Introduction to Neutrosophic Automata

International Webinar on "Neutrosophic Sets"

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Fuzzy Sets

Fuzzy Logic

- logic of graded truth or intermediate truth
- provides a way to express subtle nuances in reasoning
- successful in modeling uncertainty

Th original Zadeh's definition of a fuzzy set is:

- fuzzy subset of a set A is a function $\mu : A \to [0, 1]$, where [0,1] is the real unit closed interval.
- For $x \in A$, the membership degree $\mu_A(x)$ is interpreted as the degree of satisfaction of elements to the property corresponding to the collection.
- if $\mu_A(x)$ takes values only in the set $\{0, 1\}$, then it is treated as the ordinary crisp subset of A.

Interval-valued Fuzzy Sets[1](IVFS)

IVFS represent the membership degrees with interval values in [0,1] in order to reflect the uncertainty in assigning membership degrees.

An IVF set A is formally defined by membership functions of the form

$$A = \left\{ \left(x, \left[\mu_A^l(x), \mu_A^r(x) \right] \right) | x \in X \right\}, \quad \mu_A^l(x), \mu_A^r(x) \in [0,1].$$

Basic Operations:

$$\mu_{A\cup B}(x) = [\mu^l_{A\cup B}(x), \mu^r_{A\cup B}(x)] = \begin{cases} \mu^l_{A\cup B}(x) = \max\{\mu^l_A(x), \mu^l_B(x)\} \\ \mu^r_{A\cup B}(x) = \max\{\mu^r_A(x), \mu^r_B(x)\} \end{cases}$$

$$\mu_{A\cap B}(x) = [\mu^l_{A\cap B}(x), \mu^r_{A\cap B}(x)] = \begin{cases} \mu^l_{A\cup B}(x) = \min\{\mu^l_A(x), \mu^l_B(x)\} \\ \mu^r_{A\cap B}(x) = \min\{\mu^r_A(x), \mu^r_B(x)\} \end{cases}$$

$$\mu_{\bar{A}}(x) = [\mu^l_{\bar{A}}(x), \mu^r_{\bar{A}}(x)] = \begin{cases} \mu^l_{\bar{A}}(x) = 1 - \mu^r_A(x) \\ \mu^r_{\bar{A}}(x) = 1 - \mu^l_A(x) \end{cases}$$

[1] L.Zadeh. The concept of a linguistic variable and its application to approximate reasoning. Part 1. *Information Science*, 8 (1975), 199-249.

Intuitionistic Fuzzy Sets[2](IFS)

IFS represent the membership degrees that are a pair of membership degree and non-membership degree.

An IFS set A is formally defined by membership functions of the form

For every $x \in X$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$,

$$A = \{(x, \mu_A(x), \nu_A(x)) \, | x \in X\}, \quad \mu_A(x), \nu_A(x) \in [0, 1].$$

- The amount

$$\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x))$$

  is called the hesitation part or intuitionistic index, which may cater to either membership degree or non-membership degree.
- It means that the IFS are a representation to express the uncertainty in assigning membership degrees to elements.

[2] K.T.Atanassov. Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20 (1986), 87-96.

Intuitionistic Fuzzy Sets (IFS)

Basic Operations:

$$A \cup B = \{(x, \mu_{A\cup B}(x), \nu_{A\cup B}(x))\} = \begin{cases} \mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} \\ \nu_{A\cup B}(x) = \min\{\nu_A(x), \nu_B(x)\} \end{cases}$$

$$A \cap B = \{(x, \mu_{A\cap B}(x), \nu_{A\cap B}(x))\} = \begin{cases} \mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} \\ \nu_{A\cap B}(x) = \max\{\nu_A(x), \nu_B(x)\} \end{cases}$$

$$\bar{A} = \{(x, \mu_{\bar{A}}(x), \nu_{\bar{A}}(x) | x \in X\} = \begin{cases} \mu_{\bar{A}}(x) = \nu_A(x) \\ \nu_{\bar{A}}(x) = \mu_A(x) \end{cases}$$

Bipolar Fuzzy Sets[3](BFS)

- BFS represent the membership degrees (MD) ranges from the interval [-1,1] which is extended from [0,1].
- MD: $\mu_A(x) \in (0,1]$ – elements somewhat satisfy the property.
- MD: $\mu_A(x) = 0$ – elements are irrelevant to the corresponding property.
- MD: $\mu_A(x) \in [-1,0)$ – elements somewhat satisfy the implicit counter-property.
- Two kinds of representation: canonical and reduced.

[3]Wen-Ran Zhang, Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis, NAFIPS/IFIS/NASA '94. San Antonio, TX, USA, 1994, 305-309.

Bipolar Fuzzy Sets (BFS)

Canonical Representation

Membership degrees are expressed with a pair of a positive membership value in [0,1] and a negative membership value in [-1,0].

$$A = \{(x, (\mu_A^P(x), \mu_A^N(x)))|x \in X\}$$

where

$$\mu_A^P(x) : X \to [0,1] \quad \mu_A^N(x) : X \to [-1,0]$$

Remarks:

- $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) = 0$ – positive satisfaction.
- $\mu_A^P(x) = 0$ and $\mu_A^N(x) \neq 0$ – satisfies counter-property.
- $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) \neq 0$ – overlaps property

Bipolar Fuzzy Sets (BFS)

Basic Operations:

$$A \cup B = \{(x, \mu^P_{A\cup B}(x), \mu^N_{A\cup B}(x))\} = \begin{cases} \mu^P_{A\cup B}(x) = \max\{\mu^P_A(x), \mu^P_B(x)\} \\ \mu^N_{A\cup B}(x) = \min\{\mu^N_A(x), \mu^N_B(x)\} \end{cases}$$

$$A \cap B = \{(x, \mu^P_{A\cap B}(x), \mu^N_{A\cap B}(x))\} = \begin{cases} \mu^P_{A\cap B}(x) = \min\{\mu^P_A(x), \mu^P_B(x)\} \\ \mu^N_{A\cap B}(x) = \max\{\mu^N_A(x), \mu^N_B(x)\} \end{cases}$$

$$\bar{A} = \{(x, \mu^P_{\bar{A}}(x), \mu^N_{\bar{A}}(x)|x \in X\} = \begin{cases} \mu^P_{\bar{A}}(x) = 1 - \mu^P_A(x) \\ \mu^N_{\bar{A}}(x) = -1 - \mu^N_A(x) \end{cases}$$

Bipolar Fuzzy Sets (BFS)

Reduced Representation

Membership degrees are presented with a value in [-1,1].

$$A = \{(x, \mu^{\mathbb{R}}(x)) | x \in X\} \quad \mu_A^{\mathbb{R}} : X \to [-1, 1]$$

Member degree:

$$\mu_A^{\mathbb{R}}(x) = \begin{cases} \mu_A^P(x) & \text{if } \mu_A^N(x) = 0 \\ \mu_A^N(x) & \text{if } \mu_A^P(x) = 0 \\ f(\mu_A^P(x), \mu_A^N(x)) & \text{otherwise} \end{cases}$$

where $f(\mu_A^P(x), \mu_A^N(x))$ is an aggregation function to merge a pair of positive and negative membership values into a value.

IVFS vs. IFS

- IFS can be regarded as another expression for IVFS.
- Deduce the basic operations of IVFS and IFS have the same roles, by using the boundary values of IVFS such as

$$\mu^l_A(x) = \mu_A(x) \text{ and } \mu^r_A(x) = 1 - \nu_A(x)$$

- IVFS and IFS have the same expressive power and the same basic set operations.
- The intuitionistic fuzzy set representation is useful when there are some uncertainties in assigning membership degrees.

IFS vs. BFS

We can compare BFS with IFS under the conditions

$$\mu_A^P(x) = \mu_A(x)$$

and

$$\mu_A^N(x) = -\nu_A(x)$$

IFS vs. BFS

positive membership
satisfies the property A

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positive membership satisfies the property A

membership degree satisfies the property of A

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satisfies an implicit counter-property of A

satisfies the not-property A

Both BFS and IFS are the different extensions of fuzzy sets, since a counter-property is not usually equivalent to not-property of A.

IVFS vs. IFS vs. BFS

Element x with membership value (0,0)

- In BFS, element x does not satisfy both the property and counter-property of BFS which means that it is indifferent or neutral.
- In IFS, element x does not satisfy both the property and not-property.
- In IVFS, element with the mv (0,0) in IFS has the mv [0,1] in IVFS which means that no knowledge about the element.

- The IFS representation is useful when there are some uncertainties in assigning membership degrees.
- The BFS representation is useful when irrelevant elements and contrary elements are needed to be discriminated.

Example: Fuzzy concept *frog's prey*

- IVFS for frog's prey:

frog's prey={(*mosqito*,[1,1]), (*dragon fly*,[0.4,0.7]), (*turtle*,[0,0]), (*snake*,[0,0])}

- IFS for frog's prey:

frog's prey={(*mosqito*,1,0), (*dragon fly*,0.4,0.3), (*turtle*,0,1), (*snake*,0,1)}

- BFS for frog's prey:

frog's prey={(*mosqito*,1,0), (*dragon fly*,0.4,0), (*turtle*,0,0), (*snake*,0,-1)}

Neutrosophic Sets

In Neutrosophic sets, we can connect an idea with its opposite and with its neutral and get common parts.

$$\prec A \succ \wedge \prec \text{non} - A \succ = \text{nonempty set}$$

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It is true in a restricted case because most of the investigation only considers the dynamics of opposite interacts such as

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In our everyday life, we not only interact with opposite things, but with neutrals between them too

$$\prec \text{neut} - A \succ$$

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In our everyday life, we not only interact with opposite things, but with neutrals between them too

$$\prec \text{neut} - A \succ$$

For example, if you fight with a man (so you both are the opposites to each other), but neutral people around both of you (especially the police) interfere to reconcile both of you.

Characterisation of Neutrosophic Sets

A neutrosophic set is characterised by

- $\prec A \succ$ – a truth-membership function (T)
- $\prec$ anti-$A\succ$ (the opposite of $\prec A \succ$) – an indeterminancy-membership function (I)
- $\prec$ neut-$A\succ$ (the neutral between $\prec A \succ$ and $\prec$ anti-$A\succ$) interact among themselves – a falsity-membership function (F)

where T, I F are subsets of the unit interval [0,1].

- If T, I, F are crisp numbers in [0,1], then we have a single-valued neutrosophic set.
- If T, I, F are intervals included in [0,1], then we have an interval-valued neutrosophic set.

Neutrosophic logic introduces a percentage of "indeterminacy" due to unexpected parameters hidden in some propositions.

Neutrosophic Set[4]

Definition

Let X be a universe of discourse and $A \subseteq X$. The neutrosophic set is an object having the form

$$A = \{\prec x, T(x), I(x), F(x) \succ | \forall x \in X\}$$

where the functions can be defined by

$$T, I, F : X \to [0, 1]$$

with the condition

$$0 \leq T(x) + I(x) + F(x) \leq 3.$$

[4] Smarandache, F. (1999). *A unifying field in logics: Neutrosophy, neutrosophic probability, set and logic*. Rehoboth, VA: American Research Press.

Overview of Fuzzy Automata

Concept of fuzzy automata

- natural generalization of the concept of non-deterministic automata

Močkoř, Bělohlávek, Li and Pedrycz

- Močkoř-fuzzy automata represented as nested systems of non-deterministic automata
- Bělohlávek-deterministic automata with fuzzy sets of final states represented as nested systems of deterministic automata
- Li and Pedrycz-fuzzy automata represented as automata with fuzzy transition relations taking membership values in a lattice ordered monoid

Non-deterministic automaton

- a tuple $\mathcal{A} = (A, X, \delta, \sigma, \tau)$

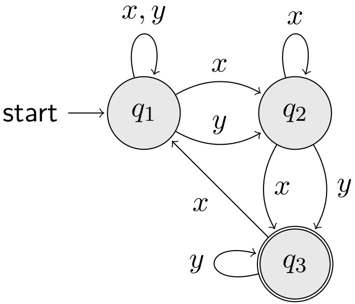
  $A \neq \emptyset$ - set of states, $X \neq \emptyset$ - input alphabet

  $\delta \subseteq A \times X \times A \quad (\delta_x \subseteq A \times A)$ - transition relation

  $(a, x, b) \in \delta \Leftrightarrow (a, b) \in \delta_x$, for all $a, b \in A$, $x \in X$

  $\sigma \subseteq A$, $\tau \subset A$ - sets of initial and terminal states

Transition relations, sets of initial and terminal states



- represented by Boolean matrices and vectors:

$$\delta_x = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \delta_y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \tau = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Fuzzy Automata

Fuzzy Automaton

- 6-tuple $\mathscr{A} = (Q, \Sigma, \delta, R, Z, \omega)$

  Q is a finite set of states, $Q = \{q_1, q_2, \cdots, q_n\}$.
  Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \cdots, a_n\}$.
  $R \in Q$ is the (possibly fuzzy) start state of Q.
  Z is a finite set of output symbols, $Z = \{b_1, b_2, \cdots, b_k\}$.
  $\delta : Q \times \Sigma \times Q \rightarrow (0, 1]$ - fuzzy transition function
  $\omega : Q \rightarrow Z$- is the output function which is used to map a (fuzzy) state to the output set.

- associated with each fuzzy transition, there is a membership value in $(0, 1]$, i.e. the weight of the transition.
- the transition from state q_i to state q_j upon input a_k is denoted by $\delta(q_i, a_k, q_j)$.

Neutrosophic Automata

- Tahir & Khan, 2016
  - the interval neutrosophic finite switchboard state machine
- Tahir, 2018
  - concepts of single-valued neutrosophic finite state machine and switchboard state machine
- Kavikumar et al, 2019
  - concepts of neutrosophic general fuzzy automata and neutrosophic general switchboard automata
- Kavikumar et al, 2020
  - concept of distinguishabilty and inverse of neutrosophic finite automata

Neutrosophic Automaton

- 5-tuple $\mathscr{N} = (Q, \Sigma, Z, \delta, \sigma)$

  Q is a finite set of states, $Q = \{q_1, q_2, \cdots, q_n\}$.
  Σ is a finite set of input symbols, $\Sigma = \{x_1, x_2, \cdots, x_n\}$.
  Z is a finite set of output symbols, $Z = \{y_1, y_2, \cdots, y_n\}$.
  δ is a neutrosophic subset of $Q \times \Sigma \times Q$ which represents neutrosophic transition function.
  σ is a neutrosophic subset of $Q \times \Sigma \times Z$ which represents neutrosophic output function.

Neutrosophic Automaton: Neutrosophic Transition Function

$$\delta = \prec \delta_1, \delta_2, \delta_3 \succ$$

is a neutrosophic subset of $Q \times \Sigma \times Q$ such that the neutrosophic transition function

$$\delta : Q \times \Sigma \times Q \to [0,1] \times [0,1] \times [0,1]$$

is defined as follows: $\forall q_i, q_j \in Q$ and $x_1, x_2 \in \Sigma$,

$$\delta_1(q_i, \Lambda, q_j) = \begin{cases} 1 & \text{if } q_i = q_j \\ 0 & \text{if } q_i \neq q_j \end{cases}$$

$$\delta_2(q_i, \Lambda, q_j) = \begin{cases} 0 & \text{if } q_i = q_j \\ 1 & \text{if } q_i \neq q_j \end{cases}$$

$$\delta_3(q_i, \Lambda, q_j) = \begin{cases} 0 & \text{if } q_i = q_j \\ 1 & \text{if } q_i \neq q_j \end{cases}$$

Neutrosophic Automaton: Neutrosophic Transition Function

$$\delta_1(q_i, x_1x_2, q_j) = \bigvee_{r\in Q} \{\delta_1(q_i, x_1, r) \wedge \delta_1(r, x_2, q_j)\}$$

$$\delta_2(q_i, x_1x_2, q_j) = \bigwedge_{r\in Q} \{\delta_2(q_i, x_1, r) \vee \delta_2(r, x_2, q_j)\}$$

$$\delta_3(q_i, x_1x_2, q_j) = \bigwedge_{r\in Q} \{\delta_3(q_i, x_1, r) \vee \delta_3(r, x_2, q_j)\}$$

Neutrosophic Automata

Neutrosophic Automaton: Neutrosophic Output Function

$$\sigma = \prec \sigma_1, \sigma_2, \sigma_3 \succ$$

is a neutrosophic subset of $Q \times \Sigma \times Z$ such that the neutrosophic output function

$$\sigma : Q \times \Sigma \times Z \to L \times L \times L$$

is defined as follows: $\forall q_i, q_j \in Q, x_1, x_2 \in \Sigma$ and $y_1, y_2 \in Z$,

$$\sigma_1(q_i, x_1, q_j) = \begin{cases} 1 & \text{if } x_1 = y_1 = \Lambda \\ 0 & \text{if } x_1 = \Lambda, y_1 \neq \Lambda \text{ or } x_1 \neq \Lambda, y_1 = \Lambda \end{cases}$$

$$\sigma_2(q_i, x_1, q_j) = \begin{cases} 0 & \text{if } x_1 = y_1 = \Lambda \\ 1 & \text{if } x_1 = \Lambda, y_1 \neq \Lambda \text{ or } x_1 \neq \Lambda, y_1 = \Lambda \end{cases}$$

$$\sigma_3(q_i, x_1, q_j) = \begin{cases} 0 & \text{if } x_1 = y_1 = \Lambda \\ 1 & \text{if } x_1 = \Lambda, y_1 \neq \Lambda \text{ or } x_1 \neq \Lambda, y_1 = \Lambda \end{cases}$$

Neutrosophic Automaton: Neutrosophic Output Function

$$\sigma_1(q_i, x_1x_2, y_1y_2) = \bigvee_{r\in Q} \{\sigma_1(q_i, x_1, y_1) \wedge \delta_1(q_i, x_1, r) \wedge \sigma_1(r, x_2, y_2)\}$$

$$\sigma_2(q_i, x_1x_2, y_1y_2) = \bigwedge_{r\in Q} \{\sigma_2(q_i, x_1, y_1) \vee \delta_2(q_i, x_1, r) \vee \sigma_2(r, x_2, y_2)\}$$

$$\sigma_3(q_i, x_1x_2, y_1y_2) = \bigwedge_{r\in Q} \{\sigma_3(q_i, x_1, y_1) \vee \delta_3(q_i, x_1, r) \vee \sigma_3(r, x_2, y_2)\}$$

Distinguishable

$\mathscr{N} = (Q, \Sigma, Z, \delta, \sigma)$ and $\mathscr{N}' = (Q', \Sigma', Z, \delta', \sigma')$ be a neutrosophic finite automata.

a pair of states (q, q') is indistinguishable if

$$\sigma(q, x, y) = \sigma'(q', x', y')$$

for evert $q_i \in Q$, $q_i' \in Q'$ and for all $x \in \Sigma$, $y \in Z$.

Rational

State $q \in Q$ is said to be rational

When the inputs $\{x_n\} \in \Sigma$ are ultimately periodic sequence which yields an ultimately periodic sequence of outputs $\{y_n\} \in Z$

$$\sigma(q, \{x_n\}, \{y_n\}) > 0 \Rightarrow \{\sigma(q_n, x_n, y_n)\} > 0$$

where $q_1 = q$ and for $n \geq 2$, $\delta(q_{n-1}, x_{n-1}, q_n) > 0$.

- It is clear that if q is a rational state of a neutrosophic finite automata and p is indistinguishable from q, then p is rational.
- To check the given $q \in Q$ is rational state it is enough to assume that the sequence $\{x_n\} \in \Sigma$ is an infinite.

THANK YOU