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Neutrosophic Hypercompositional Structures defined by Binary Relations

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Abstract: The objective of this paper is to study *neutrosophic* hypercompositional structures $H(I)_\tau$ arising from the hypercompositions derived from the binary relations τ on a *neutrosophic* set $H(I)$. We give the characterizations of τ that make $H(I)_\tau$ hypergroupoids,quasihypergroups, semihypergroups, *neutrosophic* hypergroupoids, *neutrosophic* quasihypergroups, *neutrosophic* semihypergroups and *neutrosophic* hypergroups.

Keywords: hypergroup, neutrosophic hypergroup, binary relations.

1 Introduction

The concept of hyperstructure together with the concept of hypergroup was introduced by F. Marty at the 8th Congress of Scandinavian Mathematicians held in 1934. A comprehensive review of the concept can be found in [5, 6, 12]. The concept of neutrosophy was introduced by F. Smarandache in 1995 and the concept of *neutrosophic* algebraic structures was introduced by F. Smarandache and W.B. Vasantha Kandasamy in 2006. A comprehensive review of *neutrosophy* and *neutrosophic* algebraic structures can be found in [1, 2, 3, 4, 15, 24 , 25].

One of the techniques of constructing hypergoupoids, quasi hypergroups, semihypergroups and hypergroups is to endow a nonempty set H with a hypercomposition derived from the binary relation ρ on H that give rise to a hypercompositional structure H_ρ. In this paper, we consider binary relations τ on a neutrosophic set H(I) that define hypercompositional structures $H(I)_\tau$ .Hypercompositions in H(I) considered in this paper are in the sense of Rosenberg [22], Massouros and Tsitouras [16, 17], Corsini [8, 9], and De Salvo and Lo Maro [13, 14]. We give the characterizations of τ that make $H(I)_\tau$ hypergroupoids, quasihypergroups, semihypergroups, *neutrosophic* hypergroupoids, *neutrosophic* quasihypergroups, *neutrosophic* semihypergroups, and *neutrosophic* hypergroups.

2 Preliminaries

Definition 2.1. Let H be a non-empty set, and $\circ : H\times H \to P^*(H)$ be a hyperoperation.

(1) The couple $(H,\circ)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x\in H$, we define
$$A\circ B=\bigcup_{a\in A,b\in B} a\circ b, A\circ x = A\circ\{x\} \text{ and}$$
$$x\circ B=\{x\}\circ B$$

(2) A hypergroupoid $(H,\circ)$ is called a semihypergroup if for all a,b,c of H we have $(a\circ b)\circ c = a\circ(b\circ c)$, which means that
$$\bigcup_{u\in a\circ b} u\circ c=\bigcup_{v\in b\circ c} a\circ v.$$
A hypergroupoid $(H,\circ)$ is called a quasihypergroup if for all a of H we have $a\circ H = H\circ a = H$. This condition is also called the reproduction axiom.

(3) A hypergroupoid $(H,\circ)$ which is both a semihypergroup and a quasihypergroup is called a hypergroup.

Definition 2.2. Let $(G,*)$ be any group and let $G(I)=\langle G\cup I\rangle$. The couple $(G(I),*)$ is called a neutrosophic group generated by G and I under the binary operation $*$. The indeterminancy factor I is such that $I*I=I$. If $*$ is ordinary multiplication, then $IastI*...*I=I^n=I$ and if $*$ is ordinary addition, then $I*I*I*...*I=nI$ for $n\in\mathbb{N}$.

If $a*b=b*a$ for all $a,b\in G(I)$, we say that G(I) is commutative. Otherwise, G(I) is called a non-commutative *neutrosophic* group.

Theorem 2.3. *[24] Let G(I) be a neutrosophic group. Then,*

(1) G(I) in general is not a group;

(2) G(I) always contain a group.

Example 1. [3] Let G(I)={e, a, b, c, I, aI, bI, cI} be a set, where $a^2=b^2=c^2=e$, bc=cb=a, ac=ca=b, ab=ba=c. Then (G(I),.) is a commutative *neutrosophic* group.

Definition 2.4. [4] Let $(H,\circ)$ be any hypergroup and let $H(I)=\langle H\cup I\rangle=\{(a,bI):a,b\in H\}$. The couple $(H(I),\circ)$ is called a *neutrosophic hypergroup* generated by H and I under the hyperoperation $\circ$.

For all (a,bI),(c,dI) $\in H(I)$, the composition of elements of H(I) is defined by

$$(a,bI)\circ(c,dI)=\{(x,yI):x\in a\circ c,$$
$$y\in a\circ d\cup b\circ c\cup b\circ d\}.$$

Example 2. [4] Let H(I)={a,b,(a,aI),(a,bI),(b,aI), (b,bI)} be a set and let $\circ$ be a hyperoperation on H defined in the table below.

○	a	b	(a,aI)	(a,bI)	(b,aI)	(b,bI)
a	a	b	(a,aI)	(a,bI)	(b,aI)	(b,bI)
b	b	a b	(b,bI)	(b,aI) (b,bI)	(a,bI) (b,bI)	(a,aI) (a,bI) (b,aI) (b,bI)
(a,aI)	(a,aI)	(b,bI)	(a,aI)	(a,aI) (b,bI)	(b,aI) (b,bI)	(b,bI)
(a,bI)	(a,bI)	(b,aI) (b,bI)	(a,aI) (a,bI)	(a,aI) (a,bI)	(b,aI) (b,bI)	(b,aI) (b,bI)
(b,aI)	(b,aI)	(b,bI) (a,bI)	(b,aI) (b,bI)	(b,aI) (b,bI)	(a,aI) (a,bI) (b,aI) (b,bI)	(a,aI) (a,bI) (b,aI) (b,bI)
(b,bI)	(b,bI)	(a,aI) (a,bI) (b,aI) (b,bI)	(b,bI)	(b,aI) (b,bI)	(a,aI) (a,bI) (b,aI) (b,bI)	(a,aI) (a,bI) (b,aI) (b,bI)

Then $(H(I),\circ)$ is a *neutrosophic* hypergroup.

Definition 2.5. Let H be a nonempty set and let ρ be a binary relation on H.

(1) $\rho\circ\rho=\rho^2=\{(x,y):(x,z),(z,y)\in\rho,$ for some $z\in H\}.$

(2) An element $x\in H$ is called an outer element of ρ if $(z,x)\notin\rho^2$ for some $z\in H.$ Otherwise, x is called an inner element.

(3) The domain of ρ is the set $D(\rho)=\{x\in H:(x,z)\in\rho,$ for some $z\in H\}.$

(4) The range of ρ is the set

$R(\rho)=\{x\in H:(z,x)\in\rho,$ for some $z\in H\}.$

In [22], Rosenberg introduced in H the hypercomposition

$x\circ x=\{z\in H:(x,z)\in\rho\}$ and

$x\circ y=x\circ x\cup y\circ y$ (1)

and proved the following:

Proposition 2.6. *[22] $H_\rho=(H,\circ)$ is a hypergroupoid if and only if $H=D(\rho)$.*

Proposition 2.7. *[22] H_ρ is a quasihypergroup if and only if*

(1) $H=D(\rho)$.
(2) $H=R(\rho)$.

Proposition 2.8. *[22] H_ρ is a semihypergroup if and only if*

(1) $H=D(\rho)$.
(2) $\rho\subseteq\rho^2$.
(3) $(a,x)\in\rho^2$ implies that $(a,x)\in\rho$ whenever x is an outer element of ρ.

Proposition 2.9. *[22] H_ρ is a hypergroup if and only if*

(1) $H=D(\rho)$.
(2) $H=R(\rho)$.
(3) $\rho\subseteq\rho^2$.
(4) $(a,x)\in\rho^2$ implies that $(a,x)\in\rho$ whenever x is an outer element of ρ.

In [17], Massouros and Tsitouras noted that whenever x is an outer element of ρ, then it can be deduced from condition (2) and (3) (conditions (3) and (4)) of Proposition 2.8 (Proposition 2.9) that $(a,x)\in\rho$ if and only if $(a,x)\in\rho^2$ for some $a\in H_\rho$. Hence, they restated Propositions 2.8 and 2.9 in the following equivalent forms:

Proposition 2.10. *[17] H_ρ is a semihypergroup if and only if*

(1) $H=D(\rho)$.
(2) $(a,x)\in\rho^2$ if and only if $(a,x)\in\rho$ for all $a\in H$ whenever x is an outer element of ρ.

Proposition 2.11. *[17] H_ρ is a semihypergroup if and only if*

(1) $H=D(\rho)$.
(2) $H=R(\rho)$.
(3) $(a,x)\in\rho^2$ if and only if $(a,x)\in\rho$ for all $a\in H$ whenever x is an outer element of ρ.

If H is a nonempty set and ρ is a binary on H, Massouros and Tsitouras [17] defined hypercomposition • on H as follows:

$x\bullet x=\{z\in H:(z,x)\in\rho\}$ and

$x\bullet y=x\bullet x\cup y\bullet y$ (2)

and stated that:

Proposition 2.12. *[17] If ρ is symmetric, then the hypercompositional structures $(H,\circ)$ and $(H,\bullet)$ coincide.*

Following Rosenberg's terminology in [22], Massouros and Tsitouras established the following:

Definition 2.13. [17]

(1) For $(a,b)\in\rho$, a is called a predecessor of b and b a successor of a.
(2) An element x of H is called a predecessor outer element of ρ if $(x,z)\notin\rho^2$ for some $z\in H$.

Using hypercomposition •, Massouros and Tsitouras established the following:

Proposition 2.14. [17] $H_\rho=(H,\bullet)$ is hypergroupoid if and only if $H=R(\rho)$.

Proposition 2.15. [17] $H_\rho=(H,\bullet)$ is quasihypergroup if and only if

(1) $H=D(\rho)$.
(2) $H=R(\rho)$.

Proposition 2.16. [17] $H_\rho=(H,\bullet)$ is semihypergroup if and only if

(1) $H=R(\rho)$.
(2) $(x,y)\in\rho^2$ if and only if $(x,y)\in\rho$ for all $y\in H$ whenever x is a predecessor outer element of ρ.

Proposition 2.17. [17] $H_\rho=(H,\bullet)$ is hypergroup if and only if

(1) $H=D(\rho)$.
(2) $H=R(\rho)$.
(3) $(x,y)\in\rho^2$ if and only if $(x,y)\in\rho$ for all $y\in H$ whenever x is a predecessor outer element of ρ.

If H is a nonempty set and ρ is a binary relation on H, Corsini [8, 9] introduced in H the hypercomposition:

$x*y=\{z\in H:(x,z)\in\rho$ and

$(z,y)\in\rho$ for some $z\in H\}.$ (3)

It is clear that $(H,*)$ is a partial hypergroupoid and it is a hypergroupoid if for each pair of elements $x,y\in H$, there exists $z\in H$ such that $(x,z)\in\rho$ and $(z,y)\in\rho$. Equivalently, $(H,*)$ is a hypergroupoid if and only if $\rho^2=H^2$.

If H_ρ is the hypercompositional structure defined by equation (3) , Massouros and Tsitouras [16] proved the following:

Proposition 2.18. [16] H_ρ is a quasihypergroup if and only if $(x,y)\in\rho$ for all $x,y\in H_\rho$.

Lemma 2.19. *[16] If H_ρ is a semihypergroup and $(z,z)\notin\rho$ for some $z\in H_\rho$, then $(s,z)\in\rho$ implies that $(z,s)\notin\rho$.*

Corrolary 2.20. [16] If H_ρ is a semihypergroup and ρ is not reflexive, then ρ is not symmetric.

Lemma 2.21. *If H_ρ is a semihypergroup then ρ is reflexive.*

Proposition 2.22. [16] *H_ρ is a semihypergroup if and only if $(x,y)\in\rho$ for all $x,y\in H_\rho$.*

Definition 2.23. A hyperoperation $*$ defined through ρ is said to be a total hypercomposition if and only if $(x,y)\in\rho$ for all $x,y\in H_\rho$. In other words, $*$ is said to be a total hypercomposition if $x*y=H_\rho$ for all $x,y\in H_\rho$.

Remark 1. If a hypercompositional structure H_ρ is endowed with the total hypercomposition $*$, then $(H_\rho,*)$ is a hypergroup.

Theorem 2.24. *[16] The only semihypergroup and the only quasihypergroup defined by the binary relation ρ is the total hypergroup.*

If H is a nonempty set and ρ is a binary relation on H, De Salvo and Lo Faro [13, 14] introduced in H the hypercomposition:

$$x\Diamond y=\{z\in H:(x,z)\in\rho$$
$$(x,y)\in\rho \text{ for some } z\in H\}.$$

They characterized the relations ρ which give quasihypergoups, semihypergroups and hypergroups.

3 Neutrosophic Hypercompositional Structures

3.1 Neutrosophic Hypercompositional Structures of Rosenberg Type

Let τ be a binary relation on H(I) and let $\rho=\tau\big|_H$. For all $(a,bI),(c,dI)\in H(I)$, define hypercomposition on H(I) as follows:

$$\begin{aligned}(a,bI)\circ(a,bI)&=\{(x,yI)\in H(I):x\in a\circ a,\\ &\qquad y\in a\circ a\cup b\circ b\}\\ &=\{(x,yI)\in H(I):(a,x)\in\rho,\\ &\qquad (a,y)\in\rho \text{ or } (b,y)\in\rho\}.\end{aligned}$$

(5)

$$\begin{aligned}(a,bI)\circ(c,dI)&=\{(x,yI)\in H(I):x\in a\circ a\cup c\circ c,\\ &\qquad y\in a\circ a\cup b\circ b\cup c\circ c\cup d\circ d\}\\ &=\{(x,yI)\in H(I):(a,x)\in\rho,\\ &\qquad \text{or } (c,x)\in\rho,(a,y)\in\rho\\ &\qquad \text{or } (b,y)\in\rho \text{ or } (c,y)\in\rho \text{ or } (d,y)\in\rho\}.\end{aligned} \quad (6)$$

Let $H(I)_\tau=(H(I),\circ)$ be a hypercompositional structure arising from the hypercomposition defined by equation (6).

Proposition 3.1.1. $H(I)_\tau$ is a hypergroupoid if and only if H_ρ is a hypergroupoid.

Proof. Suppose that H_ρ is a hypergroupoid. Then $H=D(\rho)$ and from equation (6) we have $(a,bI)\circ(c,dI)\subseteq H(I)_\tau$ for all $(a,bI),(c,dI)\in H(I)$. Hence $H(I)_\tau$ is a hypergroupoid. The converse is obvious.

Proposition 3.1.2. $H(I)_\tau$ is a quasihypergroup if and only if H_ρ is a quasihypergroup.

Proof. Suppose that H_ρ is a quasihypergroup. Then $H=D(\rho)=R(\rho)$. Let $(x,yI)\in(a,bI)\circ(c,dI)$ for an arbitrary $(c,dI)\in H(I)$. Then

$$(a,bI)\circ H(I)_\tau=\bigcup\{(a,bI)\circ(c,dI)\}$$

$$= \bigcup \{(x, yI) \in H(I) : (a, x) \in \rho,$$
$$\text{or } (c, x) \in \rho, (a, y) \in \rho$$
$$\text{or } (b, y) \in \rho \text{ or } (c, y) \in \rho \text{ or } (d, y) \in \rho\}.$$
$$= H(I)_\tau$$

Similarly, it can be shown that $H(I)_\tau \circ (a,bI) = H(I)_\tau$ for all $(a,bI) \in H(I)$.

Hence $(H(I)_\tau, \circ)$ is a quasihypergroup. The converse is obvious.

Lemma 3.1.1. If ρ is not reflexive, then $(a,bI) \notin (a,bI) \circ (a,bI)$ for all $(a,bI) \in H(I)$.

Proof. Suppose that ρ is not reflexive and suppose that $(a,bI) \notin (a,bI) \circ (a,bI)$ for all $(a,bI) \in H(I)$. Assuming that $(a,b) \in \rho$, we have from equation (5):

$$(a,bI) \circ (a,bI) = \{(a,bI) \in H(I) : (a,a) \in \rho,$$
$$(a,b) \in \rho \text{ or } (b,b) \in \rho\}$$
$$= \varnothing$$

a contradiction. Hence $(a,bI) \notin (a,bI) \circ (a,bI)$.

Proposition 3.1.3. $H(I)_\tau$ is a semihypergroup if ρ is reflexive and symmetric.

Proof. Suppose that ρ is reflexive and symmetric. Let $(a,bI),(b,aI) \in H(I)$ be arbitrary and let $(x,a) \in \rho$, $(x,b) \in \rho$ and $(y,a) \in \rho$. Then $(b,aI) \in (a,bI) \circ ((b,aI) \circ (a,bI))$ implies that

$$(a,bI) \circ ((b,aI) \circ (a,bI)) = \{(b,aI) \in H(I) : (a,b) \in \rho$$
$$\text{or } (x,b) \in \rho,\ (a,a) \in \rho, (b,a) \in \rho \text{ or}$$
$$(x,a) \in \rho \text{ or } (y,a) \in \rho\}$$
$$= ((a,bI) \circ (b,aI)) \circ (a,bI).$$

This shows that $(b,aI) \in ((a,bI) \circ (b,aI)) \circ (a,bI)$. Since (a,bI) and (b,aI) are arbitrary, it follows that $H(I)_\tau$ is a semihypergroup.

The following results are immediate from the hypercomposition defined by equation (6):

Proposition 3.1.4. (1) $H(I)_\tau$ is a *neutrosophic* hypergroupoid if and only if H_ρ is a hypergroupoid.

(2) $H(I)_\tau$ is a *neutrosophic* semihypergroup if and only if H_ρ is a semihypergroup.

(3) $H(I)_\tau$ is a *neutrosophic* hypergroup if and only if H_ρ is a hypergroup.

3.2 Neutrosophic Hypercompositional Structures of Massouros and Tsitouras Type

Let τ be a binary relation on H(I) and let $\rho = \tau|_H$. For all $(a,bI),(c,dI) \in H(I)$, define hypercomposition on H(I) as follows:

$$(a,bI) \bullet (a,bI) = \{(x,yI) : x \in a \bullet a,$$
$$y \in a \bullet a \cup b \bullet b\}$$
$$= \{(x,yI) : (x,a) \in \rho,$$
$$(y,a) \in \rho \text{ or } (y,b) \in \rho\} \tag{7}$$

$$(a,bI) \bullet (c,dI) = \{(x,yI) : x \in a \bullet a \cup c \bullet c,$$
$$y \in a \bullet a \cup b \bullet b \cup c \bullet c \cup d \bullet d\}$$
$$= \{(x,yI) : (x,a) \in \rho,$$
$$\text{or } (x,c) \in \rho,\ (y,a) \in \rho \text{ or}$$
$$(y,b) \in \rho \text{ or } (y,c) \in \rho \text{ or } (y,d) \in \rho\} \tag{8}$$

$H(I)_\tau = (H(I), \bullet)$ be a hypercompositional structure arising from the hypercomposition defined by equation (8).

Proposition 3.2.1. If ρ is symmetric, then hypercompositional structure $(H(I), \bullet)$ coincide with hypercompositional structure $(H(I), \circ)$.

Proof. This follows directly from equations (6) and (8).

Proposition 3.2.2. $H(I)_\tau$ is a hypergroupoid if and only if H_ρ is a hypergroupoid.

Proof. Suppose that H_ρ is a hypergroupoid. Then $H = R(\rho)$ and from equation (8) we have $(a,bI) \bullet (c,dI) \subseteq H(I)_\tau$ for all $(a,bI),(c,dI) \in H(I)$. Hence $H(I)_\tau$ is a hypergroupoid. The converse is obvious.

Proposition 3.2.3. $H(I)_\tau$ is a quasihypergroup if and only if H_ρ is aquasi hypergroup.

Proof. Suppose that H_ρ is a quasihypergroup. Then $H = D(\rho) = R(\rho)$. Let $(x,yI) \in (a,bI) \bullet (c,dI)$ for an arbitrary $(c,dI) \in H(I)$. Then

$$(a,bI) \bullet H(I)_\tau = \bigcup \{(a,bI) \bullet (c,dI)\}$$
$$= \bigcup \{(x,yI) \in H(I) : (x,a) \in \rho$$
$$\text{or } (x,c) \in \rho,\ (y,a) \in \rho \text{ or}$$
$$(y,b) \in \rho \text{ or } (y,c) \in \rho \text{ or } (y,d) \in \rho\}$$
$$= H(I)_\tau$$

Similarly, it can be shown that $H(I)_\tau \bullet (a,bI) = H(I)_\tau$ for all $(a,bI) \in H(I)$. Hence $H(I)_\tau$ is a quasihypergroup. The converse is obvious.

Lemma 3.2.1. If ρ is not reflexive, then $(a,bI) \notin (a,bI) \bullet (a,bI)$ for all $(a,bI) \in H(I)$.

Proof. The same as the proof of Lemma 3.1.1.

Proposition 3.2.4. $H(I)_\tau$ is a semihypergroup if ρ is reflexive and symmetric.

Proof. This follows from Proposition 3.1.3 and Proposition 3.2.1.

Proposition 3.2.5. (1) $H(I)_\tau$ is a *neutrosophic* hypergroupoid if and only if H_ρ is a hypergroupoid.

(2) $H(I)_\tau$ is a *neutrosophic* semihypergroup if and only if H_ρ is a semihypergroup.

(3) $H(I)_\tau$ is a *neutrosophic* hypergroup if and only if H_ρ is a hypergroup.

3.3 Neutrosophic Hypercompositional Structures of Corsini Type

Let τ be a binary relation on H(I) and let $\rho = \tau|_H$. For all $(a,bI),(c,dI) \in H(I)$, define hypercomposition on H(I) as follows:

$$(a,bI)*(c,dI) = \{(x,yI) \in H(I) : x \in a*a, \; y \in a*d \cup b*c \cup b*d\}$$
$$= \{(x,yI) \in H(I) : (a,x) \in \rho, \text{ and } (x,c) \in \rho, [(a,y) \in \rho \text{ and } (y,d) \in \rho] \text{ or } [(b,y) \in \rho \text{ and } (y,c) \in \rho] \text{ or } [(b,y) \in \rho \text{ and } (y,d) \in \rho]\}. \quad (9)$$

Let $H(I)_\tau = (H(I),*)$ be a hypercompositional structure arising from the hypercomposition defined by equation (9).

Proposition 3.3.1. $H(I)_\tau$ is a hypergroupoid if and only if H_ρ is a hypergroupoid.

Proof. Suppose that H_ρ is a hypergroupoid. Then $H^2=\rho^2$. Since $(a,c),(a,d),(b,c),(b,d) \in \rho^2$ from equation (9), it follows that $(a,bI)*(c,dI) \subseteq H(I)_\tau$ for all $(a,bI),(c,dI) \in H(I)$. Hence $H(I)_\tau$ is a hypergroupoid. The converse is obvious.

Proposition 3.3.2. $H(I)_\tau$ is a quasihypergroup if and only if H_ρ is a quasihypergroup.

Proof. Suppose that H_ρ is a quasihypergroup. Then $(x,y) \in \rho$ for all $x,y \in H$. Let $(x,yI) \in (a,bI)*(c,dI)$ for an arbitrary $(c,dI) \in H(I)$. Then

$$(a,bI)*H(I)_\tau = \bigcup\{(a,bI)*(c,dI)\}$$
$$= \{(x,yI) \in H(I) : (a,x) \in \rho, \text{ and } (x,c) \in \rho, [(a,y) \in \rho \text{ and } (y,d) \in \rho] \text{ or } [(b,y) \in \rho \text{ and } (y,c) \in \rho] \text{ or } [(b,y) \in \rho \text{ and } (y,d) \in \rho]\}.$$
$$= H(I)_\tau$$

Similarly, it can be shown that $H(I)_\tau * (a,bI) = H(I)_\tau$ for all $(a,bI) \in H(I)$. Hence $H(I)_\tau$ is a quasihypergroup. The converse is obvious.

Proposition 3.3.3. $H(I)_\tau$ is a *neutrosophic* quasihypergroup if and only if H_ρ is aquasihypergroup.

Proof. Follows directly from equation (9).

Lemma 3.3.1. If ρ is not reflexive and symmetric, then

(1) $(a,bI) \notin (a,bI)*(a,bI)$ for all $(a,bI) \in H(I)$.

(2) $(b,aI) \notin (a,bI)*(a,bI)$ for all $(a,bI),(b,aI) \in H(I)$.

(3) $(a,aI) \notin (a,bI)*(a,bI)$ for all $(a,aI),(a,bI) \in H(I)$.

(4) $(a,bI) \notin (a,bI)*(a,bI)$ for all $(a,bI),(b,aI) \in H(I)$.

(5) $(b,aI) \notin (a,bI)*(b,aI)$ for all $(a,bI),(b,aI) \in H(I)$.

(6) $(a,aI) \notin (a,bI)*(b,aI)$ for all $(a,aI),(a,bI),(b,aI) \in H(I)$.

Proof. (1) Suppose that ρ is not reflexive and symmetric and suppose that $(a,bI) \notin (a,bI)*(a,bI)$. Then

$$(a,bI)*(a,bI) = \{(a,bI) \in H(I) : (a,a) \in \rho, (b,b) \in \rho \text{ or } [(a,b) \in \rho \text{ and } (b,b) \in \rho] \text{ or } [(b,b) \in \rho \text{ and } (a,b) \in \rho]$$
$$= \varnothing$$

a contradiction. Hence $(a,bI) \notin (a,bI)*(a,bI)$. Using similar argument, (2), (3), (4), (5) and (6) can be established.

Proposition 3.3.4. $H(I)_\tau$ is a semihypergroup if ρ is reflexive and symmetric.

Proof. Suppose that ρ is reflexive and symmetric. Let $(a,bI),(b,aI) \in H(I)$ be arbitrary and let $(x,a) \in \rho$, $(x,b) \in \rho$, $(y,b) \in \rho$ and $(b,a) \in \rho$. Then $(a,bI) \in (a,bI)*((b,aI)*(a,bI))$ implies that

$$(a,bI)*((b,aI)*(a,bI)) = \{(a,bI) \in H(I) : (x,a) \in \rho \text{ and } (a,a) \in \rho, [(x,b) \in \rho \text{ and } (b,b) \in \rho] \text{ or } [(y,a) \in \rho \text{ and } (b,a) \in \rho] \text{ or } [(y,b) \in \rho \text{ and } (b,b) \in \rho]\}$$
$$= ((a,bI)*(b,aI))*(a,bI).$$

This shows that $(b,aI) \in ((a,bI)*(b,aI))*(a,bI)$. Since (a,bI) and (b,aI) are arbitrary, it follows that $H(I)_\tau$ is a semihypergroup.

Corollary 3.3.1. $H(I)_\tau$ is a semihypergroup if and only if H_ρ is a semihypergroup.

Proposition 3.3.5. If any pair of elements of H_ρ does not belong to ρ, then $H(I)_\tau$ is not a semihypergroup.

3.1 Neutrosophic Hypercompositional Structures of De Salvo and Lo Faro Type

Let τ be a binary relation on H(I) and let $\rho=\tau|_H$. For all $(a,bI),(c,dI)\in H(I)$, define hypercomposition on H(I) as follows:

$$(a,bI)\lozenge(c,dI)=\{(x,yI)\in H(I): x\in a\lozenge c, y\in a\lozenge d\cup b\lozenge c\cup b\lozenge d\}$$
$$=\{(x,yI)\in H(I):(a,x)\in\rho, \text{ or } (x,c)\in\rho,(a,y)\in\rho \text{ or } (b,y)\in\rho \text{ or } (y,c)\in\rho \text{ or } (y,d)\in\rho\}. \quad (10)$$

Let $H(I)_\tau=(H(I),\lozenge)$ be a hypercompositional structure arising from the hypercomposition defined by equation (10).

Proposition 3.4.1. If ρ is symmetric, then hypercompositional structures $(H(I),\lozenge)$, $(H(I),\circ)$ and $(H(I),\bullet)$ coincide.

Proof. Follows directly from equations (6), (8) and (10).

Proposition 3.4.2. $H(I)_\tau$ is a hypergroupoid if and only if H_ρ is a hypergroupoid.

Proof. Suppose that H_ρ is a hypergroupoid. Then H=D(ρ) or H=R(ρ) and from equation (10) we have $(a,bI)\lozenge(c,dI)\subseteq H(I)_\tau$ for all $(a,bI),(c,dI)\in H(I)$. Hence $H(I)_\tau$ is a hypergroupoid. The converse is obvious.

Proposition 3.4.3. $H(I)_\tau$ is a quasihypergroup if and only if H_ρ is a quasihypergroup.

Proof. The same as the proof of Proposition 3.2.3.

Lemma 3.4.1. If ρ is not reflexive and symmetric, then

(1) $(a,bI)\notin(a,bI)\lozenge(a,bI)$ for all $(a,bI)\in H(I)$.
(2) $(b,aI)\notin(a,bI)\lozenge(a,bI)$ for all $(a,bI),(b,aI)\in H(I)$.
(3) $(a,aI)\notin(a,bI)\lozenge(a,bI)$ for all $(a,aI),(a,bI)\in H(I)$.
(4) $(a,bI)\notin(a,bI)\lozenge(a,bI)$ for all $(a,bI),(b,aI)\in H(I)$.
(5) $(b,aI)\notin(a,bI)\lozenge(b,aI)$ for all $(a,bI),(b,aI)\in H(I)$.
(6) $(a,aI)\notin(a,bI)\lozenge(b,aI)$ for all $(a,aI),(a,bI),(b,aI)\in H(I)$.

Proof. (1) Suppose that ρ is not reflexive and symmetric and suppose that $(a,bI)\notin(a,bI)\lozenge(a,bI)$. Then

$$(a,bI)\lozenge(a,bI)=\{(a,bI)\in H(I):(a,a)\in\rho, (a,b)\in\rho \text{ or } (b,b)\in\rho \text{ or } (b,a)\in\rho\}$$
$$=\varnothing$$

a contradiction. Hence $(a,bI)\notin(a,bI)\lozenge(a,bI)$. Using similar argument, (2), (3), (4), (5) and (6) can be established.

Proposition 3.4.4. $H(I)_\tau$ is a semihypergroup if ρ is reflexive and symmetric.

Proof. Suppose that ρ is reflexive and symmetric. Let $(a,bI),(b,aI)\in H(I)$ be arbitrary and let $(a,x)\in\rho$, $(b,x)\in\rho$, $(b,y)\in\rho$ and $(a,b)\in\rho$. Then $(a,bI)\in(a,bI)\lozenge((b,aI)\lozenge(a,bI))$ implies that

$$(a,bI)\lozenge((b,aI)\lozenge(a,bI))=\{(a,bI)\in H(I): (a,a)\in\rho \text{ or } (a,x)\in\rho,(a,b)\in\rho \text{ or } (b,y)\in\rho \text{ or } (b,b)\in\rho \text{ or } (b,x)\in\rho\}$$
$$=((a,bI)\lozenge(b,aI))\lozenge(a,bI).$$

This shows that $(a,bI)\in((a,bI)\lozenge(b,aI))\lozenge(a,bI)$. Since (a,bI) and (b,aI) are arbitrary, it follows that $H(I)_\tau$ is a semihypergroup.

The following results are immediate from the hypercomposition defined by equation (10):

Proposition 3.4.5. (1) $H(I)_\tau$ is a *neutrosophic* hypergroupoid if and only if H_ρ is a hypergroupoid.

(2) $H(I)_\tau$ is a *neutrosophic* semihypergroup if and only if H_ρ is a semihypergroup.

(3) $H(I)_\tau$ is a *neutrosophic* hypergroup if and only if H_ρ is a hypergroup.

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