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Triangular Dense Fuzzy Neutrosophic Sets

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Abstract. In this study, we introduce the concept of denser property in fuzzy membership function used in neutrosophic sets. We present several new definitions and study their properties. Defuzzification methods over neutrosophic triangular dense fuzzy sets and neutrosophic triangular intuitionistic dense fuzzy sets are then given. Finally practical applicability of the methods have been discussed with graphical implications in recent times.

Keywords: Dense fuzzy set; Triangular dense fuzzy neutrosophic set; defuzzification method.

1. Introduction

For any kind of multi-attribute decision making (MADM) it is essential to have adequate crisp data, but in modern situations crisp data are inadequate. The data for which they used to rely more are basically imprecise, vague, inappropriate and piecewise untruth as a whole. Belnap [3] made an attempt to study with the four valued logic namely Truth (T), false (F), Unknown (U) and Contradiction (C). He used a bi-lattice where the four components were inter-related. Smarandache [13] founded and developed the neutrosophic set, neutrosophic logic, neutrosophic probability and neutrosophic statistics. Several researchers Ye [19], Biswas et al. [4,5], Mandal and Pramanik [10] etc. have discussed several ranking method based on current problems using neutrosophic sets (NS). Also, recently the multivalued power operator in NS has been veveloped by Peng at al.[12].Fuzzy set theory was first studied by Zadeh [21] but after few decades later the concept on hesitant fuzzy set has been grown by Torra[15]. Moreover, in intuitionistic fuzzy environment, numerous research articles have been studied by eminent practitioner. The concept on intuitionistic fuzzy sets (IFS) has been developed independently by Atanassov[1,2] and Dubois et al. [9]. Through its process,Wang et al. [17,18], Pei and Zheng [11] discussed new concepts on evidence based IFS and a novel approach for decision making respectively. However, the decision maker's (DM) are usually applying their appropriate membership grade values of the different attributes which are prior and experienced data. But in reality, the data predicted a day ago may not be useful for tomorrow and in many cases those grade values demanding changing values with the change of the dealing frequency among several monopoly enterprises or between the time gap also. Thus, it is troublesome to find the actual data (because most of the original data is in hidden and secret under some national or international law and orders). For instance, to find the information over flood victims in a particular place several opinions may come out. But the data accepted by the authorities usually vary the reality because of the limitations on governmental financial supports to be offered to the victims. However, the situation began to clear as the day passing on. To model the above situation, in this article, we first give some basic concepts on neutrosophic set (NS), and then we develop the NS under dense fuzzy environment. The fuzzy components under several compositions are discussed from the existing literature. Next, some extensions are made with proper justification.

The paper is organized as follows: Section 2 describes some basic concepts of NS for subsequent use. In section 3, we develop the NS under dense fuzzy environment. Section 4 deals with defuzzified values of NS. Section 5 improves NS assessment under dense fuzzy environment. Section 6 gives further implications of dense fuzzy in NS. Section 7 presents applications to show the practicality and feasibility of our method. Section 8 ends the paper with some concluding remarks..

2. Preliminaries

Here, we shall discuss some basic concepts and operations on neutrosophic set.

Definition 1.[4] Let X be a space of points (objects) with generic element x. Then a neutrosophic set (NS) A in X is characterize by a truth membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A. It is denoted by $N_s =< T_A, I_A, F_A >$ where the functions T_A, I_A and F_A are real standard or non-standard subsets of $]0^-, 1^+[$. That is $T_A : X \to]0^-, 1^+[$, $I_A : X \to]0^-, 1^+[$ and is $F_A : X \to]0^-, 1^+[$ satisfying the relation is $0^- \le supT_A(x) + sup\, I_A(x) + supF_A(x) \le 3^+$,

Definition 2. [13]The complementA^c of a NS A is defined as follows:

$i)\ T_{A^c}(x) = \{1^+\} - T_A(x)$,
ii) $I_{A^c}(x) = \{1^+\} - I_A(x)$;
iii) $F_{A^c}(x) = \{1^+\} - F_A(x)$

Definition 3. [13]A neutrosophic set A is contained in other neutrosophic set B i.e, $A \subseteq B$ if and only if the following results hold good for

$\forall x \in X$

i) $infT_A(x) \le infT_B(x)$, $supT_A(x) \le supT_B(x)$
ii) $infI_A(x) \ge infI_B(x)$, $supI_A(x) \ge supI_B(x)$
iii) $infF_A(x) \ge infF_B(x)$, $supF_A(x) \ge supF_B(x)$

Definition 4. [16]The complement N_s^c of a single valued NS is given by

i) $T_{N_s^c}(x) = F_{N_s^c}(x)$,
ii) ii) $I_{N_s^c}(x) = 1 - I_{N_s}(x)$;$F_{N_s^c}(x) = T_{N_s}(x)$.

Definition 5. [16]The union of two single valued neutrosophic sets A and B , denoted by $C = A \cup B$. Its truth membership , indeterminancy membership, and falsity membership functions are related to those of A and B as follows:

i) $T_C(x) = \max(T_A(x), T_B(x))$
ii) $I_C(x) = \max(I_A(x), I_B(x))$
$iii) F_C(x) = \min(F_A(x), F_B(x))$,
$\forall x \in X$

Definition 6. The intersection of two single valued neutrosophic sets A and B , denoted by $C = A \cap B$. Its truth membership , indeterminancy membership, and falsity membership functions are related to those of A and B as follows:

$i)\ T_C(x) = \min(T_A(x), T_B(x))$
$ii)\ I_C(x) = \min(I_A(x), I_B(x))$
$iii) F_C(x) = \max(F_A(x), F_B(x))$,
$\forall x \in X$.

Definition 7. The additionof two single valued neutrosophic sets A and B , denoted by $C = A \oplus B$. Its truth membership , indeterminancy membership, and falsity membership functions are related to those of A and B as follows:

$i)\ T_C(x) = T_A(x) + T_B(x) - T_A(x)T_B(x)$
ii) $I_C(x) = I_A(x)I_B(x)$
iii) $F_C(x) = F_A(x)F_B(x)$,
$\forall x \in X$

Definition 8. The multiplication of two single valued neutrosophic sets A and B , denoted by $C = A \otimes B$. Its truth membership , indeterminancy membership, and falsity membership functions are related to those of A and B as follows:

i)$T_C(x) = T_A(x)T_B(x)$
ii) $I_C(x) = I_A(x) + I_B(x) - I_A(x)I_B(x)$
$iii) F_C(x) = F_A(x) + F_B(x) - F_A(x)F_B(x)$,
$\forall x \in X$

Remark 1.Neutrosophic Cube describing IFS &NS. Jean Dezert [8] introduced the neutrosophic cube $A'B'C'D'E'F'G'H'$ to make a distinction between IFS and NS. For technical use, we take the classical interval [0,1] for the NS parameters T_A, I_A and F_A . Then the cube $ABCDEFGH$ is called technical / relative neutrosophic cube and its extension $A'B'C'D'E'F'G'H'$ is called the absolute neutrosophic cube. Now, we divide the technical neutrosophic cube into three disjoint regions. The observations from the following cube are

i) The equilateral triangle BDE, whose sides are equal to $\sqrt{2}$, it represents the geometrical locus of the points whose sum of the coordinates is 1. This triangle is known as Atanassov-Intuitionistic fuzzy set (A-IFS). Here, if q is a point on ΔBDE or inside of it then as in A-IFS, $t_q + i_q + f_q = 1$.
ii) The pyramid EABD[situated in the right side of the ΔEBD , including its faces ΔABD (base), ΔEBA and ΔEDA (lateral faces), but excluding its face ΔBDE] is the locus of the points whose sum of coordinates is less than 1. If p is point on EABD then $t_q + i_q + f_q < 1$ as in IFS with incomplete information.
iii) In the left side of ΔBDE in the cube there is the solid EFGCDEBD(excluding ΔBDE) which is the locus of points whose sum of their coordinates is greater than 1 as in the paraconsistent set . If a point r lies on EFGCDEBD, then $t_q + i_q + f_q > 1$.

Thus, we have a source which is capable to find only the degree of membership of an element; but it is unable to find the degree of non-membership. Another source is capable to find only the degree of non-membership of an element. Or, a source which only computes the indeterminacy. Putting these results we always have$t_q + i_q + f_q \neq 1$. Moreover, in information fusion, when dealing with indeterminate models (that is elements of the fusion space which are indeterminate/unknown, such as intersections we don't know if they are empty or not since we don't have enough information, similarly for complements of indeterminate elements etc) ; if we compute the believe in that element (truth), the disbelieve in that element(falsehood) and the indeterminacy part of that element, then the sum of these three components is strictly less than 1 (the difference to 1 is the missing information). This is shown in Fig. 1.

3 Triangular dense fuzzy environment

Here, we discuss the dense fuzzy environment in the all possible cases of NS.

Definition 9. [7]Let $\tilde{A}$ be the fuzzy number whose components are the elements of $\mathcal{R} \times N$, $\mathcal{R}$ being the set of real numbers and N being the set of natural numbers with the membership grade satisfying the functional relation $\mu : \mathcal{R} \times N \to [0,1]$. Now as $n \to \infty$ if $\mu(x,n) \to 1$ for some $x \in \mathcal{R}$ then we call the set $\tilde{A}$ as dense fuzzy set. If $\tilde{A}$ is triangular then it is called TDFS. Now, if for some n, $\mu(x,n)$ attains the highest membership degree 1 then the set itself is called "Normalized Triangular Dense Fuzzy Set" or NTDFS. The graphical interpretation is shown in Fig. 2.

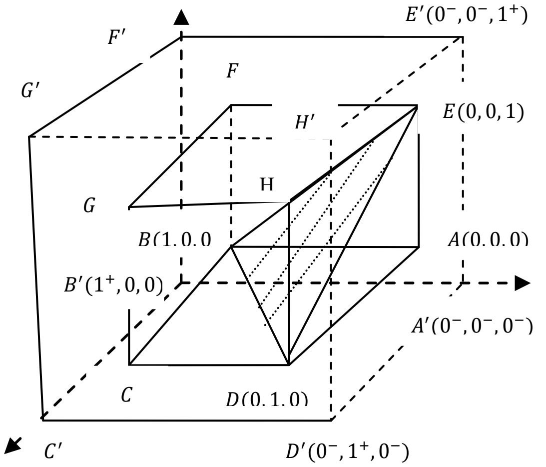

Fig-1: Geometric representation of **NeutrosophicCube**

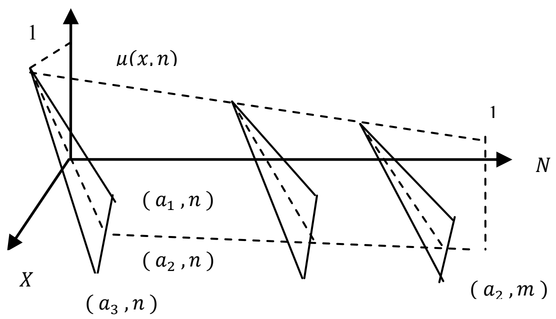

Fig.2: Membership function of NTDFS based on definition 9

Example 1. As per definitions (9) let us assume the TDFS as follows $\tilde{A} =< a_2\left(1-\frac{\rho_1}{1+n}\right), a_2, a_2\left(1+\frac{\sigma_1}{1+n}\right) >$, $for\ 0 < \rho_1, \sigma_1 < 1$(1)

The memberships function for $0 \le n$ is defined as follows:$\gamma_T(x,n) =$
lows:$\gamma_T(x,n) =$

$$\begin{cases} 0 \quad if \quad x < a_2\left(1-\frac{\rho_1}{1+n}\right) \ and\ x > a_2\left(1+\frac{\sigma_1}{1+n}\right) \\ \left\{\frac{x-a_2\left(1-\frac{\rho_1}{1+n}\right)}{\frac{\rho_1 a_2}{1+n}}\right\} \quad if \quad a_2\left(1-\frac{\rho_1}{1+n}\right) \le x \le a_2 \\ \left\{\frac{a_2\left(1+\frac{\sigma_1}{1+n}\right)-x}{\frac{\sigma_1 a_2}{1+n}}\right\} \quad if \quad a_2 \le x \le a_2\left(1+\frac{\sigma_1}{1+n}\right) \end{cases} \quad (2)$$

Similarly, here also we note that, the ordinary membership functions of falsehood and indeterminacy is given by

$$\gamma_F(x,n) = \begin{cases} 0 \quad if \quad x < b_2\left(1-\frac{\rho_2}{1+n}\right) \ and\ x > b_2\left(1+\frac{\sigma_2}{1+n}\right) \\ \left\{\frac{b_2-x}{\frac{\rho_2 b_2}{1+n}}\right\} \quad if \quad b_2\left(1-\frac{\rho_2}{1+n}\right) \le x \le b_2 \\ \left\{\frac{x-b_2}{\frac{\sigma_2 b_2}{1+n}}\right\} \quad if \quad b_2 \le x \le b_2\left(1+\frac{\sigma_2}{1+n}\right) \end{cases} \quad (3)$$

$$\gamma_I(x,n) = \begin{cases} 0 \quad if \quad x < c_2\left(1-\frac{\rho_3}{1+n}\right) \ and\ x > c_2\left(1+\frac{\sigma_3}{1+n}\right) \\ \left\{\frac{c_2-x}{\frac{\rho_3 c_2}{1+n}}\right\} \quad if \quad c_2\left(1-\frac{\rho_3}{1+n}\right) \le x \le c_2 \\ \left\{\frac{x-c_2}{\frac{\sigma_3 c_2}{1+n}}\right\} \quad if \quad c_2 \le x \le c_2\left(1+\frac{\sigma_3}{1+n}\right) \end{cases} \quad (4)$$

Definition 10: TDFS based on non-membership & indeterminate function

Let $\tilde{A}$ be the fuzzy number whose components are the elements of $\mathcal{R} \times N$ whose non-membership grade satisfying the functional relation $\vartheta : \mathcal{R} \times N \to [0,1]$. Now as $n \to \infty$ if $\vartheta(x,n) \to 0$ for some $x \in \mathcal{R}$ then we call the set $\tilde{A}$ as dense fuzzy set. If we consider the fuzzy number $\tilde{A}$ of the form $\tilde{A} = \langle a_1, a_2, a_3 \rangle$ then we call it "Triangular Dense Fuzzy Set". Now, if for n= 0 in T, $\vartheta(x,n)$ attains the highest membership degree 1 then we can express this fuzzy number as "Normalized Triangular Dense Fuzzy Set" or NTDFS.

Example-2: Let the falsity set is given by

$\tilde{B} =< b_2(1-\rho_2)e^{-n}, b_2 e^{-n}, b_2(1+\sigma_2)e^{-n} >$

$for\ 0 < \rho_2, \sigma_2 < 1$ (5)

And its non-membership function for $0 \le n$ is defined as $\vartheta(x,n) =$

$$\begin{cases} 0 \ if \quad x < b_2(1-\rho_2)e^{-n} \ and \ x > b_2(1+\sigma_2)e^{-n} \\ \left\{\frac{b_2e^{-n} - x}{\rho_2 b_2 e^{-n}}\right\} \quad if \ \ b_2(1-\rho_2)e^{-n} \le x \le b_2 e^{-n} \\ \left\{\frac{x - b_2e^{-n}}{\sigma_2 b_2 e^{-n}}\right\} \ if \ \ b_2 e^{-n} \le x \le b_2(1+\sigma_2)e^{-n} \end{cases} \quad (6)$$

The graphical representation of non-membership function is given in Fig. 3.

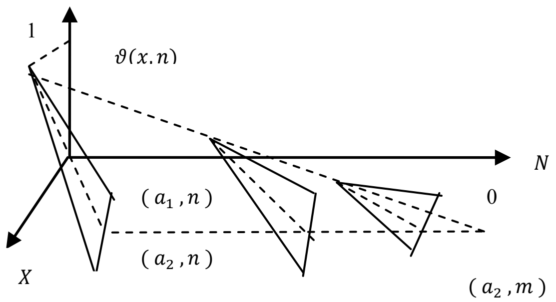

Fig. 3: Non membership function of NTDFS

Example-3: Let, the indeterminacy dense fuzzy set be of the form $\widetilde{C} =< c_2(1-\rho_3)e^{-n}, c_2e^{-n}, c_2(1+\sigma_3)e^{-n} >$ $for\ 0 < \rho_3, \sigma_3 < 1$ (7)

With the membership function

$$\pi(x,n) =$$

$$\begin{cases} 0 \quad if \quad x < c_2(1-\rho_3)e^{-n} \ and \ x > c_2(1+\sigma_3)e^{-n} \\ \left\{\frac{c_2e^{-n} - x}{\rho_3 c_2 e^{-n}}\right\} \quad if \ \ c_2(1-\rho_3)e^{-n} \le x \le c_2 e^{-n} \\ \left\{\frac{x - c_2e^{-n}}{\sigma_3 c_2 e^{-n}}\right\} \ if \quad c_2 e^{-n} \le x \le c_2(1+\sigma_3)e^{-n} \end{cases}$$

(8)

Definition-11: Let $X \times N$ be a space of points (objects) with generic element (*x, n*). Then a neutrosophic set A in $X \times N$ is characterize by a truth membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A. The functions $T_A, I_A and F_A$ are real standard or non-standard subsets of $]0^-, 1^+[$.That is $T_A : X \times N \to]0^-, 1^+[$, $I_A : X \times N \to]0^-, 1^+[$ and is $F_A : X \times N \to]0^-, 1^+[$ having the property that, as $n \to \infty$ if $T_A(x,n) \to 1$, $I_A(x,n) \to 0 \leftarrow F_A(x,n)$ And satisfying the relation is $0^- \le supT_A(x,n) + supI_A(x,n) + supF_A(x,n) \le 3^+$,

Definition-12: A Neutrosophic set A in $X \times N$ is said to be Neutrosophic Intuitionistic Dense fuzzy Set if the elements of NS, that is the functional components $T_A, I_A and F_A$ are taken from the real standard subsets of $[0,1]$. That is $T_A : X \times N \to [0,1]$, $I_A : X \times N \to [0,1]$ and is $F_A : X \times N \to [0,1]$ having the property that, as $n \to \infty$ if $T_A(x,n) \to 1$, $I_A(x,n) \to 0 \leftarrow F_A(x,n)$ satisfying the relation is $0 \le supT_A(x,n) + sup\ I_A(x,n) + supF_A(x,n) \le 3$,

Remark 2.The NS for dependency components(Ye [20])

Here we draw the simple Venn diagram for the NS with dependency components. We use this diagram to realize the overall assessment of the fuzzy components. According to probability theory, the overall score obtained from the fig.-4 is stated in (14). Note that, if $a_2 = b_2 = c_2$ holds in the fuzzy sets stated in (1), (5) and (7) then the common region of that set will be crisp one. This is shown in Fig-4

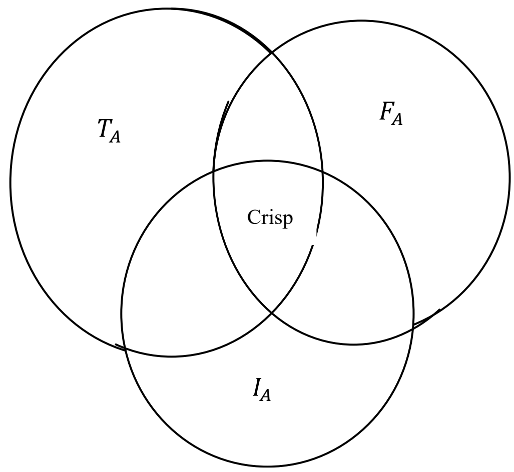

Fig.-4: Venn diagram of General Neutrosophic Set

4 Some basics over expected values of NS

Here we shall discuss over the ultimate score or expected defuzzified values for the proposed neutrosophic set$N_s =< T_A, I_A, F_A >$

i) When the components $T_A, I_A and F_A$ are independent Then as per [**Biswas et. al.[4]**] the total average expected score will be

$S(x) = \frac{1}{3}\{T_A(x) + I_A(x) + F_A(x)\}$(9)

And the truth favorite relative expected value (score) is given by

$S(x) = \frac{3\,T_A(x)}{T_A(x)+ I_A(x)+ F_A(x)}$ (10)

For information lacking in NS, this part can be divided into several sub cases.

a) The components $T_A, I_A and F_A$ all together constitute a positive skewed distribution.

In this case (truth leads in the major part), the total score will be

$S(x) = w_1T_A(x) + w_2I_A(x) + w_3F_A(x)$, for $w_1 > w_2 > w_3$ and $w_1 + w_2 + w_3 = 1$ (11)

b) The components $T_A, I_A and F_A$ all together constitute a negative skewed distribution.

In this case (falsehood leads major part), the total score will be

$S(x) = w_1T_A(x) + w_2I_A(x) + w_3F_A(x)$, for $w_1 < w_2 < w_3$ and $w_1 + w_2 + w_3 = 1$ (12)

c) The components $T_A, I_A and F_A$ all together constitute a normal distribution.

In this case (truth and falsehood are symmetric) , the total score will be

$S(x) = w_1T_A(x) + w_2I_A(x) + w_3F_A(x)$, for $w_1 < w_2 > w_3$ and $w_1 + w_2 + w_3 = 1$ (13)

ii) When the components $T_A, I_A and F_A$ are dependent

a) If the components keep the positive sign then the ultimate score is given by

$S(x) = T_A(x) + I_A(x) + F_A(x) - T_A(x)I_A(x) - I_A(x)F_A(x) - T_A(x)F_A(x)+T_A(x)I_A(x)F_A(x)$ (14)

b) For the case of IFS, the effect of $I_A(x) \to \{0\}$ or unknown and the sign of $F_A(x)$ be negative and hence the ultimate score be

$S(x) = T_A(x) - F_A(x) + T_A(x)F_A(x)$ (15)

c) [Chen and Tan [6]] Since the expected values of $T_A(x) < 1$ $and F_A(x) < 1$ so their product values $T_A(x)F_A(x) \ll 1$ and hence their effect can be ignored. In this case the score function be

$S(x) = T_A(x) - F_A(x)$ (16)

5 Improved NS assessments under dense fuzzy environment

Case-I : When all the membership functions keep their values of similar types for same number of observations/interactions or time durations. First of all, we shall discuss the fuzzy assessments under dense fuzzy developed by De and Beg [7]. The basic aim of the dense fuzzy model is that each and every fuzzy component reaches to singleton crisp value whenever we would like to experiencing with fuzzy data for a long period of time or interactions in practice. Here also we assume the learning experiences are conducted within the same elapsed time or interactions for all the fuzzy components of NS. Thus unlike Biswas et al. [4]; utilizing dense concept and $\alpha - cuts$ developed by De and Beg[7], the defuzzified value reduces from the formula $I(\widetilde{A}) = \frac{1}{2T}\iint_{\alpha=0,t=0}^{\alpha=1,t=T}\{L(\alpha,t) + R(\alpha,t)\}\,d\alpha dt = a_2\left\{1 + \frac{(\sigma-\rho)}{4T}Log(1+T)\right\}$ for time (T) dependent fuzzy membership function and that for frequency dependent membership function : $I(\widetilde{A}) = \frac{1}{2N}\sum_{n=0}^{N} a_2\left\{2 + \frac{\sigma-\rho}{2(1+n)}\right\} = \frac{a_2}{2N}\left[2N + \frac{\sigma-\rho}{2}\left\{\frac{1}{1+0} + \frac{1}{1+1} + \frac{1}{1+2} + \cdots .... + \frac{1}{1+N}\right\}\right]$and obtained as

5.1 For time dependent, (11) reduces to $I(S(x,t)) = \frac{1}{3}\{I[T_A(x,t) + I[\,I_A(x,t)] + I[F_A(x,t)]\}$

$\Rightarrow I(S) = \frac{1}{3}\Big[a_2\left\{1 + \frac{(\sigma_1-\rho_1)}{4T}Log(1+T)\right\} + b_2\left\{1 + \frac{(\sigma_2-\rho_2)}{4T}Log(1+T)\right\} + c_2\left\{1 + \frac{(\sigma_3-\rho_3)}{4T}Log(1+T)\right\}\Big]$ (17)

And that for frequency dependent, $3I(S(x,n)) = I[T_A(x,n) + I[\,I_A(x,n)] + I[F_A(x,n)]$

$\Rightarrow I(S) = \frac{1}{6N}\sum_{n=0}^{N} a_2\left\{2 + \frac{(\sigma_1-\rho_1)}{2(1+n)}\right\} + \frac{1}{6N}\sum_{n=0}^{N} b_2\left\{2 + \frac{(\sigma_2-\rho_2)}{2(1+n)}\right\} + \frac{1}{6N}\sum_{n=0}^{N} c_2\left\{2 + \frac{(\sigma_3-\rho_3)}{2(1+n)}\right\}$ (18)

5.2 For time dependent fuzzy components, (10) reduces to

$I(S) = 3a_2\left\{1 + \frac{(\sigma_1-\rho_1)}{4T}Log(1+T)\right\}/ \Big[a_2\left\{1 + \frac{(\sigma_1-\rho_1)}{4T}Log(1+T)\right\} + b_2\left\{1 + \frac{(\sigma_2-\rho_2)}{4T}Log(1+T)\right\} + c_2\left\{1 + \frac{(\sigma_3-\rho_3)}{4T}Log(1+T)\right\}\Big]$(19)

And that for frequency dependent

$$I(S) = \frac{3}{2N}\sum_{n=0}^{N} a_2\left\{2 + \frac{(\sigma_1-\rho_1)}{2(1+n)}\right\}/$$

$[\frac{1}{2N}\sum_{n=0}^{N} a_2\left\{2 + \frac{(\sigma_1-\rho_1)}{2(1+n)}\right\} + \frac{1}{2N}\sum_{n=0}^{N} b_2\left\{2 + \frac{(\sigma_2-\rho_2)}{2(1+n)}\right\} + \frac{1}{2N}\sum_{n=0}^{N} c_2\left\{2 + \frac{(\sigma_3-\rho_3)}{2(1+n)}\right\}]$ (20)

Similarly for (11-13) we get

5.3 For time dependent, (11-13) reduces to $I(S(x,t)) = w_1I[T_A(x,t) + w_2I[\,I_A(x,t)] + w_3I[F_A(x,t)]$

$\Rightarrow I(S) = w_1a_2\left\{1 + \frac{(\sigma_1-\rho_1)}{4T}Log(1+T)\right\} + w_2b_2\left\{1 + \frac{(\sigma_2-\rho_2)}{4T}Log(1+T)\right\} + w_3c_2\left\{1 + \frac{(\sigma_3-\rho_3)}{4T}Log(1+T)\right\}$with$w_1 + w_2 + w_3 = 1$

(21)

And that for frequency dependent,

$\Rightarrow I(S) = w_1 \frac{1}{2N}\sum_{n=0}^{N} a_2 \left\{2+\frac{(\sigma_1-\rho_1)}{2(1+n)}\right\} + w_2 \frac{1}{2N}\sum_{n=0}^{N} b_2 \left\{2+\frac{(\sigma_2-\rho_2)}{2(1+n)}\right\} + w_3 \frac{1}{2N}\sum_{n=0}^{N} c_2 \left\{2+\frac{(\sigma_3-\rho_3)}{2(1+n)}\right\}$ with $w_1 + w_2 + w_3 = 1$ (22)

5.4 For frequency dependent fuzzy components(14) reduces to

$$I(S) = I(T_A) + I(I_A) + I(F_A) - I(T_A I_A) - I(I_A F_A) - I(T_A F_A) + I(T_A I_A F_A)$$

$$\Rightarrow I(S) = \frac{1}{2N}\sum_{n=0}^{N} a_2\left\{2+\frac{(\sigma_1-\rho_1)}{2(1+n)}\right\} + \frac{1}{2N}\sum_{n=0}^{N} b_2\left\{2+\frac{(\sigma_2-\rho_2)}{2(1+n)}\right\} + \frac{1}{2N}\sum_{n=0}^{N} c_2\left\{2+\frac{(\sigma_3-\rho_3)}{2(1+n)}\right\}$$
$$-\frac{1}{2N}\sum_{n=0}^{N} a_2 b_2\left[2+\frac{\sigma_1+\sigma_2-\rho_1-\rho_2}{2(1+n)} + \frac{\rho_1\rho_2+\sigma_1\sigma_2}{6\,(1+n)^2}\right] - \frac{1}{2N}\sum_{n=0}^{N} b_2 c_2\left[2+\frac{\sigma_2+\sigma_3-\rho_2-\rho_3}{2(1+n)} + \frac{\rho_2\rho_3+\sigma_2\sigma_3}{3\,(1+n)^2}\right]$$
$$-\frac{1}{2N}\sum_{n=0}^{N} a_2 c_2\left[2+\frac{\sigma_1+\sigma_3-\rho_1-\rho_3}{2(1+n)} + \frac{\rho_1\rho_3+\sigma_1\sigma_3}{6\,(1+n)^2}\right] + \frac{a_2b_2c_2}{2N}\sum_{n=0}^{N}\left[2+\frac{\sigma_1-\rho_1}{1+n}+\frac{\rho_1+\sigma_2+\sigma_3-\sigma_1-\rho_2-\rho_3}{2(1+n)}\right]$$
$$+\frac{a_2b_2c_2}{2N}\sum_{n=0}^{N}\left[\frac{\rho_1(\rho_2+\rho_3)+\sigma_1(\sigma_2+\sigma_3)}{2(1+n)^2} + \frac{\sigma_1\sigma_2\sigma_3-\rho_1\rho_2\rho_3}{(1+n)^3}\right]$$
$$-\frac{a_2b_2c_2}{2N}\sum_{n=0}^{N}\left[\frac{\rho_1\rho_2+\rho_1\rho_3-\rho_2\rho_3+\sigma_1\sigma_2+\sigma_1\sigma_3-\sigma_2\sigma_3}{3\,(1+n)^2} + \frac{\sigma_1\sigma_2\sigma_3+\rho_1\rho_2\rho_3}{4(1+n)^3}\right] \quad (23)$$

(for detail, see appendix equation (A.1 - A.5)

Also, the results for time continuous fuzzy membership we have by simply integrating the above sum with respect to time and applying the dense rule and get

$$I(S) = a_2\left\{1+\frac{(\sigma_1-\rho_1)}{4T}Log(1+T)\right\} + b_2\left\{1+\frac{(\sigma_2-\rho_2)}{4T}Log(1+T)\right\} + c_2\left\{1+\frac{(\sigma_3-\rho_3)}{4T}Log(1+T)\right\}$$
$$-a_2b_2\left[1+\frac{\sigma_1+\sigma_2-\rho_1-\rho_2}{4T}Log(1+T) - \frac{\rho_1\rho_2+\sigma_1\sigma_2}{12\,T(1+T)}\right] - b_2c_2\left[1+\frac{\sigma_2+\sigma_3-\rho_2-\rho_3}{4T}Log(1+T) - \frac{\rho_2\rho_3+\sigma_2\sigma_3}{6\,T(1+T)}\right]$$
$$-a_2c_2\left[1+\frac{\sigma_1+\sigma_3-\rho_1-\rho_3}{4T}Log(1+T) + \frac{\rho_1\rho_3+\sigma_1\sigma_3}{12T\,(1+T)}\right]$$
$$+a_2b_2c_2\left[1+\frac{\sigma_1-\rho_1}{2T}Log(1+T) + \frac{\rho_1+\sigma_2+\sigma_3-\sigma_1-\rho_2-\rho_3}{4T}Log(1+T) - \frac{\rho_1(\rho_2+\rho_3)+\sigma_1(\sigma_2+\sigma_3)}{4T(1+T)}\right]$$
$$+a_2b_2c_2\left[\frac{\rho_1\rho_2+\rho_1\rho_3-\rho_2\rho_3+\sigma_1\sigma_2+\sigma_1\sigma_3-\sigma_2\sigma_3}{6T\,(1+T)} - \frac{\sigma_1\sigma_2\sigma_3-\rho_1\rho_2\rho_3}{4T(1+T)^2} + \frac{\sigma_1\sigma_2\sigma_3+\rho_1\rho_2\rho_3}{16T(1+T)^2}\right] \quad (24)$$

5.5 For time dependent fuzzy components, (15) reduces to

$$I(S) = I(T_A) + I(F_A) - I(T_A F_A) \quad \Rightarrow I(S) = a_2\left\{1+\frac{(\sigma_1-\rho_1)}{4T}Log(1+T)\right\} + b_2\left\{1+\frac{(\sigma_2-\rho_2)}{4T}Log(1+T)\right\}$$
$$-a_2b_2\left[1+\frac{\sigma_1+\sigma_2-\rho_1-\rho_2}{4T}Log(1+T) - \frac{\rho_1\rho_2+\sigma_1\sigma_2}{12\,T(1+T)}\right] \quad (25)$$

And that for frequency dependent fuzzy components,

$$\Rightarrow I(S) = \frac{1}{2N}\sum_{n=0}^{N} a_2\left\{2+\frac{(\sigma_1-\rho_1)}{2(1+n)}\right\} + \frac{1}{2N}\sum_{n=0}^{N} b_2\left\{2+\frac{(\sigma_2-\rho_2)}{2(1+n)}\right\}$$
$$-\frac{1}{2N}\sum_{n=0}^{N} a_2b_2\left[2+\frac{\sigma_1+\sigma_2-\rho_1-\rho_2}{2(1+n)} + \frac{\rho_1\rho_2+\sigma_1\sigma_2}{6\,(1+n)^2}\right] \quad (26)$$

Case-II : When all the membership functions keep their values of different types for same number of observations/interactions or time durations. Here we shall take the membership functions of T_A, I_A and F_A as stated in (2), (6) and (8) .

Now the score function of (16) under time sensitive fuzzy numbers is given by, $S(x,t) = \begin{cases} \frac{x-a_2\left(1-\frac{\rho_1}{1+t}\right)}{\frac{\rho_1 a_2}{1+t}} - \frac{b_2 e^{-t}-x}{b_2\rho_2 e^{-t}} if \quad u \le x \le v \ ,say \\ \frac{a_2\left(1+\frac{\sigma_1}{1+t}\right)-x}{\frac{\sigma_1 a_2}{1+t}} - \frac{x-b_2 e^{-t}}{b_2\sigma_1 e^{-t}} if v \le x \le w \ say \end{cases}$

$\Rightarrow S(x,t)$

$$= \begin{cases} x\left[\frac{1+t}{\rho_1 a_2}+\frac{e^t}{\rho_2 b_2}\right] - \left(\frac{1+t}{\rho_1}+\frac{1}{\rho_2}-1\right) if \quad u \le x \le v \ ,say \\ \left(\frac{1+t}{\sigma_1}+\frac{1}{\sigma_2}+1\right) - x\left[\frac{1+t}{\sigma_1 a_2}+\frac{e^t}{\sigma_2 b_2}\right] if v \le x \le w \ say \end{cases}$$

Using α −cuts and employing the index formula developed by De and Beg [7] we get, $\boldsymbol{I}\,(\boldsymbol{S}) = \frac{\mathbf{1}}{\mathbf{2T}}\iint_{\alpha=0,t=0}^{\alpha=1,t=T}\{L(\alpha,t)+R(\alpha,t)\}\,d\alpha dt$

$$= \frac{1}{2T}\int_0^T\left[\frac{\frac{1+t}{\rho_1}+\frac{1}{\rho_2}-1}{\frac{1+t}{\rho_1 a_2}+\frac{e^t}{\rho_2 b_2}} + \frac{\frac{1+t}{\sigma_1}+\frac{1}{\sigma_2}+1}{\frac{1+t}{\sigma_1 a_2}+\frac{e^t}{\sigma_2 b_2}} + \frac{1}{2}\left(\frac{1}{\frac{1+t}{\rho_1 a_2}+\frac{e^t}{\rho_2 b_2}} - \frac{1}{\frac{1+t}{\sigma_1 a_2}+\frac{e^t}{\sigma_2 b_2}}\right)\right]dt = \frac{a_2 b_2}{2T}\int_0^T\left[\frac{\left((1+t)\rho_2+\rho_1-\frac{\rho_1\rho_2}{2}\right)}{(1+t)\rho_2 b_2+e^t\rho_1 a_2} + \frac{\left((1+t)\sigma_2+\sigma_1+\frac{\sigma_1\sigma_2}{2}\right)}{(1+t)\sigma_2 b_2+e^t\sigma_1 a_2}\right]dt \qquad (27)$$

And that for discrete case, replacing *t* by *n* we write the expected score of the NS as

$$\boldsymbol{I}\,(\boldsymbol{S}\ ) = \frac{a_2 b_2}{2N}\sum_{n=0}^{N}\left[\frac{\left((1+n)\rho_2+\rho_1-\frac{\rho_1\rho_2}{2}\right)}{(1+n)\rho_2 b_2+e^n\rho_1 a_2} + \frac{\left((1+n)\sigma_2+\sigma_1+\frac{\sigma_1\sigma_2}{2}\right)}{(1+n)\sigma_2 b_2+e^n\sigma_1 a_2}\right] \qquad (28)$$

Case-III: If we assume that the learning effects are not performed in same time duration / interactions for all fuzzy components then the above defuzzification formula (17) & (18) reduces to

$$\Rightarrow 3\,I(S) = a_2\left\{1+\frac{(\sigma_1-\rho_1)}{4T_1}Log(1+T_1)\right\} + b_2\left\{1+\frac{(\sigma_2-\rho_2)}{4T_2}Log(1+T_2)\right\} + c_2\left\{1+\frac{(\sigma_3-\rho_3)}{4T_3}Log(1+T_3)\right\} (29)$$

And that for frequency dependent, $3I(S(x,m,n,p)) = I[T_A(x,m)+I[\,I_A(x,n)]+\ I[F_A(x,p)]$

$$\Rightarrow I(S) = \frac{1}{6M}\sum_{m=0}^{M} a_2\left\{2+\frac{(\sigma_1-\rho_1)}{2(1+m)}\right\} + \frac{1}{6N}\sum_{n=0}^{N} b_2\left\{2+\frac{(\sigma_2-\rho_2)}{2(1+n)}\right\} + \frac{1}{6P}\sum_{p=0}^{P} c_2\left\{2+\frac{(\sigma_3-\rho_3)}{2(1+p)}\right\} \qquad (30)$$

The detailed discussion is made in Appendix B.

6 Implication of Dense fuzzy in NS

In NS, the traditional concept of membership values of truth, falsehood and indeterminacy are either fixed or cannot be changed once it is assigned. But the present study reveals that such points are continuously changing because of learning experiences or flexibility of the human behavior and intentions. For instance, when we ask a question to a person about the life loss due to a certain train accident, then (s) he might be answered that 75% of the whole people died, 40% not died, and 30% is unknown. But, after few days later if the same question has been thrown to the same person, then obviously his/her answer might differ from the earlier statement. This may be 90%, 50% and 10% or 10%, 80%, 5%. Such kind of observation occurs due to information gathering from the society or learning experiences as soon as the time is passed/ increased frequency of human interactions within the locality/ society/ mass media etc. By this way before going to take governmental supports (Decision maker's accountability), on the basis of that prior information several enquiries committee will be formed and finally come to a concrete decision for establishing law and order in people's benefit. The earlier concepts on NS analyze the data based on first answer obtained from that person, but in our present study it analyzes the data subsequently obtained.

6.1 Procedure for the computation of defuzzified values of a Neutrosophic Sets(NSs)

Step-1: Find the NS involved in the different field of activities

Step-2: Find the appropriate membership components of that NSs

Step-3: Find the interior relationship among the different NSs

Step-4: Select the strategies involved in those NSs

a) If we are intending to find the joint performances then take their union using the definition 5.

b) If we are intending to find the common performances then take their intersection using the definition 6.

c) If the selection of one's NS might able to change the other's NS then to find the complexities involved take their multiplication using the definition 8.

d) If the selection of one's NS do not able to change the other's NS at all then to find the complexities involved take their addition using the definition 7.

Step-5: After getting the appropriate NS obtained from Step-4, use defuzzification rules which one you are going to assess.

7 Applications of dense fuzzy in NS (DFNS)

Several applications can be drawn from our day to day life problems (from science and engineering, sociology, philosophy, crime research, educational psychology etc.) The following are some major areas where the DFNS can be applied.

7.1 In any kind of decision making process

Example: Suppose, in a supply chain (SC) model the set of information $N_{ss} =< T_{As}, I_{As}, F_{As} >$ and $N_{sr} =< T_{Ar}, I_{Ar}, F_{Ar} >$ for both the supplier and retailer are available under dense fuzzy environment. First of all we have to obtain the bounds of each fuzzy components utilizing dense property; then check whether these sets are subsets of each other or not. Now applying congruency rule or similarity measures the score functions for the chain can be obtained and can be solved by the proposed defuzzification methods. Note that if these two sets are disjoint then the chain immediately gets breaking down and the decision maker will have to choose another model as well.

7.2 Psychological testing/ military selection

Example: Suppose in a psychological test there are five different attributes to be measured. The attributes are {Moral value, behavior, leadership, criminal offence, responsibility}. Among these attributes some of them have positively correlation, some of them are negatively correlation and rest of them has no relation. However we have to perform the membership functions of each attributes under dense fuzzy environment and exercising these every after some stipulated interval of time/ days. Take for instance, to gain best leadership quality, one might have to compromise with criminal activities and bad responsibility in many cases in practice. On the other hand, a person having good moral standards, (s)he might carry a good behavior and good leadership. Under these circumstances, whenever we wish to compute the total score, few of the score components of the NS became negative. Thus, to get the overall performance, the proposed defuzzification method can be applied and ranked accordingly.

7.3 Leadership assessment under social agenda

Example: Suppose an open problem on flood prone zone in a locality has been thrown before two political leaders having different ideologies. The problem itself contains three different parametric attributes, like {highly flood prone, unknown, no flood at all}. In this case, first set is an appropriate NS for the given attributes under dense fuzzy environment. Then, we ask to answer the question to the political leaders at different places and different interval of times. We usually notice that the scores obtained by them at different time are not the same. Thus, to have the actual score obtained by each of them, we might have to rely their views that were delivered at the final stage of assessment.

7.4 Assessing the age of a digital image

Let in a digital image be three parametric attributes to be measured. These attributes are {brightness, white, darkness}. First of all, with proper definitions, construct the membership functions of each of these attributes under dense fuzzy environment. Now changing time (with proper record), compute score values of the NS and compare with the values obtained from the specimen digital image each time. Continue this process until the expected value gets merged with the values of the specimen image. Finally, get the age, the time you have recorded last time.

7.5 Vulnerability/ risk assessment in disaster prone zone

Let the attributes under study for measuring the vulnerability or risk in a disaster management is {whether zone is disaster prone, insurance of life covered, valuation of the property}. Many times several attitudes may come whether a particular zone is disaster prone or not. The life involved in that area is known and hence the insurance covered is assessed but valuation of the property is quite unclear in practice. However if we think of birds/ animals like pet and firm animals' life insurance or beyond then this part also carry some information lack. Under these circumstances computing score values of the proposed NS at several years the actual risk can be measured. The existing research may be viewed in Takacs [14].

7.6 Economic, Cultural, Political, Climatic, Disease Mapping

With the help of NS we can map within Country/State even in the world also with respect to different socio economic parameters. For example, studying with {rich, mediocre, poor} in different countries of the world the economically sound/unsound such as developed, under developed, less developed countries can be identified globally and can map them accordingly. Similarly for peoples' cultural entity { true telling, no telling, lie telling} or { live to eat, no comments, eat to live} or { like dance, like song, both, none } can be measured at different time at different countries and then map accordingly. For political alliance taking 10 years data from different coun-try peoples' attitudes on {democracy, autocracy, idealism, materialism, and socialism} may be considered for better mapping of friendship. For mapping on different climatic zone the parameters like { hot, neither hot nor cold, cold} or { highly polluted, unknown, no pollution} and that for mapping of severe disease prone zone, taking 15-20 years data of public opinion on { Typhoid, Diarrhea, cholera, free of disease } etc. can be applied in developing NS.

7.7 Game theory

In this competitive world, behind any kind of activities there must have a hidden game. For each strategic player we may think of a NS having three or more fuzzy components. These components are changing for several reasons. For instance, sudden fall of share market, instant price hike of commodities, ammendment of Govt. policies etc. may cause the flexibility of fuzzy (non) membership grade. However, the players might have to change their plan within one day duration. Utilizing NS fuzzy cross product and algebraic properties the problem can be solved. To know the ultimate gain of each player our proposed defuzzification method can be applied.

7.8 Personality Test/ ability identification

For the identification of psychological ability or cognitive development, several possible attributes are taken for an individual. Then constructing different membership expected neutrosophic sets are drawn. Defuzzifying the given NS, we may easily measure the different abilities, which may guide to form a personality development.

Conclusion

In this study, we have explained the existing neutrosophic set under dense fuzzy environment. Traditionally, most of the researchers were experiencing with the (non) membership grade value directly from the study area, and took decisions through some aggregation rules or ranking scores. They do not feel the urge to defuzzify the NS earlier. Also, in some of the cases, fuzzy graph theory or fuzzy matrices have been developed to capture the decision theory. The concepts of human learning, the changing characteristics of the fuzzy membership with respect to time and the number of observations have been ignored by the founder thinkers of NS. We need to defuzzify the NS under dense fuzzy environment for the following reason:

a) The fuzzy elements in NS are obtained directly from field data only
b) The fuzzy flexibility may change with time elapsed and interactions covered
c) To know the individual value rather than membership degree, because a different value may carry the same degree, but the reverse is not true due to convexity property.
d) To realize the actual world rather than a hypothetical world.

Appendix A

To find the values of the following equation (15) $I(S) = I(T_A) + I(I_A) + I(F_A) - I(T_A I_A) - I(I_A F_A) - I(T_A F_A) \quad + I(T_A I_A F_A)$ for the cases of general membership functions

We take the help of $\alpha - cuts$ of the fuzzy components.. Now as per De and Beg[7], the left and right $\alpha - cuts$ of the fuzzy set $\tilde{A} = \langle a_1, a_2, a_3 \rangle$ or $\tilde{A} =<$

$a_2\left(1-\frac{\rho_1}{1+n}\right), a_2, a_2\left(1+\frac{\sigma_1}{1+n}\right)>$, $for\ 0<\rho_1,\sigma_1<1$ is given by

$L^{-1}(\alpha,n)=a_2\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right)$ and $R^{-1}(\alpha,n)=a_2\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)$

Now from the properties of $\alpha-cuts$ we have

$$(T_AF_A)_\alpha=(T_A)_\alpha(F_A)_\alpha$$
$$=\left\{a_2\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right), a_2\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)\right\}\left\{b_2\left(1-\frac{\rho_2\alpha}{1+n}\right), b_2\left(1+\frac{\sigma_2\alpha}{1+n}\right)\right\}$$
$$=a_2b_2\left\{\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right)\left(1-\frac{\rho_2\alpha}{1+n}\right),\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)\left(1+\frac{\sigma_2\alpha}{1+n}\right)\right\}$$
$$=a_2b_2\left[1-\frac{\rho_1}{1+n}+\alpha\left\{\frac{\rho_1\rho_2}{(1+n)^2}+\frac{\rho_1-\rho_2}{(1+n)}\right\}-\frac{\rho_1\rho_2\alpha^2}{(1+n)^2},\ 1+\frac{\sigma_1}{1+n}+\alpha\left\{\frac{\sigma_1\sigma_2}{(1+n)^2}+\frac{\sigma_2-\sigma_1}{(1+n)}\right\}-\frac{\sigma_1\sigma_2\alpha^2}{(1+n)^2}\right]$$

Note that, in above we assume the left and right $\alpha-cuts$ are increasing and decreasing functions of α respectively. If it is impossible to determine whether the above conditions hold or not, then the gross value of the $\alpha-cuts$ are given by

The left $\alpha-cut$ of $(T_AF_A)_\alpha=$

$$=Min\left\{a_2\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right), a_2\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)\right\}\left\{b_2\left(1-\frac{\rho_2\alpha}{1+n}\right), b_2\left(1+\frac{\sigma_2\alpha}{1+n}\right)\right\}$$
$$=Min\,a_2b_2\left\{\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right)\left(1-\frac{\rho_2\alpha}{1+n}\right),\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right)\left(1+\frac{\sigma_2\alpha}{1+n}\right),\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)\left(1-\frac{\rho_2\alpha}{1+n}\right),\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)\left(1+\frac{\sigma_2\alpha}{1+n}\right)\right\}$$

And the right $\alpha-cut$ of $(T_AF_A)_\alpha=$

$$=Max\,a_2b_2\left\{\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right)\left(1-\frac{\rho_2\alpha}{1+n}\right),\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right)\left(1+\frac{\sigma_2\alpha}{1+n}\right),\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)\left(1-\frac{\rho_2\alpha}{1+n}\right),\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)\left(1+\frac{\sigma_2\alpha}{1+n}\right)\right\}$$

The same rule could be applied in other cases also.

Therefore, the index value is given by

$$I(T_AF_A)=\frac{1}{2N}\sum_{n=0}^{N}a_2b_2\left[1-\frac{\rho_1}{1+n}+\frac{1}{2}\left\{\frac{\rho_1\rho_2}{(1+n)^2}+\frac{\rho_1-\rho_2}{(1+n)}\right\}-\frac{\rho_1\rho_2}{3\,(1+n)^2}+\ 1+\frac{\sigma_1}{1+n}+\frac{1}{2}\left\{\frac{\sigma_1\sigma_2}{(1+n)^2}+\frac{\sigma_2-\sigma_1}{(1+n)}\right\}-\frac{\sigma_1\sigma_2}{3\,(1+n)^2}\right]$$
$$=\frac{1}{2N}\sum_{n=0}^{N}a_2b_2\left[2+\frac{\sigma_1+\sigma_2-\rho_1-\rho_2}{2(1+n)}+\frac{\rho_1\rho_2+\sigma_1\sigma_2}{6\,(1+n)^2}\right]$$

(A.1)

Similarly,

$$I(T_AI_A)=\frac{1}{2N}\sum_{n=0}^{N}a_2c_2\left[2+\frac{\sigma_1+\sigma_3-\rho_1-\rho_3}{2(1+n)}+\frac{\rho_1\rho_3+\sigma_1\sigma_3}{6\,(1+n)^2}\right]$$

(A.2)

Also to find, $I(F_AI_A)$ we write,

$$(I_AF_A)_\alpha=(I_A)_\alpha(F_A)_\alpha$$
$$=\left\{b_2\left(1-\frac{\rho_2\alpha}{1+n}\right), b_2\left(1+\frac{\sigma_2\alpha}{1+n}\right)\right\}\left\{c_2\left(1-\frac{\rho_3\alpha}{1+n}\right), c_2\left(1+\frac{\sigma_3\alpha}{1+n}\right)\right\}$$
$$=b_2c_2\left[\left(1-\frac{\rho_2\alpha}{1+n}\right)\left(1-\frac{\rho_3\alpha}{1+n}\right),\left(1+\frac{\sigma_2\alpha}{1+n}\right)\left(1+\frac{\sigma_3\alpha}{1+n}\right)\right]$$
$$=b_2c_2\left[1-\alpha\left(\frac{\rho_2+\rho_3}{1+n}\right)+\frac{\rho_2\rho_3\alpha^2}{(1+n)^2},1+\alpha\left(\frac{\sigma_2+\sigma_3}{1+n}\right)+\frac{\sigma_2\sigma_3\alpha^2}{(1+n)^2}\right]$$

Therefore

$$I(F_AI_A)=\frac{1}{2N}\sum_{n=0}^{N}b_2c_2\left[2+\frac{\sigma_2+\sigma_3-\rho_2-\rho_3}{2(1+n)}+\frac{\rho_2\rho_3+\sigma_2\sigma_3}{3\,(1+n)^2}\right]$$

(A.3)

Moreover to find the index value of $(T_AI_AF_A)$ we write, the

$$(T_AI_AF_A)_\alpha=(T_A)_\alpha(I_A)_\alpha(F_A)_\alpha$$

$$= a_2b_2c_2\left[\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right),\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)\right]\left[1-\alpha\left(\frac{\rho_2+\rho_3}{1+n}\right)+\frac{\rho_2\rho_3\alpha^2}{(1+n)^2},1+\alpha\left(\frac{\sigma_2+\sigma_3}{1+n}\right)+\frac{\sigma_2\sigma_3\alpha^2}{(1+n)^2}\right]$$

$$= a_2b_2c_2\left[1-\frac{\rho_1}{1+n}+\alpha\left\{\frac{\rho_1-\rho_2-\rho_3}{(1+n)}+\frac{\rho_1(\rho_2+\rho_3)}{(1+n)^2}\right\}-\alpha^2\left\{\frac{\rho_1\rho_2\rho_3}{(1+n)^3}+\frac{\rho_1\rho_2+\rho_1\rho_3-\rho_2\rho_3}{(1+n)^2}\right\}+\frac{\rho_1\rho_2\rho_3\alpha^3}{(1+n)^3},\right.$$

$$\left.1+\frac{\sigma_1}{1+n}+\alpha\left\{\frac{\sigma_2+\sigma_3-\sigma_1}{(1+n)}+\frac{\sigma_1(\sigma_2+\sigma_3)}{(1+n)^2}\right\}+\alpha^2\left\{\frac{\sigma_1\sigma_2\sigma_3}{(1+n)^3}+\frac{\sigma_2\sigma_3-\sigma_1\sigma_2-\sigma_1\sigma_3}{(1+n)^2}\right\}-\frac{\sigma_1\sigma_2\sigma_3\alpha^3}{(1+n)^3}\right]$$

Thus, $I(T_A\, I_A\, F_A)$

$$= \frac{1}{2N}\sum_{n=0}^{N} a_2b_2c_2\left[2+\frac{\sigma_1-\rho_1}{1+n}+\frac{\rho_1+\sigma_2+\sigma_3-\sigma_1-\rho_2-\rho_3}{2(1+n)}+\frac{\rho_1(\rho_2+\rho_3)+\sigma_1(\sigma_2+\sigma_3)}{2(1+n)^2}-\frac{\rho_1\rho_2+\rho_1\rho_3-\rho_2\rho_3+\sigma_1\sigma_2+\sigma_1\sigma_3-\sigma_2\sigma_3}{3\,(1+n)^2}+\frac{\sigma_1\sigma_2\sigma_3-\rho_1\rho_2\rho_3}{(1+n)^3}-\frac{\sigma_1\sigma_2\sigma_3+\rho_1\rho_2\rho_3}{4(1+n)^3}\right] \quad (A.4)$$

Appendix B

Here we shall study the $\alpha-cuts$ of $(T_A,\ I_A\ and\ F_A)$ when all the components occur proper dense property of fuzzy sets[stated in (2), (6) and (8)]. In this case the $\alpha-cut$ of T_A that is $(T_A)_\alpha$ will remains the same.

But the $\alpha-cuts$ of $(F_A)_\alpha$ are given by $\left\{\frac{b_2e^{-n}-x}{\rho_2b_2e^{-n}}\right\}\geq\alpha\rightarrow x\leq b_2e^{-n}(1-\alpha\rho_2)$ and $\left\{\frac{x-b_2e^{-n}}{\sigma_2b_2e^{-n}}\right\}\geq\alpha\rightarrow x\geq b_2e^{-n}(1+\alpha\sigma_2)$

Thus $(F_A)_\alpha=[b_2e^{-n}(1-\alpha\rho_2),b_2e^{-n}(1+\alpha\sigma_2)]=b_2e^{-n}[(1-\alpha\rho_2),(1+\alpha\sigma_2)]$ (B.1)

Similarly, $(I_A)_\alpha=c_2e^{-n}[(1-\alpha\rho_3),(1+\alpha\sigma_3)]$ (B.2)

Therefore, $I(T_A)=\frac{1}{2N}\sum_{n=0}^{N}a_2\left\{2+\frac{(\sigma_1-\rho_1)}{2(1+n)}\right\}$ (B.3)

$I(I_A)=\frac{c_2}{2N}\left\{2+\frac{1}{2}(\sigma_3-\rho_3)\right\}\sum_{n=0}^{N}e^{-n}$ (B.4)

$I(F_A)=\frac{b_2}{2N}\left\{2+\frac{1}{2}(\sigma_2-\rho_2)\right\}\sum_{n=0}^{N}e^{-n}$ (B.5)

Again, $(T_A\,F_A)_\alpha=(T_A)_\alpha(F_A)_\alpha=a_2\left[\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right),\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)\right]\{b_2e^{-n}[(1-\alpha\rho_2),(1+\alpha\sigma_2)]\}$

$$= a_2b_2e^{-n}\left[\left(1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}\right)(1-\alpha\rho_2),\left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)(1+\alpha\sigma_2)\right]$$

$$= a_2b_2e^{-n}\begin{bmatrix}1-\frac{\rho_1}{1+n}+\frac{\rho_1\alpha}{1+n}-\alpha\rho_2+\frac{\alpha\rho_1\rho_2}{1+n}-\frac{\alpha^2\rho_1\rho_2}{1+n},\\ 1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}+\alpha\sigma_2+\frac{\alpha\sigma_1\sigma_2}{1+n}-\frac{\alpha^2\sigma_1\sigma_2}{1+n}\end{bmatrix}$$

$$= a_2b_2e^{-n}\begin{bmatrix}1-\frac{\rho_1}{1+n}+\alpha\left(\frac{\rho_1+\rho_1\rho_2}{1+n}-\rho_2\right)-\frac{\alpha^2\rho_1\rho_2}{1+n},\\ 1+\frac{\sigma_1}{1+n}+\alpha\left(\frac{\sigma_1\sigma_2-\sigma_1}{1+n}+\sigma_2\right)-\frac{\alpha^2\sigma_1\sigma_2}{1+n}\end{bmatrix}$$

(B.6)

Thus, $I(T_A\,F_A)=\frac{a_2b_2}{2N}\sum_{n=0}^{N}e^{-n}\left[2+\frac{\sigma_1-\rho_1}{1+n}+\frac{1}{2}\left(\frac{\rho_1-\sigma_1+\sigma_1\sigma_2+\rho_1\rho_2}{1+n}+\sigma_2-\rho_2\right)-\frac{\rho_1\rho_2+\sigma_1\sigma_2}{3(1+n)}\right]$ (B.7)

Similarly, $I(T_A\,I_A)=$

$\frac{a_2c_2}{2N}\sum_{n=0}^{N}e^{-n}\left[2+\frac{\sigma_1-\rho_1}{1+n}+\frac{1}{2}\left(\frac{\rho_1-\sigma_1+\sigma_1\sigma_3+\rho_1\rho_3}{1+n}+\sigma_3-\rho_3\right)-\frac{\rho_1\rho_3+\sigma_1\sigma_3}{3(1+n)}\right]$ (B.8)

For, $I(F_A\,I_A)$

We write, $(I_A\,F_A)_\alpha=(I_A)_\alpha(F_A)_\alpha$

$$= b_2c_2e^{-2n}[(1-\alpha\rho_2),(1+\alpha\sigma_2)][(1-\alpha\rho_3),(1+\alpha\sigma_3)]$$

$$= b_2c_2e^{-2n}[1-\alpha\rho_2-\alpha\rho_3+\alpha^2\rho_2\rho_3,\ 1+\alpha\sigma_2+\alpha\sigma_3+\alpha^2\sigma_2\sigma_3]$$

Therefore,

$I(F_A\,I_A)=$

$\frac{b_2c_2}{2N}\sum_{n=0}^{N}e^{-2n}\left[2+\frac{1}{2}(\sigma_2+\sigma_3-\rho_2-\rho_3)+\frac{\rho_2\rho_3+\sigma_2\sigma_3}{3}\right]$

$$= \frac{b_2c_2}{2N}\left[2+\frac{1}{2}(\sigma_2+\sigma_3-\rho_2-\rho_3)+\frac{\rho_2\rho_3+\sigma_2\sigma_3}{3}\right]\sum_{n=0}^{N}e^{-2n} \quad (B.9)$$

For, $I(T_A\,I_A\,F_A)$ we take,

$$(T_A I_A F_A)_\alpha = (T_A)_\alpha (I_A)_\alpha (F_A)_\alpha$$
$$= a_2 b_2 c_2 e^{-2n}[1-\alpha\rho_2-\alpha\rho_3+\alpha^2\rho_2\rho_3, 1+\alpha\sigma_2 + \alpha\sigma_3 + \alpha^2\sigma_2\sigma_3]\left[\left(1-\frac{\rho_1}{1+n} + \frac{\rho_1\alpha}{1+n}\right), \left(1+\frac{\sigma_1}{1+n}-\frac{\sigma_1\alpha}{1+n}\right)\right]$$

$$= a_2 b_2 c_2 e^{-2n}\left[\left(1-\frac{\rho_1}{1+n}\right)(1-\alpha\rho_2-\alpha\rho_3 + \alpha^2\rho_2\rho_3) + \frac{\rho_1}{1+n}(\alpha-\alpha^2\rho_2-\alpha^2\rho_3 + \alpha^3\rho_2\rho_3), \left(1+\frac{\sigma_1}{1+n}\right)(1+\alpha\sigma_2 + \alpha\sigma_3+\alpha^2\sigma_2\sigma_3) - \frac{\sigma_1}{1+n}(\alpha+\alpha^2\sigma_2+\alpha^2\sigma_3 + \alpha^3\sigma_2\sigma_3)\right]$$

Therefore,

$$I(T_A I_A F_A) = \frac{a_2b_2c_2}{2N}\sum_{n=0}^{N} e^{-2n}\left[\left(1-\frac{\rho_1}{1+n}\right)\left(1-\frac{1}{2}\rho_2-\frac{1}{2}\rho_3+\frac{1}{3}\rho_2\rho_3\right)+\frac{\rho_1}{1+n}\left(\frac{1}{2}-\frac{1}{3}\rho_2-\frac{1}{3}\rho_3+\frac{1}{4}\rho_2\rho_3\right)+\left(1+\frac{\sigma_1}{1+n}\right)\left(1+\frac{1}{2}\sigma_2+\frac{1}{2}\sigma_3+\frac{1}{3}\sigma_2\sigma_3\right)-\frac{\sigma_1}{1+n}\left(\frac{1}{2}+\frac{1}{3}\sigma_2+\frac{1}{3}\sigma_3+\frac{1}{4}\sigma_2\sigma_3\right)\right]$$
(B.10)

Hence, using (B.3)-(B.10) the expected values of the given NS can be obtained as

$I(S) = I(T_A) + I(I_A) + I(F_A) - I(T_A I_A) - I(I_A F_A) - I(T_A F_A) + I(T_A I_A F_A)$where

$$I(S) = \frac{1}{2N}\sum_{n=0}^{N} a_2\left\{2+\frac{(\sigma_1-\rho_1)}{2(1+n)}\right\} + \frac{b_2}{2N}\left\{2+\frac{1}{2}(\sigma_2-\rho_2)\right\}\sum_{n=0}^{N} e^{-n} + \frac{c_2}{2N}\left\{2+\frac{1}{2}(\sigma_3-\rho_3)\right\}\sum_{n=0}^{N} e^{-n}$$
$$-\frac{a_2b_2}{2N}\sum_{n=0}^{N} e^{-n}\left[2+\frac{\sigma_1-\rho_1}{1+n} + \frac{1}{2}\left(\frac{\rho_1-\sigma_1+\sigma_1\sigma_2+\rho_1\rho_2}{1+n}+\sigma_2 - \rho_2\right)-\frac{\rho_1\rho_2+\sigma_1\sigma_2}{3(1+n)}\right]$$
$$-\frac{b_2c_2}{2N}\sum_{n=0}^{N} e^{-2n}\left[2 + \frac{1}{2}(\sigma_2+\sigma_3-\rho_2-\rho_3) + \frac{\rho_2\rho_3+\sigma_2\sigma_3}{3}\right]$$
$$-\frac{a_2c_2}{2N}\sum_{n=0}^{N} e^{-n}\left[2+\frac{\sigma_1-\rho_1}{1+n} + \frac{1}{2}\left(\frac{\rho_1-\sigma_1+\sigma_1\sigma_3+\rho_1\rho_3}{1+n}+\sigma_3 - \rho_3\right)-\frac{\rho_1\rho_3+\sigma_1\sigma_3}{3(1+n)}\right]$$
$$+\frac{a_2b_2c_2}{2N}\sum_{n=0}^{N} e^{-2n}\left[\left(1-\frac{\rho_1}{1+n}\right)\left(1-\frac{1}{2}\rho_2-\frac{1}{2}\rho_3+\frac{1}{3}\rho_2\rho_3\right)+\frac{\rho_1}{1+n}\left(\frac{1}{2}-\frac{1}{3}\rho_2-\frac{1}{3}\rho_3+\frac{1}{4}\rho_2\rho_3\right)+\left(1+\frac{\sigma_1}{1+n}\right)\left(1+\frac{1}{2}\sigma_2+\frac{1}{2}\sigma_3+\frac{1}{3}\sigma_2\sigma_3\right)-\frac{\sigma_1}{1+n}\left(\frac{1}{2}+\frac{1}{3}\sigma_2+\frac{1}{3}\sigma_3+\frac{1}{4}\sigma_2\sigma_3\right)\right] \quad \text{(B.11)}$$

Note that, if the fuzzy components are experienced with different interactions then we shall calculate the expected score values as follows:

$$I(S) = \frac{1}{2M}\sum_{m=0}^{M} a_2\left\{2+\frac{(\sigma_1-\rho_1)}{2(1+m)}\right\} + \frac{b_2}{2N}\left\{2+\frac{1}{2}(\sigma_2-\rho_2)\right\}\sum_{n=0}^{N} e^{-n} + \frac{c_2}{2P}\left\{2+\frac{1}{2}(\sigma_3-\rho_3)\right\}\sum_{p=0}^{P} e^{-p}$$

$$-\frac{a_2b_2}{2N_1}\sum_{n=0}^{N_1}e^{-n}\left[2+\frac{\sigma_1-\rho_1}{1+n}+\frac{1}{2}\left(\frac{\rho_1-\sigma_1+\sigma_1\sigma_2+\rho_1\rho_2}{1+n}+\sigma_2-\rho_2\right)-\frac{\rho_1\rho_2+\sigma_1\sigma_2}{3(1+n)}\right]$$

$$-\frac{b_2c_2}{2N_2}\sum_{n=0}^{N_2}e^{-2n}\left[2+\frac{1}{2}(\sigma_2+\sigma_3-\rho_2-\rho_3)+\frac{\rho_2\rho_3+\sigma_2\sigma_3}{3}\right]$$

$$-\frac{a_2c_2}{2N_3}\sum_{n=0}^{N_3}e^{-n}\left[2+\frac{\sigma_1-\rho_1}{1+n}+\frac{1}{2}\left(\frac{\rho_1-\sigma_1+\sigma_1\sigma_3+\rho_1\rho_3}{1+n}+\sigma_3-\rho_3\right)-\frac{\rho_1\rho_3+\sigma_1\sigma_3}{3(1+n)}\right]$$

$$+\frac{a_2b_2c_2}{2N_4}\sum_{n=0}^{N_4}e^{-2n}\left[\left(1-\frac{\rho_1}{1+n}\right)\left(1-\frac{1}{2}\rho_2-\frac{1}{2}\rho_3+\frac{1}{3}\rho_2\rho_3\right)+\frac{\rho_1}{1+n}\left(\frac{1}{2}-\frac{1}{3}\rho_2-\frac{1}{3}\rho_3+\frac{1}{4}\rho_2\rho_3\right)+\left(1+\frac{\sigma_1}{1+n}\right)\left(1+\frac{1}{2}\sigma_2+\frac{1}{2}\sigma_3+\frac{1}{3}\sigma_2\sigma_3\right)-\frac{\sigma_1}{1+n}\left(\frac{1}{2}+\frac{1}{3}\sigma_2+\frac{1}{3}\sigma_3+\frac{1}{4}\sigma_2\sigma_3\right)\right] \quad (B.12)$$

Where $N_1 = min(M,N)$, $N_2 = min(N,P)$, $N_3 = min(M,P)$ and $N_4 = min(M,N,P)$

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