antiA>, to get a synthesis) to the neutrosophic tetrad thesis-antithesis-neutrothesis-neutrosynthesis (dynamics of <A>, <antiA>, and <neutA>, in order to get a neutrosynthesis). We do this for better reflecting our world, since the neutralities between opposites play an important role. The neutrosophic synthesis (neutrosynthesis) is more refined that the dialectical synthesis. It carries on the unification and synthesis regarding the opposites and their neutrals too.

Keywords: Thesis, Antithesis, Synthesis, Thesis-Antithesis-Neutrothesis, and Neutrosynthesis.

1. Introduction.

In neutrosophy, <A>, <antiA>, and <neutA> combined two by two, and also all three of them together form the NeutroSynthesis. Neutrosophy establishes the universal relations between <A>, <antiA>, and <neutA>.
<A> is the thesis, <antiA> the antithesis, and <neutA> the neutrothesis (neither <A> nor <antiA>, but the neutrality in between them).
In the neutrosophic notation, <nonA> (not <A>, outside of <A>) is the union of <antiA> and <neutA>.
<neutA> may be from no middle (excluded middle), to one middle (included middle), to many finite discrete middles (finite multiple included-middles), and to an infinitude of discrete or continuous middles (infinite multiple included-middles) [for example, as in color for the last one, let's say between black and white there is an infinite spectrum of middle/intermediate colors].

2. Thesis, Antithesis, Synthesis.

The classical reasoning development about evidences, popularly known as thesis-antithesis-synthesis from dialectics, was attributed to the renowned philosopher Georg Wilhelm Friedrich Hegel (1770-1831) and later it was used by Karl Marx (1818-1883) and Friedrich Engels (1820-1895). About thesis and antithesis have also written Immanuel Kant (1724-1804), Johann Gottlieb Fichte (1762-1814), and Thomas Schelling (born 1921). While in ancient Chinese philosophy the opposites yin [feminine, the moon] and yang [masculine, the sun] were considered complementary.

Thesis, Antithesis, Neutrothesis, Neutrosynthesis.

Neutrosophy is a generalization of dialectics (which is based on contradictions only, <A> and <antiA>), because neutrosophy is based on contradictions and on the neutralities between them (<A>, <antiA>, and <neutA>). Therefore, the dialectical triad thesis-antithesis-synthesis is extended to the neutrosophic tetrad thesis-antithesis-neutrothesis-neutrosynthesis. We do this not for the sake of generalization, but for better reflecting our world. A neutrosophic synthesis (neutrosynthesis) is more refined that the dialectical synthesis. It carries on the unification and synthesis regarding the opposites and their neutrals too.

Neutrosophic Dynamicity.

We have extended in [1] the Principle of Dynamic Opposition [opposition between <A> and <antiA>] to the **Principle of Dynamic Neutropposition** [which means oppositions among <A>, <antiA>, and <neutA>]. Etymologically "neutropposition" means "neutrosophic opposition".
This reasoning style is not a neutrosophic scheme, but it is based on reality, because if an idea (or notion) <A> arises, then multiple versions of this idea are spread out, let's denote them by $<A>_1$, $<A>_2$, …, $<A>_m$. Afterwards, the opposites (in a smaller or higher degree) ideas are born, as reactions to <A> and its versions $<A>_i$. Let's denote these versions of opposites by $<antiA>_1$, $<antiA>_2$, …, $<antiA>_n$. The neutrality <neutA> between these contradictories ideas may embrace various forms, let's denote them by $<neutA>_1$, $<neutA>_2$, …, $<neutA>_p$, where m, n, p are integers greater than or equal to 1.

In general, for each <A> there may be corresponding many <antiA>’s and many <neutA>’s. Also, each <A> may be interpreted in many different versions of <A>’s too.
Neutrosophic Dynamicity means the interactions among all these multi-versions of <A>’s
with their multi-<antiA>’s and their multi-<neutA>’s, which will result in a new thesis, let’s call it <A’> at a superior level. And a new cycle of <A’>, <antiA’>, and <neutA’> restarts its neutrosophic dynamicity.

Practical Example

Let’s say <A> is a country that goes to war with another country, which can be named <antiA> since it is antagonistic to the first country. But many neutral countries <neutA> can interfere, either supporting or aggressing one of them, in a smaller or bigger degree. Other neutral countries <neutA> can still remain neutral in this war. Yet, there is a continuous dynamicity between the three categories (<A>, <antiA>, <neutA.), for countries changing sides (moving from a coalition to another coalition), or simply retreating from any coalition.
In our easy example, we only wanted to emphasize the fact that <neutA> plays a role in the conflict between the opposites <A> and <antiA>, role which was ignored by dialectics.
So, the dialectical synthesis is extended to a neutrosophic synthesis, called neutrosynthesis, which combines thesis, antithesis, and neutrothesis.

Theoretical Example.

Suppose <A> is a philosophical school, and its opposite philosophical school is <antiA>. In the dispute between <A> and <antiA>, philosophers from the two contradictory groups may bring arguments against the other philosophical school from various neutral philosophical schools’ ideas (<neutA>, which were neither for <A> nor <antiA>) as well.

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Negating Four Color Theorem with Neutrosophy and Quad-stage Method

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Abstract. With the help of Neutrosophy and Quad-stage Method, the proof for negation of “the four color theorem” is given. In which the key issue is to consider the color of the boundary, thus “the two color theorem” and “the five color theorem” are derived to replace "the four color theorem".

Keywords: The four color theorem, neutrosophy, quad-stage, boundary, proof for negation, the two color theorem, the five color theorem.

1 Introduction

The four color theorem states that, given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color. Two regions are called adjacent if they share a common boundary that is not a corner, where corners are the points shared by three or more regions.

In 1976, Kenneth Appel and Wolfgang Haken published their proof of the four color theorem. It was the first major theorem to be proved using a computer.

Accordingly, this paper starts with the assumption that, in the case without considering the color of the boundary, "the four color theorem" is correct.

Figure 1 is an example of four-color map.

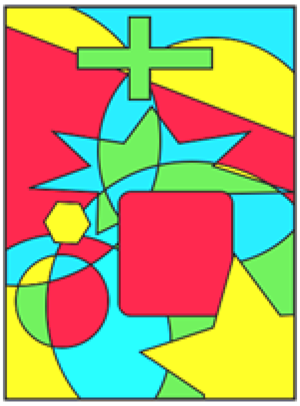

Figure 1 An example of four-color map

However, whether or not the color of the boundary should be considered? We believe that it should be taken into account.

In this paper, with the help of Neutrosophy and Quad-stage Method, the proof for negation of “the four color theorem” is given. In which the key issue is to consider the color of the boundary.

2 Basic Contents of Neutrosophy

Neutrosophy is proposed by Prof. Florentin Smarandache in 1995.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <Anti-A> and the spectrum of "neutralities" <Neut-A> (i.e. notions or ideas located between the two extremes, supporting neither <A> nor <Anti-A>). The <Neut-A> and <Anti-A> ideas together are referred to as <Non-A>.

Neutrosophy is the base of neutrosophic logic, neutrosophic set, neutrosophic probability and statistics used in engineering applications (especially for software and information fusion), medicine, military, cybernetics, and physics.

Neutrosophic Logic is a general framework for unification of many existing logics, such as fuzzy logic (especially intuitionistic fuzzy logic), paraconsistent logic, intuitionistic logic, etc. The main idea of NL is to characterize each logical statement in a 3D Neutrosophic Space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of]-0, 1+[without necessarily connection between them.

More information about Neutrosophy can be found in references [1 , 2].

3 Basic Contents of Quad-stage

Quad-stage (Four stages) is presented in reference [3], it is the expansion of Hegel's triad-stage (triad thesis, antithesis, synthesis of development). The four stages are "general theses", "general antitheses", "the most important and the most complicated universal relations", and "general syntheses". They can be stated as follows.

The first stage, for the beginning of development (thesis), the thesis should be widely, deeply, carefully and repeatedly contacted, explored, analyzed, perfected and so on; this is the stage of general theses. It should be noted that, here the thesis will be evolved into two or three, even more theses step by step. In addition, if in other stage we find that the first stage's work is not yet completed, then we may come back to do some additional work for the first stage.

The second stage, for the appearance of opposite (antithesis), the antithesis should be also widely, deeply, carefully and repeatedly contacted, explored, analyzed, perfected and so on; this is the stage of general antitheses. It should be also noted that, here the antithesis will be evolved into two or three, even more antitheses step by step.

The third stage is the one that the most important and the most complicated universal relations, namely the seedtime inherited from the past and carried on for the future. Its purpose is to establish the universal relations in the widest scope. This widest scope contains all the regions related and non-related to the "general theses", "general antitheses", and the like. This stage's foundational works are to contact, grasp, discover, dig, and even create the opportunities, pieces of information, and so on as many as possible. The degree of the universal relations may be different, theoretically its upper limit is to connect all the existences, pieces of information and so on related to matters, spirits and so on in the universe; for the cases such as to create science fiction, even may connect all the existences, pieces of information and so on in the virtual world. Obviously, this stage provides all possibilities to fully use the complete achievements of nature and society, as well as all the humanity's wisdoms in the past, present and future. Therefore this stage is shortened as "universal relations" (for other stages, the universal relations are also existed, but their importance and complexity cannot be compared with the ones in this stage).

The fourth stage, to carry on the unification and synthesis regarding various opposites and the suitable pieces of information, factors, and so on; and reach one or more results which are the best or agreed with some conditions; this is the stage of "general syntheses". The results of this stage are called "synthesized second generation theses", all or partial of them may become the beginning of the next quad-stage.

4 Negating the Four Color Theorem

The combination of Neutrosophy and Quad-stage is very useful for innovations in areas of science, technology and the like. For example, in reference [4], we expand Newton mechanics with Neutrosophy and Quad-stage method, and present New Newton Mechanics taking law of conservation of energy as unique source law.

The process of negating "the four color theorem" with Neutrosophy and Quad-stage method, and deriving "the two color theorem" and "the five color theorem" to replace "the four color theorem", can be divided into four stages.

The first stage (stage of "general theses"), for the beginning of development, the thesis (namely "the four color theorem") should be widely, deeply, carefully and repeatedly contacted, explored, analyzed, perfected and so on.

About these aspects, especially the brilliant accomplishments of proving "the four color theorem" with computer, many discussions could be found in related literatures, therefore we will not repeat them here, while the only topic we should discuss is finding the shortcomings in the existing proofs of "the four color theorem". In fact, many scholars believe that the existing proofs of "the four color theorem" with computer are not satisfactory. Therefore, many new proofs are still appeared unceasingly. For example, a very simple proof of this theorem was given in reference [5] recently.

For the different proofs of "the four color theorem", we can name the results as "the Appel-Haken's four color theorem", "the Chen Jianguo's four color theorem", and the like.

In addition, some experts still ask the question that whether or not "the four color theorem" is correct.

On other viewpoints about "the four color theorem", we will discuss them in detail below, in order to avoid duplication.

The second stage (the stage of "general antitheses"), the opposites (antitheses) should be discussed carefully.

For "the four color theorem", there are many opposites (antitheses). For example: "the two color theorem", "the three color theorem", "the five color theorem", "the six color theorem", and so on. As "the four color theorem" is denied, we will select the suitable one from these theorems.

The third stage is the one of the most important and the most complicated universal relations. The purpose of this provision stage is to establish the universal relations in the widest scope.

To link and combine with Neutrosophy, we will take into account the intermediate part; on a map, what is the "intermediate part"? After careful analyses, we can identify two main types of "intermediate part". The first one is already considered in the existing proofs of "the four color theorem", such as the third region between two regions. The second one is not considered in the existing proofs of "the four color theorem", such as the boundary between two regions. Obviously, so far the color of boundary is not considered in any existing proof also. However, whether or not the boundary and its color should be considered" We believe that they should be taken into account, because the boundary is the objective reality on a map.

The fourth stage (the stage of “general syntheses”), our purpose is to negate "the four color theorem", and reach the results that are the best or agreed with some conditions.

Firstly, we suppose that, in the case without considering the color of the boundary, "the four color theorem" is correct. In other words, in this case, "the Appel-Haken’s four color theorem", "the Chen Jianguo’s four color theorem", and the like, are all correct.

Secondly, we introduce the concept of "boundary part" (or “totality of boundary”). Because the boundary on a map has a certain width, the "boundary part" (or “totality of boundary”) can be defined as: a special connected region constituted by all the boundaries.

After considering the boundary and its color, there are two situations should be considered. The first one is that, the original color distribution (namely the color distribution on a map that is agreed with the principle of "the four color theorem" ; or the color distribution before considering the boundary) can be changed. The second one is that, the original color distribution cannot be changed.

For the first situation, only two colors will be sufficient: one color could be used for all the boundaries; another color for all the regions. For example, on a country’s black and white map, the color of all the boundaries of states or provinces could be black; while the color of all the states or provinces could be white. Obviously, this is also the general drawing method for the black and white map. At this time, "the four color theorem" is replaced by "the two color theorem".

Figure 2 is a black and white world map.

Figure 2 A black and white world map

For the second situation, because the original color distribution cannot be changed (four colors are required), the fifth color is required. Otherwise, supposing that the color of the boundaries is one of the four colors for regions, for the reason that the boundary has a certain width and can be considered as the special region, therefore the color of this special region will be the same color as at least one ordinary region, thereby the principle of color distribution will be violated (the colors of the adjacent two regions will be the same). Obviously, in this case five colors are required. For example, in Figure 1, the colors for different regions are red, yellow, green and blue respectively, while the color for boundaries is black.

In Figure 1, the number of the required colors is as follows

$$4+1=5$$

At this time, "the four color theorem" is replaced by "the five color theorem"..

Thus, we already prove that, as considering the boundary and its color, "the four color theorem" is incorrect.

Conclusion

As considering the boundary and its color, if the original color distribution (namely the color distribution on a map that is agreed with the principle of "the four color theorem" ; or the color distribution before considering the boundary) can be changed, "the four color theorem" is replaced by "the two color theorem"; if the original color distribution cannot be changed, "the four color theorem" is replaced by "the five color theorem".

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NSS

A new method of measuring similarity between two neutrosophic soft sets and its application in pattern recognition problems

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Abstract. Smarandache in 1995 introduced the concept of neutrosophic set and in 2013 Maji introduced the notion of neutrosophic soft set, which is a hybridization of neutrosophic set and soft set. After its introduction neutrosophic soft sets become most efficient tools to deals with problems that contain uncertainty such as problem in social, economic system, medical diagnosis, pattern recognition, game theory, coding theory and so on. In this work a new method of measuring similarity measure and weighted similarity measure between two neutrosophic soft sets (NSSs) are proposed. A comparative study with existing similarity measures for neutrosophic soft sets also studied. A decision making method is established for neutrosophic soft set setting using similarity measures. Lastly a numerical example is given to demonstrate the possible application of similarity measures in pattern recognition problems.

Keywords: Fuzzy sets, soft sets, neutrosophic sets, neutrosophic soft sets, similarity measure, pattern recognition.

1 Introduction

The concept of fuzzy set theory was initiated by Prof. L. A. Zadeh in1965[20]. After its introduction several researchers have extended this concept in many directions. The traditional fuzzy set is characterized by the membership value or the grade of membership value. Sometimes it may be very difficult to assign the membership value for a fuzzy set. To overcome this difficulty the concept of interval valued fuzzy sets was proposed by L.A. Zadeh in1975[21]. In some real life problems in expert system, belief system, information fusion and so on, we must consider the truth-membership as well as the falsity-membership for proper description of an object in uncertain, ambiguous environment. Neither the fuzzy sets nor the interval valued fuzzy sets is appropriate for such a situation. Intuitionistic fuzzy setsintroduced by K. Atanassov[1] in 1986 and interval valued intuitionistic fuzzy setsintroduced by K. Atanassov and G. Gargov in 1989[2] are appropriate for such a situation.But these do not handle the indeterminate and inconsistent information which exists in belief system. F. Smarandache in 1995[16,17], introduced the concept of neutrosophicset,which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. Soft set theory[7, 11] has enriched its potentiality since its introduction by Molodtsov in 1999. Using the concept of soft set theory P. K. Maji in 2013[12] introduced neutrosophic soft set**.** Neutrosophic sets and neutrosophic soft sets now become the most useful mathematical tools to deal with the problems which involve the indeterminate and inconsistent information.

Similarity measure is an important topic in the fuzzy set theory. The similarity measure indicates the degree of similarity between two fuzzy sets. P. Z. Wang[18] first introduced the concept of similarity measure of fuzzy sets and gave a computational formula. Science then, similarity measure of fuzzy sets has attracted several researchers' interest and has been investigated more. Domain of application of similarity measure of fuzzy sets are fuzzy clustering, image processing, fuzzy reasoning, fuzzy neural network, pattern recognition, medical diagnosis, game theory, coding theory and several problems that contain uncertainties. S. M. Chen[5, 6] proposedsimilarity between vague sets, similarity measure of soft sets was studied by P. Majumderet al.[8, 9, 10] and W.K. Min[13], NaimCagman and Irfan Deli[4] introduced similarity measure for intuitionistic fuzzy soft sets, severalsimilarity measures for interval-valued fuzzy soft sets were studied by A. Mukherjee and S. Sarkar[14]. Said Broumi and FlorentinSmarandache[3] introduced the concept of several similarity measures of neutrosophic sets and Jun Ye[19] introduced the concept of similarity measures between interval neutrosophic sets. Recently A. Mukherjee and S. Sarkar[15] introduced several methods of similarity measure for neutrosophic soft sets

Pattern recognition problem has been one of the fastest growing areas during the last two decades because of its usefulness and fascination. The main objective of pattern recognition problems is supervised or unsupervised classification of unknown patterns. Among the various frameworks in which pattern recognition problem has been traditionally formulated the statistical approach has been most intensively studied and used in practice.

In this paper a new method of measuring degree of similarity and weighted similarity measure between two neutrosophic soft set is proposed and some basic properties of similarity measure also are studied. A decision making method is established based on the proposed similarity measure. An illustrative numerical example is given to demonstrate the application of proposed decision making method in a supervised pattern recognition problem that is on the basis of the knowledge of the known pattern our aim is to classify the unknown pattern.

The rest of the paper is organized as --- section 2: some preliminary basic definitions are given in this section. In section 3 similarity measures, weighted similarity measure between two NSSs is defined with examples and some basic properties are studied. In section 4 a decision making method is established with an example in a pattern recognition problem. In Section 5 a comparative study of similarity measures between existing and proposed method is given. Finally in section 6 some remarks of the proposed similarity measure between NSSs and the proposed decision making method are given.

2 Preliminaries and related works

In this section we briefly review some basic definitions related to neutrosophic soft sets which will be used in the rest of the paper.

2.1 Definition[20] Let X be a non empty collection of objects denoted by x. Then a *fuzzy set* (*FS for short*) α in X is a set of ordered pairs having the form $\alpha = \{(x, \mu_\alpha(x)) : x \in X\}$,

where the function $\mu_\alpha : X \to [0,1]$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in α .The interval M = $[0,1]$ is called membership space.

2.2 Definition[21] Let D[0, 1] be the set of closed sub-intervals of the interval [0, 1]. An *interval-valued fuzzy set* in X, $X \neq \phi$ and Card(X) = n, is an expression A given by $A = \{(x, M_A(x)) : x \in X\}$, where $M_A : X \to D[0,1]$.

2.3 Definition[1] Let X be a non empty set. Then an *intuitionistic fuzzy set (IFS for short)* A is a set having the form A={(x, $\mu_A(x)$, $\gamma_A(x)$): x∈X} where the functions μ_A: X→[0,1] and γ_A: X→[0,1] represents the degree ofmembership and the degree of non-membership respectively of each element x∈X and $0 \leq \mu_A(x)+\gamma_A(x) \leq 1$ for each x∈X.

2.4 Definition[7,11] Let U be an initial universe and E be a set of parameters. Let P(U) denotes the power set of U and A⊆ E. Then the pair (F,A) is called a *soft set* over U, where F is a mapping given byF:A→P(U).

2.5 Definition[16,17] A neutrosophic set A on the universe of discourse X is defined as $A = \{(x, T_A(x), I_A(x), F_A(x)), x \in X\}$ where *T, I, F* : $X \to]^-0,1^+[$ and

$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0,1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of [0,1] that is

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3 .$$

2.6 Definition[12] Let U be the universe set and E be the set of parameters. Also let A ⊆ E and P(U) be the set of all neutrosophic sets of U. Then the collection (F, A) is called neutrosophic soft set over U, where F is a mapping given by F: A $\to$P(U).

2.7 Definition[12] Let (F,E_1) and (G,E_2) be two neutrosophic soft sets over the common universe U , where E_1, E_2 are two sets of parameters. Then (F,E_1) is said to be neutrosophic soft subset of (G,E_2) if $E_1 \subseteq E_2$ and $T_{F(e)}(x) \leq T_{G(e)}(x)$, $I_{F(e)}(x) \leq I_{G(e)}(x)$, $F_{F(e)}(x) \geq F_{G(e)}(x)$, $\forall e \in E_1, x \in U$. If (F,E_1) be neutrosophic soft subset of (G,E_2) then it is denoted by $(F, E_1) \subseteq (G, E_2)$.

2.8 Definition[12]

Let E = {e_1, e_2, e_3, ... ,e_m} be the set of parameters, then the set denoted by ˥E and defined by ˥E = {┐e_1, ┐e_2, ┐e_3, ... , ┐e_m}, where ┐e_i = not e_i , $\forall i$ is called NOT set of the set of parameters E. Where ˥ and ┐different operators.

2.9 Definition[12] The complement of a neutrosophic soft set (F,E) denoted by $(F,E)^c$ is defined as $(F,E)^c$= (F^c ,˥E), where F^c : ˥E$\to P(U)$ is amapping given by

$F^c(\alpha) =$ neutrosophic soft complement with

$T_{F^c(x)} = F_{F(x)}$, $I_{F^c(x)} = I_{F(x)}$ and $F_{F^c(x)} = T_{F(x)}$.

3 Similarity measure for neutrosophic soft sets(NSSs)

In this section we have proposed a new method for measuring similarity measure and weighted similarity measure for NSSs and some basic properties are also studied.

3.1 Similarity measure

U = {x_1, x_2, x_3, ...,x_n} be the universe of discourse and E = {e_1, e_2, e_3, ... ,e_m} be the set of parameters and (N_1,E), (N_2,E) be two neutrosophic soft sets over U(E). Then the similarity measure between NSSs (N_1,E) and (N_2,E) is denoted by $Sim(N_1, N_2)$ and is defined as follows :

$$Sim(N_1, N_1) = \frac{1}{3mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\Big(3 - \left|T_{N_1}(x_i)(e_j) - T_{N_2}(x_i)(e_j)\right| - \left|I_{N_1}(x_i)(e_j) - I_{N_2}(x_i)(e_j)\right| - \left|F_{N_1}(x_i)(e_j) - F_{N_2}(x_i)(e_j)\right|\Big) \quad \text{……………………………… (1)}$$

3.2 Theorem If $Sim(N_1, N_2)$ be the similarity measure between two NSSs (N_1,E) and (N_2,E) then

(i) $0 \le Sim(N_1, N_2) \le 1$

(ii) $Sim(N_1, N_2) = Sim(N_2, N_1)$

(iii) $Sim(N_1, N_1) = 1$

(iv) $If\ (N_1, E) \subseteq (N_2, E) \subseteq (N_3, E)\ then$ $Sim(N_1, N_3) \le Sim(N_2, N_3)$

Proof:

(i) Obvious from definition 3.1 .

(ii) Obvious from definition 3.1 .

(iii) Obvious from definition 3.1

(iv) Let U = {x_1, x_2, x_3, ..., x_n} be the universe of discourse and E = {e_1, e_2, e_3, ... ,e_m} be the set of parameters and (N_1,E), (N_2,E), (N_3,E) be three neutrosophic soft sets over U(E), such that $(N_1, E) \subseteq (N_2, E) \subseteq (N_3, E)$ .Now by definition of neutrosophic soft sub sets (Maji, 2013) we have

$$T_{N_1}(x_i)(e_j) \le T_{N_2}(x_i)(e_j) \le T_{N_3}(x_i)(e_j)$$
$$I_{N_1}(x_i)(e_j) \le I_{N_2}(x_i)(e_j) \le I_{N_3}(x_i)(e_j)$$
$$F_{N_1}(x_i)(e_j) \ge F_{N_2}(x_i)(e_j) \ge F_{N_3}(x_i)(e_j)$$

$$\Rightarrow \left|T_{N_1}(x_i)(e_j) - T_{N_3}(x_i)(e_j)\right| \ge \left|T_{N_2}(x_i)(e_j) - T_{N_3}(x_i)(e_j)\right|,$$
$$\left|I_{N_1}(x_i)(e_j) - I_{N_3}(x_i)(e_j)\right| \ge \left|I_{N_2}(x_i)(e_j) - I_{N_3}(x_i)(e_j)\right|,$$
$$\left|F_{N_1}(x_i)(e_j) - F_{N_3}(x_i)(e_j)\right| \ge \left|F_{N_2}(x_i)(e_j) - F_{N_3}(x_i)(e_j)\right|$$

$$\Rightarrow \Big(\left|T_{N_1}(x_i)(e_j) - T_{N_3}(x_i)(e_j)\right| + \left|I_{N_1}(x_i)(e_j) - I_{N_3}(x_i)(e_j)\right| + \left|F_{N_1}(x_i)(e_j) - F_{N_3}(x_i)(e_j)\right|\Big) \ge \Big(\left|T_{N_2}(x_i)(e_j) - T_{N_3}(x_i)(e_j)\right| + \left|I_{N_2}(x_i)(e_j) - I_{N_3}(x_i)(e_j)\right| + \left|F_{N_2}(x_i)(e_j) - F_{N_3}(x_i)(e_j)\right|\Big)$$

$$\Rightarrow \Big(3 - \left|T_{N_1}(x_i)(e_j) - T_{N_3}(x_i)(e_j)\right| - \left|I_{N_1}(x_i)(e_j) - I_{N_3}(x_i)(e_j)\right| - \left|F_{N_1}(x_i)(e_j) - F_{N_3}(x_i)(e_j)\right|\Big) \le \Big(3 - \left|T_{N_2}(x_i)(e_j) - T_{N_3}(x_i)(e_j)\right| - \left|I_{N_2}(x_i)(e_j) - I_{N_3}(x_i)(e_j)\right| - \left|F_{N_2}(x_i)(e_j) - F_{N_3}(x_i)(e_j)\right|\Big)$$

$$\Rightarrow \frac{1}{3mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\Big(3 - \left|T_{N_1}(x_i)(e_j) - T_{N_3}(x_i)(e_j)\right| - \left|I_{N_1}(x_i)(e_j) - I_{N_3}(x_i)(e_j)\right| - \left|F_{N_1}(x_i)(e_j) - F_{N_3}(x_i)(e_j)\right|\Big) \le \frac{1}{3mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\Big(3 - \left|T_{N_2}(x_i)(e_j) - T_{N_3}(x_i)(e_j)\right| - \left|I_{N_2}(x_i)(e_j) - I_{N_3}(x_i)(e_j)\right| - \left|F_{N_2}(x_i)(e_j) - F_{N_3}(x_i)(e_j)\right|\Big)$$

$\Rightarrow \quad Sim(N_1, N_3) \le Sim(N_2, N_3)$ [By equation (1)]

3.3 Weighted similarity measure

Let U = {x_1, x_2, x_3, ...,x_n} be the universe of discourse and E = {e_1, e_2, e_3, ... ,e_m} be the set of parameters and (N_1,E), (N_2,E) be two neutrosophic soft sets over U(E). Now if we consider weights w_i of x_i (i = 1,2,3,.....,n) then the weighted similarity measure between NSSs (N_1,E) and (N_2,E) is denoted by $WSim(N_1, N_2)$ is proposed as follows :

$$WSim(N_1, N_2) = \frac{1}{3m}\sum_{i=1}^{n}\sum_{j=1}^{m} w_i\Big(3 - \left|T_{N_1}(x_i)(e_j) - T_{N_2}(x_i)(e_j)\right| - \left|I_{N_1}(x_i)(e_j) - I_{N_2}(x_i)(e_j)\right| - \left|F_{N_1}(x_i)(e_j) - F_{N_2}(x_i)(e_j)\right|\Big)$$

.... (2)

Where $w_i \in [0,1]$, i = 1,2,3,.......,n and $\sum_{i=1}^{n} w_i = 1$. In particular if we take $w_i = \frac{1}{n}$, i = 1,2,3,... ....,n then $WSim(N_1, N_2) = Sim(N_1, N_2)$.

3.4 Theorem Let If $WSim(N_1, N_2)$ be the similarity measure between two NSSs (N_1,E) and (N_2,E) then

(i) $0 \le WSim(N_1, N_2) \le 1$

(ii) $WSim(N_1, N_2) = WSim(N_2, N_1)$

(iii) $WSim(N_1, N_1) = 1$

(iv) $If\ (N_1, E) \subseteq (N_2, E) \subseteq (N_3, E)\ then$ $WSim(N_1, N_3) \le WSim(N_2, N_3)$

Proof:

(i) Obvious from definition 3.3 .

(ii) Obvious from definition 3.3 .

(iii) Obvious from definition 3.3 .

(iv) Similar to proof of (iv) of theorem 3.2.

3.5 Example Here we consider example 3.3 of [15]. Let U={x_1,x_2,x_3} be the universal set and E={e_1,e_2, e_3} be the set of parameters. Let (N_1,E) and (N_2,E) be two neutrosophic soft sets over U such that their tabular representations are as follows:

Table 1: tabular representation of (N_1,E)

(N_1,E)	e_1	e_2	e_3
x_1	(0.2,0.4,0.7)	(0.5,0.1,0.3)	(0.4,0.2,0.3)
x_2	(0.7,0.0,0.4)	(0.0,0.4,0.8)	(0.5,0.7,0.3)
x_3	(0.3,0.4,0.3)	(0.6,0.5,0.2)	(0.5,0.7,0.1)

Table 2: tabular representation of (N_2,E)

(N_2,E)	e_1	e_2	e_3
x_1	(0.3,0.5,0.4)	(0.4,0.3,0.4)	(0.5,0.1,0.2)
x_2	(0.7,0.1,0.5)	(0.2,0.4,0.7)	(0.5,0.6,0.3)
x_3	(0.3,0.3,0.4)	(0.7,0.5,0.2)	(0.6,0.6,0.2)

Now by definition 3.1 similarity measure between (N_1,E) and (N_2,E) is given by *Sim*(N_1,N_2) = 0.91

3.6 Example

Let U={x_1,x_2,x_3} be the universal set and E={e_1,e_2, e_3} be the set of parameters. Let (A_1,E) and (A_2,E) be two neutrosophic soft sets over U such that their tabular representations are as follows:

Table 3: tabular representation of (A_1,E)

(A_1,E)	e_1	e_2	e_3
x_1	(0.1,0.2,0.1)	(0.2,0.1,0.1)	(0.1,0.1,0.2)
x_2	(0.3,0.1,0.2)	(0.2,0.2,0.3)	(0.7,0.2,0.2)
x_3	(0.9,0.3,0.1)	(0.1,0.1,0.2)	(0.2,0.3,0.8)

Table 4: tabular representation of (A_2,E)

(A_2,E)	e_1	e_2	e_3
x_1	(0.9,0.9,0.8)	(0.8,0.7,0.9)	(0.9,0.8,0.9)
x_2	(0.9,0.8,0.8)	(0.8,0.8,0.9)	(0.1,0.9,0.9)
x_3	(0.1,0.9,0.9)	(0.8,0.8,0.9)	(0.8,0.9,0.2)

Now by definition 3.1 similarity measure between (A_1,E) and (A_2,E) is given by *Sim*(A_1,A_2) = 0.32 .

3.7 Definition Let (N_1,E) and (N_2,E) be twoNSSsover the universe U. Then (N_1,E) and (N_2,E) are said be α - similar , denoted by $(N_1, E)\overset{\alpha}{\simeq}(N_2, E)$ if and only if *Sim*(N_1,N_2) > α for α ∈ (0,1). We call the two NSSs significantly similar if *Sim*(N_1,N_2) > 0.5 .

3.8 Definition Let (N_1,E) and (N_2,E) be twoNSSs over the universe U. Then (N_1,E) and (N_2,E) are said be substantially-similar if *Sim*(N_1,N_2) > 0.95 and is denoted by (N_1,E) ≡ (N_2,E) .

3.9 Definition In example 3.5 Sim(N_1,N_2) = 0.91> 0.5 , therefore (N_1,E) and (N_2,E) are significantly similar. Again in example 3.6 *S*im(A_1,A_2) = 0.32 < 0.5 , therefore (A_1,E) and (A_2, E) are not significantly similar.

3.10 Theorem

Let (N_1,E) and (N_2,E) be two neutrosophic soft sets over the universe U and $\left(N_1^c, E\right)$ and $\left(N_2^c, E\right)$ be their complements respectively. Then

i. if $Sim(N_1, N_2) = \mu$ then $Sim(N_1^c, N_2^c) = \mu$, ($0 \le \mu \le 1$) .

ii. if $WSim(N_1, N_2) = \lambda$ then $WSim(N_1^c, N_2^c) = \lambda$, $0 \le \lambda \le 1$.

Proof : Straight forward from definition **2.7**, **3.1** and **3.3** .

4 Application of similarity measure of NSSs in pattern recognition problem

In this section we developed an algorithm for pattern recognition problem in neutrosophic soft set setting using similarity measure. A numerical example is given to demonstrate the effectiveness of the proposed method.
Steps of algorithm are as follows:

Step1: construct NSS(s) $\hat{N}_i$ (i = 1,2,3........,n)as ideal pattern(s).

Step2: construct NSS(s) $\hat{M}_j$ (j = 1,2,3........,m)for sample pattern(s) which is/are to be recognized.

Step3: calculate similarity measure between NSS(s) for ideal pattern(s) and sample pattern(s).

Step4: recognize sample pattern(s) under certain predefined conditions.

4.1 Example In order to demonstrate the application of the proposed method of measuring similarity between NSSs, we consider the medical diagnosis problem discussed in example 5.1 [15] as a supervised pattern recognition problem . In this example our proposed method is applied to determine whether an ill person having some visible symptoms is suffering from cancer or not suffering from cancer. We first construct an ideal NSS (known pattern) for cancer disease and NSS(sample pattern) for the ill person(s) and we also assume that if the similarity measure between these two NSSs is greater than or equal to **0.75** then the ill person is possibly suffering from the diseases.

Let U be the universal set, which contains only two elements x_1 = severe and x_2 = mild i.e. U={x_1,x_2}. Here the set of parameters E is a set of certain visible symptoms. Let E = {e_1,e_2,e_3,e_4,e_5}, where e_1 = headache, e_2 = fatigue, e_3 = nausea and vomiting , e_4 = skin changes, e_5 = weakness.

Step 1: construct an ideal NSS ($\hat{N}$,E) for illness (cancer) which can be done with the help of medical expert.

Table 5: tabular representation of NSS ($\hat{N}$,E) for cancer.

($\hat{N}$,E)	e_1	e_2	e_3	e_4	e_5
x_1	(0.6,0.2,0.3)	(0.7,0.3,0.4)	(0.4,0.3,0.6)	(0.8,0.2,0.3)	(0.5,0.3,0.2)
x_2	(0.4,0.1,0.2)	(0.3,0.1,0.2)	(0.2,0.2,0.4)	(0.3,0.1,0.4)	(0.2,0.1,0.3)

Step 2: construct NSSs for ill persons (patients) X and Y.

Table 6: tabular representation of NSS ($\hat{M}_1$,E) for patient X.

($\hat{M}_1$,E)	e_1	e_2	e_3	e_4	e_5
x_1	(0.7,0.3,0.4)	(0.8,0.2,0.5)	(0.4,0.2,0.5)	(0.8,0.1,0.2)	(0.5,0.3,0.2)
x_2	(0.3,0.2,0.3)	(0.2,0.2,0.3)	(0.3,0.1,0.3)	(0.3,0.2,0.3)	(0.1,0.2,0.2)

Table 7: tabular representation of NSS ($\hat{M}_2$, E) for patient Y.

($\hat{M}_2$,E)	e_1	e_2	e_3	e_4	e_5
x_1	(0.2,0.5,0.8)	(0.1,0.0,0.8)	(0.8,0.6,0.1)	(0.1,0.5,0.8)	(0.9,0.6,0.8)
x_2	(0.9,0.6,0.7)	(0.7,0.5,0.6)	(0.7,0.6,0.1)	(0.8,0.7,0.9)	(0.8,0.7,0.7)

Step 3: By definition 3.1 similarity measure between ($\hat{N}$,E) and ($\hat{M}_1$,E) is given by $Sim(\hat{N},\hat{M}_1) = 0.91$ and similarity measure between ($\hat{N}$,E) and ($\hat{M}_2$,E) is given by $Sim(\hat{N},\hat{M}_2) = 0.54$.

Step 4: Since $Sim(\hat{N},\hat{M}_1) = 0.91 >$ **0.75** therefore patient X is possibly suffering from cancer. Again since $Sim(\hat{N},\hat{M}_2) = 0.54 <$ **0.75** therefore patient Y is possibly not suffering from cancer.

The result obtained here is same as the result obtained in [15].

5 Comparison of different similarity measures for NSSs

In this section effectiveness of the proposed method is demonstrated by the comparison between the proposed similarity measure and existing similarity measures in NSS setting. Here we consider NSSs of examples 3.5, 3.6 and 4.1 for comparison of similarity measures as given in table 8.

Table 8: comparison of different similarity measures

NSSs→ / Similarity measure based on ↓	(N_1,N_2)	(A_1,A_2)	$(\hat{N}, \hat{M}_1)$	$(\hat{N}, \hat{M}_2)$
Hamming distance	0.71	0.24	0.69	0.31
Set theoretic approach	0.80	0.20	0.75	0.33
Proposed method	**0.91**	**0.32**	**0.92**	**0.54**

Table 8 shows that each method has its own measuring but the results of similarity measures by proposed method are emphatic over the other.

Conclusions

In this paper we proposed a new method of measuring degree of similarity and weighted similarity between two neutrosophic soft sets and studied some properties of similarity measure. Based on the comparison between the proposed method and existing methods introduced by Mukherjee and Sarkar[15], proposed method has been found to give strong similarity measure. A decision making method is developed based on similarity measure. Finally a fictitious numerical example is given to demonstrate the application of similarity measure of NSSs in a supervised pattern recognition problem. Next research work is to develop the application in other fields.

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