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Neutrosophic Vector Spaces

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Abstract. The objective of this paper is to study *neutrosophic* vector spaces. Some basic definitions and properties of the classical vector spaces are generalized. It is shown that every *neutrosophic* vector space over a *neutrosophic* field (resp. a field) is a vector space. Also, it is shown that an element of a *neutrosophic* vector space over a *neutrosophic* field can be infinitely expressed as a linear combination of some elements of the *neutrosophic* vector space. *Neutrosophic* quotient spaces and *neutrosophic* vector space homomorphism are also studied.

Keywords: Weak *neutrosophic* vector space, strong *neutrosophic* vector space, field, *neutrosophic* field.

1 Introduction and Preliminaries

The theory of fuzzy set introduced by L. Zadeh [9] is mainly concerned with the measurement of the degree of membership and non-membership of a given abstract situation. Despite its wide range of real life applications, fuzzy set theory cannot be applied to model an abstract situation where indeterminancy is involved. In his quest to modelling situations involving indeterminates, F. Smarandache introduced the theory of *neutrosophy* in 1995. *Neutrosophic* logic is an extension of the fuzzy logic in which indeterminancy is included. In the *neutrosophic* logic, each proposition is characterized by the degree of truth in the set (T), the degree of falsehood in the set (F) and the degree of indeterminancy in the set (I) where T,F,I are subsets of] -,0,1,+ [. *Neutrosophic* logic has wide applications in science, engineering, IT, law, politics, economics, finance, etc. The concept of *neutrosophic* algebraic structures was introduced by F. Smarandache and W.B. Vasantha Kandasamy in 2006. However, for details about *neutrosophy* and *neutrosophic* algebraic structures, the reader should see [1, 2, 3, 4, 5, 6, 7, 8].

Definition 1.1. Let U be a universe of discourse and let M be a subset of U. M is called a neutrosophic set if an element $x = x(T, I, F) \in U$ belongs to M in the following way:

(1) x is t% true in M,
(2) x is i% indeterminate in M, and
(3) x is f% false in M,

where $t \in T$, $i \in I$ and $f \in F$.

It is possible to have t+i+f=1 as in the case of classical and fuzzy logics and probability. Also, it is possible to have t+i+f<1 as in the case of intuitionistic logic and as in the case of paraconsistent logic, it is possible to have t+i+f>1.

Remark 1. Statically, T,I,F are subsets of]-,0,1,+[but dynamically, they are functions/operators depending on many known or unknown parameters.

Example 1. The probability of that a student will pass his final year examination in Mathematics is 60% true according to his Mathematics Teacher from Year 1, 25 or 30-35% false according to his present poor performance, and 15 or 20% indeterminate due to sickness during his final year examination.

Definition 1.2. Let $(G,*)$ be any group and let $G(I) = \langle G \cup I \rangle$. The couple $(G(I),*)$ is called a *neutrosophic* group generated by G and I under the binary operation $*$. The indeterminancy factor I is such that $I * I = I$. If $*$ is ordinary multiplication, then $IastI * ... * I = I^n = I$ and if $*$ is ordinary addition, then $I * I * I * ... * I = nI$ for $n \in \mathbb{N}$.

If $a * b = b * a$ for all $a, b \in G(I)$, we say that G(I) is commutative. Otherwise, G(I) is called a non-commutative *neutrosophic* group.

Theorem 1.3. *[5] Let G(I) be a neutrosophic group. Then,*

(1) G(I) in general is not a group;
(2) G(I) always contain a group.

Example 2. [3] Let G(I)={e, a, b, c, I, aI, bI, cI} be a set, where $a^2=b^2=c^2=e$, bc=cb=a, ac=ca=b, ab=ba=c. Then (G(I),.) is a commutative *neutrosophic* group.

Definition 1.4. Let (K,+,.) be any field and let $K(I) = \langle K \cup I \rangle$ be a *neutrosophic* set generated by K and I. The triple (K(I),+,.) is called a *neutrosophic* field. The zero element $0 \in K$ is represented by 0+0I in K(I) and $1 \in K$ is represented by 1+0I in K(I).

Definition 1.5. Let K(I) be a neutrosophic field and let F(I) be a nonempty subset of K(I). F(I) is called a neutrosophic subfield of K(I) if F(I) is itself a neutrosophic field. It is essential that F(I) contains a proper subset which is a field.

Example 3. $\mathbb{Q}(I)$, $\mathbb{R}(I)$ and $\mathbb{C}(I)$ are respectively neutrosophic fields of rational, real and complex numbers. $\mathbb{Q}(I)$ is a neutrosophic subfield of $\mathbb{R}(I)$ and $\mathbb{R}(I)$ is a neutrosophic subfield of $\mathbb{C}(I)$.

2 Neutrosophic Vector Spaces

Definition 2.1. Let (V,+,.) be any vector space over a field K and let $V(I) = < V \cup I >$ be a *neutrosophic* set generated by V and I. The triple (V(I),+,.) is called a weak *neutrosophic* vector space over a field K. If V(I) is a *neutrosophic* vector space over a *neutrosophic* field K(I), then V(I) is called a strong *neutrosophic* vector space. The elements of V(I) are called *neutrosophic* vectors and the elements of K(I) are called *neutrosophic* scalars.

If $u = a + bI, v = c + dI \in V(I)$ where a,b,c and d are vectors in V and $\alpha = k + mI \in K(I)$ where k and m are scalars in K, we define:

$$u + v = (a + bI) + (c + dI)$$
$$= (a + c) + (b + d)I, \text{ and}$$
$$\alpha u = (k + mI).(a + bI)$$
$$= k.a + (k.b + m.a + m.b)I.$$

Example 4. (1) $\mathbb{R}(I)$ is a weak *neutrosophic* vector space over a field $\mathbb{Q}$ and it is a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{Q}(I)$.

(2) $\mathbb{R}^n(I)$ is a weak *neutrosophic* vector space over a field $\mathbb{R}$ and it is a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{R}(I)$.

(3) $M_{m\times n}(I) = \{[a_{ij}] : a_{ij} \in \mathbb{Q}(I)\}$ is a weak *neutrosophic* vector space over a field $\mathbb{Q}$ and it is a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{Q}(I)$.

Theorem 2.2. Every strong *neutrosophic* vector space is a weak *neutrosophic* vector space.

Proof. Suppose that V(I) is a strong *neutrosophic* space over a *neutrosophic* field K(I). Since $K \subseteq K(I)$ for every field K, it follows that V(I) is a weak *neutrosophic* vector space.

Theorem 2.3. *Every weak (strong) neutrosophic vector space is a vector space.*

Proof. Suppose that V(I) is a strong *neutrosophic* space over a *neutrosophic* field K(I). Obviously, (V(I),+,.) is an abelian group. Let $u = a + bI, v = c + dI \in V(I)$, $\alpha = k + mI, \beta = p + nI \in K(I)$ where $a, b, c, d \in V$ and $k, m, p, n \in K$. Then

(1) $\alpha(u + v) = (k + mI)(a + bI + c + dI)$
$= ka + kc + [kb + kd + ma + mb + mc + md]I$
$= (k + mI)(a + bI) + (k + mI)(c + dI)$
$= \alpha u + \alpha v$.

(2) $(\alpha + \beta)u = (k + mI + p + nI)(a + bI)$
$= ka + pa + [kb + pb + ma + na + mb + nb]I$
$= (k + mI)(a + bI) + (p + nI)(a + bI)$
$= \alpha u + \beta u$

(3) $(\alpha\beta)u = ((k + mI)(p + nI))(a + bI)$
$= kpa + [kpb + kna + mpa + mna + knb + mpb + mnb]I$
$= (k + mI)((p + nI)(a + bI))$
$= \alpha(\beta u)$

(4) For $1 + 1 + 0I \in K(I)$, we have
$1u = (1 + 0I)(a + bI)$
$= a(b + 0 + 0)I$
$= a + bI$.

Accordingly, V(I) is a vector space.

Lemma 2.4. *Let V(I) be a strong neutrosophic vector space over a neutrosophic field K(I) and let* u=a+bI,v=c+dI, $w = e + fI \in V(I), \alpha = k + mI \in K(I)$. *Then:*

(1) u+w=v+w implies u=v.
(2) α0=0.
(3) 0u=0.
(4) (-α)u=α(-u)=-(αu)

Definition 2.5. Let V(I) be a strong *neutrosophic* vector space over a *neutrosophic* field K(I) and let W(I) be a nonempty subset of V(I). W(I) is called a strong *neutrosophic* subspace of V(I) if W(I) is itself a strong *neutrosophic* vector space over K(I). It is essential that W(I) contains a proper subset which is a vector space.

Definition 2.6. Let V(I) be a weak *neutrosophic* vector space over a field K and let W(I) be a nonempty subset of V (I). W(I) is called a weak *neutrosophic* subspace of V(I) if W(I) is itself a weak *neutrosophic* vector space over K. It is essential that W(I) contains a proper subset which is a vector space.

Theorem 2.7. *Let V(I) be a strong neutrosophic vector space over a neutrosophic field K(I) and let W(I) be a nonempty subset of V(I). W(I) is a strong neutrosophic subspace of V(I) if and only if the following conditions hold*:

(1) $u, v \in W(I)$ *implies* $u + v \in W(I)$.
(2) $u \in W(I)$ *implies* $\alpha u \in W(I)$ for all $\alpha \in K(I)$.
(3) W(I) contains a proper subset which is a vector space.

Corollary 2.8. Let V(I) be a strong *neutrosophic* vector space over a *neutrosophic* field K(I) and let W(I) be a nonempty subset of V(I). W(I) is a strong *neutrosophic* subspace of V(I) if and only if the following conditions hold:

(1) $u, v \in W(I)$ *implies* $\alpha u + \beta v \in W(I)$ for all $\alpha, \beta \in K(I)$.
(2) W(I) contains a proper subset which is a vector space.

Example 5. Let V(I) be a weak (strong) *neutrosophic* vector space. V(I) is a weak (strong) *neutrosophic* subspace called a trivial weak (strong) *neutrosophic* subspace.

Example 6. Let $V(I) = \mathbb{R}^3(I)$ be a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{R}(I)$ and let

$$W(I) = \{(u = a + bI, v = c + dI, 0 = 0 + 0I) \in V(I) : a, b, c, d \in V\}.$$

Then W(I) is a strong *neutrosophic* subspace of V(I).

Example 7. Let $V(I) = M_{m\times n}(I) = \{[a_{ij}] : a_{ij} \in \mathbb{R}(I)\}$ be a strong *neutrosophic* vector space over $\mathbb{R}(I)$ and let

$$W(I) = A_{m\times n}(I) = \{[b_{ij}] : b_{ij} \in \mathbb{R}(I) \text{ and } trace(A) = 0\}.$$

Then W(I) is a strong *neutrosophic* subspace of V(I).

Theorem 2.9. *Let V(I) be a strong neutrosophic vector space over a neutrosophic field K(I) and let* $\{W_n(I)\}_{n\in\Lambda}$ *be a family of strong neutrosophic subspaces of V(I). Then* $\bigcap W_n(I)$ *is a strong neutrosophic subspace of V(I).*

Remark 2. Let V(I) be a strong *neutrosophic* vector space over a *neutrosophic* field K(I) and let $W_1(I)$ and $W_2(I)$ be two distinct strong *neutrosophic* subspaces of V(I). Generally, $W_1(I) \cup W_1(I)$ is not a strong *neutrosophic* subspace of V(I). However, if $W_1(I) \subseteq W_2(I)$ or $W_2(I) \subseteq W_1(I)$, then $W_1(I) \cup W_2(I)$ is a strong *neutrosophic* subspace of V(I).

Definition 2.10. Let U(I) and W(I) be any two strong *neutrosophic* subspaces of a strong *neutrosophic* vector space V(I) over a *neutrosophic* field K(I).

(1) The sum of U(I) and W(I) denoted by U(I)+W(I) is defined by the set

$$\{u + w : u \in U(I), w \in W(I)\}.$$

(2) V(I) is said to be the direct sum of U(I) and W(I) written $V(I) = U(I) \oplus W(I)$ if every element $v \in V(I)$ can be written uniquely as v=u+w where $u \in U(I)$ and $w \in W(I)$.

Example 8. Let $V(I) = \mathbb{R}^3(I)$ be a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{R}(I)$ and let

$$U(I) = \{(u, v, 0) : u, v \in \mathbb{R}(I)\} \text{ and}$$
$$W(I) = \{(0, 0, w) : w \in \mathbb{R}(I)\}.$$

Then $V(I) = U(I) \oplus W(I)$.

Lemma 2.11. *Let W(I) be a strong neutrosophic subspace of a strong neutrosophic vector space V(I) over a neutrosophic field K(I). Then:*

(1) W(I)+W(I)=W(I).
(2) *w+W(I)=W(I) for all* $w \in W(I)$.

Theorem 2.12. Let U(I) and W(I) be any two strong *neutrosophic* subspaces of a strong *neutrosophic* vector space V(I) over a *neutrosophic* field K(I). Then:

(1) U(I)+W(I) is a strong *neutrosophic* subspace of V(I).
(2) U(I) and W(I) are contained in U(I)+W(I).

Proof. (1) Obviously, U+W is a subspace contained in U(I)+W(I). Let $u, w \in U(I) + W(I)$ and let

$\alpha, \beta \in K(I)$. Then $u = (u_1 + u_2 I) + (w_1 + w_2 I)$, $w = (u_3 + u_4 I) + (w_3 + w_4 I)$ where $u_i \in U, w_i \in W$, i=1,2,3,4, $\alpha = k + mI, \beta = p + nI$ where $k, m, p, n \in K$. Now,

$$\alpha u + \beta w = [(ku_1 + pu_3) + [ku_2 + mu_1 + pu_4 + nu_3 + nu_4]I] + [(kw_1 + pw_3) + [kw_2 + mw_1 + pw_4 + nw_3 + nw_4]I] \in U(I) + W(I).$$

Accordingly, U(I)+W(I) is a strong *neutrosophic* subspace of V(I).

(2) Obvious.

Theorem 2.13. *Let U(I) and W(I) be strong neutrosophic subspaces of a strong neutrosophic vector space V(I) over a neutrosophic field K(I).* $V(I) = U(I) \oplus W(I)$ *if and only if the following conditions hold:*

(1) V(I)=U(I)+W(I) and

(2) $U(I) \cap W(I) = \{0\}$.

Theorem 2.14. *Let $V_1(I)$ and $V_2(I)$ be strong neutrosophic vector spaces over a neutrosophic field K(I). Then*

$$V_1(I) \times V_2(I) = \{(u_1, u_2) : u_1 \in V_1(I), u_2 \in V_2(I)\}$$

is a strong neutrosophic vector space over K(I) where addition and multiplication are defined by

$$(u_1 + u_2) + (v_1 + v_2) = (u_1 + v_1, u_2 + v_2),$$

$$\alpha(u_1, u_2) = (\alpha u_1, \alpha u_2).$$

Definition 2.15. Let W(I) be a strong *neutrosophic* subspace of a strong *neutrosophic* vector space V(I) over a *neutrosophic* field K(I). The quotient V(I)/W(I) is defined by the set

$$\{v + W(I) : v \in V(I)\}.$$

V(I)/W(I) can be made a strong *neutrosophic* vector space over a *neutrosophic* field K(I) if addition and multiplication are defined for all $u + W(I)), (v + W(I) \in V(I)/W(I)$ and $\alpha \in K(I)$ as follows:

$$(u + W(I)) + (v + W(I)) = (u + v) + W(I),$$

$$\alpha(u + W(I)) = \alpha u + W(I).$$

The strong *neutrosophic* vector space (V(I)/W(I),+,.) over a *neutrosophic* field K(I) is called a strong *neutrosophic* quotient space.

Example 9. Let V(I) be any strong *neutrosophic* vector space over a *neutrosophic* field K(I). Then V(I)/V(I) is strong *neutrosophic* zero space.

Definition 2.16. Let V(I) be a strong *neutrosophic* vector space over a *neutrosophic* field K(I) and let $v_1, v_2, ..., v_3 \in V(I)$.

(1) An element $v \in V(I)$ is said to be a linear combination of the $v_i s$ if

$v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$, where $\alpha_i \in K(I)$.

(2) $v_i s$ are said to be linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$$

implies that $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$. In this case, the set $\{v_1, v_2, ..., v_n\}$ is called a linearly independent set.

(3) $v_i s$ are said to be linearly dependent if

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$$

implies that not all α_i are equal to zero. In this case, the set $\{v_1, v_2, ..., v_n\}$ is called a linearly dependent set.

Definition 2.17. Let V(I) be a weak *neutrosophic* vector space over a field K(I) and let $v_1, v_2, ..., v_3 \in V(I)$.

(1) An element $v \in V(I)$ is said to be a linear combination of the $v_i s$ if

$v = k_1 v_1 + k_2 v_2 + ... + k_n v_n$, where $k_i \in K(I)$.

(2) $v_i s$ are said to be linearly independent if

$$k_1 v_1 + k_2 v_2 + ... + k_n v_n = 0$$

implies that $k_1 = k_2 = ... = k_n = 0$. In this case, the set $\{v_1, v_2, ..., v_n\}$ is called a linearly independent set.

(3) $v_i s$ are said to be linearly dependent if

$$k_1 v_1 + k_2 v_2 + ... + k_n v_n = 0$$

implies that not all α_i are equal to zero. In this case, the set $\{v_1, v_2, ..., v_n\}$ is called a linearly dependent set.

Theorem 2.18. *Let V(I) be a strong neutrosophic vector space over a neutrosophic field K(I) and let U[I] and W[I] be subsets of V(I) such that* $U[I]\subseteq W[I]$. *If U[I] is linearly dependent, then then W[I] is linearly dependent.*

Corrolary 2.19. Let V(I) be a strong *neutrosophic* vector space over a *neutrosophic* field K(I). Every subset of a linearly dependent set in V(I) is linearly dependent.

Theorem 2.20. *Let V(I) be a strong neutrosophic vector space over a neutrosophic field K(I) and let U[I] and W[I] be subsets of V(I) such that* $U[I]\subseteq W[I]$. *If U[I] is linearly independent, then then W[I] is linearly independent.*

Example 10. Let $V(I)=\mathbb{R}(I)$ be a weak *neutrosophic* vector space over a field $K(I)=\mathbb{R}(I)$. An element $v=7+24I\in V(I)$ is a linear combination of the elements $v_1=1+2I, v_2=2+3I\in V(I)$ since 7+24I= 27(1+2I)-10(2+3I).

Example 11. Let $V(I)=\mathbb{R}(I)$ be a strong *neutrosophic* vector space over a field $K(I)=\mathbb{R}(I)$. An element $v=7+24I\in V(I)$ is a linear combination of the elements $v_1=1+2I, v_2=2+3I\in V(I)$ since

$$\begin{aligned}7+24I&=(1+I)(1+2I)+(3+2I)(2+3I),\\ &\quad\text{where } 1+I, 3+2I\in K(I)\\ &=(5+(16/3)I)(1+2I)+(1-I)(2+3I),\\ &\quad\text{where } 5+(16/3)I, 1-I\in K(I)\\ &=(9+(4/3)I)(1+2I)+(-1+I)(2+3I),\\ &\quad\text{where } 9+(4/3)I, -1+I\in K(I)\\ &=(13-I)(1+2I)+(-3+2I)(2+3I),\\ &\quad\text{where } 13-I, -3+2I\in K(I)\end{aligned}$$

This example shows that the element v=7+24I can be infinitely expressed as a linear combination of the elements $v_1=1+2I, v_2=2+3I\in V(I)$. This observation is recorded in the next Theorem.

Theorem 2.21. Let V(I) be a strong *neutrosophic* vector space over a *neutrosophic* field K(I) and let $v_1, v_2, ..., v_3\in V(I)$. An element $v\in V(I)$ can be infinitely expressed as a linear combination of the $v_i s$.

Proof. Suppose that $v=\alpha_1 v_1+\alpha_2 v_2+...+\alpha_n v_n$, where $v=a+bI, v_1=a_1+b_1I, v_2=a_2+b_2I, ...,$ $v_n=a_n+b_nI$ and $\alpha_1=k_1+m_1I, \alpha_2=k_2+m_2I, ...,$ $\alpha_n=k_n+m_nI\in K(I)$. Then

$$a+bI=(k_1+m_1I)(a_1+b_1I)+(k_2+m_2I)(a_2+b_2I)$$
$$+...+(k_n+m_nI)(a_n+b_nI)$$

from which we obtain

$$a_1k_1+a_2k_2+...+a_nk_n=a,$$
$$b_1k_1+a_1m_1+b_1m_1+b_2k_2+a_2m_2+...+b_nk_n+$$
$$a_nm_n+b_nm_n=b$$

This is a linear system in unknowns $k_i, m_i, i=1,2,3,...,n$. Since the system is consistent and have infinitely many solutions, it follows that the $v_i s$ can be infinitely combined to produce v.

Remark 3. In a strong *neutrosophic* vector space V(I) over a *neutrosophic* field K(I), it is possible to have $0\neq v\in V(I), 0\neq\alpha\in K(I)$ and yet $\alpha v=0$. For instance, if v=k-kI and $\alpha=mI$ where $0\neq k, m\in K$, we have $\alpha v=mI(k-kI)=mkI-mkI=0$.

Theorem 2.22. *Let V(I) be a strong neutrosophic vector space over a neutrosophic field K(I) and let* $v_1=k_1-k_1I, v_2=k_2-k_2I, ..., v_n=k_n-k_nI$ *be elements of V(I) where* $0\neq k_i\in K$. *Then* $\{v_1, v_2, ..., v_n\}$ *is a linearly dependent set.*

Proof. Let $\alpha_1=p_1+q_1I, \alpha_2=p_2+q_2I, ...,$ $\alpha_n=p_n+q_nI$ be elements of K(I). Then

$$\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n=0$$

which implies that

$$(p_1+q_1I)(k_1-k_1I)+(p_2+q_2I)(k_2-k_2I)+...$$
$$+(p_n+q_nI)(k_n-k_nI)=0$$

from which we obtain

$$k_1p_1+k_2p_2+...+k_np_n=0.$$

This is a homogeneous linear system in unknowns $p_i, i=1,2,...,n$. It is clear that the system has infinitely many nontrivial solutions. Hence are not all zero and therefore, $\{v_1, v_2, ..., v_n\}$ is a linearly dependent set.

Example 12. (1) Let $V(I) = \mathbb{R}^n(I)$ be a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{R}(I)$. The set

$$\{v_1 = (1,0,0,...,0), v_2 = (0,1,0,...,0), ..., v_n = (0,0,0,...,1)\}$$

is a linearly independent set in V(I).

(2) Let $V(I) = \mathbb{R}^n(I)$ be a weak *neutrosophic* vector space over a *neutrosophic* field $\mathbb{R}$. The set

$$\{v_1 = (1,0,0,...,0), v_2 = (0,1,0,...,0), ..., v_k = (0,0,0,...,1), v_{k+1} = (I,0,0,...,0), v_{k+2} = (0,I,0,...,0), ..., v_n = (0,0,0,...,I)\}$$

is a linearly independent set in V(I).

Theorem 2.23. *Let V(I) be a strong neutrosophic vector space over a neutrosophic field K(I) and let X[I] be a nonempty subset of V(I). If L(X[I]) is the set of all linear combinations of elements of X[I], then:*

(1) L(X[I]) is a strong neutrosophic subspace of V(I) containing X[I].

(2) If W(I) is any strong neutrosophic subspace of V(I) containing X[I], then $L(X[I]) \subset W(I)$.

Proof. (1) Obviously, L(X[I]) is nonempty since X[I] is nonempty. Suppose that $v = a + bI \in X[I]$ is arbitrary, then for $\alpha = 1 + 0I \in K(I)$, we have $\alpha v = (1+0I)(a+bI) = a + bI \in L(X[I])$. Therefore, X[I] is contained in L(X[I]). Lastly, let $v, w \in L(X[I])$. Then

$$v = \alpha_1 v_1 + \alpha_2 v_2 + ..., \alpha_n v_n,$$
$$w = \beta_1 w_1 + \beta_2 w_2 + ..., \beta_n w_n,$$

Where $v_i, w_j \in X(I), \alpha_i, \beta_j \in K(I)$. For $\alpha, \beta \in K(I)$, it can be shown that $\alpha v + \beta w \in L(X[I])$. Since L(X) is a proper subset of L(X[I]) which is a subspace of V containing X, it follows that L(X[I]) is a strong *neutrosophic* subspace of V(I) containing X[I].

(2) Same as the classical case and omitted.

Definition 2.24. Let V(I) be a strong *neutrosophic* vector space over a *neutrosophic* field K(I).

(1) The strong neutrosophic subspace L(X[I]) of Theorem 2.23 is called the span of X[I] and it is denoted by *span*(X[I]).

(2) X[I] is said to span V(I) if V(I)=*span*(X[I]).

(3) A linearly independent subset $B[I] = \{v_1, v_2, ..., v_n\}$ of V(I) is called a basis for V(I) if $B[I]$ spans V(I).

Example 13. (1) Let $V(I) = \mathbb{R}^n(I)$ be a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{R}(I)$. The set

$$B[I] = \{v_1 = (1,0,0,...,0), v_2 = (0,1,0,...,0), ..., v_n = (0,0,0,...,1)\}$$

is a basis for V(I).

(2) Let $V(I) = \mathbb{R}^n(I)$ be a weak *neutrosophic* vector space over $\mathbb{R}$. The set

$$B = \{v_1 = (1,0,0,...,0), v_2 = (0,1,0,...,0), ..., v_k = (0,0,0,...,1), v_{k+1} = (I,0,0,...,0), v_{k+2} = (0,I,0,...,0), ..., v_n = (0,0,0,...,I)\}$$

is a basis for V(I).

Theorem 2.25. *Let V(I) be a strong neutrosophic vector space over a neutrosophic field K(I). The bases of V(I) are the same as the bases of V over a field K.*

Proof. Suppose that $B = \{v_1, v_2, ..., v_n\}$ is an arbitrary basis for V over the field K. Let v=a+bI be an arbitrary element of V(I) and let $\alpha_1 = k_1 + m_1 I$, $\alpha_2 = k_2 + m_2 I$, $..., \alpha_n = k_n + m_n I$ be elements of K(I). Then from $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$, we obtain

$$k_1 v_1 + k_2 v_2 + ... + k_n v_n = 0,$$
$$m_1 v_1 + m_2 v_2 + ... + m_n v_n = 0.$$

Since $v_i s$ are linearly independent, we have k_i=0 and m_j=0 where i,j=1,2,...,n. Hence $\alpha_i = 0, i = 1,2,...,n$. This shows that B is also a linearly independent set in V(I). To show that B spans V(I), let v=a+bI $= \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$, .Then we have

$$a = k_1 v_1 + k_2 v_2 + ... + k_n v_n,$$
$$b = m_1 v_1 + m_2 v_2 + ... + m_n v_n.$$

Since $a, b \in V$, it follows that v=a+bI can be written uniquely as a linear combination of $v_i s$. Hence, B is a basis for V(I). Since B is arbitrary, the required result follows.

Theorem 2.26. *Let V(I) be a strong neutrosophic vector space over a neutrosophic field K(I). Then the bases of V(I) over K(I) are contained in the bases of the weak neutrosophic vector space V(I) over a field K(I).*

Definition 2.27. Let V(I) be a strong *neutrosophic* vector space over a *neutrosophic* field K(I). The number of elements in the basis for V(I) is called the dimension of V(I) and it is denoted by $\dim_s(V(I))$. If the number of elements in the basis for V(I) is finite, V(I) is called a finite dimensional strong *neutrosophic* vector space. Otherwise, V(I) is called an infinite dimensional strong *neutrosophic* vector space.

Definition 2.28. Let V(I) be a weak *neutrosophic* vector space over a field K(I). The number of elements in the basis for V(I) is called the dimension of V(I) and it is denoted by $\dim_s(V(I))$. If the number of elements in the basis for V(I) is finite, V(I) is called a finite dimensional strong *neutrosophic* vector space. Otherwise, V(I) is called an infinite dimensional weak *neutrosophic* vector space.

Example 14. (1) The strong *neutrosophic* vector space of Example 12(1) is finite dimensional and $\dim_s(V(I))$ =n.

(2) The weak *neutrosophic* vector space of Example 12(2) is finite dimensional and $\dim_w(V(I)) = n$.

Theorem 2.29. *Let V(I) be a finite dimensional strong neutrosophic vector space over a field K(I). Then every basis of V(I) has the same number of elements.*

Theorem 2.30. *Let V(I) be a finite dimensional weak (strong) neutrosophic vector space over a field K(resp. over a neutrosophic field K(I)). If* $\dim_s(V(I)) = n$ *, then* $\dim_w(V(I)) = 2n$.

Theorem 2.31. Let W(I) be a strong neutrosophic subspace of a finite dimensional strong neutrosophic vector space V(I) over a neutrosophic field K(I). Then W(I) is finite dimensional and $\dim_s(W(I)) \le \dim_s(V(I))$ .If $\dim_s(W(I)) = \dim_s(V(I))$, then W(I)=V(I).

Theorem 2.32. *Let U(I) and W(I) be a finite dimensional strong neutrosophic subspaces of a strong neutrosophic vector space V(I) over a neutrosophic field K(I). Then U(I)+W(I) is a finite dimensional strong neutrosophic subspace of V(I) and*

$$\dim_s(U(I)+W(I)) = \dim_s(U(I)) + \dim_s(W(I)) - \dim_s(U(I)\cap W(I)).$$

If $V(I) = U(I) \oplus W(I)$, then

$$\dim_s(U(I)+W(I)) = \dim_s(U(I)) + \dim_s(W(I))$$

Definition 2.33. Let V(I) and W(I) be strong *neutrosophic* vector spaces over a *neutrosophic* field K(I) and let $\phi : V(I) \to W(I)$ be a mapping of V(I) into W(I). ϕ is called a *neutrosophic* vector space homomorphism if the following conditions hold:

(1) ϕ is a vector space homomorphism.

(2) $\phi(I) = I$.

If ϕ is a bijective *neutrosophic* vector space homomorphism, then ϕ is called a *neutrosophic* vector space isomorphism and we write $V(I) \cong W(I)$.

Definition 2.34. Let V (I) and W(I) be strong *neutrosophic* vector spaces over a *neutrosophic* field K(I) and let $\phi : V(I) \to W(I)$ be a *neutrosophic* vector space homomorphism.

(1) The kernel of ϕ denoted by $Ker\phi$ is defined by the set $\{v \in V(I) : \phi(v) = 0\}$.

(2) The image of ϕ denoted by $\operatorname{Im}\phi$ is defined by the set $\{w \in W(I) : \phi(v) = w$ for some $v \in V(I)\}$.

Example 15. Let V(I) be a strong *neutrosophic* vector space over a *neutrosophic* field K(I).

(1) The mapping $\phi : V(I) \to V(I)$ defined by $\phi(v) = v$ for all $v \in V(I)$ is *neutrosophic* vector space homomorphism and $Ker\phi$=0.

(2) The mapping $\phi : V(I) \to V(I)$ defined by $\phi(v) = 0$ for all $v \in V(I)$ is *neutrosophic* vector space homomorphism since $I \in V(I)$ but

$\phi(I) \neq 0$.

Definition 2.35. *Let V (I) and W(I) be strong neutrosophic vector spaces over a neutrosophic field K(I) and let* $\phi: V(I) \to W(I)$ *be a neutrosophic vector space homomorphism. Then:*

(1) $Ker\phi$ *is not a strong neutrosophic subspace of V(I) but a subspace of V.*

(2) $\operatorname{Im}\phi$ *is a strong neutrosophic subspace of W(I).*

Proof. (1) Obviously, $I \in V(I)$ but $\phi(I) \neq 0$. That $Ker\phi$ is a subspace of V is clear. (2) Clear.

Theorem 2.36. *Let V (I) and W(I) be strong neutrosophic vector spaces over a neutrosophic field K(I) and let* $\phi: V(I) \to W(I)$ *be a neutrosophic vector space homomorphism. If* $B = \{v_1, v_2, ..., v_n\}$ *is a basis for V(I), then* $\phi(B) = \{\phi(v_1), \phi(v_2), ..., \phi(v_n)\}$ *is a basis for W(I).*

Theorem 2.37. *Let W(I) be a strong neutrosophic subspace of a strong neutrosophic vector space V(I) over a neutrosophic field K(I). Let* $\phi: V(I) \to V(I)/W(I)$ *be a mapping defined by* $\phi(v) = v + W(I)$ *for all* $v \in V(I)$. *Then* ϕ *is not a neutrosophic vector space homomorphism.*

Proof. Obvious since $\phi(I) = I + W(I) = W(I) \neq I$.

Theorem 2.38. *Let W(I) be a strong neutrosophic subspace of a strong neutrosophic vector space V(I) over a neutrosophic field K(I) and let* $\phi: V(I) \to U(I)$ *be a neutrosophic vector space homomorphism from V(I) into a strong neutrosophic vector space U(I) over K(I). If* $\phi_{W(I)}: W(I) \to U(I)$ *is the restriction of* ϕ *to W(I) is defined by* $\phi_{W(I)}(w) = \phi(w)$ *for all, then:*

(1) $\phi_{W(I)}$ is a neutrosophic vector space homomorphism.

(2) $Ker\phi_{W(I)} = Ker\phi \cap W(I)$.

(3) $\operatorname{Im}\phi_{W(I)} = \phi(W(I))$.

Remark 4. If V(I) and W(I) are strong *neutrosophic* vector spaces over a *neutrosophic* field K(I) and $\phi, \psi: V(I) \to W(I)$ are *neutrosophic* vector space homomorphisms, then $(\phi + \psi)$ and $(\alpha\phi)$ are not *neutrosophic* vector space homomorphisms since $(\phi + \psi)(I) = \phi(I) + \psi(I) = I + I = 2I \neq I$ and $(\alpha\phi)(I) = \alpha\phi(I) = \alpha I \neq I$ for all $\alpha \in K(I)$. Hence, if $Hom(V(I), W(I))$ is the collection of all *neutrosophic* vector space homomorphisms from V(I) into W(I), then $Hom(V(I), W(I))$ is not a *neutrosophic* vector space over K(I). This is different from what is obtainable in the classical vector spaces.

Definition 2.39. Let U(I), V(I) and W(I) be strong *neutrosophic* vector spaces over a *neutrosophic* field K(I) and let $\phi: U(I) \to V(I), \psi: V(I) \to W(I)$ be *neutrosophic* vector space homomorphisms. The composition $\psi\phi: U(I) \to W(I)$ is defined by $\psi\phi(u) = \psi(\phi(u))$ for all $u \in U(I)$.

Theorem 2.40. *Let U(I), V(I) and W(I) be strong neutrosophic vector spaces over a neutrosophic field K(I) and let* $\phi: U(I) \to V(I), \psi: V(I) \to W(I)$ *be neutrosophic vector space homomorphisms. Then the composition* $\psi\phi: U(I) \to W(I)$ *is a neutrosophic vector space homomorphism.*

Proof. Clearly, $\psi\phi$ is a vector space homomorphism. For $u = I \in U(I)$, we have:

$$\begin{aligned}\psi\phi(I) &= \psi(\phi(I)) \\ &= \psi(I) \\ &= I.\end{aligned}$$

Hence $\psi\phi$ is a *neutrosophic* vector space homomorphism.

Corrolary 2.41. Let L(V(I)) be the collection of all *neutrosophic* vector space homomorphisms from V(I) onto V(I). Then $\phi(\psi\lambda) = (\phi\psi)\lambda$ for all $\phi, \psi, \lambda \in L(V(I))$.

Theorem 2.42. *Let U(I), V(I) and W(I) be strong neutrosophic vector spaces over a neutrosophic field K(I) and let* $\phi: U(I) \to V(I), \psi: V(I) \to W(I)$ *be neutroso-*

phic vector space homomorphisms. Then

(1) If $\psi\phi$ is injective, then ϕ is injective.

(2) If $\psi\phi$ is surjective, then ψ is surjective.

(3) If ψ and ϕ are injective, then $\psi\phi$ is injective.

Let V(I) be a strong *neutrosophic* vector space over a *neutrosophic* field K(I) and let $\phi: V(I) \to V(I)$ be a *neutrosophic* vector space homomorphism. If $B = \{v_1, v_2, ..., v_n\}$ is a basis for V(I), then each $\phi(v_i) \in V(I)$ and thus for $\alpha_{ij} \in K(I)$, we can write

$$\phi(v_1) = \alpha_{11}v_1 + \alpha_{12}v_2 + ... + \alpha_{1n}v_n$$
$$\phi(v_1) = \alpha_{21}v_1 + \alpha_{22}v_2 + ... + \alpha_{2n}v_n$$
$$...$$
$$\phi(v_n) = \alpha_{n1}v_1 + \alpha_{n2}v_2 + ... + \alpha_{nn}v_n$$

Let

$$[\phi]_B = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}$$

$[\phi]_B$ is called the matrix representation of ϕ relative to the basis B.

Theorem 2.43. *Let V(I) be a strong neutrosophic vector space over a neutrosophic field K(I) and let* $\phi: V(I) \to V(I)$ *be a neutrosophic vector space homomorphism. If B is a basis for V(I) and v is any element of V(I), then*

$$[\phi]_B[v]_B = [\phi(v)]_B.$$

Example 16. Let $V(I) = \mathbb{R}^3(I)$ be a strong *neutrosophic* vector space over a *neutrosophic* field K(I)= $\mathbb{R}(I)$ and let $v = (1+2I, I, 3-2I) \in V(I)$. If $\phi: V(I) \to V(I)$ is a *neutrosophic* vector space homomorphism defined by $\phi(v) = v$ for all $v \in V(I)$, then relative to the basis $B = (v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)\}$ for V(I), the matrix of ϕ is obtained as

$$[\phi]_B = \begin{bmatrix} 1+0I & 0+0I & 0+0I \\ 0+0I & 1+0I & 0+0I \\ 0+0I & 0+0I & 1+0I \end{bmatrix}$$

For $v = (1+2I, I, 3-2I) \in V(I)$, we have

$$\phi(v) = v = (1+2I)v_1 + Iv_2 + (3-2I)v_3$$

So that

$$[v]_B = \begin{bmatrix} 1+2I \\ I \\ 3-2I \end{bmatrix} = [\phi(v)]_B$$

And we have

$$[\phi]_B[v]_B = [\phi(v)]_B$$

Example 17. Let $V(I) = \mathbb{R}^3(I)$ be a weak *neutrosophic* vector space over a *neutrosophic* field K= $\mathbb{R}$ and let $v = (1-2I, 3-4I) \in V(I)$. If $\phi: V(I) \to V(I)$ is a *neutrosophic* vector space homomorphism defined by $\phi(v) = v$ for all $v \in V(I)$, then relative to the basis $B = (v_1 = (1,0), v_2 = (0,1), v_3 = (I,0), v_4 = (0,I)\}$ for V(I), the matrix of ϕ is obtained as

$$[\phi]_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For $v = (1-2I, 3-4I) \in V(I)$, we have

$$\phi(v) = v = v_1 + 3v_2 - 2v_3 - 4v_4.$$

Therefore,

$$[v]_B = \begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix} = [\phi(v)]_B$$

And thus

$$[\phi]_B[v]_B = [\phi(v)]_B$$

3 Conclusion

In this paper, we have studied *neutrosophic* vector spaces. Basic definitions and properties of the classical vector spaces were generalized. It was shown that every weak (strong) *neutrosophic* vector space is a vector space.

Also, it was shown that an element of a strong *neutrosophic* vector space can be infinitely expressed as a linear combination of some elements of the *neutrosophic* vector space. *Neutrosophic* quotient spaces and *neutrosophic* vector space homomorphisms were also studied. Matrix representations of *neutrosophic* vector space homomorphisms were presented.

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